# Micromechanics of granular materials: Modeling anisotropy by a hyperelastic-plastic model 

Lianxin Hu

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## Micromechanics of Granular Materials:

## Modeling Anisotropy by a Hyperelastic-plastic Model

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#### Abstract

In order to model the behavior of geometarials under complex loadings, several researches have done numerous experimental works and established relative constitutive models for decades. An important feature of granular materials is that the relationship between stress and strain especially in elastic domain is not linear, unlike the responses of typical metal or rubber. It has been also found that the stress-strain response of granular materials shows the characteristics of cross-anisotropy, as well as the non-linearities. Besides, the stress-induced anisotropy occurs expectedly during the process of disturbance on soils, for example, the loads or displacements. In this work, a new model which is a combination of Houlsby hyperelastic model and elastoplastic Plasol model was proposed. This new model took into account the non-linear response of stress and strain in both elastic and plastic domain, and the anisotropic elasticity was also well considered. Moreover, the overflow problem of plastic strain in plastic part was calibrated by a proper integration algorithm. Later, new model was verified by using numerical method and compared with laboratory experiments in axisymmetric triaxial conditions. The comparison results showed a good simulation effect of new model which just used one single set of parameters for a specific soil in different confining pressure situations. Then the analysis of new model internal variable, i.e., pressure exponent, illustrated that the value of pressure exponent which corresponds to the degree of anisotropy had an obvious effect on the stress-strain response. Moreover, this kind of effect is also affected by the density and drainage condition of samples. Basing on new model, a safety factor which refers to the second-order work criterion was adopted and tested in axisymmetric model and actual slope model. It showed that the negative value or dramatic decreasing of global normalized second-order work occurs accompanying with a local or global failure with a burst of kinetic energy. This feature of second-order work can also be affected by the variable pressure exponent. At last, new model was also compared with an elastoplastic model which considers both anisotropic elastic and anisotropic dilatancy, i.e., modified SANISAND model. Both advantages and disadvantages were illustrated in the comparison results.


## Key words

Houlsby hyperelasticity; Plasol model; Granular materials; Axisymmetric model; Slope; Drainage conditions; Modified SANISAND model; Stress-induced anisotropy; Anisotropic elasticity

## Résumé

Afin de modéliser le comportement des géométariaux sous des charges complexes, plusieurs études et travaux expérimentaux ont été réalisées afin d'établir des modèles constitutifs relatifs. Une caractéristique importante des matériaux granulaires est que la relation entre la contrainte et la déformation et ce même dans le domaine élastique n'est pas linéaire, contrairement aux réponses du métal. Il a également été constaté que la réponse contrainte-déformation des matériaux granulaires montre les caractéristiques de l'anisotropie induite, ainsi que les non-linéarités. En outre, l'anisotropie induite par la contrainte se produit pendant le processus de chargement sur les sols, par exemple, les charges ou les déplacements. Dans ce travail, un nouveau modèle qui est une combinaison de modèle hyperélastique Houlsby et modèle élastoplastique Plasol a été proposé. Ce nouveau modèle a pris en compte la réponse non linéaire de la contrainte dans le domaine élastique et plastique, et l'élasticité anisotrope a également été bien considérée. En outre, les problèmes de l'écoulement de la déformation plastique a été calibré par un algorithme d'intégration approprié. Plus tard, le nouveau modèle a été vérifié en utilisant la méthode numérique et comparé aux expériences de laboratoire dans des conditions triaxiales axisymmétriques. Les résultats de comparaison ont montré un bon effet de simulation du nouveau modèle qui a juste utilisé un seul ensemble de paramètres pour un sol spécifique dans différentes situations de contraintes. Ensuite, l'analyse de la nouvelle variable interne du modèle, c'est-à-dire l'exposant de pression, a montré que la valeur de l'exposant de pression qui correspond au degré d'anisotropie a eu un effet évident sur la réponse contrainte-déformation. De plus, ce type d'effet est également affecté par la densité et l'état de drainage des échantillons. En s'appuyant sur un nouveau modèle, un facteur de sécurité qui fait référence au critère de travail de deuxième ordre a été adopté et testé dans un modèle axisymétrique et un modèle de pente réel. Il a montré que la valeur négative ou la diminution spectaculaire du travail global normalisé de second ordre se produit lors d'une défaillance locale ou globale avec apparition d'énergie cinétique. Cette caractéristique du travail du second ordre peut également être affectée par l'exposant à pression variable. Enfin, un nouveau modèle a également été comparé à un modèle élastoplastique qui considère à la fois l'anisotropie élastique et la dilatation anisotrope, c'est-à-dire le modèle SANISAND modifié. Les avantages et les inconvénients ont été illustrés dans les résultats de comparaison.

## Mots clés

Houlsby hyperélasticité; Modèle Plasol; Matériaux granulaires; Modèle axisymmétrique; Pente; Conditions de drainage; Modèle SANISAND modifié; Anisotropie induite par le stress; Élasticité anisotrope

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## Notations

## Latin symbols

| $a, b, n$ | Parameters related with yield function in Chapter 2 |
| :---: | :---: |
| $B_{\mathrm{p}}, B_{c}$ | Hardening coefficients of the Plasol model |
| $c$ | Cohesion |
| $c_{i j k l}$ | Compliance matrix component |
| $D_{i j k l}^{e}, \quad D_{i j k l}^{p}$ | Constitutive elastic tensor component, constitutive plastic tensor component |
| $D(i j k l), D(i j)$ | Jacobian matrix (incremental stiffness matrix) component |
| $d_{\text {ijkl }}$ | Stiffness matrix component |
| $\underline{d Q}, \underline{d F}$ | Global nodal incremental displacement and force |
| E | Young's modulus |
| $E^{\prime}$ | Complementary energy (Gibbs free energy) |
| $e_{i j}$ | Deviator of strain tensor component |
| $e$ | Void ratio |
| $F$ | Strain energy (Helmholtz free energy) |
| $f$ | Yield surface |
| $G, G_{0}$ | Shear modulus, small strain shear modulus |
| $g$ | (1) Plastic potential, (2) shear stiffness factor |
| $I_{\sigma}$ | First stress invariant |
| $I I_{\widehat{\sigma}}$ | Second deviatoric stress invariant |
| $I I I_{\widehat{\sigma}}$ | Third deviatoric stress invariant |
| $J$ | Coupling modulus |
| K | Bulk modulus |
| $\underline{\underline{K}}$ | Global consistent tangent matrix |
| $k$ | (1) Over-consolidation ratio exponent (also with * superscript), (2) bulk stiffness factor |
| $k_{0}$ | Permeability |
| $m, m^{\prime}, k^{\prime}$ | Parameters as function of cohesion, friction angle or dilatancy angle |
| $n 1$ | Pressure exponent (also with * superscript) |
| $\underline{\underline{N}}$ | Rate-independent constitutive operator |
| $\underline{\underline{N}}$ | Symmetric part of $\underline{\underline{N}}$ |
| $p, p^{\prime}$ | Effective mean pressure |
| $p_{a}$ | Atmospheric pressure |
| $q$ | Deviator stress (invariant) |
| $R$ | Over-consolidation ratio in terms of mean effective stress (also with subscript) |
| $r, r_{C}, r_{E}$ | Reduced radius, reduced radius in compression, in extension |
| $S$ | Dimensionless stiffness factor (also with * superscript and $\eta$ subscript) |
| $S_{i j}$ | Anisotropic stiffness factor |
| $s_{i j}$ | Deviator of effective stress tensor component |
| $u$ | Displacement |


| Val | Scalar related with equivalent plastic strain |
| :--- | :--- |
| $W_{2}, W_{2 n}$ | Global second-order work, integrated second-order work |
| $w_{2}, w_{2 n}$ | Local second-order work, normalized second-order work |
| $x_{i}$ | Position vector |

## Greek letters

| $\Delta()$ | Increment |
| :--- | :--- |
| $\beta$ | Lode angle |
| $\delta_{\mathrm{ij}}$ | Kronecker tensor (unit tensor) |
| $\varepsilon_{i j}$ | Strain tensor |
| $\varepsilon_{v}$ | Volumetric strain |
| $\varepsilon_{v}^{*}$ | Modified volumetric strain |
| $\varepsilon_{v}^{0}$ | Function of strains |
| $\varepsilon_{s}$ | Shear strain |
| $\dot{\varepsilon}_{i j}, \dot{\varepsilon}_{i j}^{e}, \quad \dot{\varepsilon}_{i j}^{p}$ | Components of strain rate, elastic strain rate and plastic strain rate |
| $\varepsilon_{e q}^{p}$ | Equivalent plastic strain |
| $\varphi_{\mathrm{C}}$ | Friction angle under triaxial compression path |
| $\varphi_{\mathrm{E}}$ | Friction angle under triaxial extension path |
| $\eta$ | Triaxial stress ratio |
| $\kappa$ | Internal variables |
| $\kappa^{\prime}$ | Slope of swelling line in a consolidation plot |
| $\lambda$ | Lame's first parameter |
| $\dot{\lambda}$ | Plastic multiplier |
| $\mu$ | Lame's second parameter |
| $v$ | Poisson's ratio |
| $\rho$ | Density |
| $\sigma_{i j}, \dot{\sigma}_{i j}$ | Components of stress tensor (effective) and stress rate |
| $\sigma_{N}$ | Normal stress |
| $\hat{\sigma}, \hat{\sigma}_{i j}$ | Deviatoric stress tensor, deviatoric stress tensor component |
| $\sigma_{1}{ }^{*}, \sigma_{2}{ }^{*}, \sigma_{3}{ }^{*}$ | Projection of the principal stress on the deviatoric plane |
| $\sigma_{0}$ | Confining pressure |
| $\tau$ | Tangential stress |
| $\psi$ | Dilatancy angle |

## General introduction

Nowadays, the constitutive modeling work of the geomaterials has been causing various geotechnical researchers' attention for more than 50 years, as well as the detecting and definition problems of failure. The granular material which is a representative material of general geomaterials always shows a non-linear stress and strain response in both elastic part and plastic part. Therefore, when numerically simulating the actual behavior of granular materials, the non-linearization should be taken into account. Besides, another important fact that natural soils show characteristics of cross-anisotropy (or transverse isotropy) requires also attention (CAS 17). Anisotropy could be divided into two parts, namely inherent anisotropy and induced anisotropy depending on the formation condition, which is due to the process of natural deposition and later disturbance respectively (ART 72; ODA 72c). Also, the anisotropy could also be composed into fabric anisotropy, stress or stiffness anisotropy and permeability anisotropy (KUH 15) from the manifestations aspect which are closely related with the external loading. Therefore, this research work, our main purpose focuses on the stress-induced anisotropy and its influence on the response of stress and strain.

In order to simulate the appropriate behavior of granular materials, various authors present some anisotropic elastoplastic models. For normally consolidated or lightly overconsolidated soil, an even small increase of stresses is likely to cause yielding. It is also easily concluded that the plastic deformation is likely to dominate for most problems of practical interest, while the elastic strain is relatively unimportant (WHE 03). At the meantime, when simulating this kind of behavior, the elasticity part in elastoplastic model has an influence on the response of stress and strain (CHA 05). In that case, the feature of anisotropy in elastic domain in the whole process of deformation is way important even though some elastoplastic models neglect the anisotropic characteristic in the elastic part for the reason of simplification (SCH 68; ROS 68; WRO 80). Besides, the hyperelastic formulations which are energy conservative and thermodynamically consistent present the feature that the elastic behavior can be derived from a relative potential energy function (HOU 00; EIN 04; GAJ 08). That means that a theoretical relation between stress and strain can be derived from this kind of formulation. Furthermore, another point needs to be focused on is that the stiffness matrix derived from this formulation varies as a function of current stress state, i.e. the pressure dependency (HOU
05). Moreover, the induced anisotropy which occurs when non-zero value of off diagonal components exist in the stiffness matrix shows a good relation with the natural response of granular materials.

The problem of observation and definition of failure of soil has also been a hot issue for decades. The complex properties of granular materials determine that the analysis work is hard and elusive for both theoretical and physical definition of failure. Despite the earliest but outdated definition that failure occurs accompanied with the existence of a limit stress state which is impossible to exceed for any possible monotonous loading path, the basic but classical Lyapunov's definition of stability shows a different and contradictory theory with the limit stress state theory mentioned before (LYA 07). Moreover, Lyapunov's definition clearly states that the materials instabilities can be expected in elastoplastic media. But Lyapunov's definition has limitation because it does not provide a well defined mathematical equation for a proper media. Thus the Hill's condition of stability is taken into account (HIL 58). Hill states that a stress strain state is unstable if one loading direction which can be pursed in an infinitesimal manner exists and there is no external energy input in this direction. It means that the deformation could proceed itself without any input external loading or energy. Basing on Hill's theory, the equation of second-order work which is the product of incremental strain and incremental stress is established and used for determine the existence of specific failure. According to Daouadji's synthesis work (DAO 10), the equations of bifurcation domain in the stress space and of cones of unstable loading directions with proper control parameters are established from experimental, theoretical and numerical points. It is also far important that an essential feature of failure can be the outburst of kinetic energy accompanied with the drastically increased strains and decreased stresses. Furthermore, three necessary and sufficient conditions for an effective failure are proposed, namely: (1) the stress state within bifurcation domain, (2) the loading direction within cones of unstable directions, and (3) the proper parameters set in right place (LAO 02; PRU 009; NIC 09).

In order to apply the second-order work criterion basing on an appropriate constitutive relation, several works should be done in advance, and these works will be shown in the following Chapters.

In Chapter one, the literature research about anisotropy is firstly presented, which includes the different components of inherent and induced anisotropy in the way of generation, and the fabric and stress/stiffness anisotropy. As well, some latest anisotropic elastoplastic models are shown. Moreover, a hyperelastic formulation which is closely relevant with actual engineering situation is introduced later. In the latest, the works performed related to the second-order work are also introduced in
detailed.
In the second Chapter, an elastoplastic constitutive model 'Plasol' which contains an implicit backward Euler integration algorithm is firstly presented. Much emphasis is put on the fact that this Plasol model involves a general yield criterion which could be anyone of classical failure criterion, such as Mohr-Coulomb, or Drücker Prager and so on. Secondly, a hyperelastic strain energy function which is thermodynamically consistent is taking into account. The elastic strain energy is expressed in terms of strain invariants so that related stress invariants as a function of the strains can be derived from this function, as well as the incremental stiffness matrix. Thirdly, the combination and replacement work of Plasol model and the elastic stress strain relation mentioned above is implemented. Thus a new constitutive relation which is much appropriate theoretically for the behavior of non-associated granular materials is established. Last, another constitutive model which also considers the anisotropic elasticity will be shown in detail.

The works concerning numerical simulations will be described in the following Chapter 3. First of all, the equations of models mentioned before, including the original Plasol model, Modified SANISAND model and the new model we proposed, will be written in the User-Defined Material (abbreviated as 'UMAT') tool in FORTRAN which could be used as external model in ABAQUS simulation. Later, a simplified but representative axisymmetric Abaqus model will be built and run with these models. After, the comparison results between the simulations and experimental data will be shown, as well as the relevant evolution of tangent modulus. Then, the undrained triaxial tests will be implemented to analyze the influence of parameter namely pressure exponent, and the effect of different densities of samples with different Young's moduli and Poisson's ratios will be taken into account to test on the influence of pressure exponent. At the end, the simulation by modified SANISAND model will be compared with new model.

In the fourth Chapter, the second-order work criterion will be adopted with this new model written into the Umat format. The use of global or integrated second-order work criterion used as a safety factor will be discussed.

Finally, the last Chapter about conclusion and further perspectives will be drawn.

## Chapter 1 - Literature review

### 1.1 Research on anisotropy

### 1.1.1 Inherent and induced anisotropy

In materials science, anisotropy is a material's directional dependence of a physical property. It is a critical consideration for materials selection in engineering applications. Tensor descriptions of material properties can be used to determine the directional dependence of that property. In other words, the diverse directional dependence of internal structure determines the anisotropic physical property, as well as the application in engineering problems.

Anisotropy can be composed by two forms: inherent anisotropy and stress induced anisotropy. The former one is formed because of preferred particles and contacts orientations that develop in the process of deposition, while the latter one is due to the displacement or the loaded stress during later activities (ART 72; ODA 72c). According to Arthur and Menzies (ART 72), the inherent anisotropy of non-cohesive granular materials is described. Samples are set up in different directions of deposition referring to the sample axes and then loaded in principal stress directions. It is concluded that inherent anisotropy can emerge in the direction corresponding to strength and pre-failure stress-strain anisotropy. Furthermore, the coincidence of principal stress axes and direction of strain increment is shown up, as well as the anisotropic effects on stress-strain response. Oda shows that the fabric reconstruction of initial fabric is continuous and is caused mostly by the sliding along unstable contacts between the neighboring particles and partly by the rotation of particle during the process of axial strain increment. Note that a material's fabric describes the spatial and geometric configuration of all the elements that make it up.

Arthur et al. (ART 77) also publishes a work about the pre-failure induced anisotropy in dense granular media. After the plane strain test with controlled changes of principal stress directions, it is found that the induced anisotropy has less influence on the angle of shearing resistance but large effect on the secant modulus on reloading after a principal stress rotation. Moreover, the different directions between the present axes of induced anisotropy and previous axes leads to slight and diminishing deviation between axes of stress and strain rates.

This theory is also mentioned and used as fundament by later researchers. Hu et al. (HU 10) emphasize the fact that the anisotropy of structure of granular materials can influence the response of stress and strain, and its feature of two kinds of components, i.e. inherent and induced anisotropy. In their paper, the fabric tensor is used to explicitly present the interactions between individual particles, and its eigenvalues can be treated as a measure of the fabric anisotropy, so does the coordination number to the packing density of material. Here, the coordination number is described as the average of the contacts of all grains of an assembly. The Discrete Element Method (DEM) in two-dimensional is used to simulate the evolution of fabric in cyclic loading condition. It is found that the inherent anisotropy decreases in the process of isotropic consolidation, but increases in the process of anisotropic consolidation. The anisotropy which is induced by the loading cyclic stress path has a dependency on maximum and minimum values of cyclic loadings. Furthermore, the ratio of normal to shear springs stiffnesses can influence the development of anisotropy, with the fact that higher value of this ratios corresponding to lower degree of anisotropy induced by anisotropic consolidation.

Induced anisotropy could also be decomposed into fabric anisotropy, stress (or stiffness) anisotropy and permeability anisotropy (KUH 15). According to Kuhn et al., the anisotropies of granular particles (fabric) and of materials' strength, stiffness and permeability are induced by the external loading on samples. For fabric anisotropy measurements, there are four categories which are preferred orientation of particle, particle surface, normal contacts and void space can be developed. It is found that the measures of particle's orientation are the most representative one to loading. For stiffness anisotropy, it is concluded that this kind of stiffness increases along the initial compressive loading direction and reduces along the extension direction. Furthermore, it is closely matched with a special measure of contact fabric. For permeability anisotropy, the results that the permeability is negatively linked with directional mean free path and is positively linked with pore width show that the induced permeability anisotropy is caused by the changes in the direction of directional hydraulic radius.

### 1.1.2 Fabric and stress/stiffness anisotropy

For geotechnical materials, especially natural soil, the response of stress and strain shows a non-linear behavior, which could be described by using elastoplastic models for instance which can be
phenomenological or micro-mechanical-based. When simulating this kind of behavior, the elasticity part in elastoplastic model has an influence on the response of stress and strain (CHA 05). According to Chang and Hicher, the inter-particle stiffness is closely related to inter-particle elastic constants and proportional to the mean particle size. This stiffness can directly affect the response of stress and strain under an externally applied stress for an assembly of particles. In this paper, an elastoplastic model which is a combination of Hertz-Mindlin's elastic law and Mohr-Coulomb's plastic law is established. They also conclude that the anisotropic samples which are placed in different loading directions show the model's qualification and consideration of the influence of inherent anisotropy on the behavior of stress and strain in drained triaxial loading test.

Oda et al. (ODA 85) presented that three main factors affecting and quantifying the fabric anisotropy which are the: (1) distribution of normal contacts; (2) shape of non-spherical particles; and (3) shape of associated voids. In their paper, a basis of biaxial compression tests performed on two-dimensional assemblies is used to define the second-rank fabric tensor which is representative of the corresponding anisotropy. It is found that the direction of principal axes of fabric tensor changes gradually approaching to the principal axes of stress tensor in the process of monotonic loading. Moreover, the generation of particle contacts along the maximum principal compression are closely linked with the order of column-like loading paths along the same direction, and these new contacts result in the stress-induced anisotropy and seem to be a contributing factor to the post-peak failure.

Li and Dafalias (LI 12) proved that the anisotropic critical state theory which accounts for the role of anisotropic fabric of the classical critical state theory is appropriate for large deformations. A fabric tensor is taken into account during the studies of micromechanics and experiments, and a scalar fabric anisotropy variable which is an evolving fabric tensor in the direction of loading is also proposed. It is shown that, when this variable reaches its critical state value, the dilatancy state line in the void ratio - pressure plane is coincident with the classical critical state line, and the dilatancy state parameter evaluating the contracting or dilating trends of current state changes toward to the value of dilatancy angle. Moreover, the feature of static liquefaction occurs when the dilatancy state parameter equals to zero and stress ratio reaches its critical value.

Another related factor that has to be considered is its cross-anisotropy (or transverse isotropy) due to the geological processes, which leads to a more complicated and unpredictable relationship of stress and strain (CAS 17). The authors present that natural clays have a feature of significant degree
of anisotropy in fabric because of the shape of clay platelets, deposition process and one-dimensional consolidation, and this behavior can be called as cross-anisotropy or transversely isotropy. A new constitutive model includes the anisotropic behavior of elastic and plastic features and incorporates the stress-dependent cross-anisotropic elastic behavior using three independent elastic parameters and one additional variable, namely the ratio of horizontal and vertical stiffness. The evolution of elastic anisotropy is not considered within this new model, but the noticeable variation of elastic anisotropy can be caused within a large deformation condition, which induces a more complex relationship between strain and stress.

Zdravkovic et al. (ZDR 02) state that neglecting this anisotropy of natural soil behavior can induce high inaccuracy during the predicting of stress-strain response. When designing and calculating the safety factor of embankment, the traditional way is often based on a limit equilibrium approach with a consideration of isotropic materials. However, the natural soil always shows the behavior of anisotropy in both stiffness and strength, so that the old practice would be inaccurate and uneconomic even the empirical factors are introduced into conventional design procedures.

Other studies have been done about the influence of anisotropy on the stress-strain in different situations. Toyota et al. (TOY 14) investigated the stability of slopes which were affected by the anisotropy of shear strength induced by $K_{0}$ consolidation and swelling in cohesive soils. Due to the reason that shear direction changes at each sliding points, the strength anisotropy should be taken into account in the process of stability analyses. Therefore, in this paper the undrained torsional shear tests and two-dimensional plane strain tests conducted on samples which are consolidated and swollen in different directions are implemented to assess the influence of anisotropy of shear strength. Results show that the anisotropy of undrained shear strength closely corresponding to the difference between consolidation and shearing in terms of directions and stress condition. Furthermore, the slope stability analyses also need take the anisotropy of undrained shear strength into account.

Additionally, the influence of anisotropy on the failure with localization pattern and the limit loading capacity of geostructure is investigated by Chang et al. (CHA 14). For this purpose, an extended Drücker Prager yield criterion is developed for this kind of transversely isotropic geomaterials. After simulating with Finite Element Method, the results show that the localization pattern and the critical bearing capacity of geostructure are corresponding closely to the principal direction of materials and the degree of anisotropy.

The existence and the level of anisotropy occurring in particulate materials which have nearly spherical aggregates have been shown within the work of Oboudi et al. (OBO 16). To this aim, both experimental test and theoretical support are contained in this study. The former is about the performance of series of tests at different sample orientations, while the latter is about the plasticity formulation based on a critical plane approach. The results present that the proposed framework can account for the various performance of load-induced anisotropy.

### 1.1.3 Anisotropic elastoplastic models

Before introducing anisotropic elastoplastic models, the general elastoplastic models have been accepted for decades and adopted until now. For example, the elastoplastic Plasol (BAR 98) model which contains a linear elastic part and a plastic part is a general constitutive model for granular materials which will be show detailedly in the Chapter 2. In this Section, however, the anisotropic elastoplastic models are focused on.

Several authors have done extensive experimental testing and proposed relevant elastoplastic soil models for modeling the mechanical behavior of natural soils. Dafalias (DAF 86) presents the fact that the isotropic constitutive model is inappropriate for modeling the behavior of stress and strain of soil within the framework of elastoplasticity. Then an evolution law with yield surface accompanying anisotropic features such as rotational hardening has been postulated basing on the provided expression of a rotated and distorted ellipse as the yield surface. These constitutive equations equip a significant feature of simplicity, as well as the successful comparison with experimental data.

Whittle and Kavvadas (KAV 94) proposed an effective stress model for clays normally and moderately and over-consolidated. Three components are comprised in this model which are: the elastoplastic model for clay normally consolidated and including the behavior of anisotropy and strain softening; the equations describing the nonlinearity of small strain and characterizing the hysteretic response in the process of loading and unloading; and the surface plasticity of boundary for irrecoverable, anisotropic features of overconsolidated clays. Furthermore, the complexity of model can be controlled by using specific parameters which are obtained from few standardized soil tests.

However, some of these models assume that the elastic part of the model is isotropic due to the reasons of simplification and thoughts of leading position of plastic deformation, yet it is widely
known that the natural soils exhibit anisotropy of elastic behavior (GRA 83; WHE 03). Graham and Houlsby describe the anisotropic behavior of natural clays because of the mode of deposition. Five elastic parameters which could reduce to three in the situation of triaxial tests are also proposed to describe the transverse isotropy. For example, in these required parameters, the bulk modulus, shear modulus, and cross modulus are easily identified and can be used to express the behavior of anisotropic soil between strain and stress, i.e., mean stress and shear strain, shear stress and volumetric strain specifically. The result of this anisotropic model is compared with true triaxial tests and results show that the clay is approximately 1.8 times stiffer along the horizontal direction than vertical direction which is a quite strongly anisotropic.

According to Wheeler et al. (WHE 03), an anisotropic elastoplastic model is presented on the basis of experimental data for shape of yield curve and relationship describing the influence of plastic straining on yield curve inclination. This model is called S-CLAY1 and incorporates a rotational component of hardening in the reason of influence of plastic anisotropy. This rotational hardening law contains the dependence on plastic shear and volumetric strain increment and is also validated by conventional drained triaxial tests on clay. The simulation results of new model are compared with experimental data and proved improving the performance of the Modified Cam Clay model (ROS 68). However, it also equips a shortcoming, which is that it would under-predict the post yield volumetric strain in the case of high values of stress ratio. This shortcoming could be made up by taking the bonding and destruction into account which has been shown in an extended version (KAE 05) of this model.

Schädlich and Schweiger (SCH 13) check the effect of anisotropic elasticity on the deformation behavior of deep excavations and strip footings in means of a constitutive model. A model which incorporates the feature of taking the anisotropic elasticity into account in the range of small strain is proposed and aimed to solve two simple benchmark problems mentioned above. The studies show that under the condition of relatively low displacement and strains, the influence of small strain stiffness anisotropy is more significant than the case under higher strains conditions. It is also found that this kind of stiffness anisotropy can be approximated as the average of axial stiffnesses.

However, their model is limited in a very small range of strains and can't be regarded as a full constitutive model because it doesn't consider stress-dependent stiffness and plastic strains. Based on this model, a model named S-CLAY1S is proposed by Karstunen et al. (KAE 05), and extended by

Castro and Sivasithamparam (CAS 17), avoids this problem and contains a yield surface function and stress-dependent stiffness. Based on the fact that S-CLAY1 takes the initial and plastic strain induced anisotropy into account, its extension model, i.e., S-CLAY1S also considers additionally the case of inter-particle bonding and degradation of bonds. After comparing results of these two models and isotropic Modified Cam Clay with the obtained field data, it is shown that the anisotropy is importantly and necessarily taken into account, whereas the influence of destructuration seems to be implicit on the predicted deformations. This model also had a good agreement with the field measurements, according to the work of Yildiz et al. (YIL 09) on embankment test.

The extended model based on S-CLAY1S, is proposed for situation of cross-anisotropy (CAS 17) which also shows anisotropy behavior both of elastic and plastic nature as well as the stress-independent cross-anisotropic elastic behavior. Note that the cross-anisotropy behavior could be described with three elastic parameters in this model which have been discussed by Graham and Houlsby (GRA 83). Only one additional parameter is contained in this new model, i.e., the ratio of horizontal stiffness to vertical stiffness, which can be easily obtained from conventional experimental tests. By the model, the initial non-vertical effective stress path can be analytically and easily captured, and the deviatoric strain in the process of isotropic loading and unloading can be predicted as well.

Another extended model based on S-CLAY1S was proposed by combining the anisotropy and destructuration using an elasto-viscoplastic model on the natural soft clays (YIN 11). The clays which equip the strain-rate-dependency were tested at constant strain-rate and creep in one-dimensional and triaxial conditions. With this model, the result that loading scenarios is necessary to get an accurate prediction when accounting for anisotropy and / or destructuration was revealed. Also, the fact that proposed model can successfully reproduced the time-dependent behavior of natural soft clays can be obtained by the comparisons between predicted and measured results.

### 1.2 Hyperelastic formulation

Before introducing the hyperelastic formulation in this Section, the definition of elasticity, hyperelasticity and hypoelasticity should be well described in detailed. Elasticity in materials science is the ability of a material body to resist a distorting influence and to return to its original size and shape when the influence is removed. If the materials is elastic, this material body will return to its
original shape and size after the removal of influence or force. Hyperelasticity or hyperelastic material is a type of constitutive model for ideally elastic material for which the stress-strain relationship derives from a strain energy density function. This type of material is a special case of simple elastic material. Note that the hyperelasticity equips an integrable expression because it is derived from a potential function, and it shows a conservative elastic response. Hypoelasticity or a hypoelastic material is an elastic material that has a constitutive model independent of finite strain measures except in the linearized case. Hypoelastic material models are distinct from hyperelastic material models (or standard elasticity models) in that, except under special circumstances, they cannot be derived from a strain energy density function. This kind of elasticity shows non-conservative elastic response.

An approach could be used here for the non-linear elastic response of stress and strain (HOU 85; HUE 92; BOR 97) which is related to hyperelasticity. According to Houlsby (HOU 85), the use of a shear modulus proportional to the mean effective stress is justified by measurement modeling the elastic behavior. It is also shown that this measurement has a shortcoming that it induces a non-conservative elastic behavior and is inappropriate for the case of cyclic loading. Thus, a theoretical approach using the pressure dependency shear modulus is proposed and several experimental evidences for supporting this approach are discussed. Moreover, Hueckel et al (HUE 92) proved an approach that the shear modulus is not only depending on the mean pressure but also on the over-consolidation ratio. Later, a stored energy functions including two invariants and describing hyperelastic characteristics is coupled with a critical state plasticity model by Borja et al. (BOR 97). The pressure dependency elastic shear modulus is included in the energy function and makes the function satisfying elastic behavior for any loading path and being energy conservative. After assessing the pressure dependency of shear modulus within elastic and plastic response for undrained elastic responses, this hyperelastic model shows a fundamentally accuracy simulation of elastic behavior.

Unlike constitutive models with hypoelastic formulation, the hyperelastic formulations present energy conservative behavior (elastic stress or strain can be derived from a potential function) and thermodynamics consistency (HOU 00; EIN 04; HOU 05; GAJ 08). Houlsby and Purzin (HOU 00) proposed four energy functions to describe the combinations of stress, strain and temperature, and these energy functions could be transferred to each other by using the Legendre transformations. The
irreversible behavior could be described by a dissipation function or a yield function, which is related to the degenerate case of Legendre transformation. According to Einav et al. (EIN 04), a versatile energy potential function is firstly presented within the framework of an energy conservative or non-conservative elasticity and a plasticity model. The fact that the stiffness of soil depends on the pressure is also shown in this work. The combination of this energy potential and hyperplastic constitutive relation allows for the relevant model to obey the Law of Energy Conservation for both elastic and plastic parts of soil behavior. Except that, two very important features are that this model can automatically show a stress-induced cross-anisotropy of elastic component, and that a dilatancy term can arise because of the pressure dependency shear modulus. The model with these features, which are neglected by conventional hypoelastic-plastic model, shows a significant accuracy on the prediction of undrained behaviors of overconsolidated clays. Houlsby et al. (HOU 05) also present a hyperelastic formulation which shows the non-linear elastic behavior and the dependency of elastic modulus on mean effective stress. Furthermore, the elastic model which is derived from this hyperelastic formulation allows the elastic modulus being a function of effective mean stress, and satisfies the thermodynamic acceptability. Similarly, Gajo and Bigoni (GAJ 08) also present a formulation based on hyperelasticity after showing the experimental evidence that cohesive and granular materials contain an elastic range in which the elasticity is non-linear and anisotropic. This formulation is established within the framework of elastoplastic coupling via the new proposal of elastic potentials and the combination of non-linear elasticity and dependency of fabric tensor on plastic strain. Combining the formulation in the proposed constitutive framework, the simulation results show a very accurate fitting degree on the evolution of elastic behavior with the existing experimental data.

### 1.3 Second-order work criterion

For modeling the behavior of geo-materials, several constitutive models are proposed and mentioned above. However, the observation and definition of the failure of sample is also important, so do the tools for detecting failure.

The theoretical and physical definition of failure in solid and its analysis is hard and elusive to determine in granular materials. For more than a century, much works have been done about finding a
way to detect the failure behavior, as well as its defining criterions. In fact, there is a version of failure definition at initial time that failure occurs accompanied with the existence of special limit stress states which are impossible to exceed for any possible monotonous loading path. This physical definition shows that large deformation, cracks, or fragmentation will suddenly occur if any tiny additional loading is loaded at such limit stress state. This change of materials state is called roughly 'failure'.

Around the initial definition above, two typical classes of failure modes due to the instabilities can be found in either geometric or materials, which could be expressed in column buckling, or constitutive behavior respectively based on the observations of experiments tests. Within the domain of instability, two popular criteria have emerged. The first criterion is about the vanishing of determinant value of the acoustic tensor (RUD 75) which is accompanied with the emergence of plastic strain localization, whereas the second one refers to the vanishing of the determinant of the whole constitutive tensor which involves the signals failure at plastic limit condition.

For the associated materials, such as metal materials which follow the associative flow rule, these two criteria mentioned above coincide because of the symmetry feature of the elasto-plastic tensor. However, the geomaterials are widely known as non-associated materials because of the non-symmetry of the elasto-plastic tensor. Therefore, based on the evidence of much laboratory experiments, the localization criterion can be met before the plastic limit criterion for particularly dense sands or overconsolidated clays (VAR 95). However, for the fact that the stress controlled undrained triaxial test on very loose sands show a performance of different type of failure, this different mode of failure occurring at the peak of deviatoric stress which is not described before is named as 'diffuse failure' (KHO 06) to distinguish it from the localized one (NIC 10; DAO 2010; JRA 12).

The problems of bifurcation occur due to the loss of uniqueness of the basic governing equations' solution which is caused by the instability of materials. Therefore, it is quite necessary to propose the bifurcation theory being a general framework when analyzing all kinds of failure. Bifurcation happens at the time that the system state changes suddenly following with one of the at least two possibilities, which could be either stable or unstable state, under the condition of continuous variations of state variables. For example, with the condition of proper loading, failure can happen with a state phenomenon of sudden transition from a static regime to a dynamic one accompanying with the
exponential growth of strains (DAR 07; NIC 09; SIB 09). This kind of phenomenon is closely related with the experimental phenomenon of failure. Furthermore, one of the bifurcated states' characteristics refers to the fact that failure will happen with small additional perturbations in the system. Based on this kind of property, failure can be also considered as an instability phenomenon in the Lyapunov's definition of stability (LYA 07) which will be described below. As a conclusion, any tiny additional loading at a given bifurcation state on the curve of stress and strain will result in an infinitely large responses if proper control variables are applied (direction of loading and mix loading conditions).

For non-associated materials, which relate to the non-symmetry of the elastoplastic tensor, the elastoplastic theory considering the bifurcation criteria precede the plastic limit criterion which can be represented precisely by the zero determinant value of product of elastoplastic matrix and unit matrix (BIG 91). Different bifurcation criteria exist in the literature which is relating to different modes of failure. Concerning the shear band formation by plastic strain localizations, Rice's criterion (RIC 76) is based on the description that the earliest shear band in normal direction corresponds to the vanishing values of the so-called 'acoustic tensor'. However, this phenomenon occurs before the plastic limit criterion is met for non-associated materials, and this has been demonstrated and verified in experimental tests on dense sand (DES 90). The plastic strain localization refers to the bifurcated strain mode from a diffuse one to a strictly discontinuous one. This kind of bifurcation can be called as 'discontinuous bifurcation' or localized mode of bifurcation.

As an opposition of discontinuous bifurcation, the 'continuous bifurcation' also refers to a failure mode but which does not contain strain localization behaviors. Like we present before, this diffuse failure mode is subjected to a bifurcation with the vanishing constitutive uniqueness at the bifurcation point. According to Nova's theory (NOV 94), different loading control model can lead to different response paths at the bifurcation point, and for certain control modes, the stress state is no more controllable any longer. This continuous bifurcation, as well as the related diffuse failure mode can be detected by Hill's stability condition (HIL 58), which corresponds to the vanishing value of second-order work for unstable states (DAR 09). Moreover, this vanishing value of second-order work corresponds to the vanishing value of determinant of the symmetric part of elastoplastic matrix for non-linear constitutive relation.

Now two basic definition of failure namely Lyapunov's definition of stability and Hill's stability
condition will be presented in details. Based on Lyapunov's stability definition re-visited in the field of solid mechanics, the statement is: "A stress-strain state, for a given material after a given loading history, is called stable, if for every positive scalar $\varepsilon$, a positive number $\eta(\varepsilon)$ exists such that for all incremental loading bounded by $\eta$, the associated responses remain bounded by $\varepsilon$." (LAO 02)

According to this definition of stability, all the limit stress states which are mentioned before are unstable. That means that a very small incremental reverse stress at the limit stress state could induce a small response, whereas this small incremental additional stress exceeding the limit state can produce large strain response. Moreover, Lyapunov's definition clearly states that the materials instabilities can be expected in elastoplastic media. Taking the fact into account that some limit stress states are strictly met before the Mohr-Coulomb plastic limit surface, it is possible that instabilities can occur before the Mohr-Coulomb plastic limit condition is satisfied.

However, Lyapunov's definition is inappropriate to use in the content of geomaterials. Thus, the Hill's condition of stability is taken into account. Hill states that a stress strain state is unstable if one loading direction which can be pursued in an infinitesimal manner exists and there is no external energy input in this direction. That means the deformation could proceed itself without any input external loading or energy. Indeed, in some practice situations, failure happens with external energy such as the weight or loads on the slope, whereas in some other cases, failure occurs without any additional energy input from outside such as the landslides, rockfalls and so on. Moreover, based on Hill's condition of stability, the stress state is stable if the second-order work which is the product of incremental strain and incremental stress linked by constitutive relation is strictly positive. This is the fundamental second-order work criterion, which could be expressed as $w_{2}$ by the following general expression.

$$
\begin{equation*}
w_{2}=d \sigma \cdot d \varepsilon \tag{1.1}
\end{equation*}
$$

In this expression, $w_{2}$ by presents the second-order work, while and $\sigma$ are $\varepsilon$ stress and strain respectively. The positive value of $w_{2}$ refers to a stable stress state.

Nicot and Darve (NIC 07) presented the investigation of bifurcation from the view of micro-mechanics of granular materials. The fact that the vanishing value of second-order work defined on the macroscopic scale can be viewed as a fundamental role for detecting the occurrence of bifurcation referring to the loss of sustainability is noted. Also, the relationship between the macroscopic second-order work at the sample scale and the discrete local expression which represents
the microscopic variables at particle contact scale is established. Furthermore, this relationship helps us to figure out which factor at microscopic scale should be responsible for the vanishing of macroscopic second-order work. Darve et al. (DAR 07) also do the same work from the macro-mechanics as well as the micro-mechanics points. The discrete element method is used to study this problem via the phenomenological constitutive relations and numerical simulations after showing the experimental evidence. The equations of bifurcation domain and cones of unstable direction are established, and the relationship between second-order work criterion and the diffuse failure mode is taken into consideration by means of continuum and discrete mechanics approaches. The fact turns out that the second-order work criterion can determine analytically and verify numerically the boundaries of bifurcation domain and cones of unstable direction by means of the macro- and microscopic values of second-order work. From the macroscopic point of view, the expression for macro second-order work can be expressed as an integration of second-order work in microscopic scale. In the following expression, $W_{2}$ refers to the integrated or global second-order work, while the variable $V$ means the integration range of micro second-order work.

$$
\begin{equation*}
W_{2}=\int_{V} \underline{d \varepsilon}^{t} \cdot \underline{d \sigma} d V \tag{1.2}
\end{equation*}
$$

For the three-dimensional case, Prunier et al. (PRU 09) analyzes the problems of bifurcation of granular media for the first time. Two phenomenological constitutive relations, namely the incrementally piece-wise linear and nonlinear relations proposed by Darve et al. (DAR 95) are considered for the analytical and numerical investigation of Hill's criterion. Moreover, the 3D bifurcation domain and 3D cones of unstable directions are taken into analysis for these two relations. Results show that the shape and structure of bifurcation domain is probably affected by the yield criterion, and the direction of first cones of unstable direction in the stress space is oriented close to the origin of stress plane no matter what the loading path is.

The relation of second-order work, kinetic energy and diffuse failure mode is considered by Nicot et al. (NIC 10) for granular materials. After presenting the condition that collapse of granular soils is related to the sudden burst of kinetic energy, the relation between the burst of kinetic energy and vanishing value of second-order work is numerically and theoretically obtained and experimentally verified. The same research is also done by Nguyen et al. (NGU 16) by means of numerical analysis. They find that the increasing kinetic energy is caused by the difference between external second-order work involving the external loading parameters and internal second-order work.

The dynamic response occurs when the internal stress loses the capacity to balance the external stress which relates to the value of internal second-order work is lower than the one of external second-order work.

According to Daouadji et al. (DAO 10), a synthesis of the works about the second-order work criterion done by various teams from France, Italy and Canada is presented. In this paper, the second-order work criterion with the analytical and numerical analysis via both the macro-mechanical phenomenological elastoplastic constitutive relations and micro-mechanical mode is proposed and analyzed systematically. Furthermore, the equations of bifurcation domain in the stress space and of cones of unstable loading directions with proper control parameters are established from experimental, theoretical and numerical points. The authors also present that there are three necessary and sufficient conditions for an effective failure, namely: (1) the stress state within bifurcation domain, (2) the loading direction within cones of unstable directions, and (3) the proper parameters set in right place.

