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New Methods in Forcing Iteration and Applications

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کیست که بتواند آتش بر کف دست نهد
و با یاد کوه های پر برف قفقاز خود را سرگرم کند
یا تیغ تیز گرسنگی را با یاد سفره های رنگارنگ کند کند
یا برهنه در برف دی ماه فرو غلتد
و به آفتاب تموز بیاندیشد
نه هیچ کس هیچ کس
چنین خطری را به چنان خاطره ای تاب نیاورد
از این که خیال خوبی ها درمان بدی ها نیست
بلکه صد چندان بر زشتی آنها می افزاید.

-ویلیم شکسپیر، ریچارد دوم، ترجمه حسین الهی قمشه‌ای

O, who can hold a fire in his hand
By thinking on the frosty Caucasus?
Or cloy the hungry edge of appetite
By bare imagination of a feast?
Or wallow naked in December snow
By thinking on fantastic summer's heat?
O, no! the apprehension of the good
Gives but the greater feeling to the worse:
Fell sorrow's tooth doth never rankle more
Than when he bites, but lanceth not the sore.

-William Shakespeare, Richard II

Delneveshteh

A curtain is falling while another one going up, new players step into the scene, novel stories are awaiting to be narrated. Whose turn is it? Who are the spectators? Those who know the answers need not to play, the ones do not know are doomed to go on forever while carrying a cumbersome burden on shoulders. What is being considered as an achievement is nothing but a hoax to deny the pain of burdens. The hoax is named success by people who want to call themselves persistent. Now what if one wants not to carry on? Any compensation? Well, you either move simply to a more complicated game or disappear forever. Though we all vanish in the dark eventually, there is always a bigger game, here or there, a bigger story in another universe that mention us, you and me. We are not persistent, we are the gamblers who gamble with ourselves and bet on everything, everyday, everywhere, and what we win is nothing meaningful, but the very thing we call it life.

The mathematics if not more sophisticated than life, I confess, is not less. The analogy mentioned above is still valid to my eyes nevertheless I am about, with all regrets, to tell you part of my story here, with pictures being polished and burdens being eliminated. What are included in the forthcoming chapters whether being called success or failure, pain or pain killer, are the outcome of forty three months mathematical journey under the eyes of Boban Veličković, whom I would like to express my deepest and sincerest acknowledgment and regards. Thank you Boban for all the invaluable things you taught me, for taking my hand like a kid's in the dry desert of forcings with side conditions. Thank you for pulling me back from following the mirages promising me a paradise; thank you for putting my feet on the earth when my dreams were to fly me up to the stars shining brightly above.

In the course of my PhD, I had chance to meet or become friend with many wonderful people in Paris, in particular in the Sophie Germain building. My mind is currently too far from the universe, and so I am afraid that my memory cannot compute the long list of your names correctly, so I thank you all at once; thank you very much for being there!

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My indescribable thanks go to my father and my beloved sister Azadeh, and to Fati's parents as well; I have been feeling your love constantly during the days passed far from you. While writing this, I was thinking how marvelous would it be if I had been able to dedicate the rest about you, Fati, but miserably, the words will shatter under my love for you. I can thereby do nothing, but leave this Delneveshteh open to let all the potential blank pages ahead be reserved for my true love and my thanks to you

Abstract

This thesis concerns forcing iterations using virtual models as side conditions. The ultimate goal of such techniques is to achieve a higher forcing axiom. In the first chapter, we present the necessary materials, including definitions and lemmata for the later chapters. The chapter two contains the scaffolding poset which is a warm-up for the later constructions. The notion of a virtual model and its properties are introduced and investigated extensively in the third chapter, where we also study how the virtual models of different types interact. We then introduce, in the fourth chapter, the forcing notion consisting of pure side conditions which are finite sets of countable virtual models and Magidor models. In the chapter five, we plug forcings in our construction from the fourth chapter to form an iteration using virtual models, we analyze properties of our iteration and its quotients by Magidor models such as the ω_1-approximation. The iteration indeed gives a forcing axiom for a certain class of proper forcings which is compatible with $2^{\aleph_0} > \aleph_2$. The chapter six is devoted to the study of guessing models and their specialization, we introduce certain combinatorial principles in terms of guessing models which can be considered as consequences of a higher forcing axiom. We shall show their consistency and state their consequences concerning the approachability ideal, Abraham's maximality principle etc.

Keywords

Approachability ideal, Approximation property, Forcing axiom, Forcing iteration, Guessing model, Higher forcing axiom, Magidor model, Maximality principle, Supercompact cardinal, and Virtual model.

Nouvelles Méthodes d'Itération de Forcing et Applications

Résumé

Dans cette thèse on considère l'itération de forcing en utilisant des modèles virtuels comme conditions latérales. Le but ultime de ces techniques est de trouver un axiome de forcing supérieur. Dans le premier chapitre, nous présentons les matériaux nécessaires, y compris les définitions et les lemmes pour les chapitres suivants. Le deuxième chapitre contient quelques constructions avec des conditions latérales appelées scaffolding poset; c'est un échauffement pour les constructions compliquées des chapitres suivants. La notion de modèle virtuel et ses propriétés sont introduites et étudiées en détail dans le troisième chapitre, où nous étudions également la manière dont les modèles virtuels de différents types interagissent. Nous introduisons ensuite dans le chapitre quatre la notion de forcing qui consiste à les conditions latérales pures qui sont des ensembles finis de modèles dénombrables et de modèles Magidor. Dans le chapitre cinq, nous avons intégré des forcings dans la construction du chapitre quatre pour former une itération, nous analysons les propriétés de l'itération et de ses quotients par des modèle Magidors, par exemple la propriété de ω_1-approximation. L'itération donne en effet un axiome de forcing pour une certaine classe de forcings propres qui est compatible avec $2^{\aleph_0} > \aleph_2$. Le dernier chapitre est consacré à l'étude des modèles d'estimation, nous introduisons certains principes combinatoires en termes de modèles de devinettes qui peuvent être considérés comme les conséquences d'un axiome de forcing supérieur. Nous montrons leur cohérence et énonçons leurs conséquences concernant l'idéal des points approchables, le principe de maximalité d'Abraham etc.

Mots-clés

Idéal des points approchables, Propriété d'approximation, axiome de forcing, itération de forcing, Guessing model, axiome de forcing supérieur, Modèle Magidor, Principe de Maximalité, Cardinal supercompact, et Modèle virtuel.

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Introduction

Motivation

In the late 60s and early 70s, the seminal contributions of Martin, Solovay and Tennenbaum [20, 32] towards the Suslin Hypothesis uncovered two closely related phenomena in Set Theory, *forcing axioms* and *forcing iterations*. Martin's axiom (MA), the Proper Forcing Axiom (PFA) and Martin's Maximum (MM) are celebrated examples of forcing axioms. Martin's Maximum gives a very satisfactory and detailed structural analysis of H_{ω_2}. It is very desirable to move on to H_{ω_3} or even bigger initial segments of the set theoretical universe. This is the main motivation for studying higher forcing axioms. The main problem we encounter in this project has to do with our ability to iterate forcings in a large class of forcings while staying in the same class. The main techniques of iterating forcing are countable support and revised countable support iterations, both developed by Shelah [31] over the last 30 years. These iterations preserve ω_1 but not necessarily ω_2 and the strong forcing axioms (e.g. PFA and MM) one obtains typically imply that the continuum is equal to ω_2. It is an outstanding open problem to develop analogs of these iteration techniques which allow us to preserve ω_1, ω_2, or even more, and thus obtain higher cardinal versions of strong forcing axioms. In this thesis we aim to step in this research landscape.

Forcing with Side Conditions

Using models as side conditions in forcing constructions dates back to the work of Todorčević [34] in the 80th. Since then set theorists employed this idea to make nice forcing constructions, notably Mitchell [22] and Friedman [13] made independently a breakthrough in Set Theory by showing that one can add clubs in $\aleph_2$ with finite conditions. In 2014, Neeman [27] introduced set theorists to an elegant and general framework to utilize models as side conditions in forcing constructions. In particular, he could reprove the consistency of PFA via an iterated forcing with finite support. This prompted experts to speculate that might a higher forcing axiom exist. His approach allows us to decouple the preservation of cardinals from the size of the

supports in the iteration and reduce the problem to finding a suitable condition on the type of chains of models one uses in the side condition. This approach can be useful not only for iterating proper forcings but also for adding large objects by a single forcing.

In 2015, Veličković discovered the notion of a virtual model and employed those models as side conditions to build an iteration of semi-proper forcings with finite supports. Virtual models are very powerful and seem to be very well-behaved in the design of forcing iterations. Using this type of models as side conditions allows us not only to generalize Neeman's iteration theory to semiproper forcing, but also to formulate and prove iteration theorems for certain classes of forcing notions preserving two uncountable cardinals, such as ω_1 and ω_2. This theory was presented in [37] and [38]. In this thesis we adapt the pure side condition forcing from [38] to two types of models, but replacing models of size ω_1 by models having a strong closure property that we call Magidor models (see Definition 3.2.16), or more precisely κ-Magidor models, where κ is a supercompact cardinal. We then use the forcing with pure side conditions as the skeleton of our iteration.

Suppose that $\mathfrak{A}$ is a transitive model satisfying ZFC, e.g $\mathfrak{A} = (V_\lambda, \in, \dots)$, where λ is an inaccessible cardinal. A virtual model (see Definition 3.2.3) is, roughly speaking, an elementary submodel of an elementary extension of some initial segment of $\mathfrak{A}$, and thus they are not, in general, elementary in $\mathfrak{A}$, but they are correct about elementarity to some extent. There are two central notions in the theory of virtual models, namely $\in_\alpha$ and $\simeq_\alpha$ which replace the ordinary $\in$-relation and isomorphism, where α is an ordinal in $\mathfrak{A}$ so that $\mathfrak{A}_\alpha \prec \mathfrak{A}$. This construction is too broad in the sense that we can use different hierarchies of the universe or require more properties about virtual models, once it is constructed we can think about the properties needed for the $\in_\alpha$-chains, such as closure properties, or decorations etc. This is the main subject of §3, where we introduce the notion of a virtual model, and state a number of properties of them, mostly those properties we need for our applications. We also introduce the notion of a κ-Magidor model inspired by Magidor's characterization of a supercompact cardinal κ. In order to require some degree of closure in side condition components, we introduce the notion of the meet between a countable model and κ-Magidor models (see Definition 3.2.26) to compensate the absence of the intersections, and simply in a few words, we prepare the ground for our later forcing constructions.

In §4, we give the definition of the pure side condition forcing whose conditions are finite collections of virtual models closed under the meets, and so that at each

level α of the construction they form an $\in_\alpha$-chain. We will be also able to decorate these conditions in order to add clubs to ω_2 or in general to add continuous objects in generic extensions. This device, introduced by Neeman [27], consists of attaching to each model M of an $\in$-chain a finite set $d_p(M)$ which belongs to all models N of the chain such that $M \in N$. In a stronger condition this finite set is allowed to increase. The main point is that $d_p(M)$ controls what models can be added $\in$-above M in stronger conditions. In our situation there are some complications since we have not $\in$-chains, but certain $\in_\alpha$-chains. Moreover, decoration should not intervene in the α-isomorphism. This imposes a subtle interplay between the decorations on different levels.

We are then ready to construct the building upon our skeleton in §5. We focus on the iteration where we use the pure side condition forcing obtained in §4 to iterate certain proper forcings. In [4], Asperó and Mota defined the notion of an $\aleph_{1.5}$-c.c forcing, and studied the consistency of a generalized Martin's axiom called $\mathrm{MA}^{1.5}_\lambda$. One important consequence of this forcing axiom is the failure of Moore's $\mho$ principle. The principle $\mho$ states that there is a sequence $(f_\xi)_{\xi\in\omega_1}$ of continuous functions $f_\xi : \xi \to \omega$ such that for every club $C \subseteq \omega_1$, there is some $\xi \in C$ so that for each $k < \omega$, $f_\xi^{-1}(k) \cap C$ is unbounded in $C \cap \xi$. The weak variations of this principle are $\mho_n$, where $n \geq 2$ is a natural number, which are obtained similarly by replacing ω with n. It is not known if the failure of the weakest one, namely $\mho_2$, is consistent with continuum bigger than $\aleph_2$. In this thesis we shall introduce the notion of a structured $\aleph_{1.5}$-c.c forcing (Definition 1.4.4) where instead of arbitrary finite collections of countable models in the definition of an $\aleph_{1.5}$-c.c forcing, we allow it contains models of size $\aleph_1$, but the collections form $\in$-chains that are closed under intersection. We show that the corresponding forcing axiom is consistent (see Theorem 5.1.20). In fact we start with large cardinals $\kappa < \lambda$ and iterate structured $\aleph_{1.5}$-c.c forcings using virtual models of two types as side conditions. It is shown that the resulting forcing is proper and λ-c.c, preserves κ and makes $\kappa = \aleph_2$ and $\lambda = \aleph_3$. Furthermore, if each iterant has the ω_1-approximation property, then each level of the iteration has also the ω_1-approximation property that is if every countable subset of a given set of ordinals in generic extensions is in the ground model, then that set is also in the ground model. In certain applications such as those in §6 we would like to transfer a property of large cardinals to small cardinals, we have to analyze certain quotients of the construction. One advantage of forcing with side conditions and in particular, of our construction is that we know what the quotients are. Let us illustrate the situation by an example. For example the κ-Magidor models have

the strongest approximation property and we cannot expect such a property for accessible cardinals, but yet there are weaker properties like the ω_1-approximation property which can be satisfied by accessible cardinals. In order to obtain this sort of properties in the final extension, we have to go to an intermediate model where κ-Magidor models still have the desirable property there, simply because κ is not yet an accessible cardinal in such an intermediate model, and then what remains is to show that the quotient does not kill that property. This nontechnical explanation is called quotient by a Magidor model.

Guessing models

In [42] C. Weiß formulated some combinatorial principles that capture the essence of some large cardinal properties, but can hold at small cardinals. These principles usually have two parameters, a regular uncountable cardinal κ and a cardinal $\lambda \geq \kappa$. Among them there are, in increasing strength, the principles $\mathrm{TP}(\kappa, \lambda)$, $\mathrm{ITP}(\kappa, \lambda)$, and $\mathrm{ISP}(\kappa, \lambda)$. We will write $\mathrm{P}(\kappa)$, if the property $\mathrm{P}(\kappa, \lambda)$ holds, for all $\lambda \geq \kappa$. The study of these principles was continued by M. Viale and C. Weiß in [40]. Using them they obtained a striking result saying that any standard forcing construction of a model of the Proper Forcing Axiom (PFA) requires at least a strongly compact cardinal. One important concept that emerged from this work is that of a guessing model. These models have generated considerable interest and have a number of interesting applications, see for instance [39], [7], [8], and [35].

Given the interest of these principles, it is natural to ask if they can hold simultaneously at several successive regular cardinals. In this direction L. Fontanella [10] extended the previous work of U. Abraham [2] to obtain, modulo two supercompact cardinals, a model of ZFC in which $\mathrm{ITP}(\omega_2)$ and $\mathrm{ITP}(\omega_3)$ hold simultaneously. This was later extended by Fontanella [11] and Unger [36] who generalized the result of Cummings and Foreman [9] to get a model of ZFC in which $\mathrm{ITP}(\omega_n)$ holds, for all $n \geq 2$. Now, it was shown in [42] that $\mathrm{ISP}(\omega_2)$ is strictly stronger than $\mathrm{ITP}(\omega_2)$. In fact, in the model constructed by B. König in [17], the principle $\mathrm{ITP}(\omega_2)$ holds, but $\mathrm{ISP}(\omega_2)$ fails. One can then ask if $\mathrm{ISP}(\omega_2)$ and $\mathrm{ISP}(\omega_3)$ can hold simultaneously, or more generally if there is a models of ZFC in which the principles $\mathrm{ISP}(\omega_n)$ hold, for all $n \geq 2$. Let us point out that in [35] Trang showed the consistency of $\mathrm{ISP}(\omega_3)$. However, in his model CH holds, and therefore the principle $\mathrm{ISP}(\omega_2)$ fails.

One concept closely related to the above principles is that of the approachability property on a regular uncountable cardinal λ and the associated ideal $I[\lambda]$. These

notions were introduced by Shelah implicitly in [28], and studied by him extensively over the past 40 years. For instance, in [29] he showed that if λ is a regular cardinal then $S^{<\lambda}_{\lambda^+} \in I[\lambda^+]$, and in [30] he showed that if κ is regular and $\kappa^+ < \lambda$ then $I[\lambda]$ contains a stationary subset of S^κ_λ. Shelah then asked in [29] if it is consistent to have a regular λ such that $I[\lambda^+] \restriction S^\lambda_{\lambda^+}$ is the non stationary ideal on $S^\lambda_{\lambda^+}$. This major question was finally answered by W. Mitchell [23]. He started with a cardinal κ that is κ^+-Mahlo, and built an involved forcing construction yielding a model in which $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$. One feature of his construction is that it uses $\square_\kappa$ in the ground model, and so $\omega_3 \in I[\omega_3]$ in generic extensions. It is therefore unclear if Mitchell's method can be adapted to obtain a model in which both $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ and $I[\omega_3] \restriction S^{\omega_2}_{\omega_3}$ contain only non stationary sets. The connection with the principles introduced by Weiß is the following. If κ is a regular uncountable cardinal then $\mathrm{ISP}(\kappa^+)$ implies that there is a stationary subset of $S^\kappa_{\kappa^+}$ that is not in $I[\kappa^+]$, but it does not imply Mitchell's result. The main purpose of this thesis is to formulate and show the relative consistency of a principle that we call $\mathrm{GM}^+(\omega_3, \omega_1)$. This statement implies $\mathrm{ISP}(\omega_2)$ and $\mathrm{ISP}(\omega_3)$ and hence the tree property at ω_2 and ω_3. It also implies Mitchell's result, namely that $I[\omega_2] \restriction S^{\omega_1}_{\omega_2} = \mathrm{NS}_{\omega_2} \restriction S^{\omega_1}_{\omega_2}$ (see Proposition 6.1.5). The last chapter of the thesis is devoted to the study of the above-mentioned principles and their variations. Let us state our first theorem in this chapter Theorem 6.2.1.

Theorem. *Suppose that $\kappa < \lambda$ are supercompact cardinals. Then in a generic extension $\kappa = \omega_2$, $\lambda = \omega_3$, and $\mathrm{GM}^+(\omega_3, \omega_1)$ holds.*

In fact for Mitchell's result we do not need the full strength of the principle $\mathrm{GM}^+(\omega_3, \omega_1)$, a weaker principle $\mathrm{FS}(\omega_2, \omega_1)$ suffices. In order to obtain $\mathrm{FS}(\omega_2, \omega_1)$ it is enough to assume that λ is just inaccessible.

In [8], Cox and Krueger studied the indestructible version of $\mathrm{ISP}(\omega_2)$ or in our terminology $\mathrm{GM}(\omega_2, \omega_1)$. They showed that it is consistent (even with the continuum being arbitrarily large) that for every θ sufficiently large there are stationary many indestructible ω_1-guessing models of size $\aleph_1$ in H_θ. They also gave some applications of their principle in particular the fact that it implies the Suslin Hypothesis and hence it does not follow from $\mathrm{GM}(\omega_2, \omega_1)$. To each ω_1-guessing models of size $\aleph_1$, they correspond a tree of size and height ω_1 with ω_1 many branches and show that the tree is weakly special if and only if the ω_1-guessing model is indestructible meaning that it remains ω_1-guessing in any generic extension with the same ω_1. We continue this line of research and study the specialization of internally club models. In fact

we show that the indestructible version of our $\mathrm{GM}^+(\omega_3, \omega_1)$, say $\mathrm{SGM}^+(\omega_3, \omega_1)$, is consistent modulo two supercompact cardinals, see Theorem 6.3.1.

Theorem. *Suppose that $\kappa < \lambda$ are supercompact cardinals. Then in a generic extension. $\kappa = \omega_2$, $\lambda = \omega_3$, and $\mathrm{SGM}^+(\omega_3, \omega_1)$ holds*

Todorčević showed in [33] that, under some reasonable cardinal arithmetic assumption, if every tree of size and height $\aleph_1$ with at most $\aleph_1$ cofinal branches is weakly special, then every forcing which adds a new subset of ω_1 whose initial segments belong to the ground model either collapses ω_1 or ω_2. Let us denote this consequence by $\mathrm{AMP}(\omega_1)$. This principle and its variations studied extensively by Golshani and Shelah in [15], where they showed, among other things, that for every regular cardinal κ, one can force a model of $\mathrm{AMP}(\kappa^+)$. However, it was not known if one can have it for several cardinals simultaneously. The following gives as a corollary that $\mathrm{AMP}(\omega_1)$ and $\mathrm{AMP}(\omega_2)$ are consistent simultaneously, see Theorem 6.1.16.

Theorem. *Suppose that $V \subseteq W$ are transitive models of* ZFC*. Assume in V, $\mathrm{SGM}^+(\omega_3, \omega_1)$, $2^{\aleph_0} < \aleph_{\omega_1}$ and $2^{\aleph_1} < \aleph_{\omega_2}$ hold. Suppose that W has a new subset of $\aleph_2^V$. Then either W contains a real which is not in V or some cardinal $\leq 2^{\aleph_1}$ in V is no longer a cardinal in W.*

We discussed briefly the contents of §3-§6. In the first chapter we introduce our notation and bring some lemmata and material needed for the rest of the thesis. In §2, we discuss two scaffolding posets of a forcing notion, one with chains of countable models as side conditions and the other with chains of two types models. This is the heart of our main constructions, and we expect it prepare the reader in dealing with virtual models in our forcing constructions. There are three appendices, the first one contains a diagram expressing the logical relation between the principles discussed in the last chapter, the second one is a forcing construction with finite conditions to force an instance of the Mapping Reflection Principle which somehow resembles the scaffolding construction from §2, and finally the last one contains open problems related to the content of this thesis.

Most of the results in this thesis are obtained in collaborations with the thesis supervisor Boban Veličković, [25, 24].

Chapter One

Preliminaries

This chapter is devoted to the basic materials we need in the chapters ahead. We first fix our notation.

We consider ZFC as our basic theory though we will work internally with some other set theories. By a model M we mean a set or a class such that $(M, \in)$ satisfies a sufficient fragment of ZFC. For a model M, we let $\overline{M}$ denote its transitive collapse and we let π_M be the collapse map. We let ORD denote the class of ordinals, and whenever M is a model, we let naturally $\mathrm{ORD}^{\mathrm{M}}$ denote the collection of ordinals of M. For a set of ordinals C, $\mathrm{otp}(C)$ denotes the order type of C. For a set X and a cardinal κ, we let $\mathcal{P}_\kappa(X)$ denote the set of all subsets of X with size less than κ. We say that a subset $\mathcal{S}$ of $\mathcal{P}_\kappa(X)$ is *stationary* if, for every function (algebra) $F : [X]^{<\omega} \to \mathcal{P}_\kappa(X)$, there exists $A \in \mathcal{S}$ closed under F (i.e $F(a) \subseteq A$ for every $a \in [A]^{<\omega}$). For regular[1] cardinals $\kappa < \lambda$, S^κ_λ denotes the set $\{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$ and NS_λ denotes the nonstationary ideal on λ. For a cardinal θ we let H_θ denote the collection of all sets whose transitive closure has size less than θ. If p and q are two forcing conditions, then by $p \leq q$ we mean p is stronger than q. If $\mathbb{P}$ and $\mathbb{Q}$ are two forcings, we say that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ and write $\mathbb{P} \subseteq_c \mathbb{Q}$ if $\mathbb{P}$ is a subset of $\mathbb{Q}$, and that every maximal antichain in $\mathbb{P}$ is a maximal antichain in $\mathbb{Q}$.

1.1 Approachability Ideal

The approachability property on a regular uncountable cardinal λ and the associated ideal $I[\lambda]$ were introduced by Shelah implicitly in [28], and studied by him extensively over the past 40 years. For instance, in [29] he showed that if λ is a regular cardinal

[1] In this thesis, the term "regular cardinal" always means regular infinite.

then $S^{<\lambda}_{\lambda^+} \in I[\lambda^+]$, and in [30] he showed that if κ is a regular cardinal such that $\kappa^+ < \lambda$ then $I[\lambda]$ contains a stationary subset of S^κ_λ. Let us first recall the relevant definitions from [29].

Definition 1.1.1. *Let λ be a regular cardinal. A* λ-approaching sequence *is a λ-sequence of bounded subsets of λ. If $\bar{a} = \langle a_\xi : \xi < \lambda \rangle$ is a λ-approaching sequence, we let $B(\bar{a})$ denote the set of all $\delta < \lambda$ such that there is a cofinal subset $c \subseteq \delta$ with the following properties.*

1. $\mathrm{otp}(c) < \delta$.
2. *for all $\gamma < \delta$, there exists $\eta < \delta$ such that $c \cap \gamma = a_\eta$.*

Definition 1.1.2. *Suppose that λ is a regular cardinal. Let $I[\lambda]$ be the ideal generated by NS_λ and the sets $B(\bar{a})$, for all λ-approaching sequences $\bar{a}$.*

It is straightforward to check that $I[\lambda]$ is a normal ideal on λ, but it may be non proper. $I[\lambda]$ is called the *approachability ideal* on λ. Shelah asked in [29] if it is consistent to have a regular λ such that $I[\lambda^+] \restriction S^\lambda_{\lambda^+}$ is the non stationary ideal on $S^\lambda_{\lambda^+}$. This major question was finally answered by W. Mitchell [23] in the affirmative from the optimal assumption.

Theorem 1.1.3 (Mitchell, [23])**.** *Assume that κ is a κ^+-Mahlo cardinal. Then there is a generic extension in which $I[\omega_2] \restriction S^{\omega_1}_{\omega_2} = \mathrm{NS}_{\omega_2} \restriction S^{\omega_1}_{\omega_2}$.*[2]

For convenience, let us define the Mitchell Property as follows.

Definition 1.1.4. *For a regular cardinal λ, the* Mitchell Property *at λ^+, denoted by $\mathrm{MP}(\lambda^+)$, holds if the approachability ideal on λ^+ is the nonstationary ideal modulo $S^\lambda_{\lambda^+}$.*

The following is a well-known fact.

Fact 1.1.5. *Suppose that λ is a regular cardinal.* $\mathrm{MP}(\lambda^{++})$ *implies* $2^\lambda \geq \lambda^{+++}$.

Proof. Suppose that $2^\lambda \leq \lambda^{++}$ and let $\bar{a} = \langle a_\xi : \xi < \lambda^{++} \rangle$ be an enumeration of $[\lambda^{++}]^{\leq\lambda}$. Then the set $B(\bar{a})$ belongs to $I[\lambda^{++}]$ by definition, but it is easy to see that the set of $\delta \in B(\bar{a})$ of cofinality λ^+ is stationary, which contradicts $\mathrm{MP}(\lambda^{++})$.

$\boxed{1.1.5}$

[2] It is worth mentioning that the large cardinal assumption is optimal. The result is is due to Mitchell and Shelah, independently. A proof can be found in [21].

1.2 Models and Approximation Property

Unless otherwise stated, we assume $|M| \subseteq M$ whenever $M \prec H_\theta$.

Definition 1.2.1. *Suppose that M is a model of size $\aleph_1$.*

1. *M is called an internally unbounded model (*IU-model*) if M is the union of an $\in$-increasing sequence $\langle M_\xi : \xi < \omega_1 \rangle$. Such a sequence is called an* IU-sequence *for M.*

2. *M is called an internally club model (*IC-model*) if there is a continuous* IU-*sequence for M. In this case, the sequence is called an* IC-sequence *for M.*

Notice that if M is a countable elementary submodel of H_θ, and $N \in M$ is an IC-model, then by elementarity there exists, in M, an IC-sequence $\langle N_\xi : \xi < \omega_1 \rangle$ for N. It is easy to see that $N \cap M = N_\delta$ where $\delta = M \cap \omega_1$. In particular, this shows that $N \cap M \in N$, and that one can always thin out such a sequence to obtain an IC-sequence consisting of elementary submodels of N.

For a set or class M we say that a set $x \subseteq M$ is *bounded in M* if there is $y \in M$ such that $x \subseteq y$. We now recall some relevant definitions from [39].

Definition 1.2.2. *Let γ be a regular cardinal. A set M is said to be* γ-guessing *if for any $x \subseteq M$ which is bounded in M, if x is γ-approximated in M i.e $x \cap a \in M$, for all $a \in M \cap \mathcal{P}_\gamma(M)$, then there is some $g \in M$ such that $x \cap M = g \cap M$.*

We also recall the concept of the γ-approximation property which was introduced by Hamkins in [16].

Definition 1.2.3. *Let γ be an uncountable regular cardinal. Suppose that M and N are transitive models (sets or classes), $M \subseteq N$ and $\gamma \in M$. We say the pair (M, N) satisfies the* γ-approximation property, *if the following are equivalent for every set $x \in N$ which is bounded in M.*

1. *$x \in M$.*

2. *$x \cap a \in M$, for every $a \in M$ with $M \models |a| < \gamma$.*

Definition 1.2.4. *Let γ be an uncountable regular cardinal. Suppose that M and N are transitive models (sets or classes), $M \subseteq N$ and $\gamma \in M$. We say the pair (M, N) satisfies the* γ-covering property, *if whenever $x \in N$ is bounded in M and $N \models |x| < \gamma$, there is some $y \in M$ such that $x \subseteq y$ and $M \models |y| < \gamma$.*

The above definitions are of independent interest if N is a generic extension of M.

Definition 1.2.5. *A forcing notion $\mathbb{P}$ has the γ-approximation property (γ-covering property) if for every V-generic filter $G \subseteq \mathbb{P}$ the pair $(V, V[G])$ has the γ-approximation property (γ-covering property).*

It is well-known that every proper forcing has the ω_1-covering property. It is not hard to see that if two pairs (V, W) and (W, Z) have the γ-covering property, then (V, Z) has the γ-covering property, if in addition (V, W) and (W, Z) have the γ-approximation property, then (V, Z) has the γ-approximation property. Quite recently, Krueger [18] proved that every forcing with the ω_1-approximation property has the ω_1-covering property. In fact, he proved the following nice result.

Proposition 1.2.6 (Krueger, [18])**.** *Suppose (V, W) be a pair of transitive models of "*ZFC *minus the power set axiom" with $V \subseteq W$. Let $\gamma \in V$ be an uncountable regular cardinal. Assume that for all W-cardinals $\mu < \gamma$, any subset of V which is a member of W and has W-cardinality less than μ is a member of V. If (V, W) has the γ-approximation property, then it has the γ-covering property.*

Let us call a transitive model R *powerful* if it is closed under taking subsets, i.e. if $x \in R$ and $y \subseteq x$ then $y \in R$. Cox and Krueger [7] observed that the ω_1-approximation property is connected to the notion of a guessing model.

Lemma 1.2.7. *Suppose R is a powerful model, and that $M \prec R$. Let also γ be a regular cardinal in M with $M \cap \gamma \subseteq M$. Then the following are equivalent.*

1. *M is a γ-guessing model.*
2. *The pair $(\overline{M}, V)$ has the γ-approximation property.*

1.2.7

We have the following useful corollary.

