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E-Polynomial of $\mathrm{GL}_n \rtimes <\sigma>$-Character Varieties

Cheng Shu

Thèse de doctorat de mathématiques

Dirigée par **Emmanuel Letellier**

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Résumé

Soit σ l'automorphisme par transpose-inverse de GL_n, qui définit un produit semi-direct $\mathrm{GL}_n \rtimes <\sigma>$. Soit $\tilde{X} \to X$ un revêtement double de surfaces de Riemann, qui est exactement la partie non ramifiée d'un revêtement ramifié de surfaces de Riemann compactes. L'élément non trivial de $\mathrm{Gal}(\tilde{X}/X)$ sera noté τ. A chaque point ramifié enlevé, on associe une $\mathrm{GL}_n(\mathbb{C})$-classe de conjugaison contenue dans la composante connexe $\mathrm{GL}_n(\mathbb{C}).\sigma$, et on exige que la famille $\mathcal{C}$ des classes de conjugaison soient générique. La variété de $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$-caractère que l'on a étudié est l'espace de module des pairs $(\mathcal{L}, \Phi)$ formés d'un système local $\mathcal{L}$ sur $\tilde{X}$ et d'un isomorphisme $\Phi : \mathcal{L} \xrightarrow{\sim} \tau^*\mathcal{L}^\vee$, dont les monodromies autour des points ramifiés sont déterminées par $\mathcal{C}$. On calcule le E-polynôme de cette variété de caractère. A ce fin, on utilise un théorème de Katz, ce qui nous ramème au comptage des points sur corps finis. La formule de comptage fait intervenir les caractères irréductibles de $\mathrm{GL}_n(q) \rtimes <\sigma>$, et donc la table des $\bar{\mathbb{Q}}_\ell$-caractères de ce groupe est déterminée au fur et à mesure. Le polynôme qui en résulte s'exprime comme un produit scalaire de certaines fonctions symétriques associées au produit de couronne $(\mathbb{Z}/2\mathbb{Z})^N \rtimes \mathfrak{S}_N$, avec $N = [n/2]$.

Mots-Clefs: Variété de Caractère; Polynôme de Hodge Mixte; Groupe Fini de Type de Lie; Fonctions Symétrique.

Abstract

Let σ be the transpose-inverse automorphism of GL_n so that we have a semi-direct product $\mathrm{GL}_n \rtimes <\sigma>$. Let $\tilde{X} \to X$ be a double covering of Riemann surfaces, which is exactly the unramified part of a ramified covering of compact Riemann surfaces. The non trivial covering transformation is denoted by τ. To each puncture (removed ramification point), we prescribe a $\mathrm{GL}_n(\mathbb{C})$-conjugacy class contained in the connected component $\mathrm{GL}_n(\mathbb{C}).\sigma$. And we require the collection $\mathcal{C}$ of these conjugacy classes to be generic. Our $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$-character variety is the moduli of the pairs $(\mathcal{L}, \Phi)$, where $\mathcal{L}$ is a local system on $\tilde{X}$ and $\Phi : \mathcal{L} \xrightarrow{\sim} \tau^*\mathcal{L}^\vee$ is an isomorphism, whose monodromy at the punctures are determined by $\mathcal{C}$. We compute the E-polynomial of this character variety. To this end, we use a theorem of Katz and translate the problem to point-counting over finite fields. The counting formula involves the irreducible characters of $\mathrm{GL}_n(q) \rtimes <\sigma>$, and so the $\bar{\mathbb{Q}}_\ell$-character table of $\mathrm{GL}_n(q) \rtimes <\sigma>$is determined along the way. The resulting polynomial is expressed as the inner product of certain symmetric functions associated to the wreath product $(\mathbb{Z}/2\mathbb{Z})^N \rtimes \mathfrak{S}_N$, with $N = [n/2]$.

Key words: Character Varieties; Mixed Hodge Polynomials; Finite Groups of Lie Type; Symmetric Functions.

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Chapter I

Introduction

I.1 Character Varieties

Motivation

Given a connected complex reductive algebraic group G and a Riemann surface X, one defines the associated representation variety as the affine variety

$$\mathrm{Hom}(\pi_1(X), G)$$

consisting of the G-representations of $\pi_1(X)$. Then G acts on it by conjugation on the target. The corresponding GIT quotient is called the character variety. It is a topological invariant of X. We will call G the structure group of the character variety. By Riemann-Hilbert correspondance, the character variety is complex analytically isomorphic to the moduli space of flat connections of principal G-bundles on X. And is further diffeomorphic to the moduli space of semi-stable Higgs G-bundles on X, by the non abelian Hodge correspondance. We regard these objects as the torsors under the constant group scheme $G \times X$ endowed with a Higgs field or a connection.

One can equally consider torsors under a non constant group scheme on X, possibly equipped with flat connections or Higgs fields. Indeed, this is what has been considered in [LN], where torsors under unitary group schemes with Higgs fields are considered, but on a curve defined over a finite field. There has since been growing interest in this kind of quasi-split objects, for example, [PR], [Ze1], [Ze2], [HK]. Representation theory suggests that, when we work in the complex setting, the corresponding character varieties should have a structure group of the form $G \rtimes \Gamma$, where Γ is the Galois group of a finite Galois covering of X such that the non constant group scheme lifts to a constant one.

On the other hand, motivated by the study of some exotic Stockes data, Boalch and Yamakawa have considered the moduli space of twisted Stockes representations. When the Stokes data are trivial, we are left with torsors under a non-constant local system of groups on X. Such local systems of groups are parametrised by group homomorphisms

$\psi_0 : \pi_1(X) \to \operatorname{Aut} G$, with torsors parametrised by the group cohomology $H^1(\pi_1(X), G)$. Let us fix such a ψ_0. If we further assume that ψ_0 factors through a finite quotient, denoted by $\psi : \Gamma \to \operatorname{Aut} G$, then the torsors can be identified with homomorphisms $\pi_1(X) \to G \rtimes \Gamma$, with the semi-direct product defined by ψ.

Galois Invariant Connections

The finite quotient Γ determines a finite Galois cover $\tilde{X} \to X$, and the homomorphisms $\rho : \pi_1(X) \to G \rtimes \Gamma$ can be restricted to the usual representations $\tilde{\rho} : \pi_1(\tilde{X}) \to G$ via the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\tilde{X}) \arrow[r] \arrow[d, "\tilde{\rho}"] & \pi_1(X) \arrow[r] \arrow[d, "\rho"] & \Gamma \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & G \arrow[r] & G \rtimes \Gamma \arrow[r] & \Gamma \arrow[r] & 1
\end{tikzcd}.$$

We will call $\tilde{\rho}$ the *underlying G-representation* of ρ. The representation variety that we are interested in is exactly the moduli space of those homomorphisms $\pi_1(X) \to G \rtimes \Gamma$ that make (the right hand side of) the above diagram commute, and its elements are called $G \rtimes \Gamma$-representations. The conjugation of G on $G \rtimes \Gamma$ induces an action on this variety and the corresponding GIT quotient is called the $G \rtimes \Gamma$-character variety. Instead of the usual G-character varieties associated to a topological space, we have the slogan

$G \rtimes \Gamma$-character varieties are associated to a Γ-Galois cover $\tilde{X}/X$.

There can be non isomorphic coverings with isomorphic Galois group Γ.

In fact, in the above diagram, the usual representation $\tilde{\rho}$ is *Galois invariant*. This is best understood in terms of flat connections. Suppose $(\mathcal{E}, \nabla)$ is a principal G-bundle on $\tilde{X}$ equipped with a flat connection, then with the homomorphism $\psi : \Gamma \to \operatorname{Aut} G$, each element of $\sigma \in \Gamma$ defines a twist of $(\mathcal{E}, \nabla)$, denoted by $(\sigma^*\mathcal{E}^\sigma, \nabla^\sigma)$, where the underlying bundle is obtained by first pulling back $\mathcal{E}$ by σ and then twisting the fibres by $\psi(\sigma)$, i.e. the right action of G on the fibres is twisted by this group automorphism. Fixing ψ, we say that $(\mathcal{E}, \nabla)$ is Γ-invariant if for each $\sigma \in \Gamma$, there is an isomorphism

$$\Phi_\sigma : (\mathcal{E}, \nabla) \to (\sigma^*\mathcal{E}^\sigma, \nabla^\sigma). \tag{I.1}$$

These isomorphisms must satisfy some cocycle conditions. Let us denote by Φ_* this family of isomorphisms. Our first result basically says

Theorem 1. *There is a one to one correspondence*

$$\begin{aligned} \{\Gamma\text{-invariant flat connections on } \tilde{X}\} &\longleftrightarrow \{G \rtimes \Gamma\text{-representations of } \pi_1(X)\} \\ ((\mathcal{E}, \nabla), \Phi_*) &\longleftrightarrow \rho \end{aligned}$$

And the underlying G-representation of ρ corresponds to $(\mathcal{E}, \nabla)$.

The proof is divided into Theorem III.1.2.1, Lemma III.3.1.1 and Proposition III.3.1.3.

Some Technical Issues

Since our character varieties are defined as the GIT quotient of some affine varieties, the closed orbits and stable orbits are of particular importance. As in the case of usual character varieties, they consist exactly of *semi-simple* representations and *irreducible* representations. The semi-simple and irreducible $G \rtimes \Gamma$-representations are defined in exactly the same way as the case of usual G-representations. More precisely, a representation $\rho : \pi_1(X) \to G \rtimes \Gamma$ is semi-simple if the Zariski closure of its image is a completely reducible subgroup of $G \rtimes \Gamma$ and is irreducible if the Zariski closure of its image is an irreducible subgroup of $G \rtimes \Gamma$. We recall the relevant notions for non-connected groups in §1.

Observe that the underlying G-representation of an irreducible $G \rtimes \Gamma$-representation is always semi-simple but may not be irreducible. We would like to restrict ourselves to those character varieties such that the subset $\{\rho \mid \tilde{\rho} \textit{ is irreducible}\}$ is non-empty. This gives rise to a classification problem. To see this, suppse that $\tilde{X} \to X$ is a double covering and $\psi : \Gamma \cong \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut} G$ sends the nontrivial element σ to the transpose inverse automorphism of $G = \mathrm{GL}_n(\mathbb{C})$. Let $\mathcal{E}$ be a stable vector bundle on $\tilde{X}$ and $\Phi_\sigma : \mathcal{E} \to \sigma^*\mathcal{E}^\sigma$ an isomorphism. Then the composition

$$(\sigma^*\Phi_\sigma)^\sigma \circ \Phi_\sigma \tag{I.2}$$

where the superscript σ means the induced isomorphism on the contragradient vector bundle, is an automorphism of $\mathcal{E}$ and is thus a homothety. It either equals to $+1$ or -1 due to σ-invariance. Both cases are studied in [Ze2]. Note that for a fixed $\mathcal{E}$, different choices of the isomorphism Φ_σ give the same value of $(\sigma^*\Phi_\sigma)^\sigma \circ \Phi_\sigma$, and it only depends on $\mathcal{E}$. Therefore the signs ± 1 classify different σ-invariant stable vector bundles. This is a particular case of the classification of irreducible (resp. stable) Γ-invariant G-representations (resp. flat connections) in terms of the group cohomology $H^2(\Gamma, Z_G)$, where Z_G is the centre of G. For example in the context above, the non trivial element of Γ acts as $x \mapsto x^{-1}$ on Z_G, and the sign $+1$ corresponds to the trivial cohomology class of $H^2(\Gamma, Z_G)$ and -1 corresponds to the other class. This classification results have already been obtained by Schaffhauser in [Sch].

The irreducibility of $G \rtimes \Gamma$-representations is translated into the stability condition for Γ-invariant flat connections on $\tilde{X}$. We thus morally recover the stability condition of Γ-bundles as defined by Seshadri [Ses] and the stability condition of *anti-invariant* bundles as defined by Zelaci [Ze2]. More precisely, we will show

Proposition 2. *(See Proposition III.3.2.2) Under the correspondence in Theorem 1, a $G \rtimes \Gamma$-representation is irreducible if and only if the corresponding Γ-invariant connection is stable.*

Let $G = \mathrm{GL}_n(\mathbb{C})$, $\Gamma = \mathbb{Z}/2\mathbb{Z}$, and the non-trivial element act as σ, the transpose inverse automorphism of G. By studying some certain irreducible subgroups of $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$, we

find a similar result of [Ze2, §4.1] and [Ra, Proposition 4.5], which asserts (See Proposition II.5.4.1, Proposition III.2.1.7, Remark III.2.1.8 and Remark III.3.1.2)

Proposition 3. *If $(\mathcal{E}, \nabla), \Phi_*)$ is a stable Γ-invariant flat connection on $\tilde{X}$, then its underlying flat connection $(\mathcal{E}, \nabla)$ is semi-simple with pairwise non-isomorphic factors, each one being Γ-invariant with respect to the restrction of Φ_*.*

An Important Example

Perhaps the most important examples of such character varieties are the $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$ character varieties of the fundamental groups of Riemann surfaces, where σ is an order 2 non trivial exterior automorphism of $\mathrm{GL}_n(\mathbb{C})$. Such character varieties are associated to an unramified double cover $\tilde{X} \to X$ of Riemann surfaces. And if σ is the transpose inverse automorphism, then we expect that they correspond to the moduli spaces of torsors of unitary group scheme equipped with Higgs fields. Suppose $\tilde{X} \to X$ is exactly the unramified part of a ramified double cover $\tilde{X}' \to X'$, then by restricting the monodromy around the punctures (removed ramification points) to some G-conjugacy classes of $G.\sigma$, or equivalently σ-conjugacy classes of G, denoted by $(C_j)_j$, the character variety can be written as

$$\{(A_i, B_i)_i \times (X_j)_j \in G^{2g} \times \prod_j C_j \mid \prod_{i=1}^{g} (A_i, B_i) \prod_j X_j = 1\}, \tag{I.3}$$

where g is the genus of X. Note that there are necessarily an even number of ramification points so that the product falls into the identity component. Let us remark that

> *If all of the conjugacy classes C_j are the conjugacy class of σ (in which case C_j is isomorphic to the symmetric space), this can be regarded as the $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$-character variety associated to the ramified covering $\tilde{X}' \to X'$, i.e. it parametrises Galois invariant local systems on $\tilde{X}'$.*

This is explained in §III.5.2.3.

The natural symplectic structure of these varieties can be deduced from the quasi-Hamiltonian reduction procedure, but in the twisted setting as in [BY]. The dimension of these character varieties with *generic conjugacy classes* can also be easily obtained by regarding the variety as fusion from small building blocks. In the above example, since the center of $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$ has dimension 0, the dimension of the character variety is

$$(2g-2)n^2 + \sum_j \dim C_j. \tag{I.4}$$

The notion of generic conjugacy classes is defined in §III.4.2, and is a natural generalisation of the tame case of [B, Corollary 9.7, Corollary 9.8]. Any $G \rtimes \Gamma$-representation with generic conjugacy classes is necessarily irreducible.

E-Polynomials and Related Works

The above variety looks very much the same as the usual character varieties, and its cohomology, or more specifically the E-polynomial, can be calculated following the method developped in [HLR], which involves point-counting over finite fields.

The E-polynomial is a specialisation of the mixed Hodge polynomial and can be specialised to the Euler characteristic. For character varieties, the Poincaré polynomial only overlaps with E-polynomial at the level of Euler characteristic. The early investigation of the E-polynomials of $M_{dR}(X, G)$, the moduli space of flat connections on Riemann surfaces X with G a connected reductive group, was motivated by Mirror symmetry and the speculation was that the stringy E-polynomial of $M_{dR}(X, G)$ should agree with that of $M_{dR}(X, G^\vee)$, where $G^\vee$ is the dual group. Then it was expected that the character variety version of this is also true.

The computation of the usual $\mathrm{GL}_n(\mathbb{C})$-character varieties shows that the E-polynomial satisfies $q^d E(q^{-1}) = E(q)$, where d is the dimension of the character variety. This had led to the discovery of an interesting symmetry in the weight filtration of the cohomology, called the curious Hard Lefschetz, which has only recently been proved by A. Mellit. This symmetry resembles the Hard Lefschetz theorem, but holds for the character variety which is affine. Then it was observed that this symmetry behaves like the relative Hard Lefschetz for the perverse filtration on the Higgs bundles side. It is now known as the P=W conjecture, which claims the identification via non abelian Hodge correspondence of perverse filtration for Higgs moduli and weight filtration for character varieties, in a suitable sense. The curious Hard Lefschetz is also a main ingradient in formulating a conjectural formula of the mixed Hodge polynomial of the character varieties. The resulting combinatorial formula involves the Macdonald polynomials.

Both the P=W conjecture and the conjectural mixed Hodge polynomial have been verified in small ranks with the help of explicit knowledge of the cohomology ring the moduli spaces. In our case, there is no existing knowledge of the cohomology ring and we cannot give any conjecture with supporting evidence. We will instead focus on the computation of the E-polynomials.

Point-Counting

The computation relies on a theorem of Katz, which translates the computation of E-polynomials to point-counting over finite fields. The theorem basically says the following. One first finds a finitely generated ring R contained in $\mathbb{C}$, and an R-model of the character variety. By base change to finite field $\mathbb{F}_q$, we can count the number of solutions of (V.0.0.0.1) in finite fields. If this gives a polynomial in q, then this is also the E-polynomial.

The point-counting formula in our case is the following,

$$\sum_{\chi\in\mathrm{Irr}(G)^\sigma}\left(\frac{|G|}{\chi(1)}\right)^{2g-2}\prod_{i=1}^{2k}\frac{|C_i|\tilde{\chi}(C_i)}{\chi(1)}, \tag{I.5}$$

where $G = \mathrm{GL}_n(q)$, $C_i \subset \mathrm{GL}_n(q).\sigma$, and $\mathrm{Irr}(G)^\sigma$ is the set of σ-stable irreducible characters of $\mathrm{GL}_n(q)$ and $\tilde{\chi}$ is an extension of such a character to $\mathrm{GL}_n(q)\rtimes<\sigma>$. We will explain the relevant notions in the next part of this introduction.

Our final formula for the E-polynomial is expressed as the inner product of two symmetric functions. This theorem is only proved for n odd with mild restrictions on the "eigenvalues" of C_i's, and we expect that the formula for n even is slightly different.

Theorem 4. *Assume that $n = 2N + 1$ for some $N \in \mathbb{Z}_{>0}$ and that C_i has no "eigenvalue" equal to $\sqrt{-1}$ for all i. Let d be the dimension of the character variety and let β_j be the type (encoding the multiplicities of "eigenvalues") of the semi-simple conjugacy class C_j. Then,*

$$|\mathrm{Ch}_C(\mathbb{F}_q)| = q^{\frac{1}{2}d-k(N+1)}\left\langle\frac{\mathfrak{O}(\mathbf{x},q)_1\mathfrak{O}(\mathbf{x},q)_0}{\tilde{\mathfrak{O}}(\mathbf{x},q)},\prod_{j=1}^{2k}h_{\beta_j^*}(\mathbf{x}_j)\right\rangle. \tag{I.6}$$

where $\mathfrak{O}(\mathbf{x},q)_0$, $\mathfrak{O}(\mathbf{x},q)_1$ and $\tilde{\mathfrak{O}}(\mathbf{x},q)$ are certain symmetric functions in the variables $\{\mathbf{x}_j\}_{1\le j\le 2k}$, and $h_{\beta_j^*}(\mathbf{x}_j)$ are the complete symmetric functions.

The symmetric functions $\mathfrak{O}(\mathbf{x},q)_+$, $\mathfrak{O}(\mathbf{x},q)_-$ and $\tilde{\mathfrak{O}}(\mathbf{x},q)$ only depend on the genus g and the number of punctures in the Riemann surface (that is, only depend on on the topology of the Riemann surface). Note that each infinite variable $\mathbf{x}_j$ should be understood as a pair $(\mathbf{x}_j^{(0)},\mathbf{x}_j^{(1)})$ of infinite variables since the underlying symmetry is the wreath product $(\mathbb{Z}/2\mathbb{Z})^n\rtimes\mathfrak{S}_n$.

I.2 Character Table of $\mathrm{GL}_n(q)\rtimes<\sigma>$

Irreducible Characters of $\mathrm{GL}_n(q)$

Suppose $G = \mathrm{GL}_n(k)$. Given a Frobenius of G, the associated finite group $G(q)$ is $\mathrm{GL}_n(q)$ or $\mathrm{GL}_n^-(q)$ according to whether the action of F on the Dynkin diagram is trivial or not. The $\bar{\mathbb{Q}}_\ell$-character table of $\mathrm{GL}_n(q)$ is well known since the work of Green [Gr]. Instead of the combinatorial point of view of Green, we present below a parametrisation of the irreducible characters due to Lusztig and Srinivasan, which is convenient for the problem of extending characters to $\mathrm{GL}_n(q).<\sigma>$.

For each F-stable Levi subgroup L, we denote by $\mathrm{Irr}_{reg}(L^F)$ the set of regular linear characters of L^F (See [LS, §3.1]), and denote by $\mathrm{Irr}(W_L)^F$ the set of F-stable irreducible characters of the Weyl group $W_L = W_L(T)$, $T \subset L$ being an F-stable maximal torus. We take $\theta \in \mathrm{Irr}_{reg}(L^F)$

and $\varphi \in \mathrm{Irr}(W_L)^F$. For each φ, we denote by $\tilde{\varphi}$ an extension of φ to $W_L \rtimes <F>$. We put

$$R_\varphi^G \theta = \epsilon_G \epsilon_L |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^G \theta, \tag{I.7}$$

where $\epsilon = (-1)^{rk}$ and rk is the $\mathbb{F}_q$-rank of the algebraic group concerned, and $R_T^G\theta$ is the Deligne-Lusztig induction of (T, θ).

Theorem 5. *(Lusztig, Srinivasan, [LS, Theorem 3.2]) Let $G = \mathrm{GL}_n^{\pm}(q)$. For some choice of $\tilde{\varphi}$, the virtual character $R_\varphi^G\theta$ is an irreducible character of G^F. Moreover, all irreducible characters of G^F are of the form $R_\varphi^G\theta$ for a triple (L, φ, θ). The characters associated to the triples (L, φ, θ) and (L', φ', θ') and distinct if and only if one of the following conditions is satisfied*

- *(L, θ) and (L', θ') are not G^F-conjuguate;*
- *$(L, \theta) = (L', \theta')$ and $\varphi \neq \varphi'$.*

Therefore, the calculation of the values of the irreducible characters of $\mathrm{GL}_n(q)$ is reduced to the calculation of the values of Deligne-Lusztig characters, i.e. virtual characters of the form $R_T^G\theta$.

Clifford Theory

Let σ be an automorphism of order 2 of GL_n. It defines a semi-direct product $\mathrm{GL}_n(q) \rtimes \mathbb{Z}/2\mathbb{Z}$. This group will be denoted by $\mathrm{GL}_n(q) \rtimes <\sigma>$ (or simply $\mathrm{GL}_n(q).<\sigma>$) in order to specify the action of $1 \in \mathbb{Z}/2\mathbb{Z}$. We will assume that σ is an exterior automorphism. Regarded as element of this non-connected group, $\sigma = (\mathrm{Id}, 1)$ satisfies $\sigma^2 = 1$ and $\sigma g \sigma^{-1} = \sigma(g)$, for all $g \in \mathrm{GL}_n(q)$.

The representations of $\mathrm{GL}_n(q) \rtimes <\sigma>$ are related to the representations of $\mathrm{GL}_n(q)$ by the Clifford theory in the following way. Let H be a finite group and let N be a normal subgroup of H such that $H/N \simeq \mathbb{Z}/r\mathbb{Z}$ with r prime, and let let χ be an irreducible character of H. We denote by χ_N the restriction of χ to N. Then

- Either χ_N is irreducible;
- Or $\chi_N = \bigoplus_i^r \theta_i$, where $\theta_i \in \mathrm{Irr}(N)$ are some distinct irreducible characters.

Moreover, the θ_i's form an orbit under the action of H/N on $\mathrm{Irr}(N)$. Conversely, $\chi_N \in \mathrm{Irr}(N)$ extends to an irreducible character of H if and only if it is invariant under the action of H/N by conjugaction. If χ is such an extension, we obtain all other extensions by multiplying χ by a character of H/N.

Denote by $\mathrm{Irr}(\mathrm{GL}_n(q))^\sigma$ the set of σ-stable irreducible characters, i.e. those satisfying $\chi = \chi \circ \sigma$. The irreducible characters of $\mathrm{GL}_n(q) \rtimes <\sigma>$ are either an extension of a character $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))^\sigma$ or an extension of $\chi \oplus^\sigma \chi$ with χ a non σ-stable character of $\mathrm{GL}_n(q)$. Note that

the extension of $\chi \oplus {}^{\sigma}\chi$ for χ non σ-stable vanishes on the component $\mathrm{GL}_n(q).\sigma$, whereas two extensions of $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))^{\sigma}$ differ by a sign on $\mathrm{GL}_n(q).\sigma$. Their values on $\mathrm{GL}_n(q)$ are then given by the character table of $\mathrm{GL}_n(q)$. Once we fix an extension $\tilde{\chi}$ for all $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))^{\sigma}$, it remains for us to calculate the restriction of $\tilde{\chi}$ on $\mathrm{GL}_n(q).\sigma$. If no confusion arises, we will also say that $\tilde{\chi}|_{\mathrm{GL}_n(q).\sigma}$ is an extension of χ to $\mathrm{GL}_n(q).\sigma$.

The conjugacy classes of $\mathrm{GL}_n(q).<\sigma>$ consist of the conjugacy classes of $\mathrm{GL}_n(q)$ that are stable under σ, of the unions of pairs of conjugacy classes of the form $(C, \sigma(C))$, with $C \subset \mathrm{GL}_n(q)$ non σ-stable, and of the conjugacy classes contained in $\mathrm{GL}_n(q).\sigma$. From the equality

$$\begin{aligned}
&\#\sigma\text{-stable classes} + \frac{1}{2}\#\text{non } \sigma\text{-stable classes} + \#\text{classes in } \mathrm{GL}_n(q).\sigma \\
&=\#\text{classes of } \mathrm{GL}_n(q).<\sigma> \\
&=\#\text{irreducible characters of } \mathrm{GL}_n(q).<\sigma> \\
&=2\#\sigma\text{-stable characters} + \frac{1}{2}\#\text{non } \sigma\text{-stable characters}
\end{aligned} \tag{I.8}$$

and from the fact that $\#\sigma$-stable classes = $\#\sigma$-stable characters, we deduce that

$$\#\text{classes contained in } \mathrm{GL}_n(q).\sigma = \#\sigma\text{-stable characters.} \tag{I.9}$$

So the table that we are going to calculate is a square table, its lines and columns being indexed by the σ-stable irreducible characters of $\mathrm{GL}_n(q)$ and the conjugacy classes in $\mathrm{GL}_n(q).\sigma$ respectively. However, there is no natural bijection between the classes and the characters.

Deligne-Lusztig Induction for $\mathrm{GL}_n(q) \rtimes <\sigma>$

Let σ be as above and let $\rho : \mathrm{GL}_n(q) \to V$ be a σ-stable irreducible representation. Defining an extension of ρ, say $\tilde{\rho}$, is to define an action of σ on V in such a way that $\tilde{\rho}(\sigma)^2 = \mathrm{Id}$ and that $\tilde{\rho}(\sigma)\rho(g)\tilde{\rho}(\sigma)^{-1} = \rho(\sigma(g))$ for all $g \in \mathrm{GL}_n(q)$. Except in some particular cases, we do not know how to do it. However, when σ is *quasi-central*, we have a natural action of σ on the Deligne-Lusztig varieties X_w associated to $w \in W^{\sigma}$, the subgroup of σ-fixed elements of $W = W_G(T_0)$, with T_0 being a σ-stable and F-stable maximal torus of G. This allows us to define the extensions of the Deligne-Lusztig characters $R_{T_w}^G 1$ to $\mathrm{GL}_n(q).<\sigma>$. By expressing a unipotent character of $\mathrm{GL}_n(q)$ as a linear combination of these Deligne-Lusztig characters, we can obtain an extension of this unipotent character. More concretely, if we take an F-stable and σ-stable Borel subgroup $B_0 \subset G$, the variety X_w consists of the Borel subgroups B such that $(B, F(B))$ are conjugate to $(B_0, \dot{w}B_0\dot{w}^{-1})$ by G, where $\dot{w}$ is a representative of $w \in W^{\sigma}$ in G which can be chosen to be σ-stable. The action of σ on X_w is just $B \mapsto \sigma(B)$, which induces an action on the cohomology. The character $R_{T_w}^G 1$ thus extends into the function

$$g\sigma \mapsto \mathrm{Tr}(g\sigma | H_c^*(X_w, \bar{\mathbb{Q}}_\ell)),$$

denoted by $R^{G.\sigma}_{T_w.\sigma}1$. This is a particular case of the Deligne-Lusztig induction for non-connected reductive groups developed by Digne and Michel [DM94]. More generally, given an F-stable and σ-stable Levi factor of a σ-stable parabolic subgroup, we have the maps $R^{G.\sigma}_{L.\sigma}$, that sends L^F-invariant functions on $L^F.\sigma$ to G^F-invariant functions on $G^F.\sigma$.

Each irreducible character χ of G^F is induced from an irreducible character χ_L of L^F, where L is an F-stable Levi subgroup as in the setting of Theorem 5. If L is moreover an σ-stable factor of some σ-stable parabolic subgroup, and χ_L is a σ-stable character of L^F, then χ is also σ-stable. Suppose that we know how to calculate $\tilde{\chi}_L$, an extension of χ_L to $L^F.\sigma$, then the character formula will allow us to calculate the values of $R^{G.\sigma}_{L.\sigma}\tilde{\chi}_L$, which coincides with the values of an extension of χ on $G^F.\sigma$.

Quadratic-Unipotent Characters

However, there exist some σ-stable characters of G^F that can not be obtained by the above procedure. Let us look at some examples for $n = 2, 3$ and 4. We take as σ the automorphism $g \mapsto \mathcal{J}_n {}^t g^{-1} \mathcal{J}_n^{-1}$ with $g \in \mathrm{GL}_n(k)$, where

$$\mathcal{J}_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},\ \mathcal{J}_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix},\ \mathcal{J}_4 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}. \tag{I.10}$$

If $G = \mathrm{GL}_2(k)$, and T is the maximal torus consisting of diagonal matrices, 1 the trivial character of $\mathbb{F}_q^*$, η the order 2 irreducible character of $\mathbb{F}_q^*$, and if we denote by θ the character $(1, \eta)$ of $T^F \cong \mathbb{F}_q^* \times \mathbb{F}_q^*$, then one can verify that $R^G_T\theta$ is an σ-stable irreducible character while $\theta \in \mathrm{Irr}(T^F)$ is not σ-stable.

Besides, a priori, the map $R^{G.\sigma}_{L.\sigma}$ is not defined for $L^F.<\sigma>$, but for the normaliser $N_{G.<\sigma>}(L, P)$ (the set of elements that simultaneously normalise L and P). If L is a σ-stable Levi factor of a σ-stable parabolic subgroup P, then $N_{G.<\sigma>}(L, P) = L.<\sigma>$, otherwise, the two groups are not the same, as the following examples show. In fact, what really matters is whether $N_{G.<\sigma>}(L, P)$ meets the connected component $G.\sigma$.

If $G = \mathrm{GL}_4(k)$ and $L = C_G(t)$ with $t = \mathrm{diag}(1, -1, -1, 1)$, then $L \cong \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ and $\theta := (\mathrm{Id} \circ \det, \eta \circ \det)$ is a σ-stable irreducible character of L^F and thus induces a σ-stable irreducible character of $\mathrm{GL}_4(q)$. However, L is not a σ-stable Levi factor of a σ-stable parabolic subgroup, because otherwise $L^\sigma \cong \mathrm{SL}_2(k) \times \mathrm{SL}_2(k)$ would be a Levi subgroup of $G^\sigma \cong \mathrm{Sp}_4(k)$. In this case, $N_{G.<\sigma>}(L, P) = L \cup Ln\sigma$, where $n \in N_G(L)$ permutes the two components of L. So we are back to the previous example, i.e. θ is not $n\sigma$-stable.

Now we take $L = \mathrm{GL}_3(k) \times k^*$ and a character with semi-simple part (Id, η) with respect to this direct sum. In this case, L itself is not σ-stable. In fact, it is not conjugate to any σ-stable Levi subgroup. So $N_{G.<\sigma>}(L, P) \subset G$, no matter which parabolic subgroup is P.

The above examples are typical. Let $L = G_1 \times G_2$ be a Levi subgroup of G, where

$G_1 \cong \mathrm{GL}_m(k)$ and $G_2 \cong \mathrm{GL}_{n-m}(k)$. Let χ_1(resp. χ_2) a unipotent irreducible character of G_1^F(resp. G_2^F). The character $\chi_L := \chi_1 \otimes \chi_2 \eta$ (or $\chi_1 \eta \otimes \chi_2$) always induces a σ-stable irreducible character of G^F, where we regard η as a central character of G_1^F or G_2^F. But L does not fit into Deligne-Lusztig theory for non-connected groups: either L is not conjugate to any σ-stable Levi factor of σ-stable parabolic subgroups, or χ_L is not a σ-stable character of L^F. The irreducible characters of $\mathrm{GL}_n(q)$ of the form χ_L as above are called quadratic-unipotent. They are parametrised by the 2-partitions of n. Their extensions to $\mathrm{GL}_n(q).<\sigma>$ have been computed by J.-L.Waldspurger by using character sheaves for non-connected groups. The main result is as follows.

Let (μ_+, μ_-) be a 2-partition of n, to which is associated the data $(\varphi_+, \varphi_-, h_1, h_2)$, where φ_+(resp. φ_-) is an irreducible character, determined by the 2-quotient of μ_+(resp. μ_-), of the Weyl group $\mathfrak{W}_+$(resp. $\mathfrak{W}_-$) of type C_{N_+}(resp. C_{N_-}), while h_1 and h_2 are two non negative integers related to the 2-cores of μ_+ and μ_-. We have $n = 2N_+ + 2N_- + h_1(h_1+1) + h_2^2$.

Theorem 6 (Waldspurger)**.** *The extension of the quadratic-unipotent character of* $\mathrm{GL}_n(q)$ *associated to* (μ_+, μ_-) *is given up to a sign by*

$$R_\varphi^{G.\sigma} = \frac{1}{|\mathfrak{W}_+|}\frac{1}{|\mathfrak{W}_-|}\sum_{\substack{w_+\in\mathfrak{W}_+\\ w_-\in\mathfrak{W}_-}} \varphi_+(w_+)\varphi_-(w_-)R_{L_{\mathbf{w}}.\sigma}^{G.\sigma}\phi_{\mathbf{w}}. \tag{I.11}$$

In the above expression, $L_{\mathbf{w}}$ is a σ-stable and F-stable Levi factor, isomorphic to the product $\mathrm{GL}_{h_1(h_1+1)+h_2^2}(k) \times T_{w_+} \times T_{w_-}$ with $w_\pm \in \mathfrak{W}_\pm$, of a σ-stable parabolic subgroup, and $\phi_{\mathbf{w}}$ is some kind of "tensor product" of $\phi(h_1, h_2)$, $\tilde{1}$ and $\tilde{\eta}$, where $\phi(h_1, h_2)$ is a cuspidal function on $\mathrm{GL}_{h_1(h_1+1)+h_2^2}(k).\sigma$ which is supported on an isolated conjugacy class; $\tilde{1}$(resp. $\tilde{\eta}$) is the extension of the linear character 1(resp. η) of $T_{w_+}^F$(resp. $T_{w_-}^F$). If μ_- is the empty partition and μ_+ has trivial 2-core or the 2-core (1) according to the parity of n, then $L_{\mathbf{w}}$ becomes a σ-stable maximal torus and $\mathfrak{W}_+$ is isomorphic to W_G^σ, the σ-fixed subgroup of the Weyl group of G.

Parametrisation of σ-Stable Characters of $\mathrm{GL}_n(q)$

A general σ-stable irreducible character is the product of a quadratic-unipotent component and a component that looks like induced from an σ-stable Levi factor of a σ-stable parabolic subgroup. See Proposition IV.1.2.2 and Proposition IV.1.2.3 for the details.

Suppose that χ is a σ-stable irreducible character corresponding to (M, θ, φ) as in Theorem 5. We write $\theta = (\alpha_i)_i$ with respect to the the decomposition of M^F into a product of some $\mathrm{GL}_{n_i}(q^{r_i})$'s, where α_i are some characters of $\mathbb{F}_{q^{r_i}}^*$ and we have omitted the determinant map from the notation. It is easy to see that the action of σ sends χ to the character associated to $(\sigma(M), \sigma_*\theta, \sigma_*\varphi)$. According to the parametrisation of the irreducible characters of $\mathrm{GL}_n(q)$, there exists some

$$g \in N_{G^F}(\sigma(M), M) = \{x \in G^F \mid x\sigma(M)x^{-1} = M\}$$

such that $\sigma_*\theta = \mathrm{ad}^* g\theta$. Note that the value of α_i only depends on the determinant of the corresponding factor. The action of σ inverts the determinant while the conjugation by g does not change the determinant. We can then conclude that for each α_i, its inverse α_i^{-1} is also a factor of θ. The factors satisfying $\alpha_i^{-1} = \alpha_i$ form the quadratic-unipotent part.

Fix a σ-stable maximal torus T contained in a σ-stable Borel subgroup $B \subset G$ and identify the simple roots with the Dynkin diagram of G. If I is a σ-stable subdiagram of the Dynkin diagram of G, then it defines a σ-stable Levi factor L_I of some σ-stable standard parabolic subgroup containing B. It has at most one σ-stable component, denote by L_0, and so $L_I \cong L_0 \times L_1$ so that σ non-trivially permutes the factors of L_1. All σ-stable standard Levi subgroup corresponds to such an I. We associate a quadratic-unipotent character χ_0 to L_0^F, and a pair of characters, with semi-simple parts α_i and α_i^{-1} respectively, to each pair of components of L_1 that are exchanged by σ. By defining the unipotent parts of the character in a way compatible with the action of σ, we obtain a σ-stable character of L_1^F, denoted by χ_1. Then $R_L^G(\chi_0 \otimes \chi_1)$ is a σ-stable irreducible character of G^F (for suitable α_i's), and all σ-stable irreducible characters of $\mathrm{GL}_n(q)$ are obtained this way. We will calculate the extension of $\chi_L = \chi_0 \otimes \chi_1$ to $L^F.\sigma$, and then apply the map $R_{L.\sigma}^{G.\sigma}$. Note that if we regard χ as induced from (M, θ, φ) following Theorem 5, then M is not necessarily σ-stable.

Extensions of σ-Stable Characters

The extension of the quadratic -unipotent part determined, the problem is reduced to the following.

Problem. Put $L_1 = G_0 \times G_0$, $G_0 = \mathrm{GL}_m(k)$, and let σ_0 be an automorphism of G_0 of order 2. Denote by F_0 the Frobenius of $\mathrm{GL}_m(k)$ that sends each entry to its q-th power. Define an automorphism σ of L_1 by

$$(g, h) \longmapsto (\sigma_0(h), \sigma_0(g)), \tag{I.12}$$

and a Frobenius F by

- Linear Case: $(g, h) \longmapsto (F_0(g), F_0(h))$,
- Unitary Case: $(g, h) \longmapsto (F_0(h), F_0(g))$.

The problem is to decompose the extension of a σ-stable irreducible character of L_1^F to $L_1^F.<\sigma>$ as a linear combination of Deligne-Lusztig characters.

Let us first look at the linear case. We have $L_1^F = G_0^{F_0} \times G_0^{F_0}$. Let χ be a unipotent character of $G_0^{F_0}$. Then $\chi \boxtimes \chi \in \mathrm{Irr}(L_1^F)$ is σ-stable. In order to calculate its extension, we separate σ into two automorphisms, one sending (g, h) to $(\sigma_0(g), \sigma_0(h))$, the other one, denoted by τ, sending (g, h) to (h, g). Denote by $\tilde{\chi}$ an extension of χ to $G_0^{F_0}<\sigma_0>$. Consider the τ-stable character $\tilde{\chi} \boxtimes \tilde{\chi}$ of $G_0^{F_0}<\sigma_0> \times G_0^{F_0}<\sigma_0>$. Its extension to $(G_0^{F_0}<\sigma_0> \times G_0^{F_0}<\sigma_0>) \rtimes <\tau>$ restricts to an irreducible

character $\bar{\chi}$ of $L_1^F.<\sigma>$, regarded as a subgroup of $(G_0^{F_0}<\sigma_0>\times G_0^{F_0}<\sigma_0>)\rtimes<\tau>$. This gives an extension of $\chi\otimes\chi$. Some linear algebra calculation shows that $\bar{\chi}((g,h)\sigma)=\chi(g\sigma_0(h))$. The latter is the value of a character of $\mathrm{GL}_n(q)$.

The unitary case is a little more complicated and relies on the result of the linear case. In this case, $L_1^F\cong G_0^{F_0^2}$, and the action of σ on $G_0^{F_0^2}$ is given by $g\mapsto\sigma_0F_0(g)$, which can be thought of as another Frobenius endomorphism. That is where the Shintani descent intervenes, which relates the functions on $G_0^{F_0^2}.\sigma_0F_0$ to the functions on $G_0^{\sigma_0F_0}.F_0^2$. Note that $(\sigma_0F_0)^2=F_0^2$ acts trivially on $G_0^{\sigma_0F_0}$. We know how to calculate the characters of $G_0^{\sigma_0F_0}\cong\mathrm{GL}_m^-(q)$, which extends trivially to $G_0^{\sigma_0F_0}.F_0^2$. Thus, we obtain the extension of a character to $L_1^F.\sigma$. The result is as follows.

Let θ_1 be a σ-stable regular linear character of L_1^F and let $\varphi\in\mathrm{Irr}(W_{L_1}^\sigma)^F$. Note that a σ-stable linear character extends trivially to $L_1^F.\sigma$, and that $W_{L_1}^\sigma$ is in fact a product of symmetric groups.

Theorem 7. *Let χ_{L_1} be a σ-stable irreducible character of L_1^F defined by (θ,φ). For some choice of $\tilde{\varphi}$, the extension of χ_{L_1} to $L_1^F.\sigma$ is given up to a sign by*

$$R_\varphi^{L_1.\sigma}\tilde{\theta}_1=|W_{L_1}^\sigma|^{-1}\sum_{w\in W_{L_1}^\sigma}\tilde{\varphi}(wF)R_{T_w.\sigma}^{L_1.\sigma}\tilde{\theta}_1. \tag{I.13}$$

Combined with the preceding theorem, it gives the theorem below.

According to the parametrisation of σ-stable characters, each $\chi\in\mathrm{Irr}(\mathrm{GL}_n(q))^\sigma$ is of the form $R_L^G(\chi_0\otimes\chi_1)$, where $L\cong L_0\times L_1$ is a σ-stable and F-stable Levi factor of some σ-stable parabolic subgroup, χ_0 is a quadratic-unipotent character of L_0^F and χ_1 is a σ-stable irreducible character of L_1^F whose semi-simple part and unipotent part are defined by $\theta_1\in\mathrm{Irr}_{reg}(L_1^F)$ and $\varphi\in\mathrm{Irr}(W_{L_1}^\sigma)$ respectively. In the following theorem, we use the notations of Theorem 6.

Theorem 8. *For some choice of $\tilde{\varphi}$, the extension of χ is up to a sign given by*

$$\tilde{\chi}|_{G^F.\sigma}=|W_{L_1}^\sigma\times\mathfrak{W}_+\times\mathfrak{W}_-|^{-1}\sum_{(w,w_+,w_-)\in W_{L_1}^\sigma\times\mathfrak{W}_+\times\mathfrak{W}_-}\tilde{\varphi}(wF)\varphi_+(w_+)\varphi_-(w_-)R_{(T_w\times L_{\mathbf{w}}).\sigma}^{G.\sigma}(\tilde{\theta}_1\widetilde{\boxtimes}\phi_{\mathbf{w}}). \tag{I.14}$$

Green Functions

We remark that in $R_{L_{\mathbf{w}}.\sigma}^{L_0.\sigma}\phi_{\mathbf{w}}$ appears the generalised Green functions associated to the centraliser of a semi-simple element in $L_0^F.\sigma$, which is in general a product of $\mathrm{GL}_m^\pm(q)$, $\mathrm{Sp}_{2m}(q)$, $\mathrm{SO}_{2m+1}(q)$ and $\mathrm{SO}_{2m}^\pm(q)$, where the negative sign means that the Frobenius is twisted by a graph automorphism of order 2. In [L85, V], Lusztig gives an algorithm of calculating the generalised Green functions of classical groups. The paper of Shoji [Sh83] is also good reference for Green functions. The values of the (generalised) Green functions have been computed by various people, and we will not make explicit theirs values except in the examples.

Chapter II

Preliminaries on Representations and Algebraic Groups

II.1 Notations and Generalities

We introduce the notations, terminology and some basic results that will be used later.

In this section, we work over an algebraically closed field k of arbitrary characteristic. The positive characteristic version of these results is essential to the problem of counting points of character varieties over finite fields. Whenever we work with some reductive group G, we will assume

Assumption. $\operatorname{char} k \nmid |G/G^\circ|$.

This implies that all unipotent elements of G are contained in G° and that all quasi-semi-simple elements are semi-simple.

Throughout the thesis, we will denote by $\mathfrak{i}$ a square root of (-1).

II.1.1 Generalities on Algebraic Groups

II.1.1.1 Let G be an arbitrary linear algebraic group, which is not necessarily connected. A closed subgroup H of G is a Levi factor of G if G is the semi-direct product of H and $R_u(G)$. For any linear algebraic group in characteristic 0, Levi factor exists, and any two Levi factors are conjugate under $R_u(G)$. See [Ri88] 1.2.4.

A closed subgroup $P \subset G$ is parabolic if G/P is complete. By [Spr] Lemma 6.2.4, P is parabolic in G if and only if P° is parabolic in G°. A closed subgroup of G is called a Levi subgroup if it is a Levi factor of some parabolic subgroup.

An algebraic group G is reductive if G° is reductive. Levi factors are necessarily reductive.

For $g \in G$, we will often denote $C_G(g)$ by H^g.

We denote by $\mathrm{X}_\bullet(G)$ the set of cocharacters of G and by $\mathrm{X}^\bullet(G)$ the set of characters of G. Note that $\mathrm{X}_\bullet(G) = \mathrm{X}_\bullet(G^\circ)$.

II.1.1.2 Let G be a reductive group acting on an algebraic variety X. Let $\lambda \in \mathrm{X}_\bullet(G^\circ)$ and let x be a closed point of X. The G-action induces an action of $\mathbb{G}_m$ via λ. Denote by $\lambda_x : \mathbb{G}_m \to X$, $t \mapsto \lambda(t).x$ the orbit morphism. We say the limit $\lim_{t\to 0} \lambda(t).x$ exists, if the morphism λ_x extends to $\mathbb{G}_a \to X$, and the limit is defined as the image of $0 \in \mathbb{G}_a$, denoted by $\lambda(0).x$.

Since G is reductive, there is a unique closed orbit contained in the closure of each orbit. We assume from now on that G is reductive.

Theorem II.1.1.1 (Hilbert-Mumford Theorem)**.** *Let $x \in X$ be a closed point and let $\mathcal{O}$ be the unique closed orbit in $\overline{G.x}$, Then there exist $\lambda \in \mathrm{X}_\bullet(G)$ and a closed point $y \in \mathcal{O}$ such that $\lim_{t\to 0} .x = y$.*

II.1.1.3 Let λ be a cocharacter of G°. For the G°-conjugation action on G, we put

$$\begin{aligned} P_\lambda &= \{g \in G \mid \lim_{t\to 0} \lambda(t).g \text{ exists}\} \\ L_\lambda &= \{g \in G \mid \lim_{t\to 0} \lambda(t).g = g\} \\ U_\lambda &= \{g \in G \mid \lim_{t\to 0} \lambda(t).g = 1\} \end{aligned} \tag{II.1.1.3.1}$$

Then P_λ is a parabolic subgroup of G, L_λ is a Levi factor of P_λ and U_λ is the unipotent radical of P_λ. Beware that for non-connected G, not all parabolic subgroups are of the form P_λ.

The identity component P_λ° can also be defined as the unique closed subgroup of G° whose Lie algebra is generated by weight subspaces in $\mathfrak{g} := \mathrm{Lie}(G)$ with non negative weights with respect to the adjoint action of $\mathbb{G}_m$ on $\mathfrak{g}$. Thus $U_\lambda = U_\lambda^\circ$ is associated to those subspaces with positive weights and $L_\lambda^\circ = C_{G^\circ}(\mathrm{Im}\,\lambda)$.

We see that $P_\lambda \subset N_G(P_\lambda^\circ)$ and $L_\lambda \subset N_G(L_\lambda^\circ, P_\lambda^\circ) := N_G(L_\lambda^\circ) \cap N_G(P_\lambda^\circ)$.

Proposition II.1.1.2. *([DM94, Proposition 1.5]) Let $P^\circ \subset G^\circ$ be a parabolic subgroup, L° a Levi factor of P° and U the unipotent radical of P°. Then we have the Levi decomposition*

$$N_G(P^\circ) = U \rtimes N_G(L^\circ, P^\circ).$$

Note that all Levi factors of $N_G(P^\circ)$ are necessarily of the form $N_G(L^\circ, P^\circ)$ for some Levi factor L° of P°.

Proposition II.1.1.3 ([Ri88] Proposition 2.4)**.** *Let $P^\circ \subset G^\circ$ be a parabolic subgroup and let L be a Levi factor of $P := N_G(P^\circ)$. Then there exists $\lambda \in \mathrm{X}_\bullet(G^\circ)$ such that $P = P_\lambda$, $L = L_\lambda$, and $R_u(P) = U_\lambda$.*

Given a parabolic subgroup $P^\circ \subset G^\circ$, $N_G(P^\circ)$ is the largest parabolic subgroup of G that has P° as its identity component and if $P^\circ = P_\lambda^\circ$ for some cocharacter λ, then P_λ is the union of a subset of connected components of $N_G(P^\circ)$. Note that P° itself is also a parabolic subgroup of G.

II.1.1.4 For any parabolic subgroup of the form P_λ, there is a homomorphism of algebraic groups $P_\lambda \to L_\lambda$, $p \mapsto \lambda(0).p$, which is none other than the projection with respect to the Levi decomposition $P_\lambda = U_\lambda \rtimes L_\lambda$.

II.1.1.5 In general, for an arbitrary parabolic subgroup P° of G°, $P := N_G(P^\circ)$ does not necessarily meet all connected components of G. Let us determine the connected components of G that meet P.

Let G^1 be a connected component of G and denote by $\mathcal{P}$ the G°-conjugacy class of P°. Observe that the conjugation of G^1 on G° induces a well-defined bijection from the set of G°-conjugacy classes of parabolic subgroups of G° into itself. Then P meets G^1 if and only if some element of G^1 normalises P°, if and only if G^1 leaves $\mathcal{P}$ stable. The set of conjugacy classes of parabolic subgroups of G° are in bijection with the set of subsets of vertices of the Dynkin diagram of G°. Therefore, P meets G^1 if and only if the subset of the Dynkin diagram corresponding to $\mathcal{P}$ is stable under the induced action by G^1.

II.1.2 Irreducible and Completely Reducible Subgroups

The definitions are natural generalisation of the case of connected groups.

II.1.2.1 Let us recall the notions of complete reducibility and irreducibility of algebraic subgroups. For the moment, G can be any reductive algebraic group.

Definition II.1.2.1. A closed subgroup H of G is G-completely reducible if for any parabolic subgroup $P \subset G$ containing H, there is a Levi factor of P containing H. A closed subgroup H of G is G-irreducible if it is not contained in any subgroup of of the form $N_G(P^\circ)$ with $P^\circ \subset G^\circ$ being a proper parabolic subgroup.

In particular, an irreducible subgroup is completely reducible. Clearly, if $G = G^\circ$, then the above definition coincides with the definitions for connected reductive groups.

Lemma II.1.2.2. *We have*

(i) A closed subgroup of G° is irreducible in G° if and only if it is irreducible in G;

(ii) A closed subgroup of G° is completely reducible in G° if and only if it is completely reducible in G;

(ii) If H is a completely reducible subgroup of G, then $H \cap G^\circ$ is a completely reducible subgroup of G°.

Proof. The first assertion is obvious. The rest is [BMR, Lemma 6.12]. □

II.1.2.2 The following result is well-known.

Theorem II.1.2.3. *In characteristic 0, a closed subgroup of G is reductive if and only if it is G-completely reducible.*

This follows from the following results. See [Ri88] 1.2.4(c) and [Ri88] Proposition 2.6.

Proposition II.1.2.4 (Mostow)**.** *Let H be a reductive subgroup of G over an algebraically closed field of characteristic 0. Then H is contained in the Levi factor of some parabolic subgroup of G.*

Proposition II.1.2.5. *Let H be a closed subgroup of G and let M be a Levi factor of it. Then there exists $\lambda \in \mathrm{X}_\bullet(G^\circ)$ such that $H \subset P_\lambda$, $M \subset L_\lambda$ and $R_u(H) \subset U_\lambda$.*

II.1.2.3 Given an n-tuple of elements of G, say $\mathbf{x}$, denote by $H(\mathbf{x})$ the closed algebraic subgroup of G such that $H(\mathbf{x})(k)$ is the Zariski closure of $\mathbf{x}$. We will write $\mathrm{Stab}_{G^\circ}(\mathbf{x})$ instead of $C_{G^\circ}(\mathbf{x})$. Obviously $\mathrm{Stab}_{G^\circ}(\mathbf{x}) = \mathrm{Stab}_{G^\circ}(H(\mathbf{x}))$.

Theorem II.1.2.6. *Let G be a reductive group and let $\mathbf{x} = (x_1, \ldots, x_n) \in G^n$, the direct product of n copies of G. Then the orbit $G^\circ.\mathbf{x}$ is closed if and only if $H(\mathbf{x})$ is completely reducible.*

Proof. Note that the orbit $G.\mathbf{x}$ is closed if and only if $G^\circ.\mathbf{x}$ is closed. In characteristic 0, by Theorem II.1.2.3, we can apply [Ri88] Theorem 3.6, and in positive characteristic, this a combination of [BMR, §6.3] and [Ma, Proposition 8.3]. □

We will need the following.

Proposition II.1.2.7 ([Ri77] Theorem A)**.** *Let X be an affine algebraic variety with a G-action. If the G-orbit of $x \in X$ is closed, then* $\mathrm{Stab}_G(x)$ *is a reductive group.*

Recall that for an G-action on an algebraic variety X, an orbit $G.x$, $x \in X$, is called *stable*, if it is closed and $\mathrm{Stab}_G(x)/Z_X$ is finite, where $Z_X := \cap_{x \in X} \mathrm{Stab}_G(x)$ is the kernel of the action.

Theorem II.1.2.8. *The G°-orbit of $\mathbf{x} \in G^n$ is stable if and only if $H(\mathbf{x})$ is an irreducible subgroup of G and* $\mathrm{Stab}_{Z_{G^\circ}}(\mathbf{x})^\circ = Z_G^\circ$.

For $G = G^\circ$, this is [Ri88, Theorem 4.1 and Proposition 16.7]. Let us suppose $G \neq G^\circ$. The proof will be essentially the same as the case $G = G^\circ$ but one needs to be careful when $Z_G^\circ \neq Z_{G^\circ}^\circ$. For example, if $G^\circ = \mathrm{GL}_n(k)$ and a connected component of G acts on $Z_{G^\circ} = k^*$ by $x \mapsto x^{-1}$ so that $Z_G = \{\pm 1\}$, then the irreducibility of the subgroup $H(\mathbf{x})$ of G with $\mathbf{x} \subset G^\circ$ does not imply that $G^\circ.\mathbf{x} \subset G^n$ is stable.

Proof. Let us first show that the irreducibility of $H(\mathbf{x})$ with the technical assumption implies the stability of its orbit. We prove by contradiction. Suppose that the orbit $G^\circ.\mathbf{x}$ is not closed. By the Hilbert-Mumford Theorem there exists $\lambda \in \mathrm{X}_\bullet(G)$ and $\mathbf{x}'$ in the unique closed orbit contained in the closure of $G^\circ.\mathbf{x}$, such that $\lim_{t \to 0} \lambda(t).\mathbf{x} = \mathbf{x}'$. In particular, the image of $\mathbf{x}$ must be contained in P_λ in order for the limit to exist. Since $\mathbf{x} \neq \mathbf{x}'$, $\lambda \notin \mathrm{X}_\bullet(Z_{G^\circ}^\circ)$ and P_λ° is a proper parabolic subgroup of G°. This contradicts the irreducibility of $H(\mathbf{x})$. Suppose now that the orbit $G^\circ.\mathbf{x}$ is closed but not stable. So $\mathrm{Stab}_{G^\circ}^\circ(\mathbf{x})$ strictly contains Z_G°. By Proposition II.1.2.7, there exists $\lambda \in \mathrm{X}_\bullet(\mathrm{Stab}_{G^\circ}^\circ(\mathbf{x}))$ such that $\lambda \notin \mathrm{X}_\bullet(Z_G^\circ)$. By the hypothesis in the statement of the theorem, we have in fact $\lambda \notin \mathrm{X}_\bullet(Z_{G^\circ}^\circ)$. Again, we have a proper parabolic subgroup containing $H(\mathbf{x})$, contradicting the irreducibility.

To prove the other direction, suppose that $H(\mathbf{x})$ is contained in some proper parabolic subgroup $P \subset G$. By Proposition II.1.1.3, there exists $\lambda \in \mathrm{X}_\bullet(G)$ such that $P = P_\lambda$. Let $\mathbf{x}' = \lim_{t\to 0} \lambda(t).\mathbf{x}$ and so $H(\mathbf{x}') \subset L_\lambda$. Thus $\lambda \in \mathrm{X}_\bullet(\mathrm{Stab}_{G^\circ}(\mathbf{x}'))$ and $\mathbf{x}'$ is not a stable point. If $G^\circ.\mathbf{x}$ is not closed, then by definition $\mathbf{x}$ is not stable. If $G^\circ.\mathbf{x}$ is closed, then $\mathbf{x}'$ is conjugate to $\mathbf{x}$, therefore $\mathbf{x}$ is not stable. □

II.2 Finite Classical Groups

II.2.1 Partitions and Symbols

II.2.1.1 We denote by $\mathcal{P}_n$ the set of all partitions of the integer $n \geq 0$ and by $\mathcal{P}$ the union $\cup_n \mathcal{P}_n$. A partition is written as $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, a decreasing sequence of positive integers, or as $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ where m_i is the multiplicity of i that appears in λ. Each λ_i is called a *part* of λ. We denote by $|\lambda|$ the size of λ and $l(\lambda)$ the length of λ. For any e-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(e)}) \in \mathcal{P}^e$ (direct product of e copies), its size is defined by $|\boldsymbol{\lambda}| := \sum_i |\lambda^{(i)}|$ and its length is defined by $l(\boldsymbol{\lambda}) := \sum_i l(\lambda^{(i)})$. Denote by $\mathcal{P}_n(e)$ the set of all e-partitions of n, i.e. those $\boldsymbol{\lambda}$ with $|\boldsymbol{\lambda}| = n$. In the cases that concern us, $e = 2$.

II.2.1.2 A partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ is called *symplectic* if m_i is even for any odd i. To each symplectic partition λ one associates an index $\kappa(\lambda) = \#\{i \text{ pair} | m_i > 0\}$. We denote by $\mathcal{P}_n^{sp} \subset \mathcal{P}_n$ the subset of symplectic partitions. A partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ is called *orthogonal* if m_i is even for any even i. To each orthogonal partition λ one associates an index $\kappa(\lambda) = \#\{i \text{ impair} | m_i > 0\}$. We denote by $\mathcal{P}_n^{ort} \subset \mathcal{P}_n$ the subset of orthogonal partitions. The orthogonal partitions with $\kappa = 0$ are called *degenerate*.

II.2.1.3 Given a partition λ, we take $r \geq l(\lambda)$, and we put $\delta_r = (r-1, r-2, \ldots, 1, 0)$. Let $(2y_1 > \cdots > 2y_{l_0})$ and $(2y'_1 + 1 > \cdots > 2y'_{l_1} + 1)$ be the even parts and the odd parts of $\lambda + \delta_r$, where the sum is made term by term and λ is regarded as an decreasing sequence of integers $(\lambda_i)_i$ with $\lambda_i = 0$ for $i > l(\lambda)$.. Denote by $\lambda^{(0)}$ the partition defined by $\lambda_k^{(0)} = y_k - l_0 + k$ and denote by $\lambda^{(1)}$ the partition defined by $\lambda_k^{(1)} = y'_k - l_1 + k$. Then $(\lambda^{(0)}, \lambda^{(1)})_r$ is a 2-partition that depends on r. Changing the value of r will permutes $\lambda^{(0)}$ and $\lambda^{(1)}$. The *2-quotient* of λ is then the unordered pair of partitions $(\lambda^{(0)}, \lambda^{(1)})$.

Denote by λ' the partition that has as its parts the numbers $2s + t$, $0 \leq s \leq l_t - 1$, $t = 0, 1$. We have $l(\lambda') = l(\lambda)$. The *2-core* of λ is the partition defined by $(\lambda'_k - l(\lambda) + k)_{1 \leq k \leq l(\lambda)}$. It is independent of r and is necessarily of the form $(d, d-1, \ldots, 2, 1)$, for some $d \in \mathbb{Z}_{>0}$. ① Fixing r, the above constructions give a bijection between the partitions of n with the same 2-core and the 2-partitions of $(n-r)/2$.

II.2.1.4 We refer to [L84a] for the notion of symbols.

①d

Fix an even positive interger N. A symbol of symplectic type is the equivalence class of ordered pairs (A, B) with A a finite subset of $\{0, 1, 2, \ldots\}$ and B a finite subset of $\{1, 2, 3, \ldots\}$ satisfying the following conditions

(a) For any integer i, the set $\{i, i+1\}$ is contained neither in A nor in B;

(b) $|A| + |B|$ is odd;

(c) $\sum_{a\in A} a + \sum_{b\in B} b = \frac{1}{2}N + \frac{1}{2}(|A| + |B|)(|A| + |B| - 1)$;

under the equivalence that identifies (A, B) and $(\{0\} \cup (A+2), \{1\} \cup (B+2))$. The set of these symbols is denoted by Ψ^s_N.

Fix a positive integer $N > 2$. A symbol of orthogonal type is the equivalence class of the unordered pairs (A, B) of finite subsets of $\{0, 1, 2, \ldots\}$ satisfying the following conditions

(a) For any integer i, $\{i, i+1\}$ is contained neither in A nor in B;

(c) $\sum_{a\in A} a + \sum_{b\in B} b = \frac{1}{2}N + \frac{1}{2}((|A| + |B| - 1)^2 - 1)$;

under the equivalence that identifies (A, B) and $(\{0\} \cup (A+2), \{0\} \cup (B+2))$. The set of these symbols is denote by Ψ^o_N. A symbol of orthogonal type is called *degenerate* if it is of the form (A, A). The subset of non degenerate symbols is denoted by Ψ'^o_N.

Two symbols are similar if they admit representatives (A, B) and (A', B') such that $A \cup B = A' \cup B'$ and $A \cap B = A' \cap B'$.

II.2.1.5 To each symplectic partition is associated a symbol of symplectic type in the following manner. Let λ be such a partition and let r be an integer such that $2r \geq l(\lambda)$. Denote by $(2y_1 > \cdots > 2y_r)$ and $(2y'_1 + 1 > \cdots > 2y'_r + 1)$ the even parts and the odd parts of $\lambda + \delta_{2r}$. One can verify that there are indeed r even parts and r odd parts. Put $A = \{0\} \cup \{y'_k + r + 2 - k \mid 1 \leq k \leq r\}$ and put $B = \{y_k + r + 1 - k \mid 1 \leq k \leq r\}$. Then (A, B) is a symplectic symbol, whose similarity class is independent of r.

To each orthogonal partition is associated a symbol of orthogonal type in the following manner. Let λ be such a partition and let r be an integer such that $r \geq l(\lambda)$. Denote by $(2y_1 > \cdots > 2y_{[r/2]})$ and $(2y'_1 + 1 > \cdots > 2y'_{[(r+1)/2]} + 1)$ the even parts and the odd parts of $\lambda + \delta_r$. One can verify that there are indeed $[r/2]$ even parts and $[(r+1)/2]$ odd parts. Put $A = \{0\} \cup \{y'_k + [(r+1)/2] - k \mid 1 \leq k \leq [(r+1)/2]\}$ and $B = \{y_k + [r/2] - k \mid 1 \leq k \leq [r/2]\}$. Then (A, B) is a symbol of orthogonal type, whose similarity class is independent of r.

II.2.2 Weyl Groups

Some basic facts about Weyl groups of type B_m, C_m and D_m.

II.2.2.1 Denote by w_0 the permutation $(1,-1)(2,-2)\cdots(m,-m)$ of the set

$$I = \{1,\dots,m,-m,\dots,-1\}.$$

The set of permutations of I that are invariant under w_0 is identified with $(\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathfrak{S}_m$. This is the Weyl group of type B_m and C_m, which will be denoted by $\mathfrak{W}_m^C$. We will identify $\mathbb{Z}/2\mathbb{Z}$ with μ_2, the roots of unity of order 2, and denote its elements by signs ± 1. An element of $\mathfrak{W}_m^C$ written as

$$w = ((\epsilon_1,\dots,\epsilon_m),\tau) \in (\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathfrak{S}_m, \tag{II.2.2.1.1}$$

where $((1,\dots,1,\epsilon_i,1,\dots,1),1)$, $\epsilon_i = -1$ is the permutation $(i,-i)$, and $((1,\dots,1),\tau)$ is the permutation

$$i \mapsto \tau(i), \quad -i \mapsto \tau(-i) = -\tau(i).$$

II.2.2.2 The permutation τ is decomposed into cycles $\tau = c_{I_1}\cdots c_{I_l}$, where the disjoint subsets $I_r \subset \{1,\dots,m\}$ form a partition of $\{1,\dots,m\}$ and c_{I_r} is a circular permutation of the indices in I_r. The permutation τ determines a partition $(\tau_1,\dots,\tau_l)$ of m, also denoted by τ, with $\tau_r = |I_r|$, where $l = l(\tau)$ is the length of the partition.

For all $1 \le r \le l$, put $\bar{\epsilon}_r = \prod_{k\in I_r}\epsilon_k$ and $\underline{\epsilon}_r = (\epsilon_k)_{k\in I_r}$. Define the permutations

$$\tau^{(0)} = \prod_{\bar{\epsilon}_r=1} c_{I_r}, \quad \tau^{(1)} = \prod_{\bar{\epsilon}_r=-1} c_{I_r}, \tag{II.2.2.2.1}$$

so that $\tau = \tau^{(0)}\tau^{(1)}$. Also denote by $\tau^{(0)} = (\tau_r^{(0)})$ and $\tau^{(1)} = (\tau_r^{(1)})$ the associated partitions. We then have a 2-partition $(\tau^{(0)},\tau^{(1)})$, which determines the conjugacy class of w. We sometimes call it a signed partition of n. The conjugacy classes and irreducible characters of $\mathfrak{W}_m^C$ are both parametrised by by the 2-partitions of size n. Write $l_0 = l(\tau^{(0)})$, and $l_1 = l(\tau^{(1)})$.

II.2.2.3 The Weyl group of type D_m, denoted by $\mathfrak{W}_m^D$, is the subgroup of $\mathfrak{W}_m^C$ consisting of the elements $((\epsilon_1,\dots,\epsilon_m),\tau)$ such that $\prod\epsilon_i = 1$. The parametrisation of the conjugacy classes of $\mathfrak{W}_m^D$ is given as follows. (See [Ca, Proposition 25]) Let τ be a signed partition. If each part of τ is even and the $\bar{\epsilon}_r$'s are all equal to 1, then the conjugacy class of $\mathfrak{W}_m^C$ corresponding to τ splits into two classes of $\mathfrak{W}_m^D$. Otherwise, this conjugacy class restricts to one single class of $\mathfrak{W}_m^D$..

The element $((1,\dots,1,-1),1)$ belongs to $\mathfrak{W}_m^C \setminus \mathfrak{W}_m^D$. It can be realised as an element of $\mathrm{O}_{2m}(k) \setminus \mathrm{SO}_{2m}(k)$. Its conjugation action on $\mathrm{SO}_{2m}(k)$ permutes the two simple roots $\mathbf{e}_{m-1} - \mathbf{e}_m$ and $\mathbf{e}_{m-1} + \mathbf{e}_m$, and thus induces a non-trivial graph automorphism.

II.2.3 Unipotent Classes and Centralisers

The parametrisation of unipotent conjugacy classes of finite classical groups is well known. We refer to ([LiSe, Chapter 3, Chapter 7]) for a more complete survey. In this paragraph, G will be one of the groups $\mathrm{GL}_n(k)$, $\mathrm{Sp}_n(k)$, $\mathrm{SO}_n(k)$ and $\mathrm{O}_n(k)$. If the Frobenius F is split, we denote by $\mathrm{GL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{SO}_n(q)$ and $\mathrm{O}_n(q)$ the associated finite groups, and if F induces a graph automorphism of order 2, we denote by $\mathrm{GL}_n^-(q)$, $\mathrm{SO}_n^-(k)$ and $\mathrm{O}_n^-(k)$ (only when n is even for the orthogonal groups) the associated finite groups. We will use the notations $\mathrm{SO}_n^\epsilon(k)$ and $\mathrm{O}_n^\epsilon(k)$ for all n, understanding that ϵ can be $-$ only if n is even and that $+$ corresponds to the split groups.

II.2.3.1 The unipotent classes of $\mathrm{GL}_n(k)$ are parametrised by $\mathcal{P}_n$, with the sizes of Jordan blocks given by the corresponding partition. The unipotent classes of $\mathrm{Sp}_n(k)$ are parametrised by $\mathcal{P}_n^{sp}$. These are represented by the Jordan matrices in $\mathrm{GL}_n(k)$ that belong to $\mathrm{Sp}_n(k)$. The unipotent classes of $\mathrm{O}_n(k)$ are parametrised by $\mathcal{P}_n^{ort}$. These are represented by the Jordan matrices in $\mathrm{GL}_n(k)$ that belong to $\mathrm{O}_n(k)$.

II.2.3.2 Let u be a unipotent matrix in G, associated to the partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$. In general, we have $C_G(u) = VR$, where V is the unipotent radical of $C_G(u)$, whence an affine space, and R is a reductive group given as follows. For any symplectic or orthogonal partition λ, let κ be the $\kappa(\lambda)$ defined in §II.2.1.2.

If $G = \mathrm{GL}_n(k)$, then

$$R \cong \prod_{\{i \mid m_i > 0\}} \mathrm{GL}_{m_i}.$$

In particular, $C_G(u)$ is connected.

If $G = \mathrm{Sp}_n(k)$, then

$$R \cong \prod_{i \text{ odd}} \mathrm{Sp}_{m_i} \times \prod_{i \text{ even}} \mathrm{O}_{m_i}.$$

So $C_G(u)/C_G(u)^\circ \cong (\mathbb{Z}/2\mathbb{Z})^\kappa$.

If $G = \mathrm{O}_n(k)$, then

$$R \cong \prod_{i \text{ odd}} \mathrm{O}_{m_i} \times \prod_{i \text{ even}} \mathrm{Sp}_{m_i}.$$

So $C_G(u)/C_G(u)^\circ \cong (\mathbb{Z}/2\mathbb{Z})^\kappa$.

In the case $G = \mathrm{O}_n(k)$, an element $z \in C_G(u)$ belongs to $\mathrm{SO}_n(k)$ if and only if its equivalence class $\bar{z} = (e_1, \ldots, e_\kappa) \in C_G(u)/C_G(u)^\circ$ satisfies $\prod e_i = 1$, where $e_i = \pm 1$. So $C_{\mathrm{SO}}(u) \cong (\mathbb{Z}/2\mathbb{Z})^{\kappa-1}$ if $\kappa > 0$. It follows that if the G-conjugacy class of u contains an F-stable element, then it contains one single G^F-class if $G = \mathrm{GL}_n^\epsilon(k)$, 2^κ G^F-classes if $G = \mathrm{Sp}_n(k)$, and $2^{\kappa-1}$ G^F-classes if $G = \mathrm{SO}_n^\epsilon(k)$ as long as $\kappa > 0$. The G-conjugacy class of u does not contain any element of G^F only when $G = \mathrm{O}_n^-(q)$ and $\kappa(\lambda) = 0$. If $G = \mathrm{O}_n^+(q)$ and $\kappa(\lambda) = 0$, the G-conjugacy class contains one single G^F-class, which splits into two classes for the conjugation by $\mathrm{SO}_n(q)$. These two classes are called *degenerate*. Fixing u, which corresponds to the partition λ, let

$u_{\mathbf{e}}$ be an element representing the G^F-class associated to $\mathbf{e} := (e_1, \ldots, e_\kappa) \in C_G(u)/C_G(u)^\circ$. We have $C_G(u_{\mathbf{e}})^F = V^F R^F$, where $V^F \cong \mathbb{F}_q^{\dim V}$, and R^F is given as follows.

If $G = \mathrm{GL}_n^\epsilon(q)$, then

$$R^F \cong \prod \mathrm{GL}_{m_i}^\epsilon(q).$$

If $G = \mathrm{Sp}_n(q)$, then

$$R^F \cong \prod_{i \text{ edd}} \mathrm{Sp}_{m_i}(q) \times \prod_{i \text{ even}} \mathrm{O}_{m_i}^{e_i}(q).$$

If $G = \mathrm{O}_n^\epsilon(q)$, then

$$R^F \cong \prod_{i \text{ odd}} \mathrm{O}_{m_i}^{e_i}(q) \times \prod_{i \text{ even}} \mathrm{Sp}_{m_i}(q),$$

subject to the condition that if all of the m_i's are even, then $\prod e_i = \epsilon$.

In the above formulas, we have renumbered the e_i's by the parts of λ, i.e. i is the part of λ corresponding to e_i.

II.2.3.3 We take u and λ as above. We have $\dim \mathrm{Sp}_n(k) = n(n+1)/2$, $\dim \mathrm{SO}_n(k) = n(n-1)/2$, and the following formulas.

(II.2.3.3.1) $$\dim C_{\mathrm{GL}}(u) = \sum_i i m_i^2 + 2 \sum_{i<j} i m_i m_j$$

(II.2.3.3.2) $$\dim C_{\mathrm{Sp}}(u) = \frac{1}{2} \sum_i i m_i^2 + \sum_{i<j} i m_i m_j + \frac{1}{2} \sum_{i \text{ odd}} m_i$$

(II.2.3.3.3) $$\dim C_{\mathrm{SO}}(u) = \frac{1}{2} \sum_i i m_i^2 + \sum_{i<j} i m_i m_j - \frac{1}{2} \sum_{i \text{ odd}} m_i$$

For the unipotent radicals, we have,

(II.2.3.3.4) $$\dim V(u) = \sum_i (i-1) m_i^2 + 2 \sum_{i<j} i m_i m_j, \text{ if } G = \mathrm{GL}_n(k),$$

(II.2.3.3.5) $$\dim V(u) = \frac{1}{2} \sum_i (i-1) m_i^2 + \sum_{i<j} i m_i m_j + \frac{1}{2} \sum_{i \text{ even}} m_i, \text{ if } G = \mathrm{Sp}_n(k),$$

(II.2.3.3.6) $$\dim V(u) = \frac{1}{2} \sum_i (i-1) m_i^2 + \sum_{i<j} i m_i m_j + \frac{1}{2} \sum_{i \text{ even}} m_i, \text{ if } G = \mathrm{SO}_n(k).$$

The cardinality of finite classical groups is as follows.