A proper framework based on energy conservation is built by using the second-order work criterion in the work of Nicot and Darve (NIC 15). It is also shown that the increase in kinetic energy in incremental loading condition is equal to the difference of external second-order work and internal second-order work which involves the constitutive properties of material. If an additional external pressure is loaded at the stress limit state, a dramatic increase of strain rate happens. Moreover, the plastic limit theory appears to be one particular case of second-order work theory in the theoretical framework, and the incremental external loading resulting in the sharp increasing of kinetic energy leads to the sudden collapse of specimen from the experimental results.

### 1.4 Objectives of this research

In the present work, our objectives are the following:

- Obtain a correct and accurate representation for the frictional behavior of granular materials.
- Obey the First Law of Thermodynamics or guarantee thermodynamic acceptability about the elastic part of constitutive model.
- Find a proper integration algorithm to correct the overflow problems of plastic strain in the plastic part of constitutive model.
- Emphasize the influence of anisotropy of elastic behavior even though the initial elastic
matrix is isotropic.
- Evaluate the stabilities with a proper failure criterion as the safety factor.
- Allow the large deformation of the objects of simulations, for example the landslide phenomenon of slope.

In order to be able to study the above points mentioned, some requirements must be satisfied as follows:

- The elastic stress strain relation which is derived from a proper hyperelastic energy function obeying the First Law of Thermodynamics should be established, as well as the further anisotropy of elastic behavior.
- A new constitutive model should contain the elastic model mentioned above and a plastic model which can correct the overflow of plastic strain.
- The Finite Element Method (will be abbreviated as 'FEM' in later content) tools can simulate different geometric and parameters of models.
- The new constitutive model needs to be transferred into numerical language so that the FEM tools can run it directly.
- Experimental tests should be implemented so that the comparison between laboratory and numerical simulation can be made.
- The second-order work criterion based on the new model should be calculated for detecting the failure.


### 1.5 Originality of this work

The work that is presented in this thesis aims to propose an appropriate theoretically and numerically approach to model the behavior of stress strain of frictionally granular materials. The main points are addressed in this thesis:

- A more actual constitutive model which includes a stiffness matrix depending on current stress state is established, and the anisotropy induced by stress arises as a natural consequence of this hyperelastic formulation corresponds well to real observations of soil behavior.
- Only one set of parameter is necessary for one type of soil whatever confining pressure or
consolidation conditions the soil sample is loaded. That means for different confining pressures, just one optimal set of parameter is needed and well appropriate for simulating the behavior of granular material under the framework of new constitutive relation.
- The stress-induced anisotropy which relates to the real observations is found to have influence on the stress-strain response, even change the hardening tendency of plastic strain. The degree of this kind of anisotropy is affected by one single variable with other parameters well controlled.
- From a macroscopic perspective, the density of sample is found to have effect on the degree of anisotropy which could illustrate a different response in the stress plane.
- A proper stability criterion is established basing on new model as the safety factor and can detect accurately the occurrence of failure.


## Chapter 2 - A new constitutive relation

In this Chapter, an elastoplastic constitutive model 'Plasol' which contains an implicit backward Euler integration algorithm and a hyperelastic strain energy function which is thermodynamically consistent are presented in details. Then, the proposed constitutive relation which combines properly these two relations above is established and supposed to be an appropriate theoretically relation for modeling the behavior of non-associated granular materials. Note that unless explicitly expressed, the stresses used in the following are effective stresses which are equal to the total stress for dry cases or to the difference between the total normal stresses and the pore pressure.

### 2.1 Plasol model

### 2.1.1 Incremental general elastoplastic formulation

Generally, the elastoplastic relations are formulated in rate form because of the dependency of the stress-strain response on its actual state. The sign convention of solid mechanics is used, i.e. compression is recognized as negative and traction as positive. The strain rate is divided into the elastic and plastic parts (additive decomposition):

$$
\begin{equation*}
\dot{\varepsilon}_{i j}=\dot{\varepsilon}_{i j}^{e}+\dot{\varepsilon}_{i j}^{p} \tag{2.1}
\end{equation*}
$$

The elastic part is related with the stress tensor which refers to the Hooke Law:

$$
\begin{equation*}
\dot{\sigma}_{i j}=D_{i j k l}^{e} \dot{\varepsilon}_{k l}^{e} \tag{2.2}
\end{equation*}
$$

where the constitutive elastic tensor is defined by

$$
\begin{equation*}
D_{i j k l}^{e}=\frac{E}{1+v} \delta_{i k} \delta_{j l}+\frac{E v}{(1+v)(1-2 v)} \delta_{i j} \delta_{k l} \tag{2.3}
\end{equation*}
$$

Using the definition of the compressibility modulus $K$ and the shear modulus $G$

$$
\begin{align*}
& K=\frac{E}{3(1-2 v)}  \tag{2.4}\\
& G=\frac{E}{2(1+v)} \tag{2.5}
\end{align*}
$$

the constitutive elastic tensor can be rewritten as:

$$
\begin{equation*}
D_{i j k l}^{e}=2 G \delta_{i k} \delta_{j l}+\left(K-\frac{2}{3} G\right) \delta_{i j} \delta_{k l} \tag{2.6}
\end{equation*}
$$

The limitation between elastic and plastic domain is expressed by a yield surface in the principal
stress space, and this surface is described by using the yield function $f$. Thus, it could say that the stress state is within the elastic domain if $f<0$, or in the plastic domain if $f=0(f>0$ is impossible for rate-independent materials). In the classical or associated plasticity, the rate of plastic flow is perpendicular to the yield surface and its intensity is proportional to the so called plastic multiplier $\dot{\lambda}$. During the process of plastic flow, the stress state must remain on the yield surface, i.e. $f$ $=0$.

However, the more general framework of non-associated plasticity of geomaterials is considered for limiting the dilatancy process. Then the rate of plastic flow is perpendicular to a plastic potential $g$. It is worth noting that the plastic potential $g$ could be substituted by yield function $f$ for associated materials.

$$
\begin{equation*}
\dot{\varepsilon}_{i j}^{p}=\dot{\lambda} \frac{\partial g}{\partial \sigma_{i j}} \tag{2.7}
\end{equation*}
$$

In the equation (2.7), the plastic potential $g$ is defined as the same definition method of yield surface but assuming a dilatancy angle distinct from the friction angle. The cohesion is avoided because only the derivative of $g$ is required. Then the combination of (2.1) and (2.7) is shown as:

$$
\begin{equation*}
\dot{\varepsilon}_{i j}=\dot{\varepsilon}_{i j}^{e}+\dot{\lambda} \frac{\partial g}{\partial \sigma_{i j}} \tag{2.8}
\end{equation*}
$$

The value of $\dot{\lambda}$ can be obtained from the consistency condition, while the consistency condition explicates that the stress state stays on the limit surface during the plastic flow. For a perfectly plastic law this condition yield as:

$$
\begin{equation*}
\dot{f}=\frac{\partial f}{\partial \sigma_{i j}} \dot{\sigma}_{i j}=0 \tag{2.9}
\end{equation*}
$$

When considering the general plastic flow with hardening or softening of the internal variables $\kappa$, the consistency condition can be formulated as following:

$$
\begin{equation*}
\dot{f}=\frac{\partial f}{\partial \sigma_{i j}} \dot{\sigma}_{i j}+\frac{\partial f}{\partial \kappa} \dot{\kappa}=0 \tag{2.10}
\end{equation*}
$$

It is widely known that the plastic flow is the key to induce hardening or softening of the limit surface, by introducing a hyperbolic variation of the internal variables which contain the friction angle under triaxial compression path (referred as $\varphi_{\mathrm{C}}$ ), the friction angle under triaxial extension path ( $\varphi_{\mathrm{E}}$ ) and the cohesion $c$. These internal variables are function of the Von Mises equivalent plastic strain $\varepsilon_{e q}^{p}$.

$$
\begin{equation*}
\varphi_{C}=\varphi_{C 0}+\frac{\left(\varphi_{C f}-\varphi_{C 0}\right) \varepsilon_{e q}^{p}}{B_{p}+\varepsilon_{e q}^{p}} \tag{2.11}
\end{equation*}
$$

$$
\begin{align*}
\varphi_{E} & =\varphi_{E 0}+\frac{\left(\varphi_{E f}-\varphi_{E 0}\right) \varepsilon_{e q}^{p}}{B_{p}+\varepsilon_{e q}^{p}}  \tag{2.12}\\
c & =c_{0}+\frac{\left(c_{f}-c_{0}\right) \varepsilon_{e q}^{p}}{B_{c}+\varepsilon_{e q}^{p}} \tag{2.13}
\end{align*}
$$

In these equations, the Von Mises equivalent plastic strain is obtained and calculated by integration of the Von Mises equivalent plastic strain rate $\varepsilon_{e q}^{\dot{p}}$

$$
\begin{align*}
& \varepsilon_{e q}^{p}=\int_{0}^{t} \varepsilon_{e q}^{p} d t  \tag{2.14}\\
& \varepsilon_{e q}^{\dot{p}}=\sqrt{\frac{2}{3} \dot{\varepsilon}_{i j}^{p} \dot{\varepsilon}_{i j}^{p}} \tag{2.15}
\end{align*}
$$

The coefficients $B_{\mathrm{p}}$ and $B_{\mathrm{c}}$ refer to the values of equivalent plastic strains when the half values of the hardening/softening of friction angle and cohesion is achieved respectively. This process could be illustrated in Figure 2.1 clearly.


Figure 2.1 Hardening hyperbolic relation for 2 values of coefficient $B_{\mathrm{p}}\left(\right.$ with $\left.\varphi_{\mathrm{E}}=30^{\circ}, \varphi_{\mathrm{E}}=35^{\circ}\right)(\mathrm{BAR}$ 98)

In some cases, the plastic flow might also lead to a modification of the flow surface which induces an equation proposed by Taylor (TAY 48): based on experimental evidences, the difference between friction angle and dilatancy angle is regarded as constant. That is to say, the modification of friction angle will modify the value of the dilatancy angle.

$$
\begin{equation*}
\varphi-\psi=c s t \tag{2.16}
\end{equation*}
$$

Then the consistency condition (equation (2.10)) can be modified as:

$$
\begin{equation*}
\dot{f}=\frac{\partial f}{\partial \sigma_{i j}} \dot{\sigma}_{i j}+\frac{\partial f}{\partial \varphi_{C}} \dot{\varphi}_{C}+\frac{\partial f}{\partial \varphi_{E}} \dot{\varphi}_{E}+\frac{\partial f}{\partial c} \dot{c}=0 \tag{2.17}
\end{equation*}
$$

and in this equation, the derivatives of internal variables are formulated as:

$$
\begin{align*}
\dot{\varphi}_{C} & =\frac{d \varphi_{C}}{d \varepsilon_{e q}^{p}} \dot{\varepsilon}_{e q}^{p}  \tag{2.18}\\
\dot{\varphi}_{E} & =\frac{d \varphi_{E}}{d \varepsilon_{e q}^{p}} \dot{\varepsilon}_{e q}^{p}  \tag{2.19}\\
\dot{c} & =\frac{d c}{d \varepsilon_{e q}^{p}} \dot{\varepsilon}_{e q}^{p} \tag{2.20}
\end{align*}
$$

The combination of the plastic multiplier $\dot{\lambda}$ and the equivalent plastic strain rate can be achieved from equations (2.7) and (2.15) and shown with:

$$
\begin{equation*}
\dot{\varepsilon}_{e q}^{p}=\operatorname{Val} \dot{\lambda} \tag{2.21}
\end{equation*}
$$

where the values of Val is calculated by

$$
\begin{equation*}
\text { Val }=\sqrt{\frac{2}{3}\left(\frac{\partial g}{\partial \sigma_{i j}} \frac{\partial g}{\partial \sigma_{i j}}-\frac{1}{3} \frac{\partial g}{\partial \sigma_{k k}} \frac{\partial g}{\partial \sigma_{l l}}\right)} \tag{2.22}
\end{equation*}
$$

Therefore, the former consistency condition equation can be modified as:

$$
\begin{equation*}
\dot{f}=\frac{\partial f}{\partial \sigma_{i j}} \dot{\sigma}_{i j}+\operatorname{Val} \dot{\lambda}\left(\frac{\partial f}{\partial \varphi_{C}} \frac{d \varphi_{C}}{d \varepsilon_{e q}^{p}}+\frac{\partial f}{\partial \varphi_{E}} \frac{d \varphi_{E}}{d \varepsilon_{e q}^{p}}+\frac{\partial f}{\partial c} \frac{d c}{d \varepsilon_{e q}^{p}}\right)=0 \tag{2.23}
\end{equation*}
$$

After combining the equation (2.2) and (2.8) together, it is easily concluded that

$$
\begin{equation*}
\dot{\sigma}_{i j}=D_{i j k l}^{e}\left(\dot{\varepsilon}_{k l}-\dot{\lambda} \frac{\partial g}{\partial \sigma_{k l}}\right) \tag{2.24}
\end{equation*}
$$

Then, the value of plastic multiplier $\dot{\lambda}$ could be obtained by combining the equation (2.23) and (2.24) together.

Now, the full version of incremental constitutive elastoplastic relation can be written as:

$$
\begin{equation*}
\dot{\sigma}_{i j}=\left(D_{i j k l}^{e}-D_{i j k l}^{p}\right) \dot{\varepsilon}_{k l} \tag{2.26}
\end{equation*}
$$

where the $D_{i j k l}^{p}$ is the plastic constitutive tensor and formulated as:

$$
\begin{equation*}
D_{i j k l}^{p}=\frac{\frac{\partial f}{\partial \sigma_{a b}} D_{a b k l}^{e} D_{i j c d}^{e} \frac{\partial g}{\partial \sigma_{c d}}}{\frac{\partial f}{\partial \sigma_{m n}} D_{m n o p}^{e} \frac{\partial g}{\partial \sigma_{o p}}-\operatorname{Val}\left(\frac{\partial f\left(\varphi_{c}\right.}{\partial \varphi_{c C l} \varepsilon_{e q}^{p}}+\frac{\partial f d \varphi_{E}}{\partial \varphi_{E} d \varepsilon_{e q}}+\frac{\partial f d c}{\partial c d \varepsilon_{e q}^{p}}\right)} \tag{2.27}
\end{equation*}
$$

It is worth noting that in this kind of general elastoplasticity framework, it is quite common to derive the yield surface (also named as yield function in the following Sections) $f$ and the flow surface (or plastic potential) $g$ with respect to stresses. Conversely, the derivatives of $f$ and $g$ with respect to stresses are quite suitable for these formulations.

### 2.1.2 Yield surfaces for frictional behavior of geomaterials

Modeling the behavior of geomechanical materials requires taking into account the plastic behavior of materials of different rock types, for example, the sand and sandstones, and so on. The plastic behavior is usually based on the concept of yield surface.

Experimentally, the existence of the yield surface expresses the loss of the linear stress-strain relation. The yield surface represents a bound in the stress space which cannot be overcome. As far as frictional materials are concerned, it has been observed from triaxial experiments that a linear relation exists between tangential stresses $\tau$ and normal stresses $\sigma_{\mathrm{N}}$, which led to the formulation of the so-called Mohr-Coulomb yield criterion (described in the next Section). This linearity is not only valid in the Coulomb plane ( $\tau, \sigma_{\mathrm{N}}$ ) but also in the $(p, q)$ plane. However, the shape of the yield surface in the deviatoric plane has been investigated experimentally much later (LAN 88), see Figure 2.2.


Figure 2.2 Experimental limit surfaces for Hostun sand, modified from (LAN 88)
The simple model which is namely Mohr Coulomb model, which will be presented in next Section, only considers the frictional properties of granular materials. This kind of simple model just takes the minimum and maximum principal stresses into account. However, this model is inconvenient to use during the numerical simulation because there exists the geometric singularities in the principal stress space for the plasticity surface. But this problem can be solved by using an approximation of the Mohr Coulomb criterion, which could be called Drucker Prager criterion. As the similarity of the former one, the latter one also has a main disadvantage about which in some cases the criterion makes the doubtful results. Therefore, a more sophisticated model can be chosen to approximate the Mohr Coulomb criterion more accurately (e.g. Matsuoka-Nakai (MAT 82), Van

## Eekelen (VAN 80))

The Mohr Coulomb (MC) model will be briefly presented here while the other two criterions, i.e., the Drucker Prager (DP) and Van Eekelen (VE) criterion with their full elastoplastic formulation will be described in this Section too. The formulation will be given including the isotropic hardening / softening of friction angles and cohesion.

### 2.1.2.1 Stress invariants and stress space

In this Section, $I_{\sigma}, I I_{\widehat{\sigma}}, I I I_{\widehat{\sigma}}$ and $\beta$ represent the first stress tensor invariant, the second deviatoric stress tensor invariant, the third deviatoric stress tensor invariant related to the Lode angle, respectively. In other word, the first stress tensor invariant can show the isotropic change, while the second and third invariant of the deviatoric stress tensor describe the deviatoric changes.

$$
\begin{gather*}
I_{\sigma}=\sigma_{i i}  \tag{2.28}\\
I I_{\widehat{\sigma}}=\sqrt{\frac{1}{2} \hat{\sigma}_{i j} \hat{\sigma}_{i j}}  \tag{2.29}\\
I I I_{\widehat{\sigma}}=\frac{1}{3} \hat{\sigma}_{i j} \hat{\sigma}_{j k} \hat{\sigma}_{k i}  \tag{2.30}\\
\hat{\sigma}_{i j}=\sigma_{i j}-\frac{I \sigma}{3} \delta_{i j}  \tag{2.31}\\
\beta=-\frac{1}{3} \sin ^{-1}\left(\frac{3 \sqrt{3}}{2} \frac{I I I_{\widehat{\sigma}}}{I I_{\widehat{\sigma}}{ }^{3}}\right) \tag{2.32}
\end{gather*}
$$

The three invariants $I_{\sigma}, I I_{\widehat{\sigma}}$ and $\beta$ define a cylindrical referential around the $I_{\sigma}$ axis. This referential is shown in Figure 2.3a, and the distance between any stress state ' $p$ ' and the axis $I_{\sigma}$ is presented by $I I_{\widehat{\sigma}}$, while the angular position of ' $p$ ' with respect to the pure shear line is represented by the Lode angle $\beta$ with a range of $\left(-30^{\circ},+30^{\circ}\right)$. In this Figure $2.3 \mathrm{~b}, \sigma_{1}{ }^{*}, \sigma_{2}{ }^{*}$ and $\sigma_{3}{ }^{*}$ represent the projection of the principal stress axes on the deviatoric plane. The deviatoric plane can be also called $\Pi$ plane.


Figure 2.3 a) representation of the principal stress space, b) deviatoric plane (BAR 98)

### 2.1.2.2 Stress invariants and stress space

The Mohr Coulomb failure criterion is an intrinsic curve criterion, i.e. it does not depend on the state of the material. It expresses a linear relationship between the shear stress $\tau$ and the normal stress $\sigma_{\mathrm{N}}$ in the failure plane, which is given by:

$$
\begin{equation*}
\tau=c+\sigma_{N} \cdot \tan \varphi \tag{2.33}
\end{equation*}
$$

In this relation, $c$ and $\varphi$ are the cohesion and effective friction angle respectively. This criterion can be expressed in terms of stress tensor invariant by the relation

$$
\begin{equation*}
f=\frac{I_{\sigma}}{3} \sin \varphi+I I_{\hat{\sigma}} \cos \beta-\frac{I_{\hat{\sigma}}}{\sqrt{3}} \sin \beta \sin \varphi-c \cdot \cos \varphi=0 \tag{2.34}
\end{equation*}
$$

Note that, in this criterion, the friction angle under triaxial compression paths (referred as $\varphi_{\mathrm{C}}$ ) is identical with the angle under triaxial extension path $\left(\varphi_{\mathrm{E}}\right)$, i.e., $\varphi=\varphi_{\mathrm{C}}=\varphi_{\mathrm{E}}$. It is worth starting from an isotropic stress state for this relation, the triaxial compression path results to an increasing of axial stress whereas the triaxial extension path results to a decreasing of axial stress which however still remains in the compressive state.

Geometric representation of this criterion in the principal stress space ( $\sigma_{1}, \sigma_{2}$ and $\sigma_{3}$ ) is a hexagonal pyramid which is shown in Figure 2.4. As mentioned previously, this criterion is inconvenient to implement in a classical plasticity framework because the gradient of this yield surface is undefined on the hexagon corners. Therefore it is much necessary to develop more complex integration schemes (CRI 87), which means the more continuously derivable yield surfaces functions are preferred in priority.


Figure 2.4 Limit surface for Mohr-Coulomb criterion in the deviatoric plane for $\varphi=35^{\circ}$ (BAR 98)

### 2.1.2.3 Drucker Prager criterion (DP)

An alternative and more sophisticated solution to solve this difficulty has been proposed by Drucker and Prager (DRU 52) who defined the yield function $f$ by using a linear relationship between the first stress tensor invariant and the second deviatoric stress tensor invariant.

$$
\begin{equation*}
f=I I_{\widehat{\sigma}}+m I_{\sigma}-k^{\prime}=0 \tag{2.35}
\end{equation*}
$$

In the principal stress space, the plasticity surface becomes a cone which is much easier to use in numerical algorithms. The trace of this plasticity surface on the $\Pi$ plane is then a circle which is presented in Figure 2.5.


Figure 2.5 Limit surface for Drucker Prager criterion in the deviatoric plane for $\varphi=35^{\circ}$ (BAR 98)
As mentioned before, this criterion is independent on the third stress invariant and thus on the Lode angle $\beta$, although it has been widely used in geomechanics to represent frictional behavior of granular materials. Parameters $m$ and $k^{\prime}$ can be identified on the Mohr envelop as a function of internal friction angle in compression $\varphi_{\mathrm{C}}$ and cohesion $c$ (DES 84) if the compression cone is chosen,
i.e. the DP circle is circumscribed to the MC hexagon.

$$
\begin{align*}
m & =\frac{2 \sin \varphi_{C}}{\sqrt{3}\left(3-\sin \varphi_{C}\right)}  \tag{2.36}\\
k^{\prime} & =\frac{6 c \cdot \cos \varphi_{C}}{\sqrt{3}\left(3-\sin \varphi_{C}\right)} \tag{2.37}
\end{align*}
$$

Combining the equation (2.35), (2.36) and (2.37), the Drucker Prager criterion can be rewritten as:

$$
\begin{equation*}
f=I I_{\widehat{\sigma}}+m\left(I_{\sigma}-\frac{3 c}{\tan \varphi_{c}}\right)=0 \tag{2.38}
\end{equation*}
$$

The definition of reduced radius $r$ is needed to be mentioned.

$$
\begin{equation*}
r=\frac{I I_{\widehat{\sigma}}}{I_{\sigma}} \tag{2.39}
\end{equation*}
$$

and the reduced radius in compression $\left(r_{\mathrm{C}}\right)$ and in extension $\left(r_{\mathrm{E}}\right)$ for triaxial tests can be expressed and deduced from Mohr circle and intrinsic curve.

$$
\begin{align*}
& r_{C}=\frac{1}{\sqrt{3}}\left(\frac{2 \sin \varphi_{C}}{3-\sin \varphi_{C}}\right)  \tag{2.40}\\
& r_{E}=\frac{1}{\sqrt{3}}\left(\frac{2 \sin \varphi_{E}}{3+\sin \varphi_{E}}\right) \tag{2.41}
\end{align*}
$$

When putting $r_{\mathrm{E}}=r_{\mathrm{C}}$ since the radius is constant in the Drucker Prager model, leads a relation between $\varphi_{\mathrm{C}}$ and $\varphi_{\mathrm{E}}$, and this relation equation is shown as the combination of former two equations.

$$
\begin{equation*}
\varphi_{E}=\sin ^{-1}\left(\frac{\frac{3 \sin \varphi_{C}}{3-\sin \varphi_{C}}}{1-\left(\frac{\sin \varphi_{C}}{3-\sin \varphi_{C}}\right)}\right) \tag{2.42}
\end{equation*}
$$

Since the relation between $\varphi_{\mathrm{C}}$ and $\varphi_{\mathrm{E}}$ is formulated, their relation curve can be plotted, which is illustrated in Figure 2.6. Observing directly from this Figure, the non-linear relation between them is easily found, as well as the limit values of $\varphi_{\mathrm{E}}=90^{\circ}$ for $\varphi_{\mathrm{C}} \approx 36.87^{\circ}$.


Figure $2.6 \varphi_{\mathrm{C}}$ vs $\varphi_{\mathrm{E}}$ for Drucker Prager criterion (BAR 98)
In the principal stress space, the conical limit surface based on Drucker Prager criterion leads to an angle $\theta$ :

$$
\begin{equation*}
\theta=\tan ^{-1}\left(\sqrt{2} \frac{2 \sin \varphi_{C}}{3-\sin \varphi_{C}}\right) \tag{2.43}
\end{equation*}
$$

and this angle refers to the hydrostatic axis $\left(\sigma_{1}=\sigma_{2}=\sigma_{3}\right)$.
Since the plastic criterion equation $f$ is shown by formula (2.38) for Drucker Prager, the plastic potential $g$ is needed to be defined in a similar form by

$$
\begin{equation*}
g=I I_{\widehat{\sigma}}+m^{\prime} I_{\sigma}=0 \tag{2.44}
\end{equation*}
$$

with

$$
\begin{equation*}
m^{\prime}=\frac{2 \sin \psi}{\sqrt{3}(3-\sin \psi)} \tag{2.45}
\end{equation*}
$$

with the angle $\psi$ is the dilatancy angle.
For the aim of obtaining the general elastoplastic relation mentioned before, it is necessary to derive the limit surface $f$ and plastic potential $g$ with respect to stresses.

Firstly, the derivation of limit surface of Drucker Prager is presented as following.

$$
\begin{gather*}
\frac{\partial f}{\partial \sigma_{i j}}=\frac{\partial f}{\partial I_{\sigma}} \frac{\partial I_{\sigma}}{\partial \sigma_{i j}}+\frac{\partial f}{\partial I_{\widehat{\sigma}}} \frac{\partial I I_{\widehat{\sigma}}}{\partial \sigma_{i j}}  \tag{2.46}\\
\frac{\partial f}{\partial I_{\sigma}}=m  \tag{2.47}\\
\frac{\partial I_{\sigma}}{\partial \sigma_{i j}}=\delta_{i j} \tag{2.48}
\end{gather*}
$$

$$
\begin{gather*}
\frac{\partial f}{\partial I_{\widehat{\sigma}}}=1  \tag{2.49}\\
\frac{\partial I I_{\widehat{\sigma}}}{\partial \sigma_{i j}}=\frac{\widehat{\sigma}_{i j}}{2 I I_{\widehat{\sigma}}} \tag{2.50}
\end{gather*}
$$

Thus, the derivative of limit surface of Druker Prager with respect to stresses is

$$
\begin{equation*}
\frac{\partial f}{\partial \sigma_{i j}}=m \delta_{i j}+\frac{\widehat{\sigma}_{i j}}{2 I I_{\widehat{\sigma}}} \tag{2.51}
\end{equation*}
$$

Secondly, the derivative of plastic potential with respect to stresses is

$$
\begin{equation*}
\frac{\partial g}{\partial \sigma_{i j}}=m^{\prime} \delta_{i j}+\frac{\widehat{\sigma}_{i j}}{2 I I_{\widehat{\sigma}}} \tag{2.52}
\end{equation*}
$$

In this case, the scalar Val which refers to the relation between the equivalent plastic strain and the plastic multiplier (equation (2.22)) simplifies to:

$$
\begin{equation*}
V a l=\frac{\sqrt{3}}{3} \tag{2.53}
\end{equation*}
$$

Note that, in this format of DP model, only two independent hardening variables are needed to be considered, namely the compression friction angle $\varphi_{\mathrm{C}}$ and the cohesion $c$. Now the derivatives of $f$ with respect to these two variables and the variables with respect to the equivalent plastic strain are required and shown as following:

$$
\begin{gather*}
\frac{\partial f}{\partial \varphi_{C}}=\frac{2 I_{\sigma}}{\sqrt{3}}\left(\frac{\cos \varphi_{C}}{3-\sin \varphi_{C}}+\frac{\cos \varphi_{C} \sin \varphi_{C}}{\left(3-\sin \varphi_{C}\right)^{2}}\right)-\frac{6 c}{\sqrt{3}}\left(\frac{\sin \varphi_{C}}{3-\sin \varphi_{C}}+\frac{\cos ^{2} \varphi_{C}}{\left(3-\sin \varphi_{C}\right)^{2}}\right)  \tag{2.54}\\
\frac{\partial f}{\partial c}=-\frac{3 m}{\tan \varphi_{C}}  \tag{2.55}\\
\frac{\partial \varphi_{C}}{\partial \varepsilon_{e q}^{p}}=\frac{\varphi_{C f}-\varphi_{C 0}}{B_{p}+\varepsilon_{e q}^{p}}-\varepsilon_{e q}^{p} \frac{\varphi_{C f}-\varphi_{C 0}}{\left(B_{p}+\varepsilon_{e q}^{p}\right)^{2}}  \tag{2.56}\\
\frac{\partial c}{\partial \varepsilon_{e q}^{p}}=\frac{c_{f}-c_{0}}{B_{c}+\varepsilon_{e q}^{p}}-\varepsilon_{e q}^{p} \frac{c_{f}-c_{0}}{\left(B_{C}+\varepsilon_{e q}^{p}\right)^{2}} \tag{2.57}
\end{gather*}
$$

### 2.1.2.4 Van Eekelen criterion (VE)

In order to match the Mohr Coulomb criterion more closely, a more sophisticated model by introducing the dependence of Lode angle $\beta$ into Drucker Prager model is proposed (VAN 80). It includes a more smooth plasticity surface than Mohr Coulomb model does, and it can be written in a similar way with Drucker Prager criterion (see Equation (2.36) and (2.38)).

$$
\begin{equation*}
f=I I_{\hat{\sigma}}+m\left(I_{\sigma}-\frac{3 c}{\tan \varphi_{c}}\right)=0 \tag{2.58}
\end{equation*}
$$

but with the different value of coefficient $m$

$$
\begin{equation*}
m=a(1+b \sin 3 \beta)^{n} \tag{2.59}
\end{equation*}
$$

Noting that in this equation, the $\sin 3 \beta$ is derived from equation (2.32). Moreover, it is easily found that the one and only one difference between Drucker Prager and Van Eekelen criteria is the different values of coefficient $m$ which is constant for the former criterion whereas a function of the Lode angle for the later criterion.

$$
\begin{equation*}
\beta=-\frac{1}{3} \sin ^{-1}\left(\frac{3 \sqrt{3}}{2} \frac{I I_{\partial}}{I_{\vec{\sigma}}{ }^{3}}\right) \tag{2.60}
\end{equation*}
$$

In the formula giving the coefficient $m$ (Equation (2.59)), three parameters $a, b$ and $n$ must be suited for the following conditions:

$$
\begin{gather*}
a>0  \tag{2.61}\\
b n>0  \tag{2.62}\\
-1<b<1 \tag{2.63}
\end{gather*}
$$

Moreover, the coefficients $a$ and $b$ have relationship with friction angles $\varphi_{\mathrm{C}}$ and $\varphi_{\mathrm{E}}$ actually with the formulas (2.40), (2.41) and the following two equations.

$$
\begin{gather*}
b=\frac{\left(\frac{r_{C}}{r_{E}}\right)^{\frac{1}{n}}-1}{\left(\frac{r_{C}}{r_{E}}\right)^{\frac{1}{n}}+1}  \tag{2.64}\\
a=\frac{r_{C}}{(1+b)^{n}} \tag{2.65}
\end{gather*}
$$

Note that the exponent $n$ controls the convexity of the yield surface in the formulation of Drucker, and its default value equals to -0.229 basing on the conclusion of Van Eekelen (VAN 80).

The trace of this plasticity surface of Van Eekelen criterion in the $\Pi$ plane is much smoother than the one of Mohr Coulomb, but in the other hand, it fits the Mohr Coulomb criterion much better than the Drucker Prager criterion for high friction angles. The limit surface is illustrated in Figure 2.7.


Figure 2.7 Limit surface for Van Eekelen criterion in the deviatoric plane for $\varphi=35^{\circ}(\mathrm{BAR} 98)$
Similarly, the plastic potential $g$ for Van Eekelen criterion has the same definition with $f$.

$$
\begin{equation*}
g=I I_{\widehat{\sigma}}+m^{\prime}\left(I_{\sigma}-\frac{3 c}{\tan \varphi_{C}}\right)=0 \tag{2.66}
\end{equation*}
$$

where the coefficient $m^{\prime}$ is calculated in the same equation with equation (2.59), (2.64), (2.65), (2.40) and (2.41) but containing two dilatancy angles $\psi_{\mathrm{C}}$ and $\psi_{\mathrm{E}}$ to replace the two friction angles $\varphi_{\mathrm{C}}$ and $\varphi_{\mathrm{E}}$.

Considering the derivatives of the limit surface $f$ with respect to stresses is necessary to get access to the general elastoplastic relation. Here are the equations:

$$
\begin{gather*}
\frac{\partial f}{\partial \sigma_{i j}}=\frac{\partial f}{\partial I_{\sigma}} \frac{\partial I_{\sigma}}{\partial \sigma_{i j}}+\frac{\partial f}{\partial I_{\widehat{\sigma}}} \frac{\partial I I_{\widehat{\sigma}}}{\partial \sigma_{i j}}+\frac{\partial f}{\partial \sin 3 \beta} \frac{\partial \sin 3 \beta}{\partial \sigma_{i j}}  \tag{2.67}\\
\frac{\partial f}{\partial \sin 3 \beta}=a b n(1+b \sin 3 \beta)^{n-1}\left(I_{\sigma}-\frac{3 c}{\tan \varphi_{C}}\right)  \tag{2.68}\\
\frac{\partial \sin 3 \beta}{\partial \sigma_{i j}}=-\frac{3 \sqrt{3}}{2 I I_{\widehat{\sigma}}{ }^{3}}\left(\hat{\sigma}_{i k} \hat{\sigma}_{k j}-\frac{2}{3} I I_{\widehat{\sigma}}^{2} \delta_{i j}-\frac{3 I I I_{\widehat{\sigma}}}{I I_{\widehat{\sigma}}} \frac{\partial I_{\widehat{\sigma}}}{\partial \sigma_{i j}}\right) \tag{2.69}
\end{gather*}
$$

The values of $\frac{\partial f}{\partial I_{\sigma}}, \frac{\partial I_{\sigma}}{\partial \sigma_{i j}}, \frac{\partial f}{\partial I_{\hat{\sigma}}}$ and $\frac{\partial I I_{\widehat{\sigma}}}{\partial \sigma_{i j}}$ have already been defined in the Section of Drucker Prager criterion (see Equations (2.47) to (2.50)).

In this case, the scalar Val which refers to the relation between equivalent plastic strain and the plastic multiplier has different value with Drucker Prager does but with the same calculation equation.

It is important to present that there would be three parameters, namely the compressive friction angle $\varphi_{\mathrm{C}}$, the extensive friction angle $\varphi_{\mathrm{E}}$ and the cohesion $c$, playing important roles in this criterion. The derivatives of friction angles and cohesion with respect to equivalent plastic strain are identical to the ones of Drucker Prager criterion. The derivatives of $f$ with respect to cohesion and the two independent friction angles are

$$
\begin{gather*}
\frac{\partial f}{\partial c}=-\frac{3 a(1+b \sin 3 \beta)^{n}}{\tan _{C}}  \tag{2.70}\\
\frac{\partial f}{\partial \varphi_{C}}=\frac{N u m_{1}}{\text { Deno }_{1}}  \tag{2.71}\\
\frac{\partial f}{\partial \varphi_{E}}=\frac{\text { Num }_{2}}{\text { Deno }_{2}} \tag{2.72}
\end{gather*}
$$

with

$$
\begin{gather*}
\operatorname{Num}_{1}=\frac{\sqrt{3}}{\sin \varphi_{C}}\left(1+\frac{\sin 3 \beta(\text { Pol-1) }}{1+\text { Pol }}\right)^{n}\left(A_{1}+\text { Pol } \cdot A_{2}\right)  \tag{2.73}\\
\text { Deno }_{1}=2^{n}\left(\sin \varphi_{C}-3\right)^{2}\left(\frac{\text { Pol }}{1+\text { Pol }}\right)^{n}(1-\sin 3 \beta+\operatorname{Pol}(1+\sin 3 \beta))  \tag{2.74}\\
\text { Num }_{2}=2 \sqrt{3} \frac{(\sin 3 \beta-1)}{\tan \varphi_{E}}\left(I_{\sigma} \sin \varphi_{C}-3 c \cdot \cos \varphi_{C}\right)\left(1+\frac{\sin 3 \beta(\text { Pol-1) }}{1+\text { Pol }}\right)^{n}  \tag{2.75}\\
\text { Deno }_{2}=2^{n}\left(\sin \varphi_{C}-3\right)\left(\sin \varphi_{E}+3\right)\left(\frac{\text { Pol }}{1+\text { Pol }}\right)^{n}(1-\sin 3 \beta+\operatorname{Pol}(1+\sin 3 \beta)) \tag{2.76}
\end{gather*}
$$

where

$$
\begin{gather*}
\operatorname{Pol}=\left(\frac{\sin \varphi_{C}\left(3+\sin \varphi_{E}\right)}{\sin \varphi_{E}\left(3-\sin \varphi_{C}\right)}\right)^{\frac{1}{n}}  \tag{2.77}\\
A_{1}=\left(6 c-2 c \cdot \sin \varphi_{C}\right)(1-\sin 3 \beta)  \tag{2.78}\\
A_{2}=3 c\left(1+\sin 3 \beta-\cos \left(2 \varphi_{C}\right)-\sin 3 \beta \cdot \cos \left(2 \varphi_{C}\right)\right)-2 c \cdot \sin \varphi_{C}(\sin 3 \beta+1)+I_{\sigma} \sin \left(2 \varphi_{C}\right) \\
(1+\sin 3 \beta) \tag{2.79}
\end{gather*}
$$

Comparing the derivatives of $f$ of the VE model and the DP model with respect to friction angles, the hardening part makes a big difference between them. Since the initial and final values of friction angles for the two criteria are identical, so are the intermediate values of the compressive friction angles $\varphi_{\mathrm{C}}$ in the process of hardening. But for the extensive friction angles $\varphi_{\mathrm{E}}$, the situation changes during hardening for two criteria. In DP criterion, the variation of $\varphi_{\mathrm{E}}$ is related to the change of $\varphi_{\mathrm{C}}$, whereas in VE criterion, the variation of $\varphi_{\mathrm{E}}$ is related to the equivalent plastic strain with the relation (2.12).

Regarding the derivatives of flow potential $g$ with respect to stresses, the equations are quietly similar with those obtained for the limit surface $f$ (see equations (2.67) to (2.79)). It is worth highlighting that the expression of the flow potential should contain the component of cohesion in VE model since it appears in the derivative of $g$ respect to stresses basing on the relation of $f$ and $g$ and equation (2.68).

### 2.1.2.5 Comparison between MC, DP and VE yield criteria

The traces of the three yield surfaces are plotted on Figure 2.8 for low $\left(\varphi_{\mathrm{C}}=5^{\circ}\right)$ to high friction angles values $\left(\varphi_{\mathrm{C}}=45^{\circ}\right)$, with $\varphi_{\mathrm{E}}=\varphi_{\mathrm{C}}$ for Van Eekelen one. At very low friction angles the 3 criteria are pretty much similar (see Figure 2.8a). It is clear that the differences between the DP criterion on one hand and the MC or VE criteria on the other hand increases as friction angle gets larger. This is directly related to the relation between $\varphi_{\mathrm{C}}$ and $\varphi_{\mathrm{E}}$ (see equation (2.42) and Figure 2.6). From equation (2.42) it is found that, for $\varphi_{\mathrm{C}}=20^{\circ}$, then $\varphi_{\mathrm{E}}=26^{\circ}$. However as friction angle $\varphi_{\mathrm{C}}$ gets closer to the limit value $36.89^{\circ}$, the corresponding angle $\varphi_{\mathrm{E}}$ approaches $90^{\circ}$. Therefore if low friction angles are considered (let say below $20^{\circ}$ ), the 3 criteria will give approximately the same results. However above this value of $20^{\circ}$, some significant differences can be expected between the DP criterion on one hand and the MC or VE criteria on the other hand. These 3 criteria can also be compared against experimental determination of the limit surface shape in the $\Pi$ plane obtained on sand by (LAN 88). The VE model is the best fit of these experimental results both regarding the yield surface shape and regarding the slight difference observed between the friction angle for triaxial compression and triaxial extension stress paths.


Figure 2.8 Limit surfaces for Mohr Coulomb, Drucker Prager and Van Eekelen criteria in the deviatoric plane for different friction angle value: a) $\varphi=5^{\circ}$, b) $\varphi=20^{\circ}$, c) $\varphi=35^{\circ}$, d) $\varphi=45^{\circ}(\operatorname{BAR} 98)$

### 2.1.3 Integration of elastoplastic constitutive relation

The general rate constitutive elastoplastic relation has been obtained in the previous Section (see Equation (2.26)), and the integration over time of this equation leads to an incremental form:

$$
\begin{equation*}
\Delta \sigma_{i j}=\left(D_{i j k l}^{e}-D_{i j k l}^{p}\right) \Delta \varepsilon_{k l} \tag{2.81}
\end{equation*}
$$

Where the $\Delta \sigma_{i j}$ refers to the stress increment, while the $\Delta \varepsilon_{k l}$ means the strain increment. $D_{i j k l}^{e}$ and $D_{i j k l}^{p}$ represent the elastic and plastic constitutive tensors.

A detailed analysis of the different methods used for the integration of different stress-strain relations has been presented in (CHA 87). The method used here is based on the operator split methodology (SIM 85) which consists in computing an elastic predictor and plastic corrector.

### 2.1.3.1 Elastic predictor

From a given stress state $\sigma^{\mathrm{A}}$ at beginning of the step, the stress increment $\Delta \sigma^{e}$ corresponding to a purely elastic response is computed with

$$
\begin{equation*}
\Delta \sigma_{i j}^{e}=D_{i j k l}^{e} \Delta \varepsilon \tag{2.82}
\end{equation*}
$$

which defines an elastic trial stress state $\boldsymbol{\sigma}^{\mathrm{E}}$ at point E

$$
\begin{equation*}
\sigma_{i j}^{E}=\sigma_{i j}^{A}+\Delta \sigma_{i j}^{e} \tag{2.83}
\end{equation*}
$$

When the value of trial stress $\boldsymbol{\sigma}^{\mathrm{E}}$ does not overpasses the value of yield criterion, i.e. $f\left(\boldsymbol{\sigma}^{\mathrm{E}}, \boldsymbol{\kappa}^{\mathrm{A}}\right) \leq 0$, and then this trial step is done in elastic domain. But if the stress state at point E violates the yield criterion, i.e. $f\left(\boldsymbol{\sigma}^{\mathrm{E}}, \boldsymbol{\kappa}^{\mathrm{A}}\right)>0$, then a plastic correction must be computed to turn the stress state back onto the yield surface (Figure 2.9). The stress point $\boldsymbol{\sigma}^{\mathrm{C}}$ on the yield surface means the cross point with the trial line from point A to E .


