Fundamental equivariants and symmetry preservation in multivariate interpolation and H-bases

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Equivariants fondamentaux et préservation de la symétrie dans l'interpolation multivariée et les H-bases.

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Jury :

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Résumé

La symétrie est omniprésente dans la science et l'art. Dans cette thèse, on considère les symétries décrites par la représentation d'un groupe fini pour aborder trois problèmes algébriques dans lesquels la symétrie apparait naturellement : l'interpolation multivariée, l'interpolation idéale et le calcul des invariants et équivariants fondamentaux. Les bases adaptées à la symétrie des anneaux polynomiaux sont essentielles afin de préserver et d'exploiter la symétrie dans ces calculs algébriques. On les utilise pour réduire les calculs d'un facteur qui dépend de la taille du groupe, refléter les symétries initiales sur les solutions fournies, et calculer des ensemble générateurs d'équivariants.

L'interpolation est un outil de premier ordre en calcul algébrique tandis que la symétrie est une caractéristique qualitative qui peut être plus pertinente pour un modèle mathématique que la précision numérique des paramètres. On montre comment préserver exactement la symétrie dans l'interpolation multivariée tout en l'exploitant pour alléger le coût de calcul. On revisite l'interpolation de degré minimal et la moindre interpolation avec des bases adaptées à la symétrie, plutôt que la base monomiale. Cela permet de construire des bases d'espaces d'interpolation invariants par blocs et qui capturent la redondance des calculs dûs à la symétrie. On montre que les bases d'interpolation adaptées à la symétrie ainsi construites allègent le coût de calcul de tout problème d'interpolation et préservent automatiquement toute équivariance que celui-ci pourrait avoir.

Les interpolations multivariées de Lagrange et Hermite sont des exemples d'interpolation idéale. Plus généralement, un problème d'interpolation idéal est défini par un ensemble de formes sur l'anneau polynomial, dont les noyaux se croisent en un idéal. Pour un problème d'interpolation idéal avec symétrie, on aborde le calcul d'une base adaptée à la symétrie du moindre espace d'interpolation et d'une H-base l'idéal adaptée à la symétrie. Outre sa présence manifeste dans la sortie, la symétrie est exploitée à toutes les étapes de l'algorithme.

Les bases adaptées à la symétrie sont constituées d'équivariants fondamentaux et ceux-ci forment des modules sur l'anneau des invariants. Dans cette thèse, on propose trois algorithmes pour calculer des ensembles générateurs pertinents de ces modules, ainsi que des ensembles générateurs pour l'anneau des invariants.

On montre comment la théorie de l'interpolation idéale qu'on a développée peut-être appliquée pour calculer les invariants et les équivariants générateurs d'un groupe de réflexion. Etant donné un ensemble d'invariants primaires pour toute

représentation d'un groupe fini, on applique les algorithmes des Chapitres 3 et 4 pour calculer en même temps un ensemble d'invariants secondaires et des bases libres pour tous les modules équivariants fondamentaux. On propose un nouvel algorithme pour calculer un ensemble d'invariants générateurs simultanément aux équivariants générateurs.

Mots Clés : Interpolation, Symétrie, Théorie des représentations, Action de groupe, H-base, Matrice de Macaulay, Matrice de Vandermonde, Théorie des invariants

Abstract

Symmetry is ubiquitous in science and art. In this thesis we consider symmetries described by the representation of a finite group. Symmetry adapted bases of polynomial rings are essential in order to preserve and exploit symmetry in algebraic computations. In this work we address three algebraic problems in which symmetry is naturally embedded: multivariate interpolation, ideal interpolation and generation of fundamental equivariants. We make use of symmetry adapted bases to reduce the computations by a factor depending on the size of the group, to reflect the initial symmetries on the provided solutions, and to compute generating sets of equivariants.

Interpolation is a prime tool in algebraic computation while symmetry is a qualitative feature that can be more relevant to a mathematical model than the numerical accuracy of the parameters. We show how to exactly preserve symmetry in multivariate interpolation while exploiting it to alleviate the computational cost. We revisit minimal degree and least interpolation with symmetry adapted bases, rather than monomial bases. This allows to construct bases of invariant interpolation spaces in blocks, capturing the inherent redundancy in the computations. We show that the so constructed symmetry adapted interpolation bases alleviate the computational cost of any interpolation problem and automatically preserve any equivariance of this interpolation problem might have.

Multivariate Lagrange and Hermite interpolation are examples of ideal interpolation. More generally, an ideal interpolation problem is defined by a set of linear forms, on the polynomial ring, whose kernels intersect into an ideal.

For an ideal interpolation problem with symmetry, we address the simultaneous computation of a symmetry adapted basis of the least interpolation space and the symmetry adapted H-basis of the ideal. Beside its manifest presence in the output, symmetry is exploited computationally at all stages of the algorithm.

Symmetry adapted bases are indeed made of fundamental equivariants and these form finitely generated modules over the invariant ring. In this work we offers algorithms to compute relevant sets of generators of these modules, together with generators for the ring of invariants.

We show how the ideal interpolation theory that we developed can be applied to compute the generating invariants and equivariants of a reflection group. Given a set of primary invariants for any representation of a finite group, we apply the algorithms in Chapters 3 and 4 to compute both a set of secondary invariants; and

free bases of all fundamental equivariant modules. We propose a new algorithm to compute a set of generating invariants simultaneously to the generating equivariants.

Keywords : Interpolation, Symmetry, Representation Theory, Group Action,H-basis, Macaulay matrix, Vandermonde matrix, Invariant Theory.

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Chapter 1

General introduction

This introductory chapter provides an overview of the contributions in this thesis. Since we kept the chapters of this thesis as close as possible to their original publications, more discussion on the state of the art can be found within the individual chapters.

Preserving and exploiting symmetry in algebraic computations is a challenge that has been addressed within a few topics and, mostly, for specific groups of symmetry; for instance interpolation and symmetric group (Krick et al., 2017), cubature (Collowald and Hubert, 2015), (Gatermann, 1992), global optimisation (Gatermann and Parrilo, 2004; Riener et al., 2013), equivariant dynamical systems (Gatermann, 2000; Hubert and Labahn, 2013) and solving systems of polynomial equations (Faugere and Svartz, 2013; Gatermann, 1990; Gatermann and Guyard, 1999; Hubert and Labahn, 2012, 2016; Riener and Safey El Din, 2018; Verschelde and Gatermann, 1995). Our objective in this thesis is to solve algebraic problems so that the initial symmetries are reflected in the provided solutions. In addition to be manifested in the output, we want symmetry to be exploited all along our algorithms to reduce the size of the matrices involved, and avoid sizable redundancies. We are also interested in the computation of generating sets of fundamentals invariants and equivarinats, which play a key role in a large number of applications in different mathematical disciplines.

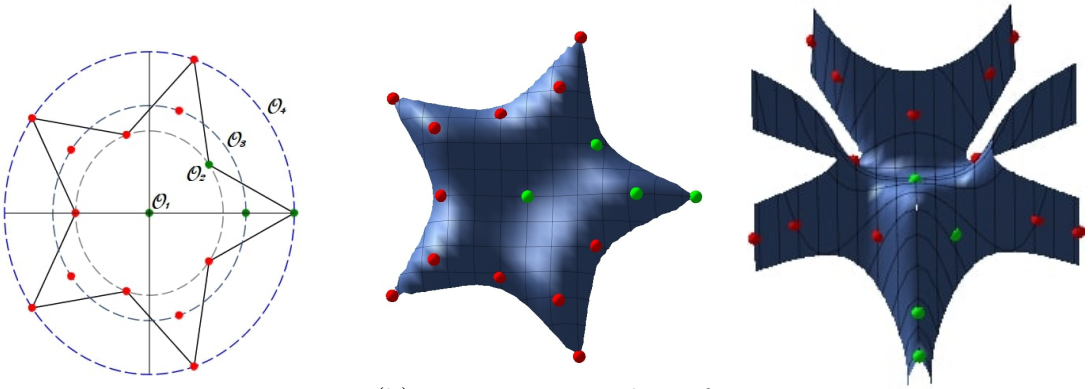
We focus on three problems: Multivariate interpolation, ideal interpolation and computation of fundamental equivariants. However, the theory developed in this work can be used as a framework for different algebraic problems in the presence of symmetries.

Multivariate Interpolation.

Due to its relevance in approximation theory and geometrical modeling, interpolation is a prime topic in algebraic computation. Polynomial interpolation in n variables has been widely studied, see (Gasca and Sauer, 2000; Lorentz, 2000) for

good surveys in the area. Unlike univariate interpolation in which there is a unique interpolation space of polynomials of degree less or equal to k for any given set of $(k+1)$ points in $\mathbb{R}$, there is no such corresponding space in the multivariate case. Then the structure of the input data needs to be taken into account.

Lagrange interpolation is what comes to mind when we speak of interpolation. For a set of r points $\xi_1, \ldots, \xi_r$ in n-space, and r values $\eta_1, \ldots, \eta_r$, Lagrange interpolation consists in finding a n-variate polynomial function p such that $p(\xi_i) = \eta_i$, for $1 \leq i \leq r$. The evaluations at the points ξ_i form a basic example of linear forms. The space they generate is invariant under a group action as soon as the set of points is a union of orbits of this group action. The above interpolation problem is invariant if $\eta_i = \eta_j$ whenever ξ_i and ξ_j belong to the same orbit. It is then natural to expect an invariant polynomial as interpolant. Yet, contrary to the univariate case, there is no unique interpolant of minimal degree and the symmetry of the interpolation problem may very well be violated in the computed solution (compare Figure 1.1c and 1.1b).



(a) interpolation data

(b) invariant interpolant of minimal degree

(c) interpolant obtained from a monomial basis

Figure 1.1: Invariant Lagrange interpolation problem

Among the several problems in multivariate interpolation, we focus on the construction of a polynomial interpolation space for a given set of linear forms. An interpolation space for a set of linear forms is a subspace of the polynomial ring that has a unique interpolant for each instantiated interpolation problem. Along this thesis we consider an interpolation problem as a pair (Λ, ϕ) where Λ is a finite dimensional linear subspace of $\mathbb{K}[x]^*$ and $\phi : \Lambda \longrightarrow \mathbb{K}$ is a $\mathbb{K}$-linear map. An interpolant, i.e., a solution to the interpolation problem, is a polynomial p such that

$$\lambda(p) = \phi(\lambda) \text{ for any } \lambda \in \Lambda.$$

In Chapter 3 we consider general sets of linear forms; instead of, or in addition to, fixing the values at the nodes, we could also impose the values of some derivatives, moment functionals, or coefficients in a given basis of functions. The new

angle on the above problems that is offered in Chapter 3 is to consider general sets of linear forms invariant under a group action and seek to compute interpolants that respect the symmetry of the interpolation problem. We mentioned invariance as an instance of symmetry, but equivariance is the more general concept. We show that the unique interpolants automatically inherit the symmetry of the problem when the interpolation space is invariant. We need to point out that an invariant interpolation space is generally not spanned by monomials, when much of the literature on algebraic computation restricts to monomial bases.

For a space of linear forms Λ, a canonical interpolation space $\Lambda_\downarrow$ also known as the *least interpolation space*, was introduced by De Boor and Ron (1990, 1992a,b). It has a desirable set of properties. We shall observe that it is invariant as soon as the space of linear forms is. In floating point arithmetics though, the computed interpolation space might fail to be exactly invariant. Yet, in mathematical modeling, symmetry is often more relevant than numerical accuracy. We shall remedy this flaw and further exploit symmetry to mitigate the cost and numerical sensitivity of computing a minimal degree or least interpolation space.

Ideal Interpolation and H-Basis.

Multivariate Lagrange and Hermite, interpolation are examples of the encompassing notion of ideal interpolation, introduced in (Birkhoff, 1979). They are defined by linear forms consisting of evaluation at some nodes, and possibly composed with differential operators, without *gaps*. More generally, a space of linear forms Λ on the polynomial ring $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$ is an ideal interpolation scheme if

$$I = \bigcap_{\lambda \in \Lambda} \ker \lambda = \{p \in \mathbb{K}[\mathrm{x}] : \lambda(p) = 0, \text{ for all } \lambda \text{ in } \Lambda\}$$

is an ideal in $\mathbb{K}[\mathrm{x}]$. In the case of Lagrange interpolation, $\mathcal{I}$ is the ideal of the nodes and is thus a radical ideal.

An interpolation space for Λ identifies with the quotient space $\mathbb{K}[\mathrm{x}]/\mathcal{I}$. Hence, a number of operations related to I can already be performed with a basis of an interpolation space for Λ: decide of membership to I, determine normal forms of polynomials modulo $\mathcal{I}$ and compute matrices of multiplication maps in $\mathbb{K}[\mathrm{x}]/\mathcal{I}$. Given a Gröbner basis of I, its corresponding set of reduced monomials span an interpolation space. The relation between the reduction module a Gröbner basis and multivariate interpolation is exploited in (Sauer, 1998; De Boor, 2004).

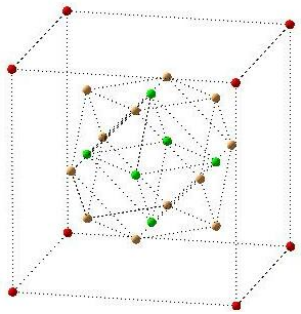
Initiated in (Möller and Buchberger, 1982), for a set Λ of point evaluations, computing a Gröbner basis of I found applications in the design of experiments (Pistone and Wynn, 1996; Pistone et al., 2000). As pointed out in (Marinari et al., 1991), one can furthermore interpret the FGLM algorithm (Faugere et al., 1993) as an instance of this problem. The linear forms are the coefficients, in the normal forms, of the reduced monomials. The alternative approach in (Faugère and Mou, 2017) can

be understood similarly. The resulting algorithm then pertains to the Berlekamp-Massey-Sakata algorithm and is related the multivariate version of Prony's problem to compute Gröbner bases, or border bases or H-basis (Berthomieu et al., 2017; Mourrain, 2017; Sauer, 2017, 2018).

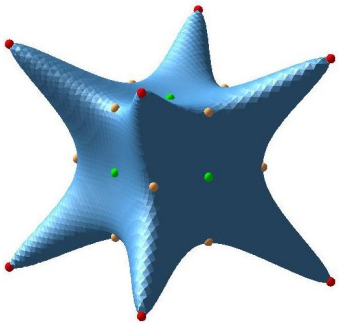
All, the above mentioned algorithms and complexity analyses heavily depend on a term order and basis of monomials. These are notoriously not suited for preserving symmetry. Our ambition in Chapter 4 is to showcase how symmetry can be embedded in the representation of both the interpolation space and the representation of the ideal. This is a marker for the more canonical representations.

In the case of ideal interpolation, the least interpolation space is a canonical representation of the quotient of the polynomial algebra by the ideal. It is the orthogonal complement of the space of leading forms of the defined ideal. It has great properties, even beyond symmetry, that cannot be achieved by a space spanned by monomials. These latter are the object of focus for using H-bases as an alternative approach in interpolation. The least interpolation space can be seen as the space induced by the reduction process module an orthogonal H-basis (Sauer, 2001). H-bases connection with multivariate polynomial interpolation has been approached in the literature (De Boor, 1994; Möller and Sauer, 2000; Sauer, 2002, 2006)

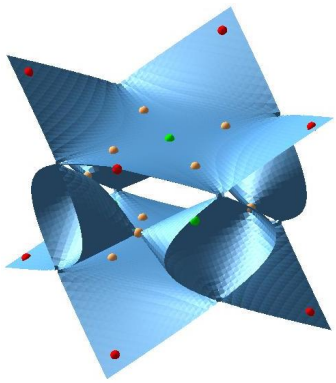
Introduced by Macaulay (1916), H-bases are more intrinsically tied to the ideal. Their computation from generators of the ideal is challenging though holds promises in a symbolic-numeric setting (Möller and Sauer, 2000; Javanbakht and Sauer, 2019). By providing the computation of a *symmetry adapted H-basis*, in the case of ideal interpolation, we demonstrate that symmetry can be naturally preserved and represented in H-bases, as well as exploited in the course of their computation (compare Figure 1.2c and 1.2b).



(a) invariant set of points



(b) Variety of the minimal degree invariant recovered from a symmetry adapted H-basis



(c) Variety of a minimal degree polynomial recovered from Gröbner basis

Figure 1.2: Lowest degree algebraic surface through an invariant set of the points

In Chapter 4 we provide an algorithm to compute simultaneously a symmetry adapted basis of the least interpolation space and a symmetry adapted H-basis of

the associated ideal. Based on QR-decomposition, the algorithm also lends itself to numerical computations. Our algorithm somehow fits in the loose sketch proposed in (De Boor, 1994). Yet we are now in a position to incorporate symmetry in a natural way, refining the algorithm to exploit it. Symmetry is preserved and exploited thanks to the block diagonal structure of the matrices at play in the algorithms.

Computation of fundamental equivariants and invariants.

The computation of invariants of group action has been an active subject since the 19th century. Celebrated mathematicians have left their mark on the subject and textbooks reporting major progress still appeared recently (Derksen and Kemper, 2015; Sturmfels, 2007).

Invariant theory has a wide range of applications. Many of these depend on computing generating set of rings of invariants. Below we mention some relevant applications in different mathematical disciplines.

In computer vision invariants are useful for classifying objects up to the action of a group of transformation. The objects are described in terms of invariants and these invariants are used to provide all the essential information related to the object. A good survey in the area can be found in (Mundy et al., 1992). In (Palmer et al., 2020) the invariants of the octahedral group are used to cut the variety of frame fields that are used to construct hexahedral meshes of volumes in computer graphics. In graph theory the computation of generating invariants for permutation representation has been used to show whether or not two given graphs are isomorphic (Aslaksen et al., 1996; Thiéry, 2000). In code theory, once a code is converted in polynomials constrains, invariant theory is used to restrict the polynomials that satisfy the constraints (Sloane, 1977; Bannai et al., 1999; Nebe et al., 2006). In quantum computing generating invariants of density operators are computed to determine whether or not two quantum systems are locally equivalent (Makhlin, 2002; Albeverio et al., 2005; King et al., 2007). Classical application for solving systems of algebraic equations and for studying the behavior of dynamical systems can be found in (Gatermann, 1990) and (Gatermann, 2000) respectively.

Several algorithms have been proposed for computing generating invariants of finite groups. Algorithms to compute sets of primary and secondary invariants can be found in (Kemper, 1999; Kemper and Steel, 1999; Gatermann, 2000; Sturmfels, 2007). For the computation of fundamental invariant we refer to (King, 2013). In Chapter 5 we give a brief overview of these different algorithms. For a complete overview of the subject as well as a more extensive comparison among the different existing techniques we refer to (Derksen and Kemper, 2015).

Semi-invariants and equivariants, sometimes called covariants, appear frequently in different mathematical models. Equivariants appear for the study of dynamical systems and their bifurcations (Chossat and Lauterbach, 2000; Gatermann, 2000; Golubitsky and Schaeffer, 1985; Golubitsky et al., 1988), mechanics (Olive et al., 2018), and cryptography (Lercier et al., 2016; Mestre, 1991). Despite their relevance

though, the few algorithms to compute equivariants (Stanley, 1979; Worfolk, 1994; Gatermann, 1996) are not as advanced.

Any polynomial equivariant can be written in terms of fundamental equivariants, i.e., equivariants for the irreducible representations of the group, the algorithms presented in this thesis thus fill a void in the subject.

In Chapter 5 we show the existing link between symmetry adapted basis and fundamental equivariants. Symmetry adapted bases are indeed made of fundamental equivariants and these form finitely generated modules over the invariant ring. We propose algorithms to simultaneously compute sets of generators of these modules, together with generators for the ring of invariants. We show how ideal interpolation can be used to compute the generating invariants and equivariants of a reflection group. We also show how to obtain a set of secondary invariants; and free bases of all fundamental equivariant for a set of primary invariants for any representation of a finite group.

SyCo (Symmetry and Computations)

Our research results and algorithms are implemented into a Maple library. We provide generic representations of linear actions over vector spaces and linear forms. In this way our library is flexible, to add new linear forms and groups, it is only necessary to inherit the provided structure. We offer procedures with multiple options to compute symmetry adapted bases, interpolation spaces, H-basis, fundamental invariants and equivariants. Our methods preserve and exploit symmetry at each step.

Much of the content of this work has been published and presented in the following papers and venues:

Multivariate Interpolation:

- Rodriguez Bazan, E., Hubert, E., 2019. Symmetry Preserving Interpolation. In: ISSAC'19. ACM, Beijing, China, pp. 34–41
- Presented in SIAM Conference on Applied Algebraic Geometry. July 11-15, 2019, Bern, Switzerland.
- Poster presentation in Summer School on Numerical Computing in Algebraic Geometry. August 13-17, 2018, Leipzig, Germany.

Ideal Interpolation:

- Rodriguez Bazan, E., Hubert, E., 2020. Ideal Interpolation, H-basis and Symmetry. In: ISSAC'20. ACM, Kalamata, Greece
- Presented in Journées nationales de calcul formel. March 2-6, 2020, CIRM, Marseille, France.

Equivariants

- Hubert, E., Rodriguez Bazan, E., Algorithms for fundamental equivariants and invariants.

This manuscript is structured as follows. We start in Chapter 2 by introducing the relevant mathematical concepts and results that will be used in our work. In Chapter 3 we show how to exactly preserve symmetry in multivariate interpolation. In Chapter 4 we address the simultaneous computation of a symmetry adapted basis of the least interpolation space and the symmetry adapted H-basis of the ideal. In Chapter 5 we offer algorithms to compute relevant sets fundamental equivariants, together with generators for the ring of invariants. Finally, in Chapter 6, we present the library we developed with the implementations of the methods and algorithms from chapters 3 to 5. We concluded with some general overviews and comments on future development.

Chapter 2

Preliminaries

2.1 Notations

In this section we introduce some of the notation we use throughout the thesis. Let $\mathbb{K}$ be either $\mathbb{C}$ or $\mathbb{R}$. $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$ denotes the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$; $\mathbb{K}[\mathrm{x}]_{\leq d}$ and $\mathbb{K}[\mathrm{x}]_d$ the $\mathbb{K}-$vector spaces of polynomials of degree at most d and the space of homogeneous polynomials of degree d respectively. The *dual* of $\mathbb{K}[\mathrm{x}]$, the set of $\mathbb{K}-$linear forms on $\mathbb{K}[\mathrm{x}]$, is denoted by $\mathbb{K}[x]^*$. By $\mathbb{K}[\mathrm{x}]^m$ we denote the space of polynomials maps with $m-$ components.

2.2 Duality

$\mathbb{K}[x]^*$ can be identified with the ring of formal power series $\mathbb{K}[[\partial]] = \mathbb{K}[[\partial_1, \ldots, \partial_r]]$ through the isomorphism $\Phi : \mathbb{K}[[\partial]] \longrightarrow \mathbb{K}[x]^*$, where for $p = \sum_\alpha p_\alpha x^\alpha \in \mathbb{K}[\mathrm{x}]$ and $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha \partial^\alpha \in \mathbb{K}[[\partial]]$

$$\Phi(f)(p) := \sum_{\alpha \in \mathbb{N}^n} f_\alpha \frac{\partial^\alpha p}{\partial x^\alpha}(0) = \sum_{\alpha \in \mathbb{N}} \alpha! f_\alpha p_\alpha.$$

For instance, the evaluation e_ξ at a point $\xi \in \mathbb{K}^n$ is represented by $e^{(\xi,\partial)} = \sum_{k \in \mathbb{N}} \frac{(\xi, \partial)^k}{k!}$, the power series expansion of the exponential function with frequency ξ.

The dual pairing

$$\begin{array}{rcl} \mathbb{K}[x]^* \times \mathbb{K}[\mathrm{x}] & \to & \mathbb{K} \\ (\lambda, p) & \to & \lambda(p) \end{array}$$

induces the *apolar product* on $\mathbb{K}[\mathrm{x}]$ by associating $p \in \mathbb{K}[\mathrm{x}]$ to $\overline{p}(\partial) \in \mathbb{K}[[\partial]]$. For $p = \sum_\alpha p_\alpha x^\alpha$ and $q = \sum_\alpha q_\alpha x^\alpha$ the apolar product between p and q is given by

$$\langle p, q \rangle := \overline{p}(\partial) q = \sum_\alpha \alpha! \overline{p}_\alpha q_\alpha \in \mathbb{K}.$$

Note that for a linear map $a : \mathbb{K}^n \to \mathbb{K}^n$, $\langle p, q \circ a \rangle = \left\langle p \circ \overline{a}^t, q \right\rangle$.

For a set of linearly independent homogeneous polynomials $\mathcal{P}$ we define the dual set $\mathcal{P}^\dagger$ to be a set of homogeneous polynomials such that $\left\langle p_i^\dagger, p_j \right\rangle = \delta_{ij}$. For instance the dual basis of the monomial basis $\{x^\alpha\}_{\alpha\in\mathbb{N}^n}$ is $\{\frac{1}{\alpha!}x^\alpha\}_{\alpha\in\mathbb{N}^n}$. Thus any linear form $\lambda \in \mathbb{K}[x]^*$ can be written as $\lambda = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\lambda(x^\alpha)\partial^\alpha \in \mathbb{K}[[\partial]]$. More generally, any linear form on $\langle\mathcal{P}\rangle$ can be written as

$$\lambda = \sum_{p\in\mathcal{P}} \lambda(p)\overline{p}^\dagger(\partial) \in \mathbb{K}[[\partial]].$$

2.3 Linear representations of finite groups

The basic material on representation of finite groups is taken from Serre (1977) and Fässler and Stiefel (1992). We deal with a finite group $\mathcal{G}$.

2.3.1 Linear representations and characters

A *linear representation* of the group $\mathcal{G}$ on the $\mathbb{C}-$vector space V is a group morphism from $\mathcal{G}$ to the group $GL(V)$ of isomorphisms from V to itself. V is called the *representation space* and n is the *dimension* (or the *degree*) of the representation $\mathfrak{r}$. If V has finite dimension n, and $\mathfrak{r}$ is a linear representation of $\mathcal{G}$ on V, upon introducing a basis $\mathcal{P}$ of V the isomorphism $\mathfrak{r}(g)$ can be described by a non-singular $n \times n$ matrix. This *representing matrix* is denoted by $[\mathfrak{r}(g)]_\mathcal{P}$. The complex-value function

$$\chi : \mathcal{G} \longrightarrow \mathbb{C}, \text{ with } \chi(g) \to \text{Trace}(\mathfrak{r}(g))$$

is the *character of the representation* $\mathfrak{r}$.

The *dual* or *contragredient representation* of $\mathfrak{r}$ is the representation $\mathfrak{r}^*$ on the dual vector space V^* defined by:

$$\mathfrak{r}^*(g)(\lambda) = \lambda \circ \mathfrak{r}(g^{-1}) \text{ for any } \lambda \in V^*. \tag{2.3.1}$$

If $\mathcal{P}$ is a basis of V and $\mathcal{P}^*$ its dual basis then $[\mathfrak{r}^*(g)]_{\mathcal{P}^*} = [\mathfrak{r}(g^{-1})]^t_\mathcal{P}$. It follows that

$$\chi_{\mathfrak{r}^*}(g) = \chi_\mathfrak{r}(g^{-1}) = \overline{\chi}_\mathfrak{r}(g).$$

A linear representation $\mathfrak{r}$ of a group $\mathcal{G}$ on a space V is *irreducible* if there is no proper nonzero subspace W of V with the property that, for every $g \in \mathcal{G}$, the isomorphism $\mathfrak{r}(g)$ maps every vector of W into W. In this case, its representation space V is also called *irreducible*. The contragredient representation $\mathfrak{r}^*$ is irreducible when $\mathfrak{r}$ is. A finite group has a finite number of inequivalent irreducible representations. Any representation of a finite group is completely reducible, meaning that it decomposes into a finite number of irreducible subspaces.

Let $\mathfrak{r}_\ell$ $(\ell = 1, \dots, \mathfrak{n})$ be the irreducible n_ℓ dimensional representations of $\mathcal{G}$. The complete reduction of the representation $\mathfrak{r}$ and its representation space are denoted by

$$\mathfrak{r} = c_1\mathfrak{r}_1 \oplus \cdots \oplus c_\mathfrak{n}\mathfrak{r}_\mathfrak{n} \text{ and } V = V^{(1)} \oplus \cdots \oplus V^{(\mathfrak{n})}. \tag{2.3.2}$$

Each invariant subspace $V^{(\ell)}$ is the direct sum of c_ℓ irreducible subspaces and the restriction of $\mathfrak{r}$ to each one is equivalent to $\mathfrak{r}_\ell$. The $(c_\ell n_\ell)$–dimensional subspaces $V^{(\ell)}$ of V are the *isotypic components*.

The decomposition of V into irreducible components is not unique, whereas its decomposition into a direct sum of isotypic components is unique (Fulton and Harris, 1991, Proposition 1.8) and is called the *isotypic decomposition* of V.

An element $v \in V$ is called invariant, if $\mathfrak{r}(g)(v) = v$, for any $g \in \mathcal{G}$. The invariant elements of V form a vector space denoted by $V^\mathcal{G}$. The invariant subspace $V^\mathcal{G}$ is the isotypic component corresponding to the unit representation $\mathfrak{r}_1$. The elements of $V^{(\ell)}$, with $\ell > 1$, are called *semi-invariants* (Gatermann, 1996).

A representation with character χ is irreducible if and only if

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \overline{\chi(g)}\chi(g) = 1.$$

With χ_ℓ the character of $\mathfrak{r}_\ell$ we determine the multiplicity c_ℓ and the projection π_ℓ onto the isotypic component $V^{(\ell)}$

$$c_\ell = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \overline{\chi_\ell(g)}\chi(g), \quad \pi_\ell = \frac{n_\ell}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} \chi_\ell(g^{-1})\mathfrak{r}(g). \tag{2.3.3}$$

2.3.2 Symmetry adapted bases

A basis that is compatible with the decomposition of a vector space V into isotypic components is called a *symmetry adapted basis*. The construction of a symmetry adapted basis is basically given by (Serre, 1977, Proposition 8) that we reproduce here for ease of reference.

Proposition 2.3.1 *(Serre, 1977, Proposition 8) Consider $\mathrm{r}_\ell(g)$ a representing matrix for an irreducible n_ℓ–dimensional representation $\mathfrak{r}_\ell$. The linear maps $\pi^{(\ell)}_{\alpha\beta} : V \to V$ defined by*

$$\pi^{(\ell)}_{\alpha\beta}(v) = \frac{n_\ell}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} [\mathrm{r}_\ell(g)]_{\beta\alpha}\left(g^{-1}\right)\mathfrak{r}(g)(v)$$

satisfy the following properties:

(1) For every $1 \le \alpha \le n_\ell$, the map $\pi^{(\ell)}_{\alpha\alpha}$ is a projection ; it is zero on the isotypic components $V^{(k)}$, $k \neq \ell$. Its image $V^{(\ell,\alpha)}$ is contained in $V^{(\ell)}$ and

$$V^{(\ell)} = V^{(\ell,1)} \oplus \cdots \oplus V^{(\ell,n_\ell)}, \tag{2.3.4}$$

while $\pi^{(\ell)} = \sum_{\alpha=1}^{n_\ell} \pi^{(\ell)}_{\alpha\alpha}$.

(2) For every $1 \le \alpha, \beta \le n_\ell$*, the linear map* $\pi^{(\ell)}_{\alpha\beta}$ *is zero on the isotypic components* $V^{(k)}$*,* $k \neq \ell$*, as well as on the subspaces* $V^{(\ell,k)}$ *for* $k \neq \ell$ *; it defines an isomorphism from* $V^{(\ell,\beta)}$ *to* $V^{(\ell,\alpha)}$*.*

(3) For any $v \in V$ *and* $1 \le \gamma \le n_\ell$ *consider* $v_\sigma = \pi^{(\ell)}_{\sigma\gamma}(v) \in V^{(\ell,\sigma)}$ *for all* $1 \le \sigma \le n_\ell$*. If non zero,* $v_1, \dots, v_{n_\ell}$ *are linearly independent and generate an invariant irreducible subspace of dimension* n_ℓ*. For each* $g \in \mathcal{G}$*, we have*

$$\mathfrak{r}(g)(v_\gamma) = \sum_{\sigma=1}^{n_\ell} [\mathrm{r}_\ell(g)]_{\sigma\gamma}\, (v_\sigma) \qquad \forall \sigma,\, \gamma = 1, \dots, n_\ell.$$

(4) For every $1 \le \alpha, \beta, \gamma, \delta \le n_\ell$*, we have*

$$\pi^{(\ell)}_{\alpha\beta} \circ \pi^{(\ell)}_{\gamma\delta} = \begin{cases} \pi^{(\ell)}_{\alpha\delta} & \text{if } \beta = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Considering $\left\{\pi^{(\ell)}_{11}(v_1), \dots, \pi^{(\ell)}_{11}(v_{c_\ell})\right\}$ a basis of the subspace $V^{(\ell,1)} = \pi^{(\ell)}_{11}(V)$, a symmetry adapted basis $\mathcal{P}^{(\ell)}$ of the isotypic component $V^{(\ell)}$ is then given by:

$$\mathcal{P}^{(\ell)} = \left\{\pi^{(\ell)}_{11}(v_1), \dots, \pi^{(\ell)}_{11}(v_{c_\ell}), \dots, \pi^{(\ell)}_{n_\ell 1}(v_1), \dots, \pi^{(\ell)}_{n_\ell 1}(v_{c_\ell})\right\}. \tag{2.3.5}$$

The union $\mathcal{P}$ of the $\mathcal{P}^{(\ell)}$ for $1 \le \ell \le \mathfrak{n}$, is a symmetry adapted basis for V. Indeed, by Proposition 2.3.4, the set $\left\{\pi^{(\ell)}_{\alpha 1}(v_1), \dots, \pi^{(\ell)}_{\alpha}(v_{c_\ell})\right\}$ is a basis of $V^{(\ell,\alpha)} = \pi^{(\ell)}_{\alpha\alpha}(V)$ and $V^{(\ell)} = V^{(\ell,1)} \oplus \cdots \oplus V^{(\ell,n_\ell)}$.

Hereafter we denote by $\mathcal{P}^{(\ell,\alpha)}$ the polynomial map defined by

$$\mathcal{P}^{(\ell,\alpha)} = \left(\pi^{(\ell)}_{\alpha 1}(v_1), \dots, \pi^{(\ell)}_{\alpha}(v_{c_\ell})\right). \tag{2.3.6}$$

A symmetry adapted basis $\mathcal{P}$ is characterized by the fact that

$$[\mathfrak{r}(g)]_{\mathcal{P}} = \mathrm{diag}\,\left(\mathrm{r}_1(g) \otimes \mathrm{I}_{c_1},\ \dots,\ \mathrm{r}_{\mathfrak{n}}(g) \otimes \mathrm{I}_{c_{\mathfrak{n}}}\right).$$

Then $[\mathfrak{r}^*(g)]_{\mathcal{P}^*} = \mathrm{diag}\,\left(\mathrm{r}_i^{-t}(g) \otimes \mathrm{I}_{c_i} \,|\, i = 1..\mathfrak{n}\right)$.

As we can perceive in (2.3.5), a symmetry adapted basis of a vector space V is not unique, yet it is fully determined by the choice of the bases for the subspaces $V^{(\ell,1)}$ with $1 \le \ell \le \mathfrak{n}$.

Proposition 2.3.2 *If* $\mathcal{P} = \cup_{i=1}^{\mathfrak{n}} \mathcal{P}^{(i)}$ *be a symmetry adapted basis of* V *where* $\mathcal{P}^{(i)}$ *spans the isotypic component associated to* $\mathfrak{r}_i$ *then its dual basis* $\mathcal{P}^* = \cup_{i=1}^{\mathfrak{n}} \left(\mathcal{P}^{(i)}\right)^*$ *in* V^* *is a symmetry adapted basis where* $\left(\mathcal{P}^{(i)}\right)^*$ *spans the isotypic component associated to* $\mathfrak{r}_i^*$*.*

Corollary 2.3.3 *If* $\mathcal{P}$ *is a symmetry adapted basis of* $\mathbb{K}[\mathrm{x}]_{\le\delta}$*, so is its dual* $\mathcal{P}^\dagger$ *with respect to the apolar product.*

Theorem 2.3.4 *Let ϑ and θ be representations of $\mathcal{G}$ on the vector space V and W respectively, and $\mathcal{P}$ and $\mathcal{Q}$ the respective symmetry adapted bases. Consider $\phi : V \longrightarrow W$ an $\vartheta - \theta$ equivariant map,* i.e., *$\phi \circ \vartheta(g) = \theta(g) \circ \phi$ for all $g \in \mathcal{G}$. The matrix Φ of ϕ in the bases $\mathcal{P}$ and $\mathcal{Q}$ has the following structure*

$$\Phi = \operatorname{diag}\left(\mathrm{I}_{n_i} \otimes \Phi_i \mid i = 1 \dots \mathfrak{n}\right), \tag{2.3.7}$$

where Φ_i is a $c_i \times c_i'$ matrix, c_i and c_i' being the multiplicities of the $i-$th irreducible representation of $\mathcal{G}$ in V and W respectively.

This is a consequence of Schur lemma (Serre, 1977, Proposition 4). The proof is an simple extension of the result for an equivariant endomorphism $\phi : V \longrightarrow V$ in (Fässler and Stiefel, 1992, Theorem 2.5).

A scalar product is $\mathcal{G}-$invariant with respect to a linear representation $\mathfrak{r}$ if

$$\langle v, w\rangle = \langle \mathfrak{r}(g)(v), \mathfrak{r}(g)(w)\rangle \text{ for any } g \in \mathcal{G} \text{ and } v, w \in V$$

If we consider unitary representing matrices $\mathrm{r}_\ell(g)$, and an orthonormal basis

$$\left\{\pi_{11}^{(\ell)}(v_1), \dots, \pi_{11}^{(\ell)}(v_{c_\ell})\right\}$$

of $V^{(\ell,1)}$ with respect to a $\mathcal{G}-$invariant inner product, then the same process leads to an *orthonormal symmetry adapted basis* (Fässler and Stiefel, 1992, Theorem 5.4).

2.3.3 Real symmetry adapted bases

Some group have irreducible representations that have no representing matrices in $\mathbb{R}$, as is the case for the cyclic group C_m, $m > 2$. This would lead to symmetry adapted bases whose components are polynomials in $\mathbb{C}[x]$. This is not desirable for an interpolation problem over $\mathbb{R}$. Fortunately one can determine a *real symmetry adapted basis* by combining the isotypic components related to conjugate irreducible representations.

Based on (Serre, 1977, Chapter 13.2), we study the linear representations on a $\mathbb{R}-$vector space from the linear representations on a $\mathbb{C}-$vector space. In fact, any linear representation $\mathfrak{r}$ on a $\mathbb{R}-$vector space V can be considered as a linear representation on the $\mathbb{C}-$vector space $V \otimes_{\mathbb{R}} \mathbb{C}$, i.e., the vector space obtained from V by extending the scalars from the real numbers to the complex numbers.