Corollary 1.2.8. *Suppose that R is a powerful model. Suppose that $M \prec R$ is an ω_1-guessing model. Assume that $\mathbb{P}$ is a forcing with the ω_1-approximation property. Then, M remains an ω_1-guessing model in generic extensions by $\mathbb{P}$.*

1.2.8

The above corollary says that the ω_1-guessing models are indestructible under forcings with the ω_1-approximation property, and hence it leads to the question that to what extent we can have such indestructibility. The notion of an indestructible guessing model was introduced and studied by Cox and Krueger in [8]. Recall that a tree $(T, <)$ of size and height ω_1 is *w*eakly special if there is a function $\sigma : T \to \omega$ such that if $\sigma(r) = \sigma(s) = \sigma(t)$ with $r < s, t$, then s and t are $<$-comparable. It is a well-known fact (see [5] and [6]) that if $(T, <)$ is a tree of size and height ω_1 with at most $\aleph_1$ many cofinal branches, then there is a c.c.c forcing $\mathbb{B}(T)$ of size $\aleph_1$ with the ω_1-approximation property so that T is weakly special in generic extensions by $\mathbb{B}(T)$. There is a stronger version of specialization which is suitable for trees without cofinal branches, that is a function $\sigma : T \to \omega$ so that if $s < t$, $\sigma(s) \neq \sigma(t)$, if such a function exists we say T is *s*pecial. Notice that if T has no cofinal branches then it is special if and only if it is weakly special. It turns out that if every tree of size and height $\aleph_1$ without cofinal branches is special, then every tree of size and height ω_1 with at most $\aleph_1$ many cofinal branches is weakly special. As a matter of fact, the above mentioned forcing $\mathbb{B}(T)$ can be sued to specialize a definable (modulo the enumeration of the cofinal branches through T) tree $T^* \subseteq T$ which has no cofinal branches i.e forcing with $\mathbb{B}(T^*)$ weakly specializes T. Notice that no forcing with the ω_1-approximation property is able to add a new cofinal branch through a tree of height ω_1. We need the following lemma from [18].

Lemma 1.2.9. *Suppose that R is a powerful model. If $M \prec R$ is an ω_1-guessing model of size $\aleph_1$, then M is internally unbounded.*

$\boxed{1.2.9}$

Definition 1.2.10. *Let M be a set. We set*

$$T_M = \{(Z, f) : Z \in M \text{ is uncountable and } f : Z \cap M \to 2\}.$$

Definition 1.2.11. *Suppose that M is an ω_1-guessing* IU*-model. Let $\langle M_\xi : \xi < \omega_1 \rangle$ be an* IU*-sequence. Let $T(M) = \bigcup_{\xi<\omega_1}(T_{M_\xi} \cap M)$. Let us define the ordering $\leq$ on $T(M)$ be letting $(Z, f) \leq (W, g)$ if and only if $Z = W$ and $f \subseteq g$.*

Notice that the definition of $T(M)$ depends on the IU-sequence, but we ignore this point as it will not lead to confusion.

Lemma 1.2.12. *Suppose that M is an ω_1-guessing* IU*-model. Then $(T(M), \leq)$ is a tree of size and height ω_1 with $\aleph_1$ cofinal branches.*

Proof. It is clear that $T(M)$ is of size $\aleph_1$ since $T(M) \subseteq M$. By the monotonicity of the IU-sequence and the fact that it is an $\in$-chain, the predecessors of any node in the tree forms a well-ordered set. On the other hand since the length of an IU-sequence is ω_1, $T(M)$ is of height ω_1. Any cofinal branch through $T(M)$ induces a countably approximated function f on $Z \cap M$ for some uncountable set $Z \in M$, but then since M is an ω_1-guessing, there is $\bar{f} \in M$ such that $\bar{f} \restriction_M = f$. There are $\aleph_1$ many such functions in M, and hence M has $\aleph_1$ cofinal branches. $\boxed{1.2.12}$

Definition 1.2.13 ([8])**.** *We say that an ω_1-guessing model M is* indestructibly ω_1-guessing *if the corresponding tree $T(M)$ is weakly special.*

To justify the term indestructibly ω_1-guessing model, we notice that if M is an indestructibly ω_1-guessing model in V, and that W is an extension of V with the same ω_1, then M remains ω_1-guessing in W, simply because there is a witness for the weak specialization of $T(M)$ in W.

Lemma 1.2.14. *Suppose that $V \subseteq W$ are two transitive models of* ZFC *with the same $\aleph_1$. Assume that M is an indestructibly ω_1-guessing model. Then M is a guessing model in W.*

Proof. We show that there can be no new cofinal branches through $T(M)$ in W. Working in W, suppose that $Z \in M$ and $f : Z \to 2$ is a function which is countably approximated in M. We shall show that $f \restriction_M$ is guessed in M. Let $t_\xi = (Z, f \restriction_{M_\xi})$. Since M is indestructibly guessing in V, there is a weak specialization σ on $T(M)$. Thus there are $n \in \omega$ and some unbounded set $S \subseteq \omega_1$ such that for every $\xi \in S$, $\sigma(t_\xi) = n$. Fix one $\xi \in S$. Working now in V, $f \restriction_M$ can be reconstructed from $\{t \in T(M) : t \geq t_\xi \text{ and } \sigma(t) = n\}$. Thus $f \restriction_M$ belongs to V. Now in V, M is an ω_1-guessing model and that $f \restriction_M$ is countably approximated in M, thus $f \restriction_M$ should be guessed in M. $\boxed{1.2.14}$

Let us now introduce a somewhat stronger version of the indestructibility of an ω_1-guessing model.

Definition 1.2.15. *Suppose that M is an* IC*-model. Let $(M_\xi)_{\xi<\omega_1}$ be an* IC*-sequence for M. Consider the above-defined tree $T(M)$ based on the sequence $(M_\xi)_{\xi<\omega_1}$. We say that M is a* special model *if there is a function $\sigma : T(M) \to \omega$ such that if $(Z, f) \leq (W, g)$ are in $T(M)$ with $\sigma(Z, f) = \sigma(W, g)$ and $(Z, f) \in T_{M_\xi}$ for some $\xi < \omega_1$, then f is guessed in M_ξ.*

We can consider such a function σ as a *s*pecializing function of M, and the former one as a *w*eak specializing function of M.

Lemma 1.2.16. *Suppose that M is a special model. Then M is ω_1-guessing.*

Proof. Suppose that σ is a specializing function of M. Let $f : Z \cap M \to 2$ be countably approximated in M, for some uncountable $Z \in M$. Let $t_\xi = (Z, f \restriction_{M_\xi})$. We have that $t_\xi \in T(M)$, for every $\xi < \omega_1$. Now, there exists $n \in \omega$ such that the set $S = \{\xi < \omega_1 : \sigma(t_\xi) = n\}$ is stationary. Since $t_\xi < t_\eta$ for every $\xi < \eta$ in S, there is, for each $\xi \in S$, some function $\bar{f}_\xi \in M_\xi$ such that $\bar{f}_\xi \restriction_{M_\xi} = f \restriction_{M_\xi}$, but then since M is an IC-model, by Fodor's lemma, there is a stationary set $T \subseteq S$ and a function $\bar{f} \in M$ so that for every $\xi \in T$, $\bar{f}_\xi = \bar{f}$. This easily implies that $f \restriction_M = \bar{f} \restriction_M$, and thus f is guessed in M. $\boxed{1.2.16}$

It is clear that if M is a special model, then in any extension with the same ω_1, it remains an ω_1-guessing model. The point is that, in contrast to indestructibility, we did not assume that M is a guessing model, however notice that if M is not an ω_1-guessing model, then $T(M)$ can have more than ω_1 cofinal branches. On the other hand, if M is not ω_1-guessing, it can never turn into an ω_1-guessing model in an outer extension, but let us point out that we can specialize an ω_1-guessing IC-model using the previously mentioned Baumgartner's forcing. Suppose that M is an ω_1-guessing model, and let $T^*(M) \subseteq T(M)$ be the tree consisting of $(Z, f) \in T_{M_\xi}$ such that f is not guessed in M_ξ. Then $T^*(M)$ has no cofinal branches as M is an ω_1-guessing model. If σ^* is a specialization function on $T^*(M)$, then it can induce a function $\sigma : T(M) \to \omega$ by letting $\sigma(Z, f) = 0$ if $(Z, f) \notin T^*(M)$, and $\sigma(Z, f) = \sigma^*(Z, f) + 1$ if $(Z, f) \in T^*(M)$. It is clear that σ is a witness for M being a special model. Thus, forcing with $\mathbb{B}(T^*(M))$, makes M a special model.

We now state a series of lemmata for later use.

Lemma 1.2.17. *Suppose that $\mathbb{P} \in H_\theta$ is a forcing notion with the ω_1-approximation property. Suppose that $M \prec H_\theta$ is countable and contains $\mathbb{P}$. Let $f : M \cap Z \to 2$ be a function which is not guessed in M, where $Z \in M$. If $p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, then*

$$p \Vdash \text{“}\check{f} \textit{ is not guessed in } M[\dot{G}]\text{.”}$$

Proof. Let G be a V-generic filter on $\mathbb{P}$ containing p. Assume towards a contradiction that $g \in M[G]$ is a function such that $g \restriction_{M[G]} = f$. Thus for each countable set $x \in M[G] \cap V = M$, $g \restriction_x = f \restriction_x \in V$, and hence by elementarity, g is countably

approximated in V, and thus g is in V by the ω_1- approximation property of $\mathbb{P}$. Since p is $(M, \mathbb{P})$-generic and belongs to the generic filter G, g should be in M, and that also $f \restriction_M = g \restriction_M$. Thus f is guessed in M, which is a contradiction. $\boxed{1.2.17}$

Lemma 1.2.18. *Suppose $\mathbb{P}$ is a forcing notion. Assume that $\dot{f}$ is a $\mathbb{P}$-name for a function form Z to 2. Suppose that M is a model containing $\dot{f}$ and Z. Let $p \in \mathbb{P}$ be an $(M, \mathbb{P})$-generic condition. Suppose that p forces that $\dot{f} \restriction_M = g$, for some function $g : M \cap Z \to 2$ in V. If g is guessed in M, then p decides $\dot{f}$.*

Proof. Suppose that $h \in M$ is a function such that $h \restriction_M = g$, we shall show that $p \Vdash \dot{f} = \check{h}$. Thus fix $q \leq p$. Assume towards a contradiction that there is no condition below q forcing $\dot{f} = \check{h}$. Thus the following set is pre-dense below q which belongs to M.

$$D = \{r \in \mathbb{P} : \exists \zeta \in Z \text{ such that } r \Vdash \check{h}(\zeta) \neq \dot{f}(\zeta)\}.$$

Since q is $(M, \mathbb{P})$-generic, we can find $r \in D \cap M$ compatible with q. Thus there exists $\zeta \in M \cap Z$ such that $r \Vdash \dot{f}(\zeta) \neq \check{h}(\zeta) = \check{g}(\zeta)$ which is contradictions as r is compatible with q and that q forces $\dot{f}(\zeta) = \check{g}(\zeta)$. $\boxed{1.2.18}$

Lemma 1.2.19. *Suppose that $M \prec H_\theta$ is countable, and $N \in M$ is an ω_1-guessing model. Assume that $f : M \cap Z \to 2$ is a function for some $Z \in N \cap M$ such that $f \restriction_{N \cap M}$ is in N. If f is guessed in M, then $f \restriction_{N \cap M}$ is guessed in $N \cap M$.*

Proof. Suppose $\bar{f} : Z \to 2$ in M is such that $\bar{f} \restriction_M = f \restriction_M$. For every countable set $z \in N \cap M$, $\bar{f} \restriction_z = f \restriction_z \in N \cap M$ since $f \restriction_{N \cap M} \in N$ and $\bar{f} \in M$. Therefore, f is countably approximated in N by the elementarity of M. Using the elementarity once more, there exists a function $g \in N \cap M$ such that $g \restriction_N = \bar{f} \restriction_N$, but it then implies that $g \restriction_{N \cap M} = f \restriction_{N \cap M}$, and therefore, $f \restriction_{N \cap M}$ is guessed in $N \cap M$. $\boxed{1.2.19}$

We now state a crucial lemma which reminisces the crux of the Baumgartner's argument in [5], regarding the countable chain condition of his forcing for specializing trees of size and height $\aleph_1$ without cofinal branches.

Lemma 1.2.20. *Assume that $M \prec H_\theta$ is countable, and that Z is a set in M. Suppose that $z \mapsto f_z$ is a function on $[Z]^\omega$ in M such that f_z is a function with $z \subseteq \mathrm{dom}(f_z)$. Assume that $N \in M$ is an ω_1-guessing model containing Z. Let $f : N \cap M \cap Z \to 2$ be a function belonging to N which is not guessed in $N \cap M$. Let $B \in M$ be a cofinal subset of $[Z]^\omega$. Then there is a set $B^* \in M$ cofinal in B such that for every $z \in B^*$, $f_z \nsubseteq f$.*

Proof. For each $\zeta \in N \cap Z$, let

$$A^{\epsilon}_{\zeta} = \{z \in B : f_z(\zeta) = \epsilon\}, \text{ where } \epsilon = 0, 1.$$

Notice that the sequence

$$\langle A^{\epsilon}_{\zeta} : \zeta \in N \cap Z, \epsilon \in \{0, 1\}\rangle$$

belongs to M. We are done if there is some $\zeta \in Z \cap N$ such that both A^0_{ζ} and A^1_{ζ} are cofinal in B, as then one can find such $\zeta \in N \cap M \cap Z$, and pick $A^{1-f(\zeta)}_{\zeta}$. Therefore, let us assume that for every $\zeta \in N \cap Z$, there is an $\epsilon \in \{0, 1\}$, which is necessarily unique, such that A^{ϵ}_{ζ} is cofinal in B. Define now h on $N \cap Z$ by letting $h(\zeta)$ be ϵ if and only if A^{ϵ}_{ζ} is cofinal in $[Z]^{\omega}$. Clearly h is in M. Lemma 1.2.19 makes sure that f is not guessed in M, and thus $h \restriction_M \neq f$. Thus, there exists $\zeta \in N \cap M \cap Z$ such that $h(\zeta) \neq f(\zeta)$, but it then implies that $A^{1-f(\zeta)}_{\zeta}$ is cofinal in B and belongs to M. Let B^* be $A^{1-f(\zeta)}_{\zeta}$. $\boxed{1.2.20}$

1.3 Strong Properness

Recall that if p is a condition in a forcing $\mathbb{P}$, and A is a set, then p is said to be $(A, \mathbb{P})$-generic, if for every $q \leq p$ and every dense set $D \subseteq \mathbb{P}$ in A, there is some $r \in A \cap D$ such that q and r are compatible.

Definition 1.3.1 ($\mathcal{S}$-properness)**.** *Let $\mathbb{P}$ be a forcing notion, and $\mathcal{S}$ a collection of sets. We say that $\mathbb{P}$ is* $\mathcal{S}$-proper *if, for every $A \in \mathcal{S}$ and $p \in A \cap \mathbb{P}$, there is $q \leq p$ that is $(A, \mathbb{P})$-generic.*

Suppose that θ is a sufficiently large regular cardinal and $\mathbb{P}$ is a forcing in H_θ. Now, if $\mathcal{S}$ is a collection of elementary submodels of H_θ, then the above notion is useful if at least $\mathcal{S}$ is unbounded, it is much more useful if $\mathcal{S}$ is stationary, and finally it is the most useful one if S contains a club, that is $\mathbb{P}$ is proper.

Definition 1.3.2. *A forcing notion $\mathbb{P}$ is called* $\aleph_1$-proper *if for every sufficiently large θ, there is some algebra F on H_θ such that $\mathbb{P}$ is $\mathcal{F}$-proper, where*

$$\mathcal{F} = \{M \prec H_\theta : \mathrm{cl}_F(M) = M, \text{ and } |M| = \aleph_1\}.$$

The notion of strong properness, introduced by Mitchell in [22], plays a key role in our constructions later. Let us recall the following definition.

Definition 1.3.3. *Let $\mathbb{P}$ be a forcing notion and A a set. We say that $p \in \mathbb{P}$ is* $(A, \mathbb{P})$-strongly generic *if for all $q \leq p$ there is a condition $q \restriction_A \in A$ such that any $r \leq q \restriction_A$ with $r \in A$ is compatible with q.*

Definition 1.3.4 (Strong properness)**.** *Let $\mathbb{P}$ be a forcing notion, and $\mathcal{S}$ a collection of sets. We say that $\mathbb{P}$ is* $\mathcal{S}$-strongly proper *if, for every $A \in \mathcal{S}$ and $p \in A \cap \mathbb{P}$, there is $q \leq p$ that is $(A, \mathbb{P})$-strongly generic.*

A forcing notion is called strongly proper, if it is $\mathcal{C}$-strongly proper, where for some sufficiently large regular cardinal θ, $\mathcal{C}$ is a club in $\mathcal{P}_{\omega_1}(H_\theta)$.

If $\mathbb{P} \in M \prec H_\theta$, it is then clear that every $(M \cap \mathbb{P}, \mathbb{P})$-strongly generic condition is $(M, \mathbb{P})$-generic, and that every strongly proper forcing is proper. It turns out that if M is sufficiently closed then every $(M, \mathbb{P})$-generic condition is $(M, \mathbb{P})$-strongly generic condition. We learned the following fact from [14].

Proposition 1.3.5. *Suppose that M is an ω-guessing model which is sufficiently elementary in some transitive model A. Suppose that $\mathbb{P} \in M$ is a forcing. Let $p \in \mathbb{P}$. Then, p is $(M, \mathbb{P})$-strongly generic if and only if it is $(M, \mathbb{P})$-generic.*

Proof. For the nontrivial implication, assume towards a contradiction that there is $q \leq p$ such that for every $r \in \mathbb{P} \cap M$, there is $s \in M$ with $s \leq r$ such that q is incompatible with s. Therefore the following set is dense in $\mathbb{P} \cap M$.

$$D = \{s \in M \cap \mathbb{P} : s \perp q\}.$$

Since $D \subseteq M$ and that M is ω-guessing, there is $D^* \in M$ such that $D^* \cap M = D$. By elementarity, D^* is dense in $\mathbb{P}$. Since q is $(M, \mathbb{P})$-generic, there exists $s \in M \cap D^*$, and hence in D, such that q and s are compatible, but this is impossible by the definition of D. A contradiction! $\boxed{1.3.5}$

We have also the following proposition which connects the approximation property with strong properness.

Proposition 1.3.6. *Let $\mathbb{P}$ be a forcing notion, and let κ be an uncountable regular cardinal. Suppose that $\mathbb{P}$ is $\mathcal{S}$-strongly proper, for some stationary $\mathcal{S} \subseteq \mathcal{P}_\kappa(\mathbb{P})$. Then $\mathbb{P}$ has the κ-approximation property.*

Proof. Work in V. Let α be an ordinal, $\dot{X}$ a $\mathbb{P}$-name, and suppose some condition $p \in \mathbb{P}$ forces that $\dot{X} \subseteq \alpha$ and $\dot{X} \cap \check{Z} \in V$, for all $Z \in V$ with $|Z|^V < \kappa$. Fix a sufficiently large regular cardinal θ. By the stationarity of $\mathcal{S}$, we can find $M \prec H_\theta$,

of size less than κ, containing $p, \mathbb{P}, \dot{X}$, and such that $M \cap \mathbb{P} \in \mathcal{S}$. Let $q \leq p$ be $(M \cap \mathbb{P}, \mathbb{P})$-strongly generic. Since $M \cap \mathbb{P}$ is of size $< \kappa$, by strengthening q if necessary, we may assume that q decides $\dot{X} \cap M$. Since $q \restriction_{(M \cap \mathbb{P})}$ and p are compatible, and M is elementary, they are compatible in M. Therefore, by replacing $q \restriction_{(M \cap \mathbb{P})}$ by a stronger condition in M, we may assume that it extends p. We now argue that $q \restriction_{(M \cap \mathbb{P})}$ decides $\dot{X}$. Otherwise, by elementarity of M, we can find $\xi \in \alpha \cap M$ and $r_0, r_1 \in M$ with $r_0, r_1 \leq q \restriction_{(M \cap \mathbb{P})}$ such that r_0 forces $\xi \in \dot{X}$ and r_1 forces $\xi \notin \dot{X}$. Now, by the $(M \cap \mathbb{P}, \mathbb{P})$-strong genericity of q, we have that r_0 and r_1 are both compatible with q. Let s_0 be a common extension of q and r_0, and s_1 a common extension of q and r_1. Then $s_0, s_1 \leq q$ and force contradictory information about $\xi \in \dot{X}$. This contradicts the fact that q decides $\dot{X} \cap M$. $\boxed{1.3.6}$

1.4 Forcing Axioms

Let us say what we mean by a forcing axiom.

Definition 1.4.1. *Fix a class of forcing notions $\mathfrak{K}$ and an infinite cardinal κ. A* forcing axiom *is a set theoretical statement* $\mathrm{FA}_\kappa(\mathfrak{K})$ *which states that for every forcing $\mathbb{P} \in \mathfrak{K}$ and every collection $\mathscr{D}$ of dense subsets of $\mathbb{P}$ with $|\mathscr{D}| \leq \kappa$, there is a $\mathscr{D}$-generic filter G on $\mathbb{P}$, i.e $G \cap D \neq \varnothing$, for every $D \in \mathscr{D}$. The forcing axiom* $\mathrm{FA}_{<\kappa}(\mathfrak{K})$ *is defined naturally.*

We shall use $\mathfrak{K}_\bullet$, where $\bullet$ expresses in a clear way a property of forcing notions, to point out that the class $\mathfrak{K}_\bullet$ consist of all forcings which have the property $\bullet$. Thus $\mathrm{FA}_{\aleph_1}(\mathfrak{K}_{\text{proper}})$ and $\mathrm{FA}_\kappa(\mathfrak{K}_{\text{ccc}})$ refer to PFA and MA_κ, respectively. In [4], Asperó and Mota studied certain generalisations of Martin's axiom. They introduced the notion of an $\aleph_{1.5}$-c.c forcing notion which is proper and lies between c.c.c-ness and $\aleph_2$-c.c-ness. Their motivation behind this notion was to get the consistency of certain consequences of PFA together with arbitrarily large continuum.

Definition 1.4.2 ([4])**.** *Suppose $\mathbb{P}$ is a forcing in H_κ, where κ is a sufficiently large regular cardinal. We say $\mathbb{P}$ has the* $\aleph_{1.5}$-chain condition *(or is* $\aleph_{1.5}$-c.c *for short) if and only if there is a club $\mathcal{D} \subseteq \mathcal{P}_{\omega_1}(H_\kappa)$ consisting of elementary submodels of H_κ such that for every finite set $\mathcal{M} \subseteq \mathcal{D}$ and every $p \in \mathbb{P}$, if $p \in M$ for some $M \in \mathcal{M}$ with $M \cap \omega_1 = \min(N \cap \omega_1 : N \in \mathcal{M})$, then there is some condition extending p which is $(N, \mathbb{P})$-generic, for every $N \in \mathcal{M}$.*

We weaken $\aleph_{1.5}$-c.c-ness to a property which is rather structural.

Notation 1.4.3. *Suppose X is in H_θ. Let $\mathbb{M}_2(X,\theta)$ consist of finite $\mathcal{M}$ of elementary submodels of H_θ such that:*

1. *$\mathcal{M}$ is an $\in$-chain meaning that for every distinct $M, N \in \mathcal{M}$, either $M \in N$, or $N \in M$, or there is some $P \in \mathcal{M}$ such that either $M \in P \in N$ or $N \in P \in M$.*
2. *Every model in $\mathcal{M}$ has size at most $\aleph_1$ and contains X.*
3. *$\mathcal{M}$ is closed under intersections i.e if $N \in M$ are in $\mathcal{M}$, where N is of size $\aleph_1$, and M is countable, then $N \cap M$ belongs to $\mathcal{M} \cap N$.*

Definition 1.4.4. *A forcing $\mathbb{P}$ has the* structured $\aleph_{1.5}$- chain condition *(or is $\aleph^s_{1.5}$-c.c for short), if for every sufficiently large regular cardinal κ with $\mathbb{P} \in H_\kappa$ and every $\mathcal{M} \in \mathbb{M}_2(\mathbb{P},\kappa)$, if p is in $\mathbb{P} \cap M$, for some countable $M \in \mathcal{M}$ with $M \cap \omega_1 = \min(N \cap \omega_1 : N \in \mathcal{M})$, then there is a condition extending p which is $(N,\mathbb{P})$-generic for every countable model $N \in \mathcal{M}$.*

It is easily seen that every $\aleph_{1.5}$-c.c forcing is $\aleph^s_{1.5}$-c.c. We have also the following as in [4].

Lemma 1.4.5. *Every $\aleph^s_{1.5}$-c.c forcing is $\aleph_2$-c.c.*

Proof. Suppose $\mathbb{P}$ is an $\aleph^s_{1.5}$-c.c. forcing. Pick a sufficiently large regular cardinal κ such that $\mathbb{P} \in H_\kappa$. Assume towards a contradiction that $\mathbb{P}$ is not $\aleph_2$-c.c, and fix a maximal antichain $A \subseteq \mathbb{P}$ with $|A| \geq \aleph_2$. Let $N \prec H_\kappa$ be an IC-model with $A, \mathbb{P} \in N$. Thus there is some $p \in A \setminus N$. Pick a countable $M \prec H_\kappa$ such that $A, \mathbb{P}, N, p \in M$. Then $\mathcal{M} = \{N \cap M \in N \in M\}$ is in $\mathbb{M}_2(\mathbb{P},\kappa)$, and p belongs to M and $M \cap \omega_1 = N \cap M \cap \omega_1$. By $\aleph^s_{1.5}$-c.c-ness, there is some $q \leq p$ which is $(N \cap M, \mathbb{P})$-generic, thus there is some $r \in A \cap N \cap M$, compatible with q, but since A is an antichain and $q \leq p$, we should have that $r = p$, and thus $p \in N$, which is a contradiction. $\boxed{1.4.5}$

Remark 1.4.6. Using the above lemma, one can reformulate the $\aleph^s_{1.5}$-c.c-ness as follows. Suppose $\mathbb{P}$ is a forcing notion. Then $\mathbb{P}$ is $\aleph^s_{1.5}$-c.c if and only if for every sufficiently large regular cardinal κ with $\mathbb{P} \in H_\kappa$, every $\mathcal{M} \in \mathbb{M}_2(\mathbb{P},\kappa)$, if p is in $\mathbb{P} \cap M$, for some $M \in \mathcal{M}$ with $M \cap \omega_1 = \min(N \cap \omega_1 : N \in \mathcal{M})$, then there is a condition extending p which is $(N,\mathbb{P})$-generic for every $N \in \mathcal{M}$.

Letting $\mathrm{MA}^{1.5}_\kappa$ and $\mathrm{MA}^{1.5}_{<\kappa}$ be $\mathrm{FA}_\kappa(\mathfrak{K}_{1.5})$ and $\mathrm{FA}_{<\kappa}(\mathfrak{K}_{1.5})$, respectively. Asperó and Mota proved in [3] that $\mathrm{MA}^{1.5}_{<2^{\aleph_0}}$ is consistent with arbitrarily large continuum. Letting also $\mathrm{MA}^{s-1.5}_\kappa$ denote $\mathrm{FA}_\kappa(\mathfrak{K}_{s-1.5})$, it is clear that $\mathrm{MA}^{s-1.5}_\kappa$ implies $\mathrm{MA}^{1.5}_\kappa$,

and hence MA_κ. The forcing axiom $\mathrm{MA}^{s-1.5}_{\aleph_2}$ is apparently stronger than $\mathrm{MA}^{1.5}_{\aleph_2}$, but thus far we are not aware of any particular consequence of $\mathrm{MA}^{s-1.5}_{\aleph_2}$ which cannot follow from $\mathrm{MA}^{1.5}_{\aleph_2}$. We hope that such a structural requirement in the definition of $\aleph^{s-1.5}$-c.c can help us to understand, for example, the structure of proper forcings of size $\aleph_1$. Let us mention some interesting consequence of $\mathrm{MA}^{1.5}_{\aleph_1}$ which was proved in [3].

Definition 1.4.7. *The principle $\mho$ states that there is a sequence $(f_\xi)_{\xi\in\omega_1}$ of continuous functions $f_\xi : \xi \to \omega$ such that for every club $C \subseteq \omega_1$, there is some $\xi \in C$ so that for each $k < \omega$, $f_\xi^{-1}(k) \cap C$ is unbounded in $C \cap \xi$. The weak variations of this principle are $\mho_n$, where $n \geq 2$ is a natural number, which are obtained similarly by replacing ω with n. It is not known if the failure of $\mho_2$ is consistent with $2^{\aleph_0} > \aleph_2$.*

The principle $\mho$ was introduced by Moore. In [3], Asperó and Mota showed the failure of $\mho$ follows from $\mathrm{MA}^{1.5}_{\aleph_1}$, and thus it is consistent with $2^{\aleph_0} > \aleph_2$.

1.5 Large Cardinals

Recall that an inaccessible cardinal is a regular strongly limit cardinal. An inaccessible cardinal is called Mahlo if the set of regular cardinals below it is stationary.

We have the following characterization of Mahlo cardinals.

Fact 1.5.1 (Folklore)**.** *The following are equivalent for every uncountable regular cardinal κ.*

1. *κ is Mahlo.*
2. *For every sufficiently large cardinal θ and all $x \in H_\theta$, there is a model $M \prec H_\theta$ of size less than κ such that $x, \kappa \in M$, and letting π be the collapse map of M, then $\pi(\kappa)$ is inaccessible and M is closed under $< \pi(\kappa)$-sequences.*

$\boxed{1.5.1}$

Definition 1.5.2. *A model M satisfying the above properties is called a κ-Mahlo model in H_θ.*

The most important large cardinal in this thesis is the supercompact cardinal. We will need the following well-known theorem due to Magidor.

Fact 1.5.3 (Magidor, [19])**.** *The following are equivalent for a regular cardinal κ.*

1. *κ is supercompact.*

2. *For every $\gamma > \kappa$ and $x \in V_\gamma$ there exist $\bar{\kappa} < \bar{\gamma} < \kappa$, and an elementary embedding $j : V_{\bar{\gamma}} \to V_\gamma$ with critical point $\bar{\kappa}$ such that $j(\bar{\kappa}) = \kappa$ and $x \in j[V_{\bar{\gamma}}]$.*

$\boxed{1.5.3}$

Chapter Two

Warm-up

The scaffolding poset of a given forcing $\mathbb{P}$ was introduced by Veličković, inspired by [27]. Given a notion of side conditions with models, the corresponding scaffolding poset of $\mathbb{P}$ helps us to understand what requirements for $\mathbb{P}$ are sufficient in order to carry it along an iteration using our side conditions. In this chapter we discuses and further study the scaffolding posets using $\in$-chains of models.

2.1 Scaffolding Posets

Let $\mathcal{C}$ be a collection of countable elementary submodels of H_θ, and let also $\mathcal{U}$ be a collection of elementary IC-submodels of H_θ.

Scaffolding Poset with One Type Model

We first define the $\mathcal{C}$-based scaffolding poset of $\mathbb{P}$, say $\mathbb{M}(\mathbb{P}, \mathcal{C})$, where $\mathbb{P} \in M$ for every $M \in \mathcal{C}$.

