Results of compactness and regularity in a non-local Ginzburg-Landau model resulting from micromagnetism. Poincaré lemma and domain regularity

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Abstract

In this thesis, we study some boundary value problems involving micromagnetic models and differential forms.

In the first part, we consider a nonlocal Ginzburg-Landau model arising in micromagnetics with an imposed Dirichlet boundary condition. The model typically involves $\mathbb{S}^2$-valued maps with an energy functional depending on several parameters, which represent physical quantities. A first question concerns the compactness of magnetizations having the energies of several Néel walls of finite length and topological defects when these parameters converge to 0. Our method uses techniques developed for Ginzburg-Landau type problems for the concentration of energy on vortex balls, together with an approximation argument of $\mathbb{S}^2$-valued vector fields by $\mathbb{S}^1$-valued vector fields away from the vortex balls. We also carry out in detail the proofs of the C^∞ regularity in the interior and $C^{1,\alpha}$ regularity up to the boundary, for all $\alpha \in (0, \frac{1}{2})$, of critical points of the model.

In the second part, we study the Poincaré lemma, which states that on a simply connected domain every closed form is exact. We prove the Poincaré lemma on a domain with a Dirichlet boundary condition under a natural assumption on the regularity of the domain: a closed form f in the Hölder space $C^{r,\alpha}$ is the differential of a $C^{r+1,\alpha}$ form, provided that the domain itself is $C^{r+1,\alpha}$. The proof is based on a construction by approximation, together with a duality argument. We also establish the corresponding statement in the setting of higher order Sobolev spaces.

Keywords: compactness, critical point, Dirichlet condition, divergence equation, Hölder and Sobolev spaces, harmonic maps, Néel wall, micromagnetics, Poincaré lemma, regularity, vortex, Ginzburg-Landau theory.

Résumé

Dans cette thèse, nous étudions des problèmes aux limites impliquant le modèle micromagnétique et les formes différentielles.

Dans la première partie, nous considérons un modèle non-local de Ginzburg-Landau apparaissant en micromagnétisme avec une condition au bord de type Dirichlet. Le modèle typique implique une fonctionelle d'énergie définie pour des applications des valeurs dans la sphère $\mathbb{S}^2$ et qui depend de plusieurs paramètres, qui représentent des quantités physiques. Une première question concerne la compacité des aimantations ayant les énergies de quelques parois de Néel de longueur finie et des défauts topologiques lorsque ces paramètres convergent vers 0. Notre méthode utilise des techniques développées pour les problèmes de type Ginzburg-Landau sur la concentration d'énergie autour des vortex, avec un argument d'approximation des champs de vecteurs dans $\mathbb{S}^2$ par des champs de vecteurs dans $\mathbb{S}^1$ éloignés des vortex. Nous effectuons également en détail la preuve de la régularité C^∞ à l'intérieur et la régularité $C^{1,\alpha}$ au bord, pour tous les $\alpha \in (0, \frac{1}{2})$, des points critiques du modèle.

Dans la deuxième partie, nous étudions le lemme de Poincaré qui affirme que sur un domaine simplement connexe chaque forme fermée est exacte. Nous prouvons le lemme de Poincaré sur un domaine avec une condition aux limites de Dirichlet sous une hypothèse naturelle sur la régularité du domaine : une forme fermée f dans l'espace $C^{r,\alpha}$ est la différentielle d'une forme $C^{r+1,\alpha}$ à condition que le domaine lui-même soit $C^{r+1,\alpha}$. La preuve est basée sur une construction par approximation, avec un argument de dualité. Nous établissons également le résultat correspondant dans le cadre d'espaces de Sobolev d'ordre supérieur.

Mots cléfs: compacité, point critique, condition de Dirichlet, équation de divergence, espaces de Hölder et de Sobolev, application harmonique, paroi de Néel, micromagnétisme, lemme de Poincaré, régularité, vortex, théorie de Ginzburg-Landau.

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Contents

Part I

Compactness and regularity results in a non-local Ginzburg-Landau model arising in micromagnetics

Chapter 1

Introduction

1.1 Micromagnetics

Micromagnetics is a field of physics that deals especially with the behavior of ferromagnetic materials at very small length scales. In this theory, the ferromagnetic material is characterized by a 3D vector-field distribution, called magnetization. The micromagnetic model consists in associating to the magnetization a micromagnetic energy, whose local minimizers represent the stable magnetization of the ferromagnetic body. The associated variational problem is non-convex and non-local which make it difficult to solve analytically. Moreover, the multi-scale complexity of the micromagnetic functional creates a lot of asymptotic regimes, depending on the relation between the material and geometrical parameters. This leads to formation of several magnetization patterns.

One of the most extensively researched topics is the qualitative and quantitative analysis of magnetization patterns. Since there are several distinct regimes, by identifying and exploring these regimes, we obtain various type of magnetic walls: 2D wall defects (*Néel wall, Bloch walls*), 1D vortex-lines (*Bloch lines*), boundary vortices. We aim to justify mathematically the physical prediction on the formation and description of these defects.

1.2 The three-dimensional ferromagnetic model with Dzyaloshinsky-Moriya interaction

The open set $\omega \subset \mathbb{R}^3$ denotes the ferromagnetic sample that will be considered later as a cylinder. The magnetization of the ferromagnet ω can be described by a three-dimensional unit-length vector field $m = (m_1, m_2, m_3) : \omega \to \mathbb{S}^2$. In the classical theory of micromagnetics (see the book of Hubert and Schäfer [28], also [19]), the free energy per unit volume of such a magnetization takes the form

$$E_{3D} = d^2 \int_\omega |\nabla m|^2 dx + Q \int_\omega \varphi(m) dx + \int_{\mathbb{R}^3} |\nabla U|^2 dx - 2 \int_\omega H_{ext}.m dx + \int_\omega w_D(m) dx. \quad (1.1)$$

Let us now explain and comment on these five terms.

(i) The first term is called the exchange energy. It penalizes spatial variations of m through the Dirichlet integral of m. The constant d is the exchange length. It is an intrinsic parameter of the material of the order of nanometers.

(ii) The second term is the anisotropy energy which refers to the fact that the properties of a magnetic material are dependent on the directions in which they are measured. The energy density φ is a non negative function called the anisotropy energy density.

It is typically a polynomial with symmetry properties inherited from the crystalline lattice. The zeros of φ stands for the preferred directions of magnetizations. For instance, $\varphi(m) = m_3^2$ favors the easy plan as the horizontal one. The constant Q is a second intrinsic parameter of the material that measures the strength of the anisotropy energy relative to the strength of the exchange and stray-field energy. According to the values of the constant Q, one distinguishes ferromagnetic materials into two broad classes: soft materials ($Q < 1$) and hard materials ($Q > 1$).

(iii) The third term is the energy of the stray field (or the magnetostatic energy), where the stray-field potential $U : \mathbb{R}^3 \to \mathbb{R}$ is generated by the magnetization m through the classical Maxwell equation for electrostatics, that is given by

$$-\Delta U = \nabla.(m\mathbb{1}_\omega) \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

$$i.e., \quad \int_{\mathbb{R}^3} \nabla U.\nabla \xi dx = -\int_\omega m.\nabla \xi dx, \quad \forall \xi \in C_0^\infty(\mathbb{R}^3).$$

In view of (1.2), there are two sources of stray field ∇U: magnetic volume charges (with volume charge density $\nabla.m$ in ω) and magnetic surface charges (with surface charge density $\nu.m$ on $\partial\Omega$, where ν is the normal component of boundary $\partial\omega$).

(iv) The fourth term is the external field energy generated by an applied external field $H_{ext} : \mathbb{R}^3 \to \mathbb{R}^3$. It favors alignment of the magnetization with the external field H_{ext}.

(v) Finally, the last term is the energy connected with the Dzyaloshinsky-Moriya interaction w_D which is considered here in the following form (see Bogdanov and Hubert [3])

$$\begin{aligned} w_D &= D_1 w_1 + D_2 w_2 + D_3 w_3 \\ &= D_1(m_1\partial_{x_2}m_3 - m_2\partial_{x_1}m_3 + m_3(\partial_{x_1}m_2 - \partial_{x_2}m_1)) \\ &+ D_2(m_3\partial_{x_1}m_1 - m_1\partial_{x_1}m_3 + m_3\partial_{x_2}m_2 - m_2\partial_{x_2}m_3) \\ &+ D_3(m_1\partial_{x_3}m_2 - m_2\partial_{x_3}m_1), \end{aligned} \tag{1.3}$$

with arbitrary coefficients D_i. The term w_1 in (1.3) favors the Bloch-like spirals. While the term w_2 favors a rotation along the propagation direction of a spiral structure, as in a Néel wall. Such a rotation is necessarily connected with the stray-field. The last term w_3 in (1.3) favors the formation of spiral structures with propagation vectors along the crystal axis (x_3- axis).

1.3 A reduced two dimensional thin-film model

In this section we will discuss thin magnetic films which involve several length scales: We assume that the ferromagnetic sample is a cylinder

$$\omega = \omega' \times (0, t)$$

of height t and we denote by l a typical length of the base $\omega' \subset \mathbb{R}^2$. This film regime means that the aspect is small, i.e.,

$$h := \frac{t}{l} \ll 1. \tag{1.4}$$

It entails that the variations of m in the third variable are strongly penalized by the energy. Heuristically, we assume that m depends only on the horizontal variable $x' = (x_1, x_2)$

$$m(x) = (m', m_3)(x') \ : \omega \to \mathbb{S}^2 \tag{1.5}$$

and

$$m \text{ varies on length scales } \gg \frac{t}{l}. \tag{1.6}$$

The external field H_{ext} is assumed being in-plane and invariant in x_3 , i.e.,

$$H_{ext}(x) = (H'_{ext}(x'), 0).$$

Notations: in this part, the prime $'$ always indicates a 2D quantity. We denote $a \ll b$ if $\frac{a}{b} \to 0$ and $a \lesssim b$ if $a \leq Cb$ for some universal constant $C > 0$.

Using configuration (1.5), the Dzyolashinsky-Moriya interaction reduces as follows:

$$\begin{aligned}\int_\omega w_D(m)dx \quad &= \int_\omega (D_1 w_1 + D_2 w_2)dx \\ &= \int_\omega \Big(D_1(m.\nabla' \times m) + D_2(m_3\nabla'.m' - m'.\nabla' m_3)\Big)dx, \end{aligned} \tag{1.7}$$

where

$$m.\nabla' \times m = -m_3\partial_{x_2}m_1 - m_2\partial_{x_1}m_3 + m_3\partial_{x_1}m_2 + m_1\partial_{x_2}m_3.$$

The change of variables $x' \mapsto \widetilde{x}' = \frac{x'}{l} \in \frac{\omega'}{l}$ rescales ω to a set $\widetilde{\omega} = \widetilde{\omega}' \times (0, h)$ with $\operatorname{diam}(\omega') = 1$, the external field to $\widetilde{H}'_{ext}(\widetilde{x}') = H'_{ext}(x')$ and the magnetization $\widetilde{m}(\widetilde{x}') = m(x')$. Then, it reduces the exchange, anisotropy, external field and the Dzyolashinsky-Moriya energies to the following form

$$\int_\omega \Big(d^2|\nabla m|^2 + Q\varphi(m) - 2H_{ext}.m + w_D(m)dx\Big)dx$$

$$= td^2\int_{\widetilde{\omega}'}|\widetilde{\nabla}'\widetilde{m}'|^2 d\widetilde{x}' + tl^2\int_{\widetilde{\omega}'}\Big(Q\varphi(\widetilde{m}) - 2\widetilde{H}'_{ext}.\widetilde{m}'\Big)d\widetilde{x}' \tag{1.8}$$

$$+tl\int_{\widetilde{\omega}'}\Big(D_1(\widetilde{m}.\widetilde{\nabla}' \times \widetilde{m}) + D_2(\widetilde{m}_3\widetilde{\nabla}'.\widetilde{m}' - \widetilde{m}'.\widetilde{\nabla}'\widetilde{m}_3)\Big)d\widetilde{x}'. \tag{1.9}$$

Since (1.5), the Maxwell equation (1.2) implies that

$$-\Delta U = \nabla'.m'\mathbb{1}_\omega - m.\nu\mathbb{1}_{\partial\omega} \text{ in } \mathbb{R}^3, \tag{1.10}$$

here ν is the unit outer normal vector on $\partial\omega$. In view of (1.10), there are two sources of stray field ∇U; that is, the magnetic volume charges which are given by the in-plane flux $\nabla'.(m'\mathbb{1}_\omega)$ and the magnetic surface charges on the top and the bottom side of the cylinder which are presented by the third component of the magnetization and the lateral charges $m'.\nu$. Moreover, since (1.10), the non-local magnetostatic energy can be computed by considering the Fourier transform in the horizontal variables,

$$\mathcal{F}(m'\mathbb{1}_{\omega'})(\xi') = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2} e^{-i\xi'.x'}m(x')\mathbb{1}_{\omega'}(x')dx' \quad \text{for } \xi' \in \mathbb{R}^2.$$

One gets (cf. Ignat [29]):

$$\int_{\mathbb{R}^3}|\nabla U|^2 dx = t\int_{\mathbb{R}^2} f(\frac{t}{2}|\xi'|)|\frac{\xi'}{|\xi'|}\cdot\mathcal{F}(m'\mathbb{1}_{\omega'})|^2 d\xi' + t\int_{\mathbb{R}^2} g(\frac{t}{2}|\xi'|)|\mathcal{F}(m_3\mathbb{1}_{\omega'})|^2 d\xi',$$

where

$$g(s) = \frac{1 - e^{-2s}}{2s} \text{ and } f(s) = 1 - g(s) \text{ for every } s \geq 0.$$

In view of (1.6), then the Fourier transform of m is concentrated on wave vectors ξ' of order t/l. Assumption (1.4) implies that the arguments of f and g are small in the range of ξ'. We then approximate $g(s) \approx 1$ and $f(s) \approx s$. Rescaling in the length scale l of ω', the stray-field energy is approximated as follows (see DeSimone, Kohn, Müller and Otto [17], Kohn and Slastikov [36]):

$$\int_{\mathbb{R}^3} |\nabla U|^2 dx \approx \frac{t^2 l}{2} ||(\widetilde{\nabla}'.\widetilde{m}')_{ac}||^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} + \frac{t^2 l}{2\pi} |\log \frac{l}{t}| \int_{\partial\widetilde{\omega}'} (\widetilde{m}'.\widetilde{\nu})^2 d\mathcal{H}^1 + tl^2 \int_{\widetilde{\omega}'} \widetilde{m}_3^2. \quad (1.11)$$

Thus, the stray-field energy asymptotically decomposes into three terms in the thin-film regime. The first one is penalizing the volume charges

$$(\widetilde{\nabla}'.\widetilde{m}')_{ac} = \widetilde{\nabla}'.\widetilde{m}' \mathbb{1}_{\widetilde{\omega}'},$$

as an homogeneous $\dot{H}^{-\frac{1}{2}}$-seminorm and induces the leading order of the energy of Néel walls. The second term counts the lateral charges $\widetilde{m}'.\widetilde{\nu}$ in the L^2-norm and it is responsible for the nucleation of boundary vortices The third term penalizes the surface charges $\widetilde{m}_3$ on the top and bottom of the cylinder and leads to interior vortices (so called Bloch lines).

Summing up (1.8), (1.11), we deduce the following reduced two dimensional thin-film energy:

$$\begin{aligned} E^{red}(\widetilde{m}) \quad &= td^2 \int_{\widetilde{\omega}'} |\widetilde{\nabla}' \widetilde{m}|^2 d\widetilde{x}' + \frac{t^2 l}{2} \int_{\mathbb{R}^2} \left| |\widetilde{\nabla}'|^{-1/2} (\widetilde{\nabla}.\widetilde{m}')_{ac} \right|^2 d\widetilde{x}' \\ &+ \frac{t^2 l}{2\pi} |\log \frac{l}{t}| \int_{\partial\widetilde{\omega}'} (\widetilde{m}'.\widetilde{\nu})^2 d\mathcal{H}^1 + tl^2 \int_{\widetilde{\omega}'} \left(\widetilde{m}_3^2 + Q\varphi(\widetilde{m}) - 2\widetilde{H}'_{ext}.\widetilde{m}' \right) d\widetilde{x}'. \\ &+ tl \int_{\widetilde{\omega}'} \left(D_1(\widetilde{m}.\widetilde{\nabla}' \times \widetilde{m}) + D_2(\widetilde{m}_3 \widetilde{\nabla}'.\widetilde{m}' - \widetilde{m}'.\widetilde{\nabla}' \widetilde{m}_3) \right) d\widetilde{x}'. \end{aligned} \quad (1.12)$$

Scaling the energy at order of td^2, and omitting , the above reduced energy can be written as the following functional:

$$\begin{aligned} E_{\epsilon,\eta,\kappa}(m) \quad &= \int_\Omega |\nabla' m|^2 dx' + \frac{1}{\eta} ||(\nabla'.m')_{ac}||^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} \\ &+ \frac{|\log \frac{2\epsilon^2}{\eta}|}{\pi\eta} \int_{\partial\Omega} (m'.\nu)^2 d\mathcal{H}^1 + \frac{1}{\epsilon^2} \int_\Omega \left(m_3^2 + Q\varphi(m) - 2H_{ext}.m' \right) dx' \\ &+ \kappa_1 \int_\Omega m.\nabla' \times m dx' + \kappa_2 \int_\Omega (m_3 \nabla'.m' - m'.\nabla' m_3) dx', \end{aligned} \quad (1.13)$$

where $\epsilon = d/l, \eta = 2d^2/(tl), \kappa = (\kappa_1, \kappa_2) = (lD_1/d^2, lD_2/d^2)$ and $\Omega = \omega'$. For the convenience, from now on, we denote by $\int_\Omega DM(m)dx'$ the last term in (1.13), that is,

$$\int_\Omega DM(m)dx' = \kappa_1 \int_\Omega m.\nabla' \times m dx' + \kappa_2 \int_\Omega (m_3 \nabla'.m' - m'.\nabla' m_3) dx'.$$

According to the specific thin-film regime, three types of singular pattern of the magnetization occur (see DeSimone, Kohn, Müller and Otto [17], Ignat [29]): Néel walls, interior vortices and boundary vortices. In fact, the formation of one of these patterns depends on the scale ordering of the three terms in the RHS of (1.11). Let us now discuss a non-local Ginzburg-Landau model which is strongly motivated by the above two-dimensional ferromagnetic energy.

1.4 A non local Ginzburg-Landau model

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with a $C^{1,1}$ boundary and let $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function satisfying

$$\deg(g, \partial\Omega) = \boldsymbol{d} > 0.$$

Here, the degree of a C^1- function $g : \partial\Omega \to \mathbb{S}^1$ is defined on the boundary $\partial\Omega$ with the unit tangential vector τ:

$$\deg(g, \partial\Omega) := \frac{1}{2\pi} \int_{\partial\Omega} \det(g, \partial_\tau g) d\mathcal{H}^1.$$

If $g : \partial\Omega \to \mathbb{R}^2$ is C^1- function with $|g|> 0$, we set $\deg(g, \partial\Omega) = \deg(\frac{g}{|g|}, \partial\Omega)$. The notion of degree can be extended to continuous fields and more generally, VMO vector fields, in particular $H^{\frac{1}{2}}(\partial\Omega, \mathbb{S}^1)$ (see Brezis and Nirenberg [10]).

We consider $m = (m_1, m_2, m_3) : \Omega \to \mathbb{S}^2$ be a vector field with the Dirichlet condition:

$$m = (m', m_3) = (g, 0) \text{ on } \partial\Omega, \tag{1.14}$$

and the following micromagnetic energy functional:

$$E_{\epsilon,\eta,\kappa}(m) = \int_\Omega |\nabla' m|^2 dx' + \frac{1}{\epsilon^2} \int_\Omega m_3^2 dx' + \frac{1}{\eta} ||(\nabla'.m')_{ac}||^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} + \int_\Omega DM(m) dx', \tag{1.15}$$

where

$$(\nabla'.m')_{ac} = \nabla'.m' \mathbb{1}_\Omega. \tag{1.16}$$

We emphasize that model (1.15) is motivated by the reduced thin-film model (1.13). In fact, we may ignore the anisotropy and external field terms - they can always be made to interact with the surviving terms by scaling Q and H'_{ext} appropriately. The third term of (1.13) is considered as the penalizing of the lateral charges $m'.\nu$ in $L^2(\partial\Omega)$-norm. Then if $m'.\nu = 0$ on $\partial\Omega$ and $Q = 0, H'_{ext} = 0$, then the reduced thin film model (1.13) can be written exactly as in model (1.15). Moreover, the first and second terms in (1.15) are reminiscent to the Ginburg-Landau energy. In the case of boundary condtion $m'.\nu = 0$ on $\partial\Omega$, the concentration of Ginzburg-Landau energy around one interior vortex or two boundary vortices is proved by Ignat and Otto (see [33, Theorem 3]). Here we want to generalize the vector fields tangent at the boundary by one satisfying the Dirichlet boundary condition (1.14).

In this work, we shall study the reduced two dimensional films with the Dzyaloshinsky-Moriya interaction term given by

$$\int_\Omega DM(m) dx' = \kappa_1 \int_\Omega (m_3 \nabla'.m' - m'.\nabla' m_3) dx' + \kappa_2 \int_\Omega m.\nabla' \times m dx', \tag{1.17}$$

where

$$m.\nabla' \times m = -m_3 \partial_{x_2} m_1 - m_2 \partial_{x_1} m_3 + m_3 \partial_{x_1} m_2 + m_1 \partial_{x_2} m_3.$$

The parameter $\kappa = (\kappa_1, \kappa_2)$ appearing in the Dzyaloshinsky-Moriya energy, stands for the Dzyaloshinsky-Moriya interaction parameter.

The principal questions we shall discuss are the compactness and regularity of minimizer of the non-local energy $E_{\epsilon,\eta,\kappa}$ in a certain regime.

The compactness is presented in Chapter 2. For that, we are interested in the asymptotic behavior of minimizers of the energy $E_{\epsilon,\eta,\kappa}$ in the regime

$$\epsilon \ll 1, \quad \eta \ll 1 \quad \text{and} \quad |\kappa| \gg 1.$$

The singular patterns expected in this context are the Néel walls together with topological defects (due to the boundary condition (1.14)) standing for interior vortices. The regime where we study corresponds to the case where topological defects is energetically more expensive than the Néel wall. Now we shall informally explain how the principle of pole avoidance leads to the formations of walls and vortices.

Vortices. The competition between the exchange energy and the penalization of the m_3 component will try to enforce the condition $m_3 = 0$. Together with the boundary condition (1.14), this explains the formation of *interior vortices*. Here m' carries topological degree, $\deg(m', \partial\Omega) = \boldsymbol{d}$. One expects the nucleation of *interior vortices* of core-scale ϵ. The scaling of the vortex energy is strong related to the Ginzburg-Landau energy (see the seminar book of Bethuel, Brézis and Hélein [2]):

$$\min_{\substack{m' \in H^1(\Omega, \mathbb{R}^2) \\ m'=g \text{ on } \partial\Omega}} \int_\Omega G_\epsilon(m') dx',$$

where the Ginzburg-Landau density energy is given by the following:

$$G_\epsilon(m') := |\nabla' m'|^2 + \frac{1}{\epsilon^2}(1 - |m'|^2)^2.$$

The energetic cost of our vortices is given by

$$2\pi \boldsymbol{d} |\log \epsilon| + O(1).$$

Néel walls. The stray field tries to enforce the divergence-free condition for m'. Moreover, the Dzyaloshinsky-Moriya term also sharpens that condition. Therefore, at the mesoscopic level of magnetization in thin films, we expect

$$|m'| = 1 \quad \text{and} \quad \nabla'.m' = 0 \text{ in } \Omega. \tag{1.18}$$

We note that (1.14) implies $m'.\nu = g.\nu$ on the boundary. In general, the combination of this condition, (1.18) are too rigid for smooth magnetization m'. This can be seen by writing $m' = \nabla'^{\perp}\phi$ with the help of a "stream function" ϕ. Then (1.18) and (1.14) turn into a Dirichlet problem for the eikonal equation in ϕ:

$$|\nabla'^{\perp}\phi| = 1 \text{ in } \Omega \quad \text{and} \quad \nabla'^{\perp}\phi.\nu = g.\nu \text{ on } \partial\Omega.$$

Hence, the divergence-free equation in (1.18) has to be interpreted in the distribution sense and it is expected to induce *line-singularities* for solutions m' . These ridges are an idealization of the wall formation in thin-film elements at the microscopic level. They are replaced by smooth transition layers where the magnetization varies very quickly, see Figure 1.1. Let us recall that the energy $E_{\epsilon,\eta,\kappa}$ per unit length of a Néel wall of angle 2θ

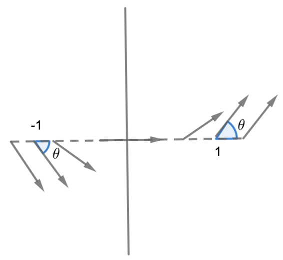

Figure 1.1: Néel wall of angle 2θ confined in $[-1, 1]$.

(with $\theta \in (0, \frac{\pi}{2}]$) is given in DeSimone, Kohn, Muller and Otto [18], Ignat and Otto [32] (see also Ignat [30]):

$$\frac{\pi(1 - \cos\theta)^2 + o(1)}{\eta|\log\eta|} \text{ as } \eta \to 0.$$

Compactness. The first problem we discuss here is the asymptotic as $\epsilon, \eta \to 0, |\kappa| \to \infty$ of families of two dimensional magnetizations when the energy $E_{\epsilon,\eta,\kappa}(m)$ is of order

$$O(\frac{1}{\eta|\log \eta|}) + 2\pi d|\log \epsilon|.$$

One of the issues we discuss is the question of the L^1 compactness of these families of magnetizations. The issue consists in rigorously justifying that the constraint $|m|= 1$ is conserved by the limit configuration as $\epsilon, \eta \to 0, 1/|\kappa| \to \infty$. We emphasize that the regime that we prove the compactness result corresponds to the case where a topological defect is more expensive than the Néel wall. More precisely, we have the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with a $C^{1,1}$ boundary and $g : \partial\Omega \to \mathbb{S}^1$ be a smooth function satisfying $\deg(g, \partial\Omega) = \boldsymbol{d} \in \mathbb{Z}\backslash\{0\}$. We consider the following regime between the parameters $\epsilon \to 0, \eta = \eta(\epsilon)$ and $\kappa = \kappa(\epsilon)$ such that:*

$$|\kappa| \lesssim \frac{1}{\epsilon|\log \epsilon|}, \quad \frac{1}{\eta|\log \eta|} \ll |\log \epsilon|.$$

For each ϵ, consider a $H^1(\Omega, \mathbb{S}^2)$ vector field: $m_\epsilon : \Omega \to \mathbb{S}^2$ such that:

$$m'_\epsilon = g \quad on \quad \partial\Omega$$

and

$$E_{\epsilon,\eta,\kappa}(m_\epsilon) - 2\boldsymbol{d}\pi|\log \epsilon| \lesssim \frac{1}{\eta|\log \eta|}.$$

Then $\{m_\epsilon\}_\epsilon$ is relatively compact in $L^p_{loc}(\Omega, \mathbb{S}^2)$ for $p \in [1, \infty)$ and any accumulation point $m : \Omega \to \mathbb{S}^2$ satisfies

$$m_3 = 0, \quad |m'|= 1 \ in\ \Omega \quad and \quad \nabla'.m' = 0 \ in\ \mathcal{D}'(\Omega).$$

The above result gives us the compactness in the interior of domain. It may be of interest to know whether the above sequences of magnetization are relatively compact on the boundary. Does their limit still satisfy the Dirichlet condition? The answer is negative in general. We shall prove this by constructing a sequence $(m_\eta)_\eta$ that satisfies the upper bound as Theorem 1.1 that has a Néel wall going to the boundary, so the boundary condition (1.14) fails to be true. In fact, we are going to construct the sequence $m_\eta : \Omega \to \mathbb{S}^1$, (so the third component of m_η vanishes). The cost of such configuration now is $O(\frac{1}{\eta|\log \eta|})$. The idea that m_η may have Néel walls tending to the boundary.

Theorem 1.2. *Let $\Omega = (0, 1) \times (-1, 1)$. In the regime $\eta \ll 1$, there exists a C^1 vector field $m_\eta : \Omega \to \mathbb{S}^1$ and $g : \Gamma = \{0\} \times (-1, 1) \subset \partial\Omega \to \mathbb{S}^1$ which satisfy*

$$m_\eta = g \quad on\ \Gamma \subset \partial\Omega,\ \forall \eta \ll 1.$$

$$m_\eta \to m \ in\ L^1_{loc}(\Omega) \ as\ \eta \to 0,$$

and

$$E_{\epsilon,\eta,\kappa} = O(\frac{1}{\eta|\log \eta|}).$$

But $m \neq g$ on $\Gamma \subset \partial\Omega$.

Regularity. In Chapter 3, we study the regularity of critical points of the energy $E_{\epsilon,\eta,\kappa}$, which are subject to the Dirichlet boundary condition (1.14). For that, we consider Ω be $C^{1,1}$ domain and a magnetization $m = (m', m_3) : \Omega \to \mathbb{S}^2$ satisfying

$$m = (g, 0) \text{ on } \partial\Omega,$$

where $g : \partial\Omega \to \mathbb{S}^1$ is a $C^{1,1}$ function.
Denote

$$H^1_g(\Omega, \mathbb{S}^2) := \{u \in H^1(\Omega, \mathbb{R}^3) \ : |u(x)|= 1 \text{ a.e }, u = (g, 0) \text{ on } \partial\Omega\}$$

and

$$E_{\epsilon,\eta,\kappa}(m) = \int_\Omega |\nabla' m|^2 dx' + \frac{1}{\epsilon^2}\int_\Omega m_3^2 dx' + \frac{1}{\eta}||(\nabla'.m')_{ac}||^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} + \int_\Omega DM(m)dx',$$

where $(\nabla'.m')ac$ is defined as in (1.16). The existence of minimizers of $E_{\epsilon,\eta,\kappa}$ follows with the help of the direct method of the calculus of variations and the compact Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q < \infty$. Critical points of $E_{\epsilon,\eta,\kappa}$ on $H^1_g(\Omega, \mathbb{S}^2)$ satisfy the Euler-Lagrange equation

$$\begin{aligned} &-\Delta m - m|\nabla' m|^2 - \frac{1}{\eta}\begin{pmatrix} H \\ 0 \end{pmatrix} + \frac{1}{\eta}(m'.H)m + \frac{1}{\epsilon^2}((0,0,m_3) - m_3^2 m) \\ &\quad + \kappa_1(Id - m \otimes m)\nabla' \times m + \kappa_2\left(\begin{pmatrix} -\nabla' m_3 \\ \nabla'.m' \end{pmatrix} + (\nabla' m_3.m' - m_3\nabla'.m')m\right) = 0 \quad (1.19) \end{aligned}$$

in Ω, where $H = \nabla' U(\cdot, 0)$ is the stray field in the plane and

$$\nabla' \times m = (\partial_{x_2} m_3, -\partial_{x_1} m_3, \partial_{x_1} m_2 - \partial_{x_2} m_1).$$

The above equation can be seen as a perturbation of the harmonic map for $\mathbb{S}^2$-valued maps equation. We shall use regularity result of Wente (see [49]) which yields the regularity in the interior of Ω.

Those proofs also show that m is continuous up to the boundary. In order to obtain a higher regularity result at the boundary, we need to handle not only the non-local term, but also the imposed boundary condition (1.14). Compared with the theories related to the harmonic maps, the non-local term does not allow to get a monotonically formula which is the principal feature in the partial regularity of the stationary critical point of harmonic maps (see Hardt, Kinderlehrer and Lin [24], also Evans [20]). In [23], Hardt and Kinderlehrer used the *almost minimizers* definition to tackle a different non-local term. For that, one assumes the natural boundary condition; that is, $\frac{\partial m}{\partial \nu} = 0$, where ν is the unit outer normal vector. Their method can not apply to our imposed boundary condition (1.14). Finally, we state the regularity results.

Theorem 1.3. *Let Ω be a domain of $C^{1,1}$, $g \in C^{1,1}(\partial\Omega, \mathbb{S}^1)$ and $m \in H^1_g(\Omega, \mathbb{S}^2)$ be a solution of* (1.19)*. Then $m \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in (0, \frac{1}{2})$.*

Chapter 2

A compactness result in a non-local Ginzburg-Landau model arising in thin ferromagnetic films

Abstract

We analyze the behavior of minimizers in an asymptotic regime for a non-local Ginzburg-Landau model arising in a thin film micromagnetics where the Dzyaloshinsky-Moriya interaction is taken into account. It consists in a free energy functional depending on parameters ϵ, η and κ and defined over vector fields $m : \Omega \to \mathbb{S}^2$ that satisfy a Dirichlet boundary condition. We are interested in the behavior of minimizers as $\epsilon, \eta, 1/|\kappa| \to 0$. They are expect to be asymptotically $\mathbb{S}^1$- valued maps away from regions of length of scale ϵ where intrinsic vortices nucleate and of vanishing divergence away of regions of lengths of η where Néel walls nucleate. We establish compactness of the magnetizations in the energetic regime where Néel walls are cheaper than vortices. We also give an example where the lack of compactness at the boundary occurs.

2.1 Introduction

In thin ferromagnetic films, variations of the magnetization in thickness direction are strongly penalized. This leads to a reduced two-dimensional variational model where the magnetization is described by a $\mathbb{S}^2$-valued map defined on a 2D domain. The aim of this Chapter is to study the asymptotic regime for thin ferromagnetic films where the Dzyaloshinsku-Moriya interation is taken into account and allows the occurrence of transition layers (Néel walls) and topological defects (vortices).

2.1.1 Model

We will focus on the following two dimensional model for thin ferromagnetic films. For that, let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with a $C^{1,1}$ boundary and $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function satisfying

$$\deg(g, \partial\Omega) = \boldsymbol{d}.$$

We consider $m = (m_1, m_2, m_3) : \Omega \to \mathbb{S}^2$ be a vector field with the Dirichlet condition

$$m = (m', m_3) = (g, 0) \text{ on } \partial\Omega, \tag{2.1}$$

where $m' = (m_1, m_2)$ is the in plane component of the magnetization m. We consider the following micromagnetic energy functional:

$$E_{\epsilon,\eta,\kappa}(m) = \int_\Omega |\nabla' m|^2 dx' + \frac{1}{\epsilon^2}\int_\Omega m_3^2 dx' + \frac{1}{\eta}\int_{\mathbb{R}^2} \left| |\nabla'|^{-1/2}(\nabla'.m')_{ac}\right|^2 dx' + \int_\Omega DM(m)dx' \tag{2.2}$$

where $\epsilon, \eta > 0$ are two small positive parameters, κ is a parameter inside the Dzyaloshinsky-Moriya interaction term and is discussed later . Here $x' = (x_1, x_2)$ are the in-plane variables with the differential operator

$$\nabla' = (\partial_{x_1}, \partial_{x_2}),$$

and the third variable is denoted by x_3.

The first term of (2.2) is called the exchange energy. The second and third terms are derived form the stray field energy (see Section 1.4). The second term penalizes the surface charges m_3 on the top and bottom of the magnetic cylinder. While the third term counts the penalization of the volume charges $\nabla'.m'$.

Using Fourier transform in the horizontal variables, the non-local term in the energy can be equivalently expressed in term of L^2-norm of the stray-field ∇U_{ac}:

$$\int_{\mathbb{R}^2} \left| |\nabla'|^{-1/2}(\nabla'.m')_{ac}\right|^2 dx' = \int_{\mathbb{R}^2} \frac{1}{|\xi|}|\mathcal{F}(\nabla'.m')_{ac}|^2 d\xi = 2\int_{\mathbb{R}^3} |\nabla U_{ac}|^2 dx$$

Here we denote

$$(\nabla'.m')_{ac} = \nabla'.m' \mathbb{1}_\Omega$$

and $U_{ac} : \mathbb{R}^3 \to \mathbb{R}$ is the stray field potential which is determined by static Maxwell's equation in weak sense:

$$\int_{\mathbb{R}^3} \nabla U_{ac}(x).\nabla\zeta(x)dx = \int_\Omega \nabla'.m'(x')\zeta(x', 0)dx' \text{ for every } \zeta \in C_0^\infty(\mathbb{R}^3). \tag{2.3}$$

The fourth term is the energy connected with the Dzyaloshinsky-Moriya interaction (shorten by DM), which is a relativistic effect stemming from spin-orbit coupling and the lack of inversion symmetry and given by

$$\begin{aligned}
&\int_\Omega DM(m)dx' = \kappa_1 \int_\Omega m.\nabla' \times m dx' + \kappa_2 \int_\Omega (m_3\nabla'.m' - m'.\nabla' m_3)dx' \\
&= \kappa_1 \int_\Omega (m_1\partial_{x_2} m_3 - m_2\partial_{x_1} m_3 + m_3\partial_{x_1} m_2 - m_3\partial_{x_2} m_1)dx' \\
&+\kappa_2 \int_\Omega (m_3(\partial_{x_1} m_1 + \partial_{x_2} m_2) - m_1\partial_{x_1} m_3 - m_2\partial_{x_2} m_3)dx',
\end{aligned} \tag{2.4}$$

where $\kappa = (\kappa_1, \kappa_2)$ arbitrary.

Essential features of this variational model are the non-convex constraint $|m|= 1$ and the non-locality of the stray field interaction. In this model, we expect asymptotically two types of singular patterns: singularity lines and vortices. These patterns result from the competition between the different contributions in the total energy $E_{\epsilon,\eta,\kappa}(m)$ with boundary condition (2.1). Let us explain these structures in the following.

Néel walls. A Néel wall is a transition layer describing a one-dimensional in-plane rotation connecting two directions of the magnetization. More precisely, it is a one-dimensional transition $m' = (m_1, m_2) : \mathbb{R} \to \mathbb{S}^1$ that minimizes the energy under boundary constraint $m(\pm x_1) = (\cos\theta, \pm\sin\theta)$, for $x_1 \geq 1, \theta \in [0, \pi/2)$:

$$E_\eta(m) = \int_\mathbb{R} \left|\frac{dm}{dx_1}\right|^2 dx_1 + \frac{1}{\eta}\int_\mathbb{R} \left| \left|\frac{d}{dx_1}\right|^{1/2} m_1\right|^2 dx_1,$$

where $\theta \in [0, \frac{\pi}{2}]$ and $\eta > 0$ stand for the angle and core of the wall, respectively.
It follows that the minimizer is a two length scale object: it has a small core with fast varying rotation and two logarithmically decaying tails. As $\eta \to 0$, the scale of the Néel core is given by $|x_1| \lesssim \omega_{core} = O(\eta)$ while the two logarithmic decaying tails scale as $\omega_{core} \lesssim |x_1| \lesssim \omega_{tail} = O(1)$, see Melcher [38]. The energetic cost (by unit length) of Néel is given in DeSimone, Kohn, Muller and Otto [18], Ignat and Otto [32] (see also Ignat [30]) by:

$$\frac{\pi(1-\cos\theta)^2 + o(1)}{\eta|\log\eta|} \quad \text{as } \eta \to 0.$$

Vortices. Vortices correspond in our model to topological singularities at the microscopic level where the magnetization points out-of-plane. The prototype of a vortex vector field is given by minimizing the energy:

$$E_\epsilon(m) = \int_\Omega |\nabla' m'(x')|^2 dx' + \frac{1}{\epsilon^2} m_3^2 dx'$$

under the constraint that $m \in H^1(\Omega, \mathbb{S}^2)$ and $m' = g$ on the boundary $\partial\Omega$. Since $m_3^2 = 1 - |m'|^2$ for $\mathbb{S}^2$-valued map m, it is strongly related to the minimal Ginzburg-Landau (GL) energy (see Bethuel, Brézis and Hélein [2]):

$$\min_{\substack{m' \in H^1(\Omega,\mathbb{R}^2) \\ m'=g \text{ on } \partial\Omega}} \int_\Omega G_\epsilon(m') dx',$$

where the GL density energy is given by the following:

$$G_\epsilon(m') := |\nabla' m'|^2 + \frac{1}{\epsilon^2}(1 - |m'|^2)^2.$$

The energetic cost of our vortices is given by

$$2\pi \boldsymbol{d} |\log\epsilon| + O(1),$$

because $\deg(m', \partial\Omega) = \boldsymbol{d}$. By Ginzburg-Landau theory, $\boldsymbol{d}$ localized regions are created, those regions are the cores of the vortex of size ϵ, where the magnetization becomes indeed perpendicular to the horizontal plane.

Compactness. We are interested in the following asymptotic regime

$$\epsilon \ll 1, \quad \frac{1}{\eta|\log\eta|} \ll |\log\epsilon| \quad \text{and} \quad |\kappa| \lesssim \frac{1}{\epsilon|\log\epsilon|}, \tag{2.5}$$

where $\kappa = (\kappa_1, \kappa_2)$.

We also consider families of magnetization m_ϵ satisfying the energy bounded

$$E_{\epsilon,\eta,\kappa}(m_\epsilon) - 2\boldsymbol{d}\pi|\log\epsilon| \lesssim \frac{1}{\eta|\log\eta|}, \tag{2.6}$$

that is satisfied particular by minimizer of $E_{\epsilon,\eta,\kappa}$. By the regime assumption, it implies that the size ϵ of the vortices is smaller exponentially than the size of the Néel wall core η. We first detect the topological defect regions, which are $\boldsymbol{d}$ vortex cores of size ϵ. Then we use an argument of approximating $\mathbb{S}^2$-valued magnetization by $\mathbb{S}^1$-valued magnetization away from these vortex cores. This result is due to Ignat and Otto [32]. We expect the limiting magnetization m satisfies $\nabla'.m' = 0$ and $m_3 = 0$ in Ω. Together with the Dirichlet boundary condition (2.1) and the expected condition $\nabla'.m' = 0$ in Ω, we shall arrive that

$$\nabla'.m' = 0 \text{ and } m_3 = 0 \text{ in } \Omega \text{ and } m'.\nu = g.\nu \text{ on } \partial\Omega. \tag{2.7}$$

We notice that the conditions $\nabla'.m' = 0$ in Ω and $m'.\nu = g.\nu$ on $\partial\Omega$ are interpreted in the distributional senses. In general, the conditions (2.7) is too rigid for smooth magnetization m. Indeed, writing $m' = -\nabla'^{\perp}\psi$ leads to the eikonal equation

$$|\nabla'\psi| = 1 \text{ in } \Omega \quad \text{and} \quad \nabla'^{\perp}\psi.\nu = -g.\nu \text{ on } \partial\Omega.$$

As $\deg(g, \partial\Omega) \neq 0$, it follows that there is no smooth solution of that problem. On the other hand, there are many continuous solutions ψ that satisfy the above equation away from a set of vanishing Lebesgue measure (in particular singularity lines).

Lack of compactness on the boundary of the domain. We are also interested in the compactness of the above magnetizations m at the boundary $\partial\Omega$. Here we note that, by the limiting condition $\nabla'.m' = 0$ and $|m'| \leq 1$ in Ω, we obtain the compactness of the *normal component* of the magnetizations on the boundary. The loss of compactness on the boundary occurs only in the *tangential component*.

2.2 Main Results

The notation: We always denote $a \ll b$ if $\frac{a}{b} \to 0$ and $a \lesssim b$ if $a \leq Cb$ for some universal constant C.
Our main result concerns the local compactness of the $\mathbb{S}^2$-valued magnetizations in a certain regime.
From now on, we always think $\boxed{U = U_{ac}, \text{ and } \nabla'.m' = (\nabla'.m')_{ac} = \nabla'.m'\mathbb{1}_{\Omega}.}$
Here U_{ac} is stray field potential which is defined as in (2.3).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain with a $C^{1,1}$ boundary and $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function satisfying $\deg(g, \partial\Omega) = \boldsymbol{d} \in \mathbb{Z}\backslash\{0\}$. We consider the following regime between the parameters $\epsilon \ll 1, \eta = \eta(\epsilon)$ and $\kappa = \kappa(\epsilon)$:*

$$|\kappa| \lesssim \frac{1}{\epsilon|\log\epsilon|}, \quad \frac{1}{\eta|\log\eta|} \ll |\log\epsilon|. \tag{2.8}$$

For each ϵ, consider a $H^1(\Omega, \mathbb{S}^2)$ vector field: $m_\epsilon : \Omega \to \mathbb{S}^2$ such that:

$$m'_\epsilon = g \quad on \quad \partial\Omega,$$

and

$$E_{\epsilon,\eta,\kappa}(m_\epsilon) - 2\boldsymbol{d}\pi|\log\epsilon| \lesssim \frac{1}{\eta|\log\eta|}. \tag{2.9}$$

Then $\{m_\epsilon\}_\epsilon$ is relatively compact in $L^p_{loc}(\Omega, \mathbb{S}^2)$ for every $p \in [1, \infty)$ and any accumulation point $m : \Omega \to \mathbb{S}^2$ satisfies

$$m_3 = 0, \quad |m'| = 1 \text{ in } \Omega \quad \text{and} \quad \nabla'.m' = 0 \text{ in } \mathcal{D}'(\Omega).$$

The proof of Theorem 2.1 is based on argument of approximating $\mathbb{S}^2$-valued vector fields by $\mathbb{S}^1$-valued vector fields away from small defect regions. This is due to Ignat and Otto [32] to detect these regions, we use some topological methods due to Jerrard [34] and Sandier [45] for the concentration of the Ginzburg-Landau energy around vortices. Away from these small regions, the energy level only allows for Néel walls. The compactness results for the $\mathbb{S}^1$-valued maps due to Ignat and Otto, (see [32]) will lead to conclusion.

Let us discuss the assumption of the regime (2.8) and (2.9). Inequality (2.9) assures that cutting out the topological defects ($\boldsymbol{d}$ vortices), the remaining energy rescaled at the energetic level of Néel walls is uniformly bounded. The regime $\frac{1}{\eta|\log\eta|} \ll |\log\epsilon|$ is imposed due to our method to detect vortices and approximate $\mathbb{S}^2$- valued vector fields by $\mathbb{S}^1$-valued vector fields away from the vortex balls. It also means that the energy of the topological defects is more expensive than the energy of Néel walls. In fact, the above assumption establishes two principal regimes; namely,

$$\frac{C}{\eta|\log \eta|} \leq 2\pi\alpha|\log \epsilon| \text{ for some } \alpha \in (0,1) \tag{2.10}$$

and

$$\epsilon^{\beta} \lesssim \eta \text{ for any } \beta \in (0, 1-\alpha). \tag{2.11}$$

If we write (2.9) as

$$E_{\epsilon,\eta,\kappa} - 2\pi\boldsymbol{d}|\log \epsilon| \leq \frac{C}{\eta|\log \eta|},$$

where C given in (2.10), then (2.9) and (2.10) yield that $E_{\epsilon,\eta,\kappa} \leq 2\pi(\boldsymbol{d}+\alpha)|\log \epsilon|$. Due to the boundary condition (2.1), we then expect to obtain exactly $\boldsymbol{d}$ vortex regions in the interior of the domain. Moreover, far from the interior vortices, starting from the values of m' on a square grid of size ϵ^{β}, the approximation argument requires zero degree of m' in each cell of the cell grid, leading to the condition $\beta < 1-\alpha$, (see Lemma 2.5). Moreover, the condition $\epsilon^{\beta} \lesssim \eta$ for any $\beta \in (0, 1-\alpha)$ is used in order that the approximating $\mathbb{S}^1$-valued vector fields induce a stray field energy of the same order of m', (see (2.48)).
The regime $|\kappa| \lesssim \frac{1}{\epsilon|\log \epsilon|}$ is rather technical: in fact, according to the boundary condition (2.1) (in particular $m_3 = 0$ on $\partial\Omega$), and the Green formula, this regime is added to ensure that the Dzyaloshinsky-Moriya energy is absorbed into the Ginzburg-Landau energy, see (2.21), (2.22) and (2.29).

In [33, Theorem 2], with a similar energy (without the Dzyaloshinsky-Moriya energy), Ignat and Otto studied the compactness in thin ferromagnetic films under the Dirichlet boundary condition for the normal component; that is,

$$m'.\nu = 0 \text{ on } \partial\Omega.$$

For such a boundary condition, the small defect region consists in either one interior vortex or two boundary vortices. The case of one interior vortex corresponds to $\boldsymbol{d} = 1$ in our Theorem 2.1. For the boundary vortices case, one needs to add more assumptions to detect those vortices, that is, $\log|\log \epsilon| \lesssim \frac{1}{\eta|\log \eta|}$. We emphasize that due to boundary condition (1.14), the boundary vortices do not occur in our case.

With the compactness result of Theorem 2.1 , we then obtain that for a subsequence, m_ϵ converges to m almost everywhere in Ω. As a consequence of the dominated convergence theorem, one has $m_\epsilon \to m$ in $L^1(\Omega)$. Together with the condition $\nabla'.m' = 0$ in Ω, one can define the *normal trace* in sense of distributions for the limiting point m and $m'.\nu = g.\nu$ on $\partial\Omega$ (see Remark 2.8). It is of interest to know whether the above sequences of magnetization are relatively compact at the boundary. Does their limit still satisfy the Dirichlet condition? The answer is negative in general. We shall prove this by constructing a sequence $(m_\eta)_\eta$ such that m_η is $\mathbb{S}^1$-valued and satisfies the upper bound as Theorem 2.1 that has a Néel wall tending to the boundary, so the boundary condition (2.1) fails in the limit $\eta \to 0$ (due to the tangential component).

Theorem 2.2. *Let $\Omega = (0,1) \times (-1,1)$. In the regime $0 < \eta \ll 1$. There exist a C^1 vector field $m_\eta : \Omega \to \mathbb{S}^1$ and $g : \Gamma = \{0\} \times (-1,1) \subset \partial\Omega \to \mathbb{S}^1$ which satisfy*

$$m_\eta = g \quad \text{on } \Gamma,\ \forall \eta \ll 1,$$

$$m_\eta \to m \text{ in } L^1_{loc}(\Omega) \text{ as } \eta \to 0,$$

and

$$E_{\epsilon,\eta,\kappa}(m_\eta) = O(\frac{1}{\eta|\log \eta|}).$$

But $m \neq g$ on Γ.

The constructed sequence $(m_\eta)_\eta$ has the third component vanishing. Therefore, our model is written as[1]

$$E_\eta(m) = \int_\Omega |\nabla' m'|^2 dx' + \frac{1}{\eta}\int_{\mathbb{R}^2} \left| |\nabla'|^{-1/2}(\nabla'.m')\mathbb{1}_\Omega \right|^2 dx'.$$

The cost of such a energy is $O(\frac{1}{\eta|\log\eta|})$. The idea of proof based the fact that $(m_\eta)_\eta$ may have the Néel walls tending to the boundary as $\eta \to 0$.

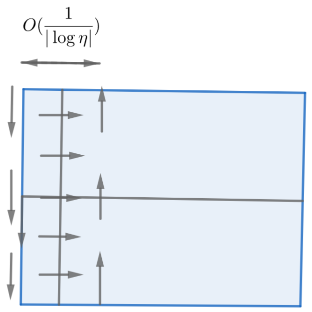

Figure 2.1: Magnetization has Néel walls tending to boundary as $\eta \to 0$.

The outline of this Chapter is as follows: in the next section, we recall some definitions on the stray field and some results that we need for the proof of our results such as: the concentration of the Ginzburg-Landau energy on vortex balls and a compactness result for $\mathbb{S}^1$-valued magnetizations.

2.3 A Few Preliminary Results

2.3.1 Preliminary Results on Existence and Uniqueness of the Stray Field

We state the existence and uniqueness results for the stray field generated by the volume charges, as well as the expression of the stray field energy. For that, we introduce the Beppo-Levi space

$$\mathcal{BL} := \{U : \mathbb{R}^3 \to \mathbb{R} : \nabla U \in L^2(\mathbb{R}^3), \frac{U(x)}{1+|x|} \in L^2(\mathbb{R}^3)\}.$$

Consequently, the space $\mathcal{BL}$ endowed by the homogeneous $\dot{H}^1$-norm, $U \mapsto ||\nabla U||_{L^2(\mathbb{R}^3)}$ is a Hilbert space, and the set $C_0^\infty(\mathbb{R}^3)$ of smooth compactly supported functions is a dense set, see Dautray and Lions [16]. Let us denote by $\mathcal{F}$ the Fourier transform of the in-plane $\mathbb{R}^2$, i.e., for every $\xi \in \mathbb{R}^2$,

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2} e^{-ix'.\xi} f(x')dx',$$

where f belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{F}$ extends to the space of tempered distributions. We have the following

[1]We note that as $m_{\eta,3} = 0$, ϵ and κ do not play any role here.

Theorem 2.3. *Let $m \in H^1(\Omega, \mathbb{R}^2)$. The variational problem (2.3) has a unique solution $U \in \mathcal{BL}$. Classically, U satisfies*

$$\begin{cases} \Delta U = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\Omega} \times \{0\} \\ \left[\frac{\partial U}{\partial x_3}\right] = -\nabla'.m' & \text{on } \Omega \times \{0\}, \\ [U] = 0 & \text{on } \mathbb{R}^2 \times \{0\}, \end{cases} \tag{2.12}$$

where $[q] = q^+ - q^-$ stands for the jump in the vertical direction x_3 of the quantity q across the horizontal plane. Moreover the stray field energy is also given by

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla U|^2 dx &= \int_\Omega U(x', 0)(\nabla'.m')(x')dx' \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \left| |\nabla'|^{-1/2}((\nabla'.m')\mathbb{1}_\Omega) \right|^2 dx' = \frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{|\xi|} |\mathcal{F}(\nabla'.m'\mathbb{1}_\Omega)(\xi)|^2 d\xi \end{aligned} \tag{2.13}$$

A proof of Theorem 2.3 is given in the appendix.

2.3.2 Preliminary Results on the concentration of the Ginburg-Landau energy

In the proof of Theorem 2.1 we will use the following result due to Jerrard for the concentration of the GL energy around vortices

Theorem 2.4. *(see [34]) Let Ω be a $C^{1,1}$ domain, $\alpha \in [0, 1)$ and $\boldsymbol{d} > 0$ be a positive integer. Let $g : \partial\Omega \to \mathbb{S}^1$ be $C^{1,1}$ with*

$$|\deg(g, \partial\Omega)| = \boldsymbol{d}.$$

There exists $R = R(\alpha, \boldsymbol{d}, \Omega) > 0$ such that for every $0 < r < R$, if $m' : \Omega \to \mathbb{R}^2$ satisfies the following conditions:

$$m' = g \text{ on } \partial\Omega$$

and

$$\int_\Omega G_\epsilon(m')dx' \leq 2\pi(\boldsymbol{d} + \alpha)|\log \epsilon|,$$

then there exist n distinct points $x_1, ..., x_n \in \Omega$ and positive integers $d_1, ..., d_n > 0$ such that the n balls $\{B(x_j, r)\}_{1 \leq j \leq n}$ are disjoint,

$$\sum_{j=1}^n d_j = \boldsymbol{d}$$

and

$$\int_{B(x_j, r)\cap\Omega} G_\epsilon(m')dx' \geq 2\pi d_j |\log \frac{r}{\epsilon}| - C(\alpha, \boldsymbol{d}, R), j = 1, ..., n,$$

where $C(\alpha, \boldsymbol{d}, R)$ is a constant which only depend on $\boldsymbol{d}, \alpha$ and R.

In Step B of the Proof of Theorem 2.2 we also use the following lemma (see [33, Lemma 2]) that also follows from Theorem 2.4. This gives a link to the condition (2.11).

Lemma 2.5. *Let $0 < \alpha < 1, 0 < \beta < 1 - \alpha, C > 0$. There exists $\epsilon_0(\alpha, \beta, C) > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ the following holds: if $Z = \left(-\frac{\epsilon^\beta}{2}, \frac{\epsilon^\beta}{2}\right)^2$ is the square cell of length ϵ^β and $m' : \overline{Z} \to \overline{\mathbb{B}^2}$ is a C^1 vector field such that:*

$$\int_{\partial Z} G_\epsilon(m')d\mathcal{H}^1 \leq \frac{C|\log \epsilon|}{\epsilon^\beta}, \quad \int_Z G_\epsilon(m')dx' \leq 2\pi\alpha|\log \epsilon|$$

then

$$|m'| \geq \frac{1}{2} \text{ on } \partial Z \quad \text{and} \quad \deg(m', \partial Z) = 0.$$

2.3.3 Compactness result in thin-film micromagnetics

The proof of Theorem 2.1 mainly uses the compactness result of $\mathbb{S}^1$-valued vector fields obtained by Ignat and Otto, see [32]. For the convenience, we state their result. We refer reader to [32, Theorem 4] for a detailed proof.

Theorem 2.6. *Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n, n = 2, 3$. For every small $\eta > 0$, let $m'_\eta : \mathbb{B}^2 \to \mathbb{S}^1$ and $h_\eta : \mathbb{B}^3 \to \mathbb{R}^3$ be related by*

$$\int_{\mathbb{B}^3} h_\eta(x).\nabla\zeta(x)dx = \int_{\mathbb{B}^2} \zeta(x', 0)\nabla'.m'_\eta(x')dx', \qquad \forall\zeta \in C_0^\infty(\mathbb{B}^3). \tag{2.14}$$

Suppose that

$$\int_{\mathbb{B}^2} |\nabla'.m'_\eta|^2 dx' + \frac{1}{\eta}\int_{\mathbb{B}^3} |h_\eta|^2 dx \leq \frac{C}{\eta|\log\eta|} \tag{2.15}$$

for some fixed constant C. Then $\{m'{}_\eta\}_{\eta\downarrow 0}$ is relatively compact in $L^1(\mathbb{B}^2)$ and any accumulation point $m' : \mathbb{B}^2 \to \mathbb{R}^2$ satisfies:

$$|m'| = 1 \text{ a.e in } \mathbb{B}^2 \qquad \text{and} \qquad \nabla'.m' = 0 \text{ in the sense of distributions.}$$

2.4 Proof of Theorem 2.1

This section is devoted to the proof of the compactness result for magnetizations in the energy regime $O(\frac{1}{\eta_k|\log\eta_k|}) + 2\pi\boldsymbol{d}|\log\epsilon_k|$.
We will work at the level of sequences of parameters $\epsilon_k, \eta_k, \kappa_k$, $(\kappa = (\kappa_{1,k}, \kappa_{2,k}))$ and a sequence of magnetization m_k satisfying the assumptions in Theorem 2.1.
By assumption, $E_{\epsilon_k,\eta_k,\kappa_k}(m_k) - 2\boldsymbol{d}\pi|\log\epsilon_k| \lesssim \frac{1}{\eta_k|\log\eta_k|}$. Then there exists $A > 0$ such that

$$E_{\epsilon_k,\eta_k,\kappa_k}(m_k) - 2\boldsymbol{d}\pi|\log\epsilon_k| \leq \frac{A}{\eta_k|\log\eta_k|}. \tag{2.16}$$

Also, by the condition (2.8), there exists $\alpha \in (0, 1)$ and $C > 0$ such that

$$\frac{A}{\eta_k|\log\eta_k|} \leq 2\pi\alpha|\log\epsilon_k|. \tag{2.17}$$

and

$$|\kappa_k| \leq \frac{C}{\epsilon_k|\log\epsilon_k|} \ll \frac{1}{\epsilon_k}. \tag{2.18}$$

We split the proof of the Theorem 2.1 in several steps.

Step A *We locate the vortex balls of m_k.*
Our strategy is to apply Theorem 2.4 to locate the vortex balls of m'_k in Ω. It remains to us prove the following claim

Claim 1.
$$\int_\Omega |\nabla' m'_k|^2 dx' + \frac{1}{\epsilon_k^2}\int_\Omega (m_{3,k})^2 dx' \leq 2\pi(\boldsymbol{d} + \alpha')|\log\epsilon_k|, \tag{2.19}$$

with some $0 < \alpha' < 1$.