Three types of irreducible representations on a $\mathbb{C}-$vector space are distinguished in (Serre, 1977, Chapter 13.2). Let $\mathfrak{r}$ be an irreducible representation of a finite group $\mathcal{G}$ on a $n-$dimensional $\mathbb{C}-$vector space V and let χ be its character. The three mutually exclusive cases are the following:

1. **Complex type** The character χ is not real-valued. Hence $\mathfrak{r}$ and its conjugate $\bar{\mathfrak{r}}$ are non equivalent irreducible representations. When restricting the scalars, $\mathfrak{r}$ defines an irreducible representation on a $\mathbb{R}-$vector space of dimension $2n$ with character $\chi + \overline{\chi}$.

2. **Absolutely irreducible** The representation $\mathfrak{r}$ can be realized by matrices having coefficients in $\mathbb{R}$. Hence $\mathfrak{r}$ defines an irreducible representation $\mathfrak{r}$ on a $\mathbb{R}-$vector space of dimension n with character χ.

3. **Quaternonian type** The character χ is a real-valued function, but the representation $\mathfrak{r}$ cannot be realized by matrices having coefficients in $\mathbb{R}$. By restriction of the scalars, $\mathfrak{r}$ defines an irreducible representation $\mathfrak{r}$ on a $\mathbb{R}-$vector space of dimension $2n$ with character 2χ.

According to (Serre, 1977, Proposition 39), an irreducible representation $\mathfrak{r}$ is of type 1, 2, 3, if and only if $\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi(g^2)$ is equal to $0, +1,$ or -1, respectively. We shall denote M the number of absolutely irreducible representations of the group $\mathcal{G}$ and $2L$ the number of irreducible representations of complex type. They will be denoted $\mathfrak{r}_1,\ldots,\mathfrak{r}_M$, and $\mathfrak{r}_{M+1},\ldots,\mathfrak{r}_{M+L}$ together with their conjugates $\bar{\mathfrak{r}}_{M+1},\ldots,\bar{\mathfrak{r}}_{M+L}$. We shall make the reasonable asumption that the group considered has no irreducible representation of quaternion type.

The construction of a real symmetry adapted basis for a real represenation $\mathfrak{r}$ on a $\mathbb{R}$-vector space V is based on the construction of a complex symmetry adapted basis as presented in Section 2.3.1. From a basis of the isotypic component associated to $\mathfrak{r}_i$, for $1 \le i \le M+L$ we construct a basis for the real irreducible representation defined by $\mathfrak{r}_i$.

Absolutely irreducible In this case we can choose representing matrices $\mathrm{r}_\ell(g)$ of the irreducible representation $\mathfrak{r}_\ell$ with real entries. Taking a real basis

$$\left\{\pi_{11}^{(\ell)}(p_1),\ldots,\pi_{11}^{(\ell)}(p_{c_\ell})\right\}$$

of the subspace $\pi_{11}^{(\ell)}(V\otimes_{\mathbb{R}}\mathbb{C})$, the basis

$$\mathcal{P}^{(\ell)} = \left\{\pi_{11}^{(\ell)}(p_1),\ldots,\pi_{11}^{(\ell)}(p_{c_\ell}),\ldots,\pi_{n_\ell 1}^{(\ell)}(p_1),\ldots,\pi_{n_\ell 1}^{(\ell)}(p_{c_\ell})\right\} \tag{2.3.8}$$

is a real symmetry adapted basis of $V^{(\ell)}$.

Complex Type Let $\mathfrak{r}_\ell$ be an irreducible representation of complex type and χ_ℓ its character. Since χ_ℓ and $\overline{\chi}_\ell$ are different characters of $\mathcal{G}$, then $\mathfrak{r}_\ell$ and $\bar{\mathfrak{r}}_\ell$ are inequivalent irreducible representations of $\mathfrak{r}$, both with multiplicity c_ℓ. A real symmetry adapted basis $\mathcal{P}^{(\ell)}$ of the space $V^{(\ell)}\oplus\overline{V}^{(\ell)}$ is given by

$$\begin{aligned}\mathcal{P}^{(\ell)} = \;& \left\{p_1+\overline{p}_1, \tfrac{1}{i}(p_1-\overline{p}_1),\ldots,p_{c_\ell}+\overline{p}_{c_\ell}, \tfrac{1}{i}\left(p_{c_\ell}-\overline{p}_{c_\ell}\right),\ldots,\right.\\ & \pi_{n_\ell 1}^{(\ell)}(p_1)+\overline{\pi}_{n_\ell 1}^{(\ell)}(\overline{p}_1), \tfrac{1}{i}\left(\pi_{n_\ell 1}^{(\ell)}(p_1)-\overline{\pi}_{n_\ell 1}^{(\ell)}(\overline{p}_1)\right),\ldots,\\ & \left.\pi_{n_\ell 1}^{(\ell)}(p_{c_\ell})+\overline{\pi}_{n_\ell 1}^{(\ell)}\left(\overline{p}_{c_\ell}\right), \tfrac{1}{i}\left(\pi_{n_\ell 1}^{(\ell)}(p_{c_\ell})-\overline{\pi}_{n_\ell 1}^{(\ell)}\left(\overline{p}_{c_\ell}\right)\right)\right\},\end{aligned} \tag{2.3.9}$$

where $\left\{\pi_{11}^{(\ell)}(p_1),\ldots,\pi_{11}^{(\ell)}(p_{c_\ell}),\ldots,\pi_{n_\ell 1}^{(\ell)}(p_1),\ldots,\pi_{n_\ell 1}^{(\ell)}(p_{c_\ell})\right\}$ is a complex symmetry adapted basis of $V^{(\ell)}$.

Considering $b_k = \begin{cases} p_k + \bar{p}_k & \text{if } k \mod 2 = 1 \\ \frac{1}{i}(p_k - \bar{p}_k) & \text{otherwise} \end{cases}$ and $\hat{\pi}_{\alpha 1}^{(\ell)} = \pi_{\alpha 1}^{(\ell)} + \overline{\pi}_{\alpha 1}^{(\ell)}$ we can write Equation (2.3.9) in a similar fashion than Equation (2.3.8) as follows

$$\mathcal{P}^{(\ell)} = \left\{b_1,\ldots,b_{2c_\ell},\ldots,\hat{\pi}_{n_\ell 1}^{(\ell)}(b_1),\ldots,\hat{\pi}_{n_\ell 1}^{(\ell)}(b_{2c_\ell})\right\}. \tag{2.3.10}$$

Notice that if $p_1,\ldots,p_{c_j}$ are ordered by degree, so are $b_1,\ldots,b_{2c_j}$, and if $\{p_1,\ldots,p_{c_\ell}\}$ is an orthonormal basis then:

$$\left\langle \pi_{\alpha 1}^{(\ell)}(b_k) \pm \overline{\pi}_{\alpha 1}^{(\ell)}(\overline{b_k}),\ \pi_{\alpha 1}^{(\ell)}(b_i) \pm \overline{\pi}_{\alpha 1}^{(\ell)}(\overline{b_i}) \right\rangle = \mp 2\delta_{ki}. \tag{2.3.11}$$

Hence $\frac{1}{\sqrt{2}}\mathcal{P}^{(\ell)}$ is an orthonormal symmetry adapted basis of V.

When the group $\mathcal{G}$ admits absolutely irreducible representations and irreducible representation of complex type, a real symmetry adapted basis for a representation space V is characterized by the fact that

$$[\mathfrak{r}(g)]_{\mathcal{P}} = \operatorname{diag}\left(\mathrm{A}_1(g),\ \ldots,\ \mathrm{A}_{M+L}(g)\right),$$

and $\mathrm{A}_j(g)$ is given by

$$\mathrm{A}_j(g) = \begin{cases} \mathrm{r}_j(g) \otimes \mathrm{I}_{c_j} & \text{if} \quad 1 \le j \le M \\ \begin{pmatrix} \mathrm{I}_{c_j}\otimes \mathrm{B}_{11}^j(g) & \mathrm{I}_{c_j}\otimes \mathrm{B}_{12}^j(g) & \cdots & \mathrm{I}_{c_j}\otimes \mathrm{B}_{1n_j}^j(g) \\ \mathrm{I}_{c_j}\otimes \mathrm{B}_{21}^j(g) & \mathrm{I}_{c_j}\otimes \mathrm{B}_{21}^j(g) & \cdots & \mathrm{I}_{c_j}\otimes \mathrm{B}_{2n_j}^j(g) \\ \cdots & \cdots & & \cdots \\ \mathrm{I}_{c_j}\otimes \mathrm{B}_{n_j 1}^j(g) & \mathrm{I}_{c_j}\otimes \mathrm{B}_{n_j 2}^j(g) & \cdots & \mathrm{I}_{c_j}\otimes \mathrm{B}_{n_j n_j}^j(g) \end{pmatrix} & \text{if} \quad M < j \le M+L \end{cases}$$

with $B_{\alpha\beta}^j(g) = \begin{pmatrix} s_{\alpha\beta}^j(g) & -t_{\alpha\beta}^j(g) \\ t_{\alpha\beta}^j(g) & s_{\alpha\beta}^j(g) \end{pmatrix}$, $\mathrm{S}^j(g) = \frac{1}{2}(\mathrm{r}_j(g) + \overline{\mathrm{r}}_j(g))$ and $\mathrm{T}^j(g) = \frac{i}{2}(\mathrm{r}_j(g) - \overline{\mathrm{r}}_j(g))$.

Proposition 2.3.5 *Let ϑ and θ be representations of $\mathcal{G}$ on the $\mathbb{R}-$vector space V and W respectively, with real symmetry adapted bases $\mathcal{P}$ and $\mathcal{Q}$. Consider $\psi : V \longrightarrow W$ a $\vartheta - \theta$ equivariant map. Then the matrix Ψ of ψ in the bases $\mathcal{P}$ and $\mathcal{Q}$ has the following structure*

$$\Psi = \operatorname{diag}\left(\mathrm{I}_{n_i} \otimes \Psi_i \mid i = 1 \ldots M+L\right). \tag{2.3.12}$$

If $\mathcal{P}$ and $\mathcal{Q}$ stem from the symmetry adapted bases $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ of $V \otimes_{\mathbb{R}} \mathbb{C}$ and $W \otimes_{\mathbb{R}} \mathbb{C}$ respectively and $\Phi = \operatorname{diag}\left(\mathrm{I}_{n_i} \otimes \Phi_i \mid i = 1 \ldots \mathfrak{n}\right)$ is the matrix in w.r.t $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{Q}}$ of

the $\vartheta-\theta$ equivariant map $\phi : V \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow W \otimes_{\mathbb{R}} \mathbb{C}$, given by $\phi(zv) \to \mathrm{Re}(z)\psi(v) + i\,\mathrm{Im}(z)\psi(v)$. Then for every $1 \le j \le M+L$ the matrix Ψ_j has the following structure

$$\Psi_j = \begin{cases} \Phi_j & \text{if} \quad 1 \le j \le M \\ \begin{pmatrix} s_{11}^{\ j} & -t_{11}^{\ j} & \dots & s_{1c_j}^{\ j} & -t_{1c_j}^{\ j} \\ t_{11}^{\ j} & s_{11}^{\ j} & \dots & t_{1c_j}^{\ j} & s_{1c_j}^{\ j} \\ \dots & \dots & & \dots & \dots \\ s_{c'_j1}^{\ j} & -t_{c'_j1}^{\ j} & \dots & s_{c'_jc_j}^{\ j} & -t_{c'_jc_j}^{\ j} \\ t_{c'_j1}^{\ j} & s_{c'_j1}^{\ j} & \dots & t_{c'_jc_j}^{\ j} & s_{c'_jc_j}^{\ j} \end{pmatrix} & \text{if} \quad M < j \le M+L \end{cases},$$

where c_i and c'_i are the multiplicities of the $i-th$ irreducible representation of $\mathcal{G}$ in V and W respectively and the matrices $\mathrm{S}^j = \left(s_{\alpha\beta}^{\ j}\right)_{\substack{1\le\alpha\le c_j \\ 1\le\beta\le c'_j}}$ and $\mathrm{T}^j = \left(t_{\alpha\beta}^{\ j}\right)_{\substack{1\le\alpha\le c_j \\ 1\le\beta\le c'_j}}$ are defined by

$$\mathrm{S}^j = \frac{1}{2}\left(\Phi_j + \overline{\Phi}_j\right) \qquad \mathrm{T}^j = \frac{i}{2}\left(\Phi_j - \overline{\Phi}_j\right).$$

PROOF: For every real vector v we have that $\psi(v) = \phi(v)$. We distinguish now two cases:

- $v_{\alpha k}^j$ is an element of a symmetry adapted basis associated to an irreducible component of absolutely irreducible type. Then $\psi(v_{\alpha k}^j) = \phi(v_{\alpha k}^j) = \sum_{\ell=1}^{c'_j} \Phi_{\ell\alpha}^{(j)} w_{\alpha\ell}^j$ and the structure of Ψ_j for $1 \le j \le M$ follows
- $v_{\alpha k}^j$ is an element of a symmetry adapted basis associated to an irreducible component of complex type. Then the elements of the real symmetry adapted basis $\mathcal{P}^{(j)}$ for $V^{(j)} \oplus \overline{V}^{(j)}$ are given by Equation (2.3.10). The following qualities holds

$$\psi\left(v_{\alpha k}^j + \overline{v}_{\alpha k}^j\right) = \sum_{\ell=1}^{c'_j} \Phi_{\ell\alpha}^{(j)} w_{\alpha\ell}^j + \sum_{\ell=1}^{c'_j} \overline{\Phi}_{\ell\alpha}^{(j)} \overline{w}_{\alpha\ell}^j = \sum_{\ell=1}^{c'_j} \left\{ s_{\ell\alpha}^j \left(w_{\alpha\ell}^j + \overline{w}_{\alpha\ell}^j\right) + t_{\ell\alpha}^j \left(\frac{w_{\alpha\ell}^j - \overline{w}_{\alpha\ell}^j}{i}\right)\right\}$$

$$\psi\left(\frac{1}{i}\left(v_{\alpha k}^j - \overline{v}_{\alpha k}^j\right)\right) = \sum_{\ell=1}^{c'_j} \left\{ -t_{\ell\alpha}^j \left(w_{\alpha\ell}^j + \overline{w}_{\alpha\ell}^j\right) + s_{\ell\alpha}^j \left(\frac{w_{\alpha\ell}^j - \overline{w}_{\alpha\ell}^j}{i}\right)\right\}$$

  and therefore the structure of Ψ_j follows for $M \le j \le M+L$.

□

Conventions We introduce the following conventions so as to have uniform statements for symmetry adapted bases of both real and complex vectors spaces. In the complex case $N = M + 2L$ is the number of inequivalent irreducible representations of the group $\mathcal{G}$ and c_i is the multiplicity of $\mathfrak{r}_i$ in the representation considered. In the real case $N = M + L$ is the number of inequivalent real irreducible representations of the group $\mathcal{G}$ and c_i is, in the representation considered,

- the multiplicity of $\mathfrak{r}_i$ if $1 \le i \le M$;
- is twice the multiplicity of $\mathfrak{r}_i$ if $M+1 \le i \le M+L$.

In either case we denote a symmetry adapted basis by $\mathcal{P} = \cup_{i=1}^{N} \mathcal{P}^{(i)}$ and say that $\mathcal{P}^{(i)}$ is determined by $p_1, \ldots, p_{c_i}$ to mean that the basis of $\mathcal{P}^{(i)}$ is

In the complex case:

$$\left\{p_1, \ldots, p_{c_i}, \ldots, \pi_{n_i 1}^{(i)}(p_1), \ldots, \pi_{n_i 1}^{(i)}(p_{c_i})\right\}$$

In the real case and $1 \le i \le M$ *:* same as above

In the real case and $M+1 \le i \le M+L$ *:*

$$\left\{p_1, \ldots, p_{c_i}, \ldots, \hat{\pi}_{n_i 1}^{(i)}(p_i), \ldots, \hat{\pi}_{n_i 1}^{(i)}(p_{c_i})\right\}.$$

2.4 Invariant theory

In this section we introduce some basic notions of invariant theory. The literature on the computation of polynomial invariants is rather extensive, for proof and more extensive discussion we shall restrict here to the book references (Derksen and Kemper, 2015; Gatermann, 2000; Sturmfels, 2007; Stanley, 1979).

The symmetries we deal with are given by the linear action of a finite group $\mathcal{G}$ on $\mathbb{K}^n$. It is thus given by a representation ϑ of $\mathcal{G}$ on $\mathbb{K}^n$. It induces a representation ρ of $\mathcal{G}$ on $\mathbb{K}[\mathrm{x}]$ given by

$$\rho(g)p(x) = p \circ \vartheta\left(g^{-1}\right), \tag{2.4.1}$$

for $p \in \mathbb{K}[\mathrm{x}]$, $g \in \mathcal{G}$. For a row vector $\mathrm{q} = [q_1, \ldots, q_m] \in \mathbb{K}[\mathrm{x}]^m$ of polynomials we write $\rho(g)(\mathrm{q})$ for the row vector $[\rho(g)(q_1), \ldots, \rho(g)(q_m)] \in \mathbb{K}[\mathrm{x}]^m$. For $\xi \in \mathbb{K}^n$, $p \in \mathbb{K}[\mathrm{x}]$ or $\mathrm{q} \in \mathbb{K}[\mathrm{x}]^m$ we occasionally write $g \cdot \xi$, $g \cdot p$, or $g \cdot \mathrm{q}$ instead of $\vartheta(g)(\xi)$, $\rho(g)(p)$ or $\rho(g)(\mathrm{q})$.

The representation $\rho : \mathcal{G} \to \mathrm{GL}(\mathbb{K}[\mathrm{x}])$ leaves invariant the finite dimensional subspace $\mathbb{K}[\mathrm{x}]_d$ spanned by the homogeneous polynomials of degree d. The isotypic components of $\mathbb{K}[\mathrm{x}]_d$ are denoted $\mathbb{K}[\mathrm{x}]_d^{(\ell)}$. We write

$$\mathbb{K}[\mathrm{x}] = \bigoplus_{d\in\mathbb{N}} \mathbb{K}[\mathrm{x}]_d = \bigoplus_{d\in\mathbb{N}} \bigoplus_{\ell=1}^{\mathfrak{n}} \mathbb{K}[\mathrm{x}]_d^{(\ell)}, \qquad \mathbb{K}[\mathrm{x}]^{(\ell)} = \bigoplus_{d\in\mathbb{N}} \mathbb{K}[\mathrm{x}]_d^{(\ell)}, \qquad \mathbb{K}[\mathrm{x}] = \bigoplus_{\ell=1}^{\mathfrak{n}} \mathbb{K}[\mathrm{x}]^{(\ell)}.$$

Similarly

$$\mathbb{K}[\mathrm{x}]^{(\ell)} = \bigoplus_{d\in\mathbb{N}} \mathbb{K}[\mathrm{x}]_d^{(\ell)} = \bigoplus_{d\in\mathbb{N}} \bigoplus_{i=1}^{n_\ell} \mathbb{K}[\mathrm{x}]_d^{(\ell,i)}, \quad \mathbb{K}[\mathrm{x}]^{(\ell,i)} = \bigoplus_{d\in\mathbb{N}} \mathbb{K}[\mathrm{x}]_d^{(\ell,i)}, \quad \mathbb{K}[\mathrm{x}]^{(\ell)} = \bigoplus_{i=1}^{n_\ell} \mathbb{K}[\mathrm{x}]^{(\ell,i)}.$$

2.4.1 Ring of invariants

An *invariant polynomial*, or simply *invariant*, is a polynomial $p \in \mathbb{K}[\mathrm{x}]$ such that

$$\rho(g)(p) = p \text{ for all } g \in \mathcal{G}.$$

The $\mathbb{K}$-vector space of invariants is the isotypic component of $\mathbb{K}[\mathrm{x}]$ associated to the trivial representation i.e. $\mathbb{K}[\mathrm{x}]^{(1)}$. But the invariants also form a ring and we shall denote this ring $\mathbb{K}[x]^{\mathcal{G}}$. The projection $\pi_{11}^{(1)}$ of $\mathbb{K}[\mathrm{x}]$ in $\mathbb{K}[\mathrm{x}]^{\mathcal{G}}$ is also known as the *Reynolds* operator.

The Hilbert's finiteness theorem (Sturmfels, 2007, Theorem 2.1.3) states that the invariant ring is finitely generated. A bound for the number of generating invariants, as well as for their degrees are given in the following theorem

Theorem 2.4.1 *(Noether's degree bound). The invariant ring of a finite matrix group $\mathcal{G}$ has an algebra basis consisting of at most $\binom{n+|\mathcal{G}|}{n}$ invariants whose degree is bounded above by the group order $|\mathcal{G}|$.*

The invariant ring is a graded algebra. It is the direct sum of the finite-dimensional $\mathbb{K}$-vector spaces $\mathbb{K}[\mathrm{x}]_d^{\mathcal{G}}$, where $\mathbb{K}[\mathrm{x}]_d^{\mathcal{G}}$ denotes the space of homogeneous invariants of degree d. The Hilbert series of $\mathbb{K}[\mathrm{x}]^{\mathcal{G}}$ defined by the generating function

$$H\left(\mathbb{K}[\mathrm{x}]_d^{\mathcal{G}}, t\right) = \sum_{d=0}^{\infty} \dim\left(\mathbb{K}[\mathrm{x}]_d^{\mathcal{G}}\right) t^d$$

gives the dimension of the vector space of invariants at each degree. An explicit formula for $H\left(\mathbb{K}[\mathrm{x}]^{\mathcal{G}}, t\right)$ in terms of the matrix group $\mathcal{G}$ is given in the following theorem.

Theorem 2.4.2 *(Molien's Theorem). The Hilbert series of the invariant ring $\mathbb{K}[\mathrm{x}]^{\mathcal{G}}$ is given by*

$$H(\mathbb{K}[\mathrm{x}]^{\mathcal{G}}, t) = \frac{1}{|\mathcal{G}|} \sum_{\Gamma \in \mathcal{G}} \frac{1}{\det\left(\mathrm{I}_n - t\Gamma\right)}.$$

An estimate of the *density* of $\mathbb{K}[\mathrm{x}]_d^{\mathcal{G}}$ in $\mathbb{K}[\mathrm{x}]_d$ is given in the following theorem:

Theorem 2.4.3 *(Kemper and Steel, 1999, Proposition 1) Let $a_d = \dim\left(\mathbb{K}[\mathrm{x}]_d^{\mathcal{G}}\right)$ and $b_d = \dim\left(\mathbb{K}[\mathrm{x}]_d\right)$. Then*

$$\lim_{N\to\infty} \frac{\sum_{d=0}^{N} a_d}{\sum_{d=0}^{N} b_d} = \frac{1}{|G|}. \tag{2.4.2}$$

Equation 2.4.2 will be useful in Chapter 3 to study the complexity of invariant interpolation.

Let $\mathbb{A}$ be a graded $\mathbb{K}$-algebra and let m be the maximal number of algebraically independent elements of $\mathbb{A}$. A set $\sigma_1 \ldots, \sigma_m$ of m homogeneous elements of positive degree is called a *homogeneous system of parameters*, if $\mathbb{A}$ is a finitely generated

module over $\mathbb{K}[\sigma_1 \ldots, \sigma_m]$. The invariant ring $\mathbb{K}[x]^{\mathcal{G}}$ has n algebraically independent invariants (Sturmfels, 2007, Theorem 2.2.1). The elements in an homogeneous system of parameters of $\mathbb{K}[x]^{\mathcal{G}}$ are called primary invariants. Algorithms to compute primary invariants can be found in (Derksen and Kemper, 2015; Gatermann, 2000; Sturmfels, 2007; Stanley, 1979). In section (5.4) we give a more detailed overview of the existing algorithms for computing primary invariants.

Since the invariant ring is Cohen–Macaulay (Sturmfels, 2007, Theorem 2.2.1), for each set of primary invariants $\{\sigma_1 \ldots, \sigma_n\}$, there exists homogeneous invariants $\eta_1, \ldots, \eta_r$ such that

$$\mathbb{K}[x]^{\mathcal{G}} = \bigoplus_{i=1}^{r} \eta_i \mathbb{K}[\sigma_1 \ldots, \sigma_n],$$

the η_i are called secondary invariants. The secondary invariants are uniquely determined by the degrees of their corresponding primary invariants, this is reflected in the following theorem

Theorem 2.4.4 *Let $d_1, \ldots, d_n$ be the degrees of a collection of primary invariants of a matrix group . The number of secondary invariants is given by*

$$m = \frac{d_1 \ldots d_n}{|\mathcal{G}|}.$$

The degrees (together with their multiplicities) of the secondary invariants are the exponents of the generating function

$$H(\mathbb{K}[x]^{\mathcal{G}}, t) \cdot \Pi_{i=1}^{n} \left(1 - t^{d_i}\right) = t^{e_1} + \ldots + t^{e_m}.$$

An explicit algorithm to compute secondary invariant can be found in (Kemper and Steel, 1999).

2.4.2 Modules of equivariant

In this section we discuss about the theory of equivariants, and its relation with linear representation. We provided the equivariants properties that are analogous to the results given in the previous section.

Isotypic components.

The infinite dimensional isotypic components $\mathbb{K}[x]^{(\ell)}$, and their components $\mathbb{K}[x]^{(\ell,i)}$, are actually $\mathbb{K}[x]^{\mathcal{G}}$-modules. Indeed, if $p \in \mathbb{K}[x]^{\mathcal{G}}$, then $\pi_{ij}^{(\ell)}(p\,q) = p\,\pi_{ij}^{(\ell)}(q)$ for any $q \in \mathbb{K}[x]$. In fact (Stanley, 1979, Theorem 1.3) asserts that $\mathbb{K}[x]^{(\ell)}$ is finitely generated by polynomials of degree less than $|\mathcal{G}|$. Since $\mathbb{K}[x]$ can be decomposed as a direct sum of its isotypic components it follows that $\mathbb{K}[x]$ is finitely generated over $\mathbb{K}[x]^{\mathcal{G}}$.

Theorem 2.4.5 *(Stanley, 1979) Let $\sigma_1, \ldots, \sigma_n$ be a set of primary invariants for $\mathbb{K}[\mathrm{x}]$. Then the isotypic components $\mathbb{K}[\mathrm{x}]^{(\ell)}$, $\ell = 2 \ldots N$ are modules finitely-generated and free over $\mathbb{K}[\sigma_1, \ldots, \sigma_n]$.*

Theorem 2.4.6 *(Stanley, 1979, Proposition 4.9) Let $\sigma_1, \ldots, \sigma_m$ be a set of the primary invariants of $\mathbb{K}[\mathrm{x}]^{\mathcal{G}}$. Then the group action on $\mathbb{K}[\mathrm{x}]/\langle\sigma_1, \ldots, \sigma_m\rangle$ is a times the regular representation.*

When $\mathcal{G}$ is a reflection group then $\mathbb{K}[\mathrm{x}]/\langle\sigma_1, \ldots, \sigma_m\rangle$ is isomorphic to the regular representation (Kane, 2001, Theorem 17.5).

A generalization of the Molien's formula (Theorem 2.4.2) for any isotypic component is given in the following theorem.

Theorem 2.4.7 *The Hilbert series of $\mathbb{K}[\mathrm{x}]^{(\ell)}$ is given by*

$$H(\mathbb{K}[\mathrm{x}]^{(\ell)}, t) = \frac{n_\ell}{|\mathcal{G}|} \sum_{\Gamma \in \mathcal{G}} \frac{\overline{\chi_\ell(\Gamma)}}{\det(\mathrm{I}_n - t\Gamma)},$$

where χ_ℓ is the character of the ℓ−th irreducible representation of $\mathcal{G}$.

Based on Theorem 2.4.7 the following proposition is proposed in (Svartz, 2014, Proposition 3.83) to estimate the density of $\mathbb{K}[\mathrm{x}]^{(\ell)}_{\leq d}$ in $\mathbb{K}[\mathrm{x}]_{\leq d}$.

Theorem 2.4.8 *Let $a_d^{(\ell)} = \dim\left(\mathbb{K}[\mathrm{x}]_d^{(\ell)}\right)$ and $b_d = \dim\left(\mathbb{K}[\mathrm{x}]_d\right)$. Then*

$$\lim_{N\to\infty} \frac{\sum_{d=0}^{N} a_d^{(\ell)}}{\sum_{d=0}^{N} b_d} = \frac{n_\ell^2}{|G|}. \tag{2.4.3}$$

Equivariant

More generally, if $\mathfrak{r} : \mathcal{G} \to \mathrm{GL}_m(\mathbb{K})$ is a m dimensional matrix representation of $\mathcal{G}$, an $\mathfrak{r}$-*equivariant* is a row vector $\mathrm{q} = (q_1, \ldots, q_m) \in \mathbb{K}[\mathrm{x}]^m$ such that $\rho(g)(\mathrm{q}) = \mathrm{q}\,\mathfrak{r}(g)$, where the left handside is a vector-matrix multiplication. The set of all $\mathfrak{r}$-equivariants forms a $\mathbb{K}[x]^{\mathcal{G}}$-module that we denote $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}}$.

A projection on $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}}$ is given by $\pi_{\rho,\mathfrak{r}} : \mathbb{K}[\mathrm{x}]^m \to \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}}$ with

$$\pi_{\rho,\mathfrak{r}}(\mathrm{q}) = \frac{1}{|\mathcal{G}|} \sum_{g\in\mathcal{G}} \rho\left(g^{-1}\right)(\mathrm{q}) \cdot \mathfrak{r}(g),\ \mathrm{q} \in \mathbb{K}[\mathrm{x}]^m.$$

The map $\pi_{\rho,\mathfrak{r}}$ is a projection on the trivial component of the representation

$$\tau : \mathcal{G} \to GL\left(\mathbb{K}[\mathrm{x}]^m\right), \quad \tau(g) = \rho\left(g^{-1}\right)(\mathrm{q})\mathfrak{r}(g).$$

A set of $\mathfrak{r}$-equivariants $\mathcal{Q} = \{\mathrm{q}_1, \ldots, \mathrm{q}_m\}$ is generating for $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module if any other $\mathfrak{r}$-equivariant q can be written as a linear combination of $\mathcal{Q}$ over $\mathbb{K}[x]^{\mathcal{G}}$: $\mathrm{q} = a_1\,\mathrm{q}_1 + \ldots + a_m\,\mathrm{q}_m,\ \ a_i \in \mathbb{K}[x]^{\mathcal{G}}$.

Theorem 2.4.9 *(Gatermann, 1996, Theorem 2.13) If $\sigma_1, \ldots, \sigma_n$ are primary invariants for $\mathbb{K}[\mathrm{x}]$, then the module of $\mathbb{K}[x]^{\mathcal{G}}_{\mathrm{t}}$-equivariants is finitely generated and free over $\mathbb{K}[\sigma_1, \ldots, \sigma_n]$.*

Chapter 3

Invariant Interpolation

This chapter is an extended version of:

(Rodriguez Bazan and Hubert, 2019) Rodriguez Bazan, E., Hubert, E., 2019. Symmetry Preserving Interpolation. In: ISSAC'19. ACM, Beijing, China, pp. 34–41

3.1 Introduction

In this chapter, we address how to deal with multivariate interpolation in the presence of symmetry.

Among the several problems in multivariate interpolation (Gasca and Sauer, 2000; Lorentz, 2000), we focus on the construction of a polynomial interpolation space for a given set of linear forms. Assuming the space generated by the linear forms is invariant under a group action, we show how to, not only, preserve exactly the symmetry, but also, exploit it throughout the computations.

An interpolation space for a set of linear forms is a subspace of the polynomial ring that has a unique interpolant for each instantiated interpolation problem. We show that the unique interpolants automatically inherit the symmetry of the problem when the interpolation space is invariant (Section 3.3). We need to point out that an invariant interpolation space is generally not spanned by monomials, when much of the literature on algebraic computation restricts to monomial bases.

A canonical interpolation space, the *least interpolation space*, was introduced by De Boor and Ron (1990, 1992a,b). We shall observe that it is invariant as soon as the space of linear forms is. The least interpolation space can be constructed by Gauss elimination in a multivariate Vandermonde (or *collocation*) matrix. A different collection of the terms allows to determine the least interpolation space. The columns of the Vandermonde matrix are traditionally indexed by monomials. We show how any other graded basis of the polynomial ring can be used. In particular there is a two fold gain in using a *symmetry adapted basis*. On one hand, the computed interpolation space will be exactly invariant independently of the accuracy of the data for the interpolation problem. On the other hand, the new Vandermonde matrix is block diagonal so that Gauss elimination can be performed independently on smaller size matrices, with better conditioning. Computational savings are obtained from identical blocks being repeated according to the dimension of the related irreducible representations of the group. Symmetry adapted bases also played a prominent role in (Collowald and Hubert, 2015; Gatermann and Parrilo, 2004; Riener et al., 2013) where it allowed the block diagonalisation of a multivariate Hankel matrix.

In Section 3.2 we define minimal degree and least interpolation space and review how to compute a basis of it with Gauss elimination. In Section 3.3 we make explicit how symmetry is expressed and the main ingredients to preserve it. In Section 3.4 we review symmetry adapted bases and show how the Vandermonde matrix becomes block diagonal. This is applied to provide an algorithm for the computation of invariant interpolation spaces in Section 3.5 together with a selection of relevant invariant and equivariant interpolation problems.

3.2 Polynomial interpolation

In this section we first review the definitions and constructions of interpolation spaces of minimal degree. By introducing general dual polynomial bases we generalize the

construction of least interpolation spaces. We shall then be in a position to work with adapted bases to preserve and exploit symmetry.

3.2.1 Interpolation space

A typical example of a linear form on $\mathbb{K}[x]$ is the evaluation e_ξ at a point ξ of $\mathbb{K}^n$. It is defined by

$$\begin{array}{rccc} \mathrm{e}_\xi : & \mathbb{K}[x] & \to & \mathbb{K} \\ & p & \mapsto & p(\xi). \end{array}$$

Other examples of linear forms on $\mathbb{K}[x]$ are given by compositions of evaluation and differentiation

$$\begin{array}{rccc} \lambda : & \mathbb{K}[x] & \to & \mathbb{K} \\ & p & \mapsto & \sum_{j=1}^{r} \mathrm{e}_{\xi_j} \circ q_j(\partial)(p), \end{array}$$

with $\xi_j \in \mathbb{K}^n, q_j \in \mathbb{K}[x]$ and $\partial^\alpha = \dfrac{\partial}{\partial x_1^{\alpha_1}} \dots \dfrac{\partial}{\partial x_n^{\alpha_n}}$.

As the most common type, *Lagrange interpolation* starts with a set of points $\xi_1, \dots, \xi_r$ in $\mathbb{K}^n$ and a set of values $\eta_1, \dots \eta_r \in \mathbb{K}$, and consists in finding, a polynomial p such that $\mathrm{e}_{\xi_j}(p) = \eta_j$, $1 \le j \le r$. More generally an *interpolation problem* is a pair (Λ, ϕ) where Λ is a finite dimensional linear subspace of $\mathbb{K}[x]^*$ and $\phi : \Lambda \longrightarrow \mathbb{K}$ is a $\mathbb{K}$-linear map. An interpolant, *i.e.,* a solution to the interpolation problem, is a polynomial p such that

$$\lambda(p) = \phi(\lambda) \text{ for any } \lambda \in \Lambda. \tag{3.2.1}$$

An *interpolation space for* Λ is a polynomial subspace P of $\mathbb{K}[x]$ such that Equation (3.2.1) has a unique solution in P for any map ϕ.

3.2.2 Vandermonde matrix

For a given linear space of linear forms Λ we introduce the *Vandermonde operator* ω as

$$\begin{array}{rccc} \omega : & \mathbb{K}[x] & \to & \Lambda^* \\ & p & \to & (\cdot\ ,\ p)\,, \end{array} \tag{3.2.2}$$

where $(\cdot\ ,\ \cdot)$ is the dual pairing between Λ and Λ^*, *i.e.*, $(\lambda\ ,\ p) = \lambda(p)$. When Λ is finite dimensional ω is a surjective map. Indeed, for every $\phi \in \Lambda^*$ let $p_\phi \in \mathbb{K}[x]$ be a solution of the interpolation problem (Λ, ϕ). Then $\phi = (\cdot, p_\phi)$ and therefore $\omega(p_\phi) = \phi$.

We denote by $\omega_d : \mathbb{K}[x]_{\le d} \to \Lambda^*$ the restriction of ω to $\mathbb{K}[x]_{\le d}$. The matrix of ω_d in the bases $\mathcal{P} = \{p_1, p_2, \dots, p_m\}$ of $\mathbb{K}[x]_{\le d}$ and the dual of the basis $\mathcal{L} = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$ of Λ is the *Vandermonde matrix*

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \begin{bmatrix} \lambda_1(p_1) & \lambda_1(p_2) & \lambda_1(p_3) & \dots & \lambda_1(p_m) \\ \lambda_2(p_1) & \lambda_2(p_2) & \lambda_2(p_3) & \dots & \lambda_2(p_m) \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_r(p_1) & \lambda_r(p_2) & \lambda_r(p_r) & \dots & \lambda_r(p_m) \end{bmatrix}. \tag{3.2.3}$$

As in the univariate case, the Vandermonde matrix appears naturally in the interpolation problem. Indeed $\langle p_1, \ldots, p_r \rangle_{\mathbb{K}}$ is an interpolation space for $\langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{K}}$ if and only if for any $\phi : \Lambda \to \mathbb{K}$ there exists a unique $a = (a_1, \ldots, a_r)^T \in \mathbb{K}^r$ such that

$$\begin{cases} \lambda_1(a_1\, p_1 + \ldots a_m\, p_m) = \phi(\lambda_1) \\ \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ \lambda_r(a_1\, p_1 + \ldots a_m\, p_m) = \phi(\lambda_r) \end{cases}.$$

Then a is the solution of the linear system $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}\ a = (\phi(\lambda_1), \ldots, \phi(\lambda_r))^T$. Therefore $\langle p_1, \ldots, p_r \rangle_{\mathbb{K}}$ is an interpolation space if and only if $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is an invertible matrix. This leads to a straightforward approach to compute an interpolation space for $\langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{K}}$. Since the elements of $\mathcal{L}$ are linearly independent, there is $\delta > 0$ such that $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$ has full row rank, where $\mathcal{P}_\delta$ is a basis of $\mathbb{K}[\mathrm{x}]_{\leq \delta}$. For Lagrange interpolation $\delta \leq |\mathcal{L}|$. Hence we can choose r linearly independent columns $j_1, j_2, \ldots j_r$ of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$ and the corresponding space $P = \langle p_{j_1}, \ldots p_{j_k} \rangle_{\mathbb{K}}$ is an interpolation space for Λ.

In order to select r linearly independent columns of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$we can use any rank revealing decomposition of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$. Singular value decomposition (SVD) and QR decomposition provide better numerical accuracy but to obtain a minimal degree interpolation space we shall resort to Gauss elimination. It produces a LU factorization of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$ where L is an invertible matrix and $\mathrm{U} = [u_{ij}]_{\substack{1 \leq i \leq r \\ 1 \leq j \leq m}}$ is in row echelon form. This means that there exists an increasing sequence $j_1, \ldots, j_r$ with $j_i \geq i$, such that u_{ij_i} is the first non-zero entry in the i–th row of U. We call $j_1, \ldots, j_r$ the *echelon index sequence* of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$. They index a maximal set of linearly independent columns of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\delta}$.