Definition 2.1.1. *Assume that $\mathbb{P}$ is $\mathcal{C}$-proper. A condition p in $\mathbb{M}(\mathbb{P}, \mathcal{C})$ is a pair $p = (\mathcal{M}_p, w_p)$, where*

1. *$\mathcal{M}_p$ is a finite $\in$-chain of models in $\mathcal{C}$.*

2. *$w_p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, for every $M \in \mathcal{M}_p$.*

We define the ordering of $\mathbb{M}(\mathbb{P}, \mathcal{C})$ by letting $p \leq q$ if and only if $\mathcal{M}_q \subseteq \mathcal{M}_p$ and $w_p \leq_{\mathbb{P}} w_q$.

If $\mathbb{P}$ is $\mathcal{C}$-proper, then for every model $M \in \mathcal{C}$ and every condition $p \in M$, one can find a condition $p^M \leq p$ such that $M \in \mathcal{M}_{p^M}$. To see this, extend w_p to a condition z so that z is $(M, \mathbb{P})$-generic. On the other hand $\mathcal{M}_p \cup \{M\}$ is an $\in$-chain. Thus the condition p^M defined by $\mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M\}$ and $w_{p^M} = z$ is a condition extending p and that $M \in \mathcal{M}_{p^M}$.

Theorem 2.1.2. *Suppose that $\theta^* > \theta$ is a sufficiently large regular cardinal and $M^* \prec H_{\theta^*}$ contains the relevant objects such that $M = M^* \cap H_\theta \in \mathcal{C}$. Let p be a condition such that $M \in \mathcal{M}_p$. Then p is $(M^*, \mathbb{M}(\mathbb{P}, \mathcal{C}))$-generic.*

Proof. Assume that $D \in M^*$ is an open dense subset of $\mathbb{M}(\mathbb{P}, \mathcal{C})$. We may assume that $p \in D$. Set

$$E = \{w \in \mathbb{P} : \exists q \in D \text{ such that } \mathcal{M}_q \supseteq \mathcal{M}_p \cap M \text{ and } w_q = w\}.$$

Notice that E belongs to M, and that $E \cup E^\perp$ is a dense subset of $\mathbb{P}$, where

$$E^\perp = \{z \in \mathbb{P} : z \perp w \ \forall w \in E\}.$$

Since w_p is $(M, \mathbb{P})$-generic, there is some $w \in (E \cup E^\perp) \cap M$ such that w is compatible with w_p. On the one hand, w_p is in E, therefore w cannot be in $E^\perp$, and hence it is in E. On the one hand by elementarity, there exists $q \in D \cap M^*$ such that $w_q = w$, but then $q \in M$. Let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$ which is easily seen to be an $\in$-chain. Let also w_r be a common extension of w_q and w_p. Therefore for every model N in $\mathcal{M}_q$, w_q is $(N, \mathbb{P})$-generic. It is now clear that $r = (\mathcal{M}_r, w_r)$ is a common extension of p and q. Thus p is $(M^*, \mathbb{M}(\mathbb{P}, \mathcal{C}))$-generic.

$\boxed{2.1.2}$

Let $\theta^* > \theta$ be a sufficiently large regular cardinal, and set

$$\mathcal{C}^* = \{M \prec H_{\theta^*} : \mathbb{P}, \theta \in M, \text{ and } M \cap H_\theta \in \mathcal{C}\}.$$

Corollary 2.1.3. *Suppose $\mathbb{P}$ is a proper forcing. Then, $\mathbb{M}(\mathbb{P}, \mathcal{C})$ is $\mathcal{C}^*$-proper. Moreover, if $\mathcal{C}$ is stationary in H_θ, then $\mathbb{M}(\mathbb{P}, \mathcal{C})$ preserves ω_1, and if $\mathcal{C}$ contains a club, then $\mathbb{M}(\mathbb{P}, \mathcal{C})$ is proper.*

$\boxed{2.1.3}$

Remark 2.1.4. For a V-generic filter G on $\mathbb{M}(\mathbb{P}, \mathcal{C})$, let $\mathcal{M}_G = \{M \in \mathcal{M}_p : p \in G\}$. It is easily seen that $\mathcal{M}_G$ is an $\in$-chain, and that $H_\theta = \bigcup \mathcal{M}_G$, and hence $\mathbb{M}(\mathbb{P}, \mathcal{C})$

collapses θ onto ω_1. Moreover, if ω_1 is preserved, then $\mathcal{M}_G$ is a sequence of length ω_1. Notice that also $w_G = \{w_p : p \in G\}$ generates a V-generic filter on $\mathbb{P}$.

In Appendix B, we shall give a forcing construction with finite conditions, roughly similar to the above construction, which forces an instance of the Mapping Reflection Principle. Notice that one can use decorations on $\mathcal{M}_p$, in order to add clubs in ω_1. A condition p in the decorated version of $\mathbb{M}(\mathbb{P}, \mathcal{C})$ is a triple $p = (\mathcal{M}_p, d_p, w_p)$, where $(\mathcal{M}_p, w_p)$ is as before, and d_p which is called the decoration is a function from $\mathcal{M}_p$ to $[H_\theta]^{<\omega}$ such that if $M \in N$ are in $\mathcal{M}_p$, then $d_p(M) \in N$. In extensions of p, we allow that $d_p(M)$ can be strengthened, for each $M \in \mathcal{M}_p$. This plausible device was introduced by Neeman in [27]. Now, let us uncover another interesting feature of the scaffolding poset.

Scaffolding for Finite Support Product

Recall that the product of two proper forcing is not necessarily proper, even it may not preserve ω_1. The situation is totally different for strongly proper forcings.

The following is easy to prove.

Lemma 2.1.5. *Assume that $\mathbb{P}$ and $\mathbb{Q}$ are forcing notions in H_θ belonging to some $M \prec H_\theta$. Suppose that $(p, q) \in \mathbb{P} \times \mathbb{Q}$. Then (p, q) is $(M, \mathbb{P} \times \mathbb{Q})$-strongly generic if and only if p is $(M, \mathbb{P})$-strongly generic and q is $(M, \mathbb{Q})$-strongly generic.*

$\boxed{2.1.5}$

It thus follows that every finite product of $\mathcal{C}$-strongly proper forcings is $\mathcal{C}$-strongly proper, but it is easily seen that the finite support infinite product of $\mathcal{C}$-strongly proper forcings is not necessarily $\mathcal{C}$-strongly proper. Now, if we think of a finite support product of forcings $\mathbb{P} = \Pi_{i \in I} \mathbb{P}_i$, where I is an infinite set, as a forcing which attempts to add generics for every finite subproduct of $\mathbb{P}$, then using a scaffolding-like construction we can roughly say that the finite support product of $\mathcal{C}$-strongly proper forcings is $\mathcal{C}$-strongly proper. In fact, we build a scaffolding poset suitable for the finite support product so that in generic extensions it adds generics, for every finite product of forcings in $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$, over the ground model while enjoying nice properties of the extension such as preservation of ω_1 and the ω_1-approximation property.

Definition 2.1.6. *Let κ be a cardinal. Suppose $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ is a sequence, in H_θ, of $\mathcal{C}$-strongly proper forcings. Let $\mathbb{P}$ be the finite support product of $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$. A condition p in $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$ is a pair $p = (\mathcal{M}_p, w_p)$ such that:*

1. *$\mathcal{M}_p$ is a finite $\in$-chain of models in $\mathcal{C}$ containing κ and $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$.*

2. *$w_p \in \mathbb{P}$ is such that for every $M \in \mathcal{M}_p$, and every $\alpha \in \operatorname{supp}(w_p) \cap M$, $w_p(\alpha)$ is $(M, \mathbb{P}_\alpha)$-strongly generic, where $\operatorname{supp}(w_p) = \{\alpha < \kappa : w_p(\alpha) \neq \mathbf{1}_{\mathbb{P}_\alpha}\}$.*

We say p is stronger than q and write $p \leq q$ if and only if $\mathcal{M}_p \supseteq \mathcal{M}_q$, and $w_p \leq_{\mathbb{P}} w_q$.

Lemma 2.1.7. *Suppose $M \in \mathcal{C}$ and p is a condition in $M \cap \mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$. Then there is a condition $p^M \leq p$ such that $M \in \mathcal{M}_{p^M}$.*

Proof. We define p^M as follows. Let $\mathcal{M}_{p^M}$ be $\mathcal{M}_p \cup \{M\}$ which is a finite $\in$-chain. Since the support of w_p is finite, it is a subset of M. Thus for each $\alpha \in \operatorname{supp}(w_p)$, we can extend $w_p(\alpha)$ to an $(M, \mathbb{P}_\alpha)$-strongly generic condition $z_\alpha \in \mathbb{P}_\alpha$. Now let w_{p^M} be defined, with the same support as w_p, by letting $w_{p^M}(\alpha) = z_\alpha$. Notice that z_α is also $(N, \mathbb{P}_\alpha)$-strongly generic for every model N in $\mathcal{M}_{p^M}$ with $\alpha \in N$. Thus p^M is a condition, and it is clear that $p^M \leq p$. $\boxed{2.1.7}$

Lemma 2.1.8. *Suppose $M \in \mathcal{C}$ contains κ and $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$. Let p be a condition in $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$ with $M \in \mathcal{M}_p$. Then p is $(M, \mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C}))$-strongly generic.*

Proof. Define $p \restriction_M$ as follows. Let $\mathcal{M}_{p\restriction_M}$ be $\mathcal{M}_p \cap M$ which is an $\in$-chain obviously, and set $\operatorname{supp}(w_{p\restriction_M}) = \operatorname{supp}(w_p) \cap M$ which belongs to M. For every $\alpha \in \sup(w_{p\restriction_M})$, let $w_{p\restriction_M}(\alpha)$ be some projection of $w_p(\alpha)$ to M, say $w_p(\alpha) \restriction_M$, so that every condition $z_\alpha \in \mathbb{P}_\alpha \cap M$ extending $w_p(\alpha) \restriction_M$ is compatible with $w_p(\alpha)$. This is guaranteed of course, by the $(M, \mathbb{P}_\alpha)$-strong genericity of $w_p(\alpha)$. Notice that $p \restriction_M$ belongs to M. Suppose now q is a condition in $\mathbb{P} \cap M$ extending $p \restriction_M$, we shall define a common extension, say r, of p and q as follows. Let simply $\mathcal{M}_r$ be $\mathcal{M}_p \cup \mathcal{M}_q$ that is a finite $\in$-chain. Let also the support of w_r be $\operatorname{supp}(w_p) \cup \operatorname{supp}(w_q)$. Now assume that α is in the support of w_r. If α is in $\operatorname{supp}(w_q) \setminus \operatorname{supp}(w_p)$, then $w_q(\alpha)$ belongs to M, and $\mathcal{M}_p \setminus M$ forms an $\in$-chain of countable models, thus we are able to extend $w_q(\alpha)$ inductively to a condition z_α which is also $(N, \mathbb{P}_\alpha)$-strongly generic for every model N in $\mathcal{M}_p \setminus \mathcal{M}_q$ containing α. On the other hand $w_q(\alpha)$, and hence z_α, is $(N, \mathbb{P}_\alpha)$-strongly generic for every model $N \in \mathcal{M}_q$ with $\alpha \in N$, and hence z_α is $(N, \mathbb{P}_\alpha)$-strongly generic for every model N in $\mathcal{M}_r$ with $\alpha \in N$. If α is in $\operatorname{supp}(w_p) \cap M$, let u_α be some condition extending $w_q(\alpha)$ and $w_p(\alpha)$, such a condition exists as $w_q(\alpha) \leq w_p(\alpha) \restriction_M$. It is clear that u_α is $(N, \mathbb{P}_\alpha)$-generic for every $N \in \mathcal{M}_r$ with $\alpha \in N$.

$$w_r(\gamma) = \begin{cases} w_p(\alpha) & \text{if } \alpha \notin M \\ z_\alpha & \text{if } \alpha \in \operatorname{supp}(w_q) \setminus \operatorname{supp}(w_p) \\ u_\alpha & \text{if } \alpha \in M \cap \operatorname{supp}(w_p) \end{cases}$$

Notice that if α is not in M, then no model in $\mathcal{M}_q$ contains α. Thus $w_p(\alpha)$ is $(N, \mathbb{P}_\alpha)$-strongly generic, for every $N \in \mathcal{M}_r$ with $\alpha \in N$. It should be clear now that r is a condition extending p and q. $\boxed{2.1.8}$

The following is immediate.

Corollary 2.1.9. *If $\mathcal{C}$ is stationary, then $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$ preserves ω_1 and has the ω_1-approximation property.*

$\boxed{2.1.9}$

As in the ordinary scaffolding poset, if $\mathcal{C}$ contains a club, then $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$ is proper. Notice that also it collapses $|H_\theta|$ to ω_1. We are now about to show that forcing with $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$ adds generic filters for finite subproducts of $\mathbb{P}$. Suppose G is a V-generic filter over $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$. For a finite set $s \subseteq \kappa$, let

$$G_s := \{w_p \restriction_s : p \in G\}.$$

Lemma 2.1.10. *G_s generates a V-generic filter over $\Pi_{\alpha\in s}\mathbb{P}_\alpha$.*

Proof. First notice that the set of conditions p with $s \subseteq \operatorname{supp}(w_p)$ is dense. To see this, let $p \in \mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$. We may also assume, without loss of generality, that for every $\alpha \in s$, there is some model in $\mathcal{M}_p$ containing α. Suppose p is a condition, and fix $\alpha \in s$ which is not in the support of w_p, Let M be the least model such that $\alpha \in M$. We can now define q_α with the same side condition as p, and set $\operatorname{supp}(w_{q_\alpha})$ to be $\operatorname{supp}(w_p) \cup \{\alpha\}$. Since α is not in $\operatorname{supp}(w_p)$, $w_p(\alpha) = \mathbf{1}_{\mathbb{P}_\alpha}$ which belongs to M, on the other hand α belongs to each model above M, thus we can extend $\mathbf{1}_{\mathbb{P}_\alpha}$ inductively to a condition w_α which is $(N, \mathbb{P}_\alpha)$- strongly generic for every $N \in \mathcal{M}_p$ containing α. We have finitely many α's, and we do not add new models to the side condition part once constructing q_α. We can therefore inductively proceed to find some extension q of p so that $\mathcal{M}_p = \mathcal{M}_q$ and $s \subseteq \operatorname{supp}(w_q)$. Thus G_s is nonempty. It is clear that every two conditions in G_s are compatible in G_s, and thus it generates a filter. We now show that the filter generated by G_s is V-generic. If $D \in V$ is a dense subset of $\Pi_{\alpha\in s}\mathbb{P}_\alpha$. Set

$$E := \{p \in \mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C}) : w_p \restriction_s \in D\}.$$

E is dense in $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$. To see this, fix q in $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$. By the above paragraph we may assume $s \subseteq \mathrm{supp}(w_q)$, then let $w \in D$ be such that $w \leq w_q \restriction_s$, we can now define r by letting $\mathcal{M}_r = \mathcal{M}_q$ and w_r be the same as w on s, and be w_q elsewhere. Clearly r is a condition belonging to E which extends q. Thus E is a dense subset of $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C})$ in V, therefore there is some $p \in G \cap E$, and hence $w_p \restriction_s \in G_s \cap D$. 2.1.10

We have thus the following.

Theorem 2.1.11. *Suppose $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ is a sequence of $\mathcal{C}$-strongly proper forcings. If $\mathcal{C}$ is a stationary, then there is a generic extension $W \supseteq V$ with the same ω_1 such that the pair (V, W) has the ω_1-approximation property, and that for every finite $s \subseteq \kappa$, there is a V-generic filter $G_s \in W$ on $\Pi_{\alpha \in s} \mathbb{P}_\alpha$.*

2.1.11

Scaffolding Poset with Two Types Models

The construction from the previous subsection can be lifted to a $\mathcal{C} \cup \mathcal{U}$-based scaffolding poset, namely $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$.

Definition 2.1.12. *A set $\mathcal{M} \subseteq \mathcal{C} \cup \mathcal{U}$ is called an $\in$-chain if letting $\in^*$ be the transitive closure of the relation $\in$ on $\mathcal{M}$, then $(\mathcal{M}, \in^*)$ is linearly ordered.*

Suppose $\mathcal{M}$ is an $\in$-chain of models in $\mathcal{C} \cup \mathcal{U}$ which is closed under the intersections. Let $\in^*$ be the transitive closure of the relation $\in$ on $\mathcal{M}$ that is $M \in^* N$ if and only if there is an $\in$-sequence $(M_i)_{i \leq k}$ in $\mathcal{M}$ such that $M_0 = M$ and $M_k = N$. For $M \in^* N$ in $\mathcal{M}$, we let

$$(M, N)_{\mathcal{M}} = \{P \in \mathcal{M} : M \in^* P \in^* N\}.$$

The intervals $[M, N]_{\mathcal{M}}$, $[M, N)_{\mathcal{M}}$ and etc are defined in the same way. When p is a condition in $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$, we let $[M, N)_p$ denote $[M, N)_{\mathcal{M}_p}$.

Definition 2.1.13 (Scaffolding poset with models of two types)**.** *Assume that $\mathbb{P}$ is a $\mathcal{C} \cup \mathcal{U}$-proper forcing. A condition p in $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ is a pair $p = (\mathcal{M}_p, w_p)$, where*

1. *$\mathcal{M}_p$ is a finite $\in$-chain of models in $\mathcal{C} \cup \mathcal{U}$ closed under the intersections.*

2. *$w_p \in \mathbb{P}$ is $(M, \mathbb{P})$-generic, for every $M \in \mathcal{M}_p$.*

We define the ordering of $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ *by letting* $p \leq q$ *if and only if* $\mathcal{M}_q \subseteq \mathcal{M}_p$ *and* $w_p \leq_{\mathbb{P}} w_q$.

It is easy to see that $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ collapses θ to ω_2, and that every V-generic filter on $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ induces a V-generic filter on $\mathbb{P}$. As in the previous section, given a model $M \in \mathcal{C} \cup \mathcal{U}$ and a condition $p \in M$, we are able to extend p to a condition p^M so that $M \in \mathcal{M}_{p^M}$. We remark that one can naturally design the decorated version of $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$.

Proposition 2.1.14. *Suppose that* $\theta^* > \theta$ *is a sufficiently large regular cardinal and* $M^* \prec H_{\theta^*}$ *contains* $\mathbb{P}, \mathcal{C}$ *and* $\mathcal{U}$*, and is such that* $M := M^* \cap H_\theta \in \mathcal{U}$*. Let* p *be a condition with* $M \in \mathcal{M}_p$*. Then* p *is* $(M^*, \mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U}))$*-generic.*

Proof. The proof is quite similar to Theorem 2.1.2. $\boxed{2.1.14}$

Lemma 2.1.15. *Suppose that* $p \in \mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$*. If* $M \in \mathcal{M}_p$ *is countable, then*

$$\mathcal{M}_p \cap M = \mathcal{M}_p \setminus \bigcup \{[N \cap M, N)_p : N \in M \cap \mathcal{U} \text{ or } N = H_\theta\}.$$

Proof. From the left-hand side to the right-hand one is easy. Let us prove the other direction. Suppose $P \in \mathcal{M}_p$ belongs to the right-hand side above. We first state a general claim.

Claim 2.1.16. *If* $Q \in^* R$ *in* $\mathcal{M}_p$ *are countable, then either* $Q \in R$ *or there is some* IC*-model* $N \in R$ *such that* $Q \in [N \cap R, N)_p$.

Proof. We prove by induction on the number of IC-models in $[Q, R]_p$. Suppose the statement holds for every countable model $Q, R \in \mathcal{M}_p$ with k many IC-models in $[Q, R]_p$. It is obvious for $k = 0$. We show that it is true for $k + 1$. Suppose N is the largest IC-model in $[Q, R]_p$. It follows that $N \in R$. If $Q \in^* N \cap R$, then there are at most k IC-models in $[Q, N \cap R]_p$, and thus by applying the inductive assumption to $N \cap R$, we should have that $Q \in N \cap R$, and hence $Q \in R$. On the other hand if $Q \notin^* N \cap R$, we should have that $N \cap R \in^* Q \in N$, and hence $Q \in [N \cap R, N)_p$. $\boxed{2.1.16}$

Let us return to the main proof. Since P is not in $[M, H_\theta)_p$, it is $\in^*$-below M i.e $P \in^* M$. By the above claim and the fact that P does not belong to any interval $[N \cap M, N)_p$, for $N \in M \cap \mathcal{U}$, it follows that $P \in M$.

$\boxed{2.1.15}$

The following lemma shows that the forcing with pure side conditions i.e when there are no working parts is $\mathcal{C} \cup \mathcal{U}$-strongly proper, we omit the proof as we will not use it later.

Lemma 2.1.17. *Let $M \in \mathcal{C} \cup \mathcal{U}$. Suppose that $p, q \in \mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$, and $q \in M$ is such that $\mathcal{M}_q \supseteq \mathcal{M}_p \cap M$. Then the closure of $\mathcal{M}_p \cup \mathcal{M}_q$ under the intersections is an $\in$-chain.*

2.1.17

Proposition 2.1.18. *Suppose that $\theta^* > \theta$ is a sufficiently large regular cardinal and $M^* \prec H_{\theta^*}$ contains $\mathbb{P}, \mathcal{C}$ and $\mathcal{U}$ and is such that $M := M^* \cap H_\theta \in \mathcal{C} \cup \mathcal{U}$. Let p be a condition with $M \in \mathcal{M}_p$. Then p is $(M^*, \mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U}))$-generic.*

Proof. Thanks to Lemma 2.1.15 and Lemma 2.1.17, a proof quite similar to Theorem 2.1.2 works also here. 2.1.18

Let $\theta^* > \theta$ be a sufficiently large regular cardinal, and set

$$\mathcal{C}^* = \{M \prec H_{\theta^*} : \mathcal{C}, \mathcal{U}, \mathbb{P}, \theta \in M \text{ and } M \cap H_\theta \in \mathcal{C}\},$$

and

$$\mathcal{U}^* = \{M \prec H_{\theta^*} : \mathcal{C}, \mathcal{U}, \mathbb{P}, \theta, \in M \text{ and } M \cap H_\theta \in \mathcal{U}\}.$$

Corollary 2.1.19. *If $\mathbb{P}$ is both $\mathcal{C}$-proper and $\mathcal{U}$-proper, then $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ is also $\mathcal{C}^*$-proper and $\mathcal{U}^*$-proper. Moreover, if both $\mathcal{C}$ and $\mathcal{U}$ are stationary, then $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ preserves ω_1 and ω_2, and if $\mathcal{C}$ contains a club, then $\mathbb{M}(\mathbb{P}, \mathcal{C} \cup \mathcal{U})$ is proper.*

2.1.19

Suppose $\langle \mathbb{P}_\alpha : \alpha < \kappa \rangle$ is a sequence of $\mathcal{C} \cup \mathcal{U}$-strongly proper forcings, we would like to construct a scaffolding poset $\mathbb{M}((\mathbb{P}_\alpha)_{\alpha<\kappa}, \mathcal{C} \cup \mathcal{U})$ as in the first section so that forcing with it preserves ω_1 and ω_2, and also adds generics for the finite subproducts of $\mathbb{P} = \Pi_{\alpha<\kappa} \mathbb{P}_\alpha$. However it turns out that the proofs given above cannot be adapted here, and we need to require more properties about $\mathbb{P}_\alpha$. Let us address one issue, for a condition p and a model $M \in \mathcal{C} \cup \mathcal{U}$, we might be able to make a condition $p \restriction_M \in M$, suppose now $q \in M$ extends $p \restriction_M$, but it may have some $\alpha \in \operatorname{supp}(w_q) \setminus \operatorname{supp}(w_p)$, as we need to amalgamate p and q, we should be able at least to extend $w_q(\alpha)$ to some condition which is generic for every relevant model in $\mathcal{M}_p \cup \mathcal{M}_q$, though this is not satisfied in general, the structured $\aleph_{1.5}$-c.c-ness is a possible solution to

it. On the other hand notice that in the definition of structured $\aleph_{1.5}$-c.c-ness, the conditions are required to be generic not strongly generic, thus if we aim to make a suitable scaffolding poset for the finite support product using chains of two types models as the side conditions, we will need to modify the definition of structured $\aleph_{1.5}$-c.c-ness to include the strong genericity of conditions, not the usual genericity . Things then work smoothly as before. Our goal here is to point out that there are some obstacles if one tries naively to iterate proper forcings using two types models as side conditions, and that structured $\aleph_{1.5}$-c.c-ness is relatively reasonable in this context. We remark that in our iteration later, we will not need strong properness.

Chapter Three

The Theory of Virtual Models

This chapter is devoted to the study of the theory of virtual models which was introduced by Veličković in [37] and [38]. In [38], he used virtual models of two types: countable models and IC-models. In the this dissertation, the forcing constructions are based on models with much stronger closure property that we call Magidor models. This chapter mainly concerns the study of Magidor virtual models and their interaction with countable virtual models. We start with a general framework, but to be more precise and avoid unnecessary sophistication, the content will be thinned out in the course of the current chapter. Some basic statements are either originally or essentially due to Veličković [37].

3.1 General Virtual Models

We consider the language $\mathcal{L}$ obtained by adding a single constant symbol $\hat{c}$ and a single predicate $\hat{U}$ to the standard language $\mathcal{L}_\epsilon$ of the set theory. Notice that it is still possible to work with other expansions of $\mathcal{L}_\epsilon$, but we do not see any gain in complicating things. Let us say that an $\mathcal{L}$-structure $\mathcal{A}$ of the form $(A, \in, \kappa, U)$ is *suitable* if A is a transitive set satisfying ZFC in the expanded language, where κ, the interpretation of the constant symbol $\hat{c}$, is an inaccessible cardinal in A, and U is the interpretation of $\hat{U}$. We shall often abuse notation and refer to the structure $\mathcal{A} = (A, \in, \kappa, U)$ simply by A or $\mathcal{A}$. We may also not mention the language $\mathcal{L}$, and thus unless explicitly mentioned, all model-theoretic concepts such as the isomorphisms, the substructures and etc will be carried out within the language $\mathcal{L}$. This language and its logic will only serve as internal affairs, and should not be confused with our external set theory which is basically usual ZFC in $\mathcal{L}_\epsilon$-theory of

sets.

Suppose that $\mathcal{A}$ is a suitable structure. If α is an ordinal in A, we let A_α denote $A \cap V_\alpha$. Finally, we let

$$E_{\mathcal{A}} = \{\alpha \in \mathrm{ORD}^A : (A_\alpha, \in, \kappa, U \cap A_\alpha) \prec \mathcal{A}\}.$$

Note that E_A is a closed, possibly empty, subset of ORD^A. It is not definable in A, but $E_A \cap \alpha$ is uniformly definable in A with parameter α, for each $\alpha \in E_A$. If $\alpha \in E_A$ we let $\mathrm{next}_A(\alpha)$ be the least ordinal in E_A above α, if such an ordinal exists. Otherwise, we leave $\mathrm{next}_A(\alpha)$ undefined. We start with a simple technical lemma.

Lemma 3.1.1. *Suppose that $M \prec A$. Then*

1. *If $\alpha \in E_A$ and $(M \cap \mathrm{ORD}^A) \setminus \alpha \neq \varnothing$, then $\min(M \cap \mathrm{ORD}^A \setminus \alpha) \in E_A$.*
2. $\sup(E_A \cap M) = \sup(E_A \cap \sup(M \cap \mathrm{ORD}^A))$.

Proof. We only give the proof of the first one as it implies the second item. Let β be the least ordinal in $M \setminus \alpha$. We need to show that A_β is an elementary submodel of A. Suppose otherwise, then by the Tarski-Vaught criterion, there is a tuple $\bar{x} \in A_\beta$ and a formula $\varphi(y, \bar{x})$ such that $A \models \exists y \varphi(y, \bar{x})$, but there is no $a \in A_\beta$ such that $A \models \varphi(a, \bar{x})$. Since $\beta \in M$ and M is an elementary submodel of A, there is such a tuple $\bar{x} \in A_\beta \cap M$. Now, β is the least ordinal in M above α, therefore $\bar{x} \in M \cap A_\alpha$. Since A_α is an elementary submodel of A, there is $a \in A_\alpha$ witnessing that $A_\alpha \models \varphi(a, \bar{x})$ and so $A \models \varphi(a, \bar{x})$. Since $\alpha \leq \beta$, it follows that $a \in A_\beta$, a contradiction. $\boxed{3.1.1}$

Definition 3.1.2. *Suppose that M is a submodel of a suitable structure A and X is a subset of A. Let*

$$\mathrm{Hull}(M, X) = \{f(\bar{x}) : f \in M, \bar{x} \in X^{<\omega}, f \text{ is a function, and } \bar{x} \in \mathrm{dom}(f)\}.$$

The main reason the Hull operation is defiend in this way is that it allows us to define the Skolem hull of M and X without referring explicitly to the ambient model $(A, \in, \kappa, U)$. We refer to the structure $(\mathrm{Hull}(M, X), \in, \kappa, \mathrm{Hull}(M, X) \cap U)$ by $\mathrm{Hull}(M, X)$.

Lemma 3.1.3. *Suppose that A is a suitable structure, M is an elementary submodel of A and X is a subset of A. Let δ be $\sup(M \cap \mathrm{ORD}^A)$, and suppose $X \cap A_\delta$ is nonempty. Then $\mathrm{Hull}(M, X)$ is the least elementary submodel of A containing M and $X \cap A_\delta$ as subsets.*

Proof. For each $\gamma \in A$, let id_γ be the identity function on A_γ. Clearly, if $\gamma \in M$ then $\mathrm{id}_\gamma \in M$. Therefore, $X \cap A_\delta$ is a subset of $\mathrm{Hull}(M, X)$. Let $\gamma \in M$ be such that $X \cap A_\gamma$ is nonempty. For each $z \in M$, the constant function c_z defined on A_γ is in M, therefore M is a subset of $\mathrm{Hull}(M, X)$. Notice that A_γ belongs to A since A is transitive the rank function is absolute between transitive structures, and hence A_γ is in M be elementary.

It remains to show that $\mathrm{Hull}(M, X)$ is an elementary submodel of A. We check the Tarski-Vaught criterion for $\mathrm{Hull}(M, X)$ and A. Let φ be an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in \mathrm{Hull}(M, X)$ such that $A \models \exists u \varphi(u, a_1, \ldots, a_n)$. Then we can find functions $f_1, \ldots, f_n \in M$ and tuples $\bar{x}_1, \ldots, \bar{x}_n \in X^{<\omega}$ such that $a_i = f_i(\bar{x}_i)$, for all i. If D_i is the domain of f_i, this implies that $\bar{x}_i \in D_i$. By the Axiom of Replacement and the Axiom of Choice in A we can find a function g defined on $D_1 \times \ldots \times D_n$ such that for every $\bar{y}_1 \in D_1, \ldots, \bar{y}_n \in D_n$, if there is u such that $A \models \varphi(u, f_1(\bar{y}_1), \ldots, f_n(\bar{y}_n))$ then $g(\bar{y}_1, \ldots, \bar{y}_n)$ is such a u. Thus A satisfies the following statement:

" There exists a function g with $\mathrm{dom}(g) = D_1 \times \cdots \times D_n$ such that for every $(\bar{y}_1, \ldots, \bar{y}_n)$ in $D_1 \times \cdots \times D_n$, if there exists u such that $\varphi(u, f_1(\bar{y}_1), \ldots, f_n(\bar{y}_n))$, then $\varphi(g(\bar{y}_1, \ldots, \bar{y}_n), f_1(\bar{y}_1), \ldots, f_n(\bar{y}_n))$ "

Therefore, by the elementarity of M, we may assume that g belongs to M. Let $a = g(\bar{x}_1, \ldots, \bar{x}_n)$. It follows that $a \in \mathrm{Hull}(M, X)$ and $A \models \varphi(a, a_1, \ldots, a_n)$. Therefore, $\mathrm{Hull}(M, X)$ is an elementary submodel of A.