Figure 2.9 Evaluation of the trial stress state at point E

### 2.1.3.2 Plastic corrector

In this phase, there are three quantities which should be determined first, namely the plastic strain increment $\Delta \varepsilon^{\mathrm{p}}$, the plastic stress increment $\Delta \sigma^{\mathrm{p}}$ which equips the opposite sign to and the hardening $/$ softening increment $\Delta \boldsymbol{\kappa}$.

In classical framework of non-associated plasticity, the plastic strain increment $\Delta \varepsilon^{p}$ is defined by its intensity and direction. The intensity has been defined before with equation (2.7), and the direction is defined by the normal $\mathbf{n}$ of the plastic flow potential $g$.

$$
\begin{equation*}
\mathbf{n}=\frac{\partial g}{\partial \sigma} \tag{2.84}
\end{equation*}
$$

Moreover, there are three choices can be made for the determination of normal $\mathbf{n}$ which are illustrated
in the following Figure 2.10.


Figure 2.10 Choice of the normal: comparison between three possible directions for a Von Mises type criterion (BAR 98)
(1) Beginning of the step, i.e. at point C (referring to the tangent rigidity). Then the method in this case is fully explicit.
(2) Middle of the step, i.e. at point M (referring to the mean normal). Then the method in this case is semi-implicit.
(3) End of the step, i.e. at point E, then the method is fully implicit. Moreover, this choice corresponds to a radial return for Von Mises type of surfaces in deviatoric plane. But for the yield surface of Van Eekelen criterion which is non circular, this choice does not coincide with a radial return. In order to obtain such a radial return, the Van Eekelen criterion could be combined with a Drucker Prager flow surface.

In sum, the explicit method is only stable for small time increments whereas the semi-implicit and fully implicit method are stable without any conditions. Moreover, the explicit and semi-implicit methods need to compute the stresses at point C for which a simple relationship can be derived from the simple criterion, such as the Drucker Prager criterion. But for more complex one, i.e. the Van Eekelen criterion, this simple relationship does not exist and an iterative method is needed to compute a solution at point C and requires additional time.

Therefore, the fully implicit backward Euler method is adopted appropriately here to integrate the constitutive relation. That means all the variable are calculated at point B in Figure 2.11, and the exact
solution is obtained by using the Newton-Raphson iterative method. Based on Newton-Raphson method, the first order Taylor development of the plasticity criterion $f$ near the value of stress situated at point $B$ is

$$
\begin{equation*}
f\left(\sigma_{i j}^{B}+\Delta \sigma_{i j}^{p}, \kappa^{B}+\Delta \kappa\right)=f\left(\sigma_{i j}^{B}, \kappa^{B}\right)+\frac{\partial f}{\partial \sigma_{i j}^{B}} \Delta \sigma_{i j}^{p}+\frac{\partial f}{\partial \kappa^{B}} \Delta \kappa \tag{2.85}
\end{equation*}
$$

with

$$
\begin{gather*}
\Delta \sigma_{i j}^{p}=-D_{i j k l}^{e} \Delta \varepsilon_{k l}^{p}  \tag{2.86}\\
\Delta \kappa=\frac{d \kappa}{d \varepsilon_{e q}^{p}} \operatorname{Val} \Delta \lambda \tag{2.87}
\end{gather*}
$$

Then the equation (2.85) could be rewritten as:

$$
\begin{equation*}
f=f\left(\sigma_{i j}^{B}, \kappa^{B}\right)-\frac{\partial f}{\partial \sigma_{i j}^{B}} D_{i j k l}^{e} \Delta \lambda \frac{\partial g}{\partial \sigma_{k l}^{B}}+\operatorname{Val} \frac{\partial f}{\partial \kappa^{B}} \frac{d \kappa}{d \varepsilon_{e q}^{p}} \Delta \lambda \tag{2.88}
\end{equation*}
$$

Based on the Newton-Raphson method and the consistency condition, the plasticity criterion $f=0$, then the increment of plastic strain is obtained as:

$$
\begin{equation*}
\Delta \lambda=\frac{f\left(\sigma_{i j}^{B}, \kappa^{B}\right)}{\frac{\partial f}{\partial \sigma_{i j}^{B}} D_{i j k l}^{e} \frac{\partial g}{\partial \sigma_{k l}^{B}}-\operatorname{Val} \frac{\partial f d \kappa}{\partial \kappa^{B} d \varepsilon_{e q}^{p}}} \tag{2.89}
\end{equation*}
$$

The corrected stress state $\sigma^{\mathrm{B}}$ is obtained from the equation below.

$$
\begin{equation*}
\sigma_{i j}^{B}=\sigma_{i j}^{B}-\Delta \lambda D_{i j k l}^{e} \frac{\partial g}{\partial \sigma_{k l}^{B}} \tag{2.90}
\end{equation*}
$$



Figure 2.11 Fully implicit scheme: stress space representation for perfect plasticity (BAR 98)
Combining the Figure 2.11 and these formulas before, it is much clear that the stress state at point $\mathrm{E}\left(\boldsymbol{\sigma}^{\mathrm{E}}, \kappa^{\mathrm{E}}\right)$ is used in the first iteration, and the values obtained from previous iteration will be corrected and used for next further iterations.

Moreover, there would be a condition defining the termination of iterations, otherwise it would not stop. For perfectly plastic circular criterion (DP or Von Mises criteria), a coherent stress state at point B in which point the condition of $f\left(\boldsymbol{\sigma}^{\mathrm{B}}, \kappa^{\mathrm{B}}\right) \approx 0$ meets is reached after one time iteration. However, for more general case of non-circular criterion, some iterations do not equip the condition
mentioned above. Also, if hardening or softening exists, some additional iterations are required to integrate $\kappa^{B}$ over the step. Therefore, a convergence condition is proposed and formulated as:

$$
\begin{equation*}
f\left(\sigma^{B}, \kappa^{B}\right) \approx 0 \tag{2.91}
\end{equation*}
$$

This convergence criterion can be rewritten in terms of the stress increments ratio between the $(n-l)^{\text {th }}$ iteration and $n^{\text {th }}$ iteration as following:

$$
\begin{equation*}
1-\text { Prec }<\max \left|\frac{\Delta \sigma_{i j}^{n}}{\Delta \sigma_{i j}^{n-1}}\right|<1+\text { Prec } \tag{2.92}
\end{equation*}
$$

A value of Prec $=10^{-3}$ is chosen, which means the convergence is achieved when the maximum stress increments ratio between two iterations is smaller than $0.1 \%$.

All these steps are summarized as a global integration algorithm seen in Figure 2.12.

Beginning of the constitutive law integration
$n=0$

1. Computer elastic predictor $\boldsymbol{\sigma}^{\mathrm{E}}$

$$
\sigma_{i j}^{E}=\sigma_{i j}^{A}+D_{i j k l}^{e} \Delta \varepsilon
$$

If $f\left(\sigma^{\mathrm{E}}, \kappa^{\mathrm{A}}\right)<0$

$$
\sigma_{i j}^{B}=\sigma_{i j}^{E}, \quad \kappa^{B}=\kappa^{A} \quad \text { (elastic stress state) }
$$

## Else

Initialize stresses and internal variables:

$$
\sigma_{i j}^{B}=\sigma_{i j}^{E}, \quad \kappa^{B}=\kappa^{A}
$$

2. Iteration $n=n+1$

Compute $\Delta \lambda$ : $\quad \Delta \lambda=\frac{f\left(\partial_{i j}^{B}, \kappa^{B}\right)}{\frac{\partial f}{\partial \sigma_{i j}^{B}} D_{i j k l}^{e} \frac{\partial g}{\partial \sigma_{k l}^{B}}-V a l \frac{\partial f ~ d \kappa}{\partial \kappa^{B} d \varepsilon_{e q}^{P}}}$
3. Up-date equivalent plastic strain

$$
\varepsilon_{e q}^{p}=\varepsilon_{e q}^{p}+\mathrm{Val} \cdot \Delta \lambda
$$

4. If hardening / softening, up-date of $\kappa^{\mathrm{B}}$
5. Compute new stresses at $\boldsymbol{\sigma}^{\mathrm{B}}$

$$
\sigma_{i j}^{B}=\sigma_{i j}^{B}-\Delta \lambda \cdot D_{i j k l}^{e} \cdot \frac{\partial g}{\partial \sigma_{k l}^{B}}
$$

6. Test on yield condition $f\left(\boldsymbol{\sigma}^{\mathrm{B}}, \kappa^{\mathrm{B}}\right) \approx 0$ :

If (1-Prec $<\max \left|\Delta \boldsymbol{\sigma}^{\mathrm{n}} / \Delta \boldsymbol{\sigma}^{\mathrm{n-1}}\right|<1+$ Prec ), then
Solution has converged

## Goto End

Else
Goto step 2
End if

Figure 2.12 Implicit backward Euler integration algorithm implemented in the elastoplastic constitutive law PLASOL (BAR 98)

### 2.2 Houlsby hyperelastic energy formula

Since several decades, the analysis of geotechnical problems is depending on a realistic representation which states that the initial stiffness equips a non-linear dependence on stress. This is most commonly achieved by adopting hypoelastic formulations (FUN 65) in the elastic-plastic framework, in which the definition of varying tangent moduli is given. For example, the following formulations are commonly used to calculate the elastic moduli in Modified Cam-Clay model. The bulk modulus $K$ is usually defined via the expression $K=p^{\prime}(1+e) / \kappa^{\prime}$, in which $\kappa^{\prime}$ means the slope of swelling line in a consolidation plot. The shear modulus $G$ is achieved by assuming a constant Poisson's ratio $v$. As a result, such model leads to a non-conservative elastic response (ZYT 78). That means that, a material which follows this kind of property will lead to continuous production of energy under the application of multiple cycles loading. However, this phenomenon is clearly physically incorrect. A numerical model adopting such property for the analysis of cyclic behavior might results in a totally unreasonable result.

Thus an alternative model is proposed to adopt the hyperelastic approach, which is based on the theory of energy potential, and the reversible response can be derived from this energy potential. This results in a conservative elastic response, obeying the First Law of Thermodynamics, and thus avoiding the problems about cyclic loading mentioned above.

### 2.2.1 Selected experimental evidence

Most of the published literatures about the small strain stiffness of soils are achieved from the dynamic laboratory tests on samples in a triaxial condition and under isotropic stress states. As a results, the small strain shear stiffness $G_{0}$ has been focused on the most literatures. Here in this Section, the symbol $G$ is adopted to replace $G_{0}$ for simplification and representing the small strain modulus, i.e. shear modulus.

A basic observation from Hardin (HAR 78) shows that the shear modulus $G$ has dependence on the current stress state, and can be expressed by the current void ratio, the mean effective pressure and the stress history loaded before. Here, the stress history can be simply expressed by the over-consolidation ratio $O C R=\sigma^{\prime}{ }_{v \max } / \sigma^{\prime}$, and the $G$ can be formulated as:

$$
\begin{equation*}
\frac{G}{p_{a}}=S \cdot f(e) \cdot\left(\frac{p}{p_{a}}\right)^{n 1} \cdot O C R^{k} \tag{2.93}
\end{equation*}
$$

In this relation, the function $f(e)$ represents an empirically defined function of the void ratio which shows a decreasing trend with the increasing void ratio; $p_{\mathrm{a}}$ is the atmospheric pressure which is adopted as the reference stress; and coefficients $S, n 1$ and $k$ are dimensionless parameters determined experimentally. In this work, parameter $n 1$ is supposed to be in the range from 0 to 1 , while values 0 and 1 are included.

It is worth noting that for most engineering applications on sands, the soil does not reveal a significant variation of the void ratio or $O C R$, and the $O C R$ is also hard to calculate in most cases. Therefore, the use of simplified expressions of initial shear stiffness and bulk stiffness are naturally and necessarily derived. In the relations below, the $g$ and $k$ are dimensionless constants.

$$
\begin{align*}
& \frac{G}{p_{a}}=g\left(\frac{p}{p_{a}}\right)^{n 1}  \tag{2.94}\\
& \frac{K}{p_{a}}=k\left(\frac{p}{p_{a}}\right)^{n 1} \tag{2.95}
\end{align*}
$$

For the applications of clay soils, few further experimental observations and their interpretation (HOU 91; VIG 92; RAM 94) indicate that the shear moduli $G$ can be expressed as a function of just two out of three variables $e, p$ and $O C R$ for isotropic stress conditions. Thus, the equation (2.93) can be written as:

$$
\begin{equation*}
\frac{G}{p_{a}}=S^{*} \cdot\left(\frac{p}{p_{a}}\right)^{n^{*}} R^{k^{*}} \tag{2.96}
\end{equation*}
$$

with the $R$ represents the over-consolidation ratio in terms of mean effective stress $R=p_{\mathrm{c}} / p$, where $p_{\mathrm{c}}$ being mean pre-consolidation pressure. Noting that the exponent of mean effective stress $n^{*}$ is different with $n 1$ because of the different meaning defined by different authors.

Also, few observations of the small strain shear stiffness of soils have been performed under anisotropic stress conditions (NI 87; HAR 89; JAM 94; RAM 97; JOV 98). These researches proved that it is necessary to modify the expression of $G$ obtained under isotropic condition for anisotropic condition. For instance, Ni (NI, 87) and Hardin \& Blandford (HAR, 89) proposed a modified expression of $G$ for sands as following:

$$
\begin{equation*}
\frac{G}{p_{a}}=S_{i j} \cdot f(e) \frac{\left(\sigma_{i} \sigma_{j}\right)^{n 1 / 2}}{p_{a}^{n 1}} O C R^{k} \tag{2.97}
\end{equation*}
$$

The principal stresses $\sigma_{i}$ and $\sigma_{j}$ are captured in the plane where $G$ is measured, while $S_{i j}$ represents anisotropic stiffness factor.

Another expression for clays which is based on the equation (2.96) was proposed by Rampello et al. (RAM, 97). This expression is obtained based on the experimental results of reconstituted clay along the radial stress loading paths performed by different values of stress ratio $\eta=q / p$.

$$
\begin{equation*}
\frac{G}{p_{a}}=S_{\eta}^{*}\left(\frac{p}{p_{a}}\right)^{n^{*}} R_{\eta}^{k^{*}} \tag{2.98}
\end{equation*}
$$

where the variable $R_{\eta}$ is the over-consolidation ratio defined regarding the anisotropic compression line.

In conclusion, the empirical expressions of small strain shear moduli mentioned above indicate that the three following main features should be taken into account when describing the reversible behavior of soil: (a) the non-linear dependence on current stress state; (b) the influence of anisotropic stress state; and (c) the stress history loaded on materials at least for clays.

### 2.2.2 Linear and non-linear isotropic hyperelasticity

The soil stiffness modeling should take into consideration the current stress state. In order to propose an expression which is thermodynamically acceptable and equips the conservative behavior, Houlsby (HOU 05) presented an expression of free energy potential from which the stresses (or strains) can be derived with respect to strains (or stresses). It is necessary and sufficient that the function of this free energy potential obey the First Law of Thermodynamics.

### 2.2.2.1 Triaxial formulation

The following will focus on the potential which is expressed using the invariants of strain or stress tensors. Thus, the behavior described will be essentially isotropic, although the concept of stress-induced anisotropy will arise under this isotropic condition.

Being appropriate for triaxial test, the elastic strain energy or internal energy or Helmholtz free energy $F$ could be written as a function of the strains $F=F\left(\varepsilon_{v}, \varepsilon_{s}\right)$ where $\varepsilon_{v}$ and $\varepsilon_{s}$ are the volumetric strain and shear strain respectively. It is easily gotten that:

$$
\begin{align*}
p & =\frac{\partial F}{\partial \varepsilon_{v}}  \tag{2.99}\\
q & =\frac{\partial F}{\partial \varepsilon_{s}} \tag{2.100}
\end{align*}
$$

and then the expression of tangent bulk modulus and shear modulus are defined as:

$$
\begin{align*}
& K=\frac{\partial p}{\partial \varepsilon_{v}}=\frac{\partial^{2} F}{\partial \varepsilon_{v}{ }^{2}}  \tag{2.101}\\
& 3 G=\frac{\partial q}{\partial \varepsilon_{s}}=\frac{\partial^{2} F}{\partial \varepsilon_{s}{ }^{2}} \tag{2.102}
\end{align*}
$$

Furthermore, the incremental stiffness matrix could be defined as following.

$$
\left[\begin{array}{l}
d p  \tag{2.103}\\
d q
\end{array}\right]=\left[\begin{array}{cc}
K & J \\
J & 3 G
\end{array}\right]\left[\begin{array}{l}
d \varepsilon_{v} \\
d \varepsilon_{s}
\end{array}\right]
$$

where the off-diagonal terms $J$ :

$$
\begin{equation*}
J=\frac{\partial p}{\partial \varepsilon_{s}}=\frac{\partial q}{\partial \varepsilon_{v}}=\frac{\partial^{2} F}{\partial \varepsilon_{v} \partial \varepsilon_{s}} \tag{2.104}
\end{equation*}
$$

It is worth noting that when $J$ has a non-zero value, the materials will behave incrementally in an anisotropic manner, although the free energy $F$ is an isotropic function of strains. This is the case of stress induced anisotropic which will be discussed in next Section.

In order to solve the inconvenient of moduli expressed in terms of strains, because of the more usually practical application moduli expressed in terms of stresses, the Legendre transform of the Helmholtz free energy function is adopted and applied to obtain the complementary energy function or Gibbs free energy function $E^{\prime}$ :

$$
\begin{equation*}
E^{\prime}=\left(p \varepsilon_{v}+q \varepsilon_{s}\right)-F \tag{2.105}
\end{equation*}
$$

The $E^{\prime}$ is expressed as a function of stresses, i.e. $E^{\prime}=(p, q)$, then the strains can be derived as:

$$
\begin{align*}
& \varepsilon_{v}=\frac{\partial E^{\prime}}{\partial p}  \tag{2.106}\\
& \varepsilon_{S}=\frac{\partial E^{\prime}}{\partial q} \tag{2.107}
\end{align*}
$$

and the compliance matrix can be shown as:

$$
\left[\begin{array}{l}
d \varepsilon_{v}  \tag{2.108}\\
d \varepsilon_{s}
\end{array}\right]=\left[\begin{array}{ll}
c_{1} & c_{3} \\
c_{3} & c_{2}
\end{array}\right]\left[\begin{array}{l}
d p \\
d q
\end{array}\right]
$$

with the each terms to be

$$
\begin{align*}
& c_{1}=\frac{3 G}{3 K G-J^{2}}=\frac{\partial \varepsilon_{v}}{\partial p}=\frac{\partial^{2} E^{\prime}}{\partial p^{2}}  \tag{2.109}\\
& c_{2}=\frac{K}{3 K G-J^{2}}=\frac{\partial \varepsilon_{s}}{\partial q}=\frac{\partial^{2} E^{\prime}}{\partial q^{2}}  \tag{2.110}\\
& c_{3}=\frac{-J}{3 K G-J^{2}}=\frac{\partial \varepsilon_{v}}{\partial q}=\frac{\partial \varepsilon_{s}}{\partial p}=\frac{\partial^{2} E^{\prime}}{\partial p \partial q} \tag{2.111}
\end{align*}
$$

Therefore, it is reasonable to modify the expression of free energy $F$ and complementary energy $E^{\prime}$ for linear elasticity in a quadratic form.

$$
\begin{equation*}
F=p_{a}\left(\frac{k}{2} \varepsilon_{v}^{2}+\frac{3 g}{2} \varepsilon_{s}^{2}\right) \tag{2.112}
\end{equation*}
$$

$$
\begin{equation*}
E^{\prime}=\frac{1}{p_{a}}\left(\frac{1}{2 k} p^{2}+\frac{1}{6 g} q^{2}\right) \tag{2.113}
\end{equation*}
$$

In these two equations, coefficients $k$ and $g$ are dimensionless constants, and the expressions each term are: $p=k p_{a} \varepsilon_{v}, q=3 g p_{\mathrm{a}} \varepsilon_{s}, K=k p_{\mathrm{a}}, G=g p_{\mathrm{a}}$ and $J=0$.

The expressions of free energy and complementary energy function which expose a non-linear elasticity (i.e. $K \propto p^{\text {n1 }}$ ) under purely isotropic stress conditions (without shear stress or shear strain terms) can be also established clearly. When the pressure exponent $n 1 \neq 1$, the expressions for $F$ and $E^{\prime}$ can be:

$$
\begin{gather*}
F=\frac{p_{a}}{k(2-n 1)}\left[k(1-n 1) \varepsilon_{v}\right]^{(2-n 1)} /(1-n 1)  \tag{2.114}\\
E^{\prime}=\frac{p^{2-n 1}}{p_{a}^{1-n 1} k(1-n 1)(2-n 1)} \tag{2.115}
\end{gather*}
$$

One can derive from anyone of these two formulas above:

$$
\begin{gather*}
k(1-n 1) \varepsilon_{v}=\left(\frac{p}{p_{a}}\right)^{1-n 1}  \tag{2.116}\\
\frac{K}{p_{a}}=k\left(\frac{p}{p_{a}}\right)^{n 1}=k\left[k(1-n 1) \varepsilon_{v}\right]^{n 1 /(1-n 1)} \tag{2.117}
\end{gather*}
$$

However, the expressions above will become singular when $n 1=1$. Moreover, a problem is inevitable that if the volumetric strain is seen as zero at the state $p=0 \mathrm{kPa}$, then the problem of infinite will happen for all finite stresses. An approval that shifting the reference point for zero volumetric strain from the origin (i.e. $p=0$ ) to $p=p_{\mathrm{a}}$ can avoid this problem, just with the modification of equation (2.114) and (2.115) to:

$$
\begin{align*}
F & =\frac{p_{a}}{k(2-n 1)}\left[k(1-n 1) \varepsilon_{v}^{*}\right]^{(2-n 1)} /(1-n 1)  \tag{2.118}\\
E^{\prime} & =\frac{p^{2-n 1}}{p_{a}^{(1-n 1)} k(1-n 1)(2-n 1)}-\frac{p}{k(1-n 1)} \tag{2.119}
\end{align*}
$$

where the variable $\varepsilon_{v}^{*}$ :

$$
\begin{equation*}
\varepsilon_{v}^{*}=\varepsilon_{v}+\frac{1}{k(1-n 1)} \tag{2.120}
\end{equation*}
$$

Also, this transition can change equations (2.116) and (2.117) to:

$$
\begin{gather*}
1+k(1-n 1) \varepsilon_{v}=\left(\frac{p}{p_{a}}\right)^{(1-n 1)}  \tag{2.121}\\
\frac{K}{p_{a}}=k\left(\frac{p}{p_{a}}\right)^{n 1}=k\left[k(1-n 1) \varepsilon_{v}^{*}\right]^{n 1 /(1-n 1)} \tag{2.122}
\end{gather*}
$$

Note that this modification does not affect the expressions of stiffness in terms of pressure. In addition, one can conclude that equations (2.112) and (2.113) are appropriate for the case of pressure
exponent $n 1=0$ (linear elasticity), while equations (2.114) and (2.115) are suitable for the case of $n 1$ $\neq 0$ (non-linear elasticity), but only on the isotropic axis. However, to obtain a more general expression which suits both for any triaxial stress states and for non-linear cases is our purpose eventually, and this expression can turn into each of those equations above in the appropriate special cases.

Combining the quadratic function of $\varepsilon_{v}^{*}, \varepsilon_{s}$ and equations (2.112) and (2.118), a generalization of function $F$ is proposed in triaxial formulation.

$$
\begin{align*}
& F=\frac{p_{a}}{k(2-n 1)}[k(1-n 1)]^{(2-n 1) /(1-n 1)} \times\left[\varepsilon_{v}^{* 2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right]^{(2-n 1) /(2-2 \cdot n 1)} \\
&=\frac{p_{a}}{k(2-n 1)}\left[k \varepsilon_{v}^{0}(1-n 1)\right]^{(2-n 1) /(1-n 1)} \tag{2.123}
\end{align*}
$$

where

$$
\begin{equation*}
\varepsilon_{v}^{0}{ }^{2}=\varepsilon_{v}^{* 2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)} \tag{2.124}
\end{equation*}
$$

The variable $\varepsilon_{v}^{*}$ (equation (2.120)) is used to replace $\varepsilon_{v}$ for moving the origin point for zero volumetric strain from $p=0$ to $p=p_{\mathrm{a}}$, for consistency with the case of $n 1=1$. Similarly, the stresses and moduli in terms of strains can be obtained with differentiation of formulation (2.123). Also, the complementary energy expression $E^{\prime}$ can be obtained by the Legendre transformation of formulation (2.123).

$$
\begin{align*}
E^{\prime} & =\frac{1}{p_{a}^{(1-n 1)} k(1-n 1)(2-n 1)}\left[p^{2}+\frac{k(1-n 1)}{3 g} q^{2}\right]^{(2-n 1) / 2}-\frac{p}{k(1-n 1)} \\
& =\frac{p_{0}^{(2-n 1)}}{p_{a}^{(1-n 1)} k(1-n 1)(2-n 1)}-\frac{p}{k(1-n 1)} \tag{2.125}
\end{align*}
$$

where

$$
\begin{equation*}
p_{0}^{2}=p^{2}+\frac{k(1-n 1) q^{2}}{3 g} \tag{2.126}
\end{equation*}
$$

### 2.2.2.2 General stress formulation

The expressions described above can be generalized into others than triaxial formulation, if the free energy $F$ is expressed by strain tensor $\varepsilon_{\mathrm{ij}}$ and the complementary energy $E^{\prime}$ by effective stress tensor $\sigma_{\mathrm{ij} .}$. Thus, the following expressions must be applied.

$$
\begin{equation*}
E^{\prime}=\sigma_{i j} \varepsilon_{i j}-F \tag{2.127}
\end{equation*}
$$

$$
\begin{align*}
\sigma_{i j} & =\frac{\partial F}{\partial \varepsilon_{i j}}  \tag{2.128}\\
\varepsilon_{i j} & =\frac{\partial E^{\prime}}{\partial \sigma_{i j}} \tag{2.129}
\end{align*}
$$

Then the stiffness matrix and compliance matrix can be re-written as:

$$
\begin{align*}
d_{i j k l} & =\frac{\partial \sigma_{i j}}{\partial \varepsilon_{k l}}=\frac{\partial F}{\partial \varepsilon_{i j} \partial \varepsilon_{k l}}  \tag{2.130}\\
c_{i j k l} & =\frac{\partial \varepsilon_{i j}}{\partial \sigma_{k l}}=\frac{\partial E^{\prime}}{\partial \sigma_{i j} \partial \sigma_{k l}} \tag{2.131}
\end{align*}
$$

Now the general form of free energy (for case of $n 1 \neq 1$ ) is written as the same with equation (2.123):

$$
\begin{equation*}
F=\frac{p_{a}}{k(2-n 1)}\left[k \varepsilon_{v}^{0}(1-n 1)\right]^{(2-n 1) /(1-n 1)} \tag{2.132}
\end{equation*}
$$

but with the different expression of $\varepsilon_{v}^{0}$ :

$$
\begin{equation*}
\varepsilon_{v}^{0^{2}}=\left[\varepsilon_{i i}+\frac{1}{k(1-n 1)}\right]\left[\varepsilon_{j j}+\frac{1}{k(1-n 1)}\right]+\frac{2 g e_{i j} e_{i j}}{k(1-n 1)} \tag{2.133}
\end{equation*}
$$

Note that we use the summation convention over a repeated index, such as: $\varepsilon_{i i}=\sum_{i=1}^{3} \varepsilon_{i i}$, and the $e_{i j}=\varepsilon_{i j}-1 / 3 \cdot \varepsilon_{v} \cdot \delta_{i j}$, where $\varepsilon_{v}=\varepsilon_{i i}$ and $\delta_{i j}$ is the Kronecker delta $\left(\delta_{i j}=1\right.$, if $i=j ; \delta_{i j}=0$, if $i \neq j$.

Similarly, the complementary energy for case of $n 1 \neq 1$ is then:

$$
\begin{equation*}
E^{\prime}=\frac{p_{0}^{(2-n 1)}}{p_{a}^{(1-n 1)} k(1-n 1)(2-n 1)}-\frac{\sigma_{k k}}{3 k(1-n 1)} \tag{2.134}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{0}^{2}=\frac{\sigma_{m m} \sigma_{n n}}{9}+\frac{k(1-n 1) s_{m n} s_{m n}}{2 g} \tag{2.135}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma_{i i}=\sum_{i=1}^{3} \sigma_{i i} \tag{2.136}
\end{equation*}
$$

The deviatoric component of stress tensor $s_{i j}$ :

$$
\begin{equation*}
s_{i j}=\sigma_{i j}-p \cdot \delta_{i j} \tag{2.137}
\end{equation*}
$$

It is easily calculated that the stiffness matrix can be written as:

$$
\begin{equation*}
d_{i j k l}=p_{a}\left(\frac{p_{0}}{p_{a}}\right)^{n 1}\left[k \cdot n 1 \frac{\sigma_{i j} \sigma_{k l}}{p_{0}^{2}}+k(1-n 1) \delta_{i j} \delta_{k l}+2 g\left(\delta_{i k} \delta_{j l}-\frac{1}{3} \delta_{i j} \delta_{k l}\right)\right] \tag{2.138}
\end{equation*}
$$

while the compliance matrix is:
$c_{i j k l}=\frac{1}{p_{a}}\left(\frac{p_{a}}{p_{0}}\right)^{n 1}\left[\begin{array}{c}\left(\frac{1}{k}+\frac{n 1 \cdot s_{m n} s_{m n}}{2 g p_{0}^{2}}\right) \frac{\delta_{i j} \delta_{k l}}{9}-\frac{n 1 \cdot \sigma_{m m}}{18 g p_{0}^{2}} \times\left(s_{i j} \delta_{k l}+\delta_{i j} s_{k l}\right)+\frac{1}{2 g}\left(\delta_{i k} \delta_{j l}-\frac{1}{3} \delta_{i j} \delta_{k l}\right)- \\ \frac{n 1 \cdot k(1-n 1)}{4 g^{2} p_{0}^{2}} s_{i j} s_{k l}\end{array}\right]$

These two expressions are applicable for the cases of pressure exponent $n 1=1$, and require just three dimensionless constants $k, g$ and $n 1$. Most importantly, the compliances or stiffness matrices can be adopted directly in, for example, a finite element program for general stress states, and make sure and express a fully conservative elastic behavior when the moduli are expressed as functions of pressure.

### 2.3 New model established

### 2.3.1 Establishment of new model

As mentioned in the two previous Sections, Plasol model is an elastoplastic model with the implicit backward Euler integration algorithm with two main parts, i.e. elastic predictor part and plastic corrector part, while Houlsby proposed a general hyperelasticity potential function for describing the non-linear elastic stiffness of soils as a function of stress with no plasticity. In the Plasol model, the current stress could be calculated by the incrementally linear evolution of strain increments, and verified and calibrated by the yield criterions accompanied with an implicit integration algorithm to make sure that the stress state remains on the plastic surface at least. As for Houlsby model, the stress-dependent small-strain moduli is adopted to account for the variation of stiffness of soils, and the hyperelastic function is adopted to meet the condition of thermodynamic acceptability. Moreover, simple free and complementary energy potentials are defined from which the small-strain stiffness can be captured realistically. Most important, the behaviour of stress-induced anisotropy is predicted within this hyperelastic formulation and it also corresponds to the empirical observation of natural soils.

Besides the characteristics of hyperelastic formulation, an invisible and significant feature derived from the stress dependency of small-strain moduli is that the current stiffness is changing with the variation of current stress state, since the formulation contains the components of consolidation pressure and current stresses. In other words, the stiffness formulations are adjustable for general consolidation and shearing conditions. Therefore, an idea that accompanying with the feature above this hyperelastic approach should be employed for describing the elastic behaviour at the heart of more complex elastic-plastic models is appeared, to combine with Plasol model. More exactly, it is an excellent theory to replace the linear incremental stress-strain relation in elastic predictor part in

Plasol model with Houlsby hyperelastic formulations. This is a reasonable and valid assumption which will be discussed below.

Therefore, a new constitutive model which is theoretically appropriate for simulating actual response of granular materials is established as a combination of Houlsby and Plasol models. The accuracy of this model will be verified with triaxial experimental results in the following Section.

Here are given the equations of Houlsby model as the elastic predictor.

$$
\begin{equation*}
\varepsilon_{i j}^{E}=\varepsilon_{i j}^{A}+\Delta \varepsilon \tag{2.140}
\end{equation*}
$$

By using the formula of strain increment which is different with the one of stress increment, the following steps are related to the strains instead of stresses. Note that in the following equations, the volumetric and shear strains ( $\varepsilon_{v}$ and $\varepsilon_{s}$ ) are basing on the trial elastic strains $\varepsilon_{i j}^{E}$. Thus, the mean effective stress and shear stress which are derived from the free energy function (equations (2.99), $(2,100)$ and $(2.123))$ are given as:

$$
\begin{align*}
p= & p_{a}[k(1-n 1)]^{\frac{1}{1-n 1}}\left\{\left[\varepsilon_{v}+\frac{1}{k(1-n 1)}\right]^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right\}^{\frac{n 1}{2(1-n 1)}} \cdot\left[\varepsilon_{v}+\frac{1}{k(1-n 1)}\right]  \tag{2.141}\\
& q=p_{a}[k(1-n 1)]^{\frac{n 1}{1-n 1}}\left\{\left[\varepsilon_{v}+\frac{1}{k(1-n 1)}\right]^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right\}^{\frac{n 1}{2(1-n 1)}} \cdot 3 g \varepsilon_{s} \tag{2.142}
\end{align*}
$$

Then the trial elastic stress is derived with respect to strain

$$
\begin{align*}
\sigma_{i j}^{E}=\frac{\partial F}{\partial \varepsilon_{i j}}=p \cdot \frac{\partial \varepsilon_{v}}{\partial \varepsilon_{i j}}+q \cdot \frac{\partial \varepsilon_{s}}{\partial \varepsilon_{i j}} & =p \cdot \delta_{i j}+\frac{2}{3} \cdot q \cdot \frac{e_{i j}}{\varepsilon_{s}} \text { with } \varepsilon_{s} \neq 0  \tag{2.143a}\\
& \text { or } \sigma_{i j}^{E}=p \cdot \delta_{i j} \quad \text { with } \varepsilon_{s}=0 \tag{2.143b}
\end{align*}
$$

Using these equations ((2.140) to (2.143)) to replace the equation in elastic predictor part in Figure 2.12, a more realistic theoretical model is built.

During the process of plastic corrector, all the equations are identity with the ones mentioned in the Section of Plasol above, except the incremental stiffness matrix which should adopt equations (2.135) and (2.138).

It can be easily shown that this stiffness matrix changes and depends on the current stress state (not just the mean effective stress). More importantly but less obviously, the incremental stiffness cannot be expressed just in terms of isotropic stiffness, which means the stiffness matrix does not simply indicate the stress-dependent values (parameters $K$ and $G$ ). This matrix can be represented by anisotropic elasticity, rather than simply expressed just in terms of isotropic elasticity. These two features are also adjustable for the new constitutive model as a consequence.

In sum, the contribution on the new model can be focused on the following three aspects: (1)
theoretical idea of combining Plasol and Houlsby based on the observation of granular material behavior; (2) derivation of formula of stress / strain / stiffness matrix from Houlsby hyperelasticity function; and (3) perfect combination and theoretically feasibility of new model.

The establishment and verification works of new model are well described and explained by Hu et al. (HU 20).

### 2.3.2 Effect of elasticity in elastoplastic model

The elastoplastic model could be commonly de-componented into elasticity region and plasticity region during the devoloping process of strain. For geomaterials, it is hard to define the limit between these two parts. Due to the granular property of geomaterials, the deformation could be seen as the internal particles' relative movement in the view of microscopic. And within an infinitesimal deformation, it is acceptable for this kind of decomposition from an asymptotic point of view in several works. However, for finite increments, any uncrystallised granular material will dissipate energy because of the friction within partiles movement. It is therefore difficult in these conditions to speak of elasticity alone. The interest of micromechanical or incrementally non-linear models is precisely to avod this pitfall of having to postulate the existence of an elastic domain.

New model, which includes the hyperelasticity formulas and their transformation formulas within the elastic part could contribute to a more effect on the plastic processes. As mentioned before in Section 2.2 , hyperelasticity is more acceptable than hypoelasticity when descriping the behaviour of geomaterials because of the obedience of First Law of Thermodynamics. The property of elastic parameters bulk modulus and shear modulus derived from hyperelasticity energy function makes them realistic and changeable during the process of deformation rather than constant. Not only the effect of Hyperelasticity is focused on the elastic part, but also on the later plastic processes. Based on the changing value of bulk and shear modulus, the stiffness matrix which connects the stresses and strains, shows the ability of calibration of stress in plastic domain. As it is shown in Figure 2.12, the changing stiffness matrix deriving from Houlsby hyperelastic energy formula (Equation (2.138)) can show the effect on the plastic part as well as on the elastic part. In other words, the hyperelasticity presents an effect on the whole strain processes.

### 2.4 Compared model

In order to compare the new constitutive model we built before, another model which uses the concept of second order fabric tensor is presented in this Section, namely modified SANISAND model (LAS 10).

Before introducing the modified SANISAND model in details, the simple conventional SANISAND model should be stated clearly. SANISAND is the name represented for a family of Simple ANIsotropic SAND models which are developed within the framework of critical state soil mechanics and bounding surface plasticity (DAF 04; MAN 97). The simple SANISAND models adopt a narrow cone-type yield surface with an apex at the origin which is suitable for the rotational kinematic hardening. Due to this basic framework of Manzari and Dafalias (MAN 97), the modified SANISAND model including a simplified elasticity theory which is capable of considering the effect of induced anisotropy on the elastic response is proposed.

According to Lashkari, Dafalias and Manzari (LAS 10; DAD 04; MAN 97), the summary of constitutive equations of SANISAND model accompanying with the modified parts in triaxial space is illustrated in the following Table 2.1.

NOTE that the symbols of all variables in Section 2.4 and the following Section 3.6 referring to the SANISAND model are only meaningful in these Sections. Even if the same symbols in these Sections (2.4 and 3.6) appear in other Chapters or Sections, their meanings are not exactly the same.

Table 2.1 Modified SANISAND model in triaxial space

| Description | Constitutive equations |  | Parameters |
| :---: | :---: | :---: | :---: |
| Strain decomposition | $\dot{\varepsilon}_{v}=\dot{\varepsilon}_{v}^{e}+\dot{\varepsilon}_{v}^{p} ; \dot{\varepsilon}_{q}=\dot{\varepsilon}_{q}^{e}+\dot{\varepsilon}_{q}^{p}$ | (2.144) |  |
| Elastic strain increments | $\dot{\varepsilon}_{q}^{e}=\frac{\dot{q}}{3 G} ; \dot{\varepsilon}_{v}^{e}=\frac{\dot{p}}{K}$ | (2.145) | $G_{0}, v$ |
| Yield function | $f=\|\eta-\alpha\|-m=0 ; \eta=q / p$ | (2.146) | $m$ |
|  | $\alpha$ is illustrated in Figure 2.13 |  |  |
| Plastic strain increments | $\dot{\varepsilon}_{q}^{p}=\langle L\rangle s=\frac{\dot{\eta}}{K_{p}} ; \dot{\varepsilon}_{v}^{p}=d\left\|\dot{\varepsilon}_{q}^{p}\right\|$ | (2.147) |  |
|  | where $s=1$ if $\eta-\alpha=m$, and $s=-1$ if $\alpha-\eta=m$, |  |  |
|  | $L$ is loading index, and |  |  |
|  | ( ) Are the Macauley brackets |  |  |


| Plastic hardening modulus | $K_{p}=h_{0}\left(1-c_{h} e\right) \frac{M^{b}-s \eta}{\left\|\eta-\eta_{i n}\right\|} \sqrt{\frac{p_{r e f}}{p}}$ | (2.148) | $h_{0}, c_{h}$ |
| :---: | :---: | :---: | :---: |
| Dilatancy function | $d=A_{d}\left(M^{d}-s \eta\right)$ | (2.149) |  |
| Fabric-dilatancy and evolution | $A_{d}=A_{0}(1+\mu\langle-s z\rangle)$ | (2.150) | $A_{0}$ |
| laws | where $\mu=1$ |  |  |
|  | $\dot{z}=c_{z}\left\langle-\dot{\varepsilon}_{v}^{p}\right\rangle\left(s z_{\max }-z\right)$ | (2.151) | $c_{z}, z_{\text {max }}$ |
| Consistency condition ( $\dot{f}=0$ ) | $\dot{\alpha}=\dot{\eta}$ | (2.152) |  |
| Bounding and dilatancy surfaces | $M^{b}=M \exp \left(-n^{b} \psi\right) ; M^{d}=M \exp \left(n^{d} \psi\right)$ | (2.153) | $M, n^{\text {b }}, n^{d}$ |
|  | see also Figure 2.13 |  |  |
| State parameter | $\psi=e-e_{c}$ | (2.154) |  |
| Critical state line e-p plane | $e_{c}=e_{0}-\lambda\left(p / p_{r e f}\right)^{\xi}$ | (2.155) | $e_{0}, \lambda, \xi$ |

In the equation (2.145), shear modulus $G$ is a function of current void ratio $e$, and mean principal effective stress $p$. According to the work of Richart et al. (RIC 70), $G$ can be written as:

$$
\begin{equation*}
G=G_{0} p_{r e f} \frac{(2.97-e)^{2}}{1+e} \sqrt{\frac{p}{p_{r e f}}} \tag{2.156}
\end{equation*}
$$

In this equation, $G_{0}$ is a material parameter and $p_{\text {ref }}$ represents the reference pressure that can refer to the atmospheric pressure. As for elastic bulk modulus $K$, a small strain Poisson ratio $v$ could be introduced in advance.

$$
\begin{equation*}
K=\frac{2}{3} G\left(\frac{1+v}{1-2 v}\right) \tag{2.157}
\end{equation*}
$$

Referring to the variables $\alpha, M^{b}, M^{d}$ and $M$ in equations (2.146) and (2.153), these basic elements of conventional SANISAND model are illustrated in the following Figure 2.13.


