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Études de fibrés en droites semipositivement métrisés sur un corps non-archimédien via la convexité holomorphe

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Thèse de doctorat de **Mathématiques**

Dirigée par Huayi CHEN

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Résumé - Ce mémoire de thèse est consacré à l'étude de fibré en droites semipositif en géométrie analytique non-Archimédienne, par un point de vue d'analyse fonctionnelle sur un corps ultramétrique en exploitant la géométrie de la convexité holomorphe.

Le premier chapitre recueille quelques préliminaires pour l'algèbre de Banach sur un corps ultramétique et la géométrie de son spectre au sens de Berkovich, le cadre dans lequel l'étude est effectuée.

Le deuxième chapitre présente la construction de base, qui encode la géométrie intervenante dans certaines algèbres de Banach. On associe une algèbre normée de section à un fibré en droites métrisé. On décrit son spectre, en le reliant avec le fibré en disques unités duals de ce fibré en droites muni de la métrique enveloppante. On encode alors la positivité métrique par la convexité holomorphe.

Le troisième chapitre consiste en deux approaches indépendantes pour le problème d'extension métrique de sections restreintes sur une sous-variété fermée. On obtient une borne supérieure pour la distorsion métrique asymptotique, qui est uniforme par rapport aux choix de sections restreintes. On utilise une propriété particulière aux norms affinoïds pour obtenir cette inégalité.

Le quatrième chapitre traite le problème de la régularité de métrique enveloppante. Avec un nouveau regard venant d'analyse holomorphe à plusieurs variables, on vise à montrer que, quand le fibré en droites est ample, la métrique enveloppante est continue si la métrique de départ l'est. On suggère une méthode tentative reposant sur un analogue non archimédien spéculatif d'un résultat sur la convexité holomorphe due à Cartan et Thullen.

Mots clefs - Espaces de Berkovich; Analyse non-archimédienne; Fibré en droites; Métrique semi-positive; Convexité holomorphe

Abstract - This thesis is devoted to the study of semi-positively metrized line bundles in non-Archimedean analytic geometry, with the point of view of functional analysis over an ultra-metric field exploiting the geometry related to holomorphic convexity. The first chapter gathers some preliminaries about Banach algebras over ultra-metric fields and the geometry of their spectrum in the sense of V. Berkovich, which is the framework of our study. The second chapter present the basic construction, which encodes the related geometric information into some Banach algebra. We associate the normed algebra of sections of a metrized line bundle. We describe its spectrum, relating it with the dual unit disc bundle of this line bundle with respect to the envelope metric. We thus encode the metric positivity into the holomorphic convexity of the spectrum. The third chapter consists of two independent for the normed extension problem for restricted sections on a sub-variety. We obtain an upper bound for the asymptotic norm distorsion between the restricted section and the extended one, which is uniform with respect to the choice of restricted sections. We use a particular property of affinoid algebras to obtain this inequality. The fourth chapter treat the problem of regularity of the envelope metric. With a new look from the holomorphic analysis of several variables, we aime at showing that on ample line bundles, the envelop metric is continuous once the original metric is. We suggest a tentative approach based on a speculative analogue of Cartan-Thullen's result in the non-Archimedean setting.

Keywords - Berkovich spaces; non-Archimedean analysis; line bundles; semi-positive metrics; holomorphic convexity

Yanbo FANG

L'ÉTUDE DE FIBRÉ EN DROITES SEMIPOSITIVEMENT METRISÉ SUR UN CORPS NON-ARCHIMÉDIEN VIA CONVEXITÉ HOLOMORPHE

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L'ÉTUDE DE FIBRÉ EN DROITES SEMIPOSITIVEMENT METRISÉ SUR UN CORPS NON-ARCHIMÉDIEN VIA CONVEXITÉ HOLOMORPHE

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CONTENTS

0. Introduction (en français) 1
0.1. Motivations 1
0.2. Résultats obtenus 5
0.3. List of notations 12

1. Preliminaries on functional analysis and geometry over non-Archimedean fields 13
1.1. Seminormed vector spaces 13
1.2. Banach algebra 16
1.3. Affinoid algebras 21
1.4. Spectral calculus 26
1.5. Analytification of scheme of finite type 27

2. Normed section algebra and its spectrum 31
2.1. Section algebra and norms 31
2.2. Fubini-Study metrics 49
2.3. Spectrum and envelope 53
2.4. Restriction to sub-varieties 58

3. Normed extension for restricted sections 61
3.1. Background and results 61
3.2. Comparison of norms 64
3.3. Geometric/Spectral approximation 65
3.4. Algebraic approximation 69

4. Continuity of envelope metric 75
4.1. Background and results 75
4.2. Holomorphic convexity of holomorphic envelope 79
4.3. Neighbourhood of dual unit disc bundle and continuity of metric 82

Bibliography 85

CHAPTER 0

INTRODUCTION (EN FRANÇAIS)

0.1. Motivations

0.1.1. Géométrie analytique non archimédienne. — La géométrie analytique non archimédienne est un analogue de la géométrie analytique complexe. Cette dernière fournit un cadre naturel dans lequel on étudie les fonctions analytiques complexes sur une variété complexe. Dans son analogue non archimédien, le corps des nombres complexes (muni de la valeur absolue usuelle) est remplacé par par un corps valué complet ultramétrique, que l'on appelle un corps non archimédien. Sa naissance répond au besoin d'avoir un bon espace topologique sur lequel on peut définir et opérer les fonctions analytiques à valeur dans un corps non archimédien.

Dans la littérature, plusieurs théories ont été proposées pour formaliser les espaces analytiques non archimédiens, notamment par Tate [59], Berkovich [3] et Huber [41]. Toutes ces théories consistent à recoller des «spectre de Gelfand» des algèbres de Banach commutatives de type fini pour construire les espaces analytiques. Cependant, les constructions de ces spectres diffèrent suivant les choix de différentes théories. Comparée à l'approche des espaces rigides à la Tate, qui est basée sur la théorie de topos, celle de Berkovich repose sur la théorie de topologie classique. En outre, les espaces analytiques de Berkovich ont souvent de bonnes propriétés topologiques. Par exemple, en général les espaces analytiques que l'on s'intéresse sont séparés. Au contraire, les espaces adiques à la Huber sont rarement séparés. De plus, pour tout schéma de type fini sur un corps non archimédien, qui est séparé (resp. connexe, propre), l'espace analytique de Berkovich associé au schéma est séparé (resp. connexe par arc, compact). Ainsi le cadre de Berkovich ressemble davantage à celui des espaces analytiques complexes. On peut espérer d'inspirer plus facilement (malgré des subtilités) les idées et résultats de la géométrie analytique complexe dans l'étude des espaces analytiques de Berkovich.

La géométrie analytique à la Berkovich a de nombreuse applications dans des contextes variés, comme par exemple géométrie d'Arakelov, dégénérescence d'une famille de variété complexe sur une base unidimensionnelle, étude de l'espace de module de courbes algébriques par la géométrie tropicale, système dynamique holomorphe, etc. Dans ces applications, on adapte souvent des résultats de la géométrie analytique complexe dans le cadre non archimédien. Cependant, cette adaptation n'est guerre automatique. En effet, bien que la théorie de formes différentielles et de courants a été introduite dans [21]

et [38] pour les espaces analytiques de Berkovich, le calcul différentiel sur les espaces de Berkovich est assez différent de celui sur les variétés analytiques complexe. En particulier, certains outils que l'on utilise couramment en géométrie analytique complexe, comme par exemple la méthode L^2 de Hörmander [40], n'ont pas encore d'analogue adéquate dans le cadre non archimédien. L'unification de méthode dans l'étude des variétés analytiques complexes et non archimédiennes est un challenge technique ayant des applications potentielles dans divers domaines mathématiques, et surtout, du point de vue conceptuel, peut donne une explication à l'analogie entre ces deux types de variétés analytiques, malgré leurs différences en topologie et en calcul différentiel. Cette thèse contribue à une tentative dans cette direction, qui repose sur l'analyse fonctionnelle de l'algèbre des sections des puissances d'un fibré en droites ample métrisé. Le résultat obtenu peut être appliqué au problème d'extension avec un contrôle de norme d'une section le long d'une sous-variété fermé d'un fibré en droites métrisé.

0.1.2. Thème central: Fibré en droites semipositif. — Dans cette thèse, on étudie les fibrés en droites semi-positivement métrisés dans le cadre de la géométrie analytique non archimédienne à la Berkovich. Soient k un corps commutatif muni d'une valeur absolue non archimédienne, qui est complet par rapport à la topologie définie par la valeur absolue. Étant donné un schéma intègre et projectif X sur $\operatorname{Spec} k$ et un $\mathcal{O}_X$-module inversible ample $\mathscr{L}$, on s'intéresse à la positivité des métriques continues ϕ sur l'analytifiée de $\mathscr{L}$, ainsi que les propriété du fibré en droites métrisé $(\mathscr{L}, \phi)$ sous l'hypothèse de positivité.

En géométrie analytique complexe, on considère souvent la positivité du courant de courbure d'un fibré en droites métrisé. L'avantage de cette approche est que le courant de courbure peut être calculé localement par rapport à la topologie analytique. Cette condition locale de positivité en géométrie complexe est centrale pour divers études du fibré en droites métrisé $(\mathscr{L}, \phi)$. Ces études se regroupent en deux directions thématiques. La première s'agit de l'étude de la géométrie algébrique d'une variété projective complexe en exploitant la structure de la variété différentielle sous-jacente, où la métrique ϕ joue un rôle auxiliaire. Plus précisément, la positivité portée par ϕ est combinée avec la technique de Bochner en géométrie différentielle, pour montrer des théorèmes d'annulation de groupe de cohomologie de faisceaux cohérents, ou théorème d'extension de sections avec un contrôle de norme, etc. La deuxième direction concerne l'étude des métriques eux-mêmes, comme par exemple le problème de métrique riemannienne canonique et la théorie pluripotentielle en variété analytique complexe.

Par la nature locale de cette définition de positivité pour le fibré métrisé $(\mathscr{L}, \phi)$, il n'est pas surprenant qu'une bonne partie de démonstrations de ces résultats portent sur l'utilisation des outils d'analyse «fines», comme par exemple l'identité de Bochner-Kodaira-Nakano pour le tenseur de courbure de ϕ, la méthode de L^2 pour l'opérateur $\overline{\partial}$ à la Andreotti-Hörmander avec poids ϕ, ou l'analyse d'EDP non-linéaire pour l'opérateur Monge-Ampère appliqué à ϕ (pour une introduction à ces techniques, voir par exemple [25]).

Sous des conditions de régularité, la positivité du courant de courbure implique que la métrique ϕ s'écrit comme la limite uniforme d'une suite de métriques de Fubini-Study, construite à partir du système linéaire gradué total du fibré inversible ample $\mathscr{L}$ ([61], [62,

Theorem 3.5]). Ces conditions (locale et globale) ont des analogues dans la géométrie analytique non archimédienne. Historiquement, l'évolution de la notion de positivité pour les fibrés en droites métrisés non archimédien est inverse à celle du cas complexe, au sens où l'aspect global de la positivité a été d'abord étudié. Une notion de semipositivité pour la métrique ϕ est définie dans [62] à l'aide de la positivité algébro-géométrique ("nef") sur les modèles entiers pour le couple $(X, \mathscr{L})$, et dans ce même article on dit qu'une métrique est "semi-ample" si elle est une limite uniforme de métriques Fubini-Study (voir aussi [36] et [22] pour la notion de semipositivité). Ces deux conditions de positivité sont toutes décrites par des objects globales, et ils sont finalement montrées d'être équivalentes dans [62][24]. La deuxième s'affranchit l'utilisation de modèles entiers et donc plus intrinsèque à la géométrie non archimédienne, c'est pour cette raison que l'on travaille avec la definition de semipositivité par la deuxième condition suivant [24].

Quant à une version locale pour la positivité, il faut d'abord construire une structure différentielle associée à l'espace analytique non archimédien. Un prélude apparaissait dans [37], qui découvre une géométrie tropicale au sein d'un espace analytique à la Berkovich. Un théorie complète sur les formes et courants sur un espace analytique à la Berkovich est achevée dans [21] (voir aussi [38]). Avec cette structure différentielle, il est possible de définir la positivité d'un fibré métrisé $(\mathscr{L}, \phi)$ de manière *locale* par la positivité de son courant de courbure, exactement comme dans le cas complexe. Pourtant, la relation entre cette positivité locale avec la version globale reste encore opaque. Plus sérieusement, à ce jour, on ne dispose pas encore de tous les outils d'analyse «fine» mentionés plus haut dans la théorie développé par [21]. Un obstacle clé est que l'opérateur $\overline{\partial}$ dans cette théorie de formes différentielle sur l'espace analytique non archimédien *ne caractérise pas* les objects holomorphes. Faute de cette liaison directe avec l'objet holomorphe, l'utilisation de cet opérateur ne peut pas être une transcription de celle du cas complexe.

Malgré ces subtilités dans la comparaison entre les géométries analytiques complexe et non archimédienne, nous arrivons à montrer que les fibrés en droites métrisés en géométrie analytique non archimédienne ont des propriétés analogues à celles de fibrés en droites métrisés en géométrie complexe, comme par exemple extension de section avec un contrôle de norme, ainsi que quelques aspects de la théorie pluripotentielle. Ces résultats sont résumés dans §0.2.

0.1.3. Stratégie: positivité métrique en termes de convexité holomorphe. — On étudie un fibré en droites métrisé non archimédien sous l'hypothèse de positivité globale. Comme mentionnés plus haut, une métrique semipositive au sens global peut être approchée uniformément par des métriques de modèles entiers [62][36] ou de manière équivalente, par des métriques de Fubini-Study générales [24]. En effet, ces deux choix correspondent à deux approches différentes pour l'étude d'un objet métrique sur un espace analytique non archimédien, dites «de Zariski» et «de Berkovich» suivant la terminologie de [14, Introduction, Table 1]. Grosso modo, le point de vue «Zariski» exploite les modèles entiers pour cet espace et utilise les techniques de la géométrie algébrique sur ces modèles, tandis que le point de vue «Berkovich» travaille sur le complété de cet espace (i.e. l'espace analytique de Berkovich associé) avec l'outil d'analyse. L'étude de fibré en droites métrisé semi-positif a déjà connu beaucoup de progrès avec une combinaison de

deux approches dans divers aspects, que l'on rappellera dans §0.2. Grace aux divers résultats bien développés en géométrie algébrique, ces progrès exploitent plutôt des techniques de nature «Zariski».

Dans cette thèse on examine la positivité globale d'un fibré en droites avec le point de vue purement «Berkovich». On verra dans la section 0.2 que ceci non seulement fournit un regard intrinsèque analytique de ces aspects, mais aussi donne l'espoir d'obtenir certaines résultats sous l'hypothèse plus générale. Par exemple, dans §0.2.3, on peut espérer traiter le problème de continuité de la métrique enveloppante dans le cas de caractéristiques mixtes avec technique de nature «Berkovich», tandis que à ce jour, les techniques de nature «Zariski» ne permettent traiter que les cas de caractéristiques égales (nulle ou positive).

Pour contourner la difficulté de manque d'une structure différentielle et d'outil d'analyse «fine» en géométrie non archimédienne, l'idée adoptée dans cette thèse est de faire la géométrie analytique de manière «synthétique» et d'utiliser systématiquement des outils d'analyse «grossière », notamment l'outil d'analyse fonctionnelle. En d'autres termes, on préfère étudier directement les aspects *normés* des objets holomorphes, notamment les fonctions analytiques ou les faisceaux cohérents, sans faisant appel aux objets auxilliares de nature différentielle comme la courbure. Avec cette démarche, il est possible de remplacer les techniques d'analyse différentielle par les techniques d'analyse fonctionnelle. Autrement dit, on va étudier l'objet «dur» (i.e. holomorphe) avec l'analyse «grossière», au lieu d'exploiter l'objet «souple» (i.e. réel lisse) avec l'analyse «fine».

En géométrie complexe, ce point de vue pour examiner les objets holomorphes normés remonte à l'époque fondatrice de la théorie d'espace analytique, par exemple dans l'étude de faisceaux analytiques par Cartan-Serre sur la finitude de cohomologies [20] et par Grauert sur la cohérence d'image directe [32]. Mais il a été légèrement marginalisé après l'invention des outils puissants d'équations différentielles partielles. En géométrie non-archimédienne (théorie de Berkovich), certaines travaux de Ducros et de Poineau reprennent les outils d'analyse fonctionnelle (voir par exemple [27][50][44]). Il se trouve que ce point de vue est parfaitement adapté à la géométrie analytique non archimédienne. On espère combiner la structure de schéma sur l'espace analytique et des outils d'analyse fonctionnelle – toutes deux bien développées dans le cadre non archimédien – pour obtenir des propriétés quantitatives analogues à celles de la géométrie complexe, parfois au prix d'optimalité.

Plus précisément, quant aux problèmes concernant un fibré en droites métrisé, on va étudier la métrique via la géométrie de son fibré en disques unités duals, tout comme l'étude d'une fonction via la géométrie de sa partie de sous-niveau. La notion clé correspondante à la positivité au sens global pour la métrique est la convexité holomorphe de son fibré en disques unités duals. L'étude de la convexité holomorphe d'une partie est bien un exemple de l'étude «d'un objet dur avec l'analyse grossière», car elle ne concerne que les fonctions holomorphes et la norme sup. C'est une démarche bien connue en géométrie complexe [31][10][51].

On va emprunter cette stratégie dans cette thèse pour obtenir des propriétés importantes concernant la positivité métrique d'un fibré en droites. On donne la construction du fibré en disques unités duals en termes de l'algèbre de section normée, on rappelle et

développe certains aspects concernant la convexité holomorphe dans l'espace analytique non archimédien, et on obtient finalement ces propriétés en géométrie non archimédienne en passant par des arguments avec cette notion.

0.2. Résultats obtenus

Cette mémoire de thèse consiste d'un chapitre de préliminaire (chapitre 1) et trois chapitres de résultats (chapitres 2-4), dont le premier (chapitre 2) donne la construction de base géométrique, le suivant (chapitre 3) traitent le problème d'extension métrique, et le dernier (chapitres 4) étudient la métrique enveloppante.

0.2.1. Chapitre 2. — Algèbre de section normée et son spectre

0.2.1.1. *Algèbre de section normée.* — On propose un cadre commode pour étudier un fibré en droites métrisé semipositif $(\mathscr{L}, \phi)$ non archimédien, en imitant notamment de constructions en géométrie analytique complexe dans [31] (voir aussi [10, Appendix]). Ce cadre nous permet de formaliser la construction de métrique Fubini-Study par des normes quotients, d'encoder la limite de ces métrique par une norme d'algèbre, et donc finalement de présenter la positivité globale comme la convexité holomorphe du spectre d'une algèbre de Banach associée.

Pour tout fibré en droites $\mathscr{L}$, l'objet fondamental algébro-géométrique est l'algèbre graduée de ses sections

$$R_\bullet(\mathscr{L}) := \bigoplus_{n\in\mathbb{N}} R_n(\mathscr{L}) = \bigoplus_{n\in\mathbb{N}} H^0(X, \mathscr{L}^{\otimes n}).$$

La donnée métrique est encodée en général par une norme graduée d'algèbre $|||\cdot|||$ (i.e. une norme graduée sous-multiplicative) sur l'algèbre de sections (voir plus bas pour les constructions naturelles d'une telle norme d'algèbre à partir d'une métrique ϕ)

$$\forall \underline{s} = (s_n)_{n\in\mathbb{N}} \in R_\bullet(\mathscr{L}),\ |||\underline{s}||| = \sup_{n\in\mathbb{N}} \|s_n\|_n,\ \|s_{n_1+n_2}\|_{n_1+n_2} \leqslant \|s_{n_1}\|_{n_1} \cdot \|s_{n_2}\|_{n_2}.$$

L'objet central d'étude est le séparé complété de l'algèbre de section par rapport à cette norme, qui est une algèbre de Banach commutative notée comme $\widehat{R}_\bullet(\mathscr{L}, |||\cdot|||)$.

Considérons le comportement du couple $(R_\bullet(\mathscr{L}), |||\cdot|||)$ par rapport aux morphismes de changement de base de X. Ces morphismes permettent de construire de nouvelles algèbres de section à partir de $R_\bullet(\mathscr{L})$ et de nouvelle normes d'algèbre à partir de $|||\cdot|||$. Par exemple, soit Y une sous-variété de X, l'immersion canonique de Y induit l'algèbre de sections restreintes notée comme $R_\bullet(\mathscr{L}_{X|Y})$, et une norme d'algèbre quotient notée comme $|||\cdot|||_{X|Y}$. On obtient alors une algèbre de Banach quotient en passant au séparé complété, notée comme $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, |||\cdot|||_{X|Y})$. Par ailleurs, pour tout point $x \in X^{\mathrm{an}}$, l'extension de scalaires valués de k vers $\widehat{\kappa}(x)$ donne une algèbre graduée $R_\bullet(\mathscr{L})(x)$ et une norme $|||\cdot|||_{X|x}$ à partir de l'ancien couple $(R_\bullet(\mathscr{L}), |||\cdot|||)$, par rapport à laquelle on construit le séparé complété $\widehat{R}_\bullet(\mathscr{L})(x)$.

0.2.1.2. *Métriques et normes d'algèbre.* — On construit quelques métriques sur $\mathscr{L}$ à partir de la norme d'algèbre $|||\cdot|||$, tout en imitant les constructions usuelles en géométrie analytique complexe. Les immersions induites par les points de X^{an} donnent un passage «norme vers métrique»: pour tout entier naturel n qui est suffisamment grand, la famille

$\{\|\cdot\|_{n,X|x}\}_{x\in X^{\mathrm{an}}}$ donne une métrique continue sur $\mathscr{L}^{\otimes n}$, notée comme $\mathrm{FS}_n(|||\cdot|||)$ (ou simplement $\mathrm{FS}(\|\cdot\|_n)$). On l'appelle la n-ième métrique Fubini-Study induite par cette norme d'algèbre. La limite de la suite de métriques Fubini-Study normalisées $\{\frac{1}{n}\mathrm{FS}(\|\cdot\|_n)\}_{n\in\mathbb{N}}$ est une métrique sur $\mathscr{L}$, que l'on appelle la métrique enveloppante, notée comme $\mathcal{P}(|||\cdot|||)$.

Réciproquement, une métrique peut induire une norme d'algèbre, ce qui donne une passage «métrique vers norme». Plus précisément, toute métrique semi-continue supérieurement ϕ sur $\mathscr{L}$ induit une norme d'algèbre $|||\cdot|||_\phi$ sur $R_\bullet(\mathscr{L})$, dont la n-ième composant est la norme sup de $n\phi$ sur $R_n(\mathscr{L})$, notée comme $\|\cdot\|_{n\phi}$. Ainsi, combinée avec la construction de métrique Fubini-Study plus haut, la positivité au sens global de ϕ est décrite par l'égalité de $\mathcal{P}(\phi)$ à ϕ.

0.2.1.3. *Positivité globale et convexité holomorphe du spectre.* — Traduisons cette description de la positivité globale de $(\mathscr{L},\phi)$ en termes de la convexité holomorphe du spectre de l'algèbre de Banach $\widehat{R}_\bullet(\mathscr{L},|||\cdot|||)$. Le cœur est d'identifier géométriquement la métrique enveloppante $\mathcal{P}(\phi)$ à partir de cette algèbre de section normée.

On part du morphisme algébro-géométrique de contraction de section zéro dans l'espace totale $\mathbf{V}(\mathscr{L})$ du fibré dual de $\mathscr{L}$, vers le cône affine $\mathbf{C}(\mathscr{L})$ de $\mathscr{L}$ sur X (i.e. le spectre premier de Zariski de l'algèbre $R_\bullet(\mathscr{L})$)

$$p:\mathbf{V}(\mathscr{L})\to\mathbf{C}(\mathscr{L}).$$

Pour étudier une métrique générale ψ sur $\mathscr{L}$, on regarde la partie compacte du fibré en disques unités duals dans $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ associé à $(\mathscr{L},\psi)$, notée comme $\overline{\mathbb{D}}^\vee(\mathscr{L},\psi)$. Son image sous p^{an} est une partie compacte dans $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$, qui encode toute information concernant la métrique.

On examine un enrichissement métrique par la norme d'algèbre $|||\cdot|||$ du morphisme p, en reliant deux parties compactes contenues dans les analytifiés de ces deux espaces. D'un côté on a le fibré en disques unités duals du fibré métrisé $(\mathscr{L},\mathcal{P}(|||\cdot|||))$, et de l'autre côté, on a le spectre de Gelfand d'algèbre de Banach, noté comme $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L},|||\cdot|||))$. L'assertion suivante traduit la positivité d'une métrique enveloppante en termes de la convexité holomorphe d'un spectre de Gelfand d'une algèbre de Banach.

***Théorème 0.2.1* (Proposition 2.3.3).** — Le diagramme suivant d'applications entre des espaces topologiques est cartésien

$$\begin{tikzcd}
\overline{\mathbb{D}}^\vee(\mathscr{L},\mathcal{P}(|||\cdot|||)) \arrow[r, two heads, "p^{\mathrm{an}}"] \arrow[d, hook] & \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L},|||\cdot|||)) \arrow[d, hook] \\
\mathbf{V}(\mathscr{L})^{\mathrm{an}} \arrow[r, two heads, "p^{\mathrm{an}}"] & \mathbf{C}(\mathscr{L})^{\mathrm{an}}
\end{tikzcd}$$

Cette construction du fibré en disques unités duals donne aussi une «géométrisation» de la construction enveloppante de $\mathcal{P}(\phi)$ à partir de ϕ. On prend la norme d'algèbre $|||\cdot|||$ comme celle induite par ϕ, et on note comme $\mathfrak{M}^-(\phi)$ la partie compacte $p^{\mathrm{an}}(\overline{\mathbb{D}}^\vee(\mathscr{L},\phi))$ et comme $\mathfrak{M}(\phi)$ la partie compacte $p^{\mathrm{an}}(\overline{\mathbb{D}}^\vee(\mathscr{L},\mathcal{P}(\phi)))$.

***Corollaire 0.2.2* (Proposition 2.3.4).** — Dans $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$, la partie compacte $\mathfrak{M}(\phi)$ est l'enveloppe holomorphe de la partie compacte $\mathfrak{M}^-(\phi)$. Ainsi une métrique continue ϕ est

semipositive si et seulement si la partie compacte $\mathfrak{M}^-(\phi)$ est convexe holomorphe dans $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$.

La construction de métrique enveloppante se comporte «fonctoriellement» par rapport aux morphismes algébro-géométriques de changement de base au sens suivant. Pour une immersion fermée $Y \to X$, on peut lui associer l'algèbre quotient $R_\bullet(\mathscr{L}_{X|Y})$ et la norme d'algèbre quotient $|||\cdot|||_{\phi_{X|Y}}$. Ce couple de quotients induit une métrique enveloppante $\mathcal{P}(\phi_{X|Y})$ sur $\mathscr{L}|_Y$. Une autre construction naturelle est de regarder l'algèbre de section $R_\bullet(\mathscr{L}|_Y)$ et la norme d'algèbre sup $|||\cdot|||_{\phi|_Y}$. Ce couple sup induit aussi une métrique enveloppante $\mathcal{P}(\phi|_Y)$ sur $\mathscr{L}|_Y$. Le résultat en haut implique que les deux soient égales.

***Théorème 0.2.3* (Proposition 2.4.3).** — Pour toute norme d'algèbre graduée $|||\cdot|||$ sur $R_\bullet(\mathscr{L})$, on a l'égalité

$$(|||\cdot|||_{X|Y})_{\mathrm{sp}} = |||\cdot|||_Y.$$

Ainsi, si ϕ est une métrique semipositive, alors $\mathcal{P}(\phi_{X|Y}) = \phi|_Y$.

0.2.2. Chapitre 3. — Extension de sections du fibré restreint avec un contrôle de norme

0.2.2.1. *Problème et ses reformulations.* — On étudie un problème d'extension de sections d'un fibré en droites semi-positivement métrisé $(\mathscr{L}, \phi)$ restreint à une sous-variété fermée. Étant donné un entier $n \in \mathbb{N}$, une sous-variété fermée Y de X, et une section t_n de $\mathscr{L}^{\otimes n}|_Y$, on cherche une section globale s_n de X dont la restriction à Y s'identifie à t_n et dont la norme sup est contrôlé en fonction de n et de la norme de t_n. L'existence d'une telle section dont la norme sup «se concentre autour de sous-variété» est très utile pour étudier le volume du fibré en droites avec un argument inductive en dimension.

Dans la pratique on s'intéresse à une version asymptotique de ce problème. Reformulé en terme de normes construites dans le chapitre 2, le problème d'extension métrique de sections restreintes devient un problème de comparaison entre deux (suites de) normes $\|\cdot\|_{n,\phi_{X|Y}}$ et $\|\cdot\|_{n,\phi|_Y}$; plus précisément, on cherche une majoration de la norme d'opérateur de l'application d'identité de $(R_n(\mathscr{L}|_Y), \|\cdot\|_{n,\phi|_Y})$ dans $(R_n(\mathscr{L}|_Y), \|\cdot\|_{n,\phi_{X|Y}})$. Avec Théorème 0.2.3, on peut facilement déduire un résultat de l'extension métrique obtenu dans [24].

***Corollaire 0.2.4* (Theorem 3.2.1).** — Soit $(\mathscr{L}, \phi)$ un fibré en droites muni d'une métrique semi-positive continue. Pour tout $\epsilon > 0$, et tout $t_m \in R_m(\mathscr{L}_{X|Y})$ il existe $N \in \mathbb{N}$ tel que, pour tout $n \geqslant N$, on peut trouver $s_{nm} \in R_{nm}(\mathscr{L})$ avec $s_{nm}|_Y = t_m^{\otimes n}$ et

$$\|s_{nm}\|_{nm\phi} \leqslant \mathrm{e}^{nm\epsilon} \cdot \|t_m\|^n_{m\phi|_Y}.$$

Notons que dans Corollaire 0.2.4, le seuil N dépend non-seulement des données géométriques fixés $(X,\ Y,\ \mathscr{L},\ \phi)$ et de la précision d'approximation prescrite ϵ, mais aussi du choix de la section restreinte t_m. On propose deux méthodes pour améliorer ce résultat, en affranchissant la dépendance de la section restreinte. La version uniforme de l'extension métrique de sections restreintes est l'assertion suivante.

***Théorème 0.2.5* (Theorem 3.3.5).** — Soit $(\mathscr{L}, \phi)$ un fibré en droites muni d'une métrique semi-positive continue. Pour tout $\epsilon > 0$, il existe $N \in \mathbb{N}$ tel que pour tout

$n \geqslant N$, et tout $t_n \in R_n(\mathscr{L}_{X|Y})$, on peut trouver $s_n \in R_n(\mathscr{L})$ avec $s_n|_Y = t_n$ et

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

Transformons ces inégalités en deux suites de norme en une seule inégalité en deux normes d'algèbre. Si on note $\phi(\epsilon)$ la métrique dilatée par e^{ϵ}, on peut reformuler cette version uniforme en termes d'une majoration de norme d'algèbre $|||\cdot|||_{\phi_{X|Y}}$ par $|||\cdot|||_{\phi|_Y(\epsilon)}$. De manière equivalente, on cherche à majorer la norme d'opérateur de l'application identité de $(R_\bullet(\mathscr{L}_{X|Y}), |||\cdot|||_{\phi|_Y(\epsilon)})$ dans $(R_\bullet(\mathscr{L}_{X|Y}), |||\cdot|||_{\phi_{X|Y}})$. C'est l'énoncé suivant que l'on vise à démontrer.

***Théorème 0.2.6* (Theorem 3.3.5).** — Soit $(\mathscr{L}, \phi)$ un fibré en droites muni d'une métrique semi-positive continue, pour tout $\epsilon > 0$, il existe $C > 0$ tel que

$$|||\cdot|||_{\phi_{X|Y}} \leqslant C \cdot |||\cdot|||_{\phi|_Y(\epsilon)}.$$

Notons que $|||\cdot|||_{\phi|_Y}$ coïncide avec la norme spectrale de $|||\cdot|||_{\phi_{X|Y}}$ d'après Théorème 0.2.3. Ainsi, l'inégalité de cet énoncé peut être vue comme une majoration d'une norme d'algèbre par sa norme d'algèbre spectrale (avec un petit facteur de dilatation). On propose deux méthodes indépendantes pour établir cette majoration. Toute les deux sont basées essentiellement sur la propriété qu'une norme d'algèbre *affinoïd* est majorée par sa norme d'algèbre spectrale (sans facteur de dilatation). Pour obtenir la majoration dans l'énoncé, on fait l'approximation de notre algèbre de Banach $\widehat{R}_\bullet(\mathscr{L}, |||\cdot|||)$ par les algèbres affinoïd, puis on combine les majorations particulières aux normes d'algèbre affinoïd. Un facteur de dilatation est introduit dans la majoration pour notre algèbre de Banach à cause de l'approximation.

0.2.2.2. *Méthode d'approximation spectrale.* — Cette méthode exploite l'approximation géométrique du spectre de $\widehat{R}_\bullet(\mathscr{L}, |||\cdot|||)$ par quelques domains affinoïds. La majoration est établie en deux étapes à l'aide de techniques spectrales. On se place dans l'espace ambient $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$. D'abord on recouvre la partie compacte $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$ par un domain spécial (i.e. une union finie de domains affinoïds) W_ϵ. On note $\mathcal{W}_\epsilon$ l'algèbre structurelle de fonction analytique et $|||\cdot|||_{W_\epsilon}$ la norme sup sur ce domain, et c'est une algèbre de Banach par rapport à cette norme. On utilise la théorie spectrale du calcul fonctionnel holomorphe pour construire un homomorphism d'algèbres de Banach

$$\mathcal{W}_\epsilon \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}).$$

La continuité de cette homomorphisme implique la majoration de $|||\cdot|||_{\phi_{X|Y}}$ par $|||\cdot|||_{W_\epsilon}$. C'est dans cette étape où intervient essentiellement la convexité holomorphe du spectre fournie par la semi-positivité du couple $(\mathscr{L}, \phi)$. En suite, grâce au Théorème 0.2.3, le domain spécial W_ϵ peut être construit de telle sorte qu'il soit contenue dans la partie compacte $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon)))$. L'inclusion spectrale

$$W_\epsilon \hookrightarrow \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon)))$$

implique la majoration de $|||\cdot|||_{W_\epsilon}$ par $|||\cdot|||_{\phi|_Y(\epsilon)}$. La combinaison de ces deux majoration implique la majoration désirée.

0.2.2.3. *Méthode d'approximation algébrique.* — Cette méthode tente de comparer directement la norme d'algèbre de Banach $|||\cdot|||_{\phi_{X|Y}}$ avec une norme d'algèbre affinoïd. D'abord on montre que la norme d'algèbre $|||\cdot|||_{\phi_{X|Y}}$ est affinoïd sous l'hypothèse que ϕ soit elle-même une métrique Fubini-Study. Avec une hypothèse supplémentaire que la valeur absolue $|\cdot|$ sur k soit discrète, on réussit à calculer explicitement la norme $|||\cdot|||_\phi$ dans le cas où X est l'espace projective, $\mathscr{L}$ est le fibré $\mathscr{O}(1)$, et ϕ est une métrique Fubini-Study de forme $\mathrm{FS}(\|\cdot\|_1)$ pour une norme ultramétrique $\|\cdot\|_1$ sur $R_1(\mathscr{L})$. Dans ce cas particulier, l'algèbre de section est isomorphe à l'algèbre symétrique sur l'espace vectoriel $R_1(\mathscr{L})$. En exploitant au fond de l'existence d'une base orthogonale non archimédienne pour $(R_1(\mathscr{L}), \|\cdot\|_1)$, on montre que la norme d'algèbre $|||\cdot|||_\phi$ est une norme de Gauss de multi-rayons donnée par les normes de cette base orthogonale. Il s'en suit que quand ϕ est une métrique Fubini-Study de forme $\mathrm{FS}(\|\cdot\|_1)$, la norme d'algèbre $|||\cdot|||_{\phi_{X|Y}}$ est affinoïd pour X, Y générales et $\mathscr{L}$ très ample. Par conséquence on a la majoration désirée dans ce cas. Si on peut généraliser cette propriété d'affinoïdité de la norme d'algèbre pour $\mathscr{L}$ ample et ϕ est une métrique n-ième Fubini-Study (de forme $\frac{1}{n}\mathrm{FS}(\|\cdot\|_n)$), alors on saura traiter le cas où la métrique semi-positive ϕ est générale par l'approximation uniforme de ϕ par les métriques n-ième Fubini-Study pour n grand.

0.2.3. Chapitre 4. — Continuité de la métrique enveloppante

0.2.3.1. *Problème de continuité de la métrique enveloppante.* — On étudie une propriété de régularité de la métrique enveloppante. Comme la métrique $\mathcal{P}(\phi)$ est une limite *ponctuelle* de métriques (Fubini-Study) continues, elle n'est pas nécessairement continue.

Pourtant, la continuité de la métrique enveloppante est très importante dans l'étude de certains fonctionnels sur l'espace de toutes les métriques sur $\mathscr{L}$, comme par exemple dans le problème de la différentiabilité du volume métrique relative. En géométrie complexe, il est connu que la continuité $\mathcal{P}(\phi)$ est impliquée par celle de ϕ, ce qui est un résultat profond concernant la régularisation globale de fonction psh [26] (voir aussi [39, Appendix]). Sa démonstration exploite au fond les techniques L^2 pour l'opérateur $\overline{\partial}$.

Pour montrer ce résultat en géométrie non archimédienne, faute d'une structure différentielle convenable et d'outil d'analyse fine, il est nécessaire de concevoir un nouveau mécanisme de régularisation pour la métrique enveloppante non archimédienne. Un tel mécanisme a été trouvé dans [13] en termes de la notion d'idéale multiplicateur en géométrie algébrique, et par conséquence la continuité de $\mathcal{P}(\phi)$ est confirmée sous l'hypothèse de continuité de ϕ. Ils considèrent l'approximation de la métrique enveloppante par des métriques provenantes des modèles entiers pour $(X, \mathscr{L})$, suivant l'ancienne definition de semipositivité; une suite d'idéaux multiplicateur est construite pour capturer le processus d'approximation; et finalement l'effet de régularisation de ce processus provient d'une propriété noetherienne pour cette suite d'idéaux. Quelques hypothèses supplémentaires sont mises sur le corps valué de base $(k, |\cdot|)$, comme par exemple de caractéristiques égales nulle, et certaines entre elles sont enlevées dans [17] grâce à l'utilisation d'idéale test en cas de caractéristiques égales positive. Puisqu'ils ont adopté un point de vue «Zariski» pour cette problème et utilise essentiellement l'outil de la géométrie algébrique sur les modèles entiers, certaines hypothèses pour le corps de base sont indispensables dans leurs approches.

0.2.3.2. *Effet de régularisation: voisinage et convexité holomorphe.* — On propose un argument géométrique pour étudier la régularité de $\mathcal{P}(\phi)$, suivant un point de vue «Berkovich». Notre approche est purement analytique et donc indépendante de modèles entiers.

***Théorème 0.2.7* (Theorem 4.3.3).** — Supposons que la Speculation 0.2.8 soit vraie. Si ϕ est continue, alors $\mathcal{P}(\phi)$ l'est aussi.

L'argument est basé sur l'observation suivante. Rappelons que pour une métrique ψ qui est semi-continue supérieurement, on regard son fibre du disc unite dual $\overline{\mathbb{D}}^{\vee}(\mathscr{L},\psi)$ dans $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ et son image $\mathfrak{M}^{-}(\psi)$ dans $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$. Pour un petit facteur $\epsilon \in \mathbb{R}_{>0}$, on considère la partie compacte $\mathfrak{M}^{-}(\psi)$ et l'inclusion dans sa dilatation le long de la direction du cône par un facteur e^{ϵ}

$$\mathfrak{M}^{-}(\psi) \hookrightarrow \mathfrak{M}^{-}(\psi(\epsilon)).$$

Une description équivalente de la continuité de ψ est que la dilatation soit un voisinage de cette partie compacte.

On utilise cette description géométrique comme un critère pour la continuité d'une métrique. Si ϕ est continue, alors la dilatation $\mathfrak{M}^{-}(\phi(\epsilon))$ est un voisinage de $\mathfrak{M}^{-}(\phi)$. On aimerait constater que la dilatation $\mathfrak{M}(\phi(\epsilon))$ est un voisinage de $\mathfrak{M}(\phi)$, d'où la continuité de $\mathcal{P}(\phi)$. Notons que par Corollaire 0.2.2, la partie compacte $\mathfrak{M}(\phi)$ (resp. sa dilatation) est l'enveloppe convexe holomorphe de $\mathfrak{M}^{-}(\phi)$ (resp. sa dilatation).

Cet argument indique que l'effet de régularisation provient de la «commutativité» au sens suivant. Deux opérations géométriques sur une partie compacte doivent commuter entre elles, une la formation de voisinage et l'autre la formation d'enveloppe convexe holomorphe. Dans notre exemple, ces deux opérations doivent relier $\mathfrak{M}^{-}(\phi)$ et $\mathfrak{M}(\phi(\epsilon))$ dans le diagram commutative suivant. Les flèches horizontales signifient les opérations d'enveloppe convex holomorphe et les flèches verticales signifient les opérations de voisinage.

$$\begin{tikzcd}
\mathfrak{M}^{-}(\phi(\epsilon)) \arrow[r, squiggly, "\widehat{\cdot}"] & \mathfrak{M}(\phi(\epsilon)) \\
\mathfrak{M}^{-}(\phi) \arrow[u, dotted, "\Subset"] \arrow[r, squiggly, "\widehat{\cdot}"] & \mathfrak{M}(\phi) \arrow[u, dotted, "\Subset"']
\end{tikzcd}$$

En géométrie analytique complexe, cette «commutativité» est un théorème classique due à Cartan et Thullen. Il constate que un domain dans $\mathbb{C}^n$ est un domain d'holomorphie si et seulement s'il est convexe holomorphe (voir par exemple [25, Theorem 6.11]). Par analogie, on propose l'assertion suivante en géométrie non archimédienne.