3.2.3 Minimal degree

It is desirable to build an interpolation space such that the degree of the interpolating polynomials be as small as possible. We shall use the definition of minimal degree solution for an interpolation problem defined by De Boor and Ron (1992a,b) and Sauer (1998).

Definition 3.2.1 *An interpolation space P for Λ is of minimal degree if for any other interpolation space Q for Λ*

$$\dim(Q \cap \mathbb{K}[\mathrm{x}]_{\leq \delta}) \leq \dim(P \cap \mathbb{K}[\mathrm{x}]_{\leq \delta}), \forall \delta \in \mathbb{N}.$$

Definition 3.2.2 *An interpolation space P for Λ is degree reducing if for any $p \in \mathbb{K}[\mathrm{x}]$ the solution of the interpolation problem (Λ, ϕ_p) has degree less or equal than degree of p, where*

$$\phi_p(\lambda) = \lambda(p) \text{ for any } \lambda \in \Lambda.$$

Proposition 3.2.3 *(De Boor and Ron, 1992a, Proposition 7.1) An interpolation space P for Λ is of minimal degree if and only if it is degree reducing.*

We say that a countable set of homogeneous polynomials $P = \{p_1, p_2, \ldots\}$ is *ordered by degree* if $i \le j$ implies that $\deg p_i \le \deg p_j$.

Proposition 3.2.4 *Let $\mathcal{L}$ be a basis of Λ. Let $\mathcal{P}_\delta$, $\delta > 0$, be a homogeneous basis of $\mathbb{K}[x]_{\le\delta}$ ordered by degree, such that $W_{\mathcal{L}}^{\mathcal{P}_\delta}$ has full row rank. Let $j_1, \ldots, j_r$ be the echelon sequence of $W_{\mathcal{L}}^{\mathcal{P}_\delta}$ obtained by Gauss elimination with partial pivoting. Then $P := \langle p_{j_1}, \ldots, p_{j_r}\rangle$ is a minimal degree interpolation space for Λ.*

PROOF: Let Q be another interpolation space for Λ. Let $q_1, q_2 \ldots q_m$ be a basis of $Q \cap \mathbb{K}[x]_{\le d}$ with $d \le \delta$. Since $\mathcal{P}_\delta$ is a homogeneous basis of $\mathbb{K}[x]_{\le\delta}$, any q_i can be written as a linear combination of elements of $\mathcal{P}_\delta \cap \mathbb{K}[x]_{\le d}$. Considering $q_i = \sum_j a_{ji} p_j$ we get that $\lambda(q_i) = \sum_j a_{ji}\lambda(p_j)$ for any $\lambda \in \Lambda$.

Let $\{p_{j_{i_1}}, p_{j_{i_2}}, \ldots p_{j_{i_n}}\}$ be the elements of P that form a basis of $P \cap \mathbb{K}[x]_{\le d}$. Gauss elimination on $W_{\mathcal{L}}^{\mathcal{P}_\delta}$ ensures that $\lambda(b)$ is a linear combination of $\lambda(p_{j_{i_1}}), \ldots \lambda(p_{j_{i_n}})$ for any $b \in \mathcal{P}_\delta \cap \mathbb{K}[x]_{\le d}$ and $\lambda \in \Lambda$. The latter implies that

$$\lambda(q_i) = \sum_{k=1}^{n} c_{ki}\lambda(p_{j_{i_k}}) \text{ for } 1 \le i \le m \text{ and } c_{ki} \in \mathbb{K}.$$

If $m > n$ then the matrix $C = (c_{ij})_{\substack{1\le i\le m\\ 1\le j\le n}}$ has linearly independent columns, and therefore there exist $d_1, d_2, \ldots d_m \in \mathbb{K}$ such that

$$\sum_{i=1}^{m} d_i \lambda(q_i) = \lambda\left(\sum_{i=1}^{m} d_i q_i\right) = 0 \text{ for any } \lambda \in \Lambda$$

which is a contradiction with the fact that Q is an interpolation space of Λ. Then we can conclude that $m \le n$ and P is a minimal degree interpolation space for Λ. □

3.2.4 Least interpolation space

For a space of linear forms $\Lambda \subset \mathbb{K}[x]^*$, a canonical interpolation space $\Lambda_\downarrow$ is introduced by De Boor and Ron (1990). It has a desirable set of properties. An algorithm to build a basis of $\Lambda_\downarrow$ based on Gauss elimination on the Vandermonde matrix is presented in (De Boor and Ron, 1992a). In this algorithm the authors consider the Vandermonde matrix associated to the monomial basis of $\mathbb{K}[x]$. The notion of dual bases introduced above allows to extend the algorithm to any graded basis of $\mathbb{K}[x]$.

The initial term of a power series $\lambda \in \mathbb{K}[[\partial]]$, denoted by $\lambda_\downarrow \in \mathbb{K}[x]$ in (De Boor and Ron, 1990, 1992a,b), is the unique homogeneous polynomial for which $\lambda - \lambda_\downarrow(\partial)$ vanishes to highest possible order at the origin. Given a linear space of linear forms Λ, we define $\Lambda_\downarrow$ as the linear span of all $\lambda_\downarrow$ with $\lambda \in \Lambda$.

Proposition 3.2.5 *Let $\mathcal{P} = \{p_1, p_2, \ldots\}$ be a homogeneous basis of $\mathbb{K}[x]$ ordered by degree and $\mathcal{L} = \{\lambda_1, \ldots, \lambda_r\}$ be a basis of Λ. Let $LU = W_{\mathcal{L}}^{\mathcal{P}}$ be the factorization*

of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ *provided by Gauss elimination with partial pivoting with* $\{j_1, j_2, \ldots, j_r\}$ *as echelon index sequence. If* $\mathrm{U} = (u_{ij})$ *consider, for* $1 \le \ell \le r$,

$$h_\ell = \sum_{\deg(p_k)=\deg(p_{j_\ell})} u_{\ell k}\ \overline{p}_k^\dagger \tag{3.2.4}$$

where $\mathcal{P}^\dagger = \{p_1^\dagger, \ldots . p_j^\dagger, \ldots\}$ *is the dual basis of* $\mathcal{P}$ *with respect to the apolar product. Then* $H = \{h_1, \ldots h_r\}$ *is a basis for* $\Lambda_\downarrow$.

PROOF: Let $\mathrm{L}^{-1} = (a_{ij})$ and consider $\varsigma_\ell = \sum_{j\in\mathbb{N}} u_{\ell j}\overline{p}_j^\dagger(\partial)$. Since $u_{\ell j} = \sum_{i=1}^{r} a_{li}\lambda_i(p_j)$, then

$$\varsigma_\ell = \sum_{j\in\mathbb{N}} \Big(\sum_{i=1}^{r} a_{li}\lambda_i(p_j)\Big)\overline{p}_j^\dagger(\partial) = \sum_{i=1}^{r} a_{li} \sum_{j\in\mathbb{N}} \lambda_i(p_j)\overline{p}_j^\dagger(\partial) = \sum_{i=1}^{r} a_{li}\lambda_i \in \Lambda.$$

Notice that $h_\ell = \varsigma_{\ell\downarrow}$ and therefore $h_\ell \in \Lambda_\downarrow$ for $1 \le \ell \le r$.

The j_i are strictly increasing so that $\{h_1, h_2, \ldots, h_r\} \subset \Lambda_\downarrow$ are linearly independent. By (De Boor and Ron, 1992b, Proposition 2.10) we have that $r = \dim \Lambda = \dim \Lambda_\downarrow$. Thus we conclude that H is a basis of $\Lambda_\downarrow$. □

In the Chapter 3 we provide an alternative construction of the least interpolation space. We shall compute an orthogonal basis of the least interpolation space by applying QR decomposition. This provides a better numerical stability then LU decomposition (Fassino and Möller, 2016) but needs to be done degree by degree.

3.3 Symmetry

We define the concepts of *invariant interpolation problem* (IIP) and *equivariant interpolation problem* (EIP). These interpolation problems have a structure that we want to be preserved by the interpolant. We show that this is automatically achieved when choosing the interpolant in an invariant interpolation space. Then the solution of an IIP is an invariant polynomial and the solution of an EIP is an equivariant polynomial map. In Section 3.5 we show that the least interpolation space is invariant and how to better compute an invariant interpolation space of minimal degree.

The symmetries we shall deal with are given by the linear group action of a finite group G on $\mathbb{K}^n$. It is thus given by a representation ϑ of G on $\mathbb{K}^n$. It induces a representation ρ of G on $\mathbb{K}[\mathrm{x}]$ given by

$$\rho(g)p(x) = p(\vartheta(g^{-1})x). \tag{3.3.1}$$

It also induces a linear representation on the space of linear forms, the dual representation of ρ :

$$\rho^*(g)\lambda(p) = \lambda(\rho(g^{-1})p),\ p \in \mathbb{K}[\mathrm{x}] \text{ and } \lambda \in \mathbb{K}[x]^*. \tag{3.3.2}$$

We shall deal with an invariant subspace Λ of $\mathbb{K}[x]^*$. Hence the restriction of ρ^* to Λ is a linear representation of G in Λ.

3.3.1 Invariance

An invariant Lagrange interpolation problem can be seen as interpolation at union of orbits of points with fixed values on their orbits, i.e., given $\xi_1, \dots, \xi_m$ with orbits $\mathcal{O}_1, \dots, \mathcal{O}_m$ and $\eta_1, \dots, \eta_m \in \mathbb{K}^n$, an interpolant $p \in \mathbb{K}[x]$ is to satisfy $p \circ \vartheta(g)(\xi_k) = \eta_k$ for any $g \in G$. This is generalized as follow.

Definition 3.3.1 *Let Λ be a space of linear forms and $\phi : \Lambda \longrightarrow \mathbb{K}$ a linear map. The pair (Λ, ϕ) defines an* invariant interpolation problem *if*

1. *Λ is closed under the action of G.*
2. *$\phi(\rho^*(g)(\lambda)) = \phi(\lambda)$ for any $g \in G$ and $\lambda \in \Lambda$.*

It is natural to expect that the solution to a invariant interpolation problem is an invariant polynomial. Yet, not all minimal degree interpolants are invariant.

Example 3.3.1 *The dihedral group D_m is the group of order $2m$ that leaves invariant the regular m-gon. It thus has a representation in $\mathbb{R}^2$ given by the matrices*

$$\vartheta_k = \left[\begin{pmatrix} \cos\left(\lfloor \frac{k}{2} \rfloor \frac{2\pi}{m}\right) & -\sin\left(\lfloor \frac{k}{2} \rfloor \frac{2\pi}{m}\right) \\ \sin\left(\lfloor \frac{k}{2} \rfloor \frac{2\pi}{m}\right) & \cos\left(\lfloor \frac{k}{2} \rfloor \frac{2\pi}{m}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^k \right], \quad 0 \le k \le 2m-1. \tag{3.3.3}$$

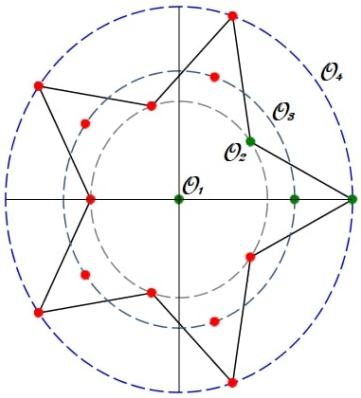
Consider $\Xi \subset \mathbb{R}^2$ a set of $1 + 3 \times 5$ points illustrated on Figure 3.1. They form four orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ of D_5 so that $\Lambda := \langle \mathrm{e}_{\xi_i} \,|\, \xi_i \in \Xi \rangle_{\mathbb{K}}$ is invariant. An invariant interpolation problem is given by the pair (Λ, ϕ) where ϕ is defined by $\phi(\mathrm{e}_\xi) = 0.1$ if $\xi \in \mathcal{O}_1$, $\phi(\mathrm{e}_\xi) = 0$ if $\xi \in \mathcal{O}_2 \cup \mathcal{O}_4$, and $\phi(\mathrm{e}_\xi) = -0.5$ if $\xi \in \mathcal{O}_3$. We show in Figure 3.1a the graph of the expected interpolant, but in Figure 3.1b we present the graph of an interpolant of minimal degree from a monomial basis; The D_5 symmetry is not respected.

Proposition 3.3.2 *Let (Λ, ϕ) be an invariant interpolation problem. Let P be an invariant interpolation space and let $p \in \mathbb{K}[x]$ be the solution of (Λ, ϕ) in P. Then $p \in \mathbb{K}[x]^G$, the ring of invariant polynomials.*

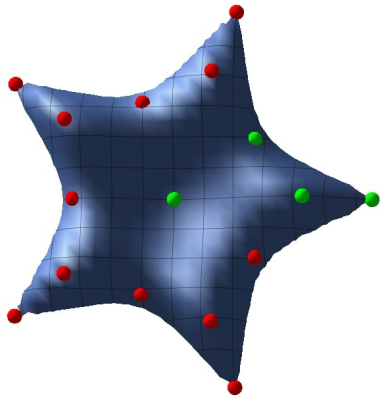
PROOF: For any $\lambda \in \Lambda$ and $g \in G$ we have that $\lambda(p) = \phi(\lambda)$ and $\rho^*(g)(\lambda)p = \phi(\rho^*(g)(\lambda))$. Since ϕ is G–invariant, we get that

$$\lambda(\rho(g^{-1})p) = \rho^*(g)(\lambda)p = \phi(\rho^*(g)(\lambda)) = \lambda(p)$$

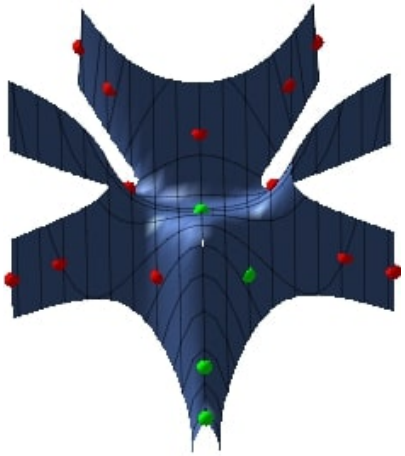
for any $\lambda \in \Lambda$. The latter implies that $\rho(g^{-1})p - p \in \cap_{\lambda \in \Lambda} \ker \lambda$. As P is closed under the action of ρ, $\rho(g^{-1})p - p \in \cap_{\lambda \in \Lambda} \ker \lambda \bigcap P$. As (Λ, P) is an interpolation space we have that $\cap_{\lambda \in \Lambda} \ker \lambda \bigcap P = \varnothing$ and therefore we can conclude that $\rho(g^{-1})p - p = 0$ for any $g \in G$, i.e., $p \in \mathbb{K}[x]^G$.□



	#Nodes	A node per orbit
$\mathcal{O}_1$	1	$\xi_1 = (0, 0)$
$\mathcal{O}_2$	5	$\xi_2 = (0.1934557, 0.1405538)$
$\mathcal{O}_3$	5	$\xi_7 = (0.4695268, 0)$
$\mathcal{O}_4$	5	$\xi_{12} = (0.6260358, 0)$



(a) invariant interpolant of minimal degree



(b) interpolant obtained from a monomial basis

Figure 3.1: Invariant Lagrange interpolation problem

3.3.2 Equivariance

Let $\mathbb{K}[\mathrm{x}]^m$ be the module of polynomial mappings with m components, and let $\theta : G \longrightarrow \mathrm{Aut}(\mathbb{K}^m)$ be a linear representation on $\mathbb{K}^m$. A polynomial mapping $f = (f_1, f_2, \ldots, f_m)^t$ is called $\vartheta - \theta$ equivariant if $f(\vartheta(g)x) = \theta(g)f(x)$ for any $g \in G$. The space of equivariant mappings over $\mathbb{K}$, denoted by $\mathbb{K}[\mathrm{x}]^{\theta}_{\vartheta}$, is a $\mathbb{K}[\mathrm{x}]^G$–module.

Equivariant maps define, for instance, dynamical systems that exhibit particularly interesting patterns and are relevant to model physical or biological phenomena (Chossat and Lauterbach, 2000; Golubitsky et al., 1988). In this context, it is interesting to have a tool to offer equivariant maps that interpolate some observed local behaviors.

Definition 3.3.3 *Let Λ be a space of linear forms on $\mathbb{K}[\mathrm{x}]$ and $\phi : \Lambda \longrightarrow \mathbb{K}^m$ a linear map. The pair (Λ, ϕ) defines a* $\vartheta - \theta$ equivariant interpolation problem *if*

1. *Λ is closed under the action of G.*

2. $\phi(\rho^*(g)(\lambda)) = \theta(g)\phi(\lambda)$ for any $g \in G$ and $\lambda \in \Lambda$.

The solution of an EIP $\big(\Lambda, \phi\big)$, is a polynomial map $f = (f_1, \ldots, f_m)^t$ such that $\lambda(f) = (\lambda(f_1), \ldots, \lambda(f_m))^t = \phi(\lambda)$ for any $\lambda \in \Lambda$. It is natural to seek f as an equivariant map. It is remarkable that any type of equivariance will be respected as soon as the interpolation space is invariant.

Proposition 3.3.4 *Let (Λ, ϕ) be an equivariant interpolation problem. Let P be an invariant interpolation space for Λ and let $f = (f_1, \ldots, f_m)^t$ be the solution of (Λ, ϕ) in P. Then $f \in \mathbb{K}[x]^{\theta}_{\vartheta}$.*

PROOF: We need to prove that $f \circ \vartheta(g^{-1}) = \theta(g)f$. As $f \in P^n$ and P is invariant, $f \circ \vartheta(g^{-1})$ belongs to P^n as does $\theta(g)f$. As P is an interpolation space for Λ, it is thus enough to prove that $\lambda\left(f \circ \vartheta(g)\right) = \lambda\left(\theta(g)f\right)$ for all $\lambda \in \Lambda$.

On one hand

$$\lambda\left(f \circ \vartheta(g)\right) = \left(\rho^*(g)\lambda\right)(f) = \phi(\rho^*(g)\lambda),$$

where the first equality is by definition of ρ and the second one stems from f being a solution of the interpolation problem. On the other hand

$$\lambda\left(\theta(g)f\right) = \theta(g)\lambda\left(f\right) = \theta(g)\phi(\lambda) = \phi\left(\rho^*(g)\lambda\right),$$

where the first equality is by linearity, the second one stems from f being a solution of the interpolation problem, and the third one comes from the definition of an equivariant interpolation problem. □

Example 3.3.2 *We consider two different equivariant problems for the groups D_3 and C_3. The cyclic group C_m is the group of order m that is generated by a single element. It has a representation in $\mathbb{R}^2$ given by m rotations matrices*

$$\tau_k = \left[\begin{pmatrix} \cos\left(\frac{2k\pi}{m}\right) & -\sin\left(\frac{2k\pi}{m}\right) \\ \sin\left(\frac{2k\pi}{m}\right) & \cos\left(\frac{2k\pi}{m}\right) \end{pmatrix}\right], \quad 0 \le k \le m-1. \tag{3.3.4}$$

The symmetries are given by the representations of D_3 and C_3 in Equations (3.3.3) and (3.3.4) respectively. For D_3 we consider the space Λ_D of linear forms spanned by the evaluations at the points of the orbits of $\xi_1 = (-\frac{5\sqrt{3}}{3}, \frac{1}{3})^t$ and $\xi_2 = (-\sqrt{3}, \frac{1}{3})^t$. We define $\phi_D : \Lambda \to \mathbb{R}^2$ by $\phi_D(\mathbb{e}_{\vartheta(g)\xi_1}) = \vartheta(g)\begin{pmatrix} v_1 \\ v_3 \end{pmatrix}$ and $\phi_D(\mathbb{e}_{\vartheta(g)\xi_2}) = \vartheta(g)\begin{pmatrix} v_2 \\ v_4 \end{pmatrix}$.

For C_3 we consider Λ_C spanned by the evaluations at the points of the orbits of $\zeta_1 = \left(-\frac{3}{2}, 0\right)^t$, $\zeta_2 = \left(0, \frac{5}{2}\right)^t$ and $\zeta_3 = \left(\frac{7}{2}, 0\right)^t$. We define $\phi_C : \Lambda \to \mathbb{R}^2$ by

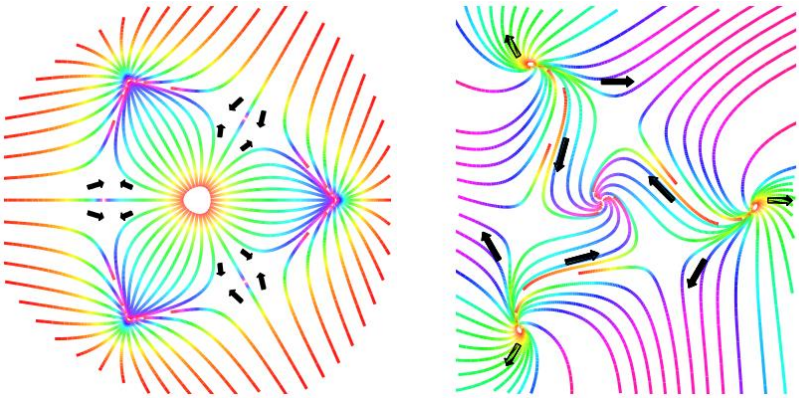
$$\phi_C(\mathbb{e}_{\tau(g)\zeta_1}) = \tau(g)\begin{pmatrix} u_1 \\ u_4 \end{pmatrix}, \quad \phi_C(\mathbb{e}_{\tau(g)\zeta_2}) = \tau(g)\begin{pmatrix} u_2 \\ u_5 \end{pmatrix} \quad \text{and} \quad \phi_C(\mathbb{e}_{\tau(g)\zeta_3}) = \tau(g)\begin{pmatrix} u_3 \\ u_6 \end{pmatrix}.$$

The thus defined interpolation problems are clearly equivariant. For each quadruplet $v \in \mathbb{R}^4$ and sextuplet $u \in \mathbb{R}^6$ it is desirable to find interpolants $(p_1, p_2)^t \in \mathbb{R}[x]^2$ and

$(q_1, q_2)^t \in \mathbb{R}[x]^2$ that are an $\vartheta - \vartheta$ and $\tau - \tau$ equivariant maps respectively. This will define the equivariant dynamical systems

$$\begin{cases} \dot{x_1}(t) = p_1(x_1(t), x_2(t)), \\ \dot{x_2}(t) = p_2(x_1(t), x_2(t)); \end{cases} \quad \text{and} \quad \begin{cases} \dot{x_1}(t) = q_1(x_1(t), x_2(t)), \\ \dot{x_2}(t) = q_2(x_1(t), x_2(t)). \end{cases}$$

whose integral curves, limit cycles and equilibrium points, will all exhibit the D_3 and C_3 symmetries respectively. In Figures 3.2a and 3.2b we draw the integral curves of the equivariant vector fields thus constructed. The data of the interpolation problem are illustrated by the black arrows



(a) D_3 (b) C_3

Figure 3.2: Integral curves for the equivariant vector fields interpolating the equivariant set of vectors shown in black.

3.4 Symmetry reduction

In this section we show how, when the space Λ of linear forms is invariant, the Vandermonde matrix can be made block diagonal. That happens when making use of *symmetry adapted bases* both for $\mathbb{K}[\mathrm{x}]_{\leq\delta}$ and Λ. as it appears in representation theory (Serre, 1977; Fässler and Stiefel, 1992) and then show how to obtain *real symmetry adapted bases*. This block diagonalisation of the Vandermonde matrix indicates how computation can be organized more efficiently, and robustly. It just draws on the invariance of the space of linear forms. So, when the evaluation points can be chosen, it makes sense to introduce symmetry among them.

We consider a linear representation ϑ of a finite group G on $\mathbb{K}^n$. It induces the representations ρ and its dual ρ^* on the space $\mathbb{K}[\mathrm{x}]$ and $\mathbb{K}[x]^*$, as made explicit in (3.3.1) and (3.3.2).

For an invariant subspace Λ of $\mathbb{K}[x]^*$ the restriction of ρ^* to Λ is a linear representation of G, the dual representation of this restriction denoted by θ^* is a linear representation of G in Λ^*.

Proposition 3.4.1 *Consider θ the restriction of ρ^* to the invariant subspace Λ of $\mathbb{K}[x]^*$, and θ^* the dual representation on Λ^*. The Vandermonde operator $\omega : \mathbb{K}[\mathrm{x}] \to \Lambda^*$ defined in (3.2.2) is $\rho - \theta^*$ equivariant.*

PROOF: This is mostly a matter of unrolling the definitions. We want to show that $\omega(\rho(g)(p)) = \theta^*(g)(\omega(p))$. The left hand side applied to any $\lambda \in \Lambda$ is equal to $\lambda(\rho(g)(p)) = (\rho^*(g^{-1})(\lambda))(p)$. The right handside applied to any $\lambda \in \Lambda$ is equal to $\omega(p)(\theta(g^{-1})(\lambda)) = (\theta(g^{-1})(\lambda))(p)$. The conclusion follows since $\theta(g^{-1})(\lambda) = \rho^*(g^{-1})(\lambda)$ by definition of θ. □

Corollary 3.4.2 *Let $\mathcal{P} = \cup_{i=1}^{\mathfrak{n}} \mathcal{P}^{(i)}$ and $\mathcal{L} = \cup_{i=1}^{\mathfrak{n}} \mathcal{L}^{(i)}$ be symmetry adapted bases of $\mathbb{K}[\mathrm{x}]_{\leq\delta}$ and Λ respectively where*

- *$\mathcal{P}^{(i)}$ determined by $\left\{p_1^{(i)}, \dots, p_{c_i}^{(i)}\right\}$ spans the isotypic component associated with the irreducible representation ρ_i*
- *$\mathcal{L}^{(i)}$ determined by $\{\lambda_1^{(i)}, \dots, \lambda_{r_i}^{(i)}\}$ spans the isotypic component associated with the irreducible representation ρ_i^**

The Vandermonde matrix $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is given by

$$\operatorname{diag}\left(\mathrm{I}_{n_j} \otimes \left(\lambda_s^{(i)}\left(p_t^{(i)}\right)\right)_{\substack{1\leq s\leq r_j \\ 1\leq t\leq c_j}},\ i = 1\dots\mathfrak{n}\right), \tag{3.4.1}$$

where $\otimes$ denotes the Kronecker product.

PROOF: According to Proposition 2.3.2, the dual basis $\mathcal{L}^*$ of $\mathcal{L}$ is symmetry adapted with its i-th component being associated to ρ_i. The matrix of the Vandermonde operator ω in $\mathcal{P}$ and $\mathcal{L}$ is $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$. Proposition 3.4.1 ensure that ω is equivariant and thus the result follows from Theorem 2.3.4. □

Note that ρ_i and ρ_i^* are not equivalent only when ρ_i is of complex type. In particular, when we deal with interpolation over the reals, and thus use a real symmetry adapted basis, there is no distinction to be made.

Example 3.4.1 *Let G be the dihedral group D_3 of order 6. A representation of G on $\mathbb{R}^2$ is given by Equation (3.3.3) with $m = 3$. D_3 has three irreducible representations, two of dimension 1 and one of dimension 2.*

Consider Ξ the orbit of the point $\xi_1 = \left(-\frac{5\sqrt{3}}{3}, \frac{1}{3}\right)^t$ in $\mathbb{R}^2$, with $\xi_i = \vartheta_{i-1}\xi_1$. Let $\Lambda = \left\langle \mathrm{e}_{\xi_i} \circ D_{\vec{\xi}_i} \right\rangle_{\mathbb{K}}$ with $D_{\vec{\xi}_i}$ the directional derivative with direction $\vec{\xi}_i$. Λ is closed under the action of G. Indeed for any $p \in \mathbb{K}[\mathrm{x}]$,

$$\rho^*(g)(\mathrm{e}_{\xi_i} \circ D_{\vec{\xi}_i})(p) = \mathrm{e}_{\xi_i} \circ D_{\vec{\xi}_i}(p(\vartheta(g^{-1}x)) = \mathrm{e}_{\vartheta(g^{-1})\xi_i} \circ D_{\vartheta(g^{-1})\xi_i}(p(x)).$$

Since $\vartheta(g^{-1})\xi_i = \xi_j$ for some $1 \leq j \leq 6$ we have $\rho^*(g)(\mathrm{e}_{\xi_i} \circ D_{\vec{\xi}_i}) = \mathrm{e}_{\xi_j} \circ D_{\vec{\xi}_j}$. Considering $\varrho_i = \mathrm{e}_{\xi_i} \circ D_{\vec{\xi}_i}$, $1 \leq i \leq 6$, a symmetry adapted basis of Λ is given by

$$\mathcal{L} := \left\{ \begin{array}{c} \{\varrho_1+\varrho_2+\varrho_3+\varrho_4+\varrho_5+\varrho_6\}, \\ \{\varrho_1-\varrho_2+\varrho_3-\varrho_4+\varrho_5-\varrho_6\}, \\ \{\{\lambda_3,\lambda_4\},\{\lambda_5,\lambda_6\}\} \end{array} \right\},$$

with

$\lambda_3 = \varrho_1+\varrho_2-\varrho_4-\varrho_5 \quad \lambda_5 = \frac{\sqrt{3}}{2}(\varrho_2-\varrho_1+\varrho_4+2\varrho_3-2\varrho_6-\varrho_5),$
$\lambda_4 = \varrho_3-\varrho_4-\varrho_5+\varrho_6, \quad \lambda_6 = \frac{\sqrt{3}}{2}(2\varrho_2-2\varrho_1-\varrho_4+\varrho_3-\varrho_5-\varrho_6).$

A symmetry adapted basis of $\mathbb{R}[x]_{\leq 3}$ *is given by*

$$\mathcal{P} := \left\{ \begin{array}{c} \{1, x_1^2+x_2^2, x_1^3-3x_1x_2^2\}, \\ \left\{x_1^2x_2-\frac{1}{3}x_2^3\right\}, \\ \{\{x_1, x_1^2-x_2^2, x_1^3+x_1x_2^2\},\{x_2,-2x_1x_2,x_1^2x_2+x_2^3\}\} \end{array} \right\}.$$

The Vandermonde matrix $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ *is block diagonal :*

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \left(\begin{array}{c|c|c} \mathrm{A}_1 & & \\ \hline & \frac{448}{9} & \\ \hline & & \mathrm{A}_3 \\ & & \mathrm{A}_3 \end{array}\right), \quad \begin{array}{l} \mathrm{A}_1 = \begin{pmatrix} 0 & \frac{304}{3} & -240\sqrt{3} \end{pmatrix} \\ \mathrm{A}_3 = \begin{pmatrix} -\frac{16\sqrt{3}}{3} & \frac{128}{3} & -\frac{1216\sqrt{3}}{9} \\ -\frac{2\sqrt{3}}{3} & -\frac{40}{3} & -\frac{152\sqrt{3}}{9} \end{pmatrix}. \end{array}$$

Example 3.4.2 *Let* G *be the cyclic group* C_3 *of order 3. A representation of* G *on* $\mathbb{R}^2$ *is given in (3.3.4).* C_3 *has* 3 *irreducible representations of dimension* 1*, one absolutely irreducible representation and a pair of conjugate irreducible representations of complex type. The real symmetry adapted bases thus have two components. Consider* Λ *the space spanned by the orbit of the points* ζ_1 *and* ζ_2 *given in Example 3.3.2. Real symmetry adapted bases of* Λ *and* $\mathbb{R}[x]_{\leq 3}$ *are given by*

$$\mathcal{L} := \left\{ \begin{array}{l} \left\{ \frac{\sqrt{3}}{3}\left(\mathrm{e}_{\zeta_1}+\mathrm{e}_{\zeta_2}+\mathrm{e}_{\zeta_3}\right), \frac{\sqrt{3}}{3}\left(\mathrm{e}_{\zeta_4}+\mathrm{e}_{\zeta_5}+\mathrm{e}_{\zeta_6}\right) \right\}, \\ \left\{ \begin{array}{l} \frac{\sqrt{6}}{6}\left(2\mathrm{e}_{\zeta_1}-\mathrm{e}_{\zeta_2}-\mathrm{e}_{\zeta_3}\right), \frac{\sqrt{6}}{6}\left(2\mathrm{e}_{\zeta_4}-\mathrm{e}_{\zeta_5}-\mathrm{e}_{\zeta_6}\right), \\ \frac{\sqrt{2}}{2}\left(\mathrm{e}_{\zeta_2}-\mathrm{e}_{\zeta_3}\right), \frac{\sqrt{2}}{2}\left(\mathrm{e}_{\zeta_5}-\mathrm{e}_{\zeta_6}\right) \end{array} \right\} \end{array} \right\} \text{ and}$$

$$\mathcal{P} := \left\{ \begin{array}{c} \left\{1, \frac{x^2}{2}+\frac{y^2}{2}\right\}, \\ \left\{x, -y, \frac{x^2}{2}-\frac{y^2}{2}, -xy\right\} \end{array} \right\}.$$

The Vandermonde matrix $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ *is block diagonal :*

$$\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \begin{pmatrix} \mathrm{A}_1 & \\ & \mathrm{A}_2 \end{pmatrix}, \ \mathrm{A}_1 = \begin{pmatrix} \sqrt{3} & \frac{9\sqrt{3}}{8} \\ \sqrt{3} & \frac{25\sqrt{3}}{8} \end{pmatrix}, \ \mathrm{A}_2 = \begin{pmatrix} \frac{3\sqrt{6}}{4} & 0 & \frac{9\sqrt{6}}{16} & 0 \\ 0 & -\frac{5\sqrt{6}}{4} & -\frac{25\sqrt{6}}{16} & 0 \\ 0 & -\frac{3\sqrt{6}}{4} & 0 & -\frac{9\sqrt{6}}{16} \\ -\frac{5\sqrt{6}}{4} & 0 & 0 & \frac{25\sqrt{6}}{16} \end{pmatrix}$$

3.5 Equivariant interpolation

In this section we shall first show how to build interpolation spaces of minimal degree that are invariant. We shall actually build symmetry adapted bases for these, exploiting the block diagonal structure of the Vandermonde matrix. Doing so we prove that the least interpolation space is invariant. We then present a selection of invariant or equivariant interpolation problems. As proved in Section 3.3, the invariance or equivariance is preserved by the interpolant when the interpolation space is invariant. The use of the symmetry adapted bases constructed allows this equivariance to be preserved exactly, independently of the numerical accuracy.

3.5.1 Constructing invariant interpolation spaces

The starting point is a representation ϑ of G on $\mathbb{K}^n$ that induces representations ρ and ρ^* on $\mathbb{K}[\mathrm{x}]$ and $\mathbb{K}[x]^*$.

Let Λ be an invariant subspace of $\mathbb{K}[x]^*$. Hereafter $\mathcal{L}$ is a symmetry adapted basis of Λ and $\mathcal{P}$ a symmetry adapted basis of $\mathbb{K}[\mathrm{x}]_{\leq\delta}$ consisting of homogeneous polynomials. The elements of $\mathcal{P}$ corresponding to the same irreducible component are ordered by degree.

According to Proposition 3.4.2, $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}} = \mathrm{diag}\,(I_{n_i} \otimes \mathrm{A}_i)$. In the factorization $\mathrm{L}_i\mathrm{U}_i := \mathrm{A}_i$ provided by Gauss elimination, let $j_1, j_2, \ldots, j_{r_j}$ be the echelon index sequence of U_i; r_i is the multiplicity of ρ_i^* in Λ. An echelon index sequence for $\mathrm{D}_i = I_{n_i} \otimes \mathrm{A}_i$ is given by

$$S_i = \bigcup_{k=0}^{c_i-1} \{j_1 + kn_i, j_2 + kn_i, \ldots, j_{r_i} + kn_i\}.$$

An echelon index sequence of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ is given by $S = \bigcup_{i=1}^{N} S_i$. Let $\mathcal{P}_\Lambda^{(i)}$ be the set of elements of $\mathcal{P}^{(i)}$ that are indexed by elements of S_i. From (2.3.5) we get that

$$\mathcal{P}_\Lambda^{(i)} = \left\{ b_{j_1}^i, \ldots, b_{j_{r_i}}^i, \ldots, \pi_{n_i 1}^{(i)}\left(b_{j_1}^i\right), \ldots, \pi_{n_i 1}^{(i)}\left(b_{j_{r_i}}^i\right) \right\}.$$

We prove the assertions made on the outputs of Algorithm 1.

Proposition 3.5.1 *The set of polynomials $\mathcal{P}_\Lambda$ built it in Algorithm 1 spans a minimal degree interpolation space for Λ that is invariant under the action of ρ. $\mathcal{P}_\Lambda$ is furthermore a symmetry adapted basis for this space.*

PROOF: Since the elements of P_Λ are indexed by the elements of S then $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_\Lambda}$ is invertible and therefore P_Λ is an interpolation space for Λ. The elements of $\mathcal{P}_\Lambda$ that correspond to the same blocks of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$ are ordered by degree. Then as a direct consequence of Proposition 3.2.4, P_Λ is a minimal degree interpolation space. We prove now that for any p in $\mathcal{P}_\Lambda$, $\rho(g)(p) \in P_\Lambda$. Considering $p = \pi_{\alpha 1}^{(j)}(b)$. By Proposition 2.3.1 we have that $\rho(g)(p) = \sum_{\beta=1}^{n_j} r_{\beta\alpha}^{\,j}(g)\pi_{\beta 1}^{(j)}(b)$. As $\pi_{\beta 1}^{(j)}(b) \in P_\Lambda$ for any

Algorithm 1 Invariant interpolation space

In: $\mathcal{P}$ and $\mathcal{L}$ s.a.b of $\mathbb{K}[x]_{\leq\delta}$ and Λ respectively.
Out: - a s.a.b $\mathcal{P}_\Lambda$ of an invariant interpolation space of min. degree
- a symmetry adapted basis $\mathcal{H}_\Lambda$ of $\Lambda_\downarrow$.