The proof of Claim 1. Using Green's formula with the fact that $m_{3,k} = 0$ on $\partial\Omega$, we rewrite the first part of the DM energy (see (2.4)) as

$$\begin{aligned}
&\kappa_{1,k}\int_\Omega m_k.\nabla' \times m_k dx' \\
&= \kappa_{1,k}\int_\Omega (m_{1,k}\partial_{x_2}m_{3,k} - m_{2,k}\partial_{x_1}m_{3,k} + m_{3,k}\partial_{x_1}m_{2,k} - m_{3,k}\partial_{x_2}m_{1,k})dx' \\
&= 2\kappa_{1,k}\int_\Omega (m_{3,k}\partial_{x_1}m_{2,k} - m_{3,k}\partial_{x_2}m_{1,k})dx'.
\end{aligned} \tag{2.20}$$

Equation (2.20) yields that

$$
\begin{aligned}
&|\kappa_{1,k} \int_\Omega m_k.\nabla' \times m_k dx'| \\
&\leq 2|\kappa_{1,k}| \int_\Omega (|m_{3,k}\partial_{x_2} m_{1,k}| + |m_{3,k}\partial_{x_1} m_{2,k}|) dx' \\
&\leq a_1 \int_\Omega (m_{3,k})^2 dx' + b_1 \int_\Omega (\partial_{x_2} m_{1,k})^2 dx' + a_1 \int_\Omega (m_{3,k})^2 dx' + b_1 \int_\Omega (\partial_{x_1} m_{2,k})^2 dx' \\
&\leq 2a_1 \int_\Omega (m_{3,k})^2 dx' + b_1 \int_\Omega |\nabla' m'_k|^2 dx', \qquad (2.21)
\end{aligned}
$$

where a_1, b_1 are positive numbers satisfying $a_1 b_1 = \kappa_{1,k}^2$, will be chosen later, see (2.25). Similarly for the second term of the DM energy

$$
\kappa_{2,k} \int_\Omega (m_{3,k}\nabla'.m'_k - m'_k.\nabla' m_{3,k}) dx' = 2\kappa_{2,k} \int_\Omega m_{3,k}\nabla'.m'_k dx'
$$

and

$$
\begin{aligned}
&\left| \kappa_{2,k} \int_\Omega (m_{3,k}(\nabla'.m'_k) - m'.\nabla' m_{3,k}) dx' \right| \\
&\leq a_2 \int_\Omega (m_{3,k})^2 dx' + b_2 \int_\Omega |\nabla' m'_k|^2 dx', \qquad (2.22)
\end{aligned}
$$

where a_2, b_2 are chosen satisfying (2.25).
From the definition of $E_{\epsilon_k,\eta_k,\kappa_k}$, it follows that

$$
\begin{aligned}
&\int_\Omega |\nabla' m'_k|^2 dx' + \frac{1}{\epsilon_k^2} \int_\Omega (m_{3,k})^2 dx' \\
&\leq E_{\epsilon_k,\eta_k,\kappa_k}(m_k) - \frac{1}{\eta_k} \int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m'_k)|^2 dx' - \int_\Omega DM(m_k) dx'. \qquad (2.23)
\end{aligned}
$$

Combining (2.21)-(2.23), then

$$
\begin{aligned}
&\int_\Omega |\nabla' m'_k|^2 dx' + \frac{1}{\epsilon_k^2} \int_\Omega (m_{3,k})^2 dx' \\
&\leq E_{\epsilon_k,\eta_k,\kappa_k}(m_k) + (2a_1 + a_2) \int_\Omega (m_{3,k})^2 dx' + (b_1 + b_2) \int_\Omega |\nabla' m'_k|^2 dx'.
\end{aligned}
$$

Together with (2.16), this implies

$$
\begin{aligned}
&(1 - b_1 - b_2) \int_\Omega |\nabla' m'_k|^2 dx' + \left(\frac{1}{\epsilon_k^2} - 2a_1 - a_2 \right) \int_\Omega (m_{3,k})^2 dx' \\
&\qquad\qquad \leq 2\pi \boldsymbol{d} |\log \epsilon_k| + \frac{A}{\eta_k |\log \eta_k|}.
\end{aligned}
$$

By (2.17), finally, we obtain

$$
\int_\Omega |\nabla' m'_k|^2 dx' + \frac{(1/\epsilon_k^2 - 2a_1 - a_2)}{1 - b_1 - b_2} \int_\Omega (m_{3,k})^2 dx' \leq \frac{2\pi(\boldsymbol{d} + \alpha)}{1 - b_1 - b_2} |\log \epsilon_k|. \qquad (2.24)
$$

To conclude Claim 1, a_1, b_1, a_2, b_2 will be chosen satisfying

$$
\begin{cases}
a_1 b_1 = \kappa_{1,k}^2, \\
a_2 b_2 = \kappa_{2,k}^2, \\
\frac{1}{\epsilon_k^2} \leq \frac{1/\epsilon_k^2 - 2a_1 - a_2}{1 - b_1 - b_2}, \qquad (1 - b_1 - b_2) > 0 \\
\frac{2\pi(\boldsymbol{d}+\alpha)}{1 - b_1 - b_2} \leq 2\pi(\boldsymbol{d} + \alpha'), \quad \text{for some } \alpha' \in (\alpha, 1) \quad (\alpha < \alpha' < 1).
\end{cases} \qquad (2.25)
$$

In fact, we choose $b_1 = b_2 = \frac{1-\alpha}{4(\boldsymbol{d}+\frac{\alpha}{2}+\frac{1}{2})} > 0$. An easy computation shows that

$$\frac{2\pi(\boldsymbol{d}+\alpha)}{1-b_1-b_2} = 2\pi(\boldsymbol{d}+\frac{\alpha}{2}+\frac{1}{2}).$$

The fourth inequality of (2.25) holds true for $\alpha' = \frac{\alpha}{2}+\frac{1}{2}$.
Setting $a_1 = \frac{\kappa_{1,k}^2}{b_1}, a_2 = \frac{\kappa_{2,k}^2}{b_2}$, therefore, we only need to check that

$$\frac{1}{\epsilon_k^2} \leq \frac{1/\epsilon_k^2 - 2a_1 - a_2}{1-b_1-b_2}.$$

This can be written equivalently to

$$\begin{aligned}
&\frac{1}{\epsilon_k^2} \leq \frac{1/\epsilon_k^2 - 2a_1 - a_2}{1-b_1-b_2} \\
\Leftrightarrow\ &\frac{b_1+b_2}{\epsilon_k^2} \geq 2a_1 + a_2 \\
\Leftrightarrow\ &\frac{b_1+b_2}{\epsilon_k^2} \geq \frac{2\kappa_{1,k}^2}{b_1} + \frac{\kappa_{2,k}^2}{b_2}.
\end{aligned}$$

The last inequality is followed from $|\kappa_k| \ll \frac{1}{\epsilon_k}$. We complete the proof of Claim 1. We note that, for the proof of Claim 1, we only use the regime assumption $|\kappa_k| \ll \frac{1}{\epsilon_k}$ (which is implied by (2.8)).

Remark 2.7. By (2.24) and (2.25), we have immediately that

$$\int_\Omega |\nabla' m_k'|^2 dx' + \frac{1}{\epsilon_k^2}\int_\Omega (m_{3,k})^2 dx' \leq 2\pi(\boldsymbol{d}+\alpha')|\log \epsilon_k|,$$

for some $\alpha' \in (0,1)$. Hence,

$$\int_\Omega |\nabla' m_k'|^2 dx' \leq 2\pi(\boldsymbol{d}+\alpha')|\log \epsilon_k| \tag{2.26}$$

and

$$\frac{1}{\epsilon_k^2}\int_\Omega (m_{3,k})^2 dx' \leq 2\pi(\boldsymbol{d}+\alpha')|\log \epsilon_k|. \tag{2.27}$$

We next apply Theorem 2.4 to m_k in domain Ω. There exist $R > 0$, n_k distinct points $x_{1,k}, ..., x_{n_k,k}$ in Ω and n_k integers $d_{1,k}, d_{2,k}, ..., d_{n_k,k} > 0, \sum_{i=1}^{n_k} d_{i,k} = \boldsymbol{d}$ such that for any $r \in (0,R)$,

$$\int_{B(x_{j,k},r)\cap\Omega} G_{\epsilon_k}(m_k') dx' \geq 2\pi d_j |\log \frac{r}{\epsilon_k}| - C(\alpha,\boldsymbol{d})$$

for $j = 1, ..., n_k$ and k sufficiently larger. We note that $x_{1,k}, ..., x_{n,k} \in \Omega$. Then, summing up by $x_{i,k}$, it yields

$$\begin{aligned}
\int_{\bigcup_{x_{j,k}} B(x_{j,k},r)\cap\Omega} G_{\epsilon_k}(m_k') dx' \quad &\geq 2\pi\boldsymbol{d}|\log \frac{r}{\epsilon_k}| - C(\alpha,\boldsymbol{d}) n_k \\
&\geq 2\pi\boldsymbol{d}|\log \epsilon_k| - C(\alpha,\boldsymbol{d},r).
\end{aligned} \tag{2.28}$$

Up to a subsequence $\{x_{1,k}\}_k, ..., \{x_{n,k}\}_k$ converge to n points $x_1, ..., x_n$ and we have for every small $r > 0$,

$$B(x_{j,k}, r) \subset B(x_i, 2r) \qquad \forall j = 1, ..., n_k, \text{ for some } i \in \{1, ..., n\},$$

for k large sufficiently. The open set $D = \bigcup_{j=1}^{n} B(x_j, 2r)$ is the location of the essential topological defects of each m'_k.

The next step is to prove that m'_k is relatively compact in $L^1(\Omega \backslash D)$.
Let $B \subset \Omega \backslash D$ be an arbitrary square. To simplify the notation, let $B = (-1,1)^2$. We prove that there exists m such that for a subsequence $m_k \to m$ in $L^1_{loc}(B)$.
The idea is to approximate m'_k away from D by $\mathbb{S}^1$-valued vector fields in L^1_{loc}, denoted by M'_k which satisfy the hypotheses of Theorem 2.6. This implies that $M'_k \to m'$ in L^1_{loc}. Therefore, we have $m'_k \to m'$ in $L^1_{loc}(B)$.

Step B. *Approximation of m'_k away from D by $\mathbb{S}^1$-valued vector fields.*
We state some inequalities. Firstly,

$$\begin{aligned}
&\int_B |\nabla' m'_k|^2 dx' + \frac{1}{\epsilon_k^2}\int_B (1-|m'_k|^2)^2 dx' + \frac{1}{\eta_k}\int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m'_k)|^2 dx' \\
&\le E_{\epsilon_k,\eta_k,\kappa_k}(m_k) - \int_D |\nabla' m'_k|^2 dx' - \frac{1}{\epsilon_k^2}\int_D (1-|m'_k|^2)^2 dx' - \int_\Omega DM(m) dx'.
\end{aligned}$$

We observe that

$$\int_\Omega DM(m_k)dx' = 2\kappa_{1,k}\int_\Omega (m_{3,k}(\partial_{x_1} m_{2,k} - \partial_{x_2} m_{1,k})dx' + 2\kappa_{2,k}\int_\Omega (m_{3,k}\nabla'.m'_k)dx'.$$

Then

$$\begin{aligned}
&\left|\int_\Omega DM(m_k)dx'\right| \\
&\le \left|2\kappa_{1,k}\int_\Omega m_{3,k}(\partial_{x_1} m_{2,k} - \partial_{x_2} m_{1,k})dx'\right| + \left|2\kappa_{2,k}\int_\Omega (m_{3,k}\nabla'.m'_k)dx'\right| \\
&\le \frac{1}{|\log\epsilon_k|}\int_\Omega |\nabla' m'_k|^2 dx' + 4(\kappa_{1,k}^2 + \kappa_{2,k}^2)|\log\epsilon_k|\int_\Omega |m_{3.k}|^2 dx'.
\end{aligned}$$

Hence,

$$\begin{aligned}
-\int_\Omega DM(m_k)dx' &\le \left|\int_\Omega DM(m_k)dx'\right| \\
&\le \frac{1}{|\log\epsilon_k|}\int_\Omega |\nabla' m'_k|^2 dx' + 4(\kappa_{1,k}^2 + \kappa_{2,k}^2)|\log\epsilon_k|\int_\Omega |m_{3.k}|^2 dx'.
\end{aligned}$$

Using Remark 2.7 and (2.18), one has

$$-\int_\Omega DM(m_k)dx' \le 2\pi(\boldsymbol{d}+\alpha')(1 + C\epsilon_k^2|\kappa_k|^2|\log\epsilon_k|^2) = O(1). \tag{2.29}$$

Combining (2.16), (2.28) and (2.29) yields that

$$\begin{aligned}
&\int_B |\nabla' m'_k|^2 dx' + \frac{1}{\epsilon_k^2}\int_B (1-|m'_k|^2)^2 dx' + \frac{1}{\eta_k}\int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m'_k)|^2 dx' \\
&\le E_{\epsilon_k,\eta_k,\kappa_k}(m_k) - \int_{\mathcal{D}} |\nabla' m'_k|^2 - \frac{1}{\epsilon_k^2}\int_{\mathcal{D}} (1-|m'_k|^2)^2 dx' + O(1) \\
&\le \frac{A}{\eta_k|\log\eta_k|} + O(1) \quad (\text{ by (2.16) and (2.28)}).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_B |\nabla' m'_k|^2 dx' + \frac{1}{\epsilon_k^2}\int_B (1-|m'_k|^2)^2 dx' + \frac{1}{\eta_k}\int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m'_k)|^2 dx' \\
&\qquad \le \frac{\widetilde{A}}{\eta_k|\log\eta_k|}
\end{aligned} \tag{2.30}$$

for k large enough and any $\widetilde{A} > A$.
Moreover, using the hypothesis (2.17), we deduce that

$$\int_B |\nabla' m_k'|^2 dx' + \frac{1}{\epsilon_k^2}\int_B (1-|m_k'|^2)^2 dx' + \frac{1}{\eta_k}\int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m_k')|^2 dx' \leq 2\pi\widetilde{\alpha}|\log\epsilon_k|, \quad (2.31)$$

for some $\widetilde{\alpha} \in (0,1)$.

Step B.1 Construction of a square grid.
The hypothesis $\frac{1}{\eta_k|\log\eta_k|} \ll |\log\epsilon_k|$, again shows that

$$\epsilon_k^\beta \leq \eta_k, \text{ for some } \beta \in (0,1) \text{ and } \beta \in (0, 1-\widetilde{\alpha}). \quad (2.32)$$

Let $\widetilde{B} \subset B$ be a compact set. For each shift $t \in [0, \epsilon_k^\beta)$, denote:

$$V_t := \{(x_1, x_2) \in B : x_2 \equiv t \,(\mathrm{mod}\ \epsilon_k^\beta)\}$$

for the net of horizontal lines at a distance ϵ_k^β in B. By Fubini theorem, there exists $t_k \in (0, \epsilon_k^\beta)$ such that

$$\int_{V_{t_k}} G_{\epsilon_k}(m_k') d\mathcal{H}^1 \leq \frac{1}{\epsilon_k^\beta}\int_B G_{\epsilon_k}(m_k') dx'.$$

If one repeats the above argument for the net of vertical lines at a distance ϵ_k^β in B, we get the square grid R^k of size ϵ_k^β such that for ϵ_k small, the convex hull of R^k covers $\widetilde{B} \subset B$ and the following estimate:

$$\int_{R^k} G_{\epsilon_k}(m_k') d\mathcal{H}^1 \leq \frac{2\widetilde{A}}{\epsilon_k^\beta \eta_k|\log\eta_k|} \qquad \text{(by (2.30))}. \quad (2.33)$$

Together with (2.17), this yields

$$\int_{R^k} G_{\epsilon_k}(m_k') d\mathcal{H}^1 \leq C|\log\epsilon_k|\epsilon_k^{-\beta}. \quad (2.34)$$

This implies that $|m_k'| > \frac{1}{2}$ on R^k for k large enough. Indeed, denoting $\rho = |m_k'|$ and $min = \min\{\rho(x') : x' \in R^k\}$. We have that

$$C|\log\epsilon_k|\epsilon_k^{-\beta} \geq \int_{R^k} G_{\epsilon_k}(m_k') d\mathcal{H}^1 \geq \int_{R^k}\left(|\partial_\tau\rho|^2 + \frac{1}{\epsilon_k^2}(1-\rho^2)^2\right) d\mathcal{H}^1 \geq \frac{\widetilde{C}}{\epsilon_k}(1-min)^2,$$

where τ is the tangent unit vector at R^k. Thus, one concludes that

$$(1-min)^2 \leq \frac{C}{\widetilde{C}}\epsilon_k^{1-\beta}|\log\epsilon_k| \ll 1.$$

Then $min > \frac{1}{2}$ for ϵ_k small enough.
Therefore, we can define the degree of m_k' on each cell of square grid R^k by:

$$\deg(m_k', \partial Z^k) := \deg\left(\frac{m_k'}{|m_k'|}, \partial Z^k\right).$$

Here, without loss of generality, we denote $Z^k = \left(-\frac{\epsilon_k^\beta}{2}, \frac{\epsilon_k^\beta}{2}\right)^2$ by the cell of length ϵ_k^β with $\partial Z_k \subset R^k$.

Step B.2 We prove that $\deg(m'_k, \partial Z^k) = 0$.
Inequality (2.31) deduces that

$$\int_{Z_k} G_{\epsilon_k}(m'_k)dx' \leq \int_B G_{\epsilon_k}(m'_k)dx' \leq 2\pi\widetilde{\alpha}|\log\epsilon_k|. \tag{2.35}$$

Moreover, by (2.34), one has

$$\int_{\partial Z_k} G_{\epsilon_k}(m'_k)d\mathcal{H}^1 \leq C\frac{|\log\epsilon_k|}{\epsilon_k^\beta}. \tag{2.36}$$

By (2.35) and (2.36), we apply Lemma 2.5 to m'_k, to establish $\deg(m_k, \partial Z^k) = 0$.

Step B.3. Construct an approximating sequence.
First, note that since Z_k is simply connected, $|m'_k|> \frac{1}{2}$ on ∂Z_k and $\deg(m_k, \partial Z^k) = 0$ then we rewrite m'_k as

$$m'_k = |m'_k|e^{i\varphi_k} = |m'_k|v_k \text{ on } \partial Z^k \text{ and } \varphi_k \in H^1(\partial Z^k, \mathbb{R}),$$

where $v_k := e^{i\varphi_k}$ on ∂Z^k. Moreover, we can lift m'_k on the grid.

On each cell Z^k of length ϵ_k^β of the grid, we define:

$$M'_k = e^{i\Phi_k} \text{ in } Z^k,$$

where Φ_k is the harmonic extension of φ_k in Z^k, i.e.

$$\begin{cases} \Delta\Phi_k = 0 & \text{in } Z^k \\ \Phi_k = \varphi_k & \text{on } \partial Z^k. \end{cases}$$

Note that we can estimate

$$\int_{Z^k}|\nabla'\Phi_k|^2dx' \leq C\epsilon_k^\beta\int_{\partial Z^k}|\nabla'\varphi_k|^2d\mathcal{H}^1.$$

Indeed, rescaling by ϵ_k^β, we can assume that

$$\begin{cases} \Delta\Phi = 0 & \text{in } B = (-1,1)^2, \\ \Phi = \varphi & \text{on } \partial B, \end{cases}$$

where $\varphi : \partial B \to \mathbb{S}^1$ satisfies $\int_{\partial B}\varphi d\mathcal{H}^1 = 0$ (up to an additive constant in $[0, 2\pi]$). We show the inequality in the unit cell B. We consider a smooth cut-off function $\Psi : [0,1] \to \mathbb{R}^1$ such that

$$\begin{cases} \Psi(t) = 0 & \text{in } t \leq 1/2, \\ \Psi(1) = 1 \end{cases}$$

and the extension Φ_{ext} of φ in B :

$$\Phi_{ext}(tx) = \Psi(t)\varphi(x') \text{ for } t \in (0,1),\ x' \in \partial B.$$

Using the Poincaré-Wirtinger inequality and the trace operator, we obtain

$$\int_B|\nabla'\Phi|^2dx' \leq \int_B|\nabla\Phi_{ext}|^2dx' \leq C\int_{\partial B}(|\nabla'\varphi|^2+\varphi^2)d\mathcal{H}^1 \leq C\int_{\partial B}|\nabla'\varphi|^2d\mathcal{H}^1.$$

It follows that:

$$\begin{aligned} \int_{Z^k}|\nabla' M'_k|^2dx' &= \int_{Z^k}|\nabla'\Phi_k|^2dx' \\ &\leq C\epsilon_k^\beta\int_{\partial Z^k}|\nabla'\varphi_k|^2d\mathcal{H}^1 = C\epsilon_k^\beta\int_{\partial Z^k}|\nabla' v_k|^2d\mathcal{H}^1 \\ &\leq 4C\epsilon_k^\beta\int_{\partial Z^k}|m'_k|^2|\nabla' v_k|^2d\mathcal{H}^1 \leq 4C\epsilon_k^\beta\int_{\partial Z^k}|\nabla' m'_k|^2d\mathcal{H}^1. \end{aligned} \tag{2.37}$$

Now we prove that M'_k is an approximation of m'_k in $L^1(\widetilde{B})$, for a compact $\widetilde{B} \subset B$.
To prove that we estimate $||(M'_k - m'_k)||_{L^2(\widetilde{B})}$. We will also estimate $||\nabla'(M'_k - m'_k)||_{L^2}(\widetilde{B})$.

Step B.4 The estimate of $||(M'_k - m'_k)||_{L^2}$.
the Poincaré-Wirtinger and (2.37) lead to

$$\int_{Z^k} \left| M'_k - \fint_{\partial Z^k} M'_k \right|^2 dx' \leq C\epsilon_k^{2\beta} \int_{Z^k} |\nabla M'_k|^2 dx' \leq C\epsilon_k^{3\beta} \int_{\partial Z^k} |\nabla' m'_k|^2 d\mathcal{H}^1 \tag{2.38}$$

and

$$\int_{Z^k} \left| m'_k - \fint_{\partial Z^k} m'_k \right|^2 dx \leq C\epsilon_k^{2\beta} \int_{Z^k} |\nabla' m'_k|^2 dx'. \tag{2.39}$$

Using Jensen's inequality, $v_k = M'_k = \frac{m'_k}{|m'_k|}$ on ∂Z^k and $|v_k|= 1$ on ∂Z^k, we get:

$$\begin{aligned} \int_{Z^k} \left| \fint_{\partial Z^k} (M'_k - m'_k) \right|^2 dx' &= \int_{Z^k} \left| \fint_{\partial Z^k} (v_k - m'_k) \right|^2 dx' \leq C\epsilon_k^{2\beta} \fint_{\partial Z^k} |v_k - m'_k|^2 d\mathcal{H}^1 \\ &\leq C\epsilon_k^{\beta} \int_{\partial Z^k} (1 - |m'_k|)^2 d\mathcal{H}^1 \leq C\epsilon_k^{\beta+2} \int_{\partial Z^k} G_{\epsilon_k}(m'_k) d\mathcal{H}^1. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{Z^k} |M'_k - m'_k|^2 dx' \\ &\leq C \int_{Z^k} \left(\left| M'_k - \fint_{\partial Z^k} M'_k \right|^2 + \left| \fint_{\partial Z^k} (M'_k - m'_k) \right|^2 + \left| m'_k - \fint_{\partial Z^k} m'_k \right|^2 \right) dx' \\ &\leq C\epsilon_k^{3\beta} \int_{\partial Z^k} |\nabla' m'_k|^2 d\mathcal{H}^1 + C\epsilon_k^{\beta+2} \int_{\partial Z^k} G_{\epsilon_k}(m'_k) d\mathcal{H}^1 + C\epsilon_k^{2\beta} \int_{Z^k} |\nabla' m'_k|^2 dx'. \end{aligned}$$

Summing up on all cells Z^k of R^k, since the convex hull of R^k covers $\widetilde{B}$, (2.30) and (2.33) we obtain:

$$\int_{\widetilde{B}} |M'_k - m'_k|^2 dx' \leq \frac{C\epsilon^{2\beta}}{\eta_k |\log \eta_k|}. \tag{2.40}$$

By (2.32), $\epsilon_k^{2\beta} \leq \eta_k^2 \ll \eta_k |\log \eta_k|$, so $||M'_k - m'_k||_{L^2(\widetilde{B})} = o(1)$ as $k \to \infty$.

Step B.5. The estimate of $||\nabla'(M'_k - m'_k)||_{L^2}$.
We have

$$\begin{aligned} \int_{Z^k} |\nabla'(M'_k - m'_k)|^2 &\leq 2 \int_{Z^k} |\nabla' M'_k|^2 + 2 \int_{Z^k} |\nabla' m'_k|^2 \\ &\leq C\epsilon_k^{\beta} \int_{\partial Z^k} |\nabla' m'_k|^2 d\mathcal{H}^1 + 2 \int_{Z^k} |\nabla' m'_k|^2 dx' \quad \text{(by (2.37))} \\ &\leq \frac{C}{\eta_k |\log \eta_k|}, \end{aligned}$$

where we have used (2.30),(2.33) in the last inequality.

Step C *Construct a stray field h_k associated to M'_k in $\mathbb{B} \subset \widetilde{B}$ such that* (2.14) *and* (2.15) *hold for the couple* (M'_k, h_k). For simplicity, we assume that $\mathbb{B} = \mathbb{B}^2$.
By Theorem 2.3, there exists $U_k \in \mathcal{BL}(\mathbb{R}^3)$ satisfying

$$\int_{\mathbb{R}^3} \nabla U_k(x) \nabla \zeta(x) dx = \int_{\Omega} \nabla'.m'_k(x') \zeta(x', 0) dx' \text{ for every } \zeta \in C_0^{\infty}(\mathbb{R}^3). \tag{2.41}$$

We note that the map $\xi \mapsto \int_{\mathbb{B}^2} \nabla'.m(x')\xi(x', 0)dx'$ is linear continuous in $H_0^1(\mathbb{B}^3)$. Indeed,

$$\begin{aligned} \int_{\mathbb{B}^2} \nabla'.m'(x')\xi(x', 0)dx' &\leq ||\nabla'.m'||_{\dot{H}^{-1/2}(\mathbb{B}^2)} ||\xi(\cdot, 0)||_{\dot{H}^{1/2}(\mathbb{B}^2)} \\ &\leq C||\nabla'.m'||_{\dot{H}^{-1/2}(\mathbb{B}^2)} ||\nabla \xi||_{L^2(\mathbb{B}^3)}. \end{aligned}$$

By Lax-Milgram's Theorem in $H_0^1(\mathbb{B}^3)$, there exists a unique solution $\overline{U}_k \in H_0^1(\mathbb{B}^3)$ of the following equation:

$$\int_{\mathbb{B}^3} \nabla\overline{U_k}(x)\nabla\zeta(x)dx = \int_{\mathbb{B}^2} \nabla.(M_k' - m_k')(x')\zeta(x',0)dx', \quad \forall\zeta \in H_0^1(\mathbb{B}^3) \tag{2.42}$$

Choosing

$$h_k := \nabla(U_k + \overline{U}_k) \tag{2.43}$$

and summing up (2.41), (2.42) we get that h_k is a stray field associated to M_k' in $\mathbb{B}^3$ and satisfies (2.14).
Now, we need to prove that:

$$\int_{\mathbb{B}^2} |\nabla' M_k'|^2 + \frac{1}{\eta_k}\int_{\mathbb{B}^3} |h_k|^2 \leq \frac{C}{\eta_k|\log\eta_k|}. \tag{2.44}$$

We observe that:

$$\int_{\mathbb{B}^2} |\nabla' M_k'|^2 \leq \frac{C}{\eta_k|\log\eta_k|} \quad \text{(by (2.37))}.$$

Using Theorem 2.3 and (2.30) we obtain

$$\int_{\mathbb{B}^3} |\nabla U_k|^2 \leq \int_{\mathbb{B}^3} |\nabla U_k|^2 = \frac{1}{2}\int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m_k')|^2 \leq \frac{C}{|\log\eta_k|}. \tag{2.45}$$

Then it is sufficient to prove:

$$\int_{\mathbb{B}^3} |\nabla\overline{U}_k|^2 \leq \frac{C}{|\log\eta_k|}. \tag{2.46}$$

Let us denote by T a linear continuous extension operator:

$$T : \dot{H}^s(\mathbb{B}^2) \to \dot{H}^s(\mathbb{R}^2), \quad s = 0, 1,$$

and let us extend $\overline{U}_k$ by 0 outside $\mathbb{B}^3$, we still denote it by $\overline{U}_k$. Then the extension $\overline{U}_k$ belongs to $H^1(\mathbb{R}^3)$ and the trace $\overline{U}_k|_{\mathbb{R}^2}$ belongs to $\dot{H}^{1/2}(\mathbb{R}^2)$. Therefore, we obtain:

$$\|\overline{U}_k|_{\mathbb{R}^2}\|^2_{\dot{H}^{1/2}(\mathbb{R}^2)} \leq \frac{1}{2}\|\overline{U}_k\|^2_{\dot{H}^1(\mathbb{R}^3)} = \frac{1}{2}\|\nabla\overline{U}_k\|^2_{L^2(\mathbb{B}^3)}. \tag{2.47}$$

Now using (2.42) with $\zeta = \overline{U}_k$ we have:

$$\begin{aligned}
\|\nabla\overline{U}_k\|^2_{L^2(\mathbb{B}^3)} &= \int_{\mathbb{B}^3} \left|\nabla\overline{U}_k\right|^2 = \int_{\mathbb{B}^2} \nabla'.(M_k' - m_k')\overline{U}_k \\
&= \int_{\mathbb{R}^2} \nabla'.T(M_k' - m_k')\overline{U}_k \leq \|\nabla'.(T(M_k' - m_k'))\|_{\dot{H}^{-1/2}(\mathbb{R}^2)}\|\overline{U}_k|_{\mathbb{R}^2}\|_{\dot{H}^{1/2}(\mathbb{R}^2)} \\
&\leq \|(T(M_k' - m_k'))\|_{\dot{H}^{1/2}(\mathbb{R}^2)}\|\overline{U}_k|_{\mathbb{R}^2}\|_{\dot{H}^{1/2}(\mathbb{R}^2)} \\
&\leq C\|T(M_k' - m_k')\|^{1/2}_{L^2(\mathbb{R}^2)}\|T(M_k' - m_k')\|^{1/2}_{\dot{H}^1(\mathbb{R}^2)}\|\nabla\overline{U}_k\|_{L^2(\mathbb{B}^3)}.
\end{aligned}$$

We have used the classical interpolation inequality and (2.47) in the last estimate.
Then

$$\begin{aligned}
\|\nabla\overline{U}_k\|_{L^2(\mathbb{B}^3)} &\leq C\|T(M_k' - m_k')\|^{1/2}_{L^2(\mathbb{R}^2)}\|T(M_k' - m_k')\|^{1/2}_{\dot{H}^1(\mathbb{R}^2)} \\
&\leq C\left(\int_{\mathbb{B}^2} |M_k' - m_k'|^2 dx'\right)^{1/2}\left(\int_{\mathbb{B}^2} |\nabla'(M_k' - m_k')|^2 dx'\right)^{1/2}.
\end{aligned}$$

In combining the results of Step B.4, Step B.5 and (2.32), this follow

$$\int_{\mathbb{B}^3} |\nabla\overline{U}_k|^2 dx \leq \frac{C\epsilon_k^\beta}{\eta_k|\log\eta_k|} \leq \frac{C}{|\log\eta_k|}. \tag{2.48}$$

By (2.45) and (2.46), we finally obtain that

$$\frac{1}{\eta_k}\int_{\mathbb{B}^3}|h_k|^2 \leq \frac{C}{\eta_k|\log\eta_k|}.$$

Step D.*Completion of the proof of Theorem 2.1.*
By Step C, we now can apply Theorem 2.6 to $\{(M'_k, h_k)\}$. Therefore, M'_k is relatively compact in $L^1(\mathbb{B}^2)$ as well as $L^p(\mathbb{B}^2)$.
Since the square B was arbitrary chosen in the complement of D and we proved the relatively compact in any ball $\mathbb{B}$ compactly included in B, by a diagonal argument, we deduce that $\{m'_k\}$ converges in $L^p(\Omega\backslash D)$ up to subsequence. Letting $r \to 0$, we obtain the conclusion of Theorem 2.1.

We finish this section by

Remark 2.8. Assume that $\{m_k\}_k$ is a sequence which satisfies the assumptions of Theorem 2.1.

(i) From Theorem 2.1, we know that, up to a subsequence, $m_k \to m$ almost everywhere in Ω. As $|m_k|= 1$ in Ω, the dominated convergence theorem implies that $m_k \to m$ in $L^1(\Omega)$.

(ii) Since $\nabla'.m' = 0$ in Ω (in the sense of distributions), then we can define the *normal trace* $(m'.\nu)$ of m' in the sense of distributions; that is,

$$\langle m'.\nu, \zeta\rangle_{\mathcal{D}'(\partial\Omega),\mathcal{D}(\partial\Omega)} := \int_\Omega m'.\nabla'\overline{\zeta}(x',0)dx', \qquad \text{for } \zeta \in C^\infty(\partial\Omega).$$

Here $\overline{\zeta}$ is the extension of ζ into $C^{1,1}(\overline{\Omega})$.

(iii) We have that $m'.\nu = g.\nu$ on $\partial\Omega$.

Indeed, for $\zeta \in C_0^\infty(\mathbb{R}^3)$, using Remark 2.8(i), one has

$$\begin{aligned}\int_\Omega m'(x').\nabla'\zeta(x',0)dx' \qquad &= \lim_{k\to\infty}\int_\Omega m_k(x').\nabla'\zeta(x',0)dx' \\ &= \int_{\partial\Omega} g.\nu\zeta d\mathcal{H}^1 - \lim_{k\to\infty}\int_\Omega \nabla'.m'_k(x')\zeta(x',0)dx'. \quad (2.49)\end{aligned}$$

Equations (2.3) and the Young inequality yield that

$$\left|\int_\Omega \nabla'.m'_k(x')\zeta(x',0)dx'\right| = \left|\int_{\mathbb{R}^3}\nabla U_k(x)\nabla\zeta(x)dx\right| \leq ||\nabla\zeta||_{L^2(\mathbb{R}^3)}||\nabla U_k||_{L^2(\mathbb{R}^3)},$$

where ∇U_k is the stray field associated with the magnetization m_k. Together with Theorem 2.3 and (2.30), this yields

$$\left|\int_\Omega \nabla'.m'_k(x')\zeta(x',0)dx'\right| \leq \frac{C}{|\log\eta_k|}.$$

Therefore, we obtain

$$m'.\nu = g.\nu \quad \text{ in } \mathcal{D}'(\partial\Omega).$$

2.5 Loss of compactness at the boundary

In this section, we aim to construct an example for the loss of compactness at the boundary, that is stated in Theorem 2.2.

The Proof of Theorem 2.2. The construction is carried in several steps. In this proof, we work at the level of sequences of parameters η_k and a sequence of magnetization m_k satisfying $m_k = (g, 0) = (0, -1, 0)$ on $\Gamma := \{0\} \times [-1, 1] \subset \partial\Omega$.

Step A. *The aim of this step is to introduce a Néel wall approximation.*
We follow Ignat [30]. Let us denote

$$\lambda_k := \eta_k |\log \eta_k|.$$

The parameter λ_k corresponds to the core size of an approximation of the Néel wall. More precisely, we consider the following 1D transition layer $(u_k, v_k) : \mathbb{R} \to \mathbb{S}^1$ that approximates a 180° Néel wall centered at the origin:

$$u_k(t) = \begin{cases} \frac{|\log \sqrt{t^2+\lambda_k^2}|}{|\log \lambda_k|} & \text{if } |t| \leq \sqrt{1-\lambda_k^2}, \\ 0 & \text{elsewhere,} \end{cases}$$

$$v_k(t) = \begin{cases} -\sqrt{1-u_k^2(t)} & \text{if } t \leq 0, \\ \sqrt{1-u_k^2(t)} & \text{if } t \geq 0. \end{cases}$$

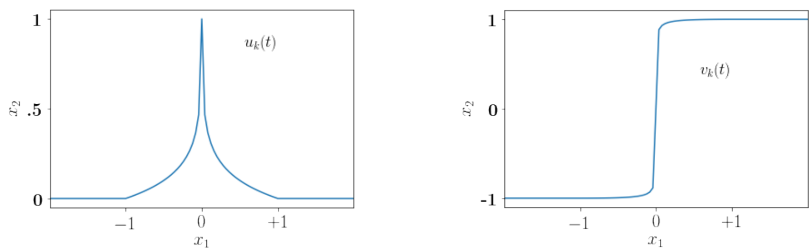

Figure 2.2: A 180° Néel wall approximation

The exchange energy corresponding to this transition as follows:

$$\begin{aligned} \int_{\mathbb{R}} \left|\frac{du_k}{dt}\right|^2 + \left|\frac{dv_k}{dt}\right|^2 \quad &= \int_{\mathbb{R}} \frac{1}{1-u_k^2}\left|\frac{du_k}{dt}\right|^2 \\ &\leq \int_{\mathbb{R}} \frac{1}{1-u_k}\left|\frac{du_k}{dt}\right|^2 \\ &= \frac{2}{|\log \lambda_k|} \int_{-\sqrt{1-\lambda_k^2}}^{\sqrt{1-\lambda_k^2}} \frac{t^2}{(t^2+\lambda^2)^2 \log \frac{t^2+\lambda^2}{\lambda^2}} dt \\ &\leq \frac{4}{\lambda_k |\log \lambda_k|} \int_0^{1/\lambda_k} \frac{s^2}{(s^2+1)^2 \log(s^2+1)} ds \\ &= O\left(\frac{1}{\lambda_k |\log \lambda_k|}\right). \end{aligned} \tag{2.50}$$

In order to estimate the stray-field energy of the transition layer, let W_k be the radial extension of u_k in $\mathbb{R}^2$:

$$W_k(x') = u_k(|x'|) \ , \ \text{for } x' \in \mathbb{R}^2.$$

By $\dot{H}^{1/2}(\mathbb{R})$-trace estimate of a $\dot{H}^1(\mathbb{R})$ - function, it follows:

$$||u_k||^2_{\dot{H}^{1/2}(\mathbb{R})} \leq \frac{1}{2}\int_{\mathbb{R}^2}|\nabla W_k|^2 dx' \leq \pi \int_0^1 r\left|\frac{du_k}{dr}\right|^2 dr.$$

Moreover,

$$\begin{aligned}\pi\int_0^1 r\left|\frac{du_k}{dr}\right|^2 dr \quad &\leq \frac{\pi}{|\log\lambda_k|^2}\int_0^1 \frac{r^3}{(r^2+\lambda_k^2)^2}dr\\ &= \frac{\pi}{|\log\lambda_k|^2}\int_0^{1/\lambda_k}\frac{s^3}{(s^2+1)^2}ds \leq \frac{\pi}{|\log\lambda_k|^2}(1+|\log\lambda_k|).\end{aligned}$$

Therefore

$$||u_k||^2_{\dot{H}^{1/2}(\mathbb{R})} \leq \frac{\pi+o(1)}{|\log\eta_k|}. \tag{2.51}$$

Step B. *Construction of sequence* $(m_k)_k$.
The sequence $m_k = (m_k', m_{3,k}) : \Omega = (0,1)\times(-1,1) \to \mathbb{S}^1$ we construct will consist of magnetization m_k that does not depend on the x_2 variable

$$m_k = m_k(x_1)$$

and

$$m_{3,k}(x') = 0 \quad \text{ in } \Omega.$$

More precisely, we have:

$$(m_{1,k}, m_{2,k}, m_{3,k})(x_1,x_2) = \left(u_k\left(\frac{x_1-\alpha_k}{\alpha_k}\right), v_k\left(\frac{x_1-\alpha_k}{\alpha_k}\right), 0\right),$$

where $\alpha_k > 0$ converges to 0 as $k\to\infty$, will be defined later, see (2.54).
Since $|m_k'|= 1$ in Ω, the full energy $E_{\epsilon_k,\eta_k,\kappa_k}(m_k)$ simplifies

$$E_{\eta_k}(m_k) = \int_\Omega |\nabla' m_k'|^2 dx' + \frac{1}{\eta_k}\int_{\mathbb{R}^2}||\nabla'|^{-1/2}(\nabla'.m_k')\mathbb{1}_\Omega|^2 dx'.$$

Observe that $m_k(x_1,x_2) = (g(x_1,x_2),0) = (0,-1,0)$ at the boundary $\{0\}\times(-1,1)$. To get the conclusion of Theorem 2.2, we will prove that

$$m_k \to (0,1,0) \text{ in } L^1(\Omega). \tag{2.52}$$

and

$$E_{\eta_k}(m_k) \leq \frac{\pi+o(1)}{\eta_k|\log\eta_k|}. \tag{2.53}$$

Step C *The proof of* (2.52) *and* (2.53). We start with (2.52).
Firstly,

$$\begin{aligned}&\int_\Omega |m_{2,k}(x')-1|dx' = \int_0^1\int_{-1}^1 |m_{2,k}(x')-1|dx_2dx_1 = 2\int_0^1 |m_{2,k}(x_1)-1|dx_1\\ &= 2\int_0^{2\alpha_k}|m_{2,k}(x')-1|dx_1 + 2\int_{2\alpha_k}^1 |m_{2,k}(x')-1|dx_1\\ &= 2\alpha_k\int_{-1}^1 |v_k(t)-1|dt + 2\alpha_k\int_1^\infty |v_k(t)-1|dt\\ &= 2\alpha_k\int_{-1}^1 |v_k(t)-1|dt \leq 8\alpha_k.\end{aligned}$$

This implies that $m_{2,k}$ converges to 1 in $L^1(\Omega)$ as $\alpha_k \to 0$.
Moreover, by the assumption $m_k(x') \in \mathbb{S}^1$ for every $x' \in \Omega$, then

$$\int_\Omega |m_{1,k}(x')|^2 dx' = \int_\Omega (1 - m_{2,k}^2(x'))dx'$$
$$\leq \int_\Omega |m_{2,k}(x') - 1||m_{2,k}(x') + 1|dx' \leq 2\int_\Omega |m_{2,k}(x') - 1|dx' \leq 16\alpha_k.$$

This implies that $m_{1,k}$ converges to 0 in $L^2(\Omega)$, so also in $L^1(\Omega)$.
The proof of (2.53)
We estimate the exchange energy of m_k.

$$\begin{aligned}\int_\Omega |\nabla' m_k(x')|^2 dx' &= \int_\Omega \left(\left|\frac{\partial m_{1,k}}{\partial x_1}(x_1)\right|^2 + \left|\frac{\partial m_{2,k}}{\partial x_1}(x_1)\right|^2 \right) dx_1 dx_2 \\ &= 2\int_0^1 \left(\left|\frac{\partial m_{1,k}}{\partial x_1}(x_1)|^2 + \right|\frac{\partial m_{2,k}}{\partial x_1}(x_1)\right|^2 \right) dx_1 \\ &\leq \frac{2}{\alpha_k}\int_\mathbb{R} \left(\left|\frac{du_k}{dt}(t)\right|^2 + \left|\frac{dv_k}{dt}(t)\right|^2 \right) dt.\end{aligned}$$

Estimate (2.50) yields that

$$\int_\Omega |\nabla' m_k(x')|^2 dx' \leq O\left(\frac{1}{\alpha_k \lambda_k |\log \lambda_k|}\right).$$

Choosing

$$\alpha_k = \frac{1}{|\log \lambda_k|^{1/2}} \to 0, \tag{2.54}$$

we then have

$$\int_\Omega |\nabla m_k(x')|^2 dx' = O\left(\frac{1}{\lambda_k |\log \lambda_k|^{1/2}}\right) = O\left(\frac{1}{\eta_k |\log \eta_k|^{3/2}}\right). \tag{2.55}$$

Estimating the stray-yield energy.
We recall the result of Theorem 2.3 that

$$\int_{\mathbb{R}^3} |\nabla U_k(x)|^2 dx = \frac{1}{2}\int_{\mathbb{R}^2} ||\nabla'|^{-1/2}(\nabla'.m_k')\mathbb{1}_\Omega|^2 dx' = \int_\Omega U_k(x',0)\nabla'.m_k'(x')dx', \tag{2.56}$$

where U_k is the stray-field associates with the magnetization m_k.
Therefore, in order to estimate the stray-field energy, it is sufficient to find an upper bound of

$$\int_\Omega U_k(x',0)\nabla'.m_k'(x')dx'.$$

The computation will be done according to $\nabla'.m_k' = \frac{\partial m_{1,k}}{\partial x_1}$.
Let us recall the homogeneous $\dot{H}^{1/2}(\mathbb{R})$ semi-norm of $v : \mathbb{R} \to \mathbb{R}$ be defined by

$$||v||^2_{\dot{H}^{1/2}(\mathbb{R})} := \int_\mathbb{R} |\xi||\mathcal{F}_1 v|^2 d\xi, \tag{2.57}$$

where $\mathcal{F}_1(v) \in \mathcal{S}'(\mathbb{R})$ stands for the Fourier transform of v.
It is known that (see Ignat [30, Proposition 7])

$$||v||^2_{\dot{H}^{1/2}(\mathbb{R})} = \frac{1}{2\pi}\int_\mathbb{R}\int_\mathbb{R} \frac{|v(s) - v(r)|^2}{|s - r|^2} dsdr, \tag{2.58}$$

also

$$||v||^2_{\dot{H}^{1/2}(\mathbb{R})} = \frac{1}{2}\min\left\{\int_{\mathbb{R}^2}|\nabla' V|^2 dx' : V(x_1,0) = v(x_1) \text{ for every } x_1 \in \mathbb{R}\right\}. \tag{2.59}$$

We now estimate $\int_\Omega U_k(x',0)\frac{\partial m_{1,k}}{\partial x_1}(x')dx'$. Using the fact that $m_{1,k}(x_1,\cdot) = 0$ for any $x_1 \notin (0, 2\alpha_k)$, (we extends $m_{1,k}(x_1,\cdot) = 0$ in $x_1 \in \mathbb{R}\backslash[0,1]$) then

$$\int_\Omega U_k(x',0)\frac{\partial m_{1,k}}{\partial x_1}(x')dx' = \int_{-1}^{1}\left(\int_{\mathbb{R}} U_k(x',0)\frac{\partial m_{1,k}}{\partial x_1}(x')dx_1\right)dx_2. \tag{2.60}$$

Without loss of generality, we consider $m_{1,k}$ as a function of the x_1-variable, then the Parseval identity and the Cauchy-Schwarz inequality yield:

$$\begin{aligned}
&\left(\int_{\mathbb{R}} U_k(x_1,x_2,0)\frac{\partial m_{1,k}}{\partial x_1}(x')dx_1\right)^2 \\
&= \left(\int_{\mathbb{R}} \mathcal{F}_1(U_k(\cdot,x_2,0))(\xi_1)\overline{\mathcal{F}_1(\frac{\partial m_{1,k}}{\partial x_1})(\xi_1)}d\xi_1\right)^2 \\
&= \left(\int_{\mathbb{R}} i\xi_1\mathcal{F}(U_k(\cdot,x_2,0))(\xi_1)\overline{\mathcal{F}_1(m_{1,k})(\xi_1)}d\xi_1\right)^2 \\
&\le \left(\int_R |\xi_1||\mathcal{F}_1(U_k(\cdot,x_2,0))(\xi_1)|^2 d\xi_1\right)\left(\int_{\mathbb{R}}|\xi_1||\mathcal{F}_1(m_{1,k})(\xi_1)|^2 d\xi_1\right) \\
&= ||m_{1,k}(\cdot)||^2_{\dot{H}^{1/2}(\mathbb{R})}||U_k(\cdot,x_2,0)||^2_{\dot{H}^{1/2}(\mathbb{R})} \\
&\le \frac{1}{2}||m_{1,k}(\cdot)||^2_{\dot{H}^{1/2}(\mathbb{R})}\int_{\mathbb{R}^2}|(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3})U_k|^2 dx_1 dx_3.
\end{aligned} \tag{2.61}$$

Here, we have use the definition (2.59) to get the last inequality.
As $\dot{H}^{1/2}(\mathbb{R})$ is scaling invariant in $\mathbb{R}$, by the definition of $m_{1,k}$, we obtain

$$||m_{1,k}(\cdot)||^2_{\dot{H}^{1/2}(\mathbb{R})} = ||u_k||^2_{\dot{H}^{1/2}(\mathbb{R})} \le \frac{\pi + o(1)}{|\log\eta_k|} \quad (\text{ by } (2.51)). \tag{2.62}$$

Combining together (2.60)-(2.62) yields that

$$\begin{aligned}
&\int_\Omega U_k(x',0)\frac{\partial m_{1,k}}{\partial x_1}(x')dx' \\
&\le \int_{-1}^{1}\left(\frac{1}{2}\int_{\mathbb{R}^2}\left|\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3}\right)U_k(x_1,x_2,x_3)\right|^2 dx_1dx_3\right)^{1/2}||u_k||_{\dot{H}^{1/2}(\mathbb{R})}dx_2 \\
&\le \sqrt{2}(\frac{\pi+o(1)}{|\log\eta_k|})^{1/2}\left(\frac{1}{2}\int_{\mathbb{R}^3}|\nabla U_k(x)|^2 dx\right)^{1/2}.
\end{aligned} \tag{2.63}$$

By (2.56), it implies that

$$\frac{1}{\eta_k}\int_{\mathbb{R}^3}|\nabla U_k(x)|^2 dx \le \frac{\pi + o(1)}{\eta_k|\log\eta_k|}. \tag{2.64}$$

Finally, summing up (2.50) and (2.64), we get the conclusion of (2.53). The proof of Theorem 2.2 is completed. □

Chapter 3

Interior and boundary regularity results in a model for thin ferromagnetic films with Dzyaloshinky-Moriya interaction

Abstract

In this chapter, we study the regularity of critical points of a non-local energy which stems from a two dimensional ferromagnetic model with Dzyaloshinky-Moriya interaction. First we show the interior regularity of critical points, like for $\mathbb{S}^2$-valued harmonic maps on $2D$ domains, the critical points are smooth in the interior of domain. We also prove a boundary regularity result that the critical points are $C^{1,\alpha}$ up to the boundary, for all $\alpha \in (0, \frac{1}{2})$. The particularity of this work is to study the non-local model within an imposed the Dirichlet boundary condition.

3.1 Introduction

In this chapter, we use the notation which are used in the previous chapter. Let $\Omega \in \mathbb{R}^2$ be a $C^{1,1}$-domain and $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function. We consider a magnetization $m = (m', m_3) : \Omega \subset \mathbb{R}^2 \to \mathbb{S}^2$ satisfying the boundary condition

$$m = (g, 0) \text{ on } \partial\Omega.$$

Denote

$$H^1_g(\Omega, \mathbb{S}^2) := \{u \in H^1(\Omega, \mathbb{R}^3) \ : \ |u(x)| = 1 \text{ a.e } , u = (g, 0) \text{ on } \partial\Omega\}$$

and

$$E_{\epsilon,\eta,\kappa}(m) = \int_\Omega |\nabla' m|^2 dx' + \frac{2}{\epsilon^2} \int_\Omega F(m) dx' + \frac{1}{\eta} \int_{\mathbb{R}^2} \left| |\nabla'|^{-1/2} (\nabla' . m')_{ac} \right|^2 dx' + \int_\Omega DM(m) dx',$$

where $\epsilon, \eta > 0$ are two small positive parameters. Here $x' = (x_1, x_2)$ are the in-plane variables with the differential operator

$$\nabla' = (\partial_{x_1}, \partial_{x_2}),$$

and the third variable is denoted by x_3.

The first term is the so-called exchange energy, it penalizes spatial variation of m through the Dirichlet integral of m. The second term is the anisotropy energy. The function $F : \mathbb{S}^2 \to \mathbb{R}_+$ is smooth whose zeros are the preferred directions of m (called easy axis[1]).

[1] The previous chapter, $F(m) = m_3^2/2$, so preferring all axis in the horizontal plane.

The third term in the energy $E_{\epsilon,\eta,\kappa}$ can be equivalently expressed in term of L^2-norm of the stray-field ∇U:

$$\int_{\mathbb{R}^2} \left| |\nabla'|^{-1/2} (\nabla'.m')_{ac} \right|^2 dx' = \int_{\mathbb{R}^2} \frac{1}{|\xi|} |\mathcal{F}(\nabla'.m')_{ac}|^2 d\xi = 2 \int_{\mathbb{R}^3} |\nabla U|^2 dx.$$

Here we denote

$$(\nabla'.m')_{ac} = \nabla'.m' \mathbb{1}_\Omega$$

and $U : \mathbb{R}^3 \to \mathbb{R}$ is the stray field potential which is determined by the static Maxwell equation in weak sense:

$$\int_{\mathbb{R}^3} \nabla U(x) \nabla \zeta(x) dx = \int_\Omega \nabla'.m'(x') \zeta(x', 0) dx' \text{ for every } \zeta \in C_0^\infty(\mathbb{R}^3). \tag{3.1}$$

The last term is the Dzyaloshinky-Moriya interaction given by

$$\begin{aligned}
&\int_\Omega DM(m) dx' = \kappa_1 \int_\Omega m.\nabla' \times m dx' + \kappa_2 \int_\Omega (m_3 \nabla'.m' - m'.\nabla' m_3) dx' \\
&= \kappa_1 \int_\Omega (m_1 \partial_{x_2} m_3 - m_2 \partial_{x_1} m_3 + m_3 \partial_{x_1} m_2 - m_3 \partial_{x_2} m_1) dx' \\
&+\kappa_2 \int_\Omega (m_3(\partial_{x_1} m_1 + \partial_{x_2} m_2) - m_1 \partial_{x_1} m_3 - m_2 \partial_{x_2} m_3) dx',
\end{aligned}$$

where $\kappa = (\kappa_1, \kappa_2)$ are arbitrary coefficients.

The principal questions we shall discuss here are the existence and regularity of critical points $E_{\epsilon,\eta,\kappa}(m)$ defined for $m \in H_g^1(\Omega, \mathbb{S}^2)$.
The existence of a critical point $m \in H_g^1(\Omega, \mathbb{S}^2)$ is presented in Section 3.2. Moreover we determiner the Euler-Lagrange equation for the energy $E_{\epsilon,\eta,\kappa}$ (as in the case of harmonic maps.)

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a $C^{1,1}$-domain and $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function. If*

$$\frac{1}{\epsilon^2} F((\xi_1, \xi_2, \xi_3)) \geq 1^+ (\kappa_1^2 + \kappa_2^2) \xi_3^2 \text{ for all } \xi = (\xi_1, \xi_2, \xi_3) : \Omega \to \mathbb{S}^2,$$

where 1^+ is any fixed number larger than 1, then there exists a minimizer $m \in H_g^1(\Omega, \mathbb{S}^2)$ of $E_{\epsilon,\eta,\kappa}$. Moreover m satisfies the following

$$\begin{aligned}
&- \Delta m - m |\nabla' m|^2 - \frac{1}{\eta} \begin{pmatrix} H \\ 0 \end{pmatrix} + \frac{1}{\eta} m'.Hm + \frac{1}{\epsilon^2} (f(m) - m.f(m)m) \\
&+ \kappa_1 (Id - m \otimes m) \nabla' \times m + \kappa_2 \left(\begin{pmatrix} -\nabla' m_3 \\ \nabla'.m' \end{pmatrix} + (\nabla' m_3.m' - m_3 \nabla'.m')m \right) = 0 \text{ in } \mathcal{D}'(\Omega),
\end{aligned}$$

where $H := \nabla' U(\cdot, 0)$ and $f = \nabla F$.

The regularity result we prove in Section 3 is that any critical point m of $E_{\epsilon,\eta,\kappa}$ over $H_g^1(\Omega)$ is smooth in the interior of domain and $C^{1,\alpha}$, for all $\alpha \in (0, \frac{1}{2})$ up to the boundary. The regularity theory for critical point of quadratic functional in dimension two has considerably progressed since the theorems of Morrey, see [41]. One of the most important results is proved by Hélein, see [26]. It concerns the regularity of harmonic maps defined in an open set of $\mathbb{R}^2$ and with values in a sphere (a Riemannian manifold). In [11], Carbou shows a result of regularity of critical points of a different ferromagnetic model (without any imposed boundary condition). For the regularity in the interior domain, we use mainly here the works of Hélein about the harmonic maps with values into $\mathbb{S}^2$, also the works of Carbou (see [11]).

The regularity at the boundary for a three dimensional ferromagnetic functional have been studied by Hardt, Kinderlehrer and Lin (see [24]), also by Huber ([27]). For that, one studies the minimizing problem

$$\overline{E}(m) = d^2 \int_{\omega} |\nabla m|^2 dx + \int_{\mathbb{R}^3} |\nabla U(x)|^2 dx$$

among all function $m \in H^1(\omega, \mathbb{S}^2)$, where the so-called stray-field potential $U : \mathbb{R}^3 \to \mathbb{R}$ is generated by the Maxwell equation:

$$-\Delta U = \nabla.(m\mathbb{1}_{\omega}) \;\text{ in } \mathbb{R}^3 \tag{3.2}$$

and $\omega \subset \mathbb{R}^3$ is the ferromagnetic sample. We remark that these minimizers satisfy $\frac{\partial m}{\partial \nu} = 0$ on the boundary, where ν is the unit outer normal vector. Thus their Neumann boundary condition is different from our imposed Dirichlet boundary condition.

The general idea used in [27] is to construct a reflection at the boundary in order to establish a situation which is similar to the setting in the interior. In the case of minimizing harmonic maps, one can follow the ideas by Schoen and Uhlenbeck, see [46] based on a monotonicity formula. Then, the higher regularity will be obtain by the interior setting tools. Moreover, a special coordinate system is introduced in [27] in order to obtain the regularity of differential in the outer normal direction which is based on a reflection construction and the Neumann boundary condition $\frac{\partial m}{\partial \nu} = 0$.

In our work, instead of using the reflection construction, we shall use delicately the Nirenberg method to obtain a higher regularity through the tangential direction. The regularity through the normal direction will be obtain by the anisotropic Sobolev embedding, see [25].

In the next section, we recall some preliminaries on the stray-field and prove the existence of minimizers. The regularity of critical points shall be given in Section 3.3. It is split into 2 subsections (the interior regularity and the boundary regularity).

3.2 Existence of minimizers and Euler-Lagrange equation

Let us first recall some important properties of the stray field $\int_{\mathbb{R}^3} |\nabla U|^2 dx$. We first recall the definition of the Beppo-Levi space

$$\mathcal{BL} := \left\{ U : \mathbb{R}^3 \to \mathbb{R} : \nabla U \in L^2(\mathbb{R}^3), \frac{U(x)}{1+|x|} \in L^2(\mathbb{R}^3) \right\}.$$

Consequently, the space $\mathcal{BL}$ endowed by the homogeneous $\dot{H}^1-$ norm, $U \mapsto ||\nabla U||_{L^2(\mathbb{R}^3)}$ is a Hilbert space, and the set $C_0^\infty(\mathbb{R}^3)$ of smooth compactly supported functions is a dense set, see Dautray and Lions [16].

Remark 3.2. If $m, l \in H^1(\Omega, \mathbb{S}^2)$ and $U = U(m), V = V(l)$ are the corresponding solutions of (3.1), then

$$\int_{\Omega} V(x',0) \nabla'.m'(x') dx' = \int_{\Omega} U(x',0) \nabla'.l'(x') dx'.$$

Indeed, by the density of $C_0^\infty(\mathbb{R}^3)$ in $\mathcal{BL}(\mathbb{R}^3)$, then (3.1) is still true for every $\zeta \in \mathcal{BL}(\mathbb{R}^3)$.
Choose $\zeta = V$, we obtain:

$$\int_{\Omega} V(x',0) \nabla'.m'(x') dx' = \int_{\mathbb{R}^3} \nabla U(x) \nabla V(x) dx' = \int_{\Omega} U(x',0) \nabla'.l'(x') dx'.$$

Remark 3.3. For $m \in H^1(\Omega, \mathbb{R}^2)$ and the stray potential $U(m)$ satisfying the Maxwell equation (3.1), one can present U under the term of the Fourier transform (see the proof of Theorem 2.3 in the appendix)

$$\mathcal{F}(U(\cdot,0))(\xi) = \frac{1}{2|\xi|}\mathcal{F}((\nabla'.m')\mathbb{1}_\Omega)(\xi) \text{ for all } \xi \in \mathbb{R}^2. \tag{3.3}$$

Hence,

$$\begin{aligned}\int_{|\mathbb{R}^2} |\nabla U(\cdot,0)|^2 dx' = \int_{\mathbb{R}^2} |\xi|^2 |\mathcal{F}(U(\cdot,0))(\xi)|^2 d\xi \qquad &= \int_{\mathbb{R}^2} |\xi|^2 \frac{1}{4|\xi|^2} |\mathcal{F}((\nabla'.m')\mathbb{1}_\Omega)(\xi)|^2 d\xi \\ &= \frac{1}{4}||(\nabla'.m')\mathbb{1}_\Omega||^2_{L^2(\mathbb{R}^2)} \\ &\le \frac{1}{4}||\nabla'.m'||^2_{L^2(\Omega)}.\end{aligned}$$

This implies that $\nabla' U(\cdot,0) \in L^2(\mathbb{R}^2)$.