(II.2.3.3.7) $$|\mathrm{GL}_n(q)| = q^{n(n-1)/2} \prod_{i=1}^n (q^i - 1),$$

(II.2.3.3.8) $$|\mathrm{GL}_n^-(q)| = q^{n(n-1)/2} \prod_{i=1}^n (q^i - (-1)^i),$$

(II.2.3.3.9) $$|\mathrm{Sp}_{2m}(q)| = q^{m^2}\prod_{i=1}^{m}(q^{2i}-1),$$

(II.2.3.3.10) $$|\mathrm{SO}_{2m+1}(q)| = q^{m^2}\prod_{i=1}^{m}(q^{2i}-1),$$

(II.2.3.3.11) $$|\mathrm{SO}_{2m}(q)| = q^{m(m-1)}(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1),$$

(II.2.3.3.12) $$|\mathrm{SO}^-_{2m}(q)| = q^{m(m-1)}(q^m+1)\prod_{i=1}^{m-1}(q^{2i}-1).$$

II.2.3.4 The unipotent classes of $\mathrm{Sp}_n(k)$ are in bijection with the symplectic partitions, and so in bijection with the similarity classes of de Ψ^s_n([L84a, 12.4(c)]). The unipotent classes of $\mathrm{O}_n(k)$ are in bijection with the orthogonal partitions, and so in bijection with the similarity classes of $\Psi^o_n(k)$. Note that a degenerate partition alone forms a similarity class. ([L84a, 13.4(c)]). The symbols in the sinilarity class associated to the symplectic partition λ are in bijection with $(\mathbb{Z}/2\mathbb{Z})^\kappa$, and so in bijection with the G^F-conjugacy classes contained in the G-class corresponding to λ. The symbols in the similarity class associated to the non degenerate orthogonal partition λ are in bijection with $(\mathbb{Z}/2\mathbb{Z})^{\kappa-1}$, and so in bijection with the G^F-conjugacy classes contained in the G-class corresponding to λ. The similarity class associated to a degenerate orthogonal partition λ consists of a single element, which corresponds to the two degenerate classes of $\mathrm{SO}_n(q)$, or the one conjugacy class of $\mathrm{O}_n(q)$, corresponding to λ. For $\mathrm{SO}^-_n(q)$, there is no G^F-conjugacy class corresponding to a degenerate partition, so the degenerate symbols do not correspond to any conjugacy classes.

II.3 Non-Connected Algebraic Groups

II.3.1 Quasi-Semi-Simple Elements

We say that a not necessarily connected algebraic group G is reductive if G° is reductive. In this section we denote by G such a group. If G is defined over $\mathbb{F}_q$, we denote by F the Frobenius endomorphism.

II.3.1.1 An automorphism of G° is *quasi-semi-simple* if it leaves stable a pair consisting of a maximal torus and a Borel subgroup containing it. An element of G is *quasi-semi-simple* if it induces by conjugation a quasi-semi-simple automorphism. Let (T°, B°) be a pair consisting of a maximal torus and a Borel subgroup containing it. Put $B = N_G(B^\circ)$ and $T = N_G(B^\circ) \cap N_G(T^\circ)$ to be the normalisers. We will write $N_G(B^\circ, T^\circ)$ instead of $N_G(B^\circ) \cap N_G(T^\circ)$. Note that the identity component of T is T°, so there is no confusion with the notation. By definition, an element of G is quasi-semi-simple if and only if it belongs to T for some B° and T°.

Proposition II.3.1.1 ([Spa] II 1.15)**.** *An element $g \in G$ is quasi-semi-simple if and only if the G°-conjugacy class of g is closed in G.*

Every semi-simple element is quasi-semi-simple ([St, Theorem 7.5]). Every element of G normalises some Borel subgroup of G°. In each connected component of G, quasi-semi-simple element exists, because all Borel subgroups and all maximal torus of G° are conjugate under G°. For a given pair (T°, B°), the group $T = N_G(T^\circ, B^\circ)$ meets all connected components of G. Let $s \in G$ be a quasi-semi-simple element, every s-stable (for the conjugation) Borel subgroup contains some s-stable maximal torus. Every s-stable parabolic subgroup of G° contains some s-stable Levi factor ([DM94, Proposition 1.11]). However, an s-stable Levi subgroup of G° is not necessarily an Levi factor of some s-stable parabolic subgroup.

Let $G^1 \neq G^\circ$ be a connected component of G, and let $s \in G^1$ be a quasi-semi-simple element. Fix an s-stable maximal torus T_0 contained in some s-stable Borel subgroup of G°. The quasi-semi-simple G°-conjugacy classes in G^1 are described as follows.

Proposition II.3.1.2. *([DM15][Proposition 7.1]) Every quasi-semi-simple G°-conjugacy class in G^1 has a representative in $C_{T_0}(s)^\circ.s$. Two elements ts and $t's$ with $t, t' \in C_{T_0}(s)^\circ$, represent the same class if and only if t and t', when passing to the quotient $T_0/[T_0, s]$, belong to the same W^s-orbit, where $[T_0, s]$ is the commutator, which is preserved by $W^s := \{w \in W_{G^\circ}(T_0) \mid sws^{-1} = w\}$.*

Note that $T_0/[T_0, s] \cong (T_0^s)^\circ/[T_0, s] \cap (T_0^s)^\circ$.

Lemma II.3.1.3. *([DM18, Lemma 1.2 (iii)]) With $T = T_0$ and s as above, we have,*

(i) $T = [T, s] \cdot (T^s)^\circ$;

(ii) $[T, s] \cap (T^s)^\circ$ *is finite.*

In particular, $\dim T = \dim(T^s)^\circ + \dim[T, s]$.

II.3.1.2 Let L° be a Levi factor of some parabolic subgroup $P^\circ \subset G^\circ$. Put $P = N_G(P^\circ)$ and $L = N_G(P^\circ, L^\circ)$ to be the normalisers. According to [Spr, Lemma 6.2.4], P is a parabolic subgroup of G, in the sens that G/P is proper. Suppose that the Levi decomposition of P° is given by $P^\circ = U \rtimes L^\circ$, where U is the unipotent radical of P°, then $P = U \rtimes L$. (See [DM94, Proposition 1.5]) In particular, L is a Levi factor of P.

II.3.1.3 If $s \in G$ is a quasi-semi-simple element, then the centraliser $H = C_G(s)^\circ$ is reductive. ([Spa, §1.17]) If the pair (T°, B°) consists of an s-stable maximal torus and an s-stable Borel subgroup of G° containing it, then $C_{B^\circ}(s)^\circ$ is a Borel subgroup of H containing the maximal torus $C_{T^\circ}(s)^\circ$ ([DM94, Théorème 1.8(iii)]). More generally, we have

Proposition II.3.1.4. *For s and H as above, we have*

(i) If (L°, P°) is a pair consisting of an s-stable Levi subgroup and an s-stable parabolic subgroup containing it as a Levi factor, then $C_{P^\circ}(s)^\circ$ is a parabolic subgroup of H, with $C_{L^\circ}(s)^\circ$ as a Levi factor.

(ii) If L' is the Levi factor of a parabolic subgroup $P' \subset H$, then there exists an s-stable parabolic subgroup $P \subset G^\circ$ such that $C_P(s)^\circ = P'$, and an s-stable Levi factor L of P such that $C_L(s)^\circ = L'$.

Remark II.3.1.5. The groups P and L in this proposition are not necessarily unique in general. See however Proposition II.3.2.1.

Proof. The first part is [DM94, Proposition 1.11]. Given L' and P', there exists a cocharacter of H, say λ, such that $L' = L'_\lambda$ and $P' = P'_\lambda$, where L'_λ and P'_λ are the Levi subgroup and parabolic subgroup associated to λ. Regarded as a cocharacter of G°, it defines a Levi subgroup and a parabolic subgroup $L_\lambda \subset P_\lambda$ of G°. They are s-stable since the image of λ commutes with s. It is clear that $L'_\lambda = L_\lambda \cap H$ and $P'_\lambda = P_\lambda \cap H$. □

If we require that the L as in (*ii*) is minimal with respect to inclusion, then it is unique.

Proposition II.3.1.6. *Given L' as in the preceding proposition, we write $L = C_{G^\circ}(Z^\circ_{L'})$. It is an s-stable Levi factor of some s-stable parabolic subgroup of G°, such that $L' = C_L(s)^\circ$. If $M \subset G^\circ$ is an s-stable Levi factor of some s-stable parabolic subgroup, such that $L' \subset M$, then $L \subset M$.*

The proof is completely analogous to [L03, §2.1] where the assertion is proved for s semi-simple.

Proof. We can find a cocharacter $\chi : k^* \to Z^\circ_{L'}$ such that $L = C_{G^\circ}(\chi(k^*))$. As in the preceding proposition, we see that L is an s-stable Levi factor of some s-stable parabolic subgroup. Since $L' = C_H(Z^\circ_{L'})$, we have $C_L(s)^\circ = L'$.

Note that $M' := (M \cap H)^\circ = C_M(s)^\circ$ is a Levi subgroup of H, and that L' is a Levi subgroup of H contained in M'. Since $(Z^\circ_M \cap H)^\circ \subset Z_{M'}$, whence $(Z^\circ_M \cap H)^\circ \subset Z^\circ_{L'}$, whence $C_G(Z^\circ_{L'}) \subset C_G((Z^\circ_M \cap H)^\circ)$. According to ([L03] §1.10), $C_G((Z^\circ_M \cap H)^\circ)^\circ = M$, so $L \subset M$. □

Remark II.3.1.7. In particular, if $T' \subset C_G(s)^\circ$ is a maximal torus, then $T := C_{G^\circ}(T')$ is the unique maximal torus of G° containing T'. It is s-stable and contained in an s-stable Borel subgroup, and we have $C_T(s)^\circ = T'$.

Remark II.3.1.8. Let M be an s-stable Levi factor of some s-stable parabolic subgroup $Q \subset G^\circ$ such that $C_M(s)^\circ = L'$. Suppose moreover that the equality $(Z^\circ_M \cap H)^\circ = Z^\circ_{L'}$ holds, i.e. the s-fixed part of the centre of M coincides with the centre of the s-fixed part of M. Then we still have $M = L$ by [L03, §1.10]. We will see in Proposition II.3.1.12 that this equality can be satisfied only if s is a quasi-isolated element of $N_G(Q) \cap N_G(M)$.

Remark II.3.1.9. It follows from the definition of L that if an element of G normalises L', then it normalises L.

II.3.1.4 A quasi-semi-simple automorphism σ of G is *quasi-central* if it satisfies the following condition.

> There exists no quasi-semi-simple automorphism of the form $\sigma' = \sigma \circ \operatorname{ad} g$ with $g \in G^\circ$ such that $\dim C_G(\sigma)^\circ < \dim C_G(\sigma')^\circ$.

A quasi-semi-simple element is *quasi-central* if it induces by conjugation a quasi-central automorphism.

A quasi-semi-simple element $\sigma \in G$ is quasi-central if and only if there exists a σ-stable maximal torus T contained in a σ-stable Borel subgroup of G° such that every σ-stable element of $N_{G^\circ}(T)/T$ has a representative in $C_G(\sigma)^\circ$([DM94, Théorème 1.15]). Considering the natural map $N_{C_G(\sigma)^\circ}(C_T(\sigma)^\circ) \to N_{G^\circ}(T)$, this simply means that $W_{G^\circ}(T)^\sigma = W_{C_G(\sigma)^\circ}(C_T(\sigma)^\circ)$.

Proposition II.3.1.10. *([DM94, Proposition 1.16]) Let $s \in G$ be a quasi-semi-simple element normalising a maximal torus T and a Borel subgroup of G° containing it. Then there exists $t \in T$ such that ts is quasi-central.*

Proposition II.3.1.11. *([DM94, Proposition 1.23]) If $\sigma \in G$ is quasi-central, and $L \subset G^\circ$ is a σ-stable Levi factor of a σ-stable parabolic subgroup of G°, then $L = C_{G^\circ}(Z^\circ_{L'})$, where $L' = C_L(\sigma)^\circ$. In particular, for any such L, $Z^\circ_{L'} = C_{Z^\circ_L}(\sigma)^\circ$.*

II.3.1.5 Let $g = g_s g_u$ be the Jordan decomposition of an element of G. Write $L'(g) = C_G(g_s)^\circ$ and $L(g) = C_{G^\circ}(Z^\circ_{L'})$. We say that g is isolated in G if $L(g) = G^\circ$. The conjugacy class of an isolated element will be called isolated. The isolated elements can be characterised as follows.

Proposition II.3.1.12. *[L03, §2.2] Let $g \in G$, and put $L' = L'(g)$ and $L = L(g)$. Then the following assertions are equivalent.*

(i) $L = G^\circ$;

(ii) $Z^\circ_{L'} = C_{Z_{G^\circ}}(g)^\circ$;

(iii) There is no g_s-stable proper parabolic subgroup $Q \subset G^\circ$ with g_s-stable Levi factor M such that $L' \subset M$.

Our definition of isolated element agrees with the definitions in the literatures, due to the following result, which is not obvious.

Proposition II.3.1.13. *([L03, IV. Proposition 18.2]) Let $s \in G$ be a semi-simple element and $u \in G$ a unipotent such that $su = us$. Then su is isolated in G (for the definition in [L03, §2]) if and only if s is isolated in G.*

Therefore, the definition of isolated element for semi-simple elements coincides with [DM18, Definition 3.1], where one fixes maximal torus T, a Borel subgroup $B \subset G^\circ$ containing it and a quasi-central element $\sigma \in N_G(T, B)$, and says that $t\sigma \in T.\sigma$ is isolated if $C_G(t\sigma)^\circ$ is not contained in a σ-stable Levi factor M of a σ-stable proper parabolic subgroup Q of G°. Note that in this definition, M necessarily contains T because $C_M(t\sigma)^\circ$ contains $C_T(t\sigma)^\circ$. Let us also recall that, an element $t\sigma$ is *quasi-isolated* if $C_G(t\sigma)$ is not contained in a σ-stable Levi factor of a σ-stable proper parabolic subgroup of G°.

II.3.2 Parabolic Subgroups and Levi Subgroups

II.3.2.1 Recall that in the setting of Proposition II.3.1.4, one does not have a bijection in general.

Proposition II.3.2.1. *([DM94, Corollaire 1.25]) Let σ be a quasi-central automorphism of G°.*

(1) The map $P \mapsto (P^\sigma)^\circ$ defines a bijection between the σ-stable parabolic subgroups of G° and the parabolic subgroups of $(G^\sigma)^\circ$.

(2) Then map $L \mapsto (L^\sigma)^\circ$ defines a bijection between the σ-stable Levi factors of σ-stable parabolic subgroups of G° and the Levi subgroups of $(G^\sigma)^\circ$.

Considering the fact that $W_{G^\circ}(T)^\sigma = W_{(G^\sigma)^\circ}((T^\sigma)^\circ)$, the bijection is obtained at the level of Weyl groups.

II.3.2.2 The following propositions will be useful.

Proposition II.3.2.2. *([DM94, Proposition 1.6]) Let σ be a quasi-semi-simple element of G and let (L°, P°) be a pair consisting of a Levi subgroup of G° and a parabolic subgroup that contains it as a Levi factor. If the G°-conjugacy class of (L°, P°) is σ-stable, then there exists $x \in G^\circ$ such that $(xL^\circ x^{-1}, xP^\circ x^{-1})$ is σ-stable.*

Now suppose that G is defined over $\mathbb{F}_q$.

Proposition II.3.2.3. *([DM94, Proposition 1.38]) Let σ be an F-stable quasi-central element of G and let (L°, P°) be a pair consisting of an F-stable Levi factor and a parabolic subgroup containing it as a Levi factor. If the $G^{\circ F}$-conjugacy class of (L°, P°) is σ-stable, then there exists $x \in G^{\circ F}$ such that $(xL^\circ x^{-1}, xP^\circ x^{-1})$ is σ-stable.*

Let L° be a Levi factor of some parabolic subgroup P° of G°, put $L = N_G(L^\circ, P^\circ)$. Let G^1 be a connected component of G. It acts by conjugation on the G°-conjugacy classes of the pairs (L°, P°). Then L meets G^1 if and only if the class of (L°, P°) is stable for this action. According to the above propositions, there is a conjugate of (L°, P°) that is σ-stable. This means that L contains σ and so $(L^\sigma)^\circ$ is a Levi subgroup of $(G^\sigma)^\circ$.

Proposition II.3.2.4. *([DM94, Proposition 1.40]) Assume that $\sigma \in G$ is quasi-central, F-stable, and G/G° is generated by the component of σ. Then the G^F-conjugacy classes of the F-stable groups $L = N_G(L^\circ, P^\circ)$ meeting the connected component $G^\circ.\sigma$ are in bijection with the $((G^\sigma)^\circ)^F$-conjugacy classes of the F-stable Levi subgroups of $(G^\sigma)^\circ$ in the following manner. Each L has has a G^F-conjugate L_1 containing σ, and the bijection associates the $((G^\sigma)^\circ)^F$-class of $((L_1)^\sigma)^\circ$ to the G^F-class of L.*

This gives in particular the classification of the G^F-conjugacy classes of the F-stable groups of the form $T = N_G(T^\circ, B^\circ)$.

II.4 Generalised Deligne-Lusztig Induction

II.4.1 Induction for Connected Groups

We recall some generalities on the Deligne-Lusztig induction for the connected reductive groups. In this section we assume G to be connected. If X is a variety over k, we denote by $H_c^i(X)$ the i-th cohomology group with compact support with coefficient in $\bar{\mathbb{Q}}_\ell$, and we denote by $H_c^*(X) = \bigoplus(-1)^i H_c^i(X)$ the virtual vector space. For a finite group H, denote by $C(H)$ the set of the invariant $\bar{\mathbb{Q}}_\ell$-valued functions on H

II.4.1.1 Let L be an F-stable Levi factor of some parabolic subgroup $P \subset G$ not necessarily F-stable. The Levi decomposition writes $P = LU$. Put $\mathcal{L}_G^{-1}(U) = \{x \in G | x^{-1}F(x) \in U\}$. Then G^F acts on $\mathcal{L}_G^{-1}(U)$ by left multiplication and L^F acts by right multiplication. This induces a $G^F \times (L^F)^{op}$-module structure on $H_c^i(\mathcal{L}_G^{-1}(U))$ for any i. Let $\theta \in C(L^F)$. The Deligne-Lusztig induction of θ, denoted by $R_L^G\theta$, is the invariant function on G^F defined by

$$R_L^G\theta(g) = |L^F|^{-1} \sum_{l \in L^F} \theta(l^{-1}) \operatorname{Tr}((g,l)|H_c^*(\mathcal{L}^{-1}(U)), \quad \text{for all } g \in G^F. \tag{II.4.1.1.1}$$

It does not depend on the choice of P if $q \neq 2$ (*cf.* [DM20, §9.2]). The functions of the form $R_L^G\theta$ with L being a maximal torus are called Deligne-Lusztig characters.

II.4.1.2 The Green function is defined on the subset of unipotent elements in the following manner.

$$\begin{aligned} Q_L^G(-,-) : G_u^F \times L_u^F &\longrightarrow \mathbb{Z} \\ (u,v) &\longmapsto \operatorname{Tr}((u,v)|H_c^*(\mathcal{L}_G^{-1}(U))). \end{aligned} \tag{II.4.1.2.1}$$

The calculation of the Deligne-Lusztig induction is reduced to the calculation of the Green functions. If $g = su$ is the Jordan decomposition of $g \in G^F$, we have the character formula ([DM91, Proposition 12.2]),

$$R_L^G\theta(g) = |L^F|^{-1}|C_G(s)^{\circ F}|^{-1} \sum_{\{h \in G^F | s \in {}^hL\}} \sum_{\{v \in C_{{}^hL}(s)_u^{\circ F}\}} Q_{C_{{}^hL}(s)^\circ}^{C_G(s)^\circ}(u, v^{-1})\, {}^h\theta(sv), \tag{II.4.1.2.2}$$

where ${}^hL = h^{-1}Lh$ and ${}^h\theta(sv) = \theta(hsvh^{-1})$. The Green functions are usually normalised in such a way that the factor $|L^F|^{-1}$ is removed from the above formula.

II.4.1.3 We will need the following simple lemmas. Let σ be an automorphism of G that commutes with F. If $L \subset G$ is an F-stable Levi subgroup, then we also denote by σ the isomorphism $L^F \to \sigma(L)^F$ and the isomorphism $W_L(T) \to W_{\sigma(L)}(\sigma(T))$ for an F-stable maximal torus T.

Lemma II.4.1.1. *Let $M \subset G$ be an F-stable Levi subgroup and let θ be a character of M^F. Then the character $(R_M^G\theta) \circ \sigma^{-1}$ is equal to $R^G_{\sigma(M)}\sigma_*\theta$.*

Proof. Let Q be a parabolic subgroup containing M such that $Q = MU_Q$. Then $\sigma(\mathcal{L}^{-1}(U_Q)) = \mathcal{L}^{-1}(\sigma(U_Q))$ as F commutes with σ, and so

$$\mathrm{Tr}((\sigma(g), \sigma(l))|H_c^*(\mathcal{L}^{-1}(\sigma(U_Q))) = \mathrm{Tr}((g, l)|H_c^*(\mathcal{L}^{-1}(U_Q))$$

for any $g \in G^F$ and $l \in M^F$. The assertion then follows from the definition of $R_M^G\theta$. □

Lemma II.4.1.2. *Assume that $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))$ is of the form $R_\varphi^G\theta$ for a triple (M, φ, θ) as in Theorem 5. Then , the character $\chi \circ \sigma^{-1}$ is of the form $R^G_{\sigma_*\varphi}\sigma_*\theta$ for a triple $(\sigma(M), \sigma_*\varphi, \sigma_*\theta)$.*

Proof. Since F commutes with σ, we can define $T_{\sigma(w)}$ to be $\sigma(T_w)$ and so by the definition of $R_\varphi^G\theta$, the lemma follows from the equality $R^G_{\sigma(T_w)}\sigma_*\theta(g) = R^G_{T_w}\theta(\sigma^{-1}(g))$. □

II.4.2 Induction for the Non Connected Groups

We will assume that G/G° is cyclic, and fix $\sigma \in G$ quasi-central and F-stable such that $G = G^\circ.<\sigma>$.

II.4.2.1 Given an F-stable Levi factor L° of some parabolic subgroup P° not necessarily F-stable that is decomposed as $P^\circ = L^\circ U$, we put L and P to be the normalisers defined in §II.3.1.1. Put $\mathcal{L}_G^{-1}(U) = \{x \in G | x^{-1}F(x) \in U\}$ and $\mathcal{L}_{G^\circ}^{-1}(U) = \{x \in G^\circ | x^{-1}F(x) \in U\}$. Then $G^F \times (L^F)^{op}$ acts on $\mathcal{L}_G^{-1}(U)$, and $H_c^*(\mathcal{L}_G^{-1}(U))$ is thus a $G^F \times (L^F)^{op}$-module. For $\theta \in \mathcal{C}(L^F)$, the (generalised) Deligne-Lusztig induction of θ is defined by,

$$R_L^G\theta(g) = |L^F|^{-1} \sum_{l \in L^F} \theta(l^{-1}) \,\mathrm{Tr}((g, l)|H_c^*(\mathcal{L}_G^{-1}(U)), \quad \forall\, g \in G^F. \tag{II.4.2.1.1}$$

It does not depend on the choice of P° if $q \neq 2$.② According to Proposition II.3.2.4, the generalised Deligne-Lusztig induction are parametrised by the pairs (L°, P°) consisting of an F-stable and σ-stable Levi factor of some σ-stable parabolic subgroup. Since only those L that meets $G.\sigma$ interest us, we can assume that $L = L^\circ.<\sigma>$. The restriction of R_L^G to $L^{\circ F}.\sigma$ is a map $\mathcal{C}(L^{\circ F}.\sigma) \to \mathcal{C}(G^{\circ F}.\sigma)$, that we denote by $R^{G^\circ.\sigma}_{L^\circ.\sigma}$. To simplify the terminology, we may also call them Deligne-Lusztig inductions.

II.4.2.2 The following lemma shows that the induction thus defined is compatible with that defined for connected groups.

②As is pointed out to be by F. Digne, for $G = G^\circ.<\sigma>$and σ non trivial, this is reduced to the case of connected groups.

Lemma II.4.2.1. *We keep the notations as above. Denote by* Res *the usual restriction of functions. If $G^F = L^F.G^{\circ F}$, then,*

$$\mathrm{Res}^{G^F}_{G^{\circ F}} \circ R^G_L = R^{G^\circ}_{L^\circ} \circ \mathrm{Res}^{L^F}_{L^{\circ F}}. \tag{II.4.2.2.1}$$

Proof. See [DM94, Corollaire 2.4 (i)]. □

II.4.2.3 The Green function is defined by

$$\begin{aligned} Q^G_L(-,-) : G^F_u \times L^F_u &\longrightarrow \mathbb{Z} \\ (u,v) &\longmapsto \begin{cases} 0 & \text{if } uv \notin G^\circ \\ \mathrm{Tr}((u,v)|H^*_c(\mathcal{L}^{-1}_{G^\circ}(U))) & \text{otherwise.} \end{cases} \end{aligned} \tag{II.4.2.3.1}$$

Note that $\mathcal{L}^{-1}_{G^\circ}(U)$ is the usual Deligne-Lusztig variety.

Proposition II.4.2.2. *(Character Formula, [DM94, Proposition 2.6]) Let L be the normaliser of the pair $(L^\circ \subset P^\circ)$ as above, and let θ be a character of L^F. Then for any $g \in G^F$ with Jordan decomposition $g = su$,*

$$R^G_L\theta(g) = |L^F|^{-1}|C_G(s)^{\circ F}|^{-1} \sum_{\{h \in G^F | s \in {}^hL\}} \sum_{\{v \in C_{{}^hL}(s)^F_u\}} Q^{C_G(s)^\circ}_{C_{{}^hL}(s)^\circ}(u, v^{-1})\, {}^h\theta(sv). \tag{II.4.2.3.2}$$

This formula will only be used in the following form.

Proposition II.4.2.3. *([DM94, Proposition 2.10]) We keep the notations as above, except that θ is now a σ-stable character of $L^{\circ F}$. Denote by $\tilde{\theta}$ an extension of θ to $L^{\circ F}.\sigma$, and let su be the Jordan decomposition of $g\sigma$, $g \in G^{\circ F}$. Then,*

$$R^{G^\circ.\sigma}_{L^\circ.\sigma}\tilde{\theta}(g\sigma) = |L^{\circ F}|^{-1}|C_G(s)^{\circ F}|^{-1} \sum_{\{h \in G^{\circ F} | s \in {}^hL\}} \sum_{\{v \in C_{{}^hL}(s)^F_u\}} Q^{C_G(s)^\circ}_{C_{{}^hL}(s)^\circ}(u, v^{-1})\, {}^h\tilde{\theta}(sv). \tag{II.4.2.3.3}$$

The Green functions are usually normalised in such a way that the above two formulas should be multiplied by $|C_{{}^hL}(s)^{\circ F}|$.

II.4.3 Uniform Functions

II.4.3.1 The irreducible characters of G^F for connected G can be expressed as linear combinations of the Deligne-Lusztig inductions of cuspidal functions on various L^F. For $G = \mathrm{GL}^\epsilon_n(\mathbb{F}_q)$, we have Theorem 5 in Introduction. It shows that in this particular case one only needs the functions $R^G_T\theta$ induced from F-stable tori, and the transition matrix is given by the characters of the Weyl group of G. In general, the transition matrix could be more complicated and the functions induced from the characters of tori are not sufficient. The invariant functions on G^F that are linear combinations of the $R^G_T\theta$'s are called *uniform functions*.

Recall that for each $w \in W_G$, the Weyl group of G, and some $\dot{w} \in G$ representing w, one can find $g \in G$ such that $g^{-1}F(g) = \dot{w}$, then $T_w := gTg^{-1}$ is an F-stable maximal torus such that $T_w^F \cong T^{F_w}$ with $F_w := \mathrm{ad}\,\dot{w} \circ F$. Then $w \mapsto T_w$ defines a bijection between the F-conjugacy classes of W_G and the G^F-conjugacy classes of F-stable maximal tori.

II.4.3.2 Now assume that G is non connected. Let $T \subset G^\circ$ be an F-stable and σ-stable maximal torus contained in a σ-stable Borel subgroup. If $\theta \in \mathrm{Irr}(T^F)$ extends into $\tilde{\theta} \in \mathrm{Irr}(T^F.<\sigma>)$, then $R_{T.\sigma}^{G^\circ.\sigma}\tilde{\theta}$ belongs to $\mathcal{C}(G^{\circ F}.\sigma)$, the set of $\bar{\mathbb{Q}}_\ell$-valued functions on $G^{\circ F}.\sigma$ that are invariant under $G^{\circ F}$. A function in $\mathcal{C}(G^{\circ F}.\sigma)$ is called *uniform* if it is a linear combination of functions of the form $R_{T.\sigma}^{G^\circ.\sigma}\tilde{\theta}$.

II.4.3.3 Let $\tilde{\chi}$ be an irreducible character of G^F. It is called *unipotent* if $\chi := \mathrm{Res}_{G^{\circ F}}^{G^F}\tilde{\chi}$ contains a unipotent character as a direct summand. In this case, $\mathrm{Res}_{G^{\circ F}}^{G^F}\chi$ is a sum of unipotent characters, as its summands are G^F-conjugate. Denote by

$$\mathcal{E}(G^{\circ F}.\sigma, (1)) = \{\tilde{\chi}|_{G^{\circ F}.\sigma} \mid \tilde{\chi} \text{ is a unipotent character}\}.$$

An element of $\mathcal{C}(G^{\circ F}.\sigma)$ is called *unipotent* if it is a linear combination of some elements of $\mathcal{E}(G^F.\sigma, (1))$, and we denote by $\mathcal{C}(G^{\circ F}.\sigma, (1))$ this subspace. It is clear that the characters $R_{T_w.\sigma}^{G^\circ.\sigma}1$ are unipotent functions, and they are parametrised by the F-conjugacy classes of W^σ. An element of $\mathcal{C}(G^{\circ F}.\sigma)$ is called *uniform-unipotent* if it is a linear combination of the functions $R_{T_w.\sigma}^{G^\circ.\sigma}1$.

II.4.3.4 A natural question is to identify those elements of $\mathcal{E}(G^{\circ F}.\sigma, (1))$ that are uniform. We have,

Theorem II.4.3.1. *([DM94, Théorème 5.2]) Put $G = \mathrm{GL}_n^\epsilon(k)$. For any $\varphi \in \mathrm{Irr}(W_G^\sigma)^F$, define*

$$R_\varphi^{G.\sigma}1 := |W_G^\sigma|^{-1} \sum_{w \in W_G^\sigma} \tilde{\varphi}(wF) R_{T_w.\sigma}^{G.\sigma}1. \tag{II.4.3.4.1}$$

Then, $R_\varphi^{G.\sigma}1$ is an extension of $R_{\pi(\varphi)}^G 1$, where we defined the injection $\pi : \mathrm{Irr}(W_G^\sigma) \to \mathrm{Irr}(W_G)$ by identifying $\mathrm{Irr}(W_G^\sigma)$ with the set of principal series unipotent representations of $\mathrm{GL}_n^-(\mathbb{F}_q)$ and identifying $\mathrm{Irr}(W_G)$ to the set of the unipotent representations of $\mathrm{GL}_n^-(\mathbb{F}_q)$.

This gives all uniform-unipotent functions on $\mathrm{GL}_n^\epsilon(\mathbb{F}_q).\sigma$. It follows that an element of $\mathcal{E}(G^{\circ F}.\sigma, (1))$ is either uniform or orthogonal to the space of uniform(-unipotent) functions.

II.4.3.5 ([DM15, Proposition 6.4]) The characteristic functions of quasi-semi-simple classes are uniform. Consequently, all non uniform characters vanish on the quasi-semi-simple classes.

II.5 The Group $\mathrm{GL}_n(k) \rtimes \mathbb{Z}/2\mathbb{Z}$

In the following, we write $G = \mathrm{GL}_n(k)$.

II.5.1 Automorphisms of $\mathrm{GL}_n(k)$

II.5.1.1 Let $\mathcal{J}_n$ be the matrix

$$(J_n)_{ij} = \delta_{i,n+1-j} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \tag{II.5.1.1.1}$$

Put $t_0 = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_i = 1$ if $i \leq [(n+1)/2]$ and $a_i = -1$ otherwise. Put $\mathcal{J}'_n = t_0 \mathcal{J}_n$. Define matrices

$$\mathcal{J} = \begin{cases} \mathcal{J}'_n & \text{if } n \text{ is even} \\ \mathcal{J}_n & \text{if } n \text{ is odd} \end{cases}, \quad \mathcal{J}' = \begin{cases} \mathcal{J}_n & \text{if } n \text{ is even} \\ \mathcal{J}'_n & \text{if } n \text{ is odd} \end{cases} \tag{II.5.1.1.2}$$

The automorphism $\sigma \in \mathrm{Aut}(\mathrm{GL}_n(k))$ that sends g to $\mathcal{J}^t g^{-1} \mathcal{J}^{-1}$ will be called the standard automorphism. We will denote by σ' the automorphism defined by replacing $\mathcal{J}$ with $\mathcal{J}'$ in the definition of σ. They are quasi-semi-simple automorphisms because the maximal torus consisting of the diagonal matrices and the Borel subgroup consisting of the upper triangular matrices are stable under the action of σ or σ'. Moreover, σ is a quasi-central involution regardless of the parity of n, while σ' is not an involution if n is odd and is not quasi-central if n is even.

II.5.1.2 The classification of the involutions and the quasi-central automorphisms is well known.([LiSe, Lemma 2.9], [DM94, Proposition 1.22]).

The conjugacy classes of the involutions are described as follows.

- If $n = 2m + 1$, the exterior involutions (exterior automorphisms of order 2) are all G-conjugate and their centralisers are of type B_m.
- If $n = 2m > 2$, there are two G-conjugacy classes of exterior involutions, with centralisers of type $C_{n/2}$ and $D_{n/2}$ respectively. If $n = 2$, the connected centralisers are $\mathrm{SL}_2(k)$ and k^* respectively.

The conjugacy classes of quasi-central automorphisms are described as follows.

- If $n = 2m + 1$, there are two classes of exterior quasi-central automorphisms, with centralisers of type B_m and of type C_m respectively.
- If $n = 2m$, there is one single class of exterior quasi-central automorphisms, with centraliser of type C_m.

Explicitly,

$$\text{(II.5.1.2.1)} \qquad \text{if } n = 2m, \begin{cases} (G^{\sigma})^{\circ} = \mathrm{Sp}_{2m}(k) \\ (G^{\sigma'})^{\circ} = \mathrm{SO}_{2m}(k) \end{cases}, \quad \text{if } n = 2m+1, \begin{cases} (G^{\sigma})^{\circ} = \mathrm{SO}_{2m+1}(k) \\ (G^{\sigma'})^{\circ} = \mathrm{Sp}_{2m}(k) \end{cases}.$$

Put $t = \mathrm{diag}(a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m})$, or $\mathrm{diag}(a_1, \ldots, a_m, 1, a_{m+1}, \ldots, a_{2m})$ if n is odd, with $a_i = \mathrm{i}$ for $i \leq m$ and $a_i = -\mathrm{i}$ for $i > m$, we have

$$\text{(II.5.1.2.2)} \qquad \begin{cases} (G^{t\sigma})^{\circ} = \mathrm{SO}_{2m}(k) & \text{si } n = 2m, \\ (G^{t\sigma})^{\circ} = \mathrm{Sp}_{2m}(k) & \text{si } n = 2m+1. \end{cases}$$

We say that an automorphism is of symplectic type or orthogonal type according to the type of its centraliser. We may write

$$\text{(II.5.1.2.3)} \qquad \sigma_o = \begin{cases} \sigma' & \text{if } n \text{ is even} \\ \sigma & \text{if } n \text{ is odd} \end{cases}, \quad \sigma_s = \begin{cases} \sigma & \text{if } n \text{ is even} \\ \sigma' & \text{si } n \text{ is odd} \end{cases}.$$

We define an automorphism $\tau \in \mathrm{Aut}(G)$ as sending g to ${}^t g^{-1}$.

II.5.1.3 We will encounter another type of quasi-central automorphism. Let ι_0 be an involution of G, not necessarily an exterior automorphism. Define an automorphism ι of $G \times G$ by $(g, h) \mapsto (\iota_0(h), \iota_0(g))$. It is easy to see from the definition that ι is quasi-central and $(G \times G)^{\iota} \cong G$.

II.5.2 The Group $\bar{G}$

II.5.2.1 A choice of an involution in $\mathrm{Aut}\, G$ defines a semi-direct product $\bar{G} := G \rtimes \mathbb{Z}/2\mathbb{Z}$. Suppose that σ_0 is such an involution, we will write $G \rtimes <\sigma_0> = G \rtimes \mathbb{Z}/2\mathbb{Z}$ to specify the action of $1 \in \mathbb{Z}/2\mathbb{Z}$ on G, in other words, σ represents the element $(e, 1) \in G \rtimes \mathbb{Z}/2\mathbb{Z}$, although σ_0 is not acturally an element of the group.

Proposition II.5.2.1. *For $G = \mathrm{GL}_n(k)$, there are three isomorphic classes of the semi-direct product $G \rtimes \mathbb{Z}/2\mathbb{Z}$ when n is even, corresponding to the inner involutions, the symplectic type outer involutions and the orthogonal type outer involutions. When n is odd, there is one isomorphic class corresponding to inner automorphisms, and only one isomorphic class corresponding to outer automorphisms.*

Proof. Let us first consider when the semi-direct products defined by two involutions σ_1 and σ_2 are isomorphic. Suppose there is an isomorphism

$$\text{(II.5.2.1.1)} \qquad \psi : G \rtimes <\sigma_1> \longrightarrow G \rtimes <\sigma_2>.$$

Let $x\sigma_2$ be the image of σ_1. It is necessary that $(x\sigma_2)^2 = 1$ and for all $g \in G$,

$$\text{(II.5.2.1.2)} \qquad \psi(\sigma_1(g)) = \psi(\sigma_1 g \sigma_1^{-1}) = x\sigma_2(\psi(g))x^{-1}.$$

Comparing the two ends of this equation, we see that σ_1 and σ_2 must lie in the same connected component of $\mathrm{Aut}\, G$.

If they are both inner automorphisms, then they must define isomorphic semi-direct product. In fact, the resulting semi-direct product is the direct product. To see this, let σ_2 be the trivial automorphism. We can always compose ψ with an automorphism of G (extending to $G \times <\sigma_2>$) such that $\psi|_G = \mathrm{Id}$, then the above equation reads $\sigma_1(g) = xgx^{-1}$ for some x satisfying $x^2 = 1$. That is, $\sigma_1 = \mathrm{ad}\, x$ for $x^2 = 1$, but these are exactly the inner involutions.

If σ_1 and σ_2 are both outer automorphisms, then we need some explicit information about the group G. But let us first note that if σ_1 and σ_2 are G-conjugate, then they define isomorphic semi-direct product. Suppose $\sigma_1 = \sigma$ and $\sigma_2 = y\sigma y^{-1}$ for some outer involution σ and $y \in G$. Then we put $x = \sigma(y)y^{-1}$. Again, assume $\psi|_G = \mathrm{Id}$, and so $(x\sigma_2)^2 = 1$ and $\sigma_1(g) = x\sigma_2(g)x^{-1}$ for all $g \in G$, as required.

We now restrict ourselves to $G = \mathrm{GL}_n(k)$. We have already said that $\mathrm{GL}_n(k)$ has two distinct conjugacy classes of outer involutions when n is even. Suppose σ_1 is of symplectic type and σ_2 is of orthogonal type and that there is an isomorphism ψ between the two semi-direct products. Then in the group $G \rtimes <\sigma_2>$, $\sigma_2^2 = 1$ by definition, and $(x\sigma_2)^2 = 1$ as the image of σ_1^2. But $x\sigma_2$ is of symplectic type as its action on G is the same as σ_1, assuming $\psi|_G = \mathrm{Id}$. We deduce that, modulo k^*, $x\sigma_2$ is conjugate to $t\sigma_2$, where t is as in (II.5.1.2.2). Since $(t\sigma_2)^2 = -1$, and for any $z \in k^*$ $(zt\sigma_2)^2 = (t\sigma_2)^2$, we have $(x\sigma_2)^2 = -1$, which is a contradiction. □

We will write ${}^s\bar{G}$ or ${}^o\bar{G}$ to indicate whether $\sigma_0 = \sigma_s$ or σ_o, and write $\bar{G}$ when there is no need to distinguish them. Note however, that the above classification also works for $\mathrm{SL}_n(k)$. But in $\mathrm{PGL}_n(k)$, since there is no difference between 1 and -1, the two isomorphism classes degenerate, and they are actually isomorphic to $\mathrm{Aut}(G)$. If no confusion arises, we can also denote by $\bar{G}$ the direct product $G \times \mathbb{Z}/2\mathbb{Z}$.

By definition, σ is a quasi-central element in $G.\sigma$.

II.5.2.2 Let H be an abstract group and let τ be an automorphism of finite order of H. By a τ-conjugacy class of H, we mean an orbit in H under the action $h : x \mapsto hx\tau(h^{-1})$, $x, h \in H$. By a τ-class function on H, we mean a function that is constant on the τ-conjugacy classes. We denote by $\mathcal{C}(H.\tau)$ the set of τ-class functions.

On the other hand, the conjugacy classes of $H \rtimes <\tau>$ contained in $H.\tau$ are identified with the H-conjugacy classes in $H.\tau$, as $\tau(h)\tau = h^{-1}(h\tau)h$, which are in turn identified with the τ-conjugacy classes of H. This justifies the notation $\mathcal{C}(H.\tau)$.

II.5.2.3 The character tables of ${}^s\bar{G}^F$ and of ${}^o\bar{G}^F$ are related in the following manner.

The G^F-conjugacy classes in ${}^s\bar{G}^F \setminus G^F$ are in bijection with the σ-conjugacy classes in G^F, which are in bijection with the $t_0\sigma'$-conjugacy classes in G^F (See §II.5.1.1 for t_0), which are in bijection with the G^F-conjugacy classes in ${}^o\bar{G}^F \setminus G^F$. More specifically, for $g \in G^F$, the G^F-class of $g\sigma \in {}^s\bar{G}^F$ corresponds to the G^F-class of $gt_0\sigma' \in {}^o\bar{G}^F$.

Since σ and σ' differ by an inner automorphism, the set of σ-stable characters coincides with that of σ'-stable characters. However, the extension of a σ-stable character to $\bar{G}^F$ behaves differently for ${}^s\bar{G}^F$ and ${}^o\bar{G}^F$. Let $\rho : G^F \to \mathrm{GL}(V)$ be a σ-stable representation of G^F. To find an extension $\tilde{\rho} \in \mathrm{Irr}({}^s\bar{G}^F)$ of ρ is to define $\tilde{\rho}(\sigma)$ in such a way that $\tilde{\rho}(\sigma)^2 = \rho(\sigma^2) = \mathrm{Id}$ and $\tilde{\rho}(\sigma)\rho(g)\tilde{\rho}(\sigma)^{-1} = \rho(\sigma(g))$ for all $g \in G^F$. Suppose that we have defined such an extension, and we would like to define $\tilde{\rho}' \in \mathrm{Irr}({}^o\bar{G}^F)$ by $\tilde{\rho}'(t_0\sigma') = \tilde{\rho}(\sigma)$. For ${}^o\bar{G}^F$, if $\rho(-1) \neq \mathrm{Id}$, the equality $\tilde{\rho}(\sigma)^2 = \mathrm{Id}$ would be violated. Consequently, we define instead $\tilde{\rho}'(t_0\sigma') = \tilde{\rho}(\sigma)\sqrt{\rho(-1)}$, where $\rho(-1)$ has as value $\pm\,\mathrm{Id}$. Replacing $\sqrt{\rho(-1)}$ by $-\sqrt{\rho(-1)}$ defines another extension of ρ to ${}^o\bar{G}^F$. We denote by $\tilde{\chi}$ and $\tilde{\chi}'$ the characters of $\tilde{\rho}$ et $\tilde{\rho}'$ respectively. Then, for all $g \in G^F$,

$$\tilde{\chi}'(gt_0\sigma') = \tilde{\chi}(g\sigma)\sqrt{\rho(-1)}, \tag{II.5.2.3.1}$$

regarding $\rho(-1)$ as a scalar.

Convention II.5.2.2. Because of the above discussion, we will also denote by σ the element $t_0\sigma' \in {}^o\bar{G}^F$. We will later parametrise the conjugacy classes in ${}^o\bar{G}^F \setminus G^F$ with respect to σ (Proposition II.3.1.2).

Remark II.5.2.3. We have $\rho(-1) = -\mathrm{Id}$ only if η has odd "multiplicity" in the semi-simple part of ρ and $q \equiv 3 \mod 4$. In particular, if $\tilde{\chi}$ is a uniform function on $G^F.\sigma$, then $\rho(-1)$ always equals to Id.

Question II.5.2.4. If $q \equiv 1 \mod 4$, then the character table of ${}^s\bar{G}^F$ and that of ${}^o\bar{G}^F$ coincide under the bijections of characters and conjugacy classes described above. Are these groups isomorphic? Working with finite groups, there might be isomorphisms that are not deduced from the underlying algebraic groups.

II.5.3 Quasi-Semi-Simple Elements

II.5.3.1 We have said in §II.3.1 that all semi-simple elements are quasi-semi-simple. For $\bar{G}$, we have

Lemma II.5.3.1. *An element of $\bar{G}$ is quasi-semi-simple if and only if it is semi-simple.*

More generally, if G is a reductive algebraic group and G/G° is semi-simple, then all quasi-semi-simple elements are semi-simple. (See [DM94, Remarque 2.7]) In positive characteristic, this is to require that $\mathrm{char}\, k \nmid |G/G^\circ|$. We give a short proof below.

Proof. It suffices to show that each quasi-semi-simple element $s\sigma \in G.\sigma$ is semi-simple. We see that $s\sigma$ is semi-simple if and only if $(s\sigma)^2 = s\sigma(s)\sigma^2$ is semi-simple, as we have assumed $\mathrm{char}\, k$ to be odd. Let (T, B) be a pair consisting of a maximal torus and a Borel subgroup containing it, both normalised by σ. Then every quasi-semi-simple element is conjugate to an element of $(T^\sigma)^\circ\sigma$ (Proposition II.3.1.2), and its square, lies in T, and so is semi-simple. □

Remark II.5.3.2. That $s\sigma$ is semi-simple does not imply that s is so. Let us fix T and B as above, and write $B = TU$, where U is the unipotent radical of B. If we take $u \in U$, then $u\sigma(u^{-1})$ is a unipotent element, whereas $u\sigma u^{-1}$ is semi-simple.

Remark II.5.3.3. There is no unipotent element in $G.\sigma$ because an odd power of $s\sigma$ lies in $G.\sigma$.

II.5.3.2 Isolated Elements Define the diagonal matrix

$$t(j) = \mathrm{diag}(\overbrace{\mathrm{i}, \dots, \mathrm{i}}^{j}, 1, \dots, 1, \overbrace{-\mathrm{i}, \dots, -\mathrm{i}}^{j}) \in \mathrm{GL}_n(k) \tag{II.5.3.2.1}$$

The elements $t(j)\sigma$, $0 \leq j \leq [n/2]$, are the representatives of the isolated elements (§II.3.1.5), except when n is even and $j = 1$, in which case $t(j)\sigma$ is quasi-isolated ([DM18, Proposition 4.2]). We have,

$$\begin{aligned} &\text{If } n \text{ is even, } C_G(t(j)\sigma) \cong \mathrm{O}_{2j}(k) \times \mathrm{Sp}_{n-2j}(k); \\ &\text{If } n \text{ is odd, } C_G(t(j)\sigma) \cong \mathrm{Sp}_{2j}(k) \times \mathrm{O}_{n-2j}(k). \end{aligned} \tag{II.5.3.2.2}$$

In particular, when n is even, $t(j)\sigma$ is quasi-central only if $j = 0$, and when $n = 2m + 1$, $t(j)\sigma$ is quasi-central only if $j = m$. Note that our choice of σ for odd n is different from that of [DM18].

II.5.3.3 Semisimple Conjugacy Classes of $\mathrm{GL}_n(k) \rtimes \mathbb{Z}/2\mathbb{Z}$ Since we assume $\mathrm{char}\, k \neq 2$, quasi-semi-simple elements are semi-simple.

At the level of the parametrisation of semi-simple G-conjugacy classes contained in $G.\sigma$, there will be no difference between ${}^{s}\bar{G}$ and ${}^{o}\bar{G}$, so we write $\bar{G} = G \cup G.\sigma$ and σ can be either σ_o or σ_s in this part. Let T be the diagonal matrices(a σ-stable maximal torus) of G and let W be the Weyl group defined by T, which admits an action of σ induced from G. Denote by W^σ the subgroup of σ-fixed points. The Borel subgroup of upper triangular matrices is also stable under σ.

Denote by $(T^\sigma)^\circ$ the connected centraliser of σ in T. It consists of matrices of the form

$$\mathrm{diag}(a_1, \dots, a_m, a_m^{-1}, \dots, a_1^{-1}),\ a_i \in k^* \tag{II.5.3.3.1}$$

if $n = 2m$, and with an extra 1 in the middle if n is odd.

Denote by $[T, \sigma]$ the commutator. It consists of

$$\mathrm{diag}(b_1, \dots, b_m, b_m, \dots, b_1),\ b_i \in k^*, \tag{II.5.3.3.2}$$

if $n = 2m$, and with an extra 1 in the middle if n is odd.

So $S := (T^\sigma)^\circ \cap [T, \sigma]$ consists of

$$\text{(II.5.3.3.3)} \qquad \mathrm{diag}(e_1, \ldots, e_m, e_m, \ldots, e_1),\ e_i \in \{\pm 1\}.$$

and modified accordingly for n odd.

By Proposition II.3.1.2, the semisimple conjugacy classes in $G.\sigma$ are parametrised by the W^σ-orbits on the quotient $(T^\sigma)^\circ/S$. The two automorphisms σ_s and σ_o have the same action on T and on W, so we see that there is indeed no difference between them. Note that W^σ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathfrak{S}_m$ so that in some basis $\{\epsilon_1, \ldots, \epsilon_m, \epsilon_{-m}, \ldots, \epsilon_{-1}\}$ diagonalising T, W^σ acts by interchanging ϵ_i and ϵ_{-i}(the $\mathbb{Z}/2\mathbb{Z}$ factors) and symmetrically permuting the vectors $\epsilon_1, \ldots, \epsilon_m$ and $\epsilon_{-m}, \ldots, \epsilon_{-1}$(the symmetric group).

In other words, the semisimple classes have representatives

$$\text{(II.5.3.3.4)} \qquad \mathrm{diag}(a_1, \ldots, a_m, a_m^{-1}, \ldots, a_1^{-1})\sigma,$$

and the following operations will leave it in the same conjugacy class:

- Interchanging a_i and a_i^{-1};
- Changing any pair (a_i, a_i^{-1}) to $(-a_i, -a_i^{-1})$;
- Symmetrically permuting the a_i's and a_i^{-1}'s.

For any $z \in k^*$, denote by $\bar{z}$ the set $\{z, -z, z^{-1}, -z^{-1}\}$, or rather, the orbit in k^* under the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, with two generators of the group sending z to $-z$ and z^{-1} respectively. Then the set $\{\bar{a}_1, \ldots, \bar{a}_m\}$ is considered as the set of eigenvalues of the above semi-simple element.

If C is any semisimple conjugacy class contained in $G.\sigma$. Then $\tilde{C} = \{(x\sigma)^2 | x\sigma \in C\}$ is a σ-stable conjugacy class in G. However, not all σ-stable semi-simple conjugacy classes of G are of this form.

II.5.4 Irreducible Subgroups of $\mathrm{GL}_n \rtimes <\sigma>$

II.5.4.1 Maximal Parabolic Subgroups of $G \rtimes \mathbb{Z}/2\mathbb{Z}$ We are only interested in those parabolic subgroups that meet both connected components of $\bar{G} = G \rtimes \mathbb{Z}/2\mathbb{Z}$. If $G \rtimes \mathbb{Z}/2\mathbb{Z} \cong G \times \mathbb{Z}/2\mathbb{Z}$, then a maximal parabolic subgroup is just the union of two copies of a maximal parabolic subgroup of G, one copy in each connected component.

Now let $G \rtimes \mathbb{Z}/2\mathbb{Z}$ be defined by some graph automorphism σ. For $G = \mathrm{GL}_n(k)$, there will be no difference between ${}^s\bar{G}$ and ${}^o\bar{G}$ for the present problem, so we will not specify the conjugacy class of σ_0. We conclude from §II.1.1.5 that if T is the maximal torus of the diagonal matrices and B is the Borel subgroup of the upper triangular matrices, then every maximal standard parabolic subgroup P of G containing B such that $N_{\bar{G}}(P)$ meets $G.\sigma_0$ is of

the following form,

$$\left(\begin{array}{c|cc|c} A & * & & * \\ \hline 0 & & & \\ 0 & & * & * \\ \hline 0 & 0 & 0 & B \end{array}\right),$$

where A and B are square matrices of the same size. Normalisers of such P's ($= P \cup P\sigma_0$) are the representatives of the G-conjugacy classes of maximal parabolic subgroups of $\bar{G}$ that meet $G.\sigma_0$.

II.5.4.2 Irreducible Subgroups of $\mathrm{GL}_n \rtimes <\sigma>$ Let $H_0 \subset \mathrm{GL}_n$ be a topologically finitely generated closed subgroup. i.e. $H_0 = H(\mathbf{x})$ for some finite tuple $\mathbf{x}$ of closed points of GL_n and let $H \subset \mathrm{GL}_n \rtimes <\sigma>$ be a closed subgroup generated by H_0 and an semi-simple element $x_0\sigma \in \mathrm{GL}_n .\sigma$ such that $H_0 = H \cap \mathrm{GL}_n$. This condition is equivalent to

- $x_0\sigma \in N_{\mathrm{GL}_n .\sigma}(H_0)$;
- $(x_0\sigma)^2 \in H_0$.

Proposition II.5.4.1. *If H is irreducible, then the natural representation k^n of GL_n is a direct sum of pairwise non isomorphic irreducible H_0-representations, say $\bigoplus_j V_j$, and the centraliser $C_{\mathrm{GL}_n}(H)$ is isomorphic to $\prod_j \mu_2$, where for each j, the elements of $\mu_2 = \{\pm \mathrm{Id}\}$ are regarded as scalar endomorphisms of V_j.*

Proof. The second statement follows from the proof of the first.

Let us first note that the centre of $\mathrm{GL}_n \rtimes <\sigma>$ is $\{\pm \mathrm{Id}\}$, so irreducibility is equivalent to having finite centraliser in GL_n by Theorem II.1.2.8. (The group H is topologically generated by finitely many, say m, elements. Then consider the conjugation action of G° on G^m, $G = \mathrm{GL}_n \rtimes <\sigma>$.) Since H is irreducible, H_0 is completely reducible in GL_n by Lemma II.1.2.2, and so k^n can be written as a direct sum of irreducible H_0-representations, say

$$k^n \cong \bigoplus_j V_j^{\oplus r_j}, \tag{II.5.4.2.1}$$

where V_j is not isomorphic to $V_{j'}$ whenever $j \neq j'$. We see that

$$C_{\mathrm{GL}_n}(H_0) \cong \prod_j \mathrm{GL}_{r_j}, \tag{II.5.4.2.2}$$

where each entry of $\mathrm{GL}_{r_j}(k)$ is identified with a scalar endomorphism of V_j.

Let us now prove that $r_j = 1$ for all j. In order for an element of $C_{\mathrm{GL}_n}(H_0)$ to centralise H, it suffices for it to commute with $x_0\sigma$. Since $x_0\sigma$ normalises H_0, it normalises $C_{\mathrm{GL}_n}(H_0)$. Also, $(x_0\sigma)^2 \in H_0$, so $x_0\sigma$ defines an order 2 automorphism of $C_{\mathrm{GL}_n}(H_0)$ as an algebraic group. Choose a $x_0\sigma$-stable maximal torus of $C_{\mathrm{GL}_n}(H_0)$ and consider its root system with respect

to this maximal torus. If its action permutes two root subgroups of $C_{\mathrm{GL}_n}(H_0)$, say U_α and U_β, $\alpha \neq \beta$. Then $C_{\mathrm{GL}_n}(H)$ would have positive dimension, which is a contradiction. So $x_0\sigma$ fixes all roots of $C_{\mathrm{GL}_n}(H_0)$. But then it would be an inner semi-simple automorphism of the derived subgroup of $C_{\mathrm{GL}_n}(H_0)$, thus fix a maximal torus of it. We deduce that the derived subgroup of $C_{\mathrm{GL}_n}(H_0)$ must have rank 0, i.e. $C_{\mathrm{GL}_n}(H_0)$ is a torus. This means that $r_j = 1$ for all j.

The semi-simplicity of $x_0\sigma$ is only needed in the following arguments. Now denote by S the torus $C_{\mathrm{GL}_n}(H_0)$. Let $M = C_{\mathrm{GL}_n}(S)$. It is a $x_0\sigma$-stable Levi subgroup. (In fact this Levi subgroup corresponds to the above decomposition of k^n into irreducible representations.) So M contains a $x_0\sigma$-stable maximal torus T, which necessarily contains its centre S. Since the action of $x_0\sigma$ on $T \cong (\mathbb{G}_m)^n$ is a combination of inversing and permuting factors, so is its action on S. The only possibility for $x_0\sigma$ to have finite centraliser in S is that all factors of S are inversed while the permutation is trivial. Hence $C_{\mathrm{GL}_n}(H) \cong \prod_j \mu_2$. $\square$

Chapter III

Character Varieties with Non-Connected Structure Groups

III.1 $G \rtimes \Gamma$-Character Varieties

Let k be an algebraically closed field.

III.1.1 (Γ, ψ)-Invariant Representations

We define and study the (Γ, ψ)-invariant k-representations of finitely generated discrete groups.

III.1.1.1 Let Π be a finitely generated discrete group and let $p : \tilde{\Pi} \to \Pi$ be a finitely generated normal subgroup with finite index, i.e. we have the short exact sequence

$$1 \longrightarrow \tilde{\Pi} \xrightarrow{p} \Pi \longrightarrow \Gamma \longrightarrow 1.$$

We choose once and for all a section $\gamma_* : \Gamma \to \Pi$. Write $\gamma_\sigma = \gamma_*(\sigma)$ for $\sigma \in \Gamma$. In general, it can only be a map of sets, but we can always require that $\gamma_1 = 1$.

Let G be a connected reductive group over k. Denote by Z_G the centre of G and denote by $G_{ad} = G/Z_G$ the corresponding group of adjoint type. Denote by $\operatorname{Aut} G$ the group of automorphisms of G, its identity component being G_{ad}. Denote by $\mathbf{A}(G)$ the component group $(\operatorname{Aut} G)/G_{ad}$. When the centre of G has dimension ≤ 1, it is a finite group. Let $\psi : \Gamma \to \operatorname{Aut} G$ be a group homomorphism.

Denote by $\operatorname{Rep}(\tilde{\Pi}, G) := \operatorname{Hom}(\tilde{\Pi}, G)$ the space of G-representations of $\tilde{\Pi}$. The conjugation by G on the target induces an action on $\operatorname{Rep}(\tilde{\Pi}, G)$. Denote by $\operatorname{Ch}(\tilde{\Pi}, G)$ the resulting GIT quotient, called the G-character variety of $\tilde{\Pi}$.

Definition III.1.1.1. For any $\tilde{\rho} \in \operatorname{Rep}(\tilde{\Pi}, G)$, we say $\tilde{\rho}$ is (Γ, ψ)-*invariant* if there exists some

cochain $h_* = (h_\sigma)_{\sigma\in\Gamma} \in C^1(\Gamma, G)$ such that for any $\sigma \in \Gamma$,

$$h_\sigma \tilde{\rho} h_\sigma^{-1} = {}^\sigma\tilde{\rho} := \psi(\sigma) \circ \tilde{\rho} \circ C_\sigma, \tag{III.1.1.1.1}$$

where C_σ is the isomorphism

$$\begin{aligned} \tilde{\Pi} &\longrightarrow \tilde{\Pi} \\ \alpha &\longmapsto \gamma_\sigma^{-1}\alpha\gamma_\sigma, \end{aligned} \tag{III.1.1.1.2}$$

and h_σ conjugates on the target of $\tilde{\rho}$. In this case, we say that $(\tilde{\rho}, h_*)$ is a (Γ, ψ)-*invariant pair*. If no confusion arises, we simply say Γ-invariant instead of (Γ, ψ)-invariant.

We will simplify the notations in what follows by writing $\psi(\sigma)$ as ψ_σ and by writing $h_\sigma \tilde{\rho} h_\sigma^{-1}$ as $h_\sigma \tilde{\rho}$. However, when we evaluate the representation at a particular element, say α, we will use the usual notation $h_\sigma \tilde{\rho}(\alpha) h_\sigma^{-1}$.

Remark III.1.1.2. Note that $\tilde{\rho} \mapsto {}^\sigma\tilde{\rho}$ does not define a group action of Γ on $\mathrm{Rep}(\tilde{\Pi}, G)$ because $\gamma_\tau\gamma_\sigma$ is not necessarily equal to $\gamma_{\tau\sigma}$, for any $\sigma, \tau \in \Gamma$. But since $\tilde{\rho}(\gamma_\tau\gamma_\sigma\gamma_{\tau\sigma}^{-1}) \in G$, we do have a G-action of Γ on $\mathrm{Ch}(\tilde{\Pi}, G)$.

III.1.1.2 We list below some basic properties.

(i) If $(\tilde{\rho}, h_*)$ is a (Γ, ψ)-invariant pair, then $(x\tilde{\rho}, (\psi_\sigma(x) h_\sigma x^{-1}))$ is also a (Γ, ψ)-invariant pair.

(ii) If $(\tilde{\rho}, h_*)$ is a (Γ, ψ)-invariant pair, then $(\tilde{\rho}, h_* z)$ is also (Γ, ψ)-invariant for any $z \in \mathrm{Stab}_G(\tilde{\rho})$.

(iii) Let $(\tilde{\rho}, h_*)$ be a (Γ, ψ)-invariant pair. Let $\psi' : \Gamma \to \mathrm{Aut}\, G$ be another homomorphism. Suppose for all $\sigma \in \Gamma$, $\psi'_\sigma \psi_\sigma^{-1} = \mathrm{ad}\, x_\sigma$, for some $x_\sigma \in G$. Then $\tilde{\rho}$ can be completed into a (Γ, ψ')-invariant pair by defining $h'_\sigma = x_\sigma h_\sigma$.

(iv) If $(\tilde{\rho}, h_*)$ is a (Γ, ψ)-invariant pair for a choice of γ_*, then for another γ'_*, the pair $(\tilde{\rho}, (\psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1})) h_\sigma)_\sigma)$ is (Γ, ψ)-invariant, with $\delta_\sigma = \gamma_\sigma^{-1}\gamma'_\sigma$.

In the case of (iii), we say that ψ and ψ' are <u>similar</u>. So if ψ and ψ' are similar, then $\tilde{\rho}$ is (Γ, ψ)-invariant if and only if it is (Γ, ψ')-invariant. Similarity classes are parametrised by the set of homomorphisms of discrete groups

$$\mathrm{Hom}(\Gamma, \mathbf{A}(G)).$$

Each such homomorphism can be lifted to a homomorphism to $\mathrm{Aut}\, G$, since the latter is the semi-direct product of $\mathbf{A}(G)$ and the inner automorphisms. We will denote by $[\psi]$ the similarity class of ψ.

III.1.1.3 Let $G \rtimes \Gamma$ be the semi-direct product defined by the given ψ. There is a natural section $s : \Gamma \to G \rtimes \Gamma$, which is a group homomorphism, satisfying

$$\psi_\sigma(g) = s_\sigma g s_\sigma^{-1}, \text{ for any } g \in G \text{ and } \sigma \in \Gamma,$$

where we write $s_\sigma = s(\sigma)$ for any $\sigma \in \Gamma$.

Definition III.1.1.3. We say a homomorphism of groups $\Pi \to G \rtimes \Gamma$ is a $G \rtimes \Gamma$-representation of Π if the right square of the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] & \tilde{\Pi} \arrow[r, "p"] \arrow[d, "\tilde{\rho}"] & \Pi \arrow[r] \arrow[d, "\rho"] & \Gamma \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\
1 \arrow[r] & G \arrow[r] & G \rtimes \Gamma \arrow[r] & \Gamma \arrow[r] & 1
\end{tikzcd}$$

where $\tilde{\rho}$ is just the restriction of ρ. We say that $\tilde{\rho}$ is the *underlying G-representation* of ρ.

We will show that the (Γ, ψ)-invariant pairs correspond exactly to $G \rtimes \Gamma$-representations.

Lemma III.1.1.4. *If $\tilde{\rho}$ is the underlying G-representation of some $G \rtimes \Gamma$-representation ρ, then there exists some cochain h_* such that $(\tilde{\rho}, h_*)$ is (Γ, ψ)-invariant.*

Proof. We calculate, for any $\alpha \in \tilde{\Pi}$,

$$\begin{aligned}
{}^\sigma\tilde{\rho}(\alpha) &= \psi_\sigma \circ \rho \circ p \circ C_\sigma(\alpha) \\
&= \psi_\sigma \circ \rho(\gamma_\sigma^{-1} \alpha \gamma_\sigma) \\
&= (\psi_\sigma \rho(\gamma_\sigma^{-1})) \cdot \tilde{\rho}(\alpha)
\end{aligned} \tag{III.1.1.3.1}$$

By the definition of $G \rtimes \Gamma$ and the commutativity of the diagram, the conjugation action of $\rho(\gamma_\sigma)$ on G differs from ψ_σ by an inner automorphism, so we can define $h_\sigma \in G$ to be any element that induces this automorphism. □

III.1.1.4 The above lemme also gives a way to extend a (Γ, ψ)-invariant $\tilde{\rho}$ to a $G \rtimes \Gamma$-representation. Indeed, if ${}^\sigma\tilde{\rho} = h_\sigma \tilde{\rho}$, comparing this equation with the above calculation suggests

$$\rho(\gamma_\sigma) := h_\sigma^{-1} s_\sigma, \tag{III.1.1.4.1}$$

since the conjugation by s_σ is just ψ_σ. Any element in Π can be uniquely written as $\eta\gamma_\sigma$ for some $\sigma \in \Gamma$ and $\eta \in \tilde{\Pi}$. We then define

$$\rho(\eta\gamma_\sigma) := \tilde{\rho}(\eta)\rho(\gamma_\sigma). \tag{III.1.1.4.2}$$

In particular, $\rho|_{\tilde{\Pi}} = \tilde{\rho}$.

Lemma III.1.1.5. *Let $(\tilde{\rho}, h_*)$ be a (Γ, ψ)-invariant pair and put $g_{\sigma_1\sigma_2} := \tilde{\rho}(\gamma_{\sigma_1}\gamma_{\sigma_2}\gamma_\sigma^{-1})$. Then the formulae (III.1.1.4.1) and (III.1.1.4.2) define a homomorphism of groups $\Pi \to G \rtimes \Gamma$ if and only if for any $\sigma_1, \sigma_2 \in \Gamma$, the equality*

$$g_{\sigma_1\sigma_2} = h_{\sigma_1}^{-1}\psi_{\sigma_1}(h_{\sigma_2}^{-1})h_{\sigma_1\sigma_2} \tag{III.1.1.4.3}$$

holds and $h_1 = 1$.

Proof. Since $\gamma_1 = 1$ and $s_1 = 1$, the equality (III.1.1.4.1) implies that h_1 must be 1. Put $\sigma = \sigma_1\sigma_2$. The claimed condition is equivalent to

$$\rho(\gamma_{\sigma_1}\gamma_{\sigma_2}\gamma_\sigma^{-1}) = \rho(\gamma_{\sigma_1})\rho(\gamma_{\sigma_2})\rho(\gamma_\sigma)^{-1}. \tag{III.1.1.4.4}$$

Note that $\gamma_{\sigma_1}\gamma_{\sigma_2}\gamma_\sigma^{-1} \in \tilde{\Pi}$. Therefore it is a necessary condition. We will show that this equality implies $\rho(\eta_1\gamma_{\sigma_1}\eta_2\gamma_{\sigma_2}) = \rho(\eta_1\gamma_{\sigma_1})\rho(\eta_2\gamma_{\sigma_2})$, for any $\sigma_1, \sigma_2 \in \Gamma$, and any $\eta_1, \eta_2 \in \tilde{\Pi}$.

First we have, for any $\eta \in \tilde{\Pi}$ and any $\sigma \in \Gamma$,

$$\begin{aligned}
&\rho(\gamma_\sigma)\tilde{\rho}(\eta)\rho(\gamma_\sigma^{-1})\\
=&h_\sigma^{-1}\psi_\sigma(\tilde{\rho}(\eta))h_\sigma\\
=&h_\sigma^{-1\sigma}\tilde{\rho}({}^{\gamma_\sigma}\eta)h_\sigma\\
=&h_\sigma^{-1}h_\sigma\tilde{\rho}({}^{\gamma_\sigma}\eta)h_\sigma^{-1}h_\sigma\\
=&\tilde{\rho}({}^{\gamma_\sigma}\eta).
\end{aligned} \tag{III.1.1.4.5}$$

where ${}^{\gamma_\sigma}\eta := \gamma_\sigma\eta\gamma_\sigma^{-1}$.

We then calculate, writing $\sigma = \sigma_1\sigma_2$,

$$\begin{aligned}
&\rho(\eta_1\gamma_{\sigma_1}\eta_2\gamma_{\sigma_2})\\
=&\rho(\eta_1({}^{\gamma_{\sigma_1}}\eta_2)\gamma_{\sigma_1}\gamma_{\sigma_2})\\
=&\rho(\eta_1({}^{\gamma_{\sigma_1}}\eta_2)\gamma_{\sigma_1}\gamma_{\sigma_2}\gamma_\sigma^{-1}\gamma_\sigma)\\
=&\tilde{\rho}(\eta_1)\tilde{\rho}({}^{\gamma_{\sigma_1}}\eta_2)\tilde{\rho}(\gamma_{\sigma_1}\gamma_{\sigma_2}\gamma_\sigma^{-1})\rho(\gamma_\sigma)\\
=&\tilde{\rho}(\eta_1)\rho(\gamma_{\sigma_1})\tilde{\rho}(\eta_2)\rho(\gamma_{\sigma_1}^{-1})\rho(\gamma_{\sigma_1})\rho(\gamma_{\sigma_2})\rho(\gamma_\sigma)^{-1}\rho(\gamma_\sigma)\\
=&\tilde{\rho}(\eta_1)\rho(\gamma_{\sigma_1})\tilde{\rho}(\eta_2)\rho(\gamma_{\sigma_2})\\
=&\rho(\eta_1\gamma_{\sigma_1})\rho(\eta_2\gamma_{\sigma_2}).
\end{aligned} \tag{III.1.1.4.6}$$

□

Lemma III.1.1.6. *Let $(\tilde{\rho}, h_*)$ be a (Γ, ψ)-invariant pair. Then $g_{\sigma_1\sigma_2}$ is equal to $h_{\sigma_1}^{-1}\psi_{\sigma_1}(h_{\sigma_2}^{-1})h_\sigma$ up to* $\mathrm{Stab}_G(\tilde{\rho})$

Proof. Let us compute $* := {}^{\sigma_1}({}^{\sigma_2}(\psi_\sigma^{-1} \circ \tilde{\rho} \circ C_\sigma^{-1}))$. Note that the equality $\psi_\sigma \circ \tilde{\rho} \circ C_\sigma = h_\sigma\tilde{\rho}$ implies $\psi_\sigma^{-1} \circ \tilde{\rho} \circ C_\sigma^{-1} = \psi_\sigma^{-1}(h_\sigma^{-1})\tilde{\rho}$.

On the one hand, using the equality ${}^\tau\tilde{\rho} = \psi_\tau \circ \tilde{\rho} \circ C_\tau$ for τ equal to $\sigma_1, \sigma_2 \in \Gamma$, and $\sigma = \sigma_1\sigma_2$,

we have,

$$* = \tilde{\rho} \circ C_{\sigma}^{-1} \circ C_{\sigma_2} \circ C_{\sigma_1}. \tag{III.1.1.4.7}$$

Since $g_{\sigma_1\sigma_2} = \tilde{\rho}(\gamma_{\sigma_1}\gamma_{\sigma_2}\gamma_{\sigma}^{-1})$, the above equation gives $* = g_{\sigma_1\sigma_2}^{-1}\tilde{\rho}$.

On the other hand, using the equality ${}^{\tau}\tilde{\rho} = h_{\tau}\tilde{\rho}$ for τ equal to $\sigma_1, \sigma_2 \in \Gamma$ and the equality at the beginning of the proof, we have,

$$* = h_{\sigma}^{-1}\psi_{\sigma_1}(h_{\sigma_2})h_{\sigma_1}\tilde{\rho}. \tag{III.1.1.4.8}$$

whence the lemma. □

Thus being Γ-invariant almost gives the equality in Lemma III.1.1.5.

III.1.1.5 Remarks on Fundamental Groups Let $p : \tilde{X} \to X$ be an unbranched covering of topological manifolds with Galois group Γ (the group of covering transformations $\mathrm{Aut}(\tilde{X}/X)$). Choose base points $\tilde{x} \in \tilde{X}$ and $x = p(\tilde{x})$. Our convention is that by a juxtaposition $\beta\alpha$ of paths we mean the path starting from α and ending along β, so that we have the short exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}) \longrightarrow \pi_1(X, x) \longrightarrow \Gamma^{op} \longrightarrow 1. \tag{III.1.1.5.1}$$

We may omit the base point in the notation of π_1 if no confusion arises. Then the general arguments apply to $\tilde{\Pi} = \pi_1(\tilde{X})$ and $\Pi = \pi_1(X)$. We choose once and for all a section (a map of sets) $\gamma_* = (\gamma_{\sigma})_{\sigma\in\Gamma}$ of the natural projection $\pi_1(X) \to \Gamma^{op}$ as in the general setting.

Let λ_{σ} be the unique lift of γ_{σ} starting from $\tilde{x}$. For any $\sigma \in \Gamma$, let σ also denote the isomorphism $\pi_1(\tilde{X}, \tilde{x}) \to \pi_1(\tilde{X}, \sigma(\tilde{x}))$ and denote by $C_{\lambda_{\sigma}}$ the isomorphism

$$\begin{aligned} \pi_1(\tilde{X}, \sigma(\tilde{x})) &\longrightarrow \pi_1(\tilde{X}, \tilde{x}) \\ \alpha &\longmapsto \lambda_{\sigma}^{-1}\alpha\lambda_{\sigma}, \end{aligned} \tag{III.1.1.5.2}$$

For any $\alpha \in \pi_1(\tilde{X}, \tilde{x})$, $C_{\lambda_{\sigma}} \circ \sigma(\alpha) = \lambda_{\sigma}^{-1}\alpha^{\sigma}\lambda_{\sigma}$ is the unique lift of $\gamma_{\sigma}^{-1}\alpha\gamma_{\sigma} \in \pi_1(X)$ in $\pi_1(\tilde{X})$, therefore $C_{\lambda_{\sigma}} \circ \sigma$ can be identified with the conjugation by γ_{σ}^{-1}. Now (Γ, ψ)-invariant G-representation should be defined by

$$h_{\sigma}\tilde{\rho}h_{\sigma}^{-1} = {}^{\sigma}\tilde{\rho} := \psi_{\sigma}^{-1} \circ \tilde{\rho} \circ C_{\lambda_{\sigma}} \circ \sigma, \tag{III.1.1.5.3}$$

This reason why we have $\psi(\sigma)^{-1}$ instead of $\psi(\sigma)$ is as follows.

Let $\psi^{op} : \Gamma^{op} \to \mathrm{Aut}\, G$ be the composition of ψ and $\Gamma^{op} \to \Gamma$, $x \mapsto x^{-1}$. It defines a semi-direct product as in the short exact sequence

$$1 \longrightarrow G \longrightarrow G \rtimes_{\psi^{op}} \Gamma^{op} \longrightarrow \Gamma^{op} \longrightarrow 1 \tag{III.1.1.5.4}$$

which comes with a homomorphism $s : \Gamma^{op} \to G \rtimes_{\psi^{op}} \Gamma^{op}$ being a section of the quotient map,

satisfying

$$\psi_{\sigma^{-1}}(g) = \psi_\sigma^{op}(g) = s_\sigma g s_\sigma^{-1}, \tag{III.1.1.5.5}$$

where we write $s(\sigma)$ as s_σ.

III.1.2 $G \rtimes \Gamma$-Character Varieties

Let us define the the character varieties that we will study.

III.1.2.1 We put

$$\overline{\mathrm{Rep}}^\Gamma(\tilde{\Pi}, G) := \{(\tilde{\rho}, (h_\sigma)) \in \mathrm{Rep}(\tilde{\Pi}, G) \times \prod_{\sigma\in\Gamma} G \mid {}^\sigma\tilde{\rho} = h_\sigma\tilde{\rho}, \text{ for all } \sigma \in \Gamma\}. \tag{III.1.2.1.1}$$

Note that if $(\tilde{\rho}, (h_\sigma)) \in \overline{\mathrm{Rep}}^\Gamma(\tilde{\Pi}, G)$, then every element of $(\tilde{\rho}, (h_\sigma . \mathrm{Stab}_G(\tilde{\rho})))$ lies in $\overline{\mathrm{Rep}}^\Gamma(\tilde{\Pi}, G)$. The cochain $(h_\sigma)_{\sigma\in\Gamma}$ will be written as h_*.