1: **for** $i = 1$ `to` $\mathfrak{n}$ **do**
2: $\quad L_i U_i := W_{\mathcal{L}^{(i,1)}}^{\mathcal{P}^{(i,1)}}$; with $U_i = \left(u_{\ell k}^{(i)}\right)_{\ell,k}$ ▷ LU factorization of A_i
3: $\quad J := (j_1, \ldots, j_{r_j})$; ▷ echelon idex sequence of U_i
4: $\quad S_i \leftarrow \bigcup_{k=0}^{c_i-1} \{j_1 + kn_i, j_2 + kn_i, \ldots, j_{r_i} + kn_i\}$;
5: $\quad \mathcal{P}_\Lambda^{(i)} \leftarrow \{p_\ell : p_\ell \in \mathcal{P}^{(i)} \text{ and } \ell \in S_i\}$;
6: $\quad \mathcal{H}_\Lambda^{(i)} \leftarrow \left\{ \sum_{d(p_k)=d(p_\ell)} u_{\ell k}^{(i)} \overline{p}_k^\dagger : p_k \in \mathcal{P}^i \text{ and } \ell \in S_i \right\}$;
7: **end for**
8: $\mathcal{P}_\Lambda \leftarrow \bigcup_{i=1}^{N} \mathcal{P}_\Lambda^{(i)}$ and $\mathcal{H}_\Lambda \leftarrow \bigcup_{i=1}^{N} \mathcal{H}_\Lambda^{(i)}$;
9: **return** $(\mathcal{P}_\Lambda, \mathcal{H}_\Lambda)$;

$1 \leq \beta \leq n_j$, we conclude that $\rho(g)(p) \in P_\Lambda$. Since $\mathcal{P}_\Lambda$ is a basis of P_Λ we can conclude that P_Λ is invariant under the action of ρ.□

Proposition 3.5.2 *The set $\mathcal{H}_\Lambda$ built it in Algorithm 1 is a symmetry adapted basis for $\Lambda_\downarrow$.*

PROOF: By Proposition 3.2.5 we get that $\mathcal{H}_\Lambda$ is a basis of $\Lambda_\downarrow$. Let $\mathcal{H}_\Lambda^{j,\alpha} = \{h_{1,\alpha}^j, \ldots, h_{m_j,\alpha}^j\} = V^{j,\alpha} \bigcap \mathcal{H}_\Lambda$ with $1 \leq \alpha \leq c_j$. By the block diagonal structure and Corollary 2.3.3 we have

$$h_{\ell,\alpha}^j = \sum_k u_{\ell k}^{(j)} \pi_{\alpha 1}^{(j)} \left(\overline{p}_k^j\right) = \pi_{\alpha 1}^{(j)} \left(\sum_k u_{\ell k}^{(j)} \overline{p}_k^j\right) = \pi_{\alpha 1}^{(j)} \left(h_{\ell,1}^j\right).$$

Therefore $\mathcal{H}_\Lambda^j$ has the following structure

$$\mathcal{H}_\Lambda^j = \left\{h_{1,1}^j, \ldots, h_{m_j,1}^j, \ldots, \pi_{n_j 1}^{(j)}\left(h_{1,1}^j\right), \ldots, \pi_{n_j 1}^{(j)}\left(h_{m_j,1}^j\right)\right\}.$$

Since for any ℓ, $h_\ell, \pi_{21}^{(j)}\left(h_\ell\right), \ldots \pi_{n_j 1}^{(j)}\left(h_\ell\right)$ form a basis of an irreducible representation of G we can conclude that $\mathcal{H}_\Lambda$ is a symmetry adapted basis of Λ. □

If ϑ is orthogonal, the apolar product is G–invariant. As pointed out in Section 2.3.1, we can construct a symmetry adapted basis $\mathcal{P}$ of $\mathbb{K}[x]_\delta$ that is orthonormal. Then $\mathcal{P} = \mathcal{P}^\dagger$ and the basis $\mathcal{P}_\Lambda$ built in Algorithm 1 is orthonormal. Moreover if

in the third step of Algorithm 1 we use Gauss Elimination by segment as in (De Boor and Ron, 1992a), then $\mathcal{H}_\Lambda$ is an orthonormal symmetry adapted basis of $\Lambda_\downarrow$.

With this construction we reproved that $\Lambda_\downarrow$ is invariant. The above approach to computing a basis of $\Lambda_\downarrow$ is advantageous in two ways. First Gauss elimination is performed only on smaller blocks. But also, when solving invariant and equivariant interpolation problems, the result will respect exactly the intended invariance or equivariance, despite possible numerical inaccuracy.

3.5.2 Computing interpolants

We consider an interpolation problem (Λ, ϕ) where Λ is a G-invariant subspace of $\mathbb{K}[x]^*$ and $\phi : \Lambda \to \mathbb{K}^m$. Take $\mathcal{P}$ to be a symmetry adapted basis of an invariant interpolation space P for Λ as obtained from Algorithm 1. The interpolant polynomial p that solves (Λ, ϕ) in P is given by

$$p = \sum_{i=1}^{\mathbb{N}} \sum_{\alpha=1}^{n_i} \mathrm{A_i}^{-1} \phi\left(\mathcal{L}^{(i,\alpha)}\right)^t \left(\mathcal{P}^{(i,\alpha)}\right)^t, \tag{3.5.1}$$

where $\mathcal{P}^{(i,\alpha)}$, $\mathcal{L}^{(i,\alpha)}$ are as in (2.3.6) and $\mathrm{A_i} = \mathrm{W}_{\mathcal{L}^{(i,1)}}^{\mathcal{P}^{(i,1)}}$. Note that we made no asumption on ϕ. The invariance of Λ allows to decompose the problem into smaller blocks, independently of the structure of ϕ. This illustrate how symmetry can be used to better organize computation : if we can choose the points of evaluation, the computational cost can be alleviated by choosing them with some symmetry.

When ϕ is invariant or equivariant, Equation (3.5.1) can be further reduced. If (Λ, ϕ) is an invariant interpolation problem, it follows from Schur's lemma that $\phi(\mathcal{L}^{(j)}) = 0$ for any $j > 1$. Therefore for solving any invariant interpolation problem we only need to compute the first block of $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}}$, i.e., the interpolant is given by $\mathrm{A_1}^{-1} \phi\left(\mathcal{L}^{(1)}\right)^t \left(\mathcal{P}^{(1)}\right)^t$.

More generally if (Λ, ϕ) is a $\vartheta - \theta$ equivariant problem, such that the irreducible representation ρ_i does not occur in θ, then $\phi\left(\mathcal{L}^{(i)}\right) = 0$. The related block can thus be dismissed.

Example 3.5.1 *Following on Example 3.3.1. Since we are interested in building an interpolation space for an invariant problem, we only need to compute bases of* Λ^G *and* $\mathbb{K}[\mathrm{x}]^G_{\leq 5}$. *We have*

$$\mathcal{L}^G = \left\{ \mathrm{e}_{\xi_1}, \sum_{i=2}^{6} \mathrm{e}_{\xi_i}, \sum_{i=7}^{11} \mathrm{e}_{\xi_i}, \sum_{i=12}^{16} \mathrm{e}_{\xi_i} \right\}$$

and

$$\mathcal{P}^G = \{1,\ x_1^2 + x_2^2,\ x_1^4 + 2x_1^2x_2^2 + x_2^4,\ x_1^5 - 10x_1^3x_2^2 + 5x_1x_2^4\}.$$

Since $\mathrm{W} = \mathrm{W}_{\mathcal{L}^G}^{\mathcal{P}^G}$ *is a square matrix with full rank,* $\langle \mathcal{P}^G \rangle_{\mathbb{K}}$ *contains a unique invariant interpolant for any invariant interpolation problem. It has to be the least interpolant.*

For ϕ given in Example 3.3.1, one finds the interpolant p by solving the 4×4 linear system $\mathrm{W}\,a = \phi(\mathcal{L}^G)$. The solution $a = (-0.3333333, 3.295689, -36.59337, 45.36692)^t$ provides the coefficients of $\mathcal{P}^G$ in p. The graph of p is shown in Figure 3.1. If p given above is only an approximation of the least interpolant, due to numerical inaccuracy, it is at least exactly invariant. Had we computed the least interpolant with the algorithm of De Boor and Ron (1992a), i.e., by elimination of the Vandermonde matrix based on the monomial basis, the least interpolant obtained would not be exactly invariant because of the propagation of numerical inacurracies.

We define the deviation from invariance (InvD) of $p = \sum_{\deg\alpha\le5} a_\alpha x^\alpha$ as

$$\sigma\left(a_{20}, a_{02}\right) + \sigma\left(a_{40}, \frac{a_{22}}{2}, a_{04}\right) + \sigma\left(a_{50}, -\frac{a_{32}}{10}, \frac{a_{14}}{5}\right) + \sum_{\beta\in\mathcal{B}}|a_\beta|$$

where σ is the standard deviation, and $\mathcal{B}$ represents the exponents of the monomials that do not belong to any of the elements in $\mathcal{P}^G$. In Table 3.1 we show the InvD for the interpolant p computed with different precisions. The obtained polynomials are somehow far from being $G-$invariant.

# Digits	10	15	20	30
InvD	72.9614	40.0289	6.0967	$< 10^{-9}$

Table 3.1: InvD values for different digits of precision

In the same spirit, let us mention that the condition number of W_Λ^M, where M is the monomial basis of $\mathbb{K}[\mathrm{x}]_{\le5}$, is more than 10^2 times the condition number of $\mathrm{W}_{\mathcal{L}^G}^{\mathcal{P}^G}$. This is an indicator that two additional digits of precision are lost in the computation.

Example 3.5.2 *Following up on Example 3.4.1. Let θ be the permutation representation of D_3 in $\mathbb{R}^3$. θ decomposes into two irreducible representations, the trivial representation and the irreducible representation ϑ of dimension 2. Let $\phi : \Lambda \to \mathbb{R}^3$ a $\vartheta - \theta$ equivariant map determined by $\phi(\varrho_1) = (1, -1, 5)^t$. For solving (Λ, ϕ) we need only consider the first and third block of the Vandermonde matrix computed in Example 3.4.1. The $\rho^* - \theta$ equivariant map that solve (Λ, ϕ) is $P = (p_1, p_2, p_3)$ with:*

$$p_1 := \frac{705}{4256}x_1^2 + \frac{135}{4256}x_2^2 + \frac{31}{56}\sqrt{3}x_1 + \frac{93}{56}x_2 - \frac{15}{112}\sqrt{3}x_1x_2$$
$$p_2 := \frac{705}{4256}x_1^2 + \frac{135}{4256}x_2^2 + \frac{31}{56}\sqrt{3}x_1 - \frac{93}{56}x_2 + \frac{15}{112}\sqrt{3}x_1x_2$$
$$p_3 := -\frac{75}{2128}x_1^2 + \frac{495}{2128}x_2^2 - \frac{31}{28}\sqrt{3}x_1.$$

In Figure 3.3 we show the image of $\mathbb{R}^2$ by P and the tangency conditions imposed by ϕ.

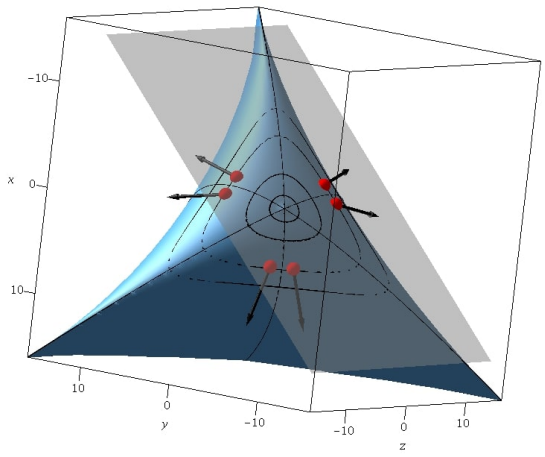


Figure 3.3: Parameterized surface with tangency constraints.

Example 3.5.3 *This example illustrate the use of an equivariant interpolating map to parameterize a surface with an symmetric set of tangency conditions. It follows up on Example 3.3.2. Since the representation ϑ of D_3 in $\mathbb{R}^2$ is irreducible, for computing any $\vartheta - \vartheta$ equivariant we only need to compute the third isotopic block in the Vandermonde matrix $\mathrm{W}_{\mathcal{L}^{(3)}}^{\mathcal{P}^{(3)}}$, where $\mathcal{P}$ is a basis for the interpolation space P_Λ built by Algorithm 1. This block is* $\mathrm{W} = \begin{pmatrix} \mathrm{A}_3 & \\ & \mathrm{A}_3 \end{pmatrix}$. *The rows correspond to*

$$\mathcal{L}^{(3)} := \left\{\mathcal{L}^{(3,1)}, \mathcal{L}^{(3,2)}\right\}, \mathcal{L}^{(3,1)} := \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \text{ and } \mathcal{L}^{(3,2)} := \{\lambda_5, \lambda_6, \lambda_7, \lambda_8\}$$

$$\begin{aligned}
\lambda_1 &= \mathrm{e}_{\xi_1} + \mathrm{e}_{\xi_2} - \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} & \lambda_5 &= \tfrac{\sqrt{3}}{2}(-\mathrm{e}_{\xi_1} + \mathrm{e}_{\xi_2} + 2\mathrm{e}_{\xi_3} + \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} - 2\mathrm{e}_{\xi_6}),\\
\lambda_2 &= \mathrm{e}_{\xi_3} - \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} + \mathrm{e}_{\xi_6}, & \lambda_6 &= \tfrac{\sqrt{3}}{2}(-2\mathrm{e}_{\xi_1} + 2\mathrm{e}_{\xi_2} + \mathrm{e}_{\xi_3} - \mathrm{e}_{\xi_4} - \mathrm{e}_{\xi_5} - \mathrm{e}_{\xi_6}).\\
\lambda_3 &= \mathrm{e}_{\xi_7} + \mathrm{e}_{\xi_8} - \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_11} & \lambda_7 &= \tfrac{\sqrt{3}}{2}(-\mathrm{e}_{\xi_7} + \mathrm{e}_{\xi_8} + 2\mathrm{e}_{\xi_9} + \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_11} - 2\mathrm{e}_{\xi_{12}}),\\
\lambda_4 &= \mathrm{e}_{\xi_9} - \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_11} + \mathrm{e}_{\xi_{12}}, & \lambda_8 &= \tfrac{\sqrt{3}}{2}(-2\mathrm{e}_{\xi_7} + 2\mathrm{e}_{\xi_8} + \mathrm{e}_{\xi_9} - \mathrm{e}_{\xi_{10}} - \mathrm{e}_{\xi_11} - \mathrm{e}_{\xi_{12}}).
\end{aligned}$$

The columns correspond to

$$\mathcal{P}^{(3)} := \left\{ \begin{array}{c} \mathcal{P}^{(3,1)} := \left\{x, x^2 - y^2, x^3 + xy^2, x^4 - y^4\right\}, \\ \mathcal{P}^{(3,2)} := \left\{y, -2xy, y(x^2 + y^2), -2xy(x^2 + y^2)\right\} \end{array} \right\}.$$

$$\mathrm{A}_3 = -\frac{2}{27}\begin{pmatrix} 72\sqrt{3} & -288 & 608\sqrt{3} & -2432 \\ 9\sqrt{3} & 90 & 76\sqrt{3} & 760 \\ 45\sqrt{3} & -90 & 140\sqrt{3} & 280 \\ 9\sqrt{3} & 18 & 28\sqrt{3} & 504 \end{pmatrix}$$

We thus determine that the equivariant interpolant for the interpolation problem described in Example 3.3.2 is given by :

$$p_1 = \frac{\alpha}{320}x + \frac{3\beta}{640}(x^2 - y^2) + \frac{9\gamma}{8960}x(x^2 + y^2) + \frac{27\delta}{17920}(x^4 - y^4)$$

$$p_2 = \frac{\alpha}{320}y - \frac{3\beta}{320}xy + \frac{9\gamma}{8960}y(x^2 + y^2) - \frac{27\delta}{8960}xy(x^2 + y^2)$$

where

$$\alpha = \sqrt{3}(25\,a - 114\,b) + 494\,d - 185\,c,\ \beta = \sqrt{3}(114\,d - 25\,c) + 38\,b - 5\,a,$$
$$\gamma = \sqrt{3}(42\,b - 25\,a) + 185\,c - 182\,d,\ \delta = \sqrt{3}(25\,c - 42\,d) + 5\,a - 14\,b.$$

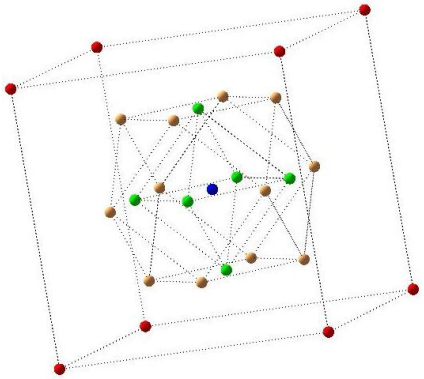
Example 3.5.4 *We now seek an invariant implicit surface through a given symmetric set of points. We thus seek an invariant polynomial that is zero on the given point. In order to obtain a non zero polynomial though, we need to select a point, for instance the origin, where the polynomial shall not be zero.*

The group here is O_h, the subgroup of the orthogonal group $\mathbb{R}^3$ that leaves the cube invariant. It has order 48 and 10 inequivalent irreducible representations whose dimensions are $(1, 1, 1, 1, 2, 2, 3, 3, 3, 3)$.

Consider $\Xi \subset \mathbb{R}^3$ the invariant set of 27 points illustrated on Figure 3.4a. They are grouped in four orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ and $\mathcal{O}_4$ of O_h. The orbit $\mathcal{O}_1$ consist of only one point, the origin of coordinates. The points in $\mathcal{O}_2$ are the vertices of cubes with the center at the origin and edge length $\sqrt{3}$. The points in $\mathcal{O}_3$ and in $\mathcal{O}_4$ are the centers of the faces and edges of a cube with the center at the origin and edge length 1. Consider $\Lambda = \langle \{ e_\xi \mid \xi \in \Xi \} \rangle_{\mathbb{K}}$. Λ is an invariant subspace. Consider the the map $\phi : \Lambda \to \mathbb{R}$ defined by $\phi(\xi) = \frac{1}{2}$ if $\xi \in \mathcal{O}_1$ and $\phi(\xi) = 0$ if $\xi \in \mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$. The pair (Λ, ϕ) is an invariant interpolation problem. The polynomial $p \in \mathbb{K}[x]^G$ given by

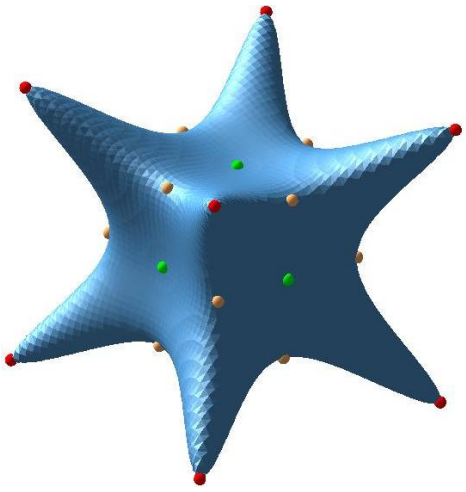
$$p = 18 + \left(x^2 + y^2 + z^2\right) + 18\left(x^2y^2 + x^2z^2 + y^2z^2\right) - 19\left(x^4 + y^4 + z^4\right)$$

is the solution of (Λ, ϕ) in $\Lambda_\downarrow$. We built p over the Vandermonde matrix $\mathrm{W}_{\mathcal{L}^G}^{\mathcal{P}^G_{\leq 6}} \in \mathbb{R}^{4\times 6}$ instead of the full Vandermonde matrix for bases of Λ and $\mathbb{K}[\mathrm{x}]_{\leq 6}$ which is a 27×84 matrix. The set $\mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$ is in the zero set of p, in Figure 3.4b we show the zero set of p, which has the symmetry of O_h. Note that we introduce a non zero value in the origin to avoid zero solutions during interpolation.



	#Nodes	A node per orbit	value
$\mathcal{O}_1$	1	$\xi_1 = (0,0,0)$	$\phi(\mathbb{e}_\xi) = 1/2$
$\mathcal{O}_2$	8	$\xi_2 = (-\sqrt{3},-\sqrt{3},-\sqrt{3})$	$\phi(\mathbb{e}_\xi) = 0$
$\mathcal{O}_3$	6	$\xi_{10} = (-1,0,0)$	$\phi(\mathbb{e}_\xi) = 0$
$\mathcal{O}_4$	12	$\xi_{16} = (0,-1,-1)$	$\phi(\mathbb{e}_\xi) = 0$

(a) Points in Ξ divided in orbits



(b) Variety of p

Figure 3.4: Interpolation data and variety of the interpolant p that goes through the points in $\mathcal{O}_2 \cup \mathcal{O}_3 \cup \mathcal{O}_4$.

Chapter 4

Ideal Interpolation

This chapter is an extended version of :

(Rodriguez Bazan and Hubert, 2020) Rodriguez Bazan, E., Hubert, E., 2020. Ideal Interpolation, H-basis and Symmetry. In: ISSAC'20. ACM, Kalamata, Greece

4.1 Introduction

In Chapter 3 we addressed multivariate interpolation, in this chapter we go further with ideal interpolation. We provide an algorithm to compute simultaneously a symmetry adapted basis of the least interpolation space and a symmetry adapted H-basis of the associated ideal. In addition to be manifested in the output, symmetry is exploited all along the algorithm to reduce the size of the matrices involved, and avoid sizable redundancies. Based on QR-decomposition (as opposed to LU-decomposition previously) the algorithm also lends itself to numerical computations.

A space of linear forms Λ on the polynomial ring $\mathbb{K}[\mathrm{x}] = \mathbb{K}[x_1, \ldots, x_n]$ is an ideal interpolation scheme if

$$I = \bigcap_{\lambda \in \Lambda} \ker \lambda = \{p \in \mathbb{K}[\mathrm{x}] : \lambda(p) = 0, \text{ for all } \lambda \text{ in } \Lambda\} \tag{4.1.1}$$

is an ideal in $\mathbb{K}[\mathrm{x}]$.

An interpolation space for Λ can be identified with the quotient space $\mathbb{K}[\mathrm{x}]/I$. In this Chapter we make use of the least interpolation space (Section 3.2.4) as a canonical representation of the quotient of the polynomial algebra by the ideal. In Proposition 3.2.4 we freed the computation of the least interpolation space from its reliance on the monomial basis by introducing *dual bases*. We pursue this approach here for the representation of the ideal by H-bases (Macaulay, 1916; Möller and Sauer, 2000). Where Gröbner bases single out leading terms with a term order, H-bases work with leading forms and the orthogonality with respect to the apolar product. The least interpolation space then reveals itself as the orthogonal complement of the ideal of leading forms.

As a result, computing a H-basis of the interpolation ideal is achieved with linear algebra in subspaces of homogeneous polynomials of growing degrees. The precise algorithm we shall offer to compute H-bases somehow fits in the loose sketch proposed in (De Boor, 1994). Yet we are now in a position to incorporate symmetry in a natural way, refining the algorithm to exploit it.

Symmetry is preserved and exploited thanks to the block diagonal structure of the matrices at play in the algorithms. This block diagonalisation, with predicted repetitions in the blocks, happens when the underlying maps are discovered to be equivariant and expressed in the related *symmetry adapted basis*. The case of the Vandermonde matrix was settled in Corollary 3.4.2. In this chapter, we also need the matrix of the prolongation map, known in the monomial basis as the Macaulay matrix. Figuring out the equivariance of this map is one of the key results of this chapter.

The chapter is organized as follows. In Section 4.2 we define ideal interpolation and explain the identification of an interpolation space with the quotient algebra. In Section 4.3 we review H-bases and discuss how they can be computed in the ideal interpolation setting. In Section 4.4 we provide an algorithm to compute simultaneously a basis of the least interpolation space and an orthogonal H-basis of the ideal.

In Section 4.5 we show how the Macaulay matrix can be block diagonalized in the presence of symmetry. This is then applied in Section 4.6 to obtain an algorithm to compute simultaneously a symmetry adapted basis of the least interpolation space and a symmetry adapted H-basis of the ideal.

4.2 Ideal interpolation

In this section, we consider the ideal interpolation problem and explain the identification of an interpolation space with the quotient algebra. We recall that the least interpolation space is the orthogonal complement of the ideal of the leading forms, I^0.

All along this chapter we shall assume that

$$I = \ker w = \cap_{\lambda\in\Lambda} \ker \lambda$$

is an ideal. When for instance $\Lambda = \langle \mathrm{e}_{\xi_1}, \ldots, \mathrm{e}_{\xi_r}\rangle_K$ then I is the ideal of the points $\{\xi_1, \ldots, \xi_r\} \subset \mathbb{K}[\mathrm{x}]$. One sees in general that $\dim \mathbb{K}[\mathrm{x}]/I = \dim \Lambda^* = \dim \Lambda =: r$.

With $\mathcal{Q} = \{q_1, \ldots, q_r\} \subset \mathbb{K}[\mathrm{x}]$, we can identify $\mathbb{K}[\mathrm{x}]/I$ with $\langle \mathcal{Q}\rangle_{\mathbb{K}}$ if $\langle \mathcal{Q}\rangle_{\mathbb{K}} \oplus I = \mathbb{K}[\mathrm{x}]$. With a slight shortcut, we say that $\mathcal{Q}$ is a basis for $\mathbb{K}[\mathrm{x}]/I$.

Proposition 4.2.1 *$\mathcal{Q} = \{q_1, \ldots, q_r\} \subset \mathbb{K}[\mathrm{x}]$ spans an interpolation space for Λ iff it is a basis for the quotient $\mathbb{K}[\mathrm{x}]/I$.*

PROOF: If $\mathcal{Q} = \{q_1, \ldots, q_r\}$ is a basis of $\mathbb{K}[\mathrm{x}]/I$ then for any $p \in \mathbb{K}[\mathrm{x}]$ there is a $q \in \langle q_1, \ldots, q_r\rangle_{\mathbb{K}}$ such that $p \equiv q \bmod I$. Hence $\lambda(p) = \lambda(q)$ for any $\lambda \in \Lambda$ and thus $\langle \mathcal{Q}\rangle_{\mathbb{K}}$ is an interpolation space for Λ. Conversely if $\langle q_1, \ldots, q_r\rangle_{\mathbb{K}}$ is an interpolation space for Λ then $\{q_1, \ldots, q_r\}$ are linearly independent modulo I and therefore a basis for $\mathbb{K}[\mathrm{x}]/I$. Indeed if $q = a_1q_1 + \ldots + a_rq_r \in I$ then any interpolation problem has multiple solutions in $\langle \mathcal{Q}\rangle_{\mathbb{K}}$, i.e, if p is the solution of (Λ, ϕ) so is $p + q$, contradicting the interpolation uniqueness on $\langle \mathcal{Q}\rangle_{\mathbb{K}}$. □

For $p \in \mathbb{K}[\mathrm{x}]$ we can find its natural projection on $\mathbb{K}[\mathrm{x}]/I$ by taking the unique $q \in \langle \mathcal{Q}\rangle_{\mathbb{K}}$ that satisfies $\lambda(q) = \lambda(p)$ for all $\lambda \in \Lambda$. From a computational point of view, q is obtained by solving the Vandermonde system, i.e.,

$$q = (q_1, \ldots, q_r)\left(\mathrm{W}_{\mathcal{L}}^{\mathcal{Q}}\right)^{-1}\begin{pmatrix}\lambda_1(p)\\ \vdots \\ \lambda_r(p)\end{pmatrix} \text{ with } \mathcal{L} = \{\lambda_1, \ldots, \lambda_r\} \text{ a basis of } \Lambda.$$

Similarly, the matrix of the multiplication map, in the basis $\mathcal{Q}$, is

$$\begin{array}{rccc} \mathfrak{m}_p : & \mathbb{K}[\mathrm{x}]/I & \to & \mathbb{K}[\mathrm{x}]/I, \\ & [q] & \mapsto & [pq] \end{array}$$

is obtained as $[\mathfrak{m}_p]_{\mathcal{Q}} = \left(\mathrm{W}_{\mathcal{L}}^{\mathcal{Q}}\right)^{-1} \mathrm{W}_{\mathcal{L}\circ m_p}^{\mathcal{Q}}$ where $\mathcal{L} \circ m_p = \{\lambda_1 \circ m_p, \ldots, \lambda_r \circ m_p\}$.

When working with Gröbner bases, one fixes a term order and focuses on leading terms of polynomials and the initial ideal of I. The basis of choice for $\mathbb{K}[x]/I$ consists of the monomials that do not belong to the initial ideal. An H-basis of I is somehow the complement of the least interpolation space $\Lambda_\downarrow$ and hence can be made to reflect the possible invariance of Λ and I. Instead of leading terms, the focus is then on the leading homogeneous forms.

Hereafter we denote by p^0 the leading homogeneous form of p, i.e., the unique homogeneous polynomial such that $\deg(p - p^0) < \deg(p)$. Given a set of polynomials P we denote $P^0 = \{p^0 \mid p \in P\}$.

Proposition 4.2.2 *Let Q be an interpolation space of minimal degree for Λ. Then $Q \oplus I^0 = \mathbb{K}[x]$.*

PROOF: We proceed by induction on the degree, i.e, we assume that any polynomial p in $\mathbb{K}[x]_{\le d}$ can be written as $p = q + l$ where $q \in Q$ and $l \in I^0$. Note that the hypothesis holds trivially when d is equal to zero.

Now let $p \in \mathbb{K}[x]_{\le d+1}$. Since $\mathbb{K}[x] = \langle Q \rangle_{\mathbb{K}} \oplus I$ there exists $q \in Q$ and $l \in I$ such that $p = q + l$. Since Q is of minimal degree, q and l are in $\mathbb{K}[x]_{\le d+1}$. Writing $l = l^0 + l_1$ he have $p = q + l^0 + l_1$ with $l_1 \in \mathbb{K}[x]_{\le d}$ then by induction $l_1 = q_1 + l_2$ with $q_1 \in Q$ and $l_2 \in I^0$ and therefore $p = q + q_1 + l^0 + l_2 \in Q \oplus I^0$. □

As a consequence we retrieve the result of (De Boor and Ron, 1992b, Theorem 4.8).

Corollary 4.2.3 *Considering orthogonality with respect to the apolar product it holds that $\Lambda_\downarrow \overset{\perp}{\oplus} I^0 = \mathbb{K}[x]$.*

PROOF: Follows from the fact that $\lambda(p) = 0 \Rightarrow \langle \lambda_\downarrow, p^0 \rangle = 0$. □

4.3 H-bases

H-bases were introduced by Macaulay (1916). The use of H-basis in interpolation has been further studied in (Möller and Sauer, 2000; Sauer, 2001). In this section we review the definitions and present the sketch of an algorithm to compute the H-basis of $I = \bigcap_{\lambda \in \Lambda} \ker \lambda$.

Definition 4.3.1 *A finite set $\mathcal{H} := \{h_1, \ldots, h_m\} \subset \mathbb{K}[x]$ is an H-basis of the ideal $I := \langle h_1, \ldots, h_m \rangle$ if, for all $p \in I$ there are $g_1, \ldots g_m$ such that,*

$$p = \sum_{i=1}^{m} h_i g_i \text{ and } \deg(h_i) + \deg(g_i) \le \deg(p), i = 1, \ldots, m.$$

Theorem 4.3.2 *(Möller and Sauer, 2000) Let $\mathcal{H} := \{h_1, \ldots, h_m\}$ and $I := \langle \mathcal{H} \rangle$. Then the following conditions are equivalent:*

1. *$\mathcal{H}$ is an H-basis of I.*

2. $I^0 := \langle \{h^0 \mid h \in I\} \rangle = \langle h_1^0, \ldots, h_m^0 \rangle$.

Hilbert Basis Theorem says that I^0 has a finite basis, hence any ideal in $\mathbb{K}[\mathrm{x}]$ has a finite H-basis. We shall now introduce the concepts of minimal, orthogonal and reduced H-basis. The notion of orthogonality is considered w.r.t the apolar product. Our definitions somewhat differ from (Möller and Sauer, 2000) as we dissociate them from the computational aspect. We need to introduce first the following vector space of homogeneous polynomials.

Definition 4.3.3 *Given a set $\mathcal{H} = \{h_1, \ldots, h_m\}$ of homogeneous polynomials in $\mathbb{K}[\mathrm{x}]$ and a degree d, we define the subspace $\mathrm{V}_d(\mathcal{H})$ as*

$$\mathrm{V}_d(\mathcal{H}) = \left\{ \sum_{i=1}^{s} g_i h_i \;\middle|\; g_i \in \mathbb{K}[\mathrm{x}]_{d-\deg(h_i)} \right\} \subset \mathbb{K}[\mathrm{x}]_d.$$

$\mathrm{V}_d(\mathcal{H})$ is the image of the linear map ψ_d:

$$\begin{array}{rccc} \psi_{d,h}: & \mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m} & \to & \mathbb{K}[\mathrm{x}]_d \\ & (g_1, \ldots, g_m) & \to & \sum_{i=1}^{m} g_i h_i \end{array}.$$

We denote by $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H})$ the matrix of ψ_d in the bases $\mathcal{M}_d$ and $\mathcal{P}_d$ of $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m}$ and $\mathbb{K}[\mathrm{x}]_d$ respectively. It is referred to as the Macaulay matrix for $\mathcal{H}$. We can write $\mathrm{V}_d(\mathcal{H})$ as

$$\mathrm{V}_d(\mathcal{H}) = \left\{ \sum_{i=0}^{|\mathcal{P}_d|} a_i p_i \;\middle|\; (a_1, \ldots, a_{|\mathcal{P}_d|}) \in \mathcal{R}\left(\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H})\right) \right\}, \tag{4.3.1}$$

where $\mathcal{R}\left(\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H})\right)$ denotes the column space of $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H})$.

We shall use the notation P_d^0 for the set of the degree d elements of P^0. In other words $P_d^0 = P^0 \cap \mathbb{K}[\mathrm{x}]_d$.

Definition 4.3.4 *We say that an H-basis $\mathcal{H}$ is minimal if, for any $d \in \mathbb{N}$, $\mathcal{H}_d^0$ is linearly independent and*

$$V_d\left(I_{d-1}^0\right) \oplus \left\langle \mathcal{H}_d^0 \right\rangle_{\mathbb{K}} = I_d^0. \tag{4.3.2}$$

Furthermore $\mathcal{H}$ is said to be orthogonal if $\left\langle \mathcal{H}_d^0 \right\rangle_{\mathbb{K}}$ is the orthogonal complement of $V_d\left(I_{d-1}^0\right)$ in I_d^0.

Note that if h_i and h_j are two elements with $\deg h_i > \deg h_j$ of an orthogonal H-basis we have

$$\left\langle h_i^0, p h_j^0 \right\rangle = 0 \text{ for all } \; p \in \mathbb{K}[\mathrm{x}]_{\deg h_i - \deg h_j}.$$

The concept of orthogonal $H-$basis introduces uniqueness of the leading homogeneous forms of a $H-$basis up to linear transformations. If $\mathcal{H}$ and $\mathcal{G}$ are two orthogonal H-basis then $\mathcal{H}_d^0$ and $\mathcal{G}_d^0$ are both bases of the orthogonal complement

of $V_d\left(I^0_{d-1}\right)$ in I^0_d, and thus $\mathcal{H}^0_d = \mathcal{G}^0_d \cdot \mathrm{Q}_d$, where Q_d is the change of basis from $\mathcal{G}^0_d$ to $\mathcal{H}^0_d$, therefore

$$\mathcal{H}^0 = \mathcal{G}^0 \cdot \mathrm{Q}, \text{ with } \mathrm{Q} = \operatorname{diag}\left(\mathrm{Q}_d, d = 1..D\right). \tag{4.3.3}$$

We can extend the previous notion of uniqueness not only for the leading homogeneous forms of a $H-$ basis but also for the complete $H-$basis by introducing the concept of *reduced* $H-$basis.

Definition 4.3.5 *Let $\mathcal{H} = \{h_1, \ldots, h_m\}$ be an orthogonal H-basis of an ideal I. The* reduced *H-basis of $\mathcal{H}$ is defined by*

$$\widetilde{\mathcal{H}} = \left\{h^0_1 - \widetilde{h^0_1}, \ldots, h^0_m - \widetilde{h^0_m}\right\} \tag{4.3.4}$$

where, for $p \in \mathbb{K}[x]$, $\tilde{p}$ is the projection of p on the orthogonal complement of I^0 parallel to I.

(Möller and Sauer, 2000, Lemma 6.2) show how $\tilde{p}$ can be computed given $\mathcal{H}$.

Proposition 4.3.6 *Let $\mathcal{H} = \{h_1, \ldots, h_m\}$ and $\mathcal{G} = \{g_1, \ldots, g_m\}$ be two reduced $H-$bases of I. Then there is a linear transformation from $\mathcal{H}$ to $\mathcal{G}$, i.e, there is a $m \times m$ matrix Q such that*

$$(h_1, \ldots, h_m) = (g_1, \ldots, g_m) \cdot \mathrm{Q}.$$

PROOF: By (4.3.3) there is a $m \times m$ matrix Q such that $\mathcal{H}^0 = \mathcal{G}^0 \cdot \mathrm{Q}$, thus

$$h^0_i = \sum_{j=1}^{m} q_{ij} g^0_j \text{ for } 1 \le i \le m.$$

By the uniqueness of $\widetilde{p}$ for any $p \in \mathbb{K}[\mathrm{x}]$ follows that $\widetilde{h^0_i} = \sum_{j=1}^m q_{ij} \widetilde{g^0_j}$ and therefore $h^0_i - \widetilde{h^0_i} = \sum_{j=1}^m q_{ij} \left(g^0_i - \widetilde{g^0_i}\right)$. Last equality implies that $\widetilde{\mathcal{H}} = \widetilde{\mathcal{G}} \cdot \mathrm{Q}$. □

Schematic computation of H-bases In the next section we elaborate on an algorithm to compute concomitantly the least interpolation space and an H-basis for the ideal associated to a set of linear forms Λ. As a way of introduction we reproduce the sketch of an algorithm as proposed by De Boor (1994) to compute an H-basis until degree D. It is based on the asumption that we have access to a basis of $I_d := I \cap \mathbb{K}[\mathrm{x}]_{\le d}$ for any d.

Algorithm 2 (De Boor, 1994) H-basis construction

Input: - a degree D.
- basis for I_d for $1 \leq d \leq D$.

Output : - an H-basis until degree D

1: $\mathcal{H} \leftarrow \{\}$;
2: **for** $d = 0$ `to` D **do**
3: $\quad \mathcal{C}_d \leftarrow$ a basis of $\mathrm{V}_d(\mathcal{H}^0)$;
4: $\quad \mathcal{B}_d \leftarrow$ a basis for the complement of $\mathrm{V}_d(\mathcal{H})$ in I_d^0;
5: $\quad \widehat{\mathcal{B}}_d \leftarrow$ projection of $\mathcal{B}_d$ in I_d
6: $\quad \mathcal{H} \leftarrow \mathcal{H} \bigcup \widehat{\mathcal{B}}_d$;
7: **end for**
8: **return** $\mathcal{H}$;

The correctness of Algorithm 2 is shown by induction. Assume that $\mathcal{H}_{d-1}$ consists of the polynomials in an H-basis of I up to degree $d-1$. Consider $p \in I$ with $\deg(p) = d$. By Step 4 in Algorithm 2 we have

$$p^0 = \sum_{h_i \in \mathcal{H}} h_i^0 g_i + \sum_{b_i \in \mathcal{B}_d} a_i b_i \tag{4.3.5}$$

with $g_i \in \mathbb{K}[\mathrm{x}]_{d-\deg(h_i)}$ and $a_i \in \mathbb{K}$. From (4.3.5) we have that $p \in I$ and $\sum_{h_i \in \mathcal{H}} h_i g_i + \sum_{b_i \in \mathcal{B}_{d+1}} a_i \hat{b}_i \in I$ have the same leading form. Thus

$$p - \sum_{h_i \in \mathcal{H}_{d-1}} h_i g_i - \sum_{b_i \in \mathcal{B}_d} a_i \hat{b}_i \in I_{d-1}$$

therefore using the induction hypothesis we get that

$$p = \sum_{h_i \in \mathcal{H}_{d-1}} h_i g_i + \sum_{b_i \in \mathcal{B}_{d+1}} a_i \hat{b}_i + \sum_{h_i \in \mathcal{H}_{d-1}} h_i q_i$$

with $q_i \in \mathbb{K}[\mathrm{x}]_{\leq d-1-\deg(h_i)}$ and therefore $\mathcal{H}$ is an H-basis.