Moreover, when m is more regular, one has

Lemma 3.4. *If $m \in H^k_{loc}(\Omega) \cap H^1(\Omega)$ then $U(\cdot,0) \in H^k_{loc}(\Omega), \forall k \ge 2$.*

The proof of Lemma 3.4 is given in the appendix.

3.2.1 The existence of minimizers

Let $\Omega \in \mathbb{R}^2$ be a $C^{1,1}$-domain and $g : \partial\Omega \to \mathbb{S}^1$ be a smooth function. In this section, we study the existence of a minimizer of the energy $E_{\epsilon,\eta,\kappa}(m)$ under the constrains $m \in H^1_g(\Omega, \mathbb{S}^2)$. For that we assume that

$$\frac{1}{\epsilon^2} F((\xi_1,\xi_2,\xi_3)) \ge 1^+(\kappa_1^2+\kappa_2^2)\xi_3^2 \text{ for all } \xi = (\xi_1,\xi_2,\xi_3) : \Omega \to \mathbb{S}^2, \tag{3.4}$$

where 1^+ is any fixed number larger than 1. Observe that

$$E_{\epsilon,\eta,\kappa}(m) > -\infty \text{ for every } m \in H^1_g(\Omega, \mathbb{S}^2). \tag{3.5}$$

Indeed, using the definition of the DM term and the boundary condition $m = (g, 0)$ on $\partial\Omega$, one has by integration by parts

$$\kappa_1 \int_\Omega m.\nabla' \times m dx' = 2\kappa_1 \int_\Omega (m_3 \partial_{x_1} m_2 - m_3 \partial_{x_2} m_1) dx'$$

and

$$\kappa_2 \int_\Omega (m_3 \nabla'.m' - m'.\nabla' m_3) dx' = 2\kappa_2 \int_\Omega m_3 \nabla'.m' dx'.$$

Therefore

$$\begin{aligned}\int_\Omega DM(m) dx' \qquad &\le 2^+(\kappa_1^2+\kappa_2^2) \int_\Omega m_3^2 dx' + 1^- \int_\Omega |\nabla' m'|^2 dx' \\ &\frac{2}{\epsilon^2}\int_\Omega F(m) dx' + 1^- \int_\Omega |\nabla' m'|^2 dx'\end{aligned}$$

where $2^+ := 2.1^+$ with 1^+ given in (3.4) and $1^- = \frac{2}{2^-}$. Combining with (3.4) and the definition of the energy, this yields (3.5).

By the above argument, we then take $(m_k)_k$ a minimizing sequence of $E_{\epsilon,\eta,\kappa}$ in $H^1_g(\Omega, \mathbb{S}^2)$ for ϵ, η, κ fixed.
Since $(0^+) \int_\Omega |\nabla m_k'|^2 \le E_{\epsilon,\eta,\kappa}(m_k)$ for some $0^+ > 0$ and (m_k) takes values in $\mathbb{S}^2$, the sequence $(m_k)_k$ is bounded in $H^1(\Omega)$.
Hence, up to a subsequence, there exists $m \in H^1(\Omega, \mathbb{R}^3)$ such that

(i) $m_k \rightharpoonup m$ weakly in $H^1(\Omega, \mathbb{R}^3)$,

(ii) $m_k \to m$ in $L^p(\Omega, \mathbb{R}^3)$ for every $1 \leq p < +\infty$

(iii) $|m|= 1$ in Ω.

The third convergence implies that m is $\mathbb{S}^2$-valued. By the trace operator, we also get that

$$m' = g \text{ in } H^{1/2}(\partial\Omega). \tag{3.6}$$

Moreover,

$$\int_\Omega |\nabla' m|^2 dx' \leq \liminf \int_\Omega |\nabla' m_k|^2 dx',$$

$$\int_\Omega F(m_k)dx' \to \int_\Omega F(m)dx'.$$

We are going to establish that

$$\int_\Omega DM(m_k)dx' \to \int_\Omega DM(m)dx'.$$

Indeed, let us consider the first term of DM energy, $(\int_\Omega m.\nabla' \times m)$. Observe that

$$m_k \to m \text{ strongly in } L^2(\Omega, \mathbb{S}^2),$$

and

$$\nabla' \times m_k \rightharpoonup \nabla' \times m \text{ weakly in } L^2(\Omega, \mathbb{R}^3).$$

Therefore, we get

$$\int_\Omega m_k.\nabla' \times m_k dx' \to \int_\Omega m.\nabla' \times m dx'. \tag{3.7}$$

Using the same argument as above for the second term of the DM energy, we obtain

$$\int_\Omega \Big(m_{3,k}\nabla'.m_k' - m_k'.\nabla' m_{3,k} \Big) dx' \to \int_\Omega \Big(m_3 \nabla'.m' - m'.\nabla' m_3 \Big) dx'.$$

Then we get the convergence of the DM energy.

We now prove the convergence of the stray-field energy. Let us call U_k, U the stray potentials associated with m_k, m satisfying (3.1).
By Theorem 2.3, we get

$$\begin{aligned}
\|\nabla(U_k - U)\|^2_{L^2(\mathbb{R}^3)} &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{|\xi|} \Big| \mathcal{F}((\nabla'.(m_k' - m'))\mathbb{1}_\Omega)(\xi) \Big|^2 d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{|\xi|} \Big| \int_{\mathbb{R}^2} e^{-ix'\xi} \nabla'.(m_k' - m')\mathbb{1}_\Omega dx' \Big|^2 d\xi.
\end{aligned}$$

Since $m_k' = m' = g$ on the boundary $\partial\Omega$, we then use the Green theorem to obtain

$$\begin{aligned}
\|\nabla(U_k - U)\|^2_{L^2(\mathbb{R}^3)} &= \frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{|\xi|} \Big| \int_{\mathbb{R}^2} -ie^{-ix'\xi}\xi.(m_k' - m')\mathbb{1}_\Omega dx' \Big|^2 d\xi. \\
&\leq \frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{|\xi|} |\xi|^2 |\mathcal{F}((m_k' - m')\mathbb{1}_\Omega)(\xi)|^2 d\xi \\
&= \frac{1}{2}\|(m_k' - m')\mathbb{1}_\Omega\|^2_{\dot{H}^{1/2}(\mathbb{R}^2)}.
\end{aligned}$$

The standard interpolation inequality implies that

$$\begin{aligned}
\|\nabla(U_k - U)\|^2_{L^2(\mathbb{R}^3)} &\leq \frac{1}{2}\|(m'_k - m')\mathbb{1}_\Omega\|_{\dot{H}^1(\mathbb{R}^2)}\|(m'_k - m')\mathbb{1}_\Omega\|_{L^2(\mathbb{R}^2)} \\
&= \frac{1}{2}\|m'_k - m'\|_{\dot{H}^1(\Omega)}\|m'_k - m'\|_{L^2(\Omega)}.
\end{aligned}$$

The first factor in the right hand side is bounded while the second one tends to 0. This proves that ∇U_k converges to ∇U in $L^2(\mathbb{R}^3)$. We conclude that m is a minimizer of $E_{\epsilon,\eta,\kappa}$ over $H^1_g(\Omega,\mathbb{S}^2)$.

3.2.2 Euler-Lagrange equation

We are going to compute the Euler-Lagrange equation satisfied by the critical points $m_k \in H^1_g(\Omega)$ of $E_{\epsilon,\eta,\kappa}$. Let Φ be an element of $C^\infty_0(\Omega,\mathbb{R}^3)$. We set

$$m_t = \frac{m + t\Phi}{|m + t\Phi|},$$

for t small (e.g. $t \leq \frac{1}{||\Phi||_{L^\infty(\Omega)}+1}$), otherwise $m + t\Phi$ may has zeros. Since $|m + t\Phi|^{-1}= \big(|m|^2+2tm.\Phi + t^2|\Phi|^2\big)^{-1/2} = 1 - tm.\Phi + O(t^2)$, then

$$m_t = m + t(Id - m \otimes m)\Phi + O(t^2).$$

Let $U_t(m_t) \in \mathcal{BL}(\mathbb{R}^3)$ be the solution of the Maxwell equation (3.1) associated with m_t. We ahve

$$\begin{aligned}
E_{\epsilon,\eta,\kappa}(m_t) = \frac{1}{2}\int_\Omega |\nabla' m_t|^2 + \frac{1}{\epsilon^2}\int_\Omega F(m_t)dx' \\
+ \frac{1}{2\eta}\int_\Omega U_t(x',0)\nabla'.m'_t(x')dx' + \int_\Omega DM(m_t)dx' \quad (3.8)
\end{aligned}$$

and $\sigma(m) = (Id - m \otimes m)\Phi$.
Since (3.1) is linear and has a unique solution, then $U_t = U + t\sigma U + O(t^2)$, where σU is the solution of (3.1) , or

$$\begin{cases}
\Delta(\sigma U) = 0 & \text{in } \mathbb{R}^3 \diagdown (\overline{\Omega} \times \{0\}), \\
\left[\frac{\partial \sigma U}{\partial x_3}\right] = -\nabla'.(\sigma(m)') & \text{on } \Omega \times \{0\}, \\
[\sigma U] = 0 & \text{on } \mathbb{R}^2 \times \{0\}.
\end{cases}$$

We prove that

$$E_{\epsilon,\eta,\kappa}(m_t) = E_{\epsilon,\eta,\kappa}(m) + t\sigma E_{\epsilon,\eta,\kappa} + O(t^2), \quad (3.9)$$

where

$$\begin{aligned}
\sigma E_{\epsilon,\eta,\kappa} \quad &:= \int_\Omega \nabla' m.\nabla'(\sigma(m))dx' + \frac{1}{\epsilon^2}\int_\Omega (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})F(m).\sigma(m)dx' \\
&+ \frac{1}{\eta}\int_\Omega U(x',0)\nabla'.(\sigma(m)'dx' \\
&+ 2\kappa_1\int_\Omega \sigma(m).\nabla' \times m dx' + 2\kappa_2\int_\Omega (m_3\nabla'.\sigma(m)' + \sigma(m)_3\nabla'.m')dx'.
\end{aligned}$$

Indeed, it is simple to check that

$$\int_\Omega |\nabla' m_t|^2 dx' = \int_\Omega |\nabla' m|^2 dx' + 2t\int_\Omega \nabla' m.\nabla'\sigma(m)dx' + O(t^2),$$

and

$$\int_\Omega F(m_t)dx' = \int_\Omega F(m)dx' + t\int_\Omega (\nabla', \partial_{x_3})F(m).\sigma(m)dx' + O(t^2) \quad \text{(by Taylor's expansion)}.$$

Remark 3.2 and definition of U_t yield that

$$\int_\Omega U_t(x',0)\nabla'.m_t'(x)dx' = \int_\Omega U(x',0)\nabla'.m'(x')dx'$$
$$+t\int_\Omega U(x',0)\nabla'.\sigma(m)'(x')dx' + t\int_\Omega \sigma U(x',0)\nabla'.m'(x')dx' + O(t^2),$$
$$= \int_\Omega U(x',0)\nabla'.m'(x')dx' + 2t\int_\Omega U(x',0)\nabla'.\sigma(m)'(x')dx' + O(t^2).$$

For the first term in the DM energy, one has

$$\kappa_1\int_\Omega m_t.\nabla'\times m_t dx'$$
$$= \kappa_1\int_\Omega m.\nabla'\times m dx' + t\kappa_1\left(\int_\Omega m.\nabla'\times\sigma(m)dx' + \int_\Omega \sigma(m).\nabla'\times m dx'\right) + O(t^2).$$

We recall that $m_3 = m_{t,3} = 0$ on the boundary, thus integration by parts implies

$$\int_\Omega m.\nabla'\times\sigma(m)dx' + \int_\Omega \sigma(m).\nabla'\times m dx' = 2\int_\Omega \sigma(m).\nabla'\times m dx'.$$

This implies that

$$\kappa_1\int_\Omega m_t.\nabla'\times m_t dx' = \kappa_1\int_\Omega m.\nabla'\times m dx' + 2t\kappa_1\int_\Omega \sigma(m).\nabla'\times m dx' + O(t^2).$$

We also have that

$$\kappa_2\int_\Omega m_{t,3}(\nabla'.m_t' - m_t'.\nabla' m_{t,3})dx' = 2\kappa_2\int_\Omega m_{t,3}\nabla'.m_t'dx'.$$

An easy computation shows that

$$2\kappa_2\int_\Omega m_{t,3}\nabla'.m_t'dx'$$
$$= 2\kappa_2\int_\Omega m_3\nabla'.m'dx' + 2t\kappa_2\int_\Omega (\sigma(m)_3\nabla'.m' + m_3\nabla'.\sigma(m)')dx' + O(t^2) \quad (3.10)$$

which proves (3.9).

We now determine the Euler-Lagrange equation.

Rewriting the first term in $\sigma E_{\epsilon,\eta,\kappa}$.

$$\begin{aligned}\int_\Omega \nabla' m.\nabla'\sigma(m)dx' &= \int_\Omega \nabla' m.\nabla'(\Phi - m.\Phi\, m)dx' \\ &= \int_\Omega \nabla' m.\nabla'\Phi dx' - \int_\Omega |\nabla' m|^2 m.\Phi dx' - \int_\Omega \nabla' m.(m\nabla'(m.\Phi)^T)dx' \\ &= \int_\Omega \nabla' m.\nabla'\Phi dx' - \int_\Omega |\nabla' m|^2 m.\Phi dx'. \qquad (3.11)\end{aligned}$$

We have used that m orthogonal to $\nabla' m$ in the last equality.
Hence

$$\int_\Omega \nabla' m.\nabla'(\sigma(m)) = \langle -\Delta m - m|\nabla' m|^2; \Phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Rewriting the second term in $\sigma E_{\epsilon,\eta,\kappa}$.

$$\int_\Omega (\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m).\sigma(m) = \langle(\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m) - m.(\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m)m;\Phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Rewriting the third term in $\sigma E_{\epsilon,\eta,\kappa}$.
Using the Green formula and the fact that $\Phi = 0$ on $\partial\Omega$, one has

$$\begin{aligned}
\int_\Omega U(x',0)\nabla'.\sigma(m)'dx' &= \int_\Omega U(x',0)\nabla'.(\Phi' - m.\Phi\, m')dx' \\
&= \int_\Omega (-\nabla' U(x',0).\Phi' + \nabla' U(x',0).m'\Phi.m)dx' \\
&= \int_\Omega (-(\nabla' U(x',0),0).\Phi + \nabla' U(x',0).m'm.\Phi)dx' \\
&= \langle -(\nabla' U(\cdot,0),0) + \nabla' U(\cdot,0).m'\, m;\Phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.
\end{aligned}$$

Rewriting the DMI term in $\sigma E_{\epsilon,\eta,\kappa}$.
The first term in the DMI.

$$\begin{aligned}
2\int_\Omega \sigma(m).\nabla' \times m dx' &= 2\Big(\int_\Omega \Phi.\nabla' \times m dx' - \int_\Omega m.\Phi\, m.\nabla' \times m\Big)dx' \\
&= \langle \Phi, 2\nabla' \times m - 2m(m.\nabla' \times m)\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.
\end{aligned}$$

The second term in the DMI. By integration by parts, it follows for $\Phi = 0$ on $\partial\Omega$

$$\begin{aligned}
&2\int_\Omega m_3\nabla'.\sigma(m)' + \sigma(m)_3\nabla'.m'dx' \\
&= 2\int_\Omega (m_3\nabla.(\Phi' - m.\Phi m') + (\Phi_3 - m.\Phi m_3)\nabla'.m')dx' \\
&= 2\int_\Omega ((m_3\nabla'.\Phi' + \Phi_3\nabla'.m') - (m_3\nabla'(m.\Phi).m' + m_3 m.\Phi\nabla'.m' + m.\Phi m_3\nabla'.m'))dx' \\
&= 2\int_\Omega ((-\nabla' m_3.\Phi' + \nabla'.m'\Phi_3) - (-\nabla'.(m_3 m')m.\Phi + 2m_3\nabla'.m'm.\Phi))dx' \\
&= 2\int_\Omega ((-\partial_{x_1}m_3, -\partial_{x_2}m_3, \nabla'.m').\Phi - (m_3\nabla'.m' - \nabla' m_3.m')m.\Phi)dx' \qquad (3.12)
\end{aligned}$$

So, the Euler-Lagrange equation is

$$\begin{aligned}
\Delta m &= -m|\nabla' m|^2 + \frac{1}{\epsilon^2}((\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m) - m.(\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m)m) \\
&+\frac{1}{\eta}\left(-\begin{pmatrix} H \\ 0 \end{pmatrix} + m'.Hm\right) + \kappa_1(Id - m\otimes m)\nabla' \times m \qquad (3.13) \\
&+\kappa_2\left(\begin{pmatrix} -\nabla' m_3 \\ \nabla'.m' \end{pmatrix} + (\nabla' m_3.m' - m_3\nabla'.m')m\right) \text{ in } \mathcal{D}(\Omega),
\end{aligned}$$

where $H := \nabla' U(\cdot,0)$.
Denote

$$\begin{aligned}
K(\cdot,m) &= \frac{-1}{\epsilon^2}((\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m) + m.(\partial_{x_1},\partial_{x_2},\partial_{x_3})F(m)m) \\
&+\frac{1}{\eta}\left(\begin{pmatrix} H \\ 0 \end{pmatrix} - m'.Hm\right) - \kappa_1(Id - m\otimes m)\nabla' \times m \\
&-\kappa_2\left(\begin{pmatrix} -\nabla' m_3 \\ \nabla'.m' \end{pmatrix} + (\nabla' m_3.m' - m_3\nabla'.m')m\right). \qquad (3.14)
\end{aligned}$$

Finally, we obtain the Euler-Lagrange equation:

$$\int_\Omega \nabla' m\nabla'\Phi dx' = \int_\Omega (m|\nabla' m|^2 + K(x', m(x'))).\Phi dx' \quad \forall\Phi \in C_0^\infty(\Omega,\mathbb{R}^3). \qquad (3.15)$$

Remark 3.5. (i) By the definition of K, the assumption of the function F and Remark 3.3, then if $m \in H^1_g(\Omega, \mathbb{S}^2)$ is a critical point of $E_{\epsilon,\eta,\kappa}$, then we have immediately that $K \in L^2(\Omega, \mathbb{R}^3)$.

(ii) Since $m \in H^1_g(\Omega, \mathbb{S}^2)$ and $K \in L^2(\Omega, \mathbb{R}^3)$, then we have $m|\nabla' m|^2 + K(x', m(x')) \in L^1(\Omega, \mathbb{R}^3)$. By a duality argument [2], this implies that (3.15) holds true for every $\Phi \in H^1_0 \cap L^\infty(\Omega, \mathbb{R}^3)$.

3.3 Regularity of critical points

In this Section, we prove

Theorem 3.6. *Let Ω be a $C^{1,1}$- domain, $F \in C^\infty(\mathbb{S}^2)$ and $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function. Let $m \in H^1_g(\Omega, \mathbb{S}^2)$ be an critical point of $E_{\epsilon,\eta,\kappa}$. Then $m \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0, \frac{1}{2})$.*

For that, we shall split the proof into two parts, the regularity of m in the interior domain and the regularity of m up to the boundary.

3.3.1 Interior regularity

We aim to prove the regularity of the solution (3.13) in the interior of the domain Ω. We follow the method used by Carbou for a slightly different ferromagnetic model in dimension 2 (see [11]), also by Jost for the interior regularity of harmonic maps into the sphere (see [35]).
For the conveniences, we have *some notational conventions:*

From now on, we will write $\nabla = (\partial_{x_1}, \partial_{x_2})$ instead of ∇', $\nabla.m' = \partial_{x_1} m_1 + \partial_{x_2} m_2$, $x \in \mathbb{R}^2$ standing for the in-plan quantity. Combining the Euler-Lagrange equation (3.15) and Remark 3.5, we has

$$\int_\Omega \nabla m \nabla \Phi dx = \int_\Omega (m|\nabla m|^2 + K(x, m(x))).\Phi dx \quad \forall \Phi \in H^1_0 \cap L^\infty(\Omega, \mathbb{R}^3). \tag{3.16}$$

Also, we shall always integrate w.r.t. to the Lebesgue measure dx and this will often be omitted.
The main goal of this section is the following theorem

Theorem 3.7. *Let Ω be a $C^{1,1}$ simply connected domain, $g : \partial\Omega \to \mathbb{S}^1$ be a $C^{1,1}$ function and $m \in H^1_g(\Omega, \mathbb{S}^2)$ be the solution of (3.13). Then $m \in C^\infty(\Omega)$.*

For the continuity of $m \in C^0(\overline{\Omega})$ we first rely on the result of Wente which is proved in [8, Lemma A.1 and Remark A.1]:

Lemma 3.8. *Assume that Ω is a $C^{1,1}$ domain in $\mathbb{R}^2$ and $f = (f_1, f_2), h = (h_1, h_2) \in H^1(\Omega, \mathbb{R}^2)$ and $u \in W^{1,1}_0(\Omega)$ is a solution of*

$$\Delta u = \sum_{i=1}^{2} \nabla f_i \nabla^\perp h_i,$$

then $u \in C^0(\overline{\Omega})$.

[2] For every such $\Phi, \exists \Phi_n \in C^\infty_0(\Omega)$ s.t. $|\Phi_n| \leq ||\Phi||_{L^\infty}, \Phi_n \to \Phi$ in $H^1(\Omega)$, $\Phi_n \to \Phi$ a.e in Ω.

Proof of Theorem 3.7. The proof is carried out in several steps.

Step A *We prove that $m \in C^0(\overline{\Omega}, \mathbb{S}^2)$.*
In this step we only use that $K \in L^2(\Omega)$. The fact $|m|= 1$ a.e implies that:

$$\sum_{i=1}^{3} m_i \partial_{x_k} m_i = 0, \text{ for } k = 1, 2.$$

Then for any $i \in \{1, 2, 3\}$, we rewrite (3.13) in the form:

$$-\Delta m_i = \sum_{k=1}^{2} \sum_{j=1}^{3} \partial_{x_k} m_j (m_i \partial_{x_k} m_j - m_j \partial_{x_k} m_i) + K_i, \tag{3.17}$$

where $K = (K_1, K_2, K_3)$ is denoted as in (3.14).
Remark that in the sense of distributions:

$$\begin{aligned} \sum_{k=1}^{2} \partial_{x_k} (m_i \partial_{x_k} m_j - m_j \partial_{x_k} m_i) &= \sum_{k=1}^{2} (\partial_{x_k} m_i \partial_{x_k} m_j + m_i \partial^2_{x_k x_k} m_j - \partial_{x_k} m_j \partial_{x_k} m_i - m_j \partial^2_{x_k x_k} m_i) \\ &= m_i \Delta m_j - m_j \Delta m_i. \end{aligned}$$

Together with (3.16), this implies that

$$\begin{aligned} \sum_{k=1}^{2} \partial_{x_k} (m_i \partial_{x_k} m_j - m_j \partial_{x_k} m_i) &= m_i(-m_j |\nabla m|^2 - K_j) - m_j(-m_i |\nabla m|^2 - K_i) \\ &= -m_i K_j + m_j K_i. \end{aligned}$$

Since m is uniformly bounded by 1 and $K \in L^2(\Omega, \mathbb{R}^3)$, then

$$-m_i K_j + m_j K_i \in L^2(\Omega).$$

Let b_{ij} be a solution in $H^1(\Omega, \mathbb{R}^2)$ of

$$\nabla . b_{ij} = -m_i K_j + m_j K_i \text{ in } \Omega.$$

Therefore we obtain:

$$\nabla.(m_i \nabla m_j - m_j \nabla m_i - b_{ij}) = 0 \text{ in } \mathcal{D}'(\Omega).$$

Applying the Poincaré lemma to $(m_i \nabla m_j - m_j \nabla m_i - b_{ij})$ in the simply connected domain Ω , then there exists $c_{ij} \in H^1(\Omega, \mathbb{R}^2)$ such that:

$$m_i \nabla m_j - m_j \nabla m_i - b_{ij} = \nabla^{\perp} c_{ij} \text{ in } \Omega.$$

Combining with (3.17), this yields

$$-\Delta m_i = \sum_{j=1}^{3} (\nabla m_j . \nabla^{\perp} c_{ij} + \nabla m_j . b_{ij}) + K_i \text{ in } \mathcal{D}'(\Omega) \text{ for any } i \in \{1, 2, 3\}.$$

Let $\alpha_i \in C^0(\overline{\Omega}) \cap H^1_0(\Omega)$ be the solution of

$$\begin{cases} -\Delta \alpha_i = \sum_{j=1}^{3} \nabla m_j . \nabla^{\perp} c_{ij} & \text{in } \Omega, \\ \alpha_i = 0 & \text{on } \partial\Omega, \end{cases}$$

(this is thanks to Theorem 2.3) and $\beta_i \in C^{0,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$, be the solution of

$$\begin{cases} -\Delta\beta_i = \sum_{j=1}^{3} \nabla m_j.b_{ij} + K_i & \text{in } \Omega \\ \beta_i = m_i = g_i & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

Indeed, since $b_{ij} \in H^1(\Omega, \mathbb{R}^2) \subset L^p(\Omega, \mathbb{R}^2)$ for all $1 \leq p < \infty$, then $\sum_{j=1}^{3} \nabla m_j.b_{ij} + K_i \in L^{2^-}(\Omega)$, where 2^- is any positive number less than 2. By the standard elliptic theory, since $g \in C^{1,1}(\partial\Omega$, there exists a unique $\beta_i \in W^{2,2^-}(\Omega)$ satisfying (3.18) . By the Morrey inequality, $\beta_i \in C^{0,\gamma}(\overline{\Omega})$ for some γ. As $\alpha_i + \beta_i$ satisfies the PDE

$$\begin{cases} -\Delta(\alpha_i + \beta_i) = \sum_{j=1}^{3} (\nabla m_j.\nabla^{\perp} c_{ij} + \nabla m_j.b_{ij}) + K_i & \text{in } \Omega, \\ \alpha_i + \beta_i = g_i & \text{on } \partial\Omega, \end{cases}$$

the uniqueness of the Poisson equation with Dirichlet condition implies that $m_i = \alpha_i + \beta_i$. So we have $m \in C^0(\overline{\Omega})$. In the next step, using $m \in C^0(\Omega)$, we are going to sharpen the interior regularity.

Step B *We prove $m \in H^2_{loc}(\Omega, \mathbb{S}^2)$.*
Step B is a consequence of the following auxiliary result which is inspired by [35, Theorem 9.6.1].

Lemma 3.9. *Suppose $m \in C^0 \cap H^1(B(x_0, R), \mathbb{S}^2)$ is a solution of* (3.16). *Then $m \in H^2(B(x_0, \frac{R}{2}))$, moreover we have the estimate:*

$$||D^2 m||_{L^2(B(x_0,\frac{R}{2}))} \leq C + C||\nabla m||_{L^2(B(x_0,R))}, \tag{3.19}$$

where $C > 0$ depends on R.

Before proving the Lemma 3.9, let us give the following Lemma.

Lemma 3.10. *Let $\Omega \subset \mathbb{R}^2$ be an open domain and $m \in H^1 \cap C^0(\Omega, \mathbb{S}^2)$ be a solution of* (3.16). *Then for every $\epsilon_0 > 0$, there exist $\rho > 0$ such that*

$$\int_{B(P,\rho)} |\nabla m(x)|^2 \eta^2(x) dx \leq \epsilon_0 \int_{B(P,\rho)} |\nabla \eta(x)|^2 dx,$$

for all $P \in \Omega, B(P, \rho) \subset \Omega$ and $\eta \in H^1_0(B(P, \rho))$.

Proof of the Lemma 3.10. We first work with $\eta \in C_0^\infty(B(P, \rho))$. Choosing $\Phi(x) = (m(x) - m(P))\eta^2(x) \in H^1_0 \cap L^\infty(\Omega, \mathbb{R}^3)$ in (3.16), we obtain

$$\int_{B(P,\rho)} \nabla m(x) \nabla((m(x) - m(P))\eta^2(x)) dx$$
$$= \int_{B(P,\rho)} (m(x)|\nabla m(x)|^2 + K)((m(x) - m(P))\eta^2(x) dx.$$

Hence,

$$
\begin{aligned}
&\int_{B(P,\rho)} |\nabla m|^2\eta^2 = -\int_{B(P,\rho)} 2\eta\nabla\eta\nabla m(m(\cdot)-m(P)) \\
&+\int_{B(P,\rho)} |\nabla m|^2 m.(m(\cdot)-m(P))\eta^2(x) + \int_{B(P,\rho)} K(\cdot,m(\cdot))(m(\cdot)-m(P))\eta^2 \\
&\leq \left| 2\int_{B(P,\rho)} \eta\nabla\eta\nabla m(m(\cdot)-m(P))\right| + \sup_{B(P,\rho)} |m(\cdot)-m(P)| \int_{B(P,\rho)} |\nabla m|^2\eta^2 \\
&+ \sup_{B(P,\rho)} |m(\cdot)-m(P)| ||K(\cdot,m(\cdot),)||_{L^2(B(P,\rho))} ||\eta||^2_{L^4(B(P,\rho))} \\
&\leq \frac{1}{2}\int_{B(P,\rho)} |\nabla m|^2\eta^2 + 2\sup_{B(P,\rho)} |m(\cdot)-m(P)|^2 \int_{B(P,\rho)} |\nabla\eta|^2 \\
&+ \sup_{B(P,\rho)} |m(\cdot)-m(P)| \int_{B(P,\rho)} |\nabla m|^2\eta^2 + C\rho \sup_{B(P,\rho)} |m(\cdot)-m(P)| \int_{B(P,\rho)} |\nabla\eta|^2,
\end{aligned}
$$

where we used the Sobolev inequality

$$
C\rho\int_{B(0,\rho)} |\nabla\eta|^2 \geq \left(\int_{B(0,\rho)} |\eta|^4\right)^{1/2},
$$

the Young inequality and the fact $K \in L^2(\Omega, \mathbb{R}^3)$ in the last inequality. The Lemma follows because we can make $\sup_{x\in B(P,\rho)} |m(x)-m(P)|$ arbitrarily small by choosing ρ sufficiently small since m is continuous.

Finally, we note that, by using the density argument, the conclusion holds for $\eta \in H^1_0(B(P,\rho))$. Indeed, for any $\eta \in H^1_0(B(P,\rho))$, there exists $\eta_k \in C^\infty_0(B(P,\rho))$ such that

$$
\begin{cases} \eta_k \to \eta & \text{a.e} \\ \eta_k \to \eta & \text{in } H^1(B(P,\rho)). \end{cases}
$$

Then by Fatou's Lemma

$$
\begin{aligned}
\int_{B(P,\rho)} |\nabla m|^2\eta^2 &\leq \liminf \int_{B(P,\rho)} |\nabla m(x)|^2\eta_k^2(x)dx \\
&\leq \liminf \epsilon_0 \int_{B(P,\rho)} |\nabla\eta_k(x)|^2 dx = \epsilon_0 \int_{B(P,\rho)} |\nabla\eta|^2.
\end{aligned}
$$

□

We continue with the proof of Lemma 3.9:

Proof of Lemma 3.9. The idea of the proof is to estimate the term

$$
\int_{B(P,\rho)} |\nabla(D^h_\gamma m)|^2\xi^2,
$$

where ξ is a good cut-off function to be defined later, $D^h_\gamma m$ is γ^{th}-difference quotient of size h defined by

$$
D^h_\gamma m(x) = \frac{m(x+he_\gamma)-m(x)}{h} \quad \text{for } \gamma = 1,2
$$

and $\bigcup_P B(P,\rho)$ cover $B(x_0,R)$.

For any $P \in \Omega$ and $R > R' > 0$ such that $B(P,R') \subset\subset B(x_0,R)$, we choose a test function in (3.16) as $\Phi = D^{-h}_\gamma(\xi^2 D^h_\gamma m)$, here $\xi \in C^\infty_0(B(P,R'))$ and $0 \leq \xi \leq 1$ will be chosen later

(see(3.26)).
Equation (3.16) implies that

$$\int_{B(P,R')} \nabla m \nabla \Phi = \int_{B(P,R')} \nabla m \nabla (D_\gamma^{-h}(\xi^2 D_\gamma^h m)) = \int_{B(P,R')} (m|\nabla m|^2 + K) D_\gamma^{-h}(\xi^2 D_\gamma^h m).$$

For h sufficiently small depending dist(supp(ζ), $\partial B(P,R')$), the integration by parts formula for difference quotients

$$\int_{B(P,R')} v D_\gamma^{-h} w = -\int_{B(P,R')} w D_\gamma^h v$$

implies that

$$\begin{aligned}\int_{B(P,R')} \nabla m \nabla (D_\gamma^{-h}(\xi^2 D_\gamma^h m)) &= \int_{B(P,R')} \nabla m D_\gamma^{-h}(\nabla(\xi^2 D_\gamma^h m)) \\ &= -\int_{B(P,R')} D_\gamma^h(\nabla m)\nabla(\xi^2 D_\gamma^h m).\end{aligned}$$

Then

$$\int_{B(P,R')} D_\gamma^h(\nabla m)\nabla(\xi^2 D_\gamma^h m) = -\int_{B(P,R')} (m|\nabla m|^2 + K) D_\gamma^{-h}(\xi^2 D_\gamma^h m).$$

Moreover:

$$D_\gamma^h(\nabla m)\nabla(\xi^2 D_\gamma^h m) = |\nabla(D_\gamma^h m)|^2 \xi^2 + \nabla(D_\gamma^h m) 2\xi \nabla\xi D_\gamma^h m.$$

Then

$$\begin{aligned}&\int_{B(P,R')} |\nabla(D_\gamma^h m)|^2 \xi^2 \\ &= -\int_{B(P,R')} (m|\nabla m|^2 + K) D_\gamma^{-h}(\xi^2 D_\gamma^h m) - \int_{B(P,R')} 2\xi\nabla\xi D_\gamma^h m \nabla(D_\gamma^h m). \quad (3.20)\end{aligned}$$

We are going to estimate the right hand side of (3.20). First, by Young's inequality,

$$\begin{aligned}&\left|\int_{B(P,R')} 2\xi\nabla\xi D_\gamma^h m \nabla(D_\gamma^h m)\right| \\ &\leq \epsilon_1 \int_{B(P,R')} |\nabla(D_\gamma^h m)|^2 \xi^2 + \frac{1}{\epsilon_1}\int_{B(P,R')} |D_\gamma^h m|^2 |\nabla\xi|^2, \quad (3.21)\end{aligned}$$

for any $\epsilon_1 > 0$.
We assume that $0 < |h| < \frac{1}{2}$dist(supp(ξ), $\partial B(P,R')$), then one has (see Evans [21])

$$\int_{B(P,R')} |D_\gamma^{-h}(\xi^2 D_\gamma^h m)|^2 \leq C \int_{B(P,R')} |\nabla(\xi^2 D_\gamma^h m)|^2. \quad (3.22)$$

The second term of RHS-(3.20) is estimated by Young's inequality and (3.22)

$$\begin{aligned}&\left|\int_{B(P,R')} K D_\gamma^{-h}(\xi^2 D_\gamma^h m)\right| \\ &\leq \frac{1}{2\epsilon_1}\int_{B(P,R')} |K|^2 + \frac{\epsilon_1}{2}\int_{B(P,R')} |D_\gamma^{-h}(\xi^2 D_\gamma^h m)|^2 \\ &\leq \frac{1}{2\epsilon_1}\int_{B(P,R')} |K|^2 + \frac{\epsilon_1}{2} C\int_{B(P,R')} |\nabla(\xi^2 D_\gamma^h m)|^2 \\ &\leq \frac{1}{2\epsilon_1}\int_{B(P,R')} |K|^2 + \frac{\epsilon_1}{2} C\int_{B(P,R')} (8\xi^2|\nabla\xi|^2|D_\gamma^h m|^2 + 2\xi^4|\nabla(D_\gamma^h m)|^2) \\ &\leq \frac{1}{2\epsilon_1}\int_{B(P,R')} |K|^2 + \frac{\epsilon_1}{2} C\int_{B(P,R')} (8\xi^2|\nabla\xi|^2|D_\gamma^h m|^2 + 2\xi^2|\nabla(D_\gamma^h m)|^2). \quad (3.23)\end{aligned}$$

Here, in the second inequality, We continue with

$$\begin{aligned}
&\left|\int_{B(P,R')} m|\nabla m|^2 D_\gamma^{-h}(\xi^2 D_\gamma^h m)\right| = \left|\int_{B(P,R')} D_\gamma^h(m|\nabla m|^2)\xi^2 D_\gamma^h m\right| \\
&\leq \int_{B(P,R')} (|D_\gamma^h m||\nabla m|^2 + |m^h|(|\nabla m| + |(\nabla m)^h||D_\gamma^h(\nabla m)|)\xi^2 |D_\gamma^h m| \\
&\leq \int_{B(P,R')} |D_\gamma^h m|^2 |\nabla m|^2 \xi^2 + \int_{B(P,R')} (|\nabla m| + |(\nabla m)^h|)|D_\gamma^h m||\nabla(D_\gamma^h m)|\xi^2 \\
&\leq \int_{B(P,R')} |D_\gamma^h m|^2 |\nabla m|^2 \xi^2 + \frac{\epsilon_1}{2}\int_{B(P,R')} |\nabla(D_\gamma^h m)|^2 \xi^2 \qquad (3.24) \\
&+ \frac{1}{2\epsilon_1}\int_{B(P,R')} (|\nabla m| + |(\nabla m)^h|)^2 |D_\gamma^h m|^2 \xi^2,
\end{aligned}$$

where $m^h(x) := m(x + he_\gamma)$ and $(\nabla m)^h(x) = \nabla m(x + he_\gamma)$.
Here we have used the formula

$$D_\gamma^h(vw) = v^h D_\gamma^h w + w D_\gamma^h v,$$

for $v^h(x) := v(x + he_\gamma)$.
Combining (3.20)-(3.24), $0 \leq \xi \leq 1$, $K \in L^2(\Omega)$ and choosing ϵ_1 small enough, this yields that:

$$\begin{aligned}
&\int_{B(P,R')} |\nabla(D_\gamma^h m)|^2 \xi^2 \leq C(1 + \int_{B(P,R')} |D_\gamma^h m|^2 |\nabla \xi|^2 \qquad (3.25) \\
&+ \int_{B(P,R')} |\nabla m|^2 |D_\gamma^h m|^2 \xi^2 + \int_{B(P,R')} |(\nabla m)^h|^2 |D_\gamma^h m|^2 \xi^2).
\end{aligned}$$

For $\epsilon_0 > 0$, we choose $R' = \rho > 0$ as in Lemma 3.10 with

$$B(P,\rho) \subset B(x_0, R)$$

and we choose $\xi \in C_0^\infty(B(P,\rho))$ such that

$$\xi = 1 \text{ in } B(P, \frac{\rho}{2}) \text{ and } 0 \leq \xi \leq 1, |\nabla \xi| \leq \frac{4}{\rho} \text{ in } B(P,\rho). \qquad (3.26)$$

Thus, all preceding integrals need to be evaluated only on $B(P,\rho)$. Applying Lemma 3.10 to $\eta = |D_\gamma^h m|\xi \in H_0^1 \cap L^\infty(B(x_0,\rho))$, we obtain:

$$\begin{aligned}
&\int_{B(P,\rho)} |\nabla m|^2 |D_\gamma^h m|^2 \xi^2 \\
&\leq \epsilon_0 \int_{B(P,\rho)} |\nabla(|D_\gamma^h m|\xi)|^2 \\
&\leq 2\epsilon_0 \int_{B(P,\rho)} |\nabla(D_\gamma^h m)|^2 \xi^2 + 2\epsilon_0 \int_{B(P,\rho)} |D_\gamma^h m|^2 |\nabla \xi|^2. \qquad (3.27)
\end{aligned}$$

Similarly, using Lemma 3.10 again with the function $m(\cdot + he_\gamma)$ we obtain

$$\int_{B(P,\rho)} |(\nabla m)^h|^2 |D_\gamma^h m|^2 \xi^2 \leq 2\epsilon_0 \int_{B(P,\rho)} |\nabla(D_\gamma^h m)|^2 \xi^2 + 2\epsilon_0 \int_{B(P,\rho)} |D_\gamma^h m|^2 |\nabla \xi|^2. \quad (3.28)$$

Since $|\nabla \xi| \leq \frac{4}{\rho}$ in $B(P,\rho)$, (3.25), (3.27), (3.28), we choose ϵ_0 small enough, then for $0 < |h| < \mathrm{dist}(\mathrm{supp}(\xi), \partial B(P,\rho))$, we get

$$\begin{aligned}
\int_{B(P,\rho)} |\nabla(D_\gamma^h m)|^2 \xi^2 \quad &\leq C(1 + \int_{B(P,\rho)} |D_\gamma^h m|^2 |\nabla \xi|^2) \\
&\leq C(1 + \frac{16}{\rho^2}\int_{B(P,\rho)} |\nabla m|^2). \qquad (3.29)
\end{aligned}$$

The properties of ξ imply that

$$\begin{aligned}\int_{B(P,\frac{\rho}{2})}|\nabla(D_\gamma^h m)|^2 &\leq \int_{B(P,\rho)}|\nabla(D_\gamma^h m)|^2\xi^2\\ &\leq C(1+\frac{16}{\rho^2}\int_{B(P,\rho)}|\nabla m|^2)\\ &\leq C+\frac{C}{\rho^2}\int_{B(P,\rho)}|\nabla m|^2. \end{aligned}\tag{3.30}$$

Covering $B(x_0,\frac{R}{2})$ by balls $B(P,\frac{\rho}{2})$ with $B(P,\rho)\subset B(x_0,R)$, we obtain the desired estimate for

$$\int_{B(x_0,\frac{R}{2})}|D^2 m|^2.$$

□

This finished the proof of Lemma 3.10. By Lemma (3.10), Step B (i.e. $m\in H^2_{loc}(\Omega,\mathbb{S}^2)$) follows immediately.

Step C *The aim of this step is to prove that $m\in H^3_{loc}(\Omega)$.*
By **Step B** and Lemma 3.4, we know that $m\in H^2_{loc}(\Omega)$, $H\in H^1_{loc}(\Omega)$, where H is defined as ($H=\nabla U(\cdot,0)$). First, we claim that $K\in H^1_{loc}(\Omega)$. Indeed, we note

$$|\nabla(m.(\nabla,\partial_{x_3})F(m)m)|\leq C(|D^2F(m)||m|+|\nabla m||(\nabla,\partial_{x_3})F(m)||m|),$$

$$|\nabla(m'Hm)|\leq C(|\nabla H||m|^2+|\nabla m||H||m|),$$

$$|\nabla((m\otimes m)\nabla\times m)|\leq C(|\nabla m|^2|m|+|D^2m||m|^2)$$

and

$$|\nabla((-\nabla m_3.m'-m_3\nabla .m')m)|\leq C(|D^2m||m|^2+|\nabla m|^2|m|).$$

Combining the definition of K (see (3.14)), these above facts and the regularities of m,H,F ($m\in H^2_{loc}\subset W^{1,p}_{loc}$ for all $1\leq p<\infty$, $H\in H^1_{loc}\subset L^p_{loc}(\Omega)$, $F\in C^\infty$), this yields $K\in H^1_{loc}(\Omega,\mathbb{R}^3)$.
We want to apply the standard interior elliptic regularity to equation

$$-\Delta m=m|\nabla m|^2+K(\cdot,m(\cdot))\ \text{ in } \mathcal{D}'(\Omega).\tag{3.31}$$

The principal difficulty of this step is to deal with the term $m|\nabla m|^2$. Observe that:

$$\nabla(m|\nabla m|^2)=\nabla m|\nabla m|^2+2mD^2m\nabla m.$$

The first term $\nabla m|\nabla m|^2\in L^p$ for all $p<+\infty$. For the second term, one has $D^2m\in L^2_{loc}(\Omega)$, $m\nabla m\in L^p_{loc}(\Omega)$ for all $1\leq p<\infty$, thus $mD^2m\nabla m\in L^q$ for all $1\leq q<2$. It then suffices to show that $|D^2m||\nabla m|\in L^2_{loc}(\Omega)$. It is a direct consequence of the following lemma:

Lemma 3.11. *Suppose $K\in H^1(B(x_0,R))$ and $m\in H^2(B(x_0,R),\mathbb{S}^2)(\subset C^0(B(x_0,R),\mathbb{S}^2))$ is a solution of* (3.16). *Then*

$$\int_{B(x_0,\frac{R}{2})}|\nabla m|^2|D^2m|^2<+\infty.\tag{3.32}$$

Moreover $m\in H^3_{loc}(B(x_0,R),\mathbb{S}^2)$.

Proof of Lemma 3.11. Since (3.16), one has

$$\begin{aligned}\int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).\nabla\Phi &= -\int_{B(x_0,R)} \nabla m.\partial_{x_\gamma}(\nabla\Phi)\\ &= -\int_{B(x_0,R)} (m|\nabla m|^2+K).\partial_{x_\gamma}\Phi \qquad (3.33)\end{aligned}$$

for all $\Phi \in C_0^\infty(B(x_0,R),\mathbb{R}^3)$ and $\gamma \in \{1,2\}$.
By a density argument, this implies that (3.33) holds true for every $\Phi \in H_0^1(B(x_0,R))$.
Denote: $\omega_L = \min\{L, |\nabla m|^2\}$, with $L > 0$.
We remark that:

$$\omega_L \in L^\infty(B(x_0,R)),$$

$$\nabla\omega_L(x) = 0 \text{ a.e. in } \{x : |\nabla m(x)|^2 > L\}.$$

It implies that

$$|\nabla\omega_L| \leq 2|D^2m|\omega_L^{1/2}, \qquad (3.34)$$

in particular $\omega_L \in H^1(B(x_0,R))$.
Choosing $\Phi = \eta^2\omega_L\partial_{x_\gamma}m \in H_0^1(B(x_0,R),\mathbb{R}^3)$ in (3.33) where $\eta \in C_0^\infty(B(x_0,R))$ with $0 \leq \eta \leq 1$ will be defined later (see (3.43)), then

$$\int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).\nabla(\eta^2\omega_L\partial_{x_\gamma}m) = -\int_{B(x_0,R)} (m|\nabla m|^2+K).\partial_{x_\gamma}(\eta^2\omega_L\partial_{x_\gamma}m). \qquad (3.35)$$

We now develop the right hand side of (3.35).
Estimate of the first term of the RHS of (3.35). The properties of the function η and (3.34) yield that

$$\begin{aligned}\int_{B(x_0,R)} m|\nabla m|^2.\partial_{x_\gamma}\Phi &= \int_{B(x_0,R)} m|\nabla m|^2.\partial_{x_\gamma}(\eta^2)\omega_L\partial_{x_\gamma}m\\ &+\int_{B(x_0,R)} m|\nabla m|^2.\eta^2\partial_{x_\gamma}\omega_L\partial_{x_\gamma}m + \int_{B(x_0,R)} m|\nabla m|^2\eta^2.\omega_L\partial^2_{x_\gamma x_\gamma}m\\ &\leq C\int_{B(x_0,R)} |\nabla\eta||\nabla m|^2\omega_L|\nabla m|\\ &+2\int_{B(x_0,R)} |\nabla m|^2|D^2m|\omega_L^{1/2}|\nabla m| + \int_{B(x_0,R)} |\nabla m|^2\eta^2\omega_L|D^2m|\\ &\leq C\int_{B(x_0,R)} |\nabla\eta||\nabla m|^5 + 3\int_{B(x_0,R)} |\nabla m|^4|D^2m|.\end{aligned}$$

Using the Young inequality, we then obtain

$$\int_{B(x_0,R)} |\nabla m|^4|D^2m| \leq \frac{1}{2}\int_{B(x_0,R)} |\nabla m|^8 + \frac{1}{2}\int_{B(x_0,R)} |D^2m|^2.$$

We deduce that

$$\int_{B(x_0,R)} m|\nabla m|^2.\partial_{x_\gamma}\Phi \leq C\int_{B(x_0,R)} |\nabla\eta||\nabla m|^5 + \frac{3}{2}\Big(\int_{B(x_0,R)} |\nabla m|^8 + \int_{B(x_0,R)} |D^2m|^2\Big) < \infty. \qquad (3.36)$$

Estimate of the second term of the RHS of (3.35). Similarly to the above estimate, since $K \in H^1(B(x_0,R))$ then $K \in L^p(B(x_0,R))$ for all $1 \le p < \infty$, then

$$\int_{B(x_0,R)} K.\partial_{x_\gamma}(\eta^2\omega_L\partial_{x_\gamma}m) \tag{3.37}$$

$$= \int_{B(x_0,R)} K.(\partial_{x_\gamma}(\eta^2)\omega_L\partial_{x_\gamma}m + \eta^2\partial_{x_\gamma}\omega_L\partial_{x_\gamma}m + \eta^2\omega_L\partial^2_{x_\gamma x_\gamma}m)$$

$$\le \int_{B(x_0,R)} |K|(C|\nabla\eta|\omega_L|\nabla m| + 2|D^2m|\omega_L^{1/2}|\nabla m| + \omega_L|D^2m|)$$

$$\le \int_{B(x_0,R)} (C|K||\nabla\eta||\nabla m|^3 + 3|K||D^2m||\nabla m|^2).$$

$$\le C\Big\||K||\nabla\eta||\nabla m|^3\Big\|_{L^1} + 3||D^2m||_{L^2}\Big\||K||\nabla m|^2\Big\|_{L^2}. \tag{3.38}$$

Therefore, all terms on the right hand side of (3.35) remain bounded as $L \to \infty$. The same then has to happen for the left hand side of (3.35). Therefore,

$$\limsup_{L\to\infty}\left|\sum_{\gamma=1}^{2}\int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).\nabla(\eta^2\omega_L\partial_{x_\gamma}m)\right| < C. \tag{3.39}$$

The expression in modulus can be written as

$$\sum_{\gamma=1}^{2}\int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).\nabla(\eta^2\omega_L\partial_{x_\gamma}m) = \sum_{\gamma=1}^{2}\Bigg(\int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).2\eta\nabla\eta\omega_L\partial_{x_\gamma}m$$
$$+ \int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).\eta^2\nabla\omega_L\partial_{x_\gamma}m + \int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).\eta^2\omega_L\nabla(\partial_{x_\gamma}m)\Bigg). \tag{3.40}$$

We denote

$$\omega := |\nabla m|^2 \text{ in } B(x_0,R).$$

Since $\partial_{x_\gamma}\omega = 2\partial_{x_\gamma}(\nabla m).\nabla m$ for $\gamma = 1,2$ and $\nabla\omega_L = 0$ in $\{\omega > L\}$, then the second integral of the right hand side of (3.40) can rewrite as

$$\int_{B(x_0,R)}\sum_{\gamma=1}^{2} \partial_{x_\gamma}(\nabla m).\eta^2\nabla\omega_L\partial_{x_\gamma}m = \frac{1}{2}\int_{\{\omega\le L\}} \eta^2|\nabla\omega|^2, \tag{3.41}$$

and it is non-negative.
For the first term in right hand side of (3.40), by the Young inequality,

$$\left|\sum_{\gamma=1}^{2}\int_{B(x_0,R)} \partial_{x_\gamma}(\nabla m).2\eta\nabla\eta\omega_L\partial_{x_\gamma}m\right|$$
$$\le \int_{B(x_0,R)} |D^2m|^2 + C\int_{B(x_0,R)} |\nabla m|^6|\nabla\eta|^2\eta^2 < +\infty. \tag{3.42}$$

Recalling

$$0 \le \eta \le 1, |\nabla\eta| \le \frac{2}{R} \text{ in } B(x_0,R), \tag{3.43}$$

and combining with (3.40) - (3.42), this implies that

$$\sum_{\gamma=1}^{2}\int_{B(x_0,\frac{R}{2})} |\partial_{x_\gamma}(\nabla m)|^2\omega_L < C$$

uniformly in $L > 0$ for $\gamma = 1, 2$
Applying Fatou's Lemma, we obtain that

$$\int_{B(x_0,\frac{R}{2})} |D^2 m|^2 |\nabla m|^2 \leq C.$$

We then apply the interior elliptic regularity to (3.31) with the the fact that the right hand side of (3.31) belongs to $H^1_{loc}(\Omega)$. Finally, we obtain that $m \in H^3_{loc}(\Omega, \mathbb{S}^2)$.

Step D *Conclude the proof of Theorem 3.7.*
We first claim that

Claim 2. *If $m \in H^k_{loc}(\Omega, \mathbb{S}^2)$ satisfies* (3.13) *and F is a smooth function, for $k \geq 3$, then $m|\nabla m|^2$ and K belong to $H^{k-1}_{loc}(\Omega)$, where K is defined as in* (3.14).

Proof of Claim 2. We assume that $m \in H^k_{loc}(\Omega, \mathbb{S}^2)$. By Lemma 3.4, we have that $H \in H^{k-1}_{loc}(\Omega)$. Therefore, $D^{k-1}m, D^{k-2}H \in L^p_{loc}$, $1 \leq p < \infty$ and $D^r m, D^{r-1}H \in L^\infty$ for $r \in \mathbb{N}$ and $r \leq k-2$.
Using the Leibniz rule, one has

$$|D^{k-1}(m|\nabla m|^2)| \leq C\Bigg(\sum_{\substack{p+q+r=k+1\\ k-1\geq p\geq 0, k\geq q,r\geq 1}} |D^p m||D^q m||D^r m|\Bigg).$$

This implies that $m|\nabla m|^2 \in H^{k-1}_{loc}(\Omega)$.
We use the same argument to estimate K given in (3.14), here we only check the term mHm, the other terms are estimated analogously:

$$|D^{k-1}(mHm)| \leq C\Bigg(\sum_{p+q+r=k-1} |D^p m||D^q H||D^r m|\Bigg).$$

This implies that $mHm \in H^{k-1}_{loc}(\Omega)$. Therefore $K \in H^{k-1}_{loc}(\Omega)$.
Applying the interior elliptic regularity to (3.13), combined with Claim 2, by bootstrap, we obtain that $m \in H^k_{loc}(\Omega)$ for every $k \in \mathbb{N}$. Finally, by the Morrey inequality, we conclude $m \in C^\infty(\Omega)$. □

3.3.2 Regularity at the boundary

In this part, we study the regularity at the boudary of the critical points of $E_{\epsilon,\eta,\kappa}$. Our strategy is firstly to adapt the method used in the interior regularity to obtain that $m \in H^2(\Omega)$. We recall that by Step A of the proof of Theorem 3.7, $m \in C^0(\overline{\Omega})$. We then expect to transfer the boundary regularity problem to the local interior regularity by a diffeomorphism mapping. In fact, it boosts the regularity becoming $\partial_\tau m \in H^1(V)$ and $\partial_\nu m \in W^{1,1}(V)$ where τ, ν are the tangent and normal vectors, respectively which are well-defined in a tabular neighborhood V of the boundary $\partial\Omega$. We split this part into some steps.

Step A *Prove that $m \in H^2(\Omega)$.* Let us fix $x_0 \in \partial\Omega$ and note that since $\partial\Omega \in C^{1,1}$, to simply notation, we may assume that $x_0 = (0,0)$ and up to rotation

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) | x_2 > \gamma(x_1)\},$$

for some $r > 0$ and some $C^{1,1}$ function $\gamma : (-r, r) \to \mathbb{R}$, $\gamma(0) = 0$.
We change coordinates near a point $x_0 = (0,0) \in \partial\Omega$ so as to "flatten out" the boundary. We define $\psi = (\psi_1, \psi_2) : \Omega \cap B(x_0, r) \to \mathbb{R}^2$ as

$$\begin{cases} \psi_1(x_1, x_2) = x_1 \\ \psi_2(x_1, x_2) = x_2 - \gamma(x_1), \end{cases}$$

and $\phi = (\phi_1, \phi_2) = \psi^{-1} : B(x_0, r) \cap \{y_2 > 0\} \to \mathbb{R}^2$

$$\begin{cases} \phi_1(y_1, y_2) = y_1 \\ \phi_2(y_1, y_2) = y_2 + \gamma(y_1). \end{cases}$$

Then $\phi = \psi^{-1}$ and the mapping ψ straightens out $\partial\Omega$ near $x_0 = (0, 0)$. Observe also that

$$\det\nabla\psi = \det\nabla\phi = 1.$$

We choose $s > 0$ so small the half-ball $B_+(x_0, s) := B(x_0, s) \cap \{y_2 > 0\}$ lies in $\psi(\Omega \cap B(x_0, r))$. We also extend the function g into $\mathbb{R}^2$, still denote g and $g \in C^{1,1}(\mathbb{R}^2)$.
Let us define $\overline{m} = m \circ \phi$ in $B_+(x_0, s)$, $\overline{g} = g \circ \phi$ on $\partial B_+(x_0, s))$. Since $\partial\Omega \in C^{1,1}$ and $g \in C^{1,1}(\partial\Omega)$, then $\Phi \in C^{1,1}$ and $\overline{g} \in C^{1,1}$.
We have the Euler-Lagrange equation in the half-ball $B_+(x_0, s)$

Lemma 3.12. *The Euler-Lagrange equation for $\overline{m}$ on $B_+(x_0, s)$ writes*

$$\int_{B_+(0,s)} \nabla\overline{m}A(y).\nabla\overline{\Phi}A(y) - \overline{m}(y)|\nabla\overline{m}A(y)|^2.\overline{\Phi}(y) - \overline{K}.\overline{\Phi}(y) = 0 \tag{3.44}$$

for all test function $\overline{\Phi} \in H_0^1 \cap L^\infty(B_+(x_0, s), \mathbb{R}^3)$, where the matrix A is defined as

$$A(y) = \nabla\psi(\phi(y))$$

and

$$\overline{K}(y) = K \circ \phi(y).$$

The proof of Lemma 3.12. Equation (3.16) and Remark 3.5 give

$$\int_{\Omega\cap B(x_0,r)} \nabla m.\nabla\Phi dx = \int_{\Omega\cap B(x_0,r)} (m|\nabla m|^2 + K)\Phi dx, \ \forall\Phi \in H_0^1 \cap L^\infty(\Omega \cap B(x_0, r)).$$

Then

$$\int_{\phi(B_+(x_0,s))} \nabla m.\nabla\Phi dx = \int_{\phi(B_+(x_0,s))} (m|\nabla m|^2 + K)\Phi dx, \ \forall\Phi \in H_0^1 \cap L^\infty(B_+(x_0, s)).$$

As $m = \overline{m} \circ \phi^{-1}$ and $\overline{\Phi} = \Phi \circ \phi$, by the change of variable $\phi(y) = x$,

$$\int_{B_+(x_0,s)} \nabla\overline{m}\nabla(\phi^{-1})(\phi).\nabla\overline{\Phi}\nabla(\phi^{-1})(\phi)|\det\nabla\phi|dy$$
$$= \int_{B_+(x_0,s)} (\overline{m}|\nabla\overline{m}\nabla(\phi^{-1})(\phi)|^2 + K \circ \phi)\overline{\Phi}|\det\nabla\phi|dy.$$

The conclusion of Lemma 3.12 is implied by the fact $|\det\nabla\phi| = 1$. □

From now on, we use m, Φ, K instead of $\overline{m}, \overline{\Phi}, \overline{K}$, respectively, and denote $B(x_0, s)$ by $B(s)$. By the smoothness of the boundary, we can assume $g \in C^{1,1}(\overline{B_+(s)}) \subset H^2(B_+(s))$.