Figure 2.13 Model surfaces in $q-p$ triaxial stress space (MAN 97; DAF 04)
The constitutive relation in Table 2.1 is the conventional SANISAND model unless the fabric tensor $\boldsymbol{Z}$ and loading index $L$ (Equation (2.147)) accounting for the anisotropy in elasticity. Based on the work of simplified process by Lashkari (LAS 10), a traceless second order fabric tensor $\boldsymbol{Z}$ which refers to the micro-structural characteristics, and a related evolving scalar fabric variable $z$ (Equation (2.151)) which indicates the degree of induced anisotropy are applied into the elastic relationship between the rates of stresses and strains in the triaxial space.

$$
\left[\begin{array}{c}
\dot{p}  \tag{2.158}\\
\dot{q}
\end{array}\right]=\left(\begin{array}{cc}
K & (3 K-2 G) z \\
(3 K-2 G) z & 3 G
\end{array}\right)\left[\begin{array}{c}
\dot{\varepsilon}_{v}^{e} \\
\dot{\varepsilon}_{q}^{e}
\end{array}\right]=\left(\begin{array}{ll}
\Gamma_{1} & \Gamma_{2} \\
\Gamma_{2} & \Gamma_{3}
\end{array}\right)\left[\begin{array}{c}
\dot{\varepsilon}_{v}^{e} \\
\dot{\varepsilon}_{q}^{e}
\end{array}\right]
$$

Assume that the plastic mechanism is activated $(L>0)$, the general form of $L$ with respect to anisotropic elasticity can be expressed as:

$$
\begin{equation*}
L=\frac{\left(\Gamma_{3}-\eta \Gamma_{2}\right) \dot{\varepsilon}_{q}+\left(\Gamma_{2}-\eta \Gamma_{1}\right) \dot{\varepsilon}_{v}}{s p K_{p}+s\left(\Gamma_{3}-\eta \Gamma_{2}\right)+d\left(\Gamma_{2}-\eta \Gamma_{1}\right)} \tag{2.159}
\end{equation*}
$$

As seen in Equation (2.158), the off-diagonal terms are non-zero (activated) in the stiffness matrix which induce the cross-coupling between volumetric and shear effects when the fabric is anisotropic. That means the elasticity tensor related with fabric affects both the volumetric and deviatoric strains rates because of the coupling effects. This feature is kindly similar with the new model proposed in this work.

In sum, based on the simulation and comparison works by Lashkari, the modified SANISAND
model taking into account the concept of elastic anisotropy can partly explain the sudden loss of mean effective stress in liquefaction tests and well imitate the behavior of sands in cyclic triaxial tests. Although for one specific geomaterials, only one single set of parameters is needed to simulate different kind of tests, especially the loading / reversal loading test, and can obtain idea results, there are still 15 parameters should be tested and calculated in advanced. These 15 parameters will be shown in the following Section 3.6.

## Chapter 3 - Numerical simulations

### 3.1 Formulation into Umat format

Since the formulation of the former models are built and described in fully format, and appropriate numerical tools are required to verify and test more conditions, we decided to adopt the numerical softwares Matlab® and Abaqus® (in later contents, 'Matlab' and 'Abaqus' represent for numerical software Matlab and Abaqus as simplified purpose) to calibrate the accuracy of our new constitutive model. It is easy to program the equations of models into codes and commands for running and testing in Matlab command window, but for combining the formulations into Abaqus model it is much more complicated.

Abaqus contains lots of different common models in its own internal database, which are verified in details and widely accepted and can run smoothly for mostly part of granular materials. However for some materials, models given by Abaqus database are not relevant so it is necessary to build a new material model which can be adopted directly in Abaqus to connect Abaqus and constitutive relation of those materials. Thus the conception of user-defined material (simplified as Umat) subroutine is revealed.

The subroutine Umat equips powerful functions and qualifies the following abilities: (1) defining the constitutive relation of materials which uses the materials not included in the ABAQUS material library for calculating to extending program functionality; (2) being suitable for any analysis process of mechanics behavior of any unit of Abaqus model. In the Umat file, it is also obligatory of providing the Jacobian matrix of constitutive model, which links the matrices of stress increment to strain increment.

Note that the basis purpose of Umat is to calculate the stress increment matrix or tensor. In the Finite Element Method, it is possible to obtain the stress in $(\mathrm{n}+1)$ step under the condition of given strain and stress in (n) step and known strain increment. Thus, it is the job of Umat to complete this calculation process since the rate of stress increment with respect to strain increment is defined. The Jacobian matrix, or called as $\operatorname{DDSDDE}(\mathrm{I}, \mathrm{J})$ unique in Umat file determines the change of $\mathrm{I}^{\text {th }}$ stress component influenced by the small change of $\mathrm{J}^{\mathrm{th}}$ strain component.

To sum up, since we know the constitutive models, the process of formulation transformation
from equation to Umat commander, and the combination of Abaqus and Umat, it is obvious and easy to verify and calibrate the new constitutive model.

As for the general format of Umat files, according to the rules that the variables statement at the beginning of Umat is used to transfer the data between the main program and Umat, even sharing the same variables, a standard writing format must be obeyed consequently. The commonly used variables in Umat file should be determined at the start of file, and the common format is like:

## SUBROUTINE UMAT(STRESS,STATEV,DDSDDE,SSE,SPD,SCD,\&

RPL,DDSDDT,DRPLDE,DRPLDT,\&
STRAN,DSTRAN,TIME,DTIME,TEMP,DTEMP,PREDEF,DPRED,CMNAME,\&
NDI,NSHR,NTENS,NSTATV,PROPS,NPROPS,COORDS,DROT,PNEWDT, \&
CELENT,DFGRDO,DFGRD1,NOEL,NPT,LAYER,KSPT,JSTEP,KINC)
!
! INCLUDE 'ABA_PARAM.INC'
!
IMPLICIT INTEGER(I-N)
IMPLICIT REAL* $8(A-H, O-Z)$
CHARACTER*80 CMNAME
DIMENSION STRESS(NTENS),STATEV(NSTATV), \&
DDSDDE(NTENS,NTENS),DDSDDT(NTENS),DRPLDE(NTENS), \&
$\operatorname{STRAN}(N T E N S), D S T R A N(N T E N S), T I M E(2), \operatorname{PREDEF}(1), \operatorname{DPRED}(1), \&$
PROPS(NPROPS),COORDS(3),DROT(3,3),DFGRDO(3,3),DFGRD1(3,3),\&
JSTEP(4)
After that, the commanders for determining the value of DDSDDE should be attached as designed by users basing on requirements of Umat, followed by the calculation of stresses, strains and other variables. Last, the whole Umat file will be ended by the following commander, then transfer the results into Abaqus database.

RETURN
END SUBROUTINE UMAT

### 3.1.1 Original Plasol model

The original Plasol model has been described in Section 2.1, and it has never been tested in Abaqus before. As a comparison, it is also obligate to transfer its equations into Umat format. Apart from the general commanders at the beginning, some settlements of key calculations in the specific orders for original Plasol model will be expressed as following.

First of all, after giving the values of Young's modulus ( $E$ ) and Poisson's ratio ( $v$ ), the values of Lamés first $(\lambda)$ and second $(\mu)$ parameters will be clearly obtained by later formulas.

$$
\begin{gather*}
\lambda=\frac{E \cdot v}{(1+v)(1-2 v)}  \tag{3.1}\\
\mu=\frac{E}{2 \cdot(1+v)} \tag{3.2}
\end{gather*}
$$

Later the important component of Umat file which is the Jacobian matrix (or incremental stiffness matrix) in matrix form for isotropic materials can be written as:

$$
D(i j k l)=\left[\begin{array}{ccccccc}
\lambda+2 \mu & \lambda & \lambda & 0 & 0 & 0  \tag{3.3}\\
\lambda & \lambda+2 \mu & \lambda & 0 & 0 & 0 \\
\lambda & & \lambda & \lambda+2 \mu & 0 & 0 & 0 \\
& 0 & 0 & 0 & & \mu & 0 \\
0 \\
& 0 & 0 & 0 & & 0 & \mu \\
0 \\
& 0 & 0 & 0 & & 0 & 0 \\
\hline
\end{array}\right]
$$

Note that the matrix above is available for the three dimensional situation, it is different for two dimensional cases. Thus, for two dimension cases, the matrix will be made as:

$$
D(i j)=\left[\begin{array}{ccccc}
\lambda+2 \mu & \lambda & 0 & 0  \tag{3.4}\\
\lambda & \lambda+2 \mu & 0 & 0 \\
0 & 0 & \lambda+2 \mu & 0 \\
0 & 0 & 0 & \mu
\end{array}\right]
$$

After finishing the orders of incremental stiffness matrix, then the most important procedure will be edited in following, i.e. the calculation and correction part of stress in later step. In case of concision, a subroutine is built naturally.

In this subroutine, see 'subroutine Original_Plasol' in Appendix A, the values of stiffness matrix, current stresses and strains, the incremental strains, the dimensional determine factor and the user definition materials properties are extracted and returned to the main routine.

Comparing with Figure 2.11, the elastic predictor in original Plasol model can be written as:

$$
\begin{equation*}
\varepsilon_{B}=\varepsilon_{B}+\Delta \varepsilon \tag{3.5}
\end{equation*}
$$

Note that the $\varepsilon_{B}$ after the equal sign is the strain in current state corresponding to the ' A ' state in

Figure 2.11, while the one before the equal sign is the strain in later state corresponding to the ' E ' state in Figure 2.11. For the accuracy and concision of writing in Umat format, both these two strains are written as the same, so does the stresses in the following equation.

$$
\begin{equation*}
\sigma_{B}=\sigma_{B}+D^{e} \Delta \varepsilon \tag{3.6}
\end{equation*}
$$

The $D^{e}$ in equation (3.6) refers to the incremental stiffness matrix in equation (3.3) or (3.4) according to the dimensional determine factor.

Then the procedure of corrector should be adopted after the trial stress and strain achieving in former predictor procedure. In this stage, a loop instruction is adopted until the loop condition meets, and those commands below should be concluded in this loop instruction to make sure that the trial stress is not beyond the yield surface.

Firstly, the yield function should be determined. Here, the Drücker Prager criterion is adopted considering the accuracy and concision. In this stage, the first stress tensor invariant, the second deviatoric stress tensor invariant, and the deviatoric stress, see equations (2.28), (2.29) and (2.31), should be calculated. After that, the parameters $m$ and $k$, as well as the yield function $f$, see equations (2.36), (2.37) and (2.38), are computed later.

Secondly, a plastic stress increment should be calculated consequently. This kind of stress increment is possible to be used or neglected according to the sign of yield function which will be discussed in later stage. In this stage, the parameters $m^{\prime}$ (equation (2.45)) which is related with the plastic potential function $g$, is written as $m g$ in the Umat file to avoid the problem of repeat. Also, the derivatives of yield function and plastic potential with respect to stress, see equation (2.46) and (2.52), are presented. Then the derivatives of yield surface with respect to the hardening variables, namely the compression friction angle and cohesion are computed, which correspond to the equations (2.54 2.57). Combining the equations (2.89), (2.90) and the value of scalar Val, the increment of plastic strain is obtained, as well as the value of plastic stress increment.

Finally, the procedure of trial and correction happen inevitably. In this phase, the sign of yield function should be recognized firstly to judge the trial stress state is within or beyond the yield surface. If the yield function is negative, which means the trial stress state is within elastic domain, the calculated plastic stress increment is useless and its value should be refreshed as ZERO, and returned back to main routine. But if the yield function is positive, which means the trial stress state is beyond the yield surface, the trial state should be corrected and returned back on yield surface. In this case,
the corrected stress state will be computed by equation (2.90). Meanwhile, compression friction angle, cohesion, and the dilatancy angle should be refreshed following the corrected equivalent plastic strain, see Figure 2.12 and equations (2.11), (2.13), (2.16) and (3.7). Symbol $\varepsilon^{p}$ expresses the equivalent plastic strain in Umat file for simplification.

$$
\begin{equation*}
\varepsilon^{p}=\varepsilon^{p}+\mathrm{Val} \cdot \Delta \lambda \tag{3.7}
\end{equation*}
$$

### 3.1.2 New model

Considering the reused part with original model, which has been introduced in detail in Chapter 2, only the different and important part will be described and detailed. All the procedures are presented in the appendix B.

Firstly, the Jacobian matrix (incremental stiffness matrix) plays a significant role in the new model, and is highly different with the constant incremental stiffness matrix in original Plasol model. According to equations (2.138), the incremental stiffness matrix is depending on the current stress state and shows an anisotropic characteristic during the increasing of stress. The constant Jacobian matrix in equation (3.3) will change into a symmetric matrix with 36 variable components ( 6 rows and 6 columns). As seen appendix C, a matrix relating to the 3D model is computed on the base of equation (2.138), which is called trial Jacobian matrix with symbol DDSDDE1. If the facing problems we will solve $D D S D D E 1$ regard to the 3D situation, the Jacobian matrix will be used directly by the value of trial Jacobian matrix. However, if it is about the 2D situation, the final Jacobian matrix will be simplified into the form of equation (3.4) on the base of trial Jacobian matrix.

Secondly, the computation of stress is also different with the original model, as it depends on the derivations of the Houlsby hyperelasticity energy function. In this part, the trial strain is firstly computed, followed by the calculation of the volumetric strain $\varepsilon_{v}$, shear strain $\varepsilon_{s}$ and deviatoric component of strain tensor $e_{i j}$, see equation (2.140) and below. Note that the $\delta_{i j}$ is the Kronecker delta tensor $\left(\delta_{i j}=1\right.$, if $i=j ; \delta_{\mathrm{ij}}=0$, if $i \neq j$ ).

$$
\begin{gather*}
\varepsilon_{v}=\varepsilon_{i i}  \tag{3.8}\\
e_{i j}=\varepsilon_{i j}-1 / 3 \cdot \varepsilon_{v} \cdot \delta_{i j}  \tag{3.9}\\
\varepsilon_{s}=\sqrt{ }\left(2 / 3 \cdot e_{i j} e_{i j}\right) \tag{3.10}
\end{gather*}
$$

Then the mean effective stress $p$ and shear stress $q$ which are derived from the free energy
function are calculated as formulas (2.141) and (2.142). The value of trial stress could be computed as equation (2.143).

Finally, after finishing the process of elastic predictor, the procedure of plastic corrector starts in the same way as the original Plasol model does.

### 3.2 Axisymmetric ABAQUS model simulation

Among those several FEM tools, the Abaqus is chosen preferentially and wisely to simulate non-linear constitutive model of geomaterials problems considering our situation. Not only because Abaqus contains lots of unit types which could reflect deeply the phenomenon of microstructures and the difference between them, the materials models which include the constitutive relationship and failure criterions, the contact and connection types which could provide more convenient tools for most types of engineering structures, but also Abaqus provide the user-defined materials properties and constitutive models under the condition of insufficient models in Abaqus itself. Combining with the Umat codes mentioned before, the Abaqus could simulate the majority of geomaterials problems.

In experimental tests, the triaxial test allows to determine some mechanical proprieties which are the disadvantages of other test methods such as the uniaxial test or direct shear test and so on. Depending on the advantages of drainage conditions and confining pressure controls, the triaxial test could obtain the highly accurate values of stress and strain at different states, as well as the variation of pore pressure. Therefore, the triaxial test is chosen naturally in Abaqus simulation.

Before establishing the Abaqus model, a difference between the actual physical triaxial test sample and Abaqus model should be mentioned firstly. For actual experimental test, the common shapes of sample are cylinders with different sizes and heights relating to different conditions. That is convenient and justified to simulate the actual condition of soil in situ. In the modeling using Abaqus, it is also possible to adopt the same full size of sample directly. But there is another more simple and concise way to imitate when considering the run process of Abaqus. Therefore, an axisymmetric model which is closely related with the actual cylinder model but with easier installation of loading and boundary conditions is established. Figure 3.1 shows the Abaqus axisymmetric model with mesh and boundary conditions of the homogeneous triaxial test. Note that the dimensions of the numerical sample are $1 \mathrm{~m} \times 1 \mathrm{~m}$, while the exact sizes of physical sample are $\phi=70 \mathrm{~mm} \times \mathrm{h}=140 \mathrm{~mm}$.


Figure 3.1 Mesh and boundary conditions of axisymmetric homogeneous triaxial test sample
For the boundary settings, the limitation in horizontal direction on the axisymmetric edge, can imitate the central axis, while the limitation in vertical direction on the bottom edge refers to the bottom surface of actual sample. It is worth noting that the values of horizontal displacement $u_{x}$ and vertical displacement $u_{y}$ vary for different drainage conditions. For drained conditions, the horizontal displacement is defined as free, but $\sigma_{x}=\sigma_{0}$ (confining pressure), while for the undrained conditions, the horizontal displacement is defined as the half value of vertical displacement to fulfill the isochoric condition, i.e., $u_{y}=-2 u_{x}$.

After the Abaqus model is established, and the user-defined constitutive model written in Umat language, the Abaqus model runs successfully with the Umat file under different parameters, boundary and loading conditions. Some reasonable results appear and will be discussed in detailed in following Sections.

### 3.3 Comparison with experimental result

### 3.3.1 Experimental and numerical test

In this part, the specific process of experimental test and Abaqus simulation will be introduced. In order to obtain a more reliable and reasonable comparison result, actual parameters for experimental triaxial test will be adopted, and then an optimal set of parameters for Abaqus simulation will be determined and used later.

### 3.3.1.1 Experimental drained triaxial test

Since several decades before, the triaxial test method has become mature and been adopted to determine the material properties, as well as the response of stress and strain. The laboratory apparatus won't be introduced here in details, but the test plan will be presented specifically.

In general, the real triaxial test contains two procedures, namely the procedures of consolidation and shearing. In our tests, there are three different values of confining pressure ( $\sigma_{0}$ ) during the consolidation process, i.e., $50 \mathrm{kPa}, 100 \mathrm{kPa}$, and 200 kPa . After that, the shearing stage is implemented on sample by loading the controlled displacement under the condition of constant confining pressure, as implemented in Figure 3.2. Note that for different values of confining pressures, the Young's moduli for shearing processes are different, as well as the values of incremental stiffness matrix in the aspects of stress strain calculation. Table 3.1 is the actual parameters for experimental samples.


Figure 3.2 Flow chart of experimental triaxial test
Table 3.1 Mechanical properties of experimental samples

|  | $E^{50}$ | 11.1 MPa |
| :---: | :---: | :---: |
| Young's modulus | $E^{100}$ | 28.9 MPa |
|  | $E^{200}$ | 36.8 MPa |
| Poisson's ratio | $v^{50}$ | 0.27 |
|  | $v^{100}$ | 0.22 |
| Failure cohesion | $v^{200}$ | 0.17 |


| Failure friction angle | $\varphi_{f}$ | $44^{\circ}$ |
| :---: | :---: | :---: |
|  | $\psi_{f}^{50}$ | $7^{\circ}$ |
| Limit dilatancy angle | $\psi_{f}^{100}$ | $5^{\circ}$ |
|  | $\psi_{f}^{200}$ | $2^{\circ}$ |

The Figure 3.3 below gives the strain-strain response under drained triaxial conditions for three initial effective mean stresses.


Figure 3.3 Experimental relationship results of stresses and strains

From the experimental results, the stress-strains response shows a typical behavior of geomaterials. While increasing the controlled axial strain, the deviatoric stress increases rapidly first and then slowly, until the failure point of sample. More specific, the value of deviatoric stress under higher confining pressure is commonly higher than the stress under lower confining pressure. As for the volumetric strains, the sample contracts first and then appears dilatation process for lower confining pressure. After the increasing of confining pressure, the sample maybe not shows the phenomenon of dilatation but only the process of contraction for the same initial density of the specimens.

### 3.3.1.2 Abaqus simulation

In the simulation process using Abaqus, the most significant advantage of new model is that only one set of parameters is necessary and determined for all confining pressure tested. These parameters could be determined using the original Plasol model, i.e., the linearly elastic incremental model. After that, this set of parameters for one specific kind of geomaterials could be utilized and combined with a proper value of pressure exponent ' $n 1$ ' in the new model to obtain a better and a more accurate simulation.

Taking the experimental tests mentioned before as sample, the original Plasol model was firstly used to try and determine the optimal parameters. The optimal set of parameters is given in Table 3.2, and then the comparison between simulation (new model) and experimental results are illustrated in Figure 3.4.

(a) Relationship curves of deviatoric stress and strain

(b) Relationship curves of volumetric and shearing strains

Figure 3.4 Comparison between experimental and numerical results obtained using new model

Table 3.2 Optimal parameters determined in original Plasol model

| Young's modulus | $E^{\text {ref }}$ | 17 MPa |
| :--- | :--- | :--- |
| Poisson's ratio | $v^{\text {ref }}$ | 0.2 |


| Inherent cohesion | $C_{o}$ | 1 kPa |
| :--- | :--- | :--- |
| Failure cohesion | $C_{f}$ | 54 kPa |
| Inherent friction angle | $\varphi_{o}$ | $1^{\circ}$ |
| Failure friction angle | $\varphi_{f}$ | $22^{\circ}$ |
| Limit dilatancy angle | $\psi_{f}$ | $8^{\circ}$ |
| Reference pressure ${ }^{\mathrm{a}}$ | $P_{a}$ | 100 kPa |
| Pressure exponent $^{\mathrm{a}}$ | $n 1$ | 0.3 |

Note: ${ }^{\text {a }}$ The reference pressure and pressure exponent are only used in new constitutive law.

Note that the reference pressure and pressure exponent mentioned above and marked by superscript ' $a$ ' are obtained and determined after the determination of other conventional parameters in the Table 3.2. In other words, those conventional parameters are attempted and decided firstly using the original Plasol model, then these parameters are utilized directly in new model and then to attempt the suitable value of pressure exponent to obtain the most accurate simulation.

### 3.3.2 Comparison of stress strain response

In the previous Section, the method to obtain the optimal parameters are introduced and shown in Table 3.2, and the final simulation results including the original Plasol model and the new model are shown in the following Figure 3.5 and Table 3.3. In the Figure 3.5 and Table 3.3, the only one set of parameters in Table 3.2 is used for the new model. From the Figure and Table, it can be found that the simulation of new model is more accurate than the original Plasol model when using the same parameters no matter the values of confining pressures. Although there are some deviatoric values in the simulation of volumetric strain in higher confining pressure conditions, it is apparent that new model with only one set of parameters is more convenient and highly efficient when comparing the old original Plasol model.

Specifically, for volumetric strain simulation in Figure 3.5 (b), the accuracy is better for the low confining pressure than high pressure. However, the accuracy in 200 kPa is not good at all in this case which dues to the constant adopted dilatancy angle in new model. In fact, this situation exists in many
regular models (ALE 05). Despite this, comparing with other linear or nonlinear elastic constitutive model, in which different sets of mechanic properties are needed for different consolidation and confining conditions, the fact that only one set of parameters is utilized for various situation with the new constitutive model is way important.

Apart from this Figure, the comparison curves in detail between the original Plasol model and new model are illustrated in Figures 3.6 and 3.7.

(b) Relationship curves of volumetric and shearing strains

Figure 3.5 Comparison results between experimental tests and numerical simulations

Table 3.3 Simulation difference between experiments (EXP), original Plasol model (ORI) model and new model (NEW)

| Initial | Strain | $q(\mathrm{kPa})$ |  |  |  |  | $\varepsilon_{v}(\%)$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| mean <br> stress | state | EXP | ORI | RE ${ }^{\text {b }}$ | NEW | RE ${ }^{\text {b }}$ | EXP | ORI | $\mathrm{RE}^{\text {b }}$ | NEW | RE ${ }^{\text {b }}$ |
| 50 kPa | 5\% | 287.74 | 221.96 | -22.86 | 245.55 | -14.66 | -0.91 | -1.05 | 15.45 | -0.88 | -3.16 |
|  | 10\% | 361.72 | 255.13 | -29.47 | 267.46 | -26.06 | 0.32 | -0.30 | 195.70 | 0.35 | 10.17 |
| 100 kPa | 5\% | 452.16 | 438.76 | -2.96 | 464.42 | 2.71 | -1.22 | -1.40 | 14.49 | -1.43 | 16.57 |
|  | 10\% | 591.92 | 505.15 | -14.66 | 535.21 | -9.58 | -0.84 | -0.76 | -9.39 | -0.79 | -5.58 |
|  | 15\% | 665.90 | 531.76 | -20.14 | 544.96 | -18.16 | 0.49 | 0.36 | -26.25 | 0.56 | 15.38 |
| 200 kPa | 5\% | 752.21 | 822.93 | 9.40 | 816.82 | 8.59 | -2.35 | -2.06 | -12.61 | -2.23 | -5.25 |
|  | 10\% | 953.66 | 977.67 | 2.52 | 973.42 | 2.07 | -2.63 | $-1.80$ | -31.82 | -2.34 | -11.17 |
|  | 15\% | 1048.21 | 1056.48 | 0.79 | 1081.85 | 3.21 | -2.38 | -0.58 | -75.69 | -0.97 | -59.11 |

Note: ${ }^{\mathrm{b}} \mathrm{RE}$ stands for relative error (unit: \%).


Figure 3.6 Comparison results between new model and original Plasol model


Figure 3.7 Comparison results between new model and original Plasol model within very small strain

Note again that the results in Figure 3.6 and 3.7 are obtained from the Abaqus simulation by using only one same set of parameters. From Figure 3.6, we can find that the value of deviatoric stress simulated by original Plasol model in small strain stage is higher than the one by new model in both these three confining pressure conditions. Then the deviatoric stress values between them are reducing with the increasing of strain. It can be also shown clearly in Figure 3.7 that the initial tangent moduli calculated in new model are lower than the values of original Plasol model.

Meanwhile, another significant factor in Figure 3.7 is that the values of tangent moduli obtained from new model for different confining pressures are different while the values from original model are identical, although the parameters adopted are the same under different confining pressure conditions. This factor is a powerful evidence proving that the stiffness matrix of new model is changing and depends on the current stress or strain state, for only one set of paraleters, which has an expression of variable tangent modulus for different conditions. The influence of tangent modulus will be presented in detailed in the Section 3.3.3.

### 3.3.3 Evolution of tangent modulus

### 3.3.3.1 Essential relation of variable tangent modulus

The fact that the initial tangent modulus, obtained in FEM is mentioned before, evolves during loading could be explained by the Houlsby hyperelastic part of the model. Since the new model is established based on the Houlsby model, the stiffness matrix is calculated directly by two times derivative of hyperelastic free energy function, which is presented in equation (2.138). From this formulation, it can be easily obtained that the initial stiffness matrix (or the Young's modulus or Poisson's ratio) for shear procedure is different for different initial consolidation conditions, and importantly it can also change during the shearing stage.

In equation (2.138), the stiffness matrix $D_{i j k l}$ is a function of variable $p_{o}$, and stress $\sigma_{i j}$, while $p_{o}$ is also a function of stress $\sigma_{i j}$, according to equation (2.135). As a conclusion, this stiffness matrix varies as power functions of the current stress state, or effective mean stress. Typically, although this kind of relation is influenced by other important coefficients, e.g. the bulk stiffness factor $k$, shear stiffness factor $g$, and pressure exponent $n 1$, the stiffness matrix with the expression of tangent modulus evolves during the variation of the current stress, which could be shown in the Figure 3.6 and 3.7.

### 3.3.3.2 Comparison with real experimental data

Note that for different confining pressure condition, the Young's modulus and stiffness matrix are different. And this kind of observation could be expressed as the evolution of tangent modulus. Therefore, a comparison results between the experimental data and simulation using new model will be made and illustrated in Figure 3.8.


Figure 3.8 Evolution of tangent modulus for experimental tests and simulations

Theoretically, the initial small-strain shear stiffness obtained using Equations (2.93) to (2.98) is function of the current stress state, expressed by mean effective stress, the current void ratio and the previous stress history experienced by the material. Such stiffness, which is expressed with tangent modulus, decreases with the increasing of shear strain.

Figure 3.8 presents the evolution of tangent modulus which is identical with shear modulus in triaxial condition. The tangent modulus obtained with numerical simulation is quite close to the experimental data in the small axial strain stage, whilst these two forms of data might not coincide well in the large strain stages although the tendencies are similar. This feature is well presented in Figure 3.5 (a) and Figure 3.8. It is worth noting that the correlation coefficient $\left(\mathrm{R}^{2}\right)$ of numerical simulation on shear modulus reaches acceptable values of 0.907.

In turn, this comparison could prove not only the stiffness matrix calculated in new model varies depending on mean stress, but also new model can well simulate the actual geomaterials behavior.

### 3.4 Analyses of pressure exponent in undrained condition and the related density

### 3.4.1 Influence of pressure exponent in undrained condition

As mentioned before, the equation (2.138) shows a relation between the stiffness matrix and the
significant variable, i.e., the pressure exponent ' $n 1$ '. In this part, the influence of pressure exponent will be discussed in detail.

Based on the formula of stiffness matrix (2.138) and the equations of components of stiffness matrix (Equations 2.101, 2.102, 2.103 and 2.104), the new formulas of components of stiffness matrix could be rewritten as following $(n 1 \neq 1)$.

$$
\begin{gather*}
K=p_{a}[k(1-n 1)]^{\frac{1}{1-n 1}} \cdot\left\{\frac{n 1}{1-n 1} \cdot\left(\varepsilon_{v}+\frac{1}{k(1-n 1)}\right)^{2} \cdot\left[\left(\varepsilon_{v}+\frac{1}{k(1-n 1)}\right)^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right]^{\frac{3 n 1-2}{2-2 n 1}}+\right. \\
\left.\left[\left(\varepsilon_{v}+\frac{1}{k(1-n 1)}\right)^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right]^{\frac{n 1}{2-2 n 1}}\right\}  \tag{3.11}\\
3 G=p_{a}[k(1-n 1)]^{\frac{n 1}{1-n 1}} \cdot 3 g \cdot\left\{\frac{n 1 \cdot 3 g \cdot \varepsilon_{s}^{2}}{k(1-n 1)^{2}} \cdot\left[\left(\varepsilon_{v}+\frac{1}{k(1-n 1)}\right)^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right]^{\frac{3 n 1-2}{2-2 n 1}}+\left[\left(\varepsilon_{v}+\right.\right.\right. \\
\left.\left.\left.\quad \frac{1}{k(1-n 1)}\right)^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right]^{\frac{n 1}{2-2 n 1}}\right\}  \tag{3.12}\\
J=p_{a}[k(1-n 1)]^{\frac{n 1}{1-n 1}} \cdot \frac{3 g \cdot n 1 \cdot \varepsilon_{s}}{1-n 1} \cdot\left[\varepsilon_{v}+\frac{1}{k(1-n 1)}\right] \cdot\left[\left(\varepsilon_{v}+\frac{1}{k(1-n 1)}\right)^{2}+\frac{3 g \varepsilon_{s}^{2}}{k(1-n 1)}\right]^{\frac{3 n 1-2}{2-2 n 1}} \tag{3.13}
\end{gather*}
$$

Due to formula (3.13), the off-diagonal components of stiffness matrix would be non-zero values when the variable pressure exponent ' $n 1$ ' is positive. Then, the induced anisotropy occurs. It is also obvious that the level of induced anisotropy differs with various values of pressure exponent, as well as the responses of stresses and strains. To sum up, the effect of pressure exponent on the level of induced anisotropy also affects the stress and strain behavior, even altering the hardening tendency of soils.

In order to present the effect of pressure exponent, a series simulation of undrained triaxial tests under 100 kPa confining pressure condition is performed. In this series of undrained triaxial simulations, the parameters are adopted in given Table 3.4, and most importantly, 8 values of pressure exponent will be adopted in the simulation, i.e., $0,0.02,0.04,0.06,0.1,0.2,0.3$ and 0.4 . The simulation results are shown in Figures 3.9 and 3.10. In Figures 3.9 and 3.10 and in Table 3.4, the sample 1 and 2 are samples with different densities which will be discussed in the next Section. It is worth noting that although there are two different relative densities of soil sample, these two samples should be seen as definetely loose ones according the absolute values of young's modulus and poisson ratio.

Table 3.4 Parameters adopted in undrained test simulation for sample 1 and sample 2

| Young's modulus | $E$ | $17 \mathrm{MPa}^{\mathrm{c}} / 7 \mathrm{MPa}$ |
| :---: | :---: | :---: |
| Poisson's ratio | $v$ | 0.2 |
| Inherent cohesion | $C_{0}$ | 1 kPa |
| Failure cohesion | $C_{f}$ | $54 \mathrm{kPa}^{\mathrm{c}} / 50 \mathrm{kPa}$ |
| Inherent friction angle | $\varphi_{0}$ | $1^{\circ}$ |
| Failure friction angle | $\varphi_{f}$ | $22^{\circ} / 15^{\circ}$ |
| Limit dilatancy angle | $\psi_{f}$ | $0^{\circ}$ |
| Reference pressure | $P_{a}$ | 100 kPa |

Note: ${ }^{\text {c }}$ The parameters for sample 1 only.

(a) Response for sample 1 ('denser' sample)

(b) Response for sample 2 ('looser' sample)

Figure 3.9 Response of stresses in undrained triaxial simulation

(a) Response for sample 1 ('denser' sample)


Figure 3.10 Evolution of ratios of shear stress to mean effective pressure in undrained triaxial simulation

More important, according to Equations (3.11) and (3.12) and the free energy expressions, the bulk and shear modulus $K$ and $G$ are functions of $p_{a}, k, g, n 1, \varepsilon_{v}$ and $\varepsilon_{s}$. Therefore, it is feasible to maintain a same bulk modulus value at the initial state of shearing procedure for 8 different values of pressure exponents, so does the value of shear modulus. Meanwhile, the void ratio doesn't change during the procedure of shearing in isochoric undrained test, and it can also be kept at a same value for all pressure exponents conditions.

In Figure 3.9 (a), based on the fact that the bulk moduli, shear moduli and void ratios are maintained at the same value for all ' $n 1$ ' conditions whilst ' $n 1$ ' is the only independent variable, the variable affecting the induced anisotropy is ' $n 1$ '. Figure 3.10 (a) shows the ratios of shear stress to mean effective pressure of sample 1, which are associated with Figure 3.9 (a). Note that the magnitude of this ratio has a connection with strain hardening pattern in plastic domain. Therefore, it is easily concluded that the induced anisotropy has a significant influence on hardening evolution even though the void ratio does not evolve at all for any value of ' $n 1$ ' during isochoric conditions. When pressure exponent ' $n 1$ ' has a small value, the sample presents a 'looser' trend, while this tendency changes to 'denser' trend with the increasing value of ' $n 1$ '. Theoretically, for small values,
the sample shows a response of isotropic elastic increment and no induced anisotropic components in stiffness matrix occurs when the pressure exponent equals to zero. However, this situation is contradiction with the theory. The shear stiffness depends on the current stress state (the mean effective stress, current void ratio and previous stress history) proposed by Hardin (HAR 78). The more general situation is that the anisotropy arises as a natural consequence of the hyperelastic formulation, which is corresponding appropriately to the observation of soil behavior with the condition of non-zero value of ' $n 1$ '.

### 3.4.2 Effect of density on the influence of pressure exponent

In order to figure out the effects of density of sample on the influence of pressure exponent on the response of stress and strain, two different values of density are adopted in the undrained test simulation, which are listed in the Table 3.4. It is worth noting that there are two sets of parameters for two samples, i.e. sample 1 with the same parameter of drained test sample presented in Section 3.3, and sample 2 with lower density (lower Young's modulus, cohesion pressure and friction angle).

The simulation results for denser sample 1 are shown and discussed in Section 3.4.1, then comes to the looser sample 2 in this Section. As a comparison, a simulation of relatively looser soil with lower density was simulated under the identical condition, as illustrated in Figure 3.9 (b) and Figure 3.10 (b). The response of looser soil with low density, which is carried out with the same ' $n 1$ ' values of denser soil with high density, is showing a different trend. The behavior of looser soil with a low value of ' $n 1$ ' presents a similar 'looser pattern' until ' $n 1$ ' reaches at 0.2 , whereas that the ' $n 1$ ' of denser soil reaches at 0.06 in this case. The evolution from 'looser' trend to 'denser' trend of high density soil can be easily reached when the value of ' $n 1$ ' increases. In general, it can be concluded that the density of soil certainly influences the effect of ' $n 1$ ' on the response of soil. In other words, the level of induced anisotropy, which corresponds to a specific value of ' $n 1$ ' and influences the stressstrain response, is closely related with soil density.

As a dimensionless experimentally determined parameter, the pressure exponent ' $n 1$ ' is closely related with the inherent fabric of soil. In new model, the off-diagonal term $J$ is expressed as Equation (3.13), which is linked tightly with pressure exponent. The anisotropic stiffness matrix arises with non-zero ' $n 1$ '. Thus the level of anisotropy which is induced by the various ' $n 1$ ' and loaded pressure,
and the stress-strain response, will also change.
The evolution of anisotropy observed during the experiments can be captured by the parameter of ' $n 1$ ', which is, for the current model, fixed. According to Kruyt (KRU 11), Pouragha and Wan (POU 17), the process of deformation is the one where fabric tensor changes. In DEM, the contact disruption, which means the granular particle contacts disconnect during the strain increment, is the dominant mechanism factor for the changing fabric tensor. More importantly, this contact disruption is strongly anisotropic. Similarly, the anisotropic response of stress-strain owing to non-zero $J$ term would affect the proportion of contact disruption during strain increment, which might alter the coordination number of the assembly of particles as well. This fabric feature can change the hardening tendency eventually.

### 3.5 Analyses of pressure exponent in drained condition

Previously, the simulations for undrained test were shown and discussed in detailed, as well as the effects of density. Theoretically and in practice, it is also necessary to do some simulations in drained condition. Note that in later simulations, the model parameters adopted are the same ones used in undrained condition (Table 3.4, sample 1) with an initial value of 0.819 for void ratio, as well as the same 100 kPa confining pressure. The simulation results are shown in Figure 3.11 and 3.12. Figure 3.11 shows the relationship of mean effective pressure and void ratio, while the stress-strain responses are illustrated in Figure 3.12. It is worth noting that the relative 'denser' sample 1 is loose sample with high value of void ratio according to the relative small value of young's modulus and poisson ratio. That is the reason why the results presented below show the absence of post-peak softening and significant contractrance.


Figure 3.11 Void ratio versus mean effective pressure in drained triaxial simulation

(a) Shearing stress versus shearing strain

(b) Volumetric strain versus shearing strain

Figure 3.12 Stress-strain responses in drained simulation for different pressure exponents

Unlike the situation of undrained simulation, it is also clear that the effect of pressure exponents on the response of stress and strain is not obvious as comparing with the undrained situation. It means that the induced anisotropy leads small impact on the response of stress and strain. Similarly, the bulk modulus $K$ and shear modulus $G$, as well as the void ratio can be kept the same for all pressure exponents at the initial state of shearing procedure. However, the one which is differing from the undrained shear condition is that the void ratio changes during the procedure of shearing because of the changing total volume. The variation of void ratio becomes the only one factor who minimize the effect of induce anisotropy comparing with the simulation with undrained triaxial test.

### 3.6 Modified SANISAND model

In Section 2.4, the simple SANISAND model and its modified model taking account for the induced elastic anisotropy were well introduced. The equations of modified SANISAND model are combined with the work done by previous researchers and re-written into an Umat file, seen in Appendix C. As comparison, some experimental data and the simulation by modified SANISAND model should be introduced firstly.

A series of experiments in undrained loading test on medium-loose and dense samples of Toyoura sand were conducted by Verdugo and Ishihara (VER 96). The initial void ratios of medium-loose and dense samples are 0.833 and 0.735 respectively. Samples are isotropically consolidated to 100 kPa firstly. Then the shear stress was applied on samples. The following tables show the physical properties of Toyoura sand and the parameters adopted in the simulation of modified SANISAND model.

It is worth mentioning that the symbols of all variables in Section 3.6 and the previous Section 2.4 referring to the SANISAND model are only meaningful in these Sections. Even if the same symbols in these Sections ( $2.4 \& 3.6$ ) appear in other Chapters or Sections, their meanings are not exactly the same.

Table 3.5 Physical properties of Toyoura sand (VER 96)

|  | Mean diameter <br> $(\mathrm{mm})$ | Uniformity <br> coefficient | Maximum void <br> ratio | Minimum void <br> ratio | Specific gravity |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Toyoura sand | 0.17 | 1.32 | 0.977 | 0.597 | 2.65 |

Table 3.6 Simulation parameters adopted in modified model (VER 96)

| Category | Parameters | Symbol | Value |
| :---: | :---: | :---: | :---: |
| Elastic | Elastic shear modulus | $G_{0}$ | 125.0 |
|  | Poisson ratio | $v$ | 0.05 |
| Yield surface | Opening of yield surface cone | $m$ | 0.01 |
| Critical state line | Critical back-stress ratio in triaxial compression | $M_{c}$ | 1.25 |
|  | Critical back-stress ratio in triaxial extension | $M_{e}$ | 0.89 |
|  | Critical state line materials constant | $\lambda$ | 0.019 |
|  | Void ratio on critical state | $e_{0}$ | 0.934 |
|  | Critical state line materials constant | $\xi$ | 0.70 |
| Plastic modulus | Positive materials constant about hardening modulus | $h_{0}$ | 881.25 |
|  | Positive materials constant about hardening modulus | $c_{h}$ | 0.968 |
|  | Material constant about stress image on boundary surface | $n^{b}$ | 1.1 |
| Dilatancy | Dilatancy material constant | $A_{0}$ | 0.704 |
|  | Material constant about stress image on dilatancy surface | $n^{d}$ | 3.5 |
| Fabric | Fabric index constant | $z_{\text {max }}$ | 3.0 |

Similarly, the FEM software Abaqus is used combining with the model of modified SANISAND for simulating the behaviour of Toyoura sand under undrained triaxial condition. The stress paths and shear stress versus shear strain curves on two different void ratios of samples implemented under undrained condition are illustrated in following Figure 3.13.

(a) Stress path for sample $e=0.833$

(b) Shear stress versus axial strain for sample $e=0.833$


Figure 3.13 Simulation by Modified SANISAND versus experiments in undrained triaxial tests on samples of Toyoura sands (LAS 10; VER 96)

Using the axisymmetric model shown in Figure 3.1, the new Houlsby-Plasol model we established is also implemented into Abaqus. Likewise, one single optimal set of parameters is
determined and shown in following Table 3.7.
Table 3.7 Optimal parameters of Toyoura sands sample for new model

| Young's modulus | $E^{\text {ref }}$ | 35 MPa |
| :--- | :--- | :--- |
| Poisson's ratio | $\nu^{\text {ref }}$ | 0.25 |
| Inherent cohesion | $C_{o}$ | 6.9 Pa |
| Failure cohesion | $C_{f}$ | 8 kPa |
| Inherent friction angle | $\varphi_{o}$ | $1^{\circ}$ |
| Failure friction angle | $\varphi_{f}$ | $30^{\circ}$ |
| Limit dilatancy angle | $\psi_{f}$ | $8^{\circ}$ |
| Reference pressure | $P_{a}$ | 100 kPa |
| Pressure exponent | $n 1$ | $0 / 0.2$ |

Note that in Table 3.7, two values of pressure exponent are adopted for the situation of two different values of void ratio. After determining the optimal parameters, the same simulation processes are conducted in Abaqus with new model. The following Figure 3.14 shows the comparison results.