Consider the morphism

$$\begin{aligned} \overline{\mathrm{Rep}}^\Gamma(\tilde{\Pi}, G) &\longrightarrow \prod_{\sigma,\tau\in\Gamma} G \\ (\tilde{\rho}, (h_\sigma)) &\longmapsto k_{\sigma\tau} = h_{\sigma\tau}^{-1}\psi_\sigma(h_\tau)h_\sigma g_{\sigma\tau}. \end{aligned} \tag{III.1.2.1.2}$$

Note that with γ_* fixed, $g_{\sigma\tau}$ only depends on $\tilde{\rho}$. Denote by $\mathrm{Rep}^\Gamma(\tilde{\Pi}, G)$ the inverse image of $(1)_{\sigma,\tau}$. The equality of Lemma III.1.1.5 implies that, for $(\tilde{\rho}, (h_\sigma)) \in \mathrm{Rep}^\Gamma(\tilde{\Pi}, G)$, $(h_\sigma)_{\sigma\in\Gamma}$ is determined by those h_σ's associated with the generators of Γ.

III.1.2.2 Denote by $\mathrm{Rep}_\Gamma(\Pi, G)$ the variety of $G \rtimes \Gamma$-representations of Π. Recall that these are the homomorphisms ρ making the following diagram commute

$$\begin{tikzcd} 1 \arrow[r] & \tilde{\Pi} \arrow[r, "p"] \arrow[d, "\tilde{\rho}"] & \Pi \arrow[r] \arrow[d, "\rho"] & \Gamma \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\ 1 \arrow[r] & G \arrow[r] & G \rtimes \Gamma \arrow[r] & \Gamma \arrow[r] & 1 \end{tikzcd}$$

As in the case of classical character varieties, this variety can be described in terms of the images of the generators (and relations) of Π with the additional constraint on the connected components they belong to.

III.1.2.3 The variety $\mathrm{Rep}^\Gamma(\tilde{\Pi}, G)$ of (Γ, ψ)-invariant pairs admits an action of G:

$$x : (\tilde{\rho}, h_*) \mapsto (x\tilde{\rho}, (\psi_\sigma(x)h_\sigma x^{-1})), \quad \text{for any } x \in G.$$

Denote by $\mathrm{Ch}^{\Gamma}(\tilde{\Pi}, G)$ the GIT quotient $\mathrm{Rep}^{\Gamma}(\tilde{\Pi}, G)/\!\!/G$.

The variety $\mathrm{Rep}_{\Gamma}(\Pi, G)$ admits the conjugation action of G. Denote by $\mathrm{Ch}_{\Gamma}(\Pi, G) = \mathrm{Rep}_{\Gamma}(\Pi, G)/\!\!/G$ the GIT quotient. This is the character variety with structure group $G \rtimes \Gamma$.

Theorem III.1.2.1. *There is an isomorphism of algebraic varieties*

$$\mathrm{Ch}^{\Gamma}(\tilde{\Pi}, G) \cong \mathrm{Ch}_{\Gamma}(\Pi, G) \tag{III.1.2.3.1}$$

Proof. By Lemma III.1.1.4 and Lemma III.1.1.5, the restriction of ρ to $\tilde{\Pi}$ defines a morphism $\mathrm{Rep}_{\Gamma}(\Pi, G) \to \mathrm{Rep}^{\Gamma}(\tilde{\Pi}, G)$. More concretely, the map sends ρ to $(\tilde{\rho}, h_*)$ with $\tilde{\rho} := \rho \circ p$ and $h_\sigma := s_\sigma \rho(\gamma_\sigma)^{-1}$. Again by Lemma III.1.1.5, there is a well-defined morphism $\mathrm{Rep}^{\Gamma}(\tilde{\Pi}, G) \to \mathrm{Rep}_{\Gamma}(\Pi, G)$. The two morphisms are obviously inverse to each other thus give an isomorphism. It can easily be checked that this isomorphism is G-equivariant, so we have an isomorphism bwtween the quotients. □

Remark III.1.2.2. In case $\tilde{\Pi} = \pi_1(\tilde{X})$ and $\Pi = \pi_1(X)$ are fundamental groups of some topological spaces, we write $\mathrm{Rep}(\tilde{X}, G)$, $\mathrm{Rep}_{\Gamma}(X, G)$. etc.

III.2 Irreducibility and Semi-Simplicity

III.2.1 Irreducible and Semi-Simple $G \rtimes \Gamma$-Representations

We will follow the notations in §1. Write $\bar{G} = G \rtimes \Gamma$.

III.2.1.1 For $\rho : \Pi \to \bar{G}$, let $\mathbf{x}$ be a tuple of elements of $\bar{G}$ which are images of a finite set of generators of Π. We put $H(\rho) := H(\mathbf{x})$.

Definition III.2.1.1. We say that ρ is semi-simple if $H(\rho)$ is a completely reducible subgroup of $\bar{G}$. We say that ρ is irreducible if $H(\rho)$ is an irreducible subgroup of $\bar{G}$.

In particular, an irreducible $\bar{G}$-representation is semi-simple. We have the following basic property.

Proposition III.2.1.2. *If ρ is a semi-simple $\bar{G}$-representation of Π, then its underlying representation $\tilde{\rho}$ is semi-simple.*

In particular, the underlying G-representation of an irreducible $\bar{G}$-representation is semi-simple. However, $\tilde{\rho}$ is not necessarily irreducible in general.

Proof. This follows from Lemma II.1.2.2. □

III.2.1.2 We will need the following notions later.

Definition III.2.1.3. Let $\tilde{\rho} : \tilde{\Pi} \to G$ be a representation. We say that $\tilde{\rho}$ is *strongly irreducible* if $\tilde{\rho}$ is irreducible and $\mathrm{Stab}_G(\tilde{\rho}) = Z_G$. We say that a (Γ, ψ)-invariant pair $(\tilde{\rho}, h_*)$ is strongly irreducible if $\tilde{\rho}$ is strongly irreducible.

Definition III.2.1.4. Let $\rho : \Pi \to \bar{G}$ be a $\bar{G}$-representation. We say that ρ is *strongly irreducible* if ρ is irreducible and $\mathrm{Stab}_G(\rho) = Z_{\bar{G}}$.

Irreducible $\mathrm{GL}_n(k)$-representations are strongly irreducible.

III.2.1.3 We are now ready to state the results on the orbits of semi-simple and irreducible $\bar{G}$-representations.

Theorem III.2.1.5. *The G-orbit of $\rho \in \mathrm{Rep}_\Gamma(\Pi, G)$ is closed if and only if ρ is a semi-simple $\bar{G}$-representation of Π.*

Proof. By Theorem II.1.2.6, the assertion holds whenever Π is a free group with n generators. Since our representation variety can be realised as a closed (See Proposition II.3.1.1) G-invariant subvariety of $\bar{G}^n$ for some n, we are done. □

Therefore, the character variety $\mathrm{Ch}_\Gamma(\Pi, G)$ is the coarse moduli space of semi-simple $\bar{G}$-representations.

Theorem III.2.1.6. *The G-orbit of $\rho \in \mathrm{Rep}_\Gamma(\Pi, G)$ is stable if and only if ρ is an irreducible $\bar{G}$-representation of Π.*

Proof. As in the proof of the previous theorem, we are reduced to the case where $\mathrm{Rep}_\Gamma(\Pi, G)$ is replaced by a direct product of finitely many $\bar{G}$. Note that the technical assumption in Theorem II.1.2.8 is always satisfied for the images of ρ as $\mathrm{Im}\,\rho$ meets all components of G. □

III.2.1.4 For $\Pi/\tilde{\Pi} \cong \mathbb{Z}/2\mathbb{Z}$, we translate Proposition II.5.4.1 into the following.

Proposition III.2.1.7. *If $\rho : \Pi \to \mathrm{GL}_n(k) \rtimes <\sigma>$ is an irreducible $\mathrm{GL}_n(k) \rtimes <\sigma>$-representation, then its underlying $\mathrm{GL}_n(k)$-representation $\tilde{\rho}$ is a direct sum of pairwise non isomorphic irreducible representations.*

With the assumptions in the proposition, we have the following remark.

Remark III.2.1.8. If $\gamma \in \Pi \setminus \tilde{\Pi}$ and $h\sigma := \rho(\gamma)$ is semi-simple, then it satisfies the assumptions on $x_0\sigma$ in Proposition II.5.4.1. According to the proof of that proposition, if $\mathrm{Im}\,\tilde{\rho}$ is decomposed into a direct sum of (pairwise non isomorphic)irreducible representations V_i, then the conjugation by $h\sigma$ leaves each factor stable. Considering $h\sigma$ as an automorphism of the group $\prod_i \mathrm{GL}(V_i)$, we may say that ρ is a direct sum of pairwise non isomorphic $\mathrm{GL}(V_i) \rtimes <h\sigma>$-representations.

III.2.1.5 Openness

Proposition III.2.1.9. *The set of irreducible representations in* $\mathrm{Rep}_\Gamma(\Pi, G)$ *or* $\mathrm{Ch}_\Gamma(\Pi, G)$ *is Zariski open.*

Proof. This is a direct consequence of Theorem III.2.1.6, since the locus of stable points is open. □

Lemma III.2.1.10. *Let ρ be a $\bar{G}$-representation with underlying G-representation $\tilde{\rho}$. If $\tilde{\rho}$ is strongly irreducible, then ρ is strongly irreducible.*

Proof. Since the image of ρ meets all connected components of $\bar{G}$, if $\tilde{\rho}$ satisfies $\mathrm{Stab}_G(\tilde{\rho}) = Z_G$, then $\mathrm{Stab}_G(\rho) = Z_{\bar{G}}$. □

We will assume that the set $\{\rho \mid \tilde{\rho} \text{ is strongly irreducible}\}$ is non-empty.

Remark III.2.1.11. By Proposition II.5.4.1, if a $\mathrm{GL}_n \rtimes <\sigma>$-representation is strongly irreducible, then the underlying GL_n-representation is necessarily (strongly) irreducible.

Proposition III.2.1.12. *The set of irreducible $\bar{G}$-representations in* $\mathrm{Rep}_\Gamma(\Pi, G)$ *or* $\mathrm{Ch}_\Gamma(\Pi, G)$ *with strongly irreducible underlying G-representations is Zariski open.*

Proof. Theorem III.1.2.1 gives an isomorphism $\mathrm{Rep}_\Gamma(\Pi, G) \cong \mathrm{Rep}^\Gamma(\tilde{\Pi}, G)$, which can be projected to $\mathrm{Rep}(\tilde{\Pi}, G)$. Then the subset in question is just the inverse image of the subset of strongly irreducibles in $\mathrm{Rep}(\tilde{\Pi}, G)$, which is open. □

III.2.2 Classification Problem

Eventually, we would like to have a representation variety such that the subset $\{\rho \mid \tilde{\rho} \text{ is irreducible}\}$ is non-empty. For this reason, we need to study the condition under which an irreducible (Γ, ψ)-invariant G-representation can be extended to some $G \rtimes \Gamma$-representation. To simplify the situation, we require instead that

$$\{\rho \in \mathrm{Rep}_\Gamma(\Pi, G) \mid \tilde{\rho} \text{ is strongly irreducible}\}$$

is non-empty.

Our reference for group cohomology is [Ser, Chapitre I, §5]. If we were to work in the setting of §III.1.1.5, we could equivalently work with the right action of Γ on Z_G with $\sigma \in \Gamma$ acting by $\psi_{\sigma^{-1}}$, but the differential map on cochains should be changed accordingly.

III.2.2.1 Denote by $\mathrm{Ch}^{s.irr}(\tilde{\Pi}, G) \subset \mathrm{Ch}(\tilde{\Pi}, G)$ the open subvariety of strongly irreducible representations.

Lemma III.2.2.1. *Let $\tilde{\rho}$ be a strongly irreducible G-representation of $\tilde{\Pi}$ and denote by $[\tilde{\rho}]$ its G-orbit. Then there exists some cochain h_* such that $(\tilde{\rho}, h_*)$ is a Γ-invariant pair if and only if $[\tilde{\rho}]$ is a point of* $\mathrm{Ch}^{s.irr}(\tilde{\Pi}, G)$ *fixed by Γ.*

Proof. Obvious. □

Remark III.2.2.2. The fixed points locus $\mathrm{Ch}^{s.irr}(\tilde{\Pi}, G)^{\Gamma}$ only depends on $[\psi]$.

III.2.2.2 By Lemma III.1.1.5 and Lemma III.1.1.6, if $Z_G = 1$, then any strongly irreducible (Γ, ψ)-invariant pair $(\tilde{\rho}, h_*)$ defines a representation $\rho : \Pi \to G \rtimes \Gamma$, and if $Z_G \neq 1$, invariance under (Γ, ψ) is not sufficient. Note that for an arbitrary (Γ, ψ)-invariant pair $(\tilde{\rho}, h_*)$, h_* is not uniquely determined by $\tilde{\rho}$, as each of its factors can be multiplied by $\mathrm{Stab}_G(\tilde{\rho})$ on the right. Therefore, the correct question is whether h_* can be chosen to define an extension ρ.

We define for any strongly irreducible $(\tilde{\rho}, h_*)$ a cochain $(k_{\sigma\tau})_{\sigma,\tau\in\Gamma} \in C^2(\Gamma, Z_G)$ by

$$k_{\sigma\tau} := h_{\sigma\tau}^{-1}\psi_\sigma(h_\tau)h_\sigma g_{\sigma\tau}. \tag{III.2.2.2.1}$$

Remark III.2.2.3. Working with Γ^{op}, we should define $k_{\sigma\tau} = h_{\sigma\tau}^{-1}\psi_\tau^{-1}(h_\sigma)h_\tau g_{\sigma\tau}$ instead.

Proposition III.2.2.4. *Choose a section $\gamma_* : \Gamma \to \Pi$ and a homomorphism $\psi : \Gamma \to \mathrm{Aut}\, G$, let $(k_{\sigma\tau})$ be the cochain associated to a given strongly irreducible (Γ, ψ)-invariant pair $(\tilde{\rho}, h_*)$ defined as above. Then,*

(i) The cochain $(k_{\sigma\tau})$ is a cocycle;

(ii) The cohomology class of $(k_{\sigma\tau})$ does not depend on h_*;

(iii) The cohomology class of $(k_{\sigma\tau})$ does not depend on the choice of γ_*;

(iv) Fixing $\tilde{\rho}$, the cochain h_* can be modified (by $\mathrm{Stab}_G(\tilde{\rho})$) to satisfy the condition in Lemma III.1.1.5 if and only if $(k_{\sigma\tau})$ is a coboundary.

We will denote by $\mathfrak{c}_\psi(\tilde{\rho}) := [k_{\sigma\tau}]$ the cohomology class of $(k_{\sigma\tau})$, which only depends on ψ and $\tilde{\rho}$. The cohomology group $H^2(\Gamma, Z_G)$ only depends on the similarity classe of ψ in the sense of §III.1.1.2.

Proof. Let us show that $(k_{\sigma\tau})$ is a cocycle. The differential $\mathrm{d} : C^2(\Gamma, A) \to C^3(\Gamma, A)$ for the left action of Γ on some abelian group A, written additively, is

$$\mathrm{d}\varphi(x, y, z) = x \cdot \varphi(y, z) - \varphi(xy, z) + \varphi(x, yz) - \varphi(x, y),$$

for any $\varphi \in C^2(\Gamma, A)$, and any $x, y, z \in \Gamma$.

Since $k_{\sigma\tau}$ is central, its factors can be permuted in a order-preserving way, i.e.

$$h_{\sigma\tau}^{-1}\psi_\sigma(h_\tau)h_\sigma g_{\sigma\tau} = \psi_\sigma(h_\tau)h_\sigma g_{\sigma\tau}h_{\sigma\tau}^{-1} = h_\sigma g_{\sigma\tau}h_{\sigma\tau}^{-1}\psi_\sigma(h_\tau) = g_{\sigma\tau}h_{\sigma\tau}^{-1}\psi_\sigma(h_\tau)h_\sigma, \tag{III.2.2.2.2}$$

In the following calculation we will put a bracket on each central element.

$$\begin{aligned}(\mathrm{d}k)_{\sigma\tau\mu} =& [\psi_\sigma(k_{\tau\mu})][k_{\sigma\tau,\mu}^{-1}][k_{\sigma,\tau\mu}][k_{\sigma\tau}^{-1}]\\ =& [\psi_\sigma(h_{\tau\mu}^{-1}\psi_\tau(h_\mu)h_\tau g_{\tau\mu})]\end{aligned}$$

$$
\begin{aligned}
&[g_{\sigma\tau,\mu}^{-1}h_{\sigma\tau}^{-1}\psi_{\sigma\tau}(h_\mu^{-1})h_{\sigma\tau\mu}]\\
&[h_{\sigma\tau\mu}^{-1}\psi_\sigma(h_{\tau\mu})h_\sigma g_{\sigma,\tau\mu}]\\
&[g_{\sigma\tau}^{-1}h_\sigma^{-1}\psi_\sigma(h_\tau^{-1})h_{\sigma\tau}]\\
=&[\psi_\sigma(h_{\tau\mu}^{-1})\psi_{\sigma\tau}(h_\mu)\psi_\sigma(h_\tau)\psi_\sigma(g_{\tau\mu})]\\
&[g_{\sigma\tau,\mu}^{-1}h_{\sigma\tau}^{-1}\psi_{\sigma\tau}(h_\mu^{-1})\psi_\sigma(h_{\tau\mu})h_\sigma g_{\sigma,\tau\mu}]\\
&[g_{\sigma\tau}^{-1}h_\sigma^{-1}\psi_\sigma(h_\tau^{-1})h_{\sigma\tau}]\\
=&[\psi_\sigma(h_{\tau\mu}^{-1})\psi_{\sigma\tau}(h_\mu)\psi_\sigma(h_\tau)\psi_\sigma(g_{\tau\mu})]\\
&[g_{\sigma,\tau\mu}g_{\sigma\tau,\mu}^{-1}h_{\sigma\tau}^{-1}\psi_{\sigma\tau}(h_\mu^{-1})\psi_\sigma(h_{\tau\mu})h_\sigma]\\
&[h_\sigma^{-1}\psi_\sigma(h_\tau^{-1})h_{\sigma\tau}g_{\sigma\tau}^{-1}]\\
=&[\psi_\sigma(h_{\tau\mu}^{-1})\psi_{\sigma\tau}(h_\mu)\psi_\sigma(h_\tau)\psi_\sigma(g_{\tau\mu})]\\
&[g_{\sigma,\tau\mu}g_{\sigma\tau,\mu}^{-1}h_{\sigma\tau}^{-1}\psi_{\sigma\tau}(h_\mu^{-1})\psi_\sigma(h_{\tau\mu})\psi_\sigma(h_\tau^{-1})h_{\sigma\tau}g_{\sigma\tau}^{-1}]\\
=&[\psi_\sigma(h_{\tau\mu}^{-1})\psi_{\sigma\tau}(h_\mu)\psi_\sigma(h_\tau)\psi_\sigma(g_{\tau\mu})]\\
&[h_{\sigma\tau}g_{\sigma\tau}^{-1}g_{\sigma,\tau\mu}g_{\sigma\tau,\mu}^{-1}h_{\sigma\tau}^{-1}\psi_{\sigma\tau}(h_\mu^{-1})\psi_\sigma(h_{\tau\mu})\psi_\sigma(h_\tau^{-1})].
\end{aligned}
$$

We calculate

$$
\begin{aligned}
&h_{\sigma\tau}g_{\sigma\tau}^{-1}g_{\sigma,\tau\mu}g_{\sigma\tau,\mu}^{-1}h_{\sigma\tau}^{-1}\\
=&h_{\sigma\tau}\tilde{\rho}((\gamma_{\sigma\tau}\gamma_\tau^{-1}\gamma_\sigma^{-1})(\gamma_\sigma\gamma_{\tau\mu}\gamma_{\sigma\tau\mu}^{-1})(\gamma_{\sigma\tau\mu}\gamma_\mu^{-1}\gamma_{\sigma\tau}^{-1}))h_{\sigma\tau}^{-1}\\
=&h_{\sigma\tau}\tilde{\rho}(\gamma_{\sigma\tau}\gamma_\tau^{-1}\gamma_{\tau\mu}\gamma_\mu^{-1}\gamma_{\sigma\tau}^{-1})h_{\sigma\tau}^{-1}\\
=&\psi_{\sigma\tau}\circ\tilde{\rho}(\gamma_\tau^{-1}\gamma_{\tau\mu}\gamma_\mu^{-1})\\
=&\psi_\sigma(h_\tau\tilde{\rho}(\gamma_{\tau\mu}\gamma_\mu^{-1}\gamma_\tau^{-1})h_\tau^{-1})\\
=&\psi_\sigma(h_\tau g_{\tau\mu}h_\tau^{-1}).
\end{aligned}
\tag{III.2.2.2.3}
$$

We then continue to calculate

$$
\begin{aligned}
(\mathrm{d}k)_{\sigma\tau\mu} =&[\psi_\sigma(h_{\tau\mu}^{-1})\psi_{\sigma\tau}(h_\mu)\psi_\sigma(h_\tau)\psi_\sigma(g_{\tau\mu})]\\
&[\psi_\sigma(h_\tau g_{\tau\mu}h_\tau^{-1})\psi_{\sigma\tau}(h_\mu^{-1})\psi_\sigma(h_{\tau\mu})\psi_\sigma(h_\tau^{-1})]\\
=&[\psi_\sigma(h_{\tau\mu}^{-1})\psi_{\sigma\tau}(h_\mu)\psi_\sigma(h_\tau)\psi_\sigma(g_{\tau\mu})]\\
&[\psi_\sigma(g_{\tau\mu})\psi_\sigma(h_\tau^{-1})\psi_{\sigma\tau}(h_\mu^{-1})\psi_\sigma(h_{\tau\mu})]\\
=&1.
\end{aligned}
\tag{III.2.2.2.4}
$$

We have shown that $(k_{\sigma\tau})$ is a cocycle.

Now if (h_σ) is replaced by $(h_\sigma x_\sigma)_{\sigma\in\Gamma}$ for $x_\sigma \in Z_G$, then $k_{\sigma\tau}$ is multiplied by $x_{\sigma\tau}^{-1}\psi_\sigma(x_\tau)x_\sigma$ which is exactly $(\mathrm{d}x_*)_{\sigma\tau}$ where the differential written additively is

$$
\mathrm{d}\varphi(x,y) = x\cdot\varphi(y) - \varphi(xy) + \varphi(x),
\tag{III.2.2.2.5}
$$

for any $\varphi \in C^1(\Gamma, Z_G)$ and any x, $y \in \Gamma$. Therefore $[k_{\sigma\tau}]$ does not depend on h_*. We also deduce from this that (h_σ) can be modified to satisfy the desired equality if and only if $(k_{\sigma\tau})$ is a coboundary.

Finally, let us show that $[k_{\sigma\tau}]$ is independent of the choice of γ_*. Let γ'_* be another section. If $(\tilde{\rho}, h_*)$ is (Γ, ψ)-invariant for γ_*, then $(\tilde{\rho}, (\psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1}))h_\sigma)_\sigma)$ is (Γ, ψ)-invariant for γ'_*, where $\delta_\sigma = \gamma_\sigma^{-1}\gamma'_\sigma \in \tilde{\Pi}$. Let $h'_\sigma = \psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1}))h_\sigma$, for any $\sigma \in \Gamma$, and let $h'_* = (h'_\sigma)_{\sigma\in\Gamma}$. We will calculate $(k'_{\sigma\tau})$ with h'_* thus defined. By the argument above, any other h''_* such that $(\tilde{\rho}, h''_*)$ is (Γ, ψ)-invariant will give the same cohomology class. Let $g'_{\sigma\tau} := \tilde{\rho}(\gamma'_\tau\gamma'_\sigma\gamma'^{-1}_{\sigma\tau})$.

We calculate

$$
\begin{aligned}
k'_{\sigma\tau} =& h'^{-1}_{\sigma\tau}\psi_\sigma(h'_\tau)h'_\sigma g'_{\sigma\tau}\\
=& h^{-1}_{\sigma\tau}\psi_{\sigma\tau}(\tilde{\rho}(\delta_{\sigma\tau}))\psi_\sigma(\psi_\tau(\tilde{\rho}(\delta_\tau^{-1})))\psi_\sigma(h_\tau)\psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1}))h_\sigma g'_{\sigma\tau}\\
=& h^{-1}_{\sigma\tau}h_{\sigma\tau}\tilde{\rho}(\gamma'_{\sigma\tau}\gamma^{-1}_{\sigma\tau})h^{-1}_{\sigma\tau}\psi_\sigma(\psi_\tau(\tilde{\rho}(\delta_\tau^{-1})))\psi_\sigma(h_\tau)\psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1}))h_\sigma g'_{\sigma\tau}\\
=& \tilde{\rho}(\gamma'_{\sigma\tau}\gamma^{-1}_{\sigma\tau})h^{-1}_{\sigma\tau}h_{\sigma\tau}\tilde{\rho}(\gamma_{\sigma\tau}\gamma'^{-1}_\tau\gamma_\tau\gamma^{-1}_{\sigma\tau})h^{-1}_{\sigma\tau}\psi_\sigma(h_\tau)\psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1}))h_\sigma g'_{\sigma\tau}\\
=& \tilde{\rho}(\gamma'_{\sigma\tau}\gamma'^{-1}_\tau\gamma_\tau\gamma^{-1}_{\sigma\tau})h^{-1}_{\sigma\tau}\psi_\sigma(h_\tau)\psi_\sigma(\tilde{\rho}(\delta_\sigma^{-1}))h_\sigma g'_{\sigma\tau}\\
=& \tilde{\rho}(\gamma'_{\sigma\tau}\gamma'^{-1}_\tau\gamma_\tau\gamma^{-1}_{\sigma\tau})h^{-1}_{\sigma\tau}\psi_\sigma(h_\tau)h_\sigma\tilde{\rho}(\gamma_\sigma\gamma'^{-1}_\sigma)g'_{\sigma\tau}\\
=& \tilde{\rho}(\gamma'_{\sigma\tau}\gamma'^{-1}_\tau\gamma_\tau\gamma^{-1}_{\sigma\tau})k_{\sigma\tau}g^{-1}_{\sigma\tau}\tilde{\rho}(\gamma_\sigma\gamma'^{-1}_\sigma)g'_{\sigma\tau}\\
=& k_{\sigma\tau}\tilde{\rho}((\gamma'_{\sigma\tau}\gamma'^{-1}_\tau\gamma_\tau\gamma^{-1}_{\sigma\tau})(\gamma_{\sigma\tau}\gamma^{-1}_\tau\gamma^{-1}_\sigma)(\gamma_\sigma\gamma'^{-1}_\sigma)(\gamma'_\sigma\gamma'_\tau\gamma'^{-1}_{\sigma\tau}))\\
=& k_{\sigma\tau}
\end{aligned}
\tag{III.2.2.2.6}
$$

□

Remark III.2.2.5. Since $k'_{\sigma\tau} = k_{\sigma\tau}$, $(\tilde{\rho}, h_*)$ defines an extension ρ if and only if $(\tilde{\rho}, h'_*)$ defines an extension ρ'. In fact, we have $\rho = \rho'$. This can be seen from the following. On the one hand, $\rho(\gamma_\sigma) = h_\sigma^{-1}s_\sigma$. On the other hand,

$$
\begin{aligned}
\rho'(\gamma'_\sigma) =& h'^{-1}_\sigma s_\sigma\\
=& h_\sigma^{-1}\psi_\sigma(\tilde{\rho}(\delta_\sigma))s_\sigma\\
=& h_\sigma^{-1}s_\sigma\tilde{\rho}(\delta_\sigma)\\
=& \rho(\gamma_\sigma)\tilde{\rho}(\gamma_\sigma^{-1}\gamma'_\sigma)\\
=& \rho(\gamma'_\sigma),
\end{aligned}
\tag{III.2.2.2.7}
$$

So the correspondence between $(\tilde{\rho}, h_*)$ and ρ is independent of the choice of γ_*.

III.2.2.3 If $\tilde{\rho}$ gives rise to some non trivial cohomology class $\mathfrak{c}_\psi(\tilde{\rho}) \in H^2(\Gamma, Z_G)$ for the chosen ψ, then there is still a chance to extend it into a $G \rtimes_{\psi'} \Gamma$-representation for some $\psi' \not\cong \psi$ (conjugate under G) but $[\psi'] = [\psi]$. As in (III.1.1.2 (iii)), we can take another homomorphism $\psi' : \Gamma \to \operatorname{Aut} G$ with $\psi'_\sigma\psi_\sigma^{-1} = \operatorname{ad} x_\sigma$, for each $\sigma \in \Gamma$, so that $h'_\sigma = c_\sigma x_\sigma h_\sigma$ defines a (Γ, ψ')-invariant pair, where c_σ is an arbitrary element of Z_G. Let $(k_{\sigma\tau})$ be the cochain defined by h_* and ψ.

Proposition III.2.2.6. *There exists some ψ' similar to ψ such that the (Γ, ψ')-invariant pair $(\tilde{\rho}, h'_*)$ defines a homomorphism $\rho : \Pi \to G \rtimes_{\psi'} \Gamma$ if and only if $[k_{\sigma\tau}]^{-1}$ lies in the image of the natural map $\delta : H^1_\psi(\Gamma, G_{ad}) \to H^2(\Gamma, Z_G)$.*

Remark III.2.2.7. While $H^1_\psi(\Gamma, G_{ad})$ depends on the choice of ψ, $H^2(\Gamma, Z_G)$ only depend on the similarity class of ψ, as similar homomorphisms give the same action of Γ on Z_G. Thus in the statement of the proposition, we have omitted ψ form $H^2(\Gamma, Z_G)$. The image of δ depends on ψ.

Proof. Given $\psi' : \Gamma \to \operatorname{Aut} G$, $(x_\sigma)_{\sigma\in\Gamma}$ and $(h'_\sigma)_{\sigma\in\Gamma}$ defined above, we define a cohomology class $[k'_{\sigma\tau}]$ with respect to ψ' by

$$k'_{\sigma\tau} := h'^{-1}_{\sigma\tau}\psi'_\sigma(h'_\tau)h'_\sigma g_{\sigma\tau}. \tag{III.2.2.3.1}$$

The exixtence of the desired ρ is equivalent to $[k'_{\sigma\tau}] = 1$. To clarify this condition, let us first note that since both ψ and ψ' are group homomorphisms, we have

$$\begin{aligned} \psi'_\sigma\psi'_\tau &= \psi'_{\sigma\tau} \\ x_\sigma\psi_\sigma x_\tau\psi_\tau &= x_{\sigma\tau}\psi_{\sigma\tau} \\ x_\sigma\psi_\sigma(x_\tau) &= x_{\sigma\tau} \end{aligned} \tag{III.2.2.3.2}$$

as automorphisms, where we have abbreviated $\operatorname{ad} x_\sigma$ by x_σ. Therefore there exists some $d_{\sigma\tau} \in Z_G$ such that

$$d_{\sigma\tau} = x^{-1}_{\sigma\tau}x_\sigma\psi_\sigma(x_\tau). \tag{III.2.2.3.3}$$

Now we compute

$$\begin{aligned} k'_{\sigma\tau} &= h'^{-1}_{\sigma\tau}\psi'_\sigma(h'_\tau)h'_\sigma g_{\sigma\tau} \\ &= h^{-1}_{\sigma\tau}x^{-1}_{\sigma\tau}c^{-1}_{\sigma\tau}x_\sigma\psi_\sigma(c_\tau x_\tau h_\tau)x^{-1}_\sigma c_\sigma x_\sigma h_\sigma g_{\sigma\tau} \\ &= d_{\sigma\tau}k_{\sigma\tau}\,\mathrm{d}c_*, \end{aligned} \tag{III.2.2.3.4}$$

where $c_* = (c_\sigma)_{\sigma\in\Gamma}$. We see that $[k'_{\sigma\tau}] = 1$ if and only if $[k_{\sigma\tau}] = [d_{\sigma\tau}]^{-1}$. If we denote by $(\bar{x}_\sigma)$ the image of (x_σ) in $C^1(\Gamma, G_{ad})$ under the natural map induced by $G \to G_{ad}$, then by the definition of δ, $[d_{\sigma\tau}] = \delta([\bar{x}_\sigma])$. □

We will see some examples in §III.5.

III.3 Flat Connections

In this section we work over $\mathbb{C}$. We fix a Galois covering of complex manifolds $\tilde{X} \to X$ and apply the previous results to $\tilde{\Pi} = \pi_1(\tilde{X})$ and $\Pi = \pi_1(X)$, but with $\Gamma^{op} \cong \tilde{\Pi}/\Pi$ as in §III.1.1.5.

III.3.1 (Γ, ψ)-Invariant Flat Connections

Fix a homomorphism $\psi : \Gamma \to \operatorname{Aut} G$. Let us begin by explaining the relation with (Γ, ψ)-Invariant Representations.

III.3.1.1 Twisted Principal G-Bundles Given a principal G-bundles $\mathcal{E}$ with a flat connection ∇ on $\tilde{X}$, for any $\sigma \in \Gamma$, we can define another principal bundle by

$$\mathcal{E}^\sigma := \mathcal{E} \times_G F, \tag{III.3.1.1.1}$$

where $F := G.s_\sigma^{-1}$ is the connected component of $G \rtimes \Gamma^{op}$ corresponding to σ^{-1}. It is a left and right G-space with the multiplication actions. The quotient $\mathcal{E} \times_{G,\psi_\sigma} F$ is defined by the relation

$$(pg, x) \sim (p, gx),$$

for any $p \in \mathcal{E}$, $x \in F$ and $g \in G$. The flat connection ∇^σ is defined accordingly. We have, $\mathcal{E}^{\sigma\tau} \cong (\mathcal{E}^\sigma)^\tau$.

We say that a flat connection $(\mathcal{E}, \nabla)$ is (Γ, ψ)-invariant if for each $\sigma \in \Gamma$, there is an isomorphism of flat connections

$$\Phi_\sigma : (\mathcal{E}, \nabla) \xrightarrow{\sim} (\sigma^*\mathcal{E}^\sigma, \sigma^*\nabla^\sigma). \tag{III.3.1.1.2}$$

If $\tilde{\rho} : \pi_1(\tilde{X}) \to G$ corresponds to $(\mathcal{E}, \nabla)$ under the Riemann-Hilbert correspondence, then $(\mathcal{E}, \nabla)$ is (Γ, ψ)-invariant if and only if $\tilde{\rho}$ is (Γ, ψ)-invariant.

III.3.1.2 As before, let $\tilde{x}$ denote the base point of the complex manifold $\tilde{X}$. We choose a base point e of the homogeneous space $\mathcal{E}_{\tilde{x}}$. It determines a base point $es_\sigma^{-1} \in \mathcal{E}^\sigma_{\tilde{x}}$. If $f : \mathcal{E}_{\tilde{x}} \to \mathcal{E}_{\tilde{x}}$ is a morphism of homogeneous spaces that sends e to eg, $g \in G$, then the induced morphism $f^\sigma : \mathcal{E}^\sigma_{\tilde{x}} \to \mathcal{E}^\sigma_{\tilde{x}}$ sends es_σ^{-1} to egs_σ^{-1}.

Lemma III.3.1.1. *Suppose that $(\mathcal{E}, \nabla)$ corresponds to $\tilde{\rho}$, both being (Γ, ψ)-invariant. There is a natural bijection between the set of families of isomorphisms $(\Phi_\sigma)_{\sigma\in\Gamma}$ such that $((\mathcal{E}, \nabla), \Phi_*)$ is a (Γ, ψ)-invariant pair and the set of cochains $(h_\sigma)_{\sigma\in\Gamma}$ such that $(\tilde{\rho}, h_*)$ is a (Γ, ψ)-invariant pair. The bijection is given as follows.*

With the chosen base points in the fibres $\mathcal{E}_{\tilde{x}}$ and $\mathcal{E}^\sigma_{\tilde{x}}$, for each $\sigma \in \Gamma$, the morphism of homogeneous G-spaces $\mathcal{E}_{\tilde{x}} \to \mathcal{E}^\sigma_{\tilde{x}}$, $e \mapsto es_\sigma^{-1}h_\sigma$ corresponds to the following morphism

$$\mathcal{E}_{\tilde{x}} \xrightarrow{\Phi_{\sigma,\tilde{x}}} \mathcal{E}^\sigma_{\sigma(\tilde{x})} \xrightarrow{(\mathcal{E}^\sigma_{\lambda_\sigma})^{-1}} \mathcal{E}^\sigma_{\tilde{x}}, \tag{III.3.1.2.1}$$

where $\Phi_{\sigma,\tilde{x}}$ is the restriction of Φ_σ on the fibres, and $\mathcal{E}^\sigma_{\lambda_\sigma}$ is the isomorphism associated to the path λ_σ (§III.1.1.5).

Proof. For any $\eta \in \pi_1(\tilde{X}, \tilde{x})$, we have a commutative diagram

$$\begin{tikzcd} \mathcal{E}_{\tilde{x}} \arrow[r, "\Phi_{\sigma,\tilde{x}}"] \arrow[d, "\tilde{\rho}(\eta)"'] & \mathcal{E}^{\sigma}_{\sigma(\tilde{x})} \arrow[r, "(\mathcal{E}^{\sigma}_{\lambda_\sigma})^{-1}"] \arrow[d, "\mathcal{E}^{\sigma}_{\sigma(\eta)}"] & \mathcal{E}^{\sigma}_{\tilde{x}} \arrow[d, "es_\sigma^{-1} \mapsto e(\tilde{\rho} \circ C_{\lambda_\sigma} \circ \sigma(\eta))s_\sigma^{-1}"] \\ \mathcal{E}_{\tilde{x}} \arrow[r, "\Phi_{\sigma,\tilde{x}}"'] & \mathcal{E}^{\sigma}_{\sigma(\tilde{x})} \arrow[r, "(\mathcal{E}^{\sigma}_{\lambda_\sigma})^{-1}"'] & \mathcal{E}^{\sigma}_{\tilde{x}}. \end{tikzcd} \tag{III.3.1.2.2}$$

Indeed, with the choice of a base point in $\mathcal{E}_{\tilde{x}}$, the isomorphism $\mathcal{E}_\eta$ induced along η is identified with $\tilde{\rho}(\eta)$; the left square commutes because Φ_σ is an isomorphism of flat connections and $(\sigma^*\mathcal{E}^\sigma)_\eta = \mathcal{E}^\sigma_{\sigma(\eta)}$; conjugation by $\mathcal{E}^\sigma_{\lambda_\sigma}$ gives the isomorphism $\mathcal{E}^\sigma_{C_{\lambda_\sigma}\circ\sigma(\eta)}$, which is the isomorphism of homogeneous spaces induced from $\mathcal{E}_{C_{\lambda_\sigma}\circ\sigma(\eta)}$.

Since $(\mathcal{E}^\sigma_{\lambda_\sigma})^{-1} \circ \Phi_{\sigma,\tilde{x}}$ is a morphism of homogeneous space, it is uniquely determined by $(\mathcal{E}^\sigma_{\lambda_\sigma})^{-1} \circ \Phi_{\sigma,\tilde{x}}(e)$, which can be written as $es_\sigma^{-1}h_\sigma$ for a unique $h_\sigma \in G$. Then

$$((\mathcal{E}^\sigma_{\lambda_\sigma})^{-1} \circ \Phi_{\sigma,\tilde{x}})^{-1}(es_\sigma^{-1}) = eh_\sigma^{-1},$$

which is sent to $e\tilde{\rho}(\eta)h_\sigma^{-1}$ by $\mathcal{E}_\eta$. Further,

$$(\mathcal{E}^\sigma_{\lambda_\sigma})^{-1} \circ \Phi_{\sigma,\tilde{x}}(e\tilde{\rho}(\eta)h_\sigma^{-1}) = es_\sigma^{-1}h_\sigma\tilde{\rho}(\eta)h_\sigma^{-1}.$$

Compared with $\mathcal{E}^\sigma_{C_{\lambda_\sigma}\circ\sigma(\eta)}$, we have

$$e(\tilde{\rho} \circ C_{\lambda_\sigma} \circ \sigma(\eta))s_\sigma^{-1} = es_\sigma^{-1}\tilde{\rho}^\sigma(\eta) = es_\sigma^{-1}h_\sigma\tilde{\rho}(\eta)h_\sigma^{-1}. \tag{III.3.1.2.3}$$

We see that such $(h_\sigma)_{\sigma\in\Gamma}$ define a (Γ, ψ)-invariant pair $(\tilde{\rho}, h_*)$.

Reversing the arguments, the h_σ's give a family of isomorphisms $\Phi_{\sigma,\tilde{x}}$ that are compatible with the isomorphisms $\mathcal{E}_\eta$ and $(\sigma^*\mathcal{E}^\sigma)_\eta$, therefore define the desired isomorphisms of flat principle G-bundles. □

Remark III.3.1.2. By Lemma III.3.1.1, we can identify $\tilde{\rho}$ with $(\mathcal{E}, \nabla)$ and identify $(h_\sigma s_\sigma)^{-1}$ with an isomorphism $\Phi : \mathcal{E} \to \sigma^*\mathcal{E}^\sigma$. Therefore in the setting of Remark III.2.1.8, the flat connection $(\mathcal{E}, \nabla)$ can be decomposed into a direct sum of pairwise non isomorphic irreducibles, and Φ induces an isomorphism on each such factor. This should be compared with [Ze2, §4.1] and [Ra, Proposition 4.5].

III.3.1.3 For any σ, $\tau \in \Gamma$, the isomorphism $\Phi_\sigma : \mathcal{E} \to \sigma^*\mathcal{E}^\sigma$ induces an isomorphism $\tau^*\Phi_\sigma : \tau^*\mathcal{E} \to \tau^*\sigma^*\mathcal{E}^\sigma$, and thus an isomorphism $(\tau^*\Phi_\sigma)^\tau : \tau^*\mathcal{E}^\tau \to (\sigma\tau)^*\mathcal{E}^{\sigma\tau}$. Combined with $\Phi_\tau : \mathcal{E} \to \tau^*\Phi_\tau$, this gives an isomorphism $\mathcal{E} \to (\sigma\tau)^*\mathcal{E}^{\sigma\tau}$.

Proposition III.3.1.3. *Let $(\tilde{\rho}, h_*)$ be the (Γ, ψ)-invariant pair corresponding to $((\mathcal{E}, \nabla), \Phi_*)$. Then for any σ, $\tau \in \Gamma$,*

$$(\tau^*\Phi_\sigma)^\tau \circ \Phi_\tau = \Phi_{\sigma\tau} \tag{III.3.1.3.1}$$

if and only if $k_{\sigma\tau} = h_{\sigma\tau}^{-1}\psi_\tau^{-1}(h_\sigma)h_\tau g_{\sigma\tau} = 1$.

Proof. We first note that there is a commutative diagram for any σ, τ:

$$\begin{tikzcd} \mathcal{E}_{\tilde{x}} \arrow[r, "\Phi_{\sigma,\tilde{x}}"] \arrow[d, "\mathcal{E}_{\lambda_\tau}"'] & (\sigma^*\mathcal{E}^\sigma)_{\tilde{x}} \arrow[d, "(\sigma^*\mathcal{E}^\sigma)_{\lambda_\tau}"] \\ \mathcal{E}_{\tau(\tilde{x})} \arrow[r, "\Phi_{\sigma,\tau(\tilde{x})}"'] & (\sigma^*\mathcal{E}^\sigma)_{\tau(\tilde{x})}, \end{tikzcd} \tag{III.3.1.3.2}$$

so $\Phi_{\sigma,\tau(\tilde{x})} = \mathcal{E}^\sigma_{\sigma\lambda_\tau} \circ \Phi_{\sigma,\tilde{x}} \circ \mathcal{E}_{\lambda_\tau^{-1}}$, which gives

$$(\tau^*\Phi_\sigma)^\tau_{\tilde{x}} = \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ (\Phi_{\sigma,\tilde{x}})^\tau \circ \mathcal{E}^\tau_{\lambda_\tau^{-1}}. \tag{III.3.1.3.3}$$

Using Lemma III.3.1.1, we calculate

$$\begin{aligned} &(\Phi_{\sigma\tau})^{-1}_{\tilde{x}} \circ (\tau^*\Phi_\sigma)^\tau_{\tilde{x}} \circ \Phi_{\tau,\tilde{x}} \\ =&(\Phi_{\sigma\tau})^{-1}_{\tilde{x}} \circ \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ (\Phi_{\sigma,\tilde{x}})^\tau \circ \mathcal{E}^\tau_{\lambda_\tau^{-1}} \circ \Phi_{\tau,\tilde{x}} \\ =&(\Phi_{\sigma\tau})^{-1}_{\tilde{x}} \circ \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ (\Phi_{\sigma,\tilde{x}})^\tau \circ s_\tau^{-1}h_\tau \\ =&(\Phi_{\sigma\tau})^{-1}_{\tilde{x}} \circ \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ \mathcal{E}^{\sigma\tau}_{\lambda_\sigma} \circ \mathcal{E}^{\sigma\tau}_{\lambda_\sigma^{-1}} \circ (\Phi_{\sigma,\tilde{x}})^\tau \circ s_\tau^{-1}h_\tau \\ =&(\Phi_{\sigma\tau})^{-1}_{\tilde{x}} \circ \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ \mathcal{E}^{\sigma\tau}_{\lambda_\sigma} \circ s_\sigma^{-1}s_\tau^{-1}\psi_\tau^{-1}(h_\sigma)h_\tau \\ =&(\Phi_{\sigma\tau})^{-1}_{\tilde{x}} \circ \mathcal{E}^{\sigma\tau}_{\lambda_{\sigma\tau}} \circ \mathcal{E}^{\sigma\tau}_{\lambda_{\sigma\tau}^{-1}} \circ \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ \mathcal{E}^{\sigma\tau}_{\lambda_\sigma} \circ s_\sigma^{-1}s_\tau^{-1}\psi_\tau^{-1}(h_\sigma)h_\tau \\ =&h_{\sigma\tau}^{-1}s_{\sigma\tau} \circ \mathcal{E}^{\sigma\tau}_{\lambda_{\sigma\tau}^{-1}} \circ \mathcal{E}^{\sigma\tau}_{\sigma\lambda_\tau} \circ \mathcal{E}^{\sigma\tau}_{\lambda_\sigma} \circ s_\sigma^{-1}s_\tau^{-1}\psi_\tau^{-1}(h_\sigma)h_\tau \\ =&h_{\sigma\tau}^{-1} \circ \psi_{\sigma\tau}^{-1}(\tilde{\rho}(\gamma_{\sigma\tau}^{-1}\gamma_\tau\gamma_\sigma)) \circ \psi_\tau^{-1}(h_\sigma) \circ h_\tau \\ =&h_{\sigma\tau}^{-1} \circ h_{\sigma\tau}\tilde{\rho}(\gamma_\tau\gamma_\sigma\gamma_{\sigma\tau}^{-1})h_{\sigma\tau}^{-1} \circ \psi_\tau^{-1}(h_\sigma) \circ h_\tau \\ =&g_{\sigma\tau} \circ h_{\sigma\tau}^{-1} \circ \psi_\tau^{-1}(h_\sigma) \circ h_\tau, \end{aligned} \tag{III.3.1.3.4}$$

whence the proposition. □

Remark III.3.1.4. We will always assume $\Phi_1 = \mathrm{Id}$ in accordance with the assumption $h_1 = 1$, so that the proposition gives an alternative condition for a (Γ, ψ)-invariant pair to descend to X.

With this proposition, we can say the $\bar{G}$-representation ρ corresponding to the pair $((\mathcal{E}, \nabla), \Phi_*)$ with Φ_* satisfying the cocycle conditions.

III.3.2 Stability Condition

III.3.2.1 Let P be a proper parabolic subgroup of G and let $\mathcal{E}$ be a principal G-bundle. Recall that a reduction of $\mathcal{E}$ to P is a principal P-bundle $\mathcal{P}$ and an isomorphism $\mathcal{E} \cong \mathcal{P} \times_P G$, where P acts on G by left multiplication.

For $\sigma \in \Gamma$, the G-conjugacy class of P is σ-stable if $\psi_\sigma(P)$ is G-conjugate to P. If the G-conjugacy class of P is σ-stable for all $\sigma \in \Gamma$, then we say that it is Γ-stable. The G-conjugacy class of P is σ-stable if and only if $N_{\bar{G}}(P)$ meets $G.s_\sigma^{-1}$. In this case, denote by P_σ the connected component of $N_{\bar{G}}(P)$ contained in $G.s_\sigma^{-1}$. If $\mathcal{P}$ is a principal P-bundle for some σ-stable P, then we can define $\mathcal{P}^\sigma$ as $\mathcal{P} \times_P P_\sigma$.

Definition III.3.2.1. A (Γ, ψ)-invariant pair $((\mathcal{E}, \nabla), \Phi_*)$ is unstable if there is

- a proper parabolic subgroup $P \subset G$ whose G-conjugacy class is Γ-stable (for the action $\psi : \Gamma \to \operatorname{Aut} G$) and
- a reduction of $\mathcal{E}$ to P, specified by an isomorphism $\mathcal{E} \cong \mathcal{P} \times_P G$,

such that for all $\sigma \in \Gamma$, Φ_σ has a reduction to P, i.e. such that Φ_σ is induced from some isomorphism $\mathcal{P} \to \sigma^*\mathcal{P}^\sigma$.

Recall that if $\gamma_* : \Gamma \to \pi_1(X)$ is the fixed section and ρ is a $\bar{G}$-representation, then $\rho(\gamma_\sigma) = h_\sigma^{-1} s_\sigma$.

Proposition III.3.2.2. *Let ρ be the $\bar{G}$-representation corresponding to $((\mathcal{E}, \nabla), \Phi_*)$. Then ρ is irreducible if and only if $((\mathcal{E}, \nabla), \Phi_*)$ is stable.*

Proof. A $\bar{G}$-representation ρ is irreducible if and only if $\operatorname{Im} \tilde{\rho}$ is not contained in any proper parabolic subgroup $P \subset G$ such that $h_\sigma^{-1} s_\sigma$ normalise P for all $\sigma \in \Gamma$, if and only if $(\mathcal{E}, \nabla)$ admits no reduction to proper parabolic P such that all $h_\sigma^{-1} s_\sigma$ normalise P, if and only if $(\mathcal{E}, \nabla)$ admits no P-reduction $(\mathcal{P}, \nabla)$ such that all Φ_σ restricts to $\mathcal{P} \to \sigma^*\mathcal{P}^\sigma$. (For any σ, $s_\sigma^{-1} h_\sigma$ normalise P if and only if $e \mapsto e s_\sigma^{-1} h_\sigma$ defines a morphism of homogeneous spaces $\mathcal{P}_{\tilde{x}} \to \mathcal{P}^\sigma_{\tilde{x}}$, which by the arguments in the proof of Lemma III.3.1.1 is equivalent to having a morphism of principal P-bundles $\mathcal{P} \to \sigma^*\mathcal{P}^\sigma$. Then this morphism induces Φ_σ.) □

III.3.2.2 Example Let $\psi : \Gamma \to \operatorname{Aut} G$ be the trivial homomorphism and let $G = \mathrm{GL}_n(\mathbb{C})$. In this case, any parabolic subgroup $P \subset \bar{G}$ is just $\cup_{\sigma \in \Gamma} P^\circ.s_\sigma$ since any s_σ normalise P°. The image of ρ is generated by $\rho(\pi_1(\tilde{X}))$ and $\rho(\gamma_\sigma) = h_\sigma^{-1} s_\sigma$, for all $\sigma \in \Gamma$. Therefore $\operatorname{Im} \rho \subset P$ if and only if $\rho(\pi_1(\tilde{X})) \subset P^\circ$ and $h_\sigma^{-1} \in P^\circ$ for any $\sigma \in \Gamma$. We see that $\operatorname{Im} \rho$ is not contained in any P if and only if $\operatorname{Im} \tilde{\rho}$ is not contained in any P° such that $h_\sigma \in P^\circ$ for any $\sigma \in \Gamma$.

Now let $\mathcal{L}$ be the local system on $\tilde{X}$ corresponding to $\tilde{\rho}$, equipped with isomorphisms $\Phi_\sigma : \mathcal{L} \to \sigma^*\mathcal{L}$ for all $\sigma \in \Gamma$. Let $\mathcal{E}$ be a local subsystem of $\mathcal{L}$ and let P be the maximal parabolic subgroup defined as the stabiliser of $\mathcal{E}_{\tilde{x}}$ in $\mathrm{GL}(\mathcal{L}_{\tilde{x}}) \cong \mathrm{GL}_n(\mathbb{C})$. By Lemma III.3.1.1, $h_\sigma \in P$ if and only if Φ_σ maps $\mathcal{E}_{\tilde{x}}$ to $(\sigma^*\mathcal{E})_{\tilde{x}}$ which is equivalent to that Φ_σ maps $\mathcal{E}$ to $\sigma^*\mathcal{E}$. The irreducibility of ρ is translated into the condition that there is no local subsystem $\mathcal{E}$ that is invariant under Φ_σ for all $\sigma \in \Gamma$.

This morally recovers the stability of Γ-bundles defined by Seshadri: A Γ-vector bundle V on $\tilde{X}$ is Γ-stable if its underlying vector bundle is semi-stable and for every proper Γ-subbundle W of V, we have

$$\mu(W) < \mu(V). \tag{III.3.2.2.1}$$

See [Ses] Chapter II, §1.

III.3.2.3 Example Let $G = \mathrm{GL}_n(\mathbb{C})$ and $\bar{G} = G \rtimes <\sigma>$ be defined by the transpose inverse σ and consider the condition for a $\bar{G}$-representation ρ to be irreducible. The definition says that $\mathrm{Im}\,\rho$ is not contained in any maximal proper parabolic subgroup of $\bar{G}$. Suppose $\mathrm{Im}\,\rho \subset P$ for some parabolic subgroup $P \subset \bar{G}$. Up to a conjugation by G, we can assume that P° is of the form given in (II.5.4.1). Then $P = P^\circ \cup P^\circ J s_1$, where s_1 acts on G by σ and $(J)_{ij} = \delta_{i,n+1-j}$. Recall that $\rho(\gamma_\sigma) = h_\sigma^{-1} s_\sigma$. The irreducibility means that for any standard σ-stable proper parabolic P° such that $h_\sigma \in JP^\circ$, $\mathrm{Im}\,\tilde{\rho}$ is not contained in P°.

On the other hand, Zelaci's work on vector bundles suggests the following definition of stable (Γ, ψ)-invariant local system. See [Ze2] Definition 4.1. Let $\mathcal{L}$ be a (Γ, ψ)-invariant local system on $\tilde{X}$. Given the defining isomorphism $\Phi : \mathcal{L} \to \sigma^*\mathcal{L}^\vee$, and a local subsystem $\mathcal{E} \hookrightarrow \mathcal{L}$, we put $\mathcal{E}^\perp$ to be the kernel of the surjection

$$\mathcal{L} \xrightarrow{\Phi} \sigma^*\mathcal{L}^\vee \twoheadrightarrow \sigma^*\mathcal{E}^\vee. \tag{III.3.2.3.1}$$

We say that $\mathcal{E}$ is isotropic if $\mathcal{E} \subset \mathcal{E}^\perp$. Note that this definition depends on Φ. In this case, we say that

$$0 \subset \mathcal{E} \subset \mathcal{E}^\perp \subset \mathcal{L} \tag{III.3.2.3.2}$$

is an isotropic flag. Obviously, giving such a flag is equivalent to giving an isotropic local subsystem. We then define that $\mathcal{L}$ is stable if it admits no isotropic flag. We will show that if $(\mathcal{L}, \Phi)$ corresponds to ρ, then this is equivalent to that ρ is irreducible.

Indeed, an isotropic flag gives a maximal standard σ-stable proper parabolic subgroup P° by taking the stabiliser of the flag

$$0 \subset \mathcal{E}_{\tilde{x}} \subset \mathcal{E}_{\tilde{x}}^\perp \subset \mathcal{L}_{\tilde{x}}. \tag{III.3.2.3.3}$$

Moreover, this flag is stable under Jh_σ. To see this, we must fix dual basis

$$\begin{aligned} &\{e_1, \ldots, e_n\} \subset \mathcal{L}_{\tilde{x}}, \\ &\{\epsilon_1, \ldots, \epsilon_n\} \subset \mathcal{L}_{\tilde{x}}^\vee, \end{aligned} \tag{III.3.2.3.4}$$

so that $\mathcal{E}_{\tilde{x}}$ is spanned by the first r basis vectors, $\mathcal{E}_{\tilde{x}}^\perp$ is spanned by the first $n-r$ basis vectors, the endomorphism of J is the permutation $e_i \leftrightarrow e_{n-r+1}$, and transpose inverse is in the usual sense of matrices. Denote by $\psi : \mathcal{L}_{\tilde{x}}^\vee \to \mathcal{L}_{\tilde{x}}$ the isomorphism defined by the dual basis. Note that Lemma III.3.1.1 in the current context means that h_σ is equal to the map

$$\mathcal{L}_{\tilde{x}} \xrightarrow{\Phi} \mathcal{L}_{\sigma(\tilde{x})}^\vee \xrightarrow{\mathcal{L}_{\lambda_\sigma^{-1}}} \mathcal{L}_{\tilde{x}}^\vee \xrightarrow{\psi} \mathcal{L}_{\tilde{x}}.$$

By definition, the composition

$$\mathcal{E}^{\perp} \hookrightarrow \mathcal{L} \xrightarrow{\Phi} \sigma^*\mathcal{L}^{\vee} \twoheadrightarrow \sigma^*\mathcal{E}^{\vee} \tag{III.3.2.3.5}$$

gives 0, which implies that the map

$$\mathcal{E}^{\perp}_{\tilde{x}} \hookrightarrow \mathcal{L}_{\tilde{x}} \xrightarrow{\Phi} \mathcal{L}^{\vee}_{\sigma(\tilde{x})} \xrightarrow{\mathcal{L}_{\lambda_\sigma^{-1}}} \mathcal{L}^{\vee}_{\tilde{x}} \xrightarrow{\psi} \mathcal{L}_{\tilde{x}} \xrightarrow{pr} \mathcal{E}_{\tilde{x}} \tag{III.3.2.3.6}$$

is 0, by the commutativity of

$$\begin{tikzcd} \mathcal{L}^{\vee}_{\sigma(\tilde{x})} \arrow[r, "\mathcal{L}_{\lambda_\sigma^{-1}}"] \arrow[d] & \mathcal{L}^{\vee}_{\tilde{x}} \arrow[r, "\psi"] \arrow[d] & \mathcal{L}_{\tilde{x}} \arrow[d, "pr"] \\ \mathcal{E}^{\vee}_{\sigma(\tilde{x})} \arrow[r, "\mathcal{E}_{\lambda_\sigma^{-1}}"] & \mathcal{E}^{\vee}_{\tilde{x}} \arrow[r, "\psi"] & \mathcal{E}_{\tilde{x}}. \end{tikzcd} \tag{III.3.2.3.7}$$

That is, h_σ maps $\mathcal{E}^{\perp}_{\tilde{x}}$ into the kernel of pr, which is spanned by the vectors $\{e_{r+1}, \ldots, e_n\}$. Therefore Jh_σ preserves $\mathcal{E}^{\perp}_{\tilde{x}}$. Dually, we have the 0 map

$$\mathcal{E} \hookrightarrow \mathcal{L} \xrightarrow{\sigma^* {}^t\Phi} \sigma^*\mathcal{L}^{\vee} \twoheadrightarrow \sigma^*(\mathcal{E}^{\perp})^{\vee}, \tag{III.3.2.3.8}$$

but $\sigma^* {}^t\Phi = \Phi$. So by the same argument we see that Jh_* preserves $\mathcal{E}_{\tilde{x}}$. We conclude that the isotropic flag is automatically preserved by Jh_σ, thus $h_\sigma^{-1} \in P^\circ J$.

Conversely, given any maximal standard σ-stable proper parabolic subgroup P° preserved by Jh_σ, we obtain a flag $0 \subset \mathcal{E} \subset \mathcal{F} \subset \mathcal{L}$ with $\mathcal{F}$ identified with $\mathcal{E}^{\perp}$ by tracing back the above reasoning.

III.4 Monodromy on Riemann Surfaces

III.4.1 Monodromy in Twisted Conjugacy Classes

Suppose the topological manifolds are Riemann surfaces. We introduce punctures on the Riemann surfaces and study the local monodromy. Let us first fix some notations.

III.4.1.1 Notations Let $p' : \tilde{X}' \to X'$ be a possibly ramified Galois covering of compact Riemann surfaces with $\mathrm{Aut}(\tilde{X}'/X') \cong \Gamma$. Denote by h the genus of X' and g the genus of $\tilde{X}'$. Let $\mathcal{R} \subset X'$ be a finite set of points such that p' is unramified over $X := X' \setminus \mathcal{R}$. Let I be the index set of the elements of $\mathcal{R}$ so that each point of $\mathcal{R}$ is written as x_j, $j \in I$. Denote by $\tilde{\mathcal{R}} \subset \tilde{X}'$ the inverse image of $\mathcal{R}$ and write $\tilde{X} := \tilde{X}' \setminus \tilde{\mathcal{R}}$. Denote by p the restriction of p' to $\tilde{X}$. We fixe the base points $\tilde{x} \in \tilde{X}$ and $x = p(\tilde{x}) \in X$ as before. For each $x_j \in \mathcal{R}$ and some $\tilde{x}_j \in p'^{-1}(x_j)$, put $n_j = |\mathrm{Stab}_\Gamma(\tilde{x}_j)|$ with $\mathrm{Stab}_\Gamma(\tilde{x}_j) = <\sigma_j>$, so $\sigma_j \in \Gamma$ is of order n_j. It only depends on x_j. Thus p' is unramified over those x_j with $n_j = 1$.

For each $j \in I$, there is a small neighbourhood V_j homeomorphic to $\mathbb{C}$ of x_j such that each point in $p'^{-1}(x_j)$ has a small neighbourhood homeomorphic to $\mathbb{C}$ on which the restriction of p' is $z \mapsto z^{n_j}$. For each j, choose a point $y_j \in V_j$ and a loop l_j around x_j based at y_j, all with the same orientation. We can choose paths λ_j lying in X from x to y_j such that $\gamma_j := \lambda_j^{-1} l_j \lambda_j$, together with the generators α_i, β_i associated to the genus, generate $\pi_1(X)$ and satisfy the relation

$$\prod_{i=1}^{h}(\alpha_i, \beta_i) \prod_{j \in I} \gamma_j = 1. \tag{III.4.1.1.1}$$

The choice of the path λ_j determines a point $\tilde{y}_j$ over y_j and thus the connected component of $p'^{-1}(V_j)$ containing $\tilde{y}_j$. Let us denote this connected component by U_j and denote by $\tilde{x}_j$ the point in U_j over x_j. All other connected components of $p'^{-1}(V_j)$ are of the form $\tau(U_j)$ for some $\tau \in \Gamma$. For each such component, we fix a τ as above and thus a point $\tau(\tilde{y}_j)$ in it. The associated objects will be indicated by a subscript (j, τ), for example $U_{j,\tau} = \tau(U_j)$, $\tilde{y}_{j,\tau} = \tau(\tilde{y}_j)$, and in particular, $U_{j,1} = U_j$. If r_j is the lift of l_j starting from $\tilde{y}_j$, then

$$\tilde{l}_j := {}^{\sigma_j^{n_j-1}} r_j \cdots {}^{\sigma_j} r_j r_j \tag{III.4.1.1.2}$$

is a loop in U_j based at $\tilde{y}_j$, where ${}^{\sigma_j} r_j$ is the image of r_j under σ_j. Let $\tilde{l}_{j,\tau} = \tau(\tilde{l}_j)$. Again, we can choose paths $\tilde{\lambda}_{j,\tau}$ for $\tilde{x}$ to $\tilde{y}_{j,\tau}$ lying in $\tilde{X}$ such that $\tilde{\gamma}_{j,\tau} := \tilde{\lambda}_{j,\tau}^{-1} \tilde{l}_{j,\tau} \tilde{\lambda}_{j,\tau}$ together with $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $1 \le i \le g$, associated to the genus of $\tilde{X}$, generate $\pi_1(\tilde{X})$ and satisfy a similar relation as for γ_j's. Note that the $\tilde{\lambda}_{j,1}$'s are not necessarily the lifts of the λ_j's.

III.4.1.2 Monodromy of $\rho : \pi_1(X) \to G \rtimes \Gamma^{op}$ Let ρ be as in III.1.2.2. Fix $\psi : \Gamma \to \operatorname{Aut} G$ and write $\bar{G} = G \rtimes_{\psi^{op}} \Gamma^{op}$ and denote by $\tilde{\rho}$ the underlying G-representation. Since the end point of the lift of γ_j is $\sigma_j(\tilde{x})$, γ_j belongs to the coset in $\pi_1(X)$ corresponding to σ_j. With the specific choices of the γ_j's as above, we say that the monodromy of a $\bar{G}$-representation ρ at the puncture x_j is the element $\rho(\gamma_j)$, which lies in the connected component $G.s_{\sigma_j}$. In the $\bar{G}$-character variety, its G-conjugacy class is well-defined, say C_j. A different choice of λ_j results in a conjugation of γ_j in $\pi_1(X)$, whence a conjugation by $\bar{G}$ of $\rho(\gamma_j)$, whence a conjugation of C_j by the group of connected components Γ^{op}. However, as we can see below, even if we fix a particular λ_j, we still need to consider the $\bar{G}$-conjugates of C_j when we go up to $\tilde{X}$. So it is natural to consider the $\bar{G}$-conjugacy class of $\rho(\gamma_j)$.

Now we consider what happens on $\tilde{X}$. The lift of $\gamma_j^{n_j}$ is conjugate to $\tilde{\gamma}_{j,1}$, therefore $\tilde{\rho}(\tilde{\gamma}_{j,1})$ must lie in the G-conjugacy class $\tilde{C}_j := C_j^{n_j} := \{g^n \mid g \in C_j\} \subset G$. Now we take τ to be the element that takes U_j to some $U_{j,\tau}$ that does not meet U_j. The lift of $\gamma_\tau^{-1} \gamma_j^{n_j} \gamma_\tau$ (γ_τ is given by the fixed section $\Gamma \to \pi_1(X)$) is conjugate to $\tilde{\lambda}_{j,\tau}^{-1} \tilde{l}_{j,\tau} \tilde{\lambda}_{j,\tau} = \tilde{\gamma}_{j,\tau}$ in $\pi_1(\tilde{X})$, therefore

$$\tilde{\rho}(\tilde{\gamma}_{j,\tau}) \text{ is conjugate to } \rho(\gamma_\tau)^{-1} \rho(\gamma_j)^{n_j} \rho(\gamma_\tau) \text{ by } \operatorname{Im} \tilde{\rho} \tag{III.4.1.2.1}$$

which is in the conjugacy class $\rho(\gamma_j)^{-1}\tilde{C}_j\rho(\gamma_j) = s_\tau^{-1}h_\tau\tilde{C}_jh_\tau^{-1}s_\tau = \psi_\tau(\tilde{C}_j) =: \tilde{C}_{j,\tau}$. Similarly, $\rho(\gamma_\tau^{-1}\gamma_j\gamma_\tau) \in s_\tau^{-1}(C_j)s_\tau =: C_{j,\tau}$ which belongs to the connected component $G.s_{\tau\sigma_j\tau^{-1}}$. We have $C_{j,\tau}^{n_j} = \tilde{C}_{j,\tau}$. Note that $\gamma_\tau^{-1}\gamma_j\gamma_\tau$ is in the coset corresponding to $\tau\sigma_j\tau^{-1}$.

III.4.1.3 Let us summarize the above discussions as follows.

Definition III.4.1.1. With a loop l_j based at y_j in a small neighbourhood of the puncture x_j, the local monodromy class of a $\bar{G}$-representation ρ at x_j is the $\bar{G}$-conjugacy class $\bar{C}_j$ of $\rho(\lambda_j^{-1}l_j\lambda_j)$ for some path λ_j from x to y_j. It does not depend on the choice of λ_j. A particular choice of λ_j singles out a G-conjugacy class C_j contained in $\bar{C}_j$, also called the local monodromy class of ρ at x_j.

We also have,

Lemma III.4.1.2. *Suppose that we are given a particular path λ_j as in the above definition so that $\rho(\lambda_j^{-1}l_j\lambda_j) \in C_j$, and $\tilde{x}_j$ is the point over x_j which has a small neighbourhood containing the end point of the lift of λ_j. Then the local monodromy class of $\tilde{\rho}$ at $\tilde{x}_j$ is $C_j^{n_j}$. Moreover, for any other point $\tau(\tilde{x}_j)$ over x_j, $\tau \in \Gamma$, the local monodromy class of $\tilde{\rho}$ at $\tau(\tilde{x}_j)$ is $\psi_\tau(C_j^{n_j})$.*

Remark III.4.1.3. Fixing the monodromy classes $\tilde{C}_{j,\tau}$ on $\tilde{X}$ does not uniquely determine the monodromy classes C_j, since in general there can be many conjugacy classes C_j' such that $(C_j')^{n_j} = \tilde{C}_{j,1}$.

Remark III.4.1.4. Suppose that ψ is the trivial homomorphism. A $\bar{G}$-conjugacy class in $\bar{G}$ is just a union of copies of a particular G-conjugacy class in $\bar{G}$, say C_j, with one copy in each connected component corresponding to the elements of some conjugacy class of Γ. On the other hand, at all points lying over x_j, the monodromy classes of $\tilde{\rho}$ are the same, and only depend on x_j, say $\tilde{C}_j$. If $\tilde{C}_j = \{1\}$, then C_j is morally the local type of a Γ-invariant vector bundle over x_j as defined by Balaji and Seshadri. See [BS, Definition 2.2.6].

III.4.2 Generic Conjugacy Classes

In this section. we assume that the image of

$$\Gamma \xrightarrow{\psi} \operatorname{Aut} G \to (\operatorname{Aut} G)/G_{ad} = \mathbf{A}(G)$$

is contained in a cyclic subgroup. If G is GL_n or almost-simple with root system not of type D_4, then this is always satisfied.

III.4.2.1 Fix a maximal torus T contained in a Borel subgroup B of G. In each connected component $G.s_\sigma$, $\sigma \in \Gamma$, we choose a quasi-central element $\mathbf{s}_\sigma \in N_{\bar{G}}(T,B)$, so that $N_{\bar{G}}(T,B) = \cup_{\sigma\in\Gamma}T.\mathbf{s}_\sigma$. The semi-simple G-conjugacy classes in $G.s_\sigma$ are parametrised by the $W^{\mathbf{s}_\sigma}$-orbits in

$$\tilde{\mathbf{T}}_\sigma := T/[T,\mathbf{s}_\sigma] \cong (T^{\mathbf{s}_\sigma})^\circ/(T^{\mathbf{s}_\sigma})^\circ \cap [T,\mathbf{s}_\sigma].$$

with $t \in (T^{\mathbf{s}_\sigma})^\circ$ representing the class of $t\mathbf{s}_\sigma$. Denote by $\mathbf{T}_\sigma$ the quotient $\tilde{\mathbf{T}}_\sigma / W^{\mathbf{s}_\sigma}$. When $\sigma = \sigma_j$ for some $j \in I$, we will write $\mathbf{s}_j$, $\tilde{\mathbf{T}}_j$ and $\mathbf{T}_j$ instead of $\mathbf{s}_{\sigma_j}$, $\tilde{\mathbf{T}}_{\sigma_j}$ and $\mathbf{T}_{\sigma_j}$.

Let $\mathcal{C} = (C_j)_{j\in I}$ be a tuple of G-conjugacy classes of $\bar{G}$, with C_j contained in $G.s_{\sigma_j}$. Denote by $\mathrm{Rep}_{\Gamma,\mathcal{C}}(X, G) \subset \mathrm{Rep}_\Gamma(X, G)$ the locally closed subvariety consisting of ρ satisfying $\rho(\gamma_j) \in C_j$, for all $j \in I$. If $\mathcal{C}$ are semi-simple classes, $\mathrm{Rep}_{\Gamma,\mathcal{C}}(X, G)$ is closed in $\mathrm{Rep}_\Gamma(X, G)$ and these tuples are parametrised by $\prod_j \mathbf{T}_j$. We will define a non-empty subset $\mathbb{T}^\circ \subset \prod_j \mathbf{T}_j$ so that $\rho \in \mathrm{Rep}_\Gamma(X, G)$ is irreducible whenever $\rho \in \mathrm{Rep}_{\Gamma,\mathcal{C}}(X, G)$, for some tuple $\mathcal{C}$ whose semi-simple parts correspond to a point of $\mathbb{T}^\circ$.

III.4.2.2 Let P be a parabolic subgroup containing B and let L be the unique Levi factor of P containing T. In this case we will simply say that (L, P) contains (T, B). There are only finitely many such pairs (L, P). Suppose that $N_{\bar{G}}(L, P)$ meets all connected components of $\bar{G}$. This implies that for any $\sigma \in \Gamma$, the G-conjugacy class of $(L \subset P)$ is stable under $\mathbf{s}_\sigma$. But $(\mathbf{s}_\sigma(L), \mathbf{s}_\sigma(P))$ also contains (T, B), so it is necessary that $(\mathbf{s}_\sigma(L), \mathbf{s}_\sigma(P)) = (L, P)$ for any $\sigma \in \Gamma$. Therefore $N_{\bar{G}}(L, P) = \cup_{\sigma\in\Gamma} L.\mathbf{s}_\sigma$.

Write $\bar{L} = N_{\bar{G}}(L, P)$.

Lemma III.4.2.1. *Let L and P be as above. If $L \neq G$, then* $\dim Z_{\bar{L}} > \dim Z_{\bar{G}}$.

Proof. For any $\sigma \in \Gamma$, the action of $\mathbf{s}_\sigma$ on T are determined by the connected component of $\mathrm{Aut}\, G$ that the automorphism $\mathrm{ad}\, \mathbf{s}_\sigma$ belongs to. By the assumption at the beginning of this section, there is some $\sigma_0 \in \Gamma$ such that $\mathrm{ad}\, \mathbf{s}_{\sigma_0}$ generates the image of

$$\Gamma \xrightarrow{\psi} \mathrm{Aut}\, G \to \mathbf{A}(G).$$

Therefore $Z_{\bar{L}} = C_{Z_L}(\mathbf{s}_{\sigma_0})$ and $Z_{\bar{G}} = C_{Z_G}(\mathbf{s}_{\sigma_0})$, since Z_L and Z_G are contained in T.

Now $L' := C_L(\mathbf{s}_{\sigma_0})^\circ$ is a Levi subgroup of $G' := C_G(\mathbf{s}_{\sigma_0})^\circ$. By Proposition II.3.2.1, if $L \neq G$, then $L' \neq G'$. By Proposition II.3.1.11, $Z^\circ_{L'} = C_{Z^\circ_L}(\mathbf{s}_{\sigma_0})^\circ$ and $Z^\circ_{G'} = C_{Z^\circ_G}(\mathbf{s}_{\sigma_0})^\circ$. Then the lemma follows from the result for usual Levi subgroups. □

III.4.2.3 For any connected reductive algebraic group H, denote by

$$\mathbf{D}_H : H \longrightarrow Z^\circ_H / (Z^\circ_H \cap [H, H])$$

the projection, identifying $H/[H, H] \cong Z^\circ_H/(Z^\circ_H \cap [H, H])$. For $H = \mathrm{GL}_n$, this is the determinant.

Each element $\mathbf{t}_\sigma \in \mathbf{T}_\sigma$ is an $W^{\mathbf{s}_\sigma}$-orbit. Each element $t_\sigma \in \mathbf{t}_\sigma$ is a coset in $(T^{\mathbf{s}_\sigma})^\circ$. We fix a representative in $(T^{\mathbf{s}_\sigma})^\circ$ of each such t_σ, also denoted by t_σ. The choice of such representative will not matter. If $\sigma = \sigma_j$ for some $j \in I$, then we write t_j and $\mathbf{t}_j$ instead of t_{σ_j} and $\mathbf{t}_{\sigma_j}$.

Definition III.4.2.2. A tuple of semi-simple conjugacy classes parametrised ty $(\mathbf{t}_j)_{j\in I}$ is generic if the following condition is satisfied. For

- any(L, P) containing (T, B) with $P \neq G$ such that $N_{\bar{G}}(L, P)$ meets all connected components of $\bar{G}$, and

- any tuple $(t_j)_{j\in I}$ with $t_j \in \mathbf{t}_j$,

the element

$$\prod_{j\in I}\prod_{\tau\in\Gamma/<\sigma_j>} \mathbf{D}_L(\mathbf{s}_\tau(t_j^{n_j}\mathbf{s}_j^{n_j})) \in Z_L^\circ/(Z_L^\circ \cap [L,L]) \tag{III.4.2.3.1}$$

is not equal to the identity, where $\mathbf{s}_\tau$ acts on T by conjugation. A tuple of conjugacy classes C is generic if the tuple of the conjugacy classes of the semi-simple parts of C is generic.