Algorithm 2 can be applied in the ideal interpolation scheme. In this setting a basis of I_d can be computed for any d using Linear Algebra techniques due to the following relation.

$$I_d = \left\{ \sum_{i=1}^{|\mathcal{P}_{\leq d}|} a_i p_i \mid \left(a_1, \ldots, a_{|\mathcal{P}_{\leq d}|}\right)^t \in \ker\left(\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_{\leq d}}\right) \text{ and } p_i \in \mathcal{P}_{\leq d} \right\},$$

for any basis $\mathcal{P}_{\leq d}$ of $\mathbb{K}[\mathrm{x}]_{\leq d}$.

In the next section we will give an efficient and detailed version of Algorithm 2 in the ideal interpolation case. We will integrate the computations of an H-basis for $I = \cap_{\lambda \in \Lambda} \ker \lambda$ and a basis for $\Lambda_{\downarrow}$.

When the ideal is given by a set of generators it is also possible to compute an H-basis with linear algebra if you know a bound on the degree of the syzygies of the generators. A numerical approach, using singular value decomposition, was introduced in (Javanbakht and Sauer, 2019). Alternatively an extension of Buchberger's algorithm is presented in (Möller and Sauer, 2000). It relies, at each step, on the computation of a basis for the module of syzygies of a set of homogeneous polynomials.

4.4 Simultaneous computation of the H-basis and least interpolation space

In this section we present an algorithm to compute both a (orthogonal) basis of $\Lambda_\downarrow$ and an orthogonal H-basis $\mathcal{H}$ of the ideal $I = \cap_{\lambda\in\Lambda} \ker \lambda$. We proceed degree by degree. At each iteration of the algorithm we compute a basis of $\Lambda_\downarrow \cap \mathbb{K}[x]_d$ and the set $\mathcal{H}^0_d = \mathcal{H}^0 \cap \mathbb{K}[x]_d$. Recall from Corollary 4.2.3, Theorem 4.3.2, and Definition 4.3.4 that

$$\mathbb{K}[x] = \Lambda_\downarrow \overset{\perp}{\oplus} I^0, \quad I^0 = \langle \mathcal{H}^0 \rangle, \quad \text{and} \quad I^0_d = V_d\left(I^0_{d-1}\right) \overset{\perp}{\oplus} \left\langle \mathcal{H}^0_d \right\rangle_{\mathbb{K}}.$$

I is the kernel of the Vandermonde operator while $\Lambda_\downarrow$ can be inferred from a rank revealing form of the Vandermonde matrix. With orthogonality prevailing in the objects we compute it is natural that the QR-decomposition plays a central role in our algorithm.

For a $m \times n$ matrix M, the QR-decomposition is M = QR where Q is a $m \times m$ orthogonal matrix and R is a $m \times n$ upper triangular matrix. If r is the rank of M the first r columns of Q form an orthogonal basis of the column space of M and the remaining $m - r$ columns of Q form an orthogonal basis of the kernel of M^T (Golub and Van Loan, 1996, Theorem 5.2.1). We thus often denote the QR-decomposition of a matrix M as

$$[\,Q_1 \mid Q_2\,] \cdot \begin{bmatrix} R \\ 0 \end{bmatrix} = M$$

where $Q_1 \in \mathbb{K}^{m\times r}, Q_2 \in \mathbb{K}^{m\times(m-r)}$ and $R \in \mathbb{K}^{r\times n}$. Algorithms to compute the QR-decomposition can be found for instance in (Golub and Van Loan, 1996).

In the Lagrange interpolation case, Fassino and Möller (2016) already used the QR-decomposition to propose a variant of the BM-algorithm (Möller and Buchberger, 1982) so as to compute a monomial basis of an interpolation space, the complement of the initial ideal for a chosen term order. They furthermore study the gain in numerical stability for perturbed data. We shall use QR-decomposition to further obtain a homogeneous basis of $\Lambda_\downarrow$ and an orthogonal H-basis of the ideal.

Due to Corollary 4.2.3 the reduction $\tilde{p}$ of p that appeared in Definition 4.3.5 is the unique interpolant of p in $\Lambda_\downarrow$.

Definition 4.4.1 *Given a space of linear forms Λ, we denote by $\Lambda_{\geq d}$ the subspace*

of Λ *given by*

$$\Lambda_{\geq d} = \{\lambda \in \Lambda \mid \lambda_{\downarrow} \in \mathbb{K}[x]_{\geqslant d}\} \cup \{0\}.$$

Hereafter we organize the elements of the bases of $\mathbb{K}[x]$, Λ, or their subspaces, as row vectors. In particular $\mathcal{P}$ and $\mathcal{P}^\dagger$ are dual homogeneous bases for $\mathbb{K}[x]$ according to the apolar product. Their degree part $\mathcal{P}_d$ and $\mathcal{P}_d^\dagger$ are dual bases of $\mathbb{K}[x]_d$.

A basis $\mathcal{L}_{\geq d}$ of $\Lambda_{\geq d}$ can be computed inductively thanks to the following observation.

Proposition 4.4.2 *Assume* $\mathcal{L}_{\geq d}$ *is a basis of* $\Lambda_{\geq d}$*. Consider the QR-decomposition*

$$W_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d} = [Q_1 \mid Q_2] \cdot \begin{bmatrix} R_d \\ 0 \end{bmatrix}$$

and the related change of basis $[\mathcal{L}_d \mid \mathcal{L}_{\geq d+1}] = \mathcal{L}_{\geq d} \cdot [Q_1 \mid Q_2]$.

Then

- $\mathcal{L}_{\geq d+1}$ *is a basis of* $\Lambda_{\geq d+1}$*;*
- $R_d = W_{\mathcal{L}_d}^{\mathcal{P}_d}$ *has full row rank;*
- *The components of* $\mathcal{L}_{d\downarrow} = \mathcal{P}_d^\dagger \cdot R_d^T$ *form a basis of* $\Lambda_{\downarrow} \cap \mathbb{K}[x]_d$.

We shall furthermore denote by $\mathcal{L}_{\leq d} = \bigcup_{i=0}^{d} \mathcal{L}_i$ the thus constructed basis of a complement of $\Lambda_{\geq d+1}$ in Λ.

PROOF: It all follows from the fact that a change of basis $\mathcal{L}' = \mathcal{L}Q$ of Λ implies that $W_{\mathcal{L}'}^{\mathcal{P}} = Q^T W_{\mathcal{L}}^{\mathcal{P}}$. In the present case $Q = [Q_1 \mid Q_2]$ is orthogonal and hence $Q^T = Q^{-1}$.

The last point simply follows from the fact that, for $\lambda \in \Lambda$, $\lambda = \sum_{p \in \mathcal{P}} \lambda(p) p^\dagger(\partial)$. Hence if $T = W_{\mathcal{L}}^{\mathcal{P}}$ then the j-th component of $\mathcal{L}$ is $\sum_i t_{ji} p^\dagger(\partial)$. □

This construction gives us a basis of $\Lambda_{\downarrow} \cap \mathbb{K}[x]_d$ in addition to a basis of $\Lambda_{\geq d+1}$ to pursue the computation at the next degree. Before going there, we need to compute a basis $\mathcal{H}_d^0$ for the complement of $V_d(\mathcal{H}_{<d}^0)$ in I_d^0. For that we shall use an additional QR-decomposition as explained in Proposition 4.4.5, after two preparatory lemmas.

Lemma 4.4.3 *Let* $d \geq 0$ *and let* $\mathcal{P}_d$ *be a basis of* $\mathbb{K}[x]_d$ *then:*

$$I_d^0 = \left\{ \sum_{i=1}^{|\mathcal{P}_d|} a_i p_i \mid (a_1, \ldots, a_{|\mathcal{P}_d|})^t \in \ker\left(W_{\mathcal{L}_d}^{\mathcal{P}_d}\right) \text{ and } p_i \in \mathcal{P}_d \right\}$$

.

PROOF: Recall that I is the kernel of the Vandermonde operator, and $W_{\mathcal{L}}^{\mathcal{P}}$ is the matrix of this latter. The Vandermonde submatrix $W_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}}$ can be written as follows

$$W_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}} = W_{[\mathcal{L}_{\leq d-1} \mid \mathcal{L}_d]}^{\mathcal{P}_{\leq d}} = \begin{pmatrix} W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq d-1}} & W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_d} \\ 0 & W_{\mathcal{L}_d}^{\mathcal{P}_d} \end{pmatrix} \tag{4.4.1}$$

where $W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq d-1}}$ has full row rank.

Assume first that p is a polynomial in I_d^0. Then there is $q \in I$ of degree d such that $q^0 = p$. Let $q = \begin{pmatrix} q_{\leq d-1} \\ q_d \end{pmatrix}$ and $p = q_d$ be the coefficients of q and p respectively in the basis $\mathcal{P}$. As $q \in I_d$ we have that

$$W_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}} \cdot q = \begin{pmatrix} W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq (d-1)}} \cdot q_{\leq d-1} + W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_d} \cdot q_d \\ W_{\mathcal{L}_d}^{\mathcal{P}_d} \cdot q_d \end{pmatrix} = 0$$

and therefore $p = q_d$ is in kernel of $W_{\mathcal{L}_d}^{\mathcal{P}_d}$. Now let v a vector in the kernel of $W_{\mathcal{L}_d}^{\mathcal{P}_d}$. A vector u such that $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{K}^{\binom{n+d}{d}}$ and $W_{\mathcal{L}_{\leq d}}^{\mathcal{P}_{\leq d}} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = 0$ can be found as the solution of the following equation.

$$W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq (d-1)}} u = W_{\mathcal{L}_d}^{\mathcal{P}_d} v - W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_d} v. \tag{4.4.2}$$

As $W_{\mathcal{L}_{\leq d-1}}^{\mathcal{P}_{\leq (d-1)}}$ has full row rank, Equation 4.4.2 always has a solution. Then $\mathcal{P}_{\leq d} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \in I$ and therefore $\mathcal{P}_d \cdot v \in I_d^0$. □

Lemma 4.4.4 *Consider the row vector* q *of coefficients of a polynomial* q *of* $\mathbb{K}[x]_d$ *in the basis* $\mathcal{P}_d$*. The polynomial* q *is in the orthogonal complement of* $V_d(\mathcal{H})$ *in* $\mathbb{K}[x]_d$ *if and only if the row vector* q *is in the left kernel of* $M_{\mathcal{M}_d, \mathcal{P}_d^\dagger}(\mathcal{H})$*.*

PROOF: The columns of $M_{\mathcal{M}_d, \mathcal{P}_d^\dagger}$ are the vectors of coefficients, in the basis $\mathcal{P}_d^\dagger$, of polynomials that span $V_d(\mathcal{H})$. The membership of q in the left kernel of $M_{\mathcal{M}_d, \mathcal{P}_d^\dagger}(\mathcal{H})$ translates as the apolar product of q with these vectors to be zero. And conversely. □

Proposition 4.4.5 *Consider the QR-decomposition*

$$\begin{bmatrix} \left(W_{\mathcal{L}_d}^{\mathcal{P}_d}\right)^T & M_{\mathcal{M}_d, \mathcal{P}_d^\dagger}(\mathcal{H}) \end{bmatrix} = [Q_1 \mid Q_2] \cdot \begin{bmatrix} R \\ 0 \end{bmatrix}$$

The components of the row vector $\mathcal{P}_d \cdot Q_2$ *span the orthogonal complement of* $V_d(\mathcal{H})$ *in* I_d^0*.*

PROOF: The columns in Q_2 span $\ker W_{\mathcal{L}_d}^{\mathcal{P}_d} \cap \ker \left(M_{\mathcal{M}_d, \mathcal{P}_d^\dagger}\right)^t$. The result thus follows from Lemmas 4.4.3 and 4.4.4. □

We are now able to show the correctness and termination of Algorithm 3.

Algorithm 3

Input: - $\mathcal{L}$ a basis of Λ ($r = |\mathcal{L}| = \dim(\Lambda)$)
- $\mathcal{P}$ a basis of $\mathbb{K}[x]_{\leq r}$
- $\mathcal{P}^\dagger$ the dual basis of $\mathcal{P}$ w.r.t the apolar product.

Output: - $\mathcal{H}$ a reduced H-basis for $I := \ker \Lambda$
- $\mathcal{P}_\Lambda$ a basis of the least interpolation space of Λ.

$\mathcal{H}^0 \leftarrow \{\}, \quad \mathcal{P}_\Lambda \leftarrow \{\}$
$d \leftarrow 0$
$\mathcal{L}_{\leq 0} \leftarrow \{\}, \quad \mathcal{L}_{\geq 0} \leftarrow \mathcal{L}$

while $\mathcal{L}_{\geq d} \neq \{\}$ **do**

$\quad Q \cdot \begin{bmatrix} R_d \\ 0 \end{bmatrix} = W_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$ ▷ QR-decomposition of $W_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$

$\quad \mathcal{P}_\Lambda \leftarrow \mathcal{P}_\Lambda \bigcup \mathcal{P}_d^\dagger \cdot R_d^T$

$\quad [\mathcal{L}_d \mid \mathcal{L}_{\geq d+1}] \leftarrow \mathcal{L}_{\geq d} \cdot Q^T$ ▷ Note that $R_d = W_{\mathcal{L}_d}^{\mathcal{P}_d}$

$\quad \mathcal{L}_{\leq d+1} \leftarrow \mathcal{L}_{\leq d} \cup \mathcal{L}_d$

$\quad [Q_1 \mid Q_2] \cdot R = \begin{bmatrix} R_d^T & M_{\mathcal{M}_d, \mathcal{P}_d^\dagger}(\mathcal{H}) \end{bmatrix}$

$\quad \mathcal{H}^0 \leftarrow \mathcal{H}^0 \bigcup \mathcal{P}_d \cdot Q_2$

$\quad d \leftarrow d + 1$

end while

for all $p \in \mathcal{H}^0$ **do**

$\quad \mathcal{H} \leftarrow \mathcal{H} \bigcup \left\{ p - \mathcal{P}_\Lambda \left(W_{\mathcal{L}_{\leq d}}^{\mathcal{P}_\Lambda} \right)^{-1} (\mathcal{L}_{\leq d})^T \right\}$

end for

return $(\mathcal{H}, \mathcal{P}_\Lambda)$

Correctness In the spirit of Algorithm 2, Algorithm 3 proceeds degree by degree. At the iteration for degree d we first compute a basis for $\Lambda_{\geq d+1}$ by splitting $\mathcal{L}_{\geq d}$ into $\mathcal{L}_{\geq d+1}$ and $\mathcal{L}_d$. As explained in Proposition 4.4.2, this is obtained through the QR-decomposition of $\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d}$. From this decomposition we also obtain a basis for $\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d$ as well as $\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$. We then go after $\mathcal{H}_d^0$, which spans the orthogonal complement of $\mathrm{V}_d(\mathcal{H}_{\leq d-1}^0)$ in I_d^0. The elements of $\mathcal{H}_d^0$ are computed via intersection of $\ker \mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$ and $\ker\left(\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}\right)^t$ as showed in Proposition 4.4.5. Algorithm 3 stops when we reach a degree δ such that $\mathcal{L}_{\geq\delta}$ is empty. Notice that for $d \geq \delta$ the matrix $\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}$ is an empty matrix and therefore its kernel is the full space $\mathbb{K}[\mathrm{x}]_d$. Then as a consequence of Lemma 4.4.3, for all $d > \delta$ we have that $V_d(I_{d-1}^0) = I_d^0$ hence $\langle \mathcal{H}_d^0 \rangle$ is an empty set. The latter implies that when the algorithm stops we have computed the full H-basis $\mathcal{H}^0$ for I^0.

We then obtain an H-basis of I by finding the projections, onto $\Lambda_\downarrow$ and parallel to I, of the elements of $\mathcal{H}^0$. These are the polynomials of $\Lambda_\downarrow$ interpolating the elements of $\mathcal{H}^0$ according to Λ.

Termination Considering $r := \dim(\Lambda)$ we have that $\mathcal{L}_{\geq r}$ is an empty set, this implies that in the worst case our algorithm stops after r iterations.

Complexity The most expensive computational step in Algorithms 3 is the computation of the kernel of the matrix $\left[\left(\mathrm{W}_{\mathcal{L}_d}^{\mathcal{P}_d}\right)^T \ \mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})\right]$, with number of columns and rows given by

$$\begin{aligned} \mathrm{row}(d) &= \tbinom{d+n-1}{n-1} = \tfrac{d^{n-1}}{(n-1)!} + \mathcal{O}\left(d^{n-1}\right) \\ \mathrm{col}(d) &= \textstyle\sum_{i=1}^{|\mathcal{H}|} \tbinom{d-d_i+n-1}{n-1} + |\mathcal{L}_d| = \tfrac{|\mathcal{H}|d^{n-1}}{(n-1)!} + \mathcal{O}\left(d^{n-1}\right) \end{aligned} \tag{4.4.3}$$

where $d_1, \ldots, d_{|\mathcal{H}|}$ are the degrees of the elements of the computed H-basis until degree d. Then the computational complexity of Algorithm 3 relies on the method used for the kernel computation of $VM(d)$, which in our case is the QR-decomposition.

We are giving a frame for the simultaneous computation of an H-basis and the Least interpolation space, but there is still room for improving the performance of Algorithm 3. The structure of the Macaulay matrix might be taken into account to alleviate the linear algebra operations as for instance in (Berthomieu et al., 2017). We can also consider different variants of Algorithm 3. In Proposition 4.4.6 we show that orthogonal bases for $\mathbb{K}[\mathrm{x}]_d \cap \Lambda_\downarrow$ and I_d^0 can be simultaneously computed by applying QR-decomposition in the Vandermonde matrix $(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d})^T$. Therefore we can split Step 9 in two steps. First we do a QR-decomposition $(\mathrm{W}_{\mathcal{L}_{\geq d}}^{\mathcal{P}_d})^T$ to obtain orthogonal bases of $\mathbb{K}[\mathrm{x}]_d \cap \Lambda_\downarrow$ and I_d^0. Once that we have in hand a basis of I_d^0 we obtain the elements of $\mathcal{H}_d$ as its complement in the column space of $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d^\dagger}(\mathcal{H})$.

Proposition 4.4.6 *Let* $[\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \begin{bmatrix} \mathrm{R}_d \\ 0 \end{bmatrix} = \left(\mathrm{W}^{\mathcal{P}_d}_{\mathcal{L}_{\geq d}}\right)^T$ *be a QR-decomposition of* $\left(\mathrm{W}^{\mathcal{P}_d}_{\mathcal{L}_{\geq d}}\right)^T$. *Let* r *be the rank of* $\left(\mathrm{W}^{\mathcal{P}_d}_{\mathcal{L}_{\geq d}}\right)^T$. *Let* $\{q_1 \dots q_r\}$ *and* $\{q_{r+1} \dots q_m\}$ *be the columns of* Q_1 *and* Q_2 *respectively. Then the following holds:*

1. $\mathcal{P}_{\Lambda,d} = \left\{\mathcal{P}^\dagger_d \cdot q_1, \dots, \mathcal{P}^\dagger_d \cdot q_r\right\}$ *is a basis of* $\mathbb{K}[\mathrm{x}]_d \bigcap \Lambda_\downarrow$.
2. $\mathcal{N} = \{\mathcal{P}_d \cdot q_{r+1}, \dots, \mathcal{P}_d \cdot q_m\}$ *is a basis of* I^0_d.
3. *If* $p \in \mathcal{P}_{\Lambda,d}$ *and* $q \in \mathcal{N}$ *then* $\langle p, q\rangle = 0$, *i.e.,* $\mathbb{K}[\mathrm{x}] = (\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d) \overset{\perp}{\oplus} I^0_d$.

In the case where $\mathcal{P}$ *is orthonormal with respect to the apolar product, i.e.* $\mathcal{P} = \mathcal{P}^\dagger$, *then* $\mathcal{P}_{\Lambda,d}$ *and* $\mathcal{N}$ *are also orthonormal bases.*

PROOF: Let D such that $\mathcal{L}_{\geq D} = \{\}$ and let $\mathcal{L}_{\leq D} = \bigcup_{d \leq D} \mathcal{L}_d$ be a basis of Λ. Then the matrix $\mathrm{W}^{\mathcal{P}_{\leq D}}_{\mathcal{L}_{\leq D}}$ is block upper triangular with non singular diagonal blocks. Consider $\{a_1, \dots a_\ell\} \in \mathbb{K}^{|\mathcal{P}_{\leq D}|}$ the rows of $\mathrm{W}^{\mathcal{P}_{\leq D}}_{\mathcal{L}_{\leq D}}$. By Proposition Proposition 3.2.4 we have that $\mathcal{P}_\Lambda \left\{\left(\mathcal{P}^\dagger_{\leq D} \cdot a^t_1\right)_\downarrow, \dots, \left(\mathcal{P}^\dagger_{\leq D} \cdot a^t_\ell\right)_\downarrow\right\}$ is a basis of $\Lambda_\downarrow$, we can rewrite $\mathcal{P}_\Lambda$ as $\bigcup_{d=1}^{D} \left\{\mathcal{P}^\dagger_d \cdot b^t_1, \dots, \mathcal{P}^\dagger_d \cdot b^t_{\ell_d}\right\}$ where $\{b_1, \dots, b_{\ell_d}\}$ is a basis of the row space of $\left(\mathrm{W}^{\mathcal{P}_d}_{\mathcal{L}_d}\right)$. Since $\mathcal{P}_\Lambda$ is a graded basis then $\left\{\mathcal{P}^\dagger_d \cdot b^t_1, \dots, \mathcal{P}^\dagger_d \cdot b^t_{\ell_d}\right\}$ is a basis $\mathbb{K}[\mathrm{x}]_d \cap \Lambda_\downarrow$.

Part (2) in the proposition is a direct consequence of Lemma 4.4.3 and the fact that the columns of Q_2 form a basis of the kernel of $\mathrm{W}^{\mathcal{P}_d}_{\mathcal{L}_{\geq d}}$. Let now $q \in \mathcal{P}_{\Lambda,d}$ and $p \in \mathcal{N}$. Then,

$$\langle p, q\rangle = \left\langle \sum_{p_i \in \mathcal{P}_d} a_i p_i, \sum_{q_i \in \mathcal{P}^\dagger_d} b_i q_i \right\rangle = \sum_{i=1} a_i b_i = 0.$$

Last equality stems from a and b being different rows in Q. □

4.5 Symmetry reduction

In the above Algorithm 3, to compute an H-basis of $I = \ker w$, we use the Vandermonde and Macaulay matrices. We showed in Section 3.4 how the Vandermonde matrix can be block diagonalized using appropriate symmetry adapted bases of $\mathbb{K}[\mathrm{x}]$ and Λ. We show here how to obtain such a block diagonalization on the Macaulay matrix when the space spanned by $\mathcal{H}$ is invariant under the induced action of a group G on $\mathbb{K}[\mathrm{x}]$. The key relies on exhibiting the equivariance of the prolongation map $\Psi_{d,\mathrm{h}}$ defined in Section 4.3.

Consider now a set $\mathcal{H} = \{h_1, \dots, h_l\}$ of homogeneous polynomials of $\mathbb{K}[\mathrm{x}]$. We denote $d_1, \dots, d_\ell$ their respective degrees and $\mathrm{h} = [h_1, \dots, h_\ell]$ the row vector of $\mathbb{K}[x]^\ell$. Associated to h, and a degree d, is the map introduced in Section 4.3

$$\begin{array}{rccc} \psi_{d,\mathrm{h}}: & \mathbb{K}[\mathrm{x}]_{d-d_1} \times \dots \times \mathbb{K}[\mathrm{x}]_{d-d_\ell} & \to & \mathbb{K}[\mathrm{x}]_d \\ & \mathrm{f} = [f_1, \dots, f_\ell]^t & \to & \mathrm{h} \cdot \mathrm{f}. \end{array} \tag{4.5.1}$$

We assume that $\mathcal{H}$ forms a basis of an invariant subspace of $\mathbb{K}[\mathrm{x}]$ and we call θ the restriction of the representation ρ to this subspace, while Θ is the matrix representation in the basis $\mathcal{H}$: $\Theta(g) = [\theta(g)]_{\mathcal{H}}$. Then

$$[\rho(g)(h_1), \ldots, \rho(g)(h_\ell)] = \mathrm{h} \circ \vartheta(g^{-1}) = \mathrm{h} \cdot \Theta(g).$$

Note that, since the representation ρ on $\mathbb{K}[\mathrm{x}]$ preserves degree, $\deg h_i \neq \deg h_j \quad \Rightarrow \Theta_{ij}(g) = 0, \ \forall g \in G$.

Proposition 4.5.1 *Consider* $\mathrm{h} = [h_1, \ldots, h_\ell] \in \mathbb{K}[x]_{d_1} \times \ldots \times \mathbb{K}[x]_{d_l}$ *and assume that* $\mathrm{h} \circ \vartheta(g^{-1}) = \mathrm{h} \cdot \Theta(g)$*, for all* $g \in G$*. For any* $d \in \mathbb{N}$*, the map* $\psi_{d,\mathrm{h}}$ *is* $\tau - \rho$ *equivariant for the representation* τ *on* $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_\ell}$ *defined by* $\tau(g)(\mathrm{f}) = \Theta(g) \cdot \mathrm{f} \circ \vartheta(g^{-1})$.

PROOF: $(\rho(g) \circ \psi_{d,\mathrm{h}})(\mathrm{f}) = \rho(g)(\mathrm{h} \cdot \mathrm{f}) = \mathrm{h} \circ \vartheta(g^{-1}) \cdot \mathrm{f} \circ \vartheta(g^{-1}) = \mathrm{h} \cdot \Theta(g) \cdot \mathrm{f} \circ \vartheta(g^{-1}) = (\psi_{\mathrm{h}} \circ \tau(g))(\mathrm{f})$. □

By application of (Fässler and Stiefel, 1992, Theorem 2.5), the matrix of $\psi_{d,\mathrm{h}}$ is block diagonal in symmetry adapted bases of $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_\ell}$ and $\mathbb{K}[\mathrm{x}]_d$. Yet, in the algorithm to compute symmetry adapted H-basis, the set $\mathcal{H}$ increases with d at each iteration and τ changes accordingly. We proceed to discuss how to hasten the computation of a symmetry adapted basis of the evolving space $\mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_\ell}$.

The set $\mathcal{H} = \mathcal{H}^1 \cup \ldots \mathcal{H}^N$ that we shall build, degree by degree, is actually a symmetry adapted basis. In particular, for $1 \leq i \leq N$, $\mathcal{H}^i$ spans the isotypic component associated to the irreducible representation ρ_i. If the multiplicity of the latter, in the span of $\mathcal{H}$, is ℓ_i then the cardinality of $\mathcal{H}^i$ is $\ell_i n_i$. The matrices of the representation θ in this basis are $\Theta(g) = \mathrm{diag}\,(R_i(g) \otimes I_{\ell_i} \,|\, i = 1 \ldots N)$.

Assume $\mathcal{H}^i$ is determined by $h_{i,1}, \ldots, h_{i,\ell_i}$, of respective degrees $d_{i,1}, \ldots, d_{i,\ell_i}$. In other words, for $1 \leq l \leq \ell_i$,

$$\mathrm{h}_{i,l} = \left[h_{i,l}, \pi^{(i)}_{21}(h_{i,l}), \ldots, \pi^{(i)}_{n_i 1}(h_{i,l})\right]$$

is such that $\mathrm{h}_{i,l} \circ \vartheta(g^{-1}) = \mathrm{h}_{i,l} \cdot R_i(g)$. Hence the related product subspace $\mathbb{K}[\mathrm{x}]^{n_i}_{d-d_{i,l}}$ is invariant under τ. The symmetry adapted bases for all these subspaces can be combined into a symmetry adapted basis for the whole product space

$$\left(\mathbb{K}[\mathrm{x}]_{d_{1,1}} \times \mathbb{K}[\mathrm{x}]_{d_{1,\ell_1}}\right)^{n_1} \times \ldots \times \left(\mathbb{K}[\mathrm{x}]_{d_{N,1}} \times \mathbb{K}[\mathrm{x}]_{d_{1,\ell_N}}\right)^{n_N}.$$

Note that the components $\mathbb{K}[\mathrm{x}]^{n_i}_e$ with representation $\tau_{i,e}$ defined by $\tau_{i,e}(g)(\mathrm{f}) = R_i(g) \cdot f \circ \vartheta(g^{-1})$ are bound to reappear several times in the overall algorithm of next section. Hence the symmetry adapted bases for the evolving τ can be computed dynamically.

We can actually go further in the combination of symmetry adapted bases of smaller size, down to the symmetry adapted bases for the tensor product representations $\rho_i \otimes \rho_j$.

Let us first observe what is the matrix of τ for any $\mathcal{H}$. We take the same set $\mathcal{P}_e$ as a basis for each component $\mathbb{K}[x]_e$ and call $\mathcal{P}$ the natural basis for the product $\mathbb{K}[x]_{d-d_1} \times \ldots \times \mathbb{K}[x]_{d-d_\ell}$. Let $\Gamma_e = \left[\rho_{|\mathbb{K}[x]_e}\right]_{\mathcal{P}_e}$ be the matrix representation of the restriction of ρ to $\mathbb{K}[x]_e$ in the basis $\mathcal{P}_e$. The matrix of τ is then described by blocks. The block in position (i,j) is zero if $d-d_i \neq d-d_j$ and $\Theta_{ij}(g)\,\Gamma_{d-d_j}(g)$.

If we choose $\mathcal{P}_e$ to be a symmetry adapted basis of $\mathbb{K}[x]_e$ then

$$\Gamma_e(g) = \operatorname{diag}\left(R_j(g) \otimes I_{c_{j,e}} \,|\, 1 \leq j \leq N\right), \text{ where } c_{j,e}$$

is the multiplicity of the irreducible representation ρ_j in $\mathbb{K}[x]_e$. Hence the matrices of the restriction of τ to the invariant product subspace $\mathbb{K}[x]^{n_i}_{d-d_{i,l}}$ are

$$R_i(g) \otimes \Gamma_{d-d_{i,l}}(g) = R_i(g) \otimes \operatorname{diag}\left(R_j(g) \otimes I_{c_{j,d-d_{i,l}}} \,|\, 1 \leq j \leq N\right).$$

With a permutation of the basis elements of $\mathbb{K}[x]^{n_i}_{d-d_{i,l}}$, the matrices of the restriction of τ can be brought to a diagonal of blocks $R_i(g) \otimes R_j(g)$. Assuming that the symmetry adapted bases for these tensorial representations of $\rho_i \otimes \rho_j$ are precomputed, a symmetry adapted basis for the restriction of τ, and then of τ, can be assembled from the symmery adapted bases of the components $\mathbb{K}[x]_e$.

4.6 Constructing Symmetry Adapted H-Basis

In this section we show, when the space Λ is invariant, an orthogonal equivariant H-basis $\mathcal{H}$ can be computed. In this setting, we exploit the symmetries of Λ to build $\mathcal{H}$. A robust and symmetry adapted version of Algorithm 3 is presented. The block diagonal structure of the Vandermonde and Macaulay matrices allow to reduce the size of the matrices to deal with. The H-basis obtained as the output of Algorithm 4 inherits the symmetries of Λ.

Proposition 4.6.1 *Let $I = \cap_{\lambda\in\Lambda} \ker\lambda$ and $d \in \mathbb{N}$. If Λ is invariant, then so are I, I^0, I^0_d , $V_d\left(I^0_{<d}\right)$. Also, if $\mathcal{H}$ is an orthogonal H-basis of I, then $\langle\mathcal{H}^0_d\rangle_{\mathbb{K}}$ is invariant.*

PROOF: Let $p \in I$ and $g \in G$, since Λ is closed under the action of G, $\lambda(\rho(g)(p)) = \rho^*(g) \circ \lambda(p) = 0$ for all $\lambda \in \Lambda$ therefore $\rho(g)(p) \in I$ implying the invariance of I. Considering d the degree of p we can write p as $p = p^0 + p_1$, with $p_1 \in \mathbb{K}[x]_{<d}$. Then we have that $\rho(g)p = \rho(g)p^0 + \rho(g)p_1 \in I$, as ρ is degree preserving then $\rho(g)p^0 \in I^0_d$ and the invariance of I^0 follows. Now for every $q = \sum_{h_i\in I^0_{d-1}} q_i h_i \in V_d\left(I^0_{\leq d}\right)$, it holds that $\rho(g)q = \sum_{h_i\in I^0_{d-1}} \rho(g)q_i\rho(g)h_i \subset V_d\left(I^0_{\leq d}\right)$, thus $V_d\left(I^0_{\leq d}\right)$ is an invariant subspace. Finally recalling (4.3.2) we conclude that $\langle\mathcal{H}^0_d\rangle_{\mathbb{K}}$ is also G-invariant for being the orthogonal complement of a G−invariant subspace. □

Algorithm 4 is a symmetry adapted version of Algorithm 3. In any iteration we compute $\mathcal{H}^0_d$ as a symmetry adapted basis of the orthogonal complement of $V_d(\mathcal{H}^0_{<d})$ in I^0.

This structure is obtained degree by degree. Assuming that the elements of $\mathcal{H}^0_{<d}$ form a symmetry adapted basis it follows from Corollary 3.4.2 and Proposition 4.5.1 that the matrices $\mathrm{W}_{\mathcal{L}}^{\mathcal{P}_d}$ and $\mathrm{M}_{\mathcal{M}_d,\mathcal{P}_d}(\mathcal{H}^0_{<d})$ are block diagonal. Computations over the symmetry blocks leads to the symmetry adapted structure of $\mathcal{H}^0_d$. For any degree d we only need to consider the matrices $\mathrm{W}^{\mathcal{P}^{i,1}_d}_{\mathcal{L}^{i,1}_{\geq d}}$ and $\mathrm{M}^i_d(\mathcal{H}^0_{<d})$, i.e., only one block per irreducible representation.

Once we have in hand $\mathcal{H}^0 = \left[h^1_{11}, \ldots, h^1_{1n_1}, \ldots, h^{\mathfrak{n}}_{c_{\mathfrak{n}}n_{\mathfrak{n}}}\right]^T$ and a symmetry adapted basis for $\Lambda_\downarrow$, we compute $\mathcal{H}$ by interpolation. Since $\mathcal{H}^0 \in \mathbb{K}[\mathrm{x}]^\theta_\vartheta$, by (**?**, Proposition 3.5), its interpolant in $\Lambda_\downarrow$ is also $\vartheta - \theta$ equivariant. Therefore

$$\mathcal{H} = \left[h^1_{11} - \widetilde{h^1_{11}}, \ldots, h^1_{1n_1} - \widetilde{h^1_{1n_1}}, \ldots, h^{\mathfrak{n}}_{c_{\mathfrak{n}}n_{\mathfrak{n}}} - \widetilde{h^{\mathfrak{n}}_{c_{\mathfrak{n}}n_{\mathfrak{n}}}}\right]^T \in \mathbb{K}[\mathrm{x}]^\theta_\vartheta.$$

The set $\mathcal{H}$ of its component is thus a symmetry adapted basis. The correctness and termination of Algorithm 4 follow from the same arguments exposed for Algorithm 3. Note that both Macaulay and Vandermonde matrices split in $\sum_{i=1}^{\mathfrak{n}} n_i$ blocks. Thanks to Proposition 2.4.8 we can approximate the dimensions of the blocks by

$$\frac{\dim \mathrm{M}^i(\mathcal{H}^0)}{\dim \mathrm{M}(\mathcal{H}^0)} \approx \frac{\dim \mathrm{W}_{\mathcal{L}^i}^{\mathcal{P}^i}}{\dim \mathrm{W}_{\mathcal{L}}^{\mathcal{P}}} \approx \frac{n_i^2}{|G|}.$$

Therefore depending on the size of G the dimensions of the matrices to deal with in Algorithm 4 can be considerably reduced.

Example 4.6.1 *Consider $\Xi \subset \mathbb{R}^3$ the invariant set of 26 points given by the union of the orbits $\mathcal{O}_2, \mathcal{O}_3$ and $\mathcal{O}_4$ in Example 3.5.4. Consider $\Lambda = \langle \{\mathrm{e}_\xi \mid \xi \in \Xi\} \rangle$. Λ is an invariant subspace and therefore $I = \bigcap_{\lambda \in \Lambda} \ker \lambda$ is an invariant ideal under the action of O_h. Applying Algorithm 4 to I we wet the following orthogonal symmetry adapted H-basis $\mathcal{H}$.*

$$\mathcal{H} = \begin{cases} \mathcal{H}^{(1)} = \{p := -19\left(x^4+y^4+z^4\right) + 18\left(x^2y^2+x^2z^2+y^2z^2\right) + x^2+y^2+z^2+18\} \\ \mathcal{H}^{(5)} = \left\{x^4 - 2y^4 + z^4 - x^2 + 2y^2 - z^2, \sqrt{3}\left(x^4 - z^4 - x^2 + z^2\right)\right\} \\ \mathcal{H}^{(7)} = \left\{yz\left(y^2-z^2\right), -xz\left(x^2-z^2\right), xy\left(x^2-y^2\right)\right\} \\ \mathcal{H}^{(9)} = \left\{-yz\left(4x^2-3y^2-3z^2+6\right), xz\left(3x^2-4y^2+3z^2-6\right), xy\left(3x^2+3y^2-4z^2-6\right)\right\} \end{cases}$$

From the structure of $\mathcal{H}$ it follows that p is the minimal degree invariant polynomial of I. Notice that as a difference with Example 3.5.4 we do not need an artificial point to get an invariant interpolant which its zero set contains Ξ. In Figure 4.1 we show the zero surface of p which is O_h invariant.

Algorithm 4

Input: - $\mathcal{L}$ a s.a.b of Λ ($r = |\mathcal{L}| = \dim(\Lambda), r_i = |\mathcal{L}^{i,1}|$)
- $\mathcal{P}$ an orthonormal graded s.a.b of $\mathbb{K}[\mathrm{x}]_{\leq r}$
- $\mathcal{M}_i$ a graded s.a.b of $\mathbb{K}[\mathrm{x}]^{n_i}_{\leq r}$, $1 \leq i \leq \mathfrak{n}$

Output: - $\mathcal{H}$ an orthogonal equivariant H-basis for $I := \ker \Lambda$
- $\mathcal{P}_\Lambda$ a s.a.b of the least interpolation space for Λ.