The minimality of $\mathrm{Hull}(M, X)$ follows from the fact that $M \subseteq A_\delta$, and thus $\mathrm{Hull}(M, X) = \mathrm{Hull}(M, X \cap A_\delta)$ $\boxed{3.1.3}$

3.2 Virtual Models in V_λ

We are now about to restrict ourselves to a particular suitable structure. Let us fix an inaccessible cardinal κ, and a cardinal $\lambda > \kappa$ such that V_λ satisfies ZFC. From now on, we shall write E instead of E_{V_λ} and $\mathrm{next}(\alpha)$ instead of $\mathrm{next}_{V_\lambda}(\alpha)$ in this chapter and elsewhere in this manuscript, whenever it is clear form the context. For each $\alpha \in E$, we shall define certain families $\mathscr{C}_\alpha, \mathscr{U}_\alpha \in V_\lambda$, as well as relations $\in_\alpha$ and $\cong_\alpha$, and the projection operation $M \mapsto M \restriction \alpha$. These families, relations and operations will be defined by a Σ_1-formula with parameter V_α. If A is another suitable structure and $\alpha \in E_A$ we can interpret these formulas in A and obtain families $\mathscr{C}_\alpha^A, \mathscr{U}_\alpha^A$, and we can interpret the relations $\in_\alpha$, $\cong_\alpha$ and the projection map $M \mapsto M \restriction \alpha$ inside

A. We shall only consider suitable A such that the interpretation of the constant symbol c is κ and $A \subseteq V_\lambda$. Note that if we have such an A and $\alpha \in E_A \cap E$ with $A_\alpha = V_\alpha$ then $\mathscr{C}_\alpha^A \subseteq \mathscr{C}_\alpha$, and $\mathscr{U}_\alpha^A \subseteq \mathscr{U}_\alpha$. Moreover, if $A \models M \in_\alpha M'$ then $M \in_\alpha M'$, and similarly for $\cong_\alpha$.

Definition 3.2.1. *Suppose $\alpha \in E$. We let $\mathscr{A}_\alpha$ denote the set of all suitable structures A that are elementary extensions of V_α and have the same cardinality as V_α.*

Note that if $A \in \mathscr{A}_\alpha$ and $\alpha \in A$ then $E_A \cap \alpha = E \cap \alpha$. If $A \in \mathscr{A}_\alpha$ we will refer to V_α as the *standard part* of A. Note that if A has nonstandard elements then $\alpha \in E_A$.

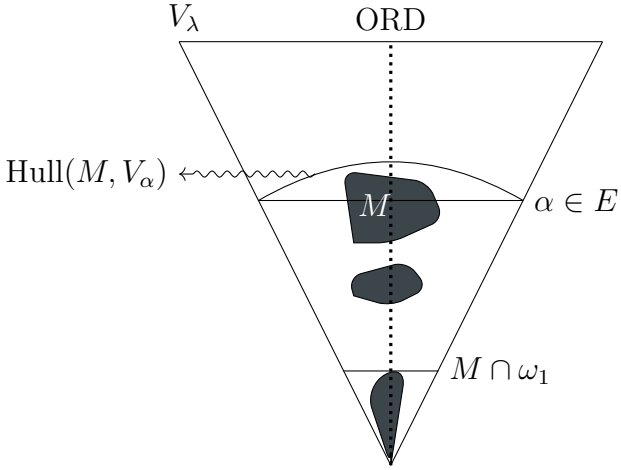
Definition 3.2.2. *Suppose $\alpha \in E$. We let $\mathscr{V}_\alpha$ denote the collection of all submodels M of V_λ of size less than κ such that, if we let $A = \mathrm{Hull}(M, V_\alpha)$, then $A \in \mathscr{A}_\alpha$ and M is an elementary submodel of A.*

Definition 3.2.3. *We refer to the members of $\mathscr{V}_\alpha$ as the α-models. We write $\mathscr{V}_{<\alpha}$ for $\bigcup\{\mathscr{V}_\gamma : \gamma \in E \cap \alpha\}$. Collections $\mathscr{V}_{\leq\alpha}$ and $\mathscr{V}_{\geq\alpha}$ are defined in the obvious way. We will write $\mathscr{V}$ for $\mathscr{V}_{<\lambda}$. If $M \in \mathscr{V}$, we then write $\eta(M)$ for the unique ordinal α such that $M \in \mathscr{V}_\alpha$.*

To see that why $\eta(M)$ is unique, we recall that $|\mathrm{Hull}(M, V_{\eta(M)})| = |V_{\eta(M)}|$, thus if $\alpha < \beta$ are in E, and M is both α-model and β-model, we would then have $|V_\alpha| = |V_\beta|$, which is impossible. Note that also if $M \in \mathscr{V}_\alpha$ then $\sup(M \cap \mathrm{ORD}) \geq \alpha$. This is because if $M \cap \lambda$ is bounded below α, say $\sup(M \cap \lambda) = \delta$, we then may work in the suitable structure V_δ, and compute $\mathrm{Hull}(M, V_\delta)$ which is equal to $\mathrm{Hull}(M, V_\alpha)$, but then $|V_\delta| = |V_\alpha|$, which is a contradiction.

In general, M is not elementary in V_λ, in fact, this only happens if $M \subseteq V_\alpha$. To see this, towards a contradiction we suppose that $M \prec V_\lambda$ is an α-model in which $M \nsubseteq V_\alpha$, indeed $\mathrm{Hull}(M, V_\alpha)$ is elementary in V_λ and that it contains some ordinal above α, thus by Lemma 3.1.1, there is some ordinal $\beta \in \mathrm{Hull}(M, V_\alpha) \cap E \setminus \alpha$, but then $V_\beta \subseteq \mathrm{Hull}(M, V_\alpha)$, and hence $|V_\beta| \leq |V_\alpha|$ which is impossible. Thus it is rational to say that M is a *standard α-model* if and only if $M \prec V_\alpha$.

We refer to members of $\mathscr{V}$ as *virtual models*. Notice that if $A \subseteq V_\lambda$ is a suitable structure, we can still talk about virtual models of A, so it is convenient to let $\mathscr{V}^A$ denote the set of all virtual models in the sense of A. We face frequently with the situation where M is a virtual model i.e it is in $\mathscr{V}$, and thus $\mathrm{Hull}(M, V_{\eta(M)})$ is a suitable structure, and we may work carefully with virtual models in $\mathrm{Hull}(M, V_{\eta(M)})$. In this case, we also write $\mathscr{V}^M$ to denote $M \cap \mathscr{V}^{\mathrm{Hull}(M, V_{\eta(M)})}$.



M is a countable α-model.

Definition 3.2.4. *Suppose $M, N \in \mathscr{V}$ and $\alpha \in E$. An isomorphism $\sigma : M \to N$ is called an* α-isomorphism *if there is an isomorphism $\bar{\sigma} : \mathrm{Hull}(M, V_\alpha) \to \mathrm{Hull}(N, V_\alpha)$ extending σ. We say that M and N are* α-isomorphic *and write $M \cong_\alpha N$ if there is an α-isomorphism between them.*

Clearly, $\cong_\alpha$ is an equivalence relation, for every $\alpha \in E$, and that if σ and $\bar{\sigma}$ exist, they are unique. Note that also if $M \in \mathscr{V}_\gamma$, for some $\gamma < \alpha$, then the only model α-isomorphic to M is M itself since $\mathrm{Hull}(N, V_\gamma) = \mathrm{Hull}(M, V_\alpha)$.

Proposition 3.2.5. *Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. if $M, N \in \mathscr{V}$ are β-isomorphic, then they are α-isomorphic.*

Proof. We may assume that $\eta(M), \eta(N) \geq \beta$. Let $\bar{\sigma} : \mathrm{Hull}(M, V_\beta) \to \mathrm{Hull}(N, V_\beta)$ be the β-isomorphism between them. Notice that $V_\alpha \subseteq \mathrm{Hull}(M, V_\beta), \mathrm{Hull}(N, V_\beta)$. It is now evident that $\bar{\sigma} \restriction_{\mathrm{Hull}(M,V_\alpha)}$ is an α-isomorphism. $\boxed{3.2.5}$

We will now see that, if $\alpha < \beta$, then for every β-model M there is a canonical representative of the $\cong_\alpha$-equivalence class of M which is an α-model.

Definition 3.2.6. *Suppose $\alpha, \beta \in E$ and M is a β-model. Let $\overline{\mathrm{Hull}(M, V_\alpha)}$ be the transitive collapse of $\mathrm{Hull}(M, V_\alpha)$, and let π be the collapse map. We define $M \restriction_\alpha$ to be $\pi[M]$, i.e. the image of M under the collapse map of $\mathrm{Hull}(M, V_\alpha)$.*

Remark 3.2.7. Note that if $\beta < \alpha$ then $M \restriction_\alpha = M$, but M is not an α-model. If $\beta \geq \alpha$, then $\overline{\mathrm{Hull}(M, V_\alpha)}$ belongs to $\mathscr{A}_\alpha$, so $M \restriction_\alpha$ is an α-model which is α-isomorphic to M. Note also that if $\beta = \alpha$, then $M \restriction_\alpha = M$ since $\mathrm{Hull}(M, V_\alpha)$

is already transitive. Note that also if $A \in \mathscr{A}_\alpha$ then $\mathscr{V}_\alpha^A \subseteq \mathscr{V}_\alpha$. Therefore, if $A, B \in \mathscr{A}_\alpha$, $M \in \mathscr{V}^A$, and $N \in \mathscr{V}^B$, we can still write $M \cong_\alpha N$ if $M \restriction_\alpha = N \restriction_\alpha$. This is of course equivalent to the existence of an α-isomorphism between M and N.

The following is straightforward.

Proposition 3.2.8. *Suppose $\alpha \leq \beta$ are in E, and $M \in \mathscr{V}$. Then $(M \restriction_\beta) \restriction_\alpha = M \restriction_\alpha$.*

3.2.8

We also need to define a version of the membership relation, for every α in E.

Definition 3.2.9. *Suppose $M, N \in \mathscr{V}$ and $\alpha \in E$. We write $M \in_\alpha N$ if there is $M' \in N$ with $M' \in \mathscr{V}^N$ such that $M' \cong_\alpha M$. If this happens, we say that M is α-in N.*

Note that if M is α-in N with $M \subseteq V_\alpha$, this simply means that $M \in N$. To see this, we may assume $\eta(M) = \alpha$ and work in the suitable structure $\mathrm{Hull}(N, V_{\eta(N)})$. Now there is M' such that $M' \restriction_\alpha \subseteq V_\alpha$, but it then means that M' has no element of rank above α, hence $M = M'$. We notice that in general, we may have $M \in_\alpha N$ even if the rank of M is higher than the rank of N. We shall often use the following simple facts without mentioning them.

Proposition 3.2.10. *Suppose $M, N \in \mathscr{V}$ with $M \in N$. Let $\alpha \in E$, and suppose $N' \in \mathscr{V}^A$, for some $A \in \mathscr{A}_\alpha$, and $\sigma : N \to N'$ is an α-isomorphism. Then M and $\sigma(M)$ are α-isomorphic.*

Proof. Since $|M| < \kappa < |V_\alpha|$, we conclude that $M \subseteq \mathrm{Hull}(N, V_\alpha)$, and hence $\mathrm{Hull}(M, V_\alpha) \subseteq \mathrm{Hull}(N, V_\alpha)$. Let $\bar{\sigma}$ be the extension of σ to $\mathrm{Hull}(N, V_\alpha)$. It follows that $\bar{\sigma} \restriction_{\mathrm{Hull}(M,V_\alpha)}$ is an isomorphism between $\mathrm{Hull}(M, V_\alpha)$ and $\mathrm{Hull}(\sigma(M), V_\alpha)$. Hence $\bar{\sigma} \restriction_M$ is an α-isomorphism between M and $\sigma(M)$. 3.2.10

Proposition 3.2.11. *Let $\alpha, \beta \in E$ with $\alpha \leq \beta$. Suppose $M, N \in \mathscr{V}_{\geq\beta}$ and $M \in_\beta N$. Then $M \restriction_\alpha \in_\alpha N \restriction_\alpha$.*

Proof. Fix some $M' \in N$ with $M \cong_\beta M'$. Since $\alpha \leq \beta$, we have that $M \cong_\alpha M'$. Letting π be the collapse map of $\mathrm{Hull}(N, V_\alpha)$, then $\pi(M') \in N \restriction_\alpha$. On the other hand, since $|M'| < \kappa < |V_\alpha|$, we have that $\mathrm{Hull}(M', V_\alpha) \subseteq \mathrm{Hull}(N, V_\alpha)$ and that $\pi[M'] = \pi(M')$. It follows that $\pi \restriction_{\mathrm{Hull}(M',V_\alpha)}$ is an isomorphism between $\mathrm{Hull}(M', V_\alpha)$ and $\mathrm{Hull}(\pi(M'), V_\alpha)$. Therefore, $M \cong_\alpha \pi(M') \in N \restriction_\alpha$. 3.2.11

We refer to the following proposition as the *continuity* of the α-isomorphism.

Proposition 3.2.12. *Let α be a limit point of E. Suppose $N, M \in \mathscr{V}$ and $M \cong_\gamma N$ for unboundedly many γ below α. Then $M \cong_\alpha N$.*

Proof. For each $\gamma \in E \cap \alpha$, let σ_γ be the unique isomorphism between $\mathrm{Hull}(M, V_\gamma)$ and $\mathrm{Hull}(N, V_\gamma)$ such that $\sigma_\gamma[M] = N$. If $\gamma < \gamma'$, we have that $\sigma_{\gamma'} \restriction_{\mathrm{Hull}(M,V_\gamma)} = \sigma_\gamma$. Let $\sigma = \bigcup\{\sigma_\gamma : \gamma \in E \cap \alpha\}$. Then σ witnesses that M and N are α-isomorphic. $\boxed{3.2.12}$

Proposition 3.2.13. *Let α be a limit point of E of uncountable cofinality. Assume that $M, N \in \mathscr{V}$ and N is countable. Suppose that $M \in_\gamma N$ for unboundedly many $\gamma < \alpha$. Then $M \in_\alpha N$.*

Proof. Since N is countable and α is of uncountable cofinality, there is $M' \in N$ with $M' \in \mathscr{V}^N$ such that $M \cong_\gamma M'$, for unboundedly many $\gamma \in E \cap \alpha$. By Proposition 3.2.12 we have that $M \cong_\alpha M'$, and hence $M \in_\alpha N$. $\boxed{3.2.13}$

Definition 3.2.14. *For $\alpha \in E$, we let $\mathscr{C}_\alpha$ denote the collection of countable models in $\mathscr{V}_\alpha$. We define similarly $\mathscr{C}_{<\alpha}$, $\mathscr{C}_{\leq\alpha}$ and $\mathscr{C}_{\geq\alpha}$. We write $\mathscr{C}$ for $\mathscr{C}_{<\lambda}$, and $\mathscr{C}_{\mathrm{st}}$ for the collection of standard models in $\mathscr{C}$.*

Proposition 3.2.15. *Suppose λ is of uncountable cofinality. Then $\mathscr{C}_{\mathrm{st}}$ contains a club in $\mathcal{P}_{\omega_1}(V_\lambda)$.*

Proof. First note that since λ is of uncountable cofinality E is unbounded and thus club in λ. Suppose M is a countable elementary submodel of $(V_\lambda, \in, \kappa, U, E)$. Let $\alpha = \sup(M \cap E)$. Note that $M \cap \mathrm{ORD}$ is unbounded in α. Hence M is a standard α-model. $\boxed{3.2.15}$

In our forcing constructions in the later chapters, we will use two types of virtual models, the countable ones and some nice models of size less than κ defined below.

The following definition is motivated by Magidor's reformulation of supercompactness, see Fact 1.5.3.

Definition 3.2.16. *We say that a model M is a κ-*Magidor model *if, letting $\overline{M}$ be the transitive collapse of M and π the collapse map, $\overline{M} = V_{\bar{\gamma}}$, for some $\bar{\gamma} < \kappa$ with $\mathrm{cof}(\bar{\gamma}) \geq \pi(\kappa)$, and $V_{\pi(\kappa)} \subseteq M$. If κ is clear from the context, we then simply say M is a Magidor model.*

Proposition 3.2.17. *Suppose κ is supercompact and $\mu > \kappa$ with $\mathrm{cof}(\mu) \geq \kappa$. Then the set of κ-Magidor models is stationary in $\mathcal{P}_\kappa(V_\mu)$.*

Proof. Fix a function $F : [V_\mu]^{<\omega} \to V_\mu$. We have to find a κ-Magidor model closed under F. Let $\gamma > \mu$ be such that V_γ satisfies a sufficient fragment of ZFC. Since κ is supercompact, by Fact 1.5.3 we can find $\bar{\kappa} < \bar{\gamma} < \kappa$ and an elementary embedding $j : V_{\bar{\gamma}} \to V_\gamma$ with critical point $\bar{\kappa}$ such that $j(\bar{\kappa}) = \kappa$ and such that $F \restriction_{[V_\mu]^{<\omega}} \in j[V_{\bar{\gamma}}]$. Note that V_μ is in $j[V_{\bar{\gamma}}]$. Let $\bar{\mu}$ be such that $j(\bar{\mu}) = \mu$. Since $\mathrm{cof}(\mu) \geq \kappa$, by elementarity we must have that $\mathrm{cof}(\bar{\mu}) \geq \bar{\kappa}$. Let $N = j[V_{\bar{\mu}}]$. Then N is a κ-Magidor elementary submodel of V_μ that is closed under F, as required. $\boxed{3.2.17}$

Definition 3.2.18. *Let $\mathscr{U}^\kappa_\alpha$ be the collection of all $M \in \mathscr{V}_\alpha$ that are κ-Magidor models. We define $\mathscr{U}^\kappa_{<\alpha}$, $\mathscr{U}^\kappa_{\leq\alpha}$, and $\mathscr{U}^\kappa_{\geq\alpha}$ in the obvious way. We write $\mathscr{U}^\kappa$ for $\mathscr{U}^\kappa_{<\lambda}$. When κ is clear from the context, we omit it. We also write $\mathscr{U}_{\mathrm{st}}$ for the standard models in $\mathscr{U}$.*

Remark 3.2.19. Suppose M is a κ-Magidor α-model. Let $V_{\bar{\gamma}}$ be its transitive collapse, and let j be the inverse of the collapse map π. Let also $A = \mathrm{Hull}(M, V_\alpha)$. Note that $j : V_{\bar{\gamma}} \to A$ is an elementary embedding with critical point $\bar{\kappa} = \pi(\kappa)$ and $j(\bar{\kappa}) = \kappa$.

By Proposition 3.2.17 we have the following immediate corollary.

Corollary 3.2.20. *Suppose κ is supercompact and λ is inaccessible. Then $\mathscr{U}_{\mathrm{st}}$ is stationary in $\mathcal{P}_\kappa(V_\lambda)$.*

$\boxed{3.2.20}$

Note that both classes $\mathscr{C}$ and $\mathscr{U}$ of virtual models are closed under projections meaning that if $M \in \mathscr{C}$ (respectively $\mathscr{U}$) and $\alpha \in E$, then $M \restriction_\alpha \in \mathscr{C}$ (respectively $\mathscr{U}$). We shall study some particular finite collections of these two types of models. We start by establishing the following easy fact.

Proposition 3.2.21. *Let $\alpha \in E$. Suppose $M, N, P \in \mathscr{V}$ and $M \in_\alpha N \in_\alpha P$. If either N is countable or P is a Magidor model then $M \in_\alpha P$.*

Proof. Pick $N' \in P$ with $N' \in \mathscr{V}^P$ which is α-isomorphic to N. We first establish that $N' \subseteq P$. If N is countable this is immediate. Suppose P is a Magidor model. Let $\overline{N'}$ be the transitive collapse of N', and let π be the collapse map. Then $\overline{N'} \in V_\kappa \cap P$ since $|N'| < \kappa$. Since P is a Magidor model, we know that $V_\kappa \cap P$ is transitive, and hence $\overline{N'} \subseteq P$, but then also $N' \subseteq P$. Let σ be an α-isomorphism between N and N', and let $M' \in N$ with $M' \in \mathscr{V}^N$ be a model that is α-isomorphic to M. By Proposition 3.2.10 we know that $\sigma(M')$ is α-isomorphic to M', and also to M by the transitivity of $\cong_\alpha$. On the other hand $\sigma(M') \in N' \subseteq P$ and thus $M \in_\alpha P$, as desired. $\boxed{3.2.21}$

Our next goal is to say when a virtual model M is active at some $\alpha \in E$.

Definition 3.2.22. *Let $M \in \mathscr{V}$. We say that M is* active *at $\alpha \in E$ if $\eta(M) \geq \alpha$ and* $\mathrm{Hull}(M, V_{\kappa_M}) \cap E \cap \alpha$ *is unbounded in $E \cap \alpha$, where $\kappa_M = \sup(M \cap \kappa)$. We say that M is* strongly active *at α if $\eta(M) \geq \alpha$ and $M \cap E \cap \alpha$ is unbounded in $E \cap \alpha$.*

We are primarily interested in the case $M \in \mathscr{C} \cup \mathscr{U}$. First note that if M is a Magidor model, then $V_{\kappa_M} \subseteq M$, hence M is active at some $\alpha \in E$ if and only if it is strongly active at α. The situation is quite different for countable models. If M is countable, then the set of $\alpha \in E$ at which M is strongly active is at most countable, while the set of $\alpha \in E$ at which M is active can be of size $|V_{\kappa_M}|$. One feature of our definition is that if $N \in_\alpha M$, then for all $\gamma \in E \cap \alpha$, if N is active at γ then so is M. To see this let $N' \in M$ be such that $N' \cong_\alpha N$. Then N' is also active at γ. Since $\mathrm{Hull}(N', V_{\kappa_{N'}}) \subseteq \mathrm{Hull}(M, V_{\kappa_M})$, it follows that M is also active at γ. Notice that the assumption $\eta(M) \geq \alpha$ will be essential later as M needs enough room to contain copies of models on the α-chain below M, for example if $\alpha = \mathrm{next}(\beta)$, and M is a β-model such that $\beta \in M$, then $M \cap E \cap \alpha$ is cofinal in $E \cap \alpha$, but even $\mathrm{Hull}(M, V_\beta)$ has size less than $|V_\alpha|$. On the other hand, since we require $\eta(M) \geq \alpha$ where M is strongly active at α, we must have $\beta \in M$ as $\beta = \max(E \cap \alpha)$. We must also have $\sup(M \cap \mathrm{ORD}) \geq \alpha$ since $\eta(M) \geq \alpha$. If $\sup(M \cap \mathrm{ORD}) = \alpha$ then M is a standard model. If $\sup(M \cap \mathrm{ORD}) > \alpha$, let $\gamma = \min(M \cap \mathrm{ORD} \setminus \alpha)$, and let $A = \mathrm{Hull}(M, V_{\eta(M)})$. Then by Lemma 3.1.1 $\gamma \in E_A$. Since $\gamma \in M$, we have that $E_A \cap (\gamma + 1) \in M$ and therefore we can compute α in M as the the next element of $E_A \cap (\gamma + 1)$ above β. Thus, in this case we have $\alpha \in M$.

It will be convenient to have also the following definition.

Definition 3.2.23. *Suppose $M \in \mathscr{V}$. Let $a(M) = \{\alpha \in E : M$ is active at $\alpha\}$ and $\alpha(M) = \max(a(M))$. We may also let $a^A(M)$ denote the set a^A relativized to a suitable structure A such that $M \in \mathscr{V}^A$.*

Note that $a(M)$ is a closed subset of E of size at most $|\mathrm{Hull}(M, V_{\kappa_M})|$, which is less than κ if M is in $\mathscr{C} \cup \mathscr{U}$.

Proposition 3.2.24. *Let $M \in \mathscr{V}$ and $N \in \mathscr{U}$. Suppose $\alpha \in E$, M and N are active at α, and $M \in_\alpha N$. Then $\alpha \in N$.*

Proof. We may assume that M and N are α-models. Let $A = \mathrm{Hull}(N, V_\alpha)$. Then $A \in \mathscr{A}_\alpha$. Fix $M^* \in N$ with $M^* \in \mathscr{V}^A$ which is α-isomorphic to M. Since M^* is α-equivalent to M, we have that $\alpha \in a^A(M^*)$. On the other hand, we have

$a^A(M^*) \cap \eta(M^*) \in N$ and has size less than κ_N, hence $a^A(M^*) \cap \eta(M^*) \subseteq N$. It follows that $\alpha \in N$. $\boxed{3.2.24}$

Proposition 3.2.25. *Let $M \in \mathscr{V}$ and $N \in \mathscr{U}$. Suppose $\alpha \in a(M)$ is a limit point of E and $M \in_\gamma N$, for all $\gamma \in E \cap \alpha$. Then $\alpha \in N$ and $M \in_\alpha N$, and hence $M \restriction_\alpha \in N$.*

Proof. Let $a = a(M) \cap N \cap \alpha$. Note that a is unbounded in α and has size less κ_N. Since N is closed under $< \kappa_N$-sequences, it follows that $a \in N$, and hence $\alpha = \sup(a) \in N$. For $\gamma < \alpha$, let $M_\gamma = M \restriction_\gamma$. For $\gamma \in a$, we have that $M \in_\gamma N$, and hence $M_\gamma \in N$. Let $A_\gamma = \mathrm{Hull}(M_\gamma, V_\gamma)$. For $\gamma, \delta \in a$ with $\gamma < \delta$, we have that $M_\delta \restriction_\gamma = M_\gamma$. In other words, A_γ is the transitive collapse of $\mathrm{Hull}(M_\delta, V_\gamma)$, and if $\sigma_{\gamma,\delta}$ is the inverse of the collapse map, we have $\sigma_{\gamma,\delta}[M_\gamma] = M_\delta$. Each of the maps $\sigma_{\gamma,\delta}$ is definable from M_δ and γ, and hence it belongs to N. Now, N is closed under $< \kappa_N$-sequences and therefore the whole system $(A_\gamma, \sigma_{\gamma,\delta} : \gamma \leq \delta \in a)$ belongs to N. Let A be the direct limit of this system, and let σ_γ be the canonical embedding of A_γ to A. If we let π_γ be the collapse map of $\mathrm{Hull}(M, V_\gamma)$ to A_γ, we then have that, for every $\gamma < \delta$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hull}(M, V_\gamma) \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi_\gamma"'] & \mathrm{Hull}(M, V_\delta) \arrow[d, "\pi_\delta"] \\
A_\gamma \arrow[r, "\sigma_{\gamma,\delta}"] & A_\delta
\end{tikzcd}$$

Since $\mathrm{Hull}(M, V_\alpha) = \bigcup\{\mathrm{Hull}(M, V_\gamma) : \gamma \in a\}$, we have that A is isomorphic to $\mathrm{Hull}(M, V_\alpha)$. Therefore, its transitive collapse is $A_\alpha = \mathrm{Hull}(M \restriction_\alpha, V_\alpha)$, and if we let π be the collapse map, $\pi[M] = M \restriction_\alpha$. We can therefore identify A with A_α, and we get that $\sigma_\gamma[M_\gamma] = M \restriction_\alpha$, for any $\gamma \in a$. Thus $M \restriction_\alpha \in N$, as required. $\boxed{3.2.25}$

From now on, we will concentrate exclusively on virtual models in $\mathscr{C} \cup \mathscr{U}$. We define an operation, that we call the *meet*, that will play the role of intersection for virtual models. We only define the meet of two models of different types. Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapse map. Note that if $\overline{N} \in M$, then $\overline{N} \cap M$ is a countable elementary submodel of $\overline{N}$. Then $\overline{N} \cap M \in \overline{N}$ since $\overline{N}$ is closed under countable sequence. Note that $\pi^{-1}(\overline{N} \cap M) = \pi^{-1}[\overline{N} \cap M]$, and this model is elementary in N.

Definition 3.2.26. *Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Let $\alpha = \max(a(N) \cap a(M))$. We shall define $N \wedge M$ if $N \in_\alpha M$. Let $\overline{N}$ be the transitive collapse of N, and let π be*

the collapse map. Set

$$\eta = \sup(\sup(\pi^{-1}[\overline{N} \cap M] \cap \mathrm{ORD}) \cap E \cap (\alpha + 1)).$$

We define the meet of N and M to be $N \wedge M = \pi^{-1}[\overline{N} \cap M] \restriction_\eta$.

Notice that $\min(E)$ belongs to $a(N) \cap a(M)$, and hence α is well-defined. To make sense of the above definition, we need to prove the following.

Proposition 3.2.27. *Under the assumptions of the above definition, $N \wedge M \in \mathscr{C}_\eta$.*

Proof. Since $\eta(N) \geq \alpha$, we can form the model $A = \mathrm{Hull}(N, V_\alpha)$ and, we thus have $N \prec A$ and $V_\alpha \prec A$. Since $\overline{N} \in M$, we have that $\overline{N} \cap M \prec \overline{N}$. Therefore, we have $\pi^{-1}[\overline{N} \cap M] \prec N$. Now, $\eta \in E \cap (\alpha + 1)$ and so $V_\eta \prec V_\alpha \prec A$. Moreover, we should have $\sup(\pi^{-1}[\overline{N} \cap M] \cap \mathrm{ORD}) \geq \eta$. By Lemma 3.1.3 we have that $\mathrm{Hull}(\pi^{-1}[\overline{N} \cap M], V_\eta) \prec A$ and $V_\eta \subseteq \mathrm{Hull}(\pi^{-1}[\overline{N} \cap M], V_\eta)$. It follows that the transitive collapse of $\mathrm{Hull}(\pi^{-1}[\overline{N} \cap M], V_\eta)$ belongs to $\mathscr{A}_\eta$. On the other hand the predicate of $N \wedge M$, say $U^{N \wedge M}$, comes through the isomorphisms above. To be more precise we first consider the predicate of N, U^N, and then compute the predicates $U^{\overline{N}}$, $U^{\overline{N}} \cap M$ and so on so forth. Thus the image of $\pi^{-1}[\overline{N} \cap M]$ under the collapse map belongs to $\mathscr{C}_\eta$. $\boxed{3.2.27}$

The following proposition justifies that the meet is not too far from the ordinary intersection.