***Speculation 0.2.8* (Speculation 4.2.1).** — Supposons que $(k, |\cdot|)$ est sphériquement complet. Soit Z un espace analytique affine qui est normal. Soit K une partie compacte dans Z, et soit U un voisinage de K. Alors $\widehat{U}$ est un voisinage de $\widehat{K}$ (les enveloppes convexes holomorphes sont prise dans Z).

0.3. List of notations

$\mathcal{B}$	a Banach algebra	p.16
$\mathfrak{M}(\mathcal{B})$	its Berkovich spectrum	p.17
X	a projective variety	p.31
Y	a closed sub-variety	p.32
$\mathscr{L}$	a line bundle over X	p.31
$\mathscr{L}\vert_Y$	its restriction on Y	p.32
$R_n(\mathscr{L})$	the space of global sections of $\mathscr{L}^{\otimes n}$	p.31
$R_n(\mathscr{L}_{X\vert Y})$	the space of restricted sections of $\mathscr{L}^{\otimes n}$ on Y	p.32
$R_\bullet(\mathscr{L})$	the algebra of sections of $\mathscr{L}$	p.31
$R_\bullet(\mathscr{L}_{X\vert Y})$	the algebra of restricted sections of $\mathscr{L}$ on Y	p.32
$\Vert\cdot\Vert_n$	an ultrametric norm on $R_n(\mathscr{L})$	p.36
$\vert\vert\vert\cdot\vert\vert\vert$	an ultrametric algebra norm on $R_\bullet(\mathscr{L})$	p.35
$\mathrm{FS}(\Vert\cdot\Vert_n)$	the Fubini-Study metric on $(\mathscr{L}^{\otimes n})^{an}$	p.45
$\mathcal{P}(\vert\vert\vert\cdot\vert\vert\vert)$	the Fubini-Study envelope metric on $\mathscr{L}^{\mathrm{an}}$	p.46
ϕ	a metric on $\mathscr{L}^{\mathrm{an}}$	p.31
$\phi(\epsilon)$	its dilated metric by a factor of e^ϵ	p.31
$\mathcal{P}(\phi)$	its semipositive envelope metric	p.31
$\Vert\cdot\Vert_{n\phi}$	the sup norm on $R_n(\mathscr{L})$ induced by $n\phi$	p.47
$\Vert\cdot\Vert_{n\phi,X\vert Y}$	its quotient norm on $R_n(\mathscr{L}_{X\vert Y})$	p.48
$\vert\vert\vert\cdot\vert\vert\vert_\phi$	the sup algebra norm on $R_\bullet(\mathscr{L})$ induced by ϕ	p.47
$\vert\vert\vert\cdot\vert\vert\vert_{\phi,X\vert Y}$	its quotient algebra norm on $R_\bullet(\mathscr{L}_{X\vert Y})$	p.48
$\vert\vert\vert\cdot\vert\vert\vert^\diamond_{\phi,X\vert Y}$	an affinoid algebra norm on $R_\bullet(\mathscr{L}_{X\vert Y})$ that dominates $\vert\vert\vert\cdot\vert\vert\vert_{\phi,X\vert Y}$	p.57
$\widehat{R}_\bullet(\mathscr{L},\phi)$	the Banach algebra of sections of $\mathscr{L}$	p.48
$\widehat{R}_\bullet(\mathscr{L}_{X\vert Y},\phi_{X\vert Y})$	its quotient Banach algebra of restricted sections of $\mathscr{L}$ on Y	p.48
$\mathbf{V}(\mathscr{L})$	the total space of the dual of $\mathscr{L}$	p.32
$\mathbf{C}(\mathscr{L})$	the affine cone of $\mathscr{L}$	p.32
p	the morphism of contraction of zero section $\mathbf{V}(\mathscr{L})\to\mathbf{C}(\mathscr{L})$	p.32
$\overline{\mathbb{D}}^\vee(\mathscr{L},\phi)$	the dual unit disc bundle of $(\mathscr{L},\phi)$ in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$	p.44
$\mathfrak{M}^-(\phi)$	its image under p^{an} in $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$	p.56
$\mathfrak{M}(\phi)$	the Berkovich spectrum of $\widehat{R}_\bullet(\mathscr{L},\phi)$ in $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$	p.56
K	a compact set in some affinoid space Z	p.26
$\widehat{K}$ (or $\widehat{K}^Z$)	its holomorphic convex envelope set in Z	p.26
$\mathrm{Int}^t(\cdot)$	topological interior	

CHAPTER 1

PRELIMINARIES ON FUNCTIONAL ANALYSIS AND GEOMETRY OVER NON-ARCHIMEDEAN FIELDS

In this section, one recalls some facts in functional analysis of normed vector spaces and normed algebras over a complete non-Archimedean valued field, following [3], [8], [29] and [60]. Besides the well-known facts, the results presented in §1.1.2, §1.2.4, §1.3.3 and §1.4 are most important for the thesis.

Throughout the section, one fixes a field k equipped with a *non-Archimedean* and non-trivial absolute value $|\cdot|$ and we assume that k equipped with the topology defined by the absolute value is complete. Denote by k° the valuation ring of $(k, |\cdot|)$, by $k^{\circ\circ}$ the maximal ideal of k°, and by $\widetilde{k}$ the residual field $k^\circ/k^{\circ\circ}$. Denote by $H(k, |\cdot|)$ the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ generated by the set of numbers $\log|k^\times|$. One says that n numbers $\{p_1, \ldots, p_n\}$ of $\mathbb{R}_{>0}$ are $(k, |\cdot|)$*-free if the images of* $\{\log p_1, \ldots, \log p_n\}$ *in the quotient* $\mathbb{Q}$*-vector space* $\mathbb{R}/H(k, |\cdot|)$ *are* $\mathbb{Q}$*-linearly independent therein.*Unless specified, all k-algebras are supposed to be commutative and unitary (with $0 \neq 1$), and by convention all homomorphism of k-algebras are supposed to preserve the units.

1.1. Seminormed vector spaces

1.1.1. Basic constructions. — Let V be a vector space over k. We call a *seminorm* on V any map $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ which satisfies the following conditions:

(1) for any $(a, s) \in k \times V$, $\|as\| = |a| \cdot \|s\|$,

(2) (triangle inequality) for any $(s, s') \in V \times V$, $\|s + s'\| \leqslant \|s\| + \|s'\|$.

The couple $(V, \|\cdot\|)$ is called a *seminormed vector space* over k. If the following *strong triangle inequality* is satisfied

$$\forall\, (s, s') \in V \times V, \quad \|s + s'\| \leqslant \max\{\|s\|, \|s'\|\},$$

we say that the seminorm $\|\cdot\|$ is ultrametric. If $\|s\| > 0$ for any non-zero vector s of V, we say that the seminorm $\|\cdot\|$ is a norm and that $(V, \|\cdot\|)$ is a *normed vector space.*

Let $(V, \|\cdot\|)$ be a seminormed vector space over k. The seminorm $\|\cdot\|$ induces a topology on V, a topological basis of which is given by the family of open balls

$$\{y \in V \mid \|y - x\| < r\}, \quad x \in V,\ r > 0.$$

Clearly $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ is a continuous map with respect to this topology. Denote by $\mathfrak{n}(\|\cdot\|)$ the inverse image of $\{0\}$ by $\|\cdot\|$. As a consequence of the triangle inequality, $\mathfrak{n}(\|\cdot\|)$

is a vector subspace of V. It is moreover closed since $\|\cdot\|$ is continuous, and is called the *null space* of $\|\cdot\|$. Note that there exists a unique norm on $V/\mathfrak{n}(\|\cdot\|)$, the composition of which with the projection map $V \to V/\mathfrak{n}(\|\cdot\|)$ identifies with the seminorm $\|\cdot\|$. We call this norm the *quotient norm* induced by the seminorm $\|\cdot\|$.

Let $(V, \|\cdot\|)$ be a seminormed vector space over k. Note that if $\|\cdot\|$ is ultrametric, then the equality

$$\|x + y\| = \max\{\|x\|, \|y\|\}$$

holds whenever $\|x\| \neq \|y\|$.

We say that a seminormed (resp. normed) vector space $(V, \|\cdot\|)$ is *complete*, or $\|\cdot\|$ is a *complete seminorm* (resp. *complete norm*) on V, if any Cauchy sequence in V with respect to the seminorm $\|\cdot\|$ admits a limit. A complete normed vector space over k is called a *Banach space* over k. Any finite-dimensional normed space $(V, \|\cdot\|)$ is complete ([16, 1.2.3 Theorem 2]).

Let $(V, \|\cdot\|_V)$ be a seminormed vector space over k. Let $\widetilde{V}_c$ be the vector space of all Cauchy sequences in V with respect to $\|\cdot\|_V$. We define a seminorm $\|\cdot\|_c$ on $\widetilde{V}_c$ which sends any Cauchy sequence $\{v_i\}_{i\in\mathbb{N}}$ to $\lim_{i\to+\infty}\|v_i\|_V$. Denote by V_c the quotient vector space $\widetilde{V}_c/\mathfrak{n}(\|\cdot\|_c)$. Then the vector space V_c equipped with the norm induced by $\|\cdot\|_c$ forms a Banach space over k, called the *separated completion* of $(V, \|\cdot\|_V)$. Tautologically it can be shown that this Banach space is canonically isomorphic to the completion of $V/\mathfrak{n}(\|\cdot\|_V)$ equipped with the quotient norm induced by the seminorm $\|\cdot\|_V$.

Definition 1.1.1. — Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be seminorms on V. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* if there exist two constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant C_2\|\cdot\|_1$. Note that this condition holds if and only if the seminorms $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology on the vector space V ([16, Corollaire I.3.3.1])(note that the absolute value $|\cdot|$ is supposed to be non-trivial).

Definition 1.1.2. — Let $(V, \|\cdot\|)$ be a seminormed vector space over k. If W is a vector subspace of V, then map $(x \in W) \mapsto \|x\|$ defines a seminorm on W, called the *restriction* of $\|\cdot\|$ on W. If Q is a quotient vector space of V and $\pi : V \to Q$ is the quotient map, then the map $(q \in Q) \mapsto \inf_{x\in\pi^{-1}(\{q\})}\|x\|$ defines a seminorm on Q, called the *quotient* of $\|\cdot\|$ on Q.

Definition 1.1.3. — Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be seminormed vector spaces over k, and $f : V \to W$ be a k-linear map. We say that f is *bounded* if there exists a constant $C > 0$ such that $\|f(x)\|_W \leqslant C\|x\|_V$ for any $x \in V$. Note that this condition holds if and only if f is continuous with respect to the topologies on V and W induced by the seminorms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively. If on the image of f, the quotient norm of $\|\cdot\|_V$ is equivalent to the subspace norm of $\|\cdot\|_W$, then f is said to be *admissible*.

We recall below several fundamental results in functional analysis and refer to [16, Theorem 1.3.3.1, Corollary 1.3.3.1, 1.3.3.2, 1.3.3.5] for more details.

Theorem 1.1.4. — Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be Banach spaces over k, and $f : V \to W$ be a k-linear map. Suppose that the absolute value $|\cdot|$ on k is non-trivial. Then

(1) The k-linear map f is bounded if and only if its graph in $V \times W$ is closed under the product topology.

(2) Assume that f is bounded and surjective, then f is an open map. In particular, the quotient norm of $\|\cdot\|_V$ on W is equivalent to $\|\cdot\|_W$.

Theorem 1.1.5. — Let V be a vector space over k and $\|\cdot\|_1$ and $\|\cdot\|_2$ be complete norms on V. Suppose that the absolute value $|\cdot|$ on k is non-trivial. If there exists $C > 0$ such that $\|\cdot\|_2 \leqslant C\|\cdot\|_1$, then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Using this norm equivalence theorem for Banach spaces over k, we have immediately the following.

Corollary 1.1.6. — Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be Banach spaces over k, and $f : V \to W$ be a bounded k-linear map with closed image. Then f is admissible.

Definition 1.1.7. — Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space. The *dual norm of* $\|\cdot\|^\vee$ on the dual vector space $V^\vee := \mathrm{Hom}_k(V, k)$ is defined by

$$\forall \ell \in V^\vee,\ \|\ell\|^\vee := \sup_{v \in V \setminus \{0\}} \frac{|\ell(v)|}{\|v\|}.$$

Remark 1.1.8. — The norm $\|\cdot\|^\vee$ is ultrametric, and $\|\cdot\|^{\vee\vee} = \|\cdot\|$ if and only if $\|\cdot\|$ is ultrametric. ([24, Section 2.2.3])

Definition 1.1.9. — Let $(V, \|\cdot\|)$ be a normed vector space. Let $(k', |\cdot|')$ be a complete valued field extension of $(k, |\cdot|)$. Set $V_{k'}$ to be $V \otimes_k k'$, which can be identified with $\mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k')$. The norm

$$\forall v' \in V_{k'},\ \|v'\|_{k'} := \sup\left\{\frac{|(\ell \otimes 1)(v')|'}{\|\ell\|^\vee},\ \ell \in V^\vee \setminus \{0\}\right\}$$

defined via this identification is called the *scalar extension of* $\|\cdot\|$.

Remark 1.1.10. — If $\|\cdot\|$ is ultrametric, then $\|\cdot\|_{k'}$ is the largest ultrametric norm on $V_{k'}$ extending $\|\cdot\|$. ([24, Definition 2.4])

Lemma 1.1.11. — Let $f : V \to W$ be a surjective k-linear map of finite-dimensional vector spaces, with $\dim_k W = 1$. Let $\|\cdot\|_V$ be a norm on V and let $\|\cdot\|_W$ be its quotient norm by f. Then the norm $\|\cdot\|_{W,k'}$ identifies with the quotient norm of $\|\cdot\|_{V,k'}$ induced by the surjective k'-linear map $f \otimes \mathrm{id}_{k'} : V_{k'} \to W_{k'}$. ([24, Lemma 2.5])

1.1.2. Orthogonal basis. —

Definition 1.1.12. — Let $(V, \|\cdot\|)$ be a finite-dimensional normed vector space over k. A basis $\{e_i\}_{i \in \{1,\dots,n\}}$ of V is called *orthogonal* (with respect to $\|\cdot\|$) if

$$\forall (c_1, \dots, c_r) \in k^n, \quad \Big\| \sum_{i \in \{1,\dots,n\}} c_i e_i \Big\| = \max_{i \in \{1,\dots,n\}} \|c_i e_i\|.$$

Moreover, it is said to be *orthonormal* if in addition $\|e_i\| = 1$ for all $i \in \{1, \dots, n\}$. A basis $\{e_i\}_{i\in\{1,\dots,n\}}$ of V is called *ρ-orthogonal* (for some $\rho \in \mathbb{R}_{\geqslant 1}$) if

$$\forall (c_1, \dots, c_r) \in k^n, \quad \rho \cdot \Big\| \sum_{i\in\{1,\dots,n\}} c_i e_i \Big\| \geqslant \max_{i\in\{1,\dots,n\}} \|c_i e_i\|$$

Lemma 1.1.13. — Let $(V, \|\cdot\|)$ be a finite-dimensional ultrametrically normed vector space over k. If $\{v_i\}_{i\in\{1,\dots,n\}}$ is a finite set of elements of V such that $\{\|v_i\|\}_{i\in\{1,\dots,n\}}$ are distinct in $\mathbb{R}_+$. Then $\|\sum_{i\in\{1,\dots,n\}} v_i\| = \max_{i\in\{1,\dots,n\}} \|v_i\|$.

Proof. — If $n = 2$, this is clear from the ultrametric inequality. The general case then follows by induction on n. □

Corollary 1.1.14. — Let $(V, \|\cdot\|_V)$ be a finite-dimensional ultrametrically normed vector space over k. Suppose that $(k, |\cdot|)$ is discretely valued. If $\{e_i\}_{i\in\{1,\dots,n\}}$ is a basis of V such that $\{\|e_i\|\}_{i\in\{1,\dots,n\}}$ are $(k, |\cdot|)$-free, then $\{e_i\}_{i\in\{1,\dots,n\}}$ is an orthogonal basis.

Proof. — For any $f = (f_1, \dots, f_n) \in (k^\times)^n$, the numbers $\{|f_i| \cdot \|e_i\|\}_{i\in\{1,\dots,r\}}$ are distinct, otherwise there exist $i, j \in \{1, \dots, n\}, i \neq j$ such that

$$\log\|e_i\| - \log\|e_j\| = \log\Big|\frac{f_i}{f_j}\Big| \in \log|k^\times|$$

which contradicts the assumption of their $\mathbb{Q}$-freeness for $(k, |\cdot|)$. Hence by Lemma 1.1.13

$$\Big\| \sum_{i\in\{1,\dots,n\}} f_i e_i \Big\| = \max_{0\leqslant i\leqslant n} \|f_i \cdot e_i\|.$$

□

Definition 1.1.15. — A valued field $(k, |\cdot|)$ is said to be *spherically complete*, if the intersection of every decreasing sequence of balls $\mathbb{B}(\lambda_n, r_n) = \{\lambda \in k,\ |\lambda - \lambda_n| \leqslant r_n\}$ is non-empty.

Proposition 1.1.16. — Let $(V, \|\cdot\|_V)$ be a finite-dimensional ultrametrically normed vector space over k. If $|\cdot|$ is non-trivial and $(k, |\cdot|)$ is spherically complete, then $(V, \|\cdot\|_V)$ admits an orthogonal basis ([8, Proposition 2.4.4.2]).

Proposition 1.1.17. — Let $(V, \|\cdot\|_V)$ be a finite-dimensional ultrametrically normed vector space over k. If $|\cdot|$ is non-trivial, then for any $\rho \in \mathbb{R}_{\geqslant 1}$, $(V, \|\cdot\|_V)$ admits a ρ-orthogonal basis ([8, Proposition 2.6.1.1]).

1.2. Banach algebra

1.2.1. Basic constructions. —

Definition 1.2.1. — Let A be a k-algebra (the unit of which is denoted by $\mathbf{1}$) and $\|\cdot\|$ be a seminorm on A (viewed as a vector space over k).

(1) The seminorm $\|\cdot\|$ is said to be *sub-multiplicative* if for any $(a, b) \in A \times A$ one has $\|ab\| \leqslant \|a\| \cdot \|b\|$.

(2) The seminorm $\|\cdot\|$ is called *power-multiplicative* if $\|a^n\| = \|a\|^n$ for any $a \in A$ and any $n \in \mathbb{N}\backslash\{0\}$.

(3) The seminorm $\|\cdot\|$ is called *multiplicative* if $\|ab\| = \|a\| \cdot \|b\|$ for any $(a, b) \in A^2$.

A *k-algebra seminorm* (resp. *k-algebra norm*) on A is defined to be a sub-multiplicative seminorm (resp. sub-multiplicative norm) $\|\cdot\|$ on A such that $\|\mathbf{1}\| = 1$. We denote by $|||\cdot|||$ an algebra seminorm. Any k-algebra equipped with a complete k-algebra norm is called a *Banach k-algebra*.

We use letters in calligraphic style to denote Banach algebras and Banach modules (defined below) and use the same letters in literary style to denote the underlying k-algebra or the underlying module of a k-algebra. For example, a Banach k-algebra $(A, |||\cdot|||)$ is denoted by $\mathcal{A}$. If A' is a sub-k-algebra of A, then the restriction of $|||\cdot|||$ on A' is a k-algebra norm. If this norm is complete, we say that $\mathcal{A}'$ (A' equipped with the restricted norm) is a Banach k-sub-algebra of $\mathcal{A}$. Similarly, if Q is a quotient k-algebra of A, then the quotient of the norm $|||\cdot|||$ on Q is a sub-multiplicative seminorm. If it is a complete norm, we say that $\mathcal{Q}$ (Q equipped with the quotient norm) is a Banach quotient k-algebra of $\mathcal{A}$.

Example 1.2.2. — Let $\mathcal{A}$ be a Banach k-algebra. The *Tate k-Banach algebra* over $\mathcal{A}$ of multiradius $\boldsymbol{r} = (r_1, \ldots, r_n) \in (\mathbb{R}_{>0})^N$ is defined as the following algebra over k (for $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$, we denote $\prod_{i\in\{1,\ldots,n\}} T_i^{j_i}$ by $\boldsymbol{T}^J$ and $\prod_{i\in\{1,\ldots,n\}} r_i^{j_i}$ by $\boldsymbol{r}^J$)

$$\Big\{ \sum_{J\in\mathbb{N}^n} a_J \boldsymbol{T}^J,\ a_J \in A \text{ and } \lim_{|J|\to\infty} |||a_J||| \cdot \boldsymbol{r}^J = 0 \Big\}$$

with a complete k-algebra norm defined by

$$\Big|\Big|\Big| \sum_{J\in\mathbb{N}^n} a_J \boldsymbol{T}^J \Big|\Big|\Big|_{\mathcal{T}_{\mathcal{A}}(\boldsymbol{r})} := \sup_J |||a_J||| \cdot \boldsymbol{r}^J$$

This Banach algebra is denoted by $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$, and is called an Tate $\mathcal{A}$-algebra of multiradius $\boldsymbol{r}$.

Definition 1.2.3. — Let $\mathcal{A}_1, \mathcal{A}_2$ be two Banach k-algebras, and $\phi : A_1 \to A_2$ be a homomorphism of k-algebras. We say that ϕ is a *homomorphism of Banach k-algebras* if it is bounded as a k-linear map. A homomorphism of Banach k-algebra ϕ is often denoted by $\phi : \mathcal{A}_1 \to \mathcal{A}_2$. A homomorphism of Banach k-algebra ϕ is called an *isomorphism of Banach k-algebras* if there exists a homomorphism of Banach k-algebras $\psi : \mathcal{A}_2 \to \mathcal{A}_1$ such that $\phi \circ \psi = \mathrm{Id}_{A_2}$ and $\psi \circ \phi = \mathrm{Id}_{A_1}$. Note that we only require ϕ to be a topological homeomorphism but not necessarily an isometry.

1.2.2. Spectrum. — Let $\mathcal{A} = (A, |||\cdot|||)$ be a Banach k-algebra. Let $|||\cdot|||'$ be a k-algebra seminorm on A. One says that $|||\cdot|||'$ is *bounded* (with respect to $\mathcal{A}$) if there exists $C > 0$ such that $|||\cdot|||' \leqslant C|||\cdot|||$. Its null-space is a closed ideal I of A; the quotient k-algebra norm of $|||\cdot|||'$ on the quotient k-algebra A/I is bounded with respect to the quotient k-algebra norm of $|||\cdot|||$ ([3, Remark 1.2.2.i]).

Definition 1.2.4. — Let $\mathcal{A}$ be a k-Banach algebra. The *Berkovich spectrum* $\mathfrak{M}(\mathcal{A})$ is the following topological space. As a set, $\mathfrak{M}(\mathcal{A})$ is defined as the set of all multiplicative seminorms on A which are bounded with respect to $\mathcal{A}$. If z is an element of $\mathfrak{M}(\mathcal{A})$, we use the notation $|\cdot|_z$ to denote the corresponding seminorm on A. For any element f of A, we denote by $|f| : \mathfrak{M}(\mathcal{A}) \to \mathbb{R}_{\geqslant 0}$ the function which sends $z \in \mathfrak{M}(\mathcal{A})$ to $|f|_z$. We

equip the set $\mathfrak{M}(\mathcal{A})$ with the *Berkovich topology*, namely the most coarse topology which makes all functions $|f|$ continuous, where $f \in A$. For any subset V of $\mathfrak{M}(\mathcal{A})$, we denote by $\mathrm{Int}^t(V)$ the topological interior of V. This topological interior is to be compared with the notion of *interior of an affinoid subdomain in an affinoid domain*, where the notation Int is adpoted (see [3, Definition 2.5.7]).

Remark 1.2.5. — A basis of the Berkovich topology is given by *basic open sets*, which are sets of the form

$$U(f;p,q) := \{z \in \mathfrak{M}(\mathcal{A}) : p < |f|_z < q\}$$

indexed by $(p,q) \in \mathbb{R}^2$ and $f \in \mathcal{A}$. A general open set is thus a union of finite intersections of basic open sets.

Proposition 1.2.6. — Let $\mathcal{A}$ be a k-Banach algebra. Then $\mathfrak{M}(\mathcal{A})$ is a non-empty compact Hausdorff topological space. ([3, Theorem 1.2.1])

For any point $z \in \mathfrak{M}(A)$, let $\mathfrak{p}_z$ be the closed ideal $\mathfrak{n}(|\cdot|_z)$ of A, which is a prime ideal. The *residual field* at z is defined to be the fraction field of $A/\mathfrak{p}_z$, denoted by $\kappa(z)$. The quotient ring $A/\mathfrak{p}_z$ is equipped with a quotient norm of $|\cdot|_z$, which is still multiplicative and hence induces an absolute value on $\kappa(z)$ extending $|\cdot|$ on k, which we also denote by $|\cdot|_z$ by abuse of notation. The *completed residual field* at z is defined to be the completion of $\kappa(z)$ with respect to this quotient norm $|\cdot|_z$, denoted as $\widehat{\kappa}(z)$. The canonical homomorphism of k-algebras from A to $(\widehat{\kappa}(z), |\cdot|_z)$ is denoted by χ_z. It is a homomorphism of Banach k-algebras.

Proposition 1.2.7. — The set of points of $\mathfrak{M}(\mathcal{A})$ is in canonical bijection with the set of pairs $(\mathfrak{p}, |\cdot|')$ where $\mathfrak{p} \in \mathrm{Spec}(\mathcal{A})$ and $|\cdot|'$ is an absolute value on the residue field at $\mathfrak{p}$, which is bounded from above by the quotient algebra norm induced by $\chi : \mathcal{A} \to \mathcal{A}/\mathfrak{p}$ on $\mathcal{A}/\mathfrak{p}$.

Proof. — A point z of $\mathfrak{M}(\mathcal{A})$ corresponds to such a pair $(\widehat{\kappa}(z), |\cdot|_z)$. Conversely, for such a pair $(\mathfrak{p}, |\cdot|')$, the seminorm on $\mathcal{A}$ given by the composition $|\chi(\cdot)|'$ is obviously multiplicative and bounded from above by $|||\cdot|||$. These canonical maps are inverse to each other. □

Definition 1.2.8. — Let $\mathcal{A}$ be a Banach k-algebra. For any $f \in A$, the *Gelfand transform* of f is defined as the element $\widehat{f} := (f(z))_{z \in \mathfrak{M}(\mathcal{A})}$ in

$$\prod_{z \in \mathfrak{M}(\mathcal{A})} \hat{\kappa}(z).$$

Proposition 1.2.9. — Let $\mathcal{A}$ be a Banach k-algebra. An element $f \in \mathcal{A}$ is invertible if and only if $f(z) \neq 0$ for any $z \in \mathfrak{M}(\mathcal{A})$ ([3, Corollary 1.2.4]).

1.2.3. Continuous map. —

Definition 1.2.10. — Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism of Banach k-algebras. One denotes by $\phi^\star : \mathfrak{M}(\mathcal{A}_2) \to \mathfrak{M}(\mathcal{A}_1)$ the map sending any element $z \in \mathfrak{M}(\mathcal{A}_2)$ to the point corresponding to the seminorm $|\phi(\cdot)|_z$, called the map associated with the homomorphism of k-Banach algebras ϕ. The map $\phi^\star$ is continuous (see [3, Remark 1.2.2 (iii)]).

Proposition 1.2.11. — If $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism of Banach k-algebras with dense image, then $\phi^\star$ is an injective map whose image is closed.

Proof. — The map $\phi^\star$ is injective, since for any two points z_1 and z_2 in $\mathfrak{M}(\mathcal{A}_2)$ whose image under $\phi^\star$ are the same, the restriction of $|\cdot|_1$ and $|\cdot|_2$ on the image of ϕ are equal, hence the two multiplicative seminorms on $\mathcal{A}_1$ $|\chi_1(\cdot)|_1$ and $|\chi_2(\cdot)|_2$ are equal by the density of image.

Let $z \in \mathfrak{M}(\mathcal{A}_1)$ which is not in the image of $\phi^\star$, then $\ker(\phi) \nsubseteq \mathfrak{p}_z$; otherwise the character $\mathcal{A}_1/\ker(\phi) \to \hat{\kappa}(z)$ extends to a character $\mathcal{A}_2 \to \hat{\kappa}(z)$ by the density of image of ϕ, which contradicts $z \notin \mathrm{Im}(\phi^\star)$. Now there exists $f \in \ker(\phi)\backslash\mathfrak{p}_z$, so $|f|_z \neq 0$. For small enough $\epsilon > 0$, the basic open set $U(f; |f|_z - \epsilon, |f|_z + \epsilon) \subset \mathfrak{M}(\mathcal{A}_1)$ is a neighbourhood of z which is not contained in the image of $\phi^\star$. So the image of $\phi^\star$ is a closed subset in $\mathfrak{M}(\mathcal{A}_1)$. □

1.2.4. Spectral seminorm. —

Definition 1.2.12. — Let A be a k-algebra and $|||\cdot|||$ be a k-algebra seminorm on A. We define a map $|||\cdot|||_{\mathrm{sp}} : A \to \mathbb{R}_{\geqslant 0}$ by

$$\forall f \in A, \quad |||f|||_{\mathrm{sp}} := \lim_{n\to\infty} |||f^n|||^{\frac{1}{n}}.$$

Note that $|||\cdot|||_{\mathrm{sp}}$ is a k-algebra seminorm on A. In particular, the triangle inequality for $|||\cdot|||_{\mathrm{sp}}$ follows from the sub-multiplicativity of $|||\cdot|||$. We call $|||\cdot|||_{\mathrm{sp}}$ the *spectral algebra seminorm* of $|||\cdot|||$. In general, $|||\cdot|||_{\mathrm{sp}}$ is only a seminorm even if $|||\cdot|||$ is a norm.

Remark 1.2.13. — The existence of limit is guaranteed by the (multiplicative) Fekete lemma for the sub-multiplicative sequence $\{|||f^n|||\}_{n\in\mathbb{N}}$. The spectral seminorm is sub-multiplicative, and is bounded from above by the original seminorm $|||\cdot|||$. Moreover, it is power-multiplicative by construction.

Proposition 1.2.14. — Let $\mathcal{A}$ be a k-Banach algebra. For any $f \in A$, one has ([3, Theorem 1.3.1])

$$|||f|||_{\mathrm{sp}} = \max_{z\in\mathfrak{M}(A)} |f|(z).$$

Definition 1.2.15. — Let $\mathcal{A} = (A, |||\cdot|||)$ be a Banach k-algebra. The *radical* of $\mathcal{A}$ is defined as the null-space of its spectral seminorm $|||\cdot|||_{\mathrm{sp}}$, denoted by $\mathrm{rad}(\mathcal{A})$ or by $\mathrm{rad}(|||\cdot|||)$. Note that

$$\mathrm{rad}(\mathcal{A}) = \big\{f \in A, \lim_{n\to+\infty} f^n = 0\big\}.$$

A Banach k-algebra with radical equal to $\{0\}$ is said to be *semi-simple*. Elements in the radical are said to be *quasi-nilpotent* (or *topological nilpotent*).

Remark 1.2.16. — The radical of $\mathcal{A}$ contains the nil-radical of A; in other words, nilpotent elements are quasi-nilpotent. If $\mathcal{A}$ is semi-simple, then A is reduced. The converse may not be true.

Let $\mathcal{A} = (A, |||\cdot|||)$ be a k-Banach algebra. The spectral seminorm $|||\cdot|||_{\mathrm{sp}}$ defines a quotient norm on the quotient k-algebra $\mathcal{A}/\mathrm{rad}(\mathcal{A})$, which is still denoted by $|||\cdot|||_{\mathrm{sp}}$. The quotient norm $|||\cdot|||_{\mathrm{sp}}$ is bounded from above by the quotient norm of $|||\cdot|||$. The *uniformization* $\mathcal{A}^u$ of $\mathcal{A}$ is defined to be the Banach k-algebra of separated completion of $(\mathcal{A}/\mathrm{rad}(\mathcal{A}), |||\cdot|||_{\mathrm{sp}})$.

Conversely, if $|||\cdot|||$ is a power-multiplicative Banach algebra norm on A with radical $\{0\}$, then it is said to be *uniform*.

Obviously, $|||\cdot|||_{\mathrm{sp}}$ is bounded from above by $|||\cdot|||$. It is important to note that in general $|||\cdot|||$ may not be bounded from above by $|||\cdot|||_{\mathrm{sp}}$. In other words, $|||\cdot|||_{\mathrm{sp}}$ may **not** be complete on $\mathcal{A}/\mathrm{rad}(\mathcal{A})$, hence the two norms may not be equivalent on this reduced algebra. (One says that $\mathcal{A}$ is a *Banach function algebra* if the two norms are indeed equivalent [8, Definition 3.8.3.1].) In general, one still has the following statement

Proposition 1.2.17. — $\mathfrak{M}(\mathcal{A})$ is canonically homeomorphic to $\mathfrak{M}(\mathcal{A}^u)$. ([3, Corollary 1.3.3, 1.3.4])

1.2.5. Banach module. — One can also consider seminorms on modules over Banach algebra. Let $\mathcal{A}$ be a Banach k-algebra. A *(semi)normed $\mathcal{A}$-module* is defined to be an A-module M with a (semi)norm $\|\cdot\|$ such that $(M, \|\cdot\|)$ is a (semi)normed vector space over k (denoted by $\mathcal{M}$), and that the multiplication is bounded, in the sense that there exists $C > 0$ such that

$$\forall a \in \mathcal{A},\ \forall m \in M, \quad \|a \cdot m\| \leqslant C|||a||| \cdot \|m\|$$

One calls a *Banach $\mathcal{A}$-module* a normed $\mathcal{A}$-module $(M, \|\cdot\|)$ whose norm is complete.

Let $\mathcal{M}_1 = (M_1, \|\cdot\|_1)$, $\mathcal{M}_2 = (M_2, \|\cdot\|_2)$ be Banach $\mathcal{A}$-modules and $\phi : M_1 \to M_2$ be a homomorphism of A-modules. It is called *bounded* if there exists $C > 0$ such that $\|\phi(m_1)\|_2 \leqslant C\|m_1\|_1$ for any $m_1 \in M_1$. In this case ϕ is said to be a *homomorphism of Banach $\mathcal{A}$-modules*, and is denoted by $\phi : \mathcal{M}_1 \to \mathcal{M}_2$. In addition, the homomorphism ϕ of Banach $\mathcal{A}$-modules is called *admissible* if it is admissible as linear map between normed-vector spaces over k (see Definition 1.1.3).

Definition 1.2.18. — Let $\mathcal{M}$ be a Banach $\mathcal{A}$-module. It is called a *Banach finite* $\mathcal{A}$-module if there exists $l \in \mathbb{N}_+$ and a surjective homomorphism of Banach $\mathcal{A}$-modules $\mathcal{A}^{\oplus l} \to \mathcal{M}$ where $\mathcal{A}^{\oplus l}$ is the Banach $\mathcal{A}$-module corresponding to the A-module $A^{\oplus l}$ equipped with the norm $(a_1, \ldots, a_l) \mapsto \max|||a_i|||$. (Note that such a homomorphism is necessarily admissible.)

Proposition 1.2.19. — Let $\mathcal{A}$ be a Banach k-algebra and $\mathcal{M}$ be a Banach $\mathcal{A}$-module. If A is Noetherian as a k-algebra and M is finitely generated as A-module, then any $\mathcal{A}$-sub-module of $\mathcal{M}$ is closed, and $\mathcal{M}$ is a Banach finite $\mathcal{A}$-module. ([29, Lemma 1.2.3]

Definition 1.2.20. — Let $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ be a homomorphism between Banach k-algebras. It is said to be *Banach finite* if $\mathcal{A}_2$ is a Banach finite $\mathcal{A}_1$-module. In this case $\mathcal{A}_2$ is called a Banach finite $\mathcal{A}_1$-algebra.

Remark 1.2.21. — Suppose that the absolute value $|\cdot|$ on k is non-trivial. If a k-Banach algebra homomorphism ϕ is finite as homomorphism of k-algebra, and $\mathcal{A}_1$ is Noetherian, then ϕ is automatically Banach finite: there is a surjective $\mathcal{A}_1$-module homomorphism $p : \mathcal{A}_1^{\oplus n} \to \mathcal{A}_2$, by Proposition 1.2.19 $\ker(p)$ is closed. Then p is continuous hence is admissible by Corollary 1.1.5. So $\mathcal{A}_2$ is a Banach finite $\mathcal{A}_1$-module.

1.3. Affinoid algebras

Affinoid algebras are a special kind of k-Banach algebras possessing good finiteness properties. These features allows one to endow a locally ringed space structure on their Berkovich spectra, namely the affinoid spaces. As a consequence, the Banach algebra norm of an affinoid algebra is equivalent to its spectral seminorm whenever the later is actually a norm.

1.3.1. Basic constructions. — Affinoid algebras are k-Banach algebras that are quotient algebras of Tate algebras. Among them are strict affinoid algebras which have good finiteness properties such as Noetherianity. Some good properties pass to general affinoid algebra by a technique enlarging the base valued field which makes the affinoid algebra strict.

Definition 1.3.1. — For a multi-radius $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$, the algebra

$$k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} = \{f = \sum_{J \in \mathbb{N}^n}^{\infty} a_J \boldsymbol{T}^J : a_J \in k, |a_J|\boldsymbol{r}^J \to 0 \text{ as } |J| \to \infty\}$$

is called the *Tate algebra* over k with multi-radius $\boldsymbol{r}$. Denote it by $\mathcal{T}_n(\boldsymbol{r})$. It is a k-Banach algebra with respect to the *Gauss norm of multi-radius* $\boldsymbol{r}$ defined by

$$|||f|||_{\mathcal{T}_n(\boldsymbol{r})} = \max_J |a_J|\boldsymbol{r}^J$$

If $\boldsymbol{r}$ is of the form $(1, \ldots, 1)$, the corresponding Tate algebra $\mathcal{T}_n(\boldsymbol{r})$ is also denoted by $\mathcal{T}_n$.

Remark 1.3.2. — This Gauss norm is obviously sub-multiplicative. It is in fact multiplicative by an argument as in the proof of Gauss Lemma.

Definition 1.3.3. — A k-Banach algebra $\mathcal{A}$ is called an *affinoid algebra* if there exists an admissible surjective homomorphism from some Tate algebra $k\{\boldsymbol{r}^{-1}\boldsymbol{T}\}$ to $\mathcal{A}$. The Banach algebra norm on an affinoid algebra $\mathcal{A}$ is called an *affinoid algebra norm*. If one can take $\boldsymbol{r}$ with $r_i = 1$ for all $i \in \{1, \ldots, n\}$, then $\mathcal{A}$ is called a *strict* affinoid algebra.

Remark 1.3.4. — An affinoid algebra norm and the quotient algebra norm of Gauss algebra norm by the defining admissible surjective homomorphism are just equivalent (see Theorem 1.1.4) but not necessarily equal.

Example 1.3.5. — The quotient Banach algebra of an affinoid algebra is an affinoid algebra.

Proposition 1.3.6. — Suppose that the absolute value $|\cdot|$ on k is non-trivial. Let $\mathcal{C}$ be a k-Banach algebra which is finite over an affinoid algebra $\mathcal{A}$, then $\mathcal{C}$ itself is an affinoid algebra. ([8, Proposition 6.1.1.5])

Proposition 1.3.7. — Let $\mathcal{B}$ be a Banach k-algebra with its Banach algebra norm $|||\cdot|||$. Suppose that there exists a finitely generated k-algebra B which is dense in $\mathcal{B}$, then there exists an affinoid algebra norm $|||\cdot|||^{\diamond}$ on B such that $|||\cdot|||^{\diamond} \geqslant |||\cdot|||$.

Proof. — Let $\{b_i\}_{i\in\{1,\dots,m\}}$ be any set of generators of B, and write B as a quotient of $k[T_1,\dots,T_m]$ by the homomorphism

$$k[T_1,\dots,T_m]\to\mathcal{B},\quad T_i\mapsto b_i$$

For any multi-index $J\in\mathbb{N}^m$ denote by $\boldsymbol{b}^J$ the monomial $\prod_{i\in\{1,\dots,m\}}(b_i)^{j_i}$. For each $i\in\{1,\dots,m\}$, denote $|||b_i|||$ by r_i and denote the multi-radius consisting of $\{r_i\}_{i\in\{1,\dots,m\}}$ by $\boldsymbol{r}$. Consider the Tate algebra norm $|||\cdot|||_{\mathcal{T}_{\boldsymbol{r}}}$ on $k[T_1,\dots,T_m]$, by the ultra-metricity of $|||\cdot|||$ and the definition of $\boldsymbol{r}$, one has

$$\forall J\in\mathbb{N}^m,\forall f_J\in k,|||\sum_{J\in\mathbb{N}^m}f_J\cdot\boldsymbol{b}^J|||_{\mathcal{B}}\leqslant|||\sum_{J\in\mathbb{N}^m}f_J\cdot\boldsymbol{b}^J|||_{\mathcal{T}_{\boldsymbol{r}}}.$$

One can take $|||\cdot|||^{\diamond}$ as the quotient norm of the restriction of $|||\cdot|||_{\mathcal{T}_{\boldsymbol{r}}}$ on $k[T_1,\dots,T_m]$ to B. □

To make an affinoid algebra strict, one can enlarge the base field.