Remark 3.13. We now remark that equation (3.44) can be considered as

$$\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m\partial_\beta\Phi = \int_{B_+(s)} (m|\nabla mA|^2 + K)\Phi \text{ for all } \Phi \in H_0^1 \cap L^\infty(B_+(s)), \tag{3.45}$$

where $(a^{\alpha\beta})_{\alpha\beta} = AA^T$. We have the following:

(i) $(a^{\alpha\beta})_{\alpha,\beta}$ is Lipschitz continuous on $B_+(s)$

(ii) $\lambda^{-1}|\xi|^2 \leq a^{\alpha\beta}\xi_\alpha\xi_\beta \leq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^2$.

Lemma 3.14. *Suppose $m \in C^0(\overline{B_+(s)}) \cap H^1(B_+(s))$ is a solution of (3.45) with $m = (g, 0)$ on $\partial B_+(s) \cap \mathbb{R}$. Then $\partial_{x_1} m \in H^1(B_+(\rho/2))$, for some $\rho > 0$.*

Proof. The idea of the proof is similar to the interior case, we estimate the term $\int_{B_+(s)} |\nabla(D_1^h m)|^2 \xi^2$, where ξ is a good cut-off function. We choose a test function in (3.45) as

$$\Phi = D_1^{-h}(\xi^2 D_1^h(m-g)) \in H_0^1 \cap L^\infty(B_+(s))$$

where $\xi \in C_0^\infty(B(s))$, $0 \leq \xi \leq 1$ will be chosen later (see (3.57)) and D_1^h is defined by

$$D_1^h m(x) := \frac{m(x+he_1) - m(x)}{h}.$$

By a discrete integration by parts and (3.45), we have that

$$\begin{aligned}
&\int_{B_+(s)} D_1^h(a^{\alpha\beta}\partial_\alpha m)\partial_\beta(\xi^2 D_1^h(m-g)) = -\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m . D_1^{-h}(\partial_\beta(\xi^2 D_1^h(m-g))) \\
&= -\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m . \partial_\beta(D_1^{-h}(\xi^2 D_1^h(m-g)) \\
&= -\int_{B_+(s)} (m|\nabla m A|^2 + K) . D_1^{-h}(\xi^2 D_1^h(m-g)). \qquad (3.46)
\end{aligned}$$

Moreover

$$\begin{aligned}
&D_1^h(a^{\alpha\beta}\partial_\alpha m)\partial_\beta(\xi^2 D_1^h(m-g)) \qquad (3.47)\\
&= D_1^h a^{\alpha\beta}\partial_\alpha m 2\xi\partial_\beta\xi D_1^h(m-g) + D_1^h a^{\alpha\beta}\partial_\alpha m\xi^2\partial_\beta(D_1^h(m-g)) \\
&+ a^{\alpha\beta,h} D_1^h(\partial_\alpha m)2\xi\partial_\beta\xi D_1^h(m-g) + a^{\alpha\beta,h} D_1^h(\partial_\alpha m)\xi^2\partial_\beta(D_1^h(m-g)),
\end{aligned}$$

where $a^{\alpha\beta,h}(x) = a^{\alpha\beta}(x+he_1)$.
Combining with (3.46), this yields that

$$\begin{aligned}
&\int_{B_+(s)} a^{\alpha\beta,h}\partial_\alpha(D_1^h m)\partial_\beta(D_1^h m)\xi^2 \\
&= -\int_{B_+(s)} (m|\nabla m A|^2 + K) D_1^{-h}(\xi^2 D_1^h(m-g)) + \int_{B_+(s)} a^{\alpha\beta,h}\partial_\alpha(D_1^h m)\partial_\beta(D_1^h g)\xi^2 \\
&- \int_{B_+(s)} D_1^h a^{\alpha\beta}\partial_\alpha m 2\xi\partial_\beta\xi D_1^h(m-g) - \int_{B_+(s)} D_1^h a^{\alpha\beta}\partial_\alpha m\xi^2\partial_\beta(D_1^h(m-g)) \\
&- \int_{B_+(s)} a^{\alpha\beta,h} D_1^h(\partial_\alpha m)2\xi\partial_\beta\xi D_1^h(m-g). \qquad (3.48)
\end{aligned}$$

We estimate the second term of RHS of (3.48) as follows

$$\begin{aligned}
&\left|\int_{B_+(s)} a^{\alpha\beta,h}\partial_\alpha(D_1^h m)\partial_\beta(D_1^h g)\xi^2\right| \\
&\leq ||a^{\alpha\beta,h}||_{L^\infty}\left(\epsilon_0\int_{B_+(s)} |\nabla(D_1^h m)|^2\xi^2 + \frac{1}{4\epsilon_0}\int_{B_+(s)} |\nabla(D_1^h g)|^2\xi^2\right) \qquad (3.49)\\
&\leq C\left(\epsilon_0\int_{B_+(s)} |\nabla(D_1^h m)|^2\xi^2 + C_0(\epsilon_0)\right).
\end{aligned}$$

As for the third term,

$$
\begin{aligned}
&\left|\int_{B_+(s)} D_1^h a^{\alpha\beta}\partial_\alpha m 2\xi\partial_\beta\xi D_1^h(m-g)\right| \\
&\leq ||D_1^h a^{\alpha\beta}||_{L^\infty}\left(\int_{B_+(s)}|\nabla\xi|^2|\nabla m|^2+\int_{B_+(s)}|D_1^h(m-g)|^2\xi^2\right) \qquad (3.50)\\
&\leq ||\nabla a^{\alpha\beta}||_{L^\infty}\left(\int_{B_+(s)}\frac{C}{s^2}|\nabla m|^2+2\int_{B_+(s)}|D_1^h m|^2\xi^2+2\int_{B_+(s)}|D_1^h g|^2\xi^2\right)\\
&\leq C_0.
\end{aligned}
$$

Here we have used the fact that $|\nabla\xi|\leq \frac{C}{s}$ in $B_+(s)$ (see (3.57)).
For the fourth term

$$
\begin{aligned}
&\left|\int_{B_+(s)} D_1^h(a^{\alpha\beta})\partial_\alpha m\xi^2\partial_\beta(D_1^h(m-g))\right| \\
&\leq ||D_1^h a^{\alpha\beta}||_{L^\infty}\left(\int_{B_+(s)}\frac{1}{4\epsilon_0}|\nabla m|^2\xi^2+\int_{B_+(s)}\epsilon_0|\nabla(D_1^h(m-g))|^2\xi^2\right)\\
&\leq ||\nabla a^{\alpha\beta}||_{L^\infty}\left(\int_{B_+(s)}\frac{1}{4\epsilon_0}|\nabla m|^2\xi^2+\int_{B_+(s)}2\epsilon_0|\nabla(D_1^h m)|^2\xi^2\right.\\
&\left.+\int_{B_+(s)}2\epsilon_0|\nabla(D_1^h g)|^2\xi^2\right)\\
&\leq C_0\left(\int_{B_+(s)}2\epsilon_0|\nabla(D_1^h m)|^2\xi^2+C(\epsilon_0)\right). \qquad (3.51)
\end{aligned}
$$

We now estimate the last term

$$
\begin{aligned}
&\left|\int_{B_+(s)} a^{\alpha\beta,h}D_1^h(\partial_\alpha m)2\xi\partial_\beta\xi D_1^h(m-g)\right| \\
&\leq ||a^{\alpha\beta}||_{L^\infty}\left(\epsilon_0\int_{B_+(s)}|D_1^h(\nabla m)|^2\xi^2+\frac{2}{\epsilon_0}\int_{B_+(s)}|\nabla\xi|^2(|D_1^h m|^2+|D_1^h g|^2)\right)\\
&\leq C_0\left(\epsilon_0\int_{B_+(s)}|D_1^h(\nabla m)|^2\xi^2+C(\epsilon_0)\right) \quad (\text{ by } (3.57)). \qquad (3.52)
\end{aligned}
$$

We proceed to estimating the first term of RHS of (3.48), firstly,

$$
\begin{aligned}
&\int_{B_+(s)} K.D_1^{-h}(\xi^2 D_1^h(m-g))\\
&\leq \frac{1}{4\epsilon_0}\int_{B_+(s)}|K|^2+\epsilon_0\int_{B_+(s)}|D_1^{-h}(\xi^2D_1^h(m-g))|^2\\
&\leq \frac{1}{2\epsilon_0}\int_{B_+(s)}|K|^2+\epsilon_0\int_{B_+(s)}|\nabla(\xi^2D_1^h(m-g))|^2\\
&\leq \frac{1}{4\epsilon_0}\int_{B_+(s)}|K|^2+2\epsilon_0\int_{B_+(s)}\xi^4|\nabla(D_1^h(m-g))|^2+8\epsilon_0\int_{B_+(s)}\xi^2|\nabla\xi|^2|D_1^h(m-g)|^2\\
&\leq \frac{1}{4\epsilon_0}\int_{B_+(s)}|K|^2+4\epsilon_0\int_{B_+(s)}\xi^2|\nabla(D_1^h(m))|^2+4\epsilon_0\int_{B_+(s)}\xi^2|\nabla(D_1^h(g))|^2\\
&+16\epsilon_0\int_{B_+(s)}|\nabla\xi|^2|D_1^h(m)|^2+16\epsilon_0\int_{B_+(s)}|\nabla\xi|^2|D_1^h(g)|^2.
\end{aligned}
$$

As $K \in L^2(B_+(s)$, and $|\nabla\xi| \leq \frac{C}{s}$ in $B_+(s)$, (see (3.57)), then we obtain

$$\int_{B_+(s)} K D_1^{-h}(\xi^2 D_1^h(m-g)) \leq 4\epsilon_0 \int_{B_+(s)} \xi^2 |\nabla(D_1^h(m))|^2 + 16\epsilon_0 \int_{B_+(s)} |\nabla\xi|^2 |D_1^h(m)|^2 + C(\epsilon_0). \tag{3.53}$$

By the discrete integration by parts, we estimate the remain term in RHS of (3.48):

$$\left|\int_{B_+(s)} m|\nabla mA|^2 D_1^{-h}(\xi^2 D_1^h(m-g))\right| = \left|\int_{B_+(s)} D_1^h(m|\nabla mA|^2)\xi^2 D_1^h(m-g)\right|$$
$$\leq \int_{B_+(s)} \Big(|D_1^h m||\nabla mA|^2 + |D_1^h(\nabla mA)|\Big(|\nabla mA| + |(\nabla mA)^h|\Big)\Big)\xi^2 |D_1^h(m-g)|,$$

where $(\nabla m)^h(x) := \nabla m(x + he_1)$ and $(\nabla mA)^h(x) := (\nabla mA)(x + he_1)$.
We estimate:

$$\int_{B_+(s)} |D_1^h m||\nabla mA|\xi^2 |D_1^h(m-g)| \leq C \int_{B_+(s)} \Big(|D_1^h m|^2 |\nabla m|^2 + |D_1^h g|^2 |\nabla m|^2\Big),$$

$$\int_{B_+(s)} |D_1^h(\nabla mA)||\nabla mA|\xi^2 |D_1^h(m-g)|$$
$$\leq \frac{\epsilon_0}{2} \int_{B_+(s)} |D_1^h(\nabla mA)|^2 + \frac{1}{C\epsilon_0}\left(\int_{B_+(s)} |\nabla m|^2 |D_1^h m|^2 \xi^2 + |\nabla m|^2 |D_1^h m|^2 \xi^2\right)$$

and

$$\int_{B_+(s)} |D_1^h(\nabla mA)||(\nabla mA)^h|\xi^2 |D_1^h(m-g)|$$
$$\leq \frac{\epsilon_0}{2} \int_{B_+(s)} |D_1^h(\nabla mA)|^2 + \frac{1}{C\epsilon_0}\left(\int_{B_+(s)} |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2 + |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2\right)$$

for any $\epsilon_0 > 0$. Here we have used Remark 3.13 and Young's inequality to obtain the above estimates.
The above estimates imply that

$$\left|\int_{B_+(s)} m|\nabla mA|^2 D_1^{-h}(\xi^2 D_1^h(m-g))\right| \leq \epsilon_0 \int_{B_+(s)} |D_1^h(\nabla m)|^2 \xi^2$$
$$+ C(\epsilon_0)\Big(\int_{B_+(s)} |\nabla m|^2 |D_1^h m|^2 \xi^2 + \int_{B_+(s)} |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2\Big) + C(\epsilon_0). \tag{3.54}$$

From (3.49)-(3.54), choose ϵ_0 small enough, we then obtain

$$\int_{B_+(s)} |\nabla(D_1^h m)|^2 \xi^2$$
$$\leq C\left(1 + \int_{B_+(s)} |\nabla m|^2 |D_1^h m|^2 \xi^2 + \int_{B_+(s)} |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2\right). \tag{3.55}$$

Similar to the interior regularity, it remains to estimate the term

$$\int_{B_+(s)} |\nabla m|^2 |D_1^h m|^2 \xi^2 + \int_{B_+(s)} |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2 \tag{3.56}$$

to conclude the proof of the Lemma 3.14.

For $\epsilon_1 > 0$, we choose $\rho > 0$ as in the Lemma 3.15 below and choose $\xi \in C_0^\infty(B(s))$ with $0 \leq \xi \leq 1$ in $B(s)$ satisfying

$$\xi = 1 \text{ in } B(\frac{s}{2}) \quad \text{ and } |\nabla\xi| \leq \frac{4}{s} \text{ in } B(s). \tag{3.57}$$

Applying the Lemma 3.15 to $\eta = |D_1^h m|\xi \in H_0^1 \cap L^\infty(B_+(\rho))$,

$$\begin{aligned}\int_{B_+(\rho)} |\nabla m|^2 |D_1^h m|^2 \xi^2 &\leq \epsilon_1 \int_{B_+(\rho)} |\nabla[(D_1^h m)\xi]|^2 + C\int_{B_+(\rho)} |D_1^h m|^2 \xi^2 \\ &\leq 2\epsilon_1 \int_{B_+(\rho)} |\nabla(D_1^h m)|^2 \xi^2 + 2\epsilon_1 \int_{B_+(\rho)} |D_1^h m|^2 |\nabla \xi|^2 \\ &+ C\int_{B_+(\rho)} |D_1^h m|^2 \xi^2. \end{aligned} \tag{3.58}$$

The properties of η yields that

$$\int_{B_+(\rho)} |D_1^h m|^2 |\nabla\xi|^2 \leq \frac{16}{\rho^2} \int_{\mathrm{supp}(\xi)\cap B_+(\rho)} |D_1^h m|^2 \leq C\int_{B_+(\rho)} |\nabla m|^2,$$

and

$$\int_{B_+(\rho)} |D_1^h m|^2 \xi^2 \leq \int_{\mathrm{supp}(\xi)\cap B_+(\rho)} |D_1^h m|^2 \leq C\int_{B_+(\rho)} |\nabla m|^2$$

for $0 < |h| < \frac{1}{2}(\mathrm{dist}(\mathrm{supp}(\xi), (-\rho,\rho)\times\{0\}))$.
This implies that

$$\int_{B_+(\rho)} |\nabla m|^2 |D_1^h m|^2 \xi^2 \leq 2\epsilon_1 \int_{B_+(\rho)} |\nabla(D_1^h m)|^2 \xi^2 + C\int_{B_+(\rho)} |\nabla m|^2.$$

These arguments apply similarly to the term $\int_{B_+(s)} |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2$. We then obtain that

$$\int_{B_+(\rho)} |(\nabla m)^h|^2 |D_1^h m|^2 \xi^2 \leq 2\epsilon_1 \int_{B_+(\rho)} |\nabla(D_1^h m)|^2 \xi^2 + C\int_{B_+(\rho)} |\nabla m|^2.$$

Combining with(3.55) and choosing ϵ_1 small enough, it yields

$$\int_{B_+(\rho/2)} |\nabla(D_1^h m)|^2 \leq \int_{B_+(\rho)} |\nabla(D_1^h m)|^2 \xi^2 \leq C + C\int_{B_+(\rho/2)} |\nabla m|^2.$$

The proof of Lemma 3.14 is completed. □

We note that, by (3.45), one gets

$$\sum_{\alpha,\beta} -\partial_\beta(a^{\alpha\beta}\partial_\alpha m) = (m|\nabla m A|^2 + K) \text{ in } B_+(\rho). \tag{3.59}$$

Using $a^{\alpha\beta} \in W^{1,\infty}(B_+(\rho)), \partial_1 m \in H^1(B_+(\rho/2))$ and the right hand side of (3.59) belong to $L^1(B_+(\rho))$, we then obtain that $a^{22}\partial_{22}m \in L^1(B_+(\rho/2))$. We recall that by Remark 3.13, $0 < \frac{1}{\lambda} < a^{22}$. Therefore $\partial_{22}m \in L^1(B_+(\rho/2))$. We know that $\partial_{22}m \in L^1(B_+(\rho/2)), \partial_{12}m \in L^2(B_+(\rho/2))$. Using the anisotropic Sobolev embedding (see [25, Theorem 1], also [42, Theorem 2]), we get $\partial_2 m \in L^4(B_+(\rho/2))$. Thus the RHS of (3.59) belongs to $L^2(B_+(\rho/2))$. The standard elliptic regularity deduces that $m \in H^2(B_+(\rho/2))$. Finally, we get $m \in H^2(\Omega)$.

Step B *Hölder regularity.* Up to now, we know that $m \in H^2(\Omega, \mathbb{S}^2)$, that implies $m \in C^{0,\alpha}(\overline{\Omega}, \mathbb{S}^2)$, for some $0 < \alpha < 1$. We now want to improve the regularity at the boundary to $C^{1,\alpha}(\overline{\Omega})$, $\forall\, 0 < \alpha < \frac{1}{2}$.

Since $m \in H^2(\Omega, \mathbb{S}^2)$, then $\nabla m \in L^q, \forall\ 1 \leq q < \infty$. In particular, extending by 0 outside Ω, one has $(\nabla m)\mathbb{1}_\Omega \in H^s(\mathbb{R}^2)$ for $0 < s < \frac{1}{2}$. These above facts imply that $\nabla(m|\nabla m|^2) \in W^{1,q}(\Omega)$, $\forall 1 \leq q < 2$.
Indeed, one has

$$\nabla(m|\nabla m|^2) = \nabla m|\nabla m'|^2 + 2mD^2m\nabla m \in L^q(\Omega),\ \forall 1 \leq q < 2.$$

We note that if $\zeta \in W^{1,q}(\Omega)$ for all $1 \leq q < 2$, then by the Sobolev embedding, $\zeta \in H^s(\Omega)$ for all $0 < s < 1$. Therefore, we obtain that $m|\nabla m|^2 \in H^s(\Omega)$ for all $0 < s < 1$.
We now use again the formula

$$\mathcal{F}(U(\cdot,0))(\xi) = \frac{1}{2|\xi|}\mathcal{F}((\nabla.m')\mathbb{1}_\Omega)(\xi)$$

to obtain that for $s < \frac{1}{2}$:

$$\begin{aligned}\int_{\mathbb{R}^2} |\xi|^{2+2s}|\mathcal{F}(U(\cdot,0))(\xi)|^2 \quad &= \int_{\mathbb{R}^2} \frac{1}{4}|\xi|^{2s}|\mathcal{F}((\nabla.m')\mathbb{1}_\Omega)(\xi)|^2 \\ &= \frac{1}{4}||(\nabla'.m)\mathbb{1}_\Omega||^2_{\dot{H}^s(\mathbb{R}^2)} < \infty. \end{aligned} \tag{3.60}$$

This leads to $\nabla U \in H^s(\mathbb{R}^2)$, therefore $H = \nabla U(\cdot,0)\mathbb{1}_\Omega \in H^s(\mathbb{R}^2)$ for $s < \frac{1}{2}$. Then $m'Hm \in H^s(\Omega)$ for all $0 < s < \frac{1}{2}$.
Back to (3.13), since $m \in H^2(\Omega)$, F is smooth, $H, m'Hm \in H^s$ for all $0 < s < \frac{1}{2}$, then we deduce the right hand side of (3.13) belongs to $H^s(\Omega)$ for all $0 < s < \frac{1}{2}$. Using the elliptic regularity with the fact that $m = (g, 0) \in C^{1,1}(\partial\Omega)$, we then obtain that $m \in H^{s+2}(\Omega)$ for all $0 < s < \frac{1}{2}$.
The Morrey embedding leads that $m \in C^{1,\alpha}(\overline{\Omega})$ for all $0 < \alpha < \frac{1}{2}$.

We finish this Chapter by

Lemma 3.15. *Support $m \in C^0(\overline{B_+(s)}) \cap H^1(B_+(s))$ is a solution of* (3.45) *with $m = (g, 0)$ on $\partial B_+(s)$. Then for every $\epsilon > 0$, there exists $\rho > 0$ such that*

$$\int_{B_+(\rho)} |\nabla m|^2\eta^2 \leq \epsilon \int_{B_+(\rho)} |\nabla\eta|^2 + C\int_{B_+(\rho)} \eta^2, \tag{3.61}$$

for all $\eta \in H^1_0 \cap L^\infty(B_+(s))$ and $\eta = 0$ on $B_+(s)\backslash B_+(s/2)$.

Proof of Lemma 3.15. Choose $\Phi(x) = (m(x) - g(x))\eta^2 \in H^1_0 \cap L^\infty(B_+(s))$ in (3.45), we have

$$\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m\partial_\beta((m-g)\eta^2) = \int_{B_+(s)} (m|\nabla mA|^2 + K)((m-g)\eta^2). \tag{3.62}$$

Then

$$\begin{aligned}\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m\partial_\beta m\eta^2 = &-\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m 2\eta\partial_\beta\eta(m-g) + \int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m\eta^2\partial_\beta g \\ &+ \int_{B_+(s)} (m|\nabla mA|^2 + K)(m-g)\eta^2. \end{aligned} \tag{3.63}$$

We estimate the first term in the RHS

$$\begin{aligned}
&\left|\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m 2\eta\partial_\beta\eta(m-g)\right| \\
&\leq ||a^{\alpha\beta}||_{L^\infty}\left(\epsilon\int_{B_+(s)}|\nabla m|^2\eta^2+\frac{4}{\epsilon}\int_{B_+(s)}|\nabla\eta|^2(m-g)^2\right).
\end{aligned}$$

As for the second term, one has

$$\left|\int_{B_+(s)} a^{\alpha\beta}\partial_\alpha m\partial_\beta g\eta^2\right| \leq ||a^{\alpha\beta}||_{L^\infty}\left(\frac{\epsilon}{2}\int_{B_+(s)}|\nabla m|^2\eta^2+\frac{1}{2\epsilon}\int_{B_+(s)}|\nabla g|^2\eta^2\right).$$

Moreover

$$\left|\int_{B_+(s)} m|\nabla mA|^2(m-g)\eta^2\right| \leq \sup_{B_+(s)}|(m-g)|\int_{B_+(s)}|\nabla mA|^2\eta^2,$$

$$\left|\int_{B_+(s)} K(m-g)\eta^2\right| \leq \sup_{B_+(s)}|(m-g)|||K||_{L^2}||\eta||^2_{L^4}\leq Cs\sup_{B_+(s)}|(m-g)||K||_{L^2}||\nabla\eta||^2_{L^2}.$$

Here, we have used the Sobolev inequality

$$C\rho\int_{B_+(s)}|\nabla\eta|^2\geq\left(\int_{B_+(s)}|\eta|^4\right)^{1/2}$$

in the last line. The proof of Lemma follows because $m=g$ on $\partial B_+(s)$ and we can make $\sup_{B_+(s)}|m-g|$ arbitrary small by choosing s sufficiently small, and s small enough. $\square$

Part II

On the Poincaré Lemma on domains

Chapter 4

Introduction and statements of the main results

Abstract

This chapter is based on a work (see [7]) in collaboration with my adviser Pierre Bousquet. We give motivations as well as results on the Poincaré lemma. We also introduce basic methodologies to tackle our problems.

4.1 An overview of the Poincaré lemma

The central theme of this work is the Poincaré lemma on a domain with a Dirichlet boundary condition. The Poincaré lemma amounts to saying that a closed differential form is exact. Our interest is in a sharp version of the Poincaré lemma regarding the regularity of the domain. To formulate the motivation, we start from the *divergence equation* on a bounded domain Ω in $\mathbb{R}^n$, under the Dirichlet boundary condition. Given $p \in (1, +\infty)$ and a function $f \in L^p(\Omega)$, we look for a vector field $X \in W^{1,p}(\Omega, \mathbb{R}^n)$ which satisfies the two following conditions:

$$\begin{cases} \operatorname{div} X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In view of the Dirichlet boundary condition, a necessary condition for the existence of a solution X is

$$\int_\Omega f\, dx = \int_\Omega \operatorname{div} X\, dx = \int_{\partial\Omega} \langle X, \nu \rangle\, d\sigma = 0.$$

Here, we assume that Ω is at least Lipschitz regular, in order to use the integration by parts formula.

A standard way of tackling this equation is to solve the Poisson equation $\Delta u = f$ in Ω to get a solution u in $W^{2,p}(\Omega)$, which classically requires that Ω be $C^{1,1}$, see e.g. [22, Theorem 9.15]. The vector field $X = \nabla u$ then satisfies $\operatorname{div} X = \Delta u = f$ and belongs to $W^{1,p}(\Omega)$. If one further imposes a Neumann boundary condition for u, namely $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, then the normal component of X vanishes on $\partial\Omega$. It is then possible to modify X to cancel its tangential component, see e.g. [13, Theorem 9.2, Remark 9.3 (iii)]. In a similar way, if f belongs to the Hölder space $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, then one gets a $C^{1,\alpha}(\overline{\Omega})$ solution X by the elliptic regularity theory in these spaces, see [22, Theorem 6.31]. Here, Ω is assumed to be $C^{2,\alpha}$.

The main drawback of this approach is that it requires a stronger regularity assumption for the domain Ω than the one naturally expected. This leads to the two following questions: for $f \in L^p(\Omega)$, is it possible to solve (4.1) when Ω is merely Lipschitz? And when $f \in C^{0,\alpha}(\overline{\Omega})$, is it enough to assume that Ω is $C^{1,\alpha}$ to get a solution $X \in C^{1,\alpha}(\overline{\Omega})$?

There are several alternative strategies to prove that the answer to the first question is positive, see e.g. [13, Remark 9.3] and [44] for references. In the setting of Hölder spaces, we are not aware of any such result in the literature. Naturally, one can formulate similar questions in higher order Sobolev and Hölder spaces. For instance, given a nonnegative integer r and $f \in C^{r,\alpha}(\overline{\Omega})$ such that $\int_\Omega f\, dx = 0$, can one find a solution $X \in C^{r+1,\alpha}(\overline{\Omega})$ to (4.1) provided that Ω is $C^{r+1,\alpha}$?

The divergence equation can be seen as a particular case of the Poincaré lemma, when the right hand side f is identified to an n form. More generally, when f is a differential form of degree $k \in \{1, \ldots, n\}$, we can consider the differential form equation:

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{d}$ is the exterior derivative operator. Now, a solution X is a $(k-1)$ form on Ω. Assuming that f has $C^{r,\alpha}(\overline{\Omega})$ coefficients, one expects to find a solution X with $C^{r+1,\alpha}(\overline{\Omega})$ coefficients, provided some necessary conditions are satisfied: f should be a closed form, satisfy certain boundary conditions and be orthogonal to a certain set of harmonic forms on Ω (in connection with a possible non trivial topology of the domain).

Once again, it is possible to solve the Poincaré lemma in the scale of Sobolev or Hölder spaces, by relying on the elliptic regularity theory. However, to get a solution in $C^{r+1,\alpha}(\overline{\Omega})$, this strategy requires that Ω be at least a $C^{r+2,\alpha}$ domain, namely one degree of differentiability higher than the solution itself. In this article, we establish that the Poincaré lemma holds true in the scale of Hölder and Sobolev spaces when the domain has the *same* order of differentiability as the expected solutions. Hence, the solvability of the divergence equation in $W_0^{1,p}(\Omega)$ under the natural assumption that Ω be Lipschitz is not a peculiarity of the 0 order Sobolev case: it remains true in the setting of differential forms, in Hölder spaces as well as higher order Sobolev spaces.

4.2 The statements of the main results

4.2.1 The divergence problem

Our first result answers the divergence problem in the scale of Hölder spaces. Assume that the right hand side of equation (4.1) belongs to the Banach space

$$C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}) := \left\{ f \in C^{r,\alpha}(\overline{\Omega}) \ : \ \int_\Omega f(x)dx = 0 \right\}.$$

We look for a vector field X in the Banach space

$$C^{r+1,\alpha}_{z}(\overline{\Omega}, \mathbb{R}^n) := \left\{ X \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) : X = 0 \text{ on } \partial\Omega \right\}$$

such that $\operatorname{div} X = f$. Moreover, we expect that the solution X can be chosen continuously and linearly with respect to f. This is not obvious since such a solution X, when it exists, is not unique. In other words, we address the existence problem: does there exist a right inverse to div : $C^{r+1,\alpha}_{z}(\overline{\Omega}, \mathbb{R}^n) \to C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega})$?
This is indeed our first result, under the mere assumption that Ω has the *same regularity* as X itself.

Theorem 4.1. *Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let Ω be a bounded connected open $C^{r+1,\alpha}$ set in $\mathbb{R}^n$. Then, given any $f \in C^{r,\alpha}(\overline{\Omega})$ such that*

$$\int_\Omega f(x)dx = 0,$$

there exists $X \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ *verifying*

$$\begin{cases} \operatorname{div} X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Furthermore, the correspondence $f \mapsto X$ *can be chosen linear and there exists* $C = C(r, \alpha, \Omega) > 0$ *such that*

$$||X||_{C^{r+1,\alpha}(\overline{\Omega})} \leq C||f||_{C^{r,\alpha}(\overline{\Omega})}. \tag{4.3}$$

As we have already said, the divergence equation is a particular case of the differential form equation. Correspondingly, the above result is a particular case of our study of the Poincaré lemma (cf. Chapter 6). However, the proof is much more elementary in this case, which requires only that $\int_\Omega f = 0$. We shall indeed take the shortest route to approach the existence of solutions of the divergence problem which is inspired by the result of Bourgain and Brezis, see [4, Theorem 2']. For the convenience of the reader, we have gathered and proved the results in Chapter 5. This chapter will be carried out quite comprehensively. This helps to better understand the Poincaré Lemma in Chapter 6.

4.2.2 The Poincaré lemma in Hölder spaces

In order to state the second result, we need to introduce some notations. The set of k forms on Ω with $C^{r,\alpha}(\overline{\Omega})$ coefficients will be denoted by $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$. We introduce the Banach space

$$C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k) := \Big\{ f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k) : \boldsymbol{d}f = 0 \text{ in } \overline{\Omega}, \nu \wedge f = 0 \text{ on } \partial\Omega, \\ \int_\Omega \langle f, \chi \rangle = 0, \forall \chi \in \mathcal{H}^k_T(\Omega) \Big\}, \tag{4.4}$$

where $\mathcal{H}^k_T(\Omega)$ is the set of the Dirichlet *harmonic fields* of order k, defined as

$$\mathcal{H}^k_T(\Omega) = \{h \in L^2(\Omega, \Lambda^k) : \boldsymbol{\delta} h = 0 \text{ in } \Omega, \boldsymbol{d}(h_z) = 0 \text{ in } \mathbb{R}^n\}.$$

Here, h_z means the extension of h by zero outside Ω. The identity $\boldsymbol{d}(h_z) = 0$ must be understood in the sense of distributions:

$$\forall \theta \in C^\infty_c(\mathbb{R}^n, \Lambda^k), \qquad \int_{\mathbb{R}^n} \langle h, \boldsymbol{\delta}\theta \rangle \, dx = 0.$$

The outer unit normal ν to Ω is identified to a 1 form: we set $\nu = \nu_1 dx_1 + \cdots + \nu_n dx_n$ if $\nu_1, \ldots, \nu_n$ are the coordinates of ν in the standard basis of $\mathbb{R}^n$, where $\boldsymbol{\delta}$ is the adjoint of $\boldsymbol{d}$ which is defined as in (6.9), Chapter 6.

For the Poincaré lemma, we look for a $(k-1)$ form X in the Banach space

$$C^{r+1,\alpha}_z(\overline{\Omega}, \Lambda^{k-1}) := \Big\{ X \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}) : X = 0 \text{ on } \partial\Omega \Big\} \tag{4.5}$$

such that $\boldsymbol{d}X = f$, where $f \in C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k)$ is given. We also prove the existence of the right inverse to the exterior derivative operator $\boldsymbol{d} : C^{r+1,\alpha}_z(\overline{\Omega}, \Lambda^{k-1}) \to C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k)$. It is stated in the following

Theorem 4.2. *Let* $r \geq 0$ *be an integer and* $0 < \alpha < 1$. *Let* Ω *be a bounded open* $C^{r+1,\alpha}$ *set in* $\mathbb{R}^n$. *Let* $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k), 1 \leq k \leq n$, *be such that*

$$\begin{cases} \boldsymbol{d}f = 0 & \text{in } \overline{\Omega}, \\ \nu \wedge f = 0 & \text{on } \partial\Omega, \end{cases}$$

and for every $\chi \in \mathcal{H}_T^k(\Omega)$*,*

$$\int_\Omega \langle f, \chi \rangle \, dx = 0. \tag{4.6}$$

Then there exists $X \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$ *such that*

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the correspondence $f \mapsto X$ *can be chosen linear and there exists* $C = C(r, \alpha, \Omega) > 0$ *such that*

$$||X||_{C^{r+1,\alpha}(\overline{\Omega},\Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}(\overline{\Omega},\Lambda^k)}.$$

We emphasize that the assumptions on f are necessary to obtain a solution. Indeed, if $X \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, then $\boldsymbol{d}(\boldsymbol{d}X) = 0$. If in addition, $X = 0$ on $\partial\Omega$, then $\nu \wedge X = 0$ and thus $\nu \wedge \boldsymbol{d}X = 0$ on $\partial\Omega$, see [13, Theorem 3.23]. When $r = 0$, these conditions must be understood in the sense of distributions, namely:

$$\forall \theta \in C_c^\infty(\mathbb{R}^n, \Lambda^k), \int_{\mathbb{R}^n} \langle f_z, \boldsymbol{\delta}\theta \rangle \, dx = \int_\Omega \langle f, \boldsymbol{\delta}\theta \rangle \, dx = \int_\Omega \langle \boldsymbol{d}X, \boldsymbol{\delta}\theta \rangle \, dx = 0,$$

where the last line follows from the integration by parts formula, see Proposition 6.9, in Chapter 6. The last assumption (4.6) also follows from the integration by parts formula. In fact, for every $\chi \in L^2(\Omega)$ such that $\boldsymbol{\delta}\chi = 0$ in the sense of distributions, one has for every $X \in C_c^\infty(\Omega, \Lambda^{k-1})$,

$$\int_\Omega \langle \boldsymbol{d}X, \chi \rangle \, dx = \int_{\partial\Omega} \langle X \wedge \nu, \chi \rangle = 0.$$

This remains true by density for $X \in W_0^{1,2}(\Omega, \Lambda^{k-1})$, and thus in particular for $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$. Finally, we formulate the corresponding result of the Poincaré lemma in the scale of Sobolev spaces.

4.2.3 The Poincaré lemma in Sobolev spaces

In the setting of Sobolev spaces, given two integers $r \geq 0$, $k \in \{1, \ldots, n\}$, and $p \in (1, \infty)$, we introduce the sets

$$W_\nu^{r,p}(\Omega, \Lambda^k) = \{f \in W^{r,p}(\Omega, \Lambda^k), \boldsymbol{d}f = 0 \text{ on } \Omega, \nu \wedge f = 0 \text{ on } \partial\Omega\},$$

$$W_z^{r+1,p}(\Omega, \Lambda^{k-1}) = \{X \in W^{r+1,p}(\Omega, \Lambda^{k-1}), X = 0 \text{ on } \partial\Omega\}.$$

One expects to obtain the conclusion for the Sobolev setting, namely: the existence of a right inverse $\boldsymbol{d} : W_z^{r+1,p}(\Omega, \Lambda^{k-1}) \to W_{\mathcal{H}}^{r,p}(\Omega, \Lambda^k)$, where

$$W_{\mathcal{H}}^{r,p}(\Omega, \Lambda^k) := \left\{ f \in W_\nu^{r,p}(\Omega, \Lambda^k) \ : \ \int_\Omega \langle f, \chi \rangle = 0 \ , \ \forall \, \chi \in \mathcal{H}_T^k(\Omega) \right\}.$$

We remark that *a priori* the quantity $\int_\Omega \langle f, \chi \rangle$does not necessarily make sense for every $f \in W^{r,p}(\Omega, \Lambda^k)$ and $\chi \in \mathcal{H}_T^k(\Omega)$. This is the reason why in the next statement, we assume that the Dirichlet harmonic fields on Ω are regular enough. In view of the above facts, we state the following statement.

Theorem 4.3. *Let $r \in \mathbb{N}$, $k \in \{1, \ldots, n\}$ and $p \in (1, \infty)$. Let Ω be a bounded $C^{r,1}$ domain in $\mathbb{R}^n$. Assume also that $\mathcal{H}_T^k(\Omega) \subset L^{p'}(\Omega, \Lambda^k)$. Let $f \in W^{r,p}(\Omega, \Lambda^k)$ be such that*

$$\begin{cases} \boldsymbol{d}f = 0 & \text{in } \Omega, \\ \nu \wedge f = 0 & \text{on } \partial\Omega, \end{cases}$$

and for every $h \in \mathcal{H}_T^k(\Omega)$,

$$\int_\Omega \langle f, h \rangle \, dx = 0. \tag{4.7}$$

Then there exists $X \in W^{r+1,p}(\Omega, \Lambda^{k-1})$ such that

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the correspondence $f \mapsto X$ can be chosen linear and there exists $C = C(r, p, \Omega) > 0$ such that

$$||X||_{W^{r+1,p}(\Omega,\Lambda^{k-1})} \leq C||f||_{W^{r,p}(\Omega,\Lambda^k)}.$$

4.3 Methodology

Our strategy is greatly inspired from the proof of [4, Theorem 2] which considers the divergence equation (4.1) for a right hand side in $L^p(\Omega)$ satisfying $\int_\Omega f dx = 0$, $p \in (1, \infty)$. In this setting, Bourgain and Brezis rely on two main ingredients. First, they observe that the range of the differential operator

$$\text{div} : W_0^{1,p}(\Omega, \mathbb{R}^n) \to L_{\mathcal{H}}^p(\Omega)$$

is dense. Here, we have denoted by $L_{\mathcal{H}}^p(\Omega)$ the set of those $f \in L^p(\Omega)$ such that $\int_\Omega f \, dx = 0$. Actually, the *dual* operator of div is simply the gradient

$$\nabla : (L_{\mathcal{H}}^p(\Omega))^* \subset (W_0^{1,p}(\Omega, \mathbb{R}^n))^*.$$

One can identify $(L_{\mathcal{H}}^p(\Omega))^*$ to $L_{\mathcal{H}}^{p'}(\Omega)$. It is then easily shown that the kernel of ∇ is trivial. Equivalently, the range of div is dense in $L_{\mathcal{H}}^p(\Omega)$. In the setting of Hölder spaces, this argument is less obvious. We just mention here that the dual space of $C_{\mathcal{H}}^{0,\alpha}$ cannot be identified to a subspace of the distributions on Ω. For instance, given $a \in \partial\Omega$, the Dirac mass δ_a located at a is a non trivial element of $(C_{\mathcal{H}}^{0,\alpha})^*$ but its restriction to $C_c^\infty(\Omega)$ is trivial. However, this duality approach can be generalized to any higher order Sobolev spaces or Hölder spaces, and this is probably one of the main achievements of this part of the thesis to do so. The strategy adopted to [4] can be adapted to various equations and spaces (see [6],[5]).

The second ingredient used in [4] is the construction of an approximate solution to the divergence equation. More precisely, Bourgain and Brezis construct two linear operators

$$S : L_{\mathcal{H}}^p(\Omega) \to W_0^{1,p}(\Omega) \qquad , \qquad K : L_{\mathcal{H}}^p(\Omega) \to L_{\mathcal{H}}^p(\Omega)$$

such that S is continuous, K is compact, and for every $f \in L_{\mathcal{H}}^p(\Omega)$, $f = \text{div}(Sf) + Kf$. Hence, up to the compact perturbation Kf, Sf is a right inverse to the divergence equation. In order to perform this construction, one localizes the problem on small balls intersecting $\partial\Omega$, where Ω can be seen as the epigraph of a Lipschitz function. In such a situation, it is possible to define an *exact* right inverse to the divergence operator. However, when gluing together all these local constructions, an error term is produced, which gives birth to the perturbation operator K.

This argument can be extended with some care to the Hölder framework. The construction on each small ball does not require a *rectification* of the boundary by local charts to reduce the problem to the case when Ω is a half space. Instead one uses an approximation argument reminiscent of the proof of the open map theorem. This is a crucial fact for our purposes. Indeed, if α is a $C^{r+1,\alpha}$ form of degree $k \in \{1, \dots, n\}$ and ϕ is a $C^{r+1,\alpha}$ local chart, then the pullback $\phi^*\alpha$ is merely $C^{r,\alpha}$ since the pullback introduces partial derivatives of ϕ. On the contrary, the approximation argument allows not to lose one order of differentiability.

Once the operators S and K are constructed, one relies on the following functional analysis statement to obtain a true, global right inverse to the divergence operator:

Lemma 4.4. *Let E, F be two Banach spaces and let T be a bounded operator from E to F. Assume that*

$$\ker(T^*) = \{0\}$$

and that there exists a bounded operator S from F to E and a compact operator K from F into itself such that

$$T \circ S = Id + K.$$

Then T admits a right inverse.

The above Lemma is applied to $E = W_0^{1,p}(\Omega, \mathbb{R}^n), F = L^p_{\mathcal{H}}$, $T = \operatorname{div}$, where the condition $\ker(T^*) = \{0\}$ amounts to the first ingredient described above.
Dealing with the Poincaré lemma, in the case of Hölder spaces, we will construct such a right inverse operator S as in the following Theorem.

Theorem 4.5. *For every integer $r \geq 0$, there is a bounded operator*

$$S : C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k) \to C^{r+1,\alpha}_{z}(\overline{\Omega}, \Lambda^{k-1})$$

such that for every $f \in C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k)$

$$f - \boldsymbol{d}(Sf) \in C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k)$$

and

$$||f - \boldsymbol{d}(Sf)||_{C^{r+1,\alpha}_{z}(\overline{\Omega}, \Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}, \Lambda^k)}.$$

Chapter 5

On the divergence equation in Hölder spaces

Abstract

This chapter studies the solution X of the equation

$$\begin{cases} \operatorname{div} X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is given. It is devoted to the proof of Theorem 4.1. This result is a particular case of the study on the Poincaré lemma (see Chapter 6). However, the proofs are much simpler in this case, when we consider the data f as a function instead of a differential form of degree n. In this case, the boundary condition of f will be ignored, whereas it will be taken into account in the next chapter in order to prove Theorem 4.2.

5.1 The main theorem

The main result of this chapter is the following

Theorem 5.1. *Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let Ω be a bounded connected open $C^{r+1,\alpha}$ set in $\mathbb{R}^n$. Then, given any $f \in C^{r,\alpha}(\overline{\Omega})$ such that*

$$\int_\Omega f(x)dx = 0,$$

there exists $X \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ verifying

$$\begin{cases} \operatorname{div} X = f & \textit{in } \Omega, \\ X = 0 & \textit{on } \partial\Omega. \end{cases} \tag{5.1}$$

Furthermore, the correspondence $f \mapsto X$ can be chosen linear and there exists $C = C(r, \alpha, \Omega) > 0$ such that

$$||X||_{C^{r+1,\alpha}(\overline{\Omega})} \leq C||f||_{C^{r,\alpha}(\overline{\Omega})}.$$

If Ω is not connected, then the condition $\int_\Omega f = 0$ has to hold on each connected component of Ω.

The main point of Theorem 5.1 is the assumption on the domain Ω which is assumed to be only $C^{r+1,\alpha}$. In the case Ω of class $C^{r+2,\alpha}$, equation (5.1) can be reduced to an elliptic problem for which standard techniques apply. For completeness, let us state the result in [13].

Theorem 5.2. *([13, Theorem 9.2]) Let $r \geq 0$ be an integer and $0 < \alpha < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,\alpha}$ set. The following conditions are then equivalent:*

(i) the function $f \in C^{r,\alpha}(\overline{\Omega})$ satisfies

$$\int_\Omega f = 0.$$

(ii) there exists $X \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ verifying

$$\begin{cases} \operatorname{div} X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the correspondence $f \mapsto X$ can be chosen linear and there exists $C = C(r, \alpha, \Omega) > 0$ such that

$$||X||_{C^{r+1,\alpha}} \leq C||f||_{C^{r,\alpha}}.$$

Similar results hold for $f \in L^p, 1 < p < \infty$, finding $X \in W^{1,p}$. However the result is false if $p = 1$ or $p = \infty$. In [4], Bourgain and Brezis have proved that the divergence equation div $X = f$ has not necessarily a solution in $W^{1,1}$ (respect $W^{1,\infty}$) when $f \in L^1$(respect $f \in L^\infty$) even when Ω is a smooth domain. It is also false for $C^{0,\alpha}$ when $\alpha = 0$ or $\alpha = 1$, see Dacorogna, Fusco and Tartar [15], and McMullen [37].

5.2 The idea of the proof

In the spirit of the proof of [4, Theorem 2], with some modifications, our argument relies heavily on the following Lemma

Lemma 5.3. *([4, Lemma 8]) Let E, F be two Banach spaces and let T be a bounded operator from E to F. Assume that*

$$\ker(T^*) = \{0\}$$

and that there exists a bounded operator S from F to E and a compact operator K from F into itself such that

$$T \circ S = Id + K.$$

Then T admits a right inverse.

More precisely, we establish Theorem 5.1 by proving the existence of a right inverse to

$$T : C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) \to C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}),$$

where

$$C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n), C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega})$$

are defined by

$$C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) := \{X \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) : X = 0 \text{ on } \partial\Omega\}$$

and

$$C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}) := \{f \in C^{r,\alpha}(\overline{\Omega}) : \int_\Omega f dx = 0\}.$$

In order to prove Theorem 5.1, we shall apply Lemma 5.3 to $E = C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n), F = C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega})$ and $T = \operatorname{div}$. In the Hölder setting of the divergence problem, such a right inverse operator S will be constructed as in the following Theorem.

Theorem 5.4. *For every integer $r \geq 0$, there is a bounded operator $S : C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega}) \to C^{r+1,\alpha}_z(\overline{\Omega}, \mathbb{R}^n)$ such that for every $f \in C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega})$*

$$f - \operatorname{div}(Sf) \in C^{r+1,\alpha}_z(\overline{\Omega})$$

and

$$||f - \operatorname{div} Sf||_{C^{r+1,\alpha}(\overline{\Omega})} \leq C||f||_{C^{r,\alpha}(\overline{\Omega})}.$$

Let us mention that to tackle the regularity of the divergence equation, the standard elliptic theories are still important features. In the proof of Theorem 5.4 below, we will handle the lack of regularity of the domain by localizing the problem on small domains intersecting $\partial\Omega$ (denoted $(V_i)_i$) which are $C^{r+1,\alpha}$ - diffeomorphic to cubes in $\mathbb{R}^n$. We first study the divergence equation in the cube where we require the boundary condition of the solutions only on one side of the cube. This problem can be treated easily by using the smooth domain version of the divergence problem (Theorem 5.2). By using the local charts, solutions in a cube give us local constructions of the bounded operators. We then glue all these local constructions, an error term is produced, which gives birth to the perturbation operator K.

We emphasize that the composition of the solutions of the divergence equation in the cube and diffeomorphisms does not imply directly the existence of solutions of the divergence equation in a neighborhood of the boundary, because the diffeomorphisms are not linear in general. Moreover, the perturbation operator ($Kf = f - Sf$) is required to belong to $C^{r+1,\alpha}$ while f only belongs to $C^{r,\alpha}$. Therefore, design and choice of the coordinate maps need great care. In fact, we shall consider locally Ω as the epigraph of a function $\psi : Q'_1 \subset \mathbb{R}^{n-1} \to \mathbb{R}$ and consider the local chart defined by

$$\Phi : (x', x_n) \in Q_1 = Q'_1 \times (0,1) \mapsto (x', x_n + \psi(x')) \in V_i.$$

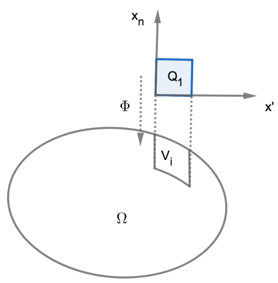

Figure 5.1

To complete the proof of Theorem 5.1, we have to verify that $\ker(T^*) = \{0\}$. As we said in the methodology section, in the Hölder setting, the proof of $\ker(T^*) = \{0\}$ is less trivial than in the Sobolev setting.

This chapter is organized as follows: In the next section, we recall some definitions and preliminaries on Hölder spaces. The proof of Theorem 5.4 will be given in Section 5.4. Then we get the conclusion of Theorem 5.1 by proving that $\ker(T^*) = \{0\}$. Some proofs for the preliminary results on Hölder spaces will be given in the appendix.

5.3 Definitions and Preliminaries

In this section, we recall some elementary properties of Hölder spaces. We refer to [13, Chapter 16] for more refined statements.

Let Ω be a non-empty bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $\alpha \in (0, 1]$. Given $f : \Omega \to \mathbb{R}$, we define the Hölder semi-norm:

$$[f]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For every $r \in \mathbb{N}$, we denote by $C^{r,\alpha}(\overline{\Omega})$ the set of those continuous maps $f : \overline{\Omega} \to \mathbb{R}$ which have continuous derivatives on Ω up to the order r and such that for every multi-index $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ of length $|\beta|(:= \beta_1 + \dots + \beta_n) = r$, one has

$$[D^\beta f]_{C^{0,\alpha}(\overline{\Omega})} < \infty.$$

Here, we have denoted

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}.$$

We observe that the derivatives up to the order r of a map $f \in C^{r,\alpha}(\overline{\Omega})$ can be continuously extended to $\overline{\Omega}$. The set $C^{r,\alpha}(\overline{\Omega})$ is a Banach space when equipped with the norm

$$\|f\|_{C^{r,\alpha}(\overline{\Omega})} = \|f\|_{C^r(\overline{\Omega})} + \max_{\substack{\beta \in \mathbb{N}^n \\ |\beta| = r}} [D^\beta f]_{C^{0,\alpha}(\overline{\Omega})}$$

where

$$\|f\|_{C^r(\overline{\Omega})} = \max_{\substack{\beta \in \mathbb{N}^n \\ |\beta| \leq r}} \sup_{x \in \Omega} |D^\beta f(x)|.$$

We can readily generalize the above definitions to the case of vector-valued functions: given $m \in \mathbb{N}$, we define the space $C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^m)$ as the set of those $f = (f_1, \dots, f_m) : \overline{\Omega} \to \mathbb{R}^m$ such that each component f_i belongs to $C^{r,\alpha}(\overline{\Omega})$. We also use the norm

$$\|f\|_{C^{r,\alpha}(\overline{\Omega}; \mathbb{R}^m)} = \sum_{i=1}^m \|f_i\|_{C^{r,\alpha}(\overline{\Omega})}.$$

Finally, when $\alpha = 0$, we set $C^{r,0}(\overline{\Omega}, \mathbb{R}^m) = C^r(\overline{\Omega}, \mathbb{R}^m)$. In this case, we write

$$[D^\beta u]_{C^{0,0}(\overline{\Omega})} = 0, \text{ for all } |\beta| = r$$

and

$$||u||_{C^{r,0}(\overline{\Omega})} = ||u||_{C^r(\overline{\Omega})}.$$

Given $x, y \in \overline{\Omega}$, we denote by $d_\Omega(x, y)$ the distance between x and y relative to Ω:

$$d_\Omega(x, y) = \inf_{\substack{\gamma \in W^{1,\infty}([0,1], \overline{\Omega}), \\ \gamma(0) = x, \gamma(1) = y}} \int_0^1 |\gamma'(t)| \, dt.$$

Here, $|\gamma'(t)|$ is the Euclidean norm of the vector $\gamma'(t)$ in $\mathbb{R}^n$. We also define the corresponding diameter of Ω

$$d_\Omega = \sup_{x,y \in \overline{\Omega}} d_\Omega(x, y) \tag{5.2}$$

as well as

$$\delta_\Omega = \sup_{\substack{x,y \in \overline{\Omega}, \\ x \neq y}} \frac{d_\Omega(x, y)}{|x - y|}. \tag{5.3}$$

When Ω is a Lipschitz domain, one can easily prove that δ_Ω is finite, see Remark 5.11. By Lipschitz set, we mean that $\overline{\Omega}$ is locally the epigraph of a Lipschitz continuous function of $n - 1$ variables in an appropriate coordinate system.

We now recall some properties of the Hölder spaces.

Proposition 5.5. *[13, Theorem 16.11] Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a linear extension operator*

$$E : \bigcup_{\substack{r\in\mathbb{N}\\ \alpha\in[0,1]}} C^{r,\alpha}(\overline{\Omega}) \to \bigcup_{\substack{r\in\mathbb{N}\\ \alpha\in[0,1]}} C^{r,\alpha}(\mathbb{R}^n).$$

More precisely, for every $r \in \mathbb{N}$, there exists a constant $C = C(r, \Omega) > 0$ such that for every $\alpha \in [0, 1]$ and every $f \in C^{r,\alpha}(\overline{\Omega})$, one has

$$E(f)|_{\overline{\Omega}}= f, \qquad supp\,[E(f)] \text{ is compact}, \qquad ||E(f)||_{C^{r,\alpha}(\mathbb{R}^n)}\leq C||f||_{C^{r,\alpha}(\overline{\Omega})}.$$

The space $C^{r,\alpha}(\overline{\Omega})$ is an algebra:

Proposition 5.6. *Let $r \in \mathbb{N}$ and $\alpha \in (0, 1]$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We denote by d_Ω the diameter of Ω. Then there exists a constant $C = C(r, n) > 0$ such that for every $f, g \in C^{r,\alpha}(\overline{\Omega})$*

$$||fg||_{C^{r,\alpha}(\overline{\Omega})}\leq C(\delta_\Omega + d_\Omega)^r||f||_{C^{r,\alpha}(\overline{\Omega})}||g||_{C^{r,\alpha}(\overline{\Omega})}.$$

The above proposition is a variant of [13, Theorem 16.28]. The latter is sharper regarding the norms of f and g in the right hand side. However, it allows a dependence of the constant C with respect to the set Ω which is not explicitly stated. This is the reason why we have formulated the above result in this form, in order to be more precise on this dependence. A proof of Proposition 5.6 is given in the appendix.

Finally, under suitable assumptions on r and α, Hölder continuous functions are stable with respect to composition. In the sequel, we need a result that we have not been able to find in the literature in this form.

Proposition 5.7. *Let $r \in \mathbb{N} \setminus \{0\}$ and $\alpha \in (0, 1)$. Let $\Omega \subset \mathbb{R}^n, O \subset \mathbb{R}^m$ be bounded Lipschitz domains, Then there exists a constant $C = C(r, n) > 0$ such that for every $f \in C^{r,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ and $g \in C^{r,\alpha}(\overline{O}, \mathbb{R})$ with $f(\overline{\Omega}) \subset \overline{O}$, one has*

$$||g \circ f||_{C^{r,\alpha}(\overline{\Omega})}\leq C(\delta_\Omega + d_\Omega)^{r^2}||g||_{C^{r,\alpha}(\overline{O})}(||Df||^{r+\alpha}_{C^{r-1,\alpha}(\overline{\Omega})}+1).$$

Remark 5.8. When $g \in C^{0,\alpha}(\overline{O}, \mathbb{R})$ and $f \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^m)$, we will use the following elementary inequality:

$$||g \circ f||_{C^{0,\alpha}(\overline{\Omega})}\leq \delta_\Omega||g||_{C^{0,\alpha}(\overline{\Omega})}(||Df||^{\alpha}_{C^0(\overline{\Omega})}+1).$$

The proofs of Proposition 5.7 and Remark 5.8 are given in the appendix.

5.4 The proof of Theorem 5.4

Throughout this section, the constants are all denoted by the same letter C and only depend on the parameters r, α, n and ρ which are introduced below. We will not systematically mention this dependence. On the contrary, when the constants depend on other quantities, we will emphasize this dependence.

5.4.1 Solution of the divergence equation on a cube

As explained in the introduction, the first step in the proof of Theorem 5.4 is to solve the divergence equation when Ω is a cube and the boundary condition is only required on one side of the cube.

In the following, for every $\delta > 0$, we denote by Q_δ the cube $(0,\delta)^n$ while Q'_δ is the lower side of Q_δ, namely $Q'_\delta = (0,\delta)^{n-1} \times \{0\}$. We will often identify Q'_δ with $(0,\delta)^{n-1}$.

Lemma 5.9. *Let $r \in \mathbb{N}$, $\alpha \in (0,1)$, $\rho > 0$ and $f \in C^{r,\alpha}(\overline{Q_\rho})$. Then there exists $X \in C^{r+1,\alpha}(\overline{Q_\rho};\mathbb{R}^n)$ such that $\operatorname{div} X = f$ on Q_ρ and $X = 0$ on Q'_ρ. Furthermore, there exists $C = C(r,\alpha,\rho,n) > 0$ such that*

$$||X||_{C^{r+1,\alpha}(\overline{Q_\rho})} \leq C||f||_{C^{r,\alpha}(\overline{Q_\rho})}.$$

Proof of Lemma 5.9. Without loss of generality, by a dilation argument, one can assume that $\rho = 1$ (the constant C in the statement is allowed to depend on ρ). By the extension property on Hölder spaces, see Proposition 5.6, there exists $\overline{f} \in C^{r,\alpha}(\mathbb{R}^n)$ such that

$$||\overline{f}||_{C^{r,\alpha}(\mathbb{R}^n)} \leq C||f||_{C^{r,\alpha}(\overline{Q_1})}. \tag{5.4}$$

Let ω be a smooth bounded open set such that $\overline{Q_1} \subset \omega \subset \mathbb{R}^{n-1} \times (0,+\infty)$. In particular, $Q'_1 \subset \partial\omega$ and ω can be chosen such that its volume $|\omega|$ depends only n . Let also $\theta \in C_c^\infty(\omega)$ be such that $\operatorname{supp}\theta \cap Q_1 = \emptyset$ and $\int_\omega \theta = 1$. We then define

$$\widetilde{f} := \overline{f} - \left(\int_\omega \overline{f}\right)\theta.$$

Observe that $\int_\omega \widetilde{f} = 0$ and $\widetilde{f} := \overline{f}$ on Q_1. Theorem 5.2 applied to $\widetilde{f}$ on ω yields a vector field $\widetilde{X} \in C^{r+1,\alpha}(\overline{\omega};\mathbb{R}^n)$ such that $\operatorname{div}\widetilde{X} = \widetilde{f}$ on ω and $\widetilde{X} = 0$ on $\partial\omega$. Moreover, there exists $C = C(r,\alpha,\omega) > 0$ such that

$$||\widetilde{X}||_{C^{r+1,\alpha}(\overline{\omega})} \leq C||\widetilde{f}||_{C^{r,\alpha}(\overline{\omega})}.$$

We next observe that

$$||\widetilde{f}||_{C^{r,\alpha}(\overline{\omega})} \leq ||\overline{f}||_{C^{r,\alpha}(\overline{\omega})} + |\omega|||\overline{f}\|_{C^0(\overline{\omega})}\|\theta\|_{C^{r,\alpha}(\overline{\omega})} \leq C||\overline{f}||_{C^{r,\alpha}(\overline{\omega})}.$$

In view of (5.4), this implies that

$$||\widetilde{f}||_{C^{r,\alpha}(\overline{\omega})} \leq C||f||_{C^{r,\alpha}(\overline{Q_1})}.$$

Then $X := \widetilde{X}|_{\overline{Q_1}}$ satisfies all the required properties.