Proposition 3.2.28. *Let $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Suppose $\alpha \in E$ and the meet $N \wedge M$ is defined and active at α. Then $(N \wedge M) \cap V_\alpha = N \cap M \cap V_\alpha$.*

Proof. Let $\beta = \max(a(N) \cap a(M))$. Since the meet of N and M is defined we must have $N \in_\beta M$. Since $N \wedge M$ is active at α, we must have $\alpha \leq \beta$. Let $N' \in \mathscr{V}^M$ be such that $N' \cong_\beta N$. Let σ be the β-isomorphism between N and N'. Notice that σ is the identity on $N \cap V_\beta$ and thus also on $N \cap V_\alpha$. Let $\overline{N}$ denote the common transitive collapse of N and N', and let π and π' be the collapse maps. Then the following diagram commutes.

$$\begin{tikzcd}
N \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & N' \arrow[dl, "\pi'"] \\
& \overline{N} &
\end{tikzcd}$$

Since $N' \in M$, $\pi'^{-1}[\overline{N} \cap M] = N' \cap M$. On the other hand σ is the identity on $N \cap V_\alpha$. Therefore we have the following.

$$\begin{aligned}(N \wedge M) \cap V_\alpha &= \pi^{-1}[\overline{N} \cap M] \cap V_\alpha \\ &= [\sigma^{-1}\pi'^{-1}[\overline{N} \cap M]] \cap V_\alpha \\ &= \sigma^{-1}[\overline{N} \cap M \cap V_\alpha] \\ &= N \cap M \cap V_\alpha\end{aligned}$$

3.2.28

Proposition 3.2.29. *Let $\alpha \in E$. Suppose $N \in \mathscr{U}$ and $M \in \mathscr{C}$, the meet $N \wedge M$ is defined, and N and M are strongly active at α. Then $N \wedge M$ is strongly active at α.*

Proof. Let $\beta = \max(a(N) \cap a(M))$. Since both N and M are active at α, we must have $\alpha \leq \beta$. Let $N' \in M$ with $N' \in \mathscr{V}^M$ be such that $N' \cong_\beta N$. Let σ be the β-isomorphism between N' and N. Then $\sigma \restriction_{N' \cap V_\beta}$ is the identity. Note that $N' \cap M \cap E \cap \alpha$ is unbounded in $E \cap \alpha$. Since $N' \cap V_\alpha = N \cap V_\alpha$, we must have that $N \cap M \cap E \cap \alpha$ is also unbounded in $E \cap \alpha$. By Proposition 3.2.28, $N \wedge M$ is strongly active at α. 3.2.29

The next proposition states the meet operation commutes with the projections.

Proposition 3.2.30. *Let $N \in \mathscr{U}$ and $M \in \mathscr{C}$. Suppose $\alpha \in E$ and the meet $N \wedge M$ is defined and active at α. Then $(N \wedge M) \restriction_\alpha = N \restriction_\alpha \wedge M \restriction_\alpha$.*

Proof. First note that if $N \wedge M$ is active at α, then $\alpha \in a(N) \cap a(M)$. It follows that α is the maximum of $a(N \restriction_\alpha) \cap a(M \restriction_\alpha)$. Then note that $N \wedge M$ depends only on $\max(a(N) \cap a(M))$, N, and $M \cap \overline{N}$, where $\overline{N}$ is the transitive collapse of N. Now, $\overline{N}$ is also the transitive collapse of $N \restriction_\alpha$. In fact, if σ is the α-isomorphism between N and $N \restriction_\alpha$, and π and π' are the collapse maps of N and $N \restriction_\alpha$ respectively, then $\pi = \pi' \circ \sigma$. Therefore, $\sigma \restriction_{\pi^{-1}[\overline{N} \cap M]}$ is an α-isomorphism between $\pi^{-1}[\overline{N} \cap M]$ and $\pi'^{-1}[\overline{N} \cap M \restriction_\alpha]$. It follows that $(N \wedge M) \restriction_\alpha = N \restriction_\alpha \wedge M \restriction_\alpha$. 3.2.30

Proposition 3.2.31. *Let $\alpha \in E$. Suppose $N \in \mathscr{U}$, $M \in \mathscr{C}$, both are active at α and $N \in_\alpha M$. Let P be another virtual model also active at α. Then $P \in_\alpha N \wedge M$ if only if $P \in_\alpha N$ and $P \in_\alpha M$.*

Proof. By Proposition 3.2.30 we may assume that N, M and P are all α-models. Assume first that $P \in_\alpha N \wedge M$. In particular this means that $N \wedge M$ is active at α.

In particular we have that $N \wedge M \subseteq N$, and hence $P \in_\alpha N$. Fix $N' \in \mathscr{V}^M$ which is α-isomorphic to N. Let $\overline{N}$ be the transitive collapse of both N and N' and let π and π' be the respective collapse maps. Note that $\sigma = \pi'^{-1} \circ \pi$ is the α-isomorphism between N and N'. Then $\sigma[N \wedge M] = N' \cap M$. Pick also $P' \in N \wedge M$ which is α-isomorphic to P. By Proposition 3.2.10, P' and $\sigma(P')$ are also α-isomorphic. Since $\sigma(P') \in M$, by the transitivity of $\cong_\alpha$ we get that P is α-isomorphic to $\sigma(P')$. This implies that $P \in_\alpha M$.

Now assume $P \in_\alpha N$ and $P \in_\alpha M$. By Proposition 3.2.24 we know that $\alpha \in N$. Since P is an α-model, we conclude that $P \in N$. If also $\alpha \in M$, we have that $N, P \in M$ and $N \wedge M = N \cap M$. Therefore, $P \in N \wedge M$. Assume now that $\alpha \notin M$ and let $\alpha^* = \min(M \cap \lambda \setminus \alpha)$. Let $A = \mathrm{Hull}(M, V_\alpha)$. Since we assumed that M is an α-model, we have that $A \in \mathscr{A}_\alpha$ and $\alpha \in E_A$. By Lemma 3.1.1 we also have that $\alpha^* \in E_A$. Fix $P^*, N^* \in M$ that are α-isomorphic to P and N respectively. By projecting them to α^* if necessary, we may assume $P^*, N^* \in \mathscr{V}^A_{\alpha^*}$. Moreover, N^* is a Magidor model from the point of view of A. Since $P^* \in_\alpha N^*$ and α^* is the least ordinal in M above α we have

$$M \models \text{“}\forall \delta \in E_A \cap \alpha^* \quad P^* \in_\delta N^*\text{”}.$$

Moreover, $M \models$ “P^* is active at α^*”. Since α^* is a limit point of E_A, we can apply Proposition 3.2.25 in A and conclude that $\alpha^* \in N^*$ and $P^* \in N^*$. Hence $P^* \in N^* \cap M$. Let now σ witness the α-isomorphism between N^* and N. Then $\sigma[N^* \cap M] = N \wedge M$. Hence $\sigma(P^*)$, that is α-isomorphic to P, belongs to $N \wedge M$. It follows that $P \in_\alpha N \wedge M$. $\boxed{3.2.31}$

One feature of the meet is the following absorption property.

Proposition 3.2.32. *Suppose $N \in \mathscr{U}$, $M \in \mathscr{C}$, and the meet $N \wedge M$ is defined. Let $\alpha \in E$, and suppose P is a Magidor α-model active at α such that $P \in_\alpha N \wedge M$. Then $P \wedge M = P \wedge (N \wedge M)$.*

Proof. Since $P \in_\alpha N \wedge M$ and P is active at α, so is $N \wedge M$, and hence both N and M are active at α as well. Let $\overline{P}$ be the transitive collapse of P. Then $\overline{P} \in N \cap V_\kappa$, and since $N \cap V_\kappa$ is transitive, we have $\overline{P} \subseteq N$. Hence $\overline{P} \cap (N \wedge M) = \overline{P} \cap M$. It follows that $P \wedge M = P \wedge (N \wedge M)$. $\boxed{3.2.32}$

Proposition 3.2.33. *Let $\alpha \in E$. Suppose $N \in \mathscr{U}$, $M \in \mathscr{C}$ and the meet $N \wedge M$ is defined and active at α. Suppose $P \in \mathscr{V}$ and $N, M \in_\alpha P$. Then $N \wedge M \in_\alpha P$.*

Proof. We may assume M, N and P are all α-models. If $\alpha \in P$ then $N, M \in P$, and hence also $N \wedge M \in P$. Suppose now $\alpha \notin P$. Let $A = \mathrm{Hull}(P, V_\alpha)$ and let $\alpha^* = \min(P \cap \mathrm{ORD} \setminus \alpha)$. Note that α^* has uncountable cofinality in A since otherwise $\alpha = \alpha^* \in P$. By Lemma 3.1.1 we have $\alpha^* \in E_A$. We can find $N^*, M^* \in P$ such that $N^* \cong_\alpha N$ and $M^* \cong_\alpha M$. We may assume that $N^* \in \mathscr{U}^A_{\alpha^*}$ and $M^* \in \mathscr{C}^A_{\alpha^*}$. Work for a moment in A. Since $N^* \in_\alpha M^*$, α^* is the least ordinal of P above α, and $N^*, M^* \in P$, we have

$$A \models \text{“}\forall \gamma \in E_A \cap \alpha^* \quad N^* \in_\gamma M^*\text{”}.$$

By applying Proposition 3.2.13 inside A we have that $N^* \in_{\alpha^*} M^*$, and hence A can compute the meet, say Q, of N^* and M^*. Then $Q \in P$, and by applying Proposition 3.2.30 inside A, we get $Q \restriction_\alpha = N^* \restriction_\alpha \wedge M^* \restriction_\alpha$. Hence we should have $Q \cong_\alpha N \wedge M$. $\boxed{3.2.33}$

Final Remark

We may mention that almost all facts about and properties of the virtual models stated in this chapter are still valid if one uses a Mahlo cardinal instead of the supercompact cardinal one. To be precise, one can use Fact 1.5.1 to obtain stationary many κ-Mahlo models.

Chapter Four

Forcing with Virtual Models of Two Types

We are now about to present the skeleton of our forcing construction. As we mentioned earlier, this relies on considering finite collections of virtual models in a coherent way, and as expected, we should be able somehow to design a virtual version of $\in$-chains of models.

4.1 Pure Side Conditions without Decorations

We start with the definition of an α-chain.

Definition 4.1.1. *Let $\alpha \in E$ and let $\mathcal{M}$ be a subset of $\mathscr{U} \cup \mathscr{C}$. We say $\mathcal{M}$ is an* α-chain *if for all distinct $M, N \in \mathcal{M}$, either $M \in_\alpha N$ or $N \in_\alpha M$, or there is a $P \in \mathcal{M}$ such that either $M \in_\alpha P \in_\alpha N$ or $N \in_\alpha P \in_\alpha M$.*

The following proposition demonstrates that our intuitive comprehension of finite α-chains is similar to the ordinary $\in$-chains.

Proposition 4.1.2. *Suppose $\alpha \in E$ and $\mathcal{M}$ is a finite subset of $\mathscr{U} \cup \mathscr{C}$. Then $\mathcal{M}$ is an α-chain if and only if there is an enumeration $\langle M_i : i < n \rangle$ of $\mathcal{M}$ such that $M_0 \in_\alpha M_1 \in_\alpha \cdots \in_\alpha M_{n-1}$.*

Proof. Suppose first $\mathcal{M}$ is an α-chain. Define the relation $<$ on $\mathcal{M}$ by letting $M < N$ iff $\kappa_M < \kappa_N$. It is straightforward to see that $<$ is a total ordering on $\mathcal{M}$. We can then let $\langle M_i : i < n \rangle$ be the $<$-increasing enumeration of $\mathcal{M}$. Conversely, suppose $\langle M_i : i < n \rangle$ is the enumeration such that $M_0 \in_\alpha M_1 \in_\alpha \cdots \in_\alpha M_{n-1}$. Let $i < j < n$.

If $j = i+1$ then $M_i \in_\alpha M_j$. Suppose $j > i+1$. If M_j is a Magidor model or if there are no Magidor models between M_i and M_j by Proposition 3.2.21 we conclude that $M_i \in_\alpha M_j$. Otherwise let $k < j$ be the largest index such that M_k is a Magidor model. Then again by Proposition 3.2.21, we conclude that $M_i \in_\alpha M_k \in_\alpha M_j$. $\boxed{4.1.2}$

Though the definition of an α-chain is natural and simple, yet it is not exactly what we want to utilize as side conditions. Our side conditions are simply finite collections of virtual models with no disciplines, but once we observe them from the viewpoint of some level $\alpha \in E$, they sit in an $\in_\alpha$-chain. To this end we need to introduce some notation to see how a collection of virtual models can induce $\in_\alpha$-chains, these chains are exactly those objects carry information in our forcing constructions.

Notation 4.1.3. *Let $\alpha \in E$ and let $\mathcal{M}$ be a set of virtual models. We let*

$$\mathcal{M} \restriction_\alpha = \{M \restriction_\alpha : M \in \mathcal{M}\}$$

and

$$\mathcal{M}^\alpha = \{M \restriction_\alpha : \ M \in \mathcal{M} \textit{ is active at } \alpha\}.$$

Let $\alpha \in E$ and let $\mathcal{M}$ be an α-chain. Let $\in_\alpha^*$ be the transitive closure of $\in_\alpha$. Then $\in_\alpha^*$ is a total ordering on $\mathcal{M}$. For $M, N \in \mathcal{M}$, we say M is *α-below* N in $\mathcal{M}$, or equivalently N is *α-above* M in $\mathcal{M}$, if $M \in_\alpha^* N$ in $\mathcal{M}$. Now using the transitivity of $\in_\alpha^*$ we can form intervals in $\mathcal{M}$. Let

$$(M, N)^\alpha_{\mathcal{M}} = \{P \in \mathcal{M} : M \in_\alpha^* P \in_\alpha^* N\}.$$

Similarly we can define $[M, N]^\alpha_{\mathcal{M}}$, $[M, N)^\alpha_{\mathcal{M}}$, etc. For convenience we also allow that the endpoints of the intervals to be $\varnothing$ or V_λ; let $(\varnothing, N)^\alpha_{\mathcal{M}}$ be $\{P \in \mathcal{M} : P \in_\alpha^* N\}$ in the first case, and let $(N, V_\lambda)^\alpha_{\mathcal{M}}$ be $\{P \in \mathcal{M} : \ N \in_\alpha^* P\}$ in the second case.

We fix an inaccessible cardinal κ and a cardinal $\lambda > \kappa$ with $\mathrm{cof}(\lambda) \geq \kappa$ such that $(V_\lambda, \in, \kappa)$ is suitable. We may assume that U is an arbitrary predicate over V_λ so that $(V_\lambda, \in, \kappa, U)$ is suitable though we are not going to work with U in this chapter, so let us simply ignore it. Let also $E = E_{(V_\lambda, \in, \kappa, U)}$. We start with the definition of $\mathbb{M}^\kappa_\alpha$, for all $\alpha \in E \cup \{\lambda\}$.

Definition 4.1.4 (pure side conditions)**.** *Suppose $\alpha \in E$. We say that $p = \mathcal{M}_p$ belongs to $\mathbb{M}^\kappa_\alpha$ if:*

1. *$\mathcal{M}_p$ is a finite subset of $\mathscr{C}_{\leq\alpha} \cup \mathscr{U}^\kappa_{\leq\alpha}$ that is closed under meets,*

2. *$\mathcal{M}^\delta_p$ is a δ-chain, for all $\delta \in E \cap (\alpha+1)$.*

We let $\mathcal{M}_q \leq \mathcal{M}_p$ if for all $M \in \mathcal{M}_p$ there is $N \in \mathcal{M}_q$ such that $N \restriction_{\eta(M)} = M$. Finally, let $\mathbb{M}^\kappa_\lambda = \bigcup\{\mathbb{M}^\kappa_\alpha : \alpha \in E\}$ with the same ordering.

Remark 4.1.5. Conditions (1) and (2) can be merged to a single condition. Let us say that a δ-chain $\mathcal{M}$ consisting of models active at δ is *closed under meets* if for every $M, N \in \mathcal{M}$, if the meet $M \wedge N$ is defined and active at δ then $M \wedge N \in \mathcal{M}$. Thus we can simply say that $\mathcal{M}^\delta_p$ is a δ-chain closed under meets, for all $\delta \in E \cap (\alpha+1)$. The order is natural since if $N \restriction_{\eta(M)} = M$, then N carries all the information that M does.

Let us explain what happens after forcing with $\mathbb{M}^\kappa_\lambda$, when $\kappa < \lambda$ are supercompact cardinals. Suppose G is generic over $\mathbb{M}^\kappa_\lambda$. Then ω_1 is preserved, but κ becomes ω_2 and λ becomes ω_3 in $V[G]$. Let $\mathcal{M}_G = \bigcup G$, and let $G_\alpha = G \cap \mathbb{M}^\kappa_\alpha$, for $\alpha \in E$. One can show that for every $\alpha \in E$ and $\beta > \alpha$, $V_\beta[G_\alpha]$ is an ω_1-guessing model in $V[G]$. To see this fix some $\delta \in E \setminus \beta$ with $\mathrm{cof}(\delta) < \kappa$. One shows that if M is a Magidor model in $\mathcal{M}^\delta_G$ then $M[G_\alpha]$ is an ω_1-guessing model in $V[G]$. Moreover, if M is a Magidor model which is a limit of Magidor models in the δ-chain $\mathcal{M}^\delta_G$ then $M \cap V_\delta$ is covered by the union of the previous models in $\mathcal{M}^\delta_G$. Therefore, if we let $\mathcal{G} = \{(M \cap V_\beta)[G_\alpha] : M \in \mathcal{M}^\delta_G \cap \mathcal{U}^\kappa_\delta\}$, then $\mathcal{G}$ is an increasing sequence of ω_1-guessing models which is continuous at uncountable limits and the union of this sequence is $V_\beta[G_\alpha]$. We will actually present a proof not for the forcing $\mathbb{M}^\kappa_\lambda$, but for a slight variation $\mathfrak{M}^\kappa_\lambda$. We would like to arrange that in addition the set $\{\sup(M \cap \kappa) : M \in \mathcal{M}^\delta_G\}$ be a club in κ, for all $\delta \in E$ with $\mathrm{cof}(\delta) < \kappa$. In order to achieve this we will add *decorations* to the conditions of $\mathbb{M}^\kappa_\lambda$. This device consists of attaching to each model M of an $\in$-chain a finite set $d_p(M)$ which belongs to all models N of the chain such that $M \in N$. In a stronger condition this finite set is allowed to increase. The main point is that $d_p(M)$ controls what models can be added $\in$-above M in stronger conditions. In our situation there are some complications. First, we have not $\in$-chain, but a δ-chain, for each $\delta \in E$. It is therefore reasonable to have decorations for each level $\delta \in E$. Now, models from a higher level project to lower levels at which they are active, but also in order to arrange strong properness for countable models, some models from lower levels will be *lifted* to higher levels and put on the chain. This imposes a subtle interplay between the decorations on different levels. In order to describe this precisely, we need to make some preliminary definitions.

4.2 Pure Side Conditions with Decorations

Notation 4.2.1. *Suppose $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$. Let*

$$\mathcal{L}(\mathcal{M}_p) = \{M \restriction_\alpha : M \in \mathcal{M}_p \text{ and } \alpha \in a(M)\}.$$

Definition 4.2.2. *Suppose $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$. We say that $M \in \mathcal{L}(\mathcal{M}_p)$ is $\mathcal{M}_p$-*free *if every $N \in \mathcal{M}_p$ with $M \in_{\eta(M)} N$ is strongly active at $\eta(M)$. Let $\mathcal{F}(\mathcal{M}_p)$ denote the set of all $M \in \mathcal{L}(\mathcal{M}_p)$ that are $\mathcal{M}_p$-free.*

Note that if $\mathcal{M}_q \leq \mathcal{M}_p$ then $\mathcal{L}(\mathcal{M}_p) \subseteq \mathcal{L}(\mathcal{M}_q)$ and $\mathcal{F}(\mathcal{M}_q) \cap \mathcal{L}(\mathcal{M}_p) \subseteq \mathcal{F}(\mathcal{M}_p)$. In other words, a node $M \in \mathcal{L}(\mathcal{M}_p)$ that is not $\mathcal{M}_p$-free is not $\mathcal{M}_q$-free, for any $\mathcal{M}_q \leq \mathcal{M}_p$. We are now ready to decorate $\mathbb{M}^\kappa_\lambda$.

Definition 4.2.3 (side conditions with decorations)**.** *Suppose $\alpha \in E \cup \{\lambda\}$. We say that a pair $p = (\mathcal{M}_p, d_p)$ belongs to $\mathfrak{M}^\kappa_\alpha$ if $\mathcal{M}_p \in \mathbb{M}^\kappa_\alpha$, d_p is a finite partial function from $\mathcal{F}(\mathcal{M}_p)$ to $\mathcal{P}_\omega(V_\kappa)$, and*

$$(\divideontimes) \qquad \text{if } M \in \mathrm{dom}(d_p),\ N \in \mathcal{M}_p, \text{ and } M \in_{\eta(M)} N, \text{ then } d_p(M) \in N.$$

We say that $q \leq p$ if $\mathcal{M}_q \leq \mathcal{M}_p$, and that for every $M \in \mathrm{dom}(d_p)$ there exists some $\gamma \in E \cap (\eta(M) + 1)$ such that $M \restriction_\gamma \in \mathrm{dom}(d_q)$ and $d_p(M) \subseteq d_q(M \restriction_\gamma)$.

We refer to d_p as the decoration of p. The point is that if $M \in \mathrm{dom}(d_p)$ is a δ-model then $d_p(M)$ constraints what models N with $M \in_\delta N$ can be put on $\mathcal{M}^\delta_q$, for any $q \leq p$. In general, M may not be $\mathcal{M}_q$-free, in which case $M \notin \mathrm{dom}(d_q)$, but then we have some $\gamma \leq \delta$ such that $M \restriction_\gamma$ is $\mathcal{M}_q$-free and $d_p(M) \subseteq d_q(M \restriction_\gamma)$. Note that then we must have $d_p(M) \in N$, for any $N \in \mathcal{M}_q$ such that $M \in_\delta N$.

The ordering on $\mathfrak{M}^\kappa_\lambda$ is clearly transitive. We will say that q is *stronger* than p if q forces that p belongs to the generic filter, in order words, any $r \leq q$ is compatible with p. We write $p \sim q$ if each of p and q is stronger than the other. We identify equivalent conditions, often without saying it. Our forcing does not have greatest lower bounds, but if p and q do have a greatest lower bound we will denote it by $p \wedge q$. To be precise we should refer to $p \wedge q$ as the $\sim$-equivalence class of a greatest lower bound, but we ignore this point since it should not cause any confusion. Note that if $p \in \mathfrak{M}^\kappa_\alpha$ and $M \in \mathcal{M}_p$ is a δ-model that is not active at δ, we may replace M by $M \restriction_{\alpha(M)}$[1] and we get an equivalent condition. Thus, if $\alpha \in E$ and $\mathrm{cof}(\alpha) \geq \kappa$, then $\mathfrak{M}^\kappa_\alpha$ is forcing equivalent to $\bigcup\{\mathfrak{M}^\kappa_\gamma : \gamma \in E \cap \alpha\}$.

[1] Recall that $\alpha(M) = \max(a(M))$

Convention 4.2.4. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $\delta \in E$. If $M, N \in \mathcal{M}^\delta_p$ with $M \in^*_\delta N$, we will write $(M, N)^\delta_p$ for the interval $(M, N)^\delta_{\mathcal{M}_p}$, and similarly, for $[M, N)^\delta_p$, $(M, N]^\delta_p$, etc.*

Suppose $\alpha, \beta \in E$ and $\alpha \leq \beta$. For every $p \in \mathfrak{M}^\kappa_\beta$, we let $\mathcal{M}_{p\restriction_\alpha} = \mathcal{M}_p \restriction_\alpha$ and $d_{p\restriction_\alpha} = d_p \restriction_{\mathcal{F}(\mathcal{M}_p\restriction_\alpha)}$. It is easily seen that $p \restriction_\alpha = (\mathcal{M}_{p\restriction_\alpha}, d_{p\restriction_\alpha}) \in \mathfrak{M}^\kappa_\alpha$. The following is straightforward.

Lemma 4.2.5. *Suppose $\alpha, \beta \in E$ with $\alpha \leq \beta$. Let $p \in \mathfrak{M}^\kappa_\beta$ and let $q \in \mathfrak{M}^\kappa_\alpha$ be such that $q \leq p \restriction_\alpha$. Then there exists $r \in \mathfrak{M}^\kappa_\beta$ such that $r \leq p, q$.*

Proof. We let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. Note that $\mathcal{M}_r$ is closed under meets. We define d_r by letting $d_r(M) = d_q(M)$ if $M \in \mathrm{dom}(d_q)$, and $d_r(M) = d_p(M)$ if $M \in \mathrm{dom}(d_p)$ with $\eta(M) > \alpha$. It is straightforward that r is as required. $\boxed{4.2.5}$

Remark 4.2.6. The condition r from the previous lemma is the greatest lower bound of p and q, so we will write $r = p \wedge q$.

Corollary 4.2.7. *Suppose $\alpha \leq \beta \in E \cup \{\lambda\}$. Then $\mathfrak{M}^\kappa_\alpha$ is a complete suborder of $\mathfrak{M}^\kappa_\beta$.*

$\boxed{4.2.7}$

Our goal is to prove that our poset $\mathfrak{M}^\kappa_\lambda$ is strongly proper for an appropriate class of models. We start by showing that if a condition p belongs to a model M we can always add M to $\mathcal{M}_p$ and form a new condition.

Lemma 4.2.8. *Let $p \in \mathfrak{M}^\kappa_\lambda$ and $M \in \mathscr{C} \cup \mathscr{U}$ be such that $p \in M$. Then there is a weakest condition $p^M \leq p$ with $M \in \mathcal{M}_{p^M}$.*

Proof. Suppose first that M is a Magidor model. Then we let $\mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M\}$ and $d_{p^M} = d_p$. It is straightforward that $p^M = (\mathcal{M}_{p^M}, d_{p^M})$ is as required.

Now assume that M is countable. We let $\mathcal{M}_{p^M}$ be the closure of $\mathcal{M}_p \cup \{M\}$ under meets. Fix $\delta \in E$. We show that $\mathcal{M}^\delta_{p^M}$ is an $\in_\delta$-chain. We may assume that M is active at δ since otherwise $\mathcal{M}^\delta_{p^M} = \varnothing$. By Proposition 3.2.32 we know that the only models added to $\mathcal{M}^\delta_p$ in order to form $\mathcal{M}^\delta_{p^M}$ are $M \restriction_\delta$ and $N \wedge M \restriction_\delta$ for $N \in \mathcal{M}^\delta_p$ such that $N \wedge M$ is active at δ. Suppose $N \in \mathcal{M}^\delta_p$ is such a model, and let P be the $\in_\delta$-predecessor of N in $\mathcal{M}^\delta_p$, if it exists. First note that $N \cap M \restriction_\delta \in N$ since N is closed under countable sequence. Therefore, $N \wedge M \restriction_\delta \in N$. Moreover, if P exists by Proposition 3.2.31 we have that $P \in_\delta N \wedge M \restriction_\delta$. This establishes that $\mathcal{M}^\delta_{p^M}$ is a δ-chain.

Let us now define the decoration d_{p^M}. Suppose $N \in \mathrm{dom}(d_p)$ is a δ-model. Then $\delta \in M$. If M is strongly active at δ, then by Proposition 3.2.28, for every Magidor model $P \in \mathcal{M}_p$ if $P \wedge M$ is active at δ then it is strongly active at δ. Hence N is $\mathcal{M}_{p^M}$-free. We then keep N in $\mathrm{dom}(d_{p^M})$ and let $d_{p^M}(N) = d_p(N)$. Now, suppose M is not strongly active at δ. This means that δ has uncountable cofinality in M. Let $\bar{\delta} = \sup(M \cap \delta)$ and note that $\bar{\delta}$ is a limit point of E. We claim that $N \restriction_{\bar{\delta}}$ is $\mathcal{M}_{p^M}$-free. Indeed, if there is $P \in \mathcal{M}_{p^M}$ such that $N \in_{\bar{\delta}} P$ and P is not strongly active at $\bar{\delta}$, then $P \in M$, and hence $\eta(P) \geq \delta$. Moreover, P is active but not strongly active at δ as well. Since $N \in_{\bar{\delta}} P$ and $N, P \in M$ it follows that $N \in_\gamma P$, for unboundedly many $\gamma \in E \cap \delta \cap M$. But then by Proposition 3.2.13 applied in M we conclude that $N \in_\delta P$, and hence N is not $\mathcal{M}_p$-free, a contradiction. Notice also that if $P \in \mathcal{M}_p$ and $N \in_{\bar{\delta}} P$ then by Proposition 3.2.13 again we must have that $N \in_\delta P$ and thus $d_p(N) \in P$. Therefore, we can replace N by $N \restriction_{\bar{\delta}}$ and let $d_{p^M}(N \restriction_{\bar{\delta}}) = d_p(N)$. It is straightforward to check that p^M is a weakest extension of p such that $M \in \mathcal{M}_{p^M}$. $\boxed{4.2.8}$

Notation 4.2.9. *For virtual models N, M, we set $\alpha(N, M) = \max(a(N) \cap a(M))$.*

We are now about to give the restriction of a condition to a given model. We start with Magidor models.

Definition 4.2.10. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. For $N \in \mathcal{M}_p$, we let $N \restriction_M = N \restriction_{\alpha(N,M)}$ if $\kappa_N < \kappa_M$, otherwise $N \restriction_M$ is undefined. Let*

$$\mathcal{M}_{p\restriction_M} = \{N \restriction_M : N \in \mathcal{M}_p\}.$$

Let $d_{p\restriction_M} = d_p \restriction_{(\mathrm{dom}(d_p)\cap M)}$, and let $p \restriction_M = (\mathcal{M}_{p\restriction_M}, d_{p\restriction_M})$.