One can verify that $\mathbf{D}_L(\mathbf{s}_\tau(t^{n_j}\mathbf{s}_j^{n_j}))$ has constant value for $t \in [T, \mathbf{s}_j]$ so it only depends on the coset t_j.

Remark III.4.2.3. If we denote by $\mathbf{D}_L$ the homomorphism $\bar{L} \to \bar{L}/[L,L]$, then

$$\prod_{\tau\in\Gamma/<\sigma_j>} \mathbf{D}_L(\mathbf{s}_\tau(t_j^{n_j}\mathbf{s}_j^{n_j})) = \prod_{\tau\in\Gamma} \mathbf{D}_L(\mathbf{s}_\tau(t_j\mathbf{s}_j)),$$

where $\mathbf{s}_\tau$ acts on $N_{\bar{G}}(T,B)$ by conjugation.

Remark III.4.2.4. The morphism of varieties $(T^{\mathbf{s}_j})^\circ\mathbf{s}_j \to T$, $t\mathbf{s}_j \to (t\mathbf{s}_j)^{n_j}$ surjects onto a connected component of $T^{\mathbf{s}_j}$. Indeed,

$$(t\mathbf{s}_j)^{n_j} = t\mathbf{s}_j(t)\cdots\mathbf{s}_j^{n_j-1}(t)\mathbf{s}_j^{n_j} \in T$$

and is obviously fixed by $\mathbf{s}_j$.

Put

$$(Z_L^\circ)^\Gamma = \{z \in Z_L^\circ \mid \mathbf{s}_\tau(z) = z, \text{ for all } \tau \in \Gamma\}.$$

We have $(Z_L^\circ)^\Gamma = Z_L \cap Z_L^\circ$. It has the same dimension as Z_L.

Lemma III.4.2.5. *For any $j \in I$,*

$$\prod_{\tau\in\Gamma/<\sigma_j>} \mathbf{D}_L \circ \mathbf{s}_\tau : T^{\mathbf{s}_j} \longrightarrow (Z_L^\circ)^\Gamma/((Z_L^\circ)^\Gamma \cap [L,L])$$

is a surjective group homomorphism.

Proof. Let $t \in T^{\mathbf{s}_j}$ and $\sigma \in \Gamma$. We have

$$\mathbf{s}_\sigma\Big(\prod_{\tau\in\Gamma/<\sigma_j>} \mathbf{D}_L(\mathbf{s}_\tau(t))\Big) = \prod_{\tau\in\Gamma/<\sigma_j>} \mathbf{D}_L(\mathbf{s}_\sigma\mathbf{s}_\tau(t)).$$

Note that $\mathbf{s}_\sigma\mathbf{s}_\tau$ differs from $\mathbf{s}_{\sigma\tau}$ by an element of T, and therefore they have the same action on T. Also, all elements in a coset $\tau<\sigma_j>$have the same action on $T^{\mathbf{s}_j}$. The right hand side of the equality is thus equal to $\prod_{\tau\in\Gamma/<\sigma_j>}\mathbf{D}_L(\mathbf{s}_\tau(t))$. So the image is Γ-invariant.

Let $z \in (Z_L^\circ)^\Gamma$. Then

$$\prod_{\tau \in \Gamma/<\sigma_j>} \mathbf{D}_L(\mathbf{s}_\tau(z)) = z^{|\Gamma|/n_j}.$$

Since char $k \nmid |\Gamma|$, this surjects onto $(Z_L^\circ)^\Gamma$. □

III.4.2.4 Denote by $\tilde{\mathbb{T}} \subset \prod_j \tilde{\mathbf{T}}_j$ the closed subvariety defined by

$$\prod_{j \in I} \prod_{\tau \in \Gamma/<\sigma_j>} \mathbf{D}_G(\mathbf{s}_\tau(t_j^{n_j} \mathbf{s}_j^{n_j})) = 1, \tag{III.4.2.4.1}$$

with $t_i \in \tilde{\mathbf{T}}_j$. One can also define $\mathbb{T} \subset \prod_j \mathbf{T}_j$ by the same equation while choosing for each orbit $\mathbf{t}_j$ an element t_j in it. This is well-defined.

Proposition III.4.2.6. *The subset of generic semi-simple conjugacy classes $\mathbb{T}^\circ \subset \mathbb{T}$ is Zariski open and non-empty.*

Thus we can regard $\mathbb{T}^\circ$ as an open subvariety.

Proof. Let $Z \subset \prod_j \tilde{\mathbf{T}}_j$ be the closed subset defined by: (t_j) belongs to Z if for some $(L, P) \supset (T, B)$ with $P \neq G$,

$$\prod_{j \in I} \prod_{\tau \in \Gamma/<\sigma_j>} \mathbf{D}_L(\mathbf{s}_\tau(t_j^{n_j} \mathbf{s}_j^{n_j})) = 1. \tag{III.4.2.4.2}$$

By Lemme III.4.2.1, Remark III.4.2.4 and Lemma III.4.2.5, we have $\dim Z < \dim \tilde{\mathbb{T}}$.

The finite group $\mathbf{W} := \prod_j W^{\mathbf{s}_j}$ acts on $\prod_j \tilde{\mathbf{T}}_j$ and preserves the closed subsets $\tilde{\mathbb{T}}$ and $\cup_{w \in \mathbf{W}} w.Z$. Define

$$\tilde{\mathbb{T}}^\circ = \tilde{\mathbb{T}} \setminus \bigcup_{w \in \mathbf{W}} w.Z.$$

It is a $\mathbf{W}$-invariant open subset of $\tilde{\mathbb{T}}$, and is non-empty for dimension reason. Then by definition $(\mathbf{t}_j) \in \mathbb{T}^\circ$ if and only if all of its fibres are contained in $\tilde{\mathbb{T}}^\circ$ and so $\mathbb{T}^\circ = \tilde{\mathbb{T}}^\circ/\mathbf{W}$ is open. □

III.4.2.5 Generic conjugacy classes of $\mathrm{GL}_n(k) \rtimes <\sigma>$ By Hurwitz formula, there can only be an even number of ramification points in a double covering of Riemann surfaces. Let the punctures be exactly the ramification points. In this case $I = \{1, \ldots, 2k\}$ and let $C = \{C_1, \ldots, C_{2k}\}$ be a $2k$-tuple of semi-simple σ-conjugacy classes. That is, G-conjugacy classes contained in $G.\sigma$. We write $n = 2m$ or $n = 2m + 1$ according to the parity. In either case, C_j is determined by an m-tuple of eigenvalues $\mathbf{A}_j = \{\bar{a}_{j,1}, \ldots, \bar{a}_{j,m}\}$. (See §II.5.3.3) Put $\Lambda = \{1, \ldots, m\}$. We write $\tilde{\mathbf{A}}_j = \{\bar{a}_{j,1}^2, \ldots, \bar{a}_{j,m}^2\}$, where for any orbit $\bar{z}$, $\bar{z}^2 := \{x^2 | x \in \bar{z}\}$. We have $\bar{1}^2 = \{1\}$, $\bar{i}^2 = \{-1\}$ ($i = \sqrt{-1}$), and for any other orbit $\bar{z}^2 = \{z^2, z^{-2}\}$. For any j and any subset $J_j \subset \Lambda$, denote by $\tilde{\mathbf{A}}_{J_j} \subset \tilde{\mathbf{A}}_j$ a tuple of the form $(z_k)_{k \in J_j}$, $z_k \in \bar{a}_{j,k}^2$.

We write

(III.4.2.5.1) $$[\tilde{\mathbf{A}}_{J_i}] = \prod_{k \in J_j} z_k.$$

We see that C is generic if and only if for any $1 \leq l \leq m$, any J_j, $j \in I$, such that $|J_j| = l$, and any $\tilde{\mathbf{A}}_{J_j}$, the following relation holds,

(III.4.2.5.2) $$[\tilde{\mathbf{A}}_{J_1}] \cdots [\tilde{\mathbf{A}}_{J_{2k}}] \neq 1.$$

We will say that C is *strictly generic* if we require further that

(III.4.2.5.3) $$[\tilde{\mathbf{A}}_{J_1}] \cdots [\tilde{\mathbf{A}}_{J_{2k}}] \neq -1,$$

for any l and J_j as above. This notion will only be used to simplify some calculations in the point-counting problem.

III.4.2.6 We conclude this section by the following proposition.

Proposition III.4.2.7. *Suppose C is a tuple of generic conjugacy classes. Then every element of* $\mathrm{Rep}_{\Gamma,C}(X, G)$ *is an irreducible $\bar{G}$-representation.*

Proof. Fix $T \subset B$ and $\mathbf{s}_\sigma$ as in §III.4.2.1. Suppose $\rho \in \mathrm{Rep}_{\Gamma,C}(X, G)$ is not irreducible. Then there exists some proper parabolic subgroup $P \subset G$ such that $N_{\bar{G}}(P)$ meets all connected components of $\bar{G}$ and $\mathrm{Im}\,\rho \subset N_{\bar{G}}(P)$. Up to a G-conjugation we can assume that P contains B. Let L be the unique Levi factor of P containing T.

Put $c_j := \rho(\gamma_j) \in P.\mathbf{s}_j$, then $c_j^{n_j} \in P$. Let $\tilde{c}_j = \tilde{\rho}(\tilde{\gamma}_j)$, then it is P-conjugate to $c_j^{n_j}$. For τ representing a coset in $\Gamma/<\sigma_j>$, $\tilde{c}_{j,\tau} := \tilde{\rho}(\tilde{\gamma}_{j,\tau})$ is P-conjugate to $\mathbf{s}_\tau(c_j^{n_j})$ by (III.4.1.2.1). Let $\pi_L : P \to L$ be the natural projection. Using a presentation of $\pi_1(\tilde{X})$ by the $\tilde{\gamma}_{j,\tau}$'s, we find

(III.4.2.6.1) $$\prod_{j \in I} \prod_{\tau \in \Gamma/<\sigma_j>} \mathbf{D}_L \circ \pi_L(\mathbf{s}_\tau(c_j^{n_j})) = 1.$$

Note that the value of $\mathbf{D}_L$ only depends on the semi-simple parts. The semi-simple part $c_{j,s}$ of c_j is contained in $G.\mathbf{s}_j$ because we have assumed char $k \nmid |\Gamma|$ and all unipotents elements are thus contained in G. In particular, $c_{j,s} \in P.\mathbf{s}_j$. It is therefore P-conjugate to an element of $N_{\bar{G}}(T, B)$ and is further L-conjugate to an element of $(T^{\mathbf{s}_j})^\circ \mathbf{s}_j$ by Proposition II.3.1.2. Now, the above relation contradicts the definition of generic conjugacy classes. □

III.5 Double Coverings

III.5.1 $G \rtimes \mathbb{Z}/2\mathbb{Z}$-Character Varieties

These are among the most important examples. We will reproduce some of the general results above in these particular cases with more explicit computations. All coverings are assumed to be connected.

III.5.1.1 Let $p : \tilde{X} \to X$ be a double covering of Riemann surfaces. Denote by θ the non-trivial covering transformation of p. Interesting structure groups include connected complex reductive group of type A, D or E_6. Other types are possible but do not have non trivial graph automorphism. We will denote the structure group by G. Since a homomorphism $\psi : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut} G$ is determined by the image of $1 \in \mathbb{Z}/2\mathbb{Z}$, we will denote its image by σ and say the (Γ, ψ)-invariant pair concerned is <u>σ-invariant</u>. Therefore for a G-representation $\tilde{\rho}$ of $\pi_1(\tilde{X})$, we write $\tilde{\rho}^\sigma := \sigma \circ \tilde{\rho} \circ C_\lambda \circ \theta$, where λ is the unique lift of $\gamma := \gamma_\theta$. Denote by $\tilde{\gamma}$ the path ${}^\theta\lambda\lambda$ which lifts γ^2. Let us look at various σ-invariant local systems on $\tilde{X}$.

We are concerned with two cases.

(i) σ is the trivial automorphism of G, and $\tilde{\rho}^\sigma = h\tilde{\rho}$ for some $h \in G$. In this case, we simply say that $(\tilde{\rho}, h)$ is an invariant pair.

(ii) σ is a graph automorphism of order 2, and $\tilde{\rho}^\sigma = h\tilde{\rho}$ for some $h \in G$. We say that $(\tilde{\rho}, h)$ is a σ-invariant pair.

III.5.1.2 If $G = \mathrm{GL}_n(\mathbb{C})$, of particular interest is the automorphism $\sigma_o : g \mapsto {}^t g^{-1}$, $g \in G$. Let $\mathcal{L} \in \operatorname{Rep}(\tilde{X}, \mathrm{GL}_n(\mathbb{C}))$ be a local system corresponding to some representation $\tilde{\rho}$, then $\tilde{\rho}^\sigma$ corresponds to $\sigma^*\mathcal{L}$ in case (i) and corresponds to $\sigma^*\mathcal{L}^\vee$ in case (ii), where $\mathcal{L}^\vee$ is the dual local system. Therefore $\mathcal{L}$ is invariant if there exists an isomorphism $\Phi : \mathcal{L} \to \sigma^*\mathcal{L}$ and that $\mathcal{L}$ is σ_o-invariant if there exists isomorphism $\Phi : \mathcal{L} \to \sigma^*\mathcal{L}^\vee$. If σ is a non-trivial order 2 inner automorphism, then it is equivalent to having the trivial automorphism, as we have seen before (§II.5.2.1). We will see later that, when n is even and $\sigma = \sigma_s$, the symplectic type automorphism, the corresponding σ-invariant pairs also have a simple description in terms of local systems.

Suppose that $\mathcal{L}$ is an irreducible invariant local system. Given an isomorphism $\Phi : \mathcal{L} \to \sigma^*\mathcal{L}$, the composition $\sigma^*\Phi \circ \Phi : \mathcal{L} \to \mathcal{L}$ is necessarily a homothety. The isomorphism Φ can be modified in such a way that this homothety is the identity, in which case Φ is called a linearisation (in [Ze2]).

Let $\mathcal{L}$ be an irreducible σ_o-invariant local system. Given an isomorphism $\Phi : \mathcal{L} \to \sigma^*\mathcal{L}^\vee$, the composition $\sigma^{*t}\Phi^{-1} \circ \Phi : \mathcal{L} \to \mathcal{L}$ is necessarily a homothety, denoted by $k \in \mathbb{C}^*$, Since a homothety is invariant under transpose and pullback by σ, we have $k = k^{-1}$. We define the signature of $\mathcal{L}$ as the value of k, denoted by $\epsilon_{\mathcal{L}}$.

III.5.1.3 We also have the equivalent description in terms of representations, but for general reductive groups.

Let $(\tilde{\rho}, h)$ be a strongly irreducible σ-invariant pair. Evaluating both sides of $\tilde{\rho}^\sigma = h\tilde{\rho}$ at $\tilde{\gamma}$ gives $\sigma(\tilde{\rho}(\tilde{\gamma})) = h\tilde{\rho}(\tilde{\gamma})h^{-1}$. Let $g := \tilde{\rho}(\tilde{\gamma})$, then

$$\sigma(g) = hgh^{-1}. \tag{III.5.1.3.1}$$

Now $(\tilde{\rho}^\sigma)^\sigma = g^{-1}\tilde{\rho} = \sigma(h)h\tilde{\rho}$, by the definition of $\tilde{\rho}^\sigma$ and by σ-invariance respectively. It follows that $kg^{-1} = \sigma(h)h$, for some $k \in Z_G$. Consider the action of σ on this equality. On the one hand, $\sigma(\sigma(h)h) = h\sigma(h)$ which is conjugated to $\sigma(h)h$ by h^{-1}. On the other hand, $\sigma(kg^{-1}) = \sigma(k)\sigma(g^{-1})$ which is conjugated to $\sigma(k)g^{-1}$ by h^{-1}, using (III.5.1.3.1). We find that $k = \sigma(k)$.

In the case of $\mathrm{GL}_n(\mathbb{C})$ and $\sigma = \sigma_o$ this means $k^2 = 1$, thus we recover the signature of $\tilde{\rho}$, denoted by $\epsilon_{\tilde{\rho}}$, which is the only value of k such that $kg^{-1} = \sigma(h)h$. Indeed, by Proposition III.3.1.3, we have $\epsilon_{\mathcal{L}} = \epsilon_{\tilde{\rho}}$ if $\mathcal{L}$ corresponds to $\tilde{\rho}$.

Suppose $\sigma = \mathrm{Id}$ and $\tilde{\rho}^\sigma = h\tilde{\rho}$. From the calculation above, we deduce that $gh^2 = k \in Z_G$, and h can be modified in such a way that $gh^2 = 1$. Such h corresponds to the linearisation Φ of an invariant pair.

III.5.1.4 The above calculation of signatures agree with our previous classification of (Γ, ψ)-invariant strongly irreducible representations in terms of group cohomology. For $G = \mathrm{GL}_n(\mathbb{C})$, $Z_G = \mathbb{C}^*$. The action of σ on $\mathbb{C}^*$ is either trivial or the inversion $x \mapsto x^{-1}$, depending whether σ is an inner or outer automorphism. Let us calculate $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$. A cochain $\varphi \in C^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$ is represented by

$$\begin{aligned} &(0,0) \mapsto a; \quad (0,1) \mapsto b; \\ &(1,0) \mapsto c; \quad (1,1) \mapsto d. \end{aligned} \tag{III.5.1.4.1}$$

The differential $\mathrm{d}\colon C^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) \to C^3(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$ is given by

$$\mathrm{d}\varphi(x, y, z) = \varphi(y, z)\varphi(x + y, z)^{-1}\varphi(x, y + z)(\varphi(x, y)^{-1} \cdot z). \tag{III.5.1.4.2}$$

We find that if the action of σ is trivial, then (a, b, c, d) is a cocycle if

$$b = a; \quad c = a; \quad \text{no restriction on } d, \tag{III.5.1.4.3}$$

and if σ acts by inversion, then (a, b, c, d) is a cocycle if

$$b = a^{-1}; \quad c = a; \quad d^2 = a^{-2}. \tag{III.5.1.4.4}$$

The differential $\mathrm{d}\colon C^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) \to C^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$ is given by

$$\mathrm{d}\varphi(x, y) = \varphi(y)\varphi(x + y)^{-1}(\varphi(x) \cdot y), \tag{III.5.1.4.5}$$

Let us represent a cochain $\varphi \in C^1(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$ by

$$0 \mapsto a; \quad 1 \mapsto b. \tag{III.5.1.4.6}$$

Then if σ acts trivially, then $\mathrm{d}\varphi$ is

$$\begin{aligned} &(0,0) \mapsto a; \quad (0,1) \mapsto a; \\ &(1,0) \mapsto a; \quad (1,1) \mapsto b^2a^{-1}, \end{aligned} \tag{III.5.1.4.7}$$

and if σ acts by inversion, then $\mathrm{d}\varphi$ is

$$\begin{aligned} &(0,0) \mapsto a; \quad (0,1) \mapsto a^{-1}; \\ &(1,0) \mapsto a; \quad (1,1) \mapsto a^{-1}. \end{aligned} \tag{III.5.1.4.8}$$

Therefore, if σ acts trivially, then $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*)$ is trivial and if σ acts by inversion, then $H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{C}^*) \cong \mu_2$ is the two-element group. The two cohomology classes, distinguished by $d = \pm a^{-1}$, correspond to the signatures of σ_o-invariant pairs. Indeed, the cochain $(k_{\sigma\tau})$ associated with $(\tilde{\rho}, h)$ is

$$\begin{aligned} &k_{(0,0)} = 1; \quad k_{(0,1)} = 1; \\ &k_{(1,0)} = 1; \quad k_{(1,1)} = k = \sigma(h)hg. \end{aligned} \tag{III.5.1.4.9}$$

Therefore $k = 1$ gives the trivial cohomology class whereas $k = -1$ gives the other cohomology class.

III.5.1.5 We have seen in Proposition III.2.2.6 that (Γ, ψ)-invariant pairs corresponding to a non-trivial cohomology class may be regarded as (Γ, ψ')-invariant pairs corresponding to the trivial cohomology class for some ψ' similar but not equal to ψ. In fact, the σ_o-invariant $\mathrm{GL}_n(\mathbb{C})$-local systems of signature -1 are exactly the σ_s-invariant local systems of signature $+1$.

Let $(\tilde{\rho}, h)$ be σ-invariant, then $(\tilde{\rho}, Jh)$ is σ_s-invariant. Suppose $\sigma(h)hg = -1$, we calculate

$$\begin{aligned} &\sigma_s(Jh)Jhg \\ =&J\sigma(Jh)J^{-1}Jhg \\ =&J\sigma(J)\sigma(h)hg \\ =&1 \end{aligned} \tag{III.5.1.5.1}$$

That is, $(\tilde{\rho}, Jh)$ has signature $+1$ with respect to σ_s.

III.5.2 Explicit Form of $G \rtimes \mathbb{Z}/2\mathbb{Z}$-Character Varieties

We are now ready to write down the explicit equations defining $G \rtimes \mathbb{Z}/2\mathbb{Z}$-character Varieties.

III.5.2.1 Unramified Covering of Compact Surfaces Suppose that $p : \tilde{X} \to X$ is an unramified double covering of compact Riemann surfaces. Let g and h be the genus of $\tilde{X}$ and of X respectively. We have $g = 2h - 1$. Choose generators α_i, β_i, $1 \le i \le h$, of $\pi_1(X)$ satisfying $\prod_i^h(\alpha_i, \beta_i) = 1$, where (α, β) is the commutator. As an index 2 subgroup, $\pi_1(\tilde{X})$ is generated by $\tilde{\alpha}_j$, $\tilde{\beta}_j$, where

$$\begin{aligned} &\tilde{\alpha}_1 = \alpha_1, \tilde{\beta}_1 = \beta_1^2, \\ &\tilde{\alpha}_j = \beta_1\alpha_j\beta_j^{-1},\ \tilde{\beta}_j = \beta_1\beta_j\beta_1^{-1},\ 2 \le j \le h \\ &\tilde{\alpha}_j = \alpha_{j+1-h},\ \tilde{\beta}_j = \beta_{j+1-h},\ h+1 \le j \le g. \end{aligned} \tag{III.5.2.1.1}$$

Then $\prod_j^g(\tilde{\alpha}_j, \tilde{\beta}_j) = 1$. With these choices, the character variety is

$$\begin{aligned} &\mathrm{Rep}_{\mathbb{Z}/2\mathbb{Z}}(X, G) \cong \\ &\qquad \{(A_1, B_1, \ldots, A_h, B_h) \in (G \times G.s_1) \times (G \times G)^{h-1} | (A_1, B_1) \cdots (A_h, B_h) = 1\}. \end{aligned}$$

where s_1 is the image of $1 \in \mathbb{Z}/2\mathbb{Z}$ in $\bar{G}$. Note that $(a, bs_1) = ab\psi_1(a^{-1})b^{-1}$, for any $a, b \in G$. We observe that

- If ψ_1, for $1 \in \mathbb{Z}/2\mathbb{Z}$, is the trivial automorphism, we obtain the usual G-character variety of $\pi_1(X)$.

In the case $G = \mathrm{GL}_n(\mathbb{C})$,

- If ψ_1, for $1 \in \mathbb{Z}/2\mathbb{Z}$, is the exterior automorphism of orthogonal type σ_o, then $G.s_1 \subset {}^o\bar{G}$. This is the moduli space of σ_o-invariant local systems of signature +1.
- If n is even and ψ_1, for $1 \in \mathbb{Z}/2\mathbb{Z}$, is the exterior automorphism of symplectic type σ_s, then $G.s_1 \subset {}^s\bar{G}$. This is the moduli space of σ_o-invariant local system of signature -1.

The groups ${}^s\bar{G}$ and ${}^o\bar{G}$ are defined in §II.5.2.1.

III.5.2.2 Unramified Covering of Noncompact Surfaces Now suppose that $p' : \tilde{X}' \to X'$ be a ramified double covering of compact Riemann surfaces. There must be an even number of ramification points. We follow the notation of §III.4.1.1. The fundamental group $\pi_1(X)$ is generated by α_i, β_i, $1 \le i \le h$, and γ_j, $j \in I$, with the only relation $\prod_i(\alpha_i, \beta_i)\prod_j \gamma_j = 1$. Note that in the particular case of double covering, each α_i, β_i is the image of some $\tilde{\alpha}_i$ or $\tilde{\beta}_i$. Therefore the images of the α_i's and β_i's under ρ must lie in $\mathrm{Im}\,\tilde{\rho} \subset G$. Also note that for each γ_j with $n_j = 1$, γ_j is conjugate to $\tilde{\gamma}_{j,1}$. (Here $1 \in <\sigma> \cong \mathbb{Z}/2\mathbb{Z} \cong \Gamma$, is written multiplicatively. We could write $\tilde{\gamma}_{j,0}$ and $\tilde{\gamma}_{j,1}$ instead of $\tilde{\gamma}_{j,1}$ and $\tilde{\gamma}_{j,\sigma}$ respectively.) For j such that $n_j = 2$, we simply write $\tilde{\gamma}_j := \tilde{\gamma}_{j,1}$.

Let $C = (C_j)_{j\in\mathcal{R}}$ be a tuple of semisimple G-conjugacy classes in $\bar{G}$ such that C_j with $n_j = 1$ is contained in G and C_j with $n_j = 2$ is contained in $\bar{G}\backslash G$ and there are even number of classes C_j with $n_j = 2$. These classes determine a tuple cf conjugacy classes $\tilde{C} := (\tilde{C}_{j,1}, \tilde{C}_{j,\sigma})_{n_j=1}(\tilde{C}_j)_{n_j=2}$ in

G, where $\tilde{C}_{j,1} = C_j$, $\tilde{C}_{j,\sigma} = \psi_1(C_j)$ and $\tilde{C}_j = C_j^2$. Denote by $\mathrm{Rep}_{\mathbb{Z}/2\mathbb{Z},C}(X)$ the closed subvariety of $\bar{G}$-representations under which the image of each γ_j lies in C_j. Then we have

$$\mathrm{Rep}_{\mathbb{Z}/2\mathbb{Z},C}(X) \cong \{(A_i, B_i)_i \times (X_j)_j \in G^{2h} \times \prod_j C_j | \prod_{i=1}^{h}(A_i, B_i) \prod_{j\in\mathcal{R}} X_j = 1\},$$

Under the isomorphism given in the proof of Theorem III.1.2.1, it is isomorphic to a closed subvariety of $\mathrm{Rep}^{\Gamma,\tilde{C}}(\tilde{X}, G)$, with $\Gamma = \mathbb{Z}/2\mathbb{Z}$, where $\mathrm{Rep}^{\Gamma,\tilde{C}}(\tilde{X}, G)$ consists of those representation $\tilde{\rho} : \pi_1(\tilde{X}) \to G$ such that $\tilde{\rho}(\tilde{\gamma}_{j,\tau}) \in \tilde{C}_{j,\tau}$ for $\tau = 1$ or σ and $\tilde{\rho}(\tilde{\gamma}_j) \in \tilde{C}_j$. Since for a given conjugacy class $\tilde{C} \subset G$, the conjugacy class $C \subset G.\sigma$ such that $C^2 = \tilde{C}$ may not be unique, the closed subvariety $\mathrm{Rep}_{\mathbb{Z}/2\mathbb{Z},C}(X)$ in general is not equal to $\mathrm{Rep}^{\Gamma,\tilde{C}}(\tilde{X}, G)$.

III.5.2.3 Ramified Covering of Compact Surfaces We apply the previous discussions to ramified coverings. Let $J \subset I$ be a subset indexing some ramified points. Write $\tilde{X}_J = \tilde{X} \cup_{j\in J} \{\tilde{x}_j\}$ and $X_J = X \cup_{j\in J} \{x_j\}$. If we put $\tilde{C}_j = \{1\}$, for any $j \in J$, then $\mathrm{Rep}^{\Gamma,\tilde{C}}(\tilde{X}, G)$ can be regarded as the variety of (Γ, ψ)-invariant representations with respect to the ramified covering $\tilde{X}_J \to X_J$, with monodromy classes at the punctures $\{x\}_j$, $j \notin J$, given by $\tilde{C}_j$, $j \in I \setminus J$. We are interested in the case where $J = I$ and $\mathcal{R}$ are exactly the ramification points. Now $\tilde{C}_j = \{1\}$ for all j. In the case of $G = \mathrm{GL}_n(\mathbb{C})$, we observe that

(a) If ψ_1, $1 \in \mathbb{Z}/2\mathbb{Z}$, is the trivial automorphism, each C_j is identified with a conjugacy class of involutions in G. These classes are parametrised by partitions of n of length at most 2, say $\boldsymbol{\mu} = \{\mu^{(j)}\}_{j\in J}$ with $\mu^{(j)} = (n_1^{(j)}, n_2^{(j)})$, $n_1^{(j)} + n_2^{(j)} = n$. Then we write $\mathrm{Rep}_{\Gamma,C}(X, G)$ as $\mathrm{Rep}_{\Gamma,\boldsymbol{\mu}}(X, G)$. The variety $\mathrm{Rep}^{\Gamma,\tilde{C}}(\tilde{X}, G)$ is a finite union of the $\mathrm{Rep}_{\Gamma,\boldsymbol{\mu}}(X, G)$'s for $\boldsymbol{\mu}$ running over the partitions above. Following Remark III.4.1.4, we say that a (Γ, ψ)-invariant representation $\tilde{\rho}$ on the ramified overing $\tilde{X}' \to X'$ has local type $\boldsymbol{\mu}$ over $\mathcal{R}$ if the corresponding $\bar{G}$-representation ρ belongs to $\mathrm{Rep}_{\Gamma,\boldsymbol{\mu}}(X, G)$.

(b) If ψ_1, $1 \in \mathbb{Z}/2\mathbb{Z}$, is the an exterior automorphism σ of orthogonal type or symplectic type, then by (II.5.3.3), the map $C \mapsto C^2$ gives an injection from the set of semisimple G-conjugacy classes contained in $G.s_1$ into the set of σ-stable semisimple conjugacy classes in G, so in particular, C_j is the conjugacy class of s_1 for all j. In this case, we have

$$\mathrm{Rep}^{\Gamma,\tilde{C}}(\tilde{X}, G) \cong \mathrm{Rep}_{\Gamma,C}(X, G).$$

Remark III.5.2.1. For general reductive group G, the conjugacy class of s_1 is isomorphic to the symmetric space G/G^{ψ_1}.

Chapter IV

The Character Table of $\mathrm{GL}_n(q) \rtimes <\sigma>$

In this chapter k denotes the algebraic closure of $\mathbb{F}_q$. We assume that $q > n$, which is assumed in the theorem of Waldspurger, and which also ensures that the Deligne-Lusztig induction does not depend on the parabolic subgroup containing a given Levi subgroup. In §IV.1 and §IV.2, G will denote $\mathrm{GL}_n(k)$. In §IV.3, G will denote a connected reductive group over k and in §IV.4, G will denote a not necessarily connected reductive group over k.

IV.1 Parametrisation of Characters

IV.1.1 F-Stable Levi Subgroups

Recall the parametrisation of the F-stable Levi subgroups of $G = \mathrm{GL}_n(k)$.

IV.1.1.1 Notations Denote by $T \subset G$ the maximal torus consisting of the diagonal matrices and denote by $B \subset G$ the Borel subgroup consisting of the upper triangular matrices. The Frobenius F of G sends each entry of an matrix to its q-th power. Denote by Φ the root system defined by T and $\Delta \subset \Phi$ the set of simple roots determined by B. Denote by $W := W_G(T) = N_G(T)/T$ the Weyl group of G. The Frobenius acts trivially on W.

Given a subset $I \subset \Delta$, we denote by $W_I \subset W$ the parabolic subgroup generated by the simple reflections $\{s_\alpha | \alpha \in I\}$. Denote by P_I the parabolic subgroup defined by W_I et B, and L_I the unique Levi factor of P_I containing T. Denote by $\Phi_I \subset \Phi$ the root subsystem assocated to L_I and its Weyl group is W_I. A Levi subgroup of the form L_I is called a standard Levi subgroup. Every Levi subgroup is conjugated to a standard Levi subgroup

Proposition IV.1.1.1. *The set of the G^F-conjugacy classes of the F-stable Levi subgroups of $G = \mathrm{GL}_n(k)$ is in bijection with the set of the unordered sequence of pairs of positive integers $(r_1, d_1) \cdots (r_s, d_s)$, satisfying $\sum_i r_i d_i = n$.*

Proof. The G-conjugacy classes of the Levi subgroups are in bijection with the equivalence classes of the subsets $I \subset \Delta$. Two subsets I and I' are equivalent if there is an element $w \in W$ such that $I' = wI$. It suffices for us to fix $I \subset \Delta$ and only consider the G^F-conjugacy classes of

L_I. The G^F-conjugacy classes of the G-conjugates of L_I are in bijection with the F-conjugacy classes of $N_G(L_I)/L_I$. Note that $N_G(L_I)/L_I \cong N_{W_G}(W_I)$ and F acts trivially on W_G and so on $N_G(L_I)/L_I$.

Let Γ_I be a finite set parametrising the positive integers n_i, $i \in \Gamma_I$ such that $L_I \cong \prod_{i\in\Gamma_I} \mathrm{GL}_{n_i}$. For any $r \in \mathbb{Z}_{>0}$, put $\Gamma_{I,r} := \{i \in \Gamma_I \mid n_i = r\}$ and put $N_r = |\Gamma_{I,r}|$. Then the equivalence classes of I is specified by the sequence $(r, N_r)_{r>0}$. We are going to show that the conjugacy classes of $N_G(L_I)/L_I$ are determined by the partitions of N_r. The desired sequence $(r_1, d_1)\cdots(r_s, d_s)$ will then be defined in such a way that $(d_i)_{i\in\{1\le i\le s|r_i=r\}}$ forms the partition of N_r.

Put $S := Z_{L_I}$. It is a subtorus of T. Denote by $\Phi(G, S)$ the subset of the nontrivial weights of the action of S on $\mathfrak{g} := \mathrm{Lie}\, G$. It is in bijection with $\{(i, j) \in \Gamma_I^2 \mid i \neq j\}$ and so we denote by $\alpha_{i,j}$ the weight corresponding to (i, j). The weight subspace $\mathfrak{g}_{\alpha_{i,j}}$ is of dimension $n_i n_j$. The subset $\Phi_I = \{\alpha_{i,j} \in \Phi(G, S) \mid n_i = n_j\}$ is obviously a root system that is decomposed into some irreducible factors $\Phi_{I,r} = \{\alpha_{i,j} \in \Phi(G, S) \mid n_i = n_j = r\}$ with $r \in \mathbb{Z}_{>0}$, each one of type A_{N_r-1}. The adjoint action of $w \in N_G(L_I)/L_I$ sends $\mathfrak{g}_{\alpha_{i,j}}$ onto $\mathfrak{g}_{w.\alpha_{i,j}}$. In fact, the action of $N_G(L_I)/L_I$ can be lifted to an action of $N_G(T)/T$, thus we see that $\mathfrak{g}_{i,j}$ is conjugate to $\mathfrak{g}_{i',j'}$ only if $\{n_i, n_j\} = \{n_{i'}, n_{j'}\}$ as sets. Consequently, the action of $N_G(L_I)/L_I$ preserves each $\Phi_{I,r}$. We have then a homomorphism $N_G(L_I)/L_I \to \prod_r W(\Phi_{I,r}) \cong \prod_r \mathfrak{S}(\Gamma_{I,r})$.

For any permutation belonging to $\prod_r \mathfrak{S}(\Gamma_{I,r})$, there obviously exists a permutation matrix normalising L_I that induces it, so this homomorphism is surjective. If $w \in N_G(T)$ representing an element of $N_G(L_I)/L_I$ acts trivially on Φ_I, then its action on S is trivial, since S is contained in T and w acts on T by permuting the factors. So $w \in L_I$, as $L_I = C_G(S)$, and the homomorphism is ths injective. We then have an isomorphism $N_G(L_I)/L_I \xrightarrow{\sim} \prod_r \mathfrak{S}(\Gamma_{I,r})$.

Let $wL_I \in N_G(L_I)/L_I$ be a class representing a G^F-class of F-stable Levi subgroup. For each $r \in \mathbb{Z}_{>0}$, denote by $\Lambda_{I,r}(w)$ the set of the orbits in $\Gamma_{I,r}$ under the action of w and write $\Lambda_I(w) = \cup_r \Lambda_{I,r}(w)$. For each $i \in \Lambda_I(w)$, put $r_i = r$ if $i \in \Lambda_{I,r}(w)$ and define d_i to be the cardinality of i. The d_i's, for $i \in \Lambda_{I,r}(w)$, form a partition of $|\Gamma_{I,r}|$. The integer s in the statement of the proposition is equal to $|\Lambda_I(w)|$. □

If the class of wW_I corresponds to $(n_1, d_1)\cdots(n_s, d_s)$ and L is an F-stable Levi subgroup corresponding to wW_I, then (L, F) is isomorphic to a standard Levi subgroup

$$L_I \cong \prod_i \mathrm{GL}_{n_i}(k)^{d_i}, \tag{IV.1.1.1.1}$$

equipped with $F_{\dot{w}}$, acting on each factor $\mathrm{GL}_{n_i}(k)^{d_i}$ in the following manner,

$$\begin{aligned} \mathrm{GL}_{n_i}(k)^{d_i} &\longrightarrow \mathrm{GL}_{n_i}(k)^{d_i} \\ (g_1, g_2, \ldots, g_{d_i}) &\longmapsto (F_0(g_{d_i}), F_0(g_1), \ldots, F_0(g_{d_i-1})), \end{aligned} \tag{IV.1.1.1.2}$$

where F_0 is the Frobenius of $\mathrm{GL}_{n_i}(k)$ that sends each entry to its q-th power.

IV.1.2 σ-Stable Characters

Fix T and B as above. We could equally work with any F-stable and σ-stable maximal torus contained in a σ-stable Borel subgroup.

IV.1.2.1 Quadratic-Unipotent Characters Given a 2-partition (μ_1, μ_2), we define an irreducible character $\chi_{(\mu_1,\mu_2)} \in \mathrm{Irr}(\mathrm{GL}_n(q))$ as follows. Put $m_1 = |\mu_1|$ and $m_2 = |\mu_2|$ and so $m_1 + m_2 = n$. Let $M \subset \mathrm{GL}_n(k)$ be a standard Levi subgroup isomorphic to $\mathrm{GL}_{m_1}(k) \times \mathrm{GL}_{m_2}(k)$, i.e.

$$M = \left(\begin{array}{c|c} \mathrm{GL}_{m_1} & 0 \\ \hline 0 & \mathrm{GL}_{m_2} \end{array}\right). \tag{IV.1.2.1.1}$$

Denote by $W_M := W_M(T)$ the Weyl group of M. It is isomorphic to $\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$. The Frobenius F acts trivially on W_M. With respect to the isomorphism $M^F \cong \mathrm{GL}_{m_1}(q) \times \mathrm{GL}_{m_2}(q)$, we define a linear character $\theta \in \mathrm{Irr}(M^F)$ to be $(\mathrm{Id} \circ \det, \eta \circ \det)$, where Id is the trivial character of $\mathbb{F}_q^*$ and η is the order 2 irreducible character of $\mathbb{F}_q^*$. It is regular in the sense of [LS, §3.1]. We define $\varphi \in \mathrm{Irr}(W_M)$ by a 2-partition (μ_1, μ_2) in such a way that the factor corresponding to $\mathfrak{S}_{m_1}$ is defined by the partition μ_1, and the other by μ_2. According to Theorem 5, the triple (M, θ, φ) gives an irreducible character of $\mathrm{GL}_n(q)$, which we denote by $\chi_{(\mu_1,\mu_2)}$. The irreducible characters of $\mathrm{GL}_n(q)$ thus obtained are called quadratic-unipotents.

Lemma IV.1.2.1. *For any 2-partition (μ_1, μ_2), the character $\chi_{(\mu_1,\mu_2)}$ is σ-stable.*

Proof. The operation $\chi \mapsto \chi \circ \sigma^{-1}$ defines an involution of $\mathrm{Irr}(\mathrm{GL}_n(q))$. Denote by ${}^\sigma\chi_{(\mu_1,\mu_2)}$ the image of $\chi_{(\mu_1,\mu_2)}$ under this involution. If χ is of the form $R_\varphi^G\theta$ for a triple (M, θ, φ), then ${}^\sigma\chi_{(\mu_1,\mu_2)}$ is of the form $R^G_{\sigma_*\varphi}\sigma_*\theta$ for the triple $(\sigma(M), \sigma_*\theta, \sigma_*\varphi)$ according to Lemma II.4.1.2. Explicitly,

$$\sigma(M) \cong \left(\begin{array}{c|c} \mathrm{GL}_{m_2} & 0 \\ \hline 0 & \mathrm{GL}_{m_1} \end{array}\right),$$

$\sigma_*\theta$ associates to the factor GL_{m_1} the trivial character of $\mathbb{F}_q^*$ and η to the other factor, and $\sigma_*\varphi$ associates to the factor GL_{m_1} the character of $\mathfrak{S}_{m_1}$ corresponding to μ_1 and to the other factor that corresponding to μ_2. Then the conjugation by $\mathcal{J}_n$ sends $(\sigma(M), \sigma_*\theta, \sigma_*\varphi)$ to (M, θ, φ), and so ${}^\sigma\chi_{(\mu_1,\mu_2)} = \chi_{(\mu_1,\mu_2)}$ according to Theorem 5 □

IV.1.2.2 Now we construct some more general σ-stable irreducible characters. If $I \subset \Delta$ is a σ-stable subset, then it defines a σ-stable standard Levi subgroup. Every σ-stable standard

Levi subgroup is of this form. We are going to use the notations of §IV.1.1.1 and of Proposition IV.1.1.1. Denote by 0 the unique element of Γ_I fixeed by σ. For any $i \in \Gamma_I$, denote by i^* its image under σ. The standard Levi subgroup L_I is decomposed as $\prod_{i\in\Gamma_I} L_i$. For any $i \in \Gamma_I$, write $n_i = r$ and so $L_i \cong \mathrm{GL}_r(k)$ if $i \in \Gamma_{I,r}$. Schematically, L_I is equal to

(IV.1.2.2.1)
$$\begin{pmatrix} L_{i_1} & & & & & & \\ & \ddots & & & & & \\ & & L_{i_s} & & & & \\ & & & \mathrm{GL}_{n_0} & & & \\ & & & & L_{i_s^*} & & \\ & & & & & \ddots & \\ & & & & & & L_{i_1^*} \end{pmatrix}.$$

IV.1.2.3 For any $r \in \mathbb{Z}_{>0}$, we write $N_r = |\Gamma_{I,r}|$. And $W_G(L_I) := N_G(L_I)/L_I$ is isomorphic to $\prod_r \mathfrak{S}_{N_r}$. Write $N_r' = N_r/2$ if $n_0 \neq r$ and write $N_r' = (N_r-1)/2$ if $n_0 = r$. Then, $W_G(L_I)^\sigma \cong \prod_r \mathfrak{W}^C_{N_r'}$. Regarded as a block permutation matrix, an element of $\mathfrak{W}^C_{N_r'}$ typically acts on $\prod_{\{i|n_i=r\}} L_i$ in the following two ways.

(IV.1.2.3.1)
$$\begin{pmatrix} L_{i_1} & & & & & \\ & \ddots & & & & \\ & & L_{i_t} & & & \\ & & & L_{i_t^*} & & \\ & & & & \ddots & \\ & & & & & L_{i_1^*} \end{pmatrix}, \quad \begin{pmatrix} L_{i_1} & & & & & \\ & \ddots & & & & \\ & & L_{i_t} & & & \\ & & & L_{i_t^*} & & \\ & & & & \ddots & \\ & & & & & L_{i_1^*} \end{pmatrix},$$

corresponding to a cycle of positive sign and a cycle of negative sign respectively.

If w is an element of $W_G(L_I)^\sigma$, then a block permutation matrix that represents it, denoted by $\dot{w}$, obviously can be chosen to be σ-stable, and so there is some $g \in (G^\sigma)^\circ$ such that $g^{-1}F(g) = \dot{w}$. Put $M_{I,w} := gL_Ig^{-1}$. It is an F-stable and σ-stable Levi factor of a σ-stable parabolic subgroup, and there is an isomorphism $\mathrm{ad}\, g : L_I^{F_w} \cong M_{I,w}^F$. A character of $M_{I,w}^F$ is σ-stable if and only if it is identified with a σ-stable character of $L_I^{F_w}$ by $\mathrm{ad}\, g$ as g is σ-stable. We are going to construct some σ-stable irreducible characters of $L_I^{F_w}$.

Write $\Lambda = \Lambda_I(w)$, following the proof the of Proposition IV.1.1.1. The action of σ on Γ_I induces an action on Λ as w commutes with σ, which justifies the notation i^* for $i \in \Lambda$. If i, $j \in \Gamma_I$ belong to the same orbit of w, then $n_i = n_j$, which justfies the notation n_i for $i \in \Lambda$. Also

note that $n_i = n_{i^*}$. For any $i \in \Lambda$, denote by d_i the cardinality of the orbit i. Write

$$\Lambda_1 = \{i \in \Lambda | i^* \neq i\}/\sim \tag{IV.1.2.3.2}$$

where the equivalence identifies i to i^*. We will say that i belongs to Λ_1 if $i \neq i^*$ and the equivalence class of i belongs to Λ_1. Write

$$\Lambda_2 = \{i \in \Lambda | i = i^*\} \setminus \{0\}. \tag{IV.1.2.3.3}$$

We can choose an isomorphism

$$L_I \cong \prod_{i \in \Lambda_1} (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}})^{d_i} \times \prod_{i \in \Lambda_2} (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}})^{d_i/2} \times \mathrm{GL}_{n_0} \tag{IV.1.2.3.4}$$

such that F_w and σ act on L_I in the following manner.

The action of F_w is given by:

$$\begin{aligned} i = 0: \quad & \mathrm{GL}_{n_0} \longrightarrow \mathrm{GL}_{n_0} && \text{(IV.1.2.3.5)} \\ & A \longmapsto F_0(A); \\ i \in \Lambda_1: \quad & (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}})^{d_i} \longrightarrow (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}})^{d_i} && \text{(IV.1.2.3.6)} \\ & (A_1, B_1, A_2, B_2 \ldots, A_{d_i}, B_{d_i}) \longmapsto \\ & (F_0(A_{d_i}), F_0(B_{d_i}), F_0(A_1), F_0(B_1) \ldots F_0(A_{d_i-1}), F_0(B_{d_i-1})); \\ i \in \Lambda_2: \quad & (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}})^{d_i/2} \longrightarrow (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}})^{d_i/2} && \text{(IV.1.2.3.7)} \\ & (A_1, B_1, A_2, B_2 \ldots, A_{d_i/2}, B_{d_i/2}) \longmapsto \\ & (F_0(B_{d_i/2}), F_0(A_{d_i/2}), F_0(A_1), F_0(B_1) \ldots F_0(A_{d_i/2-1}), F_0(B_{d_i/2-1})). \end{aligned}$$

where F_0 is the Frobenius of $\mathrm{GL}_r(k)$, for an arbitrary r, that sends each entry to its q-th power

The action of σ is given by:

$$\begin{aligned} i = 0: \quad & \mathrm{GL}_{n_0} \longrightarrow \mathrm{GL}_{n_0} && \text{(IV.1.2.3.8)} \\ & A \longmapsto \sigma_0(A); \\ i \neq 0: \quad & \mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}} \longrightarrow \mathrm{GL}_{n_i} \times \mathrm{GL}_{n_{i^*}} && \text{(IV.1.2.3.9)} \\ & (A, B) \longmapsto (\sigma_i(B), \sigma_i(A)) \end{aligned}$$

where σ_0 is the standard automorphism of GL_{n_0} (§II.5.1.1) and σ_i is the automorphism of GL_{n_i} that sends g to $\mathcal{J}_{n_i}\, {}^t g^{-1} \mathcal{J}_{n_i}^{-1}$ no matter what the parity of n_i is (§II.5.1.1). We have

$$L_I^{F_w} \cong \prod_{i \in \Lambda_1} (\mathrm{GL}_{n_i}(q^{d_i}) \times \mathrm{GL}_{n_{i^*}}(q^{d_i})) \times \prod_{i \in \Lambda_2} \mathrm{GL}_{n_i}(q^{d_i}) \times \mathrm{GL}_{n_0}(q) \tag{IV.1.2.3.10}$$

and σ acts on it in the following manner,

$$\begin{aligned} i = 0: \quad & \mathrm{GL}_{n_0}(q) \longrightarrow \mathrm{GL}_{n_0}(q) && \text{(IV.1.2.3.11)}\\ & A \longmapsto \sigma_0(A); \\ i \in \Lambda_1: \quad & \mathrm{GL}_{n_i}(q^{d_i}) \times \mathrm{GL}_{n_{i^*}}(q^{d_i}) \longrightarrow \mathrm{GL}_{n_i}(q^{d_i}) \times \mathrm{GL}_{n_{i^*}}(q^{d_i}) && \text{(IV.1.2.3.12)}\\ & (A, B) \longmapsto (\sigma_i(B), \sigma_i(A)); \\ i \in \Lambda_2: \quad & \mathrm{GL}_{n_i}(q^{d_i}) \longrightarrow \mathrm{GL}_{n_i}(q^{d_i}) && \text{(IV.1.2.3.13)}\\ & A \longmapsto \sigma_i F_0^{d_i/2}(A). \end{aligned}$$

Denote by L_1 the product of the factors of L_I except L_0. If no confusion arises, we may also denote by F_w and σ their restrictions on L_1. With respect to the decomposition of $L_I^{F_w}$ as above, a linear character θ_1 of $L_1^{F_w}$ can be written as

$$\text{(IV.1.2.3.14)} \qquad \prod_{i\in\Lambda_1}(\alpha_i, \alpha_{i^*}) \prod_{i\in\Lambda_2} \alpha_i \in \prod_{i\in\Lambda_1}(\mathrm{Irr}(\mathbb{F}^*_{q^{d_i}}) \times \mathrm{Irr}(\mathbb{F}^*_{q^{d_i}})) \times \prod_{i\in\Lambda_2} \mathrm{Irr}(\mathbb{F}^*_{q^{d_i}}).$$

The set $\mathrm{Irr}(W_{L_1})^{F_w}$ is in bijection with the irreducible characters of

$$\text{(IV.1.2.3.15)} \qquad \prod_{i\in\Lambda_1}(\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_{i^*}}) \times \prod_{i\in\Lambda_2} \mathfrak{S}_{n_i}.$$

Such a character can be written as $\varphi_1 = \prod_{i\in\Lambda_1}(\varphi_i, \varphi_{i^*}) \prod_{i\in\Lambda_2} \varphi_i$.

IV.1.2.4 Suppose that the factors of θ_1 and φ_1 satisfy

$$\begin{aligned} &\text{for any } i \in \Lambda \setminus \{0\}, \quad \alpha_i \neq \mathrm{Id} \text{ or } \eta, && \text{(IV.1.2.4.1)}\\ &\text{for any } i \in \Lambda_1, \quad \alpha_{i^*} = \alpha_i^{-1}, && \text{(IV.1.2.4.2)}\\ &\text{for any } i \in \Lambda_2, \quad \alpha^{q^{d_i/2}} = \alpha_i^{-1}, && \text{(IV.1.2.4.3)}\\ &\text{for any } i \in \Lambda_1, \quad \varphi_{i^*} = \varphi_i. && \text{(IV.1.2.4.4)} \end{aligned}$$

We choose $\tilde{\varphi}_1$, an extension of φ_1 to $W_{L_1}.<F_w>$, in such a way that

$$\text{(IV.1.2.4.5)} \qquad \chi_1 = R^G_{\varphi_1}\theta_1 = |W_{L_1}|^{-1} \sum_{v\in W_{L_1}} \tilde{\varphi}_1(vF_w) R^{L_1}_{T_v}\theta_1,$$

is an irreducible character of $L_1^{F_w}$.

Proposition IV.1.2.2. *Let χ_0 be a quadratic-unipotent character of $L_0^{F_0}$. Then, $\chi_1 \boxtimes \chi_0$ is a σ-stable irreducible character of $L_I^{F_w}$. Identified with a character of $M^F_{I,w}$, its induction $R^G_{M_{I,w}}(\chi_1 \boxtimes \chi_0)$ is a σ-stable irreducible character of* $\mathrm{GL}_n(q)$.

Proof. By the hypothesis on θ_1 and φ_1, χ_1 is σ-stable, and so $\chi_1 \boxtimes \chi_0$ is σ-stable. It follows from the definition of $R^G_{M_{I,w}}$ that if $\chi_1 \boxtimes \chi_0$ is σ-stable, then $R^G_{M_{I,w}}(\chi_1 \boxtimes \chi_0)$ is σ-stable. □

IV.1.2.5 The rest of this section is to prove the following proposition.

Proposition IV.1.2.3. *Every σ-stable irreducible character of* $\mathrm{GL}_n(q)$ *is of the form given by Proposition IV.1.2.2.*

IV.1.2.6 Suppose that $\chi \in \mathrm{Irr}(\mathrm{GL}_n(q))$ is an irreducible character defined by the triple (M, φ, θ) following Theorem 5. We will use the notations in the proof of Proposition IV.1.1.1. There exists a standard Levi subgroup L_J for some subset $J \subset \Delta$, $w \in N_{W_G}(W_J)$, $\dot{w} \in G$ a representative of w, and $g \in G$ such that $M = gL_Jg^{-1}$ and $g^{-1}F(g) = \dot{w}$. We can further assume that $\dot{w}$ is a block permutation matrix that permutes the factors of L_J. We have an isomorphism $\mathrm{ad}\, g : L_J^{F_w} \cong M^F$.

If M is an F-stable Levi subgroup corresponding to $(n_1, d_1) \cdots (n_s, d_s)$ by Proposition IV.1.1.1, then we have $M \cong \prod_{i\in\Lambda} \mathrm{GL}_{n_i}(k)^{d_i}$, and $M^F \cong \prod_{i\in\Lambda} \mathrm{GL}_{n_i}(q^{d_i})$, where $\Lambda := \Lambda_J(w)$. With respect to this decomposition, we write $\theta = \prod_i \alpha_i$ with $\alpha_i \in \mathrm{Hom}(\mathbb{F}^*_{q^{d_i}}, \bar{\mathbb{Q}}^*_\ell)$, where we have abbreviated $\alpha_i \circ \det$ as α_i. An element $m \in M^F$ can be written as $\prod_i l_i$ with $F_0^{d_i}(l_i) = l_i$.

Proposition IV.1.2.4. *In order for χ to be σ-stable, it is necessary that for each $i \in \Lambda$, there exist $i^* \in \Lambda$ and $0 \le c < d_{i^*}$ such that $n_i = n_{i^*}$ and $\alpha_i^{-1} = \alpha_{i^*}^{q^c}$.*

Proof. Assume that χ is σ-stable. Theorem 5 implies that there exists $x \in G^F$ such that

$$\mathrm{ad}\, x(M) = \sigma(M), \tag{IV.1.2.6.1}$$

$$(\mathrm{ad}\, x)^* \sigma_* \theta = \theta. \tag{IV.1.2.6.2}$$

Consider the maps

$$L_J \xrightarrow{\mathrm{ad}\, g} M \underset{\sigma}{\overset{\mathrm{ad}\, x}{\rightrightarrows}} \sigma(M). \tag{IV.1.2.6.3}$$

For any $l \in L_J$, the element

$$(\mathrm{ad}\, g)^{-1} \circ \sigma^{-1} \circ (\mathrm{ad}\, x) \circ (\mathrm{ad}\, g)(l), \tag{IV.1.2.6.4}$$

belongs to L_J and is equal to $n^t l^{-1} n^{-1}$ with $n := g^{-1}\mathcal{J}({}^t x^{-1})({}^t g^{-1})$. If $l \in L_J^{F_w}$, then $\bar{l} := n^t l^{-1} n^{-1} \in L_J^{F_w}$. Regarded as the elements of L_J, l and $\bar{l}$ can be written as

$$l = \prod_{i\in\Lambda} (l_i, F_0(l_i), \ldots, F_0^{d_i-1}(l_i)), \quad \bar{l} = \prod_{i\in\Lambda} (\bar{l}_i, F_0(\bar{l}_i), \ldots, F_0^{d_i-1}(\bar{l}_i)) \tag{IV.1.2.6.5}$$

with $F_0^{d_i}(l_i) = l_i$ and $F_0^{d_i}(\bar{l}_i) = \bar{l}_i$.

We index the factors of each element $l' \in L_J$ by (i, c), $i \in \Lambda$ and $0 \le c < d_i$, so that $l'_{ic} = F_0^c(l'_i)$ if $l' \in L_J^{F_w}$. Since n normalises L_J, it can be written as $n = vy$ with $y \in L_J$ and v being a block permutation matrix that permutes the factors of L_J. By the definition of $\bar{l}$, for any i, there

exists $i^* \in \Lambda$ and c_{i^*} such that

$$\bar{l}_i = y_{i^* c_{i^*}} F_0^{c_{i^*}} ({}^t l_{i^*}^{-1}) y_{i^* c_{i^*}}^{-1}. \tag{IV.1.2.6.6}$$

Therefore, v induces a permutation $i \mapsto i^*$ of Λ. (In this proof, we do not require that $(i^*)^* = i$.)

We have

$$\begin{aligned}(\mathrm{ad}\, g)^*(\mathrm{ad}\, x)^*\sigma_*\theta(l) &= (\mathrm{ad}\, g)^*\theta(n^t l^{-1} n^{-1}) \\ &= (\mathrm{ad}\, g)^*\theta(\prod_i y_{i^* c_{i^*}} F_0^{c_{i^*}} ({}^t l_{i^*}^{-1}) y_{i^* c_{i^*}}^{-1}).\end{aligned} \tag{IV.1.2.6.7}$$

If we write $(\mathrm{ad}\, g)^*(\theta) = \prod_i \alpha_i$, with $\alpha_i \in \mathrm{Irr}(\mathbb{F}_{q^{d_i}}^*)$, then for all $l \in L_J^{F_w}$,

$$\begin{aligned}(\mathrm{ad}\, g)^*(\mathrm{ad}\, x)^*\sigma_*\theta(l) &= \prod_i \alpha_i(F_0^{c_{i^*}} ({}^t l_{i^*}^{-1})) \\ &= \prod_i \alpha_i^{-q^{c_{i^*}}}(l_{i^*}).\end{aligned} \tag{IV.1.2.6.8}$$

On the other hand, from the equality (IV.1.2.6.2) we deduce that for all $l \in L_J^{F_w}$,

$$(\mathrm{ad}\, g)^*(\mathrm{ad}\, x)^*\sigma_*\theta(l) = \prod_i \alpha_i(l_i). \tag{IV.1.2.6.9}$$

We complete the proof by comparing the two equalities. □

Remark IV.1.2.5. It is necessary that $(i^*)^* = i$. Suppose $(i^*)^* = j \neq i$, then there exists an integer d such that $\alpha_j = \alpha_i^{q^d}$, which contradicts the regularity of θ.

Remark IV.1.2.6. If $i^* = i$, then it is necessary that d_i is an even number, and $\alpha_i^{-1} = \alpha_i^{q^{d_i/2}}$.

Remark IV.1.2.7. There are at most two $i \in \Lambda$ such that $\alpha_i^2 = 1$ in order for θ to be regular. We denote them by $\pm$. It is necessary that $d_+ = d_- = 1$. Also denote by $\pm$ the corresponding two elements of Γ.

IV.1.2.7 The previous proposition allows us to define the sets

$$\Lambda_1 = \{i \in \Lambda | i^* \neq i\}/\sim \tag{IV.1.2.7.1}$$

$$\Lambda_2 = \{i \in \Lambda | i = i^*\} \setminus \{\pm\}, \tag{IV.1.2.7.2}$$

in a way similar to (IV.1.2.3.2) and (IV.1.2.3.3). There exists a permutation matrix x such that $L_{J'} := xL_Jx^{-1}$ is of the form

$$\begin{pmatrix} L_{i_1} & & & & & & \\ & \ddots & & & & & \\ & & L_{i_s} & & & & \\ & & & L_+ & & & \\ & & & & L_- & & \\ & & & & & L_{i'_s} & \\ & & & & & & \ddots \\ & & & & & & & L_{i'_1} \end{pmatrix}.$$

where $L_{i'_j} \cong L_{i_j}$ for any j. The Frobenius concerned is $F_{xwx^{-1}}$, which fixes L_+ and L_- as $d_+ = d_- = 1$. We can further conjugate by a permutation matrix that normalises $L_{J'}$, say y, in such a way that $F_{yxwx^{-1}y^{-1}}$ acts on $L_{J'}$ according to (IV.1.2.3.1), and so the α_i's satisfy the hypothesis §IV.1.2.4. Denote by $\theta_{J'}$ and $\varphi_{J'}$ the characters associated to $L_{J'}$ that come from θ and φ via $\operatorname{ad} g$, $\operatorname{ad} x$ and $\operatorname{ad} y$.

Choose a block permutation matrix $\dot{v}$ that realises the permutation of $\Gamma_{J'}$ defined by $v := yxwx^{-1}y^{-1}$. It can be chosen to be σ-stable. Let $h \in (G^\sigma)^\circ$ be such that $h^{-1}F(h) = \dot{v}$. Then $M_{J',v} = hL_{J'}h^{-1}$ is an F-stable Levi subgroup that is G^F-conjugate to M. In particular, if we identify $\theta_{J'}$ and $\varphi_{J'}$ with the characters associated to $M_{J',v}$ by $\operatorname{ad} h$, then χ is equal to the induction $R^G_{\varphi_{J'}}\theta_{J'}$ for a triple $(M_{J',v}, \varphi_{J'}, \theta_{J'})$.

Define L_I to be the standard Levi subgroup of the form (IV.1.2.2.1) such that $n_0 = n_+ + n_-$ and that L_I coincides with $L_{J'}$ away from GL_{n_0}. We see that $M_{I,v} = hL_Ih^{-1}$ is a σ-stable and F-stable Levi factor of a σ-stable parabolic subgroup. Moreover, it contains $M_{J',v}$ and $\sigma(M_{J',v})$.

Note that ${}^\sigma\chi$ is defined by the triple $(\sigma(M_{J',v}), \sigma_*\varphi_{J'}, \sigma_*\theta_{J'})$. Since $\operatorname{ad} h$ commutes with σ, we can work with L_I and find a permutation matrix that conjugates $L_{J'}$ to $\sigma(L_{J'})$ and conjugates θ to $\sigma_*\theta$. Indeed, it is effectively a permutation matrix in GL_{n_0} as $\sigma_*\theta_{J'}$ is already equal to $\theta_{J'}$ away from GL_{n_o} by definition. By Theorem 5, in order for χ to be σ-stable, it is necessary that $\sigma_*\varphi_{J'} = \varphi_{J'}$ away from GL_{n_0}. And it suffices. This completes the proof of Proposition IV.1.2.3.

IV.1.2.8 The type of a σ-stable irreducible character consists of some non negative integers $n_\pm$, and some positive integers n_i, d_i, n'_j, and d'_j parametrised by the finite sets Λ_1 and Λ_2, denoted by

(IV.1.2.8.1) $$\mathfrak{t} = n_+n_-(n_i, d_i)_{i\in\Lambda_1}(n'_j, d'_j)_{j\in\Lambda_2},$$

satisfying

(IV.1.2.8.2) $$n = n_+ + n_- + \sum_i 2n_i d_i + \sum_j 2n'_j d'_j.$$

If $\tilde{\mathfrak{t}} = \tilde{n}_+ \tilde{n}_- (\tilde{n}_i, \tilde{d}_i)_{i \in \tilde{\Lambda}_1} (\tilde{n}'_j, \tilde{d}'_j)_{j \in \tilde{\Lambda}_2}$ is another sequence of integers, we regard it as the same as $\mathfrak{t}$ if and only if there exist some bijections $\Lambda_1 \cong \tilde{\Lambda}_1$ and $\Lambda_2 \cong \tilde{\Lambda}_2$ such that the integers are matched. (In particular, $n_+ = \tilde{n}_+$ and $n_- = \tilde{n}_-$.)

We denote by $\mathfrak{T}_\chi$ the set of the types of the σ-stable irreducible characters of $\mathrm{GL}_n(q)$.

Given $\mathfrak{t} \in \mathfrak{T}_\chi$, denote by $\bar{\mathfrak{T}}_\chi(\mathfrak{t})$ the set of the data

(IV.1.2.8.3) $$\bar{\mathfrak{t}} = \lambda_+ \lambda_- (\lambda_i, \hat{\alpha}_i)_{i \in \Lambda_1} (\lambda'_j, \hat{\alpha}'_j)_{j \in \Lambda_2}$$

satisfying

- $\lambda_\pm \in \mathcal{P}_{n_\pm}$, $\lambda_i \in \mathcal{P}_{n_i}$, $\lambda'_j \in \mathcal{P}_{n'_j}$, for some integers n_+, n_-, n_i and n'_j;
- $\hat{\alpha}_i \subset \mathrm{Irr}(\bar{\mathbb{F}}_q^*)$ is an F-orbit of order d_i that is not stable under inversion;
- $\hat{\alpha}'_j \subset \mathrm{Irr}(\bar{\mathbb{F}}_q^*)$ is an F-orbit of order $2d'_j$ that is stable under inversion;
- $\hat{\alpha}_i \neq \hat{\alpha}_{i'}$ if $i \neq i'$ and $\hat{\alpha}_i$ is different from $\pm 1 \in \mathrm{Irr}(\bar{\mathbb{F}}_q^*)$, and similarly for the $\hat{\alpha}'_j$'s.

Denote

$$\bar{\mathfrak{T}}_\chi = \cup_{\mathfrak{t} \in \mathfrak{T}_\chi} \bar{\mathfrak{T}}_\chi(\mathfrak{t}).$$

By Proposition IV.1.2.2 and Proposition IV.1.2.3, the σ-stable irreducible characters are in bijection with $\bar{\mathfrak{T}}_\chi$.

Given an element of $\bar{\mathfrak{T}}_\chi$ as above, put $n_0 = n_+ + n_-$, then $n_0 \prod_1 (n_i, d_i) \prod_j (n'_j, d'_j)$ determines a standard Levi subgroup L_I of the form (IV.1.2.2.1) and a permutation of the factors of L_I. We can recover the corresponding character by following §IV.1.2.3 and §IV.1.2.4.

IV.2 Parametrisation of Conjugacy Classes

IV.2.1 F-Stable Quasi-Semi-Simples Classes

A G-conjugacy class contains some G^F-conjugacy classes if and only if it is F-stable. We will give the parametrisation of the F-stable quasi-semi-simple conjugacy classes in $G.\sigma$. Recall that in ${}^0\bar{G}^F$, we denote by σ the element $t_0\sigma'$ (*cf.* Convention II.5.2.2).

IV.2.1.1 We begin with the parametrisation of the quasi-semi-simple G-conjugacy classes. We take for T the maximal torus consisting of the diagonal matrices, then $(T^\sigma)^\circ$ consists of

the matrices

$$\mathrm{diag}(a_1, a_2, \ldots, a_m, a_m^{-1}, \ldots, a_2^{-1}, a_1^{-1}), \text{if } n = 2m, \tag{IV.2.1.1.1}$$

$$\mathrm{diag}(a_1, a_2, \ldots, a_m, 1, a_m^{-1}, \ldots, a_2^{-1}, a_1^{-1}), \text{if } n = 2m+1, \tag{IV.2.1.1.2}$$

with $a_i \in \bar{\mathbb{F}}_q^*$ for all i, and the commutator (T, σ) consists of the matrices

$$\mathrm{diag}(b_1, b_2, \ldots, b_m, b_m, \ldots, b_2, b_1), \text{if } n = 2m, \tag{IV.2.1.1.3}$$

$$\mathrm{diag}(b_1, b_2, \ldots, b_m, b_{m+1}, b_m, \ldots, b_2, b_1), \text{if } n = 2m+1, \tag{IV.2.1.1.4}$$

with $b_i \in \bar{\mathbb{F}}_q^*$ for all i. So the elements of $S := [T, \sigma] \cap (T^\sigma)^\circ$ are the matrices

$$\mathrm{diag}(e_1, e_2, \ldots, e_m, e_m, \ldots, e_2, e_1), \text{if } n = 2m, \tag{IV.2.1.1.5}$$

$$\mathrm{diag}(e_1, e_2, \ldots, e_m, 1, e_m, \ldots, e_2, e_1), \text{if } n = 2m+1, \tag{IV.2.1.1.6}$$

with $e_i = \pm 1$ for all i.

We index the entries of a diagonal matrix by the set $\{1, 2, \ldots, m, -m, \ldots, -2, -1\}$ or the set $\{1, 2, \ldots, m, 0, -m, \ldots, -2, -1\}$ according to the parity of n so that every matrix in $(T^\sigma)^\circ$ satisfies $a_{-i} = a_i^{-1}$ for all i.

Notation IV.2.1.1. We will abbreviate an element of $(T^\sigma)^\circ$ as $[a_1, \ldots, a_m]$ regardless of the parity of n.

We have the following proposition.

Proposition IV.2.1.2. *([DM18, Proposition 1.16]) The quasi-semi-simple classes in $G.\sigma$ are in bijection with the W^σ-orbits in $T/(T, \sigma) = (T^\sigma)^\circ/S$.*

That is, the class of $[a_1, \ldots, a_m]\sigma$ is invariant under the following operations,

- Permutation of the a_i's;
- $a_i \mapsto a_i^{-1}$, for any i;
- $a_i \mapsto -a_i$, for any i,

and $[b_1, \ldots, b_m]\sigma$ belongs to the same class if it only differs from $[a_1, \ldots, a_m]\sigma$ by these operations. For another description of these conjugacy classes, see also [DM15, Example 7.3].

IV.2.1.2 Denote by $\hat{k}$ the quotient of k^* by the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$:

$$(1, 0) : a \mapsto a^{-1}. \quad (0, 1) : a \mapsto -a. \tag{IV.2.1.2.1}$$

For any $a \in k$, denote by $\hat{a}$ the set $\{a, -a, a^{-1}, -a^{-1}\}$.

Remark IV.2.1.3. The conjugacy class of $t\sigma = [a_1, \dots, a_m]\sigma$ is determined by the set $\{\hat{a}_1, \dots, \hat{a}_m\}$. We may regard its elements as the eigenvalues of $t\sigma$.

The action of F on k induces an action on $\hat{k}$, given by

$$\hat{a} \mapsto \hat{a}^q := \{a^q, -a^q, a^{-q}, -a^{-q}\}.$$

Let $\hat{\alpha} \subset \hat{k}$ be an orbit of F and take $\hat{a} \in \hat{\alpha}$. Write $d = |\hat{\alpha}|$. Denote by e and ϵ the signs such that $a^{q^d} = ea^\epsilon$, for any $a \in \hat{a}$. Note that e and ϵ are independent of the choice of $\hat{a} \in \hat{\alpha}$ or the choice of $a \in \hat{a}$. We say that $\hat{\alpha}$ is an orbit of type (d, ϵ, e).

There is some ambiguity with the type thus defined. Let $\hat{\alpha}_1$ be an orbit of type (d_1, ϵ_1, e_1) and let $\hat{\alpha}_2$ be an orbit of type (d_2, ϵ_2, e_2). Obviously if $d_1 > d_2$, then $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are distinct orbits. Suppose $d_1 = d_2 = d$. If x satisfies both of the two equations $x^{q^{d_1}} = e_1 x^{\epsilon_1}$ and $x^{q^{d_2}} = e_2 x^{\epsilon_2}$, then $e_1 e_2 x^{\epsilon_1 - \epsilon_2} = 1$. In order for this equaiton to be solvable, either $e_1 = e_2$, $\epsilon_1 = \epsilon_2$, that is, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ coincide, or $e_1 = e_2$, $\epsilon_1 = -\epsilon_2$, which gives $x^2 = 1$, or $e_1 = -e_2$, $\epsilon_1 = -\epsilon_2$, which gives $x^2 = -1$. The latter ones are the orbits $\{1, -1\}$ and $\{\mathrm{i}, -\mathrm{i}\}$, so $d = 1$, and they are said to be of type (1) and of type (i) respectively. Except these cases, orbits of different types are all distinct. In the following, these two orbits are treated separately and so there will be no confusion among types.

Fix an orbit $\hat{\alpha}$ of type (d, ϵ, e) and an element $\hat{a} \in \hat{\alpha}$. For any sequence of signs $\underline{\epsilon} = (\epsilon_0, \dots, \epsilon_{d-1})$ and $\underline{e} = (e_0, \dots, e_{d-1})$ such that $\prod_k \epsilon_k = \epsilon$ and $\prod_k e_k = e$, a representative of type $(d, \underline{\epsilon}, \underline{e})$ of $\hat{\alpha}$ is a sequence $\alpha := (a_0, a_1, \dots, a_{d-1}) \in (k^*)^d$ satisfying $a_i = e_i a_{i-1}^{q^{\epsilon_i}}$, $i \in \mathbb{Z}/d\mathbb{Z}$.

Remark IV.2.1.4. Fixing the values of ϵ and of e, there are several choices of $\underline{\epsilon}$ and $\underline{e}$. If we put $e_{d-1} = e$, $e_k = 1$ for $0 \le k < d-1$, and similarly for $\underline{\epsilon}$, the representative α is of the form $\{a, a^q, \dots, a^{q^{d-1}}\}$.

IV.2.1.3 Let us define some combinatorial data that parametrise the F-stable quasi-semi-simple conjugacy classes.

We call the type of an F-stable quasi-semi-simple conjugacy class the data consisting of some non negative integers n_+, n_-, with the parity of n_+ being that of n, some positive integers n_i, d_i and some signs e_i and ϵ_i, parametrised by a finite set Λ, denoted by

$$\mathfrak{t} = n_+ n_- (n_i, d_i, , \epsilon_i, e_i)_{i \in \Lambda}, \tag{IV.2.1.3.1}$$

satisfying

$$\sum_{i \in \Lambda} 2 n_i d_i + n_+ + n_- = n. \tag{IV.2.1.3.2}$$

If $\tilde{\mathfrak{t}} = \tilde{n}_+ \tilde{n}_- (\tilde{n}_i, \tilde{d}_i, , \tilde{\epsilon}_i, \tilde{e}_i)_{i \in \tilde{\Lambda}}$ is another sequence of integers, we regard it as the same as $\mathfrak{t}$ if and only if there exists a bijection $\Lambda \leftrightarrows \tilde{\Lambda}$ such that the integers and the signs are mathched, and moreover, $n_+ = \tilde{n}_+$ and $n_- = \tilde{n}_-$.

We denote by $\mathfrak{T}_{C,s}$ the set of the types of the F-stable quasi-semi-simple conjugacy classes.

Given $\mathfrak{t} \in \mathfrak{T}_{C,s}$, denote by $\bar{\mathfrak{T}}_{C,s}(\mathfrak{t})$ the set of the data

$$\bar{\mathfrak{t}} = n_+ n_-(n_i, \hat{\alpha}_i)_{i\in\Lambda}, \tag{IV.2.1.3.3}$$

satisfying

- for any i, $\hat{\alpha}_i$ is an orbit of type $(d_i, \epsilon_i, e_i) \neq (1)$ or (i);
- if $i \neq i'$, then $\hat{\alpha}_i \neq \hat{\alpha}_{i'}$.

If $\bar{\mathfrak{s}} = m_+ m_-(m_i, \hat{\beta}_i)_{i\in\tilde{\Lambda}}$ is another such datum, we regard it the same as $\bar{\mathfrak{t}}$ if and only if there exists a bijection $\Lambda \leftrightarrows \tilde{\Lambda}$ such that the integers and the orbits are mathched, and moreover, $n_+ = m_+$ and $n_- = m_-$.

It will sometimes be convenient to distinguish between the i's with $\epsilon_i = 1$ and the i's with $\epsilon_i = -1$. Put $\Lambda_1 \subset \Lambda$ to be the subset of the i's such that $\epsilon_i = 1$ and put $\Lambda_2 = \Lambda \setminus \Lambda_1$. The following notations will also be used,

$$\begin{aligned} \mathfrak{t} &= n_+ n_-(n_i, d_i, e_i)_{i\in\Lambda_1}(n'_j, d'_j, e'_j)_{j\in\Lambda_2}, \\ \bar{\mathfrak{t}} &= n_+ n_-(n_i, \bar{\alpha}_i)_{i\in\Lambda_1}(n'_j, \bar{\alpha}'_j)_{j\in\Lambda_2}. \end{aligned} \tag{IV.2.1.3.4}$$

Proposition IV.2.1.5. *The F-stable quasi-semi-simple conjugacy classes in $G.\sigma$ are in bijection with $\bar{\mathfrak{T}}_{C,s}$.*

This follows from Lemma IV.2.1.6 and Lemma IV.2.1.7 below.