Lemma 1.3.8. — Let $\boldsymbol{r}=(r_1,\dots,r_n)$ be a multi-radius such $\{r_i\}_{i\in\{1,\dots,n\}}$ are $(k,|\cdot|)$-free. Then the k-affinoid algebra

$$K_{\boldsymbol{r}}:=k\{\boldsymbol{r}^{-1}\boldsymbol{T},\boldsymbol{r}\boldsymbol{T}^{-1}\}=k\{\boldsymbol{r}^{-1}\boldsymbol{T},\boldsymbol{r}\boldsymbol{S}\}/(T_1S_1-1,\dots,T_nS_n-1)$$

is a field. ([3, Definition 2.1.1])

Lemma 1.3.9. — Let $\mathcal{T}_n(\boldsymbol{r})$ be a k-Tate alegbra. It is strict if and only if $r_i\in\sqrt{|k^\times|}$ for all i. ([3, Corollary 2.1.6])

Corollary 1.3.10. — Let $\mathcal{T}_n(\boldsymbol{r})$ be a k-Tate algebra. Let $I\subseteq\{1,\dots,n\}$ be a subset of indices such that $\{r_i\}_{i\in I}$ are $(k,|\cdot|)$-free and $|I|$ is maximal for this independence property. Let $\boldsymbol{r}_I=(r_{i_1},\dots,r_{i_1})$, then $K_{\boldsymbol{r}_I}\widehat{\otimes}_k\mathcal{T}_n(\boldsymbol{r})$ is a strict $K_{\boldsymbol{r}_I}$-Tate algebra.

Corollary 1.3.11. — For any k-affinoid algebra $\mathcal{A}$, there exists a multi-radius $\boldsymbol{r}_I=(r_i)_{i\in I}$ such that $\{r_i\}_{i\in I}$ are $(k,|\cdot|)$-free and $K_{\boldsymbol{r}_I}\widehat{\otimes}_k\mathcal{A}$ is a $K_{\boldsymbol{r}_I}$-strict affinoid algebra. ([3, Proposition 2.1.2])

1.3.2. Algebraic structures: Noetherianity. — Let $\mathcal{A}$ be a Banach k-algebra, one denotes by $\mathcal{A}^\circ$ the k°-algebra $\{f\in\mathcal{A}\mid|||f|||_{\mathcal{A},\mathrm{sp}}\leqslant 1\}$, and by $\mathcal{A}^{\circ\circ}$ the ideal of $\mathcal{A}^\circ$ constituting of elements $|||f|||_{\mathcal{A},\mathrm{sp}}<1$. The $\widetilde{k}$-algebra $\mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$ is called the *reduction* of $\mathcal{A}$. It can be shown that $\widetilde{\mathcal{T}_n}$ is isomorphic to $\widetilde{k}[T_1,\dots,T_n]$. ([8, Proposition 5.1.2.2])

Definition 1.3.12. — An element $f\in\mathcal{T}_n$ with $|||f|||_{\mathcal{T}_n}=1$ is said to be *regular in z_n of degree d* if its reduction $\bar{f}=\lambda(z_n)^d+\sum_{0\leqslant i\leqslant d-1}c_i(z_n)^{d-i}$ in $\bar{\mathcal{T}_n}$ where $\lambda\in k^\times$ and $c_i\in\bar{k}[z_1,\dots,z_{n-1}]$.

***Proposition 1.3.13* (Weierstrass division).** — Let $\mathcal{T}_n$ be the k-Tate algebra of multiradius $(1,\dots,1)$, then

(1) Let $f\in\mathcal{T}_n$ be an distinguished element in z_n of degree d, and $g\in\mathcal{T}_n$ be any element. Then there exists unique $r\in\mathcal{T}_{n-1}[z_n]$ of degree less than d in z_n and $q\in\mathcal{T}_n$ such that $g=q\cdot f+r$. Moreover $|||g|||_{\mathcal{T}_n}=\max\{|||q|||_{\mathcal{T}_n},|||r|||_{\mathcal{T}_n}\}$

(2) Let $f\in\mathcal{T}_n$ with $|||f|||_{\mathcal{T}_n}=1$. Then there exists a k-algebra automorphism τ of $\mathcal{T}_n$ such that $\tau(f)$ is regular in z_n.

([8, Theorem 5.2.1.2], [29, Theorem 3.1.1])

Proposition 1.3.14. — The Tate algebra $\mathcal{T}_n$ is Noetherian. All of its ideals are closed. ([8, Theorem 5.2.6.1, Corollary 5.2.7.2], [29, Theorem 3.2.1])

Corollary 1.3.15. — Any affinoid algebra is Noetherian. All of its ideals are closed ([3, Propositon 2.1.3]).

***Proposition 1.3.16* (Noether normalization).** — For strict affinoid algebra $\mathcal{A}$, there exists an injective and admissible Banach algebra homomorphism $\mathcal{T}_d \to \mathcal{A}$ for some $d > 0$, which endows $\mathcal{A}$ with a structure of *finite* algebra over $\mathcal{T}_d$. Moreover, d equals the Krull dimension of $\mathcal{A}$. ([8, Theorem 6.1.2.1], [29, Theorem 3.2.1])

Corollary 1.3.17. — Let $\mathfrak{m}$ be a maximal ideal of strict affinoid algebra $\mathcal{A}$, then $\mathcal{A}/\mathfrak{m}$ is a finite extension of k.

Proposition 1.3.18. — Let $\gamma : \mathcal{T}_d \to \mathcal{B}$ be an integral torsion-free homomorphism of Banach algebras. Then for each $b \in \mathcal{B}$, one has

$$|||b|||_{\mathcal{B},\mathrm{sp}} = \max_{1\leqslant i\leqslant n} |||a_i|||_{\mathcal{T}_d,\mathrm{sp}}^{1/i},$$

where $P(T) = T^n + a_1X^{n-1} + \cdots + a_b \in \mathcal{A}[T]$ is the minimal polynomial of b over $\mathcal{T}_d$. ([8, Proposition 3.8.2.5])

Corollary 1.3.19. — Let $\gamma : \mathcal{A} \to \mathcal{B}$ be an integral homomorphism of affinoid algebras. Then the same formula holds as in this Proposition. ([8, Proposition 6.2.2.4])

1.3.3. Topological structures: the spectral norm. — The Gauss norm on Tate algebra is equal to its spectral norm. For a general strict redueced affinoid algebra, the Banach algebra norm is equivalent to its spectral seminorm, thanks to the compatibility of Banach algebra norms with algebraic structures.

One studies the spectral norm of the Tate algebra case by direct calculation. For any affinoid algebra $\mathcal{A}$, the maximal spectrum, denoted by $\mathrm{Max}(\mathcal{A})$ is the topological subspace of $\mathfrak{M}(\mathcal{A})$ consisting of points corresponding to maximal ideals. The sup norm on this subspace is denoted by $|||\cdot|||_{\mathcal{A},\mathrm{spM}}$

Proposition 1.3.20. — For any $f \in \mathcal{T}_n$, there exists $z \in \mathrm{Max}(\mathcal{T}_n)$ such that $|f|_z = |||f|||_{\mathcal{T}_n}$ ([8, Proposition 5.1.4.3]). On $\mathcal{T}_n$, the three norms are equal: $|||\cdot|||_{\mathcal{T}_n} = |||\cdot|||_{\mathcal{T}_n,\mathrm{sp}} = |||\cdot|||_{\mathcal{T}_n,\mathrm{spM}}$.

One then uses Noether normalization to investigate the spectral seminorm of general affinoid algebra.

Proposition 1.3.21. — Let $\mathcal{A}$ be a reduced strict affinoid algebra. Then its spectral norm $|||\cdot|||_{\mathcal{A},\mathrm{sp}}$ is a complete norm on $\mathcal{A}$. It is equivalent to the Banach algebra norm $|||\cdot|||_{\mathcal{A}}$. ([29, Theorem 3.4.9], [8, Theorem 6.2.4.1])

Corollary 1.3.22. — Let $\mathcal{A}$ be a reduced general affinoid algebra. Then there exists $C > 0$ such that $|||f||| \leqslant C|||f|||_{\mathrm{sp}}$ for all $f \in \mathcal{A}$. In particular, $|||\cdot|||_{\mathrm{sp}}$ is complete on $\mathcal{A}$, and is equivalent to $|||\cdot|||$. ([3, Proposition 2.1.4.ii])

1.3.4. Affinoid space as locally ringed space. — The Berkovich spectrum of affinoid algebras are called affinoid spaces. It is possible to put locally ringed space structures on them. The construction of structural sheaf goes first with a Grothendieck topology generated by closed compact subsets of affinoid domains, then passes to the canonical topology by a limit process approximating an open set by these compact sets.

1.3.4.1. *Affinoid domains and structural algebra.* —

Definition 1.3.23. — Let $\mathcal{A}$ be an affinoid algebra. An *affinoid domain* is a closed subset V of $\mathfrak{M}(\mathcal{A})$, which is homeomorphic to $(\iota_V)^\star(\mathfrak{M}(\mathcal{A}_V))$ for some affinoid algebra $\mathcal{A}_V$ and Banach algebra homomorphism $\iota_V : \mathcal{A} \to \mathcal{A}_V$, and satisfies the *universal mapping property*: for any Banach algebra homomorphism $\phi : \mathcal{A} \to \mathcal{C}$ between affinoid algebras with $\phi^\star(\mathfrak{M}(\mathcal{C})) \subseteq V$, there exists a unique Banach algebra homomorphism $\psi : \mathcal{A}_V \to \mathcal{C}$ with $\phi = \psi \circ \iota_V$

Proposition 1.3.24. — Let V be an affinoid domain in $\mathfrak{M}(\mathcal{A})$. Then V is homeomorphic to $\mathfrak{M}(\mathcal{A}_V)$. Moreover $\mathcal{A}_V$ is a flat $\mathcal{A}$-algebra. ([3, Proposition 2.2.4])

Proposition 1.3.25. — Let V be an affinoid domain in $\mathfrak{M}(\mathcal{A})$, and let W be an affinoid domain in V. Then W is an affinoid domain in $\mathfrak{M}(\mathcal{A})$.

Example 1.3.26. — Given $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_n)$ tuples of elements of $\mathcal{A}$, $p = (p_1, \dots, p_m) \in (\mathbb{R}_{>0})^m$ and $q = (q_1, \dots, q_n) \in (\mathbb{R}_{>0})^n$, the closed subset

$$V = \mathfrak{M}(\mathcal{A})(p^{-1}f, qg^{-1}) := \{z \in \mathfrak{M}(\mathcal{A}), |f_i|_z \leqslant p_i \ |g_j|_z \geqslant q_j\}$$

is an affinoid domain. The corresponding homomorphism of affinoid algebras is

$$\mathcal{A} \to \mathcal{A}_V = \mathcal{A}\{p_1^{-1}T_1, \dots, p_m^{-1}T_m, q_1S_1, \dots, q_nS_n\}/(T_i - f_i, g_jS_j - 1)$$

Such domains are called *Laurent domains*. If $n = 0$, they are called *Weierstrass domains*.

Example 1.3.27. — Given $f_1, \dots, f_m, g \in \mathcal{A}$ without common zero in $\mathfrak{M}(\mathcal{A})$, and $p = (p_1, \dots, p_m) \in (\mathbb{R}_{>0})^m$, the closed subset

$$V = \mathfrak{M}(\mathcal{A})(p^{-1}\frac{f}{g}) := \{z \in \mathfrak{M}(\mathcal{A}), |f_i|_z \leqslant p_i|g|_z\}$$

is an affinoid domain with its affinoid algebra given by

$$\mathcal{A}_V = \mathcal{A}\{p_1^{-1}T_1, \dots, p_m^{-1}T_m\}/(gT_i - f_i)$$

V is called a *rational domain*.

Proposition 1.3.28. — A finite intersection of affinoid domains is an affinoid domain. ([3, Remark 2.2.2.iv])

Corollary 1.3.29. — Any point $z \in \mathfrak{M}(\mathcal{A})$ has a fundamental system of (closed) neighbourhoods consisting of affinoid domains. ([3, Proposition 2.2.3])

1.3.4.2. *Special domains and acyclicity of structural presheaf.* —

Definition 1.3.30. — The union of any finite family of affinoid domains in $\mathfrak{M}(\mathcal{A})$ is called a *special domain* in $\mathfrak{M}(\mathcal{A})$.

Definition 1.3.31. — The *Grothendieck topology* on $\mathfrak{M}(\mathcal{A})$ is the one with special domains as admissible open sets and finite coverings of special domain as admissible coverings. One denotes by $\mathfrak{M}(\mathcal{A})_G$ the space equipped with this G-topology.

Remark 1.3.32. — Indeed, by Proposition 1.3.28 and Proposition 1.3.25, the admissible coverings defined here verifies the required axioms.

Definition 1.3.33. — Let $\mathfrak{V}$ be an admissible covering of $\mathfrak{M}(\mathcal{A})$ by affinoid domains $\{V_i\}_{i\in I}$, where I is a finite set. Then for a Banach finite $\mathcal{A}$-module $\mathcal{M}$, the Čech complex of $\mathcal{M}$ with respect to V_i is defined to be the complex of Banach $\mathcal{A}$-modules, equipped with the usual Čech differentials

$$C^\bullet(\mathcal{M},\mathfrak{V}):\ 0\to\mathcal{M}\xrightarrow{d_0}\prod_{J\in I,|J|=1}\mathcal{M}_J\xrightarrow{d_1}\prod_{J\in I,|J|=2}\mathcal{M}_J\xrightarrow{d_2}\dots,\ \mathcal{M}_J:=\widehat{\bigotimes}_{i\in J}\mathcal{A}_i\hat{\otimes}_{\mathcal{A}}\mathcal{M}$$

$$d_l:\prod_{|J|=l}a_J\mapsto\prod_{|J'|=l+1}b_{J'},\ b_{J'}=\sum_{i\in I,J'=J\cup\{i\}}(-1)^i a_J|_{V_i}$$

The exactness of the complex $C^\bullet(\mathcal{M},\mathfrak{V})$ allows for a standard construction of a structural sheaf on $\mathfrak{M}(\mathcal{A})_G$.

Theorem 1.3.34. — Let $\mathcal{A}$ be a strict affinoid algebra and $\mathfrak{V}$ an admissible covering by strict affinoid domains for $\mathfrak{M}(\mathcal{A})$. Then the differentials d_l have closed images and the complex $C^\bullet(\mathcal{A},\mathfrak{V})$ is exact. ([8, Proposition 8.2.2.5])

Corollary 1.3.35. — For general affinoid domain $\mathfrak{M}(\mathcal{A})$ with general affinoid domains covering $\mathfrak{V}$, the complex $C^\bullet(\mathcal{A},\mathfrak{V})$ is exact where the differentials have closed images. So is $C^\bullet(M,\mathfrak{V})$ for any finite Banach $\mathcal{A}$-module M. ([3, Proposition 2.2.5]).

Definition 1.3.36. — Let V be any special domain in $\mathfrak{M}(\mathcal{A})$. Fix a way of writing V as $\bigcup_{i\in I}V_i$ where I is a finite set and $V_i=\mathfrak{M}(\mathcal{A}_{V_i})$ are affinoid algebras, one associated to V the k-algebra

$$\mathcal{A}_V:=\ker(\prod_{i\in I}\mathcal{A}_{V_i}\to\prod_{i,j\in I}\mathcal{A}_{V_i\cap V_j}),$$

which is a Banach algebra equipped with the sup norm on V. The *structural pre-sheaf of affinoid algebras* $\mathscr{O}_{\mathfrak{M}(\mathcal{A})_G}$ on $\mathfrak{M}(\mathcal{A})_G$ (**with respect to the G-topology**) is the one assigning V the k-Banach algebra $\mathcal{A}_V$. It is a sheaf thanks to Corollary 1.3.35.

Remark 1.3.37. — The k-Banach algebra $\mathscr{O}_{\mathfrak{M}(\mathcal{A})_G}(V)$ does not depend on the way of being a union of affinoid domains.

Definition 1.3.38. — For any open subset U of $\mathfrak{M}(\mathcal{A})$, let $\mathscr{O}_{\mathfrak{M}(\mathcal{A})}$ be the pre-sheaf of k-algebras (**with respect to the canonical topology**) which assigns U the limit

$$\mathscr{O}_{\mathfrak{M}(\mathcal{A})}(U):=\varprojlim_{V\subset U,\text{V special domain}}\mathcal{A}_V$$

It is also a sheaf thanks to the compactness of special domains under canonical topology. This is called the *structural sheaf* of $\mathfrak{M}(\mathcal{A})$.

Proposition 1.3.39. — $\mathscr{O}_{\mathfrak{M}(\mathcal{A})}$ is a sheaf of local rings. The topological space $\mathfrak{M}(\mathcal{A})$ has a structure of locally ringed space given by the sheaf $\mathscr{O}_{\mathfrak{M}(\mathcal{A})}$. ([3, Section 2.3])

1.3.4.3. *Change of base valued field.* — Let $\mathcal{A}$ be a k-affinoid algebra and X be the affinoid space $\mathfrak{M}(\mathcal{A})$. Let $(k', |\cdot|')$ be a ultrametric valued field extension of $(k, |\cdot|)$. Denote by $\mathcal{A}'$ the k'-affinoid algebra $\mathcal{A}\widehat{\otimes}_k k'$, it has also a structure of k-Banach algebra. Denote by $i_{k'/k} : \mathcal{A} \to \mathcal{A}'$ the canonical homomorphism of k-Banach algebras.

Definition 1.3.40. — The base valued field change of the affinoid space X is the k'-affinoid space $\mathfrak{M}(\mathcal{A}')$, denoted by X'. The base change map is defined as canonical homomorphism of topological spaces $X' \to X$ induced by $i_{k'/k}$ is denote by $\pi_{k'/k}$.

Proposition 1.3.41. — The map $\pi_{k'/k}$ is surjective.

Proof. — The inverse image of $x \in X$ is the image of $\mathfrak{M}(\widehat{\kappa}(x)\widehat{\otimes}\mathcal{A})$ in X', which is a non-empty subset by Proposition 1.2.6. □

1.4. Spectral calculus

Gelfand-Shilov theory allows one to do multi-variable spectral calculus for (commutative) Banach algebras over $\mathbb{C}$. In particular, one can localize a homomorphism between Banach algebras onto a neighbourhood of its spectrum. Similar theory, as developed in [3, Chapter 7], exists in the non-Archimedean base field setting.

1.4.1. Holomorphic convex envelope. — The holomorphic convexity of spectrum of a homomorphism of Banach k-algebra depends on the denseness of its image. In case where the spectrum of a homomorphism is not holomorphic convex, one can add variables to the source algebra so that spectrum of extended homomorphism is holomorphically convex.

Definition 1.4.1. — Let $\mathcal{A}$ and $\mathcal{B}$ be Banach k-algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach algebras. We call *spectrum of homomorphism* ϕ and we denote by Σ_ϕ the image of $\mathfrak{M}(\mathcal{B})$ in $\mathfrak{M}(\mathcal{A})$ under $\phi^\star$.

Definition 1.4.2. — Let $\mathcal{A}$ be a Banach k-algebra. Let K be a compact subset of $\mathfrak{M}(\mathcal{A})$. The *holomorphic convex envelope* of K in $\mathfrak{M}(\mathcal{A})$ is defined as the subset

$$\widehat{K} := \{z \in \mathfrak{M}(\mathcal{A}) \mid \forall f \in \mathcal{A}, |f|_z \leqslant \sup_{z' \in K} |f|_{z'}\}$$

The subset K is said to be *holomorphically convex* if $\widehat{K} = K$.

Lemma 1.4.3. — The intersection of all Weierstrass neighbourhoods of K in $\mathfrak{M}(\mathcal{A})$ coincide with $\widehat{K}$. ([3, Proposition 2.6.1])

Proposition 1.4.4. — Let $\mathcal{A}$ be a k-affinoid algebra, $\mathcal{B}$ be a Banach k-algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach k-algebras. Let $\mathcal{B}'$ be the closed sub-algebra generated by the image of ϕ of $\mathcal{A}$ in $\mathcal{B}$ and let $\phi' : \mathcal{A} \to \mathcal{B}'$ be the restricted homomorphism. Then $\widehat{\Sigma_\phi} = \Sigma_{\phi'}$. ([3, Proposition 7.3.1])

Corollary 1.4.5. — Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach algebras from an affinoid algebra to a Banach algebra with dense image. Then Σ_ϕ is holomorphically convex.

One has the following analogue of Arens-Calderon theorem, which holomorphically convexifies the spectrum of a homomorphism of Banach k-algebras by adding variables on the source algebra.

Proposition 1.4.6. — Let $\mathcal{A}$ be a k-affinoid algebra, $\mathcal{B}$ be a Banach k-algebra. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach k-algebras. Then for any open neighbourhood U in $\mathfrak{M}(\mathcal{A})$ of the spectrum Σ_ϕ, there exists a homomorphism of Banach algebras extending ϕ

$$\phi^\uparrow : \mathcal{A}^\uparrow := \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$$

such that $\mathrm{pr}(\widehat{\Sigma_\phi}) \subseteq U$, where $\mathrm{pr} : \mathfrak{M}(\mathcal{A}^\uparrow) \to \mathfrak{M}(\mathcal{A})$ is the canonical map of projection. ([3, Proposition 7.3.3])

1.4.2. Holomorphic functional calculus. — It is easy to localize the homomorphism to holomorphic convex neighbourhood of its spectrum. For a spectrum of homomorphism which is not holomorphically convex, one uses Proposition 1.4.6 to localize the homomorphism to any neighbourhood of it.

Lemma 1.4.7. — Let $\phi : \mathcal{A} \to \mathcal{B}$ be a Banach algebra homomorphism from an affinoid algebra $\mathcal{A}$ to a Banach algebra $\mathcal{B}$. Then for any Laurent domain neighbourhood V of Σ_ϕ, ϕ extends to a unique Banach algebra homomorphism $\phi_V : \mathcal{A}_V \to \mathcal{B}$. ([3, Corollary 2.5.16])

Theorem 1.4.8. — Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach algebras from an affinoid algebra to a Banach algebra. Let $V \subseteq \mathfrak{M}(\mathcal{A})$ be any special domain containing Σ_ϕ. Then there exists a Banach algebra homomorphism

$$\theta_\phi : \Gamma(V, \mathscr{O}_{\mathfrak{M}(\mathcal{A})}) \to \mathcal{B}$$

satisfying $\phi = \theta_\phi \circ \iota_V$, where $\iota_V : \mathcal{A} \to \mathcal{A}_V = \Gamma(V, \mathscr{O}_{\mathfrak{M}(\mathcal{A})})$ is the Banach algebra homomorphism corresponding to the inclusion $V \subseteq \mathfrak{M}(\mathcal{A})$. ([3, Theorem 7.3.2])

1.5. Analytification of scheme of finite type

There is a construction of Berkovich spectrum for a k-algebra similar to the one for k-Banach algebra, giving rise to analytification of k-schemes of locally finite type, as developped in [3, Section 3.4].

1.5.1. Local situation. — For affine varieties, the topological space of its analytification is defined in the same way as the spectrum of Banach algebra, except that boundedness requirement of seminorms are dropped. They enjoy similar basic properties as the spectrum of Banach algebra. Proofs are of same spirit hence are omitted.

Definition 1.5.1. — Let $Z = \mathrm{Spec}(A_Z)$ be an affine k-scheme of finite type, where A_Z is a k-algebra of finite type. Its Berkovich analytification Z^{an} is the topological space constituting of all *multiplicative* seminorms $|\cdot|_z$ on A_Z as points and with the canonical topology (the weakest topology making every function $z \mapsto |f|_z$ continuous for any $f \in A_Z$). ([3, Remark 3.4.2])

Proposition 1.5.2. — There is a bijective map from the set of points of Z^{an} to the set pairs $(\mathfrak{p}, |\cdot|_z)$ where $\mathfrak{p} \in \mathrm{Spec}(A_Z)$ and $|\cdot|_z$ is an absolute value on the residue field at $\mathfrak{p}$.

Proposition 1.5.3. — Let $\phi : A_Z \to A_W$ be a homomorphism of k-algebras of finite type where $Z = \mathrm{Spec}(A_Z)$ and $W = \mathrm{Spec}(A_W)$. Then there is an induced continuous map $\phi^\star : W^{an} \to Z^{an}$, which sends a multiplicative seminorm $|\cdot|_w$ to $|\phi(\cdot)|_w$.

Proposition 1.5.4. — If ϕ is surjective, then $\phi^\star$ is injective and is a closed map; if ϕ is finite, then $\phi^\star$ is surjective. ([3, Proposition 3.46 (6)(7)])

Proposition 1.5.5. — Let $Z = \mathrm{Spec}(A_Z)$ be an affine k-variety, $|||\cdot|||$ an algebra norm on A_Z and $\mathcal{A}_Z$ be the k-Banach algebra obtained by completing A_Z with respect to $|||\cdot|||$. Then the canonical homomorphism of k-algebras from A_Z to $\mathcal{A}_Z$ induces a continuous map which embeds the Berkovich spectrum $\mathfrak{M}(\mathcal{A}_Z)$ into Z^{an} as a compact subspace (and is closed since Z^{an} is Hausdorff), and the Berkovich topology coincides with the induced topology from Z^{an}.

Proof. — For any $z \in \mathfrak{M}(\mathcal{A}_Z)$, the multiplicative algebra seminorm (or the corresponding character) χ_z on $\mathcal{A}_Z$ corresponds to a unique multiplicative algebra seminorm on A_Z by restriction. Since A_Z is dense in $\mathcal{A}_Z$, the family of open sets $\{U(f; p, q),\ f \in A_Z,\ p, q \in \mathbb{R}\}$ form a basis for topology on $\mathfrak{M}(\mathcal{A}_Z)$, hence the inherited topology coincides with the originial topology. So the embedding is continuous, and the image of $\mathfrak{M}(\mathcal{A}_Z)$ is compact in Z^{an}. Since the topology on Z^{an} is Hausdorff, the image of $\mathfrak{M}(\mathcal{A}_Z)$ is closed. □

Definition 1.5.6. — An analytic function on open set $U \subseteq (\mathrm{Spec}\, A_Z)^{an}$ is a map $h : U \to \coprod_{z \in U} \hat{\kappa}(z)$ which is a local uniform limit of rational functions: every $z \in U$ has an open neighbourhood $U' \subseteq U$ such that for every $\epsilon > 0$, there exists $f_{U'}, g_{U'} \in A_Z$ with $|h - \frac{f_{U'}}{g_{U'}}|_z < \epsilon$ and $g(z) \neq 0$ for all $z \in U'$. Denote by $\mathscr{O}^{\mathrm{an}}(U)$ the k-algebra of all analytic functions on U.

Definition 1.5.7. — The structural sheaf $\mathscr{O}_{Z^{\mathrm{an}}}$ on Z^{an} is the one assigning $\mathscr{O}^{\mathrm{an}}(U)$ to an open set U.

Proposition 1.5.8. — The pair $(Z^{\mathrm{an}}, \mathscr{O}_{Z^{\mathrm{an}}})$ is a locally ringed space.

1.5.2. Global situation. — One can analytify a scheme of finite type defined over k by glueing local constructions.

Definition 1.5.9. — Let X be a scheme of finite type over $\operatorname{Spec} k$, and write X as $\bigcup X_i$ where $X_i = \operatorname{Spec} A_{X_i}$ are affine charts. The Berkovich analytification of $(X, \mathscr{O}_X)$ is defined as the locally ringed space obtained by gluing the Berkovich analytification $((X_i)^{an}, \mathscr{O}_{(X_i)^{an}})$ of each $(X_i, \mathscr{O}_{X_i})$.

Proposition 1.5.10. — Let $\phi : X \to Y$ be a morphism of schemes of locally finite type over $\operatorname{Spec} k$. Then it induces a continuous map $\phi^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. And ϕ is (1) separated, (2) injective, (3) surjective, (4) an open immersion and (5) an isomorphism if and only if ϕ^{an} has the same property. ([3, Proposition 3.4.6])

Theorem 1.5.11. — If X is proper, then X^{an} is Hausdorff and compact. ([3, Theorem 3.4.8])

CHAPTER 2

NORMED SECTION ALGEBRA AND ITS SPECTRUM

In this chapter, we lay the framework of our study of positively metrized line bundle.

Our basic principle is to study the positivity properties of a metrized line bundle $(\mathscr{L}, \phi)$ by globally defined objects, such as the global sections of $\mathscr{L}$ and the sup norm induced by ϕ. The algebraic geometry of an ample line bundle $\mathscr{L}$ is encoded by its ring of global sections; similarly a metric ϕ acquires positivity if it is constructed from some norms on the spaces of global sections. In practice, we gather the normed vector spaces of global sections and global norm induced by $(\mathscr{L}^{\otimes n}, n\phi)$, to form a normed algebra of sections $\widehat{R}_\bullet(\mathscr{L}, \phi)$.

The fondamental point of view is to interpret geometrically the positivity of a metrized line bundle as the holomorphic convexity of its dual unit disc bundle. We show that the Berkovich spectrum of the normed algebra of sections is a holomorphically convex set in the (analytification of) the Zariski spectrum of the algebra of sections. We relate this set with the dual unit disc bundle associated with the semipositive envelope metric $\mathcal{P}(\phi)$.

2.1. Section algebra and norms

In this section we provide the framework of our study. The first part collects basic algebro-geometric objects of study, namely the section algebra and its spectrum. The second part discusses the metrization of these algebras.

2.1.1. Section algebras. — Let k be a field. Let $\pi : X \to \operatorname{Spec} k$ be a scheme of finite type over $\operatorname{Spec} k$.

2.1.1.1. *Space of global sections.* — For any invertible $\mathscr{O}_X$-module $\mathscr{L}$ and any $n \in \mathbb{N}_{\geqslant 1}$, one denotes by $R_n(\mathscr{L})$ the k-vector space of global sections $H^0(X, \mathscr{L}^{\otimes n})$, and set $R_0(\mathscr{L})$ to be k. Denote by $R_\bullet(\mathscr{L})$ the graded algebra of sections $\bigoplus_{n\in\mathbb{N}} R_n(\mathscr{L})$. If Y is a subscheme of X, then one can restrict $\mathscr{L}$ to Y to get an invertible $\mathscr{O}_Y$-module which is denoted by $\mathscr{L}|_Y$. As above, we use the expression $R_\bullet(\mathscr{L}|_Y) = \bigoplus_{n\in\mathbb{N}} R_n(\mathscr{L}|_Y^{\otimes n})$ to denote the graded algebra of sections associated with $\mathscr{L}|_Y$.

If $\mathscr{L}^{\otimes n}$ is globally generated, namely the canonical homomorphism of $\mathscr{O}_X$-modules $\pi^*(H^0(X, \mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes}$ is surjective, the canonical homomorphism $\pi^*(H^0(X, \mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ induces a morphism of k-schemes

$$\iota_n : \ X \longrightarrow \mathbb{P}(H^0(X, \mathscr{L}^{\otimes n}))$$

2.1.1.2. *Pull back by a k-morphism.* — Let $f : Y \to X$ be a morphism of k-schemes. The morphisms of $\mathscr{O}_X$-modules $\mathscr{L}^{\otimes n} \to f_* f^*(\mathscr{L}^{\otimes n})$ induce linear maps of k-vector spaces $R_n(\mathscr{L}) \to R_n(f^*\mathscr{L})$ and a homomorphism of graded k-algebras $R_\bullet(\mathscr{L}) \to R_\bullet(f^*\mathscr{L})$ (for $n = 0$, we consider the identity map on k). For any $n \in \mathbb{N}$, we denote by $R_n(\mathscr{L}_{X|Y})$ the image of the canonical k-linear map

$$H^0(X, \mathscr{L}^{\otimes n}) \longrightarrow H^0(Y, f^*(\mathscr{L})^{\otimes n}).$$

Then

$$R_\bullet(\mathscr{L}_{X|Y}) := \bigoplus_{n\in\mathbb{N}} R_n(\mathscr{L}_{X|Y})$$

(with $R_0(\mathscr{L}_{X|Y})$ set to be k) forms a graded sub-k-algebra of $R_\bullet(f^*(\mathscr{L}))$.

2.1.1.3. *Total space and affine cone.* — One denotes by $\mathbf{V}(\mathscr{L})$ the scheme $\operatorname{Spec}(\operatorname{Sym}_{\mathscr{O}_X} \mathscr{L})$ over $\operatorname{Spec} k$, and by $\pi_{\mathscr{L}}$ the canonical morphism of k-schemes $\mathbf{V}(\mathscr{L}) \to X$. Let $\mathbb{O}_{\mathscr{L}} : X \to \mathbb{V}(\mathscr{L})$ be the closed immersion which corresponds to the homomorphism of $\mathscr{O}_X$-algebras

$$\operatorname{Sym}_{\mathscr{O}_X}(\mathscr{L}) = \bigoplus_{n\in\mathbb{N}} \mathscr{L}^{\otimes n} \longrightarrow \mathscr{O}_X$$

of projection to the homogeneous component of degree 0. Let $\mathbf{C}(\mathscr{L})$ be the affine cone $\operatorname{Spec} R_\bullet(\mathscr{L})$ of $\mathscr{L}$ and denote by $\varpi_{\mathscr{L}} : \mathbf{C}(\mathscr{L}) \to \operatorname{Spec} k$ the canonical morphism. One denotes by $\mathbf{0}_L : \operatorname{Spec} k \to \mathbf{C}(\mathscr{L})$ the rational point of $\mathbf{C}(\mathscr{L})$ corresponding to the maximal ideal

$$R_{\geqslant 1}(\mathscr{L}) := \bigoplus_{n\in\mathbb{N},\, n\geqslant 1} R_n(\mathscr{L}).$$

If U is an affine open subset of X on which $\mathscr{L}$ trivializes, then $\pi_{\mathscr{L}}^{-1}(U)$ identifies with

$$\operatorname{Spec}\Big(\bigoplus_{n\in\mathbb{N}} H^0(U, \mathscr{L}^{\otimes n})\Big).$$

Therefore the restriction maps $R_n \to H^0(U, \mathscr{L}^{\otimes n})$ induce a morphism of k-schemes from $\pi_{\mathscr{L}}^{-1}(U)$ to $\mathbf{C}(\mathscr{L})$. The gluing of such morphisms gives a morphism of k-schemes from $\mathbf{V}(\mathscr{L})$ to $\mathbf{C}(\mathscr{L})$, which we denote by $p_{\mathscr{L}}$. These schemes are related as follows.

Proposition 2.1.1. — Assume that X is separated and quasi-compact, and $\mathscr{L}$ is ample. Then the following diagram is commutative.

$$\begin{tikzcd}
X \arrow[r, "\mathbb{O}_{\mathscr{L}}"] \arrow[d, "\pi"'] & \mathbf{V}(\mathscr{L}) \arrow[r, "\pi_{\mathscr{L}}"] \arrow[d, "p_{\mathscr{L}}"] & X \arrow[d, "\pi"] \\
\operatorname{Spec} k \arrow[r, "\mathbf{0}_{\mathscr{L}}"'] & \mathbf{C}(\mathscr{L}) \arrow[r, "\varpi_{\mathscr{L}}"'] & \operatorname{Spec} k
\end{tikzcd}$$

Moreover, the restriction of $p_{\mathscr{L}}$ to $\mathbf{V}(\mathscr{L})\backslash\mathbb{O}_{\mathscr{L}}(X)$ is an open immersion, whose image is contained in $\mathbf{C}(\mathscr{L})\backslash\mathbf{0}_{\mathscr{L}}$. If X is proper, then $p_{\mathscr{L}}$ restricted to $\mathbf{V}(\mathscr{L})\backslash\mathbb{O}_{\mathscr{L}}(X)$ defines an isomorphism between $\mathbf{V}(\mathscr{L})\backslash\mathbb{O}_{\mathscr{L}}(X)$ and $\mathbf{C}(\mathscr{L})\backslash\mathbf{0}_{\mathscr{L}}$. ([33, Proposition 8.8.2, Remark 8.8.3])

In general, let $S_\bullet(\mathscr{L})$ be a k-subalgebra of $R_\bullet(\mathscr{L})$ with $S_0 = k$. We denote by $\mathbf{C}(\mathscr{L}_{S_\bullet})$ the k-scheme of affine cone $\operatorname{Spec} S_\bullet(\mathscr{L})$, and by $\mathbf{0}_{\mathscr{L}_{S_\bullet}} : \operatorname{Spec} k \to \mathbf{C}(\mathscr{L}_{S_\bullet})$ the rational point of $\mathbf{C}(\mathscr{L}_{S_\bullet})$ corresponding to the maximal ideal $S_{\geqslant 1}(\mathscr{L})$. The inclusion morphism of

algebras $S_\bullet(\mathscr{L}) \to R_\bullet(\mathscr{L})$ induces a morphism of affine k-schemes $\mathbf{C}(\mathscr{L}) \to \mathbf{C}(\mathscr{L}_{S_\bullet})$ which is denoted by $i_{S_\bullet}$. We can construct a morphism of k-schemes from $\mathbf{V}(\mathscr{L})$ to $\mathbf{C}(\mathscr{L}_{S_\bullet})$, which we denote by $p_{\mathscr{L}_{S_\bullet}}$, by gluing the following morphisms: if U is an affine open subset of Y on which $\mathscr{L}$ trivializes, then $\pi_{\mathscr{L}}^{-1}(U)$ identifies with

$$\operatorname{Spec}\Big(\bigoplus_{n\in\mathbb{N}} H^0(U, \mathscr{L}^{\otimes n})\Big).$$

Therefore the restriction maps $S_n(\mathscr{L}) \to H^0(U, \mathscr{L}^{\otimes n})$ induce a morphism of k-schemes from $\pi_{\mathscr{L}}^{-1}(U)$ to $\mathbf{C}(\mathscr{L}_{S_\bullet})$. From the composition of morphisms of k-algebras

$$S_\bullet(\mathscr{L}) \to R_\bullet(\mathscr{L}) \to \bigoplus_{n\in\mathbb{N}} H^0(U, \mathscr{L}^{\otimes n}),$$

one sees easily that the composition of $\pi_{\mathscr{L}}$ and $i_{S_\bullet}$ is $p_{\mathscr{L}_{S_\bullet}}$

$$\mathbf{V}(\mathscr{L}) \xrightarrow{\pi_{\mathscr{L}}} \mathbf{C}(\mathscr{L}) \xrightarrow{i_{S_\bullet}} \mathbf{C}(S_\bullet).$$

In general, if $R_\bullet$ is a graded k-algebra and $S_\bullet$ is a k-subalgebra of $R_\bullet$, we say that the inclusion homomorphism is *a (TN)-isomorphism* and that $S_\bullet$ is *(TN)-isomorphic* to $R_\bullet$ ([33, §2.7.2]) if there exists $N \in \mathbb{N}$ such that for any $n \geqslant N$, the inclusion $S_n \to R_n$ is an isomorphism of k-linear space.

As above, we consider the invertible $\mathscr{O}_Y$ module $\mathscr{L}|_Y$. We denote by $\mathbf{V}(\mathscr{L}|_Y)$ the scheme $\operatorname{Spec}_{\mathscr{O}_Y} \operatorname{Sym} \mathscr{L}|_Y$ and by $\pi_{\mathscr{L}|_Y}$ the canonical morphism of k-schemes $\mathbf{V}(\mathscr{L}|_Y) \to Y$. We denote by $\mathbf{C}(\mathscr{L}|_Y)$ the affine cone $\operatorname{Spec} R_\bullet(\mathscr{L}|_Y)$ and by $\mathbf{0}_{\mathscr{L}|_Y}$ the rational point of $\mathbf{C}(\mathscr{L}|_Y)$ coorsponding to the maximal ideal $R_{\geqslant 1}(\mathscr{L}|_Y)$. We denote by $p_{\mathscr{L}|_Y}$ the morphism from $\mathbf{V}(\mathscr{L}|_Y)$ to $\mathbf{C}(\mathscr{L}|_Y)$.

Moreover, we consider the subalgebra $R_\bullet(\mathscr{L}_{X|Y})$ of $R_\bullet(\mathscr{L}|_Y)$. We denote by $\mathbf{C}(\mathscr{L}_{X|Y})$ the affine cone $\operatorname{Spec} R_\bullet(\mathscr{L}_{X|Y})$, by $\mathbf{0}_{\mathscr{L}_{X|Y}}$ the rational point of $\mathbf{C}(\mathscr{L}_{X|Y})$ coorsponding to the maximal ideal $R_{\geqslant 1}(\mathscr{L}_{X|Y})$. The morphism $\mathbf{C}(\mathscr{L}|_Y) \to \mathbf{C}(\mathscr{L}_{X|Y})$ associated with the canonical inclusion $R_\bullet(\mathscr{L}_{X|Y}) \to R_\bullet(\mathscr{L}|_Y)$ is denoted by $i_{\mathscr{L}_{X|Y}}$. If L is ample and X is projective, then $R_\bullet(\mathscr{L}_{X|Y})$ is an (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L}|_Y)$.

Lemma 2.1.2. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. The restriction of $i_{S_\bullet}$ on $\mathbf{C}(\mathscr{L})\backslash\mathbf{0}$ has image in $\mathbf{C}(\mathscr{L}_{S_\bullet})\backslash\mathbf{0}$, and it is an isomorphism of schemes

$$i_{S_\bullet} : \mathbf{C}(\mathscr{L}|_Y)\backslash\mathbf{0}_{\mathscr{L}|_Y} \to \mathbf{C}(\mathscr{L}_{S_\bullet})\backslash\mathbf{0}_{\mathscr{L}_{S_\bullet}}.$$

In particular, assume that L is ample, one can consider the (TN)-isomorphic subalgebra $R_\bullet(\mathscr{L}_{X|Y})$ of $R_\bullet(\mathscr{L}|_Y)$, then the morphism

$$i_{\mathscr{L}_{X|Y}} : \mathbf{C}(\mathscr{L}|_Y)\backslash\mathbf{0}_{\mathscr{L}|_Y} \to \mathbf{C}(\mathscr{L}_{X|Y})\backslash\mathbf{0}_{\mathscr{L}_{X|Y}}$$

is an isomorphism.