Lemma 4.2.11. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. Then $p \restriction_M \in \mathfrak{M}^\kappa_\lambda \cap M$ and $p \leq p \restriction_M$.*

Proof. Since p is a condition, we have that if $N \in \mathcal{M}_p$ and $\kappa_N < \kappa_M$, then $N \in^*_\gamma M$, for all $\gamma \in a(N) \cap a(M)$. By Proposition 3.2.21 we then conclude that $N \in_\gamma M$, for all such γ. By Proposition 3.2.24 we have that $\alpha(N, M) \in M$, and hence $N \restriction_{\alpha(N,M)} \in M$. We also have that $d_{p\restriction_M} \in M$, thus $p \restriction_M \in M$. Let us check that $\mathcal{M}_{p\restriction_M} \in \mathbb{M}^\kappa_\lambda$. Suppose $\delta \in E$. If M is not active at δ then $\mathcal{M}^\delta_{p\restriction_M}$ is empty, otherwise it is equal to $(\varnothing, M \restriction_\delta)^\delta_p$, which is obviously a δ-chain. To check that $\mathcal{M}_{p\restriction_M}$ is closed under meets, suppose $N \restriction_M, P \restriction_M \in \mathcal{M}_{p\restriction_M}$ and their meet is defined. Note that then $N \wedge P$ is also defined and, by Proposition 3.2.30 $(N \wedge P) \restriction_M = N \restriction_M \wedge P \restriction_M$. It is straightforward

to check that every $N \in \mathrm{dom}(d_{p\restriction_M})$ is $\mathcal{M}_{p\restriction_M}$-free, and $(\divideontimes)$ from Definition 4.2.3 holds. Finally, the fact that $p \leq p \restriction_M$ follows from the definition. $\boxed{4.2.11}$

Lemma 4.2.12. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a Magidor model. Suppose $q \in M \cap \mathfrak{M}^\kappa_\lambda$ extends $p \restriction_M$. Then q is compatible with p and the meet $p \wedge q$ exists.*

Proof. We define $r \in \mathfrak{M}^\kappa_\lambda$ and check that it is a weakest condition extending p and q. Let $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$. We check that if $\delta \in E$, then $\mathcal{M}^\delta_r$ is a δ-chain closed under meets, meaning if $P, Q \in \mathcal{M}^\delta_r$ and the meet $P \wedge Q$ is defined and active at δ then $P \wedge Q \in \mathcal{M}^\delta_r$. Fix such $\delta \in E$. If M is not active at δ, then $\mathcal{M}^\delta_r = \mathcal{M}^\delta_p$ and thus has the required property since p is a condition. Now, suppose M is active at δ. If $R \in \mathcal{M}^\delta_r$ and $R \in^*_\delta M$, then by Proposition 3.2.21 we know that $R \in_\delta M$, and by Proposition 3.2.24 we get that $\delta \in M$. Hence $R \in M$ and therefore $R \in \mathcal{M}^\delta_q$. Therefore, $\mathcal{M}^\delta_r$ is the union of $\mathcal{M}^\delta_q$ and $[M \restriction_\delta, V_\lambda)^\delta_p$, and hence is a δ-chain. Now suppose $P, Q \in \mathcal{M}^\delta_r$ and their meet is defined and active at δ. We need to check that $Q \wedge P \in \mathcal{M}^\delta_r$. If both P and Q belong either to $\mathcal{M}^\delta_q$ or $\mathcal{M}^\delta_p$, this follows from the fact that p and q are conditions. Since $Q \in_\delta P$ and $\mathcal{M}^\delta_q$ is an $\in^*_\delta$-initial segment of $\mathcal{M}^\delta_r$, we may assume $Q \in \mathcal{M}^\delta_q$ and $P \in \mathcal{M}^\delta_p \setminus \mathcal{M}^\delta_q$. The proof goes by induction on the number of Magidor models on the δ-chain $[M \restriction_\delta, P)^\delta_p$. If $M \in_\delta P$ then $M \wedge P \in \mathcal{M}^\delta_p$ and is δ-below $M \restriction_\delta$, hence belongs to $\mathcal{M}^\delta_q$. On the other hand, by Proposition 3.2.32 we have $Q \wedge P = Q \wedge (M \wedge P)$, and since $\mathcal{M}^\delta_q$ is closed under meets we get that $Q \wedge P \in \mathcal{M}^\delta_q$. In general, if N is the $\in^*_\delta$-largest Magidor model in $[M \restriction_\delta, P)^\delta_p$, by Proposition 3.2.21, we have that $Q \in_\delta N \in_\delta P$. In particular, $N \wedge P$ is defined and by Proposition 3.2.32 we have that $Q \wedge P = Q \wedge (N \wedge P)$. Now, we are done if $N \wedge P \in \mathcal{M}^\delta_q$ as q is a condition. Otherwise, it belongs to the interval $[M \restriction_\delta, P)^\delta_p$. Then there are fewer Magidor models in $[M \restriction_\delta, N \wedge P)^\delta_p$ and thus we can use the induction hypothesis.

Let $d_r = d_q \cup d_p \restriction_{(\mathrm{dom}(d_p) \setminus M)}$. Let us check that every $N \in \mathrm{dom}(d_r)$ is $\mathcal{M}_r$-free. For simplicity, let $\eta = \eta(N)$. If $N \in \mathrm{dom}(d_p) \setminus M$, then there is no $P \in \mathcal{M}_q$ such that $N \in_\eta P$, and hence the conclusion follows from the fact that p is a condition. Suppose now $N \in \mathrm{dom}(d_q)$ and $P \in \mathcal{M}_r$ is such that $N \in_\eta P$. We have to check that P is strongly active at η. We may assume that P is a countable model. If $P \restriction_\eta$ is η-below M, then $P \restriction_M$ is defined and $P \restriction_M \cong_\eta P$, therefore, the conclusion follows from the fact that q is a condition. If $P \restriction_\eta$ is η-above M, then $M \wedge P$ is defined and belongs to $\mathcal{M}_p$. Moreover, by Proposition 3.2.31, $N \in_\eta M \wedge P$. Now $(M \wedge P) \restriction_M$ is defined, and belongs to $\mathcal{M}^{\alpha(M \wedge P, M)}_q$, and is strongly active at η since N is $\mathcal{M}_q$-free. Therefore, P is also strongly active at η. The fact that d_r satisfies

condition $(*)$ from Definition 4.2.3 is straightforward. Finally, the fact that r is the weakest common extension of p and q follows readily from the definition. $\boxed{4.2.12}$

By Lemma 4.2.8 and Lemma 4.2.12 we immediately get the following.

Theorem 4.2.13. *The forcing $\mathfrak{M}^\kappa_\lambda$ is $\mathscr{U}$-strongly proper.*

$\boxed{4.2.13}$

We now proceed to define an analogue of $p \restriction_M$ for countable models $M \in \mathcal{L}(\mathcal{M}_p)$. The situation here is more subtle since $p \restriction_M$ may not belong to the original forcing, only its version as defined in M. We first analyze the part involving $\mathcal{M}_p$. It will be useful to make the following definition.

Definition 4.2.14. *Let $\mathcal{M}$ be a subset of $\mathscr{C} \cup \mathscr{U}$ and $M \in \mathscr{C}$. For $\delta \in E$, we let $(\mathcal{M} \restriction_M)^\delta = \{N \in \mathcal{M}^\delta : N \in_\delta M\}$.*

Lemma 4.2.15. *Let $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$ and $\delta \in E$. Suppose $M \in \mathcal{M}^\delta_p$ is countable. Then $(\mathcal{M}_p \restriction_M)^\delta$ is a δ-chain closed under meets and*

$$(\mathcal{M}_p \restriction_M)^\delta = (\varnothing, M \restriction_\delta)^\delta_p \setminus \bigcup \{[N \wedge M, N)^\delta_p : N \in (\mathcal{M}_p \restriction_M)^\delta \text{ and is a Magidor model}\}.$$

Here, if $N \wedge M$ is defined and not active at δ, by $[N \wedge M, N)^\delta_p$ we mean $(\varnothing, N)^\delta_p$.

Proof. It is clear that $(\mathcal{M}_p \restriction_M)^\delta \subseteq (\varnothing, M)^\delta_p$. Suppose $P \in \mathcal{M}^\delta_p$ and $P \in_\delta M$. Then, for any Magidor model $N \in (P, M)^\delta_p$, we have $P \in_\delta N$ by Proposition 3.2.21. Then by Proposition 3.2.31 we have that $P \in_\delta N \wedge M$. Conversely, suppose P is in $(\varnothing, M)^\delta_p$, but not in $(\mathcal{M}_p \restriction_M)^\delta$. Then, by Proposition 3.2.21 again, there must be a Magidor model $N \in \mathcal{M}^\delta_p$ such that $P \in_\delta N \in_\delta M$. Let N be the $\in^*_\delta$-least such model. If $N \wedge M$ is not active at δ, then $P \in (\varnothing, N)^\delta_p$. Suppose $N \wedge M$ is active at δ. We have to show that either $P = N \wedge M$ or $N \wedge M \in^*_\delta P$. Indeed, otherwise we have $P \in^*_\delta N \wedge M$. Note that there cannot be a Magidor model $Q \in \mathcal{M}^\delta_p$ with $P \in_\delta Q \in_\delta N \wedge M$ since then we would have $Q \in_\delta M$ as well, and this contradicts the minimality of N. Since $\mathcal{M}^\delta_p$ is a δ-chain, by Proposition 3.2.21 we conclude that $P \in_\delta N \wedge M$, but then also $P \in_\delta M$, a contradiction. The fact that $(\mathcal{M}_p \restriction_M)^\delta$ is a δ-chain follows from the above analysis. By Proposition 3.2.33 it is also closed under meets. $\boxed{4.2.15}$

Lemma 4.2.16. *Let $p \in \mathfrak{M}^\kappa_\lambda$ and $M, N \in \mathcal{M}_p$. If there is $\gamma \in a(M) \cap a(N)$ such that $N \in_\gamma M$, then $N \in_\delta M$, for all $\delta \in a(M) \cap a(N)$.*

Proof. Let $\alpha = \alpha(M, N)$. If $N \in_\alpha M$ then $N \in_\gamma M$, for all $\gamma \in a(M) \cap a(N)$, by Proposition 3.2.11. Now suppose $N \notin_\alpha M$. If M is a Magidor model we have $\kappa_M \leq \kappa_N$, and hence there is no γ such that $N \in_\gamma M$. Assume now that M is countable. Then by Lemma 4.2.15 there is a Magidor model $P \in \mathcal{M}_p^\alpha$ with $P \in_\alpha M$ such that $N \restriction_\alpha$ is in the interval $[P \wedge M, P)_p^\alpha$. Now, $P \wedge M$ is active at all $\gamma \in a(N) \cap \alpha$ and $N \restriction_\gamma$ is in the interval $[(P \wedge M) \restriction_\gamma, P \restriction_\gamma)_p^\gamma$, for all such γ. But then, by Lemma 4.2.15 again, $N \notin_\gamma M$, for all $\gamma \in a(M) \cap a(N)$. $\boxed{4.2.16}$

It would be useful to introduce some notation.

Notation 4.2.17. *Suppose $M \in \mathscr{V}$ and $\mathcal{M}$ is a finite subset of $\mathscr{V}$. Let $\alpha \in E$. We write $\mathcal{M} \in_\alpha M$ if $N \in_\alpha M$, for all $N \in \mathcal{M}$.*

Lemma 4.2.18. *Suppose $\mathcal{M}_p \in \mathbb{M}_\lambda^\kappa$ and $\delta \in E$. Suppose $M \in \mathcal{M}_p^\delta$ is a countable model, $\mathcal{M} \in_\delta M$ is a finite δ-chain closed under meets, and $(\mathcal{M}_p \restriction_M)^\delta \subseteq \mathcal{M}$. Then the closure of $\mathcal{M}_p^\delta \cup \mathcal{M}$ under meets that are active at δ is a δ-chain.*

Proof. Let us first show that $\mathcal{M}_p^\delta \cup \mathcal{M}$ is a δ-chain. Indeed, by Lemma 4.2.15 it is obtained by adding to $\mathcal{M}$ the intervals $[N \wedge M, N)_p^\delta$, where $N \in (\mathcal{M}_p \restriction_M)^\delta$ is a Magidor model, and the interval $[M, V_\lambda)_p^\delta$. Consider one such interval, say $[N \wedge M, N)_p^\delta$. If P is the last model of $\mathcal{M}$ before N then $P \in_\delta M$ by the assumption that $\mathcal{M} \in_\delta M$, and $P \in_\delta N$ by Proposition 3.2.21. Hence by Proposition 3.2.30 we have that $P \in_\delta N \wedge M$. It follows that $\mathcal{M}_p^\delta \cup \mathcal{M}$ is a δ-chain.

Let us now consider what happens when we close $\mathcal{M}_p^\delta \cup \mathcal{M}$ under meets that are active at δ. Suppose $Q, P \in \mathcal{M}_p^\delta \cup \mathcal{M}$, Q is a Magidor model, P is countable, and $Q \in_\delta P$. If $P \in \mathcal{M}$ then $Q \in_\delta P \in_\delta M$, and hence by Proposition 3.2.21 $Q \in_\delta M$, and so $Q \in \mathcal{M}$ as well. Since $\mathcal{M}$ is closed under meets, we have that $Q \wedge P \in \mathcal{M}$. Now suppose $P \in \mathcal{M}_p^\delta \setminus \mathcal{M}$. By Lemma 4.2.15 we have that $P \in [M, V_\lambda)_p^\delta$ or $P \in [N \wedge M, N)_p^\delta$, for some Magidor model $N \in (\mathcal{M}_p \restriction_M)^\delta$. The two cases are only notationally different, so let us assume that there is a Magidor model $N \in (\mathcal{M}_p \restriction_M)^\delta$ such that $P \in [N \wedge M, N)_p^\delta$. We may assume that $Q \in \mathcal{M} \setminus \mathcal{M}_p^\delta$. Note that $Q \in_\delta N$ and $Q \in_\delta M$, hence by Proposition 3.2.31 $Q \in_\delta N \wedge M$. If there is a Magidor model $R \in \mathcal{M}_p^\delta$ such that $Q \in_\delta^* R \in_\delta^* P$, let R be the $\in_\delta^*$-largest such model. By Proposition 3.2.21 we have that $R \in_\delta P$ and hence $R \wedge P$ is defined and is below P on the δ-chain $\mathcal{M}_p^\delta$. Moreover, since $\mathcal{M}_p^\delta \cup \mathcal{M}$ is a δ-chain, also by Proposition 3.2.21, we have $Q \in_\delta R$. Now, by Proposition 3.2.31 we have that $Q \in_\delta R \wedge P$, and by Proposition 3.2.32 we have $Q \wedge P = Q \wedge (R \wedge P)$. Therefore, we may assume that there are no Magidor models $R \in \mathcal{M}_p^\delta$ with $Q \in_\delta R \in_\delta P$. Now, let $\{P_i : i < k\}$

list all countable models on the chain $[N \wedge M, N)_p^\delta$ below the first Magidor model, if it exists. Then $P_0 = N \wedge M$ and $P = P_j$, for some j. Note that $Q \in_\delta P_i$, for all $i < k$, again by Proposition 3.2.21. Now let S be the $\in_\delta^*$-predecessor of Q on the δ-chain $\mathcal{M}_p^\delta \cup \mathcal{M}$, if it exists, otherwise let S be $\varnothing$. Note that $S \in_\delta N \wedge M$. Indeed, if $S \in \mathcal{M}_p^\delta$ this follows from Proposition 3.2.21, and the fact that there are no Magidor models in $(S, N \wedge M)_p^\delta$. If $S \in \mathcal{M}$ then $S \in_\delta M$ and thus $S \in_\delta N \wedge M$. Now, by Proposition 3.2.21 we have that $S \in_\delta P_i$, for all $i < k$. Hence, by Proposition 3.2.31 we have $S \in_\delta Q \wedge P_i$, for all i. By Proposition 3.2.33 we have $Q \wedge P_i \in_\delta P_{i+1}$, for all $i < k-1$. Since Q is a Magidor model, we also have that $Q \wedge P_i \in_\delta Q$, for all $i < k$. By Proposition 3.2.31 again, we have that $Q \wedge P_i \in_\delta Q \wedge P_{i+1}$, for all $i < k-1$. Therefore, $S \in_\delta Q \wedge P_0 \in_\delta \ldots \in_\delta Q \wedge P_{k-1} \in_\delta Q$, and $Q \wedge P$ appears on this chain. If $S = \varnothing$ then an initial segment of this chain may be nonactive at δ, but the remainder is still a δ-chain. $\boxed{4.2.18}$

Now, suppose $p \in \mathfrak{M}_\lambda^\kappa$ and $M \in \mathcal{M}_p$ is a countable β-model, for some $\beta \in E$. Let $A = \mathrm{Hull}(M, V_\beta)$. Then $A \in \mathscr{A}_\beta$. Note that $E_A \cap \beta = E \cap \beta$, and if $\beta \in A$ then $\beta \in E_A$. Also, note that the definitions of $\mathfrak{M}_\alpha^\kappa$ and the order relation are Σ_1 with parameter V_α. For $\alpha \in E_A$, let $(\mathfrak{M}_\alpha^\kappa)^A$ be the version of $\mathfrak{M}_\alpha^\kappa$ as defined in A. Then $(\mathfrak{M}_\alpha^\kappa)^A = \mathfrak{M}_\alpha^\kappa$ if $\alpha < \beta$, and $(\mathfrak{M}_\beta^\kappa)^A \subseteq \mathfrak{M}_\beta^\kappa$. We will let $\mathscr{V}_\alpha^M = \mathscr{V}_\alpha^A \cap M$, and $(\mathfrak{M}_\alpha^\kappa)^M = (\mathfrak{M}_\alpha^\kappa)^A \cap M$, if $\alpha \in E_A \cap M$. Suppose $N \in \mathcal{M}_p$ and $N \in_\delta M$, for some $\delta \in a(M) \cap a(N)$, Then by Lemma 4.2.16, $N \in_\alpha M$, where $\alpha = \alpha(M, N)$. Note that if M is a standard β-model then $\alpha < \beta$. It may be that $\alpha \notin M$, but then, if we let $\alpha^* = \min(M \cap \mathrm{ORD} \setminus \alpha)$, we have that $\alpha^* \in E_A \cap M$, and α^* is of uncountable cofinality in A. By the previous remarks, if M is a standard β-model or $\beta \in M$ then $\alpha^* \in E \cap (\beta + 1)$, otherwise α^* may be in the nonstandard part of M. Since $N \in_\alpha M$, there is an α^*-model $N^* \in M$ with $N^* \in \mathscr{V}^A$ which is α-isomorphic to N. Now, M can compute $N^* \restriction_{\alpha^*}$, hence we may assume $N^* \in \mathscr{V}_{\alpha^*}^A$. Moreover, such N^* is unique. Indeed, if there is another model $N^{**} \in M$ with the same property, since α^* is the least ordinal in M above α and $N^* \cong_\alpha N^{**}$ we would have that $N^* \cong_\delta N^{**}$, for all $\delta \in E_A \cap \alpha^* \cap M$. Hence, by Proposition 3.2.12 applied in M, we would have that $N^* = N^{**}$. This justifies the following definition.

Definition 4.2.19. *Suppose $p \in \mathfrak{M}_\lambda^\kappa$ and let $M \in \mathcal{L}(\mathcal{M}_p)$ be a countable β-model, for some $\beta \in E$. Suppose that $N \in \mathcal{M}_p$, and let $\alpha = \alpha(M, N)$. If $N \in_\alpha M$ we let $\alpha^* = \min(M \cap \mathrm{ORD} \setminus \alpha)$. We define $N \restriction_M$ to be the unique $N^* \in \mathscr{V}_{\alpha^*}^M$ such that*

$N^* \cong_\alpha N$. Otherwise we leave $N \restriction_M$ undefined. Let

$$\mathcal{M}_{p\restriction_M} = \{N \restriction_M : N \in \mathcal{M}_p\}, \text{ and}$$

$$\mathrm{dom}(d_{p\restriction_M}) = \{N \restriction_M : N \in \mathrm{dom}(d_p) \text{ and } N \in_{\eta(N)} M\}.$$

If $N \in \mathrm{dom}(d_p)$ and $N \in_{\eta(N)} M$, let $d_{p\restriction_M}(N \restriction_M) = d_p(N)$. Let $p \restriction_M = (\mathcal{M}_{p\restriction_M}, d_{p\restriction_M})$.

Remark 4.2.20. Suppose $N \in \mathrm{dom}(d_p)$ and let $\eta = \eta(N)$. If $N \in_\eta M$ then M is strongly active at η since N is $\mathcal{M}_p$-free. If $\eta \in M$ then we put N in $\mathrm{dom}(d_{p\restriction_M})$ and keep the same decoration at N. If $\eta \notin M$ we lift N to the least level η^* of M above η, we put the resulting model N^* in $\mathrm{dom}(d_{p\restriction_M})$ and copy the decoration of N to N^*. If $P \in \mathcal{M}_p$ is such that $P \restriction_\eta = N$ then $(P \restriction_M) \restriction_{\eta^*} = N^*$. Moreover, from N^* we can recover N as $N^* \restriction_{\sup(\eta^* \cap M)}$. Thus, the function $d_{p\restriction_M}$ is well defined. Note also that $p \restriction_M \in M$.

Proposition 4.2.21. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a countable β-model, for some $\beta \in E$. Let*

$$\alpha = \max\{\alpha(N, M) : N \in_{\alpha(N,M)} M \text{ and } N \in \mathcal{M}_p\}.$$

Let $\alpha^* = \min(M \cap \mathrm{ORD} \setminus \alpha)$. Then $p \restriction_M \in (\mathfrak{M}^\kappa_{\alpha^*})^M$.

Proof. Let $A = \mathrm{Hull}(M, V_\beta)$ and work in A. It is clear that $\mathcal{M}_{p\restriction_M}$ is a finite subset of $\mathscr{C}^A_{\leq\alpha^*} \cup \mathscr{U}^A_{\leq\alpha^*}$. We first show that $\mathcal{M}^\gamma_{p\restriction_M}$ is a γ-chain closed under meets, for all $\gamma \in E_A \cap (\alpha^* + 1)$. Fix such γ and let $\delta = \min(M \cap \mathrm{ORD} \setminus \gamma)$ and $\bar{\delta} = \sup(M \cap \delta)$. If $\bar{\delta} = \delta$ then $\gamma = \delta$, and hence $\gamma \in M$. Since $\mathcal{M}^\gamma_{p\restriction_M} = \mathcal{M}^\gamma_{p\restriction_M}$the conclusion follows from the fact that p is a condition and Lemma 4.2.15. Let us assume now that $\bar{\delta} < \delta$. Note that then $\bar{\delta}, \delta \in E$, δ is of uncountable cofinality in M, and is a limit point of E. Note that if $P \in \mathcal{M}_{p\restriction_M}$ is a δ-model that is active at γ then $a(P)$ is cofinal in δ. Moreover, $a(P) \in M$ and since $\bar{\delta} = \sup(M \cap \delta)$ we have that $\bar{\delta} \in a(P)$. This implies that $\mathcal{M}^\gamma_{p\restriction_M}\restriction_{\bar{\delta}} = \mathcal{M}^{\bar{\delta}}_p\restriction_M$. Therefore, by Lemma 4.2.15 it is a $\bar{\delta}$-chain closed under active meets. Now, suppose $N, P \in \mathcal{M}^\delta_{p\restriction_M}$ and $N \in_{\bar{\delta}} P$. Since $\bar{\delta} = \sup(M \cap \delta)$ we have that $N \in_\xi P$, for unboundedly many $\xi \in E \cap \delta$. We conclude that $N \in_\delta P$. Indeed, if P is countable this follows from Proposition 3.2.13 applied in A, and if P is a Magidor model this follows from Proposition 3.2.25, again applied in A. Moreover, assuming N is a Magidor model and P is countable, and $N \restriction_\gamma \wedge P \restriction_\gamma$ is defined and active at γ then, by Proposition 3.2.30, $N \wedge P$ is defined and active at unboundedly many $\xi \in E \cap \delta$, and hence it is also active at δ and $\bar{\delta}$. It follows that

$\mathcal{M}^{\delta}_{p\restriction_M}$ is a δ-chain closed under meets, and hence $\mathcal{M}^{\gamma}_{p\restriction_M}$ is a γ-chain closed under meets as well.

Let us check that every $P^* \in \mathrm{dom}(d_{p\restriction_M})$ is $\mathcal{M}_{p\restriction_M}$-free. If $P^* \in \mathrm{dom}(d_p)$ this is immediate. Otherwise, P^* is of the form $P\restriction_M$, for some $P \in \mathrm{dom}(d_p)$ such that $\eta(P) \notin M$. Let $\eta = \eta(P)$ and $\eta^* = \eta(P^*)$. Note that M is strongly active at η and η^* is the least ordinal of M above η. Suppose $N \in \mathcal{M}_{p\restriction_M}$ is such that $P^* \in_{\eta^*} N$. Then $N\restriction_\eta \in \mathcal{L}(\mathcal{M}_p)$ and $P \in_\eta N$. Since P is $\mathcal{M}_p$-free, N must be strongly active at η. Since $\eta = \sup(M \cap \eta^*)$ and $N \in M$ we must have that N is strongly active at η^* as well. This also establishes ($\divideontimes$) from Definition 4.2.3. Indeed, if $P^* \in_{\eta^*} N$ then $P \in_\eta N\restriction_\eta$, and hence $d_p(P) \subseteq N$, since $N\restriction_\eta \in \mathcal{L}(\mathcal{M}_p)$, and p is a condition. This completes the proof that $p\restriction_M \in (\mathfrak{M}^\kappa_{\alpha^*})^A$. $\boxed{4.2.21}$

Note that if $p, q \in \mathfrak{M}^\kappa_\lambda$ are such that $q \leq p$ and $M \in \mathcal{M}_p$ then $q\restriction_M \leq p\restriction_M$.

We are planning to show that if p is a condition and $M \in \mathcal{M}_p$ is a countable β-model then, for any $q \leq p\restriction_M$ with $q \in M$, p and $q\restriction_\beta$ are compatible, and in fact the meet $p \wedge q\restriction_\beta$ exists. Before that we show the following special case of this statement.

Lemma 4.2.22. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $\delta \in E$. Suppose $M \in \mathcal{M}^\delta_p$ is a countable model, $\mathcal{M} \in_\delta M, \mathcal{M} \in \mathbb{M}^\kappa_\delta$, and $(\mathcal{M}_p\restriction_M)^\gamma \subseteq \mathcal{M}^\gamma$, for all $\gamma \in E \cap (\delta+1)$. Suppose further that $P \notin_{\eta(P)} M$, for all $P \in \mathrm{dom}(d_p)$. Let $\mathcal{M}_q$ be the closure of $\mathcal{M}_p \cup \mathcal{M}$ under meets and let $d_q = d_p$. Finally, let $q = (\mathcal{M}_q, d_q)$. Then $q \in \mathfrak{M}^\kappa_\lambda$.*

Proof. Let us first check that $\mathcal{M}^\gamma_q$ is a γ-chain closed under active meets, for all $\gamma \in E$. Fix $\gamma \in E$. If M is not active at γ then $\mathcal{M}^\gamma_q = \mathcal{M}^\gamma_p$, so this follows from the fact that p is a condition. If M is active at γ then this follows from Lemma 4.2.18.

Thus, it remains to check that every $P \in \mathrm{dom}(d_p)$ is $\mathcal{M}_q$-free and $d_p(P) \in Q$, for all $Q \in \mathcal{M}_q$ such that $P \in_{\eta(P)} Q$. Now, fix one such $P \in \mathrm{dom}(d_p)$ and let $\eta = \eta(P)$. If M is not active at η, then no model of $\mathcal{M}$ is active at η, and hence $Q \in \mathcal{M}_p$, for all $Q \in \mathcal{M}_q$ such that $P \in_\eta Q$. The conclusion then follows from the fact that p is a condition and d_p is its decoration. Suppose now that M is active at η, but P is either equal to $M\restriction_\eta$ or is above $M\restriction_\eta$ on the η-chain $\mathcal{M}^\eta_p$. Then, again any $Q \in \mathcal{M}_q$ such that $P \in_\eta Q$ is in $\mathcal{M}_p$, and the conclusion follows as above. Suppose now that M is active at η and $P \in^*_\eta M\restriction_\eta$. Note that $\mathcal{M}^\eta_q$ is obtained by closing $\mathcal{M}^\eta_p \cup \mathcal{M}^\eta$ under meets that are active at η. Suppose P is below $M\restriction_\eta$ on $\mathcal{M}^\eta_p$. By the assumption, $P \notin_\eta M$, hence by Lemma 4.2.15, there must be a Magidor model $N \in (\mathcal{M}_p\restriction_M)^\eta$ such that P is in the interval $[N \wedge M\restriction_\eta, N)^\eta_p$. By Proposition 3.2.21, we have that $P \in_\eta N$ and thus $d_q(P) \in N$. Note that P also belongs to the interval

$[N \wedge M \restriction_\eta, N)^\eta_q$. Suppose $Q \in \mathcal{M}_q$ and $P \in_\eta Q$. By replacing Q with $Q \restriction_\eta$, we may assume that $Q \in \mathcal{M}^\eta$. Note that Q cannot be a countable model since then we would have $P \in_\eta M$. If Q is a Magidor model in $\mathcal{M}^\eta$ then Q cannot be below N since then it would be below $N \wedge M \restriction_\eta$ on $\mathcal{M}^\eta_q$. Therefore, Q must be either equal to N or above N on the η-chain $\mathcal{M}^\eta$. Then we would have $N \cap V_\kappa \subseteq Q \cap V_\kappa$, and hence $d_p(P) \in Q$. If $Q \in \mathcal{M}_p$ then Q is strongly active at η and $d_p(P) \in Q$, since p is a condition. It remains to consider the case when Q is of the form $R \wedge S$, for some Magidor model $R \in \mathcal{M}^\eta$ and countable $S \in \mathcal{M}^\eta_p \setminus \mathcal{M}^\eta$. Now, we must have $R = N$ or $N \in_\eta R$ since otherwise R, and hence also $R \wedge S$, would be below $N \wedge M \restriction_\eta$. Since $d_p(P) \in N$, we must have $d_p(P) \in R$. Moreover, since $S \in \mathcal{M}^\eta_p$, and d_p is the decoration of p, S must be strongly active at η and $d_p(P) \in S$. By Proposition 3.2.29, $R \wedge S$ is strongly active at η. By Proposition 3.2.28, we have $R \wedge S \cap V_\eta = R \cap S \cap V_\eta$, and hence $d_p(P) \in R \wedge S = Q$. $\boxed{4.2.22}$

Lemma 4.2.23. *Suppose $p \in \mathfrak{M}^\kappa_\lambda$ and $M \in \mathcal{L}(\mathcal{M}_p)$ is a countable β-model, for some $\beta \in E$. Let $\alpha^* \in M$ be such that $p \restriction_M \in (\mathfrak{M}^\kappa_{\alpha^*})^M$. Then for any $q \in (\mathfrak{M}^\kappa_{\alpha^*})^M$ with $q \leq p \restriction_M$, p and $q \restriction_\beta$ are compatible, and the meet $p \wedge q \restriction_\beta$ exists.*

Proof. Let $\mathcal{M}_r$ be the closure of $\mathcal{M}_p \cup \mathcal{M}_{q\restriction_\beta}$ under meets. By Lemma 4.2.18 we already know that $\mathcal{M}^\delta_r$ is a δ-chain, for all $\delta \in E$. Hence $\mathcal{M}_r \in \mathbb{M}^\kappa_\lambda$. It remains to define the decoration d_r, and check that it satisfies $(\divideontimes)$ from Definition 4.2.3. Let

$$D_p = \{P \in \mathrm{dom}(d_p) : P \notin_{\eta(P)} M\}.$$

Now, suppose $P \in \mathrm{dom}(d_q)$. Let $\delta(P)$ be the largest ordinal $\gamma \in E \cap (\eta(P)+1)$ such that M is strongly active at γ. Let

$$D_q = \{P \restriction_{\delta(P)} : P \in \mathrm{dom}(d_q)\}.$$

Note that, for every $P \in \mathrm{dom}(d_q)$, we have $(P \restriction_{\delta(P)}) \restriction_M = P$, and P is active at $\delta(P)$. Observe that D_p and D_q are disjoint. Let $\mathrm{dom}(d_r) = D_p \cup D_q$ and define d_r by:

$$d_r(P) = \begin{cases} d_p(P) & \text{if } P \in D_p \\ d_q(P \restriction_M) & \text{if } P \in D_q \text{ and } \eta(P) \leq \beta \end{cases}$$

We have to check that every $P \in \mathrm{dom}(d_r)$ is $\mathcal{M}_r$-free and condition $(\divideontimes)$ holds. By Lemma 4.2.22 we have that $(\mathcal{M}_r, d_p \restriction_{D_p})$ is already a condition, so we may assume $P \in D_q$. Fix one such $P \in D_q$, and let $\eta = \eta(P)$. Note that it suffices to show

that the least model, say R, on the η-chain $\mathcal{M}_r^\eta$ above P is strongly active at η, and $d_r(P) \in R$. By Lemma 4.2.15 either $R \in_\eta M$ or $R = N \wedge M \restriction_\eta$, for some Magidor model $N \in (\mathcal{M}_p \restriction_M)^\eta$. Now, if R is of the form $N \wedge M \restriction_\eta$, then, since N and $M \restriction_\eta$ are strongly active at η, by Proposition 3.2.29, so is R. Moreover, $N \restriction_M \in \mathcal{L}(\mathcal{M}_q)$ and $d_q(P \restriction_M) \in M \cap N$. It follows that $d_r(P) \in R$. Suppose now that $R \in_\eta M$. Let $\rho = \min(E \cap M \setminus \eta)$. Then $P \restriction_M$ and $R \restriction_M$ are ρ-models, $R \restriction_M \in \mathcal{L}(\mathcal{M}_q)$, and $P \restriction_M \in_\rho R \restriction_M$. Therefore, $R \restriction_M$ is strongly active at ρ, and $d_q(P \restriction_M) \in R \restriction_M$. Since $(R \restriction_M) \cap V_\kappa = R \cap V_\kappa$, we get that $d_q(P \restriction_M) \in R$, and hence $d_r(P) \in R$. Moreover, since $R \restriction_M$ is strongly active at ρ, it follows that R is strongly active at η. This shows that all the models in $\mathrm{dom}(d_r)$ are $\mathcal{M}_r$-free and condition $(\divideontimes)$ holds for r. The fact that $r \leq p, q \restriction_\beta$ and is in fact the weakest such condition follows from the definition. $\boxed{4.2.23}$

Remark 4.2.24. Suppose $p \in \mathfrak{M}_\lambda^\kappa$ and $M \in \mathcal{M}_p$ is a countable β-model, for some $\beta \in E$. If either M is standard or $\beta \in M$ we have that $p \restriction_M \in \mathfrak{M}_\lambda^\kappa$. In particular, Lemma 4.2.23 shows that if $p \in \mathfrak{M}_\lambda^\kappa$ then p and $p \restriction_M$ are compatible. Now, we have already observed that, if $q \in \mathfrak{M}_\lambda^\kappa$ and $q \leq p$, then $q \restriction_M \leq p \restriction_M$. Therefore, even though it may not be the case that $p \leq p \restriction_M$, every p forces $p \restriction_M$ to belong to the generic filter, and hence p is stronger than $p \restriction_M$.