IV.2.1.4 Define a map

$$\psi : \bar{\mathfrak{T}}_{C,s} \longrightarrow \{F\text{-stable quasi-semi-simple classes in } G.\sigma\} \tag{IV.2.1.4.1}$$

as follows.

Write $\bar{\mathfrak{t}} = n_+ n_-(n_i, \hat{\alpha}_i)_{i\in\Lambda}$ with $\hat{\alpha}_i$ of type (d_i, ϵ_i, e_i). Let us define an element of $(T^\sigma)^\circ$ from $\bar{\mathfrak{t}}$ and denote it by $t = [a_1, \ldots, a_m]$. Basically, we are going to regard the elements of $\hat{\alpha}_i$'s as eigenvalues and the n_i's as their multiplicities.

(i). Take $[n_+/2]$ subsets of $\{1, \ldots, m\}$, each consisting of a point, which will be called of type (1), and then $(n_-/2)$ subsets, each consisting of a point, which will be called of type (i), and take for each $i \in \Lambda$, n_i subsets of cardinality d_i. These subsets, combined with $\{0\}$ if n is odd, form a partition of $\{1, \ldots, m\}(\cup\{0\})$ and we denote it by $(I_r)_r$.

(ii). Choose for each r an identification $I_r \cong \mathbb{Z}/d_r\mathbb{Z}$, where $d_r := d_i$ if I_r comes from i.

(iii). For each $i \in \Lambda$, take a sequence $\underline{\epsilon}_i = (\epsilon_k)_{k\in\mathbb{Z}/d_i\mathbb{Z}}$ such that $\epsilon_i = \prod \epsilon_k$ and a sequence $\underline{e}_i = (e_k)_{k\in\mathbb{Z}/d_i\mathbb{Z}}$ such that $e_i = \prod e_k$, for example the choice in Remark IV.2.1.4.

(iv). For each $i \in \Lambda$, take a representative of type $(d_i, \underline{\epsilon}_i, \underline{e}_i)$ of $\hat{\alpha}_i$, denoted by α_i.

(v). If I_r comes from $i \in \Lambda$ by the procedure (i), and $\alpha_i = \{a, a^q, \ldots, a^{q^{d_i-1}}\}$, define for all $k \in I_r$, $a_k := a^{q^k}$, under the identification $I_r \cong \mathbb{Z}/d_r\mathbb{Z}$. If I_r is of type (1)(resp. (i)), we define the only entry of t corresponding to I_r to be 1(resp. i). (We require that $\{0\} \in (I_r)_r$ is of type

(1).) Thus, we have defined an element $t \in (T^\sigma)^\circ$, whence a quasi-semi-simple geometric conjugacy class in $G.\sigma$, denoted by $\psi(\bar{\mathfrak{t}})$.

Lemma IV.2.1.6. *ψ is well defined.*

Proof. The class $\psi(\bar{\mathfrak{t}})$ does not depend on the choices of the subsets I_r or the identifications $I_r \cong \mathbb{Z}/d_r\mathbb{Z}$ due to the conjugation by W^σ. It does not depend on the choices of the representativesα_i, due to the conjugation by W^σ and the multiplication by S.

Let us show that $\psi(\bar{\mathfrak{t}})$ is F-stable. Under the identifications $I_r \cong \mathbb{Z}/d_r\mathbb{Z}$, the permutation $i \mapsto i+1$ induces the circular permutation of the elements of I_r. We define a permutation τ of $\{1,\dots,m\}(\cup\{0\})$ by the c_{I_r}'s. For $I_r = \{i_r\}$ (i.e. those consisting of one single point), put $\epsilon_{i_r} = 1$, $e_{i_r} = 1$, unless $q \equiv 3 \mod 4$ and I_r is of type (i), in which case we put $\epsilon_{i_r} = -1$, $e_{i_r} = 1$. We define $\underline{\epsilon}_{I_r} = (\epsilon_k)_{k\in I_r}$ by $\underline{\epsilon}_i$ and ϵ_{i_r}, and define $\underline{e}_{I_r} = (e_k)_{k\in I_r}$ by $\underline{e}_i$ and e_{i_r}, thus define w and s. We have $swF(t) = t$. Observe that the class of $t\sigma$ is F-stable if and only if there exists $w \in W^\sigma$ and $s \in S$ such that $swF(t) = t$. □

IV.2.1.5 Now we prove the following.

Lemma IV.2.1.7. *ψ is bijective.*

Proof. The injectivity follows immediately from the definition of $\bar{\mathfrak{T}}_{C,s}$ and Remark IV.2.1.3. Let us show the surjectivity.

Let $t = [a_1,\dots,a_m] \in (T^\sigma)^\circ$ be an element of an F-stable conjugacy class. Then there exists w and s such that $swF(t) = t$. We can write $w = ((\epsilon_1,\dots,\epsilon_m),\tau)$ with $\tau = c_{I_1}\cdots c_{I_l}$ and $s = [e_1,\dots,e_m]$. For all r, put $\bar{e}_r = \prod_{k\in I_r} e_k$ and put $\underline{e}_r = (e_k)_{k\in I_r}$(See §II.2.2.2). Similarly, we define $\bar{\epsilon}_r$ and $\underline{\epsilon}_r$. The action of wF on $(T^\sigma)^\circ$ is given by

$$[a_1,\dots,a_m] \longmapsto [a_{\tau^{-1}(1)}^{q\epsilon_1},\dots,a_{\tau^{-1}(m)}^{q\epsilon_m}]. \tag{IV.2.1.5.1}$$

For all r, we can index the elements of I_r by $\mathbb{Z}/d_r\mathbb{Z}(d_r = |I_r|)$ in such a way that the action of τ sends $i \in I_r$ to $i+1$. An element $[a_1,\dots,a_m]$ is fixed by sF_w, if in every orbit I_r, we have $a_i = e_i a_{i-1}^{q\epsilon_i}$, $i \in \mathbb{Z}/\tau_r\mathbb{Z}$. In particular, it is necessary that the a_i's satisfy the equation $x^{q^{\tau_r}} = \bar{e}_r x^{\bar{\epsilon}_r}$.

Assume that under the identification $I_1 \cong \mathbb{Z}/\tau_1\mathbb{Z}$, the a_i's, $i \in \mathbb{Z}/d_1\mathbb{Z}$ satisfy $a_i = e_i a_{i-1}^{q\epsilon_i}$. Let τ_1 be the smallest positive integer such that $a_{\tau_1} \in \pm a_0^{\pm 1}$. If $\tau_1 = d_1$, then there is nothing to be done. Suppose $\tau_1 < d_1$. Put $I_1' = \{0,1,\dots,\tau_1-1\} \subset \mathbb{Z}/d_1\mathbb{Z}$ and put $I_1'' = \{\tau_1,\tau_1+1,\dots,d_1-1\} \subset \mathbb{Z}/d_1\mathbb{Z}$, and let $c_{I_1'}$ be the permutation $i \mapsto i+1$, $\tau_1 - 1 \mapsto 0$ and $c_{I_1''}$ the permutation $i \mapsto i+1$, $d_1 - 1 \mapsto \tau_1$. We define $\tau' = c_{I_1'}c_{I_1''}c_{I_2}\cdots c_{I_l}$. By modifying ϵ_0, e_0, ϵ_{τ_1}, e_{τ_1}, ϵ_{τ_1+1}, e_{τ_1+1}, ϵ_{d_1-1} and e_{d_1-1} if necessary, we obtain $w' = ((\epsilon_1',\dots,\epsilon_m'),\tau')$ and $s' = (e_1',\dots,e_m')$ that satisfy $s'w'F(t) = t$. Continue this process if necessary and replace w by w' until this is no longer possible. Now, for each r, $\{a_k | k \in I_r\}$ form a representative α_r of an orbit $\hat{\alpha}_r \subset \hat{k}$. Collecting the same orbits and defining n_j to be the multiplicity of the corresponding orbit, we end up with $(n_1,\bar{\alpha}_1)\cdots(n_{l'},\bar{\alpha}_{l'})$. Separating the orbits of type (1) and (i), this is the image of an element of $\bar{\mathfrak{T}}_{C,s}$, whence surjectivity. □

IV.2.2 Centralisers and G^F-Classes

We will see that the centraliser of a quasi-semi-simple element is in general a product of a symplectic group, an orthogonal group and some linear groups.

IV.2.2.1 Let $\bar{C}$ be an F-stable quasi-semi-simple conjugacy class corresponding to $\bar{\mathfrak{t}} = n_+n_-(n_i, \hat{\alpha}_i)_{i\in\Lambda_1}(n_j, \hat{\alpha}_j)_{j\in\Lambda_2}$ following Proposition IV.2.1.5, and we denote its type by $\mathfrak{t} = n_+n_-(n_i, d_i, e_i)_{i\in\Lambda_1}(n'_j, d'_j, e'_j)_{j\in\Lambda_2}$. Let $t\sigma$ be a representative of $\bar{C}$ following §IV.2.1.4. Then the isomorphism class of $C_G(t\sigma)$ only depends on the type of $\bar{C}$.

Lemma IV.2.2.1. *We have,*

$$C_G(t\sigma) \cong \mathrm{Sp}_{n_+}(k) \times \mathrm{O}_{n_-}(k) \times \prod_{i\in\Lambda_1} \mathrm{GL}_{n_i}(k) \times \prod_{j\in\Lambda_2} \mathrm{GL}_{n'_j}(k), \quad \textit{if } n \textit{ is even,}$$
$$C_G(t\sigma) \cong \mathrm{Sp}_{n_-}(k) \times \mathrm{O}_{n_+}(k) \times \prod_{i\in\Lambda_1} \mathrm{GL}_{n_i}(k) \times \prod_{j\in\Lambda_2} \mathrm{GL}_{n'_j}(k), \quad \textit{if } n \textit{ is odd.}$$

The numbers n_+ and n_- are exchanged only because we have made different choices of σ for even and odd n.

Proof. If $z \in G$ commutes with $t\sigma$, then it commutes with $t\sigma t\sigma = t\sigma(t)\sigma^2$, with $\sigma^2 = \pm 1$ being central. Let us calculate $C_{C_G(t\sigma(t))}(t\sigma)$. That the $\hat{\alpha}_i$'s are pairwise distinct means that for $a_i \in \alpha_i$, $a_j \in \alpha_j$, $i \neq j$, we have $a_i^{q^c} \neq \pm a_j^{\pm 1}$, for all c, so $a_i^{2q^c} \neq a_j^{\pm 2}$, for all c. Besides, the integers n_+ and n_- become the multiplicities of 1 and -1 in $t\sigma(t)$ respectively. Consequently, the centraliser of $t\sigma(t)$ is a Levi subgroup $\bar{L}_0 := C_G(t\sigma(t))$:

$$\prod_{i\in\Lambda_1} (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i})^{d_i} \times \prod_{j\in\Lambda_2} (\mathrm{GL}_{n'_j} \times \mathrm{GL}_{n'_j})^{d'_j} \times \mathrm{GL}_{n_+} \times \mathrm{GL}_{n_-} \tag{IV.2.2.1.1}$$

with the action of σ given by

$$\begin{aligned} \sigma : \mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i} &\longrightarrow \mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i} \\ (g, h) &\longmapsto (\sigma_0(h), \sigma_0(g)), \end{aligned} \tag{IV.2.2.1.2}$$

for all $i \in \Lambda_1$, and similarly for $j \in \Lambda_2$, where $\sigma_0(g) = J^t g^{-1} J^{-1}$, with $(J)_{ab} = \delta_{a,n_i+1-b}$, for any i or j.

The action induced from $t\sigma$ on each $\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i}$ coincides with that of σ. If n is even, and $\bar{G} = {}^s\bar{G}$, the action induced by $t\sigma$ on GL_{n_+} and GL_{n_-} are respectively the automorphisms associated to J_n or J'_n defined in §II.5.1.1. It follows that in $\bar{G} = {}^s\bar{G}$,

$$L_0 := C_G(t\sigma) = \mathrm{Sp}_{n_+}(k) \times \mathrm{O}_{n_-}(k) \times \prod_{i\in\Lambda_1} \mathrm{GL}_{n_i}(k) \times \prod_{j\in\Lambda_2} \mathrm{GL}_{n'_j}(k). \tag{IV.2.2.1.3}$$

□

IV.2.2.2 Let us introduce a sign $\eta \in \{\pm 1\}$ that can be -1 only if

- n is even and $n_- > 0$, or
- n is odd and $n_+ > 0$,

or rather, if the orthogonal factor of the centraliser is non trivial.

Proposition IV.2.2.2. *The quasi-semi-simple conjugacy classes in $G^F.\sigma$ are parametrised by the data*

$$\{(\eta, \bar{\mathfrak{t}})\} \subset \{\pm 1\} \times \bar{\mathfrak{T}}_{C,s}. \tag{IV.2.2.2.1}$$

Proof. Clear. □

To simplify the notation, we will write $\eta\bar{\mathfrak{t}}$ instead of $(\eta, \bar{\mathfrak{t}})$.

If the centraliser of a semi-simple element has two connected components, then the corresponding two G^F-classes can be distinguished by the homomorphism

$$\mathrm{GL}_n(q) \rtimes <\sigma> \longrightarrow \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \longrightarrow \mu_2. \tag{IV.2.2.2.2}$$

The first map sends $g \in G(q)$ to $\det(g) \mod (\mathbb{F}_q^*)^2$ and sends σ to 1, and the second map is the nontrivial homomorphism. The value of η is defined as the image of the corresponding G^F-class under this homomorphism. In fact, the above homomorphism is the only nontrivial central character of $\mathrm{GL}_n(q).<\sigma>$that is non vanishing on $\mathrm{GL}_n(q).\sigma$, which extends the character $\eta \circ \det$ of $\mathrm{GL}_n(q)$, with η being the order 2 irreducible character of $\mathbb{F}_q^*$. This explains the notation η.

To see that the above homomorphism can distinguish the two G^F-conjugacy classes contained in the same G-conjugacy class, we argue as follows. Let $t\sigma \in G^F.\sigma$ be such that $C_G(t\sigma)$ has two connected component. Then according to our concrete description of $C_G(t\sigma)$, its two connected components are distinguished by the values ± 1 of the determinant, corresponding to the two connected components of the orthogonal factors. Let $g \in G$ be such that $g^{-1}F(g) = z \in C_G(t\sigma) \setminus C_G(t\sigma)^\circ$, then $gs\sigma g^{-1}$ is a representative of another G^F-conjugacy class. Applying the determinant to the equality $g^{-1}F(g) = z$ gives $\det(g)^{q-1} = -1$, so that $\det(g)^2 \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$; applying the above homomorphism to the element $gs\sigma g^{-1}$ gives $\det(s)\det(g)^2$, whence the claim.

IV.2.2.3 We can specify the centraliser of each quasi-semi-simple element of $G^F.\sigma$. Let $t\sigma \in G^F.\sigma$ be a quasi-semi-simple element corresponding to

$$\eta d_+ d_-(n_i, \hat{\alpha}_i)_{i\in\Lambda_1}(n'_j, \hat{\beta}_j)_{j\in\Lambda_2}. \tag{IV.2.2.3.1}$$

If n is even, then its centraliser in G^F is

$$\mathrm{Sp}_{n_+}(q) \times \mathrm{O}^{\eta}_{n_-}(q) \times \prod_{i\in\Lambda_1} \mathrm{GL}_{n_i}(q^{d_i}) \times \prod_{j\in\Lambda_2} \mathrm{GL}^-_{n'_j}(q^{d'_j}). \tag{IV.2.2.3.2}$$

If n is odd, then its centraliser in G^F is

$$\mathrm{Sp}_{n_-}(q) \times \mathrm{O}_{n_+}(q) \times \prod_{i\in\Lambda_1} \mathrm{GL}_{n_i}(q^{d_i}) \times \prod_{j\in\Lambda_2} \mathrm{GL}^-_{n'_j}(q^{d'_j}). \tag{IV.2.2.3.3}$$

Note that for odd n_+, $\mathrm{O}^+_{n_+}(q)$ is isomorphic to $\mathrm{O}^-_{n_+}(q)$.

IV.2.2.4 We refer to §II.2.3.4 for the parametrisation of the unipotent classes of finite classical groups. Let C be a semi-simple G^F-conjugacy class corresponding to

$$\eta d_+ d_-(n_i, \hat{\alpha}_i)_{i\in\Lambda_1}(n'_j, \hat{\alpha}'_j)_{j\in\Lambda_2}. \tag{IV.2.2.4.1}$$

For odd n, the G^F-classes which have C as semi-simple parts are parametrised by

$$\Lambda^s_{n_-}\Lambda^o_{n_+}(\lambda_i)_{i\in\Lambda_1}(\lambda'_j)_{j\in\Lambda_2}, \tag{IV.2.2.4.2}$$

where $\Lambda^s_{n_-} \in \Psi^s_{n_-}$, $\Lambda^o_{n_+} \in \Psi^o_{n_+}$, $\lambda_i \in \mathcal{P}_{n_i}$, $\lambda'_j \in \mathcal{P}_{n'_j}$, with each such datum corresponding to two classes if $\Lambda^o_{n_+}$ is degenerate.

For even n and $\eta = 1$, the G^F-classes which have C as semi-simple part are parametrised by

$$\Lambda^s_{n_+}\Lambda^o_{n_-}(\lambda_i)_{i\in\Lambda_1}(\lambda'_j)_{j\in\Lambda_2}, \tag{IV.2.2.4.3}$$

where $\Lambda^s_{n_+} \in \Psi^s_{n_+}$, $\Lambda^o_{n_-} \in \Psi^o_{n_-}$, $\lambda_i \in \mathcal{P}_{n_i}$, $\lambda'_j \in \mathcal{P}_{n'_j}$, with each such datum corresponding to two classes if $\Lambda^o_{n_-}$ is degenerate. If $\eta = -1$, the classes are parametrised by the same data except that $\Lambda^o_{n_-} \in \Psi'^o_{n_-}$.

IV.3 Shintani Descent

Now G denote a connected reductive group over k.

IV.3.1 Eigenvalues of the Frobenius

In this part, we collect some results on the eigenvalues of the Frobenius endomorphism acting on the ℓ-adic cohomology of the Deligne-Lusztig variety X_w. We will write $X_{w,F}$ if it is necessary to specify the Frobenius that is involved. Recall that X_w is the subvariety of the flag variety $\mathcal{B}$ consisting of the Borel subgroups B such that $(B, F(B))$ are conjugate to

$(B_0, \dot{w}B_0\dot{w}^{-1})$ by G, where $\dot{w} \in G$ is a representative of $w \in W_G$ and B_0 is some fixed F-stable Borel subgroup.

IV.3.1.1 The Deligne-Lusztig character $R_{T_w}^G 1$ is realised by the virtual representation

$$\bigoplus_i (-1)^i H_c^i(X_w, \bar{\mathbb{Q}}_\ell).$$

Recall that the Lusztig series $\mathcal{E}(G^F, (1))$ consists of the irreducible representations that appear as a direct summand of some $R_{T_w}^G 1$, or equivalently of a vector space $H_c^i(X_w, \bar{\mathbb{Q}}_\ell)$. Denote by $\mathbf{H}^i(\bar{X}_w, \bar{\mathbb{Q}}_\ell)$ the intersection cohomology of X_w. By [L84b, Corollary 2.8], each element of $\mathcal{E}(G^F, (1))$ is also an irreducible G^F-subrepresentation of $\mathbf{H}^i(\bar{X}_w, \bar{\mathbb{Q}}_\ell)$ for some i and w. We write

$$\begin{aligned} M_i(w, F) &:= \mathbf{H}^i(\bar{X}_{w,F}, \bar{\mathbb{Q}}_\ell) \\ H_i(w, F) &:= \mathrm{End}_{G^F}(M_i(w, F)) \end{aligned} \tag{IV.3.1.1.1}$$

If $w = 1$, $M_i(1, F)$ is just the ℓ-adic cohomology of X_1 and its simple factors are the principal series representations, which are in bijection with the irreducible representations of $H_i(1, F)$. If F is split, $H_i(1, F) \cong \bar{\mathbb{Q}}_\ell[W]$, and if F is twisted by a graph automorphism σ, $H_i(1, F) \cong \bar{\mathbb{Q}}_\ell[W^\sigma]$. In what follows, we fix the Frobenius F and $w \in W$, and write M_i and H_i instead of $M_i(w, F)$ and $H_i(w, F)$.

IV.3.1.2 Denote by F_0 a split Frobenius over $\mathbb{F}_q$, and denote by F the Frobenius defining the $\mathbb{F}_q$-structure of G. Assume that some power of F_0 is a power of F. Let b be the smallest integer such that F_0^b is a power of F. In the case that interests us, $b = 1$ or 2. Note that F and F_0 commute. The action of F_0 on $\mathcal{B}$ induces an isomorphism of M_i as a vector space. Moreover F_0 induces by conjugation an algebra automorphism of H_i, still denoted by F_0, which is unipotent ([L84b, Theorem 2.18]). Let ρ be an irreducible representation of G^F that appears in M_i. Denote by $M_{i,\rho}$ the isotypic component corresponding to ρ. By [L84b, Proposition 2.20], the action of F_0 respects the isotypic decomposition, i.e. $F_0(M_{i,\rho}) = M_{i,\rho}$. The algebra H_i is decomposed into some simple algebras $H_{i,\rho} = \mathrm{End}_{G^F}(M_{i,\rho})$.

IV.3.1.3 Now assume that ρ is fixed. Denote by $[\rho]$ the vector space on which G^F acts by the representation ρ. There exists a $\bar{\mathbb{Q}}_\ell$-space V such that $M_{i,\rho} \cong [\rho] \otimes V$ and that $H_{i,\rho} \cong \mathrm{End}_{\bar{\mathbb{Q}}_\ell} V$. Since $H_{i,\rho}$ is a simple algebra, we have $F_0 = \phi_G \otimes \phi_H$ with $\phi_G \in \bar{\mathbb{Q}}_\ell[G^F]$ and $\phi_H \in H_{i,\rho}$, which are invertible as F_0 is. Consider the adjoint representation,

$$\begin{aligned} \mathrm{GL}(V) &\longrightarrow \mathrm{GL}(H_{i,\rho}) \\ \phi_H &\longmapsto \mathrm{ad}\, \phi_H. \end{aligned} \tag{IV.3.1.3.1}$$

Since $\operatorname{ad}\phi_H = \operatorname{ad}F_0$, which is unipotent, we see that ϕ_H is a unipotent endomorphism up to a scalar. Modifying ϕ if necessary, we can assume that ϕ_H is unipotent. Choose a basis $\{e_1, \dots, e_s\}$ of V in such a way that in this basis ϕ_H is triangular. Each $M_{i,\rho,r} = [\rho] \otimes e_r$ provides the representation ρ. By [L84b, Proposition 2.20], the representation $\rho : G^F \to \mathrm{GL}(M_{i,\rho,r})$ is F_0-stable and extends into a representation of $G^F.<F_0>$, denoted by $\tilde{\rho}$,with F_0^b acting trivially on G^F. The action of F_0, regarded as an element of $G^F.<F_0>$, on $M_{i,\rho,r}$ is defined by $(\lambda'_\rho)^{-1}q^{-i/2}\phi$, where λ'_ρ is a root of unity. Another choice of $\tilde{\rho}$ corresponds to a multiple of λ'_ρ by a b-th root of unity. The value of λ'_ρ only depends on ρ and a choice of $\tilde{\rho}$, and does not depend on w or i. In other words, $F_0 = \tilde{\rho} \otimes \varphi'$, where φ' is a unipotent endomorphism multiplied by $\lambda'_\rho q^{i/2}\varphi$.

IV.3.1.4 If we consider a Frobenius F'_0 that is not necessarily split, it may happen that the action of F'_0 does not respect the isotypic components of M_i. However, we can nevertheless consider those components that are preserved by F'_0. In fact, a component $M_{i,\rho}$ is F'_0-stable if and only if the character $\theta_\rho \in \mathrm{Irr}(H_i)$ associated is F'_0-stable. Let $M_{i,\rho}$ be such a component, we still have $M_{i,\rho} \cong [\rho] \otimes V$ and ρ extends into a representation of $G^F.<F'_0>$.

IV.3.1.5 Now we consider the action of F_0 on the ℓ-adic cohomology. By [Sh85, Lemma 1.4], the eigenvalues of F_0 on $H^i_c(X_w, \bar{\mathbb{Q}}_\ell)$ are λ'_ρ times a power of $q^{b/2}$ which is not necessarily $q^{ib/2}$. Let $\mu = \lambda'_\rho q^{kb/2}$ be such an eigenvalue. Then, the subspace $M_{i,\rho,\mu} \subset M_{i,\rho}$ of eigenvalue μ is F_0-stable and there exists a decomposition $M_{i,\rho,\mu} \cong [\rho] \otimes V_\mu$ such that the action of F_0 on $M_{i,\rho,\mu}$ is decomposed as $\tilde{\rho} \otimes \varphi_\mu$ where φ_μ is $\lambda'_\rho q^{k/2}$ times a unipotent endomorphism of V_μ. Once again, λ'_ρ only depends on ρ and a choice of $\tilde{\rho}$.

IV.3.1.6 There are two particular cases that interest us.

Theorem IV.3.1.1. *([L77, Theorem 3.34]) Let $i \in \mathbb{Z}$ and $w \in W_G$ be arbitrary. If (G, F) is of type A_n, $n \geq 1$, then all of the eigenvalues of F on $H^i_c(X_w, \bar{\mathbb{Q}}_\ell)$ are powers of q. If (G, F) is of type ${}^2A_n$, $n \geq 2$, then all of the eigenvalues of F^2 on $H^i_c(X_w, \bar{\mathbb{Q}}_\ell)$ are powers of $(-q)$.*

IV.3.2 Shintani Descent

IV.3.2.1 Let F_1 and F_2 be two commuting Frobenius endomorphism. Denote by $\mathcal{K}(G^{F_1}.F_2)$ the F_2-conjugacy classes of G^{F_1} and by $\mathcal{K}(G^{F_2}.F_1)$ the F_1-conjugacy classes of G^{F_2}, and we denote by $\mathcal{C}(G^{F_2}.F_1)$ and $\mathcal{C}(G^{F_1}.F_2)$ the set of functions that are constant on the F_1-conjugacy classes of G^{F_2} and the functions that are constant on the F_2-conjugacy classes of G^{F_1} respectively.

Define a map $N_{F_1/F_2} : \mathcal{K}(G^{F_1}, F_2) \to \mathcal{K}(G^{F_2}.F_1)$ as follows. For $g \in G^{F_1}$, there exists $x \in G$ such that $xF_2(x^{-1}) = g$. Then $g' := x^{-1}F_1(x) \in G^{F_2}$, and its F_1-conjugacy class is well defined. This defines a bijection $N_{F_1/F_2} : \mathcal{K}(G^{F_1}, F_2) \xrightarrow{\sim} \mathcal{K}(G^{F_2}.F_1)$. We write $g' = N_{F_1/F_2}(g)$ by abuse of notation. Denote by $\mathrm{Sh}_{F_2/F_1} : \mathcal{C}(G^{F_2}.F_1) \xrightarrow{\sim} \mathcal{C}(G^{F_1}.F_2)$ the induced bijection. It is easy to check that $\mathrm{Sh}_{F_2/F_1} \circ \mathrm{Sh}_{F_1/F_2} = \mathrm{Id}$ and that $\mathrm{Sh}_{F/F}$ is an involution that may not be the identity.

IV.3.3 Action on the Irreducible Characters

IV.3.3.1 Let U be a unipotent character of G^{F_1} that extends to $G^{F_1}.<F_2>$. We denote the restriction to $G^{F_1}.F_2$ of its extension by $E_{F_2}(U) \in \mathcal{C}(G^{F_1}.F_2)$. The Shintani descent sends it into $\mathcal{C}(G^{F_2}.F_1)$. On the other hand, the unipotent irreducible characters of G^{F_2} that extends to $G^{F_2}.F_1$, which we denote by $\mathcal{E}(G^{F_2},(1))^{F_1}$, have as extensions some elements of $\mathcal{C}(G^{F_2}.F_1)$. We will see that the functions $\mathrm{Sh}_{F_1/F_2}\, E_{F_2}(U)$ can be expressed as linear combinations of the extensions of the elements of $\mathcal{E}(G^{F_2},(1))^{F_1}$.

IV.3.3.2 Let B be an F_1-stable and σ-stable Borel subgroup. Put

$$H := \mathrm{End}_{G^{F_1}}(\mathrm{Ind}_{B^{F_1}}^{G^{F_1}} 1) \cong \mathrm{End}_{G^{F_1}}(\bar{\mathbb{Q}}_\ell[\mathcal{B}^{F_1}]) \cong \bar{\mathbb{Q}}_\ell[W^{F_1}]. \tag{IV.3.3.2.1}$$

The irreducible characters of H are in bijection with the principal series representations of G^{F_1}. For $\psi \in \mathrm{Irr}(H)$, denote by $U_\psi \in \mathrm{Irr}(G^{F_1})$ the corresponding character. By §IV.3.1.4, U_ψ extends to $G^{F_1}.<F_2>$ if ψ is F_2-stable, in which case ψ itself extends to $H.<F_2>$ in such a way that the action of F_2 on $\mathrm{Ind}_{B^{F_1}}^{G^{F_1}} 1$ is decomposed into $F_2 = E_{F_2}(U_\psi)(F_2) \otimes E_{F_2}(\psi)(F_2)$, where we denote by $E_{F_2}(U_\psi)$ and $E_{F_2}(\psi)$ the extensions of U_ψ and ψ respectively.

IV.3.3.3 Let $\rho \in \mathrm{Irr}(G^{F_2})$ be unipotent. The ρ-isotypic component of $H_c^i(X_{w,F_2})$ is of the form $[\rho] \otimes V$. The action of the split Frobenius F_0 on this component can be written as $\tilde{\rho}(F_0) \otimes \varphi$, where φ is λ'_ρ times a power of $q^{1/2}$ and a unipotent endomorphism, according to §IV.3.1.5. We denote by Ω_{F_2} the isomorphism of the space $\mathcal{C}(G^{F_2},(1))$ that multiplies ρ by λ'_ρ. Denote by $E_{F_0}(\rho)$ the restriction of $\tilde{\rho}$ to $G^{F_2}.F_0$.

IV.3.3.4 Fix the split Frobenius F_0 and the order 2 quasi-central automorphism σ. In what follows, we only consider $(F_1, F_2) = (\sigma_1 F_0^m, \sigma_2 F_0)$,where $m \in \mathbb{Z}_{>0}$ and $\sigma_i = 1$ or σ. Take $\rho \in \mathcal{E}(G^{F_2},(1))^{\sigma_1}$, i.e. a σ_1-stable representation, and denote by $E_{\sigma_1}(\rho)$ an extension of ρ to $G^{F_2}.<\sigma_1>$. Since F_0 acts as σ_2^{-1} on G^{F_2}, we can define the extension $E_{F_0}(\rho)(F_0)$ to be an extension $E_{\sigma_2}(\rho)(\sigma_2^{-1})$, which commutes with $E_{\sigma_1}(\rho)(\sigma_1)$ because either one of σ_1 and σ_2 is 1 or they are equal. This allows us to define an extension $E_{F_1}(\rho)$ of ρ to $G^{F_2}.F_1$ by requiring $E_{F_1}(\rho)(\sigma_1 F_0^m) = E_{\sigma_1}(\rho)(\sigma_1) E_{F_0}(\rho)(F_0^m)$. It is well defined. In addition, E_{σ_1} defines an isomorphism of vector spaces

$$\begin{aligned} \bar{\mathbb{Q}}_\ell[\mathcal{E}(G^{F_2},(1))^{\sigma_1}] &\longrightarrow \mathcal{C}(G^{F_2}.\sigma_1,(1)) \\ \rho &\longmapsto E_{\sigma_1}(\rho). \end{aligned} \tag{IV.3.3.4.1}$$

IV.3.3.5 The following theorem makes explicit the transition matrix.

Theorem IV.3.3.1. *([DM94, Théorème 5.6]) We keep the above notations. For any $m \in \mathbb{Z}_{>0}$, and*

any $\psi \in \mathrm{Irr}(W^{\sigma_1})^{\sigma_2}$ *we have*

$$\begin{aligned}\mathrm{Sh}_{\sigma_1 F_0^m/\sigma_2 F_0} E_{\sigma_2 F_0}(U_\psi) &= \sum_{\rho \in \mathcal{E}(G^{\sigma_2 F_0},1)^{\sigma_1}} \langle R_\psi^{G^{\sigma_2 F}.\sigma_1}, E_{\sigma_1}(\rho)\rangle_{G^{\sigma_2 F_0}.\sigma_1} \lambda_\rho'^m E_{\sigma_1 F_0^m}(\rho) \\ &= E_{\sigma_1 \sigma_2^{-m}}(\Omega^m_{\sigma_2 F_0} E_{\sigma_1}^{-1}(R_\psi^{G^{\sigma_2 F_0}.\sigma_1})).\end{aligned} \tag{IV.3.3.5.1}$$

(See [DM94, Définition 5.1] for the definition of $R_\psi^{G^{\sigma_2 F_0}.\sigma_1}$ *or more generally in IV.5.2.2.2.)*

IV.3.4 Commutation with the Deligne-Lusztig Induction

IV.3.4.1 The following proposition due to Digne shows how the Deligne-Lusztig induction commutes with Shintani Descent.

Proposition IV.3.4.1. *([Di, Proposition 1.1]) Let G be a connected reductive group defined over* $\mathbb{F}_q$*, equipped with the Frobenius endomorphism F and let* σ *be a quasi-central automorphism of G. Let* $L \subset P$ *be an F-stable and* σ*-stable Levi factor of a* σ*-stable parabolic subgroup. Then*

$$\mathrm{Sh}_{\sigma F/F} \circ R^{G^{\sigma F}.\sigma^{-1}}_{L^{\sigma F}.\sigma^{-1}} = R^{G^F.\sigma}_{L^F.\sigma} \circ \mathrm{Sh}_{\sigma F/F}\,. \tag{IV.3.4.1.1}$$

IV.4 Character Sheaves

In this section, G denote a not necessarily connected reductive group. By local system, we mean a local system of $\bar{\mathbb{Q}}_\ell$-vector spaces. If X is a variety over k, we denote by $\mathcal{D}(X)$ the bounded derived category of constructible $\bar{\mathbb{Q}}_\ell$-sheaves on X. For any $g \in G$, denote by $g_s g_u$ the Jordan decomposition of g, with g_s being semi-simple and g_u unipotent.

IV.4.1 Character Sheaves for Groups Not Necessarily Connected

IV.4.1.1 If G^1 is a connected component of G, define

$$Z^\circ_{G^\circ,G^1} := C_{Z_{G^\circ}}(g)^\circ, \quad \text{for any } g \in G^1.$$

It does not depend on the choice of $g \in G^1$. An *isolated stratum* of G^1 is an orbit of isolated elements under the action of $Z^\circ_{G^\circ,G^1} \times G^\circ$ given by

$$(z,x) : g \longmapsto zxgx^{-1}.$$

(See [L03, I, §1.21 (d), §3.3 (a)])

Example IV.4.1.1. For the group $\bar{G}$ defined in §II.5.2.1, we have $Z^\circ_{G,G.\sigma} = \{1\}$, and so an isolated stratum of $G.\sigma$ is an isolated G-conjugacy class.

Given a stratum S, denote by $\mathcal{S}(S)$ the category of local systems on S invariant under the action of $Z^\circ_{G^\circ,G^1} \times G^\circ$ given by

$$(z,x) : g \longmapsto z^n x g x^{-1},$$

for some integer $n > 0$. We refer to [L03, I, §6] for the definition of *cuspidal local system (for G)*. If $\mathcal{E} \in \mathcal{S}(S)$ is cuspidal, we say that $(S, \mathcal{E})$ is a *cuspidal pair (for G)*.

IV.4.1.2 Let L be a Levi subgroup of G° and let S be an isolated stratum of $N_G(L,P)$ for a parabolic subgroup P with Levi factor L. (See [L03, I, §2.2 (a), §3.5]) Define,

$$S_{reg} = \{g \in S \mid C_G(g_s)^\circ \subset L\}.$$

Define

$$Y_{L.S} = \bigcup_{x \in G^\circ} x S_{reg} x^{-1},$$

and

$$\tilde{Y}_{L,S} = \{(g, xL) \in G \times G^\circ/L \mid x^{-1} g x \in S_{reg}\},$$

equipped with the action by G°, $h : (g, xL) \mapsto (hgh^{-1}, hxL)$, and

$$\hat{Y}_{L.S} = \{(g,x) \in G \times G^\circ \mid x^{-1} g x \in S_{reg}\},$$

equipped with the action by $G^\circ \times L$, $(h,l) : (g,x) \mapsto (hgh^{-1}, hxl^{-1})$. Consider the morphisms

$$S \xleftarrow{\alpha} \hat{Y}_{L,S} \xrightarrow{\beta} \tilde{Y}_{L,S} \xrightarrow{\pi} Y_{L,S},$$

where $\alpha(g,x) = x^{-1}gx$, $\beta(g,x) = (g, xL)$ and $\pi(g, xL) = g$. Note that β is a principal L-bundle and that π is a principal $\mathcal{W}_S$-bundle, with $\mathcal{W}_S = \tilde{\mathcal{W}}_S/L$, where

$$\tilde{\mathcal{W}}_S := \{n \in N_{G^\circ}(L) \mid nSn^{-1} = S\}.$$

It is a finite group. (See [L03, I, §3.13]) If $\mathcal{E} \in \mathcal{S}(S)$ is irreducible and cuspidal for $N_G(L,P)$, put

$$\tilde{\mathcal{W}}_{\mathcal{E}} := \{n \in \tilde{\mathcal{W}}_S \mid \mathrm{ad}(n)^* \mathcal{E} \cong \mathcal{E}\},$$

and $\mathcal{W}_{\mathcal{E}} = \tilde{\mathcal{W}}_{\mathcal{E}}/L$.

IV.4.1.3 Fix $\mathcal{E} \in \mathcal{S}(S)$. There exists a G°-equivariant local system $\tilde{\mathcal{E}}$ on $\tilde{Y}_{L,S}$ such that $\beta^*\tilde{\mathcal{E}} \cong \alpha^*\mathcal{E}$. Denote by $\mathbf{E} = \mathrm{End}(\pi_!\tilde{\mathcal{E}})$, the endomorphism algebra of $\pi_!\tilde{\mathcal{E}}$. We have a canonical decomposition ([L03, II, §7.10 (a); IV, §21.6])

$$\mathbf{E} = \bigoplus_{w \in \mathcal{W}_{\mathcal{E}}} \mathbf{E}_w,$$

where the factors $\mathbf{E}_w := \mathrm{Hom}(\mathrm{ad}(n_w)^*\mathcal{E}, \mathcal{E})$, each one defined by some representative n_w of w, are of dimension 1 and satisfy $\mathbf{E}_w\mathbf{E}_v = \mathbf{E}_{wv}$. Choose base $\{b_w \mid w \in \mathcal{W}_{\mathcal{E}}\}$ of $\mathbf{E}$ with $b_w \in \mathbf{E}_w$ for any w.

Define

$$K = \mathbf{IC}(\bar{Y}_{L,S}, \pi_!\tilde{\mathcal{E}}),$$

where $\bar{Y}_{L,S}$ is the closure of $Y_{L,S}$ in G. There exists a canonical isomorphism $\mathbf{E} \cong \mathrm{End}(K)$. Let Λ' be a finite set parametrising the isomorphism classes of the irreducible representations of $\mathbf{E}$ and for each $i \in \Lambda'$, we denote by V_i a corresponding representation. Then, we have the canonical decompositions

$$\pi_!\tilde{\mathcal{E}} \cong \bigoplus_{i\in\Lambda'} V_i \otimes (\pi_!\tilde{\mathcal{E}})_i, \quad K \cong \bigoplus_{i\in\Lambda'} V_i \otimes K_i,$$

where

$$(\pi_!\tilde{\mathcal{E}})_i = \mathrm{Hom}_{\mathbf{E}}(V_i, \pi_!\tilde{\mathcal{E}}), \quad K_i = \mathrm{Hom}_{\mathbf{E}}(V_i, K)$$

are the simple factors. Moreover, $K_i \cong \mathbf{IC}(\bar{Y}_{L,S}, (\pi_!\tilde{\mathcal{E}})_i)$.

IV.4.1.4 Assume that $F(L) = L$, $F(S) = S$ and $F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$, where F is the Frobenius of G. We fix an isomorphism $\phi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$. It induces an isomorphism $\tilde{\phi} : F^*\pi_!\tilde{\mathcal{E}} \xrightarrow{\sim} \pi_!\tilde{\mathcal{E}}$ and an isomorphisme $\phi : F^*K \xrightarrow{\sim} K$. Recall that, given a variety X/k equipped with the Frobenius F, a complex $A \in \mathcal{D}(X)$ and an isomorphism $\phi : F^*A \xrightarrow{\sim} A$, the *characteristic function* $\chi_{A,\phi} : X^F \to \bar{\mathbb{Q}}_\ell$ is defined by

$$\chi_{A,\phi}(x) = \sum_{i\in\mathbb{Z}} (-1)^i \,\mathrm{Tr}(\phi, \mathcal{H}^i_x A), \tag{IV.4.1.4.1}$$

where $\mathcal{H}^i_x A$ is the stalk at x of the cohomology sheaf in degree i of A.

The characteristic function of a cuspidal local system will be called a cuspidal function.

Theorem IV.4.1.2. *([L03, III, Theorem 16.14, §16.5, §16.13]) Let s and $u \in G^F$ be a semi-simple element and a unipotent element such that $su = us \in \bar{Y}_{L,S}$. Then,*

$$\chi_{K,\phi}(su) = \sum_{\substack{h\in G^{\circ F};\\ h^{-1}sh\in S_s}} \frac{|L_h^F|}{|C_G(s)^{\circ F}||L^F|} Q_{L_h, C_G(s), \mathfrak{c}_h, \mathcal{F}_h, \phi_h}(u), \tag{IV.4.1.4.2}$$

where S_s is the set of the semi-simple parts of the elements of S, and $Q_{L_h, C_G(s), \mathfrak{c}_h, \mathcal{F}_h, \phi_h}$ is the generalised Green function (See §IV.4.1.5 below) associated to the data L_h, $C_G(s)$, $\mathfrak{c}_h$, $\mathcal{F}_h$, ϕ_h defined by

- *$L_h := hLh^{-1} \cap C_G(s)^\circ$;*
- *$\mathfrak{c}_h := \{v \in C_G(s) \mid v$ unipotent, $h^{-1}svh \in S\}$;*
- *$\mathcal{F}_h$, inverse image of $\mathcal{E}$ under the embedding $\mathfrak{c}_h \to S$, $v \mapsto h^{-1}svh$;*

- $\phi_h : F^*\mathcal{F}_h \xrightarrow{\sim} \mathcal{F}_h$, *an isomorphism induced from* ϕ_0 *under the above embedding.*

Denote by L^1 the connected component of $N_G(L)$ containing S. If we define

- $\Sigma_h := hZ^\circ_{L,L^1}h^{-1}\mathfrak{c}_h$;
- $\mathcal{E}_h$ inverse image of $\mathcal{E}$ by the embedding $\Sigma_h \to S$, $v \mapsto h^{-1}svh$,
- $\phi'_h : F^*\mathcal{E}_h \xrightarrow{\sim} \mathcal{E}_h$ an isomorphism induced from ϕ_0 under the above embedding,

then the above embedding $\mathfrak{c}_h \to S$ factors through the inclusion $\mathfrak{c}_h \to \Sigma_h$, $\mathcal{F}_h$ is the inverse image of $\mathcal{E}_h$ and ϕ_h is induced from ϕ'_h under this inclusion. The point is that, Σ_h is a finite union of isolated strata, which has $\mathfrak{c}_h$ as the subset of unipotent elements, so that these data fit into the following definition of generalised Green functions.

IV.4.1.5 Generalised Green Functions. Given

- G a reductive algebraic group,
- $L \subset G^\circ$ an F-stable Levi subgroup,
- Σ^u the set of the unipotent elements of a finite union of isolated strata Σ of $N_G(L)$ satisfying $F(\Sigma) = \Sigma$, $F(\Sigma^u) = \Sigma^u$,
- $\mathcal{F}$ an L-equivariant local system on Σ^u, and
- $\phi_1 : F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ an isomorphism,

we choose a local system $\mathcal{E}$ on Σ that restricts to $\mathcal{F}$ under the inclusion $\Sigma^u \to \Sigma$ and an isomorphism (which always exists) $\phi'_1 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ that induces ϕ_1. Define $K = \mathbf{IC}(\tilde{Y}_{L,\Sigma}, \pi_!\tilde{\mathcal{E}})$, where $Y_{L,\Sigma}$, $\tilde{Y}_{L,\Sigma}$ and $\pi : \tilde{Y}_{L,\Sigma} \to Y_{L,\Sigma}$ are defined by the procedure §IV.4.1.2, and denote by $\phi : F^*K \xrightarrow{\sim} K$ the isomorphism induced from ϕ'_1.

The generalised Green function associated to G, L, Σ^u, $\mathcal{F}$ and ϕ_1, denoted by $Q_{L,G,\Sigma^u,\mathcal{F},\phi_1}$, is defined by ([L03, III, §15.12])

$$\begin{aligned} G^F_u = \{\text{unipotent elements of } G^F\} &\longrightarrow \bar{\mathbb{Q}}_\ell \\ u &\longmapsto \chi_{K,\phi}(u). \end{aligned} \tag{IV.4.1.5.1}$$

It does not depend on the choice of $\mathcal{E}$ and ϕ'_1.

IV.4.1.6 The isomorphism ϕ of §IV.4.1.4 induces an algebra isomorphism $\iota : \mathbf{E} \xrightarrow{\sim} \mathbf{E}$. There exists a subset $\Lambda \subset \Lambda'$ and some isomorphisms $\iota_i : V_i \xrightarrow{\sim} V_i$, $\phi_i : F^*K_i \xrightarrow{\sim} K_i$, for $i \in \Lambda$, such that the isomorphism $b_w\phi : F^*K \xrightarrow{\sim} K$, with respect to the decomposition $K = \bigoplus_{i\in\Lambda'} V_i \otimes K_i$ is of the form

- $b_w\iota_i \otimes \phi_i$, if $i \in \Lambda$;

- $V_i \otimes F^*K_i \to V_j \otimes K_j$, with $j \neq i$, if $i \notin \Lambda$.

(See [L03, IV, §20.3, §21.6]) Consequently,

(IV.4.1.6.1) $$\chi_{K,b_w\phi} = \sum_{i\in\Lambda} \mathrm{Tr}(b_w\iota_i, V_i)\chi_{K_i,\phi_i}.$$

IV.4.1.7 Take a representative $n_w \in \tilde{\mathcal{W}}_{\mathcal{E}}$ of w, and an element $g_w \in G^\circ$ such that $g_w^{-1}F(g_w) = n_w$. Define $L_w = g_wLg_w^{-1}$, $S_w = g_wSg_w^{-1}$ and $\mathcal{E}_w = \mathrm{ad}(g_w^{-1})^*\mathcal{E}$. Then L_w and S_w are F-stable and the isomorphism $\phi_0 : F^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ induces an isomorphism $\phi_{0,w} : F^*\mathcal{E}_w \xrightarrow{\sim} \mathcal{E}_w$. These allow us to define Y_{L^w,S^w}, $\tilde{Y}_{L^w,S^w}$, $\pi_w : \tilde{Y}_{L^w,S^w} \to Y_{L^w,S^w}$, $\tilde{\mathcal{E}}_w$, K_w and $\phi_w : F^*K_w \xrightarrow{\sim} K_w$ by the same procedure. It can be checked that (See [L03, IV, §21.6])

(IV.4.1.7.1) $$\chi_{K,b_w\phi} = \chi_{K_w,\phi_w}.$$

IV.4.1.8 Write $\mathcal{W} = \mathcal{W}_{\mathcal{E}}$, denote by $\bar{\mathcal{W}}$ a set of representatives of the *effective* F-conjugacy classes ([L03, IV, §20.4]), and write $\mathcal{W}_w = \{v \in \mathcal{W} \mid F^{-1}(v)wv^{-1} = w\}$. If w is not in some effective F-conjugacy class and $i \in \Lambda$, then $\mathrm{Tr}(b_w\iota_i, V_i) = 0$ ([L03, §20.4 (a)]). We have for all i, $j \in \Lambda$ ([L03, IV, §20.4 (c)])

(IV.4.1.8.1) $$\sum_{w\in\bar{\mathcal{W}}} |\mathcal{W}_w|^{-1}\,\mathrm{Tr}(b_w\iota_i, V_i)\,\mathrm{Tr}(\iota_j^{-1}b_w^{-1}, V_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. In fact, $|\bar{\mathcal{W}}| = |\Lambda|$ and $(\mathrm{Tr}(b_w\iota_i, V_i))_{i\in\Lambda, w\in\bar{\mathcal{W}}}$ is an invertible square matrix ([L03, §20.4 (e), (f), (g)]).

This, combined with the equalities (IV.4.1.6.1) and (IV.4.1.7.1), gives

(IV.4.1.8.2) $$\chi_{K_i,\phi_i} = \frac{1}{|\mathcal{W}_{\mathcal{E}}|} \sum_{\substack{w\in\mathcal{W}_{\mathcal{E}} \\ w \text{ effective}}} \mathrm{Tr}(\iota_i^{-1}b_w^{-1}, V_i)\chi_{K_w,\phi_w}.$$

IV.5 Extensions of σ-Stable Characters

IV.5.1 Some Elementary Lemmas

When dealing with a Levi subgroup L of GL_n, one often regards it as a direct product of smaller $\mathrm{GL}_{n'}$'s and reduces the problem to these direct factors. However, if σ is an automorphism of L, then $L \rtimes <\sigma>$ is not actually the direct product of groups of the form $\mathrm{GL}_{n'} \rtimes <\sigma'>$. We give some lemmas that allow us to apply arguments in the same spirit. Let H denote a finite group in this part, which could either be a finite group of Lie type or a Weyl group. Let σ be an automorphism of H, which could be induced from the automorphism of an algebraic group or the Frobenius. Denote by $H.<\sigma>$ the semi-direct product of H and the cyclic group generated by σ, with the generator acting as σ on H.

Lemma IV.5.1.1. *Let $H = H_1 \times \cdots \times H_s$ be a product of finite groups and let $\sigma = \sigma_1 \times \cdots \times \sigma_s$ be the product of some automorphisms of the direct factors. Then $H.<\sigma>$ is a subgroup of $\prod_i H_i.<\sigma_i>$. Moreover, if the characters $\chi_i \in \mathrm{Irr}(H_i)$ extend to $\tilde{\chi}_i \in \mathrm{Irr}(H_i.<\sigma_i>)$, then $\bar{\chi} := (\boxtimes_i \tilde{\chi}_i)|_{H.<\sigma>}$ is irreducible and restricts to $\boxtimes_i \chi_i \in \mathrm{Irr}(H)$.*

Proof. We define a map by

$$\begin{aligned} H.<\sigma> &\longrightarrow \prod_i H_i.<\sigma_i> \\ (h_1, \ldots, h_s)\sigma^i &\longmapsto (h_1\sigma_1^i, \ldots, h_s\sigma_s^i), \end{aligned} \tag{IV.5.1.0.1}$$

which is obviously a homomorphism and is injective. Then the assertion on $\bar{\chi}$ is immediate. □

We define an exterior tensor product that is "twisted" by σ.

Definition IV.5.1.2. Let $H = H_1 \times H_2$ be a product of finite groups and $\sigma = \sigma_1 \times \sigma_2$ a product of automorphisms. For $i = 1$ and 2, let f_i be a function on $H_i.\sigma_i$ that is invariant under the conjugation by H_i. The function $f_1 \widetilde{\boxtimes} f_2$ on $H.\sigma$ is defined as the restriction of

$$f_1 \boxtimes f_2 \ (= \mathrm{pr}_1^* f_1 \cdot \mathrm{pr}_2^* f_2) \in C(H_1.<\sigma_1> \times H_2.<\sigma_2>)$$

to $H.<\sigma>$.

Lemma IV.5.1.3. *Let $H = K \times \cdots \times K$ be the direct product of d copies of a finite group K. Let ψ be an automorphism of K, let $\zeta \in \mathfrak{S}_d$ be a circular permutation, and let $(n_1, \ldots, n_d)$ be a d-tuple of integers. With these data, we can define an automorphism Ψ of H by*

$$\begin{aligned} \Psi : H &\longrightarrow H \\ (k_1, \ldots, k_d) &\longmapsto (\psi^{n_1}(k_{\zeta(1)}), \ldots, \psi^{n_d}(k_{\zeta(d)})). \end{aligned} \tag{IV.5.1.0.2}$$

Denote by $\mathcal{H}$ the direct product $K \rtimes <\psi> \times \cdots \times K \rtimes <\psi>$, and let ζ act by permuting the components:

$$\zeta : (k_1, \ldots, k_d) \longmapsto (k_{\zeta(1)}, \ldots, k_{\zeta(d)}), \quad k_i \in K \rtimes <\psi>,$$

Let χ be a ψ-stable irreducible character of K and denote by $\tilde{\chi}$ an extension of χ to $K \rtimes <\psi>$. Then,

(i) $H \rtimes <\Psi>$ is a subgroup of $\mathcal{H}$;

(ii) The character $\tilde{\chi} \otimes \cdots \otimes \tilde{\chi}$ of $\mathcal{H}$ extends to a character of $\mathcal{H} \rtimes <\zeta>$. Its restriction $\bar{\chi}$ to $H \rtimes <\Psi>$ is irreducible;

(iii) Write $\zeta = (i_1, \ldots, i_d)$ as a permutation of $\{1, \ldots, d\}$, so that $i_r \overset{\zeta}{\mapsto} i_{r+1}$. For all $h = (k_1, \ldots, k_d) \in H$, we have

$$\bar{\chi}(h\Psi) = \tilde{\chi}(k_{i_1}\psi^{n_{i_1}} k_{i_2}^{n_{i_2}} \psi \cdots k_{i_d}\psi^{n_{i_d}}).$$

Proof. For each $i \in \{1, \dots, d\}$ and $r \in \mathbb{Z}_{>0}$, put $n_i(r) = \sum_{0 \le p \le r-1} n_{\zeta^p(i)}$. Define a map

$$\begin{aligned} H.<\Psi> &\longrightarrow K.<\psi> \times \cdots \times K.<\psi>).<\zeta> \\ (k_1, \dots, k_d) &\longmapsto (k_1, \dots, k_d) \\ (k_1, \dots, k_d)\Psi^r &\longmapsto (k_1\psi^{n_1(r)}, \dots, k_d\psi^{n_d(r)})\zeta^r. \end{aligned} \tag{IV.5.1.0.3}$$

One can verify that it is an injective group homomorphism. The character $\bar{\chi}$ is irreducible as its restriction to H is so.

Let us compute the value of $\bar{\chi}$. Let $\rho : K \to \mathrm{GL}(V)$ be a representation that realises the character χ. Let $\tilde{\rho}$ denote its extension to $K.<\psi>$. Then $V^{\otimes d}$ is a representation of $\mathcal{H}$, defining the action of ζ by $v_1 \otimes \cdots \otimes v_d \mapsto v_{\zeta(1)} \otimes \cdots \otimes v_{\zeta(d)}$. We use an argument of linear algebra. Take $A^{(1)}, \dots, A^{(d)} \in \mathrm{GL}(V)$ and let ζ act on $V^{\otimes d}$ as above. Then we have

$$\mathrm{Tr}(A^{(1)} \otimes \cdots \otimes A^{(d)} \circ \zeta | V \otimes \cdots \otimes V) = \mathrm{Tr}(A^{(i_1)} \cdots A^{(i_d)} | V). \tag{IV.5.1.0.4}$$

We conclude the proof by taking $\tilde{\rho}(k_i\psi^{n_i})$ for $A^{(i)}$. □

IV.5.2 Uniform Extensions

IV.5.2.1 Denote by L the algebraic group defined over $\mathbb{F}_q$

$$L = \prod_{i \in \Lambda_1} (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i})^{d_i} \times \prod_{i \in \Lambda_2} (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i})^{d_i} \tag{IV.5.2.1.1}$$

for some finite sets Λ_1 and Λ_2, endowed with the Frobenius F acting on it as F_w in (IV.1.2.3.6) and (IV.1.2.3.7), and with the automorphism σ acting on it as in (IV.1.2.3.9). Let $T \subset L$ be an F-stable and σ-stable maximal torus and we wirte $W_L := W_L(T)$. For all $i \in \Lambda_1 \cup \Lambda_2$, we write $L_i := (\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i})^{d_i}$, and denote by T_i the corresponding direct factor of T. Write $W_i := W_{L_i}(T_i)$. Then we have

$$W_i \cong (\mathfrak{S}_{n_i} \times \mathfrak{S}_{n_i})^{d_i}, \quad W_i^\sigma \cong \mathfrak{S}_{n_i}^{d_i} \tag{IV.5.2.1.2}$$

for all $i \in \Lambda_1 \cup \Lambda_2$. These are some direct factors of W_L and of W_L^σ that are stable under F.

Define an injection

$$\begin{aligned} \mathrm{Irr}(W_L^\sigma)^F &\hookrightarrow \mathrm{Irr}(W_L)^F \\ \varphi &\longmapsto [\varphi] \end{aligned} \tag{IV.5.2.1.3}$$

in the following manner.

For each $i \in \Lambda_1$, we have the bijections

$$\mathrm{Irr}(W_i)^F \cong \mathcal{P}_{n_i} \times \mathcal{P}_{n_i}, \quad \mathrm{Irr}(W_i^\sigma)^F \cong \mathcal{P}_{n_i}. \tag{IV.5.2.1.4}$$

We define $\mathrm{Irr}(W_i^\sigma)^F \to \mathrm{Irr}(W_i)^F$ to be sending φ to (φ, φ).

For each $i \in \Lambda_2$, we have

(IV.5.2.1.5) $$\mathrm{Irr}(W_i)^F \cong \mathcal{P}_{n_i}, \quad \mathrm{Irr}(W_i^\sigma)^F \cong \mathcal{P}_{n_i}.$$

We define $\mathrm{Irr}(W_i^\sigma)^F \to \mathrm{Irr}(W_i)^F$ to be sending φ to φ.

For any $\varphi \in \mathrm{Irr}(W_L^\sigma)^F$, we denote by $\tilde{\varphi}$ an extension of φ to $W_L^\sigma \rtimes <F>$ and denote by $\widetilde{[\varphi]}$ an extension of $[\varphi]$ to $W_L \rtimes <F>$, where F is regarded as an automorphism of finite order of the Weyl group. Eventually, these choices need to be specified.

Denote by $\mathrm{Irr}^\sigma(L^F)$ the set of the σ-stable linear characters of L^F. For any θ in $\mathrm{Irr}^\sigma(L^F)$, we denote by $\tilde{\theta}$ its trivial extension to $L^F.<\sigma>$. We also denote by the same letter the restriction of $\tilde{\theta}$ to $L^F.\sigma$.

IV.5.2.2 Given $\varphi \in \mathrm{Irr}(W_L^\sigma)^F$ and $\theta \in \mathrm{Irr}^\sigma(L^F)$, by Theorem 5, there is a particular choice of the extension $\widetilde{[\varphi]}$ such that

(IV.5.2.2.1) $$\chi_1 := R_{[\varphi]}^L \theta = |W_L|^{-1} \sum_{w \in W_L} \widetilde{[\varphi]}(wF) R_{T_w}^L \theta.$$

is a character. Obviously, it is a σ-stable character of L^F. Denote by $\tilde{\chi}_1$ an extension of χ_1 to $L^F.<\sigma>$.

For any choice of the extension $\tilde{\varphi}$, put

(IV.5.2.2.2) $$R_\varphi^{L.\sigma} \tilde{\theta} = |W_L^\sigma|^{-1} \sum_{w \in W_L^\sigma} \tilde{\varphi}(wF) R_{T_w.\sigma}^{L.\sigma} \tilde{\theta}.$$

It is an L^F-invariant function on $L^F.\sigma$.

Theorem IV.5.2.1. *For a particular choice of the extension $\tilde{\varphi}$, we have*

(IV.5.2.2.3) $$\tilde{\chi}_1|_{L^F.\sigma} = \pm R_\varphi^{L.\sigma} \tilde{\theta}.$$

We will prove this theorem in the following section.

IV.5.3 The Proof

The proof is to reduce the problem to smaller and smaller factors of L, until we can apply the known results on $\mathrm{GL}_{n'}^{\pm}(q)$, for various n'. The choice of the extension $\tilde{\varphi}$ will also be reduced to the smaller components until the choices are clear.

IV.5.3.1 Reduction to the Unipotent Characters Let $\chi_1 = R_{[\varphi]}^L 1$ be an irreducible character of L^F, which is necessarily σ-stable. Denote by $\tilde{\chi}_1 \in \mathrm{Irr}(L^F.<\sigma>)$ such that $\tilde{\chi}_1|_{L^F} = R_{[\varphi]}^L 1$. Assume that for some choice of $\tilde{\varphi}$,

(IV.5.3.1.1) $$\tilde{\chi}_1|_{L^F.\sigma} = R_\varphi^{L.\sigma} 1,$$

where we denote by 1 the trivial extension of the trivial character. Since $(R^L_{[\varphi]}1)\otimes\theta = R^L_{[\varphi]}\theta$ and $(R^{L.\sigma}_{\varphi})1\otimes\tilde{\theta} = R^{L.\sigma}_{\varphi}\tilde{\theta}$, we have

$$(\tilde{\chi}_1\otimes\tilde{\theta})|_{L^F} = R^L_{[\varphi]}\theta, \quad (\tilde{\chi}_1\otimes\tilde{\theta})|_{L^F.\sigma} = R^{L.\sigma}_{\varphi}\tilde{\theta}. \tag{IV.5.3.1.2}$$

So it suffices to prove the theorem for the unipotent characters.

IV.5.3.2 Reduction with Respect to the Action of F and σ We have decomposed L into a product of the L_i's for $i\in\Lambda_1\sqcup\Lambda_2$, each one being F-stable and σ-stable. Let us show that it suffices to prove the theorem for the L_i's.

Write $F = (F_i)_{i\in\Lambda_1\sqcup\Lambda_2}$ and $\sigma = (\sigma_i)_{i\in\Lambda_1\sqcup\Lambda_2}$, where for each i, F_i and σ_i are respectively a Frobenius and an automorphism of the corresponding direct factor. The given φ can be written as $(\varphi_i)_{i\in\Lambda_1\sqcup\Lambda_2}$, with $\varphi_i\in\mathrm{Irr}(W_i^{\sigma_i})^{F_i}$, then $[\varphi] = ([\varphi_i])_i$. Suppose that $R^{L_i}_{[\varphi_i]}1$ are some irreducible characters, denoted by χ_i, each one being σ-stable, and they extend to $\tilde{\chi}_i\in\mathrm{Irr}(L_i^{F_i}.<\sigma_i>)$. We will show that if for some choices of the extensions $\tilde{\varphi}_i\in\mathrm{Irr}(W_i^{\sigma_i}.<F_i>)$, the following equality holds

$$\tilde{\chi}_i|_{L_i^{F_i}.\sigma_i} = R^{L_i}_{\varphi_i}1,$$

then there is some choice of $\tilde{\varphi}\in\mathrm{Irr}(W_L^{\sigma}.<F>)$ such that

$$\tilde{\chi}_1|_{L^F.\sigma} = R^{L.\sigma}_{\varphi}1.$$

Given the extensions of the factors $\tilde{\chi}_i$, we can obtain an extension $\tilde{\chi}_1$ following Lemma IV.5.1.1. By definition, for any $l\sigma\in L^F.\sigma$ which can be identified with $\prod_i l_i\sigma_i\in\prod_i L_i^{F_i}.<\sigma_i>$, we have,

$$\tilde{\chi}_1(l\sigma) = \prod_i R^{L_i.\sigma_i}_{\varphi_i}1(l_i\sigma_i). \tag{IV.5.3.2.1}$$

On the one hand,

$$\begin{aligned}\prod_i R^{L_i.\sigma_i}_{T_{w_i}.\sigma_i}1(l_i\sigma_i) &= \prod_i \mathrm{Tr}(l_i\sigma_i|H^*_c(X_{w_i}))\\ &= \mathrm{Tr}(\prod_i l_i\sigma_i|\bigotimes_i H^*_c(X_{w_i}))\\ &= \mathrm{Tr}(l\sigma|H^*_c(X_w)) = R^{L.\sigma}_{T_w.\sigma}1(l\sigma).\end{aligned} \tag{IV.5.3.2.2}$$

where the T_{w_i} and T_w are defined with respect to T_i and T.

On the other hand, applying Lemma IV.5.1.1 to the Weyl groups and the Frobenius, we obtain an extension $\tilde{\varphi}$, such that for any $w = \prod_i w_i\in W_L^{\sigma}$, we have,

$$\tilde{\varphi}(wF) = (\prod_i\tilde{\varphi}_i)|_{W_L^{\sigma}.F}(\prod_i w_iF_i) = \prod_i\tilde{\varphi}_i(w_iF_i). \tag{IV.5.3.2.3}$$

Consequently, $\tilde{\chi}_1|_{L^F.\sigma} = R_\varphi^{L.\sigma}1$. (One may also check that $\boxtimes_i R_{[\varphi_i]}^{L_i}1 = R_{[\varphi]}^{L}1$ with some similar but simpler arguments.)

IV.5.3.3 The Linear Part, I In this part we fix $i \in \Lambda_1$. Write $M = \mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i}$, equipped with the Frobenius

$$F_M : (g, h) \mapsto (F_0(g), F_0(h))$$

with F_0 being the standard Frobenius of GL_n and the automorphism

$$\tau(g, h) = (\sigma_0(h), \sigma_0(g))$$

of the form §II.5.1.3, with σ_0 commuting with F_0. Then $L_i = M \times \cdots \times M$ is a direct product of d_i copies of M equipped with the automorphism $\sigma = \tau \times \cdots \times \tau$ consisting of d_i copies of τ. We will fix i and write $d = d_i$. We may assume that the maximal torus $T_i \subset L_i$, which is F_i-stable and σ_i-stable, is of the form $T_M \times \cdots \times T_M$, where $T_M \subset M$ is an F_M-stable and τ-stable maximal torus. Note that F_M acts trivially on $W_M := W_M(T_M)$. The Frobenius F_i acts on L_i in the following manner

$$\begin{aligned} M \times \cdots \times M &\longrightarrow M \times \cdots \times M \\ (m_1, \ldots, m_d) &\longmapsto (F_M(m_d), F_M(m_1), \ldots, F_M(m_{d-1})). \end{aligned} \tag{IV.5.3.3.1}$$

We have a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}(W_M^\tau) & \longrightarrow & \mathrm{Irr}(W_i^{\sigma_i})^{F_i} \\ \downarrow & & \downarrow \\ \mathrm{Irr}(W_M) & \longrightarrow & \mathrm{Irr}(W_i)^{F_i} \end{array} \tag{IV.5.3.3.2}$$

where the upper horizontal bijective map $\varphi_M \longmapsto \varphi_i = (\varphi_M, \ldots, \varphi_M)$ identifies each element of $\mathrm{Irr}(W_i^{\sigma_i})^{F_i}$ with d identical copies of an element of $\mathrm{Irr}(W_M^\tau)$. Denote by $[\varphi_M]$ and $[\varphi_i]$ the images of the vertical maps defined as in (IV.5.2.1.3), which are matched under the lower horizontal map.

Endow M with the Frobenius F_M^d. Suppose that for some choice of $\tilde{\varphi}_M$,

$$R_{\varphi_M}^{M.\tau}1 = |W_M^\tau|^{-1} \sum_{w \in W_M^\tau} \tilde{\varphi}_M(wF_M^d) R_{T_w.\tau}^{M.\tau}1 \tag{IV.5.3.3.3}$$

where T_w is defined with respect to T_M, is an extension of the irreducible character $R_{[\varphi_M]}^{M}1$ of $M^{F_M^d}$. Let us show that

$$R_{\varphi_i}^{L_i.\sigma_i}1 = |W_i^{\sigma_i}|^{-1} \sum_{w \in W_i^{\sigma_i}} \tilde{\varphi}_i(wF_i) R_{T_w.\sigma}^{L_i.\sigma_i}1 \tag{IV.5.3.3.4}$$

is an extension of the irreducible character $R^{L_i}_{[\varphi_i]}1$ of $L_i^{F_i}$. In fact, there is a natural isomorphism $M^{F_M^d} \cong L_i^{F_i}$ compatible with the action of τ and σ. We are going to show that $R^{M.\tau}_{\varphi}1$ coincides with $R^{L_i.\sigma_i}_{\varphi_L}1$ under this isomorphism and they are an extension of the same character of $M^{F_M^d} = L_i^{F_i}$ corresponding to $[\varphi] = [\varphi_L]$.

Applying Lemma IV.5.1.3 to $K = W_M^\tau$, $H = W_i^{\sigma_i}$, $\psi = F_M$, and $\zeta = (d, \dots, 2, 1) \in \mathfrak{S}_d$, we deduce from $\tilde{\varphi}_M$ an extension $\tilde{\varphi}_i$ such that for $w = (w_1, \dots, w_d) \in (W_M^\tau)^d \cong W_i^{\sigma_i}$,

$$\begin{aligned}\tilde{\varphi}_i(wF_i) &= \tilde{\varphi}_M(w_d F_M w_{d-1} F_M \dots w_1 F_M) \\ &= \tilde{\varphi}_M(w_d w_{d-1} \dots w_1 F_M^d).\end{aligned} \tag{IV.5.3.3.5}$$

Write $w_1' = w_d w_{d-1} \dots w_1$. Then, w is $F_{1,i}$-conjugate to $w' = (w_1', 1, \dots, 1)$, so for any $l \in L_i^{F_i}$, we have

$$R^{L_i.\sigma_i}_{T_w.\sigma_i}1(l\sigma_i) = R^{L_i.\sigma_i}_{T_{w'}.\sigma_i}1(l\sigma_i) = \mathrm{Tr}(l\sigma_i | H_c^*(X_{w'}))$$

We can write $l = (m, F_M(m), \dots, F_M^{d-1}(m))$ with m satisfying $F_M^d(m) = m$. Since the two varieties

$$\begin{aligned}X_{w'} &= \{B \in \mathcal{B}_{L_i} | (B, F_i(B)) \in \mathcal{O}(w')\}, \\ X_{w_1'} &= \{B \in \mathcal{B}_M | (B, F_M^d(B)) \in \mathcal{O}(w_1')\}\end{aligned}$$

are isomorphic, and the actions of $l\sigma_i$ and of $m\tau$ on the two varieties are compatible, we have

$$\mathrm{Tr}(l\sigma_i | H_c^*(X_{w'})) = \mathrm{Tr}(m\tau | H_c^*(X_{w_1'})) = R^{M.\tau}_{T_{w_1'}.\tau}1(m\tau),$$

Consequently, the value of $\tilde{\varphi}_i(wF_i)R^{L_i.\sigma_i}_{T_w.\sigma_i}1$ only depends on $w_1' \in W_M^\tau$ and is equal to $\tilde{\varphi}_M(w_1' F_M^d)R^{M.\tau}_{T_{w_1'}.\tau}1$, This, together with the fact that $|W_i^{\sigma_i}| = |W_M^\tau|^d$, shows that $R^{M.\tau}_{\varphi}1 = R^{L_i.\sigma_i}_{\varphi_L}1$. (Similar arguments show that $R^M_{[\varphi]}1 = R^{L_i}_{[\varphi_i]}1$.)

IV.5.3.4 The Unitary Part, I In this part we require that $i \in \Lambda_2$. We keep the same notations as above except that F_i acts on L_i in the following manner

$$\begin{aligned}M \times \dots \times M &\longrightarrow M \times \dots \times M \\ (m_1, \dots, m_d) &\longmapsto (F_M'(m_d), F_M(m_1), \dots, F_M(m_{d-1})),\end{aligned} \tag{IV.5.3.4.1}$$

where

$$F_M' : (g, h) \mapsto (F_0(h), F_0(g)).$$

Denote by $F_M'' : M \to M$ the Frobenius

$$F_M' F_M^{d-1} : (g, h) \mapsto (F_0^d(h), F_0^d(g)).$$

We still have a natural identification

$$\begin{aligned} \mathrm{Irr}(W_M^\tau)^{F''_M} &\longrightarrow \mathrm{Irr}(W_i^{\sigma_i})^{F_i} \\ \varphi &\longmapsto \varphi_L = (\varphi, \ldots, \varphi) \end{aligned} \tag{IV.5.3.4.2}$$

(F'_M acts trivially on $\mathrm{Irr}(W_M^\tau)$ since σ_0 induces an inner automorphism on the Weyl group) and an isomorphism $M^{F''_M} \cong L_i^{F_i}$. In a way similar to §IV.5.3.3, from the equality (IV.5.3.3.3), with F_M^d replaced by F''_M, one deduces the equality (IV.5.3.3.4), with F_i defined in the present setting.

IV.5.3.5 From now on we write $M = G \times G$, $G = \mathrm{GL}_n^\epsilon(q)$ with $\epsilon = \pm$. Denote by F_0 the split Frobenius of G and by F'_0 the Frobenius of G corresponding to ϵ, and denote by σ_0 an order 2 automorphism of G, which commutes with the Frobenius endomorphisms.

IV.5.3.6 The Linear Part, II. It is essential that we allow G to be $\mathrm{GL}_n^-(q)$, which will be applied to the unitary part later. Define

$$\begin{aligned} \sigma : M &\longrightarrow M & F : M &\longrightarrow M \\ (g,h) &\longmapsto (\sigma_0(h), \sigma_0(g)) & (g,h) &\longmapsto (F'_0(g), F'_0(h)). \end{aligned}$$

Let χ_G be a unipotent irreducible character of $G^{F'_0}$ corresponding to some $\varphi \in \mathrm{Irr}(W_G)$, it defines a character $\chi_M = \chi_G \otimes \chi_G \in \mathrm{Irr}(M^F)$ which is invariant under the action of σ and so extends to $M^F.<\sigma>$, denoted by $\tilde{\chi}_M$. Every σ-stable irreducible unipotent character of M^F is of the form $\chi_G \otimes \chi_G$. Regarding φ as a character of W_M^σ, we show that up to a sign,

$$\tilde{\chi}_M|_{M^F.\sigma} = R_\varphi^{M.\sigma} 1 := |W_M^\sigma|^{-1} \sum_{w \in W_M^\sigma} \tilde{\varphi}(wF) R_{T_w.\sigma}^{M.\sigma} 1, \tag{IV.5.3.6.1}$$

for some choice of the extension $\tilde{\varphi}$.

We apply Lemma IV.5.1.3 by taking $G^{F'_0}$ for K and obtain

$$\tilde{\chi}_M((g,h)\sigma) = \chi_G(g\sigma_0(h))$$

for any $(g,h) \in M^F$. By Theorem 5, the irreducible unipotent character of $G^{F'_0}$ can be expressed as

$$|W_G|^{-1} \sum_{w \in W_G} \tilde{\varphi}_G(wF'_0) R_{T_w}^G 1,$$

for some choice of $\tilde{\varphi}_G$. The extension $\tilde{\varphi}$ is then defined by $\tilde{\varphi}_G$ under the isomorphism $W_G \cong W_M^\sigma$, noticing that the action of F is compatible with the action of F'_0 under this isomorphism. Comparing this expression with $R_\varphi^{M.\sigma} 1$, we are reduced to show that for any

$(g,h) \in M^F$

(IV.5.3.6.2) $$R^{M.\sigma}_{T_{w_M}.\sigma}1((g,h)\sigma) = R^G_{T_w}1(g\sigma_0(h))$$

with $w_M = (w, \sigma_0(w)) \in W^\sigma_M$. Observe that $(g,h)\sigma \mapsto R^G_{T_w}1(g\sigma_0(h))$ defines a function on $M^F.\sigma$ invariant under the conjugation by M^F.

IV.5.3.7 Let us prove a more general assertion. Let I be an F'_0-stable Levi subgroup of G. Then $J := I \times \sigma_0(I)$ is a σ-stable Levi factor of a σ-stable parabolic subgroup of M, which justifies the functor $R^{M.<\sigma>}_{J.<\sigma>}$. Let χ_I be an irreducible character of $I^{F'_0}$, it defines a character $\chi_J = \chi_I \otimes \sigma_0(\chi_I) \in \mathrm{Irr}(J^F)$ which is invariant under the action of σ and thus extends to $J^F.<\sigma>$. Denoted by $\tilde{\chi}_J$ a choice of such extension. The following lemma with $I = T_w$ and $\chi_I = 1$ proves the above assertion.