□

5.4.2 Solution of the divergence equation on an epigraph

Given $\rho > 0$, let $\psi \in C^{r+1,\alpha}(\overline{Q'_\rho})$. We introduce the $C^{r+1,\alpha}$ diffeomorphism

$$\Phi = (\Phi^1, ..., \Phi^n) : x = (x', x_n) \in \overline{Q_\rho} \mapsto (x', x_n + \psi(x'))$$

and define the open set

$$U = \Phi(Q_\rho) = \{(x', x_n) \in Q'_\rho \times (0,+\infty) : \psi(x') < x_n < \psi(x') + \rho\}. \tag{5.5}$$

Since we will apply Propositions 5.6 and 5.7 on U, we first need to estimate the geometrical quantities d_U and δ_U, see (5.2) and (5.3).

Lemma 5.10. *For every $x, y \in \overline{U}$,*

$$d_U(x,y) \leq 3(1 + \|\nabla\psi\|_\infty)|x - y|.$$

In particular, $\delta_U \leq 3(1 + \|\nabla\psi\|_\infty)$.

Proof. For every $x = (x', x_n), y = (y', y_n) \in \overline{U}$, the Lipschitz map

$$\gamma : t \in [0,1] \mapsto \Big((1-t)x' + ty', \psi((1-t)x' + ty') + (1-t)(x_n - \psi(x')) + t(y_n - \psi(y'))\Big)$$

takes its values into $\overline{U}$ and satisfies $\gamma(0) = x$, $\gamma(1) = y$. Hence,

$$d_U(x,y) \leq \int_0^1 |\gamma'(t)|\, dt.$$

For a.e. $t \in [0,1]$,

$$\gamma'(t) = (y' - x', \langle \nabla\psi((1-t)x' + ty'), y' - x'\rangle + (y_n - x_n) + (\psi(x') - \psi(y')),$$

so that

$$\begin{aligned} \gamma'(t)|^2 \quad &\leq |y' - x'|^2 + 3\|\nabla\psi\|_{L^\infty}^2 |y' - x'|^2 + 3|\psi(x') - \psi(y')|^2 + 3|x_n - y_n|^2 \\ &\leq (3 + 6\|\nabla\psi\|_{L^\infty}^2)|x - y|^2. \end{aligned}$$

Hence,

$$d_U(x,y) \leq (3 + 6\|\nabla\psi\|_\infty^2)^{\frac{1}{2}} |x - y| \leq 3(1 + \|\nabla\psi\|_\infty)|x - y|$$

and the assertion of the lemma follows. □

Remark 5.11. The proof of the above lemma shows that when Ω is the epigraph of a Lipschitz function $\psi : \Omega' \to \mathbb{R}$, where Ω' is a convex open set in $\mathbb{R}^{n-1}$, then the intrinsic distance $d_\Omega(x,y)$ between two points x and y in Ω is not larger than $3(1 + \|\nabla\psi\|_\infty)|x - y|$. This implies that when Ω is *any* Lipschitz domain in $\mathbb{R}^n$, then $\delta_\Omega < \infty$.

Proof of Remark 5.11. Indeed, assume by contradiction that there are two sequences of points $(x_i)_i$ and $(y_i)_i$ in $\overline{\Omega}$ such that $x_i \neq y_i$ for every i and

$$\lim_{i \to +\infty} \frac{d_\Omega(x_i, y_i)}{|x_i - y_i|} = +\infty.$$

Then, by compactness of $\overline{\Omega}$, up to extraction (we do not relabel), $(x_i)_i$ and $(y_i)_i$ converge to the same boundary point $z \in \partial\Omega$. Since Ω is Lipschitz, there exists an open set U in $\mathbb{R}^n$ such that $\Omega \cap U$ is, in an appropriate system of coordinates, the epigraph of a Lipschitz continuous function ψ defined on a ball $B' \subset \mathbb{R}^{n-1}$. For every i sufficiently large, x_i and y_i belong to U. Hence,

$$d_\Omega(x_i, y_i) \leq d_{\Omega \cap U}(x_i, y_i) \leq 3(1 + |\nabla\psi|_{L^\infty})|x_i - y_i|.$$

This proves that $\lim_{i \to +\infty} \frac{d_\Omega(x_i, y_i)}{|x_i - y_i|} \leq 3(1 + |\nabla\psi|_{L^\infty})$: a contradiction. We can thus conclude that δ_Ω is finite. □

In the spirit of [4, Lemma 6] and [6, Lemma 7.4], we have the following lemma.

Lemma 5.12. *Let $r \in \mathbb{N}$, $\alpha \in (0,1)$ and $\rho > 0$. There exists $\epsilon \in (0,1)$ such that if $\psi \in C^{r+1,\alpha}(\overline{Q'_\rho})$ satisfies $||\nabla\psi||_{C^{r,\alpha}} \leq \epsilon$, then for every $f \in C^{r,\alpha}(\overline{U})$, there exists $X \in C^{r+1,\alpha}(\overline{U}, \mathbb{R}^n)$ which satisfies*

$$\operatorname{div} X = f \text{ in } U \text{ and } X = 0 \text{ on } \{(x', \psi(x')) : x' \in Q'_\rho\}.$$

Moreover, there exists $C = C(r, \alpha, \rho, n) > 0$ such that

$$||X||_{C^{r+1,\alpha}(\overline{U})} \leq C||f||_{C^{r,\alpha}(\overline{U})}.$$

Proof. Without loss of generality, one can assume that $\rho = 1$. For $x = (x', x_n) \in \overline{Q_1}$, we define the function $\widetilde{f} := f \circ \Phi$. Then $\widetilde{f} \in C^{r,\alpha}(\overline{Q_1})$ and

$$||\widetilde{f}||_{C^{r,\alpha}(\overline{Q_1})} \leq C||f||_{C^{r,\alpha}(\overline{U})}, \tag{5.6}$$

for some $C = C(r, \alpha, n)$. Indeed, if $r = 0$, then Remark 5.8 implies

$$\|\widetilde{f}\|_{C^{0,\alpha}} \leq (1 + \delta_{Q_1})||f||_{C^{0,\alpha}}(||D\Phi||^{\alpha}_{C^0} + 1)$$

while if $r \geq 1$, then Proposition 5.7 gives

$$||\widetilde{f}||_{C^{r,\alpha}} \leq C(1 + \delta_{Q_1} + d_{Q_1})||f||_{C^{r,\alpha}}\left(||D\Phi||^{r+\alpha}_{C^{r-1,\alpha}} + 1\right). \tag{5.7}$$

By definition of Φ, one has

$$\left(\frac{\partial \Phi^j}{\partial x_i}\right)_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ \frac{\partial \psi}{\partial x_i} & \text{if } i < n \text{ and } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$||D\Phi||^{r+\alpha}_{C^{r-1,\alpha}} \leq C(1 + ||\nabla\psi||_{C^{r-1,\alpha}(\overline{Q'_1})})^{r+\alpha}. \tag{5.8}$$

Then, the proof of (5.6) is a consequence of (5.7), (5.8) and the fact that by Lemma B.1,

$$||\nabla\psi||_{C^{r-1,\alpha}} \leq C(1 + \delta_{Q_1} + d_{Q_1})||\nabla\psi||_{C^{r,\alpha}} \leq C\epsilon \leq C.$$

Applying Lemma 5.9 to $\widetilde{f} \in C^{r,\alpha}(\overline{Q_1})$, there exists $\widetilde{X} \in C^{r+1,\alpha}(\overline{Q_1}, \mathbb{R}^n)$ such that

$$\operatorname{div} \widetilde{X} = \widetilde{f} \text{ in } Q_1, \qquad \widetilde{X} = 0 \text{ on } Q'_1,$$
$$||\widetilde{X}||_{C^{r+1,\alpha}(\overline{Q_1})} \leq C||\widetilde{f}||_{C^{r,\alpha}(\overline{Q_1})}. \tag{5.9}$$

We now consider $X_0 : U \to \mathbb{R}^n$ defined by $X_0(x) = \widetilde{X}(\Phi^{-1}(x)) = \widetilde{X}(x', x_n - \psi(x'))$ and rewrite $\widetilde{X}$ as $\widetilde{X} = (\widetilde{X}^1, ..., \widetilde{X}^n) = (\widetilde{X}', \widetilde{X}^n)$. Then we have

$$\begin{aligned} \operatorname{div} X_0(x) &= \operatorname{div}(\widetilde{X} \circ \Phi^{-1}(x)) = \sum_{i=1}^n \frac{\partial[\widetilde{X}(\Phi^{-1}(x))]^i}{\partial x_i} \\ &= \sum_{i,j=1}^n \frac{\partial \widetilde{X}^i}{\partial x_j} \circ \Phi^{-1}(x) \frac{\partial(\Phi^{-1})^j}{\partial x_i}(x) \\ &= \sum_{i=1}^n \frac{\partial \widetilde{X}^i}{\partial x_i} \circ \Phi^{-1}(x) \frac{\partial(\Phi^{-1})^i}{\partial x_i}(x) + \sum_{1 \leq i \neq j \leq n} \frac{\partial \widetilde{X}^i}{\partial x_j} \circ \Phi^{-1}(x) \frac{\partial(\Phi^{-1})^j}{\partial x_i}(x). \end{aligned}$$

Using again the definition of Φ, we have

$$\left(\frac{\partial(\Phi^{-1})^j}{\partial x_i}\right)_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ -\frac{\partial\psi}{\partial x_i} & \text{if } i < n \text{ and } j = n, \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$\operatorname{div} X_0 = (\operatorname{div} \widetilde{X}) \circ \Phi^{-1} - \sum_{i=1}^{n-1} \frac{\partial \widetilde{X}^i}{\partial x_n}(\Phi^{-1}(x))\frac{\partial\psi}{\partial x_i}(x').$$

Hence,

$$\operatorname{div} X_0(x) - f(x) = \sum_{i=1}^{n-1} \frac{\partial \widetilde{X}^i}{\partial x_n}(\Phi^{-1}(x))\frac{\partial\psi}{\partial x_i}(x').$$

Using Proposition 5.6, one gets

$$\begin{aligned} ||\operatorname{div} X_0 - f||_{C^{r,\alpha}(\overline{U})} &\leq \sum_{i=1}^{n-1} \left\| \frac{\partial \widetilde{X}^i}{\partial x_n} \circ \Phi^{-1} \frac{\partial\psi}{\partial x_i} \right\|_{C^{r,\alpha}(\overline{U})} \\ &\leq C(1 + \delta_U + d_U)^r \sum_{i=1}^{n-1} \left\| \frac{\partial \widetilde{X}^i}{\partial x_d} \circ \Phi^{-1} \right\|_{C^{r,\alpha}(\overline{U})} \left\| \frac{\partial\psi}{\partial x_i} \right\|_{C^{r,\alpha}(\overline{Q_1'})}. \end{aligned}$$

Hence, according to the assumption $||\nabla\psi||_{C^{r,\alpha}(\mathbb{R}^{n-1})} \leq \epsilon$, one gets

$$||\operatorname{div} X_0 - f||_{C^{r,\alpha}(\overline{U})} \leq C\epsilon \sum_{i=1}^{n-1} \left\| \frac{\partial \widetilde{X}^i}{\partial x_n} \circ \Phi^{-1} \right\|_{C^{r,\alpha}(\overline{U})}. \tag{5.10}$$

From Proposition 5.7, it follows that

$$\left\| \frac{\partial \widetilde{X}^i}{\partial x_n} \circ \Phi^{-1} \right\|_{C^{r,\alpha}(\overline{U})} \leq C(1 + \delta_U + d_U)^{r^2} \left\| \frac{\partial \widetilde{X}^i}{\partial x_n} \right\|_{C^{r,\alpha}(\overline{Q_1})} \left(\|D\Phi^{-1}\|^{r+\alpha}_{C^{r-1,\alpha}(\overline{U})} + 1 \right).$$

Therefore

$$\left\| \frac{\partial \widetilde{X}^i}{\partial x_n} \circ \Phi^{-1} \right\|_{C^{r,\alpha}(\overline{U})} \leq C \left\| \frac{\partial \widetilde{X}^i}{\partial x_n} \right\|_{C^{r,\alpha}(\overline{Q_1})}. \tag{5.11}$$

Combining (5.6),(5.9),(5.10) and (5.11), we get

$$||\operatorname{div} X_0 - f||_{C^{r,\alpha}(\overline{U})} \leq \epsilon\overline{C}||f||_{C^{r,\alpha}(\overline{U})},$$

for some $\overline{C}$ which only depends on r, α and n. Using again Proposition 5.7, we obtain

$$\begin{aligned} &||X_0||_{C^{r+1,\alpha}(\overline{U})} \\ &\leq C(1 + \delta_U + d_U)^{(r+1)^2} ||\widetilde{X}||_{C^{r+1,\alpha}(\overline{Q_1})} \left(\|D\Phi^{-1}\|^{r+1+\alpha}_{C^{r,\alpha}(\overline{U})} + 1 \right) \leq C||\widetilde{X}||_{C^{r+1,\alpha}(\overline{Q_1})}. \end{aligned} \tag{5.12}$$

Combining (5.6),(5.9) and (5.12), we find

$$||X_0||_{C^{r+1,\alpha}(\overline{U})} \leq C||f||_{C^{r,\alpha}(\overline{\Omega})}. \tag{5.13}$$

We now fix $\epsilon \in (0, 1)$ in such a way that $\lambda := \overline{C}\epsilon < 1$. Let us summarize the current state of the proof as follows: we have proved that given $f \in C^{r,\alpha}(\overline{U})$, there exists $X_0 \in C^{r+1,\alpha}(\overline{U}, \mathbb{R}^n)$ such that

$$X_0 = 0 \text{ on } \{(x', \psi(x')) : x' \in Q_1'\}$$

and

$$||\mathrm{div}\, X_0 - f||_{C^{r,\alpha}(\overline{U})} \leq \lambda ||f||_{C^{r,\alpha}(\overline{U})} \quad , \quad ||X_0||_{C^{r+1,\alpha}(\overline{U})} \leq C_0 ||f||_{C^{r,\alpha}(\overline{U})},$$

with $\lambda \in (0,1)$ and $C_0 = C_0(r,n,\alpha)$. We now construct by induction a sequence $(X_i)_{i\in\mathbb{N}} \subset C^{r+1,\alpha}(\overline{U},\mathbb{R}^n)$ such that for every $i \in \mathbb{N}$,

$$\left\| \mathrm{div}\, X_i - (f - \mathrm{div} \sum_{j=0}^{i-1} X_j) \right\|_{C^{r,\alpha}(\overline{U})} \leq \lambda \left\| f - \mathrm{div} \sum_{j=0}^{i-1} X_j \right\|_{C^{r,\alpha}(\overline{U})}, \tag{5.14}$$

$$X_i = 0 \text{ on } \{(x', \psi(x')) : x' \in Q'_1\}, \tag{5.15}$$

$$\|X_i\|_{C^{r+1,\alpha}(\overline{U})} \leq C_0 \left\| f - \mathrm{div} \sum_{j=0}^{i-1} X_j \right\|_{C^{r,\alpha}(\overline{U})}. \tag{5.16}$$

The vector field X_0 has been constructed above. Assuming that $X_0, \dots, X_{i-1}$ have been defined for some $i \in \mathbb{N}$, then we define X_i exactly as we have done for X_0 except that we replace f by $f - \mathrm{div} \sum_{j=0}^{i-1} X_j$. Then X_i satisfies the three properties above. This completes the proof of the existence of the sequence $(X_i)_{i\in\mathbb{N}}$.

We deduce from (5.14) that

$$\left\| f - \mathrm{div} \sum_{j=0}^{i-1} X_j \right\|_{C^{r,\alpha}(\overline{U})} \leq \lambda \left\| f - \mathrm{div} \sum_{j=0}^{i-2} X_j \right\|_{C^{r,\alpha}(\overline{U})} \leq \ldots \leq \lambda^{i-1} \|f\|_{C^{r,\alpha}(\overline{U})}. \tag{5.17}$$

Together with (5.16), this implies that

$$\|X_i\|_{C^{r+1,\alpha}(\overline{U})} \leq C_0 \lambda^{i-1} \|f\|_{C^{r,\alpha}(\overline{U})}.$$

It follows that the sum $\sum_{i\in\mathbb{N}} X_i$ converges in the space $C^{r+1,\alpha}(\overline{U})$ to some vector field X such that $X = 0$ on $\{(x', \psi(x')) : x' \in Q'_1\}$ and $\|X\|_{C^{r+1,\alpha}(\overline{U})} \leq C \|f\|_{C^{r,\alpha}(\overline{U})}$. Moreover, by (5.17), one has

$$\mathrm{div}\, X = f.$$

This completes the proof of Lemma 5.12. □

Next, we remove the smallness condition on ψ. Let us consider $\psi \in C^{r+1,\alpha}(\overline{Q'_\rho})$ and

$$U_\delta = \{(x', x_n) \in Q'_\delta \times \mathbb{R} : \psi(x') < x_n < \psi(x') + \rho\}.$$

Lemma 5.13. *With the above notation, there exists $\delta > 0$ which depends only on r, α, n, ρ and $\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q'_\rho})}$ with the following property: Given any $f \in C^{r,\alpha}(\overline{U_\delta})$, there is some $X \in C^{r+1,\alpha}(\overline{U_\delta})$ satisfying*

$$\mathrm{div}\, X = f \ \text{ in } U_\delta, \qquad X = 0 \ \text{ on } \{(x', \psi(x')) : x' \in Q'_\delta\}$$

and

$$||X||_{C^{r+1,\alpha}(\overline{U_\delta})} \leq C ||f||_{C^{r,\alpha}(\overline{U_\delta})},$$

where $C > 0$ depends on r, α, n, ρ and $\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q'_\rho})}$.

Proof. We take again $\rho = 1$. Given $\delta > 0$ (which will be subject to subsequent restrictions) and $f \in C^{r,\alpha}(\overline{U_\delta})$, let us define for every $x = (x', x_n) \in \overline{U_1}$, $\psi_\delta(x') = \psi(\delta x')$ and $f_\delta(x', x_n) = f(\delta x', x_n)$. Then $\psi_\delta \in C^{r+1,\alpha}(\overline{Q'_1})$ and

$$\|\nabla\psi_\delta\|_{C^{r,\alpha}(\overline{Q'_1})} = \delta\|(\nabla\psi)(\delta\cdot)\|_{C^{r,\alpha}(\overline{Q'_1})}$$

so that by Proposition 5.7,

$$\|\nabla\psi_\delta\|_{C^{r,\alpha}(\overline{Q'_1})} \leq C\delta\|\nabla\psi\|_{C^{r,\alpha}((\overline{Q'_1}))}(\delta^{r+\alpha} + 1) \leq C\delta\|\nabla\psi\|_{C^{r,\alpha}((\overline{Q'_1}))}.$$

Similarly,

$$||f_\delta||_{C^{r,\alpha}(\overline{U_1})} \leq C||f||_{C^{r,\alpha}(\overline{U_\delta})}.$$

There exists $\delta_0 = \delta_0(r, \alpha, d, \|\nabla\psi\|_{C^{r,\alpha}((\overline{Q'_1}))})$ such that for every $0 < \delta < \delta_0$, one has $||\nabla\psi_\delta||_{C^{r,\alpha}(\overline{Q'_1})} < \epsilon$ where ϵ is given by Lemma 5.12.

Applying Lemma 5.12 to $\psi_\delta, f_\delta, U_1$, we get a vector field $X_\delta \in C^{r+1,\alpha}(\overline{U}_1)$ satisfying:

$$\operatorname{div} X_\delta = f_\delta \text{ in } U_1 \qquad , \qquad X_\delta = 0 \text{ on } \{(x', \psi_\delta(x')) : x' \in Q'_1\}$$

and

$$||X_\delta||_{C^{r+1,\alpha}(\overline{U_1})} \leq C||f_\delta||_{C^{r,\alpha}(\overline{U_1})}.$$

We now set for every $(x', x_d) \in U_\delta$,

$$X(x', x_n) = \left(\delta X'_\delta\left(\frac{x'}{\delta}, x_n\right), X^n_\delta\left(\frac{x'}{\delta}, x_n\right)\right).$$

Then

$$\operatorname{div} X = f \text{ in } U_\delta, \qquad X = 0 \text{ on } \{(x', \psi(x')) : x' \in Q'_\delta\}$$

and

$$||X||_{C^{r+1,\alpha}(\overline{U_\delta})} \leq C||f||_{C^{r,\alpha}(\overline{U_\delta})},$$

where $C = C(r, \alpha, d, \delta, \rho)$. The proof of Lemma 5.13 is complete. □

Remark 5.14. The vector field X_δ constructed in the proof of Lemma 5.13 also satisfies the following estimates: for every $s \in \{0, \ldots, r\}$,

$$\|X_\delta\|_{C^{s+1,\alpha}(U_\delta)} \leq C\|f\|_{C^{s,\alpha}(U_\delta)}$$

where C only depends on r, α, n, ρ and $\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q'_\rho})}$.

This easily follows from Proposition 5.6 and Proposition 5.7, exactly as in the proof of Lemmata 5.9, 5.12 and 5.13.

5.4.3 Conclusion of Theorem 5.4

We now present the

Proof of Theorem 5.4 . Since Ω is $C^{r+1,\alpha}$, for every $x \in \partial\Omega$, there exists an open neighborhood $W \subset \mathbb{R}^n$ of x and a positive number $\rho > 0$ such that

- $W \cap \Omega$ is isometric to $\{(y', y_n) \in Q'_\rho \times \mathbb{R} : \psi(y') < y_n < \psi(y') + \rho\}$,
- $W \cap \partial\Omega$ is isometric to $\{(y', y_n) \in Q'_\rho \times \mathbb{R} : \psi(y') = y_n\}$

where $\psi \in C^{r,\alpha}(\overline{Q'_\rho})$.

There exists a parameter $\delta > 0$ depending on r, α, n, ρ, ψ such that Lemma 5.13 gives a solution to the divergence equation on the set $\{(y', y_n) \in Q'_\delta \times \mathbb{R} : \psi(y') < y_n < \psi(y') + \rho\}$. We deduce therefrom that there exists an open neighborhood V of x contained in W and a vector field $X \in C^{r+1,\alpha}(V \cap \Omega; \mathbb{R}^n)$ such that

$$\operatorname{div} X = f \text{ in } V \cap \Omega, \qquad X = 0 \text{ on } V \cap \partial\Omega$$

and

$$\|X\|_{C^{r+1,\alpha}(\overline{V\cap\Omega})} \leq C\|f\|_{C^{r,\alpha}(\overline{V\cap\Omega})}.$$

Here, the constant C depends on r, α, n and Ω.

By compactness of $\partial\Omega$, one can find a covering of $\partial\Omega$ by such open sets V_i, $i = 1, \ldots, k$. In particular, for every $i \in \{1, \ldots, k\}$, there exists $X_i \in C^{r+1,\alpha}(V_i; \mathbb{R}^n)$ such that

$$\operatorname{div} X_i = f \text{ in } \Omega \cap V_i, \qquad X_i = 0 \text{ on } \partial\Omega \cap V_i.$$

and

$$||X_i||_{C^{r+1,\alpha}(\overline{V_i})} \leq C||f||_{C^{r,\alpha}(\overline{V_i})}. \tag{5.18}$$

Let also $V_0 \subset \Omega$ such that V_0 is a $C^{r+2,\alpha}$ domain and $\Omega \subset \bigcup_{i=0}^k V_i$. We then solve $\operatorname{div} X_0 = f$ in $V_0 \subset \Omega$, for example $X_0 = \nabla(\Delta^{-1} f)$, where Δ^{-1} is defined with the zero Dirichlet boundary condition on ∂V_0. Moreover, there exists a constant $C = C(r, \alpha, V_0) > 0$ such that

$$||X_0||_{C^{r+1,\alpha}(\overline{V_0})} \leq C||f||_{C^{r,\alpha}(\overline{V_0})}. \tag{5.19}$$

To the covering $(V_i)_{0\leq i\leq k}$ of Ω, we associate a partition of unity $(\theta_i)_{0\leq i\leq k}$ such that

$$\sum_{i=0}^k \theta_i = 1 \text{ in } \overline{\Omega}, \text{ and } \theta_i \in C_c^\infty(V_i) \text{ for } i = 0, 1, ..., k.$$

We set

$$Sf = \sum_{i=0}^k \theta_i X_i. \tag{5.20}$$

Then

$$\operatorname{div}\, Sf = f + \sum_{i=0}^k \nabla\theta_i \cdot X_i. \tag{5.21}$$

This implies

$$\|f - Sf\|_{C^{r+1,\alpha}(\overline{\Omega})} \leq \sum_{i=0}^k \|\nabla\theta_i \cdot X_i\|_{C^{r+1,\alpha}(\overline{\Omega})} \leq C \sum_{i=0}^k \|X_i\|_{C^{r+1,\alpha}(\overline{\Omega})},$$

where C depends on r, α and Ω. We deduce from (5.18) and (5.19) that

$$\|f - Sf\|_{C^{r+1,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{r,\alpha}(\overline{\Omega})}.$$

Finally, since supp $(\nabla\theta_0) \Subset \Omega$ and $X_i = 0$ on supp $\theta_i \cap \partial\Omega$, one has $f - Sf = 0$ on $\partial\Omega$. The proof is complete. □

Remark 5.15. From the above proof, let us remark that, for any $f \in C^{r,\alpha}(\overline{\Omega})$, we can rewrite f in the following form

$$f = -\operatorname{div} Sf + \sum_{i=0}^k \nabla\theta_i \cdot X_i,$$

where $Sf = \sum_{i=0}^k \theta_i X_i \in C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ and $\sum_{i=0}^k \nabla\theta_i \cdot X_i \in C_z^{r+1,\alpha}(\overline{\Omega})$.

According to Remark 5.14, the local solutions X_i arising in the proof of Theorem 5.4 have the following additional property:

Remark 5.16. For every $s \in 0, \ldots, r$,

$$\|X_i\|_{C^{s+1,\alpha}(\overline{\Omega \cap V_i})} \leq C\|f\|_{C^{s,\alpha}(\overline{\Omega})}$$

for some $C = C(r, \alpha, n, \Omega)$.

5.5 Proof of $\ker(T^*) = \{0\}$ and the conclusion of Theorem 5.1.

We finally turn to the proof of Theorem 5.1. We want to apply Lemma 5.3 to the linear continuous map $T = \operatorname{div}$ from the set

$$C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) = \{X \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) : X = 0 \text{ on } \partial\Omega\}$$

into

$$C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}) = \left\{ f \in C^{r,\alpha}(\overline{\Omega}) : \int_\Omega f = 0 \right\}.$$

We define the linear map K by $K(f) = -f + Sf$ where S is given by Theorem 5.4. By construction, $Kf = \sum_{i=0}^k \nabla\theta_i \cdot X_i$, with $X_i \in C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ and $\theta_i \in C_c^\infty(\mathbb{R}^n)$ for all $i \in \{0, ..., k\}$. Since the embedding $C^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n) \subset C^{r,\alpha}(\overline{\Omega}, \mathbb{R}^n)$ is compact, we deduce that the operator K is compact from $C_{\mathcal{H}}^{r,\alpha}$ into $C_{\mathcal{H}}^{r,\alpha}$.

It remains to prove that $\ker T^* = \{0\}$. This is the content of the following

Theorem 5.17. *The dual operator $T^* : (C_{\mathcal{H}}^{r,\alpha})^* \to (C_z^{r+1,\alpha})^*$ has a trivial kernel:*

$$\ker(T^*) = \{0\}.$$

Proof of Theorem 5.17. Let $v \in \ker T^*$. By definition of the adjoint operator, this means

$$\langle v, TX \rangle_{(C_{\mathcal{H}}^{r,\alpha})^*, C_{\mathcal{H}}^{r,\alpha}} = 0, \qquad \forall X \in C_z^{r+1,\alpha}(\overline{\Omega}, \mathbb{R}^n). \tag{5.22}$$

We claim that

$$\langle v, \varphi \rangle_{(C_{\mathcal{H}}^{r,\alpha})^*, C_{\mathcal{H}}^{r,\alpha}} = 0, \quad \forall \varphi \in C_c^\infty(\Omega) \text{ with } \int_\Omega \varphi = 0. \tag{5.23}$$

Indeed, let us define the distribution $\overline{v}$ by

$$\overline{v} : \varphi \in C_c^\infty(\Omega) \mapsto \langle v, \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \rangle_{(C_{\mathcal{H}}^{r,\alpha})^*, C_{\mathcal{H}}^{r,\alpha}}.$$

For every $X \in C_c^\infty(\Omega)$, $\int_\Omega TX = \int_\Omega \operatorname{div} X = 0$ and thus using (5.22), one has

$$\langle v, TX - \frac{1}{|\Omega|} \int_\Omega TX \rangle_{(C_{\mathcal{H}}^{r,\alpha})^*, C_{\mathcal{H}}^{r,\alpha}} = \langle v, TX \rangle_{(C_{\mathcal{H}}^{r,\alpha})^*, C_{\mathcal{H}}^{r,\alpha}} = 0.$$

It follows that the distribution $\overline{v}$ vanishes on the set $\{\operatorname{div} X : X \in C_c^\infty(\Omega)\}$, which implies that $\overline{v}$ is a constant distribution. Hence, there exists $c \in \mathbb{R}$ such that for every $\varphi \in C_c^\infty(\Omega)$,

$$\langle v, \varphi - \frac{1}{|\Omega|} \int_\Omega \varphi \rangle_{(C_{\mathcal{H}}^{r,\alpha})^*, C_{\mathcal{H}}^{r,\alpha}} = c \int_\Omega \varphi \tag{5.24}$$

and (5.23) follows.

We next prove that for every $v \in (C^{r,\alpha}_{\mathcal{H}})^*$, there exists $C > 0$ such that for every $f \in C^{r,\alpha}_{\mathcal{H}}$,

$$\langle v, f\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, C^{r,\alpha}_{\mathcal{H}}} \leq C\|f\|_{C^{0,\alpha}}.$$

Indeed, given $f \in C^{r,\alpha}_{\mathcal{H}}(\overline{\Omega})$, we rely on Remark 5.15 to get the decomposition

$$f = \operatorname{div} X + \sum_{i=0}^{k} \nabla\theta_i \cdot X_i, \tag{5.25}$$

where $X \in C^{r+1,\alpha}_z(\overline{\Omega}, \mathbb{R}^n)$ and the sum $\sum_{i=0}^k \nabla\theta_i \cdot X_i$, that we denote by g, also belongs to $C^{r+1,\alpha}_z$, see (5.21).

Using equations (5.22) and (5.25), we obtain

$$\langle v, f\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, (C^{r,\alpha}_{\mathcal{H}})} = \langle v, g\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, (C^{r,\alpha}_{\mathcal{H}})} \leq \|v\|_{(C^{r,\alpha}_{\mathcal{H}})^*}\|g\|_{C^{r,\alpha}_{\mathcal{H}}}.$$

Since $g = \sum_{i=0}^k \nabla\theta_i \cdot X_i$, one has

$$\|g\|_{C^{r,\alpha}_{\mathcal{H}}} = \|g\|_{C^{r,\alpha}(\overline{\Omega})} \leq C\sum_{i=0}^{k}\|X_i\|_{C^{r,\alpha}(\overline{V_i\cap\Omega})}.$$

The open set V_i is the domain of the local solution X_i introduced in the proof of Theorem 5.4. We now rely on Remark 5.16 with $s = r-1$ to estimate:

$$\|X_i\|_{C^{r,\alpha}(\overline{V_i\cap\Omega})} \leq C\|f\|_{C^{r-1,\alpha}(\overline{\Omega})}.$$

Hence, we get

$$\|g\|_{C^{r,\alpha}_{\mathcal{H}}} \leq C\|f\|_{C^{r-1,\alpha}(\overline{\Omega})},$$

which implies

$$\langle v, f\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, C^{r,\alpha}_{\mathcal{H}}} \leq C\|v\|_{(C^{r,\alpha}_{\mathcal{H}})^*}\|f\|_{C^{r-1,\alpha}(\overline{\Omega})}.$$

We can repeat the above argument taking into account this new estimate that we apply to g instead of f (observe that $g = f - \operatorname{div} X$ also belongs to $C^{r,\alpha}_{\mathcal{H}}$):

$$\langle v, f\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, C^{r,\alpha}_{\mathcal{H}}} = \langle v, g\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, C^{r,\alpha}_{\mathcal{H}}} \leq C\|v\|_{(C^{r,\alpha}_{\mathcal{H}})^*}\|g\|_{C^{r-1,\alpha}(\overline{\Omega})}.$$

Using Remark 5.14 again with $s = r-2$, one has

$$\|g\|_{C^{r-1,\alpha}(\overline{\Omega})} \leq C\sum_{i=0}^{k}\|X_i\|_{C^{r-1,\alpha}(\overline{V_i\cap\Omega})} \leq C\|f\|_{C^{r-2,\alpha}(\overline{\Omega})}.$$

We deduce therefrom that

$$\langle v, f\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, C^{r,\alpha}_{\mathcal{H}}} \leq C\|f\|_{C^{r-2,\alpha}(\overline{\Omega})}.$$

Iterating this calculation, we finally obtain

$$\langle v, f\rangle_{(C^{r,\alpha}_{\mathcal{H}})^*, C^{r,\alpha}_{\mathcal{H}}} \leq C\|f\|_{C^{0,\alpha}(\overline{\Omega})} \tag{5.26}$$

where C depends on r, α, Ω and v.

Applying Lemma 5.18 below to g, there exists a sequence $(g_m)_m$, with $g_m \in C^\infty_c(\Omega)$, $\int_\Omega g_m = 0$ such that

$$g_m \to g \text{ in } C^{0,\alpha}(\overline{\Omega}).$$

Now we use (5.23) and (5.26) to obtain that

$$\begin{aligned}\langle v,f\rangle_{(C^{r,\alpha}_{\mathcal H})^*,C^{r,\alpha}_{\mathcal H}}&=\langle v,g\rangle_{(C^{r,\alpha}_{\mathcal H})^*,C^{r,\alpha}_{\mathcal H}}\\&=\lim_{m\to+\infty}\langle v,g-g_m\rangle_{(C^{r,\alpha}_{\mathcal H})^*,C^{r,\alpha}_{\mathcal H}}\\&\le\|v\|_{(C^{r,\alpha}_{\mathcal H})^*}\limsup_{m\to+\infty}||g-g_m||_{C^{0,\alpha}}=0.\end{aligned}$$

One deduces that $\langle v,f\rangle_{(C^{r,\alpha}_{\mathcal H})^*,C^{r,\alpha}_{\mathcal H}}=0$ for all $f\in C^{r,\alpha}_{\mathcal H}$ and thus $v=0$. This completes the proof of the lemma. □

We finish this section by Lemma 5.18:

Lemma 5.18. *Let $r\in\mathbb{N}^*$, $f\in C^{r,\alpha}(\overline{\Omega})$ with $f=0$ on $\partial\Omega$ and $\int_\Omega f=0$. Then there exists a sequence $(f_m)_m\subset C^\infty_c(\Omega)$ such that $\int_\Omega f_m=0$ for every $m\in\mathbb{N}$ and*

$$f_m\to f \text{ in } C^{0,\alpha}(\overline{\Omega}).$$

Proof of Lemma 5.18 . We split the proof of Lemma 5.18 into two steps:

Step 1 : Let $g\in C^{r,\alpha}(\overline{\Omega})$ with $g=0$ on $\partial\Omega$ and $\int_\Omega g=0$. Then there exists a sequence $(g_m)_m\subset C^{0,1}_c(\Omega)$ such that $\int_\Omega g_m=0$ for every $m\in\mathbb{N}$ and

$$g_m\to g \text{ in } C^{0,\alpha}(\overline{\Omega}).$$

Step 2 : Let $g\in C^{0,1}_c(\Omega)$ with $\int_\Omega g=0$. Then there exists a sequence $(g_m)_m\subset C^\infty_c(\Omega)$ such that $\int_\Omega g_m=0$ for every $m\in\mathbb{N}$ and

$$g_m\to g \text{ in } C^{0,\alpha}(\overline{\Omega}).$$

We easily get the conclusion from the two above steps.

Proof of Step 1. Let us define the function $\theta_\epsilon:\mathbb{R}\longrightarrow\mathbb{R}$ by:

$$\theta_\epsilon(t)=\begin{cases}0 & \text{if } -\epsilon\le t\le\epsilon,\\ 2t-2\epsilon & \text{if } \epsilon<t<2\epsilon,\\ 2t+2\epsilon & \text{if } -2\epsilon<t<-\epsilon,\\ t & \text{otherwise,}\end{cases}$$

and $g_\epsilon=\theta_\epsilon\circ g$. Then $g_\epsilon\in C^{0,1}_c(\Omega)$. We next prove that g_ϵ converges to g in $C^{0,\alpha}(\overline{\Omega})$. First of all, for every $x\in\overline{\Omega}$

$$\begin{aligned}|g_\epsilon(x)-g(x)|&\le(|g_\epsilon(x)|+|g(x)|)\mathbb{1}_{\{|g|\le2\epsilon\}}\\&\le2|g(x)|\mathbb{1}_{\{|g|\le2\epsilon\}}\le4\epsilon.\end{aligned}\tag{5.27}$$

Then g_ϵ converges to g uniformly in $C^0(\overline{\Omega})$.
We now estimate the Holder-semi norm

$$\begin{aligned}&\sup_{x\ne y}\frac{|(g-g_\epsilon)(x)-(g-g_\epsilon)(y)|}{|x-y|^\alpha}\\&\quad\le\sup_{|x-y|\ge\epsilon}\frac{|(g-g_\epsilon)(x)-(g-g_\epsilon)(y)|}{|x-y|^\alpha}+\sup_{0<|x-y|<\epsilon}\frac{|(g-g_\epsilon)(x)-(g-g_\epsilon)(y)|}{|x-y|^\alpha}.\end{aligned}\tag{5.28}$$

We estimate the first term of (5.28) using (5.27):

$$\begin{aligned}\sup_{|x-y|\ge\epsilon}\frac{|(g-g_\epsilon)(x)-(g-g_\epsilon)(y)|}{|x-y|^\alpha}&\le\sup_{|x-y|\ge\epsilon}\frac{|(g-g_\epsilon)(x)|+|(g-g_\epsilon)(y)|}{|x-y|^\alpha}\\&\le\sup_{|x-y|\ge\epsilon}\frac{8\epsilon}{|x-y|^\alpha}\le8\epsilon^{1-\alpha}.\end{aligned}\tag{5.29}$$

As for the second term of (5.28), one has

$$\begin{aligned}\sup_{0<|x-y|<\epsilon}\frac{|(g-g_\epsilon)(x)-(g-g_\epsilon)(y)|}{|x-y|^\alpha} &\leq \sup_{0<|x-y|<\epsilon}\frac{|g(x)-g(y)|+|g_\epsilon(x)-g_\epsilon(y)|}{|x-y|^\alpha}\\ &\leq \sup_{0<|x-y|<\epsilon}\frac{||Dg||_{L^\infty}|x-y|+||Dg_\epsilon||_{L^\infty}|x-y|}{|x-y|^\alpha}.\end{aligned}$$

Using that $||\theta'_\epsilon||_{L^\infty}\leq 2$ in $\mathbb{R}$, which implies that $\|Dg_\varepsilon\|_{L^\infty}\leq 2\|Dg\|_{L^\infty}$, one gets

$$\begin{aligned}\sup_{0<|x-y|<\epsilon}\frac{|(g-g_\epsilon)(x)-(g-g_\epsilon)(y)|}{|x-y|^\alpha} &\leq \sup_{0<|x-y|<\epsilon}\frac{3||Dg||_{L^\infty}|x-y|}{|x-y|^\alpha}\\ &\leq \sup_{0<|x-y|<\epsilon} 3||Dg||_{L^\infty}|x-y|^{1-\alpha}\\ &\leq 3||Dg||_{L^\infty}\epsilon^{1-\alpha}. \end{aligned}\tag{5.30}$$

Combining (5.29) and (5.30), we get

$$[g-g_\epsilon]_{C^{0,\alpha}(\overline{\Omega})}\leq 8\epsilon^{1-\alpha}+3||Dg||_{L^\infty}\epsilon^{1-\alpha}\leq C\varepsilon^{1-\alpha}.$$

By uniform convergence,

$$\lim_{\varepsilon}\int_\Omega g_\varepsilon=\int_\Omega g=0.$$

Let $\theta\in C_c^\infty(\Omega)$ such that $\int\theta=1$. Then the family $(\widetilde{g}_\varepsilon)_{\varepsilon>0}$ defined by

$$\widetilde{g}_\varepsilon:=g_\epsilon-\left(\frac{1}{|\Omega|}\int_\Omega g_\epsilon\right)\theta$$

satisfies all the required properties.

Proof of Step 2. In this step, we start with a function $g\in C_c^{0,1}(\Omega)$ satisfying $\int_\Omega g=0$. We still denote by g the extension by 0 of g on the whole $\mathbb{R}^n$.

Let $\zeta\in C_c^\infty(\mathbb{R}^n)$ such that $\text{supp}(\zeta)\subset\overline{B(0,1)}, \zeta\geq 0$ and

$$\int_{\mathbb{R}^n}\zeta(x)dx=1.$$

The desired g_ϵ is then given by

$$g_\epsilon=\zeta_\epsilon * g,$$

where

$$\zeta_\epsilon(x)=\frac{1}{\epsilon^n}\zeta(\frac{x}{\epsilon}).$$

Then for every ϵ small enough, $g_\epsilon\in C_c^\infty(\Omega)$ and by the Fubini theorem, $\int_\Omega g_\epsilon=0$.

Moreover, for any $\beta\in(0,1)$,

$$|g-g_\epsilon|_{C^0}\leq C\epsilon^\beta||g||_{C^{0,\beta}}.$$

Indeed, by definition of the convolution, one has

$$g_\epsilon(x)-g(x)=\int_{\mathbb{R}^n}\frac{1}{\epsilon^n}\zeta(\frac{y}{\epsilon})[g(x-y)-g(x)]dy=\int_{\mathbb{R}^n}\zeta(z)[g(x-\epsilon z)-g(x)]dz.$$

Then using that supp $\zeta\subset B(0,1)$ and $|\zeta|^\beta\leq 1$ on $B(0,1)$, one gets

$$||g-g_\epsilon||_{C^0}\leq\epsilon^\beta\|g\|_{C^{0,\beta}}\int_{\mathbb{R}^n}\zeta(z)|z|^\beta dz\leq\epsilon^\beta\|g\|_{C^{0,\beta}}.\tag{5.31}$$

By writing for every $x, y \in \Omega$, $x \neq y$,

$$\begin{aligned}
&\frac{|(g_\epsilon - g)(x) - (g_\epsilon - g)(y)|}{|x-y|^\alpha} \\
&= \left(\frac{|(g_\epsilon - g)(x) - (g_\epsilon - g)(y)|}{|x-y|}\right)^\alpha |(g_\epsilon - g)(x) - (g_\epsilon - g)(y)|^{1-\alpha} \\
&\leq 2^{1-\alpha}[g_\epsilon - g]^\alpha_{C^{0,1}} \|g_\epsilon - g\|^{1-\alpha}_{C^0},
\end{aligned}$$

one gets

$$[g - g_\epsilon]_{C^{0,\alpha}} \leq 2^{1-\alpha}[g_\epsilon - g]^\alpha_{C^{0,1}} \|g_\epsilon - g\|^{1-\alpha}_{C^0}$$

and thus

$$||g_\epsilon - g||_{C^{0,\alpha}} = \|g_\epsilon - g\|_{C^0} + [g_\epsilon - g]_{C^{0,\alpha}} \leq \|g_\epsilon - g\|_{C^0} + 2^{1-\alpha}||g - g_\epsilon||^\alpha_{C^{0,1}}||g - g_\epsilon||^{1-\alpha}_{C^0}. \quad (5.32)$$

Moreover, we have that

$$||g - g_\epsilon||_{C^{0,1}} \leq ||g_\epsilon||_{C^{0,1}} + ||g||_{C^{0,1}} \leq 2||g||_{C^{0,1}}. \quad (5.33)$$

Combining (5.31)-(5.33), we obtain the conclusion. □

Chapter 6

On the Poincaré Lemma on domains

Abstract

This chapter is based on a work (see [7]) in collaboration with my adviser Pierre Bousquet. We are interested in the Poincaré lemma on a bounded domain, under a Dirichlet boundary condition

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is a differential form of degree k and $\boldsymbol{d}$ is the exterior derivative operator. We prove the existence of a solution under a sharp regularity assumption on the domain Ω and in Hölder spaces. This result generalizes Theorem 5.1 to the differential form equation. Finally, our results cover the whole scale of Sobolev spaces.

6.1 Statements of the main results

For the convenience of the reader, let us recall some notations and state again the main result which is introduced in Chapter 4. We formulate the differential form equation in terms of functions spaces.

Let Ω be a domain (namely a connected open set) in $\mathbb{R}^n$, with $n \geq 2$. Given $k \in \{1, \dots, n\}$, $r \in \mathbb{N}$ and $\alpha \in (0,1)$, we define the set $C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ of those k differential forms with $C^{r,\alpha}$ coefficients in $\overline{\Omega}$. Given $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, we look for a $(k-1)$ form X in the Banach space

$$C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}) := \left\{ X \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}) : X = 0 \text{ on } \partial\Omega \right\}$$

such that

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Observe that any $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$ satisfies $\boldsymbol{d}(\boldsymbol{d}X) = 0$ (when $r = 0$, this condition must be understood in the sense of distributions). Moreover, the boundary condition $X = 0$ on $\partial\Omega$ implies that $\nu \wedge \boldsymbol{d}X = 0$ on $\partial\Omega$, see [13, Theorem 3.23]. In this identity, the outer unit normal ν to Ω is identified to a 1 form: we set $\nu = \nu_1 dx_1 + \cdots + \nu_n dx_n$ if $\nu_1, \dots, \nu_n$ are the coordinates of ν in the standard basis of $\mathbb{R}^n$. We deduce therefrom two necessary conditions on a k form $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ for the existence of a solution X to (6.1): f should be a closed form, and satisfy the boundary condition $f \wedge \nu = 0$ on $\partial\Omega$. In case when Ω is topologically nontrivial, one must add a further requirement. Let us introduce the set $\mathcal{H}_T^k(\Omega)$ of the Dirichlet *harmonic fields* of order k, defined as

$$\mathcal{H}_T^k(\Omega) = \{h \in L^2(\Omega, \Lambda^k) : \boldsymbol{\delta} h = 0, \boldsymbol{d} h_z = 0\}. \tag{6.2}$$

Here, h_z means the extension of h by zero outside Ω. The two conditions $\boldsymbol{\delta} h = 0, \boldsymbol{d} h_z = 0$ must be understood in the sense of distributions:

$$\forall \eta \in C_c^\infty(\Omega, \Lambda^{k-1}), \int_\Omega \langle h, \boldsymbol{d}\eta \rangle \, dx = 0,$$

$$\forall \theta \in C_c^\infty(\mathbb{R}^n, \Lambda^{k+1}), \int_{\mathbb{R}^n} \langle h, \boldsymbol{\delta}\theta \rangle \, dx = 0.$$

The latter condition is a weak formulation of the fact that $\boldsymbol{d}h = 0$ on Ω and $h \wedge \nu = 0$ on $\partial\Omega$. The set $\mathcal{H}_T^k(\Omega)$ is closely related to the topology of Ω, see e.g. [40, Chapter 11].

Now, for every $h \in \mathcal{H}_T^k$, one has $\boldsymbol{\delta} h = 0$ and thus, for every $X \in C_c^\infty(\Omega, \Lambda^{k-1})$,

$$\int_\Omega \langle \boldsymbol{d}X, h \rangle \, dx = 0.$$

This remains true by density for $X \in W_0^{1,2}(\Omega, \Lambda^{k-1})$, in particular for $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$. This yields a third necessary condition on the right hand side f of (6.1) to ensure the existence of a solution X: f should be orthogonal to any element of $\mathcal{H}_T^k(\Omega)$. We are thus led to introduce the Banach space

$$C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k) := \Big\{ f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k) : \boldsymbol{d}f = 0 \text{ in } \overline{\Omega}, \nu \wedge f = 0 \text{ on } \partial\Omega,$$
$$\int_\Omega \langle f, h \rangle = 0, \forall h \in \mathcal{H}_T^k(\Omega) \Big\}.$$

We can now formulate (6.1) as follows: is it true that any $f \in C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ is the differential of some $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$? Actually, we expect that the solution X can be chosen continuously and linearly with respect to f. This is not obvious since such a solution X, when it exists, is not unique. In other words, does there exist a right inverse to $\boldsymbol{d} : C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}) \to C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$?

This is indeed our main result, under the assumption that Ω has the same regularity as X itself.

Theorem 6.1. *Let $r \in \mathbb{N}$, $k \in \{1, \dots, n\}$ and $\alpha \in (0,1)$. Let Ω be a bounded $C^{r+1,\alpha}$ domain in $\mathbb{R}^n$. Let $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ be such that*

$$\begin{cases} \boldsymbol{d}f = 0 & \text{in } \overline{\Omega}, \\ \nu \wedge f = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

and for every $h \in \mathcal{H}_T^k(\Omega)$,

$$\int_\Omega \langle f, h \rangle \, dx = 0. \tag{6.4}$$

Then there exists $X \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$ such that

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

Furthermore, the correspondence $f \mapsto X$ can be chosen linear and there exists $C = C(r, \alpha, \Omega) > 0$ such that

$$||X||_{C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})} \leq C ||f||_{C^{r,\alpha}(\overline{\Omega}, \Lambda^k)}.$$

When $r = 0$, the condition $\boldsymbol{d}f = 0$ has to be understood in the sense of distributions in $\mathcal{D}'(\Omega)$ (the assumption $f \wedge \nu = 0$ has a classical pointwise meaning since f is continuous on the closure of Ω).

Remark 6.2. We can allow more general boundary conditions. Namely, if one replaces the homogeneous Dirichlet boundary condition $X = 0$ on $\partial\Omega$ by $X = X_0$ for some $X_0 \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, then the corresponding statement holds true under the necessary and sufficient conditions: $\boldsymbol{d}f = 0$ in Ω, $\nu \wedge f = \nu \wedge \boldsymbol{d}X_0$ on $\partial\Omega$, and for every $h \in \mathcal{H}_T^k(\Omega)$,

$$\int_\Omega \langle f, h\rangle - \int_{\partial\Omega} \langle \nu \wedge X_0, h\rangle = 0.$$

Indeed, for $X_0 \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, the differential form $f_1 = f - \boldsymbol{d}X_0 \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ satisfies

$$\boldsymbol{d}f_1 = 0 \text{ in } \Omega \quad , \quad \nu \wedge f_1 = 0 \text{ on } \partial\Omega$$

and

$$\int_\Omega \langle f_1, h\rangle \, dx = 0, \qquad \forall h \in \mathcal{H}_T^k(\Omega).$$

By Theorem 6.1, there exists $X_1 \in C_z^{r+1}(\overline{\Omega}, \Lambda^{k-1})$ such that $\boldsymbol{d}X_1 = f_1$ in Ω. Thus, $X = X_0 + X_1$ satisfies all the required properties.

The proof of Theorem 6.1 shares some features with the argument used in the previous Chapter. For that we look for a right inverse to the exterior derivative operator $\boldsymbol{d} : C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}) \to C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$. It is a consequence of Lemma 5.3 and the following Theorem

Theorem 6.3. *For every integer $r \geq 0$, there is a bounded operator*

$$S : C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k) \to C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$$

such that for every $f \in C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$

$$f - \boldsymbol{d}(Sf) \in C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$$

and

$$||f - \boldsymbol{d}(Sf)||_{C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})} \leq C||f||_{C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)}.$$

The proof of Theorem 6.3 will follow the same strategy as the proof of Theorem 5.4; that is, we localize the problem on small balls and perform the construction of the bounded operator S by the gluing process. Compared with the divergence equation, the assumptions on data entail several difficulties, namely:

- *The boundary condition.* In the case of n forms, the boundary condition of the data f is ignored. But that boundary condition will be taken into account in the general case. It is one of the difficulties we need to handle when we proceed the matching of the boundary condition of the data in the cube and those in the neighborhood of the boundary of Ω. It is well-known that if ϕ is a local chart, the pullback of the local chart commutes with the exterior derivative operator; namely, for any k differential form f, one has $\phi^*(df) = d(\phi^* f)$. Here $\phi^*(f)$ is the pullback of f by ϕ. However, the normal vector of the boundary is not necessarily conserved by the pullback of a diffeomorphism.

- *The closeness.* The closeness of the data is the second issue we meet. It follows from the fact that, the closeness of the n differential form f is automatically satisfied, ($\int_\Omega f = 0$) while in general, $df = 0$ has the classical (pointwise) sense. This fact involve new technical difficulties for equation (6.1) in the cube, as well as for the proof of $\ker(\boldsymbol{d})^* = \{0\}$, where

$$(\boldsymbol{d})^* : (C_{\mathcal{H}}^{r,\alpha})^* \to (C_z^{r+1,\alpha})^*$$

  is the adjoint operator of $\boldsymbol{d}$ and $(C_z^{r+1,\alpha})^*, (C_{\mathcal{H}}^{r,\alpha})^*$ are the dual spaces of $C_z^{r+1,\alpha}, C_{\mathcal{H}}^{r,\alpha}$ (respectively).

Remark 6.4. 1. When Ω is merely Lipschitz, Mitrea, Mitrea and Taylor [40, Chapter 11] prove that $\mathcal{H}_T^k(\Omega)$ coincides with the space $\mathcal{H}_\wedge^{k,2}(\Omega)$ of those maps $h \in C^1(\Omega, \Lambda^k)$ which satisfy $\boldsymbol{d}h = 0$, $\boldsymbol{\delta}h = 0$, $\nu \wedge h = 0$ and such that a certain *trace* of h (in an appropriate sense) belongs to $L^2(\partial\Omega)$.

2. When Ω is Lipschitz, there exists $q_0 > 2$ such that $\mathcal{H}_\wedge^{k,2}(\Omega)$ is contained in $L^q(\Omega, \Lambda^k)$ for every $q \in [1, q_0)$, see [40, Theorem 11.2].

3. Finally, the set of harmonic fields with vanishing tangential components has finite dimension, see [40, Theorem 11.1].

One can formulate a twin result of Theorem 6.1 in the scale of Sobolev spaces. Given $p \in (1, \infty)$ and a Lipschitz bounded domain Ω, it is possible to find a solution $X \in W_0^{1,p}(\Omega, \Lambda^{k+1})$ to (6.1) when $f \in L^p(\Omega, \Lambda^k)$ satisfies the necessary condition $\boldsymbol{d}f_z = 0$ in $\mathcal{D}'(\mathbb{R}^n)$ (remember that f_z is the extension of f by 0 outside Ω) and an orthogonality assumption with respect to $\mathcal{H}_T^k(\Omega)$. A more regular solution exists provided that the set Ω is regular enough, in the following precise way:

Theorem 6.5. *Let $r \in \mathbb{N}$, $k \in \{1, \ldots, n\}$ and $p \in (1, \infty)$. Let Ω be a bounded $C^{r,1}$ domain in $\mathbb{R}^n$. Assume also that $\mathcal{H}_T^k(\Omega) \subset L^{p'}(\Omega, \Lambda^k)$. Let $f \in W^{r,p}(\Omega, \Lambda^k)$ be such that*

$$\begin{cases} \boldsymbol{d}f = 0 & \text{in } \Omega, \\ \nu \wedge f = 0 & \text{on } \partial\Omega, \end{cases}$$

and for every $h \in \mathcal{H}_T^k(\Omega)$,

$$\int_\Omega \langle f, h\rangle \, dx = 0.$$

Then there exists $X \in W^{r+1,p}(\Omega, \Lambda^{k-1})$ such that

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \Omega, \\ X = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, the correspondence $f \mapsto X$ can be chosen linear and there exists $C = C(r, p, \Omega) > 0$ such that

$$||X||_{W^{r+1,p}(\Omega,\Lambda^{k-1})} \leq C||f||_{W^{r,p}(\Omega,\Lambda^k)}.$$

The assumption that $\mathcal{H}_T^k(\Omega) \subset L^{p'}(\Omega, \Lambda^k)$ has been introduced to guarantee that the quantity $\int_\Omega \langle f, h\rangle \, dx$ is well-defined for every $h \in \mathcal{H}_T^k$. This requirement is automatically satisfied when $p \geq 2$.

In the case when $\mathcal{H}_T^k(\Omega) \not\subset L^{p'}(\Omega, \Lambda^k)$ (which may happen when $r = 0$ and p is close to 1), one can rely on the following weaker form of Theorem 6.5:

Remark 6.6. Let $k \in \{1, \ldots, n\}$ and $p \in (1, \infty)$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Assume also that $\mathcal{H}_T^k(\Omega) \subset L^p(\Omega, \Lambda^k)$. Let $f \in L^p(\Omega, \Lambda^k)$ be such that $\boldsymbol{d}f_z = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. Then there exists $X \in W_0^{1,p}(\Omega, \Lambda^{k-1})$ and $h \in \mathcal{H}_T^k(\Omega)$ such that $\boldsymbol{d}X + h = f$ in Ω. Furthermore, the correspondence $f \mapsto (X, h)$ can be chosen linear and there exists $C = C(p, \Omega) > 0$ such that

$$||X||_{W^{1,p}(\Omega,\Lambda^{k-1})} + \|h\|_{L^2(\Omega,\Lambda^k)} \leq C||f||_{L^p(\Omega,\Lambda^k)}.$$

The space $\mathcal{H}_T^k(\Omega)$ is finite dimensional, see Remark 6.4. Hence, all the norms are equivalent on this set. In the last estimate, one can thus replace the L^2 norm of h by $\|h\|_{L^p(\Omega,\Lambda^k)}$.

Remark 6.7. In the limiting case $p = 1$, there exist closed forms $f \in L^1(B^n, \Lambda^k)$, where B^n is the unit ball in $\mathbb{R}^n$, which cannot be written as $f = \boldsymbol{d}X$, for any $X \in W^{1,1}(B^n, \Lambda^{k-1})$, see [4] for the case $k = n$ and [14] for $k \in \{1, \ldots, n-1\}$. The same assertion holds true for $p = +\infty$.

6.1.1 Comparison with previous results

The proofs of our results rely on a version of the Poincaré lemma for *smooth* open sets. Under such an assumption, one can exploit the elliptic regularity theory to construct solutions to (6.1) by using the *Hodge-Morrey decomposition.* This approach is detailed by Csató, Dacorogna and Kneuss in [13], where Theorem 6.1 is stated for Ω of class $C^{r+3,\alpha}$ and Theorem 6.5 holds true provided that Ω is C^{r+3}, see [13, Theorem 8.16]. According to [41, Theorem 7.7.8 (ii)], it seems enough to assume that Ω is in $C^{r+2,\alpha}$ to write the Hodge-Morrey decomposition in the scale of Hölder spaces. In the specific setting of Sobolev spaces, see also [47, Section 3.3] (where Ω is assumed to be smooth).

Remark 6.8. Let us observe that the set $\mathcal{H}^k_T(\Omega)$ as defined in (6.2) slightly differs from the one introduced in [13] and [41], where harmonic fields are required to be in $W^{1,2}$ instead of L^2. In [1, Theorem 4.10], Axelsson and McIntosh establish that these two sets coincide when Ω is C^2.