Proof. — Since the canonical inclusion is a (TN)-isomorphism, there exist $N \in \mathbb{N}$ such that $S_{\geqslant N}(\mathscr{L})$ is equal to $R_{\geqslant N}(\mathscr{L})$. For any $f \in R_{\geqslant 1}(\mathscr{L})$, one has $f^N \in R_{\geqslant N}(\mathscr{L})$, so $f^N \in S_{\geqslant N}(\mathscr{L})$. One consider the localization morphism of k-algebras induced by the canonical inclusion

$$\iota_{S_\bullet}(f) : S_\bullet(\mathscr{L})[f^{-N}] \to R_{\geqslant 1}(\mathscr{L})[f^{-N}] \simeq R_\bullet(\mathscr{L})[f^{-1}].$$

This an isomorphism of k-algebras. It is injective, and is surjective: as for any $a \in R_\bullet(\mathscr{L})$, one has $a \cdot f^N \in R_{\geqslant N}$, so $a \cdot f^N \in S_{\geqslant N}$ and $a = a \cdot f^N \cdot f^{-N} \in S_\bullet[f^{-N}]$. Hence the induced morphism of schemes

$$i_{S_\bullet}(f) : \operatorname{Spec}(R_\bullet(\mathscr{L})[f^{-1}]) \to S_\bullet(\mathscr{L})[f^{-N}]$$

is an isomorphism. Moreover, let $g \in R_{\geqslant 1}(\mathscr{L})$, one sees easily that the restrictions of $\iota(f)$ and $\iota(g)$ on $S_{\geqslant 1}(\mathscr{L})[f^{-N}][g^{-N}]$ are the same morphism of algebras

$$\iota(fg) : S_\bullet(\mathscr{L})[f^{-N}][g^{-N}] \to R_\bullet(\mathscr{L})[f^{-N}][g^{-N}] \simeq R_{\geqslant 1}(\mathscr{L})[f^{-1}][g^{-1}],$$

so the isomorphisms of schemes $\{i_{S_\bullet}(f)\}_{f \in R_{\geqslant 1}(\mathscr{L})}$ glue together into an isomorphism of schemes

$$\operatorname{Spec} R_\bullet(\mathscr{L}) \simeq \bigcup_{f \in R_{\geqslant 1}(\mathscr{L})} \operatorname{Spec} R_\bullet(\mathscr{L})[f^{-1}] \to \bigcup_{f^N \in S_{\geqslant N}(\mathscr{L})} \operatorname{Spec} S_\bullet(\mathscr{L})[f^{-N}] \simeq \operatorname{Spec} S_\bullet(\mathscr{L}).$$

□

2.1.1.4. *Fiber of total space.* — For any $x \in X$, we consider the base change morphism $\operatorname{Spec} \kappa(x) \to X$. We denote by $\mathscr{L}(x)$ the invertible sheaf $\mathscr{L} \otimes_{\mathscr{O}_X} \kappa(x)$ over $\operatorname{Spec} \kappa(x)$, which can be seen as a rank one $\kappa(x)$-free module. The fiber product $\mathbf{V}(\mathscr{L}) \otimes_{\mathscr{O}_X} \kappa(x)$ is an affine $\kappa(x)$-scheme whose coordinate ring is the graded $\kappa(x)$-algebra $\operatorname{Sym}_{\kappa(x)} \mathscr{L}(x)$. We denote by $\mathbf{V}(\mathscr{L})(x)$ this $\kappa(x)$-scheme and by $\pi_{\mathscr{L}(x)} : \mathbf{V}(\mathscr{L})(x) \to \operatorname{Spec} \kappa(x)$ the structural morphism, which inscribes into a cartesian diagram of morphism of k-schemes

$$\begin{tikzcd} \mathbf{V}(\mathscr{L})(x) \arrow[r] \arrow[d, "\pi_{\mathscr{L}(x)}"'] & \mathbf{V}(\mathscr{L}) \arrow[d, "\pi_L"] \\ \operatorname{Spec} \kappa(x) \arrow[r, "x"'] & X \end{tikzcd}$$

For any $x \in X$, we shall denote by $R_\bullet(\mathscr{L})(x)$ the graded $\kappa(x)$-algebra $R_\bullet(\mathscr{L}) \otimes_k \kappa(x)$. If U is an affine open subset of X which contains x, then the restriction morphism of graded k-algebras

$$R_\bullet(\mathscr{L}) \to \bigoplus_{n \in \mathbb{N}} H^0(U, \mathscr{L}^{\otimes n})$$

induces a morphism of $\kappa(x)$-algebras

$$\gamma_{\mathscr{L}}(x) : \ R_\bullet(\mathscr{L})(x) \longrightarrow \operatorname{Sym}_{\kappa(x)} \mathscr{L}(x).$$

Similarly, in considering $\mathscr{L}|_Y$, for any $y \in Y$, we denote by $\gamma_{\mathscr{L}|_Y}(y)$ the corresponding morphism of graded $\kappa(y)$-algebras. For any $e \in \mathscr{L}(x) \backslash \{0\}$, one can identify the graded $\kappa(x)$-algebra $\operatorname{Sym}_{\kappa(x)} \mathscr{L}(x)$ with the one-variate polynomial algebra $\kappa(x)[T]$ by identifying e with T.

In a similar way, for any subalgebra $S_\bullet(\mathscr{L})$ of $R_\bullet(\mathscr{L})$, we denote by $S_\bullet(\mathscr{L})(x)$ the graded $\kappa(x)$-algebra $S_\bullet(\mathscr{L}) \otimes_k \kappa(x)$, and by $\gamma_{\mathscr{L}_{S_\bullet}}$ the composition morphism of greded $\kappa(x)$-algebras

$$\gamma_{\mathscr{L}_{S_\bullet}} : \ S_\bullet(\mathscr{L})(x) \longrightarrow R_\bullet(\mathscr{L})(x) \longrightarrow \operatorname{Sym}_{\kappa(x)} \mathscr{L}(x).$$

In particular, for the subalgebra $R_\bullet(\mathscr{L}_{X|Y})$ of $R_\bullet(\mathscr{L}|_Y)$ and a point $y \in Y$, we denote by $\gamma_{\mathscr{L}_{X|Y}}(y)$ the corresponding morphism of $\kappa(y)$-algebra.

Proposition 2.1.3. — Assume that there exist $N \in \mathbb{N}$ such that for any $n \geqslant N$, $\mathscr{L}^{\otimes n}$ is globally generated. If $S_\bullet(\mathscr{L})$ is a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$, then for any $x \in X$, the morphism of $\kappa(x)$-algebra $\gamma_{\mathscr{L}_{S_\bullet}}(x)$ is a (TN)-isomorphism. In particular, $\gamma_{\mathscr{L}}(x)$, $\gamma_{\mathscr{L}|_Y}(y)$ and $\gamma_{\mathscr{L}_{X|Y}}(y)$ are (TN)-isomorphic.

Proof. — If $\mathscr{L}^{\otimes n}$ is globally generated, then for any $x \in X$, the $\kappa(x)$-linear map

$$\gamma_L : \ R_n(\mathscr{L})(x) \longrightarrow \mathscr{L}^{\otimes n}(x)$$

is surjective. So the composed linear map

$$\gamma_{\mathscr{L}_{S_\bullet}} : \ S_n(\mathscr{L})(x) \longrightarrow \mathscr{L}^{\otimes n}(x)$$

is also surjective for sufficiently large n. Since $\mathscr{L}^{\otimes n}(x)$ is of rank one and $S_n(\mathscr{L})(x)$ is of rank at most one, this linear map is an isomorphism for sufficiently large n.. So $\gamma_{\mathscr{L}_{S_\bullet}}$ is a (TN)-isomorphism. The rest of the assertion follows from this main one. □

In the rest of the thesis, we make the following assumptions for algebro-geometric data. Let X be an integral projective scheme over $\operatorname{Spec} k$ of pure dimension d, Y be a reduced closed sub-scheme of X with its canonical closed immersion $i_Y : Y \to X$, and $\mathscr{L}$ be an ample invertible $\mathscr{O}_X$-module. We fix an integer $M \in \mathbb{N}$ such that $\mathscr{L}^{\otimes M}$ is very ample and that, for any $n \geqslant M$, the restriction map from $R_n(\mathscr{L})$ to $R_n(\mathscr{L}|_Y)$ is surjective, namely $R_n(\mathscr{L}|_Y) = R_n(\mathscr{L}_{X|Y})$.

2.1.2. Global norms and metrics. — We consider additional metric structures on the algebraic objects, in order to measure their size. To study line bundles, there are basically two ways of metrization: either one puts a metric on the line bundle, measuring the length of any section at each point in a local manner, or one puts a norm on the space of global sections, measuring their sizes in a global manner on X.

Proposition 2.1.4. — Let $R_\bullet$ be a graded k-algebra. If $\|\cdot\|$ is an ultrametric algebra seminorm (see Definition 1.2.1) on $R_\bullet$, then the map $\|\cdot\|^+ : R_\bullet \to \mathbb{R}_{\geqslant 0}$, defined as

$$\|(s_n)_{n\in\mathbb{N}}\|^+ := \sup_{n\in\mathbb{N}} \|s_n\|,$$

is also an ultrametric algebra seminorm on $R_\bullet$. One has $\|\cdot\| \leqslant \|\cdot\|^+$, and $\|\cdot\|^+$ is the largest ultrametric algebra seminorm on $R_\bullet$ whose restriction on each homogeneous pieces R_n coincides with the restriction of $\|\cdot\|$.

Proof. — It's obvious that $\|\cdot\|^+$ is ultrametric. Let $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be elements in $R_\bullet$. Let n_1 and n_2 be the smallest integers such that

$$\|s_{n_1}\| = \sup_{n\in\mathbb{N}} \|s_n\|, \ \|t_{n_2}\| = \sup_{n\in\mathbb{N}} \|t_n\|.$$

Let $(r_n)_{n\in\mathbb{N}}$ be the product element $(r_n)_{n\in\mathbb{N}} \cdot (t_n)_{n\in\mathbb{N}}$. Thanks to the ultrametricity of $\|\cdot\|$, for any $m \in \mathbb{N}$, one has

$$\begin{aligned}\|r_m\| &= \|\sum_{n\in\mathbb{N}} s_n \cdot t_{m-n}\| \leqslant \sup_{n\in\mathbb{N}}\{\|s_n\| \cdot \|t_{m-n}\|\} \\ &\leqslant \|s_{n_1}\| \cdot \|t_{n_2}\| = \|(s_n)_{n\in\mathbb{N}}\|^+ \cdot \|(t_n)_{n\in\mathbb{N}}\|^+.\end{aligned}$$

From this inequality one deduces that $\|\cdot\|^+$ is sub-multiplicative. It's obvious that $\|1\|^+ = \|1\| = 1$. Hence $\|\cdot\|^+$ is an ultrametric algebra seminorm. Besides, if $\|\cdot\|'$ is any other

ultrametric algebra seminorm on $R_\bullet$ whose restriction on each homogeneous pieces R_n coincides with the restriction of $|||\cdot|||$, then one has $|||\cdot|||' \leqslant |||\cdot|||^+$ by the construction. $\square$

Definition 2.1.5. — We keep the notation of the above proposition. If the equality $|||\cdot||| = |||\cdot|||^+$ holds, we say that $|||\cdot|||$ is *graded*.

Definition 2.1.6. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded k-algebra. For any $n \in \mathbb{N}$, let $\|\cdot\|_n$ be a seminorm on R_n. We say that the family of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is *sub-multiplicative* if they satisfy the property

$$\forall\,(m,n) \in \mathbb{N}^2,\ s_m \in R_m,\ s_n \in R_n, \quad \|s_m \cdot s_n\|_{m+n} \leqslant \|s_m\|_m \cdot \|s_n\|_n; \quad \|1\|_0 = 1.$$

It is said to be *power-multiplicative* if

$$\forall\, n \in \mathbb{N},\ s_n \in R_n,\ l \in \mathbb{N}, \quad \|s_n^l\|_{ln} = \|s_n\|_n^l.$$

It is said to be *multiplicative* if

$$\forall\,(m,n) \in \mathbb{N}^2,\ s_m \in R_m,\ s_n \in R_n, \quad \|s_m \cdot s_n\|_{m+n} = \|s_m\|_m \cdot \|s_n\|_n; \quad \|1\|_0 = 1.$$

Proposition 2.1.7. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded k-algebra and $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be a sub-multiplicative family of ultrametric seminorms on $(R_n)_{n\in\mathbb{N}}$. Then the map $|||\cdot||| : R_\bullet \to \mathbb{R}_{\geqslant 0}$ defined by $|||(s_n)_{n\in\mathbb{N}}||| = \sup_{n\in\mathbb{N}} \|s_n\|_n$ is the unique graded ultrametric algebra seminorm on $R_\bullet$, whose restriction on R_n coincides with $\|\cdot\|_n$. It is power-multiplicative (resp. multiplicative) if and only if the norm family $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is power-multiplicative (resp. multiplicative).

Proof. — The unicity is obvious. To show the properties concerning multiplication, let $s = (s_n)_{n\in\mathbb{N}}$ and $t = (t_n)_{n\in\mathbb{N}}$ be two elements of $R_\bullet$ and $r = (r_n)_{n\in\mathbb{N}}$ be their product. One has

$$r_n = \sum_{m\in\{0,\dots,n\}} s_m \cdot t_{n-m}.$$

If the family $(\|\cdot\|)_{n\in\mathbb{N}}$ is sub-multiplicative, by the ultrametricity assumption on $\|\cdot\|_n$, one obtains that

$$\|r_n\|_n \leqslant \max_{m\in\{0,\dots,n\}} \|s_m \cdot t_{n-m}\|_n \leqslant \max_{m\in\{0,\dots,n\}} \|s_m\|_m \cdot \|t_{n-m}\|_{n-m},$$

so $|||r|||$ is bounded from above by $|||s||| \cdot |||t|||$.

If the family $(\|\cdot\|)_{n\in\mathbb{N}}$ is power-multiplicative, for any $l \in \mathbb{N}$, we already know that the norm $|||\cdot|||$ is sub-multiplicative, so

$$|||s^l||| \leqslant |||s|||^l.$$

Let $n_0 \in \mathbb{N}$ be the smallest integer for which $\|s_{n_0}\|_{n_0} = \max_{n\in\mathbb{N}} \|s_n\|_n$, hence $|||s||| = \|s_{n_0}\|_{n_0}$. By the choice of n_0, for any $(j_0, \dots, j_m) \in \mathbb{N}^{m+1}$ such that

$$\sum_{i\in\{0,\dots,m\}} j_i = l, \qquad \sum_{i\in\{0,\dots,m\}} i \cdot j_i = ln_0$$

one has

$$\Big\| \prod_{i\in\{0,\dots,m\}} (s_i)^{j_i} \Big\|_{ln_0} \leqslant \|(s_{n_0})^l\|_{ln_0}$$

and the equality holds if and only if $(j_0, \dots, j_m) = (0, \dots, 0, l, 0, \dots, 0)$ where l is on the n_0-th place for the exponent of s_{n_0}, so by the construction of $|||\cdot|||$ and the fact that it is ultrametric, one gets

$$|||s^l||| \geqslant \|(s^l)_{ln_0}\|_{ln_0} \geqslant \|(s_{n0})^l\|_{ln_0} = \|s_{n_0}\|^l = |||s|||^l,$$

hence the equality $|||s^l||| = |||s|||^l$ holds, and $|||\cdot|||$ is power-multiplicative.

If the family $(\|\cdot\|)_{n\in\mathbb{N}}$ is multiplicative, one already knows that

$$|||r||| \leqslant |||s||| \cdot |||t|||.$$

To see the reverse inequality, let i_0 and j_0 be two smallest elements of $\mathbb{N}$ verifying the following property

$$\|s_{i_0}\|_{i_0} = \max_{n\in\mathbb{N}} \big\{\|s_n\|_n\big\},\ \|t_{j_0}\|_{j_0} = \max_{n\in\mathbb{N}} \big\{\|t_n\|_n\big\}.$$

Then by the choice of i_0 and j_0 and the fact that the family is multiplicative, for any $(i, j) \in \mathbb{N}^2$ satisfying $i + j = i_0 + j_0$, one has

$$\|s_i \cdot t_j\|_{i_0+j_0} = \|s_i\|_i \cdot \|t_j\|_j \leqslant \|s_{i_0}\|_{i_0} \cdot \|t_{j_0}\|_{j_0},$$

and the equality holds only if $(i, j) = (i_0, j_0)$. Since $\|\cdot\|_n$ is ultrametric, one has

$$\|r_{i_0+j_0}\|_{i_0+j_0} = \Big\|\sum_{i+j=i_0+j_0} s_i \cdot t_j\Big\|_{i_0+j_0} = \|s_{i_0}\|_{i_0} \cdot \|t_{j_0}\|_{j_0}.$$

By the construction, one has

$$|||s \cdot t||| \geqslant \|r_{i_0+j_0}\|_{i_0+j_0} = \|s_{i_0}\|_{i_0} \cdot \|t_{j_0}\|_{j_0} = |||s||| \cdot |||t|||.$$

So $|||\cdot|||$ is multiplicative. □

Proposition 2.1.8. — Let S, R and T be k-algebras, and let $g : R \to T$ be a surjective homomorphism of algebras, $f : S \to R$ be an injective homomorphism of algebras. Let $|||\cdot|||_R$ be an ultrametric algebra seminorm on R. Then the quotient seminorm $|||\cdot|||_T$ on T induced by g, and the sub-seminorm $|||\cdot|||_S$ on S induced by f is also an ultrametric algebra seminorm.

Proof. — The ultrametricity of seminorm is inherited by the quotient one or the sub one. It suffices to verify the sub-multiplicativity. The sub-seminorm is obviously sub-multiplicative. For the quotient seminorm, let t_1 and t_2 be elements in T. Then by the definition of quotient norm, for any $\epsilon \in \mathbb{R}_{>0}$ one can find s_1 and s_2 in S such that

$$f(s_i) = t_i,\ |||s_i|||_S \leqslant \mathrm{e}^{\epsilon}|||t_i|||_T,\ i = 1, 2.$$

Thus one has

$$|||t_1 \cdot t_2|||_T \leqslant |||s_1 \cdot s_2|||_S \leqslant |||s_1|||_S \cdot |||s_2|||_S \leqslant \mathrm{e}^{2\epsilon}|||t_1|||_T \cdot |||t_2|||_T,$$

let ϵ tends to 0, one gets the sub-multiplicativity of $|||\cdot|||_T$. □

Definition 2.1.9. — We keep the notation of the above proposition. The algebra norm $|||\cdot|||$ defined by $|||(s_n)_{n\in\mathbb{N}}||| = \sup_{n\in\mathbb{N}} \|s_n\|_n$ is called the *orthogonal sum* of the sub-multiplicative family $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of ultrametric norms on $(R_n)_{n\in\mathbb{N}}$.

Corollary 2.1.10. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded k-algebra and $|||\cdot|||$ be an ultrametric seminorm on $R_\bullet$. If $|||\cdot|||$ is sub-multiplicative (resp. power-multiplicative, resp. multiplicative), then also is $|||\cdot|||^+$.

2.1.3. Completion. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded k-algebra and $|||\cdot|||$ be an ultrametric seminorm on $R_\bullet$. Denote by $\widehat{R}(|||\cdot|||)$ the separated completion of $\bigoplus_{n\in\mathbb{N}} R_n$ with respect to $|||\cdot|||$. The seminorm $|||\cdot|||$ on $\bigoplus_{n\in\mathbb{N}} R_n$ induces by continuity an ultrametric norm on $\widehat{R}(|||\cdot|||)$ which we still denote by $|||\cdot|||$ by abuse of notation.

Proposition 2.1.11. — If $|||\cdot|||$ is an ultrametric and graded norm, the Banach algebra $\widehat{R}(|||\cdot|||)$ can be identified with the Banach algebra consisting of sequences $(s_n)_{n\in\mathbb{N}}$ of $\prod_{n\in\mathbb{N}} R_n$ such that the sequence $(|||s_n|||)_{n\in\mathbb{N}}$ converges to 0, with the norm given by $|||\cdot|||$.

Proof. — We shall construct a bijection which is an isomorphism of algebras. In fact, any element $(s_n)_{n\in\mathbb{N}}$ of $\prod_{n\in\mathbb{N}} R_n$ gives rise to a sequence

$$(s_0, \ldots, s_m, 0, \ldots, 0, \ldots), \quad m \in \mathbb{N}$$

in $R_\bullet$ which is a Cauchy sequence with respect to $|||\cdot|||$ if the sequence $(|||s_n|||)_{n\in\mathbb{N}}$ converges to zero thanks to the ultrametricity of $|||\cdot|||$. Conversely, let

$$(s_n^{(m)})_{n\in\mathbb{N}}, \quad m \in \mathbb{N}$$

be a Cauchy sequence of $R_\bullet$ with respect to $|||\cdot|||$. Since the norm $|||\cdot|||$ is graded, one deduces that, for any $n \in \mathbb{N}$, the sequence $(s_n^{(m)})_{m\in\mathbb{N}}$ is a Cauchy sequence in R_n, which converges to an element $s_n^{(\infty)}$ in R_n, and that the convergence is uniform for $n \in \mathbb{N}$. Then one can change the limit in the following calculation:

$$\lim_{n\to\infty} \|s_n^{(\infty)}\|_n = \lim_{n\to\infty} \lim_{m\to\infty} \|s_n^{(m)}\|_n = \lim_{m\to\infty} \lim_{n\to\infty} \|s_n^{(m)}\|_n = \lim_{m\to\infty} 0 = 0.$$

The third equality holds since $s^{(m)}$ is an element of $R_\bullet$ so it has finitely many homogeneous components $s_n^{(m)}$ which are non-zero. It is easy to verify that this bijection gives a homomorphism of algebras. □

2.1.4. Spectral seminorm and radical. — Recall that (see §1.2.12), from any algebra seminorm $|||\cdot|||$ on a k-algebra A, one can construct the spectral seminorm $|||\cdot|||_{\mathrm{sp}}$ on A as

$$\forall a \in A, \ |||a|||_{\mathrm{sp}} = \lim_{n\to\infty} |||a^n|||^{\frac{1}{n}},$$

it is power-multiplicative.

Lemma 2.1.12. — If $|||\cdot|||$ is ultrametric, then $|||\cdot|||_{\mathrm{sp}}$ is also ultrametric.

Proof. — For any $(a, b) \in A^2$, by the definition of spectral seminorm, for any $\epsilon > 0$, there exists $C_\epsilon \in \mathbb{R}_{>0}$ such that for any $n \in \mathbb{N}$ one has

$$|||a^n||| \leqslant C_\epsilon \cdot (|||a|||_{\mathrm{sp}} + \epsilon)^n, \ |||b^n||| \leqslant C_\epsilon \cdot (|||b|||_{\mathrm{sp}} + \epsilon)^n.$$

Then for any $\epsilon > 0$, one has

$$\begin{aligned}
\|\!|a+b|\!\|_{\mathrm{sp}} &= \lim_{n\to\infty} \|\!|(a+b)^n|\!\|^{\frac{1}{n}} = \lim_{n\to\infty} \|\!|\sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i|\!\|^{\frac{1}{n}} \\
&\leqslant \limsup_{n\to\infty} \big(\max_{i\in\{0,\dots,n\}} \big\{ \|\!|a^{n-i}|\!\| \, \|\!|b^i|\!\| \big\} \big)^{\frac{1}{n}} \\
&\leqslant \limsup_{n\to\infty} \max_{i\in\{0,\dots,n\}} \Big\{ \frac{n-i}{n} \|\!|a^{n-i}|\!\|^{\frac{1}{n-i}} + \frac{i}{n} \|\!|b^i|\!\|^{\frac{1}{i}} \Big\} \\
&\leqslant \limsup_{n\to\infty} \max_{i\in\{0,\dots,n\}} \Big\{ \frac{n-i}{n} C_\epsilon^{\frac{1}{n-i}} (\|\!|a|\!\|_{\mathrm{sp}} + \epsilon) + \frac{i}{n} C_\epsilon^{\frac{1}{i}} (\|\!|b|\!\|_{\mathrm{sp}} + \epsilon) \Big\} \\
&\leqslant \max\big\{ \|\!|a|\!\|_{\mathrm{sp}}, \|\!|b|\!\|_{\mathrm{sp}}, \epsilon \big\}
\end{aligned}$$

where the second inequality comes from the inequality of arithmetic and geometric means. Thus $\|\!|\cdot|\!\|_{\mathrm{sp}}$ is ultrametric. □

In view of Proposition 2.1.7, one obtains that $\|\!|\cdot|\!\|_{\mathrm{sp}}$ is ultrametric and power-multiplicative.

Proposition 2.1.13. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be an ultrametric graded k-algebra and $\|\!|\cdot|\!\|$ be a graded algebra seminorm on $R_\bullet$. Then the kernel $\mathfrak{n}(\|\!|\cdot|\!\|)$ and the radical $\mathrm{rad}(\|\!|\cdot|\!\|)$ (see Definition 1.2.15) are both homogeneous ideals of $R_\bullet$. Moreover, the quotient seminorm of $\|\!|\cdot|\!\|$ on $R_\bullet/\mathfrak{n}(\|\!|\cdot|\!\|)$ is also graded.

Proof. — Let $s = (s_n)_{n\in\mathbb{N}}$ be an element in the kernel. Then

$$0 = \|\!|s|\!\| = \max_{n\in\mathbb{N}} \big\{ \|s_n\|_n \big\},$$

so for every $n \in \mathbb{N}$, s_n is also in the kernel. Hence the kernel is a homogeneous ideal.

Let $s = (s_n)_{n\in\mathbb{N}}$ be an element in the radical. Let $i_0 \in \mathbb{N}$ be the smallest integer such that $s_i \neq 0$. For any $l \in \mathbb{N}$, since $\|\!|\cdot|\!\|$ is graded, one has

$$\|\!|s^l|\!\| \geqslant \|s_{i_0}^l\|_{i_0 l} = \|\!|s_{i_0}^l|\!\|,$$

let l tends to ∞, one obtains that

$$0 = \lim_{l\to\infty} \|\!|s^l|\!\| \geqslant \lim_{l\to\infty} \|\!|s_{i_0}^l|\!\| \geqslant 0.$$

Hence s_{i_0} is also in the radical, and so is $s - s_{i_0}$. By induction one sees that any s_n is in the radical. So the radical is a homogeneous ideal. □

Furthermore, if $\|\!|\cdot|\!\|$ is graded, so is $\|\!|\cdot|\!\|_{\mathrm{sp}}$. This follows from the interpretation of spectral seminorm as the supremum seminorm over the spectrum of the completion Banach algebra (Theorem 1.2.14).

Lemma 2.1.14. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded k-algebra and $\|\!|\cdot|\!\|$ be a graded algebra seminorm on $R_\bullet$. Let $|\cdot|_z$ be an ultrametric multiplicative seminorm on $R_\bullet$. Then $|\cdot|_z^+$ is also an ultrametric multiplicative seminorm. If $|\cdot|_z$ is bounded from above by $\|\!|\cdot|\!\|$, so is $|\cdot|_z^+$. If $|\cdot|_z$ is a point in $\mathfrak{M}(\widehat{R}_\bullet, \|\!|\cdot|\!\|)$, then $|\cdot|_z^+$ corresponds also to a point in $\mathfrak{M}(\widehat{R}_\bullet, \|\!|\cdot|\!\|)$.

Proof. — The first assertion follows from Corollary 2.1.10. For the second assertion, note that there exists $C > 0$ such that for any $s = (s_n)_{n\in\mathbb{N}}$, since $|\cdot|_z$ is bounded by $|||\cdot|||$, one has

$$\forall n \in \mathbb{N},\ |s_n|_z \leqslant C \cdot \|s_n\|_n,$$

since $|||\cdot|||$ is graded, one has

$$|s|_z^+ = \max_{n\in\mathbb{N}} |s_n|_z \leqslant C \cdot \max_{n\in\mathbb{N}} \|s_n\|_n = C \cdot |||s|||.$$

So $|\cdot|_z^+$ is also bounded from above by $|||\cdot|||$, and it corresponds to a point in $\mathfrak{M}(\widehat{R}_\bullet, |||\cdot|||)$. □

Proposition 2.1.15. — Let $R_\bullet = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded k-algebra and $|||\cdot|||$ be an ultrametric graded algebra seminorm on $R_\bullet$, then its spectral seminorm $|||\cdot|||_{\mathrm{sp}}$ is also graded.

Proof. — By Corollary 2.1.10, since $|||\cdot|||_{\mathrm{sp}}$ is ultrametric and power-multiplicative, the seminorm $|||\cdot|||_{\mathrm{sp}}^+$ is also power-multiplicative, and one has $|||\cdot|||_{\mathrm{sp}} \leqslant |||\cdot|||_{\mathrm{sp}}^+$.

For the reverse inequality, for $s = \sum_{i\in\{0,\dots,m\}} s_i \in R_\bullet$, let $n_0 \in \mathbb{N}$ be the smallest integer such that $|||s_{n_0}|||_{\mathrm{sp}}$ is maximal among all $\{|||s_n|||_{\mathrm{sp}}\}_{n\in\mathbb{N}}$. Then for any n one has $|||s_n|||_{\mathrm{sp}} \leqslant |||s_{n_0}|||_{\mathrm{sp}}$ and the inequality is strict for $n < n_0$. Since there are only finitely many terms in s, one can find $\rho < 1$ and $N \in \mathbb{N}$ and for any $\epsilon > 0$ a constant $C_\epsilon \in \mathbb{R}_{>0}$, such that

$$\forall i, \forall j_i \in \mathbb{N},\ |||s_i^{j_i}||| \leqslant C_\epsilon \cdot \mathrm{e}^{\epsilon j_i} |||s_{n_0}|||_{\mathrm{sp}}^{j_i}; \quad \forall i < n_0, \forall j_i \geqslant N,\ |||s_i^{j_i}||| \leqslant \rho^{j_i} \cdot |||s_{n_0}|||_{\mathrm{sp}}^{j_i}.$$

Take $l \in \mathbb{N}$ and consider the monomial terms in $(s^l)_{l\cdot n_0}$. They are of form $\prod s_i^{j_i}$ where $\sum_i j_i = l$ and $\sum_i i \cdot j_i = l n_0$. By the above estimates, for one such term one has

$$|||\prod_i s_i^{j_i}||| \leqslant \prod_i |||s_i^{j_i}||| = \prod_{i\in A} |||s_i^{j_i}||| \cdot \prod_{i\in B} |||s_i^{j_i}||| \leqslant C_\epsilon^{|A|} \cdot \mathrm{e}^{\epsilon \sum_{i\in A} j_i} \cdot \rho^{\sum_{i\in B} j_i} \cdot |||s_{n_0}|||_{\mathrm{sp}}^l \tag{1}$$

where B is the subset of i in $\{0, \dots, m\}$ such that $i < n_0$ and $j_i \geqslant N$, and A is its complement with $|A| \leqslant m + 1$. Write $j_{n_0} = l \cdot \sigma$ for some $\sigma \in [0, 1]$, then one has

$$\sum_{i<n_0} j_i + \sum_{i>n_0} j_i = l(1-\sigma),\ \sum_{i<n_0} i \cdot j_i + \sum_{i>n_0} i \cdot j_i = l(1-\sigma)n_0.$$

Each summand is non-negative, thus one gets

$$l(1-\sigma)n_0 \geqslant \sum_{i>n_0} i \cdot j_i \geqslant (n_0+1) \sum_{i>n_0} j_i \geqslant (n_0+1)[(1-\sigma)l - \sum_{i<n_0} j_i],$$

which implies that $\sum_{i<n_0} j_i \geqslant \frac{n_0}{n_0+1}(1-\sigma)l$. Thus one has $\sum_{i\in B} j_i \geqslant \frac{n_0}{n_0+1}(1-\sigma)l - (n_0 - 1) \cdot N$. Denote by τ the number $\frac{n_0}{n_0+2}(1-\sigma)$.

If $j_{n_0} < l$, namely $\sigma < 1$, then $\tau \neq 0$ and for large $l \in \mathbb{N}$, one has $\sum_{i\in B} j_i \geqslant \tau \cdot l$ thus $\sum_{i\in A} j_i \leqslant (1-\tau) \cdot l$. Choose ϵ such that $\mathrm{e}^{\epsilon(1-\tau)} \cdot \rho^\tau < 1$, thus for large l in (1) one gets $|||\prod_i s_i^{j_i}||| < |||s_{n_0}|||_{\mathrm{sp}}^l$. The term where $j_{n_0} = l$ is the single one $s_{n_0}^l$ and one has $|||s_{n_0}^l||| \geqslant |||s_{n_0}|||_{\mathrm{sp}}^l$. Thus by the ultrametricity one gets $|||(s^l)_{ln_0}||| = |||s_{n_0}^l|||$ for large l and by the gradedness of $|||\cdot|||$ one has

$$|||s^l||| \geqslant |||(s^l)_{ln_0}||| \geqslant |||s_{n_0}|||_{\mathrm{sp}}^l,$$

so $|||s|||_{\mathrm{sp}} \geqslant |||s_{n_0}|||_{\mathrm{sp}}$. Hence $|||\cdot|||_{\mathrm{sp}} \geqslant |||\cdot|||_{\mathrm{sp}}^+$. Finally $|||\cdot|||_{\mathrm{sp}}$ is graded. □

Proposition 2.1.16. — Let $|||\cdot|||$ be an ultrametric power-multiplicative graded seminorm on $R_\bullet = k[T]$. Then it identifies with the Gauss-Tate algebra norm of radius $|||T|||$, namely

$$\forall s = \sum_{n\in\mathbb{N}} \lambda_n \cdot T^n,\ |||s||| = \max_{n\in\mathbb{N}} \left\{|\lambda_n| \cdot |||T|||^n\right\}.$$

The spectrum $\mathfrak{M}(\widehat{R}_\bullet)$ is isomorphic to the closed disc of radius $|||T|||$ in $\operatorname{Spec} k[T]^{\mathrm{an}}$. A point $z \in \operatorname{Spec} k[T]^{\mathrm{an}}$ belongs to this disc if and only if $|T|_z \leqslant |||T|||$.

Proof. — For any $s \in k[T]$, since the norm is ultrametric and power-multiplicative, one has

$$|||s||| = \max_{n\in\mathbb{N}} \left\{|||s_n|||\right\} = \max_{n\in\mathbb{N}} \left\{|\lambda_n| \cdot |||T^n|||\right\} = \max_{n\in\mathbb{N}} \left\{|\lambda_n| \cdot |||T|||^n\right\}.$$

The second assertion follows easily from this identification. For the third assertion, z belongs to the disc if and only if $|\cdot| \leqslant C \cdot |||\cdot|||$. So

$$|T|_z = \lim_{n\to\infty} |T^n|^{\frac{1}{n}} \leqslant \lim_{n\to\infty} (C \cdot |||T^n|||)^{\frac{1}{n}} = |||T|||.$$

Conversely if $|T|_z \leqslant |||T|||$, then $|T|_s \leqslant |||s|||$ for any $s \in k[T]$ thanks to the ultrametricity of $|\cdot|_z$ and the gradedness of $|||\cdot|||$. □

Remark 2.1.17. — The above assertion is not true if one replaces $k[T]$ by $k[T_1, \ldots, T_d]$ for $d \geqslant 2$. In fact, by Remark 1.3.2, the Gauss-Tate norm is multiplicative, so it is equal to its spectral norm. They are graded with respect to the grading by total degree, and even better, graded with respect to the multi-grading by the multi-degree of $k[T_1, \ldots, T_n]$. The particularity of case $n = 1$ is that the gradings by total degree and by multi-degree coincide.

Corollary 2.1.18. — Let $R_\bullet = k[T]$ be the one-variate polynomial algebra and $S_\bullet \subseteq R_\bullet$ be a (TN)-isomorphic subalgebra. Let $|||\cdot|||$ be an ultrametric graded power-multiplicative algebra seminorm on $S_\bullet$, then one can find a unique ultrametric graded power-multiplicative algebra seminorm on $R_\bullet$, denoted by $|||\cdot|||'$, such that the restriction of $|||\cdot|||'$ on $S_\bullet$ is equal to $|||\cdot|||$.

Proof. — Denote by $\left\{\|\cdot\|_n\right\}_{n\in\mathbb{N}}$ the family of norms of restrictions of $|||\cdot|||$ on $\left\{S_n\right\}_{n\in\mathbb{N}}$. One can construct a family of seminorms $\left\{\|\cdot\|'_n\right\}_{n\in\mathbb{N}}$ on $\left\{R_n\right\}_{n\in\mathbb{N}}$ as follows. One can find n_0 such that $S_{n_0} = R_{n_0}$ and define the norms as follows

$$\|T\|'_1 := \|T^{n_0}\|_{n_0}^{\frac{1}{n_0}},\ \|T^n\|'_n := (\|T\|'_1)^n.$$

Since the family $\left\{\|\cdot\|_n\right\}_{n\in\mathbb{N}}$ is power-multiplicative, one can easily verify that this assignment of $\left\{\|\cdot\|'_n\right\}_{n\in\mathbb{N}}$ does not depend on the choice of n_0, and the new family of norms is still power-multiplicative. By Proposition 2.1.7, the orthogonal sum of this family is a power-multiplicative algebra seminorm, which we denote by $|||\cdot|||'$. Its restriction on $S_\bullet$ is indeed $|||\cdot|||$. □

Corollary 2.1.19. — With the same hypothesis and notations as above, let $|\cdot|_z$ be an ultrametric power-multiplicative seminorm on $R_\bullet$. Denote still by $|\cdot|_z$ its restriction on $S_\bullet$. Then $|\cdot|_z \leqslant |||\cdot|||$ on $S_\bullet$ if and only if $|T|_z \leqslant |||T|||'$, if and only if $|\cdot|_z \leqslant |||\cdot|||'$ on $R_\bullet$,

Proof. — For the direct implication, if $|\cdot|_z \leqslant |||\cdot|||$ on $S_\bullet$, then one has $|T^{n_0}|_z \leqslant |||T^{n_0}|||$, so $|T|_z \leqslant |||T|||'$. Since both seminorms are ultrametric and power-multiplicative, one has $|\cdot|_z \leqslant |||\cdot|||'$ on $R_\bullet$. The inverse implications are easy. □

2.1.5. Analytic spaces. —

2.1.5.1. *Analytification.* — Let k be equipped with a complete ultrametric absolute value $|\cdot|$. Let X^{an} the Berkovich analytic space associated with X and denote by $j_X : X^{\mathrm{an}} \to X$ the morphism of locally ringed spaces sending any $x \in X^{\mathrm{an}}$ to its associated scheme point (see §1.5 for a reminder on the construction of X^{an} and j_X). Remember that any analytic point $x \in X^{\mathrm{an}}$ corresponds to a pair $(j(x), |\cdot|_x)$ where $|\cdot|_x$ is an absolute value on the residue field $\kappa(j(x))$. For simplicity, we shall use the notation $\kappa(x)$ instead of $\kappa(j(x))$, and we denote by $\widehat{\kappa}(x)$ the completed residue field. If $\mathscr{L}$ is an invertible $\mathcal{O}_X$-module, we denote by $\mathscr{L}^{\mathrm{an}}$ the pull-back of $\mathscr{L}$ by j_X.

2.1.5.2. *Analytic total space and affine cone.* — Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For any $x \in X^{\mathrm{an}}$, one denotes by $\widehat{\kappa}(x)$ the completed residual field at x, and by $\mathscr{L}(x)$ the rank one $\widehat{\kappa}(x)$ vector space $\mathscr{L} \otimes_{\mathscr{O}_X} \widehat{\kappa}(x)$.

Proposition 2.1.20. — One has the following commutative diagram of morphisms of analytic spaces

$$\begin{tikzcd}
X^{\mathrm{an}} \arrow[r, "\mathbb{O}^{\mathrm{an}}_{\mathscr{L}}"] \arrow[d, "\pi^{\mathrm{an}}"'] & \mathbf{V}(\mathscr{L})^{\mathrm{an}} \arrow[r, "\pi^{\mathrm{an}}_{\mathscr{L}}"] \arrow[d, "p_L"] & X^{\mathrm{an}} \arrow[d, "\pi^{\mathrm{an}}"] \\
(\operatorname{Spec} k)^{\mathrm{an}} \arrow[r, "\mathbf{0}^{\mathrm{an}}_{\mathscr{L}}"'] & \mathbf{C}(\mathscr{L})^{\mathrm{an}} \arrow[r, "\varpi^{\mathrm{an}}_{\mathscr{L}}"'] & (\operatorname{Spec} k)^{\mathrm{an}}
\end{tikzcd}$$

Moreover, the restriction of $p^{\mathrm{an}}_{\mathscr{L}}$ to $\mathbb{V}(\mathscr{L})^{\mathrm{an}} \backslash \mathbb{O}^{\mathrm{an}}_{\mathscr{L}}(X)$ is an open immersion, whose image is contained in $\mathbf{C}(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}}$. If X is proper, then p_L restricted to $\mathbb{V}(\mathscr{L})^{\mathrm{an}} \backslash \mathbb{O}^{\mathrm{an}}_{\mathscr{L}}(X)$ defines an isomorphism between $\mathbb{V}(\mathscr{L})^{\mathrm{an}} \backslash \mathbb{O}^{\mathrm{an}}_{\mathscr{L}}(X)$ and $\mathbf{C}(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}}$.