(a) Stress path for sample $e=0.833$

(b) Shear stress versus axial strain for sample $e=0.833$

(c) Stress path for sample $e=0.735$

(d) Shear stress versus axial strain for sample $e=0.735$

Figure 3.14 Comparison results between new model and modified SANISAND model ( $\mathrm{n} 1=0$ : isotropic elasticity)

Basing on the results from Figure 3.14, the comparrison in the initial state of shear state is quite good, though the simulation in later plastic process is not good as the modified SANISAND. This can be explained by that the modified SANISAND considered well the anisotropic dilatancy (plasticity) as well as the anisotropic elasticity, while in our model the elastic part is our main target. However, it cannot be concluded that our new model is inferior to the modified SANISAND model as shown in Figurre 3.14 (d). The new model based on the hyperelastic function which relating to the anisotropic elasticity due to the development of shear strain equips a good simulation effect. Besides, the total number and difficulty degree of obtaining the parameters for new model are quite less and lower than the one for modified SANISAND model.

### 3.7 Conclusions

In this Chapter, the new model, as well as the original Plasol model are firstly introduced and written into Umat format. Then the Abaqus models for comparison are established as the requirement of experimental conditions. Last but not least, the comparison between the laboratory experimental results and the Abaqus simulation are made and explained thoroughly. Moreover, some simulations
about the influence of pressure exponent for undrained and drained simulations and the effect by sample density are also performed reasonable.

From the comparison results between the experimental drained triaxial test and Abaqus simulations, the new model can well simulate the actual problems by using a single set of optimal parameters, although there are some deviatoric values during predicting the variation of volume strain in higher confining pressure conditions.

The tangent modulus is identical to the shear modulus in triaxial condition, and it is much useful in FEM calculation. From the evolution of tangent moduli' simulations, the value of numerical simulation is quite close to the experimental data in the small axial strain procedure, whilst these two forms of data might not coincide well in the large strain stages although the tendencies are similar. The acceptable values of correlation coefficient between them also show the satisfying "fitting degree" and simulation.

The incremental stiffness matrix becomes anisotropic when the experimentally determined parameter pressure exponent ' $n 1$ ' has a non-zero value. These different anisotropies in elasticity domain owing to various values of ' $n 1$ ' make a big difference on the response in terms of stresses and strains, even altering the tendency of hardening for undrained triaxial simulations. Besides, it seems that the density of soil affects the influence of stress-induced anisotropy on response of soil to a certain extent. However, for drained triaxial conditions, this kind of effect is quite tiny no matter what the density or initial confining pressure condition is. Drained triaxial responses are therefore more sensitive to plasticity model than the elastic model used.

At last, by comparing with a constitutive model with respect to anisotropic elasticity, i.e., modified SANISAND model, the simulation on Toyoura sands in undrained situation by new model is feasible. Although there are some deficiencies points at the plastic state, which the modified SANISAND model shows a better simulation effect at later stage of shearing process, our new model possesses the advantages of less total number and more accessible of parameters than modified SANISAND does. The new model shows better agreement for looser sample than for denser.

## Chapter 4 - Second-order work analyses

According to Daouadji (DAO 10), a synthesis of the works about the second-order work (expressed as $W_{2}$ ) criterion has been done by various teams from several countries, and the second-order work criterion has also been proved to be closely related with the diffuse failure and the outburst of kinetic energy accompanied with the drastically increased strains and decreased stresses in sample. Prunier et al. (PRU 16) proposed a conception of proper stability criterion as a safety factor by adopting the second-order work criterion. Lots of research works has been done to prove that the second-order work criteria can be used to detect the diffuse failure of actual problems (DAO 10; DAR 07; LAO 02; DAR 05).

It is worth noting that the second-order work criterion is implemented under some limited physical preconditions: (1) the elastoplastic constitutive model should be non-associated; (2) the stress (or strain) state should be located inside the bifurcation domain; (3) the current loading direction should be included inside or on an instability cone; (4) mixed loading conditions are imposed.

### 4.1 Formulation of $\boldsymbol{W}_{2}$

According to Hill's stability criterion (HIL 58), the fact that the failure of homogeneous samples of soils could happen strictly within the classic plasticity limit of Mohr-Coulomb can be described properly. Hill also proved that a sufficient condition for stability of elasto-plastic medium should meet the following relationship.

$$
\begin{equation*}
\int\left\{\delta s_{i j} d\left(\frac{\partial u_{j}}{\partial X_{i}}\right) d V_{0}\right\}>0 \tag{4.1}
\end{equation*}
$$

For any displacement $d u, s_{i j}$ are the components of the nominal stress tensor and $\delta s_{i j}$ is the change of $s_{i j}$ due to the arbitrary virtual displacement $\underline{d u} . X_{i}$ is the position vector in the initial configuration.

Theoretically, the formulation of the second-order work $\left(w_{2}\right)$ on local or single element is the product of strain increment and stress increment under small strain assumption, i.e.,

$$
\begin{equation*}
w_{2}=\underline{\underline{d \sigma}}: \underline{\underline{d \varepsilon}} \tag{4.2}
\end{equation*}
$$

In this formula, $\underline{\underline{d \sigma}}$ is the Cauchy stress tensor and $\underline{\underline{d \varepsilon}}$ is the linearised strain tensor, and the $w_{2}$
represents the inner term in equation (4.1) when small strains and small geometrical changes are assumed. Meanwhile, when considering homogeneous problems, Hill's condition might become as:

$$
\begin{equation*}
w_{2}=\underline{\underline{d \sigma}}: \underline{\underline{d \varepsilon}}>0 \tag{4.3}
\end{equation*}
$$

This equation (4.3) could be also written as following formula in the notations where second-order tensors of strain and stress are expressed as a six-component vector.

$$
\begin{equation*}
w_{2}=\underline{t} d \sigma \underline{\underline{N}}(\underline{\tilde{d}}) \underline{d \sigma}>0 \tag{4.4}
\end{equation*}
$$

In this equation, $\underline{\underline{N}}(\underline{\tilde{d}})$ is the rate-independent constitutive operator which connect the $\underline{d \varepsilon}$ with $\underline{d \sigma}$, while the left superscript ' $t$ ' is the transposed operator. $\underline{\tilde{d}}=d \sigma /\|d \sigma\|$ is the loading direction which is closely related with the rate-independent constitutive operator before. It is worth noting that at the stage of very small deformation, all the eigenvalues of this operator are strictly positive and the relate soil stress state is wthin stable elasticity. But this kind of stable state will disappear when one of eigenvalues of operator vanishes.

In classical elasto-plastic models, the operator is piecewise linear in the stress rate space. Supposing that the tensorial zone is used to express a part of stress rate space in which $\underline{\underline{N}}(\underline{\tilde{d}})$ is linear, which means this kind of zone is independent from $\underline{\tilde{d}}$. In such tensorial zone, we have the equation:

$$
\begin{equation*}
w_{2}=\underline{t} d \sigma \underline{\underline{N}} \underline{d \sigma}=0 \leftrightarrow \underline{t} \underline{d \sigma} \underline{\underline{N_{s}}} \underline{d \sigma}=0 \tag{4.5}
\end{equation*}
$$

This equation represents a general form of elliptical cone. $\underline{N}_{s}$ is the symmetric part of $\underline{N}$. In the principal stress rate space, the solutions of equation (4.5) depend on the eigenvalues of $\underline{\underline{N_{s}}}$ and are geometrically similar to the form displayed in Figure 4.1. For a given loading path, the solutions appear from zero in the order shown in Figure 4.1. That is to say, the solution is empty firstly because the sample is completely stable. Then a single unstable loading direction develops, accompanied by the cones of unstable loading directions growing until the plasticity limit. When a solution exists, loading paths which is included inside or on this kind of cone are unstable, while other loading paths are stable. This is the intuitive meaning of cone of unstable loading directions.


Figure 4.1 Solutions of equation: $\boldsymbol{\lambda}_{\mathbf{1}} \mathbf{X}^{\mathbf{2}}+\boldsymbol{\lambda}_{\mathbf{2}} \mathbf{Y}^{\mathbf{2}}+\boldsymbol{\lambda}_{\mathbf{3}} \mathbf{Z}^{\mathbf{2}}=\mathbf{0}$, which is a reduction of the quadric

$$
\underline{\mathbf{t}^{\mathbf{d} \boldsymbol{\sigma}}} \underline{\underline{\mathbf{N}_{\mathbf{s}}}} \underline{\mathbf{d} \boldsymbol{\sigma}}=\mathbf{0}(\mathrm{PRU} 16)
$$

Moreover, a specific set of stress points where the solutions of equation (4.5) is reduced to only one unstable direction is called bifurcation domain limit. In fact, this limit is located inside the plasticity limit and closely depends on the loading path. Therefore, the limit of bifurcation domain according to the constitutive model of Darve (DAR 04) can be plot with a set of several deviatoric stress paths in the deviatoric plane which is displayed in Figure 4.2.


Figure 4.2 Limit of the bifurcation domain plotted in deviatoric plane for constitutive models of Darve, compared with plastic limit of Morh-Coulomb (PRU 09)

In equation (4.2), this quantity of $w_{2}$ can be directly computed at each integration point of a mesh. Therefore, this kind of mathematical positiveness of $w_{2}$ is regarded as the subject of interest but not its intensity. Numerically, a real zero is not easy to obtain. Thus, a so-called normalized second-order work (expressed as $w_{2 n}$ ) has been proposed as follows.

$$
\begin{equation*}
w_{2 n}=\frac{w_{2}}{\|\underline{\underline{d \underline{d}} \|}\| \cdot\|\underline{\underline{d \varepsilon}}\|} \tag{4.6}
\end{equation*}
$$

It is easily found that the normalized second-order work ' $w_{2 n}$ ' is limited in the interval $[-1 ; 1]$ and geometrically present the cosine value of the angle between the incremental stress and strain vectors. As a consequence, the value of $w_{2 n}$ decreases monotonically when the stability of a homogeneous sample decreases. Therefore, $w_{2 n}$ can be computed and adopted at each integration point of a finite element or node analysis to study the local stability of a body.

Since the second-order work for local stability and its normalized one can be obtained by the equations above, it is necessary to discuss the global second-order work. It is worth noting that the $w_{2}$ can be numerically integrated on one element according to Prunier, et al (PRU 16), and when assembling over the mesh we can obtain the global second-order work as following:

$$
\begin{equation*}
W_{2}=\int_{V} \underline{d \varepsilon} \underline{\varepsilon} \underline{d \sigma} d V=\underline{d Q}^{t} \underline{\underline{K}} \underline{d Q}=\underline{d Q^{t}} \underline{d F} \tag{4.7}
\end{equation*}
$$

In this equation (4.7), $\underline{d Q}$ and $\underline{d F}$ mean the global nodal incremental displacement and force, respectively, while the $\underline{\underline{K}}$ represents the global consistent tangent matrix. In sum, the form of equation (4.7) is similar with equation (4.2). Then the form of global normalized second-order work can be rewritten as when comparing with the local equation (4.6):

$$
\begin{equation*}
W_{2 n}=\frac{W_{2}}{\|\underline{d Q}\|\| \| \underline{d F} \|} \tag{4.8}
\end{equation*}
$$

In fact, at a given spatial position, the components of vector of nodal displacement do not depend on the size of adjacent elements. Thus the simple Euclidean norm of $\|\underline{d Q}\|$ depends on the number of mesh elements. As a conclusion, this equation could be re-written as following expression for the integrated normalized second-order work:

$$
\begin{equation*}
W_{2 n}=\frac{W_{2}}{\int_{V}\|\underline{\underline{d} \|}\| \cdot\|\underline{\underline{d \varepsilon}}\| d V} \tag{4.9}
\end{equation*}
$$

Note that equation of global normalized second-order work ( $W_{2 n}$ ) matches the expression of local second-order work ( $w_{2 n}$ ) in equation (4.6) when studying the homogeneous cases because of the constant integrands. According to Prunier et al (PRU 16), the quantity of $W_{2 n}$ in expression (4.9) can be adopted theoretically as a safety factor to probe the global stability of a non-homogeneous problem. It means that when $W_{2 n}$ is strictly positive the structure is stable. On the contrary, the structure is unstable. In sum, an effective failure accompanying with the sudden burst of kinetic energy depends on the control variables at the boundary.

### 4.2 Stabilities analysis of axisymmetric model

In Section 3.2, an axisymmetric Abaqus model was built with meshed elements. The response of stress and strain for cylindrical samples were tested and simulated in different confining pressures and drainage conditions. Although the direct variations process of stress and strain could be obtained from the simulation, including the rapid increasing and slight decreasing of deviatoric stress during the application of shear strain in some cases, it is still uneasy to reasonably judge the current state in progress. Therefore, the idea that combining the second-order work within our new model is proposed and utilized to probe the stability states of axisymmetric model in this Section.

Similarly, the geometric size of axisymmetric model is same with the one in Figure 3.1, and the parameters used are the ones of looser sample (sample 2) in Table 3.4. Note that the formulation is purely mechanical without any effect of hydromechanics as no water is included, and the simulation is implemented with the classical undrained, i.e. isochoric, triaxial test. The response of homogeneous model simulated in terms of stress-strain relationship is shown in Figure 4.3 as followed. This figure is based on the situation of pressure exponent $n 1=0.1$, and other valuable values of pressure exponent
have been simulated and proved presenting a similar result.


Figure 4.3 Response of undrained triaxial test simulation
From the response of stresses and strain results, it is clear that this relationship presents a peak in the deviatoric stress $q$ variable, but here it does not correspond to a softening trend, which is not our purpose in our work. Combining with the value changes of integrated normalized second-order work in Figure 4.4, it can be figured out distinctly that at the $q$ peak, the value of $W_{2 \mathrm{n}}$ vanishes to zero and then turns to negative along the current loading path.

Meantime, a generalized flow rule (PRU 16) can be defined even though the current stress-strain state is strictly inside the classic plasticity limit. In this isochoric test, the constitutive relation can be defined as:

$$
\underline{\underline{P}}\left\{\begin{array}{l}
d \varepsilon_{y}  \tag{4.10}\\
d \sigma_{r}
\end{array}\right\}=\left\{\begin{array}{c}
d q \\
d \varepsilon_{v}
\end{array}\right\}
$$

with the isochoric condition:

$$
\left\{\begin{array}{c}
d \varepsilon_{y}=c s t<0  \tag{4.11}\\
d \varepsilon_{v}=0
\end{array}\right.
$$

Note that $\underline{\underline{P}}$ corresponds to the global stiffness operator. At the $q$ peak, this generalized flow rule can be rewritten as:

$$
\underline{P}\left\{\begin{array}{l}
d \varepsilon_{y}  \tag{4.12}\\
d \sigma_{r}
\end{array}\right\}=\left\{\begin{array}{l}
0 \\
0
\end{array}\right\}
$$

with

$$
\left\{\begin{array}{l}
d \varepsilon_{y}  \tag{4.13}\\
d \sigma_{r}
\end{array}\right\} \neq\left\{\begin{array}{l}
0 \\
0
\end{array}\right\}
$$

As a result, it is easy to conclude that the determinant of $\underline{\underline{P}}$ vanishes to zero at $q$ peak point.
After the $q$ peak, the $W_{2 n}$ is negative. In fact, before reaching the $q$ peak, the loading path crosses a bifurcation domain limit which is related with a specific axial strain. At this axial strain, the determinant of $\underline{\underline{N_{S}}}$ is no longer positive, thus a cone of unstable loading direction appears. Note that during the period between this axial strain and the $q$ peak, although the cone of unstable direction occurs, the loading path does not cross the cone, and the value of second-order work is still positive. Until the peak value, the second-order work vanishes and the stress-strain state becomes unstable.

(a) $q-p$ plane

(b) Global normalized second-order work along axial strain

Figure 4.4 Response of stresses and change of $W_{2 n}$ along the undrained triaxial test simulation

### 4.3 Stabilities analysis of slope model

### 4.3.1 Slope model established

In this Section, we present a finite element analysis of slope stability problem. The numerical slope is designed in the framework of an actual engineering problem, and the slope parameters are adopted from Liao's (LIA 08) previous work about the saturated and unsaturated seepage effect on the
slope stability. Hence the related slope Abaqus model is focused on a real engineering problem. The geometric shape, loading and boundary conditions of slope are illustrated in following Figure 4.5 and 4.6.

As we can see, it is also necessary to analyze this slope even though the height and length of slope is not quite huge. In order to analyze rigorously the slope stability, an expanding slope model which contains our typical real slope is established. The geometric size is illustrated in detailed.


Figure 4.5 Geometric size of slope model
Based on the real situation, some boundaries conditions should be limited on our numerical slope model. For the reason of reality, the left and right side of slope should be loaded with horizontal limitation which restrict the horizontal deformation but free the vertical deformation, while a vertical limitation is loaded at the bottom of slope for the vertical restriction. All these boundaries conditions are necessary for simulating the actual situation.


Figure 4.6 Boundary and loading conditions of slope
According to the real case (LIA 08), the mechanical parameters of slope soil are shown in the following Table 4.1. In this table, the general mechanical parameters up and down the infiltration line including the densities are presented, as well as the permeability of soil and void ratios. Besides, some other parameters, such as the reference pressures and precision limitation values are determined based on the experimental data. Note that the values of $\mathrm{B}_{\mathrm{p}}$ and $\mathrm{B}_{\mathrm{c}}$, which represent respectively the values of equivalent plastic strain for which half of the hardening / softening on friction angle and cohesion is achieved, is adopted as 0.003 and 0.01 based on the experimental data. Moreover, some different values of pressure exponent are also tried to explore different possibilities.

Table 4.1 Actual mechanics parameters of slope model (LIA 08)

| Soil parameters | Symbol | Uplevel/Down level |
| :---: | :---: | :---: |
| Young's modulus | $E$ | $17 / 10 \mathrm{MPa}$ |
| Poisson's ratio | $v$ | $0.3 / 0.35$ |
| Inherent cohesion | $C_{o}$ | 5 kPa |
| Failure cohesion | $C_{f}$ | $19 / 21.5 \mathrm{kPa}$ |
| Inherent friction angle | $\varphi_{o}$ | $5^{\circ}$ |
| Failure friction angle | $\varphi_{f}$ | $25.3^{\circ} / 31^{\circ}$ |
| Limit dilatancy angle | $\psi_{f}$ | $10^{\circ} / 11^{\circ}$ |


| Reference pressure | $P_{a}$ | 100 kPa |
| :---: | :---: | :---: |
| Precision limit | $P r e c$ | 0.001 |
|  |  |  |
| Permeability | $k_{0}$ | $1 . \mathrm{e}-20 / 1 . \mathrm{e}-8 \mathrm{~m} \cdot \mathrm{~d}^{-1}$ |
| Void ratio | $e$ | $0.35 / 0.3$ |
| Density | $\rho$ | $2337 / 2347 \mathrm{~kg} / \mathrm{m}^{3}$ |

### 4.3.2 Dry conditions

In the cases of axisymmetric model simulation, the drained condition as well as the undrained condition was tested (see Section 3.5). The results of drained condition with different values of pressure exponents are not obvious as the results under undrained conditions. This interesting phenomenon leads to a meaningful conjecture: can this effect of pressure exponent reproduce on the slope model? For this purpose, a slope model under drained condition was simulated in this Section.

In Table 4.2, the basic parameters for drained condition which means no restriction on the water flow and the volumetric changes are listed. It is also worth mentioning that the coupling between fluid and granular skeleton was not taking into account in this condition. The calculation here is in purely mechanical region. In this case, it is also related closely with the axisymmetric model mentioned in Chapter 3.

Table 4.2 Mechanical parameters for slope model under drained condition

| Soil parameters | Symbol | value |
| :---: | :---: | :---: |
| Young's modulus | $E$ | 20 MPa |
| Poisson's ratio | $v$ | 0.25 |
| Inherent cohesion | $C_{0}$ | 5 kPa |
| Failure cohesion | $C_{f}$ | 10 kPa |
| Inherent friction angle | $\varphi_{0}$ | $5^{\circ}$ |
| Failure friction angle | $\varphi_{f}$ | $40^{\circ}$ |
| Limit dilatancy angle | $\psi_{f}$ | $10^{\circ}$ |
| Density | $\rho$ | $1800 \mathrm{~kg} / \mathrm{m}^{3}$ |

Since the model geometric size, parameters and those boundary conditions are illustrated before, the slope model should be meshed into various small computing elements. Theoretically, the more quantity elements it is meshed, the more accurate the result of simulating is. But instead, the more time the computing process needs. After several tentative of computation, a better method of mesh is decided as following Figure 4.7.


Figure 4.7 Mesh graph of slope model
Note that the smaller size of mesh elements along the slope and near the top and toe of slope are made for more accurate computing results, for the reason that these areas are thought as unstable zones under loadings according to actual cases.

In this model, the parameters are adopted in the Table 4.2, and the geometric size, boundary condition and the meshing elements are shown before. Note that these parameters are related with the test within axisymmetric model which is just involved with the deformation and pressure variation. It is not closely corresponding to real slope situation which contains the complex drainage condition. Only the drained condition which is set and represented by releasing the restrictions of void ratio and saturation ratio is taken into account.

In this model, after setting up parameters, two general steps were added below the initial fixed computing step. The natural gravity should be set up in the first step for reality, and then the line pressure should be added at the top edge of slope in the second step for testing the stability of landslides. Moreover, an optional step could be added after these two necessary steps in case of insufficient pressure. For example, the loaded pressure could be enhanced when the original value is
not enough to cause some response, seen Figure 4.8. In this case, the loaded pressure is increased gently from the value of 100 kPa (step 2) to 1000 kPa (step 3). Before that, the boundary conditions (vertical and horizontal limitations) are also set up in the initial fixed computing step. The following procedure is the Mesh model which is introduced before, and considering the boundary condition and loading path, the slope simulation could be seen as computation of plane strain.

| STEP | CONDITION |
| ---: | :--- |
| Initial step: | Boundary condition (vertical + horizontal limitation) |
| Step 1 (gravity): | Boundary condition + gravity |
| Step 2 (loading 1): | Boundary condition + gravity + line loading 1 |
| Step 3 (loading 2): | Boundary condition + gravity + line loading 2 |

(a) Case 1 without concentrated pressure at the top of slope

| STEP | CONDITION |
| ---: | :--- |
| Initial step: | Boundary condition (vertical + horizontal limitation) |
| Step 1 (gravity): | Boundary condition + gravity |
| Step 2 (loading): | Boundary condition + gravity + line loading $1+$ concentrated pressure 1 |
| Step 3 (loading): | Boundary condition + gravity + line loading $2+$ concentrated pressure 2 |

(b) Case 2 with concentrated pressure at the top of slope

Figure 4.8 Slope simulation cases for drained condition

### 4.3.2.1 Slope simulation with line load only

For simulation case 1 , a series of computation were made firstly with the zero value of pressure exponent $(n 1=0)$. Typically, the stress state and deformation at the final stage of simulation were taken into account detailedly. In addition, the value of normalized second-order work which involves to the local stability of slope was also illustrated in the curves. The following Figure 4.9 presents these simulation results. Note the value of deformation factor is 1.

(b) Deformed mesh


Figure 4.9 Simulation results at the final step for drained case 1 without concentrated pressure ( $n 1=$ $0)$

In Figure 4.9c, a contour plot limit was set up for the convenience of demonstration of negative normalized second-order work value. By using this method, the positive value of $w_{2 n}$ will be shown in grey color, while the zero and negative $w_{2 n}$ will be colorfully marked instead. With this way, the colorful zone in the slope equips a negative value of $w_{2 n}$, which means the appearance of some instabilities.

Moreover, the simulation results with different value of pressure exponents are shown as follows.

(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.10 Simulation results at the final step for drained case 1 without concentrated pressure $(n 1=$

## 0.1)


(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.11 Simulation results at the final step for drained case 1 without concentrated pressure $(n 1=$ 0.2)

(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.12 Simulation results at the final step for drained case 1 without concentrated pressure $(n 1=$

(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.13 Simulation results at the final step for drained case 1 without concentrated pressure $(n 1=$ 0.4)


Figure 4.14 Evolution of normalized second-order work and mean pressure in drained case 1

From the Figures 4.9 to 4.13, and Figure 4.14, it is not hard to find out the difference of normalized second-order work $w_{2 n}$ and mean load for each node between various values of pressure exponent is not obvious under drained condition. Coincidentally, this comparison result is similar with the result of axisymmetric model simulation in the same drained condition in Section 3.5.

Note that there are some numerical noises of $w_{2 n}$ detected by the quantity at the left top surface.

This numerical problem is probably caused by the sensitivity of constitutive model with specific parameter and boundary condition. In fact, the horizontal displacement is limited at the left boundary. Therefore, some points near the surface can burden the traction force when the pressure loaded. This phenomenon can also happen no matter the existence of concentrated load or drainage condition.

### 4.3.2.2 Slope simulation with concentrated load

For more persuasive evidence, a concentrated pressure is loaded at the top of slope (node 2), i.e. case 2. In this case, a potential instability incident of slope might happen. Therefore, a similar simulation will be done and the results will be shown in the following figures. For the sake of simplicity, only three values of pressure exponent are adopted.

(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.15 Simulation results at the final step for drained case 2 with concentrated pressure $(n 1=0)$

(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.16 Simulation results at the final step for drained case 2 with concentrated pressure $(n 1=$

(a) S-Mises stress (unit: $P a$ )

(b) Deformed mesh

(c) Normalized second-order work

Figure 4.17 Simulation results at the final step for drained case 2 with concentrated pressure $(n 1=$ 0.4)


Figure 4.18 Evolution of normalized second-order work and mean load in drained case 2
From the simulation results of case 2, even though a high variable value of concentrated pressure was loaded at the top of slope and inducing a more obvious instability state, the slope did not show any sign of failure from both local and global point. In fact, the slope failure under drained condition is not our purpose here.

By combining case 1 and case 2, it is easy to find out that there is no obvious local and global failure at the final loading phase, even at the top of slope. Although a sudden change of quantity and sign of stress occurs at node 1 on the top surface when the pressures are loaded, the local node display a stable state gradually which does not affect the stable state of global slope. All these changes are also coincident with the evolution of normalized second-order work.

The evolution of $w_{2 \mathrm{n}}$ or mean pressure for both case 1 and case 2 is quite similar for different values of pressure exponent $n 1$. That means the effect of $n 1$ on the local and global state variation is not obvious under the drained condition. This is also closely related with the situation in Section 3.5 on the test of axisymmetric model.

Meanwhile, the different value of pressure exponent relates to different degree of stress-induced anisotropy based on the conclusion of Chapter 3, and then we can also conclude that the stress-induced anisotropy arising as the natural consequence of our new hyperelastic model shows small influence on the slope stability under drained condition.

### 4.3.3 Partly saturated condition

Being different with the drained condition, the undrained condition is much closer with reality than the drained condition. This situation usually happens during the rain seasons or a short construction period. All these features cause that this kind of case is complicated and hard to simulate using numerical method. Therefore, the slope for undrained situation is different from the drained one despite that the general geometric sizes of slope are the same. A saturation line is added in the centre of slope, accompanied with a different method that the model is meshed. The following figures present the geometric sizes and new mesh conditions. Note that an analysis point, namely node 3 , is placed near the slope and also above the saturation line.


Figure 4.19 Geometric size of slope for undrained test


Figure 4.20 Mesh graph of slope for undrained situation
Specifically, the different settings for undrained test are concentrated on two points. The first one is about the saturation line and saturation degree. The setting of saturation line is based on the actual engineering, and the saturation degree below the line is adopted as 1 while the one above the line is 0.5 . The second point is about the permeability and the void ratio which are detailed and illustrated in the Table 4.1.

In the following undrained test, a line pressure was loaded at the top of slope like the drained test did. Similarly, this pressure was divided and loaded step by step with a maximum value of 1000 kPa . Theoretically, the simulation won't stop until the case of numerical divergence occurs, in which situation the maximum load imposed at the final step appears. The figures below (4.21-29) show these simulation results of undrained test with different values of pressure exponent. The sub figures (a) and
(b) present the shear stress and total deformation at the end of load process respectively. The important variable equivalent plastic strain which is a consequent product of shear stress and not very good for the stability is illustrated in the (c) sub figure, while the value of normalized second-order work $w_{2 n}$ is shown in the (d) sub figure.

(a) S-Mises (unit: Pa)

(b) Deformed mesh

(c) Equivalent plastic strain

(d) Normalized second-order work

Figure 4.21 Simulation results at the final step under undrained condition $(n 1=0)$


(d) Normalized second-order work

Figure 4.22 Simulation results at the final step under undrained condition $(n 1=0.02)$



Figure 4.23 Simulation results at the final step under undrained condition $(n 1=0.04)$


(d) Normalized second-order work

Figure 4.24 Simulation results at the final step under undrained condition $(n 1=0.06)$

(c) Equivalent plastic strain


Figure 4.25 Simulation results at the final step under undrained condition $(n 1=0.08)$


(d) Normalized second-order work

Figure 4.26 Simulation results at the final step under undrained condition $(n 1=0.1)$

(c) Equivalent plastic strain

(d) Normalized second-order work

Figure 4.27 Simulation results at the final step under undrained condition $(n 1=0.2)$


(d) Normalized second-order work

Figure 4.28 Simulation results at the final step under undrained condition $(n 1=0.3)$



Figure 4.29 Simulation results at the final step under undrained condition $(n 1=0.4)$

From these figures 4.21-4.29 above, a significant difference between these simulations is obvious and clear, although only one set of parameters are adopted. In all cases, the simulation stops before the maximum value of load is imposed because of the numerical divergence, and apparently different values of pressure at the end process show up, as well as the maximum shear stress, seen Figure 4.30.

(a) Maximum imposed pressure


Figure 4.30 Maximum stresses at the end of loading process for different $n 1$
From these sub figures (c) and (d), it is easily found that the influence zone which is the area near the slope top shows an obvious equivalent plastic strain and negative value of normalized second-order work in most cases. Meantime, this influence zone is also the area where the maximum shear stress locates. For higher quantity of $n 1$, there is no negative value of $w_{2 \mathrm{n}}$ at the end of loading process, but a dramatically decreasing value of $w_{2 n}$ can also present a sudden change of slope. For example, the evolution of normalized second-order work at node 3 in the case of $n 1$ equaling to 0.1 is presented in the following Figure 4.31. Even though the final value of $w_{2 n}$ does not reach at numerically zero at all, the divergence also happens because of the drastically decreasing. This conclusion is coincident with the modeling results of Nailed wall by Prunier et al (PRU 16). Besides, few numerical noises are also detected at the left top surface of slope like the drained condition does. These are caused by the same reason that traction forces occur because of the fixed horizontal limitation at the left boundary.


Figure 4.31 Evolution of normalized second-order work at node 3 in undrained condition

In sum, for different values of pressure exponent, the failure states and the maximum shear stresses at the end of loading process are diverse, although only one set of parameters are adopted. This feature is highly consistent with the axisymmetric model simulation in the Section 3.4. As mentioned before, the non-zero value of pressure exponent is closely related with stress-induced anisotropy. Thus, different values of $n 1$ refer to various level of stress-induced anisotropy. As a result, the anisotropy which is induced during the loading process can also cause a significant influence on the response of slope like the axisymmetric does.

### 4.4 Second-order work results

### 4.4.1 Stability modeling with $W_{2 n}$

In the process of loading on the top of slope, a line normal pressure is imposed until the numerical divergence of computation in Abaqus happens, as it mentioned in last Section. This simulation can be useful to estimate the bearing capacity on the slope top in different conditions, but also verify the feature of integrated normalized second-order work. In those sub figures (a) in Section 4.3.2, note that the intensity of stress is relatively high in the influence zone near the top of slope. Coincidentally, the equivalent plastic strain in this area in sub figures (c) is also obvious comparing to other zones, which is similar but not identical to the unstable zones described by the value of local
normalized second-order work $w_{2 n}$ illustrated in sub figures (d). It is also worth noting that the total unstable zone is litter vaster than the zone of equivalent plastic strain. As a consequence, it can be told the shear mechanism which is initiated is not the only destabilizing mechanism.

Since it is well explained that the failure mechanism is qualitatively well correlated with the one described by the equivalent of $w_{2 \mathrm{n}}$, however, the weak zones depicted by normalized second-order work and equivalent plastic strain are not strictly identical. This is due to the complex behavior of soils. As a consequence, it is preferable to analyze results with the second-order work quantity. In fact the theoretical background highlighted in the Section 4.1 and numerical verification illustrated in the Section 4.2 prove this kind of physically well-founded meanings of second-order work criterion.

In this work, the post-failure states won't be discussed which remains as an open scientific question. But it can be reasonably assumed that when the unstable points fully merge, an effective failure would take place.

The following figures are the results when the pressure exponent $n 1$ equals to zero. The value of $W_{2 \mathrm{n}}$ at each step is obtained directly from the numerical simulation.


Figure 4.32 Evolution of integrated second-order work in undrained case with $n 1=0$

For the convenience of observation, another combined figure of $W_{2 \mathrm{n}}$ and the pressure imposed at the top surface is presented as follow Figure 4.33 . From this figure, it is clearly illustrated the evolution of $W_{2 \mathrm{n}}$ and the corresponding pressure imposed on the top surface. The value of maximum pressure loaded is obvious and related with the one which has been shown in the Figure 4.30 (a).

Meantime, the variation of $W_{2 n}$ during the process of load, especially at the final loading stage, is concision and distinct. Although the $W_{2 n}$ decreases monotonically during the whole loading process except few bounds in these figures, its final value does not reach at the numerical zero. However, being similar with the normalized second-order work $w_{2 \mathrm{n}}$, a dramatically decreasing of $W_{2 \mathrm{n}}$ at the final stage of loading appears, and at this moment computation divergence happens. Also, after combining Figures 4.21, 4.32 and 4.33 , the slope failure occurs at this point.


Figure 4.33 Evolution of $W_{2 n}$ with respect to the pressure imposed with $n 1=0$

In this case $(n 1=0)$, the numerical model we established and used will transform from the non-linear hyperelastic-plastic model into a linear elastic incremental elastoplastic one. And this one corresponds to the numerical and experimental work by Prunier et al (PRU, 16).

In sum, it can be concluded that during the process of loading, $W_{2 n}$ tends to decrease gradually. Nevertheless, the less obvious increase of stability described by $W_{2 \mathrm{n}}$ at the beginning of the loading phase may seem illogical. This fact can probably be explained by the adaptability of the soil due to its hardening rules during the first steps of loading. Then after this first stability gain, as the loading increase, $W_{2 n}$ decreases more or less monotonically, which seems to make sense. When the value of $W_{2 \mathrm{n}}$ becomes negative, an effective failure occurs. Another occasion, for example the undrained case in this Section, a failure can be also indicated when the divergence happens while $W_{2 \mathrm{n}}$ is not numerically zero but decreasing drastically. In this moment, a local or global failure pattern appears. To sum up, the criterion which uses the integrated second-order work to describe the stability of slope
with our new model is reasonable and feasible theoretically and experimentally.

### 4.4.2 Influence of pressure exponent on $W_{2 n}$

As it is described before, the variable pressure exponent $n 1$ in our new constitutive model equips the influence on the response of stress and strain of soil. While the variable pressure exponent corresponds to the fabric tensor of stiffness matrix in our elastoplastic model. Consequently, a stress-induced anisotropy in the stiffness matrix as well as the response appears naturally when the stress is imposed under the condition of non-zero value of pressure exponent. These features have been proved and introduced in the Chapter 3.

Under this situation, an assumption that the pressure exponent can also affect the influence of global normalized second-order work $W_{2 \mathrm{n}}$ on the stability detecting of slope is proposed. After comparing the relevant results on the axisymmetric and slope model, and under two different drainage conditions, the undrained test on the slope is adopted for the convenience of observation.


Figure 4.34 Evolution of $W_{2 n}$ in undrained slope simulation


Figure 4.35 Evolution $W_{2 \mathrm{n}}$ with respect to the pressure imposed

From Figure 4.34 and 4.35 , it is clear that before the sudden change of $W_{2 \mathrm{n}}$, the effect of different pressure exponents on the evolution of $W_{2 n}$ is not obvious, which means the slope stability degrees before failure with different values of $n 1$ are quite similar. In our case, the pressure exponents referring to the induced anisotropy cause less effect on the total slope stability. Note that, the global normalized second-order work $W_{2 n}$ is calculated based on new model, and it is also suitable for the non-homogeneous problems. This feature maybe due to the reason of complex soil mechanics that can causes less difference of global integrated second-order work on all of nodes until a local or global failure happens.

Another important result is that the final pressure imposed at the top surface of slope gets smaller when the value of $n 1$ increases to a certain extent. That means the slope failure happens in an earlier time with lower loaded pressure at the top surface when the degree of stress-induced anisotropy becomes higher. In other words, the stress-induced anisotropy on stiffness matrix based on Houlsby hyperelastic formulation causes significant difference of response of stress and strain of slope under undrained condition combined with a yet simple plasticity model. This difference may only be reflected in one local place, or possibly in the whole slope. Within this range of pressure exponent, the anisotropic stiffness matrix in new constitutive model seems to weak the stability of slope.

### 4.5 Conclusion

In this framework, the feature of second-order work, especially the global integrated second-order work are introduced and discussed detailedly. Then for the purpose of numerical application, its formulation is re-written by FORTRAN language as a user-defined subroutine command within the Umat format of new constitutive model to cooperate with Abaqus tool. Hereafter, the ability of second-order work was testified and verified firstly in the simulation of axisymmetric model, then moving to the computation of slope stability with different loading, boundary, and drainage conditions. At last, the effect of variable pressure exponent in new model on the integrated second-order work's feature was also discussed precisely to some extent.

In the Section 4.1, the local second-order work $w_{2}$ is shown and can be integrated numerically over a given volume using the finite element method. This integration leads to the original expression of Hill's stability criterion. Then a formulation to normalize this globally integrated second-order work is proposed. This normalization makes the global second-order work $W_{2}$ independent from the current incremental loading intensity. Thus, the evolution of global normalized second-order work $W_{2 n}$ with a loading program is better interpretable. This work can be used for homogeneous or non-homogeneous problems. Moreover, the feature that the normalized quantity decreases monotonically with the loss of stability theoretically makes it as a global safety factor with the limit value of stability of 0 instead of 1 . It is then more general and more physical than the numerical divergence of the computation.

In later Section, the function of $W_{2 n}$ as a detecting tool of stability was firstly used in an axisymmetric model under the undrained condition. A "loose" sample under this classical undrained triaxial test was investigated. It is found that the integrated normalized second-order work $W_{2 n}$ within our new model can well describe the sample stability. When the numerical quantity of $W_{2 n}$ decreases to zero, the failure happens.

Then, a real engineering case of slope stability detecting was presented. In this simulation, an analysis of stability of slope under drained condition was studied firstly. Although there was a concentrated pressure apart from the line pressure imposed near the top of slope, the difference on the response of stress and strain, or the local normalized second-order work in the unstable zones for different value of pressure exponents, is not obvious. This is coincidentally corresponding to the
simulation on axisymmetric drained model. Then the results under the undrained (isochoric) condition show a big diversity. It is found that obvious difference of $w_{2 \mathrm{n}}$ in the influence zones is detected for different pressure exponents. Even though the value of $w_{2 n}$ does not decrease to the numerical zero, a dramatic decreasing can also cause the divergence of computation. At this moment, a local or global failure with a burst of kinetic energy occurs.

In the last Section, the effect of pressure exponent on the feature of $W_{2 n}$ was investigated. It is found that before the occurrence of failure, the influence on the global normalized second-order work is small, but becomes obvious at the final loading stage. To some extent, the maximum sustainable pressure imposed on the top surface of slope at the failure stage drops during the increasing of pressure exponent.

## Chapter 5 - General conclusion and perspectives

### 5.1 General conclusion

In this work, our new constitutive model that gathers the main basic features of soil behavior was developed. This model allows a given soil to be described with only one set of parameters, for example from a loose to a dense state or from a normally consolidated to an over-consolidated state. This feature is useful for taking into account initial states or for observing a change in the main behavior due to a large change in the confining loading conditions. The major conclusion are drawn below in details.

In Chapter 2, our new model was established in the framework of original Plasol model introduced by Barnichon. By replacing the elastic part of Plasol model with a hyperelastic model proposed by Houlsby, the new model equips the features as follows: (1) accurate representation for the frictional behavior of granular materials; (2) obedience of the First Law of Thermodynamics or guarantee thermodynamic acceptability about the elastic part of constitutive model; (3) proper integration algorithm to correct the overflow problems of plastic strain in the plastic part of constitutive model; and (4) emphasis of the influence of anisotropy of elastic behavior even though the initial elastic matrix is isotropic.

In Chapter 3, the new model and original Plasol model are re-written into numerical language, i.e., Fortran, which is usable when implementing with Abaqus simulation. After building the similar model with experimental tests, simulations under undrained and drained conditions were conducted and compared with experimental results. The results show that the new model can well simulate the actual problems by using a single set of optimal parameters, although there are some deviatoric values during predicting the variation of volume strain in higher confining pressure conditions. Moreover, the value of tangent modulus in numerical simulation is quite close to the experimental data in the small axial strain procedure, whilst these two forms of data might not coincide well in the large strain stages although the tendencies are similar. The more important point is that when the internal variable pressure exponent ' $n 1$ ' equals to a non-zero value, the incremental stiffness matrix becomes anisotropic due to the progress of shear strain which corresponds to the natural phenomenon in geo-engineering. Also, it has been verified that different anisotropies in elasticity domain owing to
various values of ' $n 1$ ' make a big difference on the response in terms of stresses and strains, even altering the tendency of hardening for undrained triaxial simulations. Besides, it seems that the density of soil affects the influence of stress-induced anisotropy on response of soil to a certain extent. However, for drained triaxial conditions, this kind of effect is quite tiny no matter what the density or initial confining pressure condition is. Finally, the comparison between new model and modified SANISAND model, our new model equips the features of less total number and more accessible of parameters although the fitting degree in plastic domain is not as well as modified SANISAND model.

In Chapter 4, the application of new model with second-order work criterion was used in the simulation of slope safety. Firstly, the formulas of local normalized second-order work $w_{2 \mathrm{n}}$ and the global normalized second-order work $W_{2 n}$ were proposed and tested in an axisymmetric model under undrained condition. It is found that the integrated normalized second-order work $W_{2 \mathrm{n}}$ within our new model can well describe the sample stability. When the numerical quantity of $W_{2 \mathrm{n}}$ decreases to zero, the failure happens. Secondly, a real engineering case of slope stability detecting was presented. In the drained case, the difference of stress-strain response, or the change of local normalized second-order work in unstable zone is not obvious. But in undrained case, it is found that obvious difference of $w_{2 n}$ in the influence zones is detected for different pressure exponents. Even though the value of $w_{2 n}$ does not decrease to the numerical zero, a dramatic decreasing can also cause the divergence of computation. At this moment, a local or global failure with a burst of kinetic energy occurs. Last, the investigation results about the effect of pressure exponent on the feature of $W_{2 n}$ shows that before the happening of failure, the effect of ' $n 1$ ' on $W_{2 n}$ is quite low. But it becomes obvious at the final loading stage. To some extent, the maximum sustainable pressure imposed on the top surface of slope at the failure stage drops during the increasing of pressure exponent.