Lemma IV.5.3.1. *We keep the notations as above and write $\bar{J} = J.<\sigma>$. Assume that for any $(g',h') \in J^F$, we have $\tilde{\chi}_J((g',h')\sigma) = \chi_I(g'\sigma_0(h'))$. Then, for any $(g,h) \in M^F$, we have*

(IV.5.3.7.1) $$R^{M.<\sigma>}_{J.<\sigma>}\tilde{\chi}_J((g,h)\sigma) = R^G_I\chi_I(g\sigma_0(h)).$$

Proof. Let $(g,h)\sigma = \zeta\mu$ be the Jordan decomposition, with ζ semi-simple and μ unipotent, and we write $\mu = (u,w)$ and $\zeta = (s,t)\sigma$. Beware that neither s nor t is necessarily semi-simple. Also let $g\sigma_0(h) = \bar{s}\bar{u}$ be the Jordan decomposition, with $\bar{s}$ semi-simple and $\bar{u}$ unipotent. The proof will simply be comparing the following two formulas term by term.

$$R^{M.<\sigma>}_{J.<\sigma>}\tilde{\chi}_J((g,h)\sigma) = |J^F|^{-1}|C_M(\zeta)^{\circ F}|^{-1} \sum_{\substack{\{x\in M^F|\\ x\zeta x^{-1}\in J.\sigma\}}} \sum_{v\in C_{x^{-1}Jx}(\zeta)^F_u} Q^{C_M(\zeta)^\circ}_{C_{x^{-1}Jx}(\zeta)^\circ}(\mu, v^{-1})^x\tilde{\chi}_J(\zeta v)$$

$$R^G_I\chi_I(g\sigma_0(h)) = |I^{F'_0}|^{-1}|C_G(\bar{s})^{\circ F'_0}|^{-1} \sum_{\substack{\{y\in G^{F'_0}|\\ y\bar{s}y^{-1}\in I\}}} \sum_{v_1\in C_{y^{-1}Iy}(\bar{s})^{\circ F'_0}_u} Q^{C_G(\bar{s})^\circ}_{C_{y^{-1}Iy}(\bar{s})^\circ}(\bar{u}, v_1^{-1})^y\chi_I(\bar{s}v_1).$$

An element $(z_1,z_2) \in M$ commutes with $(s,t)\sigma$ if and only if $z_2 = t\sigma_0(z_1)t^{-1}$ and $z_1 = s\sigma_0(z_2)s^{-1}$, if and only if $z_1 \in C_G(s\sigma_0(t))$ and $z_2 = t\sigma_0(z_1)t^{-1}$, whence an isomorphism $C_M((s,t)\sigma) \cong C_G(s\sigma_0(t))$. An element of $M.<\sigma>$ is semi-simple if and only if its square is semi-simple since the characteristic is odd. The equality $((s,t)\sigma)^2 = (s\sigma_0(t), t\sigma_0(s))$ shows that $s\sigma_0(t)$ is semi-simple. The unipotent part (u,w) commutes with $(s,t)\sigma$, so $u \in C_G(s\sigma_0(t))$. Considering the equality

$$(g\sigma_0(h), h\sigma_0(g)) = ((g,h)\sigma)^2 = ((s,t)\sigma)^2(u,w)^2 = (s\sigma_0(t), t\sigma_(s))(u^2,w^2),$$

we have $g\sigma_0(h) = s\sigma_0(t)u^2$. This gives the Jordan decomposition of $g\sigma_0(h)$ because u commutes with $s\sigma_0(t)$ and u^2 is unipotent. Therefore, $\bar{s} = s\sigma_0(t)$, $\bar{u} = u^2$ and $|C_M(\zeta)^{\circ F}| = |C_G(\bar{s})^{\circ F'_0}|$.

Write $x = (x_1,x_2) \in M^F$. The condition $x\zeta x^{-1} \in J.\sigma$ means that $x_1 s\sigma_0(x_2^{-1}) \in I$ and that $x_2 t\sigma_0(x_1^{-1}) \in \sigma_0(I)$, which implies that $x_1 s\sigma_0(t)x_1^{-1} \in I$. Fix (x_1,x_2) satisfying $x\zeta x^{-1} \in J.\sigma$,

then every element of $(x_1, \sigma_0(I^{F'_0})x_2)$ also satisfies it. Let (x_1, x_2) and (x_1, x'_2) be two elements satisfying $x\zeta x^{-1} \in J.\sigma$, then the conditions $x_1 s\sigma_0(x_2^{-1}) \in I$ and $x_1 s\sigma_0(x_2'^{-1}) \in I$ implies that $x'_2 x_2^{-1} \in \sigma_0(I^{F'_0})$. We thus obtain a bijection of sets

$$\{x \in M^F | x\zeta x^{-1} \in J.\sigma\} \cong \{y \in G^{F'_0} | y\bar{s}y^{-1} \in I\} \times I^{F'_0},$$

with y corresponding to the factor x_1 of x. We will see that the sum over x in the character formula is invariant under multiplying x_2 by an element of $\sigma_0(I^{F'_0})$ on the left, which cancels a factor $|I^{F'_0}|$ from $|J^F| = |I^{F'_0}|^2$.

The isomorphism $C_M(\zeta)^\circ \cong C_G(\bar{s})^\circ$ restricts to an isomorphism $C_{x^{-1}Jx}(\zeta)^\circ \cong C_{y^{-1}Iy}(\bar{s})^\circ$. For characteristic reason, the unipotent elements of $C_{x^{-1}Jx}(\zeta)$ are contained in $C_{x^{-1}Jx}(\zeta)^\circ$ by [DM94, Remarque 2.7] and [DM94, Théorème 1.8 (i)], whence a bijection $C_{x^{-1}Jx}(\zeta)_u^{\circ F} \cong C_{y^{-1}Iy}(s\sigma_0(t))_u^{\circ F'_0}$, by which $v = (v_1, v_2)$ is sent to v_1.

Now we compare the characters ${}^x\tilde{\chi}_J(\zeta v)$ et ${}^y\chi_I(\bar{s}v_1)$. Write $x = (x_1, x_2)$ and $v = (v_1, v_2)$. Then

$$x\zeta v x^{-1} = (x_1 s\sigma_0(v_2)\sigma_0(x_2)^{-1}, x_2 t\sigma_0(v_1)\sigma_0(x_1)^{-1})\sigma. \tag{IV.5.3.7.2}$$

Taking into account the equality $v_2 = t\sigma_0(v_1)t^{-1}$, we have

$$\Big(x_1 s\sigma_0(v_2)\sigma_0(x_2)^{-1}\Big)\sigma_0\Big(x_2 t\sigma_0(v_1)\sigma_0(x_1)^{-1})\Big) = x_1 s\sigma_0(t)v_1^2 x_1^{-1}. \tag{IV.5.3.7.3}$$

By assumption, $\tilde{\chi}_J((g', h')\sigma) = \chi_I(g'\sigma_0(h'))$, for any $(g', h') \in J^F$, whence

$${}^x\tilde{\chi}_J(\zeta v) = \chi_I(x_1 s\sigma_0(t)v_1^2 x_1^{-1}) = {}^{x_1}\tilde{\chi}_I(\bar{s}v_1^2), \tag{IV.5.3.7.4}$$

where we also see that multiplying x_2 by an element of $\sigma_0(I^{F_0})$ on the left does not change the value. Since $v \mapsto v^2$ defines a bijection of $C_{y^{-1}Iy}(\bar{s})_u^{\circ F'_0}$ into itself, it only remains to show the first of the following equalities of Green functions

$$Q^{C_G(\bar{s})^\circ}_{C_{y^{-1}Iy}(\bar{s})^\circ}(u^2, v_1^{-2}) = Q^{C_G(\bar{s})^\circ}_{C_{y^{-1}Iy}(\bar{s})^\circ}(u, v_1^{-1}) = Q^{C_M(\zeta)^\circ}_{C_{x^{-1}Jx}(\zeta)^\circ}(\mu, v^{-1}), \tag{IV.5.3.7.5}$$

which follows from the fact that the value of the Green function only depends on the associated partition and a power prime to p does not change the Jordan blocks of a unipotent matrix. □

IV.5.3.8 The Unitary Part, II Define

$$\begin{array}{ll} \sigma : M \longrightarrow M & F : M \longrightarrow M \\ (g, h) \longmapsto (\sigma_0(h), \sigma_0(g)) & (g, h) \longmapsto (F_0(h), F_0(g)). \end{array}$$

Now, M^F is isomorphic to $G^{F_0^2}$ under the map $(g, F_0(g)) \mapsto g$, and M^σ is isomorphic to G under the map $(g, \sigma_0(g)) \mapsto g$. The Frobenius F acts on $M^\sigma \cong G$ by $g \mapsto \sigma_0 F_0(g)$. The automorphism

σ acts on M^F by

$$(g, F_0(g)) \mapsto (\sigma_0 F_0(g), \sigma_0(g)) = (\sigma_0 F_0(g), \sigma_0 F_0^2(g)),$$

or in other words, σ acts on $G^{F_0^2}$ as $\sigma_0 F_0$.

Let χ_M be a unipotent irreducible character of M^F, and let $\tilde{\chi}_M$ be an irreducible character of $M^F.<\sigma>$that extends χ_M. We are going to show that up to a sign,

$$\tilde{\chi}_M|_{M^F.\sigma} = R_\varphi^{M.\sigma} 1 = |W_M^\sigma|^{-1} \sum_{w \in W_M^\sigma} \tilde{\varphi}(wF) R_{T_w.\sigma}^{M.\sigma} 1 \tag{IV.5.3.8.1}$$

for some choice of $\tilde{\varphi}$. Under the isomorphism $W_M^\sigma \cong W_G$, the Frobenius F acts as σ_0 on W_G, and so an F-stable character of W_M^σ is equivalent to a character of W_G. We are reduced to show that if χ_G [1] is an irreducible unipotent character of $G^{F_0^2}$ corresponding to $\varphi \in \mathrm{Irr}(W_G)$, then its extension $\tilde{\chi}_G$ to $G^{F_0^2}.<\sigma_0 F_0>$is given by the above formula up to a sign.

We need the Shintani descent. Suppose that $\chi_G \in \mathrm{Irr}(G^{F_0^2})^{\sigma_0 F_0}$ is the unipotent character corresponding to $\varphi \in \mathrm{Irr}(W_G)^{\sigma_0}$. We apply Theorem IV.3.3.1 with $(\sigma_1 F_0^m, \sigma_2 F_0) = (F_0^2, \sigma_0 F_0)$, i.e. $m = 2$, $\sigma_1 = 1$ and $\sigma_2 = \sigma_0$, and deduce that

$$\begin{aligned} \tilde{\chi}_G &= E_{\sigma_0 F_0}(\chi_G) = \mathrm{Sh}_{\sigma_0 F_0 / F_0^2}\, \Omega_{\sigma_0 F_0}^2 R_\varphi^{G^{\sigma_0 F_0}} 1 \\ &= \pm\, \mathrm{Sh}_{\sigma_0 F_0 / F_0^2}\, R_\varphi^{G^{\sigma_0 F_0}} 1 = \pm\, \mathrm{Sh}_{\sigma_0 F_0 / F_0^2}\, |W_G|^{-1} \sum_{w \in W_G} \tilde{\varphi}(w \sigma_0 F_0) R_{T_w}^{G^{\sigma_0 F_0}} 1, \end{aligned} \tag{IV.5.3.8.2}$$

since $\Omega_{\sigma_0 F_0}^2 = \pm 1$ because $R_\varphi^{G^{\sigma_0 F_0}} 1$ is an irreducible unipotent character $G^{\sigma_0 F_0}$ on which $\Omega_{\sigma_0 F_0}^2$ acts as a scalar, whose value is given by §IV.3.1.6. For example, $\Omega_{\sigma_0 F_0} = 1$ on principal series representations and $\Omega_{\sigma_0 F_0}^2 = -1$ on cuspidal unipotent characters according to ([L77, Table I]). The sign (± 1) does not matter since the two extensions of χ_G only differ by a sign. It remains to show that $\mathrm{Sh}_{\sigma_0 F_0 / F_0^2}\, R_{T_w}^{G^{\sigma_0 F_0}} 1 = R_{T_{w_M}.\sigma}^{M.\sigma} 1$, where w_M is as in (IV.5.3.6.2) and M is equipped with the Frobenius F. The function $R_{T_w}^{G^{\sigma_0 F_0}} 1$ is invariant under F_0^2-conjugation as F_0^2 acts trivially on $G^{\sigma_0 F_0}$, which justifies $\mathrm{Sh}_{\sigma_0 F_0 / F_0^2} \circ R_{T_w}^{G^{\sigma_0 F_0}} 1$.

Proposition IV.3.4.1 gives

$$\mathrm{Sh}_{\sigma F / F} \circ R_{T_{w_M}.\sigma}^{M^{\sigma F}.\sigma} 1 = R_{T_{w_M}.\sigma}^{M^F.\sigma} 1. \tag{IV.5.3.8.3}$$

(One checks that with respect to a fixed F-stable and σ-stable maximal torus $T \subset M$, the maximal torus T_{w_M} of type w_M with respect to F is also of type w_M with respect to σF, using the fact that for σ quasi-central, w_M has a representative in M^σ.) Since F acts as σ on $M^{\sigma F}$, the function $R_{T_{w_M}.\sigma}^{M^{\sigma F}.\sigma} 1$ is invariant under the F-conjugation of $M^{\sigma F}$, and its Shintani descent $\mathrm{Sh}_{\sigma F / F} \circ R_{T_{w_M}.\sigma}^{M^{\sigma F}.\sigma} 1$ belongs to $\mathcal{C}(M^F.\sigma F)$. There is a natural bijection $\mathcal{C}(M^F.\sigma F) \leftrightarrows \mathcal{C}(G^{F_0^2}.\sigma_0 F_0)$. Let us show that

$$\mathrm{Sh}_{\sigma_0 F_0 / F_0^2} \circ R_{T_w}^{G^{\sigma_0 F_0}} 1 = \mathrm{Sh}_{\sigma F / F} \circ R_{T_{w_M}.\sigma}^{M^{\sigma F}.\sigma} 1,$$

[1] Unipotent characters are F_0-stable

which concludes the proof.

For $g \in G^{F_0^2}$, there exists $x \in G$ such that $x\sigma_0 F_0(x)^{-1} = g$, and so

$$N_{F_0^2/\sigma_0 F_0}(g) = x^{-1}F_0^2(x) \in G^{\sigma_0 F_0}.$$

We also have

$$(x, F_0(x))\sigma F(x^{-1}, F_0(x)^{-1}) = (g, F_0(g)),$$

and so

$$N_{F/\sigma F}((g, F_0(g)) = (x^{-1}, F_0(x)^{-1})F(x, F_0(x)) = (x^{-1}F_0^2(x), 1) \in M^{\sigma F}.$$

Therefore,

$$\begin{aligned} &\mathrm{Sh}_{\sigma F/F} \circ R_{T_{w_M}.\sigma}^{M^{\sigma F}.\sigma} 1((g, F_0(g)) \overset{①}{=} R_{T_{w_M}.\sigma}^{M^{\sigma F}.\sigma} 1((x^{-1}F_0^2(x), 1)) \\ &\overset{②}{=} R_{T_w}^{G^{\sigma_0 F_0}} 1(x^{-1}F_0^2(x)) \overset{③}{=} \mathrm{Sh}_{\sigma_0 F_0/F_0^2} \circ R_{T_w}^{G^{\sigma_0 F_0}} 1(g). \end{aligned} \tag{IV.5.3.8.4}$$

where the equality ① is the definition of Shintani descent and we have identified the functions on $M^{\sigma F}.\sigma$ to the functions on $M^{\sigma F}$ invariant under the F-conjugation. We have equality ② by (§IV.5.3.6.2) with the automorphism $\sigma(g, h) = (\sigma_0(h), \sigma_0(g))$ and the Frobenius $\sigma F(g, h) = (\sigma_0 F_0(g), \sigma_0 F_0(h))$. Equality ③ is again the definition of Shintani descent.

IV.5.4 Extensions of Quadratic-Unipotent Characters

In this section, we focus on $L_0 \cong \mathrm{GL}_{n_0}(k)$, equipped with the Frobenius F_0 which sends each entry to its q-th power. Let σ_0 and σ_0' be the automorphisms defined for $\mathrm{GL}_{n_0}(k)$ in the same way σ and σ' are defined for $\mathrm{GL}_n(k)$. Now the semi-direct product of $\mathrm{GL}_{n_0}(k)$ by σ_0 (resp. by σ_0') is denoted by ${}^s\bar{G}_0$ (resp. ${}^o\bar{G}_0$). We may regard σ_0 also as an element of ${}^o\bar{G}_0$, acting as σ_0 on $\mathrm{GL}_{n_0}(k)$ but satisfying $\sigma_0^2 = -\mathrm{Id}$. The point is that, in ${}^o\bar{G}_0$, we need to fix a quasi-central element to work with, and σ_0 is a convenient choice.

IV.5.4.1 Let (μ_+, μ_-) be a 2-partition of n_0 and fix some integers $r_+ > l(\mu_+)$ and $r_- > l(\mu_-)$. Write $n_+ = |\mu_+|$ and $n_- = |\mu_-|$. Let $(\alpha_+, \beta_+)_{r_+}$ and $(\alpha_-, \beta_-)_{r_-}$ be the 2-partitions associated to μ_+ and to μ_- respectively (See §II.2.1.3), such that the unordered 2-partitions (α_+, β_+) and (α_-, β_-) are the corresponding 2-quotients. Let m_+ and m_- be some non negative integers such that $(m_+, \ldots, 2, 1)$ and $(m_-, \ldots, 2, 1)$ are the 2-cores of μ_+ and of μ_- respectively, and write $N_\pm = (n_\pm - m_\pm(m_\pm + 1))/2$. There exists a unique pair $(h_1, h_2) \in \mathbb{N} \times \mathbb{Z}$ such that

$$\begin{aligned} m_+ &= \sup(h_1 + h_2, -h_1 - h_2 - 1) \\ m_- &= \sup(h_1 - h_2, h_2 - h_1 - 1). \end{aligned} \tag{IV.5.4.1.1}$$

Note that exchanging μ_+ and μ_- sends (h_1, h_2) to $(h_1, -h_2)$. Write

$$m = m_+(m_+ + 1)/2 + m_-(m_- + 1)/2,$$

and so $n_0 = m+2N_+ +2N_-$. We have $m = h_1(h_1+1)+h_2^2$. Assume that $r_- \equiv h_2 \mod 2$ and $r_+ \equiv h_2 + 1 \mod 2$. Each of the two 2-partitions $(\alpha_\pm, \beta_\pm)_{r_\pm}$ defines an irreducible representation of $\mathfrak{W}^{\mathrm{C}}_{N_\pm}$ respectively, denoted by ρ_+ and ρ_-. Then with the fixed r_+ and r_-, the 2-partitions (μ^+, μ^-) are in bijection with the data $(h_1, h_2, \rho_+, \rho_-)$, identifying ρ_+ and ρ_- with the 2-partitions determining their isomorphism classes.

IV.5.4.2 We also consider the data (h_1, h_2, w_+, w_-), with $w_+ \in \mathfrak{W}^{\mathrm{C}}_{N_+}$ and $w_- \in \mathfrak{W}^{\mathrm{C}}_{N_-}$, where h_1, h_2, N_+ and N_- are as above. To simplify, we write $\mathfrak{W}_+ = \mathfrak{W}^{\mathrm{C}}_{N_+}$, $\mathfrak{W}_- = \mathfrak{W}^{\mathrm{C}}_{N_-}$ and $\mathbf{w} = (h_1, h_2, w_+, w_-)$ instead. To each $\mathbf{w}$ is associated an F_0-stable and σ_0-stable Levi factor of a σ_0-stable parabolic subgroup, isomorphic to $L_{\mathbf{w}} \cong T_{w_+} \times T_{w_-} \times \mathrm{GL}_m(k)$, each factor preserved by σ_0. We write $\sigma_0 = \sigma_+ \times \sigma_- \times \sigma_{00}$ and $F_0 = F_+ \times F_- \times F_{00}$ with respect to this decomposition.

IV.5.4.3 To each $(h_1, h_2) \in \mathbb{N} \times \mathbb{Z}$ is associated a unique cuspidal function $\phi(h_1, h_2)$ on $\mathrm{GL}_m(q).\sigma_{00}$, which is supported on the $\mathrm{GL}_m(k)$-conjugacy class of $s\sigma_{00}u$, where $s\sigma_{00}$ has $\mathrm{Sp}_{h_1(h_1+1)}(k) \times \mathrm{SO}_{h_2^2}(k)$ as its connected centraliser in $\mathrm{GL}_m(k)$ and u corresponds to the (symplectic/orthogonal) partitions $(2h_1, 2h_1-2, \ldots, 2)$ and $(2|h_2|-1, 2|h_2|-3, \ldots, 1)$. It is an isolated conjugacy class. The cuspidal functions are explicitly given as below.

Put

$$s(h_2) = \begin{cases} 0, & \text{if } h_2 \geq 0, \\ 1, & \text{if } h_2 < 0, \end{cases}$$

and put

$$\delta(h_1) = \frac{|h_2^3 - h_2|}{3}, \quad \delta(h_2) = \frac{h_1(2h_1+1)(h_1+1)}{6}, \quad \delta(h_1, h_2) = \delta(h_1) + \delta(h_2).$$

Write $u = (u_1, u_2) \in \mathrm{Sp}_{h_1(h_1+1)}(k) \times \mathrm{SO}_{h_2^2}(k)$. The $\mathrm{Sp}_{h_1(h_1+1)}(q)$-conjugacy classes in the $\mathrm{Sp}_{h_1(h_1+1)}(k)$ conjugacy class of u_1 are parametrised by $\mu_2^{h_1}$, and for $\epsilon = \pm$ the $\mathrm{SO}^{\epsilon}_{h_2^2}(q)$-conjugacy classes in the $\mathrm{SO}_{h_2^2}(k)$-conjugacy class of u_2 are parametrised by $\mu_2^{|h_2|-1}$, regarded as the subset of $\mu_2^{|h_2|}$ consisting of elements $(e_i)_{i\in\{1,\ldots,|h_2|\}}$ with $\prod_i e_i = \epsilon\eta(-1)^{[|h_2|/2]}$, where η is the order 2 character of $\mathbb{F}_q^*$. Note that when we write an element of $\mu_2^{|h_2|}$ as (e_i), then e_i corresponds to the Jordan block of size $2i-1$.

Denote by ϱ the nontrivial irreducible character of μ_2. For $\mathrm{Sp}_{h_1(h_1+1)}(q)$, the unique cuspidal function ϕ_1 supported on the $\mathrm{Sp}_{h_1(h_1+1)}(k)$-conjugacy class of u_1 is given by (*cf.* [W01, §II.4])

$$\phi_1(u_{1,(e_i)}) = q^{\delta(h_1)/2} \prod_{i\in\{1,\ldots,h_1\},\ i \text{ is odd}} \varrho(e_i),$$

where $u_{1,(e_i)}$ is a representative of the class corresponding to (e_i). For $\mathrm{SO}^{\epsilon}_{h_2^2}(q)$, the unique cuspidal function ϕ_2 supported on the $\mathrm{SO}_{h_2^2}(k)$-conjugacy class of u_2 is given by (*cf.* [W01,

§II.5])

$$\phi_2(u_{2,(e'_i)}) = q^{\delta(h_2)/2} \prod_{i\in\{1,\dots,|h_2|\},\ i\equiv h_2 \mod 2} \varrho(e'_i),$$

where $u_{2,(e'_i)}$ is a representative of the class corresponding to (e'_i).

Finally, $\phi(h_1, h_2)$ is expressed as

$$q^{\delta(h_1,h_2)/2} \prod_{\substack{1\le i\le h_1\\ i \text{ odd}}} \varrho(e_i) \prod_{\substack{1\le i\le |h_2|\\ i\equiv h_1+h_2+1+s(h_2) \mod 2}} \varrho(e'_i). \tag{IV.5.4.3.1}$$

Note that the component group of the centraliser of $s\sigma_{00}u$ is isomorphic to $\mu_2^{h_1} \times \mu_2^{h_2}$ and there are therefore two cuspidal functions on $\mathrm{GL}_m(q).\sigma_{00}$ (unless $h_2 = 0$) that restrict to the unique cuspidal function on $\mathrm{Sp}_{h_1(h_1+1)}(q) \times \mathrm{SO}^{\epsilon}_{h_2^2}(q)$, under the map $u \mapsto s\sigma_{00}u$. The two cuspidal functions, in terms of representations of the component group $\mu_2^{h_1} \times \mu_2^{|h_2|}$, differ by $(\varrho, \dots, \varrho) \in \mathrm{Irr}(\mu_2^{|h_2|})$. Indeed, $\phi(h_1, h_2)$ and $\phi(h_1, -h_2)$ are two such functions, and the indicating function $s(h_2)$ is responsible for the difference $(\varrho, \dots, \varrho)$.

IV.5.4.4 Let Id be the trivial character of $T^{F_+}_{w_+}$. It trivially extends to $T^{F_+}_{w_+}.<\sigma_+>$, and so we can regard Id as a function on $T^{F_+}_{w_+}.\sigma_+$. Similarly, composing η with the homomorphism $T^{F_-}_{w_-} \to \mathbb{F}_q^*$ defined by the product of norm maps, we can regard η as an invariant function on $T^{F_-}_{w_-}.\sigma_-$, whose value at σ_- is equal to 1. Then, $\mathrm{Id}\,\widetilde{\boxtimes}\,\eta\,\widetilde{\boxtimes}\,\phi(h_1, h_2)$ is an invariant function on $L^F_{\mathbf{w}}.\sigma_0$, denoted by $\phi_{\mathbf{w}}$. The induced function $R^{L_0.\sigma_0}_{L_{\mathbf{w}}.\sigma_0}\phi_{\mathbf{w}}$ on $L_0^{F_0}.\sigma_0$ is invariant under the conjugation by $L_0^{F_0}$.

Remark IV.5.4.1. In the case of ${}^o\bar{G}_0$, the element σ_0 satisfies $\sigma_0^2 = -1$ and so is each of its component: σ_+, σ_- and σ_{00}. Then we can still extend η in such a way that its value at σ_- is equal to 1, because the value of η, regarded as a character of $T^{F_-}_{w_-}$ is always 1 at $-1 \in T^{F_-}_{w_-}$.

Denote by φ_+ and φ_- the characters of ρ_+ and of ρ_- respectively. We have then the invariant functions on $L_0^{F_0}.\sigma_0$ defined by

$$R_\rho^{L_0.\sigma_0} := \frac{1}{|\mathfrak{W}_+|}\frac{1}{|\mathfrak{W}_-|} \sum_{\substack{w_+\in\mathfrak{W}_+\\ w_-\in\mathfrak{W}_-}} \varphi_+(w_+)\varphi_-(w_-) R^{L_0.\sigma_0}_{L_{\mathbf{w}}.\sigma_0}\phi_{\mathbf{w}}. \tag{IV.5.4.4.1}$$

IV.5.4.5 We keep the notations above and assume that $p \neq 2$ and $q > n_0$. Let $\chi_{(\mu_+,\mu_-)}$ be a quadratic-unipotent character, which extends into a character $\tilde{\chi}_{(\mu_+,\mu_-)} \in \mathrm{Irr}(L_0^{F_0}.<\sigma_0>)$.

Theorem IV.5.4.2. *([W, §17]) Suppose that $L_0^{F_0}.<\sigma_0> = {}^o\bar{G}_0$ if n_0 is even. Then for any $(\mu_+, \mu_-) \in \mathcal{P}_{n_0}(2)$, we have,*

$$\tilde{\chi}_{(\mu_+,\mu_-)}|_{L_0^{F_0}.\sigma_0} = \pm R_\rho^{L_0.\sigma_0}. \tag{IV.5.4.5.1}$$

Given a quadratic-unipotent character χ_0 of $\mathrm{GL}_{n_0}(q)$, let $\rho : \mathrm{GL}_{n_0}(q) \to \mathrm{GL}(V)$ be a representation that realises it. Then $\rho(-\mathrm{Id}) = \pm \mathrm{Id}_V$, with $\rho(-\mathrm{Id}) = -\mathrm{Id}_V$ exactly when $\chi(-\mathrm{Id}) = -\chi(\mathrm{Id})$. Define the indicator

$$\gamma_\chi = \begin{cases} \mathrm{i} & \text{if } \chi(-\mathrm{Id}) = -\chi(\mathrm{Id}); \\ 1 & \textit{otherwise}. \end{cases}$$

Corollary IV.5.4.3. *Suppose that n_0 is even and $L_0^{F_0}.<\sigma_0> = {}^s\bar{G}_0$. Then for any $(\mu_+, \mu_-) \in \mathcal{P}_{n_0}(2)$, we have,*

$$\tilde{\chi}_{(\mu_+,\mu_-)}|_{L_0^{F_0}.\sigma_0} = \pm\gamma_\chi R_\rho^{L_0.\sigma_0}. \tag{IV.5.4.5.2}$$

Proof. This follows from §II.5.2.3. □

IV.6 The Formula

IV.6.1 Decomposition into Deligne-Lusztig Inductions

IV.6.1.1 By Proposition IV.1.2.2 and Proposition IV.1.2.3, every σ-stable irreducible character χ of $\mathrm{GL}_n(q)$ is of the form $R_{M_{I,w}}^G(\chi_1 \boxtimes \chi_0)$, for an F-stable Levi subgroup $M_{I,w}$ isomorphic to the σ-stable standard Levi subgroup L_I of the form (IV.1.2.2.1) equipped with the Frobenius F_w given by (IV.1.2.3.5), (IV.1.2.3.6) and (IV.1.2.3.7) and with the action of σ given by (IV.1.2.3.8) and (IV.1.2.3.9). Decomposing L_I into $L_1 \times L_0$ following §IV.1.2.3, then χ_1 and χ_0 are identified with some σ-stable characters of $L_1^{F_w}$ and $L_0^{F_0}$ respectively, where we also denote by the same letter the restriction of F_w to L_1 and by F_0 its restriction to L_0. We decompose σ into $(\sigma_1\, \sigma_0)$ with respect to $L_1 \times L_0$. Recall that χ_0 is defined by a 2-partition (μ_+, μ_-) and that χ_1 is defined by $[\varphi_1] \in \mathrm{Irr}(W_{L_1})^{F_w}$ and $\theta_1 \in \mathrm{Irr}_{reg}(L_1^{F_w})^{\sigma_1}$ satisfying the assumptions of §IV.1.2.4, where $\varphi_1 \in \mathrm{Irr}(W_{L_1}^{\sigma_1})^{F_w}$.

By Lemma II.4.2.1, an extension of χ to $G^F.\sigma$ is obtained by first extending $\chi_1 \boxtimes \chi_0$ to $M_{I,w}^F.\sigma$ and then taking the induction $R_{M_{I,w}.\sigma}^{G.\sigma}$. One can equally extend $\chi_1 \boxtimes \chi_0$, regarded as a character of $L_I^{F_w}$, to $L_I^{F_w}.\sigma$. The extension of χ_1 to $L_1^{F_w}.\sigma_1$ is given by Theorem IV.5.2.1 and the extension of χ_0 to $L_0^{F_0}.\sigma_0$ is given by Theorem IV.5.4.2. Explicitly, we have

$$\tilde{\chi}_1|_{L_1^{F_w}.\sigma_1} = |W_{L_1}^{\sigma_1}|^{-1} \sum_{w_1 \in W_{L_1}^{\sigma_1}} \tilde{\varphi}_1(w_1 F_w) R_{T_{w_1}.\sigma_1}^{L_1.\sigma_1} \tilde{\theta}_1,$$

$$\tilde{\chi}_0|_{L_0^{F_0}.\sigma_0} = \frac{1}{|\mathfrak{W}_+|}\frac{1}{|\mathfrak{W}_-|} \sum_{\substack{w_+ \in \mathfrak{W}_+ \\ w_- \in \mathfrak{W}_-}} \varphi_+(w_+)\varphi_-(w_-) R_{L_{\mathbf{w}}.\sigma_0}^{L_0.\sigma_0} \phi_{\mathbf{w}},$$

and in the second equality there is an extra γ_{χ_0} if σ_0 is of symplectic type. Now put

$$\gamma_\chi := \begin{cases} \gamma_{\chi_0} & \text{if } \sigma \text{ is of symplectic type,} \\ 1 & \text{otherwise.} \end{cases}$$

Write $L_{w_1,\mathbf{w}} = T_{w_1} \times L_{\mathbf{w}}$. It is an F_w-stable and σ-stable Levi factor of a σ-stable parabolic subgroup of L_I. Combining the above two formulas gives

$$\tilde{\chi}_1 \widetilde{\boxtimes} \tilde{\chi}_0|_{L_I^{F_w}.\sigma} = \gamma_\chi |W_{L_1}^{\sigma_1} \times \mathfrak{W}_+ \times \mathfrak{W}_-|^{-1} \sum_{\substack{(w_1,w_+,w_-) \\ \in W_{L_1}^{\sigma_1} \times \mathfrak{W}_+ \times \mathfrak{W}_-}} \tilde{\varphi}_1(w_1F_w)\varphi_+(w_+)\varphi_-(w_-) R_{L_{w_1,\mathbf{w}}}^{L_I.\sigma}(\tilde{\theta}_1 \widetilde{\boxtimes} \phi_{\mathbf{w}}).$$

Identifying $L_I^{F_w}.<\sigma> \cong M_{I,w}^F.<\sigma>$ and the subgroups $M_{w_1,\mathbf{w}}^F \leftrightarrow L_{w_1,\mathbf{w}}^{F_w}$, we deduce

Theorem IV.6.1.1. *The extension of χ is given by the following formula:*

$$\tilde{\chi}|_{G^F.\sigma} = \gamma_\chi |W_{L_1}^{\sigma_1} \times \mathfrak{W}_+ \times \mathfrak{W}_-|^{-1} \sum_{\substack{(w_1,w_+,w_-) \\ \in W_{L_1}^{\sigma_1} \times \mathfrak{W}_+ \times \mathfrak{W}_-}} \tilde{\varphi}_1(w_1F_w)\varphi_+(w_+)\varphi_-(w_-) R_{M_{w_1,\mathbf{w}}}^{G.\sigma}(\tilde{\theta}_1 \widetilde{\boxtimes} \phi_{\mathbf{w}}).$$

IV.6.1.2 Recall §IV.1.2.8 that the σ-stable irreducible characters of $\mathrm{GL}_n(q)$ are parametrised by $\bar{\mathfrak{T}}_\chi$. Denote by $\bar{\mathfrak{T}}_\chi^0 \subset \bar{\mathfrak{T}}_\chi$ the subsets of the elements

$$\lambda_+\lambda_- \prod_{i\in\Lambda_1}(\lambda_i, \alpha_i) \prod_{j\in\Lambda_2}(\lambda'_j, \alpha'_j)$$

in which at most one of $|\lambda_+|$ and $|\lambda_-|$ is odd, and $\lambda_\pm$ is a partition with trivial 2-core or with 2-core (1) according to the parity.

Corollary IV.6.1.2. *The σ-stable irreducible characters of $\mathrm{GL}_n(q)$ that extends to uniform functions on $\mathrm{GL}_n(q).\sigma$ are in bijection with $\bar{\mathfrak{T}}_\chi^0$, and the extensions of these characters constitute a base (identifying two extensions of the same character) of the space of the uniform functions on $\mathrm{GL}_n(q).\sigma$.*

Proof. We have seen in Theorem IV.6.1.1 that the extension of a general σ-stable irreducible character is decomposed into a linear combination of cuspidal functions induced from

$$M_{w_1,\mathbf{w}}^F \cong T_{w_1}^F \times T_{w_+}^F \times T_{w_-}^F \times \mathrm{GL}_m(q),$$

for various w_1 and $\mathbf{w}$. Cuspidal functions induced from $M_{w_1,\mathbf{w}}^F$ with $m > 1$ can not be uniform (see §II.4.3.2 for the definition of uniform functions). Now the condition $m \leq 1$ is equivalent to the condition that the sum of the 2-cores of λ_+ and λ_- is either empty or (1). We see that λ_+ and λ_- satisfy the assumption in the definition of $\bar{\mathfrak{T}}_\chi^0$, whence the first assertion.

For each $\bar{\mathfrak{t}} \in \bar{\mathfrak{T}}_\chi^0$, denote by $\chi_{\bar{\mathfrak{t}}}$ the corresponding character, and choose an extension $\tilde{\chi}_{\bar{\mathfrak{t}}} \in C(\mathrm{GL}_n(q).\sigma)$. Then, $\{\tilde{\chi}_{\bar{\mathfrak{t}}} \mid \bar{\mathfrak{t}} \in \bar{\mathfrak{T}}_\chi^0\}$ is a set consisting of functions orthogonal to each other.

Theorem IV.6.1.1 gives a transition matrix between the set of (generalised) Deligne-Lusztig characters and that of the $\tilde{\chi}_{\mathfrak{t}}$'s. □

Remark IV.6.1.3. The extension of an irreducible character is then either uniform, ot orthogonal to all uniform functions. Since the characteristic function of a quasi-semi-simple conjugacy class is uniform, the extension of a character corresponding to an element of $\tilde{\mathfrak{T}}_\chi \setminus \tilde{\mathfrak{T}}^0_\chi$ vanishes on every quasi-semi-simple element.

IV.6.2 Combinatorial Description of the Character Formula

In order to determine the character table of $\mathrm{GL}_n(q).<\sigma>$, it only remains for us to calculate the induced functions $R^{G.\sigma}_{M_{w_1,\mathbf{w}}.\sigma}\phi_w$.

IV.6.2.1 Let us recall how this is done for a split connected group G. Fix a split maximal torus T_0 and let W_G denote the Weyl group defined by T_0. Let us simplify the situation and assume that $M_{w_1,\mathbf{w}} = T_\tau$, $\tau \in W_G$, is a maximal torus. The character formula (*cf.* §II.4.1.2) reads

$$R^G_{T_\tau}\theta(g) = |T^F_\tau|^{-1}|C^\circ_G(s)^F|^{-1} \sum_{\{h\in G^F | s\in {}^hT_\tau\}} Q^{C^\circ_G(s)}_{C^\circ_{{}^hT_\tau}(s)}(u)\,{}^h\theta(s), \tag{IV.6.2.1.1}$$

for the Jordan decomposition $g = su$. Assume that $s \in T^F_\tau$, and put

$$A(s,\tau) := \{h \in G \mid hsh^{-1} \in T_\tau\}. \tag{IV.6.2.1.2}$$

We have to determine the set

$$A^F(s,\tau) := \{h \in A(s,\tau) \mid F(h) = h\}. \tag{IV.6.2.1.3}$$

Write $L = C_G(s)^\circ$. Define

$$\begin{aligned} B(s,\tau) :=\{&\text{The } L^F\text{-conjugacy classes of the } F\text{-stable maximal tori of } L \\ &\text{that are } G^F\text{-conjugate to } T_\tau.\} \end{aligned} \tag{IV.6.2.1.4}$$

It parametrises a subset of the Green functions of L^F. We fix s and τ and write A, A^F and B in what follows. Observe that there is a surjective map $A^F \to B$ which sends h to the class of $h^{-1}T_\tau h$. It factors through $\iota : A^F/L^F \to B$. The value of the Green function only depends on $\iota(h)$ while ${}^h\theta(s)$ is constant on each right L^F-coset of A^F. The calculation is eventually reduced to evaluating ${}^h\theta(s)$ on the fiber of ι over an element $\bar{\nu} \in B$. We may regard $\bar{\nu}$ as the F-conjugacy class of some $\nu \in W_L(T_\tau)$.

We have $A = N_G(T_\tau).L$, that is, the set of the elements nl, $n \in N_G(T_\tau)$ and $l \in L$, since for each h, there exists $l \in L$ such that $h^{-1}T_\tau h = lT_\tau l^{-1}$. We deduce from it an isomorphism

$$A^F/L^F \cong (A/L)^F \cong (N_G(T_\tau)/N_L(T_\tau))^F \cong (W_G(T_\tau)/W_L(T_\tau))^F, \tag{IV.6.2.1.5}$$

which sends $h = nl$ to the class of n. This does not depend on the choice of the n and l such that $h = nl$. We choose some $g \in G$ such that $T_\tau = gT_0g^{-1}$, and put $L_0 = g^{-1}Lg$. We can further identify the above set to $(W_G(T_0)/W_{L_0}(T_0))^\tau$. Write $W_{L_0} = W_{L_0}(T_0)$. The conjugation by τ preserves L_0 since L is F-stable. Now, a coset wW_{L_0} is τ-stable if and only if

$$w^{-1}\tau w\tau^{-1} \in W_{L_0} \tag{IV.6.2.1.6}$$

and $\iota(wW_{L_0}) \in \bar{\nu}$ if and only if

$$w^{-1}\tau w\tau^{-1} \in \bar{\nu}, \tag{IV.6.2.1.7}$$

regarding $\bar{\nu}$ as a τ-conjugacy class of $W_{L_0}(T_0)$. Indeed, if hL^F corresponds to wW_L, then the L^F-conjugacy class of $h^{-1}T_\tau h = l^{-1}T_\tau l$ is represented by $lF(l)^{-1} = n^{-1}F(n)$, which is none other than $w^{-1}\tau w\tau^{-1}$ under ad g^{-1}. The computation of the w's is completely combinatorial.

To summarise, once the Green functions have been computed (see the introduction), the calculation of the character formula goes as follows.

(i) Describe combinatorially the sets A^F/L^F and B;

(ii) Specify a surjection $\iota : A^F/L^F \to B$ and calculate the fibres of ι;

(iii) For each $\bar{\nu} \in B$ and each $hL^F \in \iota^{-1}(\bar{\nu})$, evaluate the character $\theta(hsh^{-1})$.

The summation in the character formula is decomposed into one summation over B and then one summation over the fibre of ι. We also see that the summation of the ${}^h\theta(s)$'s over each fibre of ι is just permuting the "eigenvalues" of s.

IV.6.2.2 We will change our notations in what follows. Let N and m be some non negative integers such that $2N + m = n$. Write $G = \mathrm{GL}_n(k)$, and let M_0 be the σ-stable standard Levi subgroup of G isomorphic to $\mathrm{GL}_m(k) \times (k^*)^{2N}$, then $N_G(M_0.\sigma)/M_0 \cong \mathfrak{W}_N^C$. Let $M_w \cong \mathrm{GL}_m(k) \times (k^*)^{2N}$ be an F-stable and σ-stable Levi factor of some σ-stable parabolic subgroup of G, with $w \in \mathfrak{W}_N^C$. The G^F-conjugacy class of $M_w.<\sigma>$is then parametrised by the conjugacy class of w, which we denote by τ. Assume that $M_w = gM_0g^{-1}$ for some $g \in (G^\sigma)^\circ$ such that $g^{-1}F(g) = \dot{w}$ is a representative of w. We will take M_w for $M_{w_1,\mathbf{w}}$ and calculate the induced function.

IV.6.2.3 Let $s\sigma \in G^F.\sigma$ be a semi-simple element. Define

$$\begin{aligned} A(s\sigma, \tau, h_1, h_2) = \{&h \in G \mid hs\sigma h^{-1} \in M_w.\sigma \text{ is quasi-isolated with } C_{M_w}(hs\sigma h^{-1}) \\ &\text{isomorphic to the product of } \mathrm{Sp}_{h_1(h_1+1)}(k) \times \mathrm{O}_{h_2^2}(k) \text{ and a torus}\}. \end{aligned}$$

Define

$$A^F(s\sigma, \tau, h_1, h_2) = \{h \in A(s\sigma, \tau, h_1, h_2) | F(h) = h\}.$$

If $s\sigma$ is not G^F-conjugate into $M_w.\sigma$, then $A^F(s\sigma,\tau)$ is empty, so we can assume that $s\sigma$ lies in $M_w.\sigma$. We will give a combinatorial description of this set.

Write $L' = C_G(s\sigma)^\circ$. If $K' \subset L'$ is an F-stable Levi subgroup, put $K = C_G(Z^\circ_{K'})$. By Proposition II.3.1.6, it is the smallest F-stable and $s\sigma$-stable Levi subgroup of G such that $(K \cap L')^\circ = K'$, which is a Levi factor of an $s\sigma$-stable parabolic subgroup, say Q. So $N_{\bar{G}}(K) \cap N_{\bar{G}}(Q) = K.<s\sigma>$. Moreover, from Proposition II.3.1.12 (i), we deduce that $s\sigma$ is isolated in $K.<s\sigma>$. Define

$$\begin{aligned} B(s\sigma,\tau,h_1,h_2) = \{ & \text{The } L'^F\text{-conjugacy classes of the Levi subgroups } K' \subset L' \\ & \text{isomorphic to the product of } \mathrm{Sp}_{h_1(h_1+1)}(k) \times \mathrm{O}_{h_2^2}(k) \text{ and a torus} \\ & \text{such that } K.<s\sigma> \text{is } G^F\text{-conjugate to } M_w.<\sigma>.\} \end{aligned}$$

In what follows, we fix $s\sigma$, τ, h_1 and h_2, then $A(s\sigma,\tau,h_1,h_2)$ and $B(s\sigma,\tau,h_1,h_2)$ will be denoted by A and B.

Lemma IV.6.2.1. *The map*

$$\begin{aligned} \iota : A^F/L'^F &\longrightarrow B \\ hL'^F &\longmapsto \textit{ the class of } C_{h^{-1}M_wh}(s\sigma)^\circ. \end{aligned} \tag{IV.6.2.3.1}$$

is well defined and surjective.

Proof. If $h \in A^F$, then $hs\sigma h^{-1}$ normalises M_w and a parabolic subgroup containing it, so $s\sigma$ normalises $h^{-1}M_wh$ and a parabolic subgroup containing it. It follows that $K' := C^\circ_{h^{-1}M_wh}(s\sigma)$ is an F-stable Levi subgroup of L'. We then obtain the Levi subgroup K as above. From the fact that $C_{h^{-1}M_wh}(s\sigma) = h^{-1}C_{M_w}(hs\sigma h^{-1})h$ and from the assumption on $hs\sigma h^{-1}$, we deduce that $s\sigma \in K.<s\sigma>$is isolated with centraliser isomorphic to the product of $\mathrm{Sp}_{h_1(h_1+1)}(k) \times \mathrm{O}_{h_2^2}(k)$ and a torus. Since $s\sigma$ is also isolated in $h^{-1}M_w.<\sigma>h$, by Remark II.3.1.8, we have $K = h^{-1}M_wh$, and so $K.<s\sigma> = h^{-1}M_w.<\sigma>h$. We see that the L'^F-class of K' indeed belongs to B. Obviously this map factors through the quotient A^F/L'^F. Given $K' \in B$ with $h \in G^F$ such that $hK.<s\sigma>h^{-1} = M_w.<\sigma>$, the same argument shows that $h \in A^F$, whence surjectivity. □

Lemma IV.6.2.2. *If $h_1, h_2 \in A$, then $h_2 \in N_G(M_w.\sigma)h_1L'$. In particular,*

$$A = N_G(M_w.\sigma).L'. \tag{IV.6.2.3.2}$$

Moreover, if $h_1L'^F, h_2L'^F \in A^F/L'^F$ belong to the same fibre of ι, then $h_2 \in N_{G^F}(M_w.\sigma)h_1L'^F$.

Proof. For h_1 and $h_2 \in A$, there exists $l \in L'$ such that

$$h_1^{-1}M_wh_1 \cap L' = l(h_2^{-1}M_wh_2 \cap L')l^{-1} = lh_2^{-1}M_wh_2l^{-1} \cap L'.$$

By assumption, $s\sigma$ is isolated in $h_1^{-1}M_w.<\sigma>h_1$ and in $h_2^{-1}M_w.<\sigma>h_2$, and so Remark II.3.1.8 implies that $h_1^{-1}M_wh_1 = lh_2^{-1}M_wh_2l^{-1}$. So there exists $n \in N_G(M_w)$ such that $h_2 = nh_1l$. Note

that nhl belongs to A, for any $h \in A$, $n \in N_G(M_w.\sigma)$ and $l \in L'$. If moreover h_1 and $h_2 \in A^F$ belong to the same fibre of ι, we can choose l to be in L'^F. Note that in this case n is necessarily F-stable.

Let us determine the set $N_G(M_w.\sigma)$. If n normalises $M_w.\sigma$, then it normalises $M_w.\sigma.M_w.\sigma$, so it normalises M_w. Then, $nM_w.\sigma n^{-1} = nM_w n^{-1} n\sigma(n^{-1})\sigma \in M_w.\sigma$, so $n\sigma(n^{-1}) \in M_w$. We see that $N_G(M_w.\sigma)$ consists of those components of $N_G(M_w)$ that are σ-stable. Besides, if $h_2 = nh_1 l$ with $n \in N_G(M_w)$, then

$$nh_1 ls\sigma l^{-1} h_1^{-1} n^{-1} = nh_1 s\sigma h_1^{-1} n^{-1} \in M_w.\sigma$$

implies that $n\sigma(n)^{-1} \in M_w$, i.e. nM_w is a σ-stable component of $N_G(M_w)$. □

Remark IV.6.2.3. For $h = nl \in A^F = (N_G(M_w.\sigma)L')^F$, neither n nor l is necessarily F-stable.

Define $M'_w = C_{M_w}(s\sigma)^\circ = (M_w \cap L')^\circ$. It is an F-stable Levi subgroup of L' isomorphic to the product of $\mathrm{Sp}_{h_1(h_1+1)}(k) \times \mathrm{SO}_{h_2^2}(k)$ and a torus. Define $L'' = g^{-1}L'g = C_G(g^{-1}s\sigma g)^\circ$ with g as in §IV.6.2.2, and $M'_0 = g^{-1}M'_w g = C_{M_0}(g^{-1}s\sigma g)^\circ$.

Corollary IV.6.2.4. *We have the bijections*

$$A^F/L'^F \cong (N_G(M_w.\sigma)/N_{L'}(M'_w))^F \cong (N_G(M_0.\sigma)/N_{L''}(M'_0))^{F_{\dot{w}}}. \tag{IV.6.2.3.3}$$

Remark II.3.1.9 implies that $N_{L''}(M'_w)$ is indeed a subgroup of $N_G(M_w.\sigma)$.

Proof. If $h \in A$, then there exists $l \in L'$ and $n \in N_G(M_w.\sigma)$ such that $h = nl$ by Lemma IV.6.2.2. The following map

$$\begin{aligned} A/L' &\longrightarrow N_G(M_w.\sigma)/N_{L'}(M'_w) \\ hL' &\longmapsto nN_{L'}(M'_w). \end{aligned} \tag{IV.6.2.3.4}$$

is well defined and bijective. So $A^F/L^F \cong (N_G(M_w.\sigma)/N_{L'}(M'_w))^F$. The second bijection is obvious. □

IV.6.2.4 Let us point out that the identity component of $N_G(M_0.\sigma)$ is M_0 whereas that of $N_{L''}(M'_0)$ is M'_0, so we cannot directly reduce the problem to a purely combinatorial one as in §IV.6.2.1.

Lemma IV.6.2.5. *Let $n \in N_G(M_0.\sigma)$ and let $\bar{v}$ be an $F_{\dot{w}}$-conjugacy class of $N_{L''}(M'_0)/M'_0$. Write $N'' = N_{L''}(M'_0)$. Then*

(i) The coset nN'' is $F_{\dot{w}}$-stable if and only if

$$n^{-1}\dot{w}F(n)\dot{w}^{-1} \in N'';$$

(ii) *Under the bijection (IV.6.2.3.3), the $F_{\dot{w}}$-stable Levi subgroup $h^{-1}M'_w h \subset L'$ lies in the class $\bar{v}$ if and only if*

$$n^{-1}\dot{w}F(n)\dot{w}^{-1}M'_0 \in \bar{v}.$$

Proof. Part (i) is obvious. For part (ii), we write $h = nl$ with $n \in N_G(M_w.\sigma)$ and $l \in L'$. The L'^F-class of the Levi $h^{-1}M'_w h = l^{-1}M'_w l \subset L'$ is given by $lF(l)^{-1} = n^{-1}F(n) \in N_{L''}(L'_w)$, or rather $n_0^{-1}\dot{w}F(n_0)\dot{w}^{-1} \in N_{L''}(M'_0)$, with $n_0 = g^{-1}ng$. □

We want to first solve the equations in the above Lemma at the level of Weyl groups. Denote $\bar{N}'' := N''.M_0$. It is the union of the connected components of $N_G(M_0.\sigma)$ that meet L''.

> **Fact.** Each connected component of $\bar{N}''$ contains exactly one connected component of N'', because an element of the identity component of $\bar{N}''$ must induce trivial action on the torus component of the identity component of N''.

We deduce from this fact an isomorphism $N''/M'_0 \cong \bar{N}''/M_0$. We can then regard $W'' := W_{L''}(M'_0)$ as a subgroup of $W_G(M_0)^\sigma$, with compatible action of F_w.

Lemma IV.6.2.6. *We keep the notations in the above Lemma and let v be the equivalence class of n in $W_G(L_0)^\sigma$.*

(i) *If the coset nN'' is $F_{\dot{w}}$-stable, then*

$$v^{-1}wvw^{-1} \in W'';$$

(ii) *Under the bijection (IV.6.2.3.3), if the $F_{\dot{w}}$-stable Levi subgroup $h^{-1}M'_w h \subset L'$ lies in the class $\bar{v}$, then*

$$v^{-1}wvw^{-1} \in \bar{v}.$$

Proof. Obvious. □

IV.6.2.5 For each $v \in W_G(M_0)^\sigma$, we choose a representative $\dot{v} \in (G^\sigma)^\circ$. Once we have solved the equations of Lemma IV.6.2.6, with the choices of the representatives of the elements of $W_G(M_0)^\sigma$, the equations in Lemma IV.6.2.5 are reduced to equations for $l \in M_0$, if we express n as $\dot{v}l$. However, we do not really need to solve such equations to get the value of $\tilde{\theta}_1 \widetilde{\boxtimes} \phi_{\mathbf{w}}(hs\sigma h^{-1})$, for hL'^F corresponding to nN''. In other words, for the purpose of computing the characters, solving the combinatorial equations of Lemma IV.6.2.6 suffices. This is explained as follows.

First of all, by Lang-Steinberg, if v satisfies Lemma IV.6.2.6 (i), then there exists $l \in M_0$ such that $\dot{v}l$ satisfies Lemma IV.6.2.5 (i).

Then we show that the value of the character is independent of such l. Rewrite $n = \dot{v}l$ as $l\dot{v}$ for some different $l \in M_0$. Let $h \in A^F$ be an element corresponding to nN''. Then $g^{-1}hs\sigma h^{-1}g = ng^{-1}s\sigma gn^{-1}$ and so $ng^{-1}s\sigma gn^{-1}$ is F_w-stable and is isolated in $M_0.<\sigma>$. We write $M_0 \cong \mathrm{GL}_m(k)\times T_1$, with $T_1 \cong (k^*)^{2N}$ and decompose the action of σ as (σ_{00}, σ_1) accordingly. We also index the

direct factors of T_1 by the set $\{1, \ldots, N, -N, \ldots, -1\}$. The action of σ_1 is then $(t_i, t_{-i}) \mapsto (t_{-i}^{-1}, t_i^{-1})$. By the definition of A^F, $hs\sigma h^{-1}$ lies in a given isolated stratum specified by (h_1, h_2), thus the direct factor of $ng^{-1}s\sigma gn^{-1}$ corresponding to $\mathrm{GL}_m(q).\sigma_{00}$ necessarily lies in a prescribed isolated conjugacy class. So we do not need to know the direct factor of l corresponding to $\mathrm{GL}_m(k)$. Let t_1 be the direct factor of l corresponding to T_1. Replacing t_1 by some other $t_2 \in T_1$ amounts to multiplying $ns\sigma n^{-1}$ by an element of the form $(x_1, \ldots, x_N, x_N, \ldots, x_1)$ within the factor $T_1.\sigma_1$. It is necessarily F_w-stable. We can check that the value of a σ_1-stable linear character of $T_1^{F_w}$ is always equal to 1 at an F_w-stable element of this form.

Given $v \in W_G(M_0)^\sigma$, which determines n and thus h, we can compute $\tilde{\theta}_1 \widetilde{\boxtimes} \phi_{\mathbf{w}}(hs\sigma h^{-1}v)$ following the procedure below. We first find some $t \in T_1$, such that the conjugation of the factor of $ng^{-1}s\sigma gn^{-1}$ in $T_1.\sigma_1$ by t is F_w-stable, then combine this with the prescribed isolated conjugacy class in $\mathrm{GL}_m(q).\sigma_{00}$ to form an element of $M_0^{F_w}.\sigma$, finally evaluate the character at this element.

IV.6.2.6 Finally, we give a formula of the cardinality of the inverse image of the fibre of ι in A^F.

Let $h \in A^F$ be such that $\iota(hL'^F) \in \bar{\nu}$. By Lemma IV.6.2.2, we need to calculate the cardinality of the double coset $N_{G^F}(M_w.\sigma)hL'^F$. An element of $hL'^F h^{-1} = C_G(hs\sigma h^{-1})^{\circ F}$ normalises $M_w.\sigma = M_w.hs\sigma h^{-1}$ if and only if it normalises M_w, if and only if it normalises $C_{M_w}(hs\sigma h^{-1})^\circ$ as $hs\sigma h^{-1}$ is isolated in $M_w.<\sigma>$, so

$$N_{G^F}(M_w.\sigma) \cap hL'^F h^{-1} = N_{hL'^F h^{-1}}(C_{M_w}(hs\sigma h^{-1})^\circ).$$

The identity component of $N_{hL'h^{-1}}(C_{M_w}(hs\sigma h^{-1})^\circ)$ is $C_{M_w}(hs\sigma h^{-1})^\circ$ as this is a Levi subgroup. The $hL'^F h^{-1}$-conjugacy class of $C_{M_w}(hs\sigma h^{-1})^\circ$ corresponds to the L'^F-class of $C_{h^{-1}M_w h}(s\sigma)^\circ$, which is $\bar{\nu}$. We have

$$|N_{G^F}(M_w.\sigma)hL'^F| = |L'^F||N_{G^F}(M_w.\sigma)||N_{hL'^F h^{-1}}(C_{M_w}(hs\sigma h^{-1})^\circ)|^{-1}.$$

Since $N_G(M_w.\sigma)/M_w \cong W_G(M_w)^\sigma$ and $(W_G(M_w)^\sigma, F) \cong (W_G(M_0)^\sigma, F_w)$, we have

$$|N_{G^F}(M_w.\sigma)| = |M_w^F| z_w,$$

where z_w is the cardinality of the centraliser of w in $W_G(M_w)^\sigma \cong \mathfrak{W}_N^C$. Similarly,

$$|N_{hL'^F h^{-1}}(C_{M_w}(hs\sigma h^{-1})^\circ)| = |C_{M_w}(hs\sigma h^{-1})^{\circ F}| z_\nu,$$

where z_ν is the cardinality of the centraliser of ν in $W_{L'}(M_0')$.

With respect to the decomposition $M_w = T_w \times \mathrm{GL}_m(k)$, we write $\sigma = (\sigma_1, \sigma_{00})$, and the action of $hs\sigma h^{-1}$ on M_w is decomposed as $(l_1\sigma_1, l_0\sigma_{00})$ with $l_1 \in T_w$ and $l_0 \in \mathrm{GL}_m(k)$. The action of F on the orthogonal group factor of $C_{M_w}(hs\sigma h^{-1})^\circ$ is compatible with the action of F on $C_{h^{-1}M_w h}(s\sigma)^\circ$, which is an F-stable Levi subgroup of L', and so only depends on the G^F-conjugacy class of $s\sigma$ and $\bar{\nu}$. Explicitly, if we write $C_G(s\sigma)^\circ$ as the direct product of $\mathrm{SO}_{n'}(k)$

and some symplectic groups and general linear groups, then $C_G(s\sigma)^{\circ F}$ is isomorphic to the direct product of $\mathrm{SO}^{\eta}_{n'}(q)$ and groups of other types, where η is the value of $s\sigma$ under the map (IV.2.2.2.2). We can write the Levi subgroup $C_{h^{-1}M_wh}(s\sigma)^{\circ}$ as the direct product of L_{so} and some other groups, with $L_{so} \cong \mathrm{SO}_{h_2^2}(k) \times T_{\nu_{so}}$ being an F-stable Levi subgroup of $\mathrm{SO}_{n'}(k)$, where ν_{so} is the direct factor of $\nu \in W_{L'}(M'_w)$ corresponding to $\mathrm{SO}_{n'}(k)$ and $T_{\nu_{so}}$ is the thereby determined torus. Let $\mathrm{sgn} : \mathfrak{W}^C \to \{\pm 1\}$ be the map whose kernel is $\mathfrak{W}^D$. Then we have

$$C^{\circ}_{M_w}(hs\sigma h^{-1})^F \cong C^{\circ}_{T_w}(l_1\sigma_1)^F \times \mathrm{Sp}_{h_1(h_1+1)}(q) \times \mathrm{SO}^{\eta\cdot\mathrm{sgn}(\nu_{so})}_{h_2^2}(q).$$

Besides, $C^{\circ}_{T_w}(l_1\sigma_1)^F = C^{\circ}_{T_w}(\sigma_1) =: T'^F_w$.

Remark IV.6.2.7. The above calculation shows that if n is odd, then the cardinality of $C_{h^{-1}M_wh}(s\sigma)^{\circ F}$, which a priori depends on ν, in fact only depends on w. In the case of connected groups, this is trivial as $(h^{-1}M_wh)^F \cong M^F_w$.

Proposition IV.6.2.8. *If $C_{h^{-1}M_wh}(s\sigma)^{\circ} \subset L'$ lies in the class of ν, then we have*

$$|N^F_G(M_w.\sigma)hL'^F| = |L'^F||M^F_w||T'^F_w|^{-1}|\mathrm{Sp}_{h_1(h_1+1)}(q)|^{-1}|\mathrm{SO}^{\eta\cdot\mathrm{sgn}(\nu_{so})}_{h_2^2}(q)|^{-1}z_wz_\nu^{-1},$$

where η is determined by the G^F-class of $s\sigma$. In particular, if $h_1 = 0$ and $h_2 = 0$ or 1, and so M_w is a maximal torus of G, we have

$$|N^F_G(M_w.\sigma)hL'^F| = |L'^F||M^F_w||T'^F_w|^{-1}z_\tau z_\nu^{-1}. \tag{IV.6.2.6.1}$$

Chapter V

E-polynomial of $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$-Character Varieties

Except in §V.5, we only compute the E-polynomials of the $\mathrm{GL}_n(\mathbb{C}) \rtimes <\sigma>$-character varieties for n odd. We will write $n = 2N + 1$. We expect that the case for n even can be similarly tackled, but involves Green functions associated to complex reflections groups $G(2, 2, m)$, $m \in \mathbb{Z}_{>1}$ ([Sh02]). This will be done elsewhere.

We will compute the E-polynomial of the character variety over complex numbers

$$\text{(V.0.0.0.1)} \quad \mathrm{Ch}_C = \mathrm{Rep}_C /\!/ \mathrm{GL}_n = \{(A_i, B_i) \times (X_j) \in \mathrm{GL}_n^{2g} \times \prod_{j=1}^{2k} C_j \mid \prod_{i=1}^{g} (A_i, B_i) \prod_{j=1}^{2k} X_j = 1\} /\!/ \mathrm{GL}_n,$$

where $C_j \subset \mathrm{GL}_n .\sigma$ is a GL_n-conjugacy class for any j. We assume that the tuple of conjugacy class $C = (C_j)$ is generic (§III.4.2.5) and that for every j, C_j has no "eigenvalue" equal to i. The second assumption is only needed in Lemma V.3.3.3. As is explained in the introduction, the problem is translated into counting points over finite fields. The first step is to give an R-model of the character variety, with $R \subset \mathbb{C}$ being some finitely generated ring. We then give a point-counting formula in terms of the irreducible characters of $\mathrm{GL}_n(q) \rtimes <\sigma>$. By decomposing the irreducible characters into Deligne-Lusztig characters, this formula becomes a purely combinatorial one involving Green functions of finite classical groups. Using the results of Shoji [Sh01], we are able to express the final formula in terms of the inner product of two symmetric functions, with one of them only depending on the Riemann surface, and the other one only on the prescribed conjugacy classes at the punctures.

V.1 The Point-Counting Formula

V.1.1 The R-Model

V.1.1.1 Write $G = \mathrm{GL}_n$, $\bar{G} = \mathrm{GL}_n \rtimes <\sigma>$. For each j, the conjugacy class C_j is represented by some N-tuple of complex numbers (*cf.* Notation IV.2.1.1)

$$\mathbf{A}^j = [1, \dots, 1, \mathrm{i}, \dots, \mathrm{i}, a_1^j, \dots, a_1^j, \dots, a_{m_j}^j, \dots, a_{m_j}^j],$$

with $a_r^j \notin \{a_s^j, (a_s^j)^{-1}, -a_s^j, -(a_s^j)^{-1}\}$ for any $r \neq s$, $(a_r^j)^2 \neq \pm 1$ for any r. Define an unordered sequence of positive integers $\mu^j = \mu_+^j \mu_-^j (\mu_i^j)$ in such a way that 1 has multiplicity μ_+^j, i has multiplicity μ_-^j and a_i^j has multiplicity μ_i^j. We have $N = \mu_+^j + \mu_-^j + \sum_i \mu_i^j$. We denote $\mu_1^j = (\mu_i^j)$ and $m_j = l(\mu_1^j)$, the length of the sequence. We will call μ^j the *type* of C_j.

Denote by R_0 the subring of $\mathbb{C}$ generated by $\{(a_i^j)^{\pm 1} \mid \text{all } i, j\}$ (together with i if some $\mu_-^j > 0$). For any subtuple $\mathbf{B}^j \subset \mathbf{A}^j$ and any $|\mathbf{B}^j|$-tuple of signs $\mathbf{e}^j = (e_a^j)_{a \in \mathbf{B}^j}$, $e_a^j \in \{\pm 1\}$, we write $[\mathbf{B}^j]_{\mathbf{e}^j} = \prod_{a \in \mathbf{B}^j} (a^{e_a^j})^2$. Let $S \subset R_0$ be the multiplicative subset generated by

(i) $(a_{i_1}^j)^2 - (a_{i_2}^j)^2$ and $(a_{i_1}^j)^2 - (a_{i_2}^j)^{-2}$ for all j and $i_1 \neq i_2$;

(ii) $(a_i^j)^4 - 1$ for all j and i;

(iii) For every integer M, $1 \leq M \leq N$, the elements $1 - [\mathbf{B}^1]_{\mathbf{e}^1} \cdots [\mathbf{B}^{2k}]_{\mathbf{e}^{2k}}$ with each $[\mathbf{B}^j]$ running over all M-subtuples of $\mathbf{A}^j$, and each $\mathbf{e}^j$ running over all M-tuples of signs.

Define the ring of generic eigenvalues as $R = S^{-1} R_0$. Then the character variety is defined over R. For any field k and a homomorphism of rings $\phi : R \to k$, the homomorphism $\bar{\phi} : R \to k \hookrightarrow \bar{k}$ defines a $2k$-tuple of semisimple conjugacy classes (in $\mathrm{GL}_n(\bar{k}).<\sigma>$) of the same types, with "eigenvalues" (Remark IV.2.1.3) $\phi(a_i^j)$ (resp. 1 and i, which is a choice of the square root of -1 in k) of multiplicity μ_i^j (resp μ_+^j and μ_-^j) in the corresponding conjugacy class. The $2k$-tuple of conjugacy classes thus obtained is denoted by $C^{\bar{\phi}} = (C_1^{\bar{\phi}}, \dots, C_{2k}^{\bar{\phi}})$. Note that if $q \equiv 3 \mod 4$ and there is some $\mu_-^j > 0$, then there is no homomorphism $\phi : R \to \mathbb{F}_q$.

V.1.1.2 Let $\mathcal{A}_0$ be the polynomial ring over R with $n^2(2g+2k)$ indeterminates corresponding to the entries of $n \times n$ matrices $A_1, B_1, \dots A_g, B_g, X_1, \dots X_{2k}$, with $\det A_i$, $\det B_i$, $\det X_j$, $1 \leq i \leq g$, $1 \leq j \leq 2k$, inverted. Denote by (A, B) the commutator of matrices A and B. Let $I_0 \subset \mathcal{A}_0$ be the ideal generated by

(i) The entries of $(A_1, B_1) \cdots (A_g, B_g) X_1 \sigma \cdots X_{2k} \sigma - \mathrm{Id}$ (Note that $X_1 \sigma X_2 \sigma = X_1 \sigma(X_2)$ and σ is defined over $\mathbb{Z}$);

(ii) For all $1 \leq j \leq 2k$, the entries of

$$\begin{aligned} ((X_j \sigma(X_j))^2 - \mathrm{Id})(X_j \sigma(X_j) - (a_1^j)^2 \,\mathrm{Id})(X_j \sigma(X_j) - (a_1^j)^{-2} \,\mathrm{Id}) \cdots \\ (X_j \sigma(X_j) - (a_{m_j}^j)^2 \,\mathrm{Id})(X_j \sigma(X_j) - (a_{m_j}^j)^{-2} \,\mathrm{Id}); \end{aligned} \tag{V.1.1.2.1}$$

(iii) For all $1 \leq j < 2k$, the entries of the coefficients of the following polynomial in an auxiliary variable t:

$$\det(t\,\mathrm{Id} - X_j\sigma(X_j)) - (t-1)^{\tau_+}(t+1)^{\tau_-}\prod_i^{m_j}(t-(a_i^j)^2)^{\tau_i^j}. \tag{V.1.1.2.2}$$

Denote $\mathcal{A} = \mathcal{A}_0/\sqrt{I_0}$. If we write $C_j^2 := \{x^2 \mid x \in C_j\}$, then $C_j \subset G.\sigma$ is semi-simple if and only if C_j^2 is semi-simple and by §IV.2.1.1 the map $C_j \mapsto C_j^2$ is an injection from the semi-simple classes in $G.\sigma$ to those in G. Therefore (ii) and (iii) guarantees that the base change to $\mathbb{C}$ recovers the complex representation variety with the correct conjugacy classes. Then $\mathbf{Rep}_C := \operatorname{Spec}\mathcal{A}$ is the R-model of Rep_C and $\mathbf{Ch}_C := \operatorname{Spec}\mathcal{A}^{\mathrm{GL}(R)}$ is the R-model of Ch_C, since taking invariants commutes with flat base change.

V.1.1.3 Let $\phi : R \to \mathbb{F}_q$ be any ring homomorphism and let $\bar{\phi} : R \to \bar{\mathbb{F}}_q$ be the extension to the algebraic closure. Denote by $\mathrm{Rep}^{\phi}_{C^{\phi}}$ and $\mathrm{Rep}^{\bar{\phi}}_{C^{\bar{\phi}}}$ be the varieties over $\mathbb{F}_q$ and $\bar{\mathbb{F}}_q$ respectively obtained by base change from $\mathbf{Rep}_C$. And similarly for $\mathbf{Ch}_C$. The variety $\mathrm{Rep}^{\bar{\phi}}_{C^{\bar{\phi}}}(\bar{\mathbb{F}}_q)$ is defined by the same equation as V.0.0.0.1, but over $\bar{\mathbb{F}}_q$, and then

$$\mathrm{Rep}^{\phi}_{C^{\phi}}(\mathbb{F}_q) = \{(A_i, B_i) \times (X_j) \in \mathrm{GL}_n(q)^{2g} \times \prod_{j=1}^{2k} C_j^{\phi}(\mathbb{F}_q) \mid \prod_{i=1}^{g}(A_i, B_i)\prod_{j=1}^{2k} X_j = 1\}. \tag{V.1.1.3.1}$$

Remark V.1.1.1. One should not confuse C^{ϕ} with the conjugacy classes of finite groups. Here we really consider varieties over $\mathbb{F}_q$ whose base changes to $\bar{\mathbb{F}}_q$ give $C^{\bar{\phi}}$. Note that for any j, we have $C_j^{\phi}(\mathbb{F}_q) = C_j^{\bar{\phi}}(\bar{\mathbb{F}}_q)^F$, which is a union of two conjugacy classes contained in $\mathrm{GL}_n(q).\sigma$. We denote these two classes by $C_{j,+}$ and $C_{j,-}$. The class $C_{j,+}$ has a representative of the form

$$\mathrm{diag}(x_1, x_2, \ldots, x_N, 1, x_N^{-1}, \ldots, x_2^{-1}, x_1^{-1})\sigma, \tag{V.1.1.3.2}$$

with every $x_i \in \mathbb{F}_q$, while $C_{j,-}$ is represented by

$$\mathrm{diag}(x_1, x_2, \ldots, x_N, c, x_N^{-1}, \ldots, x_2^{-1}, x_1^{-1})\sigma, \tag{V.1.1.3.3}$$

with $c \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$.

Notation V.1.1.2. For a given $2k$-tuple $C^{\bar{\phi}} = (C_j^{\bar{\phi}})$ of semi-simple conjugacy classes in $\mathrm{GL}_n(\bar{\mathbb{F}}_q).\sigma$ and a $2k$-tuple of signs $\mathbf{e} = (e_j)$, we will denote by $C_{\mathbf{e}} = (C_{j,e_j})$ the $2k$-tuple of $\mathrm{GL}_n(q)$-conjugacy classes contained in it. Denote $\bar{\mathbf{e}} = \prod_j e_j$ and $\operatorname{sgn} C_{\mathbf{e}} = \bar{\mathbf{e}}$. We may write $\operatorname{sgn} C = \operatorname{sgn} C_{\mathbf{e}}$ if C is a tuple of conjugacy classes of the form $C_{\mathbf{e}}$.

In the following Proposition, we fix the conjugacy classes and omit the script C.

Proposition V.1.1.3. *We have the following formula.*

$$|\mathrm{Ch}^{\phi}(\mathbb{F}_q)| = \frac{1}{G(\mathbb{F}_q)}|\mathrm{Rep}^{\phi}(\mathbb{F}_q)|. \tag{V.1.1.3.4}$$

Proof. By [Se77, II, Theorem 3], there is a natural bijection

$$\mathrm{Ch}^{\bar{\phi}}(\bar{\mathbb{F}}_q) \xleftrightarrow{\sim} (\mathrm{Rep}^{\bar{\phi}}/\!\!/G^{\bar{\phi}})(\bar{\mathbb{F}}_q). \tag{V.1.1.3.5}$$

Since $C^{\bar{\phi}}$ is generic, each element of $\mathrm{Rep}^{\bar{\phi}}(\bar{\mathbb{F}}_q)$ is an irreducible $\mathrm{GL}_n(\bar{\mathbb{F}}_q) \rtimes\!<\sigma>$representation, by Proposition III.4.2.7. By Proposition II.5.4.1, irreducible representations have finite stabilisers, thus every $\mathrm{GL}_n(\bar{\mathbb{F}}_q)$-orbit is closed. We then have a natural bijection of sets $(\mathrm{Rep}^{\bar{\phi}}/\!\!/G^{\bar{\phi}})(\bar{\mathbb{F}}_q) \xrightarrow{\sim} \mathrm{Rep}^{\bar{\phi}}(\bar{\mathbb{F}}_q)/G(\bar{\mathbb{F}}_q)$. Therefore,

$$\mathrm{Ch}^{\phi}(\mathbb{F}_q) \cong \mathrm{Ch}^{\bar{\phi}}(\bar{\mathbb{F}}_q)^F \cong (\mathrm{Rep}^{\bar{\phi}}(\bar{\mathbb{F}}_q)/G(\bar{\mathbb{F}}_q))^F. \tag{V.1.1.3.6}$$

Since G is connected, each F-stable $G(\bar{\mathbb{F}}_q)$-orbit in $\mathrm{Rep}^{\bar{\phi}}(\bar{\mathbb{F}}_q)$ must contain some F-stable point. We will prove that the number of F-stable points in each such orbit is exactly $|G(\mathbb{F}_q)|$.

Let O be an F-stable $G(\bar{\mathbb{F}}_q)$-orbit in $\mathrm{Rep}^{\bar{\phi}}(\bar{\mathbb{F}}_q)$, then O^F splits into some $G(\mathbb{F}_q)$-orbits according to the stabliliser in G of some F-stable point, say $x \in O^F$. By Proposition II.5.4.1, the stabiliser is a finite abelian group $H := \prod_j \mu_2$. The number of $G(\mathbb{F}_q)$-orbits in O^F is equal to the number of F-conjugacy classes in H.

Since H is abelian, each F-conjugacy class of it is of the form $\{h_0 hF(h)^{-1} \mid h \in H\}$ for some $h_0 \in H$. Again because H is abelian, the map $h \mapsto hF(h)^{-1}$ is a group homomorphism, with kernel $K = \{h \in H \mid F(h) = h\}$. Denote by I the image of this homomorphism. Then the F-conjugacy classes in H are the cosets $h_0 I$, therefore there are $|H|/|I| = |K|$ of them.