Proof. — This follow from Proposition 2.1.1 and Proposition 1.5.10. □

Proposition 2.1.21. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. Then restriction of the morphism of k-analytic spaces

$$\operatorname{Spec} R_\bullet(\mathscr{L})^{\mathrm{an}} \to \operatorname{Spec} S_\bullet(\mathscr{L})^{\mathrm{an}}$$

on $\operatorname{Spec} R_\bullet(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}}$ has image in $\operatorname{Spec} S_\bullet(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$, and this restriction is an isomorphism of analytic spaces

$$\operatorname{Spec} R_\bullet(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}} \to \operatorname{Spec} S_\bullet(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$$

Proof. — This follows from Proposition 2.1.2 and Proposition 1.5.10. □

2.1.5.3. *Analytic fiber of total space.* — For any $x \in X^{\mathrm{an}}$, one can consider the base change map $x : \mathfrak{M}(\widehat{\kappa}(x)) \to X^{\mathrm{an}}$ and the corresponding fiber product $\mathbf{V}(\mathscr{L})^{\mathrm{an}} \otimes_{\mathcal{O}_X} \widehat{\kappa}(x)$ which is a $\widehat{\kappa}(x)$-analytic space given by the analytification of the algebraic spectrum of

the graded $\widehat{\kappa}(x)$-algebra $\mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x)$. The fiber product gives rise to a commutative diagram of morphism of analytic spaces

$$\begin{tikzcd} \mathbf{V}(\mathscr{L})^{\mathrm{an}}(x) \arrow[r] \arrow[d] & \mathbf{V}(\mathscr{L})^{\mathrm{an}} \arrow[d, "\pi^{\mathrm{an}}_{\mathscr{L}}"] \\ \mathfrak{M}(\widehat{\kappa}(x)) \arrow[r, "x"] & X^{\mathrm{an}} \end{tikzcd}.$$

For any $x \in X^{\mathrm{an}}$ and any subalgebra $S_\bullet(\mathscr{L})$ of $R_\bullet(\mathscr{L})$ we shall denote by $S_\bullet(\mathscr{L})(x)$ the graded $\widehat{\kappa}(x)$-algebra $S_\bullet(\mathscr{L}) \otimes_k \widehat{\kappa}(x)$.

Let $w \in \mathbf{V}(\mathscr{L})^{\mathrm{an}}$ be a point such that $\pi^{\mathrm{an}}_{\mathscr{L}}(w) = x$, then it determines a unique point in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}(x)$ in the following way. One can find an affine Zariski open set U of X containing the schematic point $j(x)$ on which $\mathscr{L}$ trivializes. Denote by $R_\bullet(\mathscr{L}|_U)$ the graded algebra

$$\bigoplus_{n\in\mathbb{N}} H^0(U, \mathscr{L}^{\otimes n}),$$

then $\pi^{\mathrm{an}}_{\mathscr{L}}$ is identified with $\operatorname{Spec} R_\bullet(\mathscr{L}|_U)^{\mathrm{an}}$. The point w corresponds to a multiplicative seminorm on the k-algebra $R_\bullet(\mathscr{L}|_U)$. Since $\pi^{\mathrm{an}}_{\mathscr{L}}(w) = x$, one knows that $\pi_{\mathscr{L}}(j(w)) = j(x)$, and that the restriction of $|\cdot|_w$ on $\kappa(j(w))$ to $\kappa(j(x))$ is $|\cdot|_x$. Hence w induces a unique multiplicative seminorm on the $\widehat{\kappa}(x)$-algebra

$$R_\bullet(\mathscr{L}|_U) \otimes_k \widehat{\kappa}(x) = \mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x).$$

By abuse of notation, we shall denote this point still by w. conversely, if w is a point in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}(x)$, it determines a unique point in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ which we still denote by w, as the multiplicative seminorm on $R_\bullet(\mathscr{L}|_U)$ which comes from the following composition

$$R_\bullet(\mathscr{L}|_U) \to \mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x) \xrightarrow{|\cdot|_w} \mathbb{R}_+$$

2.1.6. Metrics. —

2.1.6.1. *Pseudometrics as real valued function.* — Let $\mathcal{F}_X$ be the sheaf of real-valued functions on X^{an}. By *pseudometric* on $\mathscr{L}$ one refers to a morphism of sheaves of sets $\phi : \mathscr{L} \to j_{X,*}(\mathcal{F}_X)$ such that, for any $x \in X^{\mathrm{an}}$, the map $|\cdot|_\phi(x) : \mathscr{L}(x) \to \mathbb{R}$ induced by ϕ is a seminorm on the one-dimensional vector space $\mathscr{L}(x)$ over $\widehat{\kappa}(x)$. The pair $(\mathscr{L}, \phi)$ is called a *pseudometrized invertible $\mathscr{O}_X$-module.* If, for any $x \in X^{\mathrm{an}}$, the map $|\cdot|_\phi(x)$ is a norm, one says that ϕ is a *metric.* If s is a section of $\mathscr{L}$ on an open neighbourhood of x, then $|s(x)|_\phi(x)$ is denoted by $|s(x)|_\phi$ for simplicity.

We say that a pseudometric ϕ is *upper semi-continuous* (resp. *continuous*) if, for any Zariski open subset U of X and any section $s \in \Gamma(U, \mathscr{L})$, the function

$$(x \in U^{\mathrm{an}}) \longrightarrow |s(x)|_\phi$$

is upper semi-continuous (resp. continuous), or equivalently, in additive notion, the function

$$(x \in U^{\mathrm{an}}) \longrightarrow -\log|s(x)|_\phi$$

is lower-semicontinuous (resp. continuous).

2.1.6.2. *Pull-back by a morphism.* — Let $f : Y \to X$ be a morphism of separated k-schemes of finite type. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, equipped with a pseudometric ϕ. We define a pseudometric $f^*\phi$ on $f^*(\mathscr{L})$ such that, for any section s of L on a Zariski open subset U of X, one has

$$\forall\, y \in f^{-1}(U)^{\mathrm{an}}, \quad |f^*(s)|_{f^*\phi}(y) = |s|_\phi(f^{\mathrm{an}}(y)).$$

The pseudometric $f^*\phi$ is called the *pull-back* of ϕ by f. Since $f^{\mathrm{an}} : Y^{\mathrm{an}} \to X^{\mathrm{an}}$ is continuous (Proposition 1.5.10), if the metric ϕ is continuous, also is $f^*\phi$. If Y is a subscheme of X and if $f : Y \to X$ is the canonical immersion, then the pull-back metric $f^*\phi$ is also denoted by $\phi|_Y$.

2.1.6.3. *Metrics on the trivial line bundle.* — Any map $f : X^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$ determines a pseudometric τ_f on $\mathscr{O}_X$ such that, for any regular function a of X on a Zariski open subset U, one has (with the convention $\mathrm{e}^{-\infty} = 0$)

$$\forall\, x \in U^{\mathrm{an}}, \quad |a(x)|_{\tau_f} = |a(x)| \cdot \mathrm{e}^{-f(x)}.$$

Note that $f \mapsto \tau_f$ defines a bijection between the set of maps $X^{\mathrm{an}} \to \mathbb{R}_+ \cup \{+\infty\}$ and that of pseudometrics on $\mathscr{O}_X$, which maps the set of real-valued functions bijectively to that of pseudometrics on $\mathscr{O}_X$. Moreover, the pseudometric τ_f is continuous if and only if f is continuous on X^{an}. The trivial invertible sheaf $\mathscr{O}_X$ equipped with the pseudometric τ_f is denoted by $\mathscr{O}_X(f)$. The metric corresponding to the identically vanishing function is called the *trivial metric* on $\mathscr{O}_X$.

2.1.6.4. *Distance between two metrics.* — Let ϕ_1 and ϕ_2 be two metrics on $\mathscr{L}$. The *distance* of these two pseudometrics is a generalized positive real number (in $\mathbb{R}_+ \cup \{+\infty\}$) defined as

$$\mathrm{dist}(\phi_1, \phi_2) = \sup_{x \in X^{\mathrm{an}}} \left| \log \left| \frac{\phi_1(x)}{\phi_2(x)} \right|_x \right|.$$

If X is proper and ϕ_1, ϕ_2 are continuous metrics, then $\mathrm{dist}(\phi_1, \phi_2) \in \mathbb{R}_+$.

2.1.6.5. *Tensor product pseudometrics.* — Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be invertible $\mathscr{O}_X$-modules, and ϕ_1 and ϕ_2 be pseudometrics on $\mathscr{L}_1$ and $\mathscr{L}_2$ respectively. The pseudometrics ϕ_1 and ϕ_2 induce by passing to tensor product a pseudometric $\phi_1 + \phi_2$ on $\mathscr{L}_1 \otimes \mathscr{L}_2$ such that, For any Zariski open subset U of X and any $(s_1, s_2) \in \Gamma(U, \mathscr{L}_1) \times \Gamma(U, \mathscr{L}_2)$, one has

$$\forall\, x \in U^{\mathrm{an}}, \quad |(s_1 \cdot s_2)(x)|_{\phi_1+\phi_2} = |s_1(x)|_{\phi_1} \cdot |s_2(x)|_{\phi_2}.$$

If ϕ_1 and ϕ_2 are continuous, then $\phi_1 + \phi_2$ is also continuous.

In particular, for any $\epsilon \in \mathbb{R}$, we denote by $\phi(\epsilon)$ the pseudometric $\phi + \tau_{\mathrm{e}^\epsilon}$ on $\mathscr{L} \otimes \mathscr{O}_X \cong \mathscr{L}$.

Moreover, any metric ϕ on $\mathscr{L}$ determines by passing to its dual a metric $-\phi$ on $\mathscr{L}^\vee$ such that, for any Zariski open subset U of X and any $(s, \alpha) \in \Gamma(U, \mathscr{L}) \times \Gamma(U, \mathscr{L}^\vee)$, one has

$$\forall\, x \in U^{\mathrm{an}}, \quad |\alpha(s)(x)| = |\alpha(x)|_{-\phi} \cdot |s(x)|_\phi.$$

If the metric ϕ is continuous, so is $-\phi$. Let L be an invertible $\mathscr{O}_X$-module, and $n \in \mathbb{N}\backslash\{0\}$. A pseudometric ϕ on $\mathscr{L}$ determines by tensor power a pseudometric on $\mathscr{L}^{\otimes n}$ for any $n \in \mathbb{N}\backslash\{0\}$, denoted by $n\phi$. By convention, 0ϕ denotes the trivial metric on $\mathscr{L}^{\otimes 0} \cong \mathscr{O}_X$ (see §2.1.6.3 above).

Similarly, assume given a pseudometric ϕ on $\mathscr{L}^{\otimes n}$. We denote by $\frac{1}{n}\phi$ the pseudometric on $\mathscr{L}$ such that, for any Zariski open subset U of X and any section $s \in \Gamma(U, \mathscr{L})$, one has

$$\forall\, x \in U^{\mathrm{an}}, \quad |s(x)|_{\frac{1}{n}\phi} = |s^n(x)|_{\phi}^{1/n}.$$

If the pseudometric ϕ is continuous, then so is $\frac{1}{n}\phi$.

2.1.6.6. *Dual disc bundle.* — Consider the fiber of the canonical projection π^{an} over $x \in X^{\mathrm{an}}$, which is given by $(\mathrm{Spec}\ \mathrm{Sym}_{\widehat{\kappa}(x)} \mathscr{L}(x))^{\mathrm{an}}$. Any pseudometric ϕ determines a family of ultrametric seminorms $\{|\cdot|_{n\phi}(x)\}_{n\in\mathbb{N}}$ on the graded pieces $\{\mathscr{L}^{\otimes n}(x)\}_{n\in\mathbb{N}}$. This family is multiplicative, so it corresponds to a graded ultrametric multiplicative seminorm on $\mathrm{Sym}_{\widehat{\kappa}(x)} \mathscr{L}(x)$ by Proposition 2.1.7, denoted by $|||\cdot|||_{\phi}(x)$, which is a Gauss-Tate seminorm by Proposition 2.1.16. We denote by $\widehat{\mathrm{Sym}}_{\widehat{\kappa}(x)}(\mathscr{L}(x), \phi)$ the completion $\widehat{\kappa}(x)$-Banach algebra of $\mathrm{Sym}_{\widehat{\kappa}(x)} \mathscr{L}(x)$ with respect to this seminorm. We denote by $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi)(x)$ the $\widehat{\kappa}(x)$-affinoid domain $\mathfrak{M}(\mathrm{Sym}_{\widehat{\kappa}(x)} \mathscr{L}(x))$ which is a disc in the $\widehat{\kappa}(x)$-analytic space $\mathbf{V}(x)^{\mathrm{an}}$.

Remember that any $w \in \mathbf{V}(\mathscr{L})^{\mathrm{an}}$ determines a unique point in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$, still denoted by w, which is a multiplicative seminorm on $\mathrm{Sym}_{\widehat{\kappa}(x)} \mathscr{L}(x)$ (see §2.1.5.3). Let ϕ be a pseudometric on L, the following topological subspace of $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$

$$\{w \in \mathbf{V}(\mathscr{L})^{\mathrm{an}} : \forall e_1 \in \mathscr{L}(x), |e_1|_w \leqslant |e_1|_{\phi}\ (\textit{resp.} < |e_1|_{\phi})\}$$

equipped with induced topology, is called the *dual closed (resp. open) disc bundle of radius* 1 of the pseudometrized pair $(\mathscr{L}, \phi)$. We denote it by $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi)$ (resp. $\mathbb{D}^{\vee}(\mathscr{L}, \phi)$).

Proposition 2.1.22. — Consider the restriction of the continuous map π^{an} to the topological subspace $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi)$. For any $x \in X$, the topological fiber of the restriction of π^{an} coincides with the underlying topological space of $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi)(x)$.

Proof. — Let $w \in \mathbf{V}(\mathscr{L})^{\mathrm{an}}$ with $\pi^{\mathrm{an}}(w) = x$. Since $|||\cdot|||_{\phi}(x)$ is an ultrametric graded seminorms on $\mathrm{Sym}_{\widehat{\kappa}(x)} \mathscr{L}(x)$, one has $w \in \overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi)$ if and only if $|e_1|_w \leqslant |e_1|_{\phi}$, if and only if $w \in \overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi)(x)$ by Proposition 2.1.16. □

2.1.7. Fubini-Study metrics. —

2.1.7.1. *Fubini-Study metric associated with a norm.* — Let L be an invertible $\mathscr{O}_X$-module. Let $\|\cdot\|_n$ be a norm on $R_n(\mathscr{L})$. For any $x \in X^{\mathrm{an}}$, if the evaluation map

$$R_n(\mathscr{L}) \otimes_k \widehat{\kappa}(x) \longrightarrow \mathscr{L}^{\otimes n}(x)$$

is surjective, then it induces a quotient norm on the $\widehat{\kappa}(x)$-vector space $\mathscr{L}^{\otimes n}(x)$, from the scalar extension norm $\|\cdot\|_{n,\widehat{\kappa}(x)}$ of $\|\cdot\|_n$ on $R_n(\mathscr{L}) \otimes_k \widehat{\kappa}(x)$. The quotient norm is denoted by $\|\cdot\|_{n,X|x}$. If $\mathscr{L}^{\otimes n}$ is generated by global sections, the family $\{\|\cdot\|_{n,X|x}\}_{x\in X^{\mathrm{an}}}$ gives rise to a metric on $\mathscr{L}^{\otimes n}$, which we call the *Fubini-Study metric associated with* $\|\cdot\|_n$ *on* $R_n(\mathscr{L})$*, denoted by* $\mathrm{FS}(\|\cdot\|_n)(x)$*. The metric* $\frac{1}{n}\mathrm{FS}(\|\cdot\|_n)$ *on* $\mathscr{L}$ *is called the* normalized *Fubini-Study metric associated with* $\|\cdot\|_n$ on $R_n(\mathscr{L})$.

In particular, if the norm $\|\cdot\|_n$ is ultrametric and admit an ultrametric orthogonal basis, namely there is a basis $\{s_{n,j}\}_{j\in\{1,\dots,d_n\}}$ of $R_n(\mathscr{L})$ such that

$$\forall (\lambda_1, \dots, \lambda_{d_n}) \in k^{d_n},\ \|\sum_{j=1}^{d_n} \lambda_j \cdot s_{n,j}\|_n = \max_{j\in\{1,\dots,d_n\}} \|\lambda_j \cdot s_{n,j}\|_n,$$

then the corresponding Fubini-Study metric $\mathrm{FS}(\|\cdot\|_n)$ is said to be *diagonalizable* ([11]). Previously, such metrics are already studied and heavily used in [24].

Moreover, let $S_\bullet(\mathscr{L})$ be a subalgebra of $R_\bullet(\mathscr{L})$, equipped with a family of norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ on $\{W_n\}_{n\in\mathbb{N}}$. If for some $n\in\mathbb{N}$, the evaluation map

$$S_n(\mathscr{L})\otimes_k \widehat{\kappa}(x)\to \mathscr{L}^{\otimes n}(x)$$

is surjective, one can define the Fubini-Study metric $\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)$ on $\mathscr{L}$.

2.1.7.2. *Fubini-Study envelope metric associated with an algebra norm.* — Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module which is generated by global sections. Let $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ be a family of seminorms on $\{R_n(\mathscr{L})\}_{n\in\mathbb{N}}$. If the sequence of pseudometrics $\{\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)\}_{n\in\mathbb{N}}$ converges pointwisely to a limit pseudometric, we denote it by $\mathcal{P}(\{\|\cdot\|_n\}_{n\in\mathbb{N}})$ and call it the *Fubini-Study envelope* pseudometric associated with $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$.

In particular, let $|||\cdot|||$ be an algebra seminorm on $R_\bullet(\mathscr{L})$, and let $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ be the associated sub-multiplicative family of seminorms on $\{R_n(\mathscr{L})\}_{n\in\mathbb{N}}$. From these data, one can construct an envelope metric in two equivalent ways, in terms of family of norms of restriction to homogeneous components or in terms of the algebra norm itself:

— Since the family $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ is sub-multiplicative, also is the family $\{\|\cdot\|_{n,\widehat{\kappa}(x)}\}_{n\in\mathbb{N}}$. Then the family of quotient seminorms $\{\mathrm{FS}(\|\cdot\|_n)(x)\}_{n\in\mathbb{N}}$ is also sub-multiplicative by Proposition 2.1.7 and Proposition 2.1.8. Hence the norms $\{\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)(x)\}_{n\in\mathbb{N}}$ converges to a limit seminorm on $\mathscr{L}(x)$ by Fekete Lemma. This gives rise to a pseudometric on $\mathscr{L}$, called the *Fubini-Study envelope pseudometric associated with* $|||\cdot|||$. We denote it by $\mathcal{P}(|||\cdot|||)$, thus the limit seminorm is $\mathcal{P}(|||\cdot|||)(x)$. The Fubini-Study envelope metric is not necessarily continuous.

— For any $x\in X^{\mathrm{an}}$, the evaluation map induces a $\widehat{\kappa}(x)$-algebra homomorphism

$$R_\bullet(\mathscr{L})\otimes\widehat{\kappa}(x)\longrightarrow\bigoplus_{n\in\mathbb{N}}\mathscr{L}^{\otimes n}(x)=:R_\bullet(\mathscr{L})(x).$$

The above quotient algebra homomorphism induces a quotient algebra norm on $R_\bullet(\mathscr{L})(x)$, from the scalar extension norm $|||\cdot|||_{\widehat{\kappa}(x)}$ of $|||\cdot|||$ on $R_\bullet(\mathscr{L})\otimes\widehat{\kappa}(x)$. The quotient algebra norm is denoted by $|||\cdot|||_{X|x}$. Its spectral seminorm, denoted by $|||\cdot|||_{X|x;\mathrm{sp}}$, is power-multiplicative. As the inclusion of graded $\widehat{\kappa}(x)$-algebras

$$\gamma_{\mathscr{L}}:\ R_\bullet(\mathscr{L})(x)\to\mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x)$$

is a (TN)-isomorphism, by Fact 2.1.18, one can find a unique graded power-multiplicative seminorm on $\mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x)$, denoted by $|||\cdot|||'_{X|x;\mathrm{sp}}$, whose restriction on $R_\bullet(\mathscr{L})(x)$ is $|||\cdot|||_{X|x;\mathrm{sp}}$.

Denote by $\widehat{R}_\bullet(\mathscr{L},|||\cdot|||)(x)$ the separated completion of $(R_\bullet(\mathscr{L})(x),|||\cdot|||_{X|x})$. By construction, the quotient algebra norm $|||\cdot|||_{X|x}$ is in fact the orthogonal sum algebra norm of the sub-multiplicative family $\{\|\cdot\|_{n,X|x}\}_{n\in\mathbb{N}}$.

Proposition 2.1.23. — For any $x\in X^{\mathrm{an}}$, the limit seminorm $\mathcal{P}(|||\cdot|||)(x)$ of the family $\{\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)(x)\}_{n\in\mathbb{N}}$ can be expressed by $|||\cdot|||_{X|x}$ in the following way:

$$\forall e_1\in\mathscr{L}(x)\backslash\{0\},\ |e_1|_{\mathcal{P}(|||\cdot|||)}=|||e_1|||'_{X|x;\mathrm{sp}}.$$

Proof. — Since the algebra seminorm $|||\cdot|||_{X|x}$ is the orthogonal sum of the family $\{\|\cdot\|_{n,X|x}\}_{n\in\mathbb{N}}$, it is graded, so one has

$$\begin{aligned}|e_1|_{\mathcal{P}(|||\cdot|||)} &= \lim_{n\to\infty}|e_1|_{\frac{1}{n}\operatorname{FS}(\|\cdot\|_n)} = \lim_{n\to\infty}|e_1^{\otimes n}(x)|^{\frac{1}{n}}_{\operatorname{FS}(\|\cdot\|_n)}\\ &= \lim_{n\to\infty}\|e_1^{\otimes n}(x)\|^{\frac{1}{n}}_{n,X|x} = \lim_{n\to\infty}|||e_1^{\otimes n}(x)|||^{\frac{1}{n}}_{X|x}\\ &= |||e_1^N(x)|||^{\frac{1}{N}}_{X|x;\mathrm{sp}} = |||e_1|||'_{X|x;\mathrm{sp}}.\end{aligned}$$

where N is an integer satisfying $R_N(\mathscr{L})(x) = \mathscr{L}^{\otimes N}(x)$. □

Moreover, let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. Since there exist $N \in \mathbb{N}$ such that the evaluation maps $S_n \to \mathscr{L}^{\otimes n}(x)$ are surjective for all $n \geqslant N$, one can construct in a similar way a pseudometric on $\mathscr{L}$, called the *Fubini-Study envelope pseudometric* associated with $|||\cdot|||$ on $S_\bullet(\mathscr{L})$. It is constructed either as the limit pseudometric of the sequence $\{\frac{1}{n}\operatorname{FS}(\|\cdot\|_n)\}_{n\in N_{\geqslant N}}$, or as the spectral seminorm $|||\cdot|||'_{X|x;\mathrm{sp}}$ on $\operatorname{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x)$ as the unique graded ultrametric power-multiplicative extension of $|||\cdot|||_{X|x;\mathrm{sp}}$ on $R_\bullet(\mathscr{L})(x)$.

2.1.7.3. *Semipositive metrics.* — A metric ϕ on $\mathscr{L}$ is said to be *semipositive* if it is a Fubini-Study envelope metric and the convergence is uniform for $x \in X^{\mathrm{an}}$ ([11, Definition 6.1]). Note that if the convergence is uniform for $x \in X^{\mathrm{an}}$, since Fubini-Study metrics $\{\frac{1}{n}\operatorname{FS}(\|\cdot\|_n)\}_{n\in\mathbb{N}}$ are continuous, the envelope metric will also be continuous. Conversely, suppose that X is proper over $\operatorname{Spec} k$ and that the envelope metric is continuous, then as X^{an} is Hausdorff and compact by Theorem 1.5.11, the convergence is uniform in $x \in X^{\mathrm{an}}$. semipositive metrics are thus continuous.

Note that the semipositive metric in this sense is equivalent to the notion of *asymptotically Fubini-Study* metric by the terminology of [11]. We refer to [13, §5.4] and [11, §6.1] for a clear discussion of other various notions of semipositivity that have been proposed and studied historically in [62], [36], [46], [13], [21], [19], [24], [34] and literatures therein.

2.1.8. Global norms associated with a metric. — We construct norms on spaces of global sections by taking the supremum norm of $|(*)(x)|_\phi$ associated with a metric ϕ on $\mathscr{L}$ for all $x \in X^{\mathrm{an}}$. We can also construct quotient norm of sup norm on the space of restricted global sections.

2.1.8.1. *Sup norm.* — Assume that X is proper over $\operatorname{Spec} k$. Note that X^{an} is then a compact Hausdorff space (see [3, Theorem 3.4.8]). Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and ϕ be an upper semicontinuous metric on $\mathscr{L}$. As X^{an} is compact, any upper semicontinuous function on X^{an} is bounded from above and attains its maximal value. In particular, for any $n \in \mathbb{N}$ and any $s \in R_n(\mathscr{L})$, one has

$$\|s_n\|_{n\phi} := \sup_{x\in X^{\mathrm{an}}}|s_n(x)|_\phi < +\infty.$$

This is a norm on $R_n(\mathscr{L})$. It is ultrametric since the absolute value $|\cdot|$ on k is non-Archimedean and $\mathscr{L}^{\otimes n}$ is of rank 1. This gives rise to a family of norms $\{\|\cdot\|_{n\phi}\}_{n\in\mathbb{N}}$.

Proposition 2.1.24. — The family $\{\|\cdot\|_{n\phi}\}_{n\in\mathbb{N}}$ of ultrametric norms is sub-multiplicative and power-multiplicative. The orthogonal sum norm on $R_\bullet(\mathscr{L})$,

$$\forall s = (s_n)_{n\in\mathbb{N}},\ \ |||s|||_\phi := \sup_{n\in\mathbb{N}} \|s_n\|_{n\phi}.$$

is ultrametric and graded, sub-multiplicative and power-multiplicative.

Proof. — In fact, since 0ϕ is the trivial metric on $\mathscr{O}_X$, one has $|||\mathbf{1}|||_{0\phi} = 1$, where $\mathbf{1}$ denotes the unit section of $\mathscr{O}_X$. For $s_n \in R_n(\mathscr{L})$ and $s_m \in R_m(\mathscr{L})$, we have

$$\begin{aligned}\|s_m \cdot s_n\|_{n\phi} &= \sup_{x\in X^{\mathrm{an}}} |s_m(x)|_{m\phi} \cdot |s_n(x)|_{n\phi} \\ &\leqslant \sup_{x\in X^{\mathrm{an}}} |s_m(x)|_{m\phi} \cdot \sup_{x\in X^{\mathrm{an}}} |s_n(x)|_{n\phi} = \|s_m\|_{m\phi} \cdot \|s_n\|_{n\phi}\end{aligned}$$

and

$$\|s_n^l\|_{ln\phi} = \sup_{x\in X^{\mathrm{an}}} |s_n^l(x)|_{ln\phi} = (\sup_{x\in X^{\mathrm{an}}} |s_n(x)|_{n\phi})^l = \|s_n\|_{n\phi}^l.$$

By Proposition 2.1.7, the ultrametric graded norm $|||\cdot|||_\phi$ is sub-multiplicative and power-multiplicative. It is an algebra norm. □

One denotes by $\widehat{R}_\bullet(\mathscr{L},\phi)$ the separated completion of the normed k-algebra $(R_\bullet(\mathscr{L}), |||\cdot|||_\phi)$. More generally, if $R_\bullet$ is a graded sub-k-algebra of $R_\bullet(\mathscr{L})$, by abuse of notation we still denote by $|||\cdot|||_\phi$ the restriction of the norm $|||\cdot|||_\phi$ on $R_\bullet$ and denote by $\widehat{R}_\bullet(\phi)$ the separated completion of the normed algebra $(R_\bullet, |||\cdot|||_\phi)$. The restricted norm is also power-multiplicative. In particular, for any $N \in \mathbb{N}\backslash\{0\}$, if one takes $R_\bullet$ to be $\bigoplus_{n\in\mathbb{N}} R_{nN}(\mathscr{L})$, denoted by $R_\bullet^{(N)}(\mathscr{L})$, we denote by $\widehat{R}_\bullet^{(N)}(\mathscr{L},\phi)$ the separated completion of $(R_\bullet^{(N)}(\mathscr{L}), |||\cdot|||_\phi)$.

2.1.8.2. *Quotient norm.* — Assume that X is proper over $\operatorname{Spec} k$. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and ϕ be an upper semicontinuous metric on $\mathscr{L}$. Let $f : Y \to X$ be a morphism of k-schemes of finite type. Let $\|\cdot\|_{n\phi,X|Y}$ be the quotient norm of $\|\cdot\|_{n\phi}$ on $R_n(\mathscr{L}_{X|Y})$. It is ultrametric and graded. Let $|||\cdot|||_{\phi,X|Y}$ be the quotient norm of $|||\cdot|||_\phi$ on $R_\bullet(\mathscr{L}_{X|Y})$. In fact,

$$\forall t = (t_n)_{n\in\mathbb{N}} \in R_\bullet(\mathscr{L}_{X|Y}), \quad |||t|||_{\phi,X|Y} = \sup_{n\in\mathbb{N}} \|t_n\|_{n\phi,X|Y}.$$

Proposition 2.1.25. — $|||\cdot|||_{\phi,X|Y}$ is sub-multiplicative. It is an algebra norm.

One denotes by $\widehat{R}_\bullet(\mathscr{L}_{X|Y},\phi_{X|Y})$ the separated completion of the normed k-algebra $(R_\bullet(\mathscr{L}_{X|Y}), |||\cdot|||_{\phi_{X|Y}})$. In particular, for any $N \in \mathbb{N}$, we denote by $R_\bullet^{(N)}(\mathscr{L}_{X|Y})$ the graded k-algebra $\bigoplus_{n\in\mathbb{N}} R_{nN}(\mathscr{L}_{X|Y})$. This is a sub-algebra of $R_\bullet(\mathscr{L}_{X|Y})$. The restriction of $|||\cdot|||_{\phi,X|Y}$ on this sub-algebra is still denoted by $|||\cdot|||_{\phi,X|Y}$. One denotes by $\widehat{R}_\bullet^{(N)}(\mathscr{L}_{X|Y},\phi_{X|Y})$ the separated completion of $(R_\bullet^{(N)}(\mathscr{L}_{X|Y}), |||\cdot|||_{\phi,X|Y})$. In particular, if f is the canonical immersion associated with a sub-scheme, we get a Banach k-algebra $\widehat{R}_\bullet^{(N)}(\mathscr{L}_{X|Y},\phi_{X|Y})$.

2.1.8.3. *Radical of algebra norms associated with a metric.* — Recall that the radical of an algebra seminorm $|||\cdot|||$ on A consists of the elements $a \in A$ such that $|||a|||_{\mathrm{sp}} = 0$. The radical is denoted by $\mathrm{rad}(|||\cdot|||)$.

Proposition 2.1.26. — The radicals $\mathrm{rad}(|||\cdot|||_\phi)$, $\mathrm{rad}(|||\cdot|||_{\phi,X|Y})$ and $\mathrm{rad}(|||\cdot|||_{\phi|_Y})$ are all $\{0\}$. In particular, the Banach k-algebras $\widehat{R}_\bullet(\mathscr{L},\phi)$, $\widehat{R}_\bullet(\mathscr{L}|_Y,\phi|_Y)$ and $\widehat{R}_\bullet(\mathscr{L}_{X|Y},\phi_{X|Y})$ are semi-simple (see Definition 1.2.15) and reduced.

Proof. — Since $|||\cdot|||_\phi$ and $|||\cdot|||_{\phi|_Y}$ are multiplicative norms, their radicals are $\{0\}$ by definition. Let $t = (t_n)_n$ be an element in the radical of $\mathrm{rad}(|||\cdot|||_{\phi,X|Y})$, then since $|||\cdot|||_{\phi,X|Y}$ is graded, the radical is also graded by Lemma 2.1.13. For every $n \in \mathbb{N}$, by the construction of norms associated with metric ϕ, one has

$$0 = \lim_{l\to\infty} \|t_n^l\|_{nl\phi,X|Y}^{\frac{1}{l}} \geqslant \lim_{l\to\infty} \|t_n^l\|_{nl\phi|_Y}^{\frac{1}{l}},$$

so t_n is in the radical of $|||\cdot|||_{\phi|_Y}$, which is $\{0\}$. Thus t is equal to 0.

□

Remark 2.1.27. — The reducedness of closed sub-scheme Y is necessary for the semi-simplicity of the Banach k-algebra $\widehat{R}_\bullet(\mathscr{L}_{X|Y},\phi_{X|Y})$.

In this thesis, we make the overall assumption on the metric data, that ϕ is an upper-semicontinuous metric on $\mathscr{L}$.

2.2. Fubini-Study metrics

We give explicit expressions for Fubini-Study metrics induced by ultrametric norms admitting ultrametric orthogonal basis, and list some basic properties for operations of passage between norm and metric. Many of the results here are also obtained previously in [24] or in [11, Section 6].

2.2.1. Explicite calculation for Fubini-Study metric with orthogonal basis. —

Lemma 2.2.1. — Assume that $\mathscr{L}^{\otimes n}$ is globally generated. Let $\|\cdot\|_n$ be a norm on $R_n(\mathscr{L})$ and let $\mathrm{FS}(\|\cdot\|_n)$ be the associated Fubini-Study metric. Then for any $x \in X^{\mathrm{an}}$ and $e_n \in \mathscr{L}^{\otimes n}(x)\backslash\{0\}$,

$$|e_n|_{\mathrm{FS}(\|\cdot\|_n)} = \inf_{\substack{\lambda\in\widehat{\kappa}(x)^\times,\ s_n\in R_n(\mathscr{L})\\ s_n(x)=\lambda\cdot e_n}} |\lambda|_x^{-1}\cdot\|s_n\|.$$

Proof. — This follows from Lemma 1.1.11. □

If the ultrametric norm $\|\cdot\|_n$ admits an orthogonal basis, one can calculate explicitly the associated Fubini-Study metric $\mathrm{FS}(\|\cdot\|_n)$.

Lemma 2.2.2. — Let $(k',|\cdot|')$ be a complete ultrametric valued field extension of $(k,|\cdot|)$. Then for any tuple $(r_1,\dots,r_n) \in \mathbb{R}_+^n$, one has

$$\inf_{\substack{(\kappa_1,\dots,\kappa_n)\in(k')^n\\ \kappa_1+\cdots+\kappa_n=1}} \max_{j\in\{1,\dots,n\}} \left\{|\kappa_j|'\cdot r_j\right\} = \min_{j\in\{1,\dots,n\}}\{r_j\}.$$

Proof. — On the one hand, let $j_0 \in \{1, \dots, n\}$ be an index such that

$$r_{j_0} = \min_{j \in \{1,\dots,n\}} \{r_j\}.$$

By taking $\kappa_j = 0$ for $j \neq j_0$ and $\kappa_{j_0} = 1$, one sees that

$$\inf_{\substack{(\kappa_1,\dots,\kappa_n) \in (k')^n \\ \kappa_1 + \cdots + \kappa_n = 1}} \max_{j \in \{1,\dots,d_n\}} \{|\kappa_j|' \cdot r_j\} \leqslant r_{j_0} = \min_{j \in \{1,\dots,d_n\}} \{r_j\}.$$

On the other hand, since $|\cdot|'$ is ultrametric, if $\sum_{j \in \{1,\dots,n\}} \kappa_j = 1$, then there exists at least one $j_1 \in \{1, \dots, n\}$ such that $|\kappa_{j_1}| \geqslant 1$, so

$$\begin{aligned}\inf_{\substack{(\kappa_1,\dots,\kappa_n) \in (k')^n \\ \kappa_1 + \cdots + \kappa_n = 1}} \max_{j \in \{1,\dots,n\}} \Big\{|\kappa_j|' \cdot r_j\Big\} &\geqslant \inf_{\substack{(\kappa_1,\dots,\kappa_n) \in (k')^n \\ \kappa_1 + \cdots + \kappa_n = 1}} |\kappa_{j_1}|' \cdot r_{j_1} \\ &\geqslant \inf_{\substack{(\kappa_1,\dots,\kappa_n) \in (k')^n \\ \kappa_1 + \cdots + \kappa_n = 1}} r_{j_1} = r_{j_1} \geqslant \min_{j \in \{1,\dots,n\}} \{r_j\}.\end{aligned}$$

Hence the two sides are equal. □

Proposition 2.2.3. — Assume that $\mathscr{L}^{\otimes n}$ is globally generated. Let $\|\cdot\|_n$ be an ultrametric norm on $R_n(\mathscr{L})$ which admits an ultrametric orthogonal basis $\{s_{n,j}\}_{j \in \{1,\dots,d_n\}}$. Then for any $x \in X^{\mathrm{an}}$ and $e_n \in \mathscr{L}^{\otimes n}(x) \backslash 0$,

$$|e_n|_{\mathrm{FS}(\|\cdot\|_n)} = \min_{j \in \{1,\dots,d_n\}} \Big\{\Big|\frac{e_n}{s_{n,j}(x)}\Big|_x \cdot \|s_{n,j}\|_n\Big\},$$

with the convention that $0^{-1} = +\infty$. (see also [24, Lemma 3.3])

Proof. — For $j \in \{0, \dots, d_n\}$, let $\lambda_j \in \widehat{\kappa}(x)$ be such that $s_{n,j}(x) = \lambda_j \cdot e_n$. By construction,

$$\begin{aligned}|e_n|_{\mathrm{FS}(\|\cdot\|_n)} &= \inf_{\substack{(\mu_1,\dots,\mu_{d_n}) \in \widehat{\kappa}(x)^{d_n} \\ \mu_1 s_{n,1}(x) + \cdots + \mu_{d_n} s_{n,d_n} = e_n}} \Big|\sum_{j \in \{1,\dots,d_n\}} \mu_j \cdot s_{n,j}\Big| \\ &= \inf_{\substack{(\mu_1,\dots,\mu_{d_n}) \in \widehat{\kappa}(x)^{d_n} \\ \mu_1 s_{n,1}(x) + \cdots + \mu_{d_n} s_{n,d_n} = e_n}} \max_{j \in \{1,\dots,d_n\}} \Big\{|\mu_j|_x \cdot \|s_{n,j}\|\Big\} \\ &= \inf_{\substack{(\kappa_1,\dots,\kappa_{d_n}) \in \widehat{\kappa}(x)^{d_n} \\ \kappa_1 + \cdots + \kappa_{d_n} = 1}} \max_{j \in \{1,\dots,d_n\}} \Big\{|\kappa_j \cdot \lambda_j^{-1}|_x \cdot \|s_{n,j}\|\Big\} \\ &= \inf_{\substack{(\kappa_1,\dots,\kappa_{d_n}) \in \widehat{\kappa}(x)^{d_n} \\ \kappa_1 + \cdots + \kappa_{d_n} = 1}} \max_{j \in \{1,\dots,d_n\}} \Big\{\big||\kappa_j|_x \cdot |\lambda_j|_{\widehat{\kappa}(x)}^{-1} \cdot \|s_{n,j}\|\big|\Big\} \\ &= \min_{j \in \{1,\dots,d_n\}} \Big\{|\lambda_j|_x^{-1} \cdot \|s_{n,j}\|_n\Big\}.\end{aligned}$$

The last equality is obtained with Lemma 2.2.2. □

Corollary 2.2.4. — With the same hypothesis as above, for $e_n^\vee \in (\mathscr{L}^{\otimes n})^\vee(x) \backslash \{0\}$, one has

$$|e_n^\vee|_{\mathrm{FS}(\|\cdot\|_n)^\vee} = \max\Big\{|e_n^\vee(s_{n,j})|_x \cdot \|s_{n,j}\|_n^{-1}\Big\}.$$

Proof. — It suffices to note that $e_n^\vee(s_{n,j}) = \lambda_j$. □

2.2.2. Passages between norms and metrics. — Recall that we have operations passing between space of norms and space of metrics as follows

$$\{\|\cdot\|_n \text{ on } R_n(\mathscr{L})\} \underset{\|\cdot\|_{\phi_n}}{\overset{\mathrm{FS}(\|\cdot\|_n)}{\rightleftarrows}} \{\phi_n \text{ on } \mathscr{L}^{\otimes n}\}$$

Since both these spaces are equipped with distances, we can study the quantitative features of these operations.