Now, by Lemma 4.2.8 and Lemma 4.2.23 we immediately get the following.

Theorem 4.2.25. *$\mathfrak{M}_\lambda^\kappa$ is $\mathscr{C}_{\mathrm{st}}$-strongly proper.*

Proof. Suppose $M \in \mathscr{C}_{\mathrm{st}}$ and $p \in M \cap \mathfrak{M}_\lambda^\kappa$. Let p^M be the condition defined in Lemma 4.2.8. If $q \leq p^M$ then $M \in \mathcal{L}(\mathcal{M}_q)$ and $q \restriction_M \in M$, and by Remark 4.2.24, $q \restriction_M \in \mathfrak{M}_\lambda^\kappa$. Then, by Lemma 4.2.23 any extension r of $q \restriction_M$ with $r \in M$ is compatible with q, and moreover $q \wedge r$ exists. Thus, p^M is a $(M, \mathfrak{M}_\lambda^\kappa)$-strongly generic condition extending p. $\boxed{4.2.25}$

Remark 4.2.26. A similar proof shows that the forcing $\mathfrak{M}_\alpha^\kappa$ is strongly proper for the collection of all $M \in \mathscr{C}$ such that $\alpha \in M$.

Notation 4.2.27. *Let F be a filter in $\mathfrak{M}_\lambda^\kappa$. Then we set*

$$\mathcal{M}_F = \bigcup \{\mathcal{M}_p : \ p \in F\}.$$

Let G be a $\mathfrak{M}_\lambda^\kappa$-generic filter over V. We let $G_\alpha = G \cap \mathfrak{M}_\alpha^\kappa$, for all $\alpha \in E$. The following is straightforward.

Proposition 4.2.28. *Let $\delta \in E$ with $\mathrm{cof}(\delta) < \kappa$. Then $\mathcal{M}_G^\delta$ is a δ-chain.*

4.2.28

Proposition 4.2.29. *Let $\delta \in E$ with $\mathrm{cof}(\delta) < \kappa$. Suppose $M \in \mathcal{M}_G^\delta$ is a Magidor model and is not the least model in $\mathcal{M}_G^\delta$. Then*

$$M \cap V_\delta = \bigcup \{Q \cap V_\delta : Q \in_\delta M \text{ and } Q \in \mathcal{M}_G^\delta\}.$$

Proof. It suffices to show that if $p \in \mathfrak{M}_\lambda^\kappa$ and $M \in \mathcal{M}_p^\delta$ is a Magidor model that is not the least model of $\mathcal{M}_p^\delta$, and $x \in M \cap V_\delta$, then there is $q \leq p$ and $Q \in \mathcal{M}_q$ which is active at δ such that $Q \in_\delta M$ and $x \in Q$. Let $N \in \mathcal{M}_p$ be active at δ such that $N \in_\delta M$. Fix some $Q^* \in \mathscr{C}$ such that $N, M, x \in Q^*$. By Lemma 4.2.8 there is a condition $q \leq p$ such that $Q^* \in \mathcal{M}_q$. Since $N \in_\delta M$ and $N \in_\delta Q^*$, if we let $Q = M \wedge Q^*$, by Proposition 3.2.30 $N \in_\delta Q$, and hence Q is active at δ. Moreover, $Q \in_\delta M$ and by Proposition 3.2.28 $Q \cap V_\delta = M \cap Q^* \cap V_\delta$ and hence $x \in Q$. It follows that the condition q and the model Q are as required. 4.2.29

Theorem 4.2.30. *Assume κ is supercompact. Then $\mathfrak{M}_\lambda^\kappa$ preserves ω_1 and κ, and collapses all cardinals between ω_1 and κ to ω_1.*

Proof. By Proposition 3.2.15, $\mathscr{C}_{\mathrm{st}}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$, and by Lemma 4.2.23, $\mathfrak{M}_\lambda^\kappa$ is $\mathscr{C}_{\mathrm{st}}$-strongly proper. Hence ω_1 is preserved. By Corollary 3.2.20, $\mathscr{U}$ is stationary in $\mathcal{P}_\kappa(V_\lambda)$, and by Theorem 4.2.13, $\mathfrak{M}_\lambda^\kappa$ is $\mathscr{U}$-strongly proper. Hence κ is preserved. Now, fix a cardinal $\mu < \kappa$. Let G be a $\mathfrak{M}_\lambda^\kappa$-generic filter over V. Fix $\alpha \in E$ of cofinality less than κ. A standard density argument shows that there exists a Magidor model $N \in \mathcal{M}_G^\alpha$ with $\mu \in N$. By Proposition 4.2.28 $\mathcal{M}_G^\alpha$ is an $\in_\alpha$-chain. Let N^* be the least Magidor model above N in $\mathcal{M}_G^\alpha$, and let $I = (N, N^*)^\alpha_{\mathcal{M}_G}$. Note that every model in I is countable and $\in_\alpha$ is transitive on I. Hence if $P, Q \in I$ and $P \in_\alpha Q$ then $P \cap V_\alpha \subseteq Q \cap V_\alpha$. Another standard density argument shows that, for every $x \in N \cap V_\alpha$, there is $P \in I$ such that $x \in P$. Thus, $\{P \cap V_\alpha : P \in I\}$ is an increasing chain of countable sets whose union covers $N \cap V_\alpha$. It follows that $N \cap V_\alpha$ is of cardinality at most ω_1. Since μ belongs to the transitive part of N, we also get that $|\mu| \leq \omega_1$. 4.2.30

Theorem 4.2.31. *$\mathfrak{M}_\lambda^\kappa$ collapses cardinals of the interval between κ and λ to κ.*

Proof. Let $\alpha \in E$ be of cofinality less than κ, and let G be a V-generic filter over $\mathfrak{M}_\lambda^\kappa$. Let $\mathcal{U}_G^\alpha$ be the set of Magidor models in $\mathcal{M}_G^\alpha$. By Proposition 3.2.21, we have

that $\in_\alpha$ is transitive on $\mathcal{U}_G^\alpha$. Note that if $P, Q \in U_\alpha$ then $P \cap V_\alpha \subseteq Q \cap V_\alpha$. Now, a standard density argument using the stationarity of $\mathscr{U}$ shows that, for every $x \in V_\alpha$, there is $P \in \mathcal{U}_G^\alpha$ such that $x \in P$. It follows that $\{P \cap V_\alpha : P \in \mathcal{U}_G^\alpha\}$ is an increasing family of sets of size $< \kappa$ whose union is V_α. Therefore, V_α has cardinality $\leq \kappa$ in $V[G]$. $\boxed{4.2.31}$

Theorem 4.2.32. *Suppose λ is an inaccessible cardinal. Then $\mathfrak{M}_\lambda^\kappa$ is λ-c.c.*

Proof. For each $p \in \mathfrak{M}_\lambda^\kappa$, let $a(p) = \bigcup\{a(M) : M \in \mathcal{M}_p\}$. Note that $a(p)$ is a closed subset of E of size $< \kappa$, for all p. Suppose A is a subset of $\mathfrak{M}_\lambda^\kappa$ of cardinality λ. Since λ is inaccessible, by a standard Δ-system argument, we can find a subset B of A of size λ and a subset a of E such that $a(p) \cap a(q) = a$, for all distinct $p, q \in B$. Note that a is closed, and if we let $\gamma = \max(a)$ then $\gamma \in E$. Since B has size λ, by a simple counting argument, we may assume there is $\mathcal{M} \in \mathbb{M}_\gamma^\kappa$ such that $\mathcal{M}_p \restriction_\gamma = \mathcal{M}$, for all $p \in B$. Now, pick distinct $p, q \in B$, and define $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$ and $d_r = d_p \cup d_q$. Let $r = (\mathcal{M}_r, d_r)$. It is straightforward to check that $r \in \mathfrak{M}_\lambda^\kappa$ and $r \leq p, q$. $\boxed{4.2.32}$

4.3 Adding CLUBs

Definition 4.3.1. *Suppose G is V-generic over $\mathfrak{M}_\lambda^\kappa$ and $\alpha \in E$ is of cofinality less than κ. Let $C_\alpha(G) = \{\kappa_M : M \in \mathcal{M}_G^\alpha\}$.*

Lemma 4.3.2. *Let G be a V-generic filter over $\mathfrak{M}_\lambda^\kappa$. Then $C_\alpha(G)$ is a club in κ, for all $\alpha \in E$ of cofinality $< \kappa$. Moreover, if $\alpha < \beta$ then $C_\beta(G) \setminus C_\alpha(G)$ is bounded in κ.*

Proof. Let us check the second statement first. Assume that $\alpha < \beta$ in E are of cofinality less than κ. By a standard density argument using the stationarity of $\mathscr{U}$ there is $p \in G$ and a Magidor model $M \in \mathcal{M}_p$ which is active at both α and β. Therefore, any model N above $M \restriction_\beta$ on the β-chain $\mathcal{M}_G^\beta$ is also active at α. It follows that $C_\beta(G) \setminus C_\alpha(G) \subseteq \kappa_M$.

We work in V and prove the first statement by induction on α. Let $\dot{\mathcal{M}}^\alpha$ and $\dot{C}_\alpha$ be canonical $\mathfrak{M}_\lambda^\kappa$-names for $\mathcal{M}_G^\alpha$ and $C_\alpha(G)$, for $\alpha \in E$. Now, fix $\alpha \in E$ of cofinality less than κ and suppose the statement has been proved for all $\bar{\alpha} \in E \cap \alpha$ of cofinality $< \kappa$. Suppose $\gamma < \kappa$ and $p \in \mathfrak{M}_\lambda^\kappa$ forces that γ is a limit point but not a member of $\dot{C}_\alpha$. We may assume that there is a model $M \in \mathcal{M}_p^\alpha$ such that p forces that M is the least model on the α-chain $\dot{\mathcal{M}}^\alpha$ such that $\gamma \leq \kappa_M$. Then we must have $\gamma < \kappa_M$.

Let P be the previous model on $\mathcal{M}_p^\alpha$ before M. We may assume that such a model exists since p forces that γ is a limit point of $\dot{C}_\alpha$. Note that $\kappa_P < \gamma$ since p forces that $\gamma \notin \dot{C}_\alpha$.

Case 1. Suppose M is strongly active at α. Since P is $\mathcal{M}_p$-free and we may assume that $P \in \mathrm{dom}(d_p)$, by defining $d_p(P) = \varnothing$ if necessary. Since $\gamma < \kappa_M$, we can find $\delta \in M$ such that $\gamma \leq \delta < \kappa_M$. Define a condition q as follows. Let $\mathcal{M}_q = \mathcal{M}_p$, and let $\mathrm{dom}(d_q) = \mathrm{dom}(d_p)$. Let $d_q(P) = d_p(P) \cup \{\delta\}$, and $d_q(Q) = d_p(Q)$, for any other $Q \in \mathrm{dom}(d_p)$. Let $q = (\mathcal{M}_q, d_q)$. Then q is a condition and forces that the next model of $\dot{\mathcal{M}}^\alpha$ above P contains δ. Hence, it forces that there is no element of $\dot{C}_\alpha$ between κ_P and γ, and so it forces that γ is not a limit point of $\dot{C}_\alpha$, a contradiction.

Case 2. Suppose now that M is not strongly active at α. Then M is countable. Let $A = \mathrm{Hull}(M, V_\alpha)$, let α^* be the least ordinal of M above α, and let $\bar{\alpha} = \sup(M \cap \alpha)$. Note that $\alpha^* \in E_A$, $\bar{\alpha}$ is a limit point of E of cofinality ω, and that P is also active at $\bar{\alpha}$. Now, by the proof of the second part of the lemma, p forces that $\dot{C}_\alpha \setminus \dot{C}_{\bar{\alpha}} \subseteq \kappa_P$, and so it also forces that γ is a limit point of $\dot{C}_{\bar{\alpha}}$. By the inductive assumption $\dot{C}_{\bar{\alpha}}$ is forced to be a club, so there is $q \leq p$ and some $N \in \mathcal{M}_q^{\bar{\alpha}}$ such that $\kappa_N = \gamma$. Now, for each $Q \in (\mathcal{M}_q \restriction_M)^{\bar{\alpha}}$, we can find a unique model $Q^* \in M$ with $Q^* \in \mathscr{V}_{\alpha^*}^A$ such that $Q^* \restriction_{\bar{\alpha}} = Q$. Let $\mathcal{M}^* = \{Q^* : Q \in (\mathcal{M}_q \restriction_M)^{\bar{\alpha}}\}$. Working in A, $\mathcal{M}^*$ is an α^*-chain closed under meets that are active at α^*. Let $\mathcal{M} = \{Q^* \restriction_\alpha : Q^* \in \mathcal{M}^*\}$. We then have $\mathcal{M} \in_\alpha M$, and that $\mathcal{M}$ is an α-chain closed under meets that are active at α, and $(\mathcal{M}_q \restriction_M)^\alpha \subseteq \mathcal{M}$. We now define a condition r. Let $\mathcal{M}_r$ be the closure of $\mathcal{M}_q$ and $\mathcal{M}$ under meets. By applying Lemma 4.2.18, for all levels $\delta \in E \cap (\bar{\alpha}, \alpha]$, we have that $\mathcal{M}_r \in \mathbb{M}_\lambda^\kappa$. Let $d_r = d_q$ and $r = (\mathcal{M}_r, d_r)$. Observe that $\mathcal{M}_r^\eta = \mathcal{M}_q^\eta$, for all $\eta \in E \setminus (\bar{\alpha}, \alpha]$. Also, if $R \in \mathrm{dom}(d_q)$ and $\eta(R) \in (\bar{\alpha}, \alpha]$ then $R \notin_{\eta(R)} M$, since M is not strongly active at $\eta(R)$. By Lemma 4.2.22, we conclude that r is a condition. Also, we have that $r \leq q$. Recall that $N \in \mathcal{M}_q^{\bar{\alpha}}$ and $\kappa_N = \gamma$. Let Q be the model on the $\bar{\alpha}$-chain $\mathcal{M}_r^{\bar{\alpha}}$ immediately before $M \restriction_{\bar{\alpha}}$. Then $Q^* \in \mathcal{M}^*$, and hence $Q^* \restriction_\alpha \in \mathcal{M}_r$. Let $R = Q^* \restriction_\alpha$. In other words, we lifted the model Q to level α and called this model R. Note that $\kappa_R = \kappa_Q$. Then r forces that $R \in \dot{\mathcal{M}}_\alpha$ and $\gamma \leq \kappa_R < \kappa_M$, which contradicts the fact that p forces that $\gamma \notin \dot{C}_\alpha$ and M is the least model on $\dot{\mathcal{M}}_\alpha$ with $\gamma \leq \kappa_M$. This completes the proof of the lemma. $\boxed{4.3.2}$

4.4 Quotients

In this section we analyze the quotients of $\mathfrak{M}_\alpha^\kappa$, by some G_β, for $\beta < \alpha$, or by $G_\alpha \cap N$ for some Magidor model $N \restriction \alpha \in \mathcal{M}_{G_\alpha}$.

Let us fix a V-generic filter G_α over $\mathfrak{M}^\kappa_\alpha$, and let $\mathbb{Q}_\alpha$ denote the quotient forcing. Recall that $\mathbb{Q}_\alpha$ consists of all $p \in \mathfrak{M}^\kappa_\lambda$ such that $p \restriction_\alpha \in G_\alpha$, with the induced ordering. Forcing with this poset over $V[G_\alpha]$ produces a V-generic filter G_λ for $\mathfrak{M}^\kappa_\lambda$ such that $G_\lambda \cap \mathfrak{M}^\kappa_\alpha = G_\alpha$. We first show that the pair $(V[G_\alpha], V[G_\lambda])$ has the ω_1-approximation property. We will need the following definition.

Definition 4.4.1. *In the model $V[G_\alpha]$, let $\mathscr{C}_{\mathrm{st}}[G_\alpha]$ denote the set of all $M \in \mathscr{C}_{\mathrm{st}}$ such that $\eta(M) > \alpha$, $\alpha \in M$, and $M \restriction_\alpha \in \mathcal{M}^\alpha_{G_\alpha}$.*

Lemma 4.4.2. *$\mathscr{C}_{\mathrm{st}}[G_\alpha]$ is a stationary subset of $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_\alpha]$.*

Proof. We work in V. Let $\dot{\mathscr{C}}^\alpha_{\mathrm{st}}$ and $\dot{\mathcal{M}}_\alpha$ be canonical $\mathfrak{M}^\kappa_\alpha$-names for $\mathscr{C}_{\mathrm{st}}[G_\alpha]$ and $\mathcal{M}_{G_\alpha}$. Suppose that $p \in \mathfrak{M}^\kappa_\alpha$ forces that $\dot{F}$ is an algebra on V_λ. It suffices to find some $q \leq p$ and $M \in \mathscr{C}_{\mathrm{st}}$ such that q forces M is closed under $\dot{F}$ and belongs to $\dot{\mathscr{C}}^\alpha_{\mathrm{st}}$. Let θ be a sufficiently large regular cardinal. By Proposition 3.2.15, $\mathscr{C}_{\mathrm{st}}$ is club in $\mathcal{P}_{\omega_1}(V_\lambda)$, hence we can find a countable $M^* \prec H(\theta)$ containing all the relevant objects such that letting $M = M^* \cap V_\lambda$ we have that $M \in \mathscr{C}_{\mathrm{st}}$. Let $M' = M \restriction_\alpha$. Note that $p \in M'$, so we can form the condition $p^{M'}$. Then $p^{M'}$ is $(M', \mathfrak{M}^\kappa_\alpha)$-strongly generic and $p^{M'} \leq p$. Let σ be the α-isomorphism between M and M'. Note that $\sigma(q) = q$, for all $q \in M \cap \mathfrak{M}^\kappa_\alpha$. Hence, $M \cap \mathfrak{M}^\kappa_\alpha = M' \cap \mathfrak{M}^\kappa_\alpha$. Therefore, $p^{M'}$ is also $(M, \mathfrak{M}^\kappa_\alpha)$-strongly generic, and thus it is $(M^*, \mathfrak{M}^\kappa_\alpha)$-generic. Since $\dot{F} \in M^*$, it follows that $p^{M'}$ forces that M is closed under $\dot{F}$. It also forces that M' belongs to $\dot{\mathcal{M}}_\alpha$, hence it forces that M belongs to $\dot{\mathscr{C}}^\alpha_{\mathrm{st}}$. $\boxed{4.4.2}$

Lemma 4.4.3. *Suppose that $\alpha \in E$ and let G_α be V-generic over $\mathbb{P}^\kappa_\alpha$. Then $\mathbb{Q}_\alpha$ is $\mathscr{C}_{\mathrm{st}}[G_\alpha]$-strongly proper.*

Proof. Work in $V[G_\alpha]$. Let $p \in \mathbb{Q}_\alpha$, and $M \in \mathscr{C}_{\mathrm{st}}[G_\alpha]$ be such that $p \in M$. Let p^M be the condition defined in Lemma 4.2.8. Since $\alpha \in M$ we have that $p \restriction_\alpha \in M$, and also $p \restriction_\alpha \in M \restriction_\alpha$. Note that $p^M \restriction_\alpha = (p \restriction_\alpha)^{M \restriction \alpha}$. Since $p \restriction_\alpha \in G_\alpha$ and $M \restriction_\alpha \in \mathcal{M}^\alpha_{G_\alpha}$, we have that $p^M \restriction_\alpha \in G_\alpha$, thus $p^M \in \mathbb{Q}_\alpha$. Let us show that p^M is $(M, \mathbb{Q}_\alpha)$-strongly generic. Suppose that $q \leq p^M$ and $q \restriction_\alpha \in G_\alpha$. Since $\alpha \in M \restriction_\alpha$ we should have that have $(q \restriction_M) \restriction_\alpha = (q \restriction_\alpha) \restriction_{(M \restriction_\alpha)}$, and hence $q \restriction_M \in M \cap \mathbb{Q}_\alpha$. Let $r \leq q \restriction_M$ be such that $r \in M \cap \mathbb{Q}_\alpha$. By Lemma 4.2.23, r and q are compatible in $\mathfrak{M}^\kappa_\lambda$ and the meet $r \wedge q$ exists. Now, observe that the meet of $r \restriction_\alpha$ and $q \restriction_\alpha$ exists, and $r \restriction_\alpha \wedge q \restriction_\alpha = (r \wedge q) \restriction_\alpha$. Since $r \restriction_\alpha, q \restriction_\alpha \in G_\alpha$, we conclude that $r \restriction_\alpha \wedge q \restriction_\alpha \in \mathbb{Q}_\alpha$. It follows that q and r are compatible in $\mathbb{Q}_\alpha$. $\boxed{4.4.3}$

Now, by Lemma 4.4.2, Lemma 4.4.3, and Proposition 1.3.6, we get the following.

Corollary 4.4.4. *Suppose that G_λ is V-generic over $\mathfrak{M}^\kappa_\lambda$, $\alpha \in E$ and $G_\alpha = G_\lambda \cap \mathfrak{M}^\kappa_\alpha$. Then the pair $(V[G_\alpha], V[G_\lambda])$ has the ω_1-approximation property.*

4.4.4

Suppose now $N \in \mathscr{U}$. Let $\mathbf{1}^N = (\{N\}, \varnothing)$. By Lemma 4.2.12, $\mathbf{1}^N$ is $(N, \mathfrak{M}^\kappa_\lambda)$-strongly generic. Moreover, for every $q \leq \mathbf{1}^N$ and $r \leq q \restriction_N$ with $r \in N$, q and r are compatible, and the meet $q \wedge r$ exists. Let $\mathfrak{M}_N = \mathfrak{M}^\kappa_\lambda \cap N$ and let

$$\mathfrak{M}^\kappa_\lambda \restriction_N = \{q \in \mathfrak{M}^\kappa_\lambda : N \in \mathcal{M}_q\}.$$

Then the map $p \mapsto p^N$ is a complete embedding from $\mathfrak{M}_N$ to $\mathfrak{M}^\kappa_\lambda \restriction_N$. Now, fix a V-generic filter G_N over $\mathfrak{M}_N$.

Definition 4.4.5. *Let $\mathscr{C}_{\mathrm{st}}[G_N]$ denote the set of all $M \in \mathscr{C}_{\mathrm{st}}$ such that $N \in M$ and $N \wedge M \in \mathcal{M}_{G_N}$.*

Lemma 4.4.6. *$\mathscr{C}_{\mathrm{st}}[G_N]$ is a stationary subset of $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_N]$.*

Proof. This is very similar to the proof of Lemma 4.4.2. We work in V. Let $\dot{\mathscr{C}}^N_{\mathrm{st}}$ be the canonical $\mathfrak{M}_N$-name for $\mathscr{C}_{\mathrm{st}}[G_N]$. It suffices to show that if $p \in \mathfrak{M}_N$ and $\dot{F}$ is a $\mathfrak{M}_N$-name for an algebra on V_λ then there is $q \leq p$ and $M \in \mathscr{C}_{\mathrm{st}}$ such that q forces that M belongs to $\dot{\mathscr{C}}^N_{\mathrm{st}}$ and is closed under $\dot{F}$. Let θ be a sufficiently large regular cardinal. By Proposition 3.2.15, $\mathscr{C}_{\mathrm{st}}$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$, hence we can find a countable $M^* \prec H(\theta)$ with $\dot{F}, N, \mathfrak{M}^\kappa_\lambda \in M^*$ such that letting $M = M^* \cap V_\lambda$, we have that $M \in \mathscr{C}_{\mathrm{st}}$. Since $N \in M$ the meet $N \wedge M$ is defined. Let $\eta = \eta(N \wedge M)$ and let σ be the η-isomorphism between $N \cap M$ and $N \wedge M$. Note that $\sigma(q) = q$, for all $q \in \mathfrak{M}_N$. Now, $p^{N \wedge M}$ is $(N \wedge M, \mathfrak{M}_N)$-strongly generic, hence also $(N \cap M, \mathfrak{M}_N)$-strongly generic, and therefore it is $(M^*, \mathfrak{M}_N)$-generic. Hence if $p^{N \wedge M} \in G_N$ then $M = M^*[G_N] \cap V_\lambda$ is closed under $\mathrm{val}_{G_N}(\dot{F})$. It follows that $p^{N \wedge M}$ forces that $M \in \dot{\mathscr{C}}^N_{\mathrm{st}}$ and is closed under $\dot{F}$. 4.4.6

Let $\mathbb{Q}_N$ denote the quotient forcing $(\mathfrak{M}^\kappa_\lambda \restriction_N)/G_N$.

Lemma 4.4.7. *$\mathbb{Q}_N$ is $\mathscr{C}_{\mathrm{st}}[G_N]$-strongly proper in the model $V[G_N]$.*

Proof. Work in $V[G_N]$. Let $p \in \mathbb{Q}_N$ and $M \in \mathscr{C}_{\mathrm{st}}[G_N]$ be such that $p \in M$. Let p^M be the condition defined in Lemma 4.2.8. Since $p, N \in M$, we have $p \restriction_N \in M$. Thus, $p \restriction_N \in N \cap M$. Observe that $p^M \restriction_N = (p \restriction_N)^{N \wedge M}$. Since $p \restriction_N \in G_N$ and $N \wedge M \in \mathcal{M}_{G_N}$, we have that $p^M \restriction_N \in G_N$, thus $p^M \in \mathbb{Q}_N$. Let us show that

p^M is $(M, \mathbb{Q}_N)$-strongly generic. Suppose that $q \leq p^M$ and $q \in \mathbb{Q}_N$. Observe that $(q \restriction_M) \restriction_N = (q \restriction_N) \restriction_{(N \wedge M)}$, and hence $q \restriction_M \in \mathbb{Q}_N$. Let $r \leq q \restriction_M$ be such that $r \in M \cap \mathbb{Q}_N$. By Lemma 4.2.23, r and q are compatible in $\mathfrak{M}^\kappa_\lambda$ and the meet $r \wedge q$ exists. Note that $r \restriction_N \in N \cap M \subseteq N \wedge M$, and $r \restriction_N$ extends $(q \restriction_N) \restriction_{(N \wedge M)}$. Hence, again by Lemma 4.2.23, the meet of $r \restriction_N$ and $q \restriction_N$ exists, and $r \restriction_N \wedge q \restriction_N = (r \wedge q) \restriction_N$. Since $r \restriction_N, q \restriction_N \in G_N$, we have that $r \restriction_N \wedge q \restriction_N \in G_N$. It follows that r and q are compatible in $\mathbb{Q}_N$. $\boxed{4.4.7}$

Chapter Five

Iteration and Forcing Axiom

In this chapter we shall discuss an iteration with virtual models and settle a forcing axiom for structured $\aleph_{1.5}$-c.c forcings.

5.1 Virtual Model Based Iteration

Suppose $\kappa < \lambda$ are inaccessible. Let $U : V_\lambda \to V_\lambda$ be a function, later we will work with a bookkeeping function U, but we do not need to specify it right now. We shall consider the construction form the previous chapter based on the suitable structure $\mathfrak{A} = (V_\lambda, \in, \kappa, U)$. For each $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$ and each $\gamma \in E$, we let

$$\mathcal{M}_p(\gamma) := \{M \in \mathcal{M}_p^{\mathrm{next}(\gamma)} : \gamma \in M\}.$$

Thus $\mathcal{M}_p(\gamma)$ consists of those models in $\mathcal{M}_p{\restriction}_{\mathrm{next}(\gamma)}$ which are strongly active at $\mathrm{next}(\gamma)$.

Realization

Lemma 5.1.1. *Suppose $\mathcal{M}_p \in \mathbb{M}^\kappa_\lambda$, and $\gamma \in E$. Then $\mathcal{M}_p(\gamma)$ is an $\in_{\mathrm{next}(\gamma)}$-chain which is closed under meets.*

Proof. Notice that $\mathcal{M}_p^{\mathrm{next}(\gamma)}$ is an $\in_{\mathrm{next}(\gamma)}$-chain. It is obvious that $\mathcal{M}_p(\gamma)$ is closed under meets. Looking at countable models between two successive Magidor models in $\mathcal{M}_p^{\mathrm{next}(\gamma)}$, if there is some model containing γ, then on a tail every model should contain γ, removing an initial segment of the chain gives still an $\in_{\mathrm{next}(\gamma)}$-chain together with the two Magidor models. Repeating this for every two successive Magidor models produces bunch of $\in_{\mathrm{next}(\gamma)}$-chains in which the union forms

an $\in_{\mathrm{next}(\gamma)}$-chain as well since from some Magidor model on every Magidor model should contain γ. The same situation holds for the least and the last Magidor models. $\boxed{5.1.1}$

Definition 5.1.2. *Suppose $\mathcal{M} = \langle M_i : i \leq k \rangle$ is an $\in_\alpha$-chain of models in $\mathscr{C} \cup \mathscr{U}$. A* realization *of $\mathcal{M}$ is an $\in$-chain $\langle M_i^* : i \leq k \rangle$ such that:*

- *For each $i < k$, $M_i^* \in \mathscr{V}^{M_{i+1}^*}$, and $M_k^* \in \mathscr{V}$.*
- *For each $i \leq k$, $M_i \cong_\alpha M_i^*$.*

Lemma 5.1.3. *Every $\in_\alpha$-chain has a realization.*

Proof. For each $i < k$, let $\alpha_i^* = \min(E \cap M_i \setminus \alpha)$, and let $M_i' \in M_{i+1}$ be the unique α_{i+1}^*-model such that $M_i \cong_\alpha M_i'$. Let σ_i witness such isomorphism between M_i and M_i'. We now define $\langle M_i^* : i \leq k \rangle$ as follows.