In the case when Ω is Lipschitz, Bogovskii [48] has introduced an alternative strategy to construct a solution to the divergence equation (4.1) when the right hand side f is in $L^p(\Omega)$. This approach has been subsequently extended to produce a solution to the Poincaré lemma (6.1) in the whole scale of Besov spaces $B^{s,p}_q$ (which contains all the Sobolev and Hölder spaces), see in particular [39] and [12]. Typically, this construction requires that the right hand side f is in the closure of k forms with smooth *compactly supported* coefficients. In order to emphasize the consequences of this fact, let us state a version of [39, Theorem 1.2] in the setting of standard Sobolev spaces $W^{r,p}(\Omega)$, when $r \geq 1$ and $\mathcal{H}^k_T(\Omega) = \{0\}$: there exists a solution X to (6.1) if $f \in W^{r,p}(\Omega, \Lambda^k)$ is closed and satisfies the following condition

$$\forall \alpha = (\alpha_1, \ldots, \alpha_n) \text{ with } |\alpha| \leq r-1, \qquad \mathrm{Tr}\ D^\alpha f|_{\partial\Omega} = 0. \tag{6.6}$$

We observe that when $r = 1$, this amounts to $f = 0$ on $\partial\Omega$, which is more restrictive than the condition $f \wedge \nu = 0$ of our Theorem 6.5. When $r \geq 2$, the discrepancy with our own assumptions increases, since (6.6) involves vanishing conditions on the *derivatives* of f. We should mention however that the solution obtained in [39, Theorem 1.2] is in the closure of smooth compactly supported $k+1$ forms, a property which considerably differs from the mere Dirichlet condition $X = 0$ on $\partial\Omega$. In other words, the Bogovskii approach generally requires additional assumptions on the right hand side f which are not necessary to solve the Poincaré lemma under the sole requirement that the solution X vanishes on the boundary.

In the specific case $r = 0$, one can rely on two properties that fail to be true when $r \geq 1$: first, $C^\infty_c(\Omega, \Lambda^k)$ is dense in $L^p(\Omega, \Lambda^k)$ and moreover, the set $W^{1,p}_z(\Omega, \Lambda^{k+1})$ of Sobolev forms vanishing on the boundary coincides with the closure $W^{1,p}_0(\Omega, \Lambda^{k+1})$ of $C^\infty_c(\Omega, \Lambda^{k+1})$ in $W^{1,p}(\Omega, \Lambda^{k+1})$. In such a situation, the results in [39] cover our Theorem 6.5, when $\mathcal{H}^k_T(\Omega) = \{0\}$.

The Bogovskii construction as extended in [12] can be applied to the framework of Hölder spaces, but still in the case when one considers the closure of compactly supported forms, which, once again, differs from the standard Hölder spaces that we consider. For instance, even in the case $r = 0$, a k form with $C^{r+1,\alpha}(\overline{\Omega})$ coefficients which vanish on the boundary, cannot be approximated in general by a sequence of smooth compactly supported forms in $C^{r+1,\alpha}$ (such an approximation would imply that the derivatives vanish as well).

Plan of Chapter 6: In the next section, we recall some definitions and preliminaries differential forms. The proof of Theorem 6.1 will begin in the third part, where we prove Theorem 6.3. Then we complete the proof of Theorem 6.1 by proving that $\ker(T^*) =$

$\{0\}$. In Section 6.3.5, we restrict our attention to the Sobolev setting. Finally, for the convenience of the reader, we have gathered in the appendix some technical tools.

6.2 Preliminaries

In this section, we recall some definitions and properties of differential forms.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$. A differential k form f on Ω will be denoted by

$$f = \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1 ... i_k} dx^{i_1} \wedge ... \wedge dx^{i_k},$$

where $f_{i_1...i_k} : \Omega \to \mathbb{R}$, for every $1 \le i_1 < ... < i_k \le n$. It is sometimes convenient to use the alternative notation

$$f = \sum_{I \in \mathcal{J}_{k,n}} f_I dx^I,$$

where $\mathcal{J}_{k,n} = \{I = (i_1, ..., i_k) \in \mathbb{N}^k, 1 \le i_1 < ... < i_k \le n\}$, and for $I = (i_1, \dots, i_k) \in \mathcal{J}_{k,n}$, $f_I = f_{i_1...i_k}$, $dx^I = dx^{i_1} \wedge ... \wedge x^{i_k}$. We often write $\mathcal{J}_k$ instead of $\mathcal{J}_{k,n}$ when the dimension n is obvious.

Given two k forms f, g, we define the function:

$$\langle f, g \rangle = \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1...i_k} g_{i_1...i_k}.$$

The Hodge star-operator is the linear operator $*$ mapping k forms to $n-k$ forms for every $k \in \{0, \dots, n\}$ and defined by

$$f \wedge g = \langle *f, g \rangle dx^1 \wedge ... \wedge dx^n, \tag{6.7}$$

for every $n-k$ form g.
We will use that for a k form f, one has (see [13, Theorem 2.10])

$$*\,(*f) = (-1)^{k(n-k)} f. \tag{6.8}$$

When each coefficient f_I of a k form f belongs to a certain $L^p(\Omega)$, we write $f \in L^p(\Omega, \Lambda^k)$ and we introduce the norm

$$\| f \|_{L^p(\Omega,\Lambda^k)} = \sum_{I \in \mathcal{J}_{k,n}} \| f_I \|_{L^p(\Omega)}.$$

Similar definitions are generalized to any functions spaces.

Given a k form f with C^1 or Sobolev coefficients, its exterior derivative $\boldsymbol{d}f$, is the $(k+1)$ form defined by

$$\boldsymbol{d}f = \sum_{1 \le i_1 < ... < i_{k+1} \le n} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \frac{\partial f_{i_1 ... \widehat{i_\gamma} ... i_k}}{\partial x_{i_\gamma}} \right) dx^{i_1} \wedge ... \wedge dx^{i_{k+1}} \text{ for } k < n,$$

and $\boldsymbol{d}f = 0$ for $k = n$. Here, the notation $i_1...\widehat{i_\gamma}...i_k$ means $i_1...i_{\gamma-1}i_{\gamma+1}...i_k$.

The codifferential $\boldsymbol{\delta}f$ is the $(k-1)$ form given by

$$\boldsymbol{\delta}f = \sum_{1 \le i_1 < ... < i_{k-1} \le n} \left(\sum_{j=1}^{n} \epsilon^j_{i_1...i_{k-1}} \frac{\partial f_{i_1...i_{k-1},j}}{\partial x_j} \right) dx^{i_1} \wedge ... \wedge dx^{i_{k-1}} \text{ for } k > 0, \tag{6.9}$$

where $(i_1...i_{k-1}, j)$ denotes the index rearranged increasingly and

$$\epsilon^j_{i_1\ldots i_{k-1}} = \begin{cases} 0 & \text{if } j \in \{i_1, ..., i_{k-1}\}, \\ (-1)^{\gamma-1} & \text{if } i_{\gamma-1} < j < i_\gamma. \end{cases}$$

If $k = 1$, this formula reads $\boldsymbol{\delta} f = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}$. When $k = 0$, we set $\boldsymbol{\delta} f = 0$. These two operators are related one to the other by the identity [13, Definition 3.2 (ii)]:

$$\boldsymbol{\delta} f = (-1)^{n(k-1)} * (\boldsymbol{d}(*f)). \tag{6.10}$$

We denote by $\nu = (\nu_1, \ldots, \nu_n)$ the outward unit normal to Ω that we identify with the 1 form $\nu = \sum_{i=1}^n \nu_i dx^i$. When $k < n$, we also consider on $\partial\Omega$ the tangential part $\nu \wedge f$ of f which is the $(k+1)$ form

$$\nu \wedge f = \sum_{1\le i_1<\ldots<i_{k+1}\le n} \left(\sum_{\gamma=1}^{k+1} (-1)^{\gamma-1} \nu_{i_\gamma} f_{i_1\ldots\widehat{i_\gamma}\ldots i_{k+1}} \right) dx^{i_1} \wedge \ldots \wedge dx^{i_{k+1}}.$$

If $k = n$, then $\nu \wedge f = 0$. Finally, when $k > 0$, the normal part $\nu \lrcorner f$ on $\partial\Omega$ is the following $(k-1)$ form:

$$\nu \lrcorner f = \sum_{1\le i_1<\ldots<i_{k-1}\le n} \left(\sum_{j=1}^{n} \epsilon^j_{i_1\ldots i_{k-1}} \nu_j f_{i_1\ldots i_{k-1},j} \right) dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$$

When $k = 0$, we set $\nu \lrcorner f = 0$. The normal and tangential components are related by the following identity (see e.g. [13, Proposition 3.20 (i)]):

$$f = \nu \wedge (\nu \lrcorner f) + \nu \lrcorner (\nu \wedge f). \tag{6.11}$$

We will also use the fact that $\nu \wedge f = 0$ on $\partial\Omega$ if and only if $i^*(f) = 0$, where $i : \partial\Omega \to \mathbb{R}^n$ is the inclusion map, see [13, Remark 3.22].

We conclude this section with the integration by parts formula for which we require that Ω is at least Lipschitz continuous (see e.g. [13, Theorem 3.28]):

Proposition 6.9. *Let $k \in \{1, \ldots, n\}$, $f \in C^1(\overline{\Omega}, \Lambda^{k-1})$ and $g \in C^1(\overline{\Omega}, \Lambda^k)$. Then*

$$\int_\Omega \langle \boldsymbol{d} f, g\rangle + \int_\Omega \langle f, \boldsymbol{\delta} g\rangle = \int_{\partial\Omega} \langle \nu \wedge f, g\rangle = \int_{\partial\Omega} \langle f, \nu \lrcorner g\rangle.$$

By density, the formula remains true when f and g belong to Sobolev spaces.

Remark 6.10. By definition of the exterior derivative, for any $f \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^k)$, one has

$$\begin{aligned} ||\boldsymbol{d} f||_{C^{r,\alpha}(\overline{\Omega},\Lambda^{k+1})} &= \sum_{1\le i_1<\ldots<i_{k+1}\le n} \left(||(\boldsymbol{d} f)_{i_1\ldots i_{k+1}}||_{C^r} + \max_{\beta\in\mathbb{N}^n, |\beta|=r} [D^\beta (\boldsymbol{d} f)_{i_1\ldots i_{k+1}}]_{C^{0,\alpha}} \right) \\ &\le C \sum_{1\le i_1<\ldots<i_k\le n} \left(||f_{i_1\ldots i_k}||_{C^{r+1}} + \max_{\beta\in\mathbb{N}^n, |\beta|=r+1} [D^\beta f_{i_1\ldots i_k}]_{C^{0,\alpha}} \right) \\ &\le C||f||_{C^{r+1,\alpha}(\overline{\Omega},\Lambda^k)}. \end{aligned}$$

6.3 The proof of Theorem 6.1

6.3.1 Solution of the Poincaré lemma on a cube

We follow the same strategy as in the divergence problem. Firstly, we solve the Poincaré lemma when Ω is a cube and the boundary condition is required on only one side of the cube.

In the following, for every $\rho > 0$, we denote by Q_ρ the cube $(0,\rho)^n$ while Q'_ρ is the lower side of Q_ρ, namely $Q'_\rho = (0,\rho)^{n-1} \times \{0\}$. We will often identify Q'_ρ with $(0,\rho)^{n-1}$.

Lemma 6.11. *Let $r \in \mathbb{N}, k \in \{1,\dots,n\}$, $\alpha \in (0,1), \rho > 0$, and $f \in C^{r,\alpha}(\overline{Q_\rho}, \Lambda^k)$ such that*

$$\begin{cases} \boldsymbol{d}f = 0 & \text{in } \overline{Q_\rho}, \\ dx_n \wedge f = 0 & \text{on } Q'_\rho. \end{cases} \tag{6.12}$$

Then there exists $X \in C^{r+1,\alpha}(\overline{Q}_\rho, \Lambda^{k-1})$ such that

$$\begin{cases} \boldsymbol{d}X = f & \text{in } \overline{Q_\rho}, \\ X = 0 & \text{on } Q'_\rho. \end{cases}$$

Furthermore, there exists $C = C(r,\alpha,\rho,n) > 0$ such that

$$||X||_{C^{r+1,\alpha}(\overline{Q_\rho},\Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}(\overline{Q_\rho},\Lambda^k)}. \tag{6.13}$$

Remark 6.12. (i) In the case $k = n$, conditions (6.12) are automatically satisfied for every $f \in C^{r,\alpha}(\overline{Q_\rho}, \Lambda^n)$.

(ii) If $r \geq 1$, the condition $\boldsymbol{d}f = 0$ has a classical (pointwise) sense. If $r = 0$, it is understood in the sense of distributions: for every $\varphi \in C_c^\infty(Q_\rho, \Lambda^k)$,

$$\int_{Q_\rho} \langle f, \delta\varphi \rangle \, dx = 0.$$

Proof of Lemma 6.11. We rely on the existence of a solution to $\boldsymbol{d}X = f$ on the cube Q_ρ (without boundary condition) which is established in the Appendix C. We next modify X on Q'_ρ in order to satisfy the boundary condition $X = 0$ there. This strategy is essentially the same as the one presented in [13, sections 8.3 and 8.4] where Q_ρ is replaced by the upper half space (or a smooth domain). We explain here how to extend this construction to domains with corners like the cube Q_ρ.

Without loss of generality, by a dilation argument, we can assume that $\rho = 1$. Applying Proposition C.6 to the closed form $f \in C^{r,\alpha}(\overline{Q_1}, \Lambda^k)$, there exists $\overline{X} \in C^{r+1,\alpha}(\overline{Q_1}, \Lambda^{k-1})$ such that $\boldsymbol{d}\overline{X} = f$ in $\overline{Q_1}$ and

$$||\overline{X}||_{C^{r+1,\alpha}} \leq C||f||_{C^{r,\alpha}}, \tag{6.14}$$

for some $C = C(r,\alpha,n) > 0$.

By assumption, on Q'_1,

$$dx_n \wedge \boldsymbol{d}\overline{X} = dx_n \wedge f = 0.$$

Equivalently, $i^*(\boldsymbol{d}\overline{X}) = 0$, where i is the inclusion map $x' \in \mathbb{R}^{n-1} \mapsto (x',0) \in \mathbb{R}^n$. Hence, the differential form $X' = i^*(\overline{X})$ satisfies $\boldsymbol{d}X' = 0$ on Q'_1. Observe that $X' \in C^{r+1,\alpha}(\overline{Q'_1}, \Lambda^{k-1})$ and

$$||X'||_{C^{r+1,\alpha}(\overline{Q'_1},\Lambda^{k-1})} \leq ||\overline{X}||_{C^{r+1,\alpha}(\overline{Q_1},\Lambda^{k-1})}. \tag{6.15}$$

When $k=1$, $\overline{X}$ is a function and X' is simply the restriction of $\overline{X}$ to $\overline{Q_1'}$. The fact that $\boldsymbol{d}X'=0$ implies that X' is a constant c. Observe that

$$|c|\leq \|\overline{X}\|_{C^0}\leq ||\overline{X}||_{C^{r+1,\alpha}}.$$

Setting $X=\overline{X}-c$, one has $\boldsymbol{d}X=f$ on Q_1, $X=0$ on Q_1' and

$$\|X\|_{C^{r+1,\alpha}}\leq C||f||_{C^{r,\alpha}}.$$

When $2\leq k\leq n$, we apply again Proposition C.6 to X' on Q_1': there exists $Y'\in C^{r+2,\alpha}(\overline{Q_1'},\Lambda^{k-2})$ such that $\boldsymbol{d}Y'=X'$ on $\overline{Q_1'}$ and

$$||Y'||_{C^{r+2,\alpha}}\leq C||X'||_{C^{r+1,\alpha}}. \tag{6.16}$$

We now define the $(k-2)$ form $Y=\pi^*(Y')$ where $\pi:(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mapsto x'\in\mathbb{R}^{n-1}$. Then $Y\in C^{r+2,\alpha}(\overline{Q_1},\Lambda^{k-2})$, and since $\pi\circ i=\mathrm{id}_{\mathbb{R}^{n-1}}$, one has $i^*Y=Y'$. Moreover,

$$||Y||_{C^{r+2,\alpha}(\overline{Q_1},\Lambda^{k-2})}\leq ||Y'||_{C^{r+2,\alpha}(\overline{Q_1'},\Lambda^{k-2})}. \tag{6.17}$$

The k form $\widetilde{X}=\overline{X}-\boldsymbol{d}Y$ belongs to $C^{r+1,\alpha}(\overline{Q_1},\Lambda^{k-1})$ and satisfies

$$\boldsymbol{d}\widetilde{X}=\boldsymbol{d}\overline{X}-\boldsymbol{dd}Y=\boldsymbol{d}\overline{X}=f \text{ on } \overline{Q_1}.$$

Moreover, on Q_1',

$$i^*(\widetilde{X})=i^*(\overline{X})-i^*(\boldsymbol{d}Y)=X'-\boldsymbol{d}(i^*Y)=X'-\boldsymbol{d}Y'=0.$$

In other words, $dx_n\wedge\widetilde{X}=0$ on Q_1'. By Lemma C.3 (that we apply with $c=\widetilde{X}$ and $c_i=0$ for every i), there exists $\widetilde{Y}\in C^{r+2,\alpha}(\overline{Q_1},\Lambda^{k-2})$ such that $\boldsymbol{d}\widetilde{Y}$ agrees with $\widetilde{X}$ on Q_1' and

$$||\widetilde{Y}||_{C^{r+2,\alpha}(\overline{Q_1},\Lambda^{k-2})}\leq C||\widetilde{X}||_{C^{r+1,\alpha}(Q_1',\Lambda^{k-1})}. \tag{6.18}$$

Set

$$X=\widetilde{X}-\boldsymbol{d}\widetilde{Y} \text{ in } \overline{Q_1}.$$

Then

$$\boldsymbol{d}X=\boldsymbol{d}\widetilde{X}-\boldsymbol{dd}\widetilde{Y}=f \text{ in } \overline{Q_1} \quad \text{and } X=0 \text{ on } Q_1'.$$

Inequality (6.13) is a direct consequence of the construction of X and inequalities (6.14)-(6.18). ☐

We next present a Sobolev version of the above lemma. The main difference in the proof is that the trace of a map $u\in W^{1,p}(Q_\rho)$ on Q_ρ' is not in the Sobolev space $W^{1,p}(Q_\rho')$ any more. This is in strong contrast with the Hölder case where the restriction of a Hölder continuous function is still Hölder continuous with the same exponents.

Lemma 6.13. *Let $r\in\mathbb{N},k\in\{1,\dots,n\}$, $p\in(1,\infty),\rho>0$, and $f\in W^{r,p}(Q_\rho,\Lambda^k)$ such that*

$$\begin{cases}\boldsymbol{d}f=0 & \text{in } Q_\rho,\\ dx_n\wedge f=0 & \text{on } Q_\rho'.\end{cases} \tag{6.19}$$

Then there exists $X\in W^{r+1,p}(Q_\rho,\Lambda^{k-1})$ such that

$$\begin{cases}\boldsymbol{d}X=f & \text{in } Q_\rho,\\ X=0 & \text{on } Q_\rho'.\end{cases}$$

Furthermore, there exists $C=C(r,p,\rho,n)>0$ such that

$$||X||_{W^{r+1,p}(Q_\rho,\Lambda^{k-1})}\leq C||f||_{W^{r,p}(Q_\rho,\Lambda^k)}.$$

Proof. We can assume that $\rho = 1$. Let us first consider the case $r = 0$. When f has L^p coefficients, one cannot define the trace of f as a function on Q'_ρ and one cannot apply the standard Poincaré lemma to $i^*(f)$. In fact, the conditions (6.19) have to be understood in the distributional sense:

$$\boldsymbol{d}(f_z) = 0 \text{ on } Q'_1 \times (-1,1)$$

where f_z is the extension of f by 0 on $Q'_1 \times (-1,1)$. By Proposition C.7, there exists $\overline{X} \in W^{1,p}(Q'_1 \times (-1,1), \Lambda^{k-1})$ such that $\boldsymbol{d}\overline{X} = f_z$ on $Q'_1 \times (-1,1)$, with the corresponding estimate. In particular, $\boldsymbol{d}\overline{X} = 0$ on $Q'_1 \times (-1,0)$.
When $k = 1$, $\overline{X}$ is a function, which is equal to a constant c on $Q'_1 \times (-1,0)$. Then $X = \overline{X} - c$ satisfies all the desired properties.
When $2 \leq k \leq n$, we rely on Proposition C.7 to get some $\overline{Y} \in W^{2,p}(Q'_1 \times (-1,0), \Lambda^{k-2})$ such that $\boldsymbol{d}\overline{Y} = \overline{X}$ on $Q'_1 \times (-1,0)$ with

$$\|\overline{Y}\|_{W^{2,p}(Q'_1\times(-1,0),\Lambda^{k-2})} \leq C\|\overline{X}\|_{W^{1,p}(Q'_1\times(-1,1),\Lambda^{k-1})}.$$

We extend $\overline{Y}$ to a map in $W^{2,p}(Q'_1 \times (-1,1), \Lambda^{k-2})$ still denoted by $\overline{Y}$. We finally set

$$X = (\overline{X} - \boldsymbol{d}\overline{Y})|_{Q_1}.$$

Then $\boldsymbol{d}X = \boldsymbol{d}\overline{X} = f$ on Q_1. Since $\overline{X} = \boldsymbol{d}\overline{Y}$ on $Q'_1 \times (-1,0)$, their traces coincide on Q'_1 (as $W^{1-\frac{1}{p},p}$ maps) and thus the trace of X on Q'_1 vanishes. Moreover,

$$\begin{aligned}\|X\|_{W^{1,p}(Q_1,\Lambda^{k-1})} &\leq \|\overline{X}\|_{W^{1,p}(Q_1,\Lambda^{k-1})} + \|\boldsymbol{d}\overline{Y}\|_{W^{1,p}(Q_1,\Lambda^{k-1})} \leq C\|\overline{X}\|_{W^{1,p}(Q'_1\times(-1,1),\Lambda^{k-1})}\\ &\leq C'\|f_z\|_{L^p(Q'_1\times(-1,1),\Lambda^k)} = C'\|f\|_{L^p(Q_1,\Lambda^k)}.\end{aligned}$$

The proof is complete in the case $r = 0$.

When $r \geq 1$, the proof is very similar to the proof of the Hölder case except that we rely on Proposition C.7 instead of Proposition C.6 and on Lemma C.4 instead of Lemma C.3. The main difference is that, with the notation used in the proof of Lemma 6.11, one has

$$Y' \in W^{r+2-\frac{1}{p},p}(Q'_1, \Lambda^{k-2}),$$

and we cannot set $Y = \pi^*(Y')$, because such a map would not belong to $W^{r+2,p}(Q'_1, \Lambda^{k-2})$. Instead, we extend each coefficient Y'_I, $I \in \mathcal{J}_{k-2,n-1}$, of Y' as a $W^{r+2,p}$ function Y_I on Q_1 such that

$$\|Y_I\|_{W^{r+2,p}(Q_1)} \leq C\|Y'_I\|_{W^{r+2-\frac{1}{p},p}(Q'_1)}$$

for some $C = C(r,p,n) > 0$. We next define

$$Y = \sum_{I\in\mathcal{J}_{k-2,n-1}} Y_I dx^I.$$

Then $Y \in W^{r+2,p}(Q_1, \Lambda^{k-2})$, $i^*Y = Y'$ and

$$\|Y\|_{W^{r+2,p}(Q_1,\Lambda^{k-2})} \leq C\|Y'\|_{W^{r+2-\frac{1}{p},p}(Q'_1,\Lambda^{k-2})}.$$

The rest of the proof is essentially the same and we omit it. □

6.3.2 Solution of the Poincaré lemma on an epigraph

Given $\rho > 0$, let $\psi \in C^{r+1,\alpha}(\overline{Q'_\rho})$. In this section, we establish the Poincaré lemma on the epigraph of ψ with a boundary condition along the graph of ψ. More precisely, let us define the open set

$$U = \{(x', x_n) \in Q'_\rho \times (0, +\infty) : \psi(x') < x_n < \psi(x') + \rho\}.$$

A normal vector to the graph of $\{(x', \psi(x')) : x' \in Q'_1\}$ at a point $(x', \psi(x'))$ is given by

$$\nu(x', \psi(x')) = (\nu_1, \ldots, \nu_n)(x', \psi(x')) = \left(\frac{\partial \psi}{\partial x_1}(x'), ..., \frac{\partial \psi}{\partial x_{n-1}}(x'), -1\right). \tag{6.20}$$

As usual, we identify the vector ν with the 1 differential form $\sum_{i=1}^{n} \nu_i dx^i$, which belongs to $C^{r,\alpha}(\overline{Q'_\rho}, \Lambda^1)$.
We emphasize that the geometrical quantities d_U and δ_U (is defined as in (5.2) and (5.3)) are finite. More precisely, one has, for every $x, y \in \overline{U}$,

$$d_U(x,y) \leq 3(1 + \|\nabla\psi\|_\infty)|x-y| \qquad , \qquad \delta_U \leq 3(1 + \|\nabla\psi\|_\infty). \tag{6.21}$$

The proof of (6.21) is detailed in the Chapter 5, see Lemma 5.10 and Remark 5.11.

We now proceed with the construction of a solution to the Poincaré lemma on U. In the spirit of [4, Lemma 6], see also [6, Lemma 7.4], we first consider the case when the gradient of ψ is small.

Lemma 6.14. *Let $r \in \mathbb{N}, k \in \{1, \ldots, n\}$, $\alpha \in (0,1)$ and $\rho > 0$. There exists $\epsilon \in (0,1)$ such that if $\psi \in C^{r+1,\alpha}(\overline{Q'_\rho})$ such that $||\nabla\psi||_{C^{r,\alpha}} \leq \epsilon$, then for every $f \in C^{r,\alpha}(\overline{U}, \Lambda^k)$ such that*

$$\begin{cases} \boldsymbol{d}f = 0 & \text{in } U, \\ \nu \wedge f = 0 & \text{on } \{(x', \psi(x')) : x' \in Q'_\rho\}, \end{cases}$$

there exists $X \in C^{r+1,\alpha}(\overline{U}, \Lambda^{k-1})$ which satisfies

$$\boldsymbol{d}X = f \text{ in } U \text{ and } X = 0 \text{ on } \{(x', \psi(x')) : x' \in Q'_\rho\}.$$

Moreover, there exists $C = C(r, \alpha, \rho, n) > 0$ such that

$$||X||_{C^{r+1,\alpha}(\overline{U},\Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}(\overline{U},\Lambda^k)}.$$

Proof. Without loss of generality, one can assume that $\rho = 1$. We introduce the $C^{r+1,\alpha}$ diffeomorphism

$$\Phi : x = (x', x_n) \in \overline{Q_1} \mapsto (x', x_n + \psi(x')) \in \overline{U}.$$

Observe that $U = \Phi(Q_1)$.
We next define the k form $\widetilde{f} := \Phi^*(f)$. Then $\widetilde{f} \in C^{r,\alpha}(\overline{Q_1}, \Lambda^k)$ and

$$||\widetilde{f}||_{C^{r,\alpha}(\overline{Q_1},\Lambda^k)} \leq C||f||_{C^{r,\alpha}(\overline{U},\Lambda^k)}. \tag{6.22}$$

Indeed, writing $f = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$, one has:

$$\begin{aligned} \widetilde{f} &= \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k} \circ \Phi \; d\Phi^{i_1} \wedge \ldots \wedge d\Phi^{i_k} \\ &= \sum_{1 \leq i_1 < \ldots < i_k < n} f_{i_1 \ldots i_k} \circ \Phi \; d\Phi^{i_1} \wedge \ldots \wedge d\Phi^{i_k} \\ &+ \sum_{1 \leq i_1 < \ldots < i_{k-1} < n} f_{i_1 \ldots i_{k-1} n} \circ \Phi \; d\Phi^{i_1} \wedge \ldots \wedge d\Phi^{i_{k-1}} \wedge d\Phi^n. \end{aligned}$$

The first term agrees with $\sum_{1 \leq i_1 < \ldots < i_k < n} f_{i_1 \ldots i_k} \circ \Phi \; dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ while the second term is equal to

$$\sum_{1 \leq i_1 < \ldots < i_{k-1} < n} f_{i_1 \ldots i_{k-1} n} \circ \Phi \; dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} \wedge (dx^n + \sum_{l=1}^{n-1} \frac{\partial \psi}{\partial x_l} dx^l).$$

It thus follows that

$$\widetilde{f} = \sum_{1 \le i_1 < \ldots < i_k \le n} f_{i_1 \ldots i_k} \circ \Phi dx^{i_1} \wedge \ldots \wedge dx^{i_k} + A, \tag{6.23}$$

where for every $(x', x_n) \in Q_1$,

$$A(x', x_n) = \sum_{1 \le i_1 < \ldots < i_{k-1} < n} \sum_{l=1}^{n-1} f_{i_1 \ldots i_{k-1} n} \circ \Phi(x', x_n) \frac{\partial \psi}{\partial x_l}(x') dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}} \wedge dx^l.$$

Note also that when $r \ge 1$,

$$||D\Phi||^{r+\alpha}_{C^{r-1,\alpha}} \le C(1 + ||\nabla \psi||_{C^{r-1,\alpha}(\overline{Q_1'})})^{r+\alpha} \le C',$$

where $C' = C'(r, \alpha, n) > 0$. Here, we have used the assumption $||\nabla \psi||_{C^{r,\alpha}} \le \epsilon \le 1$. When $r = 0$, we have instead

$$||D\Phi||^{\alpha}_{C^{0,\alpha}} \le C,$$

for some $C = C(n, \alpha)$. Hence, (6.22) is now a consequence of (6.23), Proposition 5.6 and Remark 5.8 for $r = 0$ and Proposition 5.7 for $r \ge 1$.

Moreover,

$$\boldsymbol{d}\widetilde{f} = 0 \text{ in } \overline{Q_1}. \tag{6.24}$$

Indeed, when $r \ge 1$,

$$\boldsymbol{d}\widetilde{f} = \boldsymbol{d}(\Phi^*(f)) = \Phi^*(\boldsymbol{d}f) = 0.$$

When $r = 0$, we observe that for every $\theta \in C_c^\infty(Q_1, \Lambda^{k+1})$,

$$\int_{Q_1} \langle \widetilde{f}, \boldsymbol{\delta}\theta \rangle \, dx = (-1)^{nk} \int_{Q_1} \langle \widetilde{f}, *\boldsymbol{d}(*\theta) \rangle \, dx = (-1)^{nk} \int_{Q_1} \boldsymbol{d}(*\theta) \wedge \widetilde{f}.$$

The first equality is a consequence of (6.10) while the second one follows from (6.7). By the change of variables formula, this gives

$$\int_{Q_1} \langle \widetilde{f}, \boldsymbol{\delta}\theta \rangle \, dx = (-1)^{nk} \int_U (\Phi^{-1})^*(\boldsymbol{d}(*\theta) \wedge \widetilde{f}) = (-1)^{nk} \int_U (\Phi^{-1})^*(\boldsymbol{d}(*\theta)) \wedge (\Phi^{-1})^*(\widetilde{f}).$$

Since $(\Phi^{-1})^*(\widetilde{f}) = f$ and $(\Phi^{-1})^*(\boldsymbol{d}(*\theta)) = \boldsymbol{d}((\Phi^{-1})^*(*\theta))$, (6.7) again implies that

$$\int_{Q_1} \langle \widetilde{f}, \boldsymbol{\delta}\theta \rangle \, dx = (-1)^{nk} \int_U \langle f, *\boldsymbol{d}((\Phi^{-1})^*(*\theta)) \rangle \, dx.$$

Next, by (6.8), $*\boldsymbol{d}((\Phi^{-1})^*(*\theta)) = (-1)^{(k+1)(n-k-1)} * \boldsymbol{d}(* * (\Phi^{-1})^*(*\theta))$ so that by (6.10), $*\boldsymbol{d}((\Phi^{-1})^*(*\theta)) = (-1)^{(k+1)(n-k-1)}(-1)^{nk}\boldsymbol{\delta}(*(\Phi^{-1})^*(*\theta))$. Finally, one obtains

$$\int_{Q_1} \langle \widetilde{f}, \boldsymbol{\delta}\theta \rangle \, dx = (-1)^{(k+1)(n-k-1)} \int_U \langle f, \boldsymbol{\delta}\Big(* (\Phi^{-1})^*(*\theta) \Big) \rangle \, dx. \tag{6.25}$$

Since $*(\Phi^{-1})^*(*\theta)$ is compactly supported in U and $\boldsymbol{d}f = 0$ in the sense of distributions on U, we can conclude that $\boldsymbol{d}\widetilde{f} = 0$ in the sense of distributions.

We next establish

$$dx_n \wedge \widetilde{f} = 0 \text{ on } Q_1'. \tag{6.26}$$

As in the calculation leading to (6.23), one has

$$\Phi^*(\nu) = \sum_{i=1}^{n} \nu_i \circ \Phi dx^i + \sum_{l=1}^{n-1} \nu_n \circ \Phi \frac{\partial \psi}{\partial x_l} dx^l.$$

Using (6.20), one gets

$$\Phi^*(\nu)(x',0)=\sum_{i=1}^{n-1}\frac{\partial\psi}{\partial x_i}(x')dx^i-dx^n-\sum_{l=1}^{n-1}\frac{\partial\psi}{\partial x_l}(x')dx^l=-dx^n.$$

Hence,

$$0=\Phi^*(\nu\wedge f)=\Phi^*(\nu)\wedge\Phi^*(f)=-dx^n\wedge\widetilde{f},$$

which proves (6.26).

In view of (6.24) and (6.26), we can apply Lemma 6.11 to $\widetilde{f}\in C^{r,\alpha}(\overline{Q_1},\Lambda^k)$: there exists $\widetilde{X}\in C^{r+1,\alpha}(\overline{Q_1},\Lambda^{k-1})$ such that

$$\boldsymbol{d}\widetilde{X}=\widetilde{f}\text{ in }\overline{Q_1},\qquad \widetilde{X}=0\text{ on }Q_1'$$

and

$$||\widetilde{X}||_{C^{r+1,\alpha}(\overline{Q_1})}\leq C||\widetilde{f}||_{C^{r,\alpha}(\overline{Q_1})}. \tag{6.27}$$

We write $\widetilde{X}=\sum_{I\in\mathcal{J}_{k-1}}\widetilde{X}_I dx^I$ and define[1]

$$X_0=\sum_{I\in\mathcal{J}_{k-1}}(X_0)_I dx^I,\qquad \text{where }(X_0)_I(x)=\widetilde{X}_I(\Phi^{-1}(x)),\quad\forall x\in\overline{U}. \tag{6.28}$$

Since for every $x=(x',x_n)\in\overline{U}$, $\Phi^{-1}(x)=(x',x_n-\psi(x'))$, one has for every $I\in\mathcal{J}_{k-1}$,

$$\frac{\partial(X_0)_I}{\partial x_l}(x)=\begin{cases}\frac{\partial\widetilde{X}_I}{\partial x_l}(x',x_n-\psi(x'))-\frac{\partial\widetilde{X}_I}{\partial x_n}(x',x_n-\psi(x'))\frac{\partial\psi}{\partial x_l}(x') & \text{if } l<n,\\ \frac{\partial\widetilde{X}_I}{\partial x_n}(x',x_n-\psi(x')) & \text{if } l=n.\end{cases}$$

Hence,

$$\begin{aligned}\boldsymbol{d}X_0(x)&=\sum_{I\in\mathcal{J}_{k-1}}\sum_{l=1}^{n}\frac{\partial(X_0)_I}{\partial x_l}(x)dx^l\wedge dx^I\\&=\sum_{I\in\mathcal{J}_{k-1}}\left(\sum_{l=1}^{n}\frac{\partial\widetilde{X}_I}{\partial x_l}(x',x_n-\psi(x'))dx^l\wedge dx^I\right)+B=\boldsymbol{d}\widetilde{X}(\Phi^{-1}(x))+B,\end{aligned}$$

where

$$B=-\sum_{I\in\mathcal{J}_{k-1}}\sum_{l=1}^{n-1}\frac{\partial\widetilde{X}^I}{\partial x_l}(x',x_n-\psi(x'))\frac{\partial\psi}{\partial x_l}(x')dx^l\wedge dx^I.$$

Using that $\boldsymbol{d}\widetilde{X}=\widetilde{f}$ and (6.23), this implies that

$$\boldsymbol{d}X_0=f+B+B', \tag{6.29}$$

where for every $x=(x',x_n)\in U$,

$$B'(x)=\sum_{1\leq i_1<\ldots<i_{k-1}<n}\sum_{l=1}^{n-1}f_{i_1\ldots i_{k-1}n}(x)\frac{\partial\psi}{\partial x_l}(x')dx^{i_1}\wedge\ldots\wedge dx^{i_{k-1}}\wedge dx^l.$$

In view of Proposition 5.6, estimates (6.21) and the assumption $||\nabla\psi||_{C^{r,\alpha}}\leq\epsilon$,

$$\|B'\|_{C^{r,\alpha}(\overline{U})}\leq C\epsilon||f||_{C^{r,\alpha}(\overline{U})},$$

[1] One could be tempted to define $X_0=(\Phi^{-1})^*(\widetilde{X})$. However, such a form would not have $C^{r+1,\alpha}$ coefficients in general, see the paragraph before Lemma 5.3.

for some $C = C(r, \alpha, n) > 0$. One gets a similar estimate for B by relying also on (6.27) and Proposition 5.7 (or Remark 5.8 when $r = 0$). We then deduce from (6.29) that

$$||\boldsymbol{d}X_0 - f||_{C^{r,\alpha}(\overline{U})} \leq C_0 \epsilon ||f||_{C^{r,\alpha}(\overline{U})}, \tag{6.30}$$

for some $C_0 = C_0(r, \alpha, n) > 0$. Using again Proposition 5.7 and the definition of X_0, see (6.28), we obtain

$$||X_0||_{C^{r+1,\alpha}(\overline{U})} \leq C ||\widetilde{X}||_{C^{r+1,\alpha}(\overline{Q_1})}. \tag{6.31}$$

Combining (6.22) and (6.27), one gets

$$||X_0||_{C^{r+1,\alpha}(\overline{U})} \leq C ||f||_{C^{r,\alpha}(\overline{U})} \tag{6.32}$$

for some $C = C(r, \alpha, n) > 0$. Moreover, the definition of X_0 also implies that $X_0 = 0$ on $\{(x', \psi(x')) : x' \in Q_1'\}$.

We now fix $\epsilon \in (0, 1)$ in such a way that $\lambda := C_0 \epsilon < 1$ where C_0 is the constant in (6.30).

Let us summarize the current state of the proof as follows: we have proved that given a closed form $f \in C^{r,\alpha}(\overline{U}, \Lambda^k)$ satisfying $\nu \wedge f = 0$ on $\{(x', \psi(x')) : x' \in Q_1'\}$, there exists $X_0 \in C^{r+1,\alpha}(\overline{U}, \Lambda^{k-1})$ such that

$$X_0 = 0 \text{ on } \{(x', \psi(x')) : x' \in Q_1'\}$$

and

$$||\boldsymbol{d}X_0 - f||_{C^{r,\alpha}(\overline{U})} \leq \lambda ||f||_{C^{r,\alpha}(\overline{U})} \quad , \quad ||X_0||_{C^{r+1,\alpha}(\overline{U})} \leq C ||f||_{C^{r,\alpha}(\overline{U})},$$

with $\lambda \in (0, 1)$ and $C = C(r, \alpha, n) > 0$. We now construct by induction a sequence $(X_i)_{i \in \mathbb{N}} \subset C^{r+1,\alpha}(\overline{U}, \mathbb{R}^n)$ such that for every $i \geq 0$,

$$\left\| \boldsymbol{d}X_i - (f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j) \right\|_{C^{r,\alpha}(\overline{U})} \leq \lambda \left\| f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j \right\|_{C^{r,\alpha}(\overline{U})}, \tag{6.33}$$

$$X_i = 0 \text{ on } \{(x', \psi(x')) : x' \in Q_1'\}, \tag{6.34}$$

$$\|X_i\|_{C^{r+1,\alpha}(\overline{U})} \leq C \left\| f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j \right\|_{C^{r,\alpha}(\overline{U})}. \tag{6.35}$$

The $(k-1)$ form X_0 has been constructed above. Assuming that $X_0, \ldots, X_{i-1}$ have been defined for some $i \geq 1$, then we define X_i exactly as we have done for X_0 except that we replace f by $f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j$. This is possible since

$$\boldsymbol{d}(f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j) = 0 \text{ in } \overline{U} \text{ and } \nu \wedge (f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j) = 0 \text{ on } \{(x', \psi(x')) : x' \in Q_1'\}.$$

The latter condition is a consequence of the fact that each $X_j = 0$ on the graph of ψ, so that $\nu \wedge \boldsymbol{d}X_j = 0$ there.

Then X_i satisfies the three properties aboves. This completes the proof of the existence of the sequence $(X_i)_{i \in \mathbb{N}}$.

We deduce from (6.33) that

$$\left\| f - \boldsymbol{d}\sum_{j=0}^{i} X_j \right\|_{C^{r,\alpha}(\overline{U})} \leq \lambda \left\| f - \boldsymbol{d}\sum_{j=0}^{i-1} X_j \right\|_{C^{r,\alpha}(\overline{U})} \leq \ldots \leq \lambda^{i+1} \|f\|_{C^{r,\alpha}(\overline{U})}. \tag{6.36}$$

Together with (6.35), this implies that

$$\|X_i\|_{C^{r+1,\alpha}(\overline{U})} \leq C\lambda^i \|f\|_{C^{r,\alpha}(\overline{U})}.$$

It follows that the sum $\sum_{i\in\mathbb{N}} X_i$ converges in the Banach space $C^{r+1,\alpha}(\overline{U})$ to some $k-1$ form X such that $X = 0$ on $\{(x',\psi(x')) : x' \in Q_1'\}$ and $\|X\|_{C^{r+1,\alpha}(\overline{U})} \leq C\|f\|_{C^{r,\alpha}(\overline{U})}$. Moreover, by (6.36), one has

$$\boldsymbol{d}X = f.$$

This completes the proof of Lemma 6.14. □

We proceed to remove the smallness condition on $\nabla\psi$, as in [6, Lemma 7.5]. Given $0 < \delta \leq \rho$, we consider $\psi \in C^{r+1,\alpha}(\overline{Q_\rho'})$ and use the notation:

$$U_\delta = \{(x',x_n) \in Q_\delta' \times \mathbb{R} : \psi(x') < x_n < \psi(x') + \rho\}.$$

Lemma 6.15. *Let $r \in \mathbb{N}, k \in \{1,\dots,n\}$, $\alpha \in (0,1), \rho > 0$. Then there exists $\delta = \delta(r,\alpha,\rho,n,\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q_\rho'})}) > 0$ with the following property: for every $f \in C^{r,\alpha}(\overline{U_\delta},\Lambda^k)$ such that*

$$\boldsymbol{d}f = 0 \text{ in } \overline{U_\delta}, \qquad \text{and } \nu \wedge f = 0 \text{ on } \{(x',\psi(x')) : x' \in Q_\delta'\},$$

there exists $X \in C^{r+1,\alpha}(\overline{U_\delta})$ such that

$$\boldsymbol{d}X = f \text{ in } U_\delta, \qquad X = 0 \text{ on } \{(x',\psi(x')) : x' \in Q_\delta'\}$$

and

$$||X||_{C^{r+1,\alpha}(\overline{U_\delta})} \leq C||f||_{C^{r,\alpha}(\overline{U_\delta})},$$

where $C = C(r,\alpha,\rho,n,\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q_\rho'})}) > 0$.

Proof. Without loss of generality, we can assume that $\rho = 1$. Let $\delta \in (0,1)$ such that $\delta\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q_1'})} < \epsilon$, where ϵ is given by Lemma 6.14. We then define $\psi_\delta(x') = \psi(\delta x'), x' \in \overline{Q_1'}$. Then

$$\|\nabla\psi_\delta\|_{C^{r,\alpha}(\overline{Q_1'})} \leq \delta\|\nabla\psi\|_{C^{r,\alpha}(\overline{Q_\delta'})} < \epsilon.$$

We also set $\Psi_\delta(x) = (\delta x', x_n)$, $x = (x',x_n) \in \overline{Q_1'} \times \mathbb{R}$ and

$$W_1 = \{(x',x_n) \in Q_1' \times \mathbb{R} : \psi_\delta(x') < x_n < \psi_\delta(x') + 1\}.$$

Observe that $\Psi_\delta(W_1) = U_\delta$. Let $f \in C^{r,\alpha}(\overline{U_\delta},\Lambda^k)$. We introduce $f_\delta = \Psi_\delta^*(f)$. Then by Remark 5.8 when $r = 0$ and Proposition 5.7 when $r \geq 1$, $f_\delta \in C^{r,\alpha}(\overline{W_1},\Lambda^k)$ and

$$||f_\delta||_{C^{r,\alpha}(\overline{W_1},\Lambda^k)} \leq C||f||_{C^{r,\alpha}(\overline{U_\delta},\Lambda^k)} \tag{6.37}$$

for some $C = C(r,\alpha,n) > 0$. For the normal to $\{(x',\psi(x')) : x' \in Q_\delta'\}$, we choose the 1 form $\nu = \sum_{i=1}^n \nu_i dx^i$ defined by

$$\nu_i(x',\psi(x')) = \frac{\partial\psi}{\partial x_i}(x') \text{ if } i < n, \qquad \nu_n(x',\psi(x')) = -1.$$

Then for every $x' \in Q_1'$,

$$\Psi_\delta^*(\nu)(x',\psi_\delta(x')) = \sum_{i=1}^{n-1} \frac{\partial\psi}{\partial x_i}(\delta x')\delta dx^i - dx^n = \sum_{i=1}^{n-1} \frac{\partial\psi_\delta}{\partial x_i}(x')dx^i - dx^n.$$

Hence, $\nu_\delta := \Psi_\delta^*(\nu)$ defines a normal to the graph of ψ_δ. Moreover, on $\{(x', \psi_\delta(x')) : x' \in Q_1'\}$,

$$\nu_\delta \wedge f_\delta = \Psi_\delta^*(\nu) \wedge \Psi_\delta^*(f) = \Psi_\delta^*(\nu \wedge f) = 0. \tag{6.38}$$

Finally, when $r \geq 1$,

$$\boldsymbol{d}(f_\delta) = \boldsymbol{d}((\Psi_\delta)^*(f)) = \Psi_\delta^*(\boldsymbol{d}f) = 0 \text{ in } \overline{W_1}.$$

When $r = 0$, the above identity can be understood in the sense of distributions.

We can thus apply Lemma 6.14 to the closed form $f_\delta \in C^{r,\alpha}(\overline{W_1})$: there exists $X_\delta \in C^{r+1,\alpha}(\overline{W_1}, \Lambda^{k-1})$ such that

$$\boldsymbol{d}X_\delta = f_\delta \text{ in } \overline{W_1}, \qquad X_\delta = 0 \text{ on } \{(x', \psi_\delta(x')) : x' \in Q_1'\}$$

and

$$||X_\delta||_{C^{r+1,\alpha}(\overline{W_1}, \Lambda^{k-1})} \leq C||f_\delta||_{C^{r,\alpha}(\overline{W_1}, \Lambda^k)}.$$

We then set

$$X = (\Psi_\delta^{-1})^*(X_\delta).$$

Hence,

$$\boldsymbol{d}X = \boldsymbol{d}((\Psi_\delta^{-1})^*(X_\delta)) = (\Psi_\delta^{-1})^*(\boldsymbol{d}X_\delta) = (\Psi_\delta^{-1})^*(f_\delta) = f \text{ in } \overline{U_\delta}$$

and

$$||X||_{C^{r+1,\alpha}(\overline{U_\delta})} \leq C||f||_{C^{r,\alpha}(\overline{U_\delta})},$$

for some $C = C(r, \alpha, \delta, n) > 0$. The boundary condition

$$X = 0 \text{ on } \{(x', \psi(x')) : x \in Q_\delta'\}$$

is satisfied since

$$X_\delta = 0 \text{ on } \{(x', \psi_\delta(x')) : x' \in Q_1'\} = \Psi_\delta^{-1}(\{(x', \psi(x')) : x' \in Q_\delta'\}).$$

The proof of Lemma 6.15 is complete. $\square$

Remark 6.16. Given a **finite** set $A \subset (0, \alpha]$, one can require that the solution X_δ given by Lemma 6.15 satisy the additional estimates: for every $s \in \{0, \ldots, r\}$ and every $\alpha' \in A$, one has

$$\|X_\delta\|_{C^{s+1,\alpha'}(U_\delta)} \leq C\|f\|_{C^{s,\alpha'}(U_\delta)}$$

where $C = C(r, A, \rho, n, \|\nabla\psi\|_{C^{r,\alpha}(\overline{Q_\rho'})}) > 0$.

Indeed, such estimates automatically hold in the setting of all the intermediate results leading to Lemma 6.15, including Lemma C.3 and Proposition C.6 in the appendix, with one exception: in the proof of Lemma 6.14, we have used an approximation scheme which relies on the choice of a parameter ε such that $C_0\varepsilon < 1$, where $C_0 = C_0(r, \alpha, n, \rho) > 0$, see inequality (6.30). By replacing this constant C_0 by a possibly larger constant C_0', we can ensure that (6.30) holds true for every $s \in \{0, \ldots, r\}$ and $\alpha' \in A$. Thus, if we decrease ε in order to have $C_0'\varepsilon < 1$, the approximation scheme of Lemma 6.14 is valid in every $C^{s,\alpha'}$ spaces, for every $s \in \{0, \ldots, r\}$ and $\alpha' \in A$. Finally, the value of δ in the proof of Lemma 6.15 must be modified accordingly, in order to satisfy the condition $\delta\|\nabla\psi\|_{C^{s,\alpha'}} < \varepsilon$ (for this new value of ε), for every $s \in \{0, \ldots, r\}$ and $\alpha' \in A$.

As a consequence of the proofs of the above lemmata, one can also ensure that

Remark 6.17. In the setting of Lemma 6.15, the correspondance $f \mapsto X$ is linear.

In the framework of Sobolev spaces, it is possible to formulate the corresponding versions of Lemma 6.14 and Lemma 6.15. Here, we only write the latter for later reference.

Lemma 6.18. *Let $r \in \mathbb{N}, k \in \{1, \dots, n\}$, $p \in (1, \infty), \rho > 0$. With the above notation, there exists $\delta = \delta(r, p, \rho, n, \|\nabla\psi\|_{W^{r,\infty}(Q'_\rho)}) > 0$ with the following property: for every $f \in W^{r,p}(U_\delta, \Lambda^k)$ such that*

$$\boldsymbol{d}f = 0 \text{ in } U_\delta, \qquad \text{and } \nu \wedge f = 0 \text{ on } \{(x', \psi(x')) : x' \in Q'_\delta\},$$

there exists $X \in W^{r+1,p}(U_\delta)$ such that

$$\boldsymbol{d}X = f \text{ in } U_\delta, \qquad X = 0 \text{ on } \{(x', \psi(x')) : x' \in Q'_\delta\}$$

and

$$||X||_{W^{r+1,p}(U_\delta)} \leq C||f||_{W^{r,p}(U_\delta)},$$

where $C = C(r, \alpha, \rho, n, \|\nabla\psi\|_{W^{r,\infty}(Q'_\rho)}) > 0$.

The proof is essentially the same as in the Hölder case except that we use Lemma 6.13 instead of Lemma 6.11. Moreover, the substitutes of Proposition 5.6 and Proposition 5.7 are given by the two following facts:
For every $f \in W^{r,p}$, $g \in W^{r,\infty}$, the Leibniz rule implies that $fg \in W^{r,p}$ and

$$\|fg\|_{W^{r,p}} \leq C\|f\|_{W^{r,p}}\|g\|_{W^{r,\infty}},$$

where $C = C(r, p, n) > 0$.
For every $f \in W^{r,p}$ and every biLipschitz homeomorphism $\Xi \in W^{r+1,\infty}$ with a Jacobian larger than a constant $c_0 > 0$, the change of variables formula implies that $f \circ \Xi \in W^{r,p}$ and

$$\|f \circ \Xi\|_{W^{r,p}} \leq \begin{cases} Cc_0^{-\frac{1}{p}}\|f\|_{W^{r,p}}(\|D\Xi\|^r_{W^{r-1,\infty}}+1) & \text{if } r \geq 1, \\ c_0^{-\frac{1}{p}}\|f\|_{L^p} & \text{if } r = 0, \end{cases}$$

where $C = C(r, p, n) > 0$. In fact, we use two types of diffeomorphisms: $\Xi(x', x_n) = (x', x_n \pm \psi(x'))$ which has a Jacobian equal to 1, and $\Xi^\pm(x', x_n) = (\delta^{\pm 1}x', x_n)$, which has a Jacobian equal to $\delta^{\pm(n-1)}$.

When $r = 0$, one cannot define the trace of f as a function on the graph $\{(x', \psi(x')) : x' \in Q'_\delta\}$. In that case, we establish (6.26) in Lemma 6.14 or (6.38) in Lemma 6.15 by relying on the distributional formulation of these conditions, more precisely by using the identity (6.25), which holds true for every $\theta \in C^1(\overline{Q_1}, \Lambda^{k+1})$.

The counterparts of Remarks 6.16 and 6.17 remain valid in the Sobolev framework as well.

6.3.3 Approximate solution on a bounded set

In this section, we construct an approximate solution to the Poincaré lemma on a bounded set. More precisely, given two integers $r \in \mathbb{N}$ and $k \in \{1, \dots, n\}$ and an exponent $\alpha \in (0, 1)$, we consider the two following spaces:
the set $C^{r,\alpha}_\nu(\overline{\Omega}, \Lambda^k)$ of those $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ such that

$$\boldsymbol{d}f = 0 \text{ in } \Omega, \qquad \nu \wedge f = 0 \text{ on } \partial\Omega,$$

the set $C^{r+1,\alpha}_z(\overline{\Omega}, \Lambda^{k-1})$ of those $X \in C^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$ such that

$$X = 0 \text{ on } \partial\Omega.$$

Lemma 6.19. *Let $r \in \mathbb{N}$, $k \in \{1, \dots, n\}$. Let Ω be a bounded $C^{r+1,\alpha}$ domain in $\mathbb{R}^n$. Then there exist two continuous linear operators*

$$S : C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) \to C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1}), \qquad K : C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) \to C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^k)$$

such that $\boldsymbol{d}(Sf) + Kf = f$ for every $f \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k)$.

Using the compact embedding $\iota : C^{r+1,\alpha} \hookrightarrow C^{r,\alpha}$, we can see that the operator $\iota \circ K$ is compact from $C_\nu^{r,\alpha}$ into itself. In that sense, Sf is indeed an *approximate solution* for the equation $f = \boldsymbol{d}X$.

Proof. Since Ω is $C^{r+1,\alpha}$, for every $x \in \partial\Omega$, there exists an open neighborhood $W \subset \mathbb{R}^n$ of x, a positive number $\rho > 0$ and a function $\psi \in C^{r+1,\alpha}(\overline{Q'_\rho})$ such that

- $W \cap \Omega$ is isometric to $\{(y', y_n) \in Q'_\rho \times \mathbb{R} : \psi(y') < y_n < \psi(y') + \rho\}$,
- $W \cap \partial\Omega$ is isometric to $\{(y', y_n) \in Q'_\rho \times \mathbb{R} : \psi(y') = y_n\}$.

There exists some $\delta > 0$ depending on r, α, n, ρ, ψ such that Lemma 6.15 gives a solution to the Poincaré lemma on the set $\{(y', y_n) \in Q'_\delta \times \mathbb{R} : \psi(y') < y_n < \psi(y') + \rho\}$, which vanishes on the lower part of the boundary $\{(y', \psi(y')) : y' \in Q'_\delta\}$. We deduce therefrom that there exists an open neighborhood V of x contained in W and $X \in C^{r+1,\alpha}(\overline{V \cap \Omega}; \Lambda^{k-1})$ such that

$$\boldsymbol{d}X = f \text{ in } V \cap \Omega, \qquad X = 0 \text{ on } V \cap \partial\Omega$$

and

$$\|X\|_{C^{r+1,\alpha}(\overline{V\cap\Omega}, \Lambda^{k-1})} \leq C\|f\|_{C^{r,\alpha}(\overline{V\cap\Omega}, \Lambda^k)}.$$

Here, the constant C depends on r, α, n and Ω.

By compactness of $\partial\Omega$, one can find a covering of $\partial\Omega$ by such open sets V_i, $i = 1, \dots, l$. We denote by $X_i \in C^{r+1,\alpha}(\overline{V_i} \cap \Omega, \Lambda^{k-1})$ the corresponding solution. In particular,

$$||X_i||_{C^{r+1,\alpha}(\overline{V_i}, \Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}(\overline{V_i}, \Lambda^k)}. \tag{6.39}$$

Let also V_0 be a smooth open subset of Ω such that $\Omega \subset \bigcup_{i=0}^l V_i$. We now rely on the classical Hodge-Morrey decomposition on a smooth domain, see e.g. [13, Theorem 6.12 (i)]: there exist $X_0 \in C^{r+1}(\overline{V_0}, \Lambda^{k-1})$, $h_0 \in \mathcal{H}_T^k(V_0) \cap W^{1,2}(V_0, \Lambda^k)$ such that

$$f = \boldsymbol{d}X_0 + h_0.$$

Moreover, X_0 and h_0 can be chosen linearly, and satisfying the estimates: for every $s \in \{0, 1, ..., r\}$, for every $\alpha' \in (0, 1)$,

$$||X_0||_{C^{s+1,\alpha'}(\overline{V_0}, \Lambda^{k-1})} \leq C||f||_{C^{s,\alpha'}(\overline{V_0}, \Lambda^k)}, \tag{6.40}$$

$$||h_0||_{C^{s,\alpha'}(\overline{V_0}, \Lambda^k)} \leq C||f||_{C^{s,\alpha'}(\overline{V_0}, \Lambda^k)},$$

for some $C = C(s, \alpha', V_0)$.

Since V_0 is smooth, the set[2] $\mathcal{H}_T^k(V_0) \cap W^{1,2}(V_0, \Lambda^k)$ is contained in $C^\infty(\overline{V_0}, \Lambda^k)$, see e.g. [13, Theorem 6.3]. Moreover, it is finite dimensional, see e.g. [13, Theorem 6.5 (i)]). In particular, all the norms are equivalent on that space. It follows that

$$\|h_0\|_{C^{s+1,\alpha'}(\overline{V_0}, \Lambda^k)} \leq C\|f\|_{C^{s,\alpha'}(\overline{V_0}, \Lambda^k)}. \tag{6.41}$$

[2]As a matter of fact, on a C^2 domain, harmonic fields have $W^{1,2}$ coefficients, see Remark 6.8. However, we do not need this (non trivial) regularity result here.

To the covering $(V_i)_{0\le i\le l}$ of Ω, we associate a partition of unity $(\theta_i)_{0\le i\le l}$ such that

$$\sum_{i=0}^{l}\theta_i = 1 \text{ in } \overline{\Omega}, \text{ and } \theta_i \in C_c^\infty(V_i) \text{ for } i = 0,1,...,l.$$

Finally, we set

$$Sf = \sum_{i=0}^{l}\theta_i X_i.$$

Then, in view of (6.39), (6.40) and Proposition 5.6, S is a continuous operator from $C_\nu^{r,\alpha}$ into $C_z^{r+1,\alpha}$ and

$$\boldsymbol{d}(Sf) = \sum_{i=0}^{l}\theta_i \boldsymbol{d}X_i + \sum_{i=0}^{l}\boldsymbol{d}\theta_i \wedge X_i.$$

Using that $\theta_0\boldsymbol{d}X_0 = \theta_0(f - h_0)$ and for every $i \in \{1,\dots,l\}$, $\theta_i\boldsymbol{d}X_i = \theta_i f$, one gets $\boldsymbol{d}(Sf) = f - Kf$ with

$$Kf = -\sum_{i=0}^{l}\boldsymbol{d}\theta_i \wedge X_i + h_0\theta_0. \tag{6.42}$$

Observe that $Kf|_{\partial\Omega}= 0$. As a consequence of (6.41) with $s = r$ and $\alpha' = \alpha$, the linear map $f \mapsto h_0$ is continuous from $C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k)$ into $C^{r+1,\alpha}(\overline{V_0},\Lambda^k)$. Hence, K is a continuous operator from $C_\nu^{r,\alpha}$ into $C_z^{r+1,\alpha}$. The proof is complete. □

According to Remark 6.16 and also (6.40), one can require that given a *finite* set $A \subset (0,\alpha]$, the local solutions X_i, arising in the proof of Lemma 6.19 have the following additional property: For every $s \in 0,\dots,r$ and $\alpha' \in A$,

$$\|X_i\|_{C^{s+1,\alpha'}(\overline{\Omega\cap V_i},\Lambda^{k-1})}\le C\|f\|_{C^{s,\alpha'}(\overline{\Omega},\Lambda^k)}$$

for some $C = C(r,A,n,\Omega)$. Relying on the explicit expression of Kf and also (6.41), one has

$$\|Kf\|_{C^{s+1,\alpha'}(\overline{\Omega},\Lambda^{k-1}}\le C\left(\sum_{i=0}^{l}\|X_i\|_{C^{s+1,\alpha'}(\overline{\Omega\cap V_i},\Lambda^{k-1})}+\|h_0\|_{C^{s+1,\alpha'}(\overline{V_0},\Lambda^k)}\right)\le C'\|f\|_{C^{s,\alpha'}(\overline{\Omega},\Lambda^k)}.$$

A similar calculation holds true for Sf. We can thus state the following:

Remark 6.20. The maps S and K are continuous from $C_\nu^{s,\alpha'}$ into $C_z^{s+1,\alpha'}$, for every $s \in \{0,\dots,r\}$ and $\alpha' \in A$.