Lemma 2.2.5. — Assume that $\mathscr{L}$ is globally generated. Let ϕ_1 and ϕ_2 be two upper-semicontinuous metrics on $\mathscr{L}$, then

$$\mathrm{dist}(\|\cdot\|_{\phi_1}, \|\cdot\|_{\phi_2}) \leqslant \mathrm{dist}(\phi_1, \phi_2).$$

(see also [11, Lemma 6.10])

Proof. — For any $s \in R_1(\mathscr{L})$, since X^{an} is compact, and the functions $|s(x)|_{\phi_1}$ and $|s(x)|_{\phi_2}$ are upper-semincontinuous, one can find a point $x_1 \in X^{\mathrm{an}}$ where the maximum $\|s\|_{\phi_1}$ of $|s(x)|_{\phi_1}$ is attained, and a point $x_2 \in X^{\mathrm{an}}$ where the maximum $\|s\|_{\phi_2}$ of $|s(x)|_{\phi_2}$ is attained. If $\|s\|_{\phi_1} \geqslant \|s\|_{\phi_2}$, one has

$$\Big|\log \frac{\|s\|_{\phi_1}}{\|s\|_{\phi_2}}\Big| \leqslant \Big|\log \Big|\frac{s(x_1)}{s(x_1)}\Big|_{x_1}\Big| \leqslant \mathrm{dist}(\phi_1, \phi_2).$$

Otherwise $\|s\|_{\phi_1} \leqslant \|s\|_{\phi_2}$, one has

$$\Big|\log \frac{\|s\|_{\phi_2}}{\|s\|_{\phi_1}}\Big| \leqslant \Big|\log \Big|\frac{s(x_2)}{s(x_2)}\Big|_{x_2}\Big| \leqslant \mathrm{dist}(\phi_1, \phi_2).$$

Hence the desired inequality holds. □

Lemma 2.2.6. — Assume that $\mathscr{L}$ is globally generated. Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on $R_1(\mathscr{L})$, then

$$\mathrm{dist}(\mathrm{FS}(\|\cdot\|), \mathrm{FS}(\|\cdot\|')) \leqslant \mathrm{dist}(\|\cdot\|, \|\cdot\|').$$

(see also [11, Equation (6.2)])

Proof. — For any $x \in X^{\mathrm{an}}$, let $e \in \mathscr{L}(x)\backslash\{0\}$. Let $s, s' \in R_1(\mathscr{L})$ and $\lambda, \lambda' \in \hat{\kappa}(x)$ be elements such that

$$s(x) = \lambda \cdot e,\ \|e\|_{X|x} = |\lambda|^{-1}\|s\|,$$
$$s'(x) = \lambda' \cdot e,\ \|e\|_{X|x} = |\lambda'|^{-1}\|s'\|'.$$

If $\|e\|_{X|x} > \|e\|'_{X|x}$, one has

$$\begin{aligned}\mathrm{dist}(\|e\|_{X|x}, \|e\|'_{X|x}) &= \Big|\log \frac{|\lambda|^{-1}\|s\|}{|\lambda'|^{-1}\|s'\|'}\Big| \\ &\leqslant \Big|\log \frac{|\lambda|^{-1}\|s\|}{|\lambda|^{-1}\|s\|'}\Big| \leqslant \mathrm{dist}(\|\cdot\|, \|\cdot\|').\end{aligned}$$

Otherwise, one has

$$\begin{aligned}\mathrm{dist}(\|e\|_{X|x}, \|e\|'_{X|x}) &= \Big|\log \frac{|\lambda|^{-1}\|s\|}{|\lambda'|^{-1}\|s'\|'}\Big| \\ &\leqslant \Big|\log \frac{|\lambda'|^{-1}\|s'\|}{|\lambda'|^{-1}\|s'\|'}\Big| \leqslant \mathrm{dist}(\|\cdot\|, \|\cdot\|').\end{aligned}$$

Varying x and taking the supremum for $x \in X^{\mathrm{an}}$, one gets the desired inequality. □

Lemma 2.2.7. — Assume that $\mathscr{L}$ is globally generated. Let $\|\cdot\|_1$ be an ultrametric norm on $R_1(\mathscr{L})$, then $\mathrm{FS}(\|\cdot\|_1)$ is a continuous metric on $\mathscr{L}$. ([24, Proposition 3.1])

Proof. — If $\|\cdot\|_1$ admit an ultrametric orthogonal basis, then from the explicite expression in Proposition 2.2.3, it's easy to see that $\mathrm{FS}(\|\cdot\|_1)$ is continuous. In general, by Proposition 1.1.17, for any $\epsilon > 0$, one can approximate $\|\cdot\|_1$ by a ultrametric norm $\|\cdot\|_1'$ admitting orthogonal basis, such that

$$\mathrm{dist}(\|\cdot\|_1, \|\cdot\|_1') \leqslant \epsilon,$$

Then by Lemma 2.2.6, one can approximate (uniform in $x \in X^{\mathrm{an}}$) the metric $\mathrm{FS}(\|\cdot\|_1)$ by continuous metrics $\mathrm{FS}(\|\cdot\|_1')$ in such a way that

$$\mathrm{dist}(\mathrm{FS}(\|\cdot\|_1), \mathrm{FS}(\|\cdot\|_1')) \leqslant \epsilon.$$

If ϵ tends to 0, one can see that the metric $\mathrm{FS}(\|\cdot\|_1)$ is also continuous. □

Lemma 2.2.8. — Assume that $\mathscr{L}$ is globally generated. For any metric ϕ on $\mathscr{L}$, and any $x \in X^{\mathrm{an}}$, one has

$$|(*)(x)|_{\mathrm{FS}(\|\cdot\|_\phi)} \geqslant |(*)(x)|_\phi.$$

Proof. — For any $x \in X^{\mathrm{an}}$ and any $e \in \mathscr{L}(x)\backslash\{0\}$, by definition, one has

$$\begin{aligned}
|e|_{\mathrm{FS}(\|\cdot\|_\phi)} &= \inf_{\substack{s_1(x)=\lambda e \\ s_1\in R_1(\mathscr{L}),\lambda\in\widehat{\kappa}(x)^\times}} |\lambda|_x^{-1} \cdot \sup_{y\in X^{an}} \inf_{\substack{s_1(y)=\mu t_1(y) \\ t_1\in R_1(\mathscr{L}),\mu\in\widehat{\kappa}(x)^\times}} |\mu|_x^{-1}\|t_1\|_\phi \\
&\geqslant \inf_{\substack{s_1(x)=\lambda e \\ s_1\in R_1(\mathscr{L}),\lambda\in\widehat{\kappa}(x)^\times}} |\lambda|_x^{-1} \cdot \inf_{\substack{s_1(x)=\mu t_1(x) \\ t_1\in R_1(\mathscr{L}),\mu\in\widehat{\kappa}(x)^\times}} |\mu|_x^{-1} \cdot \|t_1\|_\phi \\
&= |e|_\phi
\end{aligned}$$

so $|(*)(x)|_{\mathrm{FS}(\|\cdot\|_\phi)} \geqslant |(*)(x)|_\phi$. □

Proposition 2.2.9. — Assume that there exist an ultrametric norm $\|\cdot\|_1$ on $R_1(\mathscr{L})$ such that ϕ is given by $\mathrm{FS}(\|\cdot\|_1)$. Then for any $n \in \mathbb{N}$, $\mathrm{FS}(\|\cdot\|_{n\phi})$ is equal to $n\phi$. ([24, Proposition 3.6])

Proof. — First look at the case $n = 1$. For any $x \in X^{\mathrm{an}}$ and any $e \in \mathscr{L}(x)\backslash\{0\}$, by construction, one has

$$\begin{aligned}
|e|_{\mathrm{FS}(\|\cdot\|_\phi)} &= \inf_{\substack{s_1(x)=\lambda e \\ s_1\in R_1(\mathscr{L}),\lambda\in\widehat{\kappa}(x)^\times}} |\lambda|_x^{-1} \cdot \|s_1\|_\phi \\
&= \inf_{\substack{s_1(x)=\lambda e \\ s_1\in R_1(\mathscr{L}),\lambda\in\widehat{\kappa}(x)^\times}} |\lambda|_x^{-1} \cdot \sup_{y\in X^{an}} |s_1(y)|_\phi \\
&= \inf_{\substack{s_1(x)=\lambda e \\ s_1\in R_1(\mathscr{L}),\lambda\in\widehat{\kappa}(x)^\times}} |\lambda|_x^{-1} \cdot \sup_{y\in X^{an}} \inf_{\substack{s_1(y)=\mu t_1(y) \\ t_1\in R_1(\mathscr{L}),\mu\in\widehat{\kappa}(y)^\times}} |\mu|_y^{-1} \cdot \|t_1\|_\phi \\
&\leqslant \inf_{\substack{s_1(x)=\lambda e \\ s_1\in R_1(\mathscr{L}),\lambda\in\widehat{\kappa}(x)^\times}} |\lambda|_x^{-1} \cdot \|s_1\|_\phi = |e|_\phi
\end{aligned}$$

so $|*|_{\mathrm{FS}(\|\cdot\|_\phi)}(x) \leqslant |*|_\phi(x)$. Besides, by Lemma 2.2.8, one has $|*|_{\mathrm{FS}(\|\cdot\|_\phi)}(x) \geqslant |*|_\phi(x)$. Hence the two metrics are equal.

Then for general n, notice that by the above equality and Lemma 2.2.8,

$$|e^{\otimes n}|_{\mathrm{FS}(\|\cdot\|_{n\phi})} \geqslant |e^{\otimes n}|_{n\phi} = |e|_{\phi}^{n} = |e|^{n}_{\mathrm{FS}(\|\cdot\|_{\phi})},$$

But as the family $\{\|\cdot\|_{n\phi}\}$ is sub-multiplicative, the family of quotient norms $\{\mathrm{FS}(\|\cdot\|_{n\phi})\}$ is also sub-multiplicative by Fact 1.3.5, so the reverse inequality holds. Thus the two metrics $n\phi$ and $\mathrm{FS}(\|\cdot\|_{n\phi})$ are equal. □

Proposition 2.2.10. — Assume that ϕ is a semipositive metric on $\mathscr{L}$, then the envelope metric $\mathcal{P}(|||\cdot|||_{\phi})$ is equal to ϕ. (see also [11, Theorem 6.15 (iii)])

Proof. — By assumption, there exists a family of norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ such that uniformly for $x \in X^{\mathrm{an}}$,

$$\lim_{n\to\infty} \frac{1}{n}|(*)(x)|_{\mathrm{FS}(\|\cdot\|_n)} = |(*)(x)|_{\phi},$$

so for any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for any $n \geqslant N_0$

$$\mathrm{dist}(n\phi, \mathrm{FS}(\|\cdot\|_n)) \leqslant n\epsilon,$$

hence by Lemma 2.2.5 and 2.2.6,

$$\mathrm{dist}(\mathrm{FS}(\|\cdot\|_{n\phi}), \mathrm{FS}(\|\cdot\|_{\mathrm{FS}(\|\cdot\|_n)})) \leqslant n\epsilon.$$

By Proposition 2.2.9, one has $\mathrm{FS}(\|\cdot\|_{\mathrm{FS}(\|\cdot\|_n)})) = \mathrm{FS}(\|\cdot\|_n)$, so

$$\mathrm{dist}(\mathrm{FS}(\|\cdot\|_{n\phi}), \mathrm{FS}(\|\cdot\|_n)) \leqslant n\epsilon,$$

and

$$\mathrm{dist}(\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_{n\phi}), \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)) \leqslant \epsilon.$$

Taking limit for $n \to \infty$ and then for $\epsilon \to 0$, one has

$$\mathrm{dist}(\mathcal{P}(|||\cdot|||_{\phi}), \phi) = 0,$$

so the two metrics are equal. □

2.3. Spectrum and envelope

In this section, let $S_{\bullet}(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_{\bullet}(\mathscr{L})$, we describe the metric (Berkovich) spectrum of the normed algebra of sections $\widehat{S}(\mathscr{L}, |||\cdot|||_{\phi})$. It is a compact set in the analytification of the algebraic spectrum $\mathbf{C}(\mathscr{L}_{S_{\bullet}})^{\mathrm{an}}$. In the first part, we consider the continous map induced by the morphism of k-schemes

$$p^{\mathrm{an}} : \mathbf{V}(\mathscr{L})^{\mathrm{an}} \to \mathbf{C}(\mathscr{L}_{S_{\bullet}})^{\mathrm{an}},$$

and describe the pre-image of the Berkovich spectrum in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ by the dual unit disc bundle with respect to the envelope metric. In the second part, we roughly locate the position of the spectrum with respect to affinoid domains in $\mathbf{C}(\mathscr{L}_{S_{\bullet}})^{\mathrm{an}}$, which contains or is contained in the compact spectrum.

2.3.1. Dual unit disc bundle. — Recall that the dual unit disc bundle of a metrized line bundle $(\mathscr{L},\phi)$ is the following compact subset of $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$

$$\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)=\big\{w\in\mathbf{V}(\mathscr{L})^{\mathrm{an}},\ \forall e_1\in\mathscr{L}(x),\ |e_1|_w\leqslant|e_1|_\phi\big\}.$$

The image $p^{\mathrm{an}}(\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi))$ is a compact set in $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$, and one denotes it by $\mathfrak{M}^{-}(\phi)$. A point in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ is denoted by w, its image in $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$ by z and its image in X^{an} by x.

Proposition 2.3.1. — For any $s_n\in R_n(\mathscr{L})$, one has

$$\sup_{z\in\mathfrak{M}^{-}(\phi)}|s_n|_z=\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)}|s_n|_w=\sup_{x\in X^{\mathrm{an}}}|s_n(x)|_{n\phi}=\|s_n\|_{n\phi},$$

and for any $s\in R_\bullet(\mathscr{L})$, one has

$$\sup_{z\in\mathfrak{M}^{-}(\phi)}|s|_z=\max_{n\in\mathbb{N}}\|s_n\|_{n\phi}=|||s|||_\phi.$$

Proof. — The first equality follows immediately from the construction. For the second assertion, remember that we denote by $\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)(x)$ the fiber over x of

$$p^{\mathrm{an}}:\ \overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)\to X^{\mathrm{an}},$$

it is contained in $\mathrm{Spec}_{\widehat{\kappa}(x)}\mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x)^{\mathrm{an}}$ as a disc. Note that for any $s\in S_\bullet(\mathscr{L})$, one has

$$\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)}|s|_w=\sup_{x\in X^{\mathrm{an}}}\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)(x)}|s|_w.$$

Since $\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)(x)$ is a disc over $\widehat{\kappa}(x)$, by Corollary 2.1.16, the norm

$$\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)(x)}|\cdot|_w$$

is graded on the graded $\widehat{\kappa}(x)$-algebra $\mathrm{Sym}_{\widehat{\kappa}(x)}\mathscr{L}(x)$, hence is a graded norm when restricted to the $\widehat{\kappa}(x)$-algebra $R_\bullet(\mathscr{L})(x)$ and is also graded on the graded k-algebra $R_\bullet(\mathscr{L})$. Thus the norm

$$\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)}|\cdot|_z=\sup_{x\in X^{\mathrm{an}}}\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)(x)}|\cdot|_w$$

is also graded on the graded k-algebra $R_\bullet(\mathscr{L})$. Thanks to the first assertion, this gradedness of the norm implies that

$$\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)}|s|_w=\max_{n\in\mathbb{N}}\big\{\sup_{w\in\overline{\mathbb{D}}^{\vee}(\mathscr{L},\phi)}|s_n|_w\big\}=\max_{n\in\mathbb{N}}\big\{\|s_n\|_{n\phi}\big\}=|||s|||_\phi.$$

□

2.3.2. Spectrum of normed section algebra in terms of envelope metric. — We relate the Berkovich spectrum of normed section algebra with the dual unit disc bundle with respect to the envelope metric.

Proposition 2.3.2. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$, and let $|||\cdot|||$ be an ultrametric graded algebra norm on $S_\bullet(\mathscr{L})$. Let $w\in\mathbf{C}(\mathscr{L})^{\mathrm{an}}$ be a point and denote by $z\in\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$ the point $p^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$. Then $z\in\mathfrak{M}(\widehat{S}(\mathscr{L},|||\cdot|||))$ if and only if one of the following criteria holds

1. there exists $C(z)>0$ such that

$$\forall s\in S_\bullet(\mathscr{L}),\quad |s|_z\leqslant C(z)\cdot|||s|||.$$

2. there exists $C(z) > 0$ such that
$$\forall s(x) \in S_\bullet(\mathscr{L})(x), \quad |s(x)|_w \leqslant C(z) \cdot |||s(x)|||_{X|x}.$$
3. there exists $C'(z) = 1$ such that
$$\forall s(x) \in S_\bullet(\mathscr{L})(x), \quad |s(x)|_w \leqslant |||s(x)|||_{X|x;\mathrm{sp}}.$$
where $|||\cdot|||_{X|x;\mathrm{sp}}$ is the spectral algebra seminorm of $|||\cdot|||_{X|x}$.
4. there exists $C'(z) = 1$ such that
$$\forall s \in \mathrm{Sym}_{\kappa(x)} \mathscr{L}(x), \quad |s|_w \leqslant |||s|||'_{X|x;\mathrm{sp}},$$
where $|||\cdot|||'_{X|x;\mathrm{sp}}$ is the unique graded power multiplicative algebra seminorm on $\mathrm{Sym}_{\kappa(x)} \mathscr{L}(x)$ whose restriction on $S_\bullet(\mathscr{L})_{X|x}$ is $|||\cdot|||_{X|x;\mathrm{sp}}$.
5. there exists $C'(z) = 1$ such that
$$\forall e \in \mathscr{L}(x), \ |e|_z \leqslant |||e|||'_{X|x;\mathrm{sp}}.$$
6. there exists $C'(z) = 1$ such that
$$\forall e_1 \in \mathscr{L}(x), \ |e_1|_z \leqslant |e_1|_{\mathcal{P}(\phi)}.$$
Or equivalently, $w \in \overline{\mathbb{D}}^\vee(\mathscr{L}, \mathcal{P}(|||\cdot|||))$.

Proof. — The condition 1 unfolds the definition of the Berkovich spectrum.

The condition 2 is equivalent to the criterion 1, as $|||\cdot|||_{X|x}$ is the quotient algebra norm of $|||\cdot|||_{\widehat{\kappa}(x)}$ for the evaluation map $\mathrm{ev}(x)$. The condition 3 is equivalent to the condition 2 since $|\cdot|_z$ is power-multiplicative.

The conditions 3, 4, 5 and 6 are equivalent: Lemma 2.1.3 implies that the morphism of graded $\kappa(x)$-algebra
$$\gamma_{S_\bullet} : \ S_\bullet(\mathscr{L})_{X|x} \to \mathrm{Sym}_{\kappa(x)} \mathscr{L}(x)$$
is a (TN)-isomorphim. Proposition 2.1.18 then implies the equivalence of these conditions. Note that Proposition 2.1.23 implies that the algebra seminorm $|||\cdot|||'_{X|x;\mathrm{sp}}$ on $\mathrm{Sym}_{\kappa(x)} \mathscr{L}(x)$ is equal to the seminorm given by envelope pseudometric $|\cdot|_{\mathcal{P}(|||\cdot|||)}(x)$. □

Proposition 2.3.3. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. The restriction of the map $p^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$ is a continuous surjective map between topological spaces
$$\overline{\mathbb{D}}^\vee(\mathscr{L}, \mathcal{P}(|||\cdot|||)) \to \mathfrak{M}(\widehat{S}_\bullet(\mathscr{L}, |||\cdot|||))$$
A further restriction is an open map
$$\overline{\mathbb{D}}^\vee(\mathscr{L}, \mathcal{P}(|||\cdot|||)) \backslash \mathbb{O}(X)^{\mathrm{an}} \to \mathfrak{M}(\widehat{S}_\bullet(\mathscr{L}, |||\cdot|||)) \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}.$$

Proof. — By Proposition 2.3.2, the pre-image under p^{an} of $\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L}, |||\cdot|||))$ is exactly $\overline{\mathbb{D}}^\vee(\mathscr{L}, \mathcal{P}(|||\cdot|||))$, so one has the first restriction map, and it is continuous and surjective. Propositions 2.1.20 and 2.1.2 indicate that the composition map
$$p^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}} : \mathbf{V}(\mathscr{L})^{\mathrm{an}} \backslash \mathbb{O}(X)^{\mathrm{an}} \to \mathbf{C}(\mathscr{L})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}} \to \mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}} \backslash \mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$$
is an open map. The second assertion follows easily from general topology.

□

2.3.3. Spectrum of normed section algebra as geometric envelope. — In previous part we've identified the compact spectrum with the image in the affine cone of the dual unit disc bundle with respect to *envelope* metric. In this part, we describe it geometrically, as the holomorphic envelope set in the affine cone of the image of the dual unit disc bundle with repect to the *original* metric. For the abbreviation of notations, we shall henceforth denote by $\mathfrak{M}(\phi)$ the compact set $p^{\mathrm{an}}(\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \mathcal{P}(\phi)))$ and by $\mathfrak{M}^{-}(\phi)$ the compact set $p^{\mathrm{an}}(\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi))$. They are both compact sets contained in $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$, and one has $\mathfrak{M}^{-}(\phi) \subseteq \mathfrak{M}(\phi)$. Note that $\mathfrak{M}(\phi)$ is the spectrum of the Banach algebra $\widehat{R}_\bullet(\mathscr{L}, \phi)$, while $\mathfrak{M}^{-}(\phi)$ is *not* a spectrum of any Banach algebra.

Recall that for any compact set K in an affinoid domain $\mathfrak{M}(\mathcal{A})$, denoted by Z, one can construct its holomorphic envelope in Z as the following compact set

$$\widehat{K} := \big\{z \in Z,\ \forall f \in \mathcal{A}, |f|_z \leqslant \sup_{z' \in K} |f|_{z'}\big\}.$$

If $\mathcal{A}$ has a dense k-subalgebra A, then one can also use elements in A to define an envelope. It is the same envelope as the one defined by elements of $\mathcal{A}$.

Proposition 2.3.4. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. With the notations introduced as above, one has

$$\mathfrak{M}(\phi) = \widehat{\mathfrak{M}^{-}(\phi)},$$

where the envelope is taken in any affinoid domain in $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$ that contains both.

Proof. — One first shows the inclusion $\supseteq$. Since $\mathfrak{M}(\phi)$ is the spectrum of $\widehat{S}_\bullet(\mathscr{L}, \phi)$, it is holomorphic convex in $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$ by Corollary 1.4.5. Since it contains $\mathfrak{M}^{-}(\phi)$, it contains also the holomorphic envelope of $\mathfrak{M}^{-}(\phi)$. Conversely, to show the inclusion $\subseteq$, let $z \in V$ be a point in $\widehat{\mathfrak{M}(\phi)}$. By the construction of $\mathfrak{M}(\phi)$ and by Lemma 2.3.1, for any $s \in S_\bullet(\mathscr{L})$, one has

$$|s|_z \leqslant |||s|||_\phi = \sup_{n \in \mathbb{N}} \big\{\|s_n\|_{n\phi}\big\} = \sup_{z' \in \mathfrak{M}^{-}(\phi)} |s|_{z'}.$$

As $S_\bullet(\mathscr{L})$ is dense in $\widehat{S}_\bullet(\mathscr{L})$, one deduces that z belongs to $\widehat{\mathfrak{M}^{-}(\phi)}$. □

2.3.4. Affinoid domains sandwiching the spectrum. — We show that the Berkovich spectrum $\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L}), \phi)$ is contained in some affinoid domain, and it contains an affinoid domain neighbourhood of $\mathbf{0}_{\mathscr{L}_{S_\bullet}}$ in $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$. The first affinoid domain will be used to do holomorphic spectral calculus, the second one will be used to deduce the openness of the image in the affine cone of the open dual unit disc bundle.

2.3.4.1. *From outside.* — We embed the Berkovich spectrum into the analytification of the affine cone.

Fact 2.3.5. — Let $S_\bullet(\mathscr{L})$ be a graded algebra, and let $|||\cdot|||$ be an algebra norm on $S_\bullet(\mathscr{L})$. The inclusion $S_\bullet(\mathscr{L}) \to \widehat{S}(\mathscr{L}, |||\cdot|||)$ induces a continuous map

$$\widehat{S}(\mathscr{L}, |||\cdot|||) \to \mathbf{C}(\mathscr{L}_{S_\bullet})$$

which is homeomorphism from domain to image. Moreover, one can find an affinoid algebra norm on $S_\bullet(\mathscr{L})$, denoted by $|||\cdot|||^{\diamond}$, such that $|||\cdot||| \leqslant |||\cdot|||^{\diamond}$. And one has a homomorphism

of Banach algebras

$$\widehat{S}(\mathscr{L}, |||\cdot|||^{\diamond}) \to \widehat{S}(\mathscr{L}, |||\cdot|||),$$

the induced continuous map

$$\mathfrak{M}(\widehat{S}(\mathscr{L}, |||\cdot|||)) \to \mathfrak{M}(\widehat{S}(\mathscr{L}, |||\cdot|||)^{\diamond})$$

is a homeomorphism from domain to image.

Proof. — This follows directly from Proposition 1.5.5 and Proposition 1.3.7. □

Remark 2.3.6. — In particular, if one takes $R_{\bullet}(\mathscr{L}_{X|Y})$ to be the subalgebra $S_{\bullet}(\mathscr{L})$, and $|||\cdot|||_{\phi,X|Y}$ to be the algebra norm, then one can construct an affinoid algebra norm $|||\cdot|||^{\diamond}_{\phi,X|Y}$. We shall denote by $\widehat{R}(\mathscr{L}_{X|Y}, \phi^{\diamond}_{X|Y})$ the affinoid Banach algebra $\widehat{R}(\mathscr{L}_{X|Y}, |||\cdot|||^{\diamond}_{\phi,X|Y})$.

2.3.4.2. *From inside.* — One can describe the shape of the Berkovich spectrum if the metric is a Fubini-Study metric associated with a norm admitting an orthogonal basis.

Proposition 2.3.7. — Let $S_{\bullet}(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_{\bullet}(\mathscr{L})$. Let $n \in \mathbb{N}$ be an integer such that $\mathscr{L}^{\otimes n}$ is globally generated. Let $\|\cdot\|_n$ be an ultrametric norm on $S_n(\mathscr{L})$ admitting an orthogonal basis $\{s_{n,j}\}_{j\in\{0,\dots,d_n\}}$. Let $w \in \mathbf{V}(\mathscr{L})^{\mathrm{an}}$ and $z \in \mathbf{C}(\mathscr{L}_{S_{\bullet}})^{\mathrm{an}}$ be the point $p^{\mathrm{an}}(w)$, then $w \in \overline{\mathbb{D}}^{\vee}(\mathscr{L}, \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n))$ (resp.$\mathbb{D}^{\vee}(\mathscr{L}, \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n))$) if and only if

$$\forall j \in \{0, \dots, d_n\},\ |s_{n,j}|_z \leqslant \|s_{n,j}\|_n \ (\text{resp.} < \|s_{n,j}\|_n).$$

In particular, the image of $\mathbb{D}^{\vee}(\mathscr{L}, \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n))$ under p^{an} is an open subset in $\mathbf{C}(\mathscr{L}_{S_{\bullet}})^{\mathrm{an}}$.

Proof. — By construction of dual unit disc bundle, the point w belongs to $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n))$ (resp. to $\mathbb{D}^{\vee}(\mathscr{L}, \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n))$) if and only if for any point $x \in X^{\mathrm{an}}$, and any local section e_1 of L, one has

$$|e_1|_w \leqslant |e_1|_{\frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n)}.$$

or equivalently, for any local section e_n of $\mathscr{L}^{\otimes n}$, one has

$$|e_n|_w \leqslant |e_n|_{\mathrm{FS}(\|\cdot\|_n)}.$$

If this holds, then by Proposition 2.2.3, for any $j \in \{0, \dots, d_n\}$, one considers the local sections given by $s_{n,j}$, and one has

$$|s_{n,j}(x)|_w \leqslant |s_{n,j}(x)|_{\mathrm{FS}(\|\cdot\|_n)} = \min_{i\in\{1,\dots,d_n\}} \{ |\frac{s_{n,j}(x)}{s_{n,i}(x)}|_x \cdot \|s_{n,i}\|_n \} \leqslant \|s_{n,j}\|_n.$$

Conversely, if these inequality holds, then for any $j \in \{0, \dots, d_n\}$, and any local section e_n such that $e_n \neq 0$, one has

$$|e_n|_w = |\frac{e_n}{s_{n,j}(x)}|_x \cdot |s_{n,j}(x)|_w \leqslant |\frac{e_n}{s_{n,j}(x)}|_{\widehat{\kappa}(x)} \cdot \|s_{n,j}\|_n$$

which implies, by Proposition 2.2.3 that

$$|e_n|_w \leqslant \min_{j\in\{1,\dots,d_n\}} \{ |\frac{e_n}{s_{n,j}(x)}|_x \cdot \|s_{n,j}\|_n \} = |e_n|_{\mathrm{FS}(\|\cdot\|_n)}.$$

So the two conditions are equivalent. Then the openness of the image of $\mathbb{D}^{\vee}(\mathscr{L}, \frac{1}{n}\,\mathrm{FS}(\|\cdot\|_n))$ under p^{an} follows easily from these defining equations. □

Corollary 2.3.8. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. Let ϕ be a continuous metric on L. Then the image of $\mathbb{D}^\vee(\mathscr{L},\phi)$ under p^{an} is an open subset of $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$.

Proof. — Proposition 2.1.20 implies that the continuous map

$$p^{\mathrm{an}}:\ \mathbf{V}(\mathscr{L})^{\mathrm{an}}\backslash\mathbb{O}(X)^{\mathrm{an}}\to\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}\backslash\mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$$

is an open map. As ϕ is continuous, $\mathbb{D}^\vee(\mathscr{L},\phi)\backslash\mathbb{O}(X)^{\mathrm{an}}$ is an open subset of $\mathbf{V}(\mathscr{L})^{\mathrm{an}}\backslash\mathbb{O}(X)^{\mathrm{an}}$. Then the image of $\mathbb{D}^\vee(\mathscr{L},\phi)\backslash\mathbb{O}(X)^{\mathrm{an}}$ is an open subset of $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}\backslash\mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$, so it is also an open subset of $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$. It suffices to find an open neighbourhood of $\mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$ in the image of $\mathbb{D}^\vee(\mathscr{L},\phi)$. Note that $\mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$ is the contained in the image of $\mathbb{O}(X)^{\mathrm{an}}$.

As $\mathscr{L}$ is ample, there exist $n\in\mathbb{N}$ such that $\mathscr{L}^{\otimes n}$ is globally generated. Let $\{s_{n,j}\}_{j\in\{0,\dots,d_n\}}$ be a basis and construct an ultrametric norm $\|\cdot\|_n$ on S_n admitting this basis as an orthogonal basis. Deonte by ψ be the Fubini-Study metric associated with $\|\cdot\|_n$. As both $\frac{1}{n}\psi$ and ϕ are continuous and X^{an} is compact, there exists $\epsilon\in\mathbb{R}$ such that

$$\forall x\in X^{\mathrm{an}},\ \frac{1}{n}\psi(\epsilon)(x)\leqslant\phi(x),$$

so

$$p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\frac{1}{n}\psi(\epsilon)))\subseteq p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\phi)).$$

The left hand side is an open subset of $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$ by Lemma 2.3.7. It contains $\mathbf{0}^{\mathrm{an}}_{\mathscr{L}_{S_\bullet}}$. Then the union

$$p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\phi))=p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\phi)\backslash\mathbb{O}^{\mathrm{an}}(X))\cup p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\frac{1}{n}\psi(\epsilon)))$$

is equal to the image of $\mathbb{D}^\vee(\mathscr{L},\phi)$. Hence it is an open subset in $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$. □

Corollary 2.3.9. — Let $S_\bullet(\mathscr{L})$ be a (TN)-isomorphic subalgebra of $R_\bullet(\mathscr{L})$. Let ϕ be a continuous metric on $\mathscr{L}$. If $\mathcal{P}(\phi)$ is continuous, then for any $\epsilon\in\mathbb{R}_{>0}$, one has

$$\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L},\phi))\subseteq\mathrm{Int}^t_{\mathbf{C}}(\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L},\phi(\epsilon)))),$$

where $\mathrm{Int}^t_{\mathbf{C}}$ denotes the topological interior as subspace of $\mathbf{C}(\mathscr{L}_{S_\bullet})^{\mathrm{an}}$.

Proof. — By Proposition 2.3.3, the left hand side $\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L},\phi))$ is identified with $p^{\mathrm{an}}(\overline{\mathbb{D}}^\vee(\mathscr{L},\mathcal{P}(\phi)))$. Since $\mathcal{P}(\phi)$ is continuous, it is contained in the open subset $p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\mathcal{P}(\phi)(\epsilon)))$. This open subset is contained in $p^{\mathrm{an}}(\overline{\mathbb{D}}^\vee(\mathscr{L},\mathcal{P}(\phi)(\epsilon)))$ which is identified with $\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L},\phi(\epsilon)))$, hence $p^{\mathrm{an}}(\mathbb{D}^\vee(\mathscr{L},\mathcal{P}(\phi)(\epsilon)))$ is contained in the topological interior of $\mathfrak{M}(\widehat{S}_\bullet(\mathscr{L},\phi(\epsilon)))$, the right hand side. □

2.4. Restriction to sub-varieties

We study the behavior of norms under the restriction to a sub-variety.

Proposition 2.4.1. — Let ϕ be an upper-semicontinuous metric on $\mathscr{L}$. Then

$$\mathcal{P}(|||\cdot|||_\phi)|_Y=\mathcal{P}(|||\cdot|||_{\phi,X|Y}).$$

Proof. — By definition, for any $y \in Y^{\mathrm{an}}$, one has

$$\mathcal{P}(|||\cdot|||_{\phi})(y) = \lim_{n\to\infty} \frac{1}{n} \mathrm{FS}(\|\cdot\|_{n\phi})(y),$$

$$\mathcal{P}(|||\cdot|||_{\phi,X|Y})(y) = \lim_{n\to\infty} \frac{1}{n} \mathrm{FS}(\|\cdot\|_{n\phi,X|Y})(y).$$

Since the k-linear map $R_n(\mathscr{L}) \to R_n(\mathscr{L}_{X|Y})$ is surjective for all large $n \in \mathbb{N}$, and $\|\cdot\|_{n\phi,X|Y}$ is the quotient norm of $\|\cdot\|_{n\phi}$, one has

$$\mathrm{FS}(\|\cdot\|_{n\phi})(y) = \mathrm{FS}(\|\cdot\|_{n\phi,X|Y})(y).$$

Hence the two envelope metrics are equal. □

Lemma 2.4.2. — Let ϕ be a Fubini-Study envelope metric on $\mathscr{L}$. Then $\phi|_Y$ is a Fubini-Study envelope metric on $\mathscr{L}|_Y$.

Proof. — Suppose that ϕ is the pointwise limit on X^{an} of $\{\frac{1}{n}\mathrm{FS}(\|\cdot\|_n)\}$, where $\{\|\cdot\|_n\}$ are norms on $R_n(\mathscr{L})$. Then $\mathscr{L}|_Y$ is the pointwise limit of $\{\frac{1}{n}\mathrm{FS}(\|\cdot\|_{n,X|Y})\}$ on Y^{an}. □

Proposition 2.4.3. — Let ϕ be a semipositive metric on $\mathscr{L}$. Consider two algebra norms $|||\cdot|||_{\phi|_Y}$ and $|||\cdot|||_{\phi,X|Y}$ on $R_\bullet(\mathscr{L}_{X|Y})$. Then the three metrics are equal

$$\mathcal{P}(|||\cdot|||_{\phi,X|Y}) = \mathcal{P}(|||\cdot|||_{\phi|_Y}) = \phi|_Y.$$

Proof. — By Proposition 2.4.1, one has

$$\mathcal{P}(|||\cdot|||_{\phi,X|Y}) = \mathcal{P}(|||\cdot|||_{\phi})|_Y = \phi|_Y.$$

It suffices to show the second equality in the assertion. By Lemma 2.4.2, the restriction metric $\phi|_Y$ is a semipositive metric on $\mathscr{L}|_Y$. By Proposition 2.2.10,

$$\mathcal{P}(|||\cdot|||_{\phi|_Y}) = \phi|_Y.$$

□

Corollary 2.4.4. — Let ϕ be a semipositive metric on $\mathscr{L}$. Then on $R_\bullet(\mathscr{L}_{X|Y})$, the spectral algebra seminorm of $|||\cdot|||_{\phi,X|Y}$ is equal to $|||\cdot|||_{\phi|_Y}$. There exists a canonical homeomorphism

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y)).$$

Proof. — On the one hand, by Theorem 1.2.17, changing the algebra norm $|||\cdot|||_{\phi,X|Y}$ to its spectral seminorm does not modify the spectrum, so one has an isomorphism

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, |||\cdot|||_{\phi,X|Y;\mathrm{sp}})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, |||\cdot|||_{\phi,X|Y})) =: \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})),$$

on the other hand, by Proposition 2.4.3, one identifies the spectral seminorm of $|||\cdot|||_{\phi,X|Y}$ with $|||\cdot|||_{\phi|_Y}$ so one has an isomorphism

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, |||\cdot|||_{\phi,X|Y;\mathrm{sp}})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, |||\cdot|||_{\phi|_Y})) =: \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y)).$$

Combining the two one gets the desired isomorphism. Moreover, by Proposition 1.2.14, the two power-multiplicative algebra seminorms $|||\cdot|||_{\phi|_Y}$ and $|||\cdot|||_{\phi,X|Y;\mathrm{sp}}$ on $R_\bullet(\mathscr{L}_{X|Y})$ are both supremum norms on the same spectrum, so they are equal. □

CHAPTER 3

NORMED EXTENSION FOR RESTRICTED SECTIONS

In this chapter, given a closed subvariety Y of X and ample line bundle $\mathscr{L}$, we consider the problem of extension of sections of $\mathscr{L}|_Y$ to X with a control of their norms induced by ϕ. We interpret this control as an upper bound of the quotient algebra norm $|||\cdot|||_{X|Y}$ by its spectral algebra norm. We propose two methods in order to establish this inequality.

3.1. Background and results

3.1.1. The Problem. — The algebro-geometric setting is as follows: suppose given a sub-variety (reduced closed subscheme) Y of an integral projective scheme X over a field k, and let $\mathscr{I}_Y$ be the corresponding ideal sheaf. Recall that Serre's vanishing theorem (see for example [56, théorème III.2.2.1]) asserts that there exist an integer n_Y such that

$$H^1(X, \mathscr{I}_Y \otimes \mathscr{L}^{\otimes n}) = \{0\}$$

for any integer $n \geqslant n_Y$. Therefore the exact sequence of coherent $\mathscr{O}_X$-modules

$$\mathbf{0} \longrightarrow \mathscr{I}_Y \otimes \mathscr{L}^{\otimes n} \longrightarrow \mathscr{L}^{\otimes n} \longrightarrow (\mathscr{O}_X/\mathscr{I}_Y) \otimes \mathscr{L}^{\otimes n} \longrightarrow \mathbf{0}$$

induces a surjective k-linear map from $H^0(X, \mathscr{L}^{\otimes n})$ to $H^0(Y, \mathscr{L}|_Y^{\otimes n})$, which we denote by

$$(s \in H^0(X, \mathscr{L}^{\otimes n})) \longmapsto (s|_Y \in H^0(Y, \mathscr{L}|_Y^{\otimes n})),$$

where $\mathscr{L}|_Y$ denotes the restriction of $\mathscr{L}$ to Y. In other words, for any integer $n \geqslant n_Y$, any global section of $\mathscr{L}|_Y^{\otimes n}$ extends to a global section of $\mathscr{L}^{\otimes n}$. One says that s_n is an extension of t_n.

Now consider the metric analogue of the above statement. Assume that k is equipped with a complete absolute value $|\cdot|$ and let $\mathscr{L}$ be equipped with a continuous metric ϕ. Denote by $\mathscr{L}|_Y$ and $\phi|_Y$ their restrictions to Y, respectively. The metric extension problem is to find an extension $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ of a given prescribed section $t_n \in H^0(Y, \mathscr{L}|_Y^{\otimes n})$ of the restricted line bundle, such that the distorsion of their supremum norms is smallest possible. In terms of norms, the problem can be interpreted as looking for a uniform upper bound for

$$\inf_{\substack{s \in H^0(X, \mathscr{L}^{\otimes n}) \\ s|_Y = t}} \frac{\|s\|_{n\phi}}{\|t\|_{n\phi|_Y}}, \quad t \in \mathrm{Im}(H^0(X, \mathscr{L}^{\otimes n}) \xrightarrow{|_Y} H^0(Y, \mathscr{L}|_Y^{\otimes n}))\backslash\{0\}. \tag{2}$$

which can be further formulated as the comparison of two norms on one single vector space

$$\mathrm{Im}(H^0(X, \mathscr{L}^{\otimes n}) \xrightarrow{|_Y} H^0(Y, \mathscr{L}|_Y^{\otimes n})),$$

one is the quotient norm of $\|\cdot\|_{n\phi}$, denoted by $\|\cdot\|_{n\phi,X|Y}$; another is the sup norm $\|\cdot\|_{n\phi|_Y}$.

In practice, one contents with knowing just the asymptotic behavior (when $n \to \infty$) of such upper bounds.

3.1.2. Complex case. — In the complex analytic setting, namely when $(k, |\cdot|)$ is $\mathbb{C}$ equipped with the usual absolute value, the metric extension problem has been studied by different authors using various approaches. Assume that the metric ϕ is strictly positive (namely, for any local section s of $\mathscr{L}$ over an open subscheme U of X, the function $(x \in U^{\mathrm{an}}) \mapsto \log|s(x)|_\phi$ is strongly plurisubharmonic). In the case where X is smooth, by Gromov's theorem we can compare the sup norm $\|\cdot\|_{n\phi}$ to the L^2 norm $\|\cdot\|_{n\phi,L^2}$ defined as

$$\forall\, s \in H^0(X, L^{\otimes n}), \quad \|s\|_{n\phi,L^2} := \left(\int_{X^{\mathrm{an}}} |s(x)|_{n\phi}(x)\,\mathrm{d}V\right)^{\frac{1}{2}},$$

where $\mathrm{d}V$ is a probability measure on X^{an} which is locally equivalent with Lebesgue measure with a smooth Radon-Nikodym density. Therefore, in the case where X and Y are both smooth, we can apply the Hörmander L^2 technique and get an inequality (see for example [61], [47], [45], [62])

$$\|\cdot\|_{n\phi|_Y} \geqslant C'(\phi, Y, X) n^{-d} \|\cdot\|_{n\phi,X|Y}, \qquad n \geqslant n_Y, \tag{3}$$

where $C'(\phi, Y, X)$ is a positive constant. Alternatively, one can apply Grauert's argument of pseudo-convexity of the (open) dual unit disc bundle of $(\mathscr{L}, \phi)$ to produce, for any $\epsilon > 0$, a slightly weaker inequality of the form which however does not require the smoothness of the schemes X and Y

$$\|\cdot\|_{n\phi|_Y} \geqslant C_\epsilon(\phi, Y, X) \mathrm{e}^{-\epsilon n} \|\cdot\|_{n\phi,X|Y}, \qquad n \geqslant n_Y, \tag{4}$$

where $C_\epsilon(\phi, Y, X)$ is a positive constant depending on ϵ. We refer to [10] and [51] for more details.