$\mathcal{H}^0 \leftarrow \{\}, \ \mathcal{P}_\Lambda \leftarrow \{\}$
$d \leftarrow 0$
$\mathcal{L}_{\leq 0} \leftarrow \{\}, \ \mathcal{L}_{\geq 0} \leftarrow \mathcal{L}$

while $\mathcal{L}_{\geq d} \neq \{\}$ **do**

  for $i = 1$ `to` $\mathfrak{n}$ such that $\mathcal{L}^{i,1}_{\geq d} \neq \varnothing$ **do**
    $\mathrm{Q} \cdot \begin{bmatrix} \mathrm{R}_{d,i} \\ 0 \end{bmatrix} = \mathrm{W}^{\mathcal{P}^{i,1}_d}_{\mathcal{L}^{i,1}_{\geq d}}$ ▷ QR-decomposition of $\mathrm{W}^{\mathcal{P}^{i,1}_d}_{\mathcal{L}^{i,1}_{\geq d}}$
    $\left[\mathcal{L}^{i,1}_d \mid \mathcal{L}^{i,1}_{\geq d+1}\right] \leftarrow \mathcal{L}^{i,1}_{\geq d} \cdot \mathrm{Q}^T$

    $\mathcal{L}^{i,1}_{\leq d+1} \leftarrow \mathcal{L}^{i,1}_{\leq d} \cup \mathcal{L}^{i,1}_d$
    $[\mathrm{Q}_1 \mid \mathrm{Q}_2] \cdot \mathrm{R} = \begin{bmatrix} \mathrm{R}^T_{d,i} & \mathrm{M}^i_d(\mathcal{H}^0) \end{bmatrix}$
    for $\alpha = 1$ `to` n_i **do**
      $\mathcal{P}^i_\Lambda \leftarrow \mathcal{P}^i_\Lambda \cup \mathcal{P}^{i,\alpha}_d \cdot \mathrm{R}^T_{d,i}$
      $\mathcal{H}^0_i \leftarrow \mathcal{H}^0_i \bigcup \mathcal{P}^{i,\alpha}_d \cdot \mathrm{Q}_2$
    end for
  end for
  $d \leftarrow d + 1$
end while
for $i = 1$ `to` $\mathfrak{n}$ **do**
  for all $p \in \mathcal{H}^0_i$ **do**
    $\mathcal{H} \leftarrow \mathcal{H} \bigcup \left\{ p - \mathcal{P}^{i,1}_\Lambda \left(\mathrm{W}^{\mathcal{P}^{i,1}_\Lambda}_{\mathcal{L}^{i,1}_{\leq d}} \right)^{-1} \left(\mathcal{L}^{i,1}_{\leq d} \right)^T \right\}$
  end for
end for
return $(\mathcal{H}, \mathcal{P}_\Lambda)$

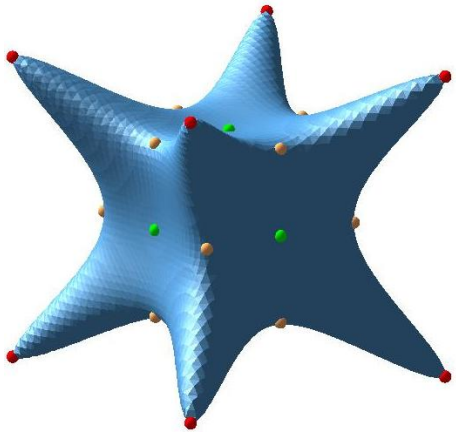


Figure 4.1: Lowest degree invariant algebraic surface through an invariant set of the points Ξ.

Example 4.6.2 *The subgroup of the orthogonal group $\mathbb{R}^3$ that leaves the regular tetrahedron invariant is commonly called T_h. It has order 24 and 5 inequivalent irreducible representations, all absolutely irreducible, whose dimensions are $(1,1,2,3,3)$. We consider the the following action of T_h in $\mathbb{R}^3$*

$$T_h = \left\{\delta^i\sigma^j\alpha^k\beta^\ell \mid 0 \le i \le 1,\ 0 \le j \le 2,\ 0 \le k \le 1,\ 0 \le \ell \le 1\right\}$$

which is defined by the matrices.

$$\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider $\Xi \subset \mathbb{R}^3$ the invariant set of 14 points illustrated on Figure 4.2a. They are gruped in three orbits $\mathcal{O}_1$ (violet points), $\mathcal{O}_2$ (brown points) and $\mathcal{O}_3$ (red points). Consider the space of linear forms Λ given by

$$\Lambda = \left\langle \{\mathbb{e}_\xi \mid \xi \in \Xi\} \cup \left\{\mathbb{e}_\xi \circ D_{\vec{\xi}} \mid \xi \in \mathcal{O}_3\right\}\right\rangle_{\mathbb{R}}$$

The fact that $I = \cap_{\lambda\in\Lambda}\lambda$ is an ideal in $\mathbb{R}[x]$ can be easily deduce from the fact that

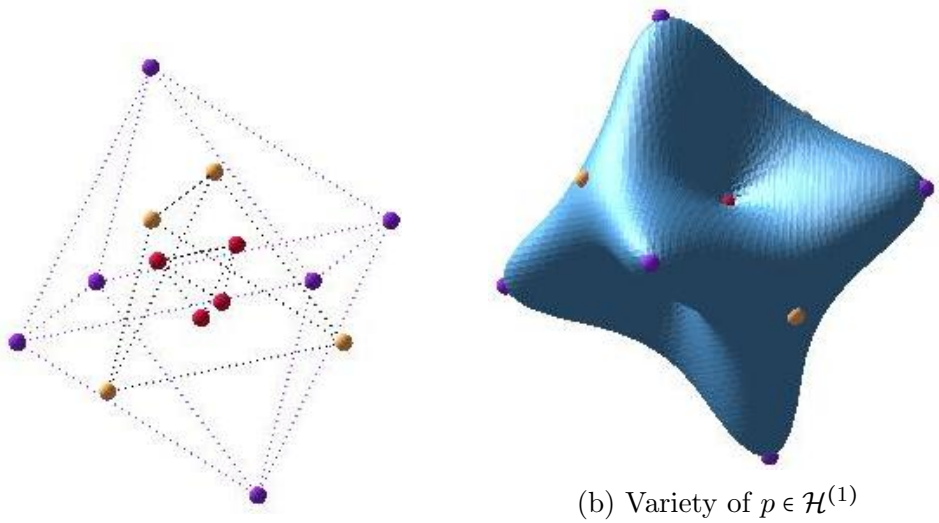
$$\mathbb{e}_\xi \circ D_{\vec{\xi}}(f \cdot g) = f(\xi) \cdot \mathbb{e}_\xi \circ D_{\vec{\xi}}(g) + g(\xi) \cdot \mathbb{e}_\xi \circ D_{\vec{\xi}}(f)$$

so if $\xi \in \mathcal{O}_3$ then for any $f \in I$ and $g \in \mathbb{R}[x]$, $\mathbb{e}_\xi \circ D_{\vec{\xi}}(f \cdot g) = 0$. Applying Algorithm 4 to I, we get a symmetry adapted H-basis $\mathcal{H}$

$$\mathcal{H} = \begin{cases} \mathcal{H}^{(1)} = \left\{ \frac{2152}{1875}\left(x^4+y^4+z^4\right) + \frac{25973}{1875}\left(x^2y^2+x^2z^2+y^2z^2\right) + \frac{45000}{25973}xyz - \frac{13750}{1875}\left(x^2+y^2+z^2\right) + 1 \right\} \\ \mathcal{H}^{(3)} = \{(4x^2+4z^2-25)(x-z)(x+z), (4y^2+4z^2-25)(y-z)(y+z)\} \\ \mathcal{H}^{(4)} = \begin{cases} 270x^2yz + 90y^3z + 90yz^3 - 8x^3 + 79xy^2 + 79xz^2 - 250yz + 50x, \\ 90x^3z + 270xy^2z + 90xz^3 + 79x^2y - 8y^3 + 79yz^2 - 250xz + 50y, \\ 90x^3y + 90xy^3 + 270xyz^2 + 79x^2z + 79y^2z - 8z^3 - 250xy + 50z \end{cases} \\ \mathcal{H}^{(5)} = \begin{cases} \mathcal{H}^{(5,1)} = \{xy^2 - xz^2, yz\ (y^2-z^2)\} \\ \mathcal{H}^{(5,2)} = \{-x^2y + yz^2, -xz\ (x^2-z^2)\} \\ \mathcal{H}^{(5,3)} = \{x^2z - y^2z, xy\ (x^2-y^2)\} \end{cases} \end{cases}$$

and a symmetry adapted basis $\mathcal{L}$ *of the least interpolation space*

$$\mathcal{L} = \begin{cases} \mathcal{L}^{(1)} = \left\{\ 1, x^2+y^2+z^2, xyz, 25973\left(x^4+y^4+z^4\right) - 12912\left(x^2y^2+x^2z^2+y^2z^2\right)\ \right\} \\ \mathcal{L}^{(3)} = \{2x^2-y^2-z^2, y^2-z^2\} \\ \mathcal{L}^{(4)} = \begin{cases} \mathcal{L}^{(5,1)} = \{x, yz, x^3, x\left(y^2+z^2\right)\} \\ \mathcal{L}^{(5,2)} = \{y, xz, y^3, y\left(x^2+z^2\right)\} \\ \mathcal{L}^{(5,3)} = \{z, yx, z^3, z\left(x^2+y^2\right)\} \end{cases} \end{cases}$$



(a) Points in Ξ divided in orbits

(b) Variety of $p \in \mathcal{H}^{(1)}$

Figure 4.2: Interpolation data and variety of the interpolant p that goes through the points in Ξ and with zero directional derivative in $\mathcal{O}_3$.

4.7 FGLM

The FGLM algorithm was introduced by Faugere et al. (1993). It performs change of orderings for Gröbner bases in a zero dimensional ideal. Variations of the FGLM to take symmetries into account can be found in (Faugère and Rahmany, 2009) and (Faugere and Svartz, 2013). Denoting by $I^{\mathcal{G}}$ the intersection of an ideal I with $\mathbb{K}[x]^{\mathcal{G}}$, an algorithm is presented in (Faugère and Rahmany, 2009) to compute a Gröbner basis of an invariant ideal $\mathbb{K}[h_1,\dots,h_m]$ with $h_i \in I^{\mathcal{G}}$ from an SAGBI basis of $I^{\mathcal{G}}$.

In (Faugere and Svartz, 2013) for invariant ideals with respect to diagonal actions, the symmetries are exploited to speed up the change of orderings. In this section we show how Algorithm 4 can be seen as a FGLM-type algorithm and as a generalization of the Diagonal-FGLM algorithm proposed in (Faugere and Svartz, 2013). From a Gröbner basis of an invariant ideal I we compute a symmetry preserving H-basis as well as a symmetry adapted basis of $\mathbb{K}[x]/I$. Having a robust and symmetry adapted basis of $\mathbb{K}[x]/I$ plays a key role for exploiting symmetries during the solution of equivariant system of equations (Gatermann, 1990; Corless et al., 2009), and for computing symmetric cubatures (Gatermann, 1988; Collowald and Hubert, 2015). An aggregated value of Algorithm 4 is that it provides a s.a.b of $\mathbb{K}[x]/I$ such that its representatives are also a s.a.b of an invariant subspace of $\mathbb{K}[x]$. We will see in Example 4.7.1 that symmetry adapted bases computed w.r.t the representation $\hat{\rho}$ does not necessarily induce a decomposition in irreducible components of $\mathbb{K}[x]$.

Proposition 4.7.1 *If I is a G-invariant ideal and $g \in \mathcal{G}$ then*

$$\begin{array}{rccc} \hat{\rho}(g): & \mathbb{K}[x]/I & \to & \mathbb{K}[x]/I \\ & [p] & \to & [\rho(g)p] \end{array} \tag{4.7.1}$$

is a linear representation of G.

PROOF: We just need to check that the definition is independent of the choice of representative for $[p]$. If $[p]=[q]$ then $p-q \in I$ so that $\rho(p-q)=\rho(p)-\rho(q) \in I$ since I is G-invariant. Hence $[\rho(p)]=[\rho(q)]$. □

Consider $n_1, n_2, \dots, n_r \in \mathbb{K}[x]$ such that $\{[n_1],\dots,[n_r]\}$ is a basis of $\mathbb{K}[x]/I$. Let $\alpha_1,\dots,\alpha_r$ be its dual basis, so that any element $[p] \in \mathbb{K}[x]/I$ can be written as

$$[p] = \sum_{i=1}^{r} \alpha_i([p])[n_i]. \tag{4.7.2}$$

As a consequence of Proposition 4.7.1, if I is G-invariant so is $(\mathbb{K}[x]/I)^*$ under the dual representation of $\hat{\rho}$. Define $\lambda_1,\dots,\lambda_r$ as elements of $\mathbb{K}[x]^*$ by $\lambda_i(p)=\alpha_i([p])$ for any $p \in \mathbb{K}[x]$.

Proposition 4.7.2 *Let $\lambda_1,\dots,\lambda_r$ be defined as above. The following holds.*

- $\Lambda = \langle \lambda_1, \ldots, \lambda_r \rangle_{\mathbb{K}}$ *is a* G*-invariant subspace of* $\mathbb{K}[x]^*$ *w.r.t* ρ^*.
- $I = \cap_{\lambda \in \Lambda} \ker \lambda$

PROOF: The first assertion is mostly a matter of unrolling the definitions. We want to show that $\rho^*(g)(\lambda) \in \Lambda$ for any $g \in G$. Applying $\rho^*(g)(\lambda)$ to any $p \in \mathbb{K}[\mathrm{x}]$ we have that $\rho^*(g)(\lambda)\, p = \lambda(\rho(g^{-1})\, p) = \alpha([\rho(g^{-1})\, p]) = \hat{\rho}^*(g)(\alpha)([p])$. Therefore as $\hat{\rho}^*(g)$ is in $(\mathbb{K}[\mathrm{x}]/I)^*$ then $\rho^*(g)(\lambda) \in \Lambda$.

For the second assertion, we have the equivalences:

$$p \in I \Leftrightarrow [p] = 0 \Leftrightarrow \alpha_i([p]) = 0, 1 \le i \le r \Leftrightarrow \lambda_i(p) = 0, 1 \le i \le r.$$

□

Let $\mathcal{G} = \{g_1, g_2, \ldots, g_m\} \subset \mathbb{K}[\mathrm{x}]$ be a Grobner basis of I, and let $\mathcal{N} = \{n_1, n_2, \ldots, n_r\} \subset \mathbb{K}[\mathrm{x}]$ the corresponding normal set of monomials. The linear forms $\lambda_1, \ldots, \lambda_r$ in $\mathbb{K}[x]^*$ are defined so that the normal form of $p \in \mathbb{K}[\mathrm{x}]$ w.r.t. $\mathcal{G}$ is $\lambda_1(p)\, n_1 + \ldots + \lambda_r(p)\, n_r$.

Example 4.7.1 *Lets consider the cyclic group* C_3*, and its action over* R^3*. It has order* 3 *and* 3 *inequivalent irreducible representations of dimension* 1*, one absolutely irreducible representation and a pair of conjugate irreducible representations of complex type. We analyze the cyclic* $n-th$ *roots system (Björck, 1990), which has been widely used as a benchmark. The cyclic* $3-th$ *roots system is defined by:*

$$C(3) = \begin{cases} x + y + z \\ xy + yz + zx \\ xyz - 1 \end{cases}$$

The associated ideal $I = \langle C(3) \rangle$ *of* $C(3)$ *is invariant under* C_3*. The reduced Gröbner basis* $\mathcal{G}$ *of* I *w.r.t the graded reverse lexicographic order and its corresponding normal set* $\mathcal{N}$ *are given by* $\mathcal{G} := \{x + y + z, y^2 + yz + z^2, z^3 - 1\}$ *and* $\mathcal{N} := \{1, z, y, z^2, yz, yz^2\}$*. Applying Algorithm 4 to the linear forms given by the coefficients of the normal forms w.r.t* $\mathcal{N}$*, we obtain a symmetry adapted H-basis*

$$\mathcal{H} = \left\{x + y + z, x^2 + y^2 + z^2, x^3 + y^3 + z^3 - 3\right\}$$

as well as a symmetry preserving and robust representation of the quotient

$$\mathcal{P} = \begin{cases} \mathcal{P}^{\mathcal{G}} &= \{1, (y-z)(x-z)(x-y)\} \\ \mathcal{P}^{(2)} &= \{x - z,\ y - z,\ (x-y)(x-2z+y),\ (y-z)(2x-y-z)\} \end{cases}$$

If instead of using Algorithm 4 we compute directly a s.a.b of $\mathbb{K}[\mathrm{x}]/I$ *whit respect to* $\hat{\rho}$ *we get the following basis*

$$\mathcal{Q} = \begin{cases} \mathcal{Q}^{\mathcal{G}} &= \{1,\ yz^2\} \\ \mathcal{Q}^{(2)} &= \{2z + y,\ yz + 2z^2,\ -z + 2y,\ -2yz + z^2\} \end{cases}.$$

We can check that the elements of $\mathcal{Q}$ *do not belong to any isotypic component of* $\mathbb{K}[\mathrm{x}]$.

Chapter 5

Fundamental Equivariants

Rodriguez Bazan, E., Hubert, E., 2020. Algorithms for fundamental equivariants and invariants.

5.1 Introduction

The main purpose of this chapter is to offer algorithms to compute relevant sets of generators of equivariant modules, together with generators for the ring of invariants.

In Section 5.3, we show that the ideal interpolation algorithms presented in Chapter 4 can be straighforwardly applied to compute the generating invariants and equivariants of a reflection group. In Section 5.4, given a set of primary invariants for any representation of a finite group $\{r_1, \ldots, r_n\}$, we apply the algorithms in Chapters 3 and 4 to compute both a set of secondary invariants; and free bases of all fundamental equivariant $\mathbb{K}[r_1, \ldots, r_n]$-modules.

In Section 5.5 we elaborate on the idea in Section 5.4 to compute a set of generating invariants simultaneously to the generating equivariants. The Section 5.2 links symmetry adapted basis and fundamental equivariants.

5.2 Symmetry adapted bases and fundamental equivariants

In polynomial rings, symmetry adapted bases can be understood to consist of fundamental equivariants. We recall these notions and we show their connections.

We recall that $\mathfrak{r}^{(1)}, \ldots, \mathfrak{r}^{(n)}$ denote the inequivalent irreducible matrix representations of $\mathcal{G}$ over $\mathbb{C}$; n_ℓ the dimension of $\mathfrak{r}^{(\ell)}$. We call *fundamental equivariant* an $\mathfrak{r}^{(\ell)}$-equivariant, for some $1 \le \ell \le \mathfrak{n}$. Hence a symmetry adapted basis of any invariant subspace of $\mathbb{K}[\mathrm{x}]$ consists of fundamental equivariants. Furthermore the $\mathfrak{r}$-equivariants, for any matrix representation $\mathfrak{r} : \mathcal{G} \to \mathrm{GL}_m(\mathbb{K})$, are linear combinations of fundamental equivariants. Indeed, let $\mathrm{q} \in \mathbb{K}[\mathrm{x}]^m$ be a $\mathfrak{r}$-equivariant and $\mathrm{P} \in \mathbb{K}^{m\times m}$ be the invertible matrix such that

$$\mathrm{P}^{-1}\,\mathfrak{r}(g)\,\mathrm{P} = \mathrm{diag}\left(I_{m_\ell} \otimes \mathfrak{r}^{(\ell)}(g) \mid 1 \le \ell \le \mathfrak{n}\right) \text{ for all } g \in \mathcal{G}.$$

Then $\mathrm{q}\,\mathrm{P}$ is a $(m_1\mathfrak{r}^{(1)} \oplus \ldots \oplus m_{\mathfrak{n}}\mathfrak{r}^{(\mathfrak{n})})$-equivariant, *i.e.,* its components are fundamental equivariants.

Proposition 5.2.1 *For any $1 \le k \le n_\ell$, the $\mathbb{K}[x]^{\mathcal{G}}$-linear maps*

$$\begin{array}{rccc} \phi_k : & \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}} & \to & \mathbb{K}[\mathrm{x}]^{(\ell,k)} \\ & [q_1, \ldots, q_{n_\ell}] & \mapsto & q_k \end{array} \quad \text{and} \quad \begin{array}{rccc} \Phi_k : & \mathbb{K}[\mathrm{x}]^{(\ell,k)} & \to & \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}} \\ & q & \mapsto & \left[\pi^{(\ell)}_{1k}(q), \ldots, \pi^{(\ell)}_{n_\ell k}(q)\right] \end{array}$$

are well defined and inverse of each other.

PROOF: The fact that ϕ_k and Φ_k are $\mathbb{K}[x]^{\mathcal{G}}$-linear is an easy observation from the definition of the maps $\pi^{(\ell)}_{ji}$. We first show that the images of ϕ_k and Φ_k are in

$\mathbb{K}[\mathrm{x}]^{(\ell,k)}$ and $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}}$ respectively. Let $\mathrm{q} = [q_1, \ldots, q_{n_\ell}]$ be an $\mathfrak{r}^{(\ell)}$-equivariant. By definition of equivariance $\mathrm{q} = \rho(g)(\mathrm{q})\,\mathfrak{r}^{(\ell)}(g)^{-1}$. Hence

$$\mathrm{q} = [\rho(g)(q_1), \ldots, \rho(g)(q_{n_\ell})]\,\mathfrak{r}^{(\ell)}(g^{-1}) = \left[\sum_{j=1}^{n_\ell} \mathfrak{r}^{(\ell)}_{j1}(g^{-1})\rho(g)(q_j), \ldots, \sum_{j=1}^{n_\ell} \mathfrak{r}^{(\ell)}_{jn_\ell}(g^{-1})\rho(g)(q_j)\right].$$

Summing over $g \in \mathcal{G}$ we see

$$|\mathcal{G}|\,\mathrm{q} = \left[\sum_{j=1}^{n_\ell}\sum_{g\in\mathcal{G}} \mathrm{r}^{(\ell)}_{j1}(g^{-1})\rho(g)(q_j), \ldots, \sum_{j=1}^{n_\ell}\sum_{g\in\mathcal{G}} \mathrm{r}^{(\ell)}_{jn_\ell}(g^{-1})\rho(g)(q_j)\right]$$

$$= \frac{|\mathcal{G}|}{n_\ell}\left[\sum_{j=1}^{n_\ell} \pi^{(\ell)}_{1j}(q_j), \ldots, \sum_{j=1}^{n_\ell} \pi^{(\ell)}_{n_\ell j}(q_j)\right].$$

Since $\pi^{(\ell)}_{jk} \circ \pi^{(\ell)}_{kl} = \pi^{(\ell)}_{jl}$ we can write

$$\mathrm{q} = \frac{1}{n_\ell}\left[\sum_{j=1}^{n_\ell} \pi^{(\ell)}_{11} \circ \pi^{(\ell)}_{1j}(q_j), \ldots, \sum_{j=1}^{n_\ell} \pi^{(\ell)}_{n_\ell 1} \circ \pi^{(\ell)}_{1j}(q_j)\right] = \left[\pi^{(\ell)}_{11}(\hat{q}), \ldots, \pi^{(\ell)}_{n_\ell 1}(\hat{q})\right], \quad (5.2.1)$$

where $\hat{q} = \dfrac{1}{n_\ell}\displaystyle\sum_{j=1}^{n_\ell} \pi^{(\ell)}_{1j}(q_j)$. It follows that $\phi_k(\mathrm{q}) \in \mathbb{K}[\mathrm{x}]^{(\ell,k)}$.

Now let $q \in \mathbb{K}[\mathrm{x}]^{(\ell,k)}$. By Proposition 2.3.1 we have

$$\rho(g)\left(\pi^{(\ell)}_{jk}(q)\right) = \sum_{i=1}^{n_\ell} \mathfrak{r}^{(\ell)}_{ij}(g)\pi^{(\ell)}_{ik}(q)$$

so that

$$\rho(g)\left(\Phi_k(q)\right) = \left[\sum_{i=1}^{n_\ell} \mathfrak{r}^{(\ell)}_{i1}(g)\pi^{(\ell)}_{ik}(q), \ldots, \sum_{i=1}^{n_\ell} \mathfrak{r}^{(\ell)}_{in_\ell}(g)\pi^{(\ell)}_{ik}(q)\right]$$

$$= \left[\pi^{(\ell)}_{1k}(q), \ldots, \pi^{(\ell)}_{n_\ell k}(q)\right]\mathfrak{r}^{(\ell)}(g) = \Phi_k(q)\,\mathfrak{r}^{(\ell)}(g).$$

Hence $\Phi_k(q) \in \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}}$.

To conclude we show that $\phi_k \circ \Phi_k$ and $\Phi_k \circ \phi_k$ are the identity maps. Let $\mathrm{q} = [q_1, \ldots, q_{n_\ell}] \in \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}}$ By Equation (5.2.1) there exists $\hat{q} \in \mathbb{K}[\mathrm{x}]$ such that $\mathrm{q} = \left[\pi^{(\ell)}_{11}(\hat{q}), \ldots, \pi^{(\ell)}_{n_\ell 1}(\hat{q})\right]$. Hence

$$\Phi_k \circ \phi_k(\mathrm{q}) = \Phi_k\left(\pi^{(\ell)}_{k1}(\hat{q})\right) = \left[\pi^{(\ell)}_{1k} \circ \pi^{(\ell)}_{k1}(\hat{q}), \ldots, \pi^{(\ell)}_{n_\ell k} \circ \pi^{(\ell)}_{k1}(\hat{q})\right] = \mathrm{q}.$$

If now $q \in \mathbb{K}[\mathrm{x}]^{(\ell,k)}$ then $\pi^{(\ell)}_{kk}(q) = q$ so that $\phi_k \circ \Phi_k(q) = q$. □

Corollary 5.2.2 *Consider a set $\mathcal{Q} = \{\mathrm{q}_1, \ldots, \mathrm{q}_m\}$ of $\mathfrak{r}^{(\ell)}$-equivariants, with $\mathrm{q}_i = [q_{i1}, \ldots, q_{in_\ell}]$. The following statements are equivalent:*

1. *$\mathcal{Q}$ is a generating set for $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}}$*

2. *for any given* $1 \le k \le n_\ell$, $\{q_{ik} \,|\, 1 \le i \le m\}$ *is a generating set for* $\mathbb{K}[\mathrm{x}]^{(\ell,k)}$

3. $\{q_{ik} \,|\, 1 \le k \le n_\ell,\ 1 \le i \le m\}$ *is a generating set for* $\mathbb{K}[\mathrm{x}]^{(\ell)}$.

All as $\mathbb{K}[x]^{\mathcal{G}}$*-module.*

PROOF:
1. $\Rightarrow$ 2. Let $q \in \mathbb{K}[\mathrm{x}]^{(\ell,k)}$. Since $\Phi_k(q) \in \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}(\ell)}$, there exist $h_1, \ldots, h_m \in \mathbb{K}[x]^{\mathcal{G}}$ such that $\Phi_k(q) = \sum_{i=1}^m h_i \mathrm{q}_i$. Applying ϕ_k on both sides we obtain $q = \sum_{i=1}^m h_i \phi_k(\mathrm{q}_i) = \sum h_i q_{ik}$.

2. $\Rightarrow$ 1. Take $\mathrm{q} \in \mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}(\ell)}$. Since $\phi_k(\mathrm{q}) \in \mathbb{K}[\mathrm{x}]^{(\ell,k)}$ there are invariant polynomials $h_1, \ldots, h_m$ such that $\phi_k(\mathrm{q}) = \sum_{i=1}^m h_i q_{ik}$. Appplying Φ_k on both sides we obtain $\mathrm{q} = \sum_{i=1}^m h_i \Phi_k(q_{ik})$ and therefore $\Phi_k(q_{1k}) = \mathrm{q}_1, \ldots, \Phi_k(q_{mk}) = \mathrm{q}_m$ form a generating set for $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}(\ell)}$.

2. $\Rightarrow$ 3. Since $\mathbb{K}[\mathrm{x}]^{(\ell)} = \bigoplus_{j=1}^{n_\ell} \mathbb{K}[\mathrm{x}]^{(\ell,j)}$, and $\pi^{(\ell)}_{jk} : \mathbb{K}[\mathrm{x}]^{(\ell,k)} \to \mathbb{K}[\mathrm{x}]^{(\ell,j)}$ is $\mathbb{K}[x]^{\mathcal{G}}$-linear and bijective, and $\pi^{(\ell)}_{jk}(q_{ik}) = q_{ij}$.

3. $\Rightarrow$ 2. Since $\pi^{(\ell)}_{kk} : \mathbb{K}[\mathrm{x}]^{(\ell)} \to \mathbb{K}[\mathrm{x}]^{(\ell,k)}$ is $\mathbb{K}[x]^{\mathcal{G}}$-linear and surjective, and $\pi^{(\ell)}_{kk}(q_{ij})$ equals to q_{ik} or 0 according to whether $j = k$ or not. □

5.3 Fundamental equivariants & invariants by interpolation

A reflection group is a subgroup of $\mathrm{GL}_n(\mathbb{K})$ that is generated by matrices that have precisely one eigenvalue different from 1. Such finite groups enjoy extensive properties (Benson and Grove, 1985; Chevalley, 1955; Kane, 2001). In this section we show that we can deduce generating invariants and equivariants for such group actions from the solution of an ideal interpolation problem as computed in Section 4.6.

For $\xi \in \mathbb{K}^n$ we consider the *evaluation map* $\mathrm{e}_\xi : \mathbb{K}[\mathrm{x}] \to \mathbb{K}$. Then $I = \bigcap_{g \in \mathcal{G}} \ker \mathrm{e}_{g\cdot\xi}$ in $\mathbb{K}[\mathrm{x}]$ is the radical ideal of the orbit of ξ. Algorithm 3 applied to the space of linear forms $\Lambda = \langle\, \mathrm{e}_{g\cdot\xi} \mid g \in \mathcal{G} \,\rangle$ determines a reduced H-basis $\mathcal{H}$ of I and a basis $\mathcal{Q}$ of the orthogonal complement of I^0, the ideal of the leading homogeneous forms of the polynomials in I. $\mathcal{Q}$ can be interpreted as a basis for the quotient $\mathbb{K}[\mathrm{x}]/I$.

For a given group, and its irreducible representations, Algorithm 4 produces furthermore a $\mathcal{H}$ and a $\mathcal{Q}$ that are symmetry adapted. We shall show in this section that a minimal set of generating invariants can be read on such a $\mathcal{H}$ and the generating fundamental equivariants are given by $\mathcal{Q}$.

Proposition 5.3.1 $\langle r - \mathrm{e}_\xi(r) \,|\, r \in \mathbb{K}[x]^{\mathcal{G}} \rangle \subset I$.

PROOF: $\mathrm{e}_{g\cdot\xi}(r - \mathrm{e}_\xi) = 0$ for all $r \in \mathbb{K}[x]^{\mathcal{G}}$ and $g \in \mathcal{G}$. □

The isotropy subgroup $\mathcal{G}_\xi$ of $\xi \in \mathbb{K}^n$ is the set of elements of $\mathcal{G}$ that leave ξ invariant: $\mathcal{G}_\xi = \{ g \in \mathcal{G} \,|\, g \cdot \xi = \xi \}$

Proposition 5.3.2 *The induced representation on $\mathbb{K}[x]/I$ is equivalent to the permutation representation of $\mathcal{G}$ associated to $\mathcal{G}/\mathcal{G}_\xi$.*

PROOF: The ideal $I = \bigcap_{g\in\mathcal{G}} \ker \mathrm{e}_{g\cdot\xi}$ is invariant under the action of $\mathcal{G}$. There is thus an action of $\mathcal{G}$ induced on $\mathbb{K}[x]/I$ that is well defined.

Let $g_1, \ldots, g_r$ be representatives for the classes of $\mathcal{G}/\mathcal{G}_\xi$. Denoting $\xi_i = \vartheta(g_i)(\xi)$ we have that $\xi_i \neq \xi_j$ for $i \neq j$ and $\{\xi_1, \ldots, \xi_r\}$ is the orbit of ξ and hence the zero set of I. Therefore for every $g \in \mathcal{G}$ there exists a permutation $\sigma(g)$ such that

$$\vartheta(g)(\xi_i) = \xi_{\sigma(g)(i)} \text{ for all } \xi_i \in V.$$

Now let $f_1, \ldots, f_r$ be polynomial such $f_i(\xi_j) = \delta_{ij}$. Such polynomials can be found via interpolation and we prove next that they form a basis of $\mathbb{K}[x]/I$. This latter is of dimension r hence we only need to prove that they are linearly independent modulo I. Let $a_1, \ldots, a_r \in \mathbb{K}$ such that $f = \sum_{i=1}^r a_i f_i \in I$. Then for any $1 \leq i \leq r$, $a_i = f(\xi_i) = 0$.

For any $1 \leq i, j \leq r$ we have

$$\rho(g)(f_i)(\xi_j) = f_i \circ \vartheta(g^{-1})(\xi_j) = f_i\left(\xi_{\sigma(g^{-1})(j)}\right) = f_{\sigma(g)(i)}(\xi_j).$$

It follows that $\rho(g)(f_i) - f_{\sigma(g)(i)}$ vanishes on the orbit of ξ and thus $\rho(g)(f_i) - f_{\sigma(g)(i)} \in I$. Hence, in the basis $f_1, \ldots, f_r$, the induced matrix representation of $\mathcal{G}$ on $\mathbb{K}[x]/I$ is given by the permutation σ. □

In particular, if $\mathcal{G}_\xi$ is restricted to the identity then the induced representation on $\mathbb{K}[x]/I$ is equivalent to the regular representation of $\mathcal{G}$ and thus has dimension $|\mathcal{G}|$.

Lemma 5.3.3 *When $\mathcal{G}$ is a reflection group and $\mathcal{G}_\xi$ is restricted to the identity, I^0 is the ideal N generated by all the homogeneous invariants of positive degree.*

PROOF: From Proposition 5.3.1 it follows that $N \subset I^0$. $\mathcal{G}$ being a reflection group, by (Kane, 2001, Theorem 24-1), and originally (Chevalley, 1955), $\mathbb{K}[x]/N$ is equivalent to the regular representation and thus of dimension $|\mathcal{G}|$. By Proposition 5.3.2, $\mathbb{K}[x]/I$, and hence $\mathbb{K}[x]/I^0$, is also of dimension $|\mathcal{G}|$. We can therefore conclude that $I^0 = N$. □

Lemma 5.3.4 *Let J be an ideal generated by k homogeneous invariants of positive degree. Then any homogeneous orthogonal H-basis of J consists of invariant polynomials, and of at most k of them.*

PROOF: We write $J_d = J \cap \mathbb{K}[x]_d$ and $J_{\leq d} = J \cap \mathbb{K}[x]_{\leq d}$ and similarly for $\mathcal{H}$ a set of homogeneous invariants generating J. Recalling $\Psi_d(\cdot)$ the map defined in (4.5.1), we have $J_d = \Psi_d(\mathcal{H}_{<d}) + \langle \mathcal{H}_d \rangle_{\mathbb{K}}$.

We show that for any degree d, any basis for the orthogonal complement Q_d of $\Psi_d(J_{<d})$ in J_d consists of homogeneous invariants in numbers less or equal to the cardinal of $\mathcal{H}_d$.

Let m be the dimension of a complement of $\Psi_d(\mathcal{H}_{<d})$ in J_d. Then $m \leq \mathrm{card}(\mathcal{H}_d)$. We can select $h_1, \ldots, h_m \in \langle \mathcal{H}_d \rangle \subset \mathbb{K}[x]_d^{\mathcal{G}}$ s.t. $J_d = \Psi_d(\mathcal{H}_{<d}) \oplus \langle h_1, \ldots, h_m \rangle_{\mathbb{K}}$.

Consider $q_1, \ldots, q_m$ any basis of Q_d. There is then a nonsingular matrix $(a_{ij}) \in \mathbb{K}^{m\times m}$ such that $q_i = \sum a_{ij} h_j + r_i$ where $r_i \in \Psi_d(\mathcal{H}_{<d})$. Since $\Psi_d(J_{<d})$, J_d and Q_d are invariant, the polynomial $q_i - \pi^{(1)}(q_i) = r_i - \pi^{(1)}(r_i)$ belongs to $Q_d \cap \Psi_d(\mathcal{H}_{<d})$. It therefore is 0, *i.e.*, $q_i = \pi^{(1)}(q_i)$, that is to say $q_i \in \mathbb{K}[x]_d^{\mathcal{G}}$. □

Theorem 5.3.5 *Consider $\mathcal{G} \subset \mathrm{GL}_n(\mathbb{K})$ a reflection group and take $\xi \in \mathbb{K}^n$ with $\mathcal{G}_\xi = \{I_n\}$. Define*

- $I = \bigcap_{g\in\mathcal{G}} \ker \mathrm{e}_{g\cdot\xi}$, *and*
- $Q = \bigcup_{\ell=1}^{\mathfrak{n}} \bigcup_{k=1}^{\mathfrak{n}_\ell} \left\{ q_{1k}^{(\ell)}, \ldots, q_{\mathfrak{n}_\ell k}^{(\ell)} \right\}$ *a symmetry adapted basis of an invariant complement of I.*

Then :

- *Any reduced H-basis of I is given by a set $\{r_1 - \mathrm{e}_\xi(r_1), \ldots, r_n - \mathrm{e}_\xi(r_n)\}$ where $\{r_1, \ldots, r_n\}$ is a set of homogeneous invariants generating $\mathbb{K}[x]^{\mathcal{G}}$*
- *$Q^{(1)} = \{1\}$ and, for $2 \leq \ell \leq \mathfrak{n}$, $Q^{(\ell)}$ consists of $\mathfrak{r}^{(\ell)}$-equivariants $\mathrm{q}_1^{(\ell)}, \ldots, \mathrm{q}_{n_\ell}^{(\ell)}$ freely generating $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module. Writing $\mathrm{q}_i = \left[q_{i1}^{(\ell)}, \ldots, q_{in_\ell}^{(\ell)} \right]$, for $1 \leq i \leq n_\ell$, we have*

$$\mathbb{K}[\mathrm{x}] = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k,i=1}^{n_\ell} \mathbb{K}[x]^{\mathcal{G}}\, q_{ik}^{(\ell)}.$$

PROOF: When $\mathcal{G}$ is a reflection group, $\mathbb{K}[x]^{\mathcal{G}}$ is generated by n algebraically independent homogeneous invariants (Chevalley, 1955), (Kane, 2001, Theorem 18-1). The ideal N generated by all homogenous invariants of positive degree is a zero dimensional ideal and thus any of its H-basis cannot have less then n elements. It thus follows from Lemma 5.3.4, that a reduced H-basis of Nconsists of exactly n homogeneous invariants. By the classical argument of Hilbert's finiteness theorem (Sturmfels, 2007, Theorem 2.1.3), a set of homogeneous invariants generating N as an ideal is also generating for $\mathbb{K}[x]^{\mathcal{G}}$ as an algebra.

$\mathcal{H}$ is a H-basis of I iff $\mathcal{H}^0$ is a H-basis of $I^0 = N$. It follows that for any orthogonal H-basis of I $\mathcal{H}^0 = \{r_1, \ldots, r_n\}$ where $r_1, \ldots, r_n$ form a generating set of homogeneous invariants. With Proposition 5.3.1 we can conclude that the shape of a reduced H-basis is as stated.