### 5.2 Perspectives

Although lots of work about the new constitutive model for granular materials has been done, few future works could be considered in these aspects:
(1) Change of new model for cyclic test. The tests of cyclic loading have been tested and verified for new model, however it does not work well because the plastic part of model which has the implicit
backward Euler integration algorithm is not suitable for cyclic loading test.
(2) Taking the anisotropic dilatancy into account. In this work, the fitting degree of new model on Toyoura sands at the plastic shearing stage is not good enough as the modified SANISAND model does. The most important reason is that the modified SANISAND model considers the anisotropic dilatancy as well as the anisotropic elasticity. Therefore, a modification taking the anisotropic dilatancy into account in new model can improve reasonably the simulation effect at the plastic shearing stage.
(3) Application of new model on various materials and drainage and boundary conditions. In this work, the application of new model was implemented on the axisymmetric model and full size slope. Strain-controlled method was mianly adopted and tested here. For more information, the stress-controlled simulation could be done for comparison as well. Moreover, the more complex drainage condition, for example, the continuous rain in the slope safety situation could be also explored as well.


| STATEV(3), STATEV (4)) | $\operatorname{EpsiB}(5)=\mathrm{dsqrt}(2 . \mathrm{d} 0) / 2 . \mathrm{d} 0 * \operatorname{EpsiB} 1(5)$ |
| :---: | :---: |
|  | $\operatorname{EpsiB}(6)=\operatorname{dsqrt}(2 . d 0) / 2 . d 0 * E p s i B 1(4)$ |
| RETURN | ! |
| END SUBROUTINE UMAT | DeltaEpsi $=$ DeltaEpsi1 |
|  | DeltaEpsi(4) $=$ dsqrt(2.d0)/2.d0*DeltaEpsi1(6) |
| subroutine Original_Plasol (Deijkl, SigB1, EpsiB1, DeltaEpsi1, | DeltaEpsi(5) $=\operatorname{dsqrt}(2 . \mathrm{d} 0) / 2 . \mathrm{d} 0 * \operatorname{DeltaEpsi1}(5)$ |
| ntens, props(1:10), Epsip, $\mathrm{kB}(1), \mathrm{kB}(2), \mathrm{kB}(3))$ | DeltaEpsi(6) $=$ dsqrt(2.d0)/2.d0*DeltaEpsi1 $(4)$ |
| !!! 'Epsip' is the only one 'state variable', i.e. 'STATEV' in | ! calculate the Stress |
| ABAQUS materials property 'depvar'. | EpsiB $=$ EpsiB + DeltaEpsi |
| implicit none | SigB $=$ SigB + Deijkl*DeltaEpsi |
| integer :: i, j, n, iter | ! start the corrector procedure |
| double precision :: Epsip, Epsiv, Epsis, p, q, Val, DeltaLamda, | $\mathrm{n}=0$ |
| \& | do |
| Isig, IIsig, f, pi | $\mathrm{n}=\mathrm{n}+1$ |
| double precision, dimension(10) :: props | call DP_in_Original_Plasol(Deijk1, props(5), props(6), |
| !!! 'props' is related to 'PROPS', it contains: E, Nu, Bp, Bc, C0, | props(4), props(8), props(7), props(3), SigB, kB, Epsip, Val, Isig, |
| Cf, PhiC0, PhiCf, Psi, Pa | IIsig, f, DeltaLamda, DeltaSigB) |
| double precision, dimension(6,6) :: Deijkl | if ( $\mathrm{f}>0 . \mathrm{d} 0$ ) then |
| double precision, dimension(6) :: DeltaSigB | Epsip $=$ Epsip + Val $*$ DeltaLamda |
| double precision, dimension(ntens) :: SigB, EpsiB, | kB (1) |
| DeltaEpsi,SigB1, EpsiB1, DeltaEpsi1 | $\operatorname{props}(5)+(\operatorname{props}(6)-\operatorname{props}(5)) * E p s i p /(\operatorname{props}(4)+E p s i p) ~$ |
| double precision, dimension(3) :: kB | kB (2) |
| $\mathrm{pi}=\mathrm{acos}(-1 . \mathrm{d} 0)$ | $\operatorname{props}(7)+(\operatorname{props}(8)-\operatorname{props}(7)) * E p s i p /(\operatorname{props}(3)+$ Epsip $)$ |
| SigB $=$ SigB1 | $\mathrm{kB}(3)=\mathrm{kB}(2)-(\operatorname{props}(8)-\operatorname{props}(9))$ |
| $\operatorname{SigB}(4)=\operatorname{dsqrt}(2 . \mathrm{d} 0) * \operatorname{SigB} 1(6)$ | ! |
| $\operatorname{SigB}(5)=\operatorname{dsqrt}(2 . \mathrm{d} 0) * \operatorname{SigB} 1(5)$ | SigB $=$ SigB-DeltaSigB |
| $\operatorname{SigB}(6)=\operatorname{dsqrt}(2 . \mathrm{d} 0) * \operatorname{SigB} 1$ (4) | else |
| ! | DeltaSigB $=(/ 0 . d 0,0 . d 0,0 . d 0,0 . d 0,0 . d 0,0 . d 0 /)$ |
| EpsiB $=$ EpsiB1 | endif |
| $\operatorname{EpsiB}(4)=\operatorname{dsqrt}(2 . \mathrm{d} 0) / 2 . \mathrm{d} 0 * \operatorname{EpsiB} 1(6)$ | $\operatorname{if}(\operatorname{maxval}(\operatorname{abs}($ DeltaSigB $))<1 . d-3$.or. $\mathrm{n}>5 . \mathrm{d} 1)$ exit |


| enddo | )) |
| :---: | :---: |
| ! | k |
| SigB1 $=$ SigB | $6 . \mathrm{d} 0 * \mathrm{kB}(1) * \cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(\mathrm{dsqrt}(3 . \mathrm{d} 0) *(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{p}$ |
| $\operatorname{SigB} 1(4)=1 . d 0 / \mathrm{dsqrt}(2 . d 0) * \operatorname{SigB}(6)$ | i/1.8d2))) |
| $\operatorname{SigB} 1(5)=1 . d 0 / \mathrm{dsqrt}(2 . d 0) * \operatorname{SigB}(5)$ | $\mathrm{f}=\mathrm{IIsig}+\mathrm{m}$ * $\mathrm{sig}-\mathrm{k}$ |
| SigB1(6) $=1 . d 0 / \mathrm{dsqrt}(2 . d 0) * \operatorname{SigB}(4)$ | $!$ |
| end subroutine Original_Plasol | mg |
|  | $2 . \mathrm{d} 0 * \sin (\mathrm{kB}(3) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(\mathrm{dsqrt}(3 . \mathrm{d} 0) *(3 . \mathrm{d} 0-\sin (\mathrm{kB}(3) * \mathrm{pi} / 1.8 \mathrm{~d} 2)$ |
| subroutine DP_in_Original_Plasol(Deijkl, props(5), props(6), | )) |
| props(4), props(8), props(7), props(3), SigB, kB, Epsip, Val, Isig, | dfdSig $=m^{*}(/ 1 . d 0, \quad 1 . d 0, \quad 1 . d 0, \quad 0 . d 0, \quad 0 . d 0$, |
| IIsig, f, DeltaLamda, DeltaSigB) | 0.d0/)+devsig/(2.d0*IIsig) |
| implicit none | $\mathrm{dgdSig} \quad=\quad \mathrm{mg}^{*}(/ 1 . \mathrm{d} 0, \quad 1 . \mathrm{d} 0, \quad 1 . \mathrm{d} 0, \quad 0 . \mathrm{d} 0, \quad 0 . \mathrm{d} 0$, |
| double precision, dimension(10), intent(in) :: props | 0.d0/)+devsig/(2.d0*IIsig) |
| double precision, intent(out) :: Val, Isig, IIsig, f, DeltaLamda | ! |
| double precision, dimension(6), intent(out) :: DeltaSigB | dfdPhiC = |
| double precision, intent(in) :: Epsip | 2.d0*Isig/dsqrt(3.d0)* $(\cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1$. |
| double precision, dimension(6), intent(in) :: SigB | 8d2) + + |
| double precision, dimension(6,6), intent(in) :: Deijkl |  |
| double precision, dimension(3), intent(in) :: kB | $\cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) * \sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8$ |
| double precision, dimension(6) :: devsig, dfdSig, dgdSig, A2 |  |
| double precision :: pi, m, mg, k, dfdPhiC, dfdC, dPhiCdEpsip, |  |
| dCdEpsip, A1 | $6 . \mathrm{d} 0 * \mathrm{kB}(1) / \mathrm{dsqrt}(3 . \mathrm{d} 0) *\left(\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /\left(3 . \mathrm{d} 0-\sin \left(\mathrm{kB}(2){ }^{*} \mathrm{pi} /\right.\right.\right.$ |
| $\mathrm{pi}=\operatorname{acos}(-1 . \mathrm{d} 0)$ |  |
| ! |  |
| $\operatorname{Isig}=\operatorname{sum}(\operatorname{SigB}(1: 3))$ |  |
| devsig $=$ SigB-Isig/3.d0*(/1.d0, 1.d0, 1.d0, 0.d0, 0.d0, 0.d0/) | $\mathrm{dfdC}=-3 . \mathrm{d} 0 * \mathrm{~m} / \tan (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2)$ |
| IIsig $=$ dsqrt( $5 . d-1 *$ dot_product(devsig, devsig) $)$ | dPhiCdEpsip = |
| ! | pi/1.8d2*(props(8)-props(7))/(props(3)+Epsip)-pi/1.8d2*Epsip* |
| m ( $=$ |  |
| $2 . \mathrm{d} 0 * \sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(\mathrm{dsqrt}(3 . \mathrm{d} 0) *(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2)$ | $(\operatorname{props}(8)-\operatorname{props}(7)) /(\operatorname{props}(3)+E p s i p) * * 2 . d 0$ |

```
dCdEpsip
=
(props(6)-props(5))/(props(4)+Epsip)-Epsip*(props(6)-props(5))/
&
    (props(4)+Epsip)**2.d0
!
Val = dsqrt(3.d0)/3.d0
!
A2 = matmul(Deijkl,dgdSig)
A1 = dot_product(dfdSig,A2)
DeltaLamda
f/(A1-Val*(dfdPhiC*dPhiCdEpsip+dfdC*dCdEpsip))

DeltaSigB \(=\) DeltaLamda*A2
end subroutine DP_in_Original_Plasol
\begin{tabular}{|c|c|}
\hline & Prec \\
\hline \multirow[t]{3}{*}{Appendix B - New model} & \(\mathrm{AE}=\mathrm{PROPS}(1)\) \\
\hline & \(\mathrm{ANu}=\mathrm{PROPS}(2)\) \\
\hline & \(\mathrm{An} 1=\operatorname{PROPS}(3)\) \\
\hline & \(\mathrm{APa}=\mathrm{PROPS}(11)\) \\
\hline & ! \\
\hline \multicolumn{2}{|l|}{! UMAT for Houlsby_Plasol} \\
\hline \multirow[t]{2}{*}{\(!========================================\)} & Akr = AE/3.dO/APa/(1.d0-2.d0 ANu) \\
\hline & \(\mathrm{Ag}=\mathrm{AE} / 2 . \mathrm{d} 0 / \mathrm{APa} /(1 . \mathrm{d} 0+\mathrm{ANu})\) \\
\hline & ! \\
\hline \multirow[t]{2}{*}{SUBROUTINE} & \\
\hline & SELECT CASE (NTENS) \\
\hline UMAT(STRESS,STATEV,DDSDDE,SSE,SPD,SCD, \& & CASE (6) \\
\hline \multirow[t]{3}{*}{RPL,DDSDDT,DRPLDE,DRPLDT,\&} & \\
\hline & S11 \(=\) STRESS(1) \\
\hline & S22 \(=\) STRESS \((2)\) \\
\hline STRAN,DSTRAN,TME,DTME,TEMP,DTEMP & S33 = STRESS(3) \\
\hline \multirow[t]{2}{*}{ED,CMNAME,\&} & \\
\hline & S12 \(=\) STRESS(4) \\
\hline \multirow[b]{2}{*}{NDI,NSHR,NTENS,NSTATV,PROPS,NPROPS,COORDS,DRO} & S23 \(=\) STRESS(5) \\
\hline & \\
\hline \multirow[t]{2}{*}{T,PNEWDT,\&} & S31 = STRESS(6) \\
\hline & CASE (4) \\
\hline \multirow{2}{*}{CELENT,DFGRD0,DFGRD1,NOEL,NPT,LAYER,KSPT,JSTEP,} & S11 = STRESS(1) \\
\hline & S22 = STRESS(2) \\
\hline \multirow[t]{2}{*}{KINC)} & \\
\hline & S33 \(=\) STRESS(3) \\
\hline & S12 \(=0 . \mathrm{D} 0\) \\
\hline \multirow[t]{2}{*}{\(!\) INCLUDE 'ABA_PARAM.INC'} & \\
\hline & S23 \(=0 . \mathrm{D} 0\) \\
\hline & S31 \(=\) STRESS(4) \\
\hline IMPLICIT INTEGER(I-N) & \\
\hline \multirow[t]{2}{*}{IMPLICIT REAL*8(A-H,O-Z)} & END SELECT \\
\hline & ! \\
\hline \multirow[t]{2}{*}{CHARACTER*80 CMNAME} & \\
\hline & SELECT CASE (NTENS) \\
\hline DIMENSION STRESS(NTENS),STATEV(NSTATV),\& & CASE (6) \\
\hline DDSDDE(NTENS,NTENS),DDSDDT(NTENS),DRPLDE(NTE & STRESS1 = STRESS \\
\hline DDSDDE(NTENS,NTENS),DDSDDT(NTENS),DRPLDE(NTE & CASE (4) \\
\hline NS),\& & STRESS1(1) = STRESS \((1)\) \\
\hline \multirow[b]{2}{*}{STRAN(NTENS),DSTRAN(NTENS),TIME(2),PREDEF(1),DP} & STRESS1(2) = STRESS(2) \\
\hline & \\
\hline \multirow[t]{2}{*}{RED(1),\&} & STRESS1(3) = STRESS \((3)\) \\
\hline & STRESS1 4 ) \(=0 . \mathrm{D} 0\) \\
\hline \multirow{3}{*}{PROPS(NPROPS),COORDS(3),DROT(3,3),DFGRD0(3,3),DFG} & STRESS1 \((5)=0 . \mathrm{D} 0\) \\
\hline & \\
\hline & \(\operatorname{STRESS} 1(6)=0 . \mathrm{D} 0\) \\
\hline RD1(3,3),\& & END SELECT \\
\hline JSTEP(4) & \\
\hline & ! \\
\hline DIMENSION Smn(6), DDSDDE1(6,6), STRESS1(6) & \\
\hline !==========================================10=10 & Smn \(=\) STRESS \(1-(\mathrm{S} 11+\mathrm{S} 22+\mathrm{S} 33) / 3 . \mathrm{d} 0 *(/ 1 . d 0,1 . d 0,1 . d 0,0 . d 0\), \\
\hline & 0.d0, \(0 . \mathrm{d} 0 /\) ) \\
\hline & P0Square \(\quad=\quad(\mathrm{S} 11+\mathrm{S} 22+\mathrm{S} 33) * * 2 . \mathrm{d} 0 / 9 . \mathrm{d} 0 \quad+\) \\
\hline Jacobian Matrix & \(\mathrm{Ak} 1 *(1 . \mathrm{d} 0-\mathrm{An} 1) / 2 . \mathrm{d} 0 / \mathrm{Ag}^{*}\) dot_product(Smn, Smn) \\
\hline & \(!{ }^{\text {a }}\) \\
\hline & \multirow[t]{2}{*}{DO \(\mathrm{J}=1,6\)
DO \(\mathrm{I}=1,6\)} \\
\hline \(!\) PROPS(12): E, Nu, n1, Bp, Bc, C0, Cf, PhiC0, PhiCf, Psi, Pa, & \\
\hline
\end{tabular}
\(\operatorname{DDSDDE}(\mathrm{I}, \mathrm{J})=0 . \mathrm{D} 0\)
ENDDO
ENDDO
!
DDSDDE1 \((1,1)=\)
\(\mathrm{APa}^{*}\left(\mathrm{P} 0 \mathrm{Square} /\left(\mathrm{APa}{ }^{* * 2 . \mathrm{d} 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 11 * \mathrm{~S} 1}\right.\right.\) 1/P0Square+Ak1* (1.d0-An1)+2.d0*Ag*2.d0/3.d0)

\section*{DDSDDE1 \((2,2)=\)}

APa ( P 0 Square/(APa**2.d0) \()^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 22 * \mathrm{~S} 2\)
2/P0Square+Ak1*
(1.d0-An1)+2.d0*Ag*2.d0/3.d0)

DDSDDE1 \((3,3)=\)
\(\mathrm{APa} *\left(\mathrm{P} 0\right.\) Square \(/\left(\mathrm{APa}{ }^{* * 2 . \mathrm{d} 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 33 * \mathrm{~S} 3}\right.\) 3/P0Square+Ak1*
(1.d0-An1)+2.d0*Ag*2.d0/3.d0)
\(\operatorname{DDSDDE} 1(1,2)=\)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S11*S2
2/P0Square+Ak1*
(1.d0-An1)-2.d0*Ag/3.d0)

DDSDDE1 \((2,1)=\)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S22*S1 1/P0Square+Ak1*
(1.d0-An1)-2.d0*Ag/3.d0)
\(\operatorname{DDSDDE} 1(1,3)=\)
\(\mathrm{APa}^{*}\left(\mathrm{P} 0 \mathrm{Square} /\left(\mathrm{APa}{ }^{* *} 2 . \mathrm{d} 0\right)\right)^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 11 * \mathrm{~S} 3\) 3/P0Square+Ak1*
(1.d0-An1)-2.d0*Ag/3.d0)

DDSDDE1 \((3,1)=\)
\(\mathrm{APa}^{*}\left(\mathrm{P} 0 \mathrm{Square} /\left(\mathrm{APa}{ }^{* *} 2 . \mathrm{d} 0\right)\right)^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 33 * \mathrm{~S} 1\) 1/P0Square+Ak1*
(1.d0-An1)-2.d0*Ag/3.d0)

DDSDDE1 \((2,3)\)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S22*S3
3/P0Square+Ak1*
(1.d0-An1)-2.d0*Ag/3.d0)

DDSDDE1(3,2)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S33*S2
2/P0Square+Ak1*
(1.d0-An1)-2.d0*Ag/3.d0)
!
DDSDDE1 \((1,4)\) =
\(\mathrm{APa}^{*}(\mathrm{P} 0\) Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S11*S1 2/P0Square)
\(\operatorname{DDSDDE} 1(4,1)=\operatorname{DDSDDE} 1(1,4)\)

DDSDDE1 \((1,5)\)
\(=\)
APa *(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S11*S2
3/P0Square)
\(\operatorname{DDSDDE} 1(5,1)=\operatorname{DDSDDE} 1(1,5)\)
DDSDDE1(1,6)
=
\(\mathrm{APa} *\left(\mathrm{P} 0 \mathrm{Square} /\left(\mathrm{APa}{ }^{* * 2 . d 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 11 * \mathrm{~S} 3}\right.\right.\)
1/POSquare)
\(\operatorname{DDSDDE} 1(6,1)=\operatorname{DDSDDE} 1(1,6)\)
\(=\operatorname{DDSDDE} 1(2,4)=\)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S22*S1
2/P0Square)
\(\operatorname{DDSDDE} 1(4,2)=\operatorname{DDSDDE} 1(2,4)\)
DDSDDE1 \((2,5)=\)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S22*S2
3/P0Square)
\(\operatorname{DDSDDE} 1(5,2)=\operatorname{DDSDDE} 1(2,5)\)
\(\operatorname{DDSDDE} 1(2,6)=\)
APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S22*S3
1/P0Square)
\(\operatorname{DDSDDE} 1(6,2)=\operatorname{DDSDDE} 1(2,6)\)
DDSDDE1 \((3,4)\)
\(\mathrm{APa} *\left(\mathrm{P} 0 \mathrm{Square} /\left(\mathrm{APa}{ }^{* * 2 . d 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 33 * \mathrm{~S} 1}\right.\right.\)
2/P0Square)
\(\operatorname{DDSDDE} 1(4,3)=\operatorname{DDSDDE} 1(3,4)\)
\(\operatorname{DDSDDE} 1(3,5)=\)
\(\mathrm{APa} *(\mathrm{P} 0 \mathrm{Square} /(\mathrm{APa} * * 2 . \mathrm{d} 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0)^{*}(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 33 * \mathrm{~S} 2\)
3/P0Square)
\(\operatorname{DDSDDE} 1(5,3)=\operatorname{DDSDDE} 1(3,5)\)
DDSDDE1 \((3,6)\)
\(=\)
\(\mathrm{APa} *(\mathrm{P} 0 \mathrm{Square} /(\mathrm{APa} * * 2 . \mathrm{d} 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0)^{*}(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 33 * \mathrm{~S} 3\)
1/P0Square)
\(\operatorname{DDSDDE} 1(6,3)=\operatorname{DDSDDE} 1(3,6)\)
\(=\quad\) !
DDSDDE1 \((4,4)\)
\(\mathrm{APa} *\left(\mathrm{P} 0 \mathrm{Square} /\left(\mathrm{APa}^{* * 2 . d 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0)^{*}(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 12 * \mathrm{~S} 1}\right.\right.\)
2/P0Square

DDSDDE1 \((5,5)\)
\(\mathrm{APa} *(\mathrm{P} 0 \mathrm{Square} /(\mathrm{APa} * * 2 . \mathrm{d} 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 23 * \mathrm{~S} 2\)
3/P0Square \(+2 . \mathrm{d} 0 * \mathrm{Ag})\)

DDSDDE1 \((6,6)\)
\(\mathrm{APa} *(\mathrm{P} 0 \mathrm{Square} /(\mathrm{APa} * * 2 . \mathrm{d} 0))^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 31 * \mathrm{~S} 3\)
1/P0Square
\[
+2 . \mathrm{d} 0 * \mathrm{Ag})
\]
\begin{tabular}{|c|c|}
\hline DDSDDE1(4,5) & 1/P0Square+Ak1* \\
\hline APa*(P0Square/(APa**2.d0))**(An1/2.d0)*(An1*Ak1*S12*S2 & (1.d0-An1)-2.d0*Ag/3.d0) \\
\hline 3/P0Square) & END SELECT \\
\hline \(\operatorname{DDSDDE} 1(5,4)=\operatorname{DDSDDE} 1(4,5)\) & ! \(=\) \\
\hline DDSDDE1 \((4,6)\) & \(=======================1\) \\
\hline APa*(P0Square/(APa**2.d0) \({ }^{*}\) * (An1/2.d0)*(An1*Ak1*S12*S3 & ! Stress calculation \\
\hline 1/P0Square) &  \\
\hline \(\operatorname{DDSDDE} 1(6,4)=\operatorname{DDSDDE} 1(4,6)\) & ========================= \\
\hline \(\operatorname{DDSDDE}(5,6)=\) & CALL Houlsby_Plasol \\
\hline \(\mathrm{APa}^{*}\left(\mathrm{P} 0\right.\) Square/(APa**2.d0) \({ }^{* * *(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 23 * \mathrm{~S} 3}\) & (DDSDDE1,STRESS,STRAN,DSTRAN,N \\
\hline 1/P0Square) & TENS,JSTEP,TIME,PROPS(1:12), Ak1, Ag, \\
\hline \(\operatorname{DDSDDE} 1(6,5)=\operatorname{DDSDDE} 1(5,6)\) & STATEV(1),STATEV(2:7),STATEV(8), \\
\hline ! & \(\operatorname{STATEV}(9), \operatorname{STATEV}(10: 12))\) \\
\hline \multicolumn{2}{|l|}{SELECT CASE (NTENS)} \\
\hline CASE (6) & RETURN \\
\hline DDSDDE \(=\) DDSDDE1 & END \\
\hline ! & ! \\
\hline CASE (4) & subroutine Houlsby_Plasol (Deijk1, SigB1, EpsiB1, DeltaEpsi1, \\
\hline DO \(\mathrm{J}=1\), NTENS & ntens, jstep, time, props, k1, g, \\
\hline DO I \(=1\), NTENS & Epsip, DeltaSig, p1, q1, kB) \\
\hline \(\operatorname{DDSDDE}(\mathrm{I}, \mathrm{J})=0 . \mathrm{D} 0\) & ! 'STATEV' in ABAQUS materials property 'depvar'. \\
\hline ENDDO & implicit none \\
\hline ENDDO & integer :: n , ntens, i \\
\hline \(\operatorname{DDSDDE}(1,1)=\) & real*8 :: Epsip, Epsiv, Epsis, p, q, p1, q1, Val, DeltaLamda, Isig, \\
\hline \(\mathrm{APa}^{*}(\mathrm{P} 0 \text { Square/(APa**2.d0) })^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 11 * \mathrm{~S} 1\) & IIsig, f, pi \\
\hline 1/P0Square+Ak1* & real*8, dimension(12) :: props \\
\hline \(\left.(1 . d 0-\mathrm{An} 1)+2 . \mathrm{d} 0 * A g^{*} 2 . d 0 / 3 . \mathrm{d} 0\right)\) & ! 'props' is related to 'PROPS', it contains: E, Nu, n1, Bp, Bc, C0, \\
\hline \(\operatorname{DDSDDE}(2,2)=\) & Cf, PhiC0, PhiCf, Psi, Pa, Prec \\
\hline \(\mathrm{APa}^{*}(\mathrm{P} 0 \text { Square/(APa**2.d0) })^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0){ }^{*}(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 22 * \mathrm{~S} 2\) & real*8, dimension(6,6) :: Deijkl, Mo \\
\hline 2/P0Square+Ak1* & real*8, dimension(6) :: DeltaSigB, DeltaSig, SigB, EpsiB, \\
\hline (1.d0-An1)+2.d0*Ag*2.d0/3.d0) & DeltaEpsi, e0 \\
\hline \(\operatorname{DDSDDE}(3,3)=\) & real*8, dimension(ntens) :: SigB1, EpsiB1, DeltaEpsi1 \\
\hline APa*(P0Square/(APa**2.d0) \({ }^{* *}\) (An \(\left.1 / 2 . \mathrm{d} 0\right) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 33 * \mathrm{~S} 3\) & real*8, dimension(3) :: kB \\
\hline 3/P0Square+Ak1* & INTEGER, dimension(4) :: jstep \\
\hline (1.d0-An1)+2.d0*Ag*2.d0/3.d0) & real*8, dimension(2) :: time \\
\hline \(\operatorname{DDSDDE}(4,4) \quad=\) & real*8 :: C0, Cf, Bc, PhiCf, PhiC0, Bp, k1, g, Pa, n1, Psi \\
\hline \(\mathrm{APa}^{*}\left(\mathrm{P} 0\right.\) Square/(APa**2.d0) \({ }^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0){ }^{*}(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 12 * \mathrm{~S} 1\) & real*8, dimension(6) :: devsig, dfdSig, dgdSig \\
\hline 2/P0Square & real*8, dimension(6) :: A2 \\
\hline \(+2 . \mathrm{d} 0 * \mathrm{Ag})\) & real*8 :: m, mg, k, dfdPhiC, dfdC, dPhiCdEpsip, dCdEpsip \\
\hline \(\operatorname{DDSDDE}(1,2)=\) & real*8 :: A1 \\
\hline APa*(P0Square/(APa**2.d0) \({ }^{* *}\) (An1/2.d0)*(An1*Ak1*S11*S2 & ! \\
\hline 2/P0Square+Ak1* & \(\mathrm{pi}=\operatorname{acos}(-1 . \mathrm{d} 0)\) \\
\hline (1.d0-An1)-2.d0*Ag/3.d0) & \(\mathrm{Pa}=\operatorname{props}(11)\) \\
\hline \(\operatorname{DDSDDE}(2,1)=\) & \(\mathrm{n} 1=\operatorname{props}(3)\) \\
\hline \(\mathrm{APa} *\left(\mathrm{P} 0\right.\) Square/(APa \(\left.\left.{ }^{*} 2 . \mathrm{d} 0\right)\right)^{* *}(\mathrm{An} 1 / 2 . \mathrm{d} 0) *(\mathrm{An} 1 * \mathrm{Ak} 1 * \mathrm{~S} 22 * \mathrm{~S} 1\) & ! \\
\hline
\end{tabular}
```

Select case(ntens)
case (6)
! 'SigB, EpsiB, DeltaEpsi' are used for calculating during the
iteration in the following context. \&
! 'SigB1, EpsiB1, DeltaEpsi1' are used for updating the value
which is related with 'abaqus'
SigB = SigB1
SigB(4) = dsqrt(2.d0)*SigB1(6)
SigB(5) = dsqrt(2.d0)*SigB1(5)
SigB(6) = dsqrt(2.d0)*SigB1(4)
!
EpsiB=EpsiB1
EpsiB(4) = dsqrt(2.d0)/2.d0*EpsiB1(6)
EpsiB(5) = dsqrt(2.d0)/2.d0*EpsiB1(5)
EpsiB(6) = dsqrt(2.d0)/2.d0*EpsiB1(4)
!
DeltaEpsi = DeltaEpsi1
DeltaEpsi(4) = dsqrt(2.d0)/2.d0*DeltaEpsi1(6)
DeltaEpsi(5) = dsqrt(2.d0)/2.d0*DeltaEpsi1(5)
DeltaEpsi(6) = dsqrt(2.d0)/2.d0*DeltaEpsi1(4)
!
case (4)
SigB(1)=SigB1(1)
SigB(2)= SigB1(2)
SigB(3)=SigB1(3)
SigB(4)=0.d0
SigB(5) = 0.d0
SigB(6) = dsqrt(2.d0)*SigB1(4)
!
EpsiB(1) = EpsiB1(1)
EpsiB(2)=EpsiB1(2)
EpsiB(3)= EpsiB1(3)
EpsiB(4) = 0.d0
EpsiB(5) = 0.d0
EpsiB(6) = dsqrt(2.d0)/2.d0*EpsiB1(4)
!
DeltaEpsi(1) = DeltaEpsi1(1)
DeltaEpsi(2) = DeltaEpsi1(2)
DeltaEpsi(3) = DeltaEpsi1(3)
DeltaEpsi(4) = 0.d0
DeltaEpsi(5) = 0.d0
DeltaEpsi(6) = dsqrt(2.d0)/2.d0*DeltaEpsi1(4)
end select
!
Mo= Deijkl

```
\begin{tabular}{|c|c|}
\hline PhiCf \(=\operatorname{props}(9)\) & dCdEpsip \\
\hline \(\operatorname{PhiC} 0=\operatorname{props}(8)\) & \((\mathrm{Cf}-\mathrm{C} 0) /(\mathrm{Bc}+\mathrm{Epsip})-\mathrm{Epsip} *(\mathrm{Cf}-\mathrm{C} 0) /(\mathrm{Bc}+\mathrm{Epsip})^{* * 2 . d 0}\) \\
\hline \(\mathrm{Bp}=\operatorname{props}(4)\) & ! \\
\hline \(\mathrm{Psi}=\operatorname{props}(10)\) & \(\mathrm{Val}=\operatorname{dsqrt}(3 . \mathrm{d} 0) / 3 . \mathrm{d} 0\) \\
\hline ! & ! \\
\hline \(\mathrm{kB}(1)=\mathrm{C} 0+(\mathrm{Cf}-\mathrm{C} 0) * \mathrm{Epsip} /(\mathrm{Bc}+\mathrm{Epsip})\) & \(\mathrm{A} 2=\operatorname{matmul}(\mathrm{Mo}, \mathrm{dgdSig})\) \\
\hline \(\mathrm{kB}(2)=\) PhiC0+(PhiCf-PhiC0)*Epsip/(Bp+Epsip) & A1 = dot_product(dfdSig,A2) \\
\hline \(\mathrm{kB}(3)=\mathrm{kB}\) (2)-(PhiCf-Psi) & DeltaLamda \\
\hline ! & f/(A1-Val*(dfdPhiC*dPhiCdEpsip+dfdC*dCdEpsip)) \\
\hline \(\mathrm{Isig}=\operatorname{sum}(\operatorname{SigB}(1: 3))\) & DeltaSigB = DeltaLamda*A2 \\
\hline devsig \(=\operatorname{SigB}-\mathrm{Isig} / 3 . \mathrm{d} 0 *(/ 1 . d 0,1 . d 0,1 . d 0,0 . d 0,0 . d 0\), & ! \\
\hline 0.d0/) & if ( \(\mathrm{f}>0 . \mathrm{d} 0\) ) then \\
\hline IIsig \(=\) dsqrt( \(0.5 \mathrm{~d} 0 *\) dot_product(devsig, devsig) \()\) & Epsip \(=\) Epsip + Val \(*\) DeltaLamda \\
\hline ! & SigB \(=\) SigB-DeltaSigB \\
\hline m = & else \\
\hline \(2 . \mathrm{d} 0 * \sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(\mathrm{dsqrt}(3 . \mathrm{d} 0) *(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2)\) & DeltaSigB \(=(/ 0 . d 0,0 . d 0,0 . d 0,0 . d 0,0 . d 0,0 . d 0 /)\) \\
\hline )) & endif \\
\hline k ( = & ! \\
\hline \(6 . \mathrm{d} 0 * \mathrm{kB}(1) * \cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(\mathrm{dsqrt}(3 . \mathrm{d} 0) *(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{p}\) & if (maxval(dabs(DeltaSigB))-1.d0 < props(12)) exit \\
\hline i/1.8d2))) & ! \\
\hline \(\mathrm{f}=\mathrm{IIsig}+\mathrm{m} *\) Isig-k & enddo \\
\hline ! & ! \\
\hline mg = & select case(ntens) \\
\hline \(2 . \mathrm{d} 0 * \sin (\mathrm{kB}(3) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(\mathrm{dsqrt}(3 . \mathrm{d} 0) *(3 . \mathrm{d} 0-\sin (\mathrm{kB}(3) * \mathrm{pi} / 1.8 \mathrm{~d} 2)\) & case (6) \\
\hline )) & ! Go back to 'SigB1'. The variable 'SigB' is just used for \\
\hline \[
\operatorname{dfdSig} \quad=\quad \mathrm{m} *(/ 1 . \mathrm{d} 0, \quad 1 . \mathrm{d} 0, \quad 1 . \mathrm{d} 0, \quad 0 . \mathrm{d} 0, \quad 0 . \mathrm{d} 0
\] & calculating in the context. \\
\hline 0.d0/)+devsig/(2.d0*IIsig) & SigB1 \(=\) SigB \\
\hline \[
\operatorname{dgdSig}=\mathrm{mg}^{*}(/ 1 . \mathrm{d} 0, \quad 1 . \mathrm{d} 0, \quad 1 . \mathrm{d} 0, \quad 0 . \mathrm{d} 0, \quad 0 . \mathrm{d} 0,
\] &  \\
\hline 0.d0/)+devsig/(2.d0*IIsig) & \(\operatorname{SigB} 1(5)=1 . d 0 / \mathrm{dsqrt}(2 . \mathrm{d} 0) * \operatorname{SigB}(5)\) \\
\hline ! & \(\operatorname{SigB} 1(6)=1 . d 0 / \mathrm{dsqrt}(2 . \mathrm{d} 0) * \operatorname{SigB}(4)\) \\
\hline dfdPhiC = & ! \\
\hline 2.d0*Isig/dsqrt(3.d0)* \(\cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1\). & EpsiB1 = EpsiB \\
\hline 8d2))+\& & \(\operatorname{EpsiB} 1(4)=\operatorname{dsqrt}(2 . \mathrm{d} 0) * \operatorname{EpsiB}(6)\) \\
\hline & \(\operatorname{EpsiB1} 15)=\operatorname{dsqrt}(2 . d 0) * \operatorname{EpsiB}(5)\) \\
\hline \(\cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) * \sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8\) & \(\operatorname{EpsiB1}(6)=\operatorname{dsqrt}(2 . d 0) * \operatorname{EpsiB}(4)\) \\
\hline d2) )**2.d0)-\& & case (4) \\
\hline & \(\operatorname{SigB} 1(1)=\operatorname{SigB}(1)\) \\
\hline \(6 . \mathrm{d} 0 * \mathrm{kB}(1) / \mathrm{dsqrt}(3 . \mathrm{d} 0) *(\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2) /(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} /\) & \(\operatorname{SigB} 1(2)=\operatorname{SigB}(2)\) \\
\hline \(1.8 \mathrm{~d} 2)+\) + & \(\operatorname{SigB1}(3)=\operatorname{SigB}(3)\) \\
\hline & \(\operatorname{SigB} 1(4)=1 . d 0 / \mathrm{dsqrt}(2 . \mathrm{d} 0) * \operatorname{SigB}(6)\) \\
\hline \((\cos (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2))^{*} 2 . \mathrm{d} 0 /(3 . \mathrm{d} 0-\sin (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2))^{* * 2 . d 0)}\) & ! \\
\hline \(\mathrm{dfdC}=-3 . \mathrm{d} 0 * \mathrm{~m} / \tan (\mathrm{kB}(2) * \mathrm{pi} / 1.8 \mathrm{~d} 2)\) & \(\operatorname{EpsiB1}(1)=\operatorname{EpsiB}(1)\) \\
\hline dPhiCdEpsip = & \(\operatorname{EpsiB1} 12)=\operatorname{EpsiB}(2)\) \\
\hline pi/1.8d2*(PhiCf-PhiC0)/(Bp+Epsip)-pi/1.8d2*Epsip* & EpsiB1(3) = EpsiB(3) \\
\hline \((\) PhiCf-PhiC0)/(Bp+Epsip)**2.d0 & EpsiB1 \((4)=\operatorname{dsqrt}(2 . \mathrm{d} 0) * \operatorname{EpsiB}(6)\) \\
\hline
\end{tabular}

\section*{end select}
end subroutine

\section*{Appendix C - \\ Modified SANISAND model}
subroutine umat(stress,statev,ddsdde,sse,spd,scd, \& rpl,ddsddt,drplde,drpldt, \&
stran,dstran,time,dtime,temp,dtemp,predef,dpred,cmname,
\& ndi,nshr,ntens,nstatv,props,nprops,coords,drot,pnewdt,
\& celent,dfgrd0,dfgrd1,noel,npt,layer,kspt,kstep,kinc) implicit none
character*80 cmname integer ntens, ndi, nshr, nstatv, nprops, noel, npt,
\& layer, kspt, kstep, kinc
double precision stress(ntens), statev(nstatv),
\& ddsdde(ntens,ntens), ddsddt(ntens), drplde(ntens),
\& \(\quad \operatorname{stran}(\) ntens \(), \quad\) dstran(ntens), time(2), predef(1), dpred(1),
\& \(\operatorname{props}(\) nprops \(), \quad \operatorname{coords}(3), \quad \operatorname{drot}(3,3), \quad \operatorname{dfgrd} 0(3,3)\), dfgrd \(1(3,3)\)
double precision sse, spd, scd, rpl, drpldt, dtime, temp,
\& dtemp, pnewdt, celent
integer prsw,elprsw,cons_lin,abaqus,chiara,check_ff,drcor integer i,error,maxnint,nfev,mario_DT_test,inittension
integer nparms,nasvdim,nfasv,nydim,nzdim
integer nasvy,nasvz,nyact,nzact,plastic,testing
double precision dot_vect
double
precision
parms(nprops),theta,tolintT,dtsub,DTmin,perturb
double precision sig_n(6),sig_np1(6),DDtan(6,6),pore,PI
double precision
deps_np1(6),depsv_np1,norm_D2,norm_D,tolintTtest
double precision eps_n(6),epsv_n,alphayield(6)
double
precision
norm_deps2,norm_deps,pp,qq,cos3t,ddum
double
precision
zero,tol_f,fact_thres,p_thres,stran_lim,eps_debug double
precision
p_atm,ptshift,phimob,tol_f_test,youngel,nuel
double
precision
avoid,apsi,aec,yf_DM,fyield,psi_void_DM,Mb
double
precision
dummy,sdev(6),I1,alpha(6),cM,tau(6),gth,etanorm
double precision sinphinorm
parameter (nasvdim \(=36\) )
parameter (nydim \(=6+14\) )
parameter (nzdim \(=14\) )
parameter (tolintT \(=1.00 \mathrm{~d}-3\) )
parameter \((\) tolintTtest \(=1.0 \mathrm{~d}-2)\)
parameter (maxnint \(=50000\) )
parameter (DTmin \(\quad=1.0 \mathrm{~d}-18\) )
parameter (perturb \(=1.0 \mathrm{~d}-4\) )
parameter (nfasv \(=1\) )
parameter (prsw \(=0\) )
parameter (cons_lin = 1 )
parameter (abaqus \(=0\) )
parameter (eps_debug \(=0.9 \mathrm{~d}-3\) )
parameter \((\) zero \(=0.0 \mathrm{~d} 0)\)
parameter (PI =
\(3.14159265358979323846264338327950288)\)
parameter (fact_thres \(=0.000000001 \mathrm{~d} 0\) )
double precision asv1(nydim-6),asv2(nzdim)
double precision
y (nydim), \(\mathrm{y} \_\mathrm{n}\) (nydim), \(\mathrm{z}(\mathrm{nzdim}), \mathrm{z} \_\mathrm{n}(\mathrm{nzdim})\)
tol_f=1.0d-6
tol_f_test=1.0d-6
check_ff=0
drcor=1
plastic \(=0\)
phimob \(=0.0 \mathrm{~d} 0\)
ptshift=0.0d0
error=0
if (ndi.ne.3) then
error=10
endif
nparms=nprops
call check_parms_DM(props,parms,nparms)
p_atm=parms(1)
p_thres=fact_thres*p_atm
elprsw \(=0\)
if (prsw .ne. 0) then
endif
call define(nasvy,nasvz)
nyact \(=6+\) nasvy
nzact \(=\) nasvz
if (nyact.gt.nydim) then
error \(=10\)
```