In the following, we will apply this remark for $A = \{\alpha,\alpha'\}$ for some $\alpha' \in (0,\alpha)$.

In the setting of Sobolev spaces, given two integers $r \ge 0$, $k \in \{1,\dots,n\}$, and $p \in (1,\infty)$, we introduce the sets

$$W_\nu^{r,p}(\Omega,\Lambda^k) = \{f \in W^{r,p}(\Omega,\Lambda^k), \boldsymbol{d}f = 0 \text{ on } \Omega, \nu\wedge f = 0 \text{ on } \partial\Omega\},$$

$$W_z^{r+1,p}(\Omega,\Lambda^{k-1}) = \{X \in W^{r+1,p}(\Omega,\Lambda^{k-1}), X = 0 \text{ on } \partial\Omega\}.$$

Then the same construction as in the Hölder case, except that one relies on Lemma 6.18 instead of Lemma 6.15, leads to

Lemma 6.21. *Let $r \in \mathbb{N}, k \in \{1,\dots,n\}$ and $p \in (1,\infty)$. Let Ω be a bounded $C^{r,1}$ domain in $\mathbb{R}^n$. Then there exist two continuous linear operators*

$$\widetilde{S} : W_\nu^{r,p}(\Omega,\Lambda^k) \to W_z^{r+1,p}(\Omega,\Lambda^{k-1}), \qquad \widetilde{K} : W_\nu^{r,p}(\Omega,\Lambda^k) \to W_z^{r+1,p}(\Omega,\Lambda^k)$$

such that $\boldsymbol{d}(\widetilde{S}f) + \widetilde{K}f = f$ for every $f \in W_\nu^{r,p}(\Omega,\Lambda^k)$.

In the same spirit as Remark 6.20, one has:

Remark 6.22. The above construction allows to require that given a **finite** family of exponents $1 < p_1 < \cdots < p_I \leq p$, the maps $\widetilde{S}$ and $\widetilde{K}$ are continuous from W_ν^{s,p_i} into W_z^{s+1,p_i}, for every $i \in \{1, \dots, I\}$ and every $s \in \{0, \dots, r\}$.

6.3.4 Proof of Theorem 6.1

We finally turn to the proof of Theorem 6.1. We want to apply Lemma 5.3 to the exterior derivative operator $T = \boldsymbol{d}$ from the set $E = C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$ into $F = C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ where

$$C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k) = \left\{ f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k) : \boldsymbol{d}f = 0 \text{ in } \Omega, \nu \wedge f = 0 \text{ on } \partial\Omega, \int_\Omega \langle f, h \rangle = 0, \forall h \in \mathcal{H}_T^k(\Omega) \right\}.$$

Remember that $\mathcal{H}_T^k(\Omega)$ is defined by

$$\mathcal{H}_T^k(\Omega) = \{h \in L^2(\Omega, \Lambda^k) : \boldsymbol{\delta} h = 0, \boldsymbol{d}(h_z) = 0\},$$

where the index z denotes the extension by zero outside Ω. In other words, $C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k) = C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) \cap (\mathcal{H}_T^k(\Omega))^\perp$, where the $\perp$ sign is related to the inner product in $L^2(\Omega, \Lambda^k)$.

We introduce the two operators S and K given by Lemma 6.19. We first observe that for every $h \in \mathcal{H}_T^k(\Omega)$,

$$\int_\Omega \langle \boldsymbol{d}X, h \rangle \, dx = 0, \qquad \forall X \in C_c^\infty(\Omega, \Lambda^{k-1}).$$

By density of $C_c^\infty(\Omega, \Lambda^{k-1})$ in $W_0^{1,2}(\Omega, \Lambda^{k-1})$, the above identity remains true for $X \in W_0^{1,2}$, and thus in particular for $X = Sf$. Hence, $Kf = f - \boldsymbol{d}(Sf)$ also belongs to $(\mathcal{H}_T^k(\Omega))^\perp$; that is,

$$K(C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)) \subset C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^k) \cap (\mathcal{H}_T^k(\Omega))^\perp.$$

Since the embedding $\iota : C^{r+1,\alpha}(\overline{\Omega}, \Lambda^k) \to C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ is compact and $K : C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) \to C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^k)$ is continuous, it follows that

$$\iota \circ K : C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k) \to C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k) \text{ is compact.}$$

In the following, in order to simplify the notation, we abbreviate $\iota \circ K$ into K. By construction, $Id = T \circ S + K$. The last assumption of Lemma 5.3 that we have to establish is $\ker T^* = \{0\}$. This is a consequence of the following:

Lemma 6.23. *Let $v \in (C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k))^*$ such that for every $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$,*

$$\langle v, \boldsymbol{d}X \rangle_{(C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k)} = 0.$$

Then $v \in \mathcal{H}_T^k(\Omega)$, in the sense that there exists $h \in \mathcal{H}_T^k(\Omega)$ such that for every $f \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k)$,

$$\langle v, f \rangle_{(C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k)} = \int_\Omega \langle h, f \rangle \, dx.$$

We first explain how Lemma 6.23 implies that $\ker T^* = \{0\}$. Let $v \in \ker T^* \subset (C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k))^*$. Then by the Hahn-Banach theorem, there exists a continuous extension $\overline{v}$ of v to $C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) \supset C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$. In particular, for every $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$,

$$\langle \overline{v}, \boldsymbol{d}X \rangle_{(C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_\nu^{r,\alpha}(\overline{\Omega},\Lambda^k)} = \langle v, \boldsymbol{d}X \rangle_{(C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega},\Lambda^k)} = 0.$$

By Lemma 6.23, $\overline{v}$ can be identified to an element of $\mathcal{H}_T^k$. This implies that for every $f \in C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$,

$$\langle v, f\rangle_{(C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega},\Lambda^k)} = \langle \overline{v}, f\rangle_{(C_{\nu}^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_{\nu}^{r,\alpha}(\overline{\Omega},\Lambda^k)} = \int_\Omega \langle \overline{v}, f\rangle \, dx.$$

By definition of $C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, $f \in (\mathcal{H}_T^k)^\perp$ and thus

$$\langle v, f\rangle_{(C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega},\Lambda^k))^*, C_{\mathcal{H}}^{r,\alpha}(\overline{\Omega},\Lambda^k)} = 0.$$

This proves that $v = 0$, as desired.
We next turn to the

Proof of Lemma 6.23. Let $v \in (C_{\nu}^{r,\alpha}(\overline{\Omega}, \Lambda^k))^*$ such that

$$\langle v, \boldsymbol{d}X\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} = 0, \qquad \forall X \in C_z^{r+1,\alpha}. \tag{6.43}$$

We fix some $\alpha' \in (0, \alpha)$. We split the proof into four steps.

Step 1. In the first step, we prove that there exists a constant $C = C(v, r, \alpha, \alpha', \Omega) > 0$ such that for every $f \in C_{\nu}^{r,\alpha}$,

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} \leq C||f||_{C^{0,\alpha'}}.$$

We first assume that $r \geq 1$. Let $f \in C_{\nu}^{r,\alpha}$. Since $Sf \in C_z^{r+1,\alpha}$, (6.43) implies

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} = \langle v, \boldsymbol{d}Sf + Kf\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} = \langle v, Kf\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}}.$$

Hence,

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} \leq \|v\|_{(C_{\nu}^{r,\alpha})^*} \|Kf\|_{C^{r,\alpha}}.$$

By Remark 6.20 with $s = r - 1$, one has

$$||Kf||_{C^{r,\alpha}} \leq C||f||_{C^{r-1,\alpha}}.$$

Hence,

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} \leq C||v||_{(C_{\nu}^{r,\alpha})^*} ||f||_{C^{r-1,\alpha}}.$$

If $r \geq 2$, we can repeat the above argument taking into account this new estimate that we apply to Kf instead of f:

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} = \langle v, Kf\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} \leq C||v||_{(C_{\nu}^{r,\alpha})^*} ||Kf||_{C^{r-1,\alpha}}.$$

Using Remark 6.20 with $s = r - 2$, we deduce that

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} \leq C||f||_{C^{r-2,\alpha}}$$

for some new constant $C = C(v, r, \alpha, \Omega)$. Iterating this calculation, we obtain that

$$\langle v, f\rangle_{(C_{\nu}^{r,\alpha})^*, C_{\nu}^{r,\alpha}} \leq C||f||_{C^{0,\alpha}}.$$

In the case $r = 0$, this estimate is obvious.
Finally, when $r \geq 0$, we rely on Remark 6.20 with $s = 0$ and $A = \{\alpha', \alpha\}$ to get

$$||Kf||_{C^{1,\alpha'}} \leq C||f||_{C^{0,\alpha'}}.$$

Hence, reasoning as above, one has

$$\begin{aligned}\langle v, f\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu} &= \langle v, Kf\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu} \\ &\le C||Kf||_{C^{0,\alpha}} \le C'||Kf||_{C^{1,\alpha'}} \le C''||f||_{C^{0,\alpha'}}.\end{aligned}$$

This completes the proof of Step 1.

Step 2. The aim of this step is to prove that there exists a constant $C > 0$ such that

$$\langle v, f\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu} \le C||f||_{L^2(\Omega,\Lambda^k)}, \text{ for all } f \in C^{r,\alpha}_\nu(\overline{\Omega}, \Lambda^k) \text{ such that } f = 0 \text{ on } \partial\Omega.$$

We define a sequence $(p_i)_{i\ge 0}$ by induction as follows: $p_0 = 2$ and

$$p_{i+1} = \begin{cases} (p_i)^* = \frac{np_i}{n-p_i} & \text{if } p_i < n, \\ \max(p_i, \frac{n}{1-\alpha}) & \text{if } p_i \ge n. \end{cases}$$

Observe that $p_{i+1} > p_i$ if $p_i \le n$ and when $p_i < n$, then

$$p_{i+1} - p_i = \frac{p_i p_{i+1}}{n} \ge \frac{1}{n}.$$

Hence, one can define p_I to be the first term of the sequence such that $p_I \ge \frac{n}{1-\alpha}$. By the Sobolev and the Morrey embeddings, we have for every $0 \le i \le I-1$,

$$W^{1,p_i} \subset L^{p_{i+1}}, \qquad W^{1,p_I} \subset C^{0,\alpha}.$$

Let $f \in C^{r,\alpha}_\nu(\overline{\Omega}, \Lambda^k)$ such that $f = 0$ on $\partial\Omega$. In particular, $f \in L^{p_I}_\nu(\Omega, \Lambda^k)$. Since Ω is $C^{r+1,\alpha}$ and thus $C^{0,1}$, we can rely on Lemma 6.21 with $r = 0$ and $p = p_I$ to write

$$f = \boldsymbol{d}(\widetilde{S}f) + \widetilde{K}f,$$

with $\widetilde{S}f \in W^{1,p_I}_0(\Omega, \Lambda^{k-1})$, $\widetilde{K}f \in W^{1,p_I}_0(\Omega, \Lambda^k)$. Since $\boldsymbol{d}f = 0$ in $\mathcal{D}'(\Omega)$, one has $\boldsymbol{d}\widetilde{K}f = \boldsymbol{d}(f - \boldsymbol{d}(\widetilde{S}f)) = 0$. Hence, $\widetilde{K}f \in W^{1,p_I}_\nu(\Omega, \Lambda^k)$.

Relying on the approximation result given in Proposition B.3, there exists a family of linear maps $\{\eta_i\}_{i\in\mathbb{N}}$ such that for every $\ell \in \{0, \dots, n\}$, η_i maps $C^{0,\alpha}_z(\overline{\Omega}, \Lambda^\ell)$ into $C^\infty_c(\Omega, \Lambda^\ell)$ and for every $g \in C^{0,\alpha}_z(\overline{\Omega}, \Lambda^\ell)$,

$$\lim_{i\to\infty} \|\eta_i(g) - g\|_{C^{0,\alpha'}} = 0.$$

Moreover, for every $h \in W^{1,1}_0(\Omega, \Lambda^\ell) \cap C^{0,\alpha}_z(\overline{\Omega}, \Lambda^\ell)$ such that $\boldsymbol{d}h \in C^{0,\alpha}_z$, one has $\eta_i(\boldsymbol{d}h) = \boldsymbol{d}(\eta_i(h))$.

We apply this last property to the map $h = \widetilde{S}f$ which belongs to $W^{1,p_I}_0 \subset W^{1,1}_0 \cap C^{0,\alpha}_z$ and satisfies $\boldsymbol{d}(\widetilde{S}f) = f - \widetilde{K}f \in C^{0,\alpha}_z$ (indeed, observe that $f \in C^{r,\alpha}_z \subset C^{0,\alpha}_z$ and $\widetilde{K}f \in W^{1,p_I}_0 \subset C^{0,\alpha}_z$). This gives $\eta_i(\boldsymbol{d}(\widetilde{S}f)) = \boldsymbol{d}(\eta_i(\widetilde{S}f))$.

Since $\eta_i(\widetilde{S}f) \in C^\infty_c(\Omega, \Lambda^{k-1})$, the assumption on v implies

$$0 = \langle v, \boldsymbol{d}(\eta_i(\widetilde{S}f))\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu}.$$

Next, by linearity of η_i,

$$\eta_i(f) = \eta_i(\boldsymbol{d}(\widetilde{S}f)) + \eta_i(\widetilde{K}f) = \boldsymbol{d}(\eta_i(\widetilde{S}f)) + \eta_i(\widetilde{K}f).$$

Hence,

$$\langle v, \eta_i(f)\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu} = \langle v, \eta_i(\widetilde{K}f)\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu}. \tag{6.44}$$

By Step 1,

$$\langle v, \eta_i(f)\rangle_{(C^{r,\alpha}_\nu)^*, C^{r,\alpha}_\nu} \le C\|\eta_i(\widetilde{K}f)\|_{C^{0,\alpha'}}. \tag{6.45}$$

In the right hand side, we use that $\widetilde{K}(f) \in W_0^{1,p_I} \subset C_z^{0,\alpha}$, to deduce that $\eta_i(\widetilde{K}f)$ converges in $C^{0,\alpha'}$ to $\widetilde{K}f$.
In the left hand side, since $f \in C_z^{0,\alpha}$ one has $\lim_{i\to\infty}\|f - \eta_i(f)\|_{C^{0,\alpha'}} = 0$ and thus by Step 1,

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} = \lim_{i\to\infty} \langle v, \eta_i(f)\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}}. \tag{6.46}$$

Hence, letting $i \to +\infty$ in (6.45), one gets

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|\widetilde{K}f\|_{C^{0,\alpha'}}.$$

By the Morrey embedding,

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|\widetilde{K}f\|_{W^{1,p_I}}.$$

By Remark 6.22, one can require that $\widetilde{K}$ is continuous from L^{p_I} into W^{1,p_I}. Hence,

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|f\|_{L^{p_I}}.$$

We start again from (6.44) but instead of relying on Step 1, we exploit the above inequality with $\eta_i(\widetilde{K}f)$ instead of f to get the following analogue of (6.45):

$$\langle v, \eta_i(f)\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|\eta_i(\widetilde{K}f)\|_{L^{p_I}}.$$

Since $\eta_i(\widetilde{K}f)$ converges to $\widetilde{K}f$ in L^{p_I} and using also (6.46), this gives

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|\widetilde{K}f\|_{L^{p_I}}.$$

In view of the Sobolev inequality $W^{1,p_{I-1}} \subset L^{p_I}$, this implies

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|\widetilde{K}f\|_{W^{1,p_{I-1}}}.$$

By Remark 6.22, we get

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|f\|_{L^{p_{I-1}}}.$$

Iterating these estimates, we finally obtain

$$\langle v, f\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} \le C\|f\|_{L^{p_0}} = C\|f\|_{L^2},$$

which is the conclusion of Step 2.

Step 3. In this step, we prove that the restriction w of v to $\{f \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) : f = 0 \text{ on } \partial\Omega\}$ belongs to $\mathcal{H}_T^k$. By the previous step, there exists $C > 0$ such that for every $f \in C_\nu^{r,\alpha}$ with $f = 0$ on $\partial\Omega$,

$$\langle w, f\rangle \le C\|f\|_{L^2}.$$

Hence, w can be continuously extended to the subset

$$\mathrm{cl}_{L^2}(\{f \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) : f = 0 \text{ on } \partial\Omega\}) \subset L^2(\Omega, \Lambda^k),$$

where cl_{L^2} denotes the closure in L^2. We then extend w by setting $w = 0$ on the orthogonal space of this subset. By the usual identification of L^2 with its dual, we can now consider w as an element of $L^2(\Omega, \Lambda^k)$.

We proceed to prove that $w \in \mathcal{H}_T^k$. First, for every $X \in C_c^\infty(\Omega, \Lambda^{k-1})$,

$$\int_\Omega \langle w, \boldsymbol{d}X\rangle\, dx = \langle v, \boldsymbol{d}X\rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} = 0,$$

where the last equality follows from the assumption on v. This proves that $\boldsymbol{\delta} w = 0$. Next, for every $X \in C^\infty(\overline{\Omega}, \Lambda^{k+1})$, for every $g \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k)$,

$$\int_\Omega \langle g, \boldsymbol{\delta} X \rangle \, dx = 0.$$

We deduce in particular that $\delta X \in \left(\mathrm{cl}_{L^2}(\{g \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k) : g = 0 \text{ on } \partial\Omega\}\right)^\perp$ and thus

$$\int_\Omega \langle w, \boldsymbol{\delta} X \rangle \, dx = 0.$$

This means that $\boldsymbol{d}(w_z) = 0$, where w_z is the extension of w by 0 outside Ω. This completes the proof of the fact that $w \in \mathcal{H}_T^k$.

Step 4. Conclusion of the proof. For every $f \in C_\nu^{r,\alpha}(\overline{\Omega}, \Lambda^k)$, we rely on Lemma 6.19 to write $f = \boldsymbol{d}(Sf) + Kf$, with $Sf \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$, $Kf \in C_\nu^{r+1,\alpha}(\overline{\Omega}, \Lambda^k)$ and $Kf = 0$ on $\partial\Omega$. It follows that

$$\langle v, f \rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} = \langle v, Kf \rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}}.$$

Since $Kf \in C_\nu^{r+1,\alpha}(\overline{\Omega}, \Lambda^k)$ with $Kf = 0$ on $\partial\Omega$, we can apply the previous step to Kf:

$$\langle v, Kf \rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} = \int_\Omega \langle w, Kf \rangle \, dx,$$

where $w \in \mathcal{H}_T^k$. We deduce therefrom that

$$\langle v, f \rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} = \int_\Omega \langle w, Kf \rangle \, dx.$$

Since $\boldsymbol{d}(Sf) \in (\mathcal{H}_T^k)^\perp$ and $w \in \mathcal{H}_T^k$, it follows that

$$\langle v, f \rangle_{(C_\nu^{r,\alpha})^*, C_\nu^{r,\alpha}} = \int_\Omega \langle w, Kf + \boldsymbol{d}(Sf) \rangle \, dx = \int_\Omega \langle w, f \rangle \, dx,$$

as desired. □

Remark 6.24. The converse of Lemma 6.23 is true: if $v \in \mathcal{H}_T^k(\Omega)$, then $\langle v, \boldsymbol{d} X \rangle = 0$ for every $X \in C_z^{r+1,\alpha}(\overline{\Omega}, \Lambda^{k-1})$.

Indeed, one can approximate such an X by a sequence $(X_i)_{i \in \mathbb{N}} \subset C_c^\infty(\Omega, \Lambda^{k-1})$ for the $W_0^{1,2}$ topology. Since $\langle v, \boldsymbol{d} X_i \rangle = 0$, we obtain the desired result when $i \to +\infty$.

Remark 6.25. In the setting of Theorem 6.1 but under the additional assumption that Ω is $C^{r+3,\alpha}$, the construction of the vector X presented in [13, Theorem 8.16] is linear and **universal**, in the following sense: there exists a linear map

$$\Xi_0 : \bigcup_{\substack{r \in \mathbb{N} \\ \alpha \in (0,1)}} C^{r,\alpha}(\overline{\Omega}, \Lambda^k) \to \bigcup_{\substack{r \in \mathbb{N} \\ \alpha \in (0,1)}} C^{r+1,\alpha}(\overline{\Omega}; \Lambda^{k-1})$$

such that for every $r \in \mathbb{N}, \alpha \in (0,1)$, for every $f \in C^{r,\alpha}(\overline{\Omega}, \Lambda^k)$ satisfying

$$\begin{cases} \boldsymbol{d} f = 0 & \text{in } \Omega, \\ \nu \wedge f = 0 & \text{on } \partial\Omega, \end{cases}$$

one has $\Xi_0(f) \in C^{r+1,\alpha}(\overline{\Omega}; \Lambda^{k-1})$,

$$\boldsymbol{d}(\Xi_0(f)) = f \text{ on } \Omega, \qquad \Xi_0(f) = 0 \text{ on } \partial\Omega,$$

$$\|\Xi_0(f)\|_{C^{r+1,\alpha}} \leq C \|f\|_{C^{r,\alpha}}$$

for some $C = C(r, \alpha, \Omega) > 0$. The same remark can be made in the framework of Lemma 6.11. The proofs of Lemma 6.14 and Lemma 6.15 are based on linear constructions as well. However, due to the restrictions on ε in Lemma 6.14 and on δ in Lemma 6.15, they are not universal in the above sense.

6.3.5 Proofs of Theorem 6.5 and Remark 6.6

Let $r \in \mathbb{N}$, $k \in \{1, \dots, n\}$ and $p \in (1, \infty)$. Let Ω be a bounded $C^{r,1}$ domain in $\mathbb{R}^n$. By the Sobolev embedding, the space $W^{r,p}(\Omega, \Lambda^k)$ is contained in $L^{p_r}(\Omega, \Lambda^k)$, where

$$p_r = \begin{cases} np/(n-rp) & \text{if } rp < n, \\ \max(2,p) & \text{if } rp \geq n. \end{cases}$$

We denote by $(p_r)'$ the Hölder exponent of p_r.

In the framework of Theorem 6.5, the main assumption is that

$$\mathcal{H}_T^k(\Omega) \subset L^{p'}(\Omega, \Lambda^k). \tag{6.47}$$

In the first two steps of the proof below, we do not need this property but assume instead that $p_r \geq 2$. The latter condition implies that $W^{r,p} \subset L^2$. In particular, the integral $\int_\Omega \langle f, h \rangle \, dx$ is well defined for every $f \in W_\nu^{r,p}(\Omega, \Lambda^k)$ and every $h \in \mathcal{H}_T^k$. In the final step, we prove Theorem 6.5 under the assumption

$$\mathcal{H}_T^k(\Omega) \subset L^{(p_r)'}(\Omega, \Lambda^k). \tag{6.48}$$

Since $p_r \geq p$, (6.48) may be seen as a less restrictive hypothesis than (6.47). Actually, when $r = 0$, namely when Ω is Lipschitz, they coincide while when Ω is C^1, it is very plausible that the methods of [40, Chapter 11] imply that $\mathcal{H}_T^k(\Omega) \subset \cap_{1<q<\infty} L^q(\Omega, \Lambda^k)$.

Proof of Theorem 6.5. **Step 1.** We first establish a Sobolev version of Lemma 6.23 under the assumption $p_r \geq 2$:

Let $v \in (W_\nu^{r,p}(\Omega, \Lambda^k))^*$ such that for every $X \in W_z^{r+1,p}(\Omega, \Lambda^{k-1})$,

$$\langle v, \boldsymbol{d}X \rangle_{(W_\nu^{r,p}(\Omega,\Lambda^k))^*, W_\nu^{r,p}(\Omega,\Lambda^k)} = 0.$$

We then claim that

$$v \in \mathcal{H}_T^k(\Omega). \tag{6.49}$$

Indeed, let $f \in W_\nu^{r,p}(\Omega, \Lambda^k)$. We introduce the two operators $\widetilde{S}$ and $\widetilde{K}$ given by Lemma 6.21. Since $\widetilde{S}f \in W_z^{r+1,p}$, $\langle v, \boldsymbol{d}(Sf) \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} = 0$ and thus

$$\langle v, f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} = \langle v, \widetilde{K}f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} \leq \|v\|_{(W_\nu^{r,p})^*} \|\widetilde{K}f\|_{W_\nu^{r,p}}.$$

Relying on Remark 6.22 with $s = r - 1$, one gets

$$\langle v, f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} \leq C \|f\|_{W_\nu^{r-1,p}},$$

for some $C = C(v, \Omega, r, p) > 0$. Iterating on $r, r-1, \dots, 0$, this leads to

$$\langle v, f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} \leq C \|f\|_{L_\nu^p}. \tag{6.50}$$

If $p \leq 2$, this proves in particular that

$$\langle v, f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} \leq C \|f\|_{L_\nu^2}.$$

Otherwise, in the case $p > 2$, we introduce as in Step 2 of the proof of Lemma 6.23, a sequence $p_0 = 2 < p_1 < \cdots < p_I = p$ such that $W^{1,p_{i-1}}(\Omega) \subset L^{p_i}(\Omega)$, for $i = 1, \dots, I$. We then exploit the estimate (6.50) for $\widetilde{K}f$ instead of f, namely

$$\langle v, f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} = \langle v, \widetilde{K}f \rangle_{(W_\nu^{r,p})^*, W_\nu^{r,p}} \leq C \|\widetilde{K}f\|_{L_\nu^p}.$$

Observing that $p = p_I$ and $W^{1,p_{I-1}} \subset L^{p_I}$, one has $\|\widetilde{K}f\|_{L^p_\nu} \leq C\|\widetilde{K}f\|_{W^{1,p_{I-1}}_\nu}$. Remark 6.22 with $s = 0$ implies that $\|\widetilde{K}f\|_{W^{1,p_{I-1}}_\nu} \leq C\|f\|_{L^{p_{I-1}}_\nu}$. This gives

$$\langle v, f\rangle_{(W^{r,p}_\nu)^*, W^{r,p}_\nu} \leq C\|f\|_{L^{p_{I-1}}}.$$

Iterating on $p_I, p_{I-1}, \ldots, p_0 = 2$, we finally obtain

$$\langle v, f\rangle_{(W^{r,p}_\nu)^*, W^{r,p}_\nu} \leq C\|f\|_{L^2}. \tag{6.51}$$

Since $p_r \geq 2$, one has $W^{r,p}_\nu \subset L^2$ and it is thus possible to extend v as a continuous linear map on $\mathrm{cl}_{L^2}(W^{r,p}_\nu(\Omega, \Lambda^k))$. We then set $v = 0$ on $(\mathrm{cl}_{L^2}(W^{r,p}_\nu(\Omega, \Lambda^k)))^\perp$. We can conclude that $v \in \mathcal{H}^k_T$, as in the Hölder case, see Step 3 of the proof of Lemma 6.23. This completes the proof of our claim (6.49).

Step 2. In this step, we prove Theorem 6.5 under the additional assumption that $p_r \geq 2$. The proof is in the same vein as the one of Theorem 6.1, but much simpler. We only indicate the main changes.

We apply Lemma 5.3 to $T = \boldsymbol{d}$ from the set $E = W^{r+1,p}_z(\Omega, \Lambda^{k-1})$ into $F = W^{r,p}_{\mathcal{H}}(\Omega, \Lambda^k)$ where

$$W^{r,p}_{\mathcal{H}}(\Omega, \Lambda^k) = \{f \in W^{r,p}(\Omega, \Lambda^k) : \boldsymbol{d}f = 0 \text{ in } \Omega, \nu\wedge f = 0 \text{ on } \partial\Omega, \int_\Omega \langle f, h\rangle = 0, \forall h \in \mathcal{H}^k_T(\Omega)\}.$$

We introduce the two operators $\widetilde{S}$ and $\widetilde{K}$ given by Lemma 6.21. Then, relying on the compact embedding $W^{r+1,p}_z \subset W^{r,p}_z$, one deduces, as in the Hölder case, that $\widetilde{K}$ is a compact map from $W^{r,p}_{\mathcal{H}}(\Omega, \Lambda^k)$ into itself. The fact that $\ker T^* = \{0\}$ now follows from Step 1 and the fact that $W^{r,p}_{\mathcal{H}} \subset (\mathcal{H}^k_T)^\perp$. We can thus apply Lemma 5.3 and get Theorem 6.5 when $p_r \geq 2$.

Step 3. We now prove Theorem 6.5 under the assumption that $\mathcal{H}^k_T(\Omega) \subset L^{(p_r)'}$.

In view of Step 2, one can assume that $p_r < 2$ (which implies in particular that $p < 2$). Let $f \in W^{r,p}_\nu(\Omega, \Lambda^k)$. We write $f = \boldsymbol{d}(\widetilde{S}f) + \widetilde{K}f$, with $\widetilde{S}f, \widetilde{K}f \in W^{r+1,p}_z$, where $p^* = np/(n-p)$ Since $W^{r+1,p}_z \subset W^{r,p^*}_z$, this means that f can be written as $\boldsymbol{d}X_1 + f_1$, with $X_1 \in W^{r+1,p}_z$ and $f_1 \in W^{r,p^*}_\nu$. If $p^* < 2$, we repeat this construction for f_1, to get $f_1 = \boldsymbol{d}Y + f_2$ with $Y \in W^{r+1,p^*}_z$ and $f_2 \in W^{r,(p^*)^*}_\nu$. This implies that $f = \boldsymbol{d}X_2 + f_2$, with $X_2 = X_1 + Y \in W^{r+1,p}_z$. Iterating this construction yields $X_\ell \in W^{r+1,p}_z$, $f_\ell \in W^{r,2}_\nu$ such that

$$f = \boldsymbol{d}X_\ell + f_\ell. \tag{6.52}$$

By the Sobolev embedding, $X_\ell \in W^{1,p_r}_0(\Omega, \Lambda^{k-1})$. Let $(Y_j)_{j\in\mathbb{N}} \subset C^\infty_c(\Omega, \Lambda^{k-1})$ converging to X_ℓ in W^{1,p_r}_0. For every $j \in \mathbb{N}$ and every $h \in \mathcal{H}^k_T(\Omega)$,

$$\int_\Omega \langle \boldsymbol{d}Y_j, h\rangle\, dx = 0.$$

Since $(\boldsymbol{d}Y_j)_{j\in\mathbb{N}}$ converges to dX in L^{p_r} and $\mathcal{H}^k_T(\Omega) \subset L^{(p_r)'}(\Omega, \Lambda^k)$, it follows that

$$\int_\Omega \langle \boldsymbol{d}X_\ell, h\rangle\, dx = 0.$$

Hence, $\boldsymbol{d}X_\ell \in (\mathcal{H}^k_T)^\perp$. We deduce therefrom that if in (6.52), we further assume that $f \in W^{r,p}_{\mathcal{H}}$, then $f_\ell \in W^{r,2}_{\mathcal{H}}$.

We can then apply Step 2 to f_ℓ: there exists $Z \in W^{r+1,2}_z(\Omega, \Lambda^{k-1}) \subset W^{r+1,p}_z(\Omega, \Lambda^{k-1})$ such that $f_\ell = \boldsymbol{d}Z$. This yields

$$f = \boldsymbol{d}(X_\ell + Z).$$

Since all the above constructions can be made continuously and linearly, this completes the proof of Theorem 6.5.

□

We end this section with the proof of Remark 6.6. In this setting, the main assumption is now that

$$\mathcal{H}_T^k(\Omega) \subset L^p(\Omega, \Lambda^k). \tag{6.53}$$

Proof of Remark 6.6. Let $f \in L^p_\nu(\Omega, \Lambda^k)$. If $p \geq 2$, then $L^p \subset L^2$ and we can introduce the L^2 orthogonal projection h of f on $\mathcal{H}_T^k(\Omega)$. Then by (6.53), $f - h \in L^p_{\mathcal{H}}(\Omega, \Lambda^k)$. Since $p \geq 2$, $\mathcal{H}_T^k(\Omega) \subset L^{p'}(\Omega, \Lambda^k)$. We can thus apply Theorem 6.5 to find some $X \in W_z^{1,p}(\Omega, \Lambda^k)$ such that $f - h = \boldsymbol{d}X$. Theorem 6.6 is proved in that case.

If $p < 2$, then we can find $X_1 \in W_z^{1,p}(\Omega, \Lambda^{k-1})$, $f_1 \in L^2_\nu(\Omega, \Lambda^k)$ such that $f = \boldsymbol{d}X_1 + f_1$, as in Step 3 of the proof of Theorem 6.5, see (6.52). By the previous argument applied to f_1 instead of f, one can write $f_1 = \boldsymbol{d}X_2 + h$, with $X_2 \in W_z^{1,2}(\Omega, \Lambda^{k-1})$ and $h \in \mathcal{H}_T^k(\Omega)$. Then $f = \boldsymbol{d}(X_1 + X_2) + h$, which completes the proof of the remark.

□

Appendix A

On the magnetostatic energy

In this appendix, we present the proofs of several elementary results on the magnetostatic energy.
Throughout this section, we use the notation

$$m = (m', m_3) : \Omega \to \mathbb{S}^2$$

and $\nabla = (\nabla', \partial_{x_3})$. The dash $'$ indicates a $2D$ quantity.

Proof of Theorem 2.3 . We apply Lax-Migram's Theorem for the variational problem (2.3) in space $\mathcal{BL}$ to obtain the unique existence. For this, we will check that the map $\xi \mapsto \int_\Omega \nabla'.m(x')\xi(x',0)dx'$ is linear continuous in our $(\mathcal{BL}, ||\cdot||_{\dot{H}^1})$. For every $\xi \in \mathcal{BL}$, we have

$$\begin{aligned} \left| \int_\Omega \nabla'.m'(x')\xi(x',0)dx' \right| &\leq ||\nabla'.m'\mathbb{1}_\Omega||_{\dot{H}^{-1/2}(\mathbb{R}^2)} ||\xi(\cdot,0)||_{\dot{H}^{1/2}(\mathbb{R}^2)} \\ &\leq C||\nabla'.m'\mathbb{1}_\Omega||_{\dot{H}^{-1/2}(\mathbb{R}^2)} ||\nabla\xi||_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{A.1}$$

Here, we have used the interpolation inequality in the first line and the trace estimate in the second line. It remains to prove that $\nabla'.m'\mathbb{1}_\Omega \in \dot{H}^{\frac{1}{@}}(\mathbb{R}^2)$
Denote $f = \nabla'.m'\ \mathbb{1}_\Omega$. We show that

$$||f||_{\dot{H}^{-1/2}(\mathbb{R}^2)} \leq C||f||_{L^2(\mathbb{R}^2)} \leq C||\nabla m||_{L^2(\Omega)} < +\infty.$$

Indeed,

$$\begin{aligned} ||f||^2_{\dot{H}^{-1/2}(\mathbb{R}^2)} &= \int_{|\zeta|>1} \frac{|\mathcal{F}(f)(\zeta)|^2}{|\xi|} d\zeta + \int_{|\zeta|\leq 1} \frac{|\mathcal{F}(f)(\zeta)|^2}{|\zeta|} d\zeta \\ &\leq ||\mathcal{F}(f)||^2_{L^2(\mathbb{R}^2)} + ||\mathcal{F}(f)||^2_{L^\infty(\mathbb{R}^2)} \int_{|\zeta|\leq 1} \frac{1}{|\zeta|} d\zeta \\ &\leq C(||f||^2_{L^2(\mathbb{R}^2)} + ||f||^2_{L^1(\mathbb{R}^2)}) \leq C||f||^2_{L^2(\mathbb{R}^2)}, \end{aligned} \tag{A.2}$$

where we used that $\mathrm{supp}(f)$ is compact in $\mathbb{R}^2$ (as Ω is bounded).

As consequence of Lax-Milgram's Theorem, the variational problem (2.3) has a unique solution of $U \in \mathcal{BL}(\mathbb{R}^3)$
The classical equation (2.12) is obtained obviously. Indeed, by choosing $\xi \in C_0^\infty(\mathbb{R}^3 \backslash (\overline{\Omega} \times \{0\}))$, then

$$\int_{\mathbb{R}^3} \nabla U(x).\nabla\xi(x)dx = 0 \quad \text{for all } \xi \in C_0^\infty(\mathbb{R}^3 \backslash (\overline{\Omega} \times \{0\})).$$

It implies that

$$\Delta U = 0 \text{ in } \mathbb{R}^3 \backslash (\overline{\Omega} \times \{0\}). \tag{A.3}$$

Moreover, for any $\eta \in C_0^\infty(\overline{\Omega} \times \{0\})$, there exists $\xi \in C_0^\infty(\mathbb{R}^3)$ such that $\xi(\cdot, 0) = \eta$. Then

$$\begin{aligned}
&\int_\Omega \nabla'.m'(x')\eta(x')dx' = \int_{\mathbb{R}^3} \nabla U(x)\nabla\xi(x)dx \\
&= \int_{\mathbb{R}^3_+} \nabla U(x)\nabla\xi(x)dx + \int_{\mathbb{R}^3_-} \nabla U(x)\nabla\xi(x)dx \\
&= \int_\Omega -\left[\frac{\partial U}{\partial x_3}\right]\xi(x',0)dx' = \int_\Omega -\left[\frac{\partial U}{\partial x_3}\right]\eta(x')dx.'
\end{aligned} \tag{A.4}$$

We then obtain the second equation of (2.12).

It remains to prove (2.13). Applying the Fourier transform with respect to the in-plane variable x' onto (A.3), we get an ODE for $\mathcal{F}(U)$ in terms of x_3 with the Fourier variable ζ as parameter:

$$\frac{\partial^2}{\partial^2 x_3}\mathcal{F}(U)(\zeta,\cdot) - |\zeta|^2\mathcal{F}(U)(\zeta,\cdot) = 0 \quad \text{for } x_3 \neq 0.\zeta \in \mathbb{R}^2. \tag{A.5}$$

The jump condition follows that:

$$\left[\frac{\partial}{\partial x_3}\mathcal{F}(U)(\zeta,\cdot)\right] = -\mathcal{F}(\nabla'.m'\mathbb{1}_\Omega)(\zeta) \quad \text{for } x_3 = 0, \zeta \neq 0. \tag{A.6}$$

Recall that $U \in \mathcal{BL}(\mathbb{R}^3)$. The trace of U is well defined, see Dautray and Lions [16]. The uniqueness of U implies that U is symmetric w.r.t. $x_3 = 0$, i.e., U is continuous on $x_3 = 0$. Then

$$[\mathcal{F}(U)(\zeta,\cdot)] = 0 \quad \text{for } x_3 = 0. \tag{A.7}$$

Equations (A.5)-(A.7) give the explicit solution, (see [30, Proposition 4]),

$$\mathcal{F}(U)(\zeta, x_3) = \frac{1}{2|\zeta|}e^{-|\zeta||x_3|}\mathcal{F}(\nabla'.m'\mathbb{1}_\Omega)(\zeta) \quad \text{for } \zeta \neq 0, x_3 \in \mathbb{R}. \tag{A.8}$$

Plancherel's identity yields

$$\begin{aligned}
\int_{\mathbb{R}^3}|\nabla U|^2dx \quad &= \int_{\mathbb{R}^2}\int_{\mathbb{R}}\left(|\zeta|^2|\mathcal{F}(U)(\zeta,x_3)|^2 + \left|\frac{\partial\mathcal{F}(U)(\zeta,x_3)}{\partial x_3}\right|^2\right)d\zeta dx_3 \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{-2|\zeta||x_3|}|\mathcal{F}(\nabla'.m'\mathbb{1}_\Omega)(\zeta)|^2 d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\frac{1}{|\zeta|}|\mathcal{F}(\nabla'.m'\mathbb{1}_\Omega)(\xi)|^2 d\xi.
\end{aligned} \tag{A.9}$$

Moreover, since $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathcal{BL}(\mathbb{R}^3)$, we obtain that

$$\int_{\mathbb{R}^3}|\nabla U(x)|^2dx = \int_\Omega \nabla'.m'(x')\nabla U(x',0)dx'$$

This completes the proof of Theorem 2.3. □

We next establish a basic regularity for the solution U of

$$\int_{\mathbb{R}^3}\nabla U(x).\nabla\zeta(x)dx = \int_\Omega \nabla'.m'(x')\zeta(x',0)dx' \text{ for every } \zeta \in C_0^\infty(\mathbb{R}^3). \tag{A.10}$$

For that we give the

Proof of Lemma 3.4. The proof is inspired from Ignat and Knüpfer [31, Lemma 3.1].

When $k \geq 2$, we assume that $m \in H^k_{loc}(\Omega)$. We take $W \subset \Omega$ a compact set and fix $\delta = \text{dist }(\partial\Omega, W)/(2k+2) > 0$. We want to prove that $U \in H^k_{loc}(\Omega)$
Choose a smooth cut-off function θ with $\theta = 1$ in W_δ and $\theta = 0$ outside of $W_{2\delta}$, where $W_{2\delta}$ is defined by

$$W_{2\delta} := \{x \in \Omega : \text{dist}(x, W) < 2\delta\}.$$

Let us denote $h = -\nabla'.m'\mathbb{1}_\Omega \in L^2(\Omega)$. Since equation (3.1) is linear and has a unique solution, then we can decompose $h = h_0 + h_1$ where $h_0 = \theta h$ and obtain $U = U_0 + U_1$ with $i \in \{1, 2\}$

$$\begin{cases} \Delta U_i = 0 & \text{in } \mathbb{R}^3 \backslash (\overline{\Omega} \times \{0\}), \\ \left[\frac{\partial U_i}{\partial x_3}\right] = h_i & \text{on } \Omega \times \{0\}, \\ [U_i] = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

Since $m \in H^k_{loc}(\Omega)$, then $m \in H^k(W_{2\delta})$. It implies that

$$h_0 = \theta h = -\theta\nabla'.m'\mathbb{1}_\Omega \in H^{k-1}(\mathbb{R}^2).$$

Using the same argument as in Remark 3.3, one has

$$\begin{aligned} \int_{\mathbb{R}^2} |\xi|^{2k} |\mathcal{F}(U_0(\cdot, 0))(\xi)|^2 d\xi &= \int_{\mathbb{R}^2} |\xi|^{2k} \frac{1}{4|\xi|^2} |\mathcal{F}(h_0)(\xi)|^2 d\xi \\ &= \frac{1}{4} ||h_0||^2_{\dot{H}^{k-1}(\mathbb{R}^2)}. \end{aligned}$$

This implies that $\nabla'^k U_0(\cdot, 0) \in L^2(\mathbb{R}^2)$. Hence, $U_0(\cdot, 0) \in H^k_{loc}(\mathbb{R}^2)$. We want to prove that $U_1 \in H^k_{loc}(W)$.
Set $V = \Delta' U_1(\cdot, 0)$ in the sense of distributions, where $\Delta' = \partial^2_{x_1x_1} + \partial^2_{x_2x_2}$. We shall prove that $V \in H^{k-2}(W)$. Using a duality argument, we claim that

$$\left| \left(V, \frac{\partial^\alpha}{\partial x^\alpha} \eta \right)_{L^2(W)} \right| \leq C(k) ||\eta||_{L^2(W)}$$

for every $\eta \in C_0^\infty(W)$ and every multi-index $\alpha \in \mathbb{N}^2, |\alpha| = k - 2$.
Indeed, using the definitions of U_1, V and the explicit expression of the $\dot{H}^{1/2}$ scalar product, we get

$$\begin{aligned} \left| (V, \frac{\partial^\alpha}{\partial x^\alpha} \eta)_{L^2(W)} \right| &= \left| (\mathcal{F}(V), \mathcal{F}(\frac{\partial^\alpha}{\partial x^\alpha} \eta))_{L^2(\mathbb{R}^2)} \right| = \left| (-\frac{|\xi|}{2} \mathcal{F}(h_1), \mathcal{F}(\frac{\partial^\alpha}{\partial x^\alpha} \eta))_{L^2(\mathbb{R}^2)} \right| \\ &= \left| \frac{-1}{2} (h_1, \frac{\partial^\alpha}{\partial x^\alpha} \eta)_{\dot{H}^{1/2}(\mathbb{R}^2)} \right| \\ &= \left| \frac{-1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(h_1(x) - h_1(y))(\frac{\partial^\alpha}{\partial x^\alpha} \eta(x) - \frac{\partial^\alpha}{\partial y^\alpha} \eta(y))}{|x - y|^3} dx dy \right| \\ &= \left| \frac{1}{2\pi} \int_{\text{supp } h_1} h_1(x) \int_{\text{supp } \eta} \frac{\frac{\partial^\alpha}{\partial y^\alpha} \eta(y)}{|x - y|^3} dy dx \right| \\ &\leq \frac{C(k)}{\delta^{3+|\alpha|}} ||h_1||_{L^2(\mathbb{R}^2)} ||\eta||_{L^2(W)}. \end{aligned}$$

We have used the Green formula, the estimate $\left| \frac{\partial^\alpha}{\partial y^\alpha} \frac{1}{|x-y|^3} \right| \leq C(k) \frac{1}{|x-y|^{3+|\alpha|}}$ and the fact that supp $(h_1) \subset W_{2\delta} \backslash W_\delta$, supp $(\eta) \subset W$ have distance $\geq \delta$. The proof is completed. □

Appendix B

Operations in Hölder spaces

This appendix is mainly devoted to the proofs of Propositions 5.6 and 5.7, and an approximation result in Hölder spaces, see Proposition B.3.

In the sequel, we will rely on the fact that, given a domain $\Omega \subset \mathbb{R}^n$, for every $r \in \mathbb{N}^*$ and every $\alpha \in (0,1)$, for every $f \in C^{r,\alpha}(\overline{\Omega})$, one has

$$\|f\|_{C^{r,\alpha}(\overline{\Omega})} \leq C(\|f\|_{C^0(\overline{\Omega})} + \|Df\|_{C^{r-1,\alpha}(\overline{\Omega};\mathbb{R}^n)}) \quad , \quad \|Df\|_{C^{r-1,\alpha}(\overline{\Omega};\mathbb{R}^n)} \leq C\|f\|_{C^{r,\alpha}(\overline{\Omega})}, \tag{B.1}$$

with $C = C(n) > 0$.

In the next estimate, we use the geometric quantities d_Ω and δ_Ω introduced in (5.2) and (5.3).

Lemma B.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a constant $C = C(n) > 0$ such that for every $r \in \mathbb{N}^*$, $\alpha \in (0,1)$ and every $f \in C^{r,\alpha}(\overline{\Omega})$, one has*

$$\|f\|_{C^{r-1,\alpha}} \leq C(\delta_\Omega + d_\Omega)\|f\|_{C^{r,\alpha}}.$$

Proof. Let $\beta \in \mathbb{N}^n$ such that $|\beta| = r-1$ and $x, y \in \Omega$. Then, by the mean value inequality, there exists $C = C(n) > 0$ such that

$$|D^\beta f(x) - D^\beta f(y)| \leq C\|f\|_{C^r} d_\Omega(x,y) \leq C\|f\|_{C^r}|x-y|^\alpha d_\Omega^{1-\alpha}\delta_\Omega^\alpha.$$

This implies that $[D^\beta f]_{C^{0,\alpha}} \leq C\delta_\Omega^\alpha d_\Omega^{1-\alpha}\|f\|_{C^r}$. Since $\|f\|_{C^{r-1}} \leq \|f\|_{C^r}$, one gets

$$\|f\|_{C^{r-1,\alpha}} \leq \|f\|_{C^r} + C\delta_\Omega^\alpha d_\Omega^{1-\alpha}\|f\|_{C^r} \leq (1 + C\delta_\Omega^\alpha d_\Omega^{1-\alpha})\|f\|_{C^{r,\alpha}}.$$

Next,

$$\begin{aligned}(1 + C\delta_\Omega^\alpha d_\Omega^{1-\alpha}) &\leq (C+1)(1 + \delta_\Omega^\alpha d_\Omega^{1-\alpha}) \\ &\leq (C+1)(1 + \max(\delta_\Omega, d_\Omega)^{\alpha+(1-\alpha)}) \\ &\leq (C+1)(1 + \delta_\Omega + d_\Omega).\end{aligned}$$

The result follows since $\delta_\Omega \geq 1$. $\square$

We proceed with the

Proof of Proposition 5.6. . We prove the result by induction on $r \in \mathbb{N}$. For $r = 0$, let $f, g \in C^{0,\alpha}(\overline{\Omega})$. Then $\|fg\|_{C^0} \leq \|f\|_{C^0}\|g\|_{C^0}$ and for every $x, y \in \Omega$, we can write

$$|(fg)(x) - (fg)(y)| \leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(y)| \leq \Big(\|f\|_{C^0}[g]_{C^{0,\alpha}} + [f]_{C^{0,\alpha}}\|g\|_{C^0}\Big)|x-y|^\alpha.$$

This implies

$$[fg]_{C^{0,\alpha}} \leq \|f\|_{C^0}[g]_{C^{0,\alpha}} + [f]_{C^{0,\alpha}}\|g\|_{C^0}$$

and thus

$$\|fg\|_{C^{0,\alpha}} \leq \|f\|_{C^0}\|g\|_{C^0} + \|f\|_{C^0}[g]_{C^{0,\alpha}} + [f]_{C^{0,\alpha}}\|g\|_{C^0} \leq \|f\|_{C^{0,\alpha}}\|g\|_{C^{0,\alpha}}.$$

We now assume that the result holds true for $r \in \mathbb{N}$ and we prove it for $r+1$. We rely on (B.1) to write

$$\|fg\|_{C^{r+1,\alpha}} \leq C(\|fg\|_{C^0} + \|D(fg)\|_{C^{r,\alpha}}) \leq C(\|f\|_{C^0}\|g\|_{C^0} + \|fDg\|_{C^{r,\alpha}} + \|gDf\|_{C^{r,\alpha}}).$$

By the induction assumption, one gets

$$\|fg\|_{C^{r+1,\alpha}} \leq C(\|f\|_{C^0}\|g\|_{C^0} + (\delta_\Omega + d_\Omega)^r\|f\|_{C^{r,\alpha}}\|Dg\|_{C^{r,\alpha}} + (\delta_\Omega + d_\Omega)^r\|g\|_{C^{r,\alpha}}\|Df\|_{C^{r,\alpha}}),$$

for some new constant $C = C(r,n) > 0$. By Lemma B.1, $\|f\|_{C^{r,\alpha}} \leq C(\delta_\Omega + d_\Omega)\|f\|_{C^{r+1,\alpha}}$. Moreover, $\|Dg\|_{C^{r,\alpha}} \leq C\|g\|_{C^{r+1,\alpha}}$, see (B.1). Hence,

$$\|f\|_{C^{r,\alpha}}\|Dg\|_{C^{r,\alpha}} \leq C(\delta_\Omega + d_\Omega)\|f\|_{C^{r+1,\alpha}}\|g\|_{C^{r+1,\alpha}}.$$

By changing the roles of f and g, we finally obtain

$$\|fg\|_{C^{r+1,\alpha}} \leq C(\delta_\Omega + d_\Omega)^{r+1}\|f\|_{C^{r+1,\alpha}}\|g\|_{C^{r+1,\alpha}}$$

possibly for a different constant $C = C(r,n) > 0$. This proves the assertion for $r+1$, completing the proof of the proposition.

$\square$

We next justify the Remark 5.8: Let $\alpha \in (0,1)$ and $f \in C^1(\overline{\Omega}, \mathbb{R}^m)$, $g \in C^{0,\alpha}(\overline{O}, \mathbb{R})$ with $f(\overline{\Omega}) \subset \overline{O}$. First,

$$\|g \circ f\|_{C^0} \leq \|g\|_{C^0}.$$

Moreover, for every $x, y \in \overline{\Omega}$, the mean value inequality implies

$$|g \circ f(x) - g \circ f(y)| \leq [g]_{C^{0,\alpha}}|f(x) - f(y)|^\alpha \leq [g]_{C^{0,\alpha}}||Df||^\alpha_{C^0}|x-y|^\alpha\delta^\alpha_\Omega.$$

Hence,

$$[g \circ f]_{C^{0,\alpha}} \leq [g]_{C^{0,\alpha}}\,||Df||^\alpha_{C^0}\delta^\alpha_\Omega.$$

Thus,

$$||g \circ f||_{C^{0,\alpha}} = ||g \circ f||_{C^0} + [g \circ f]_{C^{0,\alpha}} \leq ||g||_{C^{0,\alpha}}(||Df||^\alpha_{C^0}\delta^\alpha_\Omega + 1) \leq \delta_\Omega||g||_{C^{0,\alpha}}(||Df||^\alpha_{C^0} + 1).$$

In the last inequality, we have used the fact that $\delta_\Omega \geq 1$. This completes the proof of Remark 5.8.

We now give the

Proof of Proposition 5.7. We prove the assertion by induction on $r \geq 1$. For $r = 1$, we write that

$$\|g \circ f\|_{C^{1,\alpha}} = \max_{|\beta|\leq 1}\|D^\beta(g \circ f)\|_{C^0} + \max_{|\beta|=1}[D^\beta(g \circ f)]_{C^{0,\alpha}}.$$

By writing for every $|\beta| = 1$,

$$D^\beta(g \circ f) = \sum_{|\gamma|=1}(D^\gamma g) \circ f D^\beta f^\gamma,$$

one gets

$$\|D^\beta(g\circ f)\|_{C^0}\leq \|(Dg)\circ f\|_{C^0}\|Df\|_{C^0}\leq \|Dg\|_{C^0}\|Df\|_{C^0}$$

and also

$$[D^\beta(g\circ f)]_{C^{0,\alpha}}\leq \|(Dg)\circ f\|_{C^0}[Df]_{C^{0,\alpha}}+[(Dg)\circ f]_{C^{0,\alpha}}\|Df\|_{C^0}.$$

By Remark 5.8, one has $[(Dg)\circ f]_{C^{0,\alpha}}\leq \delta_\Omega\|Dg\|_{C^{0,\alpha}}(\|Df\|^\alpha_{C^0}+1)$. Hence,

$$\begin{aligned}[D^\beta(g\circ f)]_{C^{0,\alpha}}&\leq \|Dg\|_{C^0}[Df]_{C^{0,\alpha}}+\delta_\Omega\|Dg\|_{C^{0,\alpha}}(\|Df\|^\alpha_{C^0}+1)\|Df\|_{C^0}\\&\leq C\delta_\Omega\|g\|_{C^{1,\alpha}}(\|Df\|^{1+\alpha}_{C^{0,\alpha}}+1).\end{aligned}$$

This proves the result for $r=1$. Assuming the estimate for $r\geq 1$, let us prove it for $r+1$. We write

$$\begin{aligned}||g\circ f||_{C^{r+1,\alpha}}&\leq C(||g\circ f||_{C^0}+||D(g\circ f)||_{C^{r,\alpha}})\\&\leq C(||g||_{C^0}+\sum_{|\beta|=|\gamma|=1}||(D^\gamma g)\circ fD^\beta f^\gamma||_{C^{r,\alpha}}).\end{aligned}\tag{B.2}$$

By Proposition 5.6,

$$||(D^\gamma g)\circ fD^\beta f^\gamma||_{C^{r,\alpha}}\leq C(\delta_\Omega+d_\Omega)^r||(D^\gamma g)\circ f||_{C^{r,\alpha}}||D^\beta f^\gamma||_{C^{r,\alpha}}.$$

By the induction assumption, one has

$$||(D^\gamma g)\circ f||_{C^{r,\alpha}}\leq C(\delta_\Omega+d_\Omega)^{r^2}\|D^\gamma g\|_{C^{r,\alpha}}(\|Df\|^{r+\alpha}_{C^{r-1,\alpha}}+1)$$

and thus

$$\begin{aligned}||(D^\gamma g)\circ fD^\beta f^\gamma||_{C^{r,\alpha}}&\leq C(\delta_\Omega+d_\Omega)^{r^2+r}\|Dg\|_{C^{r,\alpha}}(\|Df\|^{r+\alpha}_{C^{r-1,\alpha}}+1)\|Df\|_{C^{r,\alpha}}\\&\leq C(\delta_\Omega+d_\Omega)^{r^2+r+r+\alpha}\|g\|_{C^{r+1,\alpha}}(\|Df\|^{r+\alpha+1}_{C^{r,\alpha}}+1).\end{aligned}$$

In the last line, we have used Lemma B.1 to write $\|Df\|_{C^{r-1,\alpha}}\leq C(\delta_\Omega+d_\Omega)\|Df\|_{C^{r,\alpha}}$ and $\|Dg\|_{C^{r,\alpha}}\leq C\|g\|_{C^{r+1,\alpha}}$. Inserting this inequality into (B.2), one obtains

$$||g\circ f||_{C^{r+1,\alpha}}\leq C(\delta_\Omega+d_\Omega)^{(r+1)^2}\|g\|_{C^{r+1,\alpha}}(\|Df\|^{r+\alpha+1}_{C^{r,\alpha}}+1).$$

This proves the assertion for $r+1$ and completes the proof of the proposition.

□

As an application of Propositions 5.6 and 5.7, we deduce the following version of Proposition 5.7 for differential forms:

Lemma B.2. *Let $r\in\mathbb{N},\alpha\in(0,1)$ and $\rho:\overline{\Omega}\to\overline{V}$ be a $C^{r+1,\alpha}$ map, where Ω and V are two bounded domains in $\mathbb{R}^n$. Then for every $k\in\{0,\dots,n\}$ and $f\in C^{r,\alpha}(\overline{V},\Lambda^k)$,*

$$\|\rho^*(f)\|_{C^{r,\alpha}(\overline{\Omega},\Lambda^k)}\leq C\|f\|_{C^{r,\alpha}(\overline{V},\Lambda^k)},$$

where C depends only on r,α,n, $\|\rho\|_{C^{r+1,\alpha}}$, and the geometrical quantities d_Ω,δ_Ω.