Note that, to obtain an estimate of the form (4) for any closed point Y, the semipositivity of the metric ϕ (in the sense of uniform approximability by Fubini-Study metrics) is actually necessary. In fact, for sufficiently positive integer n, the ample invertible $\mathscr{O}_X$-module determines a closed embedding $\iota_n : X \to \mathbb{P}(H^0(X, \mathscr{L}^{\otimes n}))$. The norm $\|\cdot\|_{n\phi}$ on $H^0(X, \mathscr{L}^{\otimes n})$ defines a Fubini-Study metric on the universal invertible sheaf of the projective space $H^0(X, \mathscr{L}^{\otimes n})$. Denote by ϕ_n the continuous metric on $\mathscr{L}$ the n-th tensor power of which identifies with the restriction of the Fubini-Study metric. Then the estimate (4) in the case where Y is a single closed point $\{x\}$ implies that the sequence of norms $|\cdot|_{\phi_n}(x)$ converges to $|\cdot|_\phi(x)$. Moreover, it can be shown the subsequence $(|\cdot|_{\phi_{2^m}}(x))_{m\in\mathbb{N}}$ is decreasing. Therefore, the semipositivity of Fubini-Study metrics implies that of the metric ϕ.

3.1.3. Non-Archimedean case. — The problem of normed extension has various applications, not only in complex analytic geometry, but also in Arakelov geometry. It is a key ingredient in the proof of the arithmetic Hilbert-Samuel theorem, see [1]. It has also been applied in the proof of Nakai-Moishezon criterion of arithmetic ampleness, cf. [62], see also [36].

From the adelic point of view of Arakelov geometry, one can replace the integral models of arithmetic objects by a family of Berkovich analytic objects (possibly equipped with metrics) parametrised by the set of finite places of a number field. The advantage of the adelic approach consists in treating all places of a number field in a uniform way. This motivates the research of the non-Archimedean analogue of notions and results in complex analytic geometry. It turns out that many usual analytic tools (such as the L^2 method for $\overline{\partial}$) do not work well in the non-Archimedean setting (see however [57] for a non-Archimedean Ohsawa-Takegoshi extension theorem in one dimension), and often new ideas are needed to develop the non-Archimedean analogue of complex analytic geometry and to unify the arguments in both settings.

In this chapter we undertake a study of the metric extension problem in (2) in the non-Archimedean analytic setting. Previously, a metric extension property of the following form is established in [24],

***Theorem 3.1.1* (Theorem 3.2.1).** — Let ϕ be a semipositive metric on $\mathscr{L}$, then for any $\epsilon > 0$, and any $t_1 \in H^0(Y, \mathscr{L}|_Y)$, there exists $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$, there exist $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ with $s_n|_Y = t_1^{\otimes n}$ and

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot (\|t_1\|_{\phi|_Y})^n.$$

A metric which is a uniform limit of model metrics is called "semi-amply metrized" in [62]. For any model metric associated with a model $(\mathfrak{X}, \mathfrak{L})$ of $(X, \mathscr{L})$ for $\mathfrak{o}_k \subseteq k$, in [24] they look at the normed vector spaces of global sections $H^0(X, \mathscr{L})$ with the norm given by a lattice $H^0(\mathfrak{X}, \mathfrak{L})$ in it. By a careful study of fine properties of norms in a vector space associated with lattices, they first established this metric extension for such model metrics. Then by the existence of non-Archimedean (almost)-orthogonal basis for arbitrary ultrametric norm, they can approximate a general Fubini-Study metric by model metrics, thus getting the final conclusion. As a by-product, they showed that one may use general Fubini-Study metrics (coming from general ultrametric norms on $H^0(X, \mathscr{L})$) instead of model metrics in the approximation process. Thus on an ample line bundle, a metric that is"semi-amplely metrized" in [62] is the same as being asymptotically Fubini-Study in [11].

Thanks to this equivalence, we can avoid the use and study of models. Our new ingredient to this metric extension problem is to investigate the normed *algebra* of sections instead of normed *vector spaces* of sections. From point of view of normed section algebras, the goal is to compare two algebra norms $|||\cdot|||_{\phi,X|Y}$ and $|||\cdot|||_{\phi_Y}$ on the graded algebra

$$R_\bullet(\mathscr{L}_{X|Y}) := \bigoplus_{n\in\mathbb{N}} \mathrm{Im}\left(H^0(X, \mathscr{L}^{\otimes n}) \xrightarrow{|_Y} H^0(Y, \mathscr{L}|_Y^{\otimes n})\right).$$

We would like to bound from the above of the quotient one by the sup one. The advantage of passing from vector spaces to an algebra is to better exploit the algebra structure and to work in an affine geometric setting than in a projective geometric one.

By virtue of Corollary 2.4.4, we know that the sup algebra norm is the spectral norm of the quotient algebra norm. This point of view is the leading thread of our methods: it suffices to concentrate on the relation between the quotient algebra norm $|||\cdot|||_{\phi,X|Y}$ and its spectral norm. First, in §1, Corollary 2.4.4 allows us to derive directly the normed extension theorem in [24]: fix an error of precision ϵ, one shows that there is a sub-exponential bound $\mathrm{e}^{n\epsilon}$ for $n \geqslant N$ with N depending on the geometry (X, Y, $\mathscr{L}$ and ϕ) and also possibly on the choice of restricted section.

We can also refine the result, dropping the dependence of N on the choice of restricted section so that the bound becomes uniform in this choice, which is our main result of this chapter

***Theorem 3.1.2* (Theorem 3.3.5, Theorem 3.4.10).** — Let ϕ be a semipositive metric on $\mathscr{L}$. Then for any $\epsilon > 0$, there exists $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$ and any $t_n \in H^0(Y, \mathscr{L}|_Y^{\otimes n})$, one can find $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ such that $s_n|_Y = t_n$ and

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

The refinement requires a finer comparison result: any (reduced) affinoid algebra norm is equivalent to its spectral norm. We devise two methods of approximation of $|||\cdot|||_{\phi,X|Y}$ by affinoid algebra norms, so as to refine the normed extension results in §2 and §3.

Note that our methods give a sub-exponential upper bound as in (4). Whether the polynomial bound of (3) is obtainable in the non-Archimedean setting remains unexplored, and evidently it requires considerations of intrinsic geometry in terms of metrized (co-)tangent sheaves and local parameters. It seems that the commutative Banach algebra techniques that we used here are insufficient to ameliorate the bound as only extrinsic geometry (i.e. embeddings into projective space) has been probed.

3.2. Comparison of norms

Recall that we shall compare two algebra norms $|||\cdot|||_{\phi,X|Y}$ and $|||\cdot|||_{\phi|_Y}$ on the restricted section algebra

$$R_\bullet(\mathscr{L}_{X|Y}) := \bigoplus_{n\in\mathbb{N}} \mathrm{Im}(H^0(X, \mathscr{L}^{\otimes n}) \xrightarrow{|_Y} H^0(Y, \mathscr{L}|_Y^{\otimes n})),$$

the second norm being the spectral algebra norm of the first by Corollary 2.4.4. We've fixed $M \in \mathbb{N}$ such that for any $n \geqslant M$, $\mathscr{L}^{\otimes n}$ is very ample, and that the restriction maps are surjective.

The starting point is the spectral relation between these two algebra norms as shown in Corollary 2.4.4,

$$|||\cdot|||_{\phi,X|Y;\mathrm{sp}} = |||\cdot|||_{\phi|_Y},$$

which is a consequence of the geometric homeomorphism of spectra

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y)).$$

Theorem 3.2.1. — Let ϕ be a semipositive metric on $\mathscr{L}$, then for any $\epsilon > 0$, and any $t_1 \in H^0(Y, \mathscr{L}|_Y)$, there exists $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$, there exists $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ with $s_n|_Y = t_1^{\otimes n}$ and

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot (\|t_1\|_{\phi|_Y})^n.$$

Proof. — For some large $l \in \mathbb{N}$, one has $t_1^{\otimes l} \in R(\mathscr{L}_{X|Y})$, thus by Corollary 2.4.4 one has

$$\lim_{n\to+\infty} \|t_1^{\otimes ln}\|_{\phi,X|Y}^{\frac{1}{n}} = \|t_1^{\otimes l}\|_{\phi|_Y}.$$

Note that $|||\cdot|||_{\phi,X|Y}$ is submultiplicative and $|||\cdot|||_{\phi|_Y}$ is power-multiplicative, so the sequence $\{\|t_1^{\otimes n}\|_{\phi,X|Y}^{\frac{1}{n}}\}_{n\geqslant M}$ has a limit and one deduces that

$$\lim_{n\to+\infty} \|t_1^{\otimes n}\|_{\phi,X|Y}^{\frac{1}{n}} = \lim_{n\to+\infty} \|t_1^{\otimes ln}\|_{\phi,X|Y}^{\frac{1}{ln}} = \|t_1^{\otimes l}\|_{\phi|_Y}^{\frac{1}{l}} = \|t_1\|_{\phi|_Y}.$$

Hence for any $\epsilon > 0$ there exists $n_Y \in \mathbb{N}$, such that for any $n \geqslant n_Y$, there exists $s_n \in R_n(\mathscr{L})$ with $s_n|_Y = t_1^{\otimes n}$ with

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_1^{\otimes n}\|_{n\phi|_Y} = \mathrm{e}^{n\epsilon} \cdot (\|t_1\|_{\phi|_Y})^n.$$

□

Remark 3.2.2. — This result is previously obtained in [24], by using approximation of ϕ by model metrics. Here we give another proof.

3.3. Geometric/Spectral approximation

Recall that $\mathbf{C}(\mathscr{L}_{X|Y})$ is the affine scheme $\operatorname{Spec} R_\bullet(\mathscr{L}_{X|Y})$. The spectra of $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})$ and $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y)$ are compact sets in $\mathbf{C}(\mathscr{L}_{X|Y})^{\mathrm{an}}$.

This approach deduces comparison of algebra norms on $R_\bullet(\mathscr{L}_{X|Y})$ from the geometric comparison of the spectra of the corresponding Banach algebras that are completions with respect to these norms. In the previous section, we've seen that a homeomorphism of spectra

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y))$$

gives rise to a weak version of comparison. To refine it to a uniform one, we examine the strict inclusion of spectra

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \hookrightarrow \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon))), \quad \forall \epsilon > 0$$

and the consequent comparison of algebra norms. Here $|||\cdot|||_{\phi|_Y(\epsilon)}$ is the sup algebra norm associated with the dilated metric $\phi(\epsilon)$.

The strategy is as follows: starting from the above inclusion of spectra, we can find a special domain W_ϵ contained between them

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \subseteq W_\epsilon \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon)))$$

which we see as a geometric approximation of $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$. We consider an intermediate Banach algebra $\mathcal{W}$, which is the structural Banach algebra of W_ϵ equipped with the sup algebra norm $|||\cdot|||_{W_\epsilon}$. We would like to translate the above inclusions of spectra to a comparison of corresponding algebra norms of the following form:

$$|||\cdot|||_{\phi,X|Y} \lesssim |||\cdot|||_{W_\epsilon} \lesssim |||\cdot|||_{\phi|_Y(\epsilon)},$$

where we use the symbol $\lesssim$ to mean $\leqslant$ which omits constants that depends only on the background geometric data (X, Y, $\mathscr{L}$ and ϕ). The first comparison is non-trivial, since it bounds a general quotient norm $|||\cdot|||_{\phi,X|Y}$ from above by a sup norm $|||\cdot|||_{W_\epsilon}$. This

comparison stems from holomorphic functional calculus, which produces a homomorphism of Banach algebras

$$\mathcal{W} \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}).$$

The comparison of algebra norms follows from the continuity of this homomorphism. The second comparison is easy: since both algebra norms $|||\cdot|||_{W_\epsilon}$ and $|||\cdot|||_{\phi|_Y(\epsilon)}$ are sup norms on W_ϵ and on $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon)))$ respectively, the desired comparison follows directly from the inclusion of spectra.

3.3.1. Localization of spectrum by affinoid domain covering. — For any $\epsilon > 0$, one considers the spectra $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$ and $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon)))$. They can be both indentified with closed subsets in $\mathbf{C}(\mathscr{L}_{X|Y})^{\mathrm{an}}$, and one denotes by $\mathrm{Int}^t_{\mathbf{C}}$ the topological interior of a set in $\mathbf{C}(\mathscr{L}_{X|Y})^{\mathrm{an}}$.

Lemma 3.3.1. — There is an affinoid algebra norm $|||\cdot|||^{\diamond_\epsilon}_{\phi(\epsilon),X|Y}$ on $R_\bullet(\mathscr{L}_{X|Y})$ such that $|||\cdot|||^{\diamond_\epsilon}_{\phi(\epsilon),X|Y} \geqslant |||\cdot|||_{\phi(\epsilon),X|Y}$. Denote by $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_\epsilon)$ the Banach algebra completion, it is an affinoid algebra, and the identity map on $R_\bullet(\mathscr{L}_{X|Y})$ induces a continuous homomorphism

$$\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_\epsilon) \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}(\epsilon)).$$

which induces an inclusion of spectra

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}(\epsilon))) \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_\epsilon)).$$

Proof. — The existence of affinoid norm is an immediate consequence of Proposition 1.3.7 applied to the Banach algebra $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}(\epsilon)$ and its finitely generated subalgebra $R_\bullet(\mathscr{L}_{X|Y})$. The induced map on spectra is injective by Proposition 1.2.11. □

Recall that a special domain is a finite union of affinoid domains (see Definition 1.3.36).

Proposition 3.3.2. — Assume that $\phi|_Y$ is semipositive (in particular, it is continuous). For any $\epsilon > 0$, there exists a special domain W_ϵ such that

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y)) \subseteq W_\epsilon \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon))).$$

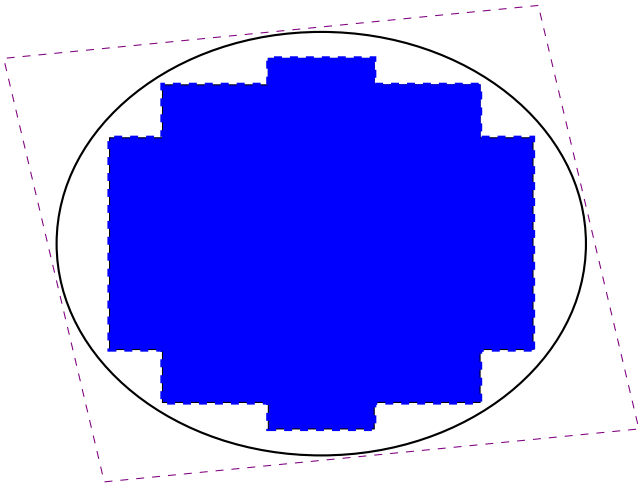
Proof. — The first isomorphism is given by Proposition 2.4.3. To construct W_ϵ, note that by Corollary 1.3.29, every point in $\mathbf{C}(R_\bullet(\mathscr{L}_{X|Y}, \phi(\epsilon)_{X|Y}))^{\mathrm{an}}$ has a neighbourhood system consisting of affinoid domains. By Proposition 2.3.9, the open set $\mathrm{Int}^t_{\mathbf{C}}(\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon))))$ is a neighbourhood of $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y))$, because the metric $\phi|_Y$ is continuous. Thus for any $z \in \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$, one may find an affinoid domain neighbourhood $V(z)$ which is contained in $\mathrm{Int}^t_{\mathbf{C}}(\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon))))$. (see the picture below)

One forms a covering by open sets

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \subseteq \bigcup_{z \in \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))} \mathrm{Int}^t_{\mathbf{C}} V(z).$$

Since the left hand side is a compact set by Proposition 1.2.6, there exist finitely many points $\{z_1, \ldots, z_m\}$ such that $\{\mathrm{Int}^t_{\mathbf{C}} V(z_i)\}_{i \in \{1,\ldots,m\}}$ form a covering of $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$. Let W_ϵ be the union of affinoid domains $\{V(z_i)\}_{i \in \{1,\ldots,m\}}$, then it is a special domain, and satisfies the desired inclusion conditions. □

Remark 3.3.3. — One denotes by $\mathcal{W}_\epsilon$ the structural Banach k-algebra $\Gamma(W_\epsilon, \mathscr{O}_{W_\epsilon})$ of the special domain W_ϵ, equipped with supremum norm $|||\cdot|||_{W_\epsilon}$ (see Definition 1.3.36).



Geometric approximation

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \subseteq W_\epsilon \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon))) \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_\epsilon)$$

3.3.2. Localization of Banach algebra homomorphism. — We shall use holomorphic functional calculus to construct Banach algebra homomorphisms. Recall that for any homomorphism of Banach algebras $f : \mathcal{A} \to \mathcal{B}$, the spectrum Σ_f of the homomorphism is the image of $\mathfrak{M}(\mathcal{B})$ in $\mathfrak{M}(\mathcal{A})$ under the map f^*. The theory of functional calculus (Theorem 1.4.8) provides a factorization of f through a localization: assume that $\mathcal{A}$ is affinoid, for any special domain W containing Σ_f, let $\mathcal{W}$ be the structural Banach algebra of W, and denote by i_W is the localization homomorphism corresponding to the inclusion $W \hookrightarrow \mathfrak{M}(\mathcal{A})$, then there exists a Banach algebra homomorphism θ which makes the following diagram commutative

$$\begin{tikzcd} \mathcal{A} \arrow[r, "f"] \arrow[d, "i_W"'] & \mathcal{B} \\ \mathcal{W} \arrow[ur, dashed, "\theta"'] & \end{tikzcd} .$$

One can apply Proposition 3.3.2 for ϵ' to construct a special domain $W_{\epsilon'}$ such that

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \subseteq W_{\epsilon'} \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon'))).$$

Proposition 3.3.4. — For any $\epsilon' > 0$, there exist a homomorphism of Banach k-algebras

$$\theta(\epsilon') : \mathcal{W}_{\epsilon'} \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})$$

which is induced by the identity map on the dense sub-k-algebra $R_\bullet(\mathscr{L}_{X|Y})$.

Proof. — By Lemma 3.3.1, for any $\epsilon' > 0$, one can construct an affinoid algebra norm $|||\cdot|||^{\diamond_{\epsilon'}}_{\phi(\epsilon'),X|Y}$ on $R_\bullet(\mathscr{L}_{X|Y})$ such that the corresponding affinoid algebra admits a (continuous) homomorphism to the original Banach algebra $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_{\epsilon'}) \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}(\epsilon'))$.

Composing it with the continuous homomorphism $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}(\epsilon')) \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})$, one gets a (continuous) homomorphism from an affinoid algebra to a Banach algebra

$$\tau(\epsilon') : \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_{\epsilon'}) \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}).$$

Its restriction on the dense subalgebra $R_\bullet(\mathscr{L}_{X|Y})$ is the identity, so this homomorphism has dense image, thus by Proposition 1.2.11 the induced continuous map

$$\tau(\epsilon')^* : \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})) \to \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond_{\epsilon'}))$$

is injective and is closed. As both spaces are compact and Hausdorff, this map is a homeomorphism of topological spaces from its domain to its image. So the spectrum $\Sigma_{\tau(\epsilon')}$ (of the homomorphism $\tau(\epsilon')$) can be identified with $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$, it is contained in $W_{\epsilon'}$.

One can perform spectral calculus for the homomorphism $\tau(\epsilon')$ with respect to the special domain $W_{\epsilon'}$, as $\Sigma_{\tau(\epsilon')}$ is contained in $W_{\epsilon'}$. By Theorem 1.4.8, there exists a homomorphism $\theta(\epsilon/2)$ of Banach k-algebras $\mathcal{W}_{\epsilon'} \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})$ which is induced by the identity map on the dense sub-k-algebra $R_\bullet(\mathscr{L}_{X|Y})$, and makes the following diagram commutative

$$\begin{tikzcd}
\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \diamond(\epsilon')) \arrow[r, "\tau(\epsilon')"] \arrow[d, "i_{W_{\epsilon'}}"'] & \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}) \\
\mathcal{W}_{\epsilon'} \arrow[ur, dashed, "\theta(\epsilon')"'] &
\end{tikzcd}.$$

□

Theorem 3.3.5. — Let ϕ be a semipositive metric on $\mathscr{L}$. Then for any $\epsilon > 0$, there exist $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$ and any $t_n \in H^0(Y, \mathscr{L}|_Y^{\otimes n})$, one can find $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ such that $s_n|_Y = t_n$ and

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

Proof. — First, fix some $\epsilon' < \epsilon$ and one constructs the homomorphism $\theta(\epsilon')_{W_{\epsilon'}}$ as in Proposition 3.3.4. The boundedness (continuity) of this homomorphism of Banach k-algebras implies that there exist $C_{\epsilon'} > 0$ such that

$$\forall t \in R_\bullet(\mathscr{L}_{X|Y}), \quad |||t|||_{\phi,X|Y} \leqslant C_{\epsilon'} \cdot |||t|||_{W_{\epsilon'}} = C_{\epsilon'} \cdot \sup_{z \in W_{\epsilon'}} |t|_z.$$

Second, by Proposition 2.4.3, one has a canonical homeomorphism

$$\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi(\epsilon')_{X|Y})) \simeq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon')))$$

from which one deduces

$$W_{\epsilon'} \subseteq \mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon'))).$$

Remember that since $|||\cdot|||_{\phi|_Y(\epsilon')}$ is power-mutliplicative, by Theorem 1.2.14, it is the supremum norm on $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon')))$. Hence by comparing supremum norms on these two closed sets, we get

$$\forall t \in R_\bullet(\mathscr{L}_{X|Y}), \quad |||t|||_{W_{\epsilon'}} \leqslant |||t|||_{\phi|_Y(\epsilon')}.$$

Combining these two inequalities, for any $t_n \in R_n(\mathscr{L}_{X|Y})$, one has

$$\begin{aligned}\|t_n\|_{n\phi,X|Y} &= |||t_n|||_{\phi,X|Y}\\ &\leqslant C_{\epsilon'} \cdot |||t_n|||_{W_{\epsilon'}}\\ &\leqslant C_{\epsilon'} \cdot |||t_n|||_{\phi|_Y(\epsilon')} = C_{\epsilon'} \cdot \mathrm{e}^{n\epsilon'} \cdot \|t_n\|_{n\phi|_Y}.\end{aligned}$$

Let n_Y be the integer $\max\{\lceil \log(C_{\epsilon'})/(\epsilon - \epsilon')\rceil, M\}$, then for any $n \geqslant n_Y$ and any $t_n \in H^0(Y, \mathscr{L}|_Y^{\otimes n})$, by the definition of quotient norm, one can find $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ such that

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

□

Remark 3.3.6. — The continuity of ϕ is necessary in this approach: it is needed in Proposition 3.3.2 to guarantee that there are enough space between $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}))$ and $\mathfrak{M}(\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi|_Y(\epsilon)))$ so as to insert a special domain W_ϵ. Otherwise, the two domains may touch at some boundary point and there would be no special domain in the larger one that is *a neighbourhood of* the smaller one.

3.4. Algebraic approximation

In this part, we approximate the algebra norm $|||\cdot|||_{\phi_{X|Y}}$ by one affinoid algebra norm on $R_\bullet(\mathscr{L}_{X|Y})$ globally. This algebraic approximation exploits subtle consequences of existence of a ultra-metric orthogonal basis in $(R_n(\mathscr{L}), \|\cdot\|_{n\phi})$ for some $n \in \mathbb{N}$. *In this section* $(k, |\cdot|)$ *is assumed to be discretely valued.*

Recall that one denotes by $H(k, |\cdot|)$ the $\mathbb{Q}$-vector space generated by $\{\log|\lambda|\}_{\lambda \in k^\times}$, and one says that a multi-radius $\boldsymbol{r} = (r_0, \ldots, r_d)$ is $(k, |\cdot|)$-free if the images of the numbers $\{\log r_i\}_{i\in\{0,\ldots,d\}}$ in the quotient $\mathbb{Q}$-vector space $\mathbb{R}/H(k, |\cdot|)$ is $\mathbb{Q}$-linearly independent. Recall that thanks to the discreteness of $|\cdot|$, any finite dimensional ultrametric normed k-vector space has an orthogonal basis. (see Proposition 1.1.16)

Our goal is to deduce an upper bound of $|||\cdot|||_{\phi,X|Y}$ by (some variant of)$|||\cdot|||_{\phi_Y}$. Recall that the second algebra norm is the *spectral norm* of the first, thanks to Corollary 2.4.4. The key is to note that such an upper bound of an algebra norm by its spectral norm holds whenever the algebra norm is an affinoid algebra norm. Namely, if $|||\cdot|||_{\phi,X|Y}$ is an affinoid algebra norm, by Corollary 1.3.22 there exists $C \in \mathbb{R}_{>0}$ such that

$$|||\cdot|||_{\phi_{X|Y}} \leqslant C \cdot |||\cdot|||_{\phi_{X|Y};\mathrm{sp}} = C \cdot |||\cdot|||_{\phi_Y}.$$

An explicite calculation of $|||\cdot|||_\phi$ is possible in the case where X is $\mathbb{P}^d$, $\mathscr{L}$ is $\mathscr{O}(1)$ and ϕ is a (genuine) Fubini-Study metric (induced from an ultrametric norm on $R_1(\mathscr{O}(1))$). In this case $|||\cdot|||_\phi$ is a Gauss-Tate norm of some multiradius. This implies that for general X, if $\mathscr{L}$ is very ample and ϕ is a (first level) Fubini-Study metric, the algebra norm $|||\cdot|||_{\phi,X|Y}$ is an affinoid algebra norm, thus the desired upper bound can be obtained in this case. It would be nice if this affinoid property also holds whenever $\mathscr{L}$ is just ample and ϕ is some m-th Fubini-Study metric, because one can treat the general case of an semipositive metric using approximations. Unfortunately we are unable to show the affinoid property of $|||\cdot|||_{\phi,X|Y}$ in this m-th Fubini-Study metric case.

3.4.1. Algebra norm induced by Fubini-Study metric. —

3.4.1.1. *Case for* $(\mathbb{P}^d, \mathscr{O}(1))$. — Let ϕ be a metric on $\mathscr{O}(1)$, one studies the algebra norm $|||\cdot|||_\phi$ on $R_\bullet(\mathscr{O}(1))$. Since $R_\bullet(\mathscr{O}(1))$ can be identified with the multivariate polynomial algebra, one would like to show that under various assumptions, the algebra norm $|||\cdot|||_\phi$ can be identified with a Gauss-Tate algebra norm on this multi-variate polynomial algebra. As a consequence, the normed section algebra $(R_\bullet(\mathscr{O}(1)), |||\cdot|||_\phi)$ will be a Tate affinoid algebra. (see Definition 1.3.1)

First, one chooses a preferred basis to do the identification. By Proposition 1.1.16, one can find an orthogonal basis $\{T_i\}_{i\in\{0,\dots,d\}}$ for the normed vector space $(R_1(\mathscr{O}(1)), \|\cdot\|_\phi)$, namely a basis of the k-vector space $R_1(\mathscr{O}(1))$ satisfying

$$\forall(\lambda_0,\dots,\lambda_d)\in k^{d+1},\quad \Big\|\sum_{i=0}^{d}\lambda_i\cdot T_i\Big\|_\phi=\max_{i\in\{0,\dots,d\}}\big\{|\lambda_i|\cdot\|T_i\|_\phi\big\}.$$

For any $i\in\{0,\dots,d\}$, one denotes by r_i the value $\|T_i\|_\phi$, and by $\boldsymbol{r}\in(\mathbb{R}_{>0})^{d+1}$ the multi-radius $(r_0,\dots,r_d)$. One fixes such an orthogonal basis, and identify the graded k-algebras

$$R_\bullet(\mathscr{O}(1))=\operatorname{Sym}R_1(\mathscr{O}(1))\simeq k[T_0,\dots,T_d],\quad T_i\mapsto T_i.$$

For any multi-index $J=(j_0,\dots,j_d)\in\mathbb{N}^{d+1}$, one denotes by $\boldsymbol{T}^J$ the monomial element $\prod_{i\in\{0,\dots,d\}}(T_i)^{j_i}\in R_{|J|}(\mathscr{O}(1))$, and by $R_J(\mathscr{O}(1))$ is the one dimensional subspace generated by the monomial section $\boldsymbol{T}^J$. With this choice of basis, one has a multi-grading by $J\in\mathbb{N}^{d+1}$ on the section algebra which refines the original grading by $n\in\mathbb{N}$

$$R_\bullet(\mathscr{O}(1))=\bigoplus_{n\in\mathbb{N}}R_n(\mathscr{O}(1))\simeq\bigoplus_{n\in\mathbb{N}}\bigoplus_{J\subseteq\mathbb{N}^{d+1},|J|=n}R_J(\mathscr{O}(1))=k[T_0,\dots,T_d].$$

Second, one would like to identify the algebra norm $|||\cdot|||_\phi$. Note that in general, since $|||\cdot|||_\phi$ is graded (with respect to the grading $n\in\mathbb{N}$), the sub-spaces $R_n(\mathscr{O}(1))$ are orthogonal with respect to $|||\cdot|||_\phi$ for different $n\in\mathbb{N}$. However, a Gauss-Tate algebra norm exhibits an orthogonality with respect to the finer multi-grading $J\subseteq\mathbb{N}^{d+1}$. In fact, for an algebra norm to be Gauss-Tate, the subspaces

$$\big\{R_J(\mathscr{O}(1))\big\}_{J\in\mathbb{N}^{d+1}}$$

should also be orthogonal.

We shall show that $|||\cdot|||_\phi$ coincide with a Gauss-Tate algebra norm $|||\cdot|||_{\mathcal{T}_{\boldsymbol{r}(\mathbf{0})}}$ of multiradius $\boldsymbol{r}$ on $R_\bullet(\mathscr{O}(1))$, in two steps:

1. The two norms coincide on each subspace $R_J(\mathscr{O}(1))$;
2. The two norms coincide on each subspace $R_n(\mathscr{O}(1))$, hence also on $R_\bullet(\mathscr{O}(1))$.

First, one calculates $|||\cdot|||_\phi$ on mononial elements.

Proposition 3.4.1. — For any $J\in\mathbb{N}^{d+1}$, one has

$$\|\boldsymbol{T}^J\|_{|J|\phi}=\prod_{i\in\{0,\dots,d\}}\|T_i\|_\phi^{j_i}.$$

Proof. — Take a complete non-Archimedean valued field extension $(k',|\cdot|')$ of $(k,|\cdot|_k)$ such that

$$\forall i\in\{0,\dots,d\},\quad r_i\in|k'^\times|',$$

hence for any $i \in \{0, \dots, d\}$, there exist elements $\kappa_i \in k'$ such that $|\kappa_i|' = r_i$. One denotes by $x(\boldsymbol{r}) \in (\mathbb{P}^d_k)^{\mathrm{an}}$ the point given by the coordinates $[\kappa_0 : \dots : \kappa_d]$.

Claim 3.4.2. — For any $i \in \{0, \dots, d\}$, one has

$$\|T_i\|_\phi = r_i = |T_i|_\phi(x(\boldsymbol{r})).$$

In other words, the maximum of the function $|T_i|_\phi(x)$ on $(\mathbb{P}^d_k)^{\mathrm{an}}$ is r_i, and the maximum values of these $d+1$ functions on $(\mathbb{P}^d_k)^{\mathrm{an}}$ can be attained **at the same point** $x(\boldsymbol{r})$.

Proof. — With the orthogonality of the basis $\{T_i\}_{i\in\{0,\dots,d\}}$, we can calculate as follows

$$\begin{aligned}
|T_i|_\phi(x(\boldsymbol{r})) &= \inf_{\substack{(\lambda_0\cdot T_0+\cdots+\lambda_d\cdot T_d)(x(\boldsymbol{r}))=(T_i)(x(\boldsymbol{r}))\\(\lambda_0,\dots,\lambda_d)\in k^{d+1}}} \big\|\lambda_0\cdot T_0+\cdots+\lambda_d\cdot T_d\big\|_\phi \\
&= \inf_{\substack{\lambda_0\cdot\kappa_0+\cdots+\lambda_d\cdot\kappa_d=\kappa_i\\(\lambda_0,\dots,\lambda_d)\in k^{d+1}}} \max_{m\in\{0,\dots,d\}} \Big\{\|\lambda_m\cdot T_m\|_\phi\Big\} \\
&= \inf_{\substack{\lambda_0\cdot\kappa_0+\cdots+\lambda_d\cdot\kappa_d=\kappa_i\\(\lambda_0,\dots,\lambda_d)\in k^{d+1}}} \max_{m\in\{0,\dots,d\}} \Big\{|\lambda_m||\kappa_m|\Big\} \\
&= \inf_{\substack{\lambda_0(\kappa_0/\kappa_i)+\cdots+\lambda_d(\kappa_d/\kappa_i)=1\\(\lambda_0,\dots,\lambda_d)\in k^{d+1}}} \max_{m\in\{0,\dots,d\}} \Big\{|\lambda_m||\kappa_m/\kappa_i|\cdot|\kappa_i|\Big\} \\
&= |\kappa_i| = r_i.
\end{aligned}$$

The last equality is obtained by Lemma 2.2.2. □

By this Claim, for any multi-index J, the function $|\boldsymbol{T}^J|_\phi(x)$ can attain its maximum value $\prod_{i\in\{0,\dots,d\}} r_i^{j_i}$ at the point $x(\boldsymbol{r}) \in (\mathbb{P}^d_k)^{an}$ as the product of maxima of the one-variate factors $\{T_i^{j_i}\}_{i\in\{1,\dots,d\}}$. By definition of sup norm, one has

$$\|\boldsymbol{T}^J\|_{|J|\phi} = \sup_{x\in(\mathbb{P}^d)^{an}} |\boldsymbol{T}^J|_\phi(x) = \prod_{i\in\{0,\dots,d\}} r_i^{j_i} = \prod_{i\in\{0,\dots,d\}} \|T_i\|_\phi^{j_i}.$$

□

Second, one calculates the algebra norm $|||\cdot|||_\phi$ on any (homogeneous) combination of monomials. For a general metric, one needs a $(k, |\cdot|)$-freeness assumption on the multiradius $\boldsymbol{r}$ to gain finer orthogonality.

Proposition 3.4.3. — Assume that $\{\|T_i\|_\phi\}_{i\in\{0,\dots,d\}}$ are $(k, |\cdot|)$-free. Let $S \subseteq \mathbb{N}^{d+1}$ be a finite set of multi-indices, then for any $J \in S$ any $\lambda_J \in k$, one has

$$\Big|\Big|\Big|\sum_{J\in S} \lambda_J\cdot\boldsymbol{T}^J\Big|\Big|\Big| = \sup_{J\in S} \|\lambda_J\cdot\boldsymbol{T}^J\|_{|J|\phi}.$$

In other words, the algebra norm $|||\cdot|||_\phi$ on $R_\bullet(\mathscr{O}(1))$ can be identified with the Gauss norm on $k[T_0, \dots, T_d]$ of multi-radius $\boldsymbol{r}$. The Banach k-algebra $\widehat{R}_\bullet(\mathscr{O}(1), \phi)$ is an affinoid algebra.

Proof. — By the $(k, |\cdot|)$-freeness assumption on $\boldsymbol{r}$ and by Proposition 3.4.1, for any two distinct multi-index J and J', and any two non-zero coefficients λ_J and $\lambda_{J'}$ in k, we have

$$\big\|\lambda_J\cdot\boldsymbol{T}^J\big\|_{|J|\phi} = |\lambda_J|\cdot\prod_{i\in\{0,\dots,d\}} r_i^{j_i} \neq |\lambda_{J'}|\cdot\prod_{i\in\{0,\dots,d\}} r_i^{j'_i} = \big\|\lambda_{J'}\cdot\boldsymbol{T}^{J'}\big\|_{|J'|\phi}.$$

By Lemma 1.1.13, the elements $\{\boldsymbol{T}^J\}_{J\in S}$ form an orthogonal basis for the normed vector space $(\bigoplus_{J\in S} k\cdot\boldsymbol{T}^J, \|\cdot\|_\phi)$. So the desired equality holds. □

Corollary 3.4.4. — With the same hypothesis, the envelope metric $\mathcal{P}(\phi)$ is a Fubini-Study metric induced by $\|\cdot\|_\phi$, and it is continuous.

If ϕ is a Fubini-Study metric, one does not need the $(k,|\cdot|)$-freeness assumption on $\boldsymbol{r}$. For any $(\delta_0,\dots,\delta_d)\in\mathbb{R}^{d+1}$, denoted by $\boldsymbol{\delta}$, one can construct a perturbed metric $\phi(\boldsymbol{\delta})$ as follows. Let $\|\cdot\|_{\phi(\boldsymbol{\delta})}$ be the following ultrametric norm on $R_1(\mathscr{O}(1))$

$$\forall(\lambda_0,\dots,\lambda_d)\in k^{d+1},\quad \Big\|\sum_{i=0}^d \lambda_i\cdot T_i\Big\| = \max_{i\in\{0,\dots,d\}}\big\{\mathrm{e}^{\delta_i}|\lambda_i|\cdot\|T_i\|_\phi\big\},$$

namely the ultrametric norm such that $\{T_i\}_{i\in\{1,\dots,d\}}$ is an orthogonal basis, and that

$$\forall i\in\{0,\dots,d\},\quad \|T_i\|_{\phi(\boldsymbol{\delta})} = \mathrm{e}^{\delta_i}\|T_i\|_\phi.$$

Let $\phi(\boldsymbol{\delta})$ be the metric $\mathrm{FS}(\|\cdot\|_{\phi(\boldsymbol{\delta})})$ on $\mathscr{O}(1)$. Denote by $|\boldsymbol{\delta}|$ the number $\max_{i\in\{0,\dots,d\}}|\delta_i|\in\mathbb{R}_{>0}$.

Lemma 3.4.5. — Assume that ϕ is a Fubini-Study metric. For any $\epsilon>0$, there exist $\boldsymbol{\delta}\in\mathbb{R}^{d+1}$ with $|\boldsymbol{\delta}|\leqslant\epsilon$ such that

$$\forall n\in\mathbb{N},\quad \mathrm{dist}(\|\cdot\|_{n\phi},\|\cdot\|_{n\phi(\boldsymbol{\delta})})\leqslant n\epsilon.$$

Proof. — Choose an arbitrary $\boldsymbol{\delta}$ with $|\boldsymbol{\delta}|\leqslant\epsilon$. Then

$$\mathrm{dist}(\|\cdot\|_\phi,\|\cdot\|_{\phi(\boldsymbol{\delta})})\leqslant\epsilon.$$

By Proposition 2.2.10, we have

$$\mathrm{dist}(\mathrm{FS}(\|\cdot\|_\phi),\mathrm{FS}(\|\cdot\|_{\phi(\boldsymbol{\delta})})) = \mathrm{dist}(\mathrm{FS}(\|\cdot\|_\phi),\phi(\boldsymbol{\delta}))\leqslant\epsilon,$$

by the assumption and Proposition 2.2.9,

$$\mathrm{FS}(\|\cdot\|_\phi)=\phi,$$

so the conlusion holds. □

Proposition 3.4.6. — Assume that ϕ is a Fubini-Study metric. The conclusion of Proposition 3.4.3 holds without the assumption of $(k,|\cdot|)$-freeness of $\big\{\|T_i\|_1\big\}_{i\in\{0,\dots,d\}}$.

Proof. — Since $|\cdot|_k$ is discrete, for any $\epsilon>0$, there exist $\boldsymbol{\delta}$ with $|\boldsymbol{\delta}|\leqslant\epsilon$ such that the elements $\big\{\|T_i\|_{\phi(\boldsymbol{\delta})}\big\}_{i\in\{0,\dots,d\}}$ are $(k,|\cdot|)$-free. By Proposition 3.4.5, for any $n\in\mathbb{N}$ and any $s_n\in R_n(\mathscr{O}(1))$ which one writes as $\sum_{|J|=n}\lambda_J\cdot\boldsymbol{T}^J$, one has

$$\mathrm{e}^{-n\epsilon}\Big\|\sum_{|J|=n}\lambda_J\cdot\boldsymbol{T}^J\Big\|_{n\phi(\boldsymbol{\delta})}\leqslant\Big\|\sum_{|J|=n}\lambda_J\cdot\boldsymbol{T}^J\Big\|_{n\phi}\leqslant\mathrm{e}^{n\epsilon}\Big\|\sum_{|J|=n}\lambda_J\cdot\boldsymbol{T}^J\Big\|_{n\phi(\boldsymbol{\delta})}.$$

By Proposition 3.4.3, one has

$$\max_{|J|=n}\big\{\mathrm{e}^{-n\epsilon}|\lambda_J|\prod_{i\in\{0,\dots,d\}}(\mathrm{e}^{\delta_i}r_i)^{j_i}\big\}\leqslant\Big\|\sum_{|J|=n}\lambda_J\cdot\boldsymbol{T}^J\Big\|_{n\phi}\leqslant\max_{|J|=n}\big\{\mathrm{e}^{n\epsilon}|\lambda_J|\prod_{i\in\{0,\dots,d\}}(\mathrm{e}^{\delta_i}r_i)^{j_i}\big\}.$$

Fix n and let ϵ tends to 0, one gets

$$\Big\|\sum_{|J|=n}\lambda_J\cdot\boldsymbol{T}^J\Big\|_{n\phi}=\max_{|J|=n}\big\{|\lambda_J|\prod_{i\in\{0,\dots,d\}}r_i^{j_i}\big\}.$$

Finally one deduces that

$$\Big\|\!\Big| \sum_{|J|<\infty} \lambda_J \cdot \boldsymbol{T}^J \Big|\!\Big\|_{n\phi} = \sup_{n\in\mathbb{N}} \max_{|J|=n} \big\{|\lambda_J| \prod_{i\in\{0,\dots,d\}} r_i^{j_i}\big\} = \max_{|J|<\infty} \big\{|\lambda_J| \prod_{i\in\{0,\dots,d\}} r_i^{j_i}\big\}.$$

Hence $\|\!|\cdot|\!\|_\phi$ can be identified with a Gauss-Tate norm of multi-radius $\boldsymbol{r}$ on $R_\bullet(\mathscr{O}(1))$. □

3.4.1.2. *Case for some general* $(X, \mathscr{L})$. —

Proposition 3.4.7. — Assume that ϕ is a (first level) Fubini-Study metric. If $\mathscr{L}$ is very ample, then $\widehat{R}_\bullet(\mathscr{L}, \phi)$, $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})$ and $\widehat{R}_\bullet(\mathscr{L}|_Y, \phi|_Y)$ are affinoid algebras.