    elseif (nzact.gt.nzdim) then
        error=10
    endif
    pore = statev(29)
    ptshift=parms(18)*parms(1)
    do i=1,3
    stress(i)= stress(i)-ptshift
    enddo
    call move_sig(stress,ntens,-1*ptshift,sig_n)
    call move_sig(stress,ntens,pore,sig_n)
    call move_eps(dstran,ntens,deps_np1,depsv_np1)
    call move_eps(stran,ntens,eps_n,epsv_n)
    norm_D2=dot_vect(2,deps_np1,deps_np1,6)
    norm_D=sqrt(norm_D2)
    if (eps_n(1).gt.eps_debug) then
        chiara=1
    end if
    if (statev(7) .lt. 0.001) then
        do i=1,6
            alphayield(i)=zero
            end do
            call deviator(sig_n,alphayield,ddum,pp)
            avoid=0
            if(parms(19) .le. 5.0) then
            avoid=parms(19)
            else if(parms(19) .gt. 5.0) then
            apsi=parms(19)-10.0d0
    aec=parms(2)-parms(3)*(pp/parms(1))**parms(4)
avoid=aec+apsi
endif
statev(7)=avoid
do i=1,6
statev(i)=alphayield(i)/pp
statev(i+14)=alphayield(i)/pp
end do
end if
do i=1,nasvy
asv1(i)=statev(i-1+nfasv)
enddo
do i=1,nasvz
asv2(i) = statev(i-1+nfasv+nasvy)
enddo
call
iniyz(y,nydim,z,nzdim,asv1,nasvy,asv2,nasvz,sig_n,ntens)

```
call push(y,y_n,nydim)
call push(z,z_n,nzdim)
if (elprsw.ne.0) then
call
wrista(3,y,nydim,deps_np1,dtime,coords,statev,nstatv,
\&
parms,nparms,noel,npt,ndi,nshr,kstep,kinc)
endif
if((dtsub.le.zero).or.(dtsub.gt.dtime)) then
dtsub = dtime
end if
testing \(=0\)
if(kstep.eq.1 .AND. kinc.eq.1) testing=1
if(norm_D.eq.0) testing=2
nfev \(=0!\) initialisation
if(testing.eq.1) then
call
rkf23_upd_DM(y,z,nyact,nasvy,nasvz,tolintTtest,maxnint,
\& DTmin,deps_np1,parms,nparms,nfev,elprsw,
\& mario_DT_test,
\& error,tol_f_test,check_ff,drcor,p_thres,plastic)
if(error.ne.0) then
\[
\text { do } \mathrm{i}=1 \text {,nyact }
\]
\(y(i)=y \_n(i)\)
end do
error=0
end if
else if(testing.eq.2) then
\[
\text { do } \mathrm{i}=1 \text {,nyact }
\]
\[
y(i)=y \_n(i)
\]
end do
else !testing.eq. 0
call
rkf23_upd_DM(y,z,nyact,nasvy,nasvz,tolintT,maxnint,
\& DTmin,deps_np1,parms,nparms,nfev,elprsw,
\& mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
end if
if(mario_DT_test.eq.1) then
call wrista(4,y,nydim,deps_np1,dtime,coords,statev,nstatv,
\&
parms,nparms,noel,npt,ndi,nshr,kstep,kinc)
endif
if(error.eq.3) then
call
wrista(2,y,nydim,deps_np1,dtime,coords,statev,nstatv,
\&
parms,nparms,noel,npt,ndi,nshr,kstep,kinc)
if(abaqus.ne.0) then
\[
\text { pnewdt }=0.25 \mathrm{~d} 0
\]
else
do \(i=1\), nyact
\(y(i)=y \_n(i)\)
end do
endif
return
elseif(error.eq.10) then
call
wrista(2,y,nydim,deps_np1,dtime,coords,statev,nstatv,
\&
parms,nparms,noel,npt,ndi,nshr,kstep,kinc)
call xit_DM
endif
if(dtsub.le.0.0d0) then
\[
\text { dtsub }=0
\]
else if(dtsub.ge.dtime) then
dtsub = dtime
end if
statev(33)=dtsub
\(\operatorname{statev}(34)=\) dfloat(nfev)
error=0
if(cons_lin.eq.0) then
norm_deps2=dot_vect(2,deps_np1,deps_np1,ntens)
norm_deps=dsqrt(norm_deps2)
theta=perturb*max(norm_deps,1.0d-6)
call
pert_DM(y_n,y,z,nyact,nasvy,nasvz,tolintT,maxnint,DTmin,
\&
deps_np1,parms,nparms,nfev,elprsw,theta,ntens,DDtan,
\& error,tol_f,check_ff,drcor,p_thres,plastic) else
call tang_stiff(y,z,nyact,nasvy,nasvz,parms,nparms,
\& DDtan,cons_lin,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
endif
if(error.ne.0) then
endif
inittension=0
call
check_RKF_DM(inittension,y,nyact,nasvy,parms,nparms)
if (inittension.ne.0) then
\[
\text { do } \mathrm{i}=1, \text { nyact }
\]
\[
\mathrm{y}(\mathrm{i})=\mathrm{y} \_\mathrm{n}(\mathrm{i})
\]
end do
end if
call solout(stress,ntens,asv1,nasvy,asv2,nasvz,ddsdde,
\&
y,nydim,z,pore,depsv_np1,parms,nparms,DDtan)
\[
\text { do } \mathrm{i}=1, \text { nasvy }
\]
\[
\operatorname{statev}(\mathrm{i}-1+\mathrm{nfasv})=\operatorname{asv} 1(\mathrm{i})
\]
end do
do \(i=1\), nasvz
\(\operatorname{statev}(\mathrm{i}-1+\mathrm{nfasv}+\mathrm{nasvy})=\operatorname{asv} 2(\mathrm{i})\)
enddo
do \(\mathrm{i}=1,6\)
sig_np1(i)=y(i)
end do
call inv_sig(sig_np1,pp,qq,cos3t)
\(\operatorname{statev}(29)=\) pore
\(\operatorname{statev}(30)=\mathrm{pp}\)
\(\operatorname{statev}(31)=\mathrm{qq}\)
\(\operatorname{statev}(32)=\cos 3 t\)
cM=parms(6)/parms(5)
alpha(1) \(=y(7)\)
\(\operatorname{alpha}(2)=y(8)\)
\(\operatorname{alpha}(3)=y(9)\)
alpha(4) \(=\mathrm{y}(10)\)
\(\operatorname{alpha}(5)=\mathrm{y}(11)\)
alpha(6) \(=\mathrm{y}(12)\)
call deviator(sig_np1,sdev,I1,pp)
do \(\mathrm{i}=1,6\)
\(\operatorname{tau}(\mathrm{i})=\operatorname{sdev}(\mathrm{i})-\mathrm{pp} * \operatorname{alpha}(\mathrm{i})\)
end do
call lode_DM(tau,cM,cos3t,gth,dummy)
etanorm=gth*qq/pp
sinphinorm \(=3 *\) etanorm/(6+etanorm)
\(\operatorname{statev}(33)=\operatorname{asin}(\) sinphinorm \() * 180 /\) PI
\(\operatorname{statev}(34)=\) nfev
do \(\mathrm{i}=1,3\)
\[
\operatorname{stress}(\mathrm{i})=\operatorname{stress}(\mathrm{i})+\mathrm{ptshift}
\]
enddo
return
end
subroutine
alpha_th_DM(flag,n,gth,psi,parms,nparms,alpha)
\begin{tabular}{|c|c|}
\hline implicit none & double precision c_z,bulk_w,sinphi,PI,sinphiext \\
\hline integer flag,nparms, 1 & double precision zero \\
\hline double precision \(\mathrm{n}(6)\),gth,psi,parms(nparms),alpha(6) & parameter(zero=0.0d0) \\
\hline double precision M_c,mm,n_b,n_d & \\
\hline double precision M,alpha_th & parameter( \(\mathrm{PI}=3.14159265358979323846264338327950288\) ) \\
\hline double precision two,three,sqrt23 & \(\mathrm{p} \_\mathrm{a}=\mathrm{props}(1)\) \\
\hline data two,three/2.0d0,3.0d0/ & \(\mathrm{e} 0=\operatorname{props}(2)\) \\
\hline sqrt23=dsqrt(two/three) & lambda=props(3) \\
\hline M_c=parms(5) & \(\mathrm{xi}=\operatorname{props}(4)\) \\
\hline \(\mathrm{mm}=\operatorname{parms}(7)\) & M_c=props(5) \\
\hline \(\mathrm{n} \_\mathrm{b}=\mathrm{parms}(12)\) & M_e=props(6) \\
\hline n _d=parms(14) & \(\mathrm{mm}=\operatorname{props}(7)\) \\
\hline if(flag.eq.1) then & \(\mathrm{G} 0=\operatorname{props}(8)\) \\
\hline \(\mathrm{M}=\mathrm{M} \_\mathrm{c}\) & \(\mathrm{nu}=\operatorname{props}(9)\) \\
\hline elseif(flag.eq.2) then & h0=props(10) \\
\hline \(\mathrm{M}=\mathrm{M} \_\mathrm{c}^{*} \operatorname{dexp}\left(-\mathrm{n} \_\mathrm{b} * \mathrm{psi}\right)\) & c_h=props(11) \\
\hline else & \(\mathrm{n} \_\mathrm{b}=\mathrm{props}(12)\) \\
\hline \(\mathrm{M}=\mathrm{M} \_\mathrm{c}^{*}\) dexp(n_d*psi) & \(\mathrm{A} 0=\mathrm{props}(13)\) \\
\hline endif & \(\mathrm{n} \_\mathrm{d}=\) props \((14)\) \\
\hline alpha_th=M*gth-mm & z_max \(=\) props(15) \\
\hline do \(\mathrm{i}=1,6\) & c_z=props(16) \\
\hline alpha(i)=sqrt23*alpha_th*n(i) & bulk_w=props(17) \\
\hline end do & call push(props,parms,nprops) \\
\hline return & if(parms(5) .gt. 5) then \\
\hline end & \(\operatorname{sinphi}=\sin (\operatorname{parms}(5) / 180 * \mathrm{PI})\) \\
\hline subroutine check_crossing(y,y_tr,n,parms,nparms,prod) & \(\operatorname{parms}(5)=6 *\) sinphi/(3-sinphi) \\
\hline implicit none & else \\
\hline integer i, n,nparms & sinphi=3*parms(5)/(6+parms(5)) \\
\hline double precision dot_vect & end if \\
\hline double precision \(\mathrm{y}(\mathrm{n}), \mathrm{y}\) _tr(n), parms(nparms) & if(parms(6) .gt. 5) then \\
\hline double precision \(\mathrm{P}(6), \mathrm{P} 1\) (6),dsig_tr(6) & sinphiext=sin(parms(6)/180*PI) \\
\hline double precision prod & parms(6) \(=6 *\) sinphiext/(3+sinphiext) \\
\hline call grad_f_DM(y,n,parms,nparms,P,P1) & else if ((parms(6) .le. 5) .and. (parms(6) .gt. 0.01)) then \\
\hline do \(\mathrm{i}=1,6\) & sinphiext=3*parms(6)/(6-parms(6)) \\
\hline dsig_tr(i)=y_tr(i)-y(i) & else \\
\hline end do ! i & \(\operatorname{parms}(6)=\operatorname{parms}(5) *(3-\sin p h i) /(3+\) sinphi \()\) \\
\hline prod=dot_vect(1,P,dsig_tr,6) & end if \\
\hline return & return \\
\hline end & end \\
\hline subroutine check_parms_DM(props,parms,nprops) & subroutine define(nasvy,nasvz) \\
\hline implicit none & implicit none \\
\hline integer nprops & integer nasvy,nasvz \\
\hline double precision props(nprops), parms(nprops) & nasvy \(=14\) \\
\hline double precision p_a,e0,lambda,xi,M_c,M_e,mm & nasvz \(=14\) \\
\hline double precision G0,nu,h0,c_h,n_b,A0,n_d,z_max & return \\
\hline
\end{tabular}
end
subroutine deviator( \(\mathrm{t}, \mathrm{s}\), trace, mean)
implicit none
double precision \(t(6), s(6)\), trace, mean
double precision one,three, onethird
data one,three/1.0d0,3.0d0/
onethird=one/three
trace \(=t(1)+t(2)+t(3)\)
mean \(=\) onethird \(*\) trace
\(s(1)=t(1)\)-mean
\(s(2)=t(2)\)-mean
\(\mathrm{s}(3)=\mathrm{t}(3)\)-mean
\(s(4)=t(4)\)
\(s(5)=t(5)\)
\(s(6)=t(6)\)
return
end
double precision function distance(alpha_k,alpha,n)
implicit none
integer i
double precision dot_vect
double precision alpha_k(6),alpha(6),n(6),delta(6)
do \(i=1,6\)
delta(i)=alpha_k(i)-alpha(i)
end do
distance=dot_vect(1,delta,n,6)
return
end
double precision function dot_vect(flag,a,b,n)
implicit none
integer i,n,flag
double precision \(\mathrm{a}(\mathrm{n}), \mathrm{b}(\mathrm{n})\)
double precision zero,half,one,two,coeff
parameter \((z e r o=0.0 \mathrm{~d} 0\), half \(=0.5 \mathrm{~d} 0\), one \(=1.0 \mathrm{~d} 0\), two \(=2.0 \mathrm{~d} 0)\)
if(flag.eq.1) then
coeff=two
elseif(flag.eq.2) then
coeff=half
else
coeff=one
end if
dot_vect=zero
do \(\mathrm{i}=1, \mathrm{n}\)
if(i.le.3) then
dot_vect \(=\) dot_vect \(+\mathrm{a}(\mathrm{i})^{*} \mathrm{~b}(\mathrm{i})\)
else
dot \(\_v e c t=\) dot \(\_v e c t+\operatorname{coeff} * a(i) * b(i)\)
end if
end do
return
end
subroutine
drift_corr_DM(y,n,z,nasvz,parms,nparms,tol,switch2,
\& mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
double precision dot_vect
integer switch2,mario_DT_test
external matmul
double precision yf_DM
integer n,nasvz,nparms,i,n_drift,max_ndrift,switch
integer iter, itermax
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
double precision \(\mathrm{y}(\mathrm{n}), \mathrm{y} 0(\mathrm{n}), \mathrm{y} 1(\mathrm{n})\),
z(nasvz), parms(nparms)
double precision \(\operatorname{gradf}(6), \operatorname{gradf} 1(6), \operatorname{gradg}(6), \operatorname{gradg} 1(6)\)
double precision
\(\operatorname{DDe}(6,6), \mathrm{UU}(6), \mathrm{VV}(6), \mathrm{h} \_\)alpha(6),Kpm1,p1,pp1
double precision
f0,tol,zero,one,denom,fnm1,p,three,onethird,f0_p
double precision factor,f1,p_atm
parameter \((\) zero \(=0.0 \mathrm{~d} 0\), one \(=1.0 \mathrm{~d} 0\), three \(=3.0 \mathrm{~d} 0)\)
parameter(max_ndrift=10000, itermax=1000)
call push \((\mathrm{y}, \mathrm{y} 0, \mathrm{n})\)
f0=yf_DM(y0,n,parms,nparms)
onethird=one/three
\(\mathrm{p}=(\mathrm{y} 0(1)+\mathrm{y} 0(2)+\mathrm{y} 0(3)) *\) onethird
n_drift=0
switch=0
f0_p=f0/p
switch2=0
do while(f0_p.gt.tol)
\(\mathrm{fnm} 1=\mathrm{f} 0\)
n_drift=n_drift+1
call el_stiff_DM(y0,n,parms,nparms,DDe,
\&
error,tol_f,check_ff,drcor,p_thres,plastic)
call grad_f_DM(y0,n,parms,nparms,gradf,gradf1)
call grad_g_DM(y0,n,parms,nparms,gradg,gradg1)
call matmul(DDe,gradg1,UU,6,6,1)
call matmul(DDe,gradf1,VV,6,6,1)
call
plast_mod_DM(y0,n,z,nasvz,parms,nparms,h_alpha,Kpm1,
\& switch2,mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
if (switch2.gt.zero) return
if(one/Kpm1.le.zero) then
error=3
return
end if
if(switch.eq.0) then
do \(\mathrm{i}=1,6\)
\(\mathrm{y} 1(\mathrm{i})=\mathrm{y} 0(\mathrm{i})-\mathrm{Kpm} 1 * \mathrm{f0} 0 \mathrm{UU}(\mathrm{i})\)
end do
do \(\mathrm{i}=1,6\)
\(\mathrm{y} 1(6+\mathrm{i})=\mathrm{y} 0(6+\mathrm{i})+\mathrm{Kpm1} * \mathrm{f} 0 * \mathrm{~h} \_\)alpha(i)
end do
do \(\mathrm{i}=13, \mathrm{n}\)
\(y 1(i)=y 0(i)\)
end do
f0=yf_DM(y1,n,parms,nparms)
if(f0.gt.fnm1) then
switch=1
\(\mathrm{p} 1=(\mathrm{y} 1(1)+\mathrm{y} 1(2)+\mathrm{y} 1(3)) *\) onethird
else
call push \((\mathrm{y} 1, \mathrm{y} 0, \mathrm{n})\)
end if
else
call push(y0,y1,n)
f0=yf_DM(y0,n,parms,nparms)
denom=dot_vect(1,gradf,gradf,6)
factor=one
\(\mathrm{f} 1=\mathrm{f} 0\)
do \(\mathrm{i}=1,6\)
\(\mathrm{yl}(\mathrm{i})=\mathrm{y} 0(\mathrm{i})-\mathrm{f0} * \operatorname{gradf}(\mathrm{i}) /\) denom/factor
end do
do \(\mathrm{i}=13\), n
\(y 1(i)=y 0(i)\)
end do
pp1 \(=(\mathrm{y} 1(1)+\mathrm{y} 1(2)+\mathrm{y} 1(3)) *\) onethird
if(pp1.lt.zero)then
switch2=1
return
endif
f1=yf_DM(y1,n,parms,nparms)
call push(y1,y0,n)
end if
f0=yf_DM(y0,n,parms,nparms)
\(\mathrm{p}=(\mathrm{y} 0(1)+\mathrm{y} 0(2)+\mathrm{y} 0(3)) *\) onethird
f0_p=f0/p
if(n_drift.gt.max_ndrift) then
f0_p=0
end if
end do
call push(y0,y,n)
return
end
subroutine el_stiff_DM(y,n,parms,nparms,DDe,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
integer i,j,n,nparms
double precision \(\mathrm{y}(\mathrm{n})\),parms(nparms)
double precision p_a,G0,nu,ratio
double precision sig1,sig2,sig3,p,void
double precision coeff1,coeff2
double precision \(\mathrm{Kt}, \mathrm{Gt}\),fe
double precision \(\operatorname{Id}(6,6), \operatorname{IxI}(6,6), \operatorname{DDe}(6,6)\)
double precision zero,half,one,two,three
double precision pp,p_thres_E,tenm3
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
parameter(zero \(=1.0 \mathrm{~d} 0\), half \(=0.5 \mathrm{~d} 0\) )
parameter \((\) one \(=1.0 \mathrm{~d} 0\), two \(=2.0 \mathrm{~d} 0\), three \(=3.0 \mathrm{~d} 0)\)
parameter(p_thres_E=0.001d0)
call pzero(Id,36)
call pzero(IxI,36)
call pzero(DDe,36)
\(\operatorname{Id}(1,1)=\) one
\(\operatorname{Id}(2,2)=\) one
\(\operatorname{Id}(3,3)=\) one
\(\operatorname{Id}(4,4)=\) half
\(\operatorname{Id}(5,5)=\) half
\(\operatorname{Id}(6,6)=\) half
\(\operatorname{IxI}(1,1)=\) one
\(\operatorname{IxI}(2,1)=\) one
\(\operatorname{IxI}(3,1)=\) one
\(\operatorname{IxI}(1,2)=\) one
\(\operatorname{IxI}(2,2)=\) one
\(\operatorname{IxI}(3,2)=o n e\)
\(\operatorname{IxI}(1,3)=\) one
\begin{tabular}{|c|c|}
\hline \(\operatorname{IxI}(2,3)=\) one & \(\mathrm{F}(1)=\) dsig_e \((1)\) \\
\hline \(\operatorname{IxI}(3,3)=\) one & \(\mathrm{F}(2)=\) dsig_e \((2)\) \\
\hline p_a=parms(1) & \(\mathrm{F}(3)=\mathrm{dsig}\) _e \((3)\) \\
\hline \(\mathrm{G} 0=\mathrm{parms}\) ( 8 ) & \(\mathrm{F}(4)=\) dsig_e \((4)\) \\
\hline \(\mathrm{nu}=\mathrm{parms}(9)\) & \(\mathrm{F}(5)=\) dsig_e \((5)\) \\
\hline \(\operatorname{sig} 1=y\) (1) & \(\mathrm{F}(6)=\) dsig_e \((6)\) \\
\hline sig2 \(=\mathrm{y}(2)\) & \(\mathrm{F}(13)=-(\) one+void)*depsv \\
\hline \(\operatorname{sig} 3=y(3)\) & return \\
\hline void= y (13) & end \\
\hline \(\mathrm{p}=(\operatorname{sig} 1+\operatorname{sig} 2+\operatorname{sig} 3) /\) three & subroutine \\
\hline \(\mathrm{pp}=\mathrm{p}\) & f_plas_DM(y,n,nasvy,z,nz,parms,nparms,deps,kRK,nfev, \\
\hline if(p.lt.p_thres)then & \& switch2,mario_DT_test, \\
\hline \(\mathrm{pp}=\mathrm{p}\) _thres & \& error,tol_f,check_ff,drcor,p_thres,plastic) \\
\hline end if & implicit none \\
\hline ratio=three*(one-two \({ }^{*}\) nu)/(two*(one+nu)) & integer n,nz,nasvy,nparms,i,nfev \\
\hline \(\mathrm{fe}=(2.97 \mathrm{~d} 0-\) void \() *(2.97 \mathrm{~d} 0-\mathrm{void}) /(\) one+void \()\) & integer switch2,mario_DT_test \\
\hline \(\mathrm{Gt}=\mathrm{G} 0 * \mathrm{p} \_\mathrm{a}\) *e \({ }^{\text {* }}\) dsqrt(pp/p_a) & double precision \\
\hline \(\mathrm{Kt}=\mathrm{Gt} /\) ratio & \(\mathrm{y}(\mathrm{n}), \mathrm{z}(\mathrm{nz}), \mathrm{kRK}(\mathrm{n}), \mathrm{parms}(\mathrm{nparms}), \operatorname{deps}(6)\) \\
\hline coeff \(1=\) Kt-two *Gt/three & double precision F_sig(6),F_q(nasvy) \\
\hline coeff2=two*Gt & integer error,check_ff,drcor,plastic \\
\hline do \(\mathrm{i}=1,6\) & double precision tol_f,p_thres \\
\hline do \(\mathrm{j}=1,6\) & double precision zero \\
\hline \(\operatorname{DDe}(\mathrm{i}, \mathrm{j})=\) coeff \(1 * \operatorname{IxI}(\mathrm{i}, \mathrm{j})+\) coeff2* \(\operatorname{Id}(\mathrm{i}, \mathrm{j})\) & parameter(zero=0.0d0) \\
\hline end do & \(n f e v=n f e v+1\) \\
\hline end do & call pzero(kRK,n) \\
\hline return & call \\
\hline end & get_F_sig_q(y,n,nasvy,z,nz,parms,nparms,deps,F_sig,F_q, \\
\hline subroutine f_hypoelas_DM(y,n,parms,nparms,deps,F, & \& switch2,mario_DT_test,error) \\
\hline \& error,tol_f,check_ff,drcor,p_thres,plastic) & if(switch2.gt.zero) return \\
\hline implicit none & if(error.eq.10) return \\
\hline external matmul & do i=1,6 \\
\hline integer \(\mathrm{n}, \mathrm{m}, \mathrm{nparms}\) & kRK(i)=F_sig(i) \\
\hline double precision y (n), parms(nparms), deps(6) & end do \\
\hline double precision depsv,void & do i=1,nasvy \\
\hline double precision F(n), \(\operatorname{De}(6,6)\),dsig_e(6) & kRK(6+i)=F_q(i) \\
\hline double precision one & end do \\
\hline integer error,check_ff,drcor,plastic & return \\
\hline double precision tol_f,p_thres & end \\
\hline data one/1.0d0/ & subroutine \\
\hline call pzero(F,n) & get_F_sig_q(y,n,nasvy,z,nz,parms,nparms,deps,F_sig,F_q, \\
\hline void \(=\mathrm{y}(13)\) & \& switch2,mario_DT_test,error) \\
\hline depsv=deps(1)+deps(2)+deps(3) & implicit none \\
\hline call el_stiff_DM(y,n,parms,nparms,De, & external matmul \\
\hline \& error,tol_f,check_ff,drcor,p_thres,plastic) & integer switch2,mario_DT_test \\
\hline call matmul(De,deps,dsig_e, \(6,6,1\) ) & integer nparms,n, nasvy,nz \\
\hline
\end{tabular}
\begin{tabular}{|c|c|}
\hline double precision \(\mathrm{y}(\mathrm{n}), \mathrm{z}(\mathrm{nz}), \operatorname{parms}(\mathrm{nparms}), \operatorname{deps}(6)\) & double precision onethird,twothird \\
\hline double precision & integer error,check_ff,drcor,plastic \\
\hline \multicolumn{2}{|l|}{\multirow[t]{2}{*}{\begin{tabular}{l}
Dep(6,6),HH(nasvy,6),F_sig(6),F_q(nasvy) \\
double precision tol_f,p_thres double precision zero
\end{tabular}}} \\
\hline & \\
\hline integer error,check_ff,drcor,plastic & parameter \((\) zero \(=0.0 \mathrm{~d} 0\), one \(=1.0 \mathrm{~d} 0\), two \(=2.0 \mathrm{~d} 0\), three \(=3.0 \mathrm{~d} 0)\) \\
\hline double precision tol_f,p_thres & parameter(tiny \(=1.0 \mathrm{~d}-15\), large \(=1.0 \mathrm{e} 15\) ) \\
\hline parameter(zero=0.0d0) & parameter(kappa=3.0d2) \\
\hline call & data m/1.0d0, 1.0d0, 1.0d0, \(0.0 \mathrm{~d} 0,0.0 \mathrm{~d} 0,0.0 \mathrm{~d} 0 /\) \\
\hline get_tan_DM(y,n,nasvy,z,nz,parms,nparms,Dep,HH,switch2, & switch2=zero \\
\hline \& mario_DT_test, & switch4=zero \\
\hline \& error,tol_f,check_ff,drcor,p_thres,plastic) & iter \(=0\) \\
\hline if(switch2.gt.zero) then & iter_max=1e3 \\
\hline return & onethird=one/three \\
\hline endif & twothird=two/three \\
\hline call matmul(Dep,deps,F_sig,6,6,1) & half=one/two \\
\hline call matmul(HH,deps,F_q,nasvy,6,1) & call pzero(Dep,36) \\
\hline return & call pzero(Hep,6*nasvy) \\
\hline end & \(\mathrm{p} \_\mathrm{a}=\mathrm{parms}(1)\) \\
\hline subroutine & \(\mathrm{e} 0=\operatorname{parms}\) (2) \\
\hline get_tan_DM(y,ny,nasvy,z,nz,parms,nparms,Dep,Hep, & lambda=parms(3) \\
\hline \& switch2,mario_DT_test, & \(\mathrm{xi}=\mathrm{parms}(4)\) \\
\hline \& error,tol_f,check_ff,drcor,p_thres,plastic) & M_c=parms(5) \\
\hline implicit none & M_e=parms(6) \\
\hline external matmul & \(\mathrm{mm}=\operatorname{parms}(7)\) \\
\hline integer & \(\mathrm{G} 0=\mathrm{parms}(8)\) \\
\hline nparms,ny,nz,nasvy,i,j,switch2,iter,iter_max,switch4 & \(\mathrm{nu}=\mathrm{parms}(9)\) \\
\hline integer mario_DT_test & \(\mathrm{h} 0=\operatorname{parms}(10)\) \\
\hline double precision dot_vect,distance,psi_void_DM & c_h=parms(11) \\
\hline double precision \(\mathrm{y}(\mathrm{ny}), \mathrm{z}(\mathrm{nz})\), parms(nparms) & \(\mathrm{n} \_\mathrm{b}=\mathrm{parms}(12)\) \\
\hline double precision \(\operatorname{De}(6,6), \operatorname{Dep}(6,6), \mathrm{Hep}(\mathrm{nasvy}, 6), \mathrm{m}(6)\) & \(\mathrm{A} 0=\mathrm{parms}(13)\) \\
\hline double precision LL(6), LL1(6), RR(6),RR1(6), U(6), V(6) & n_d=parms(14) \\
\hline double precision p_a,e0,lambda,xi,M_c,M_e,cM,mm & z_max=parms(15) \\
\hline double precision G0,nu,h0,c_h,n_b & c_z=parms(16) \\
\hline double precision A0,n_d,z_max,c_z,bulk_w & bulk_w=parms(17) \\
\hline double precision \(\operatorname{sig}(6)\),alpha(6),void,Fab(6) & cM=M_e/M_c \\
\hline double precision alpha_sr(6),alpha_b(6) & do \(\mathrm{i}=1,6\) \\
\hline double precision \(\mathrm{s}(6), \operatorname{tau}(6), \mathrm{n}(6)\) & \(\operatorname{sig}(\mathrm{i})=\mathrm{y}(\mathrm{i})\) \\
\hline double precision norm2,norm,I1,p,psi,cos3t,gth,dgdth & end do !i \\
\hline double precision b0,d_sr,hh,db & do i=1,6 \\
\hline double precision Hplas,LDeR,Kp,Kpm1 & alpha(i) \(=\mathrm{y}(6+\mathrm{i})\) \\
\hline double precision mtrR,brack_mtrR,tol_ff,tol_dil,Hvs & end do !i \\
\hline double precision & void=y(13) \\
\hline h_alpha(6),h_fab(6),HH_alpha(6,6),HH_fab(6,6) & do \(\mathrm{i}=1,6\) \\
\hline double precision yf_DM,ff0,chvoid & \(\mathrm{Fab}(\mathrm{i})=\mathrm{y}(13+\mathrm{i})\) \\
\hline double precision zero,tiny,half,one,two,three, large,kappa & end do !i \\
\hline
\end{tabular}
```

do i=1,6
alpha_sr(i)=z(i)
end do !i
call deviator(sig,s,I1,p)
do i=1,6
tau(i)=s(i)-p*alpha(i)
end do ! i
norm2=dot_vect(1,tau,tau,6)
norm=dsqrt(norm2)
if(norm.lt.tiny) then
norm=tiny
endif
do i=1,6
n(i)=tau(i)/norm
end do
call el_stiff_DM(y,ny,parms,nparms,De,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
call grad_f_DM(y,ny,parms,nparms,LL,LL1)
call matmul(De,LL1,V,6,6,1)
call grad_g_DM(y,ny,parms,nparms,RR,RR1)
call matmul(De,RR1,U,6,6,1)
if (dabs(p).gt.zero) then
chvoid=c_h*void
if(chvoid.ge.1) then
chvoid=0.99999
end if
b0=G0*h0*(one-chvoid)/dsqrt(p/p_a)
else
b0=large
end if
d_sr=distance(alpha,alpha_sr,n)
if (d_sr.lt.zero) then
call push(alpha,alpha_sr,6)
end if
if (d_sr.lt.tiny) then
d_sr=tiny
end if
hh=b0/d_sr
psi=psi_void_DM(void,p,parms,nparms)
call lode_DM(tau,cM,cos3t,gth,dgdth)
call alpha_th_DM(2,n,gth,psi,parms,nparms,alpha_b)
db=distance(alpha_b,alpha,n)
do i=1,6
h_alpha(i)=twothird*hh*(alpha_b(i)-alpha(i))
end do

```
\(m t r R=-R R(1)-R R(2)-R R(3)\)
brack_mtrR=half*(mtrR+dabs(mtrR))
do \(\mathrm{i}=1,6\)
h_fab(i)=-c_z*brack_mtrR*(z_max*n(i)+Fab(i))
end do
Hplas=twothird*hh*p*db
if(Hplas.gt.1e+15) then
endif
LDeR=dot_vect(1,LL1,U,6)
\(\mathrm{Kp}=\mathrm{LDeR}+\) Hplas
ff0=yf_DM(y,ny,parms,nparms)
if(mario_DT_test.eq.zero) then
if(LDeR.lt.zero) then
switch2=1
return
endif
if(Kp.lt.zero) then
switch2=1
return
endif
else
if(LDeR.le.zero) then
switch2=1
return
endif
endif
if(Kp.lt.zero)then
error=3
return
endif
call push(alpha_sr,z,6)
\(\mathrm{Kpm1}=\mathrm{one} / \mathrm{Kp}\)
do \(\mathrm{i}=1,6\)
\[
\text { do } \mathrm{j}=1,6
\]
\[
\operatorname{Dep}(\mathrm{i}, \mathrm{j})=\operatorname{De}(\mathrm{i}, \mathrm{j})-\mathrm{Kpm1} * \mathrm{U}(\mathrm{i}) * \mathrm{~V}(\mathrm{j})
\]
end do ! j
end do !i
do \(\mathrm{i}=1,6\)
do \(\mathrm{j}=1,6\)
HH_alpha(i, j\()=\mathrm{Kpm1} 1 * \mathrm{~h} \_\)alpha(i)*V(j)
end do ! \(j\)
end do !i
do \(\mathrm{i}=1,6\)
do \(\mathrm{j}=1,6\)
HH_fab(i,j)=Kpm1*h_fab(i)*V(j)
```

        end do !j
    end do !i
    do j=1,6
    | Hep( $1, \mathrm{j}$ ) = HH_alpha(1,j) | $!$ alpha(1) |
| :---: | :---: |
| Hep(2,j) = HH_alpha( $2, \mathrm{j}$ ) | $!$ alpha(2) |
| Hep(3,j) = HH_alpha(3,j) | $!$ alpha(3) |
| Hep(4,j) = HH_alpha(4,j) | $!$ alpha(4) |
| Hep(5,j) =HH_alpha(5,j) | $!$ alpha(5) |
| Hep(6,j) = HH_alpha(6,j) | $!$ alpha(6) |

    Hep(7,j) =-(one+void)*m(j)
    Hep(8,j)=HH_fab(1,j) ! Fab(1)
    Hep(9,j)=HH_fab(2,j) ! Fab(2)
    Hep(10,j)=HH_fab(3,j)
    Hep(11,j)=HH_fab(4,j)
    Hep(12,j)=HH_fab(5,j)
    Hep(13,j)=HH_fab(6,j)
    end do !j
return
end
subroutine grad_f_DM(y,ny,parms,nparms,gradf,gradf1)
implicit none
double precision dot_vect
integer ny,nparms,i
double
precision
parms(nparms),y(ny),gradf(6),gradf1(6),\operatorname{del(6)}
double precision mm,sig(6),s(6),r(6),I1,p
double precision alpha(6),tau(6),n(6)
double precision norm,norm2,v,vv
double precision one,two,three,sqrt23,onethird,small
double precision n1,n2
parameter(one=1.0d0,two=2.0d0,three=3.0d0)
parameter(small=1.0d-10)
parameter(n1=0.816496580927739,n2=-0.4082482904638
5)
data del/1.0d0,1.0d0,1.0d0,0.0d0,0.0d0,0.0d0/
sqrt23=dsqrt(two/three)
onethird=one/three
call pzero(n,6)
mm=parms(7)
sig(1)=y(1)
sig(2)=y(2)
sig(3)=y(3)
sig(4)=y(4)
sig(5)=y(5)
sig(6)=y(6)

```
\(\operatorname{alpha}(1)=y(7)\)
alpha(2) \(=\mathrm{y}(8)\)
\(\operatorname{alpha}(3)=y(9)\)
\(\operatorname{alpha}(4)=y(10)\)
\(\operatorname{alpha}(5)=y(11)\)
alpha(6) \(=\mathrm{y}(12)\)
call deviator(sig,s,I1,p)
do \(\mathrm{i}=1,6\)
\(\operatorname{tau}(\mathrm{i})=\mathrm{s}(\mathrm{i})-\mathrm{p} *\) alpha(i)
end do ! i
norm2=dot_vect(1,tau,tau,6)
norm=dsqrt(norm2)
if(norm.lt.small) then
norm=small
endif
do \(\mathrm{i}=1,6\)
\(\mathrm{n}(\mathrm{i})=\operatorname{tau}(\mathrm{i}) /\) norm
enddo
if(dabs(p).lt.small) then
\[
\text { do } \mathrm{i}=1,6
\]
\(\mathrm{r}(\mathrm{i})=\mathrm{s}(\mathrm{i}) / \mathrm{small}\)
enddo
else
do \(\mathrm{i}=1,6\)
\(r(i)=s(i) / p\)
enddo
endif
\(\mathrm{v}=\) dot_vect(1,r,n,6)
\(\mathrm{vv}=-\) onethird* \({ }^{*}\)
do \(\mathrm{i}=1,6\)
\(\operatorname{gradf}(\mathrm{i})=\mathrm{n}(\mathrm{i})+\mathrm{v} v^{*} \operatorname{del}(\mathrm{i})\)
if(i.le.3) then
\(\operatorname{gradf} 1(i)=\operatorname{gradf}(\mathrm{i})\)
else
gradf1(i)=two*gradf(i)
endif
enddo
return
end
subroutine grad_g_DM(y,ny,parms,nparms,gradg,gradg1)
implicit none
double precision
dot_vect,distance,psi_void,psi_void_DM
integer ny,nparms,i
double precision M_c,M_e,cM,A0

norm2 \(=\) dot_vect( 1, tau,tau,6)
norm=dsqrt(norm2)
if(norm.lt.small) then
norm=small
endif
do \(\mathrm{i}=1,6\)
\(\mathrm{n}(\mathrm{i})=\operatorname{tau}(\mathrm{i}) /\) norm
enddo
\(\mathrm{n} 2(1)=\mathrm{n}(1) * \mathrm{n}(1)+\mathrm{n}(4) * \mathrm{n}(4)+\mathrm{n}(5) * \mathrm{n}(5)\)
\(\mathrm{n} 2(2)=\mathrm{n}(4) * \mathrm{n}(4)+\mathrm{n}(2) * \mathrm{n}(2)+\mathrm{n}(6) * \mathrm{n}(6)\)
(3) \(n(6)+n(5) \cdot n(5)+n(3) * n(3)\)
\(\mathrm{n} 2(4)=\mathrm{n}(1) * \mathrm{n}(4)+\mathrm{n}(4) * \mathrm{n}(2)+\mathrm{n}(6) * \mathrm{n}(5)\)
\(\mathrm{n} 2(5)=\mathrm{n}(5) * \mathrm{n}(1)+\mathrm{n}(6) * \mathrm{n}(4)+\mathrm{n}(3) * \mathrm{n}(5)\)
\(\mathrm{n} 2(6)=\mathrm{n}(4) * \mathrm{n}(5)+\mathrm{n}(2) * \mathrm{n}(6)+\mathrm{n}(6) * \mathrm{n}(3)\)
psi=psi_void_DM(void,p,parms,nparms)
call lode_DM(tau,cM,cos3t,gth,dgdth)
temp \(1=\) one+three* \(\cos 3 \mathrm{t}\) *dgdth
temp2=-three*sqrt6*dgdth
do \(\mathrm{i}=1,6\)
Rdev(i)=temp1*n(i)+temp2*(n2(i)-onethird*del(i))
enddo
temp3=dot_vect(1,Fab,n,6)
temp4=half*(temp3+dabs(temp3))
\(\mathrm{Ad}=\mathrm{A} 0 *(\) one+temp4)
call alpha_th_DM(3,n,gth,psi,parms,nparms,alpha_d)
dd \(=\) distance(alpha_d,alpha,n)
if((psi.gt.zero).and.(dd.lt.zero)) then
dd=zero
endif
dil \(=A d * d d\)
dil3=onethird*dil
do \(\mathrm{i}=1,6\)
\(\operatorname{gradg}(1)=\operatorname{Rdev}(\mathrm{i})+\operatorname{dil} 3 * \operatorname{del}(\mathrm{i})\)
(i.le.3) then
\(\operatorname{gradg} 1(\mathrm{i})=\operatorname{gradg}(\mathrm{i})\)
\(\operatorname{gradg} 1(\mathrm{i})=t w 0^{*} \operatorname{gradg}(\mathrm{i})\)
endif
return
end
niyz(y,nydim,z,nzdim,qq1,nasvy,qq2,nasvz,sig,ntens)
implicit none
```