Proof. Let $f\in\sum_{1\leq i_1<\ldots<i_k\leq n}f_{i_1\ldots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}$. Then

$$\rho^*(f)=\sum_{1\leq i_1<\ldots<i_k\leq n}f_{i_1\ldots i_k}\circ\rho d\rho^{i_1}\wedge\ldots\wedge d\rho^{i_k},$$

where

$$d\rho^s = \sum_{i=1}^{n} \frac{\partial \rho^s}{\partial x_i} dx^i, \quad \forall s \in \{0, ..., n\}.$$

Hence,

$$\begin{aligned}
\rho^*(f) \quad &= \sum_{1 \le i_1 < ... < i_k \le n} f_{i_1 ... i_k} \circ \rho \sum_{j_l \neq j_m, \forall l \neq m} \left(\frac{\partial \rho^{i_1}}{\partial x_{j_1}} ... \frac{\partial \rho^{i_k}}{\partial x_{j_k}} \right) dx^{j_1} \wedge ... \wedge dx^{j_k} \\
&= \sum_{j_l \neq j_m, \forall l \neq m} \left(\sum_{1 \le i_1 < ... < i_k \le n} f_{i_1 ... i_k} \circ \rho . \frac{\partial \rho^{i_1}}{\partial x_{j_1}} ... \frac{\partial \rho^{i_k}}{\partial x_{j_k}} \right) dx^{j_1} \wedge ... \wedge dx^{j_k}.
\end{aligned}$$

This implies that

$$||\rho^*(f)||_{C^{r,\alpha}} \le \sum_{j_l \neq j_m, \forall l \neq m} \sum_{1 \le i_1 < ... < i_k \le n} \left\| f_{i_1 ... i_k} \circ \rho . \frac{\partial \rho^{i_1}}{\partial x_{j_1}} ... \frac{\partial \rho^{i_k}}{\partial x_{j_k}} \right\|_{C^{r,\alpha}}.$$

Using Proposition 5.6, we obtain

$$||\rho^*(f)||_{C^{r,\alpha}} \le C \sum_{1 \le i_1 < ... < i_k \le n} ||f_{i_1 ... i_k} \circ \rho||_{C^{r,\alpha}} \|D\rho\|^k_{C^{r,\alpha}},$$

where $C = C(d_\Omega, \delta_\Omega, k, n) > 0$. When $r \ge 1$, Proposition 5.7 implies

$$||\rho^*(f)||_{C^{r,\alpha}(\overline{\Omega}, \Lambda^k)} \le C \sum_{1 \le i_1 < ... < i_k \le n} ||f_{i_1 ... i_k}||_{C^{r,\alpha}} (||D\rho||^{r+\alpha}_{C^{r-1,\alpha}} + 1) ||D\rho||^k_{C^{r,\alpha}},$$

possibly for a larger constant C. In view of (B.1), this leads to the conclusion. When $r = 0$, we rely instead on Remark 5.8 and conclude similarly. $\square$

The end of this section is devoted to the proof of the following approximation result:

Proposition B.3. *Let $\alpha \in (0, 1)$ and Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^n$. Then there exists a sequence $\{\eta_i\}_{i \in \mathbb{N}}$ mapping linearly, for every $\ell \in \{0, \dots, n\}$, $C^{0,\alpha}_z(\overline{\Omega}, \Lambda^\ell)$ into $C^\infty_c(\Omega, \Lambda^\ell)$ and such that for every $g \in C^{0,\alpha}_z(\overline{\Omega}, \Lambda^\ell)$, for every $\alpha' \in (0, \alpha)$,*

$$\lim_{i \to \infty} \|\eta_i(g) - g\|_{C^{0,\alpha'}(\overline{\Omega}, \Lambda^\ell)} = 0.$$

Moreover, for every $g \in W^{1,1}_0(\Omega, \Lambda^\ell) \cap C^{0,\alpha}_z(\overline{\Omega}, \Lambda^\ell)$ such that $\boldsymbol{d}g \in C^{0,\alpha}_z(\Omega, \Lambda^{\ell+1})$, one has

$$\eta_i(\boldsymbol{d}g) = \boldsymbol{d}(\eta_i(g)).$$

The proof of this proposition is based on several technical remarks. The first one is a result on the continuity of translations in Hölder spaces.

Lemma B.4. *Let $\zeta \in C^\infty_c(\mathbb{B}(0, 2))$ such that $0 \le \zeta \le 1$ and for every $t \in \mathbb{R}$,*

$$\beta_t : x \in \mathbb{R}^n \mapsto x + \zeta(x) t \overrightarrow{e_n}.$$

Then there exist $\varepsilon > 0$ and $C > 0$ such that for every $0 < \alpha' < \alpha < 1$, for every $f \in C^{0,\alpha}(\mathbb{R}^n)$ and every $t \in [-\varepsilon, \varepsilon]$,

$$\|f \circ \beta_t - f\|_{C^{0,\alpha'}(\mathbb{R}^n)} \le C |t|^{\alpha - \alpha'} \|f\|_{C^{0,\alpha}(\overline{\mathbb{B}(0,2)})},$$

where C only depends on ζ.

Proof. There exists a ball $B \Subset \mathbb{B}(0,2)$ such that $B \Supset \operatorname{supp}\zeta$. In order to simplify the notation, we set $B' = \mathbb{B}(0,2)$. Let $\varepsilon > 0$ such that $B + \mathbb{B}(0,\varepsilon) \subset B'$. Let $f \in C^{0,\alpha}(\mathbb{R}^n)$ and $t \in [-\varepsilon,\varepsilon]$. Then $f \circ \beta_t = f$ outside B and $\beta_t(B) \subset B'$. Using the Hölder continuity of f, we thus have for every $x \in \mathbb{R}^n$,

$$|f \circ \beta_t(x) - f(x)| \leq [f]_{C^{0,\alpha}(B')} |\beta_t(x) - x|^\alpha.$$

By using the definition of β_t and the fact that $0 \leq \zeta \leq 1$, one gets

$$|f \circ \beta_t(x) - f(x)| \leq [f]_{C^{0,\alpha}(\overline{B'})} t^\alpha. \tag{B.3}$$

Next, for every $x \in B$ and $y \in \mathbb{R}^n$, we estimate the quantity

$$|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)|$$

as follows: When $|x - y| < t$, we write

$$|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| \leq |f \circ \beta_t(x) - f \circ \beta_t(y)| + |f(x) - f(y)|.$$

By Hölder continuity of f and the fact that $x, \beta_t(x), y$ and $\beta_t(y)$ all belong to B', this gives

$$|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| \leq [f]_{C^{0,\alpha}(\overline{B'})} (|\beta_t(x) - \beta_t(y)|^\alpha + |x - y|^\alpha).$$

By the definition of β_t and the mean value inequality applied to ζ, one gets

$$\begin{aligned}|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| &\leq [f]_{C^{0,\alpha}(\overline{B'})} (\|\nabla\zeta\|^\alpha_{C^0(\overline{B'})} + 2)|x - y|^\alpha \\ &\leq [f]_{C^{0,\alpha}(\overline{B'})} (\|\nabla\zeta\|^\alpha_{C^0(\overline{B'})} + 2)|t|^{\alpha-\alpha'}|x - y|^{\alpha'}.\end{aligned}$$

When $|x - y| \geq t$ instead, we write

$$|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| \leq |f \circ \beta_t(x) - f(x)| + |f \circ \beta_t(y) - f(y)|.$$

By (B.3), one gets

$$|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| \leq 2[f]_{C^{0,\alpha}(\overline{B'})} t^\alpha \leq 2[f]_{C^{0,\alpha}(\overline{B'})} |t|^{\alpha-\alpha'}|x - y|^{\alpha'}.$$

We have thus proved that for every $x \in B$, $y \in \mathbb{R}^n$,

$$|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| \leq 2[f]_{C^{0,\alpha}(\overline{B'})} (\|\nabla\zeta\|^\alpha_{C^0(\overline{B})} + 1)|t|^{\alpha-\alpha'}|x - y|^{\alpha'}.$$

The case when $x \in \mathbb{R}^n$ and $y \in B$ is similar. Finally, when $x \notin B$ and $y \notin B$, $|(f \circ \beta_t - f)(x) - (f \circ \beta_t - f)(y)| = 0$. This implies that for any $x, y \in \mathbb{R}^n$,

$$[f \circ \beta_t - f]_{C^{0,\alpha'}(\mathbb{R}^n)} \leq 2[f]_{C^{0,\alpha}(\overline{B'})} (\|\nabla\zeta\|^\alpha_{C^0(\overline{B})} + 1)|t|^{\alpha-\alpha'}.$$

The proof is complete. □

We proceed to extend the above lemma to differential forms.

Lemma B.5. *With the notation of Lemma B.4, for every $0 < \alpha' < \alpha < 1$, for every $k \in \{0, \dots, n\}$, for every $f \in C^{0,\alpha}(\mathbb{R}^n, \Lambda^k)$ and every $t \in [-1,1]$,*

$$\|\beta_t^*(f) - f\|_{C^{0,\alpha'}(\mathbb{R}^n,\Lambda^k)} \leq C|t|^{\alpha-\alpha'} \|f\|_{C^{0,\alpha}(\overline{\mathbb{B}(0,2)},\Lambda^k)},$$

where C only depends on η.

Proof. Let $f = \sum_{1\le i_1<\dots<i_k\le n} f_{i_1\dots i_k} dx^{i_1}\wedge\ldots\wedge dx^{i_k}$. Then

$$\beta_t^*(f) = \sum_{1\le i_1<\dots<i_k\le n} f_{i_1\dots i_k}\circ\beta_t d\beta_t^{i_1}\wedge\ldots\wedge d\beta_t^{i_k}.$$

By construction,

$$d\beta_t^i = \begin{cases} dx^i \text{ if } i<n,\\ dx^n + td\zeta \text{ if } i=n.\end{cases}$$

Hence,

$$\beta_t^*(f) = \sum_{1\le i_1<\dots<i_k\le n} f_{i_1\dots i_k}\circ\beta_t dx^{i_1}\wedge\ldots\wedge dx^{i_k} + t\sum_{1\le i_1<\dots<i_{k-1}<n} f_{i_1\dots i_{k-1}n}\circ\beta_t dx^{i_1}\wedge\ldots\wedge dx^{i_{k-1}}\wedge d\zeta.$$

It follows that

$$\begin{aligned}\|\beta_t^*(f)-f\|_{C^{0,\alpha'}(\mathbb{R}^n,\Lambda^k)}\le & \sum_{1\le i_1<\dots<i_k\le n}\|f_{i_1\dots i_k}\circ\beta_t - f_{i_1\dots i_k}\|_{C^{0,\alpha'}(\mathbb{R}^n)}\\ &+C|t|\sum_{1\le i_1<\dots<i_{k-1}<n}\|f_{i_1\dots i_{k-1}n}\circ\beta_t\|_{C^{0,\alpha'}(\overline{\mathbb{B}(0,2)})},\end{aligned}$$

where $C = C(\zeta, n) > 0$. Here, for the second term, we have used Proposition 5.6 and the fact that $d\zeta$ is compactly supported in $\mathbb{B}(0,2)$. Thus, by the triangle inequality,

$$\begin{aligned}\|\beta_t^*(f)-f\|_{C^{0,\alpha'}(\mathbb{R}^n,\Lambda^k)}\le & (1+C|t|)\sum_{1\le i_1<\dots<i_k\le n}\|f_{i_1\dots i_k}\circ\beta_t - f_{i_1\dots i_k}\|_{C^{0,\alpha'}(\mathbb{R}^n)}\\ &+C|t|\sum_{1\le i_1<\dots<i_{k-1}<n}\|f_{i_1\dots i_{k-1}n}\|_{C^{0,\alpha'}(\overline{\mathbb{B}(0,2)})}.\end{aligned}$$

In view of Lemma B.4 applied to each $f_{i_1\dots i_k}$, this gives

$$\begin{aligned}\|\beta_t^*(f)-f\|_{C^{0,\alpha'}(\mathbb{R}^n,\Lambda^k)} &\le C'(1+C|t|)|t|^{\alpha-\alpha'}\sum_{1\le i_1<\dots<i_k\le n}\|f_{i_1\dots i_k}\|_{C^{0,\alpha}(\overline{\mathbb{B}(0,2)})}\\ &\quad + C'|t|\sum_{1\le i_1<\dots<i_{k-1}<n}\|f_{i_1\dots i_{k-1}n}\|_{C^{0,\alpha}(\overline{\mathbb{B}(0,2)})}\\ &\le C''|t|^{\alpha-\alpha'}\|f\|_{C^{0,\alpha}(\overline{\mathbb{B}(0,2)},\Lambda^k)}.\end{aligned}$$

The proof is complete.

□

The next tool is used to approximate a differential form which vanishes on the boundary of a domain Ω by a family of forms which are compactly supported in Ω.

Lemma B.6. *Let $\alpha\in(0,1)$ and Ω be a bounded $C^{1,\alpha}$ domain. Then there exists a family $\phi_t:\mathbb{R}^n\longrightarrow\mathbb{R}^n, 0<t<T$, of $C^{1,\alpha}$ diffeomorphisms, which agree with the identity outside a compact set, and such that*

$$\phi_t(\Omega)\supset\overline{\Omega},\quad 0<t<T, \tag{B.4}$$

$$\forall f\in C^{0,\alpha}_{loc}(\mathbb{R}^n,\Lambda^k), \forall\alpha'\in(0,\alpha),\qquad \lim_{t\to0}\|\phi_t^*(f)-f\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^k)} = 0. \tag{B.5}$$

Proof. The construction below is standard in the setting of Lebesgue spaces, see e.g. [1, Lemma 4.5]. We detail the proof in order to check that it can be extended to Hölder spaces.

Since Ω is of class $C^{1,\alpha}$, one can find a finite covering $\{V_j\}_{1\leq j\leq l}$ of $\partial\Omega$ by bounded open sets, and a corresponding family of $C^{1,\alpha}$ diffeomorphisms $\rho_j : \overline{\mathbb{B}(0,2)} \to \overline{V_j}$ such that

$$\rho_j(\mathbb{B}(0,2)\cap\mathbb{R}^n_+) = V_j\cap\Omega, \quad \rho_j(\mathbb{B}(0,2)\cap(\mathbb{R}^{n-1}\times\{0\})) = V_j\cap\partial\Omega,$$

where $\mathbb{R}^n_+$ is the upper half space $\mathbb{R}^{n-1}\times(0,+\infty)$. One can further assume that $\partial\Omega \subset \bigcup_{j=1}^l \rho_j(\mathbb{B}(0,1/2))$.

Let $\zeta \in C_c^\infty(\mathbb{R}^n)$ be such that

$$0\leq\zeta\leq 1, \quad \zeta|_{\mathbb{B}(0,1)}= 1 \quad \text{and} \quad \zeta|_{\mathbb{R}^n\setminus\mathbb{B}(0,2)}= 0.$$

Define for $t\in[0,1]$, the translation map

$$\begin{aligned} \beta_t : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\mapsto x-\zeta(x)t\overrightarrow{e_n}. \end{aligned}$$

Let $\phi_t^j = \rho_j\circ\beta_t\circ\rho_j^{-1}$ extended by the identity outside V_j and let

$$\phi_t = \phi_t^l\circ ...\circ\phi_t^1.$$

Then ϕ_t is a $C^{1,\alpha}$ diffeomorphism which coincides with the identity outside $\cup_i V_i$. For every j and $t>0$ sufficiently small,

$$\phi_t^j(\Omega)\supset\Omega, \tag{B.6}$$

$$\phi_t^j(\Omega)\supset\overline{\Omega}\cap\rho_j(\mathbb{B}(0,1)). \tag{B.7}$$

It follows from (B.6) that $\phi_t(\Omega)\supset\Omega$ but for t sufficiently small, we have the stronger property

$$\phi_t(\Omega)\supset\overline{\Omega}. \tag{B.8}$$

Indeed, let $x\in\partial\Omega$. Since each ϕ_t^j is a diffeomorphism, we deduce from (B.6) that $\phi_t^j(\overline{\Omega})\supset\overline{\Omega}$. Applying this observation to $\phi_t^l,\phi_t^{l-1},\dots,\phi_t^1$ successively, one gets $l+1$ points $x = x_{l+1},\dots,x_1$ in $\overline{\Omega}$ such that $x_{i+1}=\phi_t^i(x_i)$ for every $i=l,\dots,1$. In particular, $x=\phi_t(x_1)$. It remains to prove that $x_1\in\Omega$. Since $\partial\Omega\subset\bigcup_{j=1}^l\rho_j(\mathbb{B}(0,1/2))$, there exists $i\in\{1,\dots,l\}$ such that $x\in\rho_i(\mathbb{B}(0,1/2))$. Moreover, by construction of ϕ_t^j, for every $j\in\{1,\dots,l\}$,

$$|x_{j+1}-x_j|= |\phi_t^j(x_j)-x_j|\leq|\rho_j|_{C^{0,1}}|\beta_t(\rho_j^{-1}(x_j))-\rho_j^{-1}(x_j)|\leq t|\rho_j|_{C^{0,1}}.$$

It follows that $\max_j|x-x_j|\leq lt\max_j|\rho_j|_{C^{0,1}}$.

There exists $r>0$ such that for every $i\in\{1,\dots,l\}$, for every $x\in\rho_i(B(0,\frac{1}{2}))$, for every $y\in\mathbb{R}^n$ such that $|x-y|< r$, one has $y\in\rho_i(B(0,1))$ (this can be easily seen by contradiction). Then for $t<\frac{r}{l\max_j|\rho_j|_{C^{0,1}}}$, the definition of r implies that $x_{i+1}\in\rho_i(\mathbb{B}(0,1))$. By using (B.7), we deduce that $x_i\in\Omega$. If $i=1$, then we are done. Otherwise, we repeatedly use (B.6) to get that $x_1\in\Omega$, as desired. This completes the proof of (B.8).

We now check the last assertion of the lemma. Set $f_1 = (\phi_t^2)^*...(\phi_t^l)^*(f)$. By the triangle inequality,

$$\begin{aligned} ||\phi_t^*(f)-f||_{C^{0,\alpha'}(\mathbb{R}^n)} &= ||(\phi_t^1)^*(f_1)-f||_{C^{0,\alpha'}(\mathbb{R}^n)} \\ &\leq ||(\phi_t^1)^*(f_1)-f_1||_{C^{0,\alpha'}(\mathbb{R}^n)}+||f_1-f||_{C^{0,\alpha'}(\mathbb{R}^n)}. \end{aligned}$$

Now, using that ρ_1 is a $C^{1,\alpha}$ map which is the identity outside a compact set of $\mathbb{R}^n$ and Lemma B.2

$$||(\phi_t^1)^*(f_1)-f_1||_{C^{0,\alpha'}(\mathbb{R}^n)}\leq C\|\beta_t^*\rho_1^*(f_1)-\rho_1^*(f_1)\|_{C^{0,\alpha'}(\mathbb{R}^n)}.$$

Let $\alpha_1 \in (\alpha', \alpha)$. In view of Lemma B.5, this gives

$$||(\phi_t^1)^*(f_1) - f_1||_{C^{0,\alpha'}(\mathbb{R}^n)} \leq C|t|^{\alpha_1-\alpha'}\|\rho_1^*(f_1)\|_{C^{0,\alpha_1}(\overline{\mathbb{B}(0,2)})}.$$

Using Lemma B.2 again, one gets

$$||(\phi_t^1)^*(f_1) - f_1||_{C^{0,\alpha'}(\mathbb{R}^n)} \leq C|t|^{\alpha_1-\alpha'}\|f_1\|_{C^{0,\alpha_1}(\rho_1(\overline{\mathbb{B}(0,2)})}.$$

By the triangle inequality and the fact that $||f||_{C^{0,\alpha_1}} \leq C||f||_{C^{0,\alpha}}$, $\forall\, 0 \leq \alpha_1 < \alpha \leq 1$, (this constant C does not depend on α, α'), then

$$\begin{aligned} ||(\phi_t^1)^*(f_1) - f_1||_{C^{0,\alpha'}(\mathbb{R}^n)} &\leq C|t|^{\alpha_1-\alpha'}\|f_1 - f\|_{C^{0,\alpha_1}(\mathbb{R}^n)} + C|t|^{\alpha_1-\alpha'}\|f\|_{C^{0,\alpha_1}(\rho_1(\overline{\mathbb{B}(0,2)}))} \\ &\leq C|t|^{\alpha_1-\alpha'}\|f_1 - f\|_{C^{0,\alpha_1}(\mathbb{R}^n)} + C|t|^{\alpha_1-\alpha'}\|f\|_{C^{0,\alpha}(\rho_1(\overline{\mathbb{B}(0,2)}))}. \end{aligned}$$

We have thus proved that

$$||\phi_t^*(f) - f||_{C^{0,\alpha'}(\mathbb{R}^n)} \leq C\|f_1 - f\|_{C^{0,\alpha_1}(\mathbb{R}^n)} + C|t|^{\alpha_1-\alpha'}\|f\|_{C^{0,\alpha}(\rho_1(\overline{\mathbb{B}(0,2)}))}.$$

Repeating the above estimate for $\phi_t^2, \ldots, \phi_t^l$ with a sequence of exponents $\alpha' = \alpha_0 < \alpha_1 < \cdots < \alpha_l = \alpha$, we finally get

$$||\phi_t^*(f) - f||_{C^{0,\alpha'}(\mathbb{R}^n)} \leq C\sum_{i=1}^{l}|t|^{\alpha_i-\alpha_{i-1}}\|f\|_{C^{0,\alpha}(\cup_{i=1}^n \rho_i(\overline{\mathbb{B}(0,2)}))}.$$

This implies (B.5) and completes the proof of Lemma B.6. $\square$

We have now all the ingredients to present the proof of Proposition B.3.

Proof. We introduce a family $(\xi_s)_{s\downarrow 0}$ of mollifiers such that $\operatorname{supp}\xi_s \Subset \mathbb{B}(0,s)$. We also use the family $(\phi_t)_{t\downarrow 0}$ constructed in Lemma B.6.
By (B.4), $\phi_t(\partial\Omega) \subset \mathbb{R}^n \setminus \overline{\Omega}$. Hence, for every (small) $t > 0$, there exists $s_t > 0$ such that $\phi_t(\partial\Omega + \mathbb{B}(0,s_t)) \subset \mathbb{R}^n \setminus \Omega$. Given $g \in C_z^{0,\alpha}(\overline{\Omega}, \Lambda^\ell)$, $\phi_t^*(g_z) = 0$ on $\partial\Omega + \mathbb{B}(0,s_t)$. Here, as usual, g_z is the extension of g by 0 outside Ω. Hence, $\xi_{s_t} * (\phi_t^*(g_z))$ belongs to $C_c^\infty(\Omega, \Lambda^\ell)$. Moreover, for every $g \in W_0^{1,1}(\Omega, \Lambda^\ell)$,

$$\boldsymbol{d}(\xi_{s_t} * (\phi_t^*(g_z))) = \xi_{s_t} * (\boldsymbol{d}(\phi_t^*(g_z))) = \xi_{s_t} * (\phi_t^*(\boldsymbol{d}g_z)).$$

Let $(t_i)_{i\in\mathbb{N}}$ be a sequence decreasing to 0 and $s_i = s_{t_i}$ for every $i \in \mathbb{N}$. We then set $\eta_i(g) = \xi_{s_i} * (\phi_{t_i}^*(g_z))|_{\overline{\Omega}}$. It remains to prove that

$$\lim_{i\to+\infty}\|\eta_i(g) - g\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)} = 0. \tag{B.9}$$

By the triangle inequality,

$$\|\eta_i(g) - g\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)} \leq \|\xi_{s_i} * (\phi_{t_i}^*(g_z)) - \xi_{s_i} * g_z\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)} + \|\xi_{s_i} * g_z - g\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)}.$$

Since $(\xi_{s_i})_{i\in\mathbb{N}}$ is a sequence of mollifiers and $g_z \in C^{0,\alpha}(\mathbb{R}^n, \Lambda^\ell)$,

$$\lim_{i\to+\infty}\|\xi_{s_i} * g_z - g\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)} = 0$$

and moreover, $\|\xi_{s_i} * (\phi_{t_i}^*(g_z)) - \xi_{s_i} * g_z\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)} \leq \|\phi_{t_i}^*(g_z) - g_z\|_{C^{0,\alpha'}(\overline{\Omega},\Lambda^\ell)}$. By Lemma B.6, the latter quantity converges to 0. This proves (B.9). $\square$

Appendix C

Extension of closed forms in Hölder and Sobolev spaces

Throughout this section, we still use the notation $Q_1 = (0,1)^n$ and $Q'_1 = (0,1)^{n-1} \times \{0\}$. The latter will be often identified to $(0,1)^{n-1}$.

Given a k closed form f with $C^{r,\alpha}$ coefficients on Q_1, we want to construct a $(k-1)$ form X with $C^{r+1,\alpha}$ coefficients such that $\boldsymbol{d}X = f$. We do not require any boundary condition on X.

The difficulty is that Q_1 is merely Lipschitz so that the classical Poincaré lemma in Hölder spaces does not apply. Here is our strategy. We first construct an extension $\widetilde{f}$ of f to a smooth open set ω such that $\widetilde{f}$ *is still closed* on ω. We then apply the classical Poincaré lemma on the smooth set ω: this gives $\widetilde{X} \in C^{r+1,\alpha}(\overline{Q_1}, \Lambda^{k-1})$ such that $\boldsymbol{d}\widetilde{X} = \widetilde{f}$. Then by restriction, $X := \widetilde{X}|_{\overline{Q_1}}$ is a solution of the Poincaré lemma on Q_1. In order to construct the closed form $\widetilde{f}$, we need some extensions lemma in the setting of Hölder spaces.

A Sobolev version of the above result is presented at the end of this appendix, where we rely instead on the Bogovskii construction, see Proposition C.7.

The following statement generalizes [13, Lemma 8.11], where the result is proved for $m = 1$.

Lemma C.1. *Let $r, m \in \mathbb{N}$ such that $m \leq r+1$, $\alpha \in (0,1)$ and $g \in C^{r+1-m,\alpha}(Q'_1)$. Then there exists $G \in C^{r+1,\alpha}(\overline{Q_1})$ satisfying, all over $\overline{Q'_1}$,*

$$\frac{\partial^m G}{\partial x_n^m} = g \qquad , \qquad D^\beta G = 0, \text{ for all multi-index } \beta = (\beta_1, \ldots, \beta_n) \text{ such that } \beta_n < m$$

and

$$||G||_{C^{r+1,\alpha}(\overline{Q_1})} \leq C||g||_{C^{r+1-m,\alpha}(\overline{Q'_1})}, \tag{C.1}$$

with $C = C(r, \alpha, m, n) > 0$.

Proof. Using Proposition 5.5, we first extend the function $g \in C^{r+1-m,\alpha}(\overline{Q'_1})$ to $\mathbb{R}^{n-1}$, in such a way that the resulting extension $\overline{g}$ satisfies

$$||\overline{g}||_{C^{r+1-m,\alpha}(\mathbb{R}^{n-1})} \leq C||g||_{C^{r+1-m,\alpha}(\overline{Q'_1})}. \tag{C.2}$$

Let $\varphi \in C_c^\infty(\mathbb{R}^{n-1}), \delta > 0$ be such that

$$\text{supp}\varphi \subset \mathbb{B}'(0,\delta) \qquad \text{and} \int_{\mathbb{R}^{n-1}} \varphi = 1.$$

We define
$$G(x) = G(x', x_n) = \frac{x_n^m}{m!} \int_{\mathbb{R}^{n-1}} \varphi(y')\overline{g}(x' - x_n y')dy'.$$
The function G can be rewritten as
$$G(x', x_n) = \frac{x_n^m}{m!} \int_{\mathbb{R}^{n-1}} \frac{1}{x_n^{n-1}} \varphi\left(\frac{x' - y'}{x_n}\right)\overline{g}(y')dy' \quad \text{for all } x_n > 0. \tag{C.3}$$
We compute the derivatives of G when $x_n \neq 0$. We find, for $1 \leq i \leq n-1$,
$$\begin{aligned} \frac{\partial G}{\partial x_i}(x', x_n) \quad &= \frac{x_n^m}{m!} \int_{\mathbb{R}^{n-1}} \frac{1}{x_n^{n-1}} \frac{\partial \varphi}{\partial x_i}\left(\frac{x' - y'}{x_n}\right) \frac{1}{x_n}\overline{g}(y')dy' \\ &= \frac{x_n^{m-1}}{m!} \int_{\mathbb{R}^{n-1}} \frac{\partial \varphi}{\partial x_i}(y')\overline{g}(x' - x_n y')dy', \end{aligned} \tag{C.4}$$
whereas for $i = n$,
$$\begin{aligned} \frac{\partial G}{\partial x_n}(x', x_n) &= \frac{\partial}{\partial x_n}\left(\frac{x_n^{m+1-n}}{m!} \int_{\mathbb{R}^{n-1}} \varphi(\frac{x' - y'}{x_n})\overline{g}(y')dy'\right) \\ &= \frac{(m+1-n)x_n^{m-1}}{m!} \int_{\mathbb{R}^{n-1}} \varphi(y')\overline{g}(x' - x_n y')dy' \\ &- \frac{x_n^{m-1}}{m!} \int_{\mathbb{R}^{n-1}} \langle \nabla\varphi(y'), y'\rangle \overline{g}(x' - x_n y')dy'. \end{aligned} \tag{C.5}$$

We prove by induction that for every $l \in \{1, \ldots, m\}$, there exists a function $\psi^l \in C_c^\infty(\mathbb{R}^n)$ which depends only on l, φ, m, n such that
$$\frac{\partial^l G}{\partial x_n^l}(x', x_n) = x_n^{m-l} \int_{\mathbb{R}^{n-1}} \psi^l(y')\overline{g}(x' - x_n y')dy'. \tag{C.6}$$
It follows from (C.5) that (C.6) is true when $l = 1$ with
$$\psi^1(y') = \frac{1}{m!}\Big((m+1-n)\varphi(y') - \langle \nabla\varphi(y'), y'\rangle\Big).$$
We assume that (C.6) holds for some $l \in \{1, \ldots, m-1\}$. The same computation as in the case $l = 1$ shows that
$$\begin{aligned} \frac{\partial^{l+1}}{\partial x_n^{l+1}} G(x', x_n) &= \frac{\partial}{\partial x_n}\left(x_n^{m-l} \int_{\mathbb{R}^{n-1}} \psi^l(y')\overline{g}(x' - x_n y')dy'\right) \\ &= x_n^{m-l-1} \int_{\mathbb{R}^{n-1}} \psi^{l+1}(y')\overline{g}(x' - y' x_n)dy' \end{aligned}$$
with
$$\psi^{l+1}(y') = \Big((m - l + 1 - n)\psi^l(y') - \langle \nabla\psi^l(y'), y'\rangle\Big).$$
This completes the proof of (C.6).
As in the computation leading to (C.4), one has for $(i_1, .., i_k) \in \{1, \ldots, n-1\}^k$ and $k + l \leq m$,
$$\frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G(x', x_n) = x_n^{m-k-l} \int_{\mathbb{R}^{n-1}} \psi^l_{i_1 \ldots i_k}(y')\overline{g}(x' - x_n y')dy'. \tag{C.7}$$
where
$$\psi^l_{i_1 \ldots i_k}(y') = \left(\frac{\partial^k}{\partial x_{i_1} \ldots \partial x_{i_k}} \psi^l(y')\right).$$

The formula (C.7) extends continuously to $x_n = 0$.
When $k + l < m$, then

$$\frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G(x', 0) = 0.$$

When $k + l = m$, one gets

$$\frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G(x', 0) = \overline{g}(x') \int_{\mathbb{R}^{n-1}} \psi^l_{i_1,..,i_k}(y')\, dy'. \tag{C.8}$$

Observe that $\psi^l_{i_1,..,i_k}$ does not depend on $\overline{g}$. In the particular case where $\overline{g} \equiv 1$, then by construction, $G(x', x_n) = x_n^m/m!$. Hence, when $l \leq m - 1$, (C.8) gives

$$0 = \frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} \left(\frac{x_n^m}{m!}\right)\bigg|_{x_n=0} = \int_{\mathbb{R}^{n-1}} \psi^l_{i_1,..,i_k}(y')\, dy'.$$

When $l = m$, one has instead

$$1 = \frac{\partial^m}{\partial x_n^m} \left(\frac{x_n^m}{m!}\right)\bigg|_{x_n=0} = \int_{\mathbb{R}^{n-1}} \psi^l_{i_1,..,i_k}(y')\, dy'.$$

Coming back to (C.8) for a general $\overline{g}$, this implies that when $l \leq m - 1$,

$$\frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G(x', 0) = 0$$

while when $l = m$,

$$\frac{\partial^m}{\partial x_n^m} G(x', 0) = \overline{g}(x').$$

Finally, in order to establish the estimate (C.1), one starts from (C.7) with $k + l = m$:

$$\frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G(x', x_n) = \int_{\mathbb{R}^{n-1}} \psi^l_{i_1,..,i_k}(y')\overline{g}(x' - x_n y')dy'.$$

Since $\psi^l_{i_1,..,i_k} \in C_c^\infty(\mathbb{R}^{n-1})$ and $\overline{g} \in C^{r+1-m,\alpha}(\mathbb{R}^{n-1})$, one easily deduces that the function $\frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G$ belongs to $C^{r+1-m,\alpha}_{loc}(\mathbb{R}^{n-1} \times [0, +\infty))$ with

$$\left\|\frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \frac{\partial^l}{\partial x_n^l} G\right\|_{C^{r+1-m,\alpha}(\overline{Q_1})} \leq C\|\overline{g}\|_{C^{r+1-m,\alpha}(\mathbb{R}^{n-1})} \leq C'\|g\|_{C^{r+1-m,\alpha}(\overline{Q_1'})},$$

where the last inequality follows from (C.2). Since $G \in C^\infty(\mathbb{R}^{n-1} \times (0, +\infty))$, it follows that $G \in C^{r+1,\alpha}(\mathbb{R}^{n-1} \times [0, +\infty))$ with the corresponding estimates on $\overline{Q_1}$. This completes the proof of Lemma C.1. □

We deduce from the above lemma the following statement where we prescribe each normal derivative of the extension F.

Lemma C.2. *Let $r \in \mathbb{N}$ and $\alpha \in (0, 1)$. Let $(c_i)_{1\leq i\leq r+1}$ be a family of functions such that*

$$c_i \in C^{r+1-i,\alpha}([0, 1]^{n-1} \times \{0\}).$$

Then there exists $b \in C^{r+1,\alpha}(\overline{Q_1})$ satisfying, all over $Q_1' = (0, 1)^{n-1} \times \{0\}$,

$$b = 0$$

and

$$\frac{\partial^i b}{\partial x_n^i} = c_i \quad , \qquad i = 1, ..., r+1.$$

Moreover there exists a constant $C > 0$ such that

$$||b||_{C^{r+1,\alpha}(\overline{Q_1})} \leq C\bigg(\sum_{i=1}^{r+1} ||c_i||_{C^{r+1-i,\alpha}(Q_1')}\bigg).$$

Proof of Lemma C.2. We set $F_0 = 0$. By Lemma C.1, we can construct by induction on $i = 1, \ldots, r+1$, a function $F_i \in C^{r+1,\alpha}(\overline{Q_1})$ such that

$$D^\beta F_i = 0, \text{ on } Q_1', \text{ for all multi-index } \beta \text{ such that } |\beta| < i$$

and

$$\frac{\partial^i}{\partial x_n^i} F_i = c_i - \frac{\partial^i}{\partial x_n^i}\bigg(\sum_{j=0}^{i-1} F_j\bigg).$$

Then, the function $b = \sum_{i=1}^{r+1} F_i$ satisfies all the required properties.

□

Here is a version of Lemma C.2 for differential forms.

Lemma C.3. *Let $r \in \mathbb{N}$, $k \in \{1, \ldots, n\}$ and $\alpha \in (0,1)$. Let $c \in C^{r,\alpha}(\overline{Q_1}, \Lambda^k)$ be such that $\nu \wedge c = 0$ on Q_1'. When $r \geq 1$, we also introduce a family $(c^i)_{2 \leq i \leq r+1}$ such that $c_i \in C^{r+1-i,\alpha}(\overline{Q_1}, \Lambda^{k-1})$ for each $i \in \{2, \ldots, r+1\}$. Then there exists $b \in C^{r+1,\alpha}(\overline{Q_1}, \Lambda^{k-1})$ satisfying all over Q_1' :*

$$\boldsymbol{d}b = c, \qquad \boldsymbol{\delta}b = 0, \qquad b = 0$$

and when $r \geq 1$,

$$\frac{\partial^i b}{\partial x_n^i} = c^i, \qquad \textit{for all } 2 \leq i \leq r+1. \tag{C.9}$$

Moreover, there exists a constant $C = C(r, \alpha, n) > 0$ such that

$$||b||_{C^{r+1,\alpha}(\overline{Q_1}, \Lambda^{k-1})} \leq C\bigg(||c||_{C^{r,\alpha}(\overline{Q_1'}, \Lambda^k)} + \sum_{i=2}^{r+1} ||c^i||_{C^{r+1-i,\alpha}(\overline{Q_1'}, \Lambda^{k-1})}\bigg).$$

The above statement corresponds to [13, Lemma 8.11], except for the condition (C.9) which is not required in the quoted reference.

Proof. We denote $c^j = \sum_{1 \leq i_1 < \ldots < i_{k-1} \leq n} c^j_{i_1 \ldots i_{k-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$. By Lemma C.2, for every multi-index $1 \leq i_1 < \ldots < i_{k-1} \leq n$, there exists $b_{i_1 \ldots i_{k-1}} \in C^{r+1,\alpha}(\overline{Q_1})$ such that on $\overline{Q_1'}$,

$$b_{i_1 \ldots i_{k-1}} = 0 \quad , \qquad \nabla b_{i_1 \ldots i_{k-1}} = (\nu \lrcorner c)_{i_1 \ldots i_{k-1}} \nu,$$

and when $r \geq 1$,

$$\frac{\partial^i b_{i_1 \ldots i_{k-1}}}{\partial x_n^i} = c^i_{i_1 \ldots i_{k-1}}, \qquad 2 \leq i \leq r+1. \tag{C.10}$$

Set

$$b = \sum_{1 \leq i_1 < \ldots < i_{k-1} \leq n} b_{i_1 \ldots i_{k-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}.$$

A simple computation shows that on $\overline{Q_1'}$

$$\boldsymbol{d}b = \nu \wedge (\nu \lrcorner c) \qquad \text{and } \boldsymbol{\delta}b = \nu \lrcorner (\nu \lrcorner c) = 0.$$

We combine the first equation with the hypothesis $\nu \wedge c = 0$ to get

$$\boldsymbol{d} b = \nu \wedge (\nu \lrcorner c) + \nu \lrcorner (\nu \wedge c) = c.$$

The last equality relies on (6.11). Identity (C.9) follows from (C.10). The proof is complete. $\square$

In the setting of Sobolev spaces, we will only need the following simplified version of Lemma C.3.

Lemma C.4. *Let $r \in \mathbb{N} \setminus \{0\}$, $k \in \{1, \ldots, n\}$ and $p \in (1, \infty)$. Let $c \in W^{r,p}(Q_1, \Lambda^k)$ be such that $\nu \wedge c = 0$ on Q_1'. Then there exists $b \in W^{r+1,p}(\overline{Q_1}, \Lambda^{k-1})$ satisfying all over Q_1' :*

$$\boldsymbol{d} b = c, \qquad \boldsymbol{\delta} b = 0, \qquad b = 0.$$

Moreover, there exists a constant $C = C(r, p, n) > 0$ such that

$$||b||_{W^{r+1,p}(Q_1, \Lambda^{k-1})} \leq C ||c||_{W^{r-\frac{1}{p},p}(Q_1', \Lambda^k)}.$$

The proof of Lemma C.4 is essentially based on the same ideas as the one of Lemma C.3 and we omit it. However, it relies on a Sobolev version of Lemma C.1, which is more delicate to prove than Lemma C.1 itself, see e.g. [43, Section 2.4.2].

Coming back to the setting of Hölder spaces, we deduce from Lemma C.3 an extension property for closed forms. The difficulty here is that we require that the extension remains closed.

Lemma C.5. *Let $r \in \mathbb{N}$, $k \in \{1, \ldots, n\}$ and $\alpha \in (0, 1)$. Let $f \in C^{r,\alpha}(\overline{Q_1}, \Lambda^k)$ such that $\boldsymbol{d} f = 0$ in Q_1 and $f \wedge \nu = 0$ on Q_1'. Then there exists an extension of f in $\overline{Q_1'} \times [-1, 1]$, that we denote by $\widetilde{f}$, such that*

$$\boldsymbol{d}\widetilde{f} = 0 \text{ in } \overline{Q_1'} \times [-1, 1] \qquad \text{and} \qquad \widetilde{f} = f \text{ in } \overline{Q_1}.$$

Moreover, there exists $C = C(r, \alpha, n) > 0$ such that

$$||\widetilde{f}||_{C^{r,\alpha}(\overline{Q_1'} \times [-1,1], \Lambda^k)} \leq C ||f||_{C^{r,\alpha}(\overline{Q_1}, \Lambda^k)}.$$

Proof. We write $f = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$. Since $\nu \wedge f = 0$ on Q_1', we can rely on Lemma C.3 to construct $\kappa = \sum_{1 \leq i_1 < \ldots < i_{k-1} \leq n} \kappa_{i_1 \ldots i_{k-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$ with $C^{r+1,\alpha}(\overline{Q_1'} \times [-1, 0])$ coefficients such that all over $\overline{Q_1'}$:

$$\boldsymbol{d}\kappa = f, \qquad \kappa = 0, \qquad \boldsymbol{\delta}\kappa = 0 \tag{C.11}$$

and when $r \geq 1$, for every $j \in \{1, \ldots, r\}$, for every $1 \leq i_1 < \ldots < i_{k-1} \leq n$,

$$\frac{\partial^{j+1}}{\partial {x_n}^{j+1}} \kappa_{i_1 \ldots i_{k-1}} = (-1)^{k-1} \frac{\partial^j}{\partial {x_n}^j} f_{i_1 \ldots i_{k-1} n}, \qquad \text{if } i_{k-1} < n, \tag{C.12}$$

$$\frac{\partial^{j+1}}{\partial {x_n}^{j+1}} \kappa_{i_1 \ldots i_{k-1}} = 0, \qquad \text{if } i_{k-1} = n. \tag{C.13}$$

Moreover there exists a constant $C > 0$ such that

$$||\kappa||_{C^{r+1,\alpha}(\overline{Q_1'} \times [-1,0])} \leq C ||f||_{C^{r,\alpha}(\overline{Q_1'})}.$$

If follows from the two facts $\kappa = 0$ and $\boldsymbol{\delta}\kappa = 0$ that

$$\frac{\partial}{\partial x_n} \kappa_{i_1 \ldots i_{k-2} n} = 0, \text{ for every } 1 \leq i_1 < \ldots < i_{k-2} < n. \tag{C.14}$$

Indeed, since $\boldsymbol{\delta}\kappa = 0$, one has for $i_{k-2} < n$,

$$\sum_{\gamma=1}^{k-2}(-1)^{\gamma-1}\sum_{i_{\gamma-1}<j<i_\gamma}\frac{\partial\kappa_{i_1\ldots i_{\gamma-1}ji_\gamma\ldots i_{k-2}}}{\partial x_j}+(-1)^{k-2}\sum_{j=i_{k-2}+1}^{n-1}\frac{\partial\kappa_{i_1\ldots i_{k-2}j}}{\partial x_j}+(-1)^{k-2}\frac{\partial\kappa_{i_1\ldots i_{k-2}n}}{\partial x_n}=0.$$

The first two terms of the above sum vanish, since $\kappa = 0$ on Q_1'. Hence, the third term vanishes as well. This proves (C.14). In the following, we just rely on (C.14) and not on the facts that $\kappa = 0$, $\boldsymbol{\delta}\kappa = 0$ on Q_1'.

Finally, we define the desired extension of f as follows:

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \in \overline{Q_1} \\ \boldsymbol{d}\kappa(x) & \text{if } x \in Q_1' \times [-1,0]. \end{cases} \tag{C.15}$$

In order to justify that $\overline{f}$ has $C^{r,\alpha}([0,1]^{n-1}\times[-1,1])$ coefficients, we only need to prove that for every $j \in \{0,\ldots,r\}$,

$$\frac{\partial^j}{\partial x_n^j}(\boldsymbol{d}\kappa)(x',0) = \frac{\partial^j}{\partial x_n^j}f(x',0), \qquad \forall x' \in \overline{Q_1'} \text{ and } j \le r. \tag{C.16}$$

The case $j = 0$ follows from the fact that $\boldsymbol{d}\kappa = f$ on Q_1'. When $j \ge 1$, this amounts to prove that

$$\frac{\partial^j}{\partial x_n^j}((\boldsymbol{d}\kappa)_{i_1..i_k}) = \frac{\partial^j}{\partial x_n^j}(f_{i_1\ldots i_k}) \text{ on } \overline{Q_1'}, \text{ for every } 1 \le i_k < \ldots < i_k \le n. \tag{C.17}$$

Let us first consider the case $j = 1$ (which implicitly means that we are in the case $r \ge 1$). Fix $1 \le i_1 < \ldots < i_{k-1} < i_k \le n$. By definition of the exterior differential operator, one has

$$\frac{\partial}{\partial x_n}((\boldsymbol{d}\kappa)_{i_1\ldots i_k}) = \sum_{\gamma=1}^{k}(-1)^{\gamma-1}\frac{\partial^2\kappa_{i_1\ldots\widehat{i_\gamma}\ldots i_k}}{\partial x_{i_\gamma}\partial x_n}. \tag{C.18}$$

Assume that $i_k = n$. We then rely on (C.14) to deduce that on Q_1', for every $\gamma < k$,

$$\frac{\partial^2\kappa_{i_1\ldots\widehat{i_\gamma}\ldots i_{k-1}n}}{\partial x_{i_\gamma}x_n} = 0.$$

In view of (C.18), we can conclude that

$$\frac{\partial}{\partial x_n}((\boldsymbol{d}\kappa)_{i_1..i_k}) = (-1)^{k-1}\frac{\partial^2\kappa_{i_1\ldots i_{k-1}}}{\partial x_n^2} = \frac{\partial}{\partial x_n}f_{i_1\ldots i_k}.$$

Here in the second equality, we have used (C.12) with $j = 1$.

We next assume that $i_k < n$. On $\overline{Q_1'}$, for every $\gamma \in \{1,\ldots,k\}$,

$$\begin{aligned}(\boldsymbol{d}\kappa)_{i_1\ldots\widehat{i_\gamma}\ldots i_kn} &= \sum_{\alpha=1}^{\gamma-1}(-1)^{\alpha-1}\frac{\partial\kappa_{i_1\ldots\widehat{i_\alpha}\ldots\widehat{i_\gamma}\ldots i_kn}}{\partial x_{i_\alpha}} \\ &+ \sum_{\alpha=\gamma+1}^{k}(-1)^{\alpha}\frac{\partial\kappa_{i_1\ldots\widehat{i_\gamma}\ldots\widehat{i_\alpha}\ldots i_kn}}{\partial x_{i_\alpha}} + (-1)^{k-1}\frac{\partial\kappa_{i_1\ldots\widehat{i_\gamma}\ldots i_k}}{\partial x_n}.\end{aligned}$$

We then observe that

$$\sum_{\gamma=1}^{k}\sum_{\alpha=1}^{\gamma-1}(-1)^{\gamma-1}(-1)^{\alpha-1}\frac{\partial\kappa_{i_1\ldots\widehat{i_\alpha}\ldots\widehat{i_\gamma}\ldots i_kn}}{\partial x_{i_\gamma}\partial x_{i_\alpha}} + \sum_{\gamma=1}^{k}\sum_{\alpha=\gamma+1}^{k}(-1)^{\gamma-1}(-1)^{\alpha}\frac{\partial\kappa_{i_1\ldots\widehat{i_\alpha}\ldots\widehat{i_\gamma}\ldots i_kn}}{\partial x_{i_\gamma}\partial x_{i_\alpha}} = 0.$$

In view of (C.18), this implies that

$$\frac{\partial}{\partial x_n}\Big((\boldsymbol{d}\kappa)_{i_1\dots i_k}\Big) = (-1)^{k-1}\sum_{\gamma=1}^{k}(-1)^{\gamma-1}\frac{\partial}{\partial x_{i_\gamma}}\Big((\boldsymbol{d}\kappa)_{i_1\dots\widehat{i_\gamma}..i_k n}\Big). \tag{C.19}$$

Using that on Q_1', $f_{i_1\dots\widehat{i_\gamma}\dots i_k n} = (\boldsymbol{d}\kappa)_{i_1\dots\widehat{i_\gamma}\dots i_k n}$, one gets

$$\frac{\partial}{\partial x_n}\Big((\boldsymbol{d}\kappa)_{i_1\dots i_k}\Big) = (-1)^{k-1}\sum_{\gamma=1}^{k}(-1)^{\gamma-1}\frac{\partial}{\partial x_{i_\gamma}}f_{i_1\dots\widehat{i_\gamma}..i_k n}.$$

We now exploit the fact that $(\boldsymbol{d}f)_{i_1\dots i_k n} = 0$ to deduce that the right hand side of the above equality is equal to $\frac{\partial}{\partial x_n}f_{i_1\dots i_k}$. This completes the proof of the case $i_k < n$. We have thus proved that (C.17) holds true when $j = 1$.

We now proceed by induction on $j \in \{1,\dots,r\}$. Assume that the result is true for some $j-1 \in \{1,\dots,r-1\}$, namely,

$$\frac{\partial^{j-1}}{\partial x_n^{j-1}}(\boldsymbol{d}\kappa)(x',0) = \frac{\partial^{j-1}}{\partial x_n^{j-1}}f(x',0), \quad \forall x' \in \overline{Q_1'}.$$

Let us prove it for j. Set

$$f^j = \frac{\partial^j}{\partial x_n^j}f \qquad \text{and} \qquad \kappa^j = \frac{\partial^j}{\partial x_n^j}\kappa.$$

We need to establish that

$$\frac{\partial}{\partial x_n}(\boldsymbol{d}\kappa^{j-1}) = \frac{\partial}{\partial x_n}f^{j-1}.$$

This is the same proof as in the case $j = 0$ in view of the fact that $\boldsymbol{d}f^{j-1} = 0$ on Q_1 by the Schwarz lemma and $d\kappa^{j-1} = f^{j-1}$ on Q_1' by the induction assumption. We also rely on the two following identities: for every $i_1 < \cdots < i_{k-1} \le n$,

$$\begin{aligned}\frac{\partial}{\partial x_n}(\kappa^{j-1}_{i_1\dots i_{k-2}i_{k-1}}) &= 0, \text{ if } i_{k-1} = n,\\ \frac{\partial^2}{\partial x_n^2}(\kappa^{j-1}_{i_1\dots i_{k-1}}) &= (-1)^{k-1}\frac{\partial}{\partial x_n}f^{j-1}_{i_1\dots i_{k-1}n}, \text{ if } i_{k-1} < n.\end{aligned}$$

The first equality follows from (C.13) (and is a substitute to (C.14)) while the second one is a consequence of (C.12). This completes the proof. □

We can now state the main result of this section, namely the Poincaré lemma in Hölder spaces defined on a cube, without any boundary condition:

Proposition C.6. *Let $r \in \mathbb{N}$, $k \in \{1,\dots,n\}$ and $\alpha \in (0,1)$. Let $f \in C^{r,\alpha}(\overline{Q_1},\Lambda^k)$ such that $\boldsymbol{d}f = 0$ in $\overline{Q_1}$. Then there exists $X \in C^{r+1,\alpha}(\overline{Q_1},\Lambda^{k-1})$ such that*

$$\boldsymbol{d}X = f \quad \text{in } \overline{Q_1}.$$

Moreover, the correspondence $f \mapsto X$ can be chosen linear and continuous. In particular, there exists $C = C(r,\alpha,n) > 0$ such that

$$||X||_{C^{r+1,\alpha}} \le C||f||_{C^{r,\alpha}}.$$

Proof. When $k = n$, one can rely on the extension property in Hölder spaces, see Proposition 5.5: there exists $\widetilde{f} \in C^{r,\alpha}(\mathbb{R}^n, \Lambda^n)$ such that $\widetilde{f}|_{\overline{Q_1}} = f$ and $\|\widetilde{f}\|_{C^{r,\alpha}(\mathbb{R}^n,\Lambda^n)} \leq C\|f\|_{C^{r,\alpha}(\overline{Q_1},\Lambda^n)}$ for some $C = C(r, n) > 0$. Since $\widetilde{f}$ is an n form, it is automatically closed. Applying the classical Poincaré lemma (see e.g. [13, Theorem 8.3]) on a large ball $\mathbb{B}(0, R)$ containing $\overline{Q_1}$, there exists $\widetilde{X} \in C^{r+1,\alpha}(\overline{\mathbb{B}(0, R)}, \Lambda^{n-1})$ such that $\boldsymbol{d}\widetilde{X} = \widetilde{f}$ and

$$||\widetilde{X}||_{C^{r+1,\alpha}(\overline{\mathbb{B}(0,R)},\Lambda^{n-1})} \leq C||\widetilde{f}||_{C^{r,\alpha}(\overline{\mathbb{B}(0,R)},\Lambda^{n-1})} \leq C||\widetilde{f}||_{C^{r,\alpha}(\mathbb{R}^n,\Lambda^{n-1})} \leq C'||f||_{C^{r,\alpha}(\overline{Q_1},\Lambda^{n-1})}.$$

Then $X = \widetilde{X}|_{\overline{Q_1}}$ satisfies all the desired properties. This proves Proposition C.6 when $k = n$. In particular, this settles the case $n = 1$.

We prove the result by induction on $n \in \mathbb{N}, n \geq 1$. Assume that the result is true for some $n \geq 1$. We will prove that it holds for $n + 1$. Let $k \in \{1, \dots, n + 1\}$ and $f \in C^{r,\alpha}(\overline{Q_1}, \Lambda^k)$ such that $\boldsymbol{d}f = 0$.

We introduce the two following maps:

$$i : x' \in \mathbb{R}^{n-1} \mapsto (x', 0) \in \mathbb{R}^n \qquad , \qquad \pi : (x', x_n) \in \mathbb{R}^n \mapsto x' \in \mathbb{R}^{n-1}.$$

We claim that i^*f is closed on Q_1'. This is obvious when $r \geq 1$ since one can write $\boldsymbol{d}(i^*(f)) = i^*(\boldsymbol{d}f) = 0$. When $r = 0$, we use that for every $\theta \in C_c^\infty(Q_1, \Lambda^{k+1})$,

$$\int_{Q_1} \langle f, \boldsymbol{\delta}\theta \rangle \, dx = 0.$$

We apply the above identity to $\theta(x', x_n) = \zeta(x_n)\theta'(x')$ with $\zeta \in C_c^\infty(0, 1)$, $\theta' = \sum_{I \in \mathcal{I}_{k+1,n-1}} \theta_I' dx^I \in C_c^\infty(Q_1', \Lambda^{k+1})$. Since $\boldsymbol{\delta}\theta = \zeta\boldsymbol{\delta}\theta'$, one has

$$\int_{Q_1} \zeta \langle f, \boldsymbol{\delta}\theta' \rangle \, dx = 0.$$

The Fubini theorem then implies

$$\int_0^1 \zeta(x_n) \, dx_n \int_{Q_1'} \langle f, \boldsymbol{\delta}\theta' \rangle (x', x_n) \, dx' = 0.$$

Since this is true for every $\zeta \in C_c^\infty(0, 1)$, we deduce therefrom that for every $x_n \in (0, 1)$,

$$\int_{Q_1'} \langle f, \boldsymbol{\delta}\theta' \rangle (x', x_n) \, dx' = 0.$$

By continuity of f on $\overline{Q_1}$, this implies that

$$\int_{Q_1'} \langle f, \boldsymbol{\delta}\theta' \rangle (x', 0) \, dx' = 0.$$

Moreover, θ' has no normal component, and the same is true for $\boldsymbol{\delta}\theta'$.
Hence, $\langle f, \boldsymbol{\delta}\theta' \rangle (x', 0) = \langle i^*f, \boldsymbol{\delta}\theta' \rangle (x')$. It follows that

$$\int_{Q_1'} \langle i^*f, \boldsymbol{\delta}\theta' \rangle \, dx' = 0.$$

This proves that i^*f is also closed in the sense of distributions.

We next construct a form $Y \in C^{r+1,\alpha}(\overline{Q_1'} \times [-1, 1], \Lambda^{k-1})$ such that $\nu \wedge \boldsymbol{d}Y = \nu \wedge f$ on $\overline{Q_1'}$ and moreover

$$||Y||_{C^{r+1,\alpha}(\overline{Q_1'}\times[-1,1],\Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}(\overline{Q_1'},\Lambda^k)},$$

where $C = C(r, \alpha, n) > 0$.

Applying the induction hypothesis to the closed form $i^* f$ on Q_1', there exists $X' \in C^{r+1,\alpha}(\overline{Q_1'}, \Lambda^{k-1})$ such that $\boldsymbol{d}X = i^* f$ on Q_1' with the corresponding estimate.

Set $Y = \pi^*(X')$. Then $Y \in C^{r+1,\alpha}(\overline{Q_1'} \times [-1,1], \Lambda^{k-1})$ and

$$i^*(\boldsymbol{d}Y) = \boldsymbol{d}(i^* Y) = \boldsymbol{d}(X') = i^* f,$$

or equivalently, $\nu \wedge \boldsymbol{d}Y = \nu \wedge f$ on $\overline{Q_1'}$. By definition of Y,

$$||Y||_{C^{r+1,\alpha}(\overline{Q_1'} \times [-1,1], \Lambda^{k-1})} \leq C||f||_{C^{r,\alpha}(\overline{Q_1'}, \Lambda^k)}.$$

This completes the construction of Y.

Since $\nu \wedge (f - \boldsymbol{d}Y) = 0$ on Q_1', we can rely on Lemma C.5 to construct an extension $\overline{f} \in C^{r+1,\alpha}(\overline{Q_1'} \times [-1,1], \Lambda^k)$ such that $\overline{f} = f - \boldsymbol{d}Y$ on Q_1, $\boldsymbol{d}\overline{f} = 0$ on $Q_1' \times (-1,1)$ and

$$\|\overline{f}\|_{C^{r,\alpha}(\overline{Q_1'} \times [-1,1], \Lambda^k)} \leq C\|f - \boldsymbol{d}Y\|_{C^{r,\alpha}(\overline{Q_1}, \Lambda^k)}.$$

Then the map $\widetilde{f} = \overline{f} + \boldsymbol{d}Y$ is an extension for f to $\overline{Q_1'} \times [-1,1]$. Moreover, $\boldsymbol{d}\widetilde{f} = 0$ and $\|\widetilde{f}\|_{C^{r,\alpha}(\overline{Q_1'} \times [-1,1], \Lambda^k)} \leq C\|f\|_{C^{r,\alpha}(\overline{Q_1}, \Lambda^k)}$.

We can repeat the same construction in every direction of the coordinate axes to obtain an extension to $[-1,2]^n$, see Figure C.1.

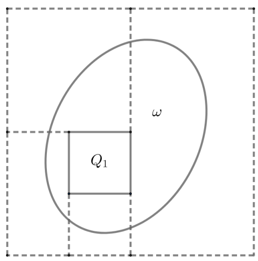

Figure C.1

We then replicate the whole construction sufficiently many times to get an extension to $[-j,j]^n$, with j large enough to ensure that $[-j,j]^n$ contains a ball $\overline{\mathbb{B}(0,R)}$ which contains $\overline{Q_1}$. This yields a k form still denoted by $\widetilde{f} \in C^{r,\alpha}(\overline{\mathbb{B}(0,R)}, \Lambda^k)$ such that $\boldsymbol{d}\widetilde{f} = 0$, $\widetilde{f} = f$ on Q_1 and $\|\widetilde{f}\|_{C^{r,\alpha}(\overline{\mathbb{B}(0,R)}, \Lambda^k)} \leq C\|f\|_{C^{r,\alpha}(\overline{Q_1}, \Lambda^k)}$, with $C = C(r, \alpha, n) > 0$.

We then apply the classical Poincaré lemma [13, Theorem 8.3] on the ball $\mathbb{B}(0,R)$ to get $\widetilde{X} \in C^{r+1,\alpha}(\overline{\mathbb{B}(0,R)}, \Lambda^{k-1})$ such that $\boldsymbol{d}\widetilde{X} = \widetilde{f}$ with the corresponding estimate. Then $X = \widetilde{X}|_{\overline{Q_1}}$ satisfies all the desired properties.

□

Finally, we state the analogue of Proposition C.6 in the setting of Sobolev spaces:

Proposition C.7. *Let $s \in [0, +\infty)$, $k \in \{1, \ldots, n\}$ and $p \in (1, \infty)$. Let $f \in W^{s,p}(Q_1, \Lambda^k)$ such that $\boldsymbol{d}f = 0$ in Q_1, $1 \leq k \leq n$. Then there exists $X \in W^{s+1,p}(Q_1, \Lambda^{k-1})$ satisfying*

$$\boldsymbol{d}X = f \quad \text{in } Q_1.$$

Moreover, the correspondence $f \mapsto X$ can be chosen linear and continuous. In particular, there exists $C = C(s, p, n) > 0$ such that

$$||X||_{W^{s+1,p}} \leq C||f||_{W^{s,p}}.$$

Proof. The above statement corresponds to [12, Proposition 4.1 (i)] where we replace the scale of H^s spaces by the scale of $W^{s,p}$ spaces, which is possible in view of [12, Remark 3.5]. □

In fact, the Bogovskii construction described in [12] can be performed in the whole scale of Besov spaces, see [12, Remark 3.5]. Since Hölder spaces can be seen as particular Besov spaces, this can be used to give an alternate proof to Proposition C.6. We have preferred to give an explicit construction, based on the Poincaré lemma on smooth domains.

[9]

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