That is, the number of $G(\mathbb{F}_q)$-orbits in O is $|K|$. On the other hand, x has K as stabiliser in $G(\mathbb{F}_q)$, so the cardinality of the $G(\mathbb{F}_q)$-orbit containing x is $|G(\mathbb{F}_q)|/|K|$. If for some $g \in G(\bar{\mathbb{F}}_q)$, $g.x$ is an F-stable point contained in another $G(\mathbb{F}_q)$-orbit, then its stabiliser in $G(\mathbb{F}_q)$ is just gKg^{-1}. Indeed, we have $g^{-1}F(g) \in \mathrm{Stab}_G(x) = H$, thus for any $k \in K$,

$$F(gkg^{-1}) = F(g)kF(g)^{-1} = g(g^{-1}F(g))k(F(g)^{-1}g)g^{-1} = gkg^{-1}. \tag{V.1.1.3.7}$$

Thus all $G(\mathbb{F}_q)$-orbits in O have cardinality $|G(\mathbb{F}_q)|/|K|$. We conclude that $|O(\mathbb{F}_q)| = |G(\mathbb{F}_q)|$, which is independent of O. Therefore,

$$|\mathrm{Ch}^{\phi}(\mathbb{F}_q)| = \frac{1}{|G(\mathbb{F}_q)|}|\mathrm{Rep}^{\phi}(\mathbb{F}_q)|. \tag{V.1.1.3.8}$$

□

V.1.2 The Counting Formula

V.1.2.1 We denote for the moment by G and $\bar{G}$ the corresponding finite groups $G(\mathbb{F}_q)$ and $\bar{G}(\mathbb{F}_q)$, and by C an arbitrary $2k$-tuple of G-conjugacy classes contained in $\bar{G} \setminus G$. We will show the following counting formula,

$$\begin{aligned} &\#\{(A_1, B_1, \cdots, A_g, B_g, X_1, \cdots, X_{2k}) \in G^{2g} \times C_1 \cdots \times C_{2k} | \\ &\qquad\qquad (A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_{2k} = \mathrm{Id}\} \\ &= |G| \sum_{\chi \in \mathrm{Irr}(G)^\sigma} \Big(\frac{|G|}{\chi(1)}\Big)^{2g-2} \prod_{i=1}^{2k} \frac{|C_i| \tilde{\chi}(C_i)}{\chi(1)}. \end{aligned} \tag{V.1.2.1.1}$$

In this expression, $\mathrm{Irr}(G)^\sigma$ is the set of σ-stable irreducible characters of G, and we choose for each $\chi \in \mathrm{Irr}(G)^\sigma$ an extension to $\bar{G}$, denoted by $\tilde{\chi}$. The two possible extensions of a given irreducible character differ by -1 on $\bar{G} \setminus G$, and consequently the value of the counting formula is independent of these choices due to the product of $2k$ terms.

V.1.2.2 Let us prepare some notations of finite groups.

Let H be a finite group. Denote by $C(H)$ the vector space of complex class functions on H. Put $\hat{H} = \mathrm{Irr}(H)$. Denote by $C(\hat{H})$ the vector space of linear combinations $\sum_{\chi \in \hat{H}} a_\chi \chi$ of irreducible characters, which is the same as $C(H)$ but will be equipped with a different product operation.

The convolution product $*$ in $C(H)$ is defined by

$$(f_1 * f_2)(x) = \sum_{yz=x} f_1(y) f_2(z), \quad f_1, f_2 \in C(H). \tag{V.1.2.2.1}$$

The dot product $\cdot$ in $C(\hat{H})$ is defined by

$$F_1 \cdot F_2(\chi) = F_1(\chi) F_2(\chi), \quad F_1, F_2 \in C(\hat{H}), \tag{V.1.2.2.2}$$

i.e. the coefficient of χ in the product is the product of coefficients of χ.

The Fourier transform $\mathcal{F} : C(H) \to C(\hat{H})$ is defined by

$$\mathcal{F}(f)(\chi) = \sum_{h \in H} \frac{f(h)\chi(h)}{\chi(1)}, \quad f \in C(H). \tag{V.1.2.2.3}$$

We also have the transform $\mathbb{F} : C(\hat{H}) \to C(H)$ defined by

$$\mathbb{F}(F)(h) = \sum_{\chi \in \hat{H}} F(\chi) \chi(1) \overline{\chi(h)}. \tag{V.1.2.2.4}$$

We have

$$\mathbb{F} \circ \mathcal{F} = |H| \cdot \mathrm{Id}_{C(H)}, \quad \mathcal{F} \circ \mathbb{F} = |H| \cdot \mathrm{Id}_{C(\hat{H})}. \tag{V.1.2.2.5}$$

They are also compatible with the product operations:

$$\mathcal{F}(f_1) \cdot \mathcal{F}(f_2) = \mathcal{F}(f_1 * f_2), \quad \mathbb{F}(F_1) * \mathbb{F}(F_2) = |H| \cdot \mathbb{F}(F_1 \cdot F_2). \tag{V.1.2.2.6}$$

V.1.2.3 From $\mathbb{F} \circ \mathcal{F} = |H| \cdot \mathrm{Id}_{C(H)}$ and the definition of the transforms, we deduce that

$$f(1) = \frac{1}{|H|} \sum_{\chi \in \hat{H}} \chi(1)^2 \mathcal{F}(f)(\chi). \tag{V.1.2.3.1}$$

for any class function f.

Now we put $H = \bar{G}$. We define the function $n^g : \bar{G} \to \mathbb{C}$ by

$$n^g(x) = \#\{(A_1, B_1, \cdots, A_g, B_g) \in G^{2g} | \prod_i (A_i, B_i) = x\}. \tag{V.1.2.3.2}$$

We find that $n^g \in C(\bar{G})$(identically zero on $\bar{G} \setminus G$) and that $n^g = n^1 * \cdots * n^1$. Denote by 1_{C_i} the characteristic function of the class C_i. Then

$$\begin{aligned} &\#\{(A_1, B_1, \cdots, A_g, B_g, X_1, \cdots, X_{2k}) \in G^{2g} \times C_1 \times \cdots \times C_{2k} | \prod_i (A_i, B_i) \prod_j X_j = \mathrm{Id}\} \\ =&(n^g * 1_{C_1} * \cdots * 1_{C_{2k}})(1). \end{aligned} \tag{V.1.2.3.3}$$

By (V.1.2.2.6) and (V.1.2.3.1), we have

$$\begin{aligned} &(n^g * 1_{C_1} * \cdots * 1_{C_{2k}})(1) \\ =&\frac{1}{2|G|} \sum_{\chi \in \mathrm{Irr}(\bar{G})} \chi(1)^2 (\mathcal{F}(n^1)(\chi))^g \prod_{i=1}^{2k} \frac{|C_i| \chi(C_i)}{\chi(1)}. \end{aligned} \tag{V.1.2.3.4}$$

It is known ([HLR, Lemma 3.1.3]) that

$$\mathcal{F}(n^1)(\chi) = \Big(\frac{|G|}{\chi(1)}\Big)^2. \tag{V.1.2.3.5}$$

Finally,

$$\begin{aligned} &\frac{1}{2|G|} \sum_{\chi \in \mathrm{Irr}(\bar{G})} \chi(1)^2 (\mathcal{F}(n^1)(\chi))^g \prod_{i=1}^{2k} \frac{|C_i| \chi(C_i)}{\chi(1)} \\ =&|G| \sum_{\chi \in \mathrm{Irr}(G)^\sigma} \left(\frac{|G|}{\chi(1)}\right)^{2g-2} \prod_{i=1}^{2k} \frac{|C_i| \tilde{\chi}(C_i)}{\chi(1)}. \end{aligned} \tag{V.1.2.3.6}$$

V.2 Symmetric Functions Associated to Wreath Products

This is a reminder of some results of [Sh01]. Some lemmas are proved that will be used later. We will freely use the notions and notations in §II.2.1 and §II.2.2. The letter q will primarily denote an indeterminate, but can appear as a prime power in a finite group of Lie type.

V.2.1 Ring of Symmetric Functions

V.2.1.1 For $k = 0$ and 1, let $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots)$ be an infinite series of variables. Denote by $\mathbf{x} = (\mathbf{x}^{(0)}, \mathbf{x}^{(1)})$ all of the variables. For any commutative ring with unity R, denote by $\mathbf{Sym}_R[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$ the ring of symmetric functions in two variables (symmetric in each $\mathbf{x}^{(\mathrm{k})}$) with coefficents in R. We will always assume that $\mathbb{Q}$ is contained in R. The usual choice of R will be the fields $\mathbb{Q}$, $\mathbb{Q}(q)$ and $\mathbb{Q}(q,t)$, i.e. the functions fields in variables q and t. If the base field is $\mathbb{Q}(q,t)$, then we may write $\mathbf{Sym}_{q,t}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$ for the corresponding ring of symmetric functions, and similarly for $\mathbb{Q}(q)$. We will omit the lower script R if there is no confusion with the base. For any $r \geq 1$, put

$$\begin{aligned} p_r^{(0)}(\mathbf{x}) &:= p_r(\mathbf{x}^{(0)}) + p_r(\mathbf{x}^{(1)}), \\ p_r^{(1)}(\mathbf{x}) &:= p_r(\mathbf{x}^{(0)}) - p_r(\mathbf{x}^{(1)}), \end{aligned} \tag{V.2.1.1.1}$$

where $p_r(\mathbf{x}^{(k)})$ is the usual power sum in $\mathbf{x}^{(\mathrm{k})}$. Put $p_0^{(k)}(\mathbf{x}) = 1$ for both values of k. For a 2-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$, with each partition $\lambda^{(k)}$ written as $\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots$, the *power sum* in $\mathbf{x}$ is defined by

$$p_{\boldsymbol{\lambda}}(\mathbf{x}) := \prod_i p^{(0)}_{\lambda_i^{(0)}}(\mathbf{x}) \prod_j p^{(1)}_{\lambda_j^{(1)}}(\mathbf{x}). \tag{V.2.1.1.2}$$

The *Schur function* is defined by

$$s_{\boldsymbol{\lambda}}(\mathbf{x}) := s_{\lambda^{(0)}}(\mathbf{x}^{(0)}) s_{\lambda^{(1)}}(\mathbf{x}^{(1)}), \tag{V.2.1.1.3}$$

the *monomial symmetric function* is defined by

$$m_{\boldsymbol{\lambda}}(\mathbf{x}) := m_{\lambda^{(0)}}(\mathbf{x}^{(0)}) m_{\lambda^{(1)}}(\mathbf{x}^{(1)}), \tag{V.2.1.1.4}$$

and the *complete symmetric function* is defined by

$$h_{\boldsymbol{\lambda}}(\mathbf{x}) := h_{\lambda^{(0)}}(\mathbf{x}^{(0)}) h_{\lambda^{(1)}}(\mathbf{x}^{(1)}), \tag{V.2.1.1.5}$$

where $s_{\lambda^{(k)}}(\mathbf{x}^{(\mathrm{k})})$, $m_{\lambda^{(k)}}(\mathbf{x}^{(\mathrm{k})})$, $h_{\lambda^{(k)}}(\mathbf{x}^{(\mathrm{k})})$ are the usual Schur functions, monomial symmetric functions and complete symmetric functions. The usual results on $\mathbf{Sym}[\mathbf{z}]$ implies that each class of these functions, with $\boldsymbol{\lambda}$ running through all 2-partitions, forms a basis of

$\mathbf{Sym}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$.

V.2.1.2 There is a family of symmetric functions $q_r^{(k)}(\mathbf{x}; q) \in \mathbf{Sym}_q[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$, $k = 0, 1$, $r \geq 0$ with the generating series

$$\begin{aligned} \sum_{r=0}^{\infty} q_r^{(0)}(\mathbf{x}, q) u^r &= \frac{\prod_i \left(1 - q u x_i^{(1)}\right)}{\prod_j \left(1 - u x_j^{(0)}\right)}, \\ \sum_{r=0}^{\infty} q_r^{(1)}(\mathbf{x}, q) u^r &= \frac{\prod_i \left(1 - q u x_i^{(0)}\right)}{\prod_j \left(1 - u x_j^{(1)}\right)}. \end{aligned} \tag{V.2.1.2.1}$$

For any 2-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \alpha^{(1)})$, we define

$$q_{\boldsymbol{\alpha}}(\mathbf{x}; q) := \prod_i q^{(0)}_{\alpha_i^{(0)}}(\mathbf{x}; q) \prod_j q^{(1)}_{\alpha_j^{(1)}}(\mathbf{x}; q). \tag{V.2.1.2.2}$$

Then $\{q_{\boldsymbol{\alpha}}(\mathbf{x}; q)\}_{\boldsymbol{\alpha} \in \mathcal{P}^2}$ forms a basis of $\mathbf{Sym}_q[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$.

V.2.1.3 Given some positive integers m_0, m_1, and some integers $r \geq s \geq 0$, let $\boldsymbol{\Lambda}^0 = (\Lambda_0, \Lambda_1)$ be the 2-partition defined by

$$\begin{aligned} &\Lambda_0 : (m_0 - 1)r \geq \cdots \geq 2r \geq r \geq 0, \\ &\Lambda_1 : (m_1 - 1)r + s \geq \cdots \geq 2r + s \geq r + s \geq s. \end{aligned} \tag{V.2.1.3.1}$$

For any 2-partition $\boldsymbol{\alpha}$, define $\boldsymbol{\Lambda}(\boldsymbol{\alpha}) := \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, with m_0 and m_1 larger than the size of $\boldsymbol{\alpha}$; we call it the symbol of type (r, s) associated to $\boldsymbol{\alpha}$. The correct definition of symbols (see for example [Sh01, §1.2]) is actually an equivalence class of the data described above, which is independent of the choice of m_0 and m_1, but it is convenient to work with a particular representative of the equivalence class. The size of $\boldsymbol{\Lambda}(\boldsymbol{\alpha})$ is defined to be the size of $\boldsymbol{\alpha}$. If $\boldsymbol{\alpha}$ is fixed, we may simply write $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$.

For any symbol $\boldsymbol{\Lambda}$ of type (r, s), by choosing a representative, we define the function

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda, \lambda' \in \boldsymbol{\Lambda}} \min(\lambda, \lambda') - \sum_{\mu, \mu' \in \boldsymbol{\Lambda}^0} \min(\mu, \mu'), \tag{V.2.1.3.2}$$

where the sums run over the entries of the corresponding symbols, and in the first sum we require that $\lambda \neq \lambda'$ if they are contained in $\Lambda^{(k)}$ for some k. (Note that $\lambda = \lambda'$ in some $\Lambda^{(k)}$ only if $r = 0$. We are however only interested in the case $r = 2$.) The value is independent of the representative of a symbol and is constant on the similarity classes of symbols. For symbols of size n of type $(2, 1)$ (resp. $(2, 0)$), each symbol $\boldsymbol{\Lambda}$ corresponds to a unipotent conjugacy class of $\mathrm{Sp}_{2n}(q)$ (resp. $\mathrm{SO}_{2n+1}(q)$, in which case there are 2 conjugacy classes corresponding to a

degenerate symbol). If we denote by u an element of this conjugacy class, then $a(\mathbf{\Lambda})$ is equal to $\dim \mathcal{B}_u$, where $\mathcal{B}_u$ is the Springer fiber over u. We may write $a(\boldsymbol{\alpha}) := a(\mathbf{\Lambda}(\boldsymbol{\alpha}))$. We will fixe once and for all a total order $\prec$ on $\mathcal{P}_n(2)$ in such a way that $a(\boldsymbol{\alpha}) \leq a(\boldsymbol{\beta})$ whenever $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ and that each similarity class (transferred from symbols) forms an interval. In particular, the element $(\varnothing, (1^n))$ corresponding to the the identity of the finite classical group, is minimal, and it is alone in its similarity class.

We have the function defined on the set of partitions: $n(\lambda) := \sum_i (i-1)\lambda_i$, for $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. This function is the analogue of $a(-)$ for partitions.

V.2.1.4 Denote by $\mathcal{A} \subset \mathbb{Q}(q)$ the subring of functions that are regular at $q = 0$.

Theorem V.2.1.1 ([Sh01] Theorem 4.4)**.** *For any $\mathbf{\Lambda} = \mathbf{\Lambda}(\boldsymbol{\alpha})$, there exists a unique function $P_{\mathbf{\Lambda}}(\mathbf{x}; q) \in \mathbf{Sym}_q[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$ satisfying the following properties.*

i) $P_{\mathbf{\Lambda}}(\mathbf{x}; q)$ can be expressed as

$$P_{\mathbf{\Lambda}}(\mathbf{x}; q) = \sum_{\beta} c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) q_{\boldsymbol{\beta}}(\mathbf{x}; q), \tag{V.2.1.4.1}$$

where $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) \in \mathbb{Q}(q)$ and $c_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) = 0$ unless $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$.

ii) $P_{\mathbf{\Lambda}}(\mathbf{x}; q)$ can be expressed as

$$P_{\mathbf{\Lambda}}(\mathbf{x}; q) = s_{\boldsymbol{\alpha}}(\mathbf{x}) + \sum_{\beta} u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) s_{\boldsymbol{\beta}}(\mathbf{x}), \tag{V.2.1.4.2}$$

where $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) \in q\mathcal{A}$ and $u_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \nsim \boldsymbol{\alpha}$.

Similarly, for any $\mathbf{\Lambda}$, there exists a unique function $Q_{\mathbf{\Lambda}}(\mathbf{x}; q)$ satisfying the following properties.

ii) $Q_{\mathbf{\Lambda}}(\mathbf{x}; q)$ can be expressed as

$$Q_{\mathbf{\Lambda}}(\mathbf{x}; q) = q_{\boldsymbol{\alpha}}(\mathbf{x}; q) + \sum_{\beta} d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) q_{\boldsymbol{\beta}}(\mathbf{x}; q), \tag{V.2.1.4.3}$$

where $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) \in \mathbb{Q}(q)$ and $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) = 0$ unless $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ ou $\boldsymbol{\beta} \nsim \boldsymbol{\alpha}$.

ii) $Q_{\mathbf{\Lambda}}(\boldsymbol{x}; t)$ can be expressed as

$$Q_{\mathbf{\Lambda}}(\mathbf{x}; q) = \sum_{\beta} w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) s_{\boldsymbol{\beta}}(\mathbf{x}), \tag{V.2.1.4.4}$$

where $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) \in \mathcal{A}$ and $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) = 0$ unless $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$ or $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$. Moreover, $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(q) \in q\mathcal{A}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$, and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(q) \in \mathcal{A}^*$.

The functions $P_{\mathbf{\Lambda}}(\mathbf{x}; q)$ and $Q_{\mathbf{\Lambda}}(\mathbf{x}; q)$ are called *Hall-Littlewood functions*. Each of these two families of functions also forms a basis of $\mathbf{Sym}_q[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$. Note that the functions $P^{\pm}_{\mathbf{\Lambda}}(\mathbf{x}; q)$ and $Q^{\pm}_{\mathbf{\Lambda}}(\mathbf{x}; q)$ defined for more general complex reflection groups degenerate into $P_{\mathbf{\Lambda}}(\mathbf{x}; q)$ and $Q_{\mathbf{\Lambda}}(\mathbf{x}; q)$ for Weyl groups of classical groups.

V.2.2 Transition Matrices

V.2.2.1 For any partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$, define $z_\lambda = \prod_i i^{m_i} m_i!$. Write λ as $(\lambda_1 \geq \lambda_2 \geq \cdots)$ and define $z_\lambda(q) = z_\lambda \prod_i (1-q^{\lambda_i})^{-1}$. For any 2-partition $\boldsymbol{\lambda} = (\lambda^{(0)}, \lambda^{(1)})$, define $z_{\boldsymbol{\lambda}} = 2^{l(\boldsymbol{\lambda})} z_{\lambda^{(0)}} z_{\lambda^{(1)}}$. For $k = 0$ and 1, we write $\lambda^{(k)} = (\lambda_1^{(k)} \geq \lambda_2^{(k)} \geq \cdots \geq \lambda_{l(\lambda^{(k)})}^{(k)})$, and define

$$
\begin{aligned}
z_{\lambda^{(0)}}(q) &= \prod_j \left(1 - q^{\lambda_j^{(0)}}\right)^{-1}, \\
z_{\lambda^{(1)}}(q) &= \prod_j \left(1 + q^{\lambda_j^{(1)}}\right)^{-1}, \\
z_{\boldsymbol{\lambda}}(q) &= z_{\boldsymbol{\lambda}} z_{\lambda^{(0)}}(q) z_{\lambda^{(1)}}(q).
\end{aligned}
\tag{V.2.2.1.1}
$$

Note that $z_{\boldsymbol{\lambda}}(0) = z_{\boldsymbol{\lambda}}$.

V.2.2.2 We have,

$$
\begin{aligned}
p_{\boldsymbol{\beta}}(\mathbf{x}) &= \sum_{\boldsymbol{\alpha} \in \mathcal{P}_{n,2}} \chi_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}}\, s_{\boldsymbol{\alpha}}(\mathbf{x}); \\
s_{\boldsymbol{\lambda}}(\mathbf{x}) &= \sum_{\boldsymbol{\tau}} \frac{1}{z_{\boldsymbol{\tau}}} \chi_{\boldsymbol{\tau}}^{\boldsymbol{\lambda}}\, p_{\boldsymbol{\tau}}(\mathbf{x}),
\end{aligned}
\tag{V.2.2.2.1}
$$

where $\chi_{\boldsymbol{\beta}}^{\boldsymbol{\alpha}}$ is the value of the character of $(\mathbb{Z}/2\mathbb{Z})^m \rtimes \mathfrak{S}_m$ ($m = |\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$) of class $\boldsymbol{\alpha}$ at an element of class $\boldsymbol{\beta}$.

V.2.2.3 The *Kostka-Foulkes polynomials* are the entries of the transition matrix

$$
s_{\boldsymbol{\beta}}(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} K_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q) P_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(\mathbf{x}; q),
\tag{V.2.2.3.1}
$$

and the *modified Kostka-Foulkes polynomials* are defined by $\tilde{K}_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q) = q^{a(\boldsymbol{\Lambda})} K_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q^{-1})$, with $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$.

The *Green function* is then defined by

$$
Q_{\boldsymbol{\beta}}^{\boldsymbol{\Lambda}}(q) = \sum_{\boldsymbol{\gamma}} \chi_{\boldsymbol{\beta}}^{\boldsymbol{\gamma}} \tilde{K}_{\boldsymbol{\gamma},\boldsymbol{\alpha}}(q).
\tag{V.2.2.3.2}
$$

V.2.2.4 The *transformed Hall-Littlewood function* is defined by

$$
\bar{H}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(\mathbf{x}; q) = \sum_{\boldsymbol{\beta}} K_{\boldsymbol{\beta},\boldsymbol{\alpha}}(q) s_{\boldsymbol{\beta}}(\mathbf{x}),
\tag{V.2.2.4.1}
$$

and the *modified Hall-Littlewood function* is defined by

$$\tilde{H}_{\mathbf{\Lambda}(\boldsymbol{\alpha})}(\mathbf{x};q) = \sum_{\beta} \tilde{K}_{\beta,\boldsymbol{\alpha}}(q) s_{\beta}(\mathbf{x}). \tag{V.2.2.4.2}$$

V.2.3 Orthogonality

V.2.3.1 Plethysm Let $\mathbb{K}$ be a base field, which will be $\mathbb{Q}$, $\mathbb{Q}(q)$ or $\mathbb{Q}(q,t)$ depending on the circumstance. Let R be a λ-ring containing $\mathbb{K}$, then the ring $\mathbf{Sym}_R[\mathbf{z}]$ of symmetric functions over R has a natural λ-ring structure $\{p_n\}_{n\in\mathbb{Z}_{>0}}$ with p_n sending $p_1(\mathbf{z})$ to $p_n(\mathbf{z})$ and acting on the coefficients according to the λ-ring structure of R. If we take $R = \mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)}]$, then $\mathbf{Sym}_R[\mathbf{x}^{(1)}] \cong \mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)},\mathbf{x}^{(1)}]$ acquires a natural λ-ring structure, and p_n sends $p_1(\mathbf{x}^{(0)})$ to $p_n(\mathbf{x}^{(0)})$ and sends $p_1(\mathbf{x}^{(1)})$ to $p_n(\mathbf{x}^{(1)})$. If $\mathbb{K}$ contains the indeterminates q or t, then p_n sends them to the n'th power. By abuse of notation, we will write $\mathbf{x}^{(\mathrm{k})} = p_1(\mathbf{x}^{(\mathrm{k})})$.

Let A be a λ-ring containg $\mathbb{K}$ as a λ-subring. Given any elements f_0 and f_1 of A, there is a unique λ-ring homomorphism $\varphi_{f_0,f_1} : \mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)},\mathbf{x}^{(1)}] \to A$ sending $\mathbf{x}^{(0)}$ to f_0 and $\mathbf{x}^{(1)}$ to f_1. For any $F \in \mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)},\mathbf{x}^{(1)}]$, its image under this map is denoted by $F[f_0,f_1]$. Concretely, by expressing F as a polynomial $f(p_n(\mathbf{x}^{(0)}), p_m(\mathbf{x}^{(1)}) \mid n,m \in \mathbb{Z}_{>0})$, one defines $F[f_0,f_1] := f(p_n(f_0), p_m(f_1) \mid n,m \in \mathbb{Z}_{>0})$. This is the plethystic substitution for F. For example, if $F = s_\alpha(\mathbf{x}^{(0)})$ is the usual Schur symmetric function in $\mathbf{x}^{(0)}$ for some partition α, and $A = \mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)},\mathbf{x}^{(1)}]$, then $F[\mathbf{x}^{(1)},\mathbf{x}^{(0)}] = s_\alpha(\mathbf{x}^{(1)})$.

V.2.3.2 The Hall inner product on $\mathbf{Sym}_{\mathbb{K}}[\mathbf{z}]$ is defined by

$$\langle p_\lambda(\mathbf{z}), p_\mu(\mathbf{z})\rangle = z_\lambda \delta_{\lambda,\mu}, \tag{V.2.3.2.1}$$

for all partitions λ and μ. Define the symmetric function

$$H(\mathbf{z},q) = \prod_i \frac{1}{1-qz_i}. \tag{V.2.3.2.2}$$

Let $\{u_\lambda\}_{\lambda\in\mathcal{P}}$ and $\{v_\lambda\}_{\lambda\in\mathcal{P}}$ be two families of symmetric functions such that for any $n \in \mathbb{Z}_{>0}$, $\{u_\lambda\}_{\lambda\in\mathcal{P}_n}$ and $\{v_\lambda\}_{\lambda\in\mathcal{P}_n}$ are $\mathbb{K}$-basis of the subspace of $\mathbf{Sym}_{\mathbb{K}}[\mathbf{z}]$ of homogenous degree n. Then the following conditions are equivalent:

- $\langle u_\lambda, v_\mu\rangle = \delta_{\lambda,\mu}$ for all λ, μ;
- $\sum_\lambda u_\lambda(\mathbf{x}) v_\lambda(\mathbf{y}) = H(\mathbf{xy},1)$.

It follows that,

$$\langle s_\lambda(\mathbf{z}), s_\mu(\mathbf{z})\rangle = \delta_{\lambda,\mu}. \tag{V.2.3.2.3}$$

Put

$$\Omega(\mathbf{x}\,\mathbf{y}, q) = \prod_{i,j} \frac{1 - qx_i y_j}{1 - x_i y_j}, \tag{V.2.3.2.4}$$

then

$$\Omega(\mathbf{x}\,\mathbf{y}, q) = H(\mathbf{xy}, 1)H(\mathbf{xy}, q)^{-1},$$

and in particular $\Omega(\mathbf{x}\,\mathbf{y}, 0) = H(\mathbf{xy}, 1)$. We deduce that

$$\Omega[(1-q)\,\mathbf{x}\,\mathbf{y}, 0] = H[(1-q)\mathbf{xy}, 1] = H(\mathbf{xy}, 1)H(\mathbf{xy}, q)^{-1} = \Omega(\mathbf{x}\,\mathbf{y}, q). \tag{V.2.3.2.5}$$

V.2.3.3 The Hall inner product on $\mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$ is defined by

$$\langle p_{\boldsymbol{\alpha}}(\mathbf{x}), p_{\boldsymbol{\beta}}(\mathbf{x})\rangle = z_{\boldsymbol{\alpha}}\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{V.2.3.3.1}$$

for any 2-partitions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

The q-inner product on $\mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)}, \mathbf{x}^{(1)}]$ is defined by

$$\langle p_{\boldsymbol{\alpha}}(\mathbf{x}), p_{\boldsymbol{\beta}}(\mathbf{x})\rangle_q = z_{\boldsymbol{\alpha}}(q)\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \tag{V.2.3.3.2}$$

for all 2-partitions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Define

$$\begin{aligned}\Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, q) :=& \prod_{i,j} \frac{1 - q\mathbf{x}_i^{(1)}\mathbf{y}_j^{(0)}}{1 - \mathbf{x}_i^{(0)}\mathbf{y}_j^{(0)}} \prod_{i,j} \frac{1 - q\mathbf{x}_i^{(0)}\mathbf{y}_j^{(1)}}{1 - \mathbf{x}_i^{(1)}\mathbf{y}_j^{(1)}} \\ =& H(\mathbf{x}^{(0)}\,\mathbf{y}^{(0)}, 1)H(\mathbf{x}^{(1)}\,\mathbf{y}^{(0)}, q)H(\mathbf{x}^{(1)}\,\mathbf{y}^{(1)}, 1)H(\mathbf{x}^{(0)}\,\mathbf{y}^{(1)}, q).\end{aligned} \tag{V.2.3.3.3}$$

Note that $\Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, 0) = \Omega(\mathbf{x}^{(0)}\,\mathbf{y}^{(0)}, 0)\Omega(\mathbf{x}^{(1)}\,\mathbf{y}^{(1)}, 0)$. We have

$$\Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, q) = \sum_{\boldsymbol{\alpha}\in\mathcal{P}^2} z_{\boldsymbol{\alpha}}(q)^{-1} p_{\boldsymbol{\alpha}}(\mathbf{x})p_{\boldsymbol{\alpha}}(\mathbf{y}). \tag{V.2.3.3.4}$$

In particular,

$$\begin{aligned}\Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, 0) &= \sum_{\boldsymbol{\alpha}} z_{\boldsymbol{\alpha}}^{-1} p_{\boldsymbol{\alpha}}(\mathbf{x})p_{\boldsymbol{\alpha}}(\mathbf{y}), \\ \Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, 0) &= \sum_{\boldsymbol{\alpha}} s_{\boldsymbol{\alpha}}(\mathbf{x})s_{\boldsymbol{\alpha}}(\mathbf{y}),\end{aligned} \tag{V.2.3.3.5}$$

where the second equality follows from the usual identity for parititions. Thus the Schur functions are orthonormal with respect to the Hall inner product. We will write

$$\Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)}, \mathbf{y}^{(1)}, q) = \tilde{\Omega}(\mathbf{x}, \mathbf{y}, q)$$

if no confusion arises.

V.2.3.4 We have

$$\langle P_{\Lambda}(\mathbf{x};q), Q_{\Lambda'}(\mathbf{x};q)\rangle_q = \delta_{\Lambda,\Lambda'}, \tag{V.2.3.4.1}$$

where $\Lambda = \Lambda(\boldsymbol{\alpha})$ and $\Lambda' = \Lambda(\boldsymbol{\alpha}')$, for all 2-partitions $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$. This follows from:

Theorem V.2.3.1. *([Sh01] Corollary 4.6) We keep the notations above. There exists a block diagonal matrix $b_{\Lambda,\Lambda'}(q)$, with each block corresponding to a similarity class, such that*

$$\begin{aligned}\tilde{\Omega}(\mathbf{x},\mathbf{y},q) &= \sum_{\Lambda,\Lambda'} b_{\Lambda,\Lambda'}(q)P_{\Lambda}(\mathbf{x};q)P_{\Lambda'}(\mathbf{y};q),\\ \tilde{\Omega}(\mathbf{x},\mathbf{y},q) &= \sum_{\Lambda} P_{\Lambda}(\mathbf{x};q)Q_{\Lambda}(\mathbf{y};q).\end{aligned} \tag{V.2.3.4.2}$$

Remark V.2.3.2. If $\Lambda = \Lambda(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = (\varnothing, (1^m))$ being the minimal element, then it is alone in its similarity class, so we can write $b_{\Lambda}(q) = b_{\Lambda,\Lambda'}(q)$. Then

$$Q_{\Lambda}(\mathbf{x};q) = b_{\Lambda}(q)P_{\Lambda}(\mathbf{x};q).$$

We may also write $b_{\boldsymbol{\alpha}}(q) = b_{\Lambda}(q)$ for such $\boldsymbol{\alpha}$.

V.2.3.5 From the definition of the transformed Hall-Littlewood functions and the orthogonality of the Schur functions(V.2.3.3.5), we see that

$$\tilde{\Omega}(\mathbf{x},\mathbf{y},0) = \sum_{\Lambda} P_{\Lambda}(\mathbf{x};q)\bar{H}_{\Lambda}(\mathbf{y};q). \tag{V.2.3.5.1}$$

In the following lemma, we write the variables of $u(\mathbf{x},q) \in \mathbf{Sym}_{\mathbb{K}}[\mathbf{x}^{(0)},\mathbf{x}^{(1)}]$ as an array:

$$u(\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}, q). \tag{V.2.3.5.2}$$

Then the expression

$$u(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}, q) = u[a\,\mathbf{x}^{(0)} + b\,\mathbf{x}^{(1)}, c\,\mathbf{x}^{(0)} + d\,\mathbf{x}^{(1)}, q], \tag{V.2.3.5.3}$$

is the plethystic substitution by $f_0 = a\,\mathbf{x}^{(0)} + b\,\mathbf{x}^{(1)}$ and $f_1 = c\,\mathbf{x}^{(0)} + d\,\mathbf{x}^{(1)}$.

Lemma V.2.3.3. *We have:*

(a) $Q_{\Lambda}(\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}, q) = \bar{H}_{\Lambda}(\begin{pmatrix} 1 & -q\\ -q & 1\end{pmatrix}\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}, q)$
for any Λ;

(b) $h_{\boldsymbol{\alpha}^*}(\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) = (-1)^m q^{-a(\boldsymbol{\alpha})-m} b_{\boldsymbol{\alpha}}(q^{-1})^{-1} \tilde{H}_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(\mathbf{x};q)$
for $\boldsymbol{\alpha} = (\varnothing, (1^m))$.

Proof. To prove (a), we show that the right hand side gives a dual basis of $\{P_{\boldsymbol{\Lambda}}(\mathbf{x};q)\}$. We calculate,

$$\begin{aligned}
&\sum_{\boldsymbol{\Lambda}} P_{\boldsymbol{\Lambda}}(\mathbf{x};q)\bar{H}_{\boldsymbol{\Lambda}}[\mathbf{y}^{(0)} - q\,\mathbf{y}^{(1)}, \mathbf{y}^{(1)} - q\,\mathbf{y}^{(0)}, q] \\
=&\Omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{y}^{(0)} - q\,\mathbf{y}^{(1)}, \mathbf{y}^{(1)} - q\,\mathbf{y}^{(0)}, 0) \\
=&\Omega(\mathbf{x}^{(0)}(\mathbf{y}^{(0)} - q\,\mathbf{y}^{(1)}), 0)\Omega(\mathbf{x}^{(1)}(\mathbf{y}^{(1)} - q\,\mathbf{y}^{(0)}), 0) \\
=&H(\mathbf{x}^{(0)}(\mathbf{y}^{(0)} - q\,\mathbf{y}^{(1)}), 1)H(\mathbf{x}^{(1)}(\mathbf{y}^{(1)} - q\,\mathbf{y}^{(0)}), 1) \\
=&H(\mathbf{x}^{(0)}\,\mathbf{y}^{(0)}, 1)H(\mathbf{x}^{(0)}\,\mathbf{y}^{(1)}, q)H(\mathbf{x}^{(1)}\,\mathbf{y}^{(1)}, 1)H(\mathbf{x}^{(1)}\,\mathbf{y}^{(0)}, q) \\
=&\tilde{\Omega}(\mathbf{x}, \mathbf{y}, q).
\end{aligned}$$

To prove (b), note that $\tilde{H}_{\boldsymbol{\Lambda}}(\mathbf{x};q) = q^{a(\boldsymbol{\alpha})}\bar{H}_{\boldsymbol{\Lambda}}(\mathbf{x};q^{-1})$ and $Q_{\boldsymbol{\Lambda}}(\mathbf{x};q) = b_{\boldsymbol{\Lambda}}(q)P_{\boldsymbol{\Lambda}}(\mathbf{x};q)$ for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} = (\varnothing, (1^m))$. We calculate

$$\begin{aligned}
&(-1)^m q^{-a(\boldsymbol{\alpha})-m} b_{\boldsymbol{\alpha}}(q^{-1})^{-1} \tilde{H}_{\boldsymbol{\Lambda}}(\mathbf{x};q) \\
=&(-q)^{-m} b_{\boldsymbol{\alpha}}(q^{-1})^{-1} Q_{\boldsymbol{\Lambda}}(\begin{pmatrix} 1 & -q^{-1} \\ -q^{-1} & 1 \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}, q^{-1}) \\
=&(-q)^{-m} P_{\boldsymbol{\Lambda}}(\begin{pmatrix} 1 & -q^{-1} \\ -q^{-1} & 1 \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}, q^{-1}) \\
\overset{①}{=}&(-q)^{-m} s_{\boldsymbol{\alpha}}(\begin{pmatrix} 1 & -q^{-1} \\ -q^{-1} & 1 \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) \\
=&(-q)^{-m} s_{\boldsymbol{\alpha}}(\frac{-q}{1-q^2} \begin{pmatrix} q & 1 \\ 1 & q \end{pmatrix} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) \\
\overset{②}{=}&(-q)^{-m} s_{(1^m)}(\frac{-q}{1-q^2}(\mathbf{x}^{(0)} + q\,\mathbf{x}^{(1)})) \\
=&s_{(m)}(\frac{\mathbf{x}^{(0)} + q\,\mathbf{x}^{(1)}}{1-q^2}) \\
=&h_m(\frac{\mathbf{x}^{(0)} + q\,\mathbf{x}^{(1)}}{1-q^2}) \\
=&h_{((m),\varnothing)}(\frac{1}{1-q^2} \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) \\
=&h_{((m),\varnothing)}(\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}).
\end{aligned}$$

In equality ① we have used the second statement about $P_{\boldsymbol{\Lambda}}$ in Theorem V.2.1.1, and in

equality $② s_{(1^m)}$ denotes the usual Schur function in one variable. □

V.2.4 More on Green Functions

V.2.4.1 The modified Kostka-Foulkes polynomials are completely determined as the entries of the matrix P in the following matrix equation,

$$P\Pi^t P = \Xi, \tag{V.2.4.1.1}$$

where Π, P and Ξ are some square matrices of degree $|\mathcal{P}_n(2)|$. The matrix Ξ is determined by the inner products of Green functions, which are known. The matrix Π is a block diagonal matrix with nonsingular blocks. The matrix P is a lower triangular block matrix with each diagonal block being the scalar $q^{a(\boldsymbol{\alpha})}$, with $\alpha \in \mathcal{P}_n(2)$.

V.2.4.2 Let $b(q)$ be the matrix $(b_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}(q))$ defined above. Denote by $\mathbb{G}_n(q)$ the cardinality of $\mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$. Define

$$\tilde{\Pi} = \frac{\mathbb{G}_n(q)}{q^n} T^{-1} b(q^{-1})^{-1} T^{-1},$$

where T is a diagonal matrix with entries $q^{a(\boldsymbol{\alpha})}$. By [Sh01, Theorem 5.4], $P = \tilde{K}(q)$ and the Π above satisfy (V.2.4.1.1).

Remark V.2.4.1. [Sh01, Theorem 5.4] states that these matrices satisfy [Sh01, (1.5.2)], which is equivalent to [Sh01, (1.4.2)] for Coxeter groups, since then the complex conjugate of an irreducible character coincides with itself.

Fact. $\tilde{\Pi}_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}} = 1$ if $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ and $\boldsymbol{\alpha} = (\varnothing, (1^n))$ is the minimal element.

For such $\boldsymbol{\alpha}$ and $\boldsymbol{\Lambda}$, we deduce that

$$\frac{1}{\mathbb{G}_n(q)} = q^{-2a(\boldsymbol{\alpha})-n} b_{\boldsymbol{\alpha}}(q^{-1})^{-1}. \tag{V.2.4.2.1}$$

For $\mathrm{GL}_n(q)$, it is known that

$$\frac{1}{|\mathrm{GL}_n(q)|} = q^{-2n((1)^n)-n} b_{(1)^n}(q^{-1})^{-1}, \tag{V.2.4.2.2}$$

where for any partition λ, $b_\lambda(q)$ is defined by $b_\lambda(q)P_\lambda(\mathbf{x},q) = Q_\lambda(\mathbf{x},q)$, i.e. the difference between the two Hall-Littlewood symmetric functions associated to the partition λ.

V.3 Miscellany of Combinatorics

V.3.1 Types

V.3.1.1 Proposition V.1.1.3 and (V.1.2.1.1) have reduced the point-counting problem to the evaluation of the following formula (*cf.* Notation V.1.1.2)

$$\sum_{\mathbf{e}=(e_j)\in\{\pm\}^{2k}} \sum_{\chi\in\mathrm{Irr}(G)^\sigma} \Big(\frac{|G|}{\chi(1)}\Big)^{2g-2} \prod_{j=1}^{2k} \frac{|C_j|\tilde{\chi}(C_{j,e_j})}{\chi(1)}. \tag{V.3.1.1.1}$$

The sum over $\mathrm{Irr}(G)^\sigma$ can be separated into a sum over $\mathfrak{T}_\chi$, the types of σ-stable irreducible characters and a sum over the σ-stable irreducible characters of the same type. It turns out that for a strictly generic (*cf.* §III.4.2.5) tuple of conjugacy classes $C = (C_i)$, only a small subset of $\mathfrak{T}_\chi$ can have irreducible characters that are non-vanishing on all of the C_i's.

Lemma V.3.1.1. *Let $(C_1, \ldots, C_{2k})$ be a strictly generic tuple of semi-simple conjugacy classes in $\mathrm{GL}_n(q).\sigma$. Assume that for each $1 \le j \le 2k$, the class C_j has a representative $s_j\sigma$ of the form (V.1.1.3.2) or of the form (V.1.1.3.3). Let M be a σ-stable and F-stable Levi factor of some σ-stable parabolic subgroup of $\mathrm{GL}_n(k)$, and denote by $n_0(n_i, d_i)_{i\in\Lambda_1}(n'_{i'}, d'_{i'})_{i'\in\Lambda_2}$ the data that determines the $\mathrm{SO}_n(q)$-conjugacy class of $(M^\sigma)^\circ$ (cf. Proposition II.3.2.4). If for all j, there exists $h_j \in \mathrm{GL}_n(q)$ such that $h_j s_j \sigma h_j^{-1} \in M.\sigma$, then $d_i = 1$, for all i, and $\Lambda_2 = \varnothing$.*

Proof. We may assume that every $s_j\sigma$ is of the form (V.1.1.3.2). More general cases only require minor modifications of the arguments.

Denote by $T \subset \mathrm{GL}_n$ the maximal torus of diagonal matrices. There is some σ-stable standard Levi subgroup L_I, an element $g \in (G^\sigma)^\circ$, and $\dot{w} = g^{-1}F(g)$ such that (M, F) is isomorphic to (L_I, F_w) via ad g. Suppose that such h_j's exist. For any j, $g^{-1}h_j s_j \sigma h_j^{-1} g$ is an element of $L_I.\sigma$. By Proposition II.3.1.2, for each j, there exists $l_j \in L_I$ such that $l_j g^{-1} h_j s_j \sigma h_j^{-1} g l_j^{-1}$ lies in $(T^\sigma)^\circ\sigma$, and moreover, for each j, there exists $w_j \in W^\sigma$ (W being the Weyl group of GL_n defined by T) and $z_j \in (T^\sigma)^\circ \cap [T, \sigma]$ such that

$$l_j g^{-1} h_j s_j \sigma h_j^{-1} g l_j^{-1} = w_j s_j w_j^{-1} z_j \sigma.$$

Now put (See §III.4.2.3 for the map $\mathbf{D}_{L_I}$)

$$l := \mathbf{D}_{L_I}\Big(\prod_{j=1}^{2k} (g^{-1}h_j s_j \sigma h_j^{-1} g)^2\Big) = \mathbf{D}_{L_I}\Big(\prod_{j=1}^{2k} w_j s_j^2 w_j^{-1}\Big), \tag{V.3.1.1.2}$$

which lies in

$$k^* \times \prod_{i\in\Lambda_1} (k^* \times k^*)^{d_i} \prod_{i'\in\Lambda_2} (k^* \times k^*)^{d'_{i'}}.$$

By the assumption of generic conjugacy classes (III.4.2.5.2) and the right hand side of (V.3.1.1.2), the direct factors of l must be pairwise distinct. However, the left hand side of

(V.3.1.1.2) shows that l is an F_w-stable element and the right hand side shows that the direct factors all belong to $\mathbb{F}_q$, which is a contradiction if any of the numbers d_i or $d'_{i'}$ are larger than 1. Finally, if there was some $d'_{i'} = 1$, then this factor of l must be equal to -1. This possibility is ruled out by the assumption (III.4.2.5.3). □

V.3.1.2 Since only those characters that are uniform on $\mathrm{GL}_n.\sigma$ can have non trivial value at a semi-simple class (see Remark IV.6.1.3), the only remaining types for irreducible characters are, up to a formal sign $\epsilon \in \{\pm\}$, of the form

$$\omega = \omega_+\omega_-(\omega_i)_{1\leq i\leq l},$$

where ω_+ and ω_- are 2-partitions, (ω_i) is an unordered sequence of nontrivial partitions, and l is a non negative integer. The sign ϵ decides whether the non trivial 2-core (1) is associated to ω_+ (if $\epsilon = +$) or to ω_- (if otherwise). We will write these data in the compact form $\epsilon\omega$. We will sometimes denote by ω_p the sequence of partitions (ω_i) in a type ω, so that $\omega = \omega_+\omega_-\omega_p$. Then l will be called the length of ω_p, denoted by $l(\omega_p)$. The size of a type is $|\omega| := |\omega_+| + |\omega_-| + \sum_i |\omega_i|$.

The following data are the types of conjugacy classes that we are interested in:

$$\beta = \beta_+\beta_-(\beta_i)_{1\leq i\leq l},$$

where (β_i) is an unordered sequence of nontrivial partitions, and $\beta_\pm$ are symbols corresponding to some irreducible characters of $\mathfrak{W}_{|\beta_\pm|}$ under the Springer correspondence (thus we may also use some 2-partitions to represent them). Again, we write $\beta_p = (\beta_i)$ and call $l(\beta_p) := l$ the length of β_p. The type of a semi-simple conjugacy class is of the form

$$\beta = (\varnothing, (1)^{m_+})(\varnothing, (1^{m_-}))((1^{m_i}))_i$$

where $m_+ = |\beta_+|$, $m_- = |\beta_-|$ and $m_i = |\beta_i|$. Since the conjugacy classes we are interested in are obtained by base change from the ring R (*cf.* §V.1.1.1) to $\mathbb{F}_q$, the only possible types of semi-simple conjugacy classes are of this form. We define its dual type by

$$\beta^* = ((m_+), \varnothing)((m_-), \varnothing)((m_i))_i.$$

There is an obvious bijection from the set of types of characters (without ϵ) to the set of types of conjugacy classes as defined above. We will denote the set of either types by $\mathfrak{T}$, and for any $\omega \in \mathfrak{T}$ we denote by $\mathbf{\Lambda}(\omega) = \mathbf{\Lambda}(\omega_+)\mathbf{\Lambda}(\omega_-)(\omega_i)$ the corresponding types of conjugacy classes. The subset $\{\omega \in \mathfrak{T} \mid \omega_+ = \omega_- = \varnothing\}$ will be denoted by $\mathfrak{T}'$. Denote by $\tilde{\mathfrak{T}}$ the set of *ordered types*, i.e. the data $\omega_+\omega_-(\omega_i)$ with (ω_i) being an ordered sequence. There is an obvious map from $\tilde{\mathfrak{T}}$ to $\mathfrak{T}$, therefore anything that can be defined for elements of $\mathfrak{T}$ is naturally defined for elements of $\tilde{\mathfrak{T}}$. Given $\alpha = \alpha_+\alpha_-(\alpha_i)_{1\leq i\leq l_1}$, $\beta = \beta_+\beta_-(\beta_i)_{1\leq i\leq l_2} \in \tilde{\mathfrak{T}}$, we write $\alpha \approx \beta$, if $l_1 = l_2 = l$ and for each $1 \leq i \leq l$, we have $|\alpha_i| = |\beta_i|$, and moreover $|\alpha_+| = |\beta_+|$ and $|\alpha_-| = |\beta_-|$.

Let us define a natural map

$$[\ ] : \mathfrak{T} \longrightarrow \mathcal{P}^2. \tag{V.3.1.2.1}$$

Regarding $\mathcal{P}^2$ as signed partitions, it sends each part of ω_i to a positively signed part with the same size, and keeps the sign and size of each part of ω_+ and ω_-. The union of these signed parts is the image of [ ] and is denoted by $[\omega]$.

Denote by $\mathring{\mathfrak{T}}$ the set of unordered sequences $\lambda_+\lambda_-(\lambda_i)_{1\leq i\leq l}$, with l being a non negative integer, where λ_+, λ_- and λ_i are partitions, with λ_+ and λ_- being possibly trivial. We may also write such a sequence as $\lambda_+\lambda_-\lambda_p$, with $\lambda_p = (\lambda_i)$. There is a natural map

$$\begin{aligned} \{\ \} : \{\pm\} \times \mathfrak{T} &\longrightarrow \mathring{\mathfrak{T}} \\ \epsilon\omega &\longmapsto \lambda_+\lambda_-\lambda_p = \{\epsilon\omega\} \end{aligned} \tag{V.3.1.2.2}$$

where $\lambda_\pm$ has $\omega_\pm$ as the 2-quotient and λ_ϵ has the 2-core (1), and each ω_i gives rise to two identical partitions $\lambda_{i_1} = \lambda_{i_2} = \omega_i$ as factors of λ_p. In reconstructing the partitions $\lambda_\pm$ from $\omega_\pm$, we have chosen integers $r_\pm$ following §IV.5.4.1.

If λ is a partition, then we define $\{\lambda\} := (\lambda_i)_{1\leq i\leq 2} \in \mathring{\mathfrak{T}}$ with $\lambda_1 = \lambda_2 = \lambda$. If $\boldsymbol{\alpha}$ is a 2-partition, then we define $\{\boldsymbol{\alpha}\}_1$ to be the partition with 2-core (1) and 2-quotient $\boldsymbol{\alpha}$, and define $\{\boldsymbol{\alpha}\}_0$ to be the partition with trivial 2-core and 2-quotient $\boldsymbol{\alpha}$, with the same choice of $r_\pm$ as above.

V.3.2 Symmetric Functions Associated to Types

Let us extend the definitions and statements of various combinatorics to types.

V.3.2.1 Given $\omega = \omega_+\omega_-(\omega_i) \in \mathfrak{T}$, we can define the Schur symmetric function associated to ω as

$$s_\omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}) := s_{\omega_+}(\mathbf{x})s_{\omega_-}(\mathbf{x}) \prod_i s_{\omega_i}(\mathbf{x}^{(0)} + \mathbf{x}^{(1)}). \tag{V.3.2.1.1}$$

Monomial symmetric functions, complete symmetric functions and power sum symmetric functions can be similarly defined. Note that for any partition λ, we have

$$p_\lambda(\mathbf{x}^{(0)} + \mathbf{x}^{(1)}) = p_{(\lambda,\varnothing)}(\mathbf{x}).$$

This implies in particular that $p_\omega(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}) = p_{[\omega]}(\mathbf{x})$.

Define

$$z_\omega(q) := z_{\omega_+}(q)z_{\omega_-}(q) \prod_i z_{\omega_i}(q), \tag{V.3.2.1.2}$$

in particular $z_\omega := z_\omega(0) = z_{\omega_+}z_{\omega_-} \prod_i z_{\omega_i}$.

Given $\alpha, \beta \in \tilde{\mathfrak{T}}$, if $\alpha \approx \beta$, define $\chi^{\alpha}_{\beta} := \chi^{\alpha_+}_{\beta_+}\chi^{\alpha_-}_{\beta_-}\prod_i \chi^{\alpha_i}_{\beta_i}$; otherwise we put $\chi^{\alpha}_{\beta} = 0$. With these definitions we have

$$\begin{aligned} p_\beta(\mathbf{x}) &= \sum_\alpha \chi^\alpha_\beta\, s_\alpha(\mathbf{x}); \\ s_\lambda(\mathbf{x}) &= \sum_\tau \frac{1}{z_\tau}\chi^\lambda_\tau\, p_\tau(\mathbf{x}). \end{aligned} \tag{V.3.2.1.3}$$

Remark V.3.2.1. One needs to be careful with the summations in these expressions. For example in the first equation, α runs over $\mathcal{P}_{|\beta_+|}(2) \times \mathcal{P}_{|\beta_-|}(2) \times \prod_i \mathcal{P}_{|\beta_i|}$, which is a subset of $\tilde{\mathfrak{T}}$. The symmetric functions above should also be regarded as associated to ordered types, although they are independent of the ordering in the types. This rule will apply to all transition matrices that involve types.

For any $\omega \in \mathfrak{T}$, define

$$a(\mathbf{\Lambda}(\omega)) = a(\mathbf{\Lambda}(\omega_+)) + a(\mathbf{\Lambda}(\omega_-)) + \sum_i n(\omega_i).$$

We may write $a(\omega) := a(\mathbf{\Lambda}(\omega))$. For any $\lambda = \lambda_+\lambda_-(\lambda_i) \in \mathring{\mathfrak{T}}$, define

$$n(\lambda) = n(\lambda_+) + n(\lambda_-) + \sum_i n(\lambda_i).$$

Define

$$P_{\mathbf{\Lambda}(\omega)}(\mathbf{x}, q) := P_{\mathbf{\Lambda}(\omega_+)}(\mathbf{x}, q)P_{\mathbf{\Lambda}(\omega_-)}(\mathbf{x}, q)\prod_i P_{\omega_i}(\mathbf{x}^{(0)} + \mathbf{x}^{(1)}, q), \tag{V.3.2.1.4}$$

and similarly for $Q_{\mathbf{\Lambda}(\omega)}(\mathbf{x}, q)$. For any $\alpha, \beta \in \tilde{\mathfrak{T}}$, define

$$K_{\beta,\alpha}(q) := K_{\beta_+,\alpha_+}(q)K_{\beta_-,\alpha_-}\prod_i K_{\beta_i,\alpha_i}(q)$$

if $\alpha \approx \beta$, and put $K_{\beta,\alpha}(q) = 0$ if otherwise. We then have

$$s_\beta(\mathbf{x}) = \sum_\alpha K_{\beta,\alpha}(q)P_{\mathbf{\Lambda}(\alpha)}(\mathbf{x}; q) \tag{V.3.2.1.5}$$

Define $\tilde{K}_{\beta,\alpha}(q) = q^{a(\alpha)}K_{\beta,\alpha}(q^{-1})$.

With the obvious definitions, we have the following identities for types:

$$Q_\beta^{\mathbf{\Lambda}(\alpha)}(q) = \sum_\gamma \chi_\beta^\gamma \tilde{K}_{\gamma,\alpha}(q), \tag{V.3.2.1.6}$$

$$\bar{H}_{\mathbf{\Lambda}(\alpha)}(\mathbf{x};q) = \sum_\beta K_{\beta,\alpha}(q) s_\beta(\mathbf{x}), \tag{V.3.2.1.7}$$

$$\tilde{H}_{\mathbf{\Lambda}(\alpha)}(\mathbf{x};q) = \sum_\beta \tilde{K}_{\beta,\alpha}(q) s_\beta(\mathbf{x}). \tag{V.3.2.1.8}$$

Lemma V.2.3.3 can also be stated for types. For any partition λ, and a symmetric function $u_\lambda(\mathbf{x})$ associated to λ, we define the notation

$$u_\lambda(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) := u_\lambda(a\,\mathbf{x}^{(0)} + b\,\mathbf{x}^{(1)} + c\,\mathbf{x}^{(0)} + d\,\mathbf{x}^{(1)}).$$

In particular

$$u_\lambda(\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) = u_\lambda(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) = u_\lambda(\mathbf{x}^{(0)} + \mathbf{x}^{(1)})$$

$$u_\lambda(\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) = u_\lambda[(1-q)(\mathbf{x}^{(0)} + \mathbf{x}^{(1)})],$$

$$u_\lambda(\begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}^{-1}\begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}) = u_\lambda[\frac{\mathbf{x}^{(0)} + \mathbf{x}^{(1)}}{1-q}],$$

which is consistent with the usual notation for symmetric functions associated to partitions. We will write

$$\mathbf{P} = \begin{pmatrix} 1 & -q \\ -q & 1 \end{pmatrix}^{-1}, \text{ and } \mathbf{x} = \begin{bmatrix} \mathbf{x}^{(0)} \\ \mathbf{x}^{(1)} \end{bmatrix}. \tag{V.3.2.1.9}$$

For example, if $\omega = \omega_+\omega_-(\omega_i)$ is a type, we have,

$$P_{\mathbf{\Lambda}}(\mathbf{P}\,\mathbf{x}, q) = P_{\mathbf{\Lambda}(\omega_+)}(\mathbf{P}\,\mathbf{x}, q) P_{\mathbf{\Lambda}(\omega_-)}(\mathbf{P}\,\mathbf{x}, q) \prod_i P_{\omega_i}(\mathbf{P}\,\mathbf{x}, q). \tag{V.3.2.1.10}$$

With these definitions, and the known identities for symmetric functions associated to partitions, we have:

Corollary V.3.2.2. *Denote by*

$$\beta = (\varnothing, (1)^{m_+})(\varnothing, (1^{m_-}))((1^{m_i}))_i$$

the type of a semi-simple conjugacy class, and put

$$b_\beta(q) = b_{\beta_+}(q) b_{\beta_-}(q) \prod_i b_{\beta_i}(q).$$

Then the following identities hold:

(a) $Q_{\mathbf{\Lambda}(\omega)}(\mathbf{x}, q) = \bar{H}_{\mathbf{\Lambda}(\omega)}(\mathbf{P}^{-1}\,\mathbf{x}, q)$;

(b) $h_{\beta^*}(\mathbf{P}\,\mathbf{x}) = (-1)^{|\beta|} q^{-a(\beta)-|\beta|} b_\beta(q^{-1})^{-1} \tilde{H}_{\mathbf{\Lambda}(\beta)}(\mathbf{x}; q)$.

Proof. This follows from Lemma V.2.3.3. □

V.3.3 Möbius Inversion Function

V.3.3.1 Let m be a positive integer and let I be a set of cardinality $x \in \mathbb{Z}_{>0}$ that is larger than m. Denote by Π the set of the partitions of $\{1, \ldots, m\}$. It is a partially ordered set. Two partitions satisfy $P_1 \prec P_2$ if P_2 refines P_1. Denote by P_0 the partition into m parts, each consisting of a single element. Then P_0 is the maximal element. Each $P \in \Pi$ can be written as a collection of disjoint union of subsets of $\{1, \ldots, m\}$, written as $p_1 \cdots p_s$. Each p_i is called a part of P and s is called the length of P, denoted by $l(P)$. For any $P \in \Pi$, denote by $(I^m)_P$ the subset of I^m consisting of the elements $(i_r)_{1 \le r \le m}$ such that $i_r = i_s$ whenever r and s are in the same part of P. Denote by $(I^m)_{P,\mathrm{reg}}$ the set of the elements (i_r) of $(I^m)_P$ such that $i_r \neq i_s$ whenever r and s are not in the same part of P. Obviously,

$$(I^m)_{P_2} = \bigcup_{P_1 \prec P_2} (I_m)_{P_1,\mathrm{reg}},$$

for any $P_2 \in \Pi$. A $\bar{\mathbb{Q}}_\ell$-valued function f defined on the set of the subsets of I^m is additive if $f(U \cup V) = f(U) + f(V)$ for any two disjoint subsets U and V. Given such a function f, we can define two functions F and F' on Π by $F(P) := f((I^m)_P)$ and $F'(P) = f((I^m)_{P,\mathrm{reg}})$. Then

$$F(P_2) = \sum_{P_1 \prec P_2} F'(P_1).$$

By the Möbius inversion formula, we have

$$F'(P_2) = \sum_{P_1 \prec P_2} \mu(P_1, P_2) F(P_1), \tag{V.3.3.1.1}$$

where $\mu(P_1, P_2)$ is the Möbius inversion function for the partially ordered set Π.

Define $c_P(x) := |(I^m)_P|$, $c'_P(x) := |(I^m)_{P,reg}|$. We have $c_P(x) = (x)^{l(P)}$ and

$$c'_P(x) = x(x-1)\cdots(x - l(P) + 1).$$

These are the functions defined by counting elements, which is additive. Inserting these expressions into the Möbius inversion formula, we have a polynomial identity in x that is

valid for all large enough x. Specialising this equality at $x = -1$, we get

$$(-1)^m m! = \sum_{\Pi < \Pi_0} \mu(\Pi, \Pi_0)(-1)^{l(\Pi)}. \tag{V.3.3.1.2}$$

V.3.3.2 Let I, m and Π be as above. Let I^* be a set in bijection with I. For any $i \in I$, let i^* denote the element of I^* corresponding i under the bijection. Write $\bar{I} = I \sqcup I^*$. For any i, $i' \in \bar{I}$, we write $[i] = [i']$ if $i' \in \{i, i^*\}$. For any $P \in \Pi$, denote by $(\bar{I}^m)_P$ the subsets of elements $(i_r)_{1\leq r\leq m}$, $i_r \in \bar{I}$, such that $[i_r] = [i_s]$ whenever r and s are in the same part of P. Denote by $(\bar{I}^m)_{P,\mathrm{reg}}$ the set of the elements (i_r) of $(\bar{I}^m)_P$ such that $[i_r] \neq [i_s]$ whenever r and s are not in the same part of P. We say that an element of $\bar{I}^m$ is regular if it is regular with respect to the maximal partition P_0. Obviously,

$$(\bar{I}^m)_{P_2} = \bigcup_{P_1 < P_2} (\bar{I}_m)_{P_1,\mathrm{reg}},$$

for any $P_2 \in \Pi$.

Now, put $\bar{I} = \mathrm{Irr}(\mathbb{F}_q^*) \setminus \{1, \eta\}$ and $\tilde{I} = (\mathrm{Irr}(\mathbb{F}_q^*) \setminus \{1, \eta\})/\sim$ with the equivalence relation that identifies α and α^{-1}. For each equivalence class we choose a representative and denote by I the set of these representatives, and so for any $\alpha \in I$, $\alpha^* = \alpha^{-1}$. Let $(a_i)_{1\leq i\leq m}$ be an m-tuple of elements of $\mathbb{F}_q^*$. Define an additive function on the set of the subsets of $\bar{I}^m$ by

$$f(J) := \sum_{(\alpha_i)_{1\leq i\leq m} \in J} \prod_{1\leq i\leq m} \alpha_i(a_i),$$

for $J \subset \bar{I}^m$. Then,

$$F'(P_0) = \sum_{P_1 < P_0} \mu(P_1, P_0) F(P_1), \tag{V.3.3.2.1}$$

with $F(P) := f((\bar{I}^m)_P)$, $F'(P) = f((\bar{I}^m)_{P,\mathrm{reg}})$ and P_0 defined in §V.3.3.1

V.3.3.3 Let s be a positive integer. Let $(a_i)_{1\leq i\leq s}$ be an s-tuple of elements of $\mathbb{F}_q^*$ such that $\prod_{1\leq i\leq s} a_i^{2e_i} \neq 1$ for any $(e_i)_{1\leq i\leq s} \in \mu_2^s$. The following lemma will be used to compute $F(P_1)$ in (V.3.3.2.1), with s being the size of a part of P_1.

Lemma V.3.3.1. *We have the identity:*

$$\sum_{\alpha\in I} \sum_{(e_i)_{1\leq i\leq s} \in \mu_2^s} \prod_{1\leq i\leq s} \alpha^{e_i}(a_i^2) = -2^s.$$

Proof. Observe that

$$\sum_{\alpha\in I} \sum_{(e_i)_{1\leq i\leq s} \in \mu_2^s} \alpha(\prod_{1\leq i\leq s} a_i^{2e_i}) = \sum_{\alpha\in I} \sum_{(e_i)_{1\leq i\leq s} \in \mu_2^s} \alpha^{-1}(\prod_{1\leq i\leq s} a_i^{2e_i}),$$

so the desired quantity is equal to

$$\frac{1}{2}\sum_{\alpha\in\bar{I}}\sum_{(e_i)_{1\le i\le s}\in\mu_2^s}\alpha(\prod_{1\le i\le s}a_i^{2e_i}).$$

Since

$$\sum_{\alpha\in\bar{I}}\alpha(\prod_{1\le i\le s}a_i^{2e_i})=-2,$$

we get $2^s\cdot(-2)/2=-2^s$. □

V.3.3.4 Let us apply the above computations to the following situation.

Fix a type $\epsilon\alpha=\epsilon\alpha_+\alpha_-(\alpha_i)_{1\le i\le l}$ of σ-stable irreducible character of $\mathrm{GL}_n(q)$, with α_+ and α_- being some 2-partitions. Put $n_+=|\alpha_+|$, $n_-=|\alpha_-|$ and $n_0=2n_++2n_-+1$ and $n_i=|\alpha_i|$. Then a σ-stable irreducible character of type $\epsilon\alpha$ is induced from some σ-stable and F-stable Levi factor M of some σ-stable parabolic subgroup, which is isomorphic to $\mathrm{GL}_{n_0}(k)\times\prod_i(\mathrm{GL}_{n_i}(k)\times\mathrm{GL}_{n_i}(k))$, following the procedure of Proposition IV.1.2.2. In view of Lemma V.3.1.1, we can assume that M is equal to L_I, some standard σ-stable Levi subgroup. Let $C=(C_1,\dots,C_{2k})$ be a tuple of semi-simple conjugacy classes in $\mathrm{GL}_n(q).\sigma$ with representatives $s_j\sigma$ as in Lemma V.3.1.1. For each $j\in\{1,\dots 2k\}$, fix $w_j\in\mathfrak{W}_{n_+}\times\mathfrak{W}_{n_-}\times\prod\mathfrak{S}_{n_i}$, regarded as an element of $W_M^\sigma(T)$, with T being the maximal torus of diagonal matrices. Let $\dot{w}_j\in(M^\sigma)^\circ$ be a representative of w_j and let $g_j\in(M^\sigma)^\circ$ be such that $g_j^{-1}F(g_j)=\dot{w}_j$. Put $T_{w_j}=g_jTg_j^{-1}$. We assume that for each j, there exists some $h_j\in\mathrm{GL}_n(q)$ such that $h_js_j\sigma h_j^{-1}$ lies in $T_{w_j}.\sigma\subset M.\sigma$.

Notation V.3.3.2. Denote by $\mathrm{Irr}^\sigma_{reg,\epsilon\alpha}$ the subset of $\prod_{i=1}^l(\mathrm{Irr}(\mathbb{F}_q^*)\setminus\{1,\eta\})$ consisting of regular elements in the sense of §V.3.3.2, where $l=l(\alpha_p)$. For all j, each element (θ_i) of $\mathrm{Irr}^\sigma_{reg,\epsilon\alpha}$ can be regarded as a character of $T^F_{w_j}$ in the following way. We write

$$w_j=(w_{j,+},w_{j.-},w_{j,i})\in\mathfrak{W}_{n_+}\times\mathfrak{W}_{n_-}\times\prod\mathfrak{S}_{n_i}.$$

To each $w_{j,i}$ is associated two subtori of T_{w_j}, isomorphic to $(k^*)^{n_i}$, equipped with the Frobenius twisted by the automorphic defined by $w_{j,i}$. Composed with the norm map, θ_i and θ_i^{-1} are regarded as the characters (of the rational points) of these two tori respectively. The factor $w_{j,+}$ acts on a factor of T_{w_j} that is isomorphic to $(k^*)^{2n_++1}$. We associate the trivial character to this torus. Then the order 2 irreducible character η is associated to the remaining factor of T_{w_j}. The resulting character of $T^F_{w_j}$ is denoted by θ_j which is σ-stable, and we denote by $\tilde{\theta}_j$ its extension to $T^F_{w_j}.<\sigma>$which equals to 1 at σ.

Lemma V.3.3.3. *Put (cf. Notation V.1.1.2)*

$$\Delta_{\epsilon,\mathrm{sgn}\,C}=\begin{cases}-1 & \textit{if } \epsilon=-,\textit{ and } \mathrm{sgn}\,C=-1;\\ 1 & \textit{otherwise.}\end{cases}\tag{V.3.3.4.1}$$

Assume that C_j has no "eigenvalue" equal to i *for every j. We have:*

$$\sum_{\theta \in \mathrm{Irr}^{\sigma}_{reg,\epsilon\alpha}} \prod_{j=1}^{2k} \tilde{\theta}_j(h_j s_j \sigma h_j^{-1}) = (-2)^l l! \Delta_{\epsilon, \mathrm{sgn}\, C}. \tag{V.3.3.4.2}$$

Remark V.3.3.4. The assumption on the "eigenvalues" morally means that, if we were working in a connected reductive group, the centraliser of the semi-simple element is a Levi subgroup.

Proof. Let us begin with evaluating the character $\tilde{\theta}_j(h_j s_j \sigma h_j^{-1})$. We can fix j for the moment and omit the lower script j, so that $\theta = \theta_j$, $h = h_j$, $s = s_j$, $w = w_j$ and $g = g_j$. Put $x = g^{-1}h$. Then $xs\sigma x^{-1} \in T.\sigma$ and it is fixed by F_w. We can then put $\tilde{\theta}_0 = \tilde{\theta} \circ \mathrm{ad}\, g$, a character of $T^{F_w}.<\sigma>$, and evaluate $\tilde{\theta}_0$ at $xs\sigma x^{-1}$. That $xs\sigma x^{-1} \in T.\sigma$ implies that $x = nl$ with $n \in N_G(T.\sigma)$ and $l \in C_G(s\sigma)^\circ$. Write n as $\dot{v}t$, with $t \in T$ and $\dot{v}$ being a permutation matrix fixed by σ. Explicitly, if $s\sigma$ is of the form (V.1.1.3.2) and $t = \mathrm{diag}(t_1, \ldots, t_{n+1}, \ldots, t_{2n+1})$, then

$$\dot{v}ts\sigma t^{-1}\dot{v}^{-1} = \dot{v}\,\mathrm{diag}(t_1 t_{2n+1} a_1, \ldots, t_n t_{n+2} a_n, t_{n+1}^2, t_n t_{n+2} a_n^{-1}, \ldots, t_1 t_{2n+1} a_1^{-1})\dot{v}^{-1}\sigma =: s'\sigma.$$

If $s\sigma$ is of the form (V.1.1.3.3), then we only need to replace t_{n+1}^2 by $t_{n+1}^2 c$. Evaluate $\tilde{\theta}_0$ at $xs\sigma x^{-1}$ is to evaluate θ_0 at s'. We claim that the contributions of the t_i's, for $i \neq n+1$, cancel out. We only show this in two typical situations.

- Suppose that a is one of the a_i's and t is equal to $t_i t_{2n+2-i}$. If F_w acts by $(ta, ta^{-1}) \mapsto (t^q a, t^q a^{-1})$, then t lies in $\mathbb{F}_q$ since s' is known to be F_w-stable. And if the corresponding two factors of θ_0 are (α, α^{-1}), then it is immediate that t does not contribute to the value of θ_0.

- Suppose that a and t are as above. If F_w acts by $(ta, ta^{-1}) \mapsto (t^q a^{-1}, t^q a)$, then it is necessary that $a^4 = 1$ and $t^{q-1} = a^2$ in order for it to be F_w-stable. Since we have assumed that there is no i among the a_i's, a^2 must be 1 and t lies in $\mathbb{F}_q$. The character $\eta \circ N_{\mathbb{F}_{q^2}/\mathbb{F}_q}$ sends (ta, ta^{-1}) to $\eta(t^2) = 1$. Again there is no contribution from t.

Note that in the second case if $a^2 = -1$, then $\eta(t^2) = -1$. Then the value of θ_0 will depend on the cycles of minus sign in w. The computation will be more complicated. This is the reason why we have excluded i from the "eigenvalues".

The remaining term t_{n+1}^2 (resp. $t_{n+1}^2 c$), which necessarily lies in $\mathbb{F}_q$, is congruent to

$$\det(s') = \det(g^{-1}hs\sigma(h)^{-1}g) = \det(hs\sigma(h^{-1})) = \det(h^2 s) \equiv \det(s) = 1 \text{ (resp. } c), \quad \mod (\mathbb{F}_q^*)^2.$$

So t does not contribute to the value of $\tilde{\theta}_0$. We conclude that

$$\tilde{\theta}_0(hs\sigma h^{-1}) = \theta_0(\dot{v}s\dot{v}^{-1}).$$

We then apply the above calculation to each j.

$$\sum_{\theta\in\mathrm{Irr}^{\sigma}_{reg,\alpha}}\prod_{j=1}^{2k}\tilde{\theta}_j(h_js_j\sigma h_j^{-1})=\sum_{\theta\in\mathrm{Irr}^{\sigma}_{reg,\alpha}}\prod_{j=1}^{2k}\theta_j\circ\mathrm{ad}\,g(\dot{v}_js_j\dot{v}_j^{-1}).$$

For each index among $\{+,-,1,\dots,l\}$, we collect the corresponding factor of $\mathrm{ad}\,g(\dot{v}_js_j\dot{v}_j^{-1})$ and apply the appropriate norm map for all j. Then we end up with an element

$$m\in\mathbb{F}_q^*\times\mathbb{F}_q^*\times\prod_{1\le i\le l}(\mathbb{F}_q^*\times\mathbb{F}_q^*).$$

Assume $\epsilon=-$, then we can write $m=(m_+=1,m_-=\mathrm{sgn}\,C,(m_i,m_i^{-1})_{1\le i\le l})$ with respect to this direct product. If $\epsilon=+$, then η always gives 1. The above computation is reduced to

$$\sum_{\substack{(\theta_1,\dots,\theta_l)\\ \text{regular}}}\prod_{i=1}^{l}\theta_i(m_i)\theta_i^{-1}(m_i^{-1})=\sum_{\substack{(\theta_1,\dots,\theta_l)\\ \text{regular}}}\prod_{i=1}^{l}\theta_i(m_i^2).$$

Note that in order for $(\theta_1,\dots,\theta_l)$ to be regular, its factors must satisfy:

- $\theta_i\neq 1$ or η, for any i;
- $\theta_i\neq\theta_j^{\pm1}$ if $i\neq j$.

Using (V.3.3.2.1), Lemma V.3.3.1 and (V.3.3.1.2), we get

$$\begin{aligned}\sum_{\substack{(\theta_1,\dots,\theta_l)\\ \text{regular}}}\prod_{i=1}^{l}\theta_i(m_i^2)&=\sum_{\Pi\prec\Pi_0}\mu(\Pi,\Pi_0)(-1)^{l(\Pi)}2^l\\&=(-2)^ll!,\end{aligned}\tag{V.3.3.4.3}$$

where Π_0 and Π are the partitions of the set $\{1,\dots,l(\alpha_1)\}$. □

V.3.4 A Combinatorial Identity

V.3.4.1 The following lemma is an analogue of [HLR, Lemma 2.3.5].

Lemma V.3.4.1. *Fix $\alpha,\beta\in\tilde{\mathfrak{T}}$. Then*

$$\langle s_\alpha(\mathbf{x}),\tilde{H}_{\mathbf{\Lambda}(\beta)}(\mathbf{x},q)\rangle=\sum_{\tau}\frac{z_{[\tau]}\chi_\tau^\alpha}{z_\tau}\sum_{\nu}\frac{Q_\nu^{\mathbf{\Lambda}(\beta)}(q)}{z_\nu},$$

where the sums run over $\tilde{\mathfrak{T}}$.

Remark V.3.4.2. The inner product only depends on the corresponding unordered types.