On the other hand, Q spans the orthogonal complement of I^0 (Corollary 4.2.3). By Lemma 5.3.3 $I^0 = N$, and thus $\mathbb{K}[\mathrm{x}] = N \oplus \langle Q \rangle_{\mathbb{K}}$. Nakayama's lemma for graded algebra (Kane, 2001, Lemma 17-5) thus implies that

$$\mathbb{K}[\mathrm{x}] = \sum_{\ell=1}^{\mathfrak{n}} \sum_{k=1}^{n_\ell} \left(\sum_{i}^{n_\ell} \mathbb{K}[x]^{\mathcal{G}} \, q_{ik}^{(\ell)} \right).$$

By (Kane, 2001, Lemma 18-3) the set $\left\{ q_{ik}^{(\ell)} \,|\, 1 \le \ell \le \mathfrak{n},\ 1 \le i,k \le n_\ell \right\}$ is furthermore algebraically independent when $\mathcal{G}$ is a reflection group. Moreover $\mathbb{K}[\mathrm{x}]^{(\ell,1)} = \sum_i^{n_\ell} \mathbb{K}[x]^{\mathcal{G}} \, q_{1i}^{(\ell)}$, and thus Proposition 5.2.2 implies that $Q^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{in_\ell}^{(\ell)} \right] \mid 1 \le i \le \mathfrak{n}_\ell \right\}$ is a freely generating $\mathbb{K}[x]^{\mathcal{G}}_{\mathfrak{r}^{(\ell)}}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module. □

Example 5.3.6 *Consider the representation ρ of the dihedral group*

$$D_8 = \left\{ \delta^\alpha \sigma^\beta \mid \delta^8 = \sigma^2 = (\sigma\delta)^2 = 1 \right\}$$

of order 16 given by the matrices

$$\vartheta(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \vartheta(\delta) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

D_8 has 7 inequivalent irreducible representations, four of dimension 1 and three of dimension 2.

Applying Algorithm 4 to $\Lambda = \langle \, \mathbb{e}_{g\cdot\xi} \mid g \in \mathcal{G} \, \rangle$ where $\xi = {}^t[a,b]$ we obtain the following H-basis $\mathcal{H}$ of I and s.a.b $\mathcal{Q} = \bigcup_{\ell=1}^{7} \mathcal{Q}^{(\ell)}$ of the orthogonal complement of I^0

$$\mathcal{H} = \left\{ x^2+y^2-(a^2+b^2),\ x^8-28x^6y^2+70x^4y^4-28x^2y^6+y^8-\left(a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8\right) \right\}$$

$$Q^{(1)} = \{1\}, \quad Q^{(2)} = \left\{ xy\left(x^6-7x^4y^2+7x^2y^4-y^6\right) \right\},$$

$$Q^{(3)} = \left\{ x^4-6x^2y^2+y^4 \right\}, \quad Q^{(4)} = \left\{ xy\left(x^2-y^2\right) \right\},$$

$$Q^{(5)} = \left\{ [x,y],\ \left[x\left(x^6-21x^4y^2+35x^2y^4-7y^6\right), y\left(-7x^6+35x^4y^2-21x^2y^4+y^6\right) \right] \right\},$$

$$Q^{(6)} = \left\{ \left[x^2-y^2, 2xy\right],\ \left[x^6-15x^4y^2+15x^2y^4-y^6, 2xy\left(3x^4-10x^2y^2+3y^4\right)\right] \right\},$$

$$Q^{(7)} = \left\{ \left[x\left(x^2-3y^2\right), y\left(3x^2-y^2\right) \right],\ \left[x\left(-x^4+10x^2y^2-5y^4\right), y\left(5x^4-10x^2y^2+y^4\right) \right] \right\}.$$

As Theorem 5.3.5 asserts $\mathcal{H}^0 = \{x^2+y^2,\ x^8-28x^6x^2+70x^4y^4-28x^2y^6+y^8\}$ is a generating set for $\mathbb{K}[\mathrm{x}]^{D_8}$ and, for $2 \le \ell \le 7$, $Q^{(\ell)}$ is a set of free generators for the $\mathfrak{r}^{(\ell)}$-equivariants.

Example 5.3.7 *Consider the action of the group of symmetries of the cube $B_3 = S_4 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$*

$$B_3 = \left\{ \delta^\ell \sigma^m \alpha \mid 0 \le i,j,k,m \le 1,\ 0 \le \ell \le 2 \right\}$$

in $\mathbb{R}^3$ *which is defined by the matrices.*

$$\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \alpha = \begin{pmatrix} (-1)^i & 0 & 0 \\ 0 & (-1)^j & 0 \\ 0 & 0 & (-1)^k \end{pmatrix}.$$

B_3 has order 48 and 10 inequivalent irreducible representations, all absolutely irreducible, whose dimensions are $\{1, 1, 1, 1, 2, 2, 3, 3, 3, 3\}$. *Applying Algorithm 4 to* $\Lambda = \langle \mathbb{e}_{g\cdot\xi} \mid g \in B_3 \rangle$ *where* $\xi = {}^t[1, 2, 3]$ *we obtain a H-basis* $\mathcal{H}$ *of* I *and a symmetry adapted basis of the orthogonal complement* $Q = \bigcup_{\ell=1}^{10} Q^{(\ell)}$ *of* I^0.

$$\mathcal{H}^0 = \left\{x^2 + y^2 + z^2, x^2y^2 + x^2z^2 + y^2z^2, x^2y^2z^2\right\}$$

is a generating set of invariants and, for $2 \le \ell \le 10$, $Q^{(\ell)}$ *freely generate the* $\mathfrak{r}^{(\ell)}$-

equivariants, with

$$Q^{(1)} = \{1\}, \quad Q^{(2)} = \{xyzp_xp_yp_z\}, \quad Q^{(3)} = \{p_xp_yp_z\}, \quad Q^{(4)} = \{xyz\},$$

$$Q^{(5)} = \left\{ \begin{array}{c} \left[-\sqrt{3}xyzp_y, xyzp_x - p_z\right], \\ \left[\begin{array}{c} \frac{\sqrt{3}}{3}\left(x^4 + 6x^2y^2 - 12x^2z^2 - 2y^4 + 6y^2z^2 + z^4\right), \\ -p_y\left(x^2 - 6y^2 + z^2\right) \end{array} \right] \end{array} \right\},$$

$$Q^{(6)} = \left\{ \begin{array}{c} \left[-\sqrt{3}xyzp_y, xyzp_x - p_z\right], \\ \left[\begin{array}{c} \frac{2\sqrt{3}}{3}p_y\left(10y^2 - 3x^2 - 3z^2\right), \\ 3x^4 + 10x^2y^2 - 30x^2z^2 - 6y^4 + 10y^2z^2 + 3z^4 \end{array} \right] \end{array} \right\},$$

$$Q^{(7)} = \left\{ \begin{array}{c} [yzp_x, xzp_y, xyp_z], \\ \left[yzp_x\left(y^2 + z^2 - 10x^2\right), xzp_y\left(z^2 + x^2 - 10y^2\right), xyp_z\left(x^2 + y^2 - 10z^2\right)\right], \\ \left[\begin{array}{c} yzp_x\left(5x^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2\right), \\ xzp_y\left(5y^4 - 3y^2z^2 - 3x^2y^2 + x^2z^2\right), \\ xyp_z\left(5z^4 - 3x^2z^2 - 3y^2z^2 + x^2y^2\right) \end{array} \right] \end{array} \right\}$$

$$Q^{(8)} = \left\{ \begin{array}{c} [x, y, z], \\ \left[x\left(3y^2 + 3z^2 - 2x^2\right), y\left(3z^2 + 3x^2 - 2y^2\right), z\left(3x^2 + 3y^2 - 2z^2\right)\right], \\ \left[\begin{array}{c} x\left(2x^4 - 5\left(y^4 + z^4\right) - 10\left(x^2\left(y^2 + z^2\right) - 6y^2z^2\right)\right), \\ y\left(2y^4 - 5\left(z^4 + x^4\right) - 10\left(y^2\left(z^2 + x^2\right) - 6x^2z^2\right)\right), \\ z\left(2z^4 - 5\left(x^4 + y^4\right) - 10\left(z^2\left(x^2 + y^2\right) - 6x^2y^2\right)\right) \end{array} \right] \end{array} \right\}$$

$$Q^{(9)} = \left\{ \begin{array}{c} [yz, xz, xy], \\ \left[yz\left(y^2 + z^2 - 6x^2\right), xz\left(z^2 + x^2 - 6y^2\right), xy\left(x^2 + y^2 - 6z^2\right)\right], \\ \left[\begin{array}{c} yz\left(20y^2z^2 + 30x^2\left(x^2 - y^2 - z^2\right) - 3\left(y^4 + z^4\right)\right), \\ xz\left(20x^2z^2 + 30y^2\left(y^2 - z^2 - x^2\right) - 3\left(z^4 + x^4\right)\right), \\ xy\left(20x^2y^2 + 30z^2\left(z^2 - x^2 - y^2\right) - 3\left(x^4 + y^4\right)\right) \end{array} \right] \end{array} \right\}$$

$$Q^{(10)} = \left\{ \begin{array}{c} [xp_x, yp_y, zp_z], \\ \left[xp_x\left(p_y - p_z\right), yp_y\left(p_z - p_x\right), zp_z\left(p_x - p_y\right)\right], \\ \left[\begin{array}{c} xp_x\left(3x^4 - 5x^2y^2 - 5x^2z^2 + 15y^2z^2\right), \\ yp_y\left(3y^4 - 5y^2z^2 - 5x^2y^2 + 15x^2z^2\right), \\ zp_z\left(3z^4 - 5x^2z^2 - 5y^2z^2 + 15x^2y^2\right) \end{array} \right] \end{array} \right\}$$

where $p_z = x^2 - y^2, p_y = z^2 - x^2, p_x = y^2 - z^2$.

We shall now illustrate that Theorem 5.3.5 fails when not applied to a reflection group.

Example 5.3.8 *Consider the three dimensional representation of D_6 given in Example 5.5.2. Contrary to the classical representation of D_6, this is not a reflection group since -1 is a double eigenvalue of $\vartheta(\sigma)$.*

The output Algorithm 4 applied to $\Lambda = \langle \mathbb{e}_{g\cdot\xi} \mid g \in \mathcal{G} \rangle$ consist of a symmetry adapted H-basis $\mathcal{H}$ of I and $\mathcal{Q}$ a s.a.b. of the orthogonal complement of I^0. For $\xi = {}^t[a, b, c]$

$$\mathcal{Q} = \left\{ \begin{array}{c} \mathcal{Q}^{(2,1)} = \{1\}, \quad \mathcal{Q}^{(2,1)} = \{z\}, \quad \mathcal{Q}^{(3,1)} = \{x^3 - 3xy^2\}, \\ \mathcal{Q}^{(4,1)} = \{3x^2y - y^3\}, \{\mathcal{Q}^{(5,1)} = \{x, yz\}, \mathcal{Q}^{(5,2)} = \{y, -xz\}\}, \\ \left\{\mathcal{Q}^{(6,1)} = \left\{x^2 - y^2, 2xyz\right\}, \mathcal{Q}^{(6,2)} = \left\{2xy, -x^2z + y^2z\right\}\right\} \end{array} \right\}$$

and

$$\mathcal{H}^{(1,1)} = \{x^2 + y^2 - (a^2 + b^2), z^2 - c^2\},$$

$$\mathcal{H}^{(2,1)} = \varnothing, \quad \mathcal{H}^{(3,1)} = \{a\left(a^2 - b^2\right)yz\left(3x^2 - y^2\right) - bc\left(3a^2 - b^2\right)x\left(x^2 - y^2\right)\},$$

$$\mathcal{H}^{(4,1)} = \{b\left(b^2 - 3a^2\right)xz\left(x^2 - 3y^2\right) - ac\left(a^2 - 3b^2\right)y\left(y^2 - 3x^2\right)\}, \quad \mathcal{H}^{(5,1)} = \varnothing,$$

$$\mathcal{H}^{(6,1)} = \left\{ \begin{array}{l} c\left(a^2 + b^2\right)^2\left(x^4 - 6x^2y^2 + y^4\right) + 4ab\left(a^2 - 3b^2\right)\left(3a^2 - b^2\right)xyz \\ +c\left(a^2 - b^2\right)\left(a^4 - 14a^2b^2 + b^4\right)\left(x^2 - y^2\right) \end{array} \right\},$$

$$\mathcal{H}^{(6,2)} = \left\{ \begin{array}{l} 2c\left(a^2 + b^2\right)^2 xy\left(x^2 - y^2\right) - ab\left(3a^2 - b^2\right)\left(a^2 - 3b^2\right)\left(x^2 - y^2\right)z \\ +c\left(a^2 - b^2\right)\left(a^4 - 14a^2b^2 + b^4\right)xy \end{array} \right\}$$

$\mathcal{Q}$ has 12 elements, as predicted by Proposition 5.3.2. Yet we saw in Example 5.5.2 that a minimal set of generating equivariants has 22 elements.

In this case $\mathcal{H}^0$ has non invariant elements. For instance

$$a\left(a^2 - b^2\right)yz\left(3x^2 - y^2\right) - bc\left(3a^2 - b^2\right)\left(x^3 - xy^2\right) \in \mathcal{H}^{(3)}.$$

Its leading form $yz\left(3x^2 - y^2\right) \in \mathcal{H}^0$ is not an invariant and one can check that it does not belong to N. Hence $N \subsetneq I^0$. Also we can see that $\left(\mathcal{H}^0\right)^{(1)}$ consists of only 2 elements so we can not obtain from $\mathcal{H}$ a fundamental set of invariants.

5.4 Fundamental equivariants from invariants

In this section we show how to determine generating invariants and fundamental equivariants assuming we have sufficiently many invariants already. We then show that this construction provides a Hironaka decomposition of $\mathbb{K}[x]^{\mathcal{G}}$ and $\mathbb{K}[x]^{(\ell)}$,

$1 \le \ell \le \mathfrak{n}$, if it is applied to a set of *primary invariants* $r_1, \dots, r_n$. Indeed, working with the concepts of *primary and secondary invariants* (Sturmfels, 2007, Section 2.3) allows to have a unique representation of invariants and equivariants, or any element of $\mathbb{K}[\mathrm{x}]$, seen as a $\mathbb{K}[r_1, \dots, r_n]$-module.

Theorem 5.4.1 *Consider $\mathcal{H} = \{h_1, \dots, h_k\} \subset \mathbb{K}[x]^{\mathcal{G}}$ a set of homogeneous invariants of positive degree such that the ideal $H = \langle h_1, \dots, h_k \rangle$ is a zero dimensional.*

Consider $\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{Q}^{(\ell)}$ a symmetry adapted basis of an invariant complement of H in $\mathbb{K}[\mathrm{x}]$, where $\mathcal{Q}^{(\ell)}$ consists of the $\mathfrak{r}^{(\ell)}$-equivariants $\mathrm{q}_i^{(\ell)} = \left[q_{i1}^{(\ell)}, \dots, q_{in_\ell}^{(\ell)}\right]$, for $1 \le i \le m_\ell$.

Then $\mathcal{Q}^{(\ell)}$ generates $\mathbb{K}[\mathrm{x}]^{(\ell)}$ as a $\mathbb{K}[h_1, \dots, h_k]$-module. In particular $\mathcal{H} \cup \mathcal{Q}^{(1)}$ is a generating set of invariants. Also $\sum_{\ell=1}^{\mathfrak{n}} n_\ell m_\ell \ge |\mathcal{G}|$.

PROOF: H is $\mathcal{G}$-invariant and thus admits an invariant complement in $\mathbb{K}[\mathrm{x}]$. As a basis of such a complement $\langle \mathcal{Q} \rangle_{\mathbb{K}}$ can be identified with $\mathbb{K}[\mathrm{x}]/H$. Thus

$$\mathbb{K}[\mathrm{x}]/H \cong \langle Q \rangle_{\mathbb{K}} = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \bigoplus_{i}^{m_\ell} \mathbb{K}\, q_{ik}^{(\ell)}.$$

By Nakayama's lemma for graded algebra (Derksen and Kemper, 2015, Lemma 3.7.1).

$$\mathbb{K}[\mathrm{x}] = \sum_{\ell=1}^{\mathfrak{n}} \sum_{k=1}^{n_\ell} \left(\sum_{i}^{m_\ell} \mathbb{K}[h]\, q_{ik}^{(\ell)} \right).$$

With the notations of (Serre, 1977), for all ℓ and k, $q_{ik}^{(\ell)}$ belongs to $\mathbb{K}[\mathrm{x}]^{(\ell,k)} = \pi_{(}^{(kk)}\mathbb{K}[\mathrm{x}])$. These subspaces are in direct sum with each other. Hence two subspaces $\left(\sum_i \mathbb{K}[x]^{\mathcal{G}}\, q_{ik}^{(\ell)}\right)$ for different k's have an intersection restricted to $\{0\}$. We can thus replace the two first $\sum$ by a $\oplus$ so that

$$\mathbb{K}[\mathrm{x}] = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \left(\sum_{i}^{n_\ell} \mathbb{K}[h]\, q_{ik}^{(\ell)} \right) \quad \text{and thus} \quad \mathbb{K}[\mathrm{x}]^{(\ell)} = \bigoplus_{k=1}^{n_\ell} \left(\sum_{i}^{n_\ell} \mathbb{K}[h]\, q_{ik}^{(\ell)} \right).$$

In particular $\mathbb{K}[x]^{\mathcal{G}} = \mathbb{K}[\mathrm{x}]^{(1)} = \sum_i^{n_\ell} \mathbb{K}[h]\, q_{ik}^{(1)}$.

Let N be the ideal generated by all the homogeneous invariants of positive degree. By (Kane, 2001, Theorem 17-5 and Propositions 18-3,4,5), $\dim_{\mathbb{K}} \mathbb{K}[\mathrm{x}]/N \ge |\mathcal{G}|$. Since $H \subset N$ the announced inequality holds. □

A particular case of this statement is when $\mathcal{H}$ is a generating set of invariants of positive degree. Then the $\mathfrak{r}^{(\ell)}$-equivariant $\mathrm{q}_i^{(\ell)}$, $1 \le i \le m_\ell$, generate $\mathbb{K}[\mathrm{x}]^{(\ell)}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module. Also $\mathcal{Q}^{(1)} = \{1\}$ then.

Corollary 5.4.2 *When $\mathcal{G}$ is a reflection group and $\mathcal{H}$ is a generating set of invariants then $m_\ell = n_\ell$ and the $\mathfrak{r}^{(\ell)}$-equivariant $\mathrm{q}_i^{(\ell)}$, $1 \le i \le n_\ell$ freely generate $\mathbb{K}[\mathrm{x}]^{(\ell)}$ so that*

$$\mathbb{K}[\mathrm{x}] = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \bigoplus_{i}^{n_\ell} \mathbb{K}[x]^{\mathcal{G}}\, q_{ik}^{(\ell)}$$

PROOF: By (Kane, 2001, Proposition 18-3,4,5 and Lemma 18-3): the homogeneous representatives of $\mathbb{K}[x]/H$ are independent over $\mathbb{K}[x]^{\mathcal{G}}$ if and only if $\mathcal{G}$ acts as a reflection group on $\mathbb{K}^n$. When $\mathcal{G}$ is a reflection group, the induced representation on $\mathbb{K}[x]/N$ is equivalent to the regular representation of $\mathcal{G}$ (Kane, 2001, Theorem 24-1). Hence then $m_\ell = n_\ell$. □

Of particular interest is the case where $\mathcal{H} = \{h_1, \ldots, h_n\}$ is a set of primary invariants, *i.e.,* a set of homogeneous parameters for $\mathbb{K}[x]^{\mathcal{G}}$. In particular $\{h_1, \ldots, h_n\}$ are then algebraically independent and $\mathbb{K}[h] = \mathbb{K}[h_1, \ldots, h_n]$ is a polynomial algebra. The secondary invariants $s_1, \ldots, s_m$ are then a free basis of $\mathbb{K}[x]^{\mathcal{G}}$ viewed as a $\mathbb{K}[h]$-module. Any $p \in \mathbb{K}[x]^{\mathcal{G}}$ can be written in a unique way as $p = p_1 s_1 + \ldots + p_m s_m$ where $p_i \in \mathbb{K}[h]$. This is a so called Hironaka decomposition of $\mathbb{K}[x]^{\mathcal{G}}$. The following results shows how we can actually obtain a Hironaka decomposition of $\mathbb{K}[\mathrm{x}]$ as a $\mathbb{K}[h]$-module.

Theorem 5.4.3 *Consider $h_1, \ldots, h_n$ a set of primary invariants. Then $|\mathcal{G}|$ divides the product of their degrees $\prod_{i=1}^n \deg(r_i)$. Let m be their quotient. If $\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{Q}^{(\ell)}$ is a symmetry adapted basis of a $\mathcal{G}$-invariant complement of the ideal $\langle h_1, \ldots, h_n \rangle$ in $\mathbb{K}[\mathrm{x}]$, then $\mathcal{Q}^{(\ell)}$ splits into $m\, n_\ell$ vectors of $\mathbb{K}[\mathrm{x}]^{n_\ell}$ that are $\mathfrak{r}^{(\ell)}$-equivariants and form a free basis of $\mathbb{K}[\mathrm{x}]^{(\ell)}$ seen as a $\mathbb{K}[h]$-module. In particular $\mathcal{Q}^{(1)}$ consists of a set of secondary invariants for $h_1, \ldots, h_n$.*

PROOF: By (Sturmfels, 2007, Proposition 2.3.6), $|\mathcal{G}|$ divides $\prod_{i=1}^n \deg(r_i)$ and the number of secondary invariants m is their quotient. With an argument based on the Molien series of the $\mathbb{K}[\mathrm{x}]^{(\ell)}$, (Stanley, 1979, Proposition 4.9) proves that $\mathbb{K}[\mathrm{x}]/\langle h_1, \ldots, h_n \rangle$ is equivalent to m times the regular representation of $\mathcal{G}$. Hence $m n_\ell^2$ is the dimension of $\mathbb{K}[\mathrm{x}]^{(\ell)}/\langle h_1, \ldots, h_n \rangle \mathbb{K}[\mathrm{x}]^{(\ell)}$ as a $\mathbb{K}$-vector space.

Nakayama's lemma for graded algebra (Derksen and Kemper, 2015, Lemma 3.7.1) ensures that $\mathcal{Q}$ generates $\mathbb{K}[\mathrm{x}]$ as a $\mathbb{K}[h]$-module. We have

$$\mathbb{K}[\mathrm{x}] = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \sum_{i=1}^{m n_\ell} \mathbb{K}[h]\, q_{ik}^{(\ell)}$$

since the components $\mathbb{K}[\mathrm{x}]^{(\ell)}$ are mutually orthogonal.

On the other hand $\mathbb{K}[\mathrm{x}]$ is finitely generated over $\mathbb{K}[x]^{\mathcal{G}}$ so that $h_1, \ldots, h_n$ also form a homogeneous system of parameters for $\mathbb{K}[\mathrm{x}]$. Since $\mathbb{K}[\mathrm{x}]$ is Cohen-Macaulay, $\mathbb{K}[\mathrm{x}]$ is a free $\mathbb{K}[h]$-module. If $\eta_1, \ldots, \eta_k$ is a free basis, then their images in $\mathbb{K}[\mathrm{x}]/\langle r_1, \ldots, r_n \rangle$ span this $m|\mathcal{G}|$-dimensional vector space. Hence the rank of $\mathbb{K}[\mathrm{x}]$ as a $\mathbb{K}[h]$-module is at least $m|\mathcal{G}|$. It follows that $\mathcal{Q}$ is a free basis. □

If we have a set of primary invariants we can compute thus the secondary invariants and fundamental equivariants with the following algorithm that is a straightforward application of the Section 4.7. Note that for reflection group we can pick the primary invariants so that $\prod_{i=1}^n \deg(r_i) = |\mathcal{G}|$ (Sturmfels, 2007, Corolary 2.4.5). Hence $m = 1$ and $\mathbb{K}[x]^{\mathcal{G}} = \mathbb{K}[r_1, \ldots, r_n]$.

Algorithm 5
Fundamental equivariants

INPUT: A set $\{h_1, \dots, h_n\}$ of primary invariants.

OUTPUT: $Q^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \dots, q_{in_\ell}^{(\ell)} \right] \mid 1 \le i \le mn_\ell \right\}$, $1 \le \ell \le \mathfrak{n}$, a set $\mathfrak{r}^{(\ell)}$-equivariant that form a free basis for $\mathbb{K}[\mathrm{x}]^{(\ell)}$ viewed as a free $\mathbb{K}[h_1, \dots, h_n]$-module.

Compute a Gröbner basis B of $\langle h_1, \dots, h_n \rangle$, according to any given term order

$\lambda_1, \dots, \lambda_{m|\mathcal{G}|} \in \mathbb{K}[x]^*$ the coefficient forms for the normal form w.r.t. B, where $m|\mathcal{G}|$ is the cardinality of the normal set of monomials w.r.t. B

Compute $Q = \bigcup_{\ell=1}^{\mathfrak{n}} Q^{(\ell)}$ a s.a.b of the least interpolation space for $\langle \lambda_1, \dots, \lambda_{m|\mathcal{G}|} \rangle$, applying Algorithm 4.

The above algorithm can also be applied to a set of invariants $\mathcal{H}$ that satisfies only the hypotheses of Theorem 5.4.1. It would still output generating sets of fundamental invariants. But they would not be freely generating.

We have to point out though that with a set of homogeneous (invariant) polynomials always form a H-basis. One can thus compute more directly a basis of the orthogonal complement of the generated ideal, with prolongation. Such an approach is taken in next section where we actually compute generating invariants *on the fly*.

There are actually several algorithms to compute primary invariants. In (Sturmfels, 2007, Algorithm 2.5.8) and (Gatermann, 2000, Algorithm 2.1.5) algorithms are proposed to extract primary invariants from a generating set. They generate invariants of increasing degree whose variety gets smaller each time until a set of algebraically independent homogeneous invariants is found. If the obtained set is composed of n elements, then it is a set of primary invariants. Otherwise sets of n elements weighted over the computed invariant are randomly generated until an algebraically independent set is found. Dade's algorithm (Sturmfels, 2007, Algorithm 2.5.14) is a more direct approach based on linear algebra. It constructs primary invariants taking products over $\mathcal{G}$–orbits of linear forms, such that all the degrees of the computed invariants are divisors of the group order. On the other hand (Kemper, 1999, Algorithm 2) looks for a set of primary invariants $\{f_1, \dots, f_n\}$ that minimize $\Pi_{i=1}^n \deg(f_i)$, and then $\sum_{i=1}^n \deg(f_i)$. The algorithm iterates over $\mathbb{N}^n$ following the described pre-order and stops when a n-tuple $d_1, \dots, d_n$ is found such that there exist a set of primary invariants with those degrees.

Example 5.4.4 *The alternating group A_n is the group of even permutations on a set of n elements. It has order $\frac{n!}{2}$. We consider here the alternating group A_4 consisting of 12 elements and generated by the cycles $\{(2,4,3), (1,3)(2,4), (1,2)(3,4)\}$.*

We consider the representation of A_4 in $\mathbb{R}^3$ given by the matrices

$$\rho((2,4,3)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \rho((1,3)(2,4)) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \rho((1,2)(3,4)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A_4 has four irreducible representations, three of dimension 1 and one of dimension 3. Two of the irreducible representations of dimension 1 are of complex type and conjugate to each other. Since we are dealing with a real representation, we shall consider the 3 irreducible real representation of A_4 given by the two absolutely irreducible ones and the one obtained by the combination of the two of complex type. A set of primary invariants for A_4 is given by

$$\mathcal{H} = \left\{x^2 + y^2 + z^2, xyz, x^2y^2 + x^2z^2 + y^2z^2\right\}.$$

The Gröbner basis and the normal sets of the ideal generated by $\mathcal{H}$ w.r.t a degree reverse lexicographical order are given by

$$\mathcal{B} = \left\{x^2 + y^2 + z^2, xyz, y^3z + yz^3, y^4 + y^2z^2 + z^4, z^5\right\}$$

$$\mathcal{N} = \left\{ \begin{array}{c} 1, z, y, x, z^2, yz, xz, y^2, xy, z^3, yz^2, xz^2, y^2z, y^3, xy^2, z^4, \\ yz^3, xz^3, y^2z^2, xy^3, yz^4, xz^4, y^2z^3, y^2z^4 \end{array} \right\}.$$

We define the linear forms $\lambda_1, \ldots, \lambda_{24}$ so that the normal form of $p \in \mathbb{K}[x]$ w.r.t $\mathcal{B}$ is $\lambda_1(p)\, n_1 + \ldots + \lambda_{24}(p) n_{24}$ where $n_i \in \mathcal{N}$ and let $\Lambda = \langle \lambda_i \mid 1 \le i \le 24 \rangle$. Then Algorithm 5 computes the secondary invariants and fundamental equivariants $\mathcal{Q}$

$$\mathcal{S} = \mathcal{Q}^{(1)} = \{1, (y^2 - z^2)(z^2 - x^2)(x^2 - y^2)\}$$

$$\mathcal{Q}^{(2)} = \{2x^2 - y^2 - z^2, y^2 - z^2, (y^2 - x^2)\, z^2, x^4 - y^4, 6x^2(y^2 - z^2) - y^4 + z^4\}$$

$$\mathcal{Q}^{(3)} = \left\{ \begin{array}{c} [x, y, z], [yz, xz, xy], [x(x^2 - 3z^2), y(y^2 - 3x^2), z(z^2 - 3y^2)], \\ [x(y^2 - z^2), y(z^2 - x^2), z(x^2 - y^2)], [yz(y^2 - z^2), xz(z^2 - x^2), xy(x^2 - y^2)], \\ [2x^3(y^2 - z^2) - x(y^4 - z^4), 2y^3(z^2 - x^2) - y(z^4 - x^4), 2z^3(x^2 - y^2) - z(x^4 - y^4)] \end{array} \right\}$$

Example 5.4.5 *Consider the action of the cyclic group C_m in $\mathbb{C}^n$ given by*

$$C_m = \left\{ \operatorname{diag}\left(e^{\frac{2ik\pi}{m}}, \ldots, e^{\frac{2ik\pi}{m}}\right), 0 \le k \le n-1 \right\}.$$

$\mathbb{C}[x]^{C_m}$ is generated by all terms of degree m (Sturmfels, 2007, Proposition 2.1.5) and $\mathcal{H} = \{x_1^m, \ldots, x_n^m\}$ is set of primary invariants. $\mathcal{H}$ is also a Gröbner basis with respect to any term order. The normal set consists of the terms $\mathcal{N} = \{x^\alpha y^\beta z^\gamma \mid 0 \le \alpha, \beta, \gamma \le m-1\}$. C_m has m inequivalent irreducible representations of dimensions 1. For $1 \le \ell \le m$, the generating fundamental equivariants are given by $\mathcal{Q}^{(\ell)} = \{x^\alpha y^\beta z^\gamma \,|\, \alpha + \beta + \gamma = m + 1 - \ell\}$. Observe $\mathcal{N} = \bigcup_{\ell=1}^{m} Q^{(\ell)}$ and this corroborates Theorem 5.4.3.

5.5 Simultaneous computation of invariants and equivariants

For a row vector of polynomials $\mathrm{h} = \begin{bmatrix} h_1 & \dots & h_m \end{bmatrix} \in \mathbb{K}[\mathrm{x}]^m$, with h_i of degree d_i, the prolongation map to degree d defined in (4.5.1) is given by

$$\begin{array}{rccc} \psi_{d\,\mathrm{h}}: & \mathbb{K}[\mathrm{x}]_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d-d_m} & \to & \mathbb{K}[\mathrm{x}]_d \\ & (g_1, \ldots, g_m) & \mapsto & \sum_{i=1}^{m} g_i h_i \end{array} .$$

The restriction of this map to $\mathbb{K}[\mathrm{x}]^{(\ell,1)}$ is

$$\begin{array}{rccc} \psi^{(\ell,1)}_{d\,\mathrm{h}}: & \mathbb{K}[\mathrm{x}]^{(\ell,1)}_{d-d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]^{(\ell,1)}_{d-d_m} & \to & \mathbb{K}[\mathrm{x}]_d \\ & (g_1, \ldots, g_m) & \to & \sum_{i=1}^{m} g_i h_i \end{array} .$$

For a set $\mathcal{H} = \{h_1, \ldots, h_m\}$ we shall denote by $\Psi_d(\mathcal{H})$ and $\Psi^{(\ell,1)}_d(\mathcal{H})$ the respective images of $\psi_{d\,\mathrm{h}}$ and $\psi^{(\ell,1)}_{d\,\mathrm{h}}$ where $\mathrm{h} = [h_1, \ldots, h_m]$. Whether $\mathcal{H} \subset \mathbb{K}[x]^{\mathcal{G}}$ or $\mathcal{Q} \subset \mathbb{K}[\mathrm{x}]^{(\ell,1)}$ both $\Psi^{(\ell,1)}_d(\mathcal{H})$ and $\Psi^{(1)}_d(\mathcal{Q})$ are contained in $\mathbb{K}[\mathrm{x}]^{(\ell,1)}_d$.

Lemma 5.5.1 *Consider $\mathcal{H} \subset \mathbb{K}[x]^{\mathcal{G}}_{\le d} \setminus \mathbb{K}$ and $\mathcal{Q} \subset \mathbb{K}[\mathrm{x}]^{(\ell,1)}_{\le d-1}$. Assume that $\Psi^{(1)}_{d-1}(\mathcal{Q}) = \mathbb{K}[\mathrm{x}]^{(\ell,1)}_{\le d-1}$ and that $\Psi^{(1)}_e(\mathcal{H}) = \mathbb{K}[x]^{\mathcal{G}}_e$ for all $1 \le e \le d$. Then $\Psi^{(\ell,1)}_d(\mathcal{H}) = \Psi^{(1)}_d(\mathcal{Q})$.*

PROOF: Let $q = \sum_{h_i \in \mathcal{H}} h_i p_i \in \Psi^{(\ell,1)}_d(\mathcal{H})$ with $p_i \in \mathbb{K}[\mathrm{x}]^{(\ell,1)}_{\le d}$. Since $\Psi^{(1)}_{d-1}(\mathcal{Q}) = \mathbb{K}[\mathrm{x}]^{(\ell,1)}_{\le d-1}$, for any i there are polynomials $r_1, \ldots, r_{|\mathcal{Q}|} \in \mathbb{K}[\mathrm{x}]^{\mathcal{G}}$ such that $p_i = \sum_{q_i \in Q} r_i q_i$ so we have

$$q = \sum_{h_i \in \mathcal{H}} h_i \sum_{q_j \in Q} r_j q_j = \sum_{q_j \in Q} q_j \sum_{h_i \in \mathcal{H}} r_j h_i \in \Psi^{(1)}_d(\mathcal{Q}),$$

hence $\Psi^{(\ell,1)}_d(\mathcal{H}) \subset \Psi^{(1)}_d(\mathcal{Q})$. Analogously any $q = \sum_{q_i \in \mathcal{Q}} p_i q_i$ in $\Psi^{(1)}_d(\mathcal{Q})$ can be written as

$$q = \sum_{q_i \in Q} \left(\sum_{h_j \in \mathcal{H}} h_j r_j \right) q_i = \sum_{h_j \in \mathcal{H}} h_j \sum_{q_i \in Q} r_j q_i \in \Psi^{(\ell,1)}_d(\mathcal{H}),$$

therefore $\Psi^{(\ell,1)}_d(\mathcal{H}) = \Psi^{(1)}_d(\mathcal{Q})$. □

Computational note. In the algorithm we need to compute a complement of $\Psi^{(\ell,1)}_d(\mathcal{H})$ in $\mathbb{K}[\mathrm{x}]^{(\ell,1)}_d$, for $\mathcal{H} \subset \mathbb{K}[x]^{\mathcal{G}}$. In general, this can be achieved with linear algebra as follows: Assume we use $\mathcal{P}^{(\ell,1)}_d$ as basis for $\mathbb{K}[\mathrm{x}]^{(\ell,1)}_d$ and let M be the matrix of $\psi^{(\ell,1)}_{d\,\mathrm{h}}$. On a column echelon form of M select the indices of the rows that

have no pivot; They index the elements of $\mathcal{P}_d^{(\ell,1)}$ that form a basis of a complement to $\Psi_d^{(\ell,1)}(\mathcal{H})$.

Alternatively, taking advantage that $\mathbb{K}$ is a subfield of $\mathbb{C}$, we can choose to compute the orthogonal complement w.r.t. the apolar product. A vector in the kernel of ${}^t\mathrm{M}$ provides the conjugate of the coefficients in the dual basis of $\mathcal{P}_d^{(\ell,1)}$ of a polynomial orthogonal to $\Psi_d^{(\ell,1)}(\mathcal{H})$. Thus a basis of the kernel of ${}^t\mathrm{M}$ provides a basis for the orthogonal complement of $\Psi_d^{(\ell,1)}(\mathcal{H})$.

Algorithm 6

Fundamental Invariants and Equivariants

INPUT: $\mathcal{P} = \bigcup_{\ell=1}^{\mathfrak{n}} \bigcup_{d=1}^{|\mathcal{G}|} \mathcal{P}_d^{(\ell,1)}$ where $\mathcal{P}_d^{(\ell,1)}$ is an orthonormal basis of $\mathbb{K}[\mathrm{x}]_d^{(\ell,1)}$.

OUTPUT: $\mathcal{H} \subset \mathbb{K}[x]^{\mathcal{G}}$ and $\mathcal{Q} = \bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{Q}^{(\ell)}$ a s.a.b. for $\mathbb{K}[\mathrm{x}]/\langle\mathcal{H}\rangle$ with the properties that

- $\mathcal{H}$ is a minimal generating set of homogeneous invariants
- $Q^{(\ell)} = \left\{ \left[q_{i1}^{(\ell)}, \ldots, q_{in_\ell}^{(\ell)} \right] \mid 1 \le i \le m_\ell \right\}$ is a generating set for $\mathbb{K}[\mathrm{x}]^{(\ell)}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module

$\mathcal{H}_0, \mathcal{Q}_0^{(2,1)}, \ldots, \mathcal{Q}_0^{(\mathfrak{n},1)} \leftarrow \varnothing$
$\mathcal{Q}^{(1)} \leftarrow \{1\}$
$a \leftarrow 1$ # *counts the polynomials of degree* d *in* $\bigcup_{\ell=1}^{\mathfrak{n}} \mathcal{Q}^{(\ell)}$
for d *from 1 to* $|\mathcal{G}|$ **while** $a > 0$ **do**
  $\mathcal{K} \leftarrow$ *a basis of a complement of* $\Psi_d^{(1)}(\mathcal{H}_{d-1})$ *in* $\mathbb{K}[x]_d^{\mathcal{G}}$
  $\mathcal{H}_d \leftarrow \mathcal{H}_{d-1} \cup \mathcal{K}$
  $a \leftarrow 0$
  for $\ell = 2$ *to* $\mathfrak{n}$ **do**
    $\mathcal{K} \leftarrow$ *a basis of a complement of the image of* $\Psi_d^{(\ell,1)}(\mathcal{H}_d)$ *in* $\mathbb{K}[\mathrm{x}]_d^{(\ell,1)}$
    $Q_d^{(\ell,1)} \leftarrow Q_{d-1}^{(\ell,1)} \cup \mathcal{K}$
    $a \leftarrow a +$ *cardinal*$(\mathcal{K})$
  end-do
end-do
return $\mathcal{H}_{d-1}$ *and* $\bigcup_{\ell=1}^{\mathfrak{n}} \left\{ \left[\pi_{11}^{(\ell)}(q), \ldots, \pi_{\mathfrak{n}_\ell 1}^{(\ell)}(q) \right] \mid q \in Q_{d-1}^{(\ell,1)} \right\}$

PROOF: We prove first that the three following properties are true at the end of each iteration of the for-while loop bearing on d.