    integer i,nydim,nzdim,nasvy,nasvz,ntens
    double precision y(nydim),z(nzdim)
    double precision qq1(nasvy),qq2(nasvz),sig(ntens)
    call pzero(y,nydim)
    call pzero(z,nzdim)
    do i=1,ntens
    y(i)=sig(i)
    enddo
do i=1,nasvy
y(6+i)=qq1(i)
enddo
do i=1,nasvz
z(i) = qq2(i)
enddo
return
end
subroutine
intersect_DM(y0,y1,y_star,n,parms,nparms,tol_ff,
\& xi,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
integer n,nparms,maxiter,kiter,i,kiter_bis,bisect
double precision yf_DM,dot_vect
double precision
parms(nparms),y0(n),y1(n),y_star(n),y05(n)
double precision tol_ff,fy_star,err,dfdxi,dfdxi_m1,xi,fy05
double precision dxi, xip1
double precision
sig0(6),sig1(6),dsig(6),P_star(6),P1_star(6)
double precision zero,one,half,three,onethird
double precision pp_star,low, fy11, fy00, xi_max, xi_i,
pp05
double precision y00(n),y11(n)
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
parameter(zero=0.0d0,one=1.0d0,half=0.5d0,three=3.0d0)
parameter(low=1.0d-10)
xi=one
maxiter=5000
kiter=0
bisect=0
kiter_bis=0
do i=1,6
sig0(i)=y0(i)

```
\(\operatorname{sig} 1(i)=y 1(i)\)
\(\operatorname{dsig}(\mathrm{i})=\operatorname{sig} 1(\mathrm{i})-\operatorname{sig} 0(\mathrm{i})\)
end do !i
call push(y1,y_star,n)
fy_star=yf_DM(y_star,n,parms,nparms)
onethird=one/three
pp_star=(y_star(1)+y_star(2)+y_star(3))*onethird
err=dabs(fy_star/pp_star)
if(pp_star.gt.one) err=dabs(fy_star)
if(bisect.eq.0) then
do while ((err.gt.tol_ff).and.(bisect.eq.0))
kiter=kiter+1
call grad_f_DM(y_star,n,parms,nparms,P_star,P1_star)
dfdxi=dot_vect(1,P_star,dsig,6)
if (dfdxi.lt.low) then
bisect=1
endif
dfdxi_m1=one/dfdxi
dxi=-dfdxi_m1*fy_star
xip1=xi+dxi
do while ((xip1.lt.zero).or.(xip1.gt.one))
dxi=half*dxi
xip1=xi+dxi
end do
xi=xip1
do \(i=1, n\)
\[
y_{-} \operatorname{star}(\mathrm{i})=\mathrm{y} 0(\mathrm{i})+\mathrm{xi}{ }^{*}(\mathrm{y} 1(\mathrm{i})-\mathrm{y} 0(\mathrm{i}))
\]
end do !i
fy_star=yf_DM(y_star,n,parms,nparms)
if (fy_star.lt.zero) then
bisect=1
else
onethird=one/three
pp_star=(y_star(1)+y_star(2)+y_star(3))*onethird
err=dabs(fy_star/pp_star)
if(pp_star.gt.one) err=dabs(fy_star)
endif
if (kiter.gt.maxiter+1) then
err \(=0\)
end if
end do ! bottom of Newton iteration
if((xi.lt.zero).and.(xi.gt.one)) then
\(x i=z e r o\)
return
endif
\begin{tabular}{|c|c|}
\hline endif & xi \(=\) xi_max \\
\hline if(bisect.eq.1) then & endif \\
\hline do \(\mathrm{i}=1, \mathrm{n}\) & return \\
\hline \(y 00(i)=y 0(i)\) & end \\
\hline \(y 11(i)=y 1(i)\) & subroutine inv_sig(sig,pp,qq, \(\cos 3 \mathrm{t}\) ) \\
\hline enddo & implicit none \\
\hline fy00 =yf_DM(y00,n,parms,nparms) & double precision \(\operatorname{sig}(6), \operatorname{sdev}(6), \mathrm{s} 2(6)\) \\
\hline fy11 =yf_DM(y11,n,parms,nparms) & double precision I1,J2bar,J2bar_sq,J3bar,trs2,trs3 \\
\hline do \(\mathrm{i}=1, \mathrm{n}\) & double precision \(\mathrm{pp}, \mathrm{qq}, \cos 3 \mathrm{t}\),numer,denom \\
\hline \(y 05(\mathrm{i})=\mathrm{y} 0(\mathrm{i})\) & double precision zero,one,two,three \\
\hline enddo & double precision onethird,half,onept5,sqrt3,tiny \\
\hline pp05=(y05(1)+y05(2)+y05(3))*onethird & double precision dot_vect \\
\hline fy05 =yf_DM(y05,n,parms,nparms) & data zero,one,two,three/0.0d0,1.0d0,2.0d0,3.0d0/ \\
\hline err=abs(fy05/pp05) & data tiny/1.0d-15/ \\
\hline if(pp05.gt.one) err=dabs(fy05) & onethird=one/three \\
\hline do while(err.gt.tol_ff) & half=one/two \\
\hline kiter_bis=kiter_bis+1 & onept5=three/two \\
\hline do \(\mathrm{i}=1,6\) & sqrt3=dsqrt(three) \\
\hline \(\mathrm{y} 05(\mathrm{i})=\operatorname{half}^{*}(\mathrm{y} 00(\mathrm{i})+\mathrm{y} 11(\mathrm{i})\) ) & \(\mathrm{I} 1=\operatorname{sig}(1)+\operatorname{sig}(2)+\operatorname{sig}(3)\) \\
\hline enddo & \(\mathrm{pp}=\) onethird*I1 \\
\hline fy05 =yf_DM(y05,n,parms,nparms) & \(\operatorname{sdev}(1)=\operatorname{sig}(1)-\mathrm{pp}\) \\
\hline \(\mathrm{pp} 05=(\mathrm{y} 05(1)+\mathrm{y} 05(2)+\mathrm{y} 05(3)) *\) onethird & \(\operatorname{sdev}(2)=\operatorname{sig}(2)-\mathrm{pp}\) \\
\hline err=abs(fy05/pp05) & \(\operatorname{sdev}(3)=\operatorname{sig}(3)-\mathrm{pp}\) \\
\hline if(pp05.gt.one) err=dabs(fy05) & \(\operatorname{sdev}(4)=\operatorname{sig}(4)\) \\
\hline if(fy05.lt.zero) then & \(\operatorname{sdev}(5)=\operatorname{sig}(5)\) \\
\hline call push(y05,y00,n) & \(\operatorname{sdev}(6)=\operatorname{sig}(6)\) \\
\hline else & trs2=dot_vect(1,sdev,sdev,6) \\
\hline call push(y05,y11,n) & J2bar=half*trs2 \\
\hline endif & \(\mathrm{qq}=\mathrm{dsqrt}(\) onept \(5 * \operatorname{trs} 2)\) \\
\hline if (kiter_bis.gt.maxiter+1) then & \(\mathrm{s} 2(1)=\operatorname{sdev}(1) * \operatorname{sdev}(1)+\operatorname{sdev}(4) * \operatorname{sdev}(4)+\operatorname{sdev}(5) * \operatorname{sdev}(5)\) \\
\hline err=0 & \(\mathrm{s} 2(2)=\operatorname{sdev}(4) * \operatorname{sdev}(4)+\operatorname{sdev}(2) * \operatorname{sdev}(2)+\operatorname{sdev}(6) * \operatorname{sdev}(6)\) \\
\hline endif & \(\mathrm{s} 2(3)=\operatorname{sdev}(6) * \operatorname{sdev}(6)+\operatorname{sdev}(5) * \operatorname{sdev}(5)+\operatorname{sdev}(3) * \operatorname{sdev}(3)\) \\
\hline enddo & \(\mathrm{s} 2(4)=\operatorname{sdev}(1) * \operatorname{sdev}(4)+\operatorname{sdev}(4) * \operatorname{sdev}(2)+\operatorname{sdev}(6) * \operatorname{sdev}(5)\) \\
\hline do \(\mathrm{i}=1, \mathrm{n}\) & \(\mathrm{s} 2(5)=\operatorname{sdev}(5) * \operatorname{sdev}(1)+\operatorname{sdev}(6) * \operatorname{sdev}(4)+\operatorname{sdev}(3) * \operatorname{sdev}(5)\) \\
\hline y_star(i)=y05(i) & \(\mathrm{s} 2(6)=\operatorname{sdev}(4) * \operatorname{sdev}(5)+\operatorname{sdev}(2) * \operatorname{sdev}(6)+\operatorname{sdev}(6) * \operatorname{sdev}(3)\) \\
\hline enddo & if(trs2.lt.tiny) then \\
\hline xi_max=zero & \(\cos 3 \mathrm{t}=\) one \\
\hline do \(i=1,6\) & else \\
\hline if((yl(i)-y0(i)).ne.zero) then & trs3=dot_vect(1,sdev,s2,6) \\
\hline \(x i \_i=(y 05(i)-y 0(\mathrm{i})) /(\mathrm{y} 1(\mathrm{i})-\mathrm{y} 0(\mathrm{i})\) ) & J3bar=onethird*trs3 \\
\hline if(xi_i.gt.xi_max) then & J2bar_sq=dsqrt(J2bar) \\
\hline xi_max = xi_i & numer=three*sqrt3*J3bar \\
\hline endif & denom=two * \({ }^{\text {(J2bar_sq**3) }}\) \\
\hline endif & \(\cos 3 \mathrm{t}=\) numer/denom \\
\hline enddo & \(\mathrm{if}(\mathrm{dabs}(\cos 3 \mathrm{t}) . \mathrm{gt.one})\) then \\
\hline
\end{tabular}
\begin{tabular}{|c|c|}
\hline \(\cos 3 \mathrm{t}=\cos 3 \mathrm{t} / \mathrm{dabs}(\cos 3 \mathrm{t})\) & dgdth \(=(1-\mathrm{cM}) * \mathrm{gth} /(\mathrm{two} * \mathrm{cM})\) \\
\hline end if & else \\
\hline end if & n_VEm1=one/n_VE \\
\hline return & tmp \(1=\) one/(two \({ }^{* *}\) n_VE) \\
\hline end & tmp2 \(=\) cM \({ }^{* *}\) n_VEm1 \\
\hline subroutine lode_DM(r,cM, cos3t,gth,dgdth) & tmp3=one+tmp2 \\
\hline implicit none & tmp4=one-tmp2 \\
\hline integer Argyris &  \\
\hline double precision r(6),r2(6) & beta \(=\) tmp4/tmp3 \\
\hline double precision trr2,trr3,J2bar,J3bar,J2bar_sq & tmp5 \(=(\) one + beta \(* \cos 3 \mathrm{t}) * *\) n_VE \\
\hline double precision cM,n_VE,n_VEm1,numer,denom, \(\cos 3 \mathrm{t}\) & tmp6=one+beta* \(\cos 3 \mathrm{t}\) \\
\hline double precision tmp1,tmp2,tmp3,tmp4,tmp5,tmp6 & gth=alpha*tmp5 \\
\hline double precision alpha,beta,gth,dgdth & dgdth=n_VE*beta/tmp6 \\
\hline double precision one,two,three & end if \\
\hline double precision onethird,half,sqrt3,tiny & return \\
\hline double precision dot_vect & end \\
\hline data one,two,three/1.0d0,2.0d0,3.0d0/ & implicit none \\
\hline data tiny,n_VE/1.0d-15,-0.25d0/ & integer \(\mathrm{i}, \mathrm{j}, \mathrm{k}, 1, \mathrm{~m}, \mathrm{n}\) \\
\hline data Argyris/0/ & do \(\mathrm{i}=1,1\) \\
\hline onethird=one/three & do \(\mathrm{j}=1, \mathrm{n}\) \\
\hline half=one/two & do \(\mathrm{k}=1, \mathrm{~m}\) \\
\hline sqrt3=dsqrt(three) & enddo \\
\hline trr2=dot_vect(1,r,r,6) & enddo \\
\hline J2bar=half*trr2 & enddo \\
\hline \(\mathrm{r} 2(1)=\mathrm{r}(1) * \mathrm{r}(1)+\mathrm{r}(4) * \mathrm{r}(4)+\mathrm{r}(5) * \mathrm{r}(5)\) & return \\
\hline \(\mathrm{r} 2(2)=\mathrm{r}(4) * \mathrm{r}(4)+\mathrm{r}(2) * \mathrm{r}(2)+\mathrm{r}(6) * \mathrm{r}(6)\) & end \\
\hline \(\mathrm{r} 2(3)=\mathrm{r}(6) * \mathrm{r}(6)+\mathrm{r}(5) * \mathrm{r}(5)+\mathrm{r}(3) * \mathrm{r}(3)\) & subroutine move_eps(dstran,ntens,deps,depsv) \\
\hline \(\mathrm{r} 2(4)=\mathrm{r}(1) * \mathrm{r}(4)+\mathrm{r}(4) * \mathrm{r}(2)+\mathrm{r}(6) * \mathrm{r}(5)\) & implicit none \\
\hline \(\mathrm{r} 2(5)=\mathrm{r}(5) * \mathrm{r}(1)+\mathrm{r}(6) * \mathrm{r}(4)+\mathrm{r}(3) * \mathrm{r}(5)\) & integer ntens, i \\
\hline \(\mathrm{r} 2(6)=\mathrm{r}(4) * \mathrm{r}(5)+\mathrm{r}(2) * \mathrm{r}(6)+\mathrm{r}(6) * \mathrm{r}(3)\) & double precision deps(6),dstran(ntens),depsv \\
\hline if(trr2.lt.tiny) then & call pzero(deps,6) \\
\hline \(\cos 3 \mathrm{t}=\) one & do \(\mathrm{i}=1\), ntens \\
\hline else & \(\operatorname{deps}(\mathrm{i})=-\mathrm{dstran}(\mathrm{i})\) \\
\hline trr3=dot_vect(1,r,r2,6) & enddo \\
\hline J3bar=onethird*tr3 & depsv=deps(1)+deps(2)+deps(3) \\
\hline J2bar_sq=dsqrt(J2bar) & return \\
\hline numer=three*sqr33*J3bar & end \\
\hline denom=two*(J2bar_sq**3) & subroutine move_sig(stress,ntens, pore,sig) \\
\hline \(\cos 3 \mathrm{t}=\) numer/denom & implicit none \\
\hline if(dabs(cos3t).gt.one) then & integer ntens, i \\
\hline \(\cos 3 \mathrm{t}=\cos 3 \mathrm{t} / \mathrm{dabs}(\cos 3 \mathrm{t})\) & double precision \(\operatorname{sig}(6)\),stress(ntens), pore \\
\hline end if & call pzero(sig,6) \\
\hline end if & do \(\mathrm{i}=1\), ntens \\
\hline if (Argyris.ne.0) then & if(i.le.3) then \\
\hline \(\mathrm{gth}=\mathrm{two} * \mathrm{cM} /((\) one +cM\()-(\) one -cM\() * \cos 3 \mathrm{t})\) & \(\operatorname{sig}(\mathrm{i})=-\mathrm{stress}(\mathrm{i})\)-pore \\
\hline
\end{tabular}
else
\[
\operatorname{sig}(\mathrm{i})=-\operatorname{stress}(\mathrm{i})
\]
end if
enddo
return
end
subroutine norm_res_DM(y_til,y_hat,ny,norm_R)
implicit none
integer ny, i
double precision y_til(ny),y_hat(ny)
double precision err(ny),norm_R2,norm_R
double precision sig_hat(6),sig_til(6),del_sig(6)
double precision alpha_hat(6),alpha_til(6),del_alpha(6)
double precision Fab_hat(6),Fab_til(6),del_Fab(6)
double precision void_hat,void_til,del_void
double precision norm_sig2,norm_alpha2,norm_Fab2
double precision norm_sig,norm_alp,norm_Fab
double precision dot_vect,zero
parameter(zero=0.0d0)
call pzero(err,ny)
do \(\mathrm{i}=1,6\)
sig_hat(i)=y_hat(i)
sig_til(i)=y_til(i)
del_sig(i)=dabs(sig_hat(i)-sig_til(i))
end do
do \(\mathrm{i}=1,6\)
alpha_hat(i)=y_hat(6+i)
alpha_til(i)=y_til(6+i)
del_alpha(i)=dabs(alpha_hat(i)-alpha_til(i))
end do
void_hat=y_hat(13)
void_til=y_til(13)
del_void=dabs(void_hat-void_til)
do \(\mathrm{i}=1,6\)
Fab_hat(i)=y_hat(13+i)
Fab_til(i)=y_til(13+i)
del_Fab(i)=dabs(Fab_hat(i)-Fab_til(i))
end do
norm_sig2=dot_vect(1,sig_hat,sig_hat,6)
norm_alpha2=dot_vect(1,alpha_hat,alpha_hat,6)
norm_Fab2=dot_vect(1,Fab_hat,Fab_hat,6)
norm_sig=dsqrt(norm_sig2)
norm_alp=dsqrt(norm_alpha2)
norm_Fab=dsqrt(norm_Fab2)
if(norm_sig.gt.zero) then
\[
\text { do } \mathrm{i}=1,6
\]
err(i)=del_sig(i)/norm_sig
end do
end if
if(norm_alp.gt.zero) then
\[
\text { do } \mathrm{i}=1,6
\]
\[
\operatorname{err}(6+i)=\text { del_alpha(i)/norm_alp }
\]
end do
end if
err(13)=del_void/void_hat
do \(\mathrm{i}=1,6\)
if((Fab_til(i).ne.zero).and.(norm_Fab.gt.zero)) then err(13+i)=del_Fab(i)/norm_Fab
end if
end do
norm_R2=dot_vect(3,err,err,ny)
norm_R=dsqrt(norm_R2)
return
end
subroutine pert_DM(y_n,y_np1,z,n,nasvy,nasvz,err_tol,
\& maxnint,DTmin,deps_np1,parms,
\&
nparms,nfev,elprsw,theta,ntens,DD,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
integer elprsw
integer ntens, \(\mathrm{jj}, \mathrm{kk}\)
integer n,nasvy,nasvz,nparms,nfev
integer maxnint,mario_DT_test
double precision
y_n(n),y_np1(n),y_star(n),z(nasvz),parms(nparms)
double precision err_tol
double precision theta,DTmin
double precision deps_np1(6),deps_star(6)
double precision \(\operatorname{dsig}(6), \mathrm{DD}(6,6)\)
double precision zero,three
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
parameter \((\) zero \(=0.0 \mathrm{~d} 0\), three \(=3.0 \mathrm{~d} 0\) )
call pzero(DD,36)
call pzero(y_star,n)
if(plastic.eq.0) then call el_stiff_DM(y_np1,n,parms,nparms,DD,
\&
error,tol_f,check_ff,drcor,p_thres,plastic)
else
do \(\mathrm{jj}=1\), ntens
call push(y_n,y_star,n)
call push(deps_np1,deps_star,ntens)
deps_star(jj) \(=\) deps_star(jij)+theta
if(error.ne.10) then
call
rkf23_upd_DM(y_star,z,n,nasvy,nasvz,err_tol,maxnint,
\&
DTmin,deps_star,parms,nparms,nfev,elprsw,
\& mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
end if
do \(\mathrm{kk}=1\), ntens
dsig(kk)=y_star(kk)-y_np1(kk)
\(\mathrm{DD}(\mathrm{kk}, \mathrm{jj})=\mathrm{dsig}(\mathrm{kk}) /\) theta
end do ! kk
end do !jj
end if
return
end
subroutine
plast_mod_DM(y,ny,z,nz,parms,nparms,h_alpha,Kpm1,
\& switch2,mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
external matmul
integer nparms,ny,nz,i,iter,switch2,mario_DT_test
double
precision
dot_vect,distance,ref_db,psi_void,psi_void_DM
double precision \(y(n y), z(n z)\), parms(nparms)
double precision \(\operatorname{De}(6,6)\), h_alpha(6)
double precision LL(6),LL1(6),RR(6),RR1(6),U(6),V(6)
double precision p_a,e0,lambda,xi,M_c,M_e,cM,mm
double precision G0,nu,h0,c_h,n_b
double precision A0,n_d,z_max,c_z,bulk_w
double precision \(\operatorname{sig}(6)\),alpha(6),void,Fab(6)
double precision alpha_sr(6),alpha_b(6)
double precision \(\mathrm{s}(6), \operatorname{tau}(6), \mathrm{n}(6), \mathrm{I} 1, \mathrm{p}, \mathrm{psi}, \cos 3 \mathrm{t}\)
double precision b0,d_sr,hh,db,gth,dgdth
double precision
HHp,LDeR,Kp,Kpm1,norm2,norm,chvoid
double precision zero,tiny,one,two,three, large
double precision twothird
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
parameter(zero \(=0.0 \mathrm{~d} 0\), one \(=1.0 \mathrm{~d} 0\), two \(=2.0 \mathrm{~d} 0\), three \(=3.0 \mathrm{~d} 0\) )
parameter(tiny \(=1.0 \mathrm{~d}-15\), large \(=1.0 \mathrm{e} 15\) )
twothird=two/three
\(\mathrm{p} \_\mathrm{a}=\mathrm{parms}(1)\)
e0=parms(2)
lambda=parms(3)
xi=parms(4)
M_c=parms(5)
M_e=parms(6)
\(\mathrm{mm}=\operatorname{parms}(7)\)
\(\mathrm{G} 0=\operatorname{parms}(8)\)
\(\mathrm{nu}=\operatorname{parms}(9)\)
\(\mathrm{h} 0=\operatorname{parms}(10)\)
c_h=parms(11)
\(\mathrm{n} \_\mathrm{b}=\mathrm{parms}(12)\)
\(\mathrm{A} 0=\operatorname{parms}(13)\)
n_d=parms(14)
z_max=parms(15)
c_z=parms(16)
bulk_w=parms(17)
\(\mathrm{cM}=\mathrm{M} \_\mathrm{e} / \mathrm{M} \_\mathrm{c}\)
switch2=zero
iter=0
do \(\mathrm{i}=1,6\)
\(\operatorname{sig}(i)=y(i)\)
end do !i
do \(\mathrm{i}=1,6\)
alpha(i) \(=\mathrm{y}(6+\mathrm{i})\)
end do !i
void \(=y(13)\)
do \(i=1,6\)
\(\operatorname{Fab}(\mathrm{i})=\mathrm{y}(13+\mathrm{i})\)
end do !i
do \(\mathrm{i}=1,6\)
alpha_sr(i)=z(i)
end do !i
call deviator(sig,s,I1,p)
do \(\mathrm{i}=1,6\)
tau(i) \(=s(\mathrm{i})-\mathrm{p} * \mathrm{alpha}(\mathrm{i})\)
end do ! i
norm2=dot_vect(1,tau,tau,6)
norm=dsqrt(norm2)
if(norm.lt.tiny) then norm=tiny
\begin{tabular}{|c|c|c|}
\hline endif & switch2=1 & \\
\hline do \(\mathrm{i}=1,6\) & return & \\
\hline \(\mathrm{n}(\mathrm{i})=\) tau(i)/norm & endif & \\
\hline end do & else & \\
\hline call el_stiff_DM(y,ny,parms,nparms,De, & if(LDeR.le.zero) then & \\
\hline \& error,tol_f,check_ff,drcor,p_thres,plastic) & switch2=1 & \\
\hline call grad_f_DM(y,ny,parms,nparms,LL,LL1) & return & \\
\hline call matmul(De,LL1,V,6,6,1) & endif & \\
\hline call grad_g_DM(y,ny,parms,nparms,RR,RR1) & endif & \\
\hline call matmul(De,RR1,U,6,6,1) & if(Kp.lt.zero)then & \\
\hline if (dabs(p).gt.zero) then & error \(=3\) & \\
\hline chvoid=c_h*void & return & \\
\hline if(chvoid.ge.1) then & endif & \\
\hline chvoid=0.99999 & call push(alpha_sr,z,6) & \\
\hline end if & \(\mathrm{Kpm1}=\mathrm{one} / \mathrm{Kp}\) & \\
\hline \(\mathrm{b} 0=\mathrm{G} 0 * \mathrm{~h} 0 *(\) one-chvoid)/dsqrt(p/p_a) & return & \\
\hline else & end & \\
\hline \(\mathrm{b} 0=\) large & double precision & function \\
\hline end if & psi_void_DM(void,p,parms,nparms) & \\
\hline d_sr=distance(alpha,alpha_sr,n) & implicit none & \\
\hline if (d_sr.lt.zero) then & integer nparms & \\
\hline call push(alpha,alpha_sr,6) & double precision void,p,parms(nparms) & \\
\hline end if & double precision p_a,e0,lambda,xi,ec & \\
\hline if (d_sr.lt.tiny) then & p_a=parms(1) & \\
\hline d_sr=tiny & \(\mathrm{e} 0=\mathrm{parms}(2)\) & \\
\hline end if & lambda=parms(3) & \\
\hline \(\mathrm{hh}=\mathrm{b} 0 / \mathrm{d}\) _sr & \(\mathrm{xi}=\mathrm{parms}(4)\) & \\
\hline psi=psi_void_DM(void,p,parms,nparms) & ec=e0-lambda* \(\left(\mathrm{p} / \mathrm{p} \_\mathrm{a}\right){ }^{*}{ }^{\text {xi }}\) & \\
\hline call lode_DM(tau,cM, cos3t,gth,dgdth) & psi_void_DM=void-ec & \\
\hline call alpha_th_DM(2,n,gth,psi,parms,nparms,alpha_b) & return & \\
\hline db=distance(alpha_b,alpha,n) & end & \\
\hline do \(\mathrm{i}=1,6\) & subroutine push(a,b,n) & \\
\hline h_alpha(i)=twothird*hh*(alpha_b(i)-alpha(i)) & implicit none & \\
\hline end do & integer i, n & \\
\hline HHp=twothird*hh*p*db & double precision \(\mathrm{a}(\mathrm{n}), \mathrm{b}(\mathrm{n})\) & \\
\hline if(HHp.gt.1e+15) then & do \(\mathrm{i}=1, \mathrm{n}\) & \\
\hline endif & \(b(i)=a(i)\) & \\
\hline LDeR=dot_vect(1,LL1,U,6) & enddo & \\
\hline \(\mathrm{Kp}=\mathrm{LDeR}+\mathrm{HHp}\) & return & \\
\hline if(mario_DT_test.eq.zero) then & end & \\
\hline if(LDeR.lt.zero) then & subroutine pzero(v,nn) & \\
\hline switch2=1 & implicit none & \\
\hline return & integer n,nn & \\
\hline endif & double precision \(\mathrm{v}(\mathrm{nn})\) & \\
\hline if(Kp.lt.zero) then & do \(\mathrm{n}=1, \mathrm{nn}\) & \\
\hline
\end{tabular}

parameter(zero \(=0.0 \mathrm{~d} 0\), one \(=1.0 \mathrm{~d} 0\), two \(=2.0 \mathrm{~d} 0\), three \(=3.0 \mathrm{~d} 0)\)
parameter(four \(=4.0 \mathrm{~d} 0\), six \(=6.0 \mathrm{~d} 0\), half \(=0.5 \mathrm{~d} 0\), ptnine \(=0.9 \mathrm{~d} 0\) )
parameter \((\mathrm{pt} 1=1.0 \mathrm{~d}-3\), ptone \(=0.1 \mathrm{~d} 0\), ten \(=1.0 \mathrm{~d} 1)\)
call pzero(y_k,n)
one6=one/six
one \(3=\) one/three
two3=two/three
plastic \(=0\)
mario \(=0\)
mario_DT=0
mario_DT_test=0
iter=iter+1
call push(y,y_k,n)
call push(z,z_k,nasvz)
p_atm=parms(1)
tol_ff=tol_f*p_atm
ff_k=yf_DM(y_k,n,parms,nparms)
onethird=one/three
pp_k=(y_k(1)+y_k(2)+y_k(3))*onethird
ff_k_pp_k=ff_k/pp_k
if(pp_k.gt.one) ff_k_pp_k=ff_k
if (ff_k_pp_k.gt.tol_ff) then
call
drift_corr_DM(y_k,n,z1,nasvz,parms,nparms,tol_ff,
\& switch2,mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
end if
deps_np1_star=deps_np1
call trial_state(y_k,n,parms,nparms,deps_np1_star,y_tr,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
pp_tr=(y_tr(1)+y_tr(2)+y_tr(3))*onethird
if((pp_k.gt.(p_thres+p_thres))) then do while(pp_tr.le.p_thres)
call
trial_state(y_k,n,parms,nparms,deps_np1_star,y_tr,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
pp_tr=\(\left(y_{-} \operatorname{tr}(1)+y_{-} \operatorname{tr}(2)+y_{-} \operatorname{tr}(3)\right) *\) onethird
deps_np1_star=deps_np1_star*half end do
elseif((pp_k.le.(p_thres+p_thres))
\&.and.(pp_tr.gt.(p_thres+p_thres))) then
deps_np1_star=deps_np1 call
trial_state(y_k,n,parms,nparms,deps_np1_star,y_tr,
\& error,tol_f,check_ff,drcor,p_thres,plastic) pp_tr=(y_tr(1)+y_tr(2)+y_tr(3))*onethird
elseif((pp_k.le.(p_thres+p_thres))
\&.and.(pp_tr.le.(p_thres+p_thres))) then call push(y_k,y_tr,n)
endif
ff_tr=yf_DM(y_tr,n,parms,nparms)
pp_tr=(y_tr(1)+y_tr(2)+y_tr(3))*onethird
ff_tr_pp_tr=ff_tr/pp_tr
if(pp_tr.gt.one) ff_tr_pp_tr=ff_tr
call check_crossing(y_k,y_tr,n,parms,nparms,prod)
```

        if (ff_tr_pp_tr.lt.tol_ff) then
        call push(y_tr,y_k,n)
        else
        if(pp_tr.lt.p_thres) then
        else
        if ((ff_k_pp_k.lt.(-tol_ff)).or.(prod.lt.zero)) then
        call
    intersect_DM(y_k,y_tr,y_star,n,parms,nparms,tol_ff,xi,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
call push(y_star,y_k,n)
plastic=1
else
xi=zero
plastic=1
end if
T_k=xi
DT_k=(one-xi)
ksubst=0
kreject=0
nfev=0
attempt=1
maxnint_1=maxnint
err_tol_1=err_tol
err_tol_n=err_tol
switch3=0
do while((T_k.lt.one).and.(mario.eq.zero)
\&.and.(mario_DT.eq.zero))
*****
ksubst=ksubst+1
if((ksubst.gt.maxnint_1).or.(switch3.eq.1)) then
if(attempt.eq.1) then
maxnint_1=2.0*maxnint
err_tol_1=1000.0*err_tol
attempt=2
DT_k=1-T_k
elseif(attempt.eq.2) then
mario=one
call push(z1,z,nasvz)
call push(y_k,y,n)
return
endif
endif
call push(z_k,z1,nasvz)
pp_kk=(y_k(1)+y_k(2)+y_k(3))*onethird

```
call f_plas_DM(y_k,n,nasvy,z1,nasvz,parms,nparms,
\(+\)
deps_np1,kRK_1,nfev,switch2,mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic) if(error.eq.10) return if (switch2.eq.zero) then
temp=half*DT_k
do \(\mathrm{i}=1, \mathrm{n}\)
\(y \_2(i)=y \_k(i)+t e m p * k R K \_1(i)\)
end do
pp_2=(y_2(1)+y_2(2)+y_2(3))*onethird
if(pp_2.gt.zero)then
call
f_plas_DM(y_2,n,nasvy,z1,nasvz,parms,nparms,
\(+\)
deps_np1,kRK_2,nfev,switch2,mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
if(error.eq.10) return
if (switch2.eq.zero) then
do \(\mathrm{i}=1, \mathrm{n}\)
y_3(i)=y_k(i)-DT_k*kRK_1(i)+two*DT_k*kRK_2(i)
end do
pp_3=(y_3(1)+y_3(2)+y_3(3))*onethird
if(pp_3.gt.zero)then
call
f_plas_DM(y_3,n,nasvy,z1,nasvz,parms,nparms,
\(+\)
deps_np1,kRK_3,nfev,switch2,mario_DT_test,
\&
error,tol_f,check_ff,drcor,p_thres,plastic) if(error.eq.10) return if (switch2.eq.zero) then do \(\mathrm{i}=1, \mathrm{n}\)
y_til(i)=y_k(i)+DT_k*kRK_2(i) \(y \_h a t(i)=y \_k(i)+D T \_k *\) \&
(one6*kRK_1(i)+two3*kRK_2(i)+one6*kRK_3(i))
end do call
norm_res_DM(y_til,y_hat,n,norm_R)

S_hull=ptnine*DT_k*(err_tol/norm_R)**one3
if(norm_R.eq.zero) then

end
```

if !(norm_R.lt.err_tol_1).and.(attempt.eq.2).and.(switch2.eq.zero

```
)
            if((norm_R.gt.err_tol).and.(attempt.ne.3)
    \& .and.(switch2.eq.zero)) then
        DT_k=max(DT_k/four,S_hull)
                        if(DT_k.lt.DTmin) then
    \&
                                    ' is too small, step
rejected'
                                    DT_k= one \(-T_{-} k\)
                                    mario2 \(=1\)
                                    switch3=1
                end if
            end if
            if((norm_R.lt.err_tol_n).and.(attempt.ne.3)
                            \& .and.(switch2.eq.zero).and.(mario2.eq.one)) then
    pp_hat=(y_hat(1)+y_hat(2)+y_hat(1))*onethird
            if (pp_hat.lt.p_thres) then
                mario=one
            endif
                if(drcor.ne.0) then
            call
drift_corr_DM(y_hat,n,z1,nasvz,parms,nparms,tol_ff,
    \& switch2,mario_DT_test,
    \& error,tol_f,check_ff,drcor,p_thres,plastic)
                end if
            if (switch2.eq.zero) then
                    call push(y_hat,y_k,n)
                                    call push(z1,z_k,nasvz)
                                    T_k=T_k+DT_k;
                                    DT_k=min(four*DT_k,S_hull)
                                    DT_k=min((one-T_k),DT_k)
            endif !switch2.eq.zero
            mario2=zero
            end if
            if (attempt.eq.3) then
            if(drcor.ne.0) then
                call
drift_corr_DM(y_k,n,z_k,nasvz,parms,nparms,tol_ff,
    \& switch2,mario_DT_test,
    \& error,tol_f,check_ff,drcor,p_thres,plastic)
        endif
        call push(y_hat,y_k,n)
        T_k=T_k+DT_k;

```

        asv1(i)= y(6+i)
    enddo
do i=1,nasvz
asv2(i) = z(i)
enddo
do j=1,ntens
do i=1,ntens
if((i.le.3).and.(j.le.3)) then
ddsdde(i,j) = DD(i,j)+bulk_w
else
ddsdde(i,j) = DD(i,j)
end if
end do
enddo
return
end
subroutine
tang_stiff(y,z,n,nasvy,nasvz,parms,nparms,DD,cons_lin,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
integer switch2,mario_DT_test
integer n,nasvy,nasvz,nparms,cons_lin
double precision y(n),z(nasvz),parms(nparms)
double precision DD(6,6),HH(nasvy,6)
double precision zero,three
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
parameter(zero=0.0d0,three=3.0d0)
call pzero(DD,36)
if(plastic.eq.1 .and. cons_lin.eq.1) then
call
get_tan_DM(y,n,nasvy,z,nasvz,parms,nparms,DD,HH,switch2,
\& mario_DT_test,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
else
call el_stiff_DM(y,n,parms,nparms,DD,
\&
error,tol_f,check_ff,drcor,p_thres,plastic)
end if
return
end
subroutine trial_state(y_k,n,parms,nparms,deps,y_tr,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
implicit none
integer n,m,nparms,mode,i
double precision y_k(n),parms(nparms),deps(6)

```
double precision \(y \_2(n), y \_3(n), y \_\operatorname{tr}(n)\)
double precision one,two,three,six
double precision \(\mathrm{kRK} \_1(\mathrm{n}), \mathrm{kRK} \_2(\mathrm{n}), \mathrm{kRK} \_3(\mathrm{n})\) double precision DT_k,DTk05,DTk2,DTk6,DTk23
integer error,check_ff,drcor,plastic
double precision tol_f,p_thres
parameter \((\) mode \(=1)\)
data one,two,three,six/1.0d0,2.0d0,3.0d0,6.0d0/
DT_k=one
call f_hypoelas_DM(y_k,n,parms,nparms,deps,kRK_1,
\& error,tol_f,check_ff,drcor,p_thres,plastic)
if (mode.eq.1) then
DTk05=DT_k/two
DTk2=two*DT_k
DTk6=DT_k/six
DTk23=two*DT_k/three
do \(\mathrm{i}=1, \mathrm{n}\)
y_2(i)=y_k(i)+DTk05*kRK_1(i)
end do ! i
call f_hypoelas_DM(y_2,n,parms,nparms,deps,kRK_2,
\&
error,tol_f,check_ff,drcor,p_thres,plastic)
do \(\mathrm{i}=1, \mathrm{n}\)
y_3(i)=y_k(i)-DT_k*kRK_1(i)+DTk2*kRK_2(i)
end do ! i
call f_hypoelas_DM(y_3,n,parms,nparms,deps,kRK_3,
\& error,tol_f,check_ff,drcor,p_thres,plastic) do \(\mathrm{i}=1, \mathrm{n}\)
y_tr(i)=y_k(i)+DTk6*kRK_1(i)+DTk23*kRK_2(i)+DTk6*kRK
_3(i)
end do ! i
else
\[
\text { do } \mathrm{i}=1, \mathrm{n}
\]
y_tr(i)=y_k(i)+DT_k*kRK_1(i)
end do ! i
end if
return
end
subroutine
wrista(mode,y,nydim,deps_np1,dtime,coords,statev,
\&
nstatv,parms,nparms,noel,npt,ndi,nshr,kstep,kinc)
implicit none
integer
mode,nydim,nstatv,nparms,noel,npt,ndi,nshr,kstep,kinc,i
```

        double precision y(nydim),statev(nstatv),parms(nparms)
        double precision deps_np1(6),coords(3),dtime
        if (mode.eq.2) then
        endif
        if(mode.ne.4) then
        & 'element: ', noel, 'Integration point: ',npt
        endif
        if (mode.eq.2) then
        & coords(3)
        do i=1,nparms
        enddo
        endif
        if ((mode.eq.2).or.(mode.eq.3)) then
        endif
        if (mode.eq.4) then
        & 'element: ', noel, 'Integration point: ',npt
        & coords(2),' x3 = ',coords(3)
        endif
        return
        end
        double precision function yf_DM(y,ny,parms,nparms)
        implicit none
        integer ny,nparms
        double precision dot_vect
        double precision y(ny),parms(nparms)
        double precision mm,zero,one,two,three,sqrt23,norm2
        double precision sig(6),s(6),trace,p,alpha(6),sbar(6)
    parameter(zero=0.0d0,one=1.0d0,two=2.0d0,three=3.0d0)
sqrt23=dsqrt(two/three)
mm=parms(7)
sig(1)=y(1)
sig(2)=y(2)
sig(3)=y(3)
sig(4)=y(4)
sig(5)=y(5)
sig(6)=y(6)
alpha(1)=y(7)
alpha(2)=y(8)
alpha(3)=y(9)
alpha(4)=y(10)
alpha(5)=y(11)
alpha(6)=y(12)
call deviator(sig,s,trace,p)
sbar(1)=s(1)-p*alpha(1)

```
\(\operatorname{sbar}(2)=s(2)-p * \operatorname{alpha}(2)\)
\(\operatorname{sbar}(3)=s(3)-\mathrm{p} *\) alpha(3)
\(\operatorname{sbar}(4)=s(4)-p * \operatorname{alpha}(4)\)
\(\operatorname{sbar}(5)=s(5)-p * \operatorname{alpha}(5)\)
\(\operatorname{sbar}(6)=s(6)-p * a l p h a(6)\)
norm2=dot_vect(1,sbar,sbar,6)
yf_DM=dsqrt(norm2)-sqrt23*mm*p
return
end
subroutine xit_DM()
stop
return
end
subroutine inv_sig_full(sig,pp,qq,cos3t,I1,I2,I3)
implicit none
double precision \(\operatorname{sig}(6), \operatorname{sdev}(6)\)
double precision eta(6),eta_d(6),eta_d2(6)
double precision \(\mathrm{xmin} 1, \mathrm{xmin} 2, \mathrm{xmin} 3\)
double precision tretadev3,pp,qq, cos3t,I1,I2,I3
double precision
norm2,norm2sig,norm2eta,numer,denom
double precision half,one,two,three,six
double precision onethird,threehalves,sqrt6,tiny
double precision dot_vect
data half,one/0.5d0,1.0d0/
data two,three,six/2.0d0,3.0d0,6.0d0/
data tiny/1.0d-18/
onethird=one/three
threehalves=three/two
sqrt6=dsqrt(six)
\(\mathrm{I} 1=\operatorname{sig}(1)+\operatorname{sig}(2)+\operatorname{sig}(3)\)
\(\mathrm{pp}=\) onethird*I1
\(\operatorname{sdev}(1)=\operatorname{sig}(1)-p p\)
\(\operatorname{sdev}(2)=\operatorname{sig}(2)-p p\)
\(\operatorname{sdev}(3)=\operatorname{sig}(3)-p p\)
\(\operatorname{sdev}(4)=\operatorname{sig}(4)\)
\(\operatorname{sdev}(5)=\operatorname{sig}(5)\)
\(\operatorname{sdev}(6)=\operatorname{sig}(6)\)
if(I1.ne.0) then
    \(\operatorname{eta}(1)=\operatorname{sig}(1) / I 1\)
    \(\operatorname{eta}(2)=\operatorname{sig}(2) / I 1\)
    \(\operatorname{eta}(3)=\operatorname{sig}(3) / I 1\)
    \(\operatorname{eta}(4)=\operatorname{sig}(4) / I 1\)
    \(\operatorname{eta}(5)=\operatorname{sig}(5) / I 1\)
    \(\operatorname{eta}(6)=\operatorname{sig}(6) /\) I1
```

        else
        eta(1)=sig(1)/tiny
        eta(2)=sig(2)/tiny
        eta(3)=sig(3)/tiny
        eta(4)=sig(4)/tiny
        eta(5)=sig(5)/tiny
        eta(6)=sig(6)/tiny
        end if
        eta_d(1)=eta(1)-onethird
        eta_d(2)=eta(2)-onethird
        eta_d(3)=eta(3)-onethird
        eta_d(4)=eta(4)
        eta_d(5)=eta(5)
        eta_d(6)=eta(6)
        norm2=dot_vect(1,sdev,sdev,6)
        norm2sig=dot_vect(1,sig,sig,6)
        norm2eta=dot_vect(1,eta_d,eta_d,6)
        qq=dsqrt(threehalves*norm2)
        I2=half*(norm2sig-I1*I1)
    eta_d2(1)=eta_d(1)*eta_d(1)+eta_d(4)*eta_d(4)+eta_d(5)*eta_d(
5)
eta_d2(2)=eta_d(4)*eta_d(4)+eta_d(2)*eta_d(2)+eta_d(6)*eta_d(
6)
eta_d2(3)=eta_d(6)*eta_d(6)+eta_d(5)*eta_d(5)+eta_d(3)*eta_d(
3)
eta_d2(4)=eta_d(1)*eta_d(4)+eta_d(4)*eta_d(2)+eta_d(6)*eta_d(
5)
eta_d2(5)=eta_d(5)*eta_d(1)+eta_d(6)*eta_d(4)+eta_d(3)*eta_d(
5)
eta_d2(6)=eta_d(4)*eta_d(5)+eta_d(2)*eta_d(6)+eta_d(6)*eta_d(
3)
if(norm2eta.lt.tiny) then
cos3t=-one
else
tretadev3=dot_vect(1,eta_d,eta_d2,6)
numer=-sqrt6*tretadev3
denom=(dsqrt(norm2eta))**3
cos3t=numer/denom
if(dabs(cos3t).gt.one) then
1 7 0

```

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RESUME : In order to model the behavior of geometarials under complex loadings, several researches have done numerous experimental works and established relative constitutive models for decades. An important feature of granular materials is that the relationship between stress and strain especially in elastic domain is not linear, unlike the responses of typical metal or rubber. It has been also found that the stress-strain response of granular materials shows the characteristics of cross-anisotropy, as well as the non-linearities. Besides, the stress-induced anisotropy occurs expectedly during the process of disturbance on soils, for example, the loads or displacements. In this work, a new model which is a combination of Houlsby hyperelastic model and elastoplastic Plasol model was proposed. This new model took into account the non-linear response of stress and strain in both elastic and plastic domain, and the anisotropic elasticity was also well considered. Moreover, the overflow problems of plastic strain in plastic part was calibrated by a proper integration algorithm. Later, new model was verified by using numerical method and compared with laboratory experiments in axisymmetric triaxial conditions. The comparison results showed a good simulation effect of new model which just used one single set of parameters for a specific soil in different confining pressure situations. Then the analysis of new model internal variable, i.e., pressure exponent, illustrated that the value of pressure exponent which corresponds to the degree of anisotropy had an obvious effect on the stress-strain response. Moreover, this kind of effect is also affected by the density and drainage condition of samples. Basing on new model, a safety factor which refers to the second-order work criterion was adopted and tested in axisymmetric model and actual slope model. It showed that the negative value or dramatic decreasing of global normalized second-order work occurs accompanying with a local or global failure with a burst of kinetic energy. This feature of second-order work can also be affected by the variable pressure exponent. At last, new model was also compared with an elastoplastic model which considers both anisotropic elastic and anisotropic dilatancy, i.e., modified SANISAND model. Both advantages and disadvantages were illustrated in the comparison results.

MOTS-CLÉS : Houlsby hyperelasticity; Plasol model; Granular materials; Axisymmetric model; Slope; Drainage conditions; Modified SANISAND model; Stress-induced anisotropy; Anisotropic elasticity

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