Proof. — By the very-ampleness assumption, the space of global sections $R_1(\mathscr{L})$ induces an projective embedding

$$\iota_1 : X \to \mathbb{P}_k^{d_1-1}$$

such that $\iota_1^*\mathscr{O}(1)$ is $\mathscr{L}$. Moreover there exist a norm $\|\cdot\|_1$ on $R_1(\mathscr{L})$ such that ϕ is given by $\mathrm{FS}(\|\cdot\|_1)$. One can view $\|\cdot\|_1$ as a norm on $R_1(\mathscr{O}(1))$, and one gets a metric $\psi = \mathrm{FS}(\|\cdot\|_1)$ on $\mathscr{O}(1)$. By construction the restriction metric $\psi|_X$ coincides with ϕ.

By Proposition 3.4.6, the Banach algebra $\widehat{R}_\bullet(\mathscr{O}(1), \|\!|\cdot|\!\|_\psi)$ is an affinoid algebra. The embedding ι_1 induces a quotient homomorphism of Banach algebras

$$\widehat{R}_\bullet(\mathscr{O}(1), \|\!|\cdot|\!\|_\psi) \to \widehat{R}_\bullet(\mathscr{L}, \|\!|\cdot|\!\|_{\psi,\mathbb{P}^{d_1}|X}).$$

Hence the quotient Banach algebra $\widehat{R}_\bullet(\mathscr{L}, \|\!|\cdot|\!\|_{\psi,\mathbb{P}^{d_1}|X})$ is an affinoid algebra.

By Proposition 2.4.4, on $\widehat{R}_\bullet(\mathscr{L}, \|\!|\cdot|\!\|_{\psi,\mathbb{P}^{d_1}|X})$, the algebra norm $\|\!|\cdot|\!\|_\phi$ is the spectral norm of $\|\!|\cdot|\!\|_{\psi,\mathbb{P}^{d_1}|X}$. The later algebra norm is an affinoid norm, since this Banach algebra is reduced, this norm is equivalent to its spectral norm by Proposition 1.3.22. So $\|\!|\cdot|\!\|_\phi$ is also an affinoid algebra norm. Thus $\widehat{R}_\bullet(\mathscr{L}, \phi)$ is an affinoid algebra.

By the same argument, one has the quotient homomorphism of Banach algebras

$$\widehat{R}_\bullet(\mathscr{L}, \phi) \to \widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y}),$$

As $\|\!|\cdot|\!\|_\phi$ is affinoid algebra norms, so is $\|\!|\cdot|\!\|_{\phi,X|Y}$. Then on $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \|\!|\cdot|\!\|_{\phi,X|Y})$, by Proposition 2.4.4, the algebra norm $\|\!|\cdot|\!\|_{\phi|_Y}$ is the spectral norm of $\|\!|\cdot|\!\|_{\phi,X|Y}$, hence is itself an affinoid algebra norm. □

Proposition 3.4.8. — Assume that ϕ is a (first level) Fubini-Study metric and that $\mathscr{L}$ is very ample. Then there exists $C(\phi, Y) > 0$ such that for any $t \in R_\bullet(\mathscr{L}_{X|Y})$, one can find $s \in R_\bullet(\mathscr{L})$ satisfying

$$\|\!|s|\!\|_\phi \leqslant C(\phi, Y, X) \cdot \|\!|t|\!\|_{\phi|_Y}.$$

In particular, for any $n \in \mathbb{N}$ and $t_n \in R_\bullet(\mathscr{L}_{X|Y})$, one can find $s_n \in R_\bullet(\mathscr{L})$ satisfying

$$\|s_n\|_{n\phi} \leqslant C(\phi, Y, X) \cdot \|t_n\|_{n\phi|_Y}.$$

Proof. — By Proposition 3.4.7 and the assumption, the Banach algebra norm $\|\!|\cdot|\!\|_{\phi_{X|Y}}$ is an affinoid algebra norm. As $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, \phi_{X|Y})$ is reduced by Proposition 2.1.26, by Proposition 1.3.22, there exist $C(\phi, Y) > 0$ such that

$$\|\!|\cdot|\!\|_{\phi_{X|Y}} \leqslant C(\phi, Y, X) \cdot \|\!|\cdot|\!\|_{\phi_{X|Y};\mathrm{sp}}. \tag{5}$$

Since $\|\!|\cdot|\!\|_{\phi_{X|Y};\mathrm{sp}} = \|\!|\cdot|\!\|_{\phi|_Y}$ by Corollary 2.4.4, one gets the desired estimate. □

Remark 3.4.9. — With the metric finiteness properties of affinoid algebra norm, here the upper bound for metric extension of a Fubini-Study metric is much better than the bound (4) or even the bound (3), for its (in)depence on $n \in \mathbb{N}$.

3.4.2. Algebra norm induced by semipositive metric. — With the extra assumption of discreteness of valuation of the base valued field, one would be able to give another proof of Theorem 3.3.5, if Proposition 3.4.7 holds for ample $\mathscr{L}$ and any m-th Fubini-Study metric ϕ. Recall that we've fixed $M \in \mathbb{N}$ such that the restriction maps $R_n(\mathscr{L}) \to R_n(\mathscr{L}|_Y)$ are surjective for all $n \geqslant M$.

Theorem 3.4.10. — Suppose that $(k, |\cdot|)$ is discretely valued. Assume that Proposition 3.4.7 holds for ample $\mathscr{L}$ and any m-th Fubini-Study metric ϕ. Let ϕ be a semipositive metric on $\mathscr{L}$. Then for any $\epsilon > 0$, there exist $n_Y \in \mathbb{N}$ such that for any $n \geqslant n_Y$ and any $t_n \in H^0(Y, \mathscr{L}|_Y^{\otimes n})$, there exits $s_n \in H^0(X, \mathscr{L}^{\otimes n})$ such that $s_n|_Y = t_n$ and

$$\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.$$

Proof. — By the semipositivity assumption, there exist norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ on $R_n(\mathscr{L})$ such that ϕ is given by $\mathcal{P}(\{\|\cdot\|_n\}_{n\in\mathbb{N}})$. Let ψ_n denote the metric $\frac{1}{n}\mathrm{FS}(\|\cdot\|)_n$ on $\mathscr{L}$. Since ϕ is continuous, the convergence of $\{\psi_n\}_{n\in\mathbb{N}}$ to the envelope metric ϕ is uniform. So for any $\epsilon > 0$, there exist $M' \geqslant M$ such that

$$\mathrm{dist}(\psi_{M'}, \phi) \leqslant \frac{1}{3}\epsilon,$$

hence for any $n \in \mathbb{N}$, one has

$$\mathrm{dist}(\|\cdot\|_{n\psi_{M'}}, \|\cdot\|_{n\phi}) \leqslant \frac{1}{3}n\epsilon, \quad \mathrm{dist}(\|\cdot\|_{n\psi_{M'},X|Y}, \|\cdot\|_{n\phi,X|Y}) \leqslant \frac{1}{3}n\epsilon.$$

By our assumption, the Banach algebra $\widehat{R}_\bullet(\mathscr{L}_{X|Y}, (\psi_{M'})_{X|Y})$ is an affinoid algebra, thus there exist $C_{M'}(X, Y, \phi) > 0$ such that

$$\forall t \in R_\bullet(\mathscr{L}_{X|Y}), \quad |||t|||_{\psi_{M'},X|Y} \leqslant C_{M'} \cdot |||t|||_{\psi_{M'},X|Y;\mathrm{sp}} = |||t|||_{\psi_{M'},Y}.$$

Combining this comparison with above estimates, for every $n \geqslant n_Y := \lceil \ln(C_{M'})/(\epsilon/3) \rceil$, one has

$$\begin{aligned}
\|t_n\|_{n\phi,X|Y} &\leqslant \mathrm{e}^{\frac{1}{3}n\epsilon} \cdot \|t_n\|_{n\psi_{M'},X|Y} \\
&\leqslant \mathrm{e}^{\frac{1}{3}n\epsilon} \cdot C_{M'} \cdot \|t_n\|_{n\psi_{M'}|_Y} \\
&\leqslant \mathrm{e}^{\frac{2}{3}n\epsilon} \cdot C_{M'} \cdot \|t_n\|_{n\phi|_Y} \\
&\leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}.
\end{aligned}$$

Hence there exist $s_n \in R_n(\mathscr{L})$ with $\|s_n\|_{n\phi} \leqslant \mathrm{e}^{n\epsilon} \cdot \|t_n\|_{n\phi|_Y}$. □

Remark 3.4.11. — The continuity assumption on ϕ is necessary in this approach, for deducing a *uniform* convergence to ϕ by m-th Fubini-Study metrics.

CHAPTER 4

CONTINUITY OF ENVELOPE METRIC

In this chapter we study the continuity of the semipositive envelope metric $\mathcal{P}(\phi)$ of a continuous metric ϕ on the ample line bundle $\mathscr{L}$. We interpret the continuity of $\mathcal{P}(\phi)$ as a neighbourhood property for the holomorphic convex set $\mathfrak{M}(\phi)$. This helps to reduce the continuity problem to a non-Archimedean version of Cartan-Thullen's result about holomorphic convexity.

4.1. Background and results

4.1.1. The Problem. — We maintain the basic setting as in previous chapters. Let k be a field, X be an integral projective scheme over $\operatorname{Spec} k$, $\mathscr{L}$ be an ample line bundle over X. The algebraic positivity of $\mathscr{L}$ plays a central role in establishing vanishing theorems and their various consequences. If k is equipped with a complete absolute value $|\cdot|$, one can construct the corresponding analytic space and put a continuous metric $\phi = (|\cdot|_\phi(x))_{x\in X^{\mathrm{an}}}$ on the analytification $\mathscr{L}^{\mathrm{an}}$. It turns out that the algebraic positivity of $\mathscr{L}$ can be captured by the metric positivity of the pair $(\mathscr{L}, \phi)$, and many problems concerning (algebraic) positive line bundle can be studied via analytic methods with the help of a positive metric ϕ.

In case where the metric ϕ is not positive, there is a way to construct a positive metric with some positivity from ϕ itself. One builds a sequence of positive metrics ψ_n on $\mathscr{L}^{\otimes n}$ from ϕ, and takes the limit metric of the (normalized) sequence of these positive metrics. The limit is a metric on $\mathscr{L}$, and it still remains to be positive (in some sense). In both Archimedean and non-Archimedean settings, Fubini-Study metrics are often used as models for ψ_n: they are constructed from an (in fact any) orthogonal basis $\{s_{n,j}\}_{j\in\{0,\dots,d_n\}}$ of the normed vector space $(H^0(X, \mathscr{L}^{\otimes n}), \|\cdot\|_n)$. If the norms $\|\cdot\|_n$ are chosen to be the norms induced by $n\phi$ (the L^2-norm with respect to a fixed volume form on X^{an} in the Archimedean setting and the supremum norm in the non-Archimedean setting), then the limit metric is called the Fubini-Study envelope metric of ϕ, and is denoted by $\mathcal{P}(\phi)$. As the positivity of $\mathcal{P}(\phi)$ is inherited from the positivity of global objects (i.e. the Fubini-Study metrics), one shall call it a global positivity.

While inheriting the global positivity from Fubini-Study metrics, the envelope metric $\mathcal{P}(\phi)$ may not be continuous on X^{an}. In general, the approximation procedure only yields a possibly singular metric, as the convergence to the limit metric is just pointwise on

each fiber of $\mathscr{L}^{\mathrm{an}}$. The continuity of envelope metric is important for many problems. For example, when X is proper, the continuity of envelope metric is equivalent to the uniformity of the approximation of $\mathcal{P}(\phi)$ by Fubini-Study metrics ψ_n. This uniformity of approximation is indispensable to establishing uniform metric extensions for restricted sections of $\mathscr{L}$ from a sub-variety Y of X (see the previous chapter). Another example concerns some functionals which are defined on the space of all *continuous* metrics such as the relative metric volume. They are devised in search for a canonical metric on $\mathscr{L}$, and there is a need to put $\mathcal{P}(\phi)$ as an argument in these functionals.

4.1.2. Complex case. — In the complex analytic setting, the positivity of a metric is defined by local objects, as the pluri-subharmonicity (psh) of the norm function $|e(x)|_\phi$ of any local regular section e of $\mathscr{L}$. This local positivity of ϕ implies the global positivity of ϕ [61][62, Theorem 3.5], namely that ϕ is a pointwise limit of Fubini-Study metrics. Concerning the regularity, a deep result of regularization of psh functions using L^2 methods for $\overline{\partial}$ from [47] implies that

Theorem 4.1.1. — Suppose that X is normal. If ϕ is continuous, then $\mathcal{P}(\phi)$ is also continuous. ([26], see also [39, Appendix])

4.1.3. Non-Archimedean case. —

4.1.3.1. *Previous results: "Zariski" approach.* — In the non-Archimedean case, semi-positivity of a metric is first defined in [62] using model metrics in the approximation. For various other definitions of semi-positivity, we refer to [13] for a clear discussion. In this vein, the envelope metric continuity has been studied in [13], [35] and [11]. In particular, the following cases are known from these works.

Theorem 4.1.2. — Suppose that X is normal. Suppose that the base field k is of equi-characteristic 0, and the valuation is discrete or trivial. If ϕ is continuous, then $\mathcal{P}(\phi)$ is also continuous. ([13],[12])

Theorem 4.1.3. — Suppose that X is normal. Suppose that the base field k is of positive equi-characteristic p, and the valuation $|\cdot|$ is discrete. Under the hypothesis of the existence of a resolution of singularity in characteristic p, if ϕ is continuous, then $\mathcal{P}(\phi)$ is also continuous. ([35])

These works adopt the "Zariski" point of view to the study of analytic spaces, emphasizing the role of integral models and using algebro-geometric techniques. In these works, the semipositivity of a metric is defined by approximation by model metrics, in the sense of [7] and [62]. In [13], a careful analysis of behavior of different models is carried out, and a wonderful framework of non-Archimedean pluripotential theory was develpped. Their general strategy is to analyze the behavior of metrics on the subset of X^{an} consisting of quasi-monomial points. First, a sequence of vertical ideals is associated with an SNC model for $(X, \mathscr{L})$, and a sequence of model metrics is thus constructed. Then the uniform convergence of this sequence of metrics to the envelope metric is captured by a sequence of multiplier ideals constructed from the integral models. Further properties of multiplier

ideals gives control on the approximation and finally lead to the uniform convergence. Additional hypothesis on the base field are imposed in order to facilitate the use of multiplier ideals and of SNC models.

In [35], some of the assumptions on the base field has been removed. They can also treat the equal positive characteristic case. Hypothesis of resolution of singularities is imposed to deal with SNC models, and multiplier ideals are replaced by test ideals for the control of approximation.

In view of these results, it is conjectured in [11, Conjecture 6.20] that $\mathcal{P}(\phi)$ can inherit the continuity of ϕ, in a general setting without much further assumptions on the valued base field. Other implications related to this conjecture concerning regularization of non-Archimedean psh functions can be found in [11, Section 6].

4.1.3.2. *Our proposal: "Berkovich" approach.* — In this chapter we would like to study the continuity problem from the "Berkovich" point of view, emphasizing the role of analysis of holomorphic functions. We propose an analytic route, which is independent of the aforementioned algebro-geometric methods. We reduce this problem to a property of holomorphic convex sets in non-Archimedean analytic space. Namely, we believe that the following assertion should hold:

***Speculation 4.1.4* (4.2.1).** — Suppose that $(k, |\cdot|)$ is spherically complete. Let Z be the analytification of a normal affine variety. For any compact set K and U in Z with $K \Subset U$, one has $\widehat{K} \Subset \widehat{U}$. In other words, for and any neighbourhood U of K in Z, the envelope $\widehat{U}$ is a neighbourhood of the envelope $\widehat{K}$. (The holomorphic envelope is taken in Z.)

The main result establishes the implication of the above property to the continuity problem.

Theorem 4.1.5. — Suppose that the base valued field $(k, |\cdot|)$ is spherically complete. Suppose that X is normal. Assume that Speculation 4.2.1 is true. Then if ϕ is continuous, the envelope metric $\mathcal{P}(\phi)$ is also continuous.

Remark 4.1.6. — Valued fields as $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, verify the assumptions. On the contrary, $\mathbb{C}_p$ is a complete valued field that is not spherically complete.

We study the continuity of the envelope metric by the geometry of holomorphic convexity. Our approach is quite different from previous ones. We benifit from [24] to use an equivalent definition of semi-positivity via approximation by general Fubini-Study metrics (instead of those which are model metrics). Thanks to this relaxation, our analysis does not concern with integral models. Further, we investigate the envelope metric from the point of view of normed *algebra* of global sections (instead of the sequence of normed *vector spaces* of sections). The basic construction is as follows, which relates the envelope metric with the spectrum of normed section algebra (see chapter II for the details).

— Normed algebra of sections. For an ample line bundle $\mathscr{L}$ over a projective variety X, we consider the graded algebra of sections

$$R_\bullet(\mathscr{L}) = \bigoplus_{n\in\mathbb{N}} R_n(\mathscr{L}) := \bigoplus_{n\in\mathbb{N}} H^0(X, \mathscr{L}^{\otimes n}),$$

If $\mathscr{L}^{\mathrm{an}}$ is equipped with an upper-semicontinuous metric, we construct a graded sub-multiplicative norm on this algebra

$$|||(s_n)|||_\phi = \sup_{n\in\mathbb{N}} \|s_n\|_{n\phi}$$

where $\|\cdot\|_{n\phi}$ are sup norms on $H^0(X, \mathscr{L}^{\otimes n})$ with respect to the metric $n\phi$. We can consider a Banach algebra, the completion of the algebra of sections with respect to this norm, which is named as the *normed algebra of sections*. This Banach algebra is denoted by $\widehat{R}_\bullet(\mathscr{L}, \phi)$.

— Spectrum. The Zariski spectrum of $R_\bullet(\mathscr{L})$ is the affine cone of $\mathscr{L}$ and is denoted by $\mathbf{C}(\mathscr{L})$. It can be identified with the total space of $\mathscr{L}^\vee$, denoted by $\mathbf{V}(\mathscr{L})$, with the zero section contracted to the origin by a morphism p. Inside $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$ one finds a compact set, the Berkovich spectrum of $\widehat{R}_\bullet(\mathscr{L}, \phi)$, denoted by $\mathfrak{M}(\mathscr{L}, \phi)$. The corresponding compact set of pre-image of $\mathfrak{M}(\mathscr{L}, \phi)$ in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ is exactly the dual unit disc bundle of $\mathscr{L}$ with respect to the envelope metric, denoted by $\overline{\mathbb{D}}^\vee(\mathscr{L}, \mathcal{P}(\phi))$. One may also look at the smaller dual unit disc bundle $\overline{\mathbb{D}}^\vee(\mathscr{L}, \phi)$ with respect to the original metric, and denote by $\mathfrak{M}^-(\mathscr{L}, \phi)$ its image under p^{an}. The spaces and relations are described by the following Cartesian digram for maps between topological spaces, where p^{an} is the analytic morphism of contraction of the zero section.

$$\begin{tikzcd}
\overline{\mathbb{D}}^\vee(\mathscr{L}, \phi) \arrow[r, two heads, "p^{\mathrm{an}}"] \arrow[d, hook] & \mathfrak{M}^-(\mathscr{L}, \phi) \arrow[d, hook] \\
\overline{\mathbb{D}}^\vee(\mathscr{L}, \mathcal{P}(\phi)) \arrow[r, two heads, "p^{\mathrm{an}}"] \arrow[d, hook] & \mathfrak{M}(\mathscr{L}, \phi) \arrow[d, hook] \\
\mathbf{V}(\mathscr{L})^{\mathrm{an}} \arrow[r, two heads, "p^{\mathrm{an}}"] & \mathbf{C}(\mathscr{L})^{\mathrm{an}}
\end{tikzcd}$$

From this result, one can recover the envelope metric $\mathcal{P}(\phi)$ directly from $\mathfrak{M}(\mathscr{L}, \phi)$, which is itself constructed from ϕ in a simple way. It then suffices to study the envelope metric from the geometry of these associated sets. Indeed, the global positivity of the metric $\mathcal{P}(\phi)$ is translated to the holomorphic convexity of the compact set $\mathfrak{M}(\mathscr{L}, \phi)$. The limit construction of $\mathcal{P}(\phi)$ from ϕ is translated to the operation of taking holomorphic envelope.

***Proposition 4.1.7* (2.3.4).** — In $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$, one has $\widehat{\mathfrak{M}^-(\phi)} = \mathfrak{M}(\phi)$.

Our study of the continuity problem is based on geometric considerations of the dual unit disc bundles. The point of attack is the effect of dilation along fibers of $(\mathscr{L}^\vee)^{\mathrm{an}}$ to the dual unit disc bundle with respect to a metric. Recall that for any $\epsilon \in \mathbb{R}$, we denote by $\phi(\epsilon)$ the dilated metric defined by

$$\forall x \in X^{\mathrm{an}},\ \forall s_n \in R_n(\mathscr{L}),\ |s_n(x)|_{n\phi(\epsilon)} := |s_n(x)|_{n\phi} \cdot \mathrm{e}^{n\epsilon}$$

Note that for $\epsilon > 0$, the dilated metric is bigger, so $\overline{\mathbb{D}}^\vee(\mathscr{L}, \psi)$ is contained in $\overline{\mathbb{D}}^\vee(\mathscr{L}, \psi(\epsilon))$. The metric ψ is upper/lower-semicontinuous if the function $x \mapsto |s_n(x)|_{n\phi}$ is so. The observation is that for an upper semicontinuous metric, the dilation on its dual unit disc bundle detects the possible discontinuity of the metric.

***Proposition 4.1.8* (4.3.1).** — Let ψ be an upper-semicontinuous metric on $\mathscr{L}$. The metric is continuous if and only if $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi(\epsilon))$ (resp. $\mathfrak{M}^{-}(\psi(\epsilon))$) is a neighbourhood in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ of $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi)$ (resp. neighbourhood in $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$ of $\mathfrak{M}^{-}(\psi)$) for any sufficiently small $\epsilon \in \mathbb{R}_{>0}$. In other words, the metric is continuous if and only if $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi)$ is contained *in the (topological) interior* of $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi(\epsilon))$.

With this geometric criterion, the continuity properties of ϕ and $\mathcal{P}(\phi)$ are related by the following diagram. Each term is seen as a topological space with the subspace topology inherited from the ambient space $\mathbf{V}(L)^{\mathrm{an}}$ (resp. $\mathbb{C}(L)^{\mathrm{an}}$). The symbol $\Subset$ means inclusion in the topological interior, and $\widehat{(\cdot)}$ means the operation of taking holomorphic convex envelope in the ambient space.

$$\begin{tikzcd}
\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi(\epsilon)) \arrow[r, squiggly, "\widehat{\cdot}"] & \overline{\mathbb{D}}^{\vee}(\mathscr{L}, \mathcal{P}(\phi)(\epsilon)) \\
\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \phi) \arrow[u, dotted, "(?)\Subset"] \arrow[r, squiggly, "\widehat{\cdot}"] & \overline{\mathbb{D}}^{\vee}(\mathscr{L}, \mathcal{P}(\phi)) \arrow[u, dotted, "(?)\Subset"']
\end{tikzcd}$$

Equivalently one may look at the following diagram, where one sends the previous diagram by the map $p^{\mathrm{an}} : \mathbf{V}(\mathscr{L})^{\mathrm{an}} \to \mathbf{C}(\mathscr{L})^{\mathrm{an}}$

$$\begin{tikzcd}
\mathfrak{M}^{-}(\phi(\epsilon)) \arrow[r, squiggly, "\widehat{\cdot}"] & \mathfrak{M}(\phi(\epsilon)) \\
\mathfrak{M}^{-}(\phi) \arrow[u, dotted, "(?)\Subset"] \arrow[r, squiggly, "\widehat{\cdot}"] & \mathfrak{M}(\phi) \arrow[u, dotted, "(?)\Subset"']
\end{tikzcd}$$

To transfer the continuity of ϕ to the continuity of $\mathcal{P}(\phi)$, it then suffices to show the "commutativity" of this diagram of operations. In fact such a phenomenon of "commutativity of holomorphic convex envelope and neighbourhood" should hold in a more general setting, for any compact set inside any ambient space of affine analytic space. This commutativity is indeed the content of Speculation 4.2.1.

With this commutativity, one easily deduces the continuity of $\mathcal{P}(\phi)$ from the continuity of ϕ.

Note that our method is of global nature, and the ampleness assumption on L is thus important.

4.2. Holomorphic convexity of holomorphic envelope

In this section we make a synopsis of the following commutativity property, listing some attempts and difficulties, without being able to give a complete proof. *This section does not contain any result and can be jumped.*

Speculation 4.2.1. — Assume that $(k, |\cdot|)$ is spherically complete. Let Z be the analytification of a normal affine variety. For any compact set K and U in Z with $K \Subset U$, one has $\widehat{K} \Subset \widehat{U}$. In other words, for and any neighbourhood U of K in Z, the envelope $\widehat{U}$ is a neighbourhood of the envelope $\widehat{K}$. (The holomorphic envelope is taken in Z.)

4.2.1. Reformulation by holomorphic convexity. — Indeed, one can reformulate the commutativity property as confirming the holomorphic convexity of $\widehat{U}$ (in Z). Namely, any relative compact subset K contained in $\mathrm{Int}^t\,\widehat{U}$ has a holomorphic envelope (in Z) $\widehat{K}$ which should be still contained in $\mathrm{Int}^t\,\widehat{U}$.

The notion of Stein spaces and holomorphic convexity in the Berkovich non-Archimedean analytic setting has been recently systematically studied in [44]. In particular, analogues of important classical results concerning Stein spaces has been established, such as Cartan theorem A and B, as well as many other fundamental intermediate results. As coherent analytic sheaf methods are used, mainly in their study, holomorphic convexity of *open* sets are dealt with.

Here for our application to the continuity problem of envelope metrics, we need to establish the holomorphic convexity for the holomorphic envelope of a *closed* set, namely $\mathfrak{M}(\phi(\epsilon))$ in our case. As we do not know much about the boundary and interior of this closed envelope set, it seems not easy to apply directly results from [44] to establish holomorphic convexity. Hence we perform another study.

4.2.2. Core issues. — To show the key result of holomorphic convexity, for any $z_0 \in \widehat{K}$, one needs to find a neighbourhood V of z_0 in Z, itself contained in $\widehat{U}$. There are two basic issues to be dealt with:

1. To characterize points such as z_0, in holomorphic envelope $\widehat{U}$, by knowledge of the original set U;
2. To propagate such characterizations to some neighbourhood around $z_0 \in \widehat{K}$, from the assumption $K \Subset U$. In other words, performing an "analytic continuation" of the characterization.

Inspired by the approaches in the complex analytic setting, one has basically two different points of view. One shall see that each of them individually encounters obstacles in the non-Archimedean setting, yet a combination of them might possibly work.

Note that in our case holomorphic envelope coincides with the polynomial envelope. Thus it suffices to use polynomials to define the envelope set.

4.2.3. Function-theoretic point of view. — From the function-theoretic point of view, points in a holomorphic envelope set, such as z, are detected by the defining boundedness conditions of the corresponding evaluation map, and the analytic continuation is simply based on the Taylor development of the function around z.

This strategy is inspired by the Cartan-Thullen's argument in the $\mathbb{C}$-analytic setting. For simplicity of presentation one may look at the case where Z is $(\mathbb{A}^n)^{\mathrm{an}}$. Let $|||\cdot|||_K$ (resp. $|||\cdot|||_U$) be the sup norm on K (resp. on U). For any polynomial $f \in k[\boldsymbol{T}]$, one can try to do the Taylor expansion at z, and gets

$$f(w) = \sum_{L\in\mathbb{N}^d} \partial_L f(z)\cdot(\boldsymbol{T}(w)-\boldsymbol{T}(z))^L,$$

where $\partial_L f(z)$ are multi-partial derivatives of $f(z)$ (Taylor coefficients) which are still polynomials. Then the coefficients $|\partial_L f(z)|$ are controlled from above by $|||\partial_L f|||_K$ thanks to the fact that $z \in \widehat{K}$. Moreover, as $K \Subset U$, by Cauchy estimates one can find $\delta > 0$ such that each term of the form $|||\partial_L f|||_K \cdot \delta^{|L|}$ is bounded from above by $|||f|||_U$. Thus for

points w in the neighbourhood of z in Z defined by the inequalities $|T_i(w) - T_i(z)| < \delta$, one has $|f(z)| \leqslant |\!|\!| f |\!|\!|_U$. Thus this neighbourhood is contained in $\widehat{U}$. Besides, one may also reformulate the above argument by considering the action of an analytic group $(\mathbb{G}_a^n)^{\mathrm{an}}$ on $(\mathbb{A}^n)^{\mathrm{an}}$. The theory developped in [3, Chapter 5] implies, by the continuity of the action, that the convolution point $\xi(\delta) * z$ with a Gauss point $\xi(\delta)$ is contained in $\widehat{U}$ if δ is small. Thus the set $\{\widehat{\xi(\delta) * z}\}$ which is just the polydisc neighbourhood of z constructed by Taylor expansion, is also contained in it.

The central difficulty to carrying out this argument in the non-Archimedean setting is the possibility of performing Taylor development at z. In general, the point z is not a rigid point of Z, this fact impeaches a Taylor development. If one uses base field change to a larger valued field $(k', |\cdot|')$ to make z a k'-rigid point in $Z_{k'}$, the holomorphic envelope behaves well with respect to base change thanks to the study in [44]. But the obtained neighbourhood on $\pi_{k'/k}^{-1}(Z)$ does not necessarily descend to a neighbourhood on Z, as the continuous surjective map $\pi_{k'/k}^{-1} : Z_{k'} \to Z$ is not *open* in general, especially when the field extension is large. In terms of the reformulation of action by $(\mathbb{G}_a^n)^{\mathrm{an}}$, the convolution point $\xi(\delta) * z$ may just coincide with z itself in case where z is itself a "big" point and δ is small (for example, one has $\xi(\delta) * \xi(1) = \xi(1)$).

4.2.4. Spectral-theoretic point of view. — From the spectral-theoretic point of view, one views the holomorphic envelope of a set as the analytic spectrum of the Banach algebra of holomorphic functions defined on this set. Points in the holomorphic envelope are characterized by the continuity of the associated evaluation homomorphism from this stuctural Banach algebra, and the analytic continuation is realized thanks to the perturbation stability of index of Fredholm operators.

More precisely, one takes again Z to be $(\mathbb{A}^n)^{\mathrm{an}}$ for illustration of ideas. One forms the k-Banach algebra $\mathcal{B}(U)$ as the completion of polynomials $k[\boldsymbol{T}]$ with respect to the sup norm $|\!|\!|\cdot|\!|\!|_U$. At any point $z \in Z$, one may look at the evaluation homomorphism $\mathrm{ev}(z)$ from (the dense subalgebra of polynomials of) $\mathcal{B}(U)$ to $\widehat{\kappa}(z)$. Points z in $\widehat{U}$ are characterized by the continuity (i.e. boundedness) of $\mathrm{ev}(z)$. In addition, in view of the Cauchy estimates, points z in the *interior* of $\widehat{U}$ should be characterized by the Fredholmness of the evaluation homomorphism $\mathrm{ev}(z)$, a property stronger than just being continuous. Indeed, since $z_0 \in \widehat{K}$, the relative compacity $K \Subset U$ implies that the evaluation homomorphism $\mathrm{ev}(z_0)$ is completely continuous, hence is Fredholm. Moreover, the index of $\mathrm{ev}(z_0)$ should be positive. As small perturbations of Fredholm operators are still Fredholm with the same index, one sees that if z is in a sufficiently small neighbourhood of $z_0 \in \widehat{K}$, the evaluation homomorphism $\mathrm{ev}(z)$ is still Fredholm with positive index. Thus this neighbourhood is in the interior of $\widehat{U}$, hence is also in $\widehat{U}$.

There are multiple difficulties to carrying out this argument in non-Archimedean setting. The first one is again about the rigidness of point z_0. Contrary to the $\mathbb{C}$-analytic case, the evaluation morphisms $\mathrm{ev}(z)$ lands in different target Banach spaces $\widehat{\kappa}(z)$. Although perturbation theory of Fredholm operators is well-developped in [55] for *fixed* source and target Banach spaces, one does not know how to formulate perturbation theory for *varying* target Banach space. A second difficulty is whether the Fredholm property (with non-zero index) of $\mathrm{ev}(z)$ really characterizes points of the *interior* of spectrum $\widehat{U}$. One would like

to use the Koszul complex of $\mathcal{B}(U)$ with respect to a regular sequence $\{T_i - T_i(z)\}_{i\in\{1,\dots,d\}}$ to study the properties of $\mathrm{ev}(z)$. The situation is not so clear even in the $\mathbb{C}$-analytic case (except in dimension 1).

4.2.5. Mixed point of view. — In view of difficulties encountered in each of the above cases, it seems plausible to treat them separately, according to the nature of the point z. On one extremity, if z is "small", namely a rigid point, one can produce a desired neighbourhood by the first method. On another extremity, if z is "large", for example a point with $\mathfrak{p}_z$ being $\{0\}$, or even more, an Abhyankar point, then one should use a method of different nature to produce the desired neighbourhood. It seems that variabilities of "small" points are geometrical while variabilities of "large" points are spectral. In the intermediate case one can decompose the variability of z into two types of directions along which z behaves as in the two extreme case. We leave this study for the future.

4.3. Neighbourhood of dual unit disc bundle and continuity of metric

One first gives a criteria of continuity of an upper-semicontinuous metric in terms of topological property of its dual unit disc bundle.

Proposition 4.3.1. — Let ψ be an upper-semicontinuous metric on $\mathscr{L}$. The metric is continuous if and only if $\overline{\mathbb{D}}^\vee(\mathscr{L},\psi(\epsilon))$ is a neighbourhood of $\overline{\mathbb{D}}^\vee(\mathscr{L},\psi)$ for any sufficiently small $\epsilon > 0$.

Proof. — The criterion is clear from the following geometric picture, while the argument is somehow cumbersome. It's obvious that if ψ is continuous, then the dilated dual unit disc bundle is a neighbourhood of the original dual unit disc bundle.

Conversely, if ψ is not continous, let $x_0 \in X^{\mathrm{an}}$ be one of the discontinuity point. Take any global section $s_n \in R_n(\mathscr{L})$ such that $s_n(x_0) \neq 0$. By the discontinuity of ψ at x_0, one has

$$\liminf_{x\to x_0}|s_n(x)|_{n\psi} < \limsup_{x\to x_0}|s_n(x)|_{n\psi}.$$

As the metric of ψ is upper-semicontinous, one has

$$\limsup_{x\to x_0}|s_n(x)|_{n\psi} \leqslant |s_n(x_0)|_{n\psi}.$$

For any $\epsilon \in \mathbb{R}$, and any $x \in X^{\mathrm{an}}$, let $\xi_x(\epsilon)$ be the Gauss point of the dual unit disc over x in $\overline{\mathbb{D}}^\vee(\mathscr{L},\psi(\epsilon))$. One has

$$|s_n(x)|_{n\psi(\epsilon)} = |s_n|_{\xi_x(\epsilon)}.$$

The upper-semicontinuity of the metric of ψ implies that there exist $\delta > 0$ such that

$$\liminf_{x\to x_0}|s_n(x)|_{n\psi} = |s_n|_{\xi_{x_0}(\delta)}.$$

Denote by R the subset of X^{an} of the points x satisfying

$$|s_n(x)|_{n\psi} \leqslant \mathrm{e}^{n\delta/2}\cdot\liminf_{x'\to x_0}|s_n(x')|_{n\psi},$$

or equivalently the points x satisfying

$$|s_n|_{\xi_x(0)} \leqslant \mathrm{e}^{n\delta/2}\cdot|s_n|_{\xi_{x_0}(-\delta)} = |s_n|_{\xi_{x_0}(-\delta/2)}.$$

Since x_0 is a discontinuous point, by construction, for any neighbourhood U of x_0 in X^{an}, one has $U \cap R \neq \varnothing$.

For any $\epsilon \in \mathbb{R}$ such that $0 < \epsilon < \delta/4$, we show that any open neighbourhood Q of $\xi^0_{x_0}$ in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$ can not be contained in $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi(\epsilon))$. One may also assume that Q is contained in the open set $\{z : |s_n|_z > |s_n|_{\xi_{x_0}(-\delta/4)}\}$ up to replacing Q by its intersection with this open set. See the picture blow.

Since the map $\pi^{\mathrm{an}} : \mathbf{V}(\mathscr{L})^{\mathrm{an}} \to X^{\mathrm{an}}$ is open, the set $\pi^{\mathrm{an}}(Q)$ is an open neighbourhood of x_0, hence has non-empty intersection with R. Let x be a point in $R \cap \pi^{\mathrm{an}}(Q)$. For any $z \in (\pi^{\mathrm{an}})^{-1}(x) \cap Q$, by the construction of Q one has

$$|s_n|_z > |s_n|_{\xi_{x_0}(-\delta/4)},$$

whereas by the construction of R and the choice of ϵ one has

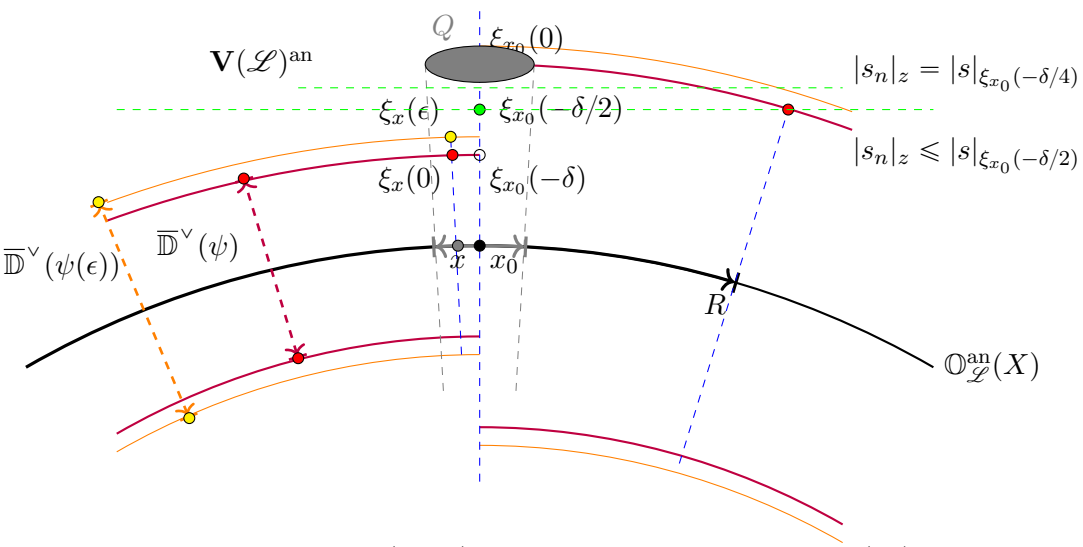
$$|s_n|_{\xi_x(\epsilon)} = \mathrm{e}^{n\epsilon} \cdot |s_n|_{\xi_x(0)} < \mathrm{e}^{n\epsilon} \cdot |s_n|_{\xi_{x_0}(-\delta/2)} < |s_n|_{\xi_{x_0}(-\delta/4)},$$

so ξ^{ϵ}_x is not contained in Q. Thus Q is not contained in $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi(\epsilon))$. From this, one deduces that $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi(\epsilon))$ is not a neighbourhood of $\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi)$.

Hence the criteria for continuity by neighbourhoodness holds. □

Recall that for any upper-semincontinuous metric ψ, we denote by $\mathfrak{M}^{-}(\psi)$ the compact set $p^{\mathrm{an}}(\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \psi))$ and by by $\mathfrak{M}(\psi)$ the compact set $p^{\mathrm{an}}(\overline{\mathbb{D}}^{\vee}(\mathscr{L}, \mathcal{P}(\psi)))$ in the affine cone $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$.

Corollary 4.3.2. — Let ψ be an upper-semicontinuous metric on $\mathscr{L}^{\mathrm{an}}$. It is continuous if and only if $\mathfrak{M}^{-}(\psi(\epsilon))$ is a neighbourhood of $\mathfrak{M}^{-}(\psi)$ for any sufficiently small $\epsilon > 0$.



Dual unit disc bundle of $(\mathscr{L}, \psi)$ and its dilation by e^{ϵ} in $\mathbf{V}(\mathscr{L})^{\mathrm{an}}$

Theorem 4.3.3. — Suppose that $(k, |\cdot|)$ is spherically complete. Suppose that X is normal. Assume that Speculation 4.2.1 holds. If ϕ is continuous, then $\mathcal{P}(\phi)$ is also continuous.

Proof. — Without loss of generality, one may assume that $\mathscr{L}$ is very ample. One takes the affine analytic space $\mathbf{C}(\mathscr{L})^{\mathrm{an}}$ as Z. For any $\epsilon \in \mathbb{R}$, consider the compact set $\mathfrak{M}^{-}(\phi(\epsilon))$. Then by Proposition 2.3.4, one has

$$\widehat{\mathfrak{M}^{-}(\phi(\epsilon))} = \mathfrak{M}(\phi(\epsilon)).$$

Note that for $\epsilon > 0$, by the continuity of ϕ, $\mathfrak{M}^{-}(\phi(\epsilon))$ is a neighbourhood of $\mathfrak{M}^{-}(\phi)$ in V. Hence by Speculation 4.2.1, the set $\mathfrak{M}(\phi(\epsilon))$ is a neighbourhood of $\mathfrak{M}(\phi)$ in Z. By Corollary 4.3.1, the envelope metric $\mathcal{P}(\phi)$ is continuous. $\square$

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