Proof. We use the base changes

$$s_\alpha(\mathbf{x}) = \sum_\tau \chi_\tau^\alpha \frac{p_\tau(\mathbf{x})}{z_\tau}.$$

and,

$$\begin{aligned}\tilde{H}_{\mathbf{\Lambda}(\beta)}(\mathbf{x}, q) &= \sum_\tau \tilde{K}_{\tau,\beta}(q) s_\tau(\mathbf{x}) \\ &= \sum_\tau \sum_\nu \chi_\nu^\tau \tilde{K}_{\tau,\beta}(q) \frac{p_\nu(\mathbf{x})}{z_\nu} \\ &= \sum_\nu Q_\nu^{\mathbf{\Lambda}(\beta)}(q) \frac{p_\nu(\mathbf{x})}{z_\nu}.\end{aligned}$$

The result follows by taking inner product, noting that

$$\langle p_\tau(\mathbf{x}), p_\nu(\mathbf{x})\rangle = \langle p_{[\tau]}(\mathbf{x}), p_{[\nu]}(\mathbf{x})\rangle = \delta_{[\tau],[\nu]} z_{[\tau]}.$$

□

V.4 Computation of the E-Polynomial

V.4.1 The Formula of E-Polynomials

V.4.1.1 The summation over the irreducible characters in the point-counting formula can be divided into two summations, one over the types of size N, the other over the σ-stable irreducible characters of the same type, as the following computation shows.

$$\begin{aligned}|\mathrm{Ch}_C(\mathbb{F}_q)| &= \sum_{\mathbf{e}=(e_j)\in\{\pm\}^{2k}} \sum_{\chi\in\mathrm{Irr}(G)^\sigma} \left(\frac{|G|}{\chi(1)}\right)^{2g-2} \prod_{j=1}^{2k} \frac{|C_{j,e_j}|\tilde{\chi}(C_{j,e_j})}{\chi(1)} \\ &= \sum_{\mathbf{e}=(e_j)\in\{\pm\}^{2k}} \sum_{|\epsilon\omega|=N} \frac{|G|^{2g-2}\prod_{j=1}^{2k}|C_{j,e_j}|}{\chi(1)^{2g+2k-2}} \sum_{\chi\in\mathrm{Irr}(G)^\sigma_{\epsilon\omega}} \prod_{j=1}^{2k} \tilde{\chi}(C_{j,e_j}) \\ &= \sum_{|\epsilon\omega|=N} \frac{|G|^{2g-2}\prod_{j=1}^{2k}|C_{j,e_j}|}{\chi(1)^{2g+2k-2}} \sum_{\mathbf{e}=(e_j)\in\{\pm\}^{2k}} \sum_{\chi\in\mathrm{Irr}(G)^\sigma_{\epsilon\omega}} \prod_{j=1}^{2k} \tilde{\chi}(C_{j,e_j}).\end{aligned}$$

In the second equality, we have used the fact that if two irreducible characters are of the same type $\epsilon\omega$, then their underlying irreducible character of $\mathrm{GL}_n(q)$ correspond to $\{\epsilon\omega\}$ (*cf.* (V.3.1.2.2)) and so have the same degree.

V.4.1.2 Let us first give an expression for the value of an irreducible character. Recall that the extension of a σ-stable irreducible character, if it is uniform, can be decomposed as

follows:

$$\tilde{\chi}|_{G^F.\sigma} = |W_{L_1}^{\sigma_1} \times \mathfrak{W}_+ \times \mathfrak{W}_-|^{-1} \sum_{\mathbf{w}:=(w_1,w_+,w_-)} \tilde{\varphi}_1(w_1F)\varphi_+(w_+)\varphi_-(w_-)R_{T_{\mathbf{w}}.\sigma}^{G.\sigma}(\tilde{\theta}), \tag{V.4.1.2.1}$$

where L_1, φ and θ are as in §IV.1.2.4. The sum over $\mathbf{w}$ only depends on its F-conjugacy class, which we denote by $\boldsymbol{\tau}$. These classes are parametrised by $\tilde{\mathfrak{T}}$. We will therefore replace $\sum_{\mathbf{w}}(-)$ by $\sum_{\boldsymbol{\tau}} |\boldsymbol{\tau}| \cdot (-)$.

If $(s\sigma)u$ is the Jordan decomposition of some element of $\mathrm{GL}_n(q).\sigma$, then the character formula reads

$$R_{T_{\mathbf{w}}.\sigma}^{G.\sigma}\tilde{\theta}(s\sigma u) = \frac{|(T_{\mathbf{w}}^{\sigma})^{\circ F}|}{|T_{\mathbf{w}}^F| \cdot |C_G(s\sigma)^{\circ F}|} \sum_{\substack{h\in G^F \\ hs\sigma h^{-1}\in T_{\mathbf{w}}.\sigma}} Q_{C^\circ_{h_{T_w}}(s\sigma)}^{C^\circ_G(s\sigma)}(u)\tilde{\theta}(hs\sigma h^{-1}). \tag{V.4.1.2.2}$$

If $\mathbf{w}$ is of class τ, denote

$$A_\tau^F = \{h \in G^F \mid hs\sigma h^{-1} \in T_{\mathbf{w}}.\sigma\},$$

and for $\nu \in \tilde{\mathfrak{T}}$, denote by $A_{\tau,\nu}^F$ the subset

$$\{h \in A_\tau^F \mid C_{h^{-1}T_{\mathbf{w}}h}(s\sigma)^\circ \text{ is of class } \nu\}.$$

Denote by α the type of the σ-stable irreducible character χ and by β the type of the conjugacy class of $s\sigma u$. Note that with the fixed L_1, φ, $s\sigma$ and u, we can make these types ordered ones. Therefore in the following expression we work with ordered types. If $C_{h^{-1}T_{\mathbf{w}}h}(s\sigma)^\circ$ is of class ν and $s\sigma u$ is of type β, we will write $Q_\nu^{\mathbf{\Lambda}(\beta)}(q) = Q_{C_{h_{T_{\mathbf{w}}}}(s\sigma)^\circ}^{C_G(s\sigma)^\circ}(u)$. Combining (V.4.1.2.1) and (V.4.1.2.2) gives

$$\begin{aligned}\tilde{\chi}(s\sigma u) &= \sum_\tau \sum_{h\in A_\tau^F} z_\tau^{-1}\chi_\tau^\alpha \frac{|(T_\tau^\sigma)^{\circ F}|}{|T_\tau^F| \cdot |C_G(s\sigma)^{\circ F}|} Q_\nu^{\mathbf{\Lambda}(\beta)}(q)\tilde{\theta}(hs\sigma h^{-1}) \\ &= \sum_\tau \sum_{\{\nu|[\nu]=[\tau]\}} z_\tau^{-1}\chi_\tau^\alpha \frac{|(T_\tau^\sigma)^{\circ F}|}{|T_\tau^F| \cdot |C_G(s\sigma)^{\circ F}|} Q_\nu^{\mathbf{\Lambda}(\beta)}(q) \sum_{h\in A_{\tau,\nu}^F} \tilde{\theta}(hs\sigma h^{-1})\end{aligned} \tag{V.4.1.2.3}$$

V.4.1.3 We calculate $\sum_{\chi\in\mathrm{Irr}(G)_{\epsilon\omega}^\sigma} \prod_{j=1}^{2k} \tilde{\chi}(C_{j,e_j})$ using the above expression. Note that a σ-stable irreducible character of a given type is completely determined by its semi-simple part. Rewrite $\omega = \omega_+\omega_-(\omega_i)$ as $\omega = \omega_+\omega_-(m_\lambda)_{\lambda\in\mathcal{P}}$, where m_λ is the multiplicity of a given partition λ that appears in the sequence (α_i), then put

$$N(\omega) = \prod_\lambda m_\lambda!. \tag{V.4.1.3.1}$$

Then we can replace $\sum_{\chi\in\mathrm{Irr}(G)_{\epsilon\omega}^\sigma}$ by $2^{-l(\omega_p)}N(\omega)^{-1}\sum_{\theta\in\mathrm{Irr}_{reg,\epsilon\omega}^\sigma}$. Recall that ω_p is the part of ω consisting of partitions.

For each j, let $s_j\sigma$ be a representative of C_{j,e_j}. We calculate

$$\sum_{\chi\in\mathrm{Irr}(G)^\sigma_{\epsilon\omega}} \prod_{j=1}^{2k} \tilde{\chi}(C_{j,e_j})$$
$$=\frac{1}{2^{l(\omega_p)}N(\omega)} \sum_{\tau_1,\dots,\tau_{2k}} \sum_{\nu_1,\dots,\nu_{2k}} \prod_{j=1}^{2k} z_{\tau_j}^{-1}\chi^\omega_{\tau_j} \frac{|(T^\sigma_{\tau_j})^{\circ F}|}{|T^F_{\tau_j}|\cdot|C_G(s\sigma)^{\circ F}|} Q^{\mathbf{\Lambda}(\beta)}_{\nu_j}(q) \sum_{\substack{(h_1,\dots,h_{2k})\in \\ A^F_{\tau_1,\nu_1}\times\cdots\times A^F_{\tau_{2k},\nu_{2k}}}} \sum_{\theta\in\mathrm{Irr}_{reg,\epsilon\omega}} \prod_{j=1}^{2k} \tilde{\theta}_j(h_j s_j\sigma h_j^{-1})$$

Since

$$\sum_{\theta\in\mathrm{Irr}_{reg,\epsilon\omega}} \prod_{j=1}^{2k} \tilde{\theta}_j(h_j s_j\sigma h_j^{-1}) = (-1)^{l(\omega_p)}2^{l(\omega_p)}l(\omega_p)!\Delta_{\epsilon,\bar{\mathbf{e}}},$$

is independent of h_j and we have already shown that (IV.6.2.6.1)

$$|A^F_{\tau,\nu}| = \frac{z_{[\tau]}}{z_\nu}\cdot\frac{|T^F_\tau|\cdot|C_G(s\sigma)^{\circ F}|}{|(T^\sigma_\tau)^{\circ F}|},$$

we find

$$\sum_{\chi\in\mathrm{Irr}(G)^\sigma_{\epsilon\omega}} \prod_{j=1}^{2k} \tilde{\chi}(C_j)$$
$$=\Delta_{\epsilon,\bar{\mathbf{e}}}\frac{K(\omega)}{N((\omega)} \prod_{j=1}^{2k} \sum_{\tau_j} \frac{z_{[\tau_j]}\chi^\omega_{\tau_j}}{z_{\tau_j}} \sum_{\nu_j} \frac{Q^{\mathbf{\Lambda}(\beta)}_{\nu_j}(q)}{z_{\nu_j}}$$

(V.4.1.3.2) $$=\Delta_{\epsilon,\bar{\mathbf{e}}}\frac{K(\omega)}{N((\omega)} \prod_{j=1}^{2k} \langle s_\omega(\mathbf{x}), \tilde{H}_{\mathbf{\Lambda}(\boldsymbol{\beta}_j)}(\mathbf{x},q)\rangle,$$

with

(V.4.1.3.3) $$K(\omega) = (-1)^{l(\omega_p)}l(\omega_p)!.$$

We remark that

(V.4.1.3.4) $$\sum_{\mathbf{e}=(e_j)\in\{\pm\}^{2k}} \Delta_{\epsilon,\bar{\mathbf{e}}} = \begin{cases} 2^{2k} & \text{if } \epsilon = +, \\ 0 & \text{if } \epsilon = -. \end{cases}$$

V.4.1.4 Recall the Hook polynomial $H_\lambda(q)$ defined for any partition λ:

(V.4.1.4.1) $$H_\lambda(q) := \prod_{x\in\lambda}(1-q^{h(x)}),$$

where λ is regarded as a Young diagram and x runs over the boxes in the diagram, and $h(x)$ is the hook length. If we denote by λ^* the dual partition of λ, then

$$\sum_{x\in\lambda} h(x) = |\lambda| + n(\lambda) + n(\lambda^*). \tag{V.4.1.4.2}$$

For any $\lambda = \lambda_+\lambda_-(\lambda_i) \in \mathring{\mathfrak{T}}$, put $H_\lambda(q) := H_{\lambda_+}(q)H_{\lambda_-}(q)\prod_i H^2_{\lambda_i}(q)$ and $\lambda^* = \lambda^*_+\lambda^*_-(\lambda^*_i)$. Then for $\chi \in \mathrm{Irr}_\lambda(\mathrm{GL}_n(q))$, we have ([Mac, Chapter IV, §6 (6.7)]):

$$\frac{|\mathrm{GL}_n(q)|}{\chi(1)} = -q^{\frac{1}{2}n(n-1)-n(\lambda^*)}H_\lambda(q). \tag{V.4.1.4.3}$$

V.4.1.5 Let $\mathbf{P}$ be the matrix (V.3.2.1.9). Introduce the following notations

$$\begin{aligned}
\mathfrak{O}(\mathbf{x},q)_e &:= \sum_{\boldsymbol{\alpha}\in\mathcal{P}^2} q^{(1-g)|\{\boldsymbol{\alpha}\}_e|}(H_{\{\boldsymbol{\alpha}\}_e}(q)q^{-n(\{\boldsymbol{\alpha}\}^*_e)})^{2g+2k-2}\prod_{j=1}^{2k} s_{\boldsymbol{\alpha}}(\mathbf{P}\,\mathbf{x}_j), \quad e = 0,\ 1;\\
\tilde{\mathfrak{O}}(\mathbf{x},q) &:= \sum_{\alpha\in\mathcal{P}} q^{(1-g)|\{\alpha\}|}(H_{\{\alpha\}}(q)q^{-n(\{\alpha\}^*)})^{2g+2k-2}\prod_{j=1}^{2k} s_{\boldsymbol{\alpha}}(\mathbf{P}\,\mathbf{x}_j)\\
&= \sum_{\alpha\in\mathcal{P}} q^{2(1-g)|\alpha|}(H_\alpha(q)^2q^{-2n(\alpha^*)})^{2g+2k-2}\prod_{j=1}^{2k} s_{\boldsymbol{\alpha}}(\mathbf{P}\,\mathbf{x}_j)
\end{aligned}$$

The summand corresponding to the trivial partition or 2-partition is equal to 1. Applying the formal expansion

$$\frac{1}{1+x} = \sum_{m\geq 0}(-1)^m x^m$$

to $1 + x = \tilde{\mathfrak{O}}(\mathbf{x},q)$ gives

$$\begin{aligned}
\frac{1}{\tilde{\mathfrak{O}}(\mathbf{x},q)} &= \sum_{m\geq 0}(-1)^m \sum_{\omega_p\leftrightarrow(m_\lambda)_\lambda} \frac{m!}{\prod_\lambda m_\lambda!} q^{(1-g)|\{\omega_p\}|}(H_{\{\omega_p\}}(q)q^{-n(\{\omega_p\}^*)})^{2g+2k-2}\prod_{j=1}^{2k} s_{\omega_p}(\mathbf{P}\,\mathbf{x}_j)\\
&= \sum_{\omega_p} \frac{K(\omega)}{N(\omega)} q^{(1-g)|\{\omega_p\}|}(H_{\{\omega_p\}}(q)q^{-n(\{\omega_p\}^*)})^{2g+2k-2}\prod_{j=1}^{2k} s_{\omega_p}(\mathbf{P}\,\mathbf{x}_j) \qquad \text{(V.4.1.5.1)}
\end{aligned}$$

V.4.1.6 The dimension of the character variety is given by:

$$\begin{aligned}
d :=& (2g-2)\dim G + \sum_{j=1}^{2k}\dim C_j\\
=& (2g-2)\dim G + \sum_{j=1}^{2k}(\dim G - \dim C_G(s_j\sigma)
\end{aligned}$$

$$\overset{①}{=}(2g-2)n^2 + 2kn^2 - \sum_{j=1}^{2k}(2a(\beta_j)+N)$$
$$=n^2(2g+2k-2) - 2kN - \sum_{j=1}^{2k} 2a(\beta_j).$$

(*cf.* I.4) In ①, we have used the fact that if H is a connected reductive group over $\mathbb{C}$ and u is the identity, then $\dim H = 2\dim \mathcal{B}_u + \operatorname{rk} H$.

V.4.1.7 We can now state the main theorem.

Theorem V.4.1.1. *The number of rational points of the character variety over a finite field is given by:*

$$|\operatorname{Ch}_C(\mathbb{F}_q)| = q^{\frac{1}{2}d-k(N+1)}\left\langle \frac{\mathfrak{O}(\mathbf{x},q)_1\mathfrak{O}(\mathbf{x},q)_0}{\tilde{\mathfrak{O}}(\mathbf{x},q)}, \prod_{j=1}^{2k} h_{\beta_j^*}(\mathbf{x}_j)\right\rangle. \tag{V.4.1.7.1}$$

Proof. We calculate

$$\begin{aligned}
&|\operatorname{Ch}_C(\mathbb{F}_q)| \\
=&\sum_{\mathbf{e}=(e_j)\in\{\pm\}^{2k}}\sum_{|\epsilon\omega|=N}\frac{|G|^{2g-2}\prod_{j=1}^{2k}|C_{j,e_j}|}{\chi(1)^{2g+2k-2}}\sum_{\chi\in\operatorname{Irr}(G)^\sigma_{\epsilon\omega}}\prod_{j=1}^{2k}\tilde{\chi}(C_{j,e_j}) \\
\overset{①}{=}&2^{2k}\sum_{\omega\in\mathfrak{T}}\frac{K(\omega)}{N(\omega)}\frac{|G|^{2g-2}\prod_{j=1}^{2k}|C_{j,+}|}{\chi(1)^{2g+2k-2}}\prod_{j=1}^{2k}\langle s_\omega(\mathbf{x}_j),\tilde{H}_{\mathbf{\Lambda}(\boldsymbol{\beta}_j)}(\mathbf{x}_j,q)\rangle \\
=&2^{2k}\sum_{\omega\in\mathfrak{T}}\frac{K(\omega)}{N(\omega)}\left(\frac{|G|}{\chi(1)}\right)^{2g+2k-2}\prod_{j=1}^{2k}\frac{|C_{j,+}|}{|G|}\prod_{j=1}^{2k}\langle s_\omega(\mathbf{x}_j),\tilde{H}_{\mathbf{\Lambda}(\boldsymbol{\beta}_j)}(\mathbf{x}_j,q)\rangle \\
=&2^{2k}\sum_{\omega\in\mathfrak{T}}\frac{K(\omega)}{N(\omega)}(H_{\{\omega\}}(q)q^{\frac{1}{2}n(n-1)-n(\{\omega\}^*)})^{2g+2k-2}\prod_{j=1}^{2k}\langle s_\omega(\mathbf{x}_j),\frac{|C_{j,+}|}{|G|}\tilde{H}_{\mathbf{\Lambda}(\boldsymbol{\beta}_j)}(\mathbf{x}_j,q)\rangle \\
=&2^{2k}q^{\frac{1}{2}(n^2(2g+2k-2)-2kn)} \\
&\cdot\left\langle\sum_{\omega\in\mathfrak{T}}\frac{K(\omega)}{N(\omega)}q^{(1-g)|\{\omega\}|}(H_{\{\omega\}}(q)q^{-n(\{\omega\}^*)})^{2g+2k-2}\prod_{j=1}^{2k}s_\omega(\mathbf{x}_j),\prod_{j=1}^{2k}\frac{|C_{j,+}|}{|G|}\tilde{H}_{\mathbf{\Lambda}(\boldsymbol{\beta}_j)}(\mathbf{x}_j,q)\rangle\right\rangle.
\end{aligned} \tag{V.4.1.7.2}$$

We have the factor 2^{2k} in ① because the summand for $\epsilon = +$ is independent of $\mathbf{e}$ and cancels out for $\epsilon = -$, and the sum can be taken over the entire $\mathfrak{T}$ because if ω was not of size N then the inner product would vanish.

Equations (V.2.4.2.1) and (V.2.4.2.2) show that

$$\frac{|C_{j,+}|}{|G|} = \frac{1}{2} q^{-2a(\beta_j)-|\beta_j|} b_{\mathbf{\Lambda}(\beta_j)}(q^{-1})^{-1}, \tag{V.4.1.7.3}$$

where the factor 1/2 comes from the two connected components of the orthogonal group. Combined with Corollary V.3.2.2, this shows

$$\frac{1}{2} q^{-a(\beta_j)} h_{\beta_j^*}(\mathbf{P}\,\mathbf{x}_j) = (-1)^{|\beta_j|} \frac{|C_{j,+}|}{|G|} \tilde{H}_{\mathbf{\Lambda}(\beta_j)}(\mathbf{x}_j; q). \tag{V.4.1.7.4}$$

Note that for any symmetric functions $u(\mathbf{x}^{(0)}, \mathbf{x}^{(1)})$ and $v(\mathbf{x}^{(0)}, \mathbf{x}^{(1)})$, we have

$$\left\langle u(\mathbf{P}\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}), v(\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}) \right\rangle = \left\langle u(\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}), v(\mathbf{P}\begin{bmatrix}\mathbf{x}^{(0)}\\ \mathbf{x}^{(1)}\end{bmatrix}) \right\rangle.$$

This can be checked on the basis of power sums. We will thus move $\mathbf{P}$ to the left hand side of the inner product.

Rewrite the left hand side of the inner product (V.4.1.7.2) as follows:

$$\begin{aligned}
&\sum_{\omega\in\mathfrak{T}} \frac{K(\omega)}{N(\omega)} q^{(1-g)|\{\omega\}|} (H_{\{\omega\}}(q) q^{-n(\{\omega\}^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_\omega(\mathbf{P}\,\mathbf{x}_j) \\
= &\left(\sum_{\omega_+\in\mathcal{P}^2} q^{(1-g)|\{\omega_+\}_1|} (H_{\{\omega_+\}_1}(q) q^{-n(\{\omega_+\}_1^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_+}(\mathbf{P}\,\mathbf{x}_j) \right) \\
&\cdot \left(\sum_{\omega_-\in\mathcal{P}^2} q^{(1-g)|\{\omega_-\}_0} (H_{\{\omega_-\}_0}(q) q^{-n(\{\omega_-\}_0^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_-}(\mathbf{P}\,\mathbf{x}_j) \right) \\
&\cdot \left(\sum_{\omega_p} \frac{K(\omega)}{N(\omega)} q^{(1-g)|\{\omega_p\}|} (H_{\{\omega_p\}}(q) q^{-n(\{\omega_p\}^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_p}(\mathbf{P}\,\mathbf{x}_j) \right)
\end{aligned}$$

Note that $K(\omega)$ and $N(\omega)$ only depend on ω_p

We deduce that

$$\sum_{\omega\in\mathfrak{T}} \frac{K(\omega)}{N(\omega)} q^{(1-g)|\{\omega\}|} (H_{\{\omega\}}(q) q^{-n(\{\omega\}^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_\omega(\mathbf{P}\,\mathbf{x}_j) = \frac{\mathfrak{O}(\mathbf{x}, q)_1 \mathfrak{O}(\mathbf{x}, q)_0}{\tilde{\mathfrak{O}}(\mathbf{x}, q)} \tag{V.4.1.7.5}$$

Therefore,

$$|\mathrm{Ch}_C(\mathbb{F}_q)| = q^{\frac{1}{2}(n^2(2g+2k-2)-2kN)-\sum_{j=1}^{2k} a(\beta_j)} q^{-k(N+1)} \cdot \left\langle \frac{\mathfrak{O}(\mathbf{x}, q)_1 \mathfrak{O}(\mathbf{x}, q)_0}{\tilde{\mathfrak{O}}(\mathbf{x}, q)}, \prod_{j=1}^{2k} h_{\beta_j^*}(\mathbf{x}_j) \right\rangle. \tag{V.4.1.7.6}$$

□

Corollary V.4.1.2. *The E-polynomial $E(q) = |\mathrm{Ch}_C(\mathbb{F}_q)|$ satisfies*

$$q^d E(q^{-1}) = E(q). \tag{V.4.1.7.7}$$

Proof. For a partition λ, we have the equalities:

$$q^{-2(1-g)|\lambda|} = q^{2(1-g)|\lambda|} \cdot q^{-4(1-g)|\lambda|}; \tag{V.4.1.7.8}$$

$$H_\lambda(q^{-1})^2 = H_\lambda(q)^2 q^{-2(|\lambda|+n(\lambda)+n(\lambda^*))}; \tag{V.4.1.7.9}$$

$$q^{2n(\lambda)} = q^{-2n(\lambda^*)} \cdot q^{2(n(\lambda)+n(\lambda^*))}; \tag{V.4.1.7.10}$$

$$s_\lambda(\frac{\mathbf{x}}{1-q^{-1}}) = s_\lambda(\frac{-q\,\mathbf{x}}{1-q}) = (-q)^{|\lambda|} s_{\lambda^*}(\frac{\mathbf{x}}{1-q}). \tag{V.4.1.7.11}$$

We deduce that

$$\begin{aligned} &q^{-(1-g)|\{\omega_p\}|}(H_{\{\omega_p\}}(q^{-1})q^{n(\{\omega_p\}^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_p}(\mathbf{P}(q^{-1})\,\mathbf{x}_j) \\ =&q^{-2k|\omega_p|} q^{(1-g)|\{\omega_p^*\}|}(H_{\{\omega_p^*\}}(q)q^{-n(\{\omega_p\})})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_p^*}(\mathbf{P}(q)\,\mathbf{x}_j). \end{aligned}$$

Since

$$\mathbf{P}(q^{-1}) = \frac{-q}{1-q^2}\begin{pmatrix} q & 1 \\ 1 & q \end{pmatrix} = -q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{P}(q),$$

for a 2-partition $\boldsymbol{\alpha}$, we have the equality:

$$s_{\boldsymbol{\alpha}}(\mathbf{P}(q^{-1})\begin{bmatrix}\mathbf{x}^{(0)} \\ \mathbf{x}^{(1)}\end{bmatrix}) = s_{\boldsymbol{\alpha}}(-q\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbf{P}\begin{bmatrix}\mathbf{x}^{(0)} \\ \mathbf{x}^{(1)}\end{bmatrix}) = s_{\boldsymbol{\alpha}^*}(-q\mathbf{P}\begin{bmatrix}\mathbf{x}^{(0)} \\ \mathbf{x}^{(1)}\end{bmatrix}) = (-q)^{|\boldsymbol{\alpha}|} s_{\boldsymbol{\alpha}^*}(\mathbf{P}\begin{bmatrix}\mathbf{x}^{(0)} \\ \mathbf{x}^{(1)}\end{bmatrix}).$$

(If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, then $\boldsymbol{\alpha}^* = (\alpha_2^*, \alpha_1^*)$.)

Using $\{\boldsymbol{\alpha}^*\}_e = \{\boldsymbol{\alpha}\}_e^*$, $e = 0, 1$, we then deduce that

$$\begin{aligned} &q^{-(1-g)|\{\omega_+\}_1|}(H_{\{\omega_+\}_1}(q^{-1})q^{n(\{\omega_+\}_1^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_+}(\mathbf{P}(q^{-1})\,\mathbf{x}_j) \\ =&q^{-2k(|\omega_+|+1)} q^{(1-g)|\{\omega_+^*\}_1|}(H_{\{\omega_+^*\}_1}(q)q^{-n(\{\omega_+\}_1)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_+^*}(\mathbf{P}(q)\,\mathbf{x}_j), \end{aligned}$$

and

$$q^{-(1-g)|\{\omega_-\}_1|}(H_{\{\omega_-\}_1}(q^{-1})q^{n(\{\omega_-\}_1^*)})^{2g+2k-2} \prod_{j=1}^{2k} s_{\omega_-}(\mathbf{P}(q^{-1})\,\mathbf{x}_j)$$

$$=q^{-2k(|\omega_-|)}q^{(1-g)|\{\omega_-^*\}_1|}(H_{\{\omega_-^*\}_1}(q)q^{-n(\{\omega_-\}_1)})^{2g+2k-2}\prod_{j=1}^{2k}s_{\omega_-^*}(\mathbf{P}(q)\,\mathbf{x}_j).$$

We conclude that inverting q on the left hand side of the inner product (V.4.1.7.2) gives rise to an extra term $q^{-2k(|\omega_p|+|\omega_-|+|\omega_+|+1)}$, which is balanced by the term $q^{-k(N+1)}$ outside the inner product.

□

V.5 Examples

In this section we will give some explicit computations in low ranks. In all these cases, we confirm that the polynomials in q are indeed palindromic. The polynomials for $\mathrm{GL}_2(q)$ have even integer coefficients with leading coefficients 2, while the polynomials for $\mathrm{GL}_3(q)$ have leading coefficients 1.

V.5.1 $G = \mathrm{GL}_2(q)$

V.5.1.1 $g = 0$, $2k = 4$. For any $j \in \{1,2,3,4\}$, let C_j be the conugacy class of $\mathrm{diag}(a_j, a_j^{-1})\sigma$, with $a_j^q = a_j$ and $a_j^2 \neq 1$ or -1, such that $C = (C_1, C_2, C_3, C_4)$ is generic. In this case we have a double covering of $\mathbb{P}^1$ by an elliptic curve with all four ramification points removed. The counting formula reads

$$|\mathrm{Ch}_C(\mathbb{F}_q)| = \sum_\chi \left(\frac{|G|}{\chi(1)}\right)^2 \prod_j \frac{C_j}{|G|} \prod_j \tilde{\chi}(C_j).$$

With the character table of $\mathrm{GL}_2(q).<\sigma>$, we get

$$\begin{aligned}|\mathrm{Ch}_C(\mathbb{F}_q)| =& \frac{1}{2}\left(\frac{q(q-1)(q^2-1)}{q+1}\right)^2 \frac{1}{(q-1)^4}\sum_\alpha \prod_j (\alpha(a_j^2)+\alpha(a_j^{-2}))\\ &+2\cdot\left(\frac{q(q-1)(q^2-1)}{1}\right)^2\frac{1}{(q-1)^4}\\ &+2\cdot\left(\frac{q(q-1)(q^2-1)}{q}\right)^2\frac{1}{(q-1)^4}\\ =& -16q^2+2q^2(q+1)^2+2(q+1)^2\\ =&2q^4+4q^3-12q^2+4q+2.\end{aligned}$$

V.5.1.2 $g = 0$, $2k = 4$. We keep C_1, C_2 and C_3 as above but put C_4 to be the conjugacy class of σ. In this case only 3 ramification points are removed. The same counting formula gives

$$\begin{aligned}
|\mathrm{Ch}_C(\mathbb{F}_q)| =& \frac{1}{2}\left(\frac{q(q-1)(q^2-1)}{q+1}\right)^2 \frac{1}{(q-1)^3}\frac{1}{q(q-1)(q+1)}\sum_{\alpha}(q+1)\prod_{j=1}^{3}(\alpha(a_j^2)+\alpha(a_j^{-2}))\\
&+2\cdot\left(\frac{q(q-1)(q^2-1)}{1}\right)^2 \frac{1}{(q-1)^3}\frac{1}{q(q-1)(q+1)}\\
&+2\cdot\left(\frac{q(q-1)(q^2-1)}{q}\right)^2 \frac{1}{(q-1)^3}\frac{1}{q(q-1)(q+1)}\cdot q\\
=& -8q^2+2q(q+1)+2(q+1)\\
=&2q^2-4q+2.
\end{aligned}$$

However, if we further assume that C_3 is also the conjugacy class of σ, that is, we keep two ramification points in the covering, then there is no generic conjugacy classes such that the character variety is non-empty.

V.5.1.3 $g = 1$, $2k = 2$. For $j \in \{1, 2\}$, let C_j be the conugacy class of $\mathrm{diag}(a_j, a_j^{-1})\sigma$, with $a_j^q = a_j$ and $a_j^2 \neq 1$ or -1, such that $C = (C_1, C_2)$ is generic. In this case we have a double covering of an elliptic curve by a curve of genus 2 with both of the two ramification points removed. The counting formula gives

$$\begin{aligned}
|\mathrm{Ch}_C(\mathbb{F}_q)| =& \sum_{\chi}\frac{|C_1||C_2|\tilde{\chi}(C_1)\tilde{\chi}(C_2)}{\chi(1)^2}\\
=&\frac{1}{2}\left(\frac{q(q-1)(q+1)}{q+1}\right)^2\sum_{\alpha}(\alpha(a_1^2)+\alpha(a_1^{-2}))(\alpha(a_2^2)+\alpha(a_2^{-2}))\\
&+2\left(\frac{q(q-1)(q+1)}{1}\right)^2\\
&+2\left(\frac{q(q-1)(q+1)}{q}\right)^2\\
=& -4q^2(q-1)^2+2q^2(q-1)^2(q+1)^2+2(q-1)^2(q+1)^2\\
=&2q^6-6q^4+8q^3-6q^2+2.
\end{aligned}$$

V.5.1.4 $g = 1$, $2k = 2$. We keep C_1 unchanged but put C_2 to be the conjugacy class of σ. In this case only 1 ramification points are removed. The same counting formula gives

$$|\mathrm{Ch}_C(\mathbb{F}_q)| = \frac{1}{2}\cdot\frac{q(q-1)^2(q+1)}{(q+1)^2}\sum_{\alpha}(q+1)(\alpha(a_1^2)+\alpha(a_1^{-2}))$$

$$
\begin{aligned}
&+2\frac{q(q-1)^2(q+1)}{1}\\
&+2\frac{q(q-1)^2(q+1)}{q^2}\cdot q\\
=&-2q(q-1)^2+2q(q-1)^2(q+1)+2(q-1)^2(q+1)\\
=&2q^4-2q^3-2q+2.
\end{aligned}
$$

V.5.2 $G = \mathrm{GL}_3(q)$

The geometric settings will be parallel to the case of $\mathrm{GL}_2(q)$.

V.5.2.1 $g = 0$, $2k = 4$. For any $j \in \{1,2,3,4\}$, let $C_{j,+}$ be the conugacy class of $\mathrm{diag}(a_j, 1, a_j^{-1})\sigma$, with $a_j^q = a_j$ and $a_j^2 \neq 1$ or -1, such that ① $\mathcal{C} = (C_{1,+}, C_{2,+}, C_{3,+}, C_{4,+})$ is generic. Let $c \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2$ and let $C_{j,-}$ be the conjugacy class of $\mathrm{diag}(a_j, c, a_j^{-1})\sigma$. Now the counting formula reads

$$
|\mathrm{Ch}_{\mathcal{C}}(\mathbb{F}_q)| = \sum_{(e_j)\in\{\pm\}^4}\sum_{\chi}\left(\frac{|G|}{\chi(1)}\right)^2\prod_j\frac{C_{j,e_j}}{|G|}\prod_j\tilde{\chi}(C_{j,e_j}).
$$

With the character table of $\mathrm{GL}_3(q).<\sigma>$, we get:

$$
\begin{aligned}
|\mathrm{Ch}_{\mathcal{C}}(\mathbb{F}_q)| =&\frac{16}{2}\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{(q+1)(q^2+q+1)}\right)^2\left(\frac{1}{2(q-1)}\right)^4\sum_{\alpha}\prod_j(\alpha(a_j^2)+\alpha(a_j^{-2}))\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{q^2+q+1}\right)^2\left(\frac{1}{2(q-1)}\right)^4\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{q(q^2+q+1)}\right)^2\left(\frac{1}{2(q-1)}\right)^4\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{1}\right)^2\left(\frac{1}{2(q-1)}\right)^4\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{q^3}\right)^2\left(\frac{1}{2(q-1)}\right)^4\\
=&-16q^6(q-1)^2+q^6(q-1)^2(q+1)^2+q^4(q-1)^2(q+1)^2\\
&+q^6(q-1)^2(q+1)^2(q^2+q+1)^2+(q-1)^2(q+1)^2(q^2+q+1)^2\\
=&q^{14}+2q^{13}+q^{12}-2q^{11}-3q^{10}-2q^9-15q^8+36q^7\\
&-15q^6-2q^5-3q^4-2q^3+q^2+2q+1.
\end{aligned}
$$

The first five lines correspond to the five types of irreducible characters: $R_T^G(\alpha, 1, \alpha^{-1})$, $R_L^G(\eta\,\mathrm{Id}_2, \mathrm{Id})$, $R_L^G(\eta\,\mathrm{St}, \mathrm{Id})$, Id_3 and χ_2, as listed in Appendix A.0.1.2. The factors 16 arise

① $C_{j,\pm}$ or $C_j^{\pm}$?

from the summation over $(e_j) \in \{\pm\}^4$, noting that for any j, $|C_{j,+}| = |C_{j,-}|$. This summation also annihilates the sum over the irreducible characters: $R_T^G(\alpha, \eta, \alpha^{-1})$, $R_L^G(\mathrm{Id}_2, \eta)$, $R_L^G(\mathrm{St}, \eta)$, $\eta\,\mathrm{Id}_3$ and $\eta\chi_2$.

V.5.2.2 $g = 0$, $2k = 4$. We keep $C_{1,\pm}$, $C_{2,\pm}$ and $C_{3,\pm}$ as above but put $C_{4,\pm}$ to be the conjugacy class of σ and $\mathrm{diag}(1, c, 1)\sigma$. The same counting formula gives

$$\begin{aligned}
|\mathrm{Ch}_C(\mathbb{F}_q)| =& \frac{16}{2}\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{(q+1)(q^2+q+1)}\right)^2\left(\frac{1}{2(q-1)}\right)^3\frac{1}{2q(q-1)(q+1)}\\
&\cdot\sum_{\alpha}\prod_{j=1}^{3}(q+1)(\alpha(a_j^2)+\alpha(a_j^{-2}))\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{q^2+q+1}\right)^2\left(\frac{1}{2(q-1)}\right)^3\frac{1}{2q(q-1)(q+1)}\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{q(q^2+q+1)}\right)^2\left(\frac{1}{2(q-1)}\right)^3\frac{1}{2q(q-1)(q+1)}\cdot q\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{1}\right)^2\left(\frac{1}{2(q-1)}\right)^3\frac{1}{2q(q-1)(q+1)}\\
&+16\left(\frac{q^3(q-1)(q^2-1)(q^3-1)}{q^3}\right)^2\left(\frac{1}{2(q-1)}\right)^3\frac{1}{2q(q-1)(q+1)}\cdot q\\
=& -8q^5(q-1)^2+q^5(q-1)^2(q+1)+q^4(q-1)^2(q+1)\\
&+q^5(q-1)^2(q+1)(q^2+q+1)^2+(q-1)^2(q+1)(q^2+q+1)^2\\
=& q^{12}+q^{11}-2q^9-q^8-7q^7+16q^6\\
&-7q^5-q^4-2q^3+q+1.
\end{aligned}$$

V.5.2.3 $g = 1$, $2k = 2$. For $j \in \{1, 2\}$, let $C_{j,\pm}$ be the conugacy class of $\mathrm{diag}(a_j, 1, a_j^{-1})\sigma$, with $a_j^q = a_j$ and $a_j^2 \neq 1$ or -1, such that $C = (C_{1,+}, C_{2,+})$ is generic. Let $C_{j,-}$ be the conjugacy class of $\mathrm{diag}(a_j, c, a_j^{-1})\sigma$. The counting formula gives

$$\begin{aligned}
|\mathrm{Ch}_C(\mathbb{F}_q)| =& \sum_{(e_j)\in\{\pm\}^2}\sum_{\chi}\frac{|C_{1,e_1}||C_{2,e_2}|\tilde{\chi}(C_{1,e_1})\tilde{\chi}(C_{2,e_2})}{\chi(1)^2}\\
=&\frac{4}{2}\left(\frac{q^3(q^2-1)(q^3-1)}{2(q+1)(q^2+q+1)}\right)^2\sum_{\alpha}\prod_{j}(\alpha(a_j^2)+\alpha(a_j^{-2}))\\
&+4\left(\frac{q^3(q^2-1)(q^3-1)}{2(q^2+q+1)}\right)^2+4\left(\frac{q^3(q^2-1)(q^3-1)}{2q(q^2+q+1)}\right)^2\\
&+4\left(\frac{q^3(q^2-1)(q^3-1)}{2}\right)^2+4\left(\frac{q^3(q^2-1)(q^3-1)}{2q^3}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= -4q^6(q-1)^4 + q^6(q-1)^4(q+1)^2 + q^4(q-1)^4(q+1)^2 \\
&\quad + q^6(q-1)^4(q+1)^2(q^2+q+1)^2 + (q-1)^4(q+1)^2(q^2+q+1)^2 \\
&= q^{16} - 2q^{14} - 2q^{13} + 2q^{12} + 2q^{11} - 2q^{10} + 16q^9 - 30q^8 \\
&\quad + 16q^7 - 2q^6 + 2q^5 + 2q^4 - 2q^3 - 2q^2 + 1.
\end{aligned}
$$

V.5.2.4 $g = 1$**,** $2k = 2$**.** We keep $C_{1,\pm}$ unchanged but put $C_{2,\pm}$ to be the conjugacy classes of σ and $\mathrm{diag}(1, c, 1)\sigma$. The same counting formula gives

$$
\begin{aligned}
|\mathrm{Ch}_C(\mathbb{F}_q)| =& \frac{4}{2}\frac{q^5(q-1)^4(q+1)(q^2+q+1)^2}{4(q+1)^2(q^2+q+1)^2}\sum_{\alpha}(q+1)(\alpha(a_j^2)+\alpha(a_j^{-2})) \\
&+ 4\frac{q^5(q-1)^4(q+1)(q^2+q+1)^2}{4(q^2+q+1)^2} + 4\frac{q^5(q-1)^4(q+1)(q^2+q+1)^2}{4q^2(q^2+q+1)^2}\cdot q \\
&+ 4\frac{q^5(q-1)^4(q+1)(q^2+q+1)^2}{4} + 4\frac{q^5(q-1)^4(q+1)(q^2+q+1)^2}{4q^6}\cdot q \\
=& -2q^5(q-1)^4 + q^5(q-1)^4(q+1) + q^4(q-1)^4(q+1) \\
&+ q^5(q-1)^4(q+1)(q^2+q+1)^2 + (q-1)^4(q+1)(q^2+q+1)^2 \\
=& q^{14} - q^{13} - q^{12} - q^{11} + 3q^{10} - q^9 + 5q^8 - 10q^7 \\
&+ 5q^6 - q^5 + 3q^4 - q^3 - q^2 - q + 1.
\end{aligned}
$$

Appendix A

Character Tables in Low Ranks

We give the character tables of $\mathrm{GL}_2 \rtimes <\sigma>$ and $\mathrm{GL}_3 \rtimes <\sigma>$. We assume that $q \equiv 1 \mod 4$ and write $\mathrm{i} = \sqrt{-1}$. Denote by μ_2 the 2-elements group, identified with $\{\pm 1\}$. Let $\eta \in \mathrm{Irr}(\mathbb{F}_q^*)$ denote the order 2 character. The σ-stable irreducible characters are specified in terms of (L, φ, θ) as in Theorem 5. Denote by T the maximal torus consisting of the diagonal matrices and denote by w the unique nontrivial element of $W_G(T)$, either for $\mathrm{GL}_2(k)$ or $\mathrm{GL}_3(k)$, that is fixed by σ. Denote by T_w a σ-stable and F-stable maximal torus corresponding to the conjugacy class of w. In the following, we will freely use the formulas in §II.2.3.3.

A.0.1 σ-stable Irreducible Characters

We specify the σ-stable irreducible characters of $\mathrm{GL}_2(q)$ and $\mathrm{GL}_3(q)$, and compute the numbers of these characters and of the quadratic-unipotent characters of $\mathrm{GL}_4(q)$ and $\mathrm{GL}_5(q)$.

A.0.1.1 Suppose $G = \mathrm{GL}_2(k)$. There are $q + 3$ σ-stable irreducible characters of $\mathrm{GL}_2(q)$, and 5 of them are quadratic-unipotent, among which one extends into a non-uniform function.

The quadratic-unipotent characters induced from $L = G$ are the following.

Id	η Id	St	η St

The only one quadratic-unipotent character induced from $L = T$ is the following.

$R_T^G(1, \eta)$

It is the unique σ-stable irreducible character with non-uniform extension.

Other σ-stable irreducible characters are either of the form,

$R_T^G(\alpha, \alpha^{-1})$

with $\alpha \in \mathrm{Irr}(\mathbb{F}_q^*)$ satisfying $\alpha^q = \alpha$, $\alpha \neq 1$ or η. There are $(q-3)/2$ of them; or of the form,

$R_{T_w}^G(\omega)$

with $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^*)$ satisfying $\omega^q = \omega^{-1}$, $\omega \neq 1$ or η. There are $(q-1)/2$ of them.

A.0.1.2 Suppose $G = \mathrm{GL}_3(k)$. There are $2q + 6$ σ-stable irreducible characters of $\mathrm{GL}_3(q)$, and 10 of them are quadratic-unipotent, among which two have non-uniform extensions.

The quadratic-unipotent characters induced from $L = G$ are the following.

Id	$\eta\,\mathrm{Id}$	χ_2	$\eta\chi_2$	χ_3	$\eta\chi_3$

The characters χ_2 and $\eta\chi_2$ are associated to the sign character of $\mathfrak{S}_3$. The characters χ_3 and $\eta\chi_3$ are associated to the degree 2 character of $\mathfrak{S}_3$, and these two characters have non-uniform extensions.

The quadratic-unipotent characters induced from $L \cong \mathrm{GL}_2(k) \times k^*$ are the following.

$R_L^G(\mathrm{Id}_2, \eta)$	$R_L^G(\eta\,\mathrm{Id}_2, \mathrm{Id})$	$R_L^G(\mathrm{St}, \eta)$	$R_L^G(\eta\,\mathrm{St}, \mathrm{Id})$

Other σ-stable irreducible characters are either of the form,

$R_T^G(\alpha, 1, \alpha^{-1})$	$R_T^G(\alpha, \eta, \alpha^{-1})$

with $\alpha \in \mathrm{Irr}(\mathbb{F}_q^*)$ satisfying $\alpha^q = \alpha$, $\alpha \neq 1$ or η. There are $q - 3$ of them; or of the form,

$R_{T_w}^G(\omega, 1, \omega^{-1})$	$R_{T_w}^G(\omega, \eta, \omega^{-1})$

with $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^*)$ satisfying $\omega^q = \omega^{-1}$, $\omega \neq 1$ or η. There are $q - 1$ of them.

A.0.1.3 Suppose $G = \mathrm{GL}_4(k)$. There are

$$\frac{1}{2}(q-2)(q-3) + 7(q-2) + 20$$

σ-stable irreducible characters of $\mathrm{GL}_4(k)$, and 20 of them are quadratic-unipotent.

The Levi subgroups $L = \mathrm{GL}_4(k)$, $L = \mathrm{GL}_3(k) \times k^*$ and $L = \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ give rise to $(5 + 3 + 2) \times 2$ quadratic-unipotent characters, knowing that $|\mathrm{Irr}(\mathfrak{S}_4)| = 5$.

The Levi subgroup $L \cong \mathrm{GL}_2(k) \times (k^* \times k^*)$ gives

$$|\text{Quad. Unip. of } \mathrm{GL}_2| \times (q-2) = 5(q-2)$$

noticing that $q - 2 = (q-3)/2 + (q-1)/2$ as in the case of $G = \mathrm{GL}_2(k)$.

The Levi subgroup $L \cong \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ gives $(q-2) \times 2$ with $2 = |\mathrm{Irr}(\mathfrak{S}_2)|$.

The maximal torus $(k^*)^2 \times (k^*)^2$ gives $\frac{1}{2}(q-2)(q-3)$.

A.0.1.4 Suppose $G = \mathrm{GL}_5(k)$. There are

$$(q-2)(q-3) + 14(q-2) + 36$$

σ-stable irreducible characters, and 36 of them are quadratic-unipotent.

The Levi subgroups $L = \mathrm{GL}_5(k)$, $L \cong \mathrm{GL}_4(k) \times k^*$ and $L \cong \mathrm{GL}_3(k) \times \mathrm{GL}_2(k)$ give $(7+5+3\times2)\times2$ quadratic-unipotent characters, knowing that $|\mathrm{Irr}(\mathfrak{S}_5)| = 7$.

The Levi subgroup $L \cong \mathrm{GL}_3(k) \times (k^*)^2$ gives

$$|\text{Quad. Unip. of } \mathrm{GL}_3| \times (q-2) = 10(q-2).$$

The Levi subgroup $L \cong k^* \times (\mathrm{GL}_2(k) \times \mathrm{GL}_2(k))$ gives $2 \times 2 \times (q-2)$, with one factor $2 = |\mathrm{Irr}(\mathfrak{S}_2)|$ and the other factor $2 = |\{1, \eta\}|$.

The maximal torus $L \cong k^* \times (k^*)^2 \times (k^*)^2$ gives $2 \times \frac{1}{2}(q-2)(q-3)$.

A.0.2 Conjugacy Classes

We present the conjugacy classes of $\mathrm{GL}_2(q).\sigma$ and of $\mathrm{GL}_3(q).\sigma$, and count the isolated classes of $\mathrm{GL}_4(q).\sigma$ and of $\mathrm{GL}_5(q).\sigma$. Denote by su the Jordan decomposition of an element of the conjugacy class concerned. Note that $\mathrm{O}_1(k) \cong \mu_2$ and $\mathrm{O}_2(k) \cong k^* \rtimes <\tau>$ with $\tau(x) = x^{-1}$.

A.0.2.1 Suppose $G = \mathrm{GL}_2(k)$. There are $q+3$ conjugacy classes, and 5 of them are isolated.

- $s = (1,1)\sigma$, $C_G(s) = \mathrm{SL}_2(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the centralisers and the G^F-classes are specified accordingly as below,

(1^2)	(2)	
$\mathrm{SL}_2(k)$	$\mathrm{O}_1(k)V$	
C_1	C_2	C_3

  where $V \cong \mathbb{A}^1$ is the unipotent radical. If we use the unit element of a root subgroup of $\mathrm{SL}_2(k)$ to represent (2), then C_2 corresponds to the identity component of the centraliser. The two components of $\mathrm{O}_1(k)V$ have as representatives the scalars $\pm\,\mathrm{Id}$.

- $s = (\mathrm{i}, -\mathrm{i})\sigma$, $C_G(s) = \mathrm{O}_2(k)$.
  Denote by C_4 the $G(q)$-class corresponding to the identity component, and C_5 the other class. The two components of $\mathrm{O}_2(k)$ have as representatives

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  respectively, and so induce the Frobenius $x \mapsto x^q$ and $x \mapsto x^{-q}$ respectively. In other words, the centralisers of C_4 and C_5 are $\mathrm{O}_2^+(q)$ and $\mathrm{O}_2^-(q)$ respectively.

- $s = (a, a^{-1})\sigma$, $C_G(s) = k^*$.
  For any value of a, the corresponding G-class contains a unique $G(q)$-class. The classes are as follows.

$a^{q-1} = 1$	$a^{q+1} = 1$	$a^{q-1} = -1$	$a^{q+1} = -1$
$C_6(a)$	$C_7(a)$	$C_8(a)$	$C_9(a)$

The Frobenius on $C_G(s) \cong k^*$ with $s \in C_6$ or C_8 is $x \mapsto x^q$, while the Frobenius on $C_G(s) \cong k^*$ with $s \in C_7$ or C_9 is $x \mapsto x^{-q}$.

We have

- $|C_1| = |G(q)|/|\mathrm{SL}_2(q)| = q - 1$;
- $|C_2| = |C_3| = |G(q)|/2|V(q)| = \frac{1}{2}(q-1)^2(q+1)$;
- $|C_4| = |G(q)|/|\mathrm{O}_2^+(q)| = \frac{1}{2}q(q+1)(q-1)$; $|C_5| = |G(q)|/|\mathrm{O}_2^-(q)| = \frac{1}{2}q(q-1)^2$;
- $|C_6| = |C_8| = |G(q)|/(q-1) = q(q+1)(q-1)$;
- $|C_7| = |C_9| = |G(q)|/(q+1) = q(q-1)^2$.

A.0.2.2 Suppose $G = \mathrm{GL}_3(k)$. There are $2q + 6$ conjugacy classes, and 10 of them are isolated. Now each semi-simple G-conjugacy class contains two $G(q)$-conjugacy classes, distinguished by the sign η (*cf.* (IV.2.2.2.2)). Depending on the value of η, we will write C^+ or C^- to represent the corresponding conjugacy class contained in a given G-conjugacy class.

Notation A.0.2.1. In what follows, we write ϵ instead of η to avoid clashing with the character of $\mathbb{F}_q^*$.

- $s = (1, 1, 1)\sigma$, $C_G(s) = \mathrm{O}_3(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the centralisers and the G^F-classes are specified accordingly as below,

(1^3)		(3)	
$\mathrm{O}_3(k)$		$\mathrm{O}_1(k).V$	
C_1^+	C_1^-	C_2^+	C_2^-

  where $V \cong \mathbb{A}^1$ is the unipotent radical.

- $s = (\mathrm{i}, 1, -\mathrm{i})\sigma$, $C_G(s) \cong \mathrm{SL}_2(k) \times \mathrm{O}_1(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the centralisers and the G^F-classes are specified accordingly as below,

(1^2)		(2)			
$\mathrm{SL}_2(k) \times \mathrm{O}_1(k)$		$\mathrm{O}_1(k) \times \mathrm{O}_1(k).V$			
C_3^+	C_3^-	C_4^+	C_5^+	C_4^-	C_5^-

  where $V \cong \mathbb{A}^1$ is the unipotent radical. If we use the unit element of a root subgroup of $\mathrm{SL}_2(k)$ to represent (2), then the correspondence between the classes C_4^+, C_5^+, C_4^-, C_5^- and the connected components of $C_G(s)$ is given as follows.

C_4^+	C_5^+	C_4^-	C_5^-
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- $s = (a, 1, a^{-1})\sigma$, $C_G(s) = k^* \times \mu_2$, identified with $\{\mathrm{diag}(x, \pm 1, x^{-1}); x \in k^*\}$.
  The conjugacy classes are as follows.

$a^{q-1} = 1$		$a^{q+1} = 1$		$a^{q-1} = -1$		$a^{q+1} = -1$	
$C_6^+(a)$	$C_6^-(a)$	$C_7^+(a)$	$C_7^-(a)$	$C_8^+(a)$	$C_8^-(a)$	$C_9^+(a)$	$C_9^-(a)$

  The Frobenius on $C_G(s) \cong k^*$ with $s \in C_6^\pm$ or $C_8^\pm$ is $x \mapsto x^q$, while the Frobenius on $C_G(s) \cong k^*$ with $s \in C_7^\pm$ or $C_9^\pm$ is $x \mapsto x^{-q}$.

We have

- $|C_1^+| = |C_1^-| = |G(q)|/|\mathrm{O}_3(q)|$;
- $|C_2^+| = |C_2^-| = |G(q)|/2|V(q)|$;
- $|C_3^+| = |C_3^-| = |G(q)|/2|\mathrm{SL}_2(q)|$;
- $|C_4^+| = |C_4^-| = |C_5^+| = |C_5^-| = |G(q)|/4|V(q)|$;
- $|C_6^\pm| = |C_8^\pm| = |G(q)|/2(q-1)$;
- $|C_7^\pm| = |C_9^\pm| = |G(q)|/2(q+1)$.

A.0.2.3 Suppose $G = \mathrm{GL}_4(k)$. There are 20 isolated conjugacy classes.

- $s = (1, 1, 1, 1)\sigma$, $C_G(s) = \mathrm{Sp}_4(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

(1^4)	$(1^2 2)$	(2^2)	(4)
$\mathrm{Sp}_4(k)$	$\mathrm{SL}_2(k) \times \mathrm{O}_1(k)$	$\mathrm{O}_2(k)$	$\mathrm{O}_1(k)$

  This gives 7 classes.

- $s = (\mathrm{i}, 1, 1, -\mathrm{i})\sigma$, $C_G(s) = \mathrm{SL}_2(k) \times \mathrm{O}_2(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

(1^2)	(2)
$\mathrm{SL}_2(k) \times \mathrm{O}_2(k)$	$\mathrm{O}_1(k) \times \mathrm{O}_2(k)$

This gives 6 classes.

- $s = (\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i})\sigma$, $C_G(s) = \mathrm{O}_4(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

(1^4)	(13)	(2^2)
$\mathrm{O}_4(k)$	$\mathrm{O}_1(k) \times \mathrm{O}_1(k)$	$\mathrm{SL}_2(k)$

  This gives 7 classes.

A.0.2.4 Suppose $G = \mathrm{GL}_5(k)$. There are 36 isolated conjugacy classes.

- $s = (1, 1, 1, 1, 1)\sigma$, $C_G(s) = \mathrm{O}_5(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

(1^5)	$(1^2 3)$	(12^2)	(5)
$\mathrm{O}_5(k)$	$\mathrm{O}_2(k) \times \mathrm{O}_1(k)$	$\mathrm{O}_1(k) \times \mathrm{SL}_2(k)$	$\mathrm{O}_1(k)$

  This gives 10 classes.

- $s = (\mathrm{i}, 1, 1, 1, -\mathrm{i})\sigma$, $C_G(s) = \mathrm{O}_3(k) \times \mathrm{SL}_2(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

$$\begin{array}{cc} (1^3) & (3) \\ \hline \mathrm{O}_3(k) & \mathrm{O}_1(k) \end{array} \times \begin{array}{cc} (1^2) & (2) \\ \hline \mathrm{SL}_2(k) & \mathrm{O}_1(k) \end{array}$$

  This gives $(2 + 2) \times (1 + 2) = 12$ classes.

- $s = (\mathrm{i}, \mathrm{i}, 1, -\mathrm{i}, -\mathrm{i})\sigma$, $C_G(s) = \mathrm{Sp}_4(k) \times \mathrm{O}_1(k)$.
  The unipotent parts are given by the partitions defined by Jordan blocks. Then the reductive parts of the centralisers are specified accordingly as below,

$$\mathrm{O}_1(k) \times \begin{array}{cccc} (1^4) & (1^2 2) & (2^2) & (4) \\ \hline \mathrm{Sp}_4(k) & \mathrm{SL}_2(k) \times \mathrm{O}_1(k) & \mathrm{O}_2(k) & \mathrm{O}_1(k) \end{array}$$

  This gives $2 \times 7 = 14$ classes.

A.0.3 The Tables

The calculation of the values of the uniform characters is reduced to the determination of the sets

$$A = A(s\sigma, T_w) = \{h \in G^F \mid hs\sigma h^{-1} \in T_w.\sigma\}$$

for various G^F-conjugacy classes of F-stable and σ-stable maximal tori T_w contained in some σ-stable Borel subgroups, and semi-simple G^F-conjugacy classes of elements $s\sigma$.

A.0.3.1 The procedure (*cf.* IV.6.2) for computing A can be summarised as follows.

Suppose $s\sigma$ is an F-fixed element contained in $T.\sigma$, and T_w can be written as gTg° for some $g \in C_G(\sigma)^\circ$. If $h \in G^F$ conjugates $s\sigma$ into $T_w.\sigma$, then there exists some $l \in C_G(s\sigma)^\circ$ such that $n := g^{-1}hl$ lies in $N_G(T.\sigma)$. Recall that $N_G(T.\sigma) \subset N_G(T)$ consists of the connected components that are stable under σ. Then $g^{-1}hs\sigma h^{-1}g = ns\sigma n^{-1}$ is an F_w-fixed element of $T.\sigma$. If $s\sigma$ is an F-fixed element contained in $T_w.\sigma$, then similar arguments show that $g^{-1}hs\sigma h^{-1}g = ng^{-1}s\sigma gn^{-1}$ with $gng^{-1} = hl$. The conjugation by n can be separated into a permutation of the "eigenvalues" and a conjugation by an element of T. For each $s\sigma$ and T_w, we will first find some $t \in T$ such that $ts\sigma t^{-1}$ (or $tg^{-1}s\sigma gt^{-1}$ if we start with some $s\sigma \in T_w.\sigma$) is fixed by F_w, then evaluate any character of $T_w^F.<\sigma>$under the isomorphism $T_w^F.<\sigma> \cong T^{F_w}.<\sigma>$. The value does not depend on the choice of t, and the permutation of the "eigenvalues" is simple.

We will use the following observation. Let $\omega \in \mathrm{Irr}(\mathbb{F}_{q^2}^*)$ be such that $\omega^q = \omega^{-1}$, and let $a \in \mathbb{F}_{q^2}^*$ be such that $a^q = a$. Then $a = b^{q+1}$ for some $b \in \mathbb{F}_{q^2}^*$, so

$$\omega(a) = \omega(b^{q+1}) = \omega^{q+1}(b) = 1.$$

A.0.3.2 Suppose $G = \mathrm{GL}_2(k)$. Consider the characters $R_T^G(1,\eta)$, $R_T^G(\alpha,\alpha^{-1})$ and $R_{T_w}^G(\omega)$.

The calculation of the extensions of $R_T^G(1,\eta)$ is a direct application of the theorem of Waldspurger. Following the notations of (IV.5.4.1), we have $(\mu_+,\mu_-) = ((1),(1))$. The 2-cores are (1) and (1), and so $m_+ = m_- = 1$. We deduce that $h_1 = 1$ and $h_2 = 0$. So the cuspidal function is supported on the class of su with $C_G(s)^\circ \cong \mathrm{SL}_2(k)$ and u corresponding to the partition (2). We find $\delta(h_1,h_2) = 1$. So the values of this character on its support are $\pm\sqrt{q}$ and vanish on all other classes.

If $s\sigma = \sigma$ and so $C_G(s\sigma) = \mathrm{SL}_2(k)$, then $h\sigma h^{-1} = \det(h).\sigma$ (regarding $\det(h)$ as a scalar matrix), which belongs to $T.\sigma$ or $T_w.\sigma$ for any h. So $A = G^F$ and $\tilde{\theta}(hs\sigma h^{-1}) = \theta(\sigma) = 1$ for any h as θ has trivial value on the scalars.

If $s\sigma = (\mathrm{i},-\mathrm{i})\sigma$ and so $C_G(s\sigma) = \mathrm{O}_2(k)$, then the elements of A are exactly those $h \in G^F$ such that $(h^{-1}Th \cap C_G(s\sigma))^\circ$ is a maximal torus of $C_G(s\sigma)^\circ = \mathrm{SO}_2(k)$ which itself is a torus whose centraliser in G is T or T_w according to whether $s\sigma \in C_4$ or $s\sigma \in C_5$. Consequently, $A(C_5,T) = A(C_4,T_w) = \varnothing$, while $A(C_4,T)$ and $A(C_5,T_w)$ are the normalisers of T and T_w respectively. It is easy to check that

$$\tilde{\theta}(hs\sigma h^{-1}) = \alpha(\mathrm{i})\alpha^{-1}(-\mathrm{i}) = \alpha(-1)$$

if $s\sigma \in C_4$. If $s\sigma \in C_5$, then we use the method at the beginning of this section. It suffices to find some $t \in T$ such that $ts\sigma t^{-1}$ is fixed by F_w. Indeed, we can take $t = \mathrm{diag}(\lambda,1)$ with $\lambda^q = -\lambda$ so that $(\mathrm{i}\lambda)^q = -\mathrm{i}\lambda$. We get $\theta(s\sigma) = \omega(\mathrm{i}\lambda)$. The value is independent of the choice of λ. We can also do it directly and explicitly, and obtain the same result. The elements of

$\mathrm{O}_2(k) \setminus \mathrm{SO}_2(k)$ are of the form

$$\begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix},$$

so they do not belong to $\mathrm{SL}_2(k)$. Let us describe T_s explicitly by choosing $g_2 \in \mathrm{SL}_2(k)$ such that

$$g_2^{-1}F(g_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and putting $T_w = g_2 T g_2^{-1}$. We choose $\lambda \in k^*$ such that $\lambda^q = -\lambda$. Put

$$g_1 = g_2 \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix},$$

such that $g_1^{-1}F(g_1) \in \mathrm{O}_2(k)$. Then the representative $s\sigma \in C_5$ is given by

$$g_1 \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \sigma g_1^{-1} = g_2 \begin{pmatrix} \mathrm{i}\lambda & 0 \\ 0 & -\mathrm{i}\lambda \end{pmatrix} g_2^{-1}\sigma \in T_w.\sigma,$$

and so $\tilde{\theta}(s\sigma) = \omega(\mathrm{i}\lambda)$. If $\omega = \eta$, then taking the norm gives λ^2 and evaluating η gives -1.

If $s\sigma = (a, a^{-1})\sigma$, and so $C_G(s\sigma) = T$ or T_w according to whether $a^q = \pm a$ or $a^q = \pm a^{-1}$, then A is equal to the normaliser of T or T_w or empty according to the G^F-class of $C_G(s\sigma)$. If $s\sigma \in C_8$, the F-stable conjugate of $s\sigma$ in $T.\sigma$ is given by $\mathrm{diag}(a\lambda, a^{-1}\lambda)\sigma$ with $\lambda^q = -\lambda$. If $\theta = \eta \circ \det|_T$, then $\tilde{\theta}((a\lambda, a^{-1}\lambda)\sigma) = \eta(\lambda^2) = -1$. If $s\sigma \in C_9$, the representative of $C_9(a)$ is given by

$$g \begin{pmatrix} a\lambda & 0 \\ 0 & a^{-1}\lambda \end{pmatrix} g^{-1}\sigma \in T_s^F.\sigma.$$

Again, if $\theta = \eta \circ \det|_{T_s}$, then $\tilde{\theta}((a\lambda, a^{-1}\lambda)\sigma) = \eta(\lambda^2) = -1$.

A.0.3.3 Suppose $G = \mathrm{GL}_3(k)$. Consider the characters χ_3, $R_T^G(\alpha, 1, \alpha^{-1})$, $R_T^G(\omega, 1, \omega^{-1})$, $R_T^G(\eta\chi_3)$, $R_T^G(\alpha, \eta, \alpha^{-1})$ and $R_T^G(\omega, \eta, \omega^{-1})$.

For χ_3, we use the theorem of Waldspurger. We have $(\mu_+, \mu_-) = ((1^3), \varnothing)$. The 2-cores are $(2, 1)$ and $\varnothing$, and so $m_+ = 2$ and $m_- = 0$. We deduce that $h_1 = 1$ and $h_2 = 1$. So the cuspidal function is supported on the class of su with $C_G(s)^\circ \cong \mathrm{SL}_2(k) \times \mathrm{O}_1(k)$ and u corresponding to the partition (2). We find $\delta(h_1, h_2) = 1$ and so the values of this character are $\pm\sqrt{q}$.

If $s\sigma = (1, 1, 1)\sigma$ and so $C_G(s\sigma) = \mathrm{O}_3(k)$, then one has to understand the set $A^F = (N_G(T.\sigma).L')^F$, with the notations of Lemma IV.6.2.2. If $h = nl \in A^F$, then the L'^F-conjugacy class of $h^{-1}Th \cap L'$ corresponds to the F-class of $n^{-1}F(n) \in N_{L'}(T \cap L')$. But $N_G(T.\sigma) \cong W_G(T)^\sigma \cong \mathfrak{S}_2$, so $n^{-1}F(n)$ necessarily belongs to $T \cap L' = (T^\sigma)^\circ$, i.e. $h^{-1}Th \cap L'$ is always L'^F-conjugate to $T \cap L'$ and the only Green function that appears in the formula of $R_T^G\theta(s\sigma u)$ is $Q_{(T^\sigma)^\circ}^{\mathrm{SO}_3(k)}(u)$. We also have a similar result for T_w. Expressing the elements $h = nl$ as some explicit matrices,

we find that $\tilde{\theta}(hs\sigma h^{-1})$ does not depend on h. It remains to calculate

$$\begin{aligned}|N_{G^F}(T.\sigma)L'^F| =& |L'^F||N_{G^F}(T.\sigma)||N_{L'^F}(C_T(\sigma)^\circ)|^{-1} \\ =& |\mathrm{SO}_3(q)| \cdot 2(q-1)^3 \cdot 2(q-1),\end{aligned} \tag{A.0.3.3.1}$$

For T_w, we have

$$|N_{G^F}(T_w.\sigma)L'^F| = |\mathrm{SO}_3(q)| \cdot 2(q-1)(q^2-1) \cdot 2(q+1). \tag{A.0.3.3.2}$$

The other $G(q)$-class contained in the G-class of σ has as representative $(1, \lambda^2, 1)\sigma$ with $\lambda^q = -\lambda$, so for example the value of $R_T^G(\alpha, \eta, \alpha^{-1})(C_2)$ differs from $R_T^G(\alpha, 1, \alpha^{-1})(C_2)$ by a sign.

The main difference between $\mathrm{GL}_3(q)$ and $\mathrm{GL}_2(q)$ is the class $(\mathrm{i}, 1, -\mathrm{i})\sigma$ (as opposed to $(\mathrm{i}, -\mathrm{i})\sigma$ for $\mathrm{GL}_2(k)$). We have $C_G(s\sigma) \cong \mathrm{SL}_2(k) \times \mathrm{O}_1(k)$ so in particular it contains representatives of each element of $W_G(T)^\sigma$. Therefore, the sets A^F are not empty either for T or for T_w. Suppose that $s\sigma$ represents C_3^ϵ and we want to evaluate $R_T^G(\alpha, \eta, \alpha^{-1})$ at $s\sigma$. Let $t \in T$ be such that $ts\sigma t^{-1}$ is fixed by F. Then $ts\sigma t^{-1}$ can be written as $\mathrm{diag}(\mathrm{i}x, y, -\mathrm{i}x)\sigma$. It is necessary that $x^q = x$ and $y^q = y$. So $\alpha(\mathrm{i})\alpha^{-1}(-\mathrm{i}) = \alpha(-1)$. Applying ϵ gives $\eta(y) = \epsilon(C_3^\epsilon)$. Therefore $(\alpha, \eta, \alpha^{-1})(hs\sigma h^{-1})$ evaluates $\epsilon\alpha(-1)$. Now we evaluate $R_T^G(\omega, \eta, \omega^{-1})$ at $s\sigma$. Again we write $ts\sigma t^{-1}$ as $\mathrm{diag}(\mathrm{i}x, y, -\mathrm{i}x)\sigma$, but which is F_w-stable. It is necessary that $x^q = -x$ and $y^q = y$. Applying ϵ gives $\eta(y) = -\epsilon(C_3^\epsilon)$ since $x^2 \notin (\mathbb{F}_q^*)^2$. Therefore $(\omega, \eta, \omega^{-1})(hs\sigma h^{-1})$ evaluates $-\epsilon\omega(\mathrm{i}\lambda)$. If $s\sigma \in C_8^\pm$, then an F-stable element $\mathrm{diag}(ax, y, a^{-1}x)\sigma = ts\sigma t^{-1}$ satisfies $x^q = -x$. and $y^q = y$. Therefore $(\alpha, \eta, \alpha^{-1})(hs\sigma h^{-1})$ evaluates $-\epsilon\alpha(a^{\pm 2})$, where the ± 2 power is due to permutation of "eigenvalues". For $(\omega, \eta, \omega^{-1})$ at $C_9^\pm$ the calculation is similar.

Table A.1 – The Character Table of $GL_2(q).<\sigma>$

	$(1,1)\sigma$			$(\mathrm{i},-\mathrm{i})\sigma$		$(a,a^{-1})\sigma$			
	(1^2)	(2)		1		$a^{q-1}=1$	$a^{q+1}=1$	$a^{q-1}=-1$	$a^{q+1}=-1$
	C_1	C_2	C_3	C_4	C_5	$C_6(a)$	$C_7(a)$	$C_8(a)$	$C_9(a)$
$R_T^G(1,\eta)$	0	$-\sqrt{q}$	$\sqrt{q}$	0	0	0	0	0	0
Id	1	1	1	1	1	1	1	1	1
$\eta\,\mathrm{Id}$	1	1	1	1	-1	1	1	-1	-1
St	q	0	0	1	-1	1	-1	1	-1
$\eta\,\mathrm{St}$	q	0	0	1	1	1	-1	-1	1
$R_T^G(\alpha,\alpha^{-1})$	$q+1$	1	1	$2\alpha(-1)$	0	$\alpha(a^2)+\alpha(a^{-2})$	0	$\alpha(a^2)+\alpha(a^{-2})$	0
$R_{T_w}^G(\omega)$	$1-q$	1	1	0	$2\omega(\mathrm{i}\lambda)$	0	$\omega(a)+\omega(a^{-1})$	0	$\omega(a\lambda)+\omega(a^{-1}\lambda)$

Table A.2 – The Character Table of $GL_3(q).<\sigma>$, (i)

	$(1,1,1)\sigma$				$(\mathrm{i},1,-\mathrm{i})\sigma$					
	(1^3)		(3)		(1^2)		(2)			
	C_1^+	C_1^-	C_2^+	C_2^-	C_3^+	C_3^-	C_4^+	C_5^+	C_4^-	C_5^-
	$(a,1,a^{-1})\sigma$									
	$a^{q-1}=1$		$a^{q+1}=1$		$a^{q-1}=-1$		$a^{q+1}=-1$			
	C_6^+	C_6^-	C_7^+	C_7^-	C_8^+	C_8^-	C_9^+	C_9^-		
χ_3	0	0	0	0	0	0	$\sqrt{q}$	$-\sqrt{q}$	$-\sqrt{q}$	$\sqrt{q}$
	0		0		0		0			
$\eta\chi_3$	0	0	0	0	0	0	$\sqrt{q}$	$-\sqrt{q}$	$\sqrt{q}$	$-\sqrt{q}$
	0		0		0		0			

Table A.3 – The Character Table of $GL_3(q).<\sigma>$, (ii)

	C_1^+	C_1^-	C_2^+	C_2^-	C_3^+	C_3^-	C_4^+	C_5^+	C_4^-	C_5^-
	C_6^+	C_6^-	C_7^+	C_7^-	C_8^+	C_8^-	C_9^+	C_9^-		
Id	1	1	1	1	1	1	1	1	1	1
	1		1		1		1			
ηId	1	-1	1	-1	1	-1	1	1	-1	-1
	ϵ		ϵ		ϵ		ϵ			
χ_2	q	q	0	0	q	q	0	0	0	0
	1		-1		1		-1			
$\eta\chi_2$	q	$-q$	0	0	q	$-q$	0	0	0	0
	ϵ		$-\epsilon$		ϵ		$-\epsilon$			

Table A.4 – The Character Table of $GL_3(q).<\sigma>$, (iii)

	C_1^+	C_1^-	C_2^+	C_2^-	C_3^+	C_3^-	C_4^+	C_5^+	C_4^-	C_5^-
	C_6^+	C_6^-	C_7^+	C_7^-	C_8^+	C_8^-	C_9^+	C_9^-		
$R_L^G(Id_2, \eta)$	1	-1	1	-1	q	$-q$	0	0	0	0
	ϵ		ϵ		$-\epsilon$		$-\epsilon$			
$R_L^G(\eta\,\mathrm{Id}, \mathrm{Id})$	1	1	1	1	q	q	0	0	0	0
	1		1		-1		-1			
$R_L^G(\mathrm{St}, \eta)$	q	$-q$	0	0	1	-1	1	1	-1	-1
	ϵ		$-\epsilon$		$-\epsilon$		ϵ			
$R_L^G(\eta\,\mathrm{St}, \mathrm{Id})$	q	q	0	0	1	1	1	1	1	1
	1		-1		-1		1			

Table A.5 – The Character Table of $GL_3(q).<\sigma>$, (iv)

	C_1^+	C_1^-	C_2^+	C_2^-	C_3^+	C_3^-	C_4^+	C_5^+	C_4^-	C_5^-
	C_6^+	C_6^-	C_7^+	C_7^-	C_8^+	C_8^-	C_9^+	C_9^-		
$(\alpha, 1, \alpha^{-1})$	$q+1$	$q+1$	1	1	$(q+1)\alpha(-1)$	$(q+1)\alpha(-1)$	$\alpha(-1)$	$\alpha(-1)$	$\alpha(-1)$	$\alpha(-1)$
	$\alpha(a^2)+\alpha(a^{-2})$		0		$\alpha(a^2)+\alpha(a^{-2})$		0			
$(\alpha, \eta, \alpha^{-1})$	$q+1$	$-q-1$	1	-1	$(q+1)\alpha(-1)$	$-(q+1)\alpha(-1)$	$\alpha(-1)$	$\alpha(-1)$	$-\alpha(-1)$	$-\alpha(-1)$
	$\epsilon(\alpha(a^2)+\alpha(a^{-2}))$		0		$-\epsilon(\alpha(a^2)+\alpha(a^{-2}))$		0			
$(\omega, 1, \omega^{-1})$	$-q+1$	$-q+1$	1	1	$(-q+1)\omega(\mathrm{i}\lambda)$	$(-q+1)\omega(\mathrm{i}\lambda)$	$\omega(\mathrm{i}\lambda)$	$\omega(\mathrm{i}\lambda)$	$\omega(\mathrm{i}\lambda)$	$\omega(\mathrm{i}\lambda)$
	0		$\omega(a)+\omega(a^{-1})$		0		$\omega(a\lambda)+\omega(a^{-1}\lambda)$			
$(\omega, \eta, \omega^{-1})$	$-q+1$	$q-1$	1	-1	$(q-1)\omega(\mathrm{i}\lambda)$	$(-q+1)\omega(\mathrm{i}\lambda)$	$-\omega(\mathrm{i}\lambda)$	$-\omega(\mathrm{i}\lambda)$	$\omega(\mathrm{i}\lambda)$	$\omega(\mathrm{i}\lambda)$
	0		$\epsilon(\omega(a)+\omega(a^{-1}))$		0		$-\epsilon(\omega(a\lambda)+\omega(a^{-1}\lambda))$			

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