$\mathbf{A}(d)$: $\Psi_e^{(1)}(\mathcal{H}_d) = \mathbb{K}[x]_e^{\mathcal{G}}$ for all $1 \le e \le d$.

$\mathbf{B}(d)$: $\Psi_d^{(1)}\left(\mathcal{Q}_d^{(\ell,1)}\right) = \mathbb{K}[\mathrm{x}]_{\le d}^{(\ell,1)}$

$\mathbf{C}(d)$: $\bigcup_{\ell=1}^{\mathfrak{n}} \bigcup_{\alpha=1}^{\mathfrak{n}_\ell} \left\{\pi_{\alpha 1}^{(\ell)}(q) \mid q \in Q_d^{(\ell,1)}\right\}$ is a basis of an invariant complement of $\bigoplus_{e=1}^{d} \Psi_e(\mathcal{H})$ in $\mathbb{K}[\mathrm{x}]_{\le d}$

Property A(0), B(0) and C(0) hold. Assume that A($d-1$), B($d-1$) and C($d-1$) hold.

Property A(d) is then secured because of Property A($d-1$) and the adjunction to $\mathcal{H}_{d-1}$ of a basis of the complement of $\Psi_d^{(1)}(\mathcal{H}_{d-1})$ in $\mathbb{K}[x]_d^{\mathcal{G}}$ to form $\mathcal{H}_d$.

By Lemma 5.5.1 and Property A(d) and B($d-1$), $\Psi_d^{(\ell,1)}(\mathcal{H}_d) = \Psi_d^{(1)}\left(\mathcal{Q}_{d-1}^{(\ell,1)}\right)$. Hence $\mathcal{Q}_d^{(\ell,1)}$ consists of $\mathcal{Q}_{d-1}^{(\ell,1)}$ and a basis of the complement of $\Psi_d^{(1)}\left(\mathcal{Q}_{d-1}^{(\ell,1)}\right)$ in $\mathbb{K}[\mathrm{x}]_d^{(\ell,1)}$, and thus Property B($d$) holds.

$\mathcal{Q}_d^{(\ell,1)}$ is a basis of the invariant complement, in $\mathbb{K}[\mathrm{x}]_{\le d}^{(l,1)}$, of $\Psi_d(\mathcal{H}_d) \cap \mathbb{K}[\mathrm{x}]_{\le d}^{(\ell,1)} = \Psi_d^{(\ell,1)}(\mathcal{H}_d)$. Recall that $\pi_{k1}^{(\ell)} : \mathbb{K}[\mathrm{x}]_{\le d}^{(\ell,1)} \to \mathbb{K}[\mathrm{x}]_{\le d}^{(\ell,k)}$ is an isomorphism such that $\pi_{k1}^{(\ell)}(rq) = r\pi_{k1}^{(\ell)}(q)$ for any $r \in \mathbb{K}[x]^{\mathcal{G}}$. It follows that $\left\{\pi_{k1}^{(\ell)}(q) \,|\, q \in \mathcal{Q}_d^{(\ell,1)}\right\}$ is a basis of the complement of $\Psi_d(\mathcal{H}_d) \cap \mathbb{K}[\mathrm{x}]_d^{(\ell,k)}$ in $\mathbb{K}[\mathrm{x}]_d^{(\ell,k)}$. As $\mathbb{K}[\mathrm{x}]_{\le d} = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \mathbb{K}[\mathrm{x}]_{\le d}^{(\ell,k)}$, we obtain Property C($d$).

With these properties at each iteration, we want now to prove that when $d = |\mathcal{G}|$ or $a = 0$ we have that $\mathcal{H}_d$ is a generating set of $\mathbb{K}[x]^{\mathcal{G}}$ and $\mathcal{Q}_d^{(\ell,1)}$ is a generating set of $\mathbb{K}[\mathrm{x}]^{(\ell,1)}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module.

Note that Property A(d) implies that any $p \in \mathbb{K}[x]_{\le d}^{\mathcal{G}}$ can be written as a polynomial in the elements of $\mathcal{H}_d$. Hence, if $d = |\mathcal{G}|$, Noether's bound, $\mathcal{H}_d$ is a generating set of invariants (Sturmfels, 2007, Theorem 2.1.4). By (Stanley, 1979, Theorem 3.1) or (Worfolk, 1994, Proposition 4.3), $\mathbb{K}[\mathrm{x}]_{|\mathcal{G}|}^{(\ell)}$ generates $\mathbb{K}[\mathrm{x}]^{(\ell)}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module. Thus Property B(d), together with the properties of the maps $\pi_{k1}^{(\ell)}$, implies that $\mathcal{Q}_d^{(\ell,1)}$ is a generating set for $\mathbb{K}[\mathrm{x}]_{|\mathcal{G}|}^{(\ell,1)}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module. So by Proposition 5.2.2, $\bigcup_{\ell=1}^{\mathfrak{n}} \left\{\left[\pi_{11}^{(\ell)}(q), \ldots, \pi_{\mathfrak{n}_\ell 1}^{(\ell)}(q)\right] \mid q \in Q_d^{(\ell,1)}\right\}$ is a generating set for $\mathbb{K}[x]_{\mathfrak{r}^\ell}^{\mathcal{G}}$ as a $\mathbb{K}[x]^{\mathcal{G}}$-module.

At the end of an iteration $a = 0$ means that $\Psi_d(\mathcal{H}_d) = \mathbb{K}[\mathrm{x}]_d$. It follows that $\Psi_e(\mathcal{H}_d) = \mathbb{K}[\mathrm{x}]_e$ for all $e \ge d$. Hence

$$\mathbb{K}[\mathrm{x}] = \langle \mathcal{H} \rangle \bigoplus \left(\bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \bigoplus_{q \in \mathcal{Q}_d^{(\ell,1)}} \mathbb{K}\pi_{k1}^{(\ell)}(q) \right).$$

Any homogeneous invariant of positive degree thus belongs to $\langle \mathcal{H} \rangle$. By the classical argument used in Hilbert's finiteness theorem (Sturmfels, 2007, Theorem 2.1.3), $\mathcal{H}_d$

is a generating set of homogeneous invariants. It is a minimal such set by construction. Then, by Nakayama's lemma for graded algebra (Derksen and Kemper, 2015, Lemma 3.7.1) and the fact that the different $\mathbb{K}[\mathrm{x}]^{(\ell,k)}$ are in direct sum we have

$$\mathbb{K}[\mathrm{x}] = \bigoplus_{\ell=1}^{\mathfrak{n}} \bigoplus_{k=1}^{n_\ell} \left(\sum_{q \in \mathcal{Q}_d^{(\ell,1)}} \mathbb{K}[x]^{\mathcal{G}} \, \pi_{k1}^{(\ell)}(q) \right).$$

It follows that $\mathbb{K}[\mathrm{x}]^{(\ell,k)} = \sum_{q \in \mathcal{Q}_d^{(\ell,1)}} \mathbb{K}[x]^{\mathcal{G}} \, \pi_{k1}^{(\ell)}(q)$. □

In the algorithm we have made use of Noether's bound to mark termination. Yet we could have use $a = 0$ as unique stopping criterion : it is indeed enough to know that $\mathbb{K}[x]^{\mathcal{G}}$ is finitely generated as an algebra and $\mathbb{K}[\mathrm{x}]^{(\ell)}$ is finitely generated as a $\mathbb{K}[x]^{\mathcal{G}}$-module.

As the algorithm in (King, 2013), for each degree d Algorithm 6, looks for invariants that are not yet in the ideal generated by the invariants of degree $d-1$. The membership in King's algorithm is decided thanks to a Gröbner basis of the ideal, which is thus computed at each step. The termination criterion is also different. King's algorithm stops when d reaches a degree bound. This latter is updated at each iteration once the ideal generated by the so far computed invariants is zero dimensional. Our algorithm terminates as soon as no new element appears in the quotient $\mathbb{K}[\mathrm{x}]/\langle \mathcal{H} \rangle$.

Prior approaches to compute generating invariants of finite groups (Derksen and Kemper, 2015) proceeded by first determining a set of primary invariants, and then a set of secondary invariants. In this approach, Molien's series is key: it provides the degrees of the invariants. These are then computed applying the Reynolds operator.

The computation of generating θ-equivariants in (Gatermann, 1996, Algorithm 3.16) and (Worfolk, 1994) is in the same spirit as the computation secondary invariants. They introduce the representation $\tau : \mathcal{G} \to \mathrm{Aut}\left(\mathbb{K}[\mathrm{x}]^m\right), \tau(g)\left(\mathrm{p}\right) = \theta(g) \cdot \mathrm{p} \circ \vartheta(g^{-1})$ and use the fact that the related the Reynolds operator is a projection on the $\mathbb{K}[x]_\theta^{\mathcal{G}}$ vector space. Note that the linear algebra operations at degree d are then made in a vector space of dimension m times the dimension of $\mathbb{K}[\mathrm{x}]_d$.

As explained in Section 5.2, any equivariant can be written as linear combinations of fundamental equivariants. Algorithm 6 computes generating sets of these with linear algebra in the vector spaces $\mathbb{K}[\mathrm{x}]_d$ of homogeneous polynomials, and not vectors of them. Furthermore we actually only compute the generators for the $\mathbb{K}[x]^{\mathcal{G}}$-modules $\mathbb{K}[\mathrm{x}]^{(\ell,1)}$, from which the generators of any $\mathbb{K}[\mathrm{x}]^{(\ell,k)}$ can be deduced.

Example 5.5.2 *Consider the representation $\vartheta : D_6 \to \mathrm{GL}_3(\mathbb{R})$ of the dihedral group $D_6 = \left\{\delta^\alpha \sigma^\beta \mid \delta^6 = \sigma^2 = (\sigma\delta)^2 = 1\right\}$ defined by*

$$\vartheta(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \vartheta(\delta) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This action was examined in (Sturmfels, 2007, Example 2.2.6). D_6 has six absolutely irreducible representations, four of dimension 1 and two of dimension 2. Applying the Fundamental Invariants and Equivariants algorithm to D_6 we obtain the followings generating invariants

$$\mathcal{H} = \left\{x^2+y^2,\, z^2, x^6-15x^4y^2+15x^2y^4-y^6,\, xyz\left(3x^4-10x^2y^2+3y^4\right)\right\},$$

and equivariants

$$\begin{aligned}
Q^{(2)} &= \left\{z,\, xy\left(3x^4-10x^2y^2+3y^4\right)\right\}, \quad Q^{(3)} = \left\{x\left(x^2-3y^2\right),\, yz\left(3x^2-y^2\right)\right\} \\
&\qquad Q^{(4)} = \left\{y\left(3x^2-y^2\right),\, xz\left(x^2-3y^2\right)\right\}, \\
Q^{(5)} &= \left\{[x,y],\, [yz,-xz],\, \left[x\left(x^4-10x^2y^2+5y^4\right), y\left(5x^4-10x^2y^2+y^4\right)\right],\right. \\
&\qquad \left.\left[yz\left(5x^4-10x^2y^2+y^4\right), xz\left(x^4-10x^2y^2+5y^4\right)\right]\right\}, \\
Q^{(6)} &= \left\{\left[x^2-y^2, 2xy\right],\, \left[2xyz, -x^2z+y^2z\right],\, \left[x^4-6x^2y^2+y^4, xy\left(x^2-y^2\right)\right],\right. \\
&\qquad \left.\left[xyz\left(x^2-y^2\right), z\left(x^4-6x^2y^2+y^4\right)\right]\right\}.
\end{aligned}$$

Example 5.5.3 *Consider the action of the full group of symmetries of the regular tetrahedron*

$$T_h = \left\{\delta^i\sigma^j\alpha^k\beta^\ell \mid 0\le i\le 1,\ 0\le j\le 2,\ 0\le k\le 1,\ 0\le \ell\le 1\right\}$$

in $\mathbb{R}^3$ which is defined by the matrices.

$$\delta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

T_h has order 24 and 5 inequivalent irreducible representations, all absolutely irreducible, whose dimensions are 1, 1, 2, 3, and 3. Applying the Fundamental Invariants and Equivariants algorithm to T_h we obtain the following generating invariants

$$\mathcal{H} = \left\{x^2+y^2+z^2,\, xyz,\, x^4+y^4+z^4-3x^2y^2-3x^2z^2-3y^2z^2\right\},$$

and equivariants

$$Q^{(1)} = \{1\}, \quad Q^{(2)} = \{(y^2 - z^2)(x^2 - z^2)(x^2 - y^2),\}$$

$$Q^{(3)} = \left\{ \begin{array}{c} [2x^2 - y^2 - z^2, y^2 - z^2], \\ \left[x^4 - 3x^2(y^2 + z^2) - \frac{1}{2}y^4 + 6y^2z^2 - \frac{1}{2}z^4, 6x^2y^2 - 6x^2z^2 - y^4 + z^4\right] \end{array} \right\}$$

$$Q^{(4)} = \left\{ \begin{array}{c} [x, y, z], [yz, xz, xy], \\ [x(2x^2 - 3y^2 - 3z^2), y(3x^2 - 2y^2 + 3z^2), z(3x^2 + 3y^2 - 2z^2)] \end{array} \right\}$$

$$Q^{(5)} = \left\{ \begin{array}{c} [xy^2 - xz^2, -x^2y + yz^2, x^2z - y^2z], \\ [yz(y^2 - z^2), xz(x^2 - z^2\}, xy\{x^2 - y^2)], \\ \left[\begin{array}{c} x(2x^2y^2 - 2x^2z^2 - y^4 + z^4), \\ y(x^4 - 2x^2y^2 + 2y^2z^2 - z^4), \\ z(x^4 - 2x^2z^2 - y^4 + 2y^2z^2) \end{array} \right] \end{array} \right\}$$

Example 5.5.4 *Consider the three-dimensional representation of the cyclic group C_4 of order 4 over $\mathbb{C}[x, y, z]$ given by the matrices*

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

This action was examined in (Sturmfels, 2007, Example 2.3.7). C_4 is an abelian group with four irreducible representations of dimension 1, two of which are absolutely irreducible and the other two being of complex type and conjugate to each other. Applying Algorithm 6 we get the following generating invariants

$$\mathcal{H} = \left\{x^2 + y^2,\ z^2,\ xyz,\ x^2z - y^2z,\ x^4 + y^4,\ xy\left(x^2 - y^2\right)\right\}$$

and equivariants

$$\begin{aligned} Q^{(2)} &= \{z,\ xy,\ x^2 - y^2\}, \\ Q^{(3)} &= \{x - iy,\ z(x + iy),\ 3x^2y + y^3 + i(x^3 + 3xy^2)\}, \\ Q^{(4)} &= \{x + iy,\ z(x - iy),\ 3x^2y + y^3 - i(x^3 + 3xy^2)\}. \end{aligned}$$

In the above, the elements of $Q^{(3)}$ and $Q^{(4)}$ are conjugate of each other. And indeed, when dealing with a real representation, pairs of conjugate irreducible representations of complex type can dealt with as a single one; If $Q^{(\ell)}$ is a generating set of $\mathfrak{r}^{(\ell)}$-equivariants then $\overline{Q^{(\ell)}}$ is a generating set of $\overline{\mathfrak{r}^{(\ell)}}$-equivariants.

Chapter 6

Implementation

The methods and algorithms we discuss in this work have been implemented in a Maple library: SyCo (Symmetry and Computations). This library can be used to construct group actions for finite groups and symmetry adapted bases. It provides algorithms to effectively preserve and exploit symmetry in multivariate interpolation (Chapter 3). We offer symmetry representation (via H-basis) for ideals given by the kernel intersection of invariant sets of linear forms (Chapter 4). The library has different methods to compute fundamental sets of invariants and equivariants (Chapter 5), either via interpolation or using direct symbolic approaches. We design SyCo to be extensible and generic. Our purpose in this chapter is to present the general design and a basic documentation of the library.

6.1 Linear Forms

All linear forms in SyCo implement an interface (LinearForm) that contains the evaluation method. This method receives as parameter a polynomial expression. Below we show the existing linear forms in our library as well as their main attributes.

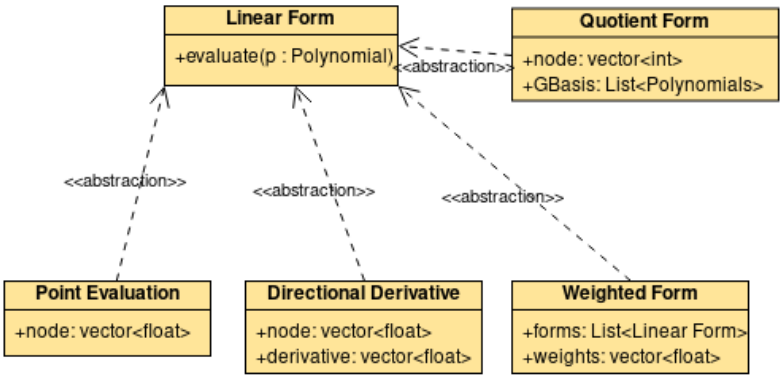


Figure 6.1: UML diagram for the implemented linear forms.

```
#Example of use
p := x^2 + y^2
Grobner_Basis  := {x^2 + y, y^3 + x};

p_eval := Object(PointEvaluation, ⟨1, 2⟩);
d_eval := Object(DirectionalDerivative, ⟨1, 2⟩, ⟨1, 0⟩);
w_eval := Object(WeightedForm, [point_eval, d_eval], ⟨1, 1⟩);
q_eval := Object(quotientForms, Grobner_Basis, ⟨0, 1, 0, 0, 0, 0⟩);

for lf in [p_eval, d_eval, w_eval] do
    print(lf:-evaluate(p));
end;
#The output is 5 , 6 , 11, -1
```

6.2 Group Actions

The group actions in SyCO are divided in two main groups: actions on $\mathbb{K}^n$ and induced representations. Any induced representation contains a group action on $\mathbb{K}^n$, and any group action is associated to a particular finite group. As we can observe in Figure 6.2, any implementation of a group action must provide:

- A method to compute linear representations for a given dimension and type.
- A method to compute a collection of non isomorphic irreducible representations in a given field ($\mathbb{R}$ or $\mathbb{C}$ for the current implementation).

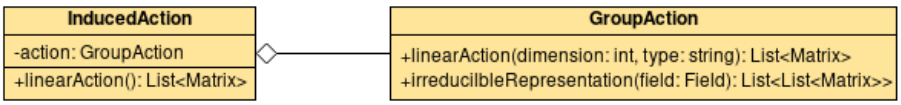


Figure 6.2: Linear actions diagram.

Table 6.1 shows the implemented group actions in our library together with the existing configuration for dimension and type. The default types for each dimension are highlighted in red.

Group action	Dimension	Type	Additional parameters
OhGroup	3	F_{1u}, F_{2u}, F_{1g}, F_{2g}	-
ThGroup	3	T_h (Example 5.5.3)	-
DihedralGroup	2 n	ortogonal (3.3.3) parametric	m:int S,R:matrices
CyclicGroup	n	permutation, diagonal	m:int
SymmetryGroup	3 n	embedded permutation	-
AlternatingGroup	3 n	embedded signedPermutation	-
DiagonalGroup	n	parametric	A:Matrix, P:List

Table 6.1: Group actions in $\mathbb{R}^n$ and $\mathbb{C}^n$.

In Table 6.1 the additional parameter m corresponds to the order of Dhiedral and Cyclic group. The matrices S and R are such that $S^2 = R^m = (S \cdot R)^2 = I_n$. The

explicit description of the parameters A and P in the diagonal group can be found in (Hubert and Labahn, 2016, Section 2.2).

We provide the following implementations for the InducedAction module:

- PolynomialAction. $\rho(g)p = p \circ \vartheta(g^{-1})$ in $\mathbb{K}[\mathrm{x}]_d$
- FormsAction. $\rho^*(g)\lambda(p) = \lambda\left(p \circ \vartheta(g^{-1})\right)$ in $\mathbb{K}[x]^*$
- PolyMapAction. $\tau(g)\mathrm{p} = \theta(g) \cdot \mathrm{p} \circ \vartheta(g^{-1})$ in $\mathbb{K}[\mathrm{x}]_d^m$
- QuotientAction. $\hat{\rho}(g)\left([p]\right) = [\rho(g)p]$ in $\mathbb{K}[\mathrm{x}]/I$

where ϑ and θ linear actions in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, $p \in \mathbb{K}[\mathrm{x}]$, $\lambda \in \mathbb{K}[x]^*$ and $\mathrm{p} \in \mathbb{K}[\mathrm{x}]^m$.

```
# Examples
DG := DihedralGroup:-linearAction(2, type=ortogonal, 3);
```

$$> \mathrm{DG} := \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1/2 & -1/2\sqrt{3}\\ 1/2\sqrt{3} & -1/2\end{pmatrix}, \begin{pmatrix}-1/2 & -1/2\sqrt{3}\\ 1/2\sqrt{3} & -1/2\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1/2 & -1/2\sqrt{3}\\ 1/2\sqrt{3} & -1/2\end{pmatrix}, \begin{pmatrix}-1/2 & -1/2\sqrt{3}\\ 1/2\sqrt{3} & -1/2\end{pmatrix}\right];$$

```
ohAction  := OhGroup:-linearAction(3, type=F_1u): #F_1u action of O_h in R^3
cycPerm   := CyclicGroup:-linearAction(3, type=permutation, 3);
```

$$> \mathrm{cycPerm} := \left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\right];$$

```
cycDiag   := CyclicGroup:-linearAction(3, type=diagonal, 3);
```

$$> \mathrm{cycDiag} := \left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}e^{\frac{2i\pi}{3}} & 0 & 0\\ 0 & e^{\frac{2i\pi}{3}} & 0\\ 0 & 0 & e^{\frac{2i\pi}{3}}\end{pmatrix}, \begin{pmatrix}e^{\frac{4i\pi}{3}} & 0 & 0\\ 0 & e^{\frac{4i\pi}{3}} & 0\\ 0 & 0 & e^{\frac{4i\pi}{3}}\end{pmatrix}\right];$$

```
#List of non-isomorphic irreducible  representations of D_3 in R^3
Irr := DihedralGroup:-irredRepresentation(3):
#Polynomial representation for the permutation action of C_3 in R[x,y]_2
PolyAction := PolynomialAction:-linearAction(cycPerm, degree=2):
```

$$> \mathrm{PolyAction} := \left[\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}\right];$$

6.3 Symmetry Adapted Bases

The effective computation of a symmetry adapted basis plays a key role in all our algorithms. Given a group $\mathcal{G}$, we provide functions to compute symmetry adapted bases for different vector spaces. The exiting options are listed below.

- SymmetricPolynomialBasis Calling sequence: SymmetricPolynomialBasis(action, irreducibles, d, W). Parameters: action a linear representation of $\mathcal{G}$, irreducibles list of all irreducible representations of $\mathcal{G}$, d degree, W a subspace of $\mathbb{K}[\mathrm{x}]_{\leq d}$ given by a set of linear generators. W is an optional parameter, by default the full space $\mathbb{K}[\mathrm{x}]_{\leq d}$ will be considered. Returns a list of symmetry adapted bases for each isotypic component, i.e., a list of lists of polynomials.

- SymmetricLinearFormBasis Calling sequence: SymmetricLinearFormBasis(action, irreducibles, $\mathbf{\Lambda}$). Parameters: action, irreducibles, $\mathbf{\Lambda}$ a set of LinearForms. Returns a list of symmetry adapted bases for each isotypic component, i.e., a list of lists of WeightedForm.

- SymmetricPolynomialMapBasis Calling sequence: SymmetricPolynomialMapBasis($\boldsymbol{\theta}$, $\boldsymbol{\vartheta}$, irreducibles, degreeMap, W). Parameters: $\boldsymbol{\theta}$, $\boldsymbol{\vartheta}$ linear actions in $\mathbb{K}^n$ and $\mathbb{K}^m$ respectively, irreducibles, degreeMap vector of degrees $d_1, \ldots, d_m$ representing the space $\mathbb{K}[\mathrm{x}]_{d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d_m}$, W a subspace of $\mathbb{K}[\mathrm{x}]_{d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d_m}$, if W is not provided the full space $\mathbb{K}[\mathrm{x}]_{d_1} \times \ldots \times \mathbb{K}[\mathrm{x}]_{d_m}$ will be considered. Returns a list of symmetry adapted bases for each isotypic component, i.e., a list of lists of polynomials maps.

- SymmetricQuotientBasis Calling sequence: SymmetricQuotientBasis(action, irreducibles, GB, degreeOrd). Parameters: action, irreducibles, GB a Gröbner basis of the associated ideal I, degreeOrd a monomial order. Returns a list of symmetry adapted bases for each isotypic component, i.e., a list of lists of polynomials. Each polynomial is representative of its associated equivalence class.

To compute a s.a.b for a generic vector space W we first look for a representation of W in $\mathbb{K}^m$ and then we call the method SymmetryAdaptedOrthogonalBasis. The method SymmetryAdaptedOrthogonalBasis is in charge of handling the computation of s.a.b for every isotypic component of W and it needs the following parameters:

action :	–	Group action in $\mathbb{K}^m$ of a group $\mathcal{G}$
irreducibles :	–	List of all irreducible representations of $\mathcal{G}$
W	–	Subspace of $\mathbb{K}^m$ given by a set of vectors

Depending on the type of each irreducible representation one of the following subroutines is called:

- IsotypicComponentAbsolutelyIrreducible Calling sequence IsotypicComponentAbsolutelyIrreducible(action, irr, W). Parameters: action, irr the i-th irreducible representation of action, and W. Returns a symmetry adapted basis of $W^{(i)}$, i.e., the i-th isotypic component of W.

- IsotypicComponentComplexTypeIsotypicComponentComplexType(action, irr, W). Parameters: action, irr, W. Returns a symmetry adapted basis of $W^{(i)} \oplus \overline{W^{(i)}}$.

6.4 Interpolation

The symmetry preserving algorithms from Chapters 3 and 4 are implemented in the SymmetryInterpolation module. We list below the precise signature of the algorithms as well as their calling sequence in our library.

- SymmetryInterpolationSpace Calling sequence: SymmetryInterpolationSpace(action, **Λ**, m, t, epsilon). Parameters: action a GroupAction, **Λ** set of LinearForm, m (optional) equation of the form method = name where name is one of 'leastInterpolation' or 'minimalDegree' methods, t (optional) equation of the form type = name where name is one of the available representations on action, epsilon (optional) integer indicating the number of digits of precision. The SymmetryInterpolationSpace first computes symmetry adapted bases of span(Λ) and $\mathbb{K}[x]_{\leq r}$, with $r \leq |\Lambda|$. Then it calls as a subroutine to Algorithm 1 with the precise method to use. Returns a symmetry preserving interpolation space for Λ, i.e., a list of polynomials.

- SymmetryIdealInterpolation Calling sequence: SymmetryInterpolationSpace(action, **Λ**, t). Parameters: action a GroupAction, **Λ** set of LinearForm such that $\cap_{\lambda\in\Lambda} \ker \lambda$ is an ideal in $\mathbb{K}[x]$, t (optional) equation of the form type = name where name is one of the available representations on action. The SymmetryIdealInterpolation method first computes symmetry adapted bases of span(Λ), $\mathbb{K}[x]_{\leq r}$ and $\mathbb{K}[x]^{n_i}_{\leq r}$, with $r \leq |\Lambda|$, $1 \leq i \leq \mathfrak{n}$. Then it calls as a subroutine to Algorithm 4. Returns a symmetry adapted H-basis for $\cap_{\lambda\in\Lambda} \ker \lambda$ and a symmetry adapted basis of $\Lambda_\downarrow$.

- ComputeInterpolant Calling sequence: ComputeInterpolant(P, **Λ**, phi, epsilon). Parameters: P an interpolation space for Λ, **Λ** set of LinearForm, phi $\in \mathbb{K}^{|\Lambda|}$ list of interpolation values, epsilon (optional) integer indicating the number of digits of precision. Returns a polynomial p which is the solution of the interpolation problem (**Λ**, phi).

```
# Examples

#Set of points closed under the orthogonal action of D_3 in R^2
THETA := [<-5*sqrt(3)/3, 1/3>, <2*sqrt(3)/3, -8/3>, <sqrt(3), 7/3>, <-5*sqrt(3)/3, -1/3>, <sqrt(3), -7/3>, <2*sqrt(3)/3, 8/3>]:

#Creation of DirectionalDerivative forms from THETA
Λ := [seq(Object(DirectionalDerivative, THETA[i], THETA[i]), i=1..nops(THETA))]:

#Computation of a s.a.b of the least interpolation space for Λ
P := SymmetryInterpolationSpace(DiagonalGroup, Λ, method='leastInterpolation',
                                type='ortogonal');
```

$> P := \left[\left[\frac{38\sqrt{6}(x^2+y^2)}{9}\right], \left[\frac{28\sqrt{6}(3x^2-y^2)y}{27}\right], \left[\left[-\frac{8\sqrt{3}x}{3}, -\frac{14\sqrt{3}(x^2-y^2)}{9}\right], \left[-\frac{8\sqrt{3}y}{3}, \frac{28\sqrt{3}xy}{9}\right]\right]\right];$

```
#Compute the interpolant  for the invariant interpolation problem (Λ, φ)
φ := ⟨1,1,1,1,1,1⟩:
p := ComputeInterpolant(Λ, φ);
```

$> p := \frac{9}{152}(x^2+y^2)$; *#Notice that p is an invariant polynomial as expected*

```
#Creation of an ideal interpolation problem
Λ1 := [seq(Object(PointEvaluation, THETA[i]), i=1..nops(THETA))]:
Λ2 := [op(Λ), op(Λ1)]:  # ⋂λ∈Λ2 ker λ is an ideal in 𝕂[x].

#Simultaneous computation of a symmetry adapted H-basis for ⋂λ∈Λ2 ker λ
#and of the least solution interpolation space of Λ2
[H, P] = SymmetryIdealInterpolation(DiagonalGroup, Λ2, type='ortogonal');
```

$> H := \left[\left[19\sqrt{3}x^3 - 57\sqrt{3}xy^2 + 135x^2 + 135y^2 - 380, (9x^2+9y^2-76)^2\right], [], [[], []]\right];$

$> P := \left[\left[1, \frac{x^2+y^2}{2}\right], \left[\frac{\sqrt{6}y(3x^2-y^2)}{12}, \frac{\sqrt{6}y(3x^4+2x^2y^2-y^4)}{48}\right], \left[\left[x, \frac{x^2-y^2}{2}, \frac{\sqrt{2}x(x^2+y^2)}{4}, \frac{\sqrt{3}(3x^4+6x^2y^2-5y^4)}{48}\right], \left[y, -xy, \frac{\sqrt{2}y(x^2+y^2)}{4}, -\frac{\sqrt{3}xy(3x^2+y^2)}{12}\right]\right]\right];$

6.5 Fundamental equivariants and invariants computation

In Chapter 5 we propose three different algorithms to compute generating sets of equivariances. The algorithms are implemented in the FundamentalEquivariants module of our library. The signature and calling sequence of each algorithm are described next.

- FundamentalEquivariants:-reflection Calling sequence: reflection(action, **Λ**, t). Parameters: action a reflection GroupAction, **Λ** set of PointEvaluation such that for each $\xi \in \Lambda$ the isotropy subgroup of ξ with respect to action is trivial, t (optional) equation of the form type = name where name is one of the available representations on action. The reflection method checks the correctness of its parameters and calls as a subroutine the SymmetryIdealInterpolation method. Returns generating sets of invariants and equivariants for action which are read from the output of the SymmetryIdealInterpolation method.

- FundamentalEquivariants:-fromInvariants Calling sequence: fromInvariants(action, primary, t) Parameters: action a GroupAction, primary a set of primary invariants for action, t (optional) equation of the form type = name where name is one of the available representations on action. The fromInvariants method is

an implementation of the Algorithm 5. Returns a set of secondary invariants and a set of fundamental equivariants of action.

- FundamentalEquivariants:-simultaneously Calling sequence: simultaneously(action, t). Parameters: action a GroupAction, t (optional) equation of the form type = name where name is one of the available representations on action. The simultaneously method starts by computing symmetry adapted bases for $\mathbb{K}[x]_{\leq d}^{(\ell,1)}$, then it calls as a subroutine to Algorithm 6. Returns a set of secondary invariants and a set of fundamental equivariants of action.

```
#Examples

# Creation of linear forms without trivial isotropy subgroup w.r.t
# the reflection group D_8 in R^2
ξ := ⟨a, b⟩;
DG    :=  DihedralGroup:-irredRepresentation(3):
forms :=  [seq(Object(PointEvaluation, DG[i] · ξ ), i=1..nops(DG))]:
# Computation of fundamental sets of invariants and equivariants for DG
# Notice that the type of the action is omitted because the default value
# for R^2 (ortogonal) is being considered
[H^0, Q] = reflection(DihedralGroup, forms);
 # H^0 and Q are showed in Example 5.3.6

# Set of primary invarinats for the action of A_4 in R^3  (Example 5.4.4)
primary := {x^2 + y^2 + z^2, xyz, x^2y^2 + x^2z^2 + y^2z^2}:
# Computation of secondary invariants and fundamental equivariants
[S, Q] = fromInvariants(AlternatingGroup, primary, type='embedded');

# Simultaneous Computation of fundamental invariants and equivariants for T_h
[H, Q] = simultaneously(ThGroup);
 # H^0 and Q are showed in Example 5.5.3
```

Chapter 7

Conclusions

In this dissertation, we addressed three main topics, symmetry preserving interpolation, ideal interpolation and the computation of generating sets for fundamental equivariant modules. A summary of our contributions is the following:

Multivariate interpolation.

- By introducing general dual polynomial bases we generalized the construction of the least interpolation space introduced in (De Boor and Ron, 1992a). As we freed the computation from its reliance on the monomial basis we are then in a position to work with symmetry adapted bases to preserve and exploit symmetry.

- We showed how, when the space Λ of linear forms is invariant, the Vandermonde matrix can be made block diagonal. The latter happens when making use of symmetry adapted bases both for $\mathbb{K}[x]_{\leq d}$ and Λ. This block diagonalisation of the Vandermonde matrix indicates how computation can be organized more efficiently and robustly. Substantial computational savings are obtained from identical blocks being repeated. It just draws on the invariance of the space of linear forms. Thus, when the evaluation points can be chosen, it makes sense to introduce symmetry among them.

- We defined the concepts of invariant interpolation problem (IIP) and equivariant interpolation problem (EIP). These interpolation problems have a structure that we want to be preserved by the interpolant. We showed that this is automatically achieved when choosing the interpolant in an invariant interpolation space. Then the solution of an IIP is an invariant polynomial and the solution of an EIP is an equivariant polynomial map. The use of the symmetry adapted bases allows symmetry to be exactly preserved, independently of the numerical accuracy.

Ideal Interpolation

- We showed $I = \cap_{\lambda \in \Lambda} \ker \lambda$ is the kernel of the Vandermonde operator while $\Lambda_\downarrow$ can be inferred from a rank revealing form of the Vandermonde matrix. We presented an algorithm to compute simultaneously a (orthogonal) basis of the least interpolation space and an orthogonal H-basis of the ideal I. We proceed degree by degree. We made use of Macaulay matrices, at each iteration of the algorithm we compute bases of $\Lambda_\downarrow \cap \mathbb{K}[\mathrm{x}]_d$ and of $I^0 \cap \mathbb{K}[\mathrm{x}]_d$.

- We showed how to obtain a block diagonalization on the Macaulay matrix when $\langle \mathcal{H} \rangle$ is invariant under the induced action of a group $\mathcal{G}$ on $\mathbb{K}[\mathrm{x}]$. The key relies on exhibiting the equivariance of the prolongation map $\psi_{d,\mathcal{H}}$. We built a robust algorithm that exploits the symmetries of Λ. The block diagonal structure of the Vandermonde and Macaulay matrices allow to reduce the size of the matrices to deal with. The H-basis obtained as the output of the algorithm inherits the symmetries of Λ. The algorithm also provides a symmetry adapted basis of $\mathbb{K}[\mathrm{x}]/I$.

- As an application of ideal interpolation we developed a FGLM-type algorithm for constructing symmetry representations for the ideal. From a Gröbner basis of an invariant ideal I we are able to compute a symmetry adapted H-basis as well as a symmetry adapted basis of $\mathbb{K}[\mathrm{x}]/I$.

Fundamental Equivariants

- We showed that a symmetry adapted basis of any invariant subspace of $\mathbb{K}[\mathrm{x}]$ consists of fundamental equivariants. Furthermore, the equivariants for any matrix representation are linear combinations of fundamental equivariants.

- When $\mathcal{G}$ is a reflection group, we showed that we can deduce generating invariants and equivariants for such group actions from the solution of an ideal interpolation problem. We showed that a minimal set of generating invariants can be read from the symmetry preserving H-basis obtained in the output of Algorithm 4, and the fundamental equivariants can be read from a symmetry adapted basis of $\Lambda_\downarrow$.

- We proposed a new approach to determine generating invariants and fundamental equivariants assuming we have primary invariants. We then showed that this construction provides a Hironaka decomposition of $\mathbb{K}[\mathrm{x}]^{\mathcal{G}}$ and $\mathbb{K}[\mathrm{x}]^{(\ell)}$, for $1 < \ell < n$.

- We presented a new algorithm to compute a set of generating invariants simultaneously to the generating equivariants. When it comes to compute generating invariants, the resulting algorithm can be compared with (King, 2013). We compute the generating sets of equivariants with linear algebra in the vector spaces $\mathbb{K}[\mathrm{x}]_d$ of homogeneous polynomials, and not a vector space of dimension m times the dimension of $\mathbb{K}[\mathrm{x}]_d$ as in previous results in the literature

(Worfolk, 1994; Gatermann, 1996). We actually only compute the generators for the $\mathbb{K}[x]^{\mathcal{G}}$-modules $\mathbb{K}[x]^{(\ell,1)}$ from which the generators of any $\mathbb{K}[x]^{(\ell,k)}$ can be deduced. The latter makes our method more efficient from a computational point of view.

Maple library *SyCo* (Symmetry and Computations) All the methods and algorithms we presented in this work were implemented in a Maple library. We described the general design of the library and gave some examples of use.

On the other hand, our results also open several ideas for further research. Some of these are the following:

- Develop more advanced algorithms for the construction of symmetry adapted basis. Find a constructive approach for building a real symmetry adapted basis of *quaternonian* type from a complex symmetry adapted basis.
- Compute symmetry adapted H-basis for invariant ideals represented by a set of polynomials generators.
- Apply our symmetry reduction techniques in the computation of resultants for equivariant systems of equations.
- Feed our algorithms in Chapter 5 with efficient criteria to separate primary and secondary invariants from a generating set of invariants.
- Extend all the theory developed in this thesis not only to $\mathbb{C}$ or $\mathbb{R}$ but also to fields of positive characteristic.

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