- $M_k^* = M_k$.
- $M_{k-1}^* = M_{k-1}'$.
- For $i < k-1$, $M_i^* = \sigma_{k-1} \circ \cdots \circ \sigma_{i+1}(M_i')$.

By Proposition 3.2.12, each M_i^* is α-isomorphic to M_i. If $i < k$, then $M_i' \in M_{i+1}$, and hence $\sigma_{i+1}(M_i') \in \sigma_{i+1}[M_{i+1}] = M_{i+1}'$. We have thus that $M_i^* \in M_{i+1}^*$. It is easily seen that $M_i^* \in \mathscr{V}^{M_{i+1}^*}$. Thus $\langle M_i^* : i \leq k \rangle$ is a realization of $\mathcal{M}$. $\boxed{5.1.3}$

Remark 5.1.4. Let us say that a realization $\langle M_i^* : i \leq k \rangle$ is a *minimal realization*, if for every realization $\langle N_i : i \leq k \rangle$ of $\mathcal{M}$, $N_i \restriction_{\eta(M_i^*)} = M_i^*$. One can show with a bit more work that every $\in_\alpha$-chain $\mathcal{M}$ has a unique minimal realization. The construction is more or less as above, but it should have this property that each initial segment of the realization is the minimal realization of the same initial segment of the original sequence. This requires working with different degrees of virtualness and applying certain projections in the course of the construction of M_i^*.

Lemma 5.1.5. *Assume that $\mathcal{M} = \langle M_i : i \leq k \rangle$ is an $\in_\alpha$-chain of models in $\mathscr{C} \cup \mathscr{U}$ which is closed under meets, and that every model in $\mathcal{M}$ is active at α. Suppose that $\mathbb{P}$ is a forcing and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-term such that $\mathbb{P}, \dot{\mathbb{Q}} \in \bigcap \mathcal{M} \cap V_\alpha$. Suppose $\mathbb{P}$ satisfies the following.*

1. $\mathbf{1}_{\mathbb{P}} \Vdash$ " *Every model in $\mathcal{M}$ is of size at most $\aleph_1$*".

2. $\mathbf{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ *is an* $\aleph^s_{1.5}$*-c.c forcing*"

Let G be a V-generic filter on $\mathbb{P}$ containing a condition which is $(M_i, \mathbb{P})$-generic, for every $i \leq k$. Let $j \leq k$ be such that $M_j \cap \omega_1 = \min(M_i \cap \omega_1 : i \leq k)$. Then every $q \in \dot{\mathbb{Q}}^G \cap M_j[G]$ has an extension, $r \leq q$ which is $(M_i[G], \mathbb{Q})$-generic, for every $i \leq k$.

Proof. Suppose $\mathcal{M}^* = \langle M^*_i : i \leq k\rangle$ is a realization of $\mathcal{M}$ with the corresponding α-isomorphisms $(\sigma_i)_{i\leq k}$. Let also $\tau \in V_\alpha \cap M^*_0$ be a $\mathbb{P}$-name such that $\mathbf{1}_{\mathbb{P}} \Vdash \tau = \mathcal{P}(\dot{\mathbb{Q}})$. Now one can pick a regular cardinal $\theta \in M^*_0 \cap \alpha$ such that $\mathbb{P}, \dot{\mathbb{Q}}, \tau \in H_\theta$. By Proposition 3.2.28 and the fact that $\sigma_i \restriction_{V_\alpha}$ is identity, $\mathcal{C} := \{M^*_i \cap H_\theta : i \leq k\}$ is an $\in$-chain which is closed under intersections. Set $\mathcal{C}[G] = \{(M^*_i \cap H_\theta)[G] : i \leq k\}$, which is an $\in$-chain of elementary submodels of $H_\theta^{V[G]}$, and moreover it is closed under intersections since G contains a condition which is $(M_i, \mathbb{P})$-generic, for every $i \leq k$. Furthermore, every model in $\mathcal{C}[G]$ contains θ and other relevant information, and that by assumption, each model in $\mathcal{C}[G]$ is of size at most $\aleph_1$. Set $\mathbb{Q} = \dot{\mathbb{Q}}^G$. Since $\mathbb{Q}$ is an $\aleph^s_{1.5}$-c.c forcing in $V[G]$, there is $r \leq q$ such that r is $((M^*_i \cap H_\theta)[G], \mathbb{Q})$-generic, for every $i \leq k$, and hence it is $(M^*_i[G], \mathbb{Q})$-generic, for every $i \leq k$. On the other one hand we can extend σ_i to an isomorphism $\bar{\sigma}_i : M_i[G] \to M^*_i[G]$, but since σ is identity on $M_i \cap H_\theta = M^*_i \cap H_\theta$, it follows that r is $(M_i[G], \mathbb{Q})$-generic, for every $i \leq k$.

$\boxed{5.1.5}$

Virtual Model Based Iteration

We define the iteration of $\aleph^s_{1.5}$-c.c forcing notions by induction along E.

Definition 5.1.6 (Iteration)**.** *Suppose $\alpha \in E$. A condition p in $\mathbb{P}^\kappa_\alpha$ is a triple $p = (\mathcal{M}_p, d_p, w_p)$, where*

1. $(\mathcal{M}_p, d_p) \in \mathfrak{M}^\kappa_\alpha$.

2. $w_p : \mathrm{dom}(w_p) \to V_\lambda$ *is a finite function with* $\mathrm{dom}(w_p) \subseteq E \cap \alpha$ *such that if* $\gamma \in \mathrm{dom}(w_p)$*, and $U(\gamma)$ is a $\mathbb{P}^\kappa_\gamma$-term in $V_{\mathrm{next}(\gamma)}$ for an $\aleph^s_{1.5}$-c.c forcing, then $w_p(\gamma)$ is a canonical $\mathbb{P}^\kappa_\gamma$-term such that*

$$\mathbf{1}_{\mathbb{P}^\kappa_\gamma} \Vdash_{\mathbb{P}^\kappa_\gamma} w_p(\gamma) \in U(\gamma).$$

Otherwise, let $w_p(\gamma)$ be a condition in the trivial forcing.

3. *For every* γ *in* $\mathrm{dom}(w_p)$ *and every* $M \in \mathcal{M}_p(\gamma)$,

$$p \restriction_\gamma \Vdash_{\mathbb{P}^\kappa_\gamma} \text{“}w_p(\gamma) \text{ is } (M[\dot{G}_\gamma], U(\gamma))\text{”-generic} ,$$

where

$$p \restriction_\gamma = (\mathcal{M}_p \restriction_\gamma, d_p \restriction_\gamma, w_p \restriction_\gamma).$$

The ordering on $\mathbb{P}^\kappa_\alpha$ *is as follows. We say* p *is stronger than* q *and write* $p \leq q$, *if and only if*

1. $(\mathcal{M}_p, d_p) \leq (\mathcal{M}_q, d_q)$ *in* $\mathfrak{M}^\kappa_\alpha$.

2. $\mathrm{dom}(w_p) \supseteq \mathrm{dom}(w_q)$, *and for every* $\gamma \in \mathrm{dom}(w_q)$,

$$p \restriction \gamma \Vdash w_p(\gamma) \leq_{U(\gamma)} w_q(\gamma).$$

Let also $\mathbb{P}^\kappa_\lambda = \bigcup_{\alpha \in E} \mathbb{P}^\kappa_\alpha$ *with the same ordering. We let also* $\mathbb{P}^\kappa_{\alpha+1} = \mathbb{P}^\kappa_\alpha * U(\alpha)$, *for every* $\alpha \in E$.

The ordering is transitive and whenever $p \leq q$ are in $\mathbb{P}^\kappa_\lambda$ and α is in E, $p \restriction_\alpha$ is a condition in $\mathbb{P}^\kappa_\alpha$, and extends $q \restriction_\alpha$. We remark that the decoration here in this construction will not play any role in the proof of properness. This is important to know since the decorations can play an auxiliary role to guarantee properness for example see the example $\mathbb{P}_\Sigma$ in Appendix B.

Remark 5.1.7. We refer to the above construction as the *virtual model based iteration.*

Remark 5.1.8. Suppose that $\mathfrak{A} = (V_\lambda, \kappa, \in, u)$ is a suitable structure, where u is a predicate on V_λ. Assume that U is a function on λ which is uniformly definable in $\mathfrak{A}$ from parameter α. We can consider the suitable structure $\mathfrak{B} = (V_\lambda, \in, \kappa, U)$. We should have that $E_{\mathfrak{A}} = E_{\mathfrak{B}}$. Notice that $E_{\mathfrak{A}} \cap \alpha$ is definable in $\mathfrak{A}$ whenever $\alpha \in E_{\mathfrak{A}}$.

Proposition 5.1.9. *Suppose* $p \in \mathbb{P}^\kappa_\beta$, *and* $\alpha \leq \beta$ *are in* E. *If* q *is a condition in* $\mathbb{P}^\kappa_\alpha$ *extending* $p \restriction \alpha$, *then* p *is compatible with* q *in* $\mathbb{P}^\kappa_\beta$.

Proof. Let (M_r, d_r) be $(\mathcal{M}_p, d_p) \wedge (\mathcal{M}_q, d_q)$ as defined in Lemma 4.2.5. Let also w_r be defined on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$ by

$$w_r(\gamma) = \begin{cases} w_q(\gamma) & \text{if } \gamma < \alpha \\ w_p(\gamma) & \text{if } \gamma \geq \alpha \end{cases}$$

If $\gamma \leq \alpha$, then $\mathcal{M}_r(\gamma) = \mathcal{M}_q(\gamma)$, and if $\gamma > \alpha$, then $\mathcal{M}_r(\gamma) = \mathcal{M}_p(\gamma)$. It is now evident that r is a condition extending p and q.

5.1.9

Remark 5.1.10. The condition r from the previous lemma is the greatest lower bound of p and q, so we will denote it by $r = p \wedge q$.

Corollary 5.1.11. *For every $\alpha \leq \beta$ in $E \cup \{\lambda\}$, $\mathbb{P}^\kappa_\alpha \subseteq_c \mathbb{P}^\kappa_\beta$.*

5.1.11

We aim to prove the properness of the iteration and the preservation of κ. This will be followed from the three lemmata below, but we need to be careful since these lemmata cannot be proved separately, basically because we have to do induction. Let us explain what may happen. First we would like to show that if p is a condition in $\mathbb{P}^\kappa_\alpha \cap M$, then p can be extended to a condition which is $(M, \mathbb{P}^\kappa_\alpha)$-generic. We do this naturally by putting M in the side condition and extending the conditions in the working part to generic conditions. If G_α is a V-generic filter, then we need to know that $N[G_\alpha]$ is of size $\aleph_1$ whenever N is a Magidor model in $\mathcal{M}_G$. Thus we need to make sure that $|N|$ gets collapsed to ω_1 and that ω_1 is preserved. On the other hand for the proof of properness we need to extend conditions inside a model M to a generic one. This seems to be circular, but as a matter of fact, Theorem 4.2.30 implies that if $\alpha \in E$, then every Magidor model in $\mathcal{M}^\alpha_G$ is of size $\aleph_1$ in $V[G_\alpha]$, and by induction one can assume the properness of the iteration up to α.

Proposition 5.1.12. *Let $p \in \mathbb{P}^\kappa_\alpha$ and $M \in \mathscr{C} \cup \mathscr{U}$ be such that $p \in M$. Then there is a condition $p^M \leq p$ with $M \in \mathcal{M}_{p^M}$.*

Proof. Let $\mathcal{M}_{p^M}$ and d_{p^M} be defined by $(\mathcal{M}_p, d_p)^M$. If M is a Magidor model, then $\mathcal{M}_{p^M} = \mathcal{M}_p \cup \{M\}$. For every $\gamma \in \mathrm{dom}(w_p)$, since $U(\gamma)$ is forced by the maximal condition to be an $\aleph_2$-c.c forcing, $p \restriction_\gamma$ forces that $w_p(\gamma)$ is $(M[G_\gamma], U(\gamma))$-generic . It is evident that $p^M = (\mathcal{M}_{p^M}, d_{p^M}, w_p)$ is a condition extending p^M.

Now, suppose that M is countable. If γ is in $\mathrm{dom}(w_p)$, then $w_p(\gamma)$ and $U(\gamma)$ are in M. Let w_{p^M} be defined on $\mathrm{dom}(w_p)$ as follows. One can find a $\mathbb{P}^\kappa_\gamma$-term τ_γ so that

$$\mathbf{1}_{\mathbb{P}^\kappa_\gamma} \Vdash \tau_\gamma \leq w_p(\gamma) \text{ and } \tau_\gamma \text{ is } (M[\dot{G}_\gamma], U(\gamma))\text{-generic.}$$

This is possible since $U(\gamma)$ is forced to be a proper forcing notion. By the $\aleph_2$-c.c-ness of $U(\gamma)$, $w_{p^M}(\gamma)$ is forced to be $(N[\dot{G}_\gamma], U(\gamma))$-generic for every Magidor model

$N \in \mathcal{M}_p(\gamma)$, and hence it is forced to be $(P[\dot{G}_\gamma], U(\gamma))$-generic for every model P in $\mathcal{M}_{p^M}(\gamma)$ since the only new models in $\mathcal{M}_{p^M}(\gamma)$ are M itself and the meets of M and the Magidor models in $\mathcal{M}_p$. Thus p^M is a condition and $p^M \leq p$. $\boxed{5.1.12}$

Proposition 5.1.13. *Suppose p is a condition in $\mathbb{P}^\kappa_\alpha$, and $M \in \mathscr{C} \cup \mathscr{U}$ is such that $\alpha \in M$ and $M \restriction_\alpha \in \mathcal{M}_p$ with $\eta(M) > \alpha$. Then p is $(M, \mathbb{P}^\kappa_\alpha)$-generic.*

Proof. By assumption, $\mathbb{P}^\kappa_\alpha$ belongs to M. Our proof relies on an induction over E. Notice that $\mathbb{P}^\kappa_{\min(E)}$ is $\mathscr{C}$-strongly proper. Suppose the conclusion holds for every ordinal in $E \cap \alpha$. Suppose $D \subseteq \mathbb{P}^\kappa_\alpha$ is an open dense set belonging to M. We may assume that p is in D. Let $\beta := \max(\mathrm{dom}(w_p) \cap M)$. Pick a V-generic filter G_β on $\mathbb{P}^\kappa_\beta$ so that $p \restriction_\beta \in G_\beta$. Let w_0 and $\mathbb{U}$ be the interpretations of $w_p(\beta)$ and of $U(\beta)$ under G_β, respectively. Let

$$\Delta = \bigcup \{P \cap \alpha : P \in \mathcal{M}_p \restriction_M \text{ is countable}\}.$$

Thus Δ is at most countable, and belongs to M. Notice that if $\gamma > \beta$ is in $\mathrm{dom}(w_p)$, then γ cannot be in Δ. Let F consist of conditions $z \in \mathbb{U}$ such that there exists $q \in D$ satisfying the following conditions.

1. $(\mathcal{M}_q, d_q) \leq (\mathcal{M}_p, d_p) \restriction_M$.
2. $M \cap \mathrm{dom}(w_p) \subseteq \mathrm{dom}(w_q)$.
3. $q \restriction_\beta \in G_\beta$.
4. $w_q^{G_\beta}(\beta) = z$.
5. $\Delta \cap \mathrm{dom}(w_q) \setminus (\beta + 1) = \varnothing$.

Notice that w_0 belongs to F. Let also $F^\perp = \{w \in \mathbb{U} : w \perp z \quad \forall z \in F\}$. Both F and $F^\perp$ belong to $M[G_\beta]$. Consider now $F^* := F \cup F^\perp$ that is pre-dense in $\mathbb{U}$. Since w_0 is $(M[G_\beta], \mathbb{U})$-generic, there is some $z \in F^* \cap M[G_\beta]$ compatible with w_0, but z cannot be in $F^\perp$ as w_0 belongs to F. Therefore, there is some $q \in D \cap M[G_\beta]$ such that $w_q^{G_\beta}(\beta) = z$, and that it satisfies the conditions above, in particular $(\mathcal{M}_q, d_q) \leq (\mathcal{M}_p \restriction_M, d_p \restriction_M)$. Notice that q is in $D \cap M$, by the inductive assumption. We shall show that p and q are compatible. Let $\dot{z}$ be in M so that $\dot{z}^{G_\beta} = z$. We may then find a condition $r \in G_\beta$ extending $p \restriction_\beta$ and $q \restriction_\beta$, and also some $\mathbb{P}^\kappa_\beta$-name τ_β so that $r \Vdash \tau_\beta \leq \dot{z}, w_p(\beta)$, and that the maximal condition in $\mathbb{P}^\kappa_\beta$ forces τ_β is in $U(\beta)$. Let $t = ((\mathcal{M}_p, d_p) \wedge (\mathcal{M}_q, d_q), \varnothing)$. Suppose that $\gamma \in \mathrm{dom}(w_q)$ is

bigger than β and is in M. Consider the $\in_\gamma$-chain $\mathcal{M}_t(\gamma)$. Since $\gamma > \beta$, it avoids Δ by the last condition above, and thus it does not belong to any countable model in $\mathcal{M}_p(\gamma)$. Now if M is countable, we have that $M \cap \omega_1 = \min(P \cap \omega_1 : P \in \mathcal{M}_t(\gamma))$. Since $\mathbb{P}^\kappa_\gamma$ is proper by the inductive assumption, if G_γ is a V-generic filter on $\mathbb{P}^\kappa_\gamma$ containing $t \restriction_\gamma$, then $U^{G_\gamma}(\gamma)$ is an $\aleph^s_{1.5}$-c.c forcing, and that the assumptions of Lemma 5.1.5 are satisfied, so one can apply it to $w_q^{G_\gamma}(\gamma)$ to find some condition $z_\gamma \leq w_q^{G_\gamma}(\gamma)$ which is $(P[G_\gamma], U^{G_\gamma}(\gamma))$-generic, for every $P \in \mathcal{M}_t(\gamma)$. We may now find a canonical $\mathbb{P}^\kappa_\gamma$-name τ_γ so that the maximal condition of $\mathbb{P}^\kappa_\gamma$ forces that $\tau_\gamma \in U(\gamma)$ and $\tau_\gamma \leq w_q(\gamma)$, and that it is forced by $t \restriction_\gamma$ to be a $(P[\dot{G}_\gamma], U(\gamma))$-generic condition for every $P \in \mathcal{M}_t(\gamma)$. On the other hand, if M is Magidor, the there is no countable model in $\mathcal{M}_p(\gamma)$ or in other words every countable model in $\mathcal{M}_t(\gamma)$ belongs to $\mathcal{M}_q(t)$. This together with $\aleph_2$-c.c-ness imply that the maximal condition forces that $w_q(\gamma)$ is $(P[\dot{G}_\gamma], U(\gamma))$-generic, for every $P \in \mathcal{M}_t(\gamma)$.

We now demonstrate how to amalgamate p and q. We define s as follows. Let $\mathcal{M}_s = \mathcal{M}_r \cup \mathcal{M}_t$ which is easily seen to be in $\mathbb{M}^\kappa_\alpha$. Let also d_s be defined on $\mathrm{dom}(d_r) \cup \mathrm{dom}(d_t)$ by letting $d_s(P) = d_r(P)$ if $\eta(P) \leq \beta$, and $d_s(P) = d_t(P)$ if $\eta(P) > \beta$. Notice that (M_s, d_s) belongs to $\mathfrak{M}^\kappa_\alpha$. It remains to define w_s. Let w_s be defined on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_q) \cup \mathrm{dom}(w_r)$ by

$$w_s(\gamma) = \begin{cases} w_r(\gamma) & \gamma < \beta \\ \tau_\gamma & \gamma \geq \beta \text{ and } \gamma \in M \\ w_p(\gamma) & \gamma > \beta \text{ and } \gamma \notin M \end{cases}$$

Notice that if $\gamma > \beta$ is not in M, then there is no countable model in M containing γ, and if $N \in \mathcal{M}_s(\gamma)$ is a Magidor model, then $\aleph_2 - c.c.$-ness implies that the maximal condition of $\mathbb{P}^\kappa_\gamma$ forces that $w_p(\gamma)$ is $(N[\dot{G}_\gamma], U(\gamma))$-generic. On the other hand, since $\mathcal{M}_s(\gamma) = \mathcal{M}_t(\gamma)$ and $s \restriction_\gamma \leq t \restriction_\gamma$, for every $\gamma \geq \beta$, τ_γ is forced by $s \restriction_\gamma$ to be $(P[\dot{G}_\gamma], U(\gamma))$-generic, for every $P \in \mathcal{M}_s(\gamma)$. It is now easily seen that s is a condition and that it extends both p and q. $\boxed{5.1.13}$

Corollary 5.1.14. *For every $\alpha \in E \cup \{\lambda\}$, $\mathbb{P}^\kappa_\alpha$ is proper.*

Proof. For $\alpha \in E$, it is easily seen by Proposition 5.1.12 and Proposition 5.1.13. Suppose $\theta > \lambda$ is a sufficiently large regular cardinal, and let M be a countable elementary submodel of H_θ such $\kappa, \lambda, E \in M$. Then $\eta = \sup(M \cap \lambda)$ is in E. It is easily seen that $M \cap V_\lambda$ is an η-model. Thus the set of such models contains a club in $\mathcal{P}_{\omega_1}(H_\theta)$. Now if p is a condition in $\mathbb{P}^\kappa_\lambda \cap M$, then it belongs to $\mathbb{P}^\kappa_\gamma$, for some $\gamma < \eta$.

Using Proposition 5.1.12, one can extend p to a condition p^M by adding $M \cap V_\lambda$ to $\mathcal{M}_p$, and then use the same argument as in Proposition 5.1.13 to show that every extension of p^M is $(M, \mathbb{P}^\kappa_\lambda)$-generic.

5.1.14

Corollary 5.1.15. *For every $\alpha \in E$, $\mathbb{P}^\kappa_\alpha$ is $\mathscr{U}$-strongly proper.*

Proof. This follows from Proposition 5.1.13 and the fact (Proposition 1.3.5) that being generic for Magidor models is the same as being strongly generic. 5.1.15

The following can be proved in a similar way as in Corollary 5.1.14 and in the last corollary.

Corollary 5.1.16. *For every θ sufficiently large, there exists some stationary set $\mathcal{U} \subseteq \mathcal{P}_\kappa(H_\theta)$ such that $\mathbb{P}^\kappa_\lambda$ is $\mathcal{U}$-strongly proper.*

5.1.16

We have also the following chain condition.

Proposition 5.1.17. *$\mathbb{P}^\kappa_\lambda$ is λ-c.c.*

Proof. We follow closely the proof of Theorem 4.2.32. We give the details for completion. Suppose A is a subset of $\mathbb{P}^\kappa_\lambda$ of cardinality λ. For each $p \in \mathfrak{M}^\kappa_\lambda$, let $a(p) = \bigcup\{a(M) : M \in \mathcal{M}_p\}$. Note that $a(p)$ is a closed subset of E of size $< \kappa$, for all p. By a standard Δ-system argument, we can find a subset B of A of size λ so that there are a and d subsets of E such that $a(p) \cap a(q) = a$ and $\mathrm{dom}(w_p) \cap \mathrm{dom}(w_q) = d$, for all distinct $p, q \in B$. Note that a is closed, and if we let $\gamma = \max(a)$ then $\gamma \in E$. Since B has size λ, by a simple counting argument, we may assume there is $\mathcal{M} \in \mathbb{M}^\kappa_\gamma$ such that $\mathcal{M}_p \restriction_\gamma = \mathcal{M}$ and that $w_p \restriction_d = w_q \restriction_d$, for all $p \in B$. Now, pick distinct $p, q \in B$, and define $\mathcal{M}_r = \mathcal{M}_p \cup \mathcal{M}_q$, $d_r = d_p \cup d_q$, and also $w_r = w_p \cup w_q$. Let $r = (\mathcal{M}_r, d_r, w_r)$. It is straightforward to check that $r \in \mathbb{P}^\kappa_\lambda$ and $r \leq p, q$. 5.1.17

Putting everything together, we have the following.

Corollary 5.1.18. *$\mathbb{P}^\kappa_\lambda$ preserves ω_1, κ and λ, and that forces $\kappa = \omega_2$ and $\lambda = \omega_3$.*

Proof. The preservation of ω_1 and κ are guaranteed by Proposition 5.1.13 and Corollary 5.1.16, respectively. The same arguments as in Theorem 4.2.30 together with Proposition 5.1.17 show that in the generic extension by $\mathbb{P}^\kappa_\lambda$, $\kappa = \omega_2$ and $\lambda = \omega_3$. 5.1.18

The following easy lemma says that we definitely iterate the relevant forcing notions given by U, and hence the above facts about $\langle \mathbb{P}^\kappa_\alpha : \alpha \in E \rangle$ are not superfluous comparing to the forcing with pure side conditions.

Lemma 5.1.19. *Suppose p is a condition in $\mathbb{P}^\kappa_\alpha$. Let $\beta \in E \cap \alpha$. Then there is a condition $q \leq p$ such that $\beta \in \mathrm{dom}(w_p)$.*

Proof. Suppose β is not $\mathrm{dom}(w_p)$. We may assume that $U(\beta)$ is forced by the maximal condition to be an $\aleph^s_{1.5}$-c.c forcing notion. We can then find a canonical $\mathbb{P}^\kappa_\beta$-term τ such that for every $M \in \mathcal{M}_p(\beta)$,

$$\mathbf{1} \Vdash_{\mathbb{P}^\kappa_\beta} \tau \text{ is } (M[\dot{G}_\beta], U(\beta)) - \text{generic}.$$

Thus we define q by letting $(\mathcal{M}_q, d_q) = (\mathcal{M}_p, d_p)$ and

$$w_q(\gamma) = \begin{cases} \tau & \text{if } \gamma = \beta \\ w_p(\gamma) & \text{if } \gamma \neq \beta \end{cases}$$

Thus q is a condition extending p such that $\beta \in \mathrm{dom}(w_q)$. $\boxed{5.1.19}$

Consistency of $\mathrm{MA}^{s-1.5}_{\aleph_2}$

In this subsection, we establish the consistency of $\mathrm{MA}^{s-1.5}_{\aleph_2}$.

Theorem 5.1.20. *Suppose it is consistent that there are two supercompact cardinals. Then, the forcing axiom $\mathrm{MA}^{s-1.5}_{\aleph_2}$ is consistent.*

Proof. Suppose $\kappa < \lambda$ are supercompact cardinals. Let $F : \lambda \to V_\lambda$ be a Laver function. We define $U : V_\lambda \to V_\lambda$ as follows. If $F(\alpha)$ is a pair, we let $U(\alpha)$ be the first coordinate of $F(\alpha)$, and otherwise let it be $\varnothing$. Notice that U is definable from F without parameters. We consider our construction $\mathbb{P}^\kappa_\lambda$ in $\mathfrak{A} = (V_\lambda, \in, \kappa, F, U)$ based on U i.e according to Definition 5.1.6 we use U as our predicate. Let G_λ be a V-generic filter on $\mathbb{P}^\kappa_\lambda$. The following lemma concludes the theorem.

Lemma 5.1.21. *Suppose $\mathbb{P}$ is a structured $\aleph_{1.5}$-c.c forcing notion in $V[G_\lambda]$, and $\mathscr{D} \in V[G_\lambda]$ is a collection of dense subsets of $\mathbb{P}$, of size $\aleph_2$ in $V[G_\lambda]$. Then there there is a $\mathscr{D}$-generic filter on $\mathbb{P}$.*

Proof. Let $\dot{\mathbb{P}}$ and $\dot{\mathscr{D}}$ be $\mathbb{P}^\kappa_\lambda$-terms for $\mathbb{P}$ and $\mathscr{D}$. Pick a sufficiently large regular cardinal $\mu > \lambda$. We may assume without loss of generality that $\mathbb{P} \subseteq \mu$. Since F

is a Laver function, there is an elementary embedding $j : V \to M$ where M is closed under μ-sequences, $j(\lambda) > \mu$ and that $F(\lambda) = (\dot{\mathbb{P}}, \dot{\mathscr{D}})$. We have also that $j(\mathfrak{A}) = (M_{j(\lambda)}, \in, \kappa, j(F), j(U))$, moreover $E \subseteq j(E)$, and hence $\lambda \in j(E)$. Thus every virtual model in the sense of $\mathfrak{A}$ is a virtual model in $j(\mathfrak{A})$, and vice versa every η-model in the sense $j(\mathfrak{A})$ with $\eta \in E$ is a virtual model in $\mathfrak{A}$. Using uniform definability of $\mathbb{P}^\kappa_\alpha$ for $\alpha \in E$, we have that $j(\mathbb{P}^\kappa_\lambda)$ is the direct limit of $\langle \mathbb{P}^\kappa_\alpha : \alpha \in j(E)\rangle$. In other words $\mathbb{P}^\kappa_{j(\lambda)}$ is the iteration of structured $\aleph_{1.5}$-c.c forcings along $j(E)$ in the structure $j(\mathfrak{A})$, and hence $\mathbb{P}^\kappa_\lambda$ is a complete suborder of $\mathbb{P}^\kappa_{j(\lambda)}$. As a matter of fact, $\mathbb{P}^\kappa_\lambda$ is the λ-th element of $\langle \mathbb{P}^\kappa_\alpha : \alpha \in j(E)\rangle$. Notice that $\mathbb{P}$ is an $\aleph^s_{1.5}$-c.c forcing in $M[G]$ since $\mathbb{P}^\kappa_\lambda$ is λ-c.c. and that M is closed under under λ-sequences. Suppose $G_{j(\lambda)}$ is a $V[G_\lambda]$-generic filter on $\mathbb{P}^\kappa_{j(\lambda)}/\mathbb{P}^\kappa_\lambda$. Thus one can naturally extend j to

$$j^* : V[G_\lambda] \to M[G_{j(\lambda)}]$$

defined by $j^*(\dot{x}^{G_\lambda}) = j(\dot{x})^{G_{j(\lambda)}}$. Since $j(F)(\lambda) = (\dot{\mathbb{P}}, \dot{\mathscr{D}})$, there is, in $M[G_{j(\lambda)}]$, a $j^*(\mathscr{D})$-generic filter on $j(\mathbb{P})$, and hence by the elementarity of j^*, there is such a filter in $V[G_\lambda]$.

5.1.21

5.1.20

The ω_1-approximation Property

We now prove that the iteration has the ω_1-approximation property if each $U(\alpha)$ is forced to have the ω_1-approximation property.

Proposition 5.1.22. *Suppose $\langle \mathbb{P}^\kappa_\alpha : \alpha \in E\rangle$ is a virtual model based iteration. Assume that for each $\alpha \in E$,*

$$\mathbf{1}_{\mathbb{P}^\kappa_\alpha} \Vdash U(\alpha) \text{ has the } \omega_1\text{-approximation property.}$$

Suppose $\alpha \leq \beta$ are in $E \cup \{\lambda\}$, and that G_β is a V-generic filter on $\mathbb{P}^\kappa_\beta$. Then the pairs $(V, V[G_\alpha])$ and $(V[G_\alpha], V[G_\beta])$ have the ω_1-approximation property.

Proof. We prove by induction on $\alpha \in E \cup \{\lambda\}$ that $\mathbb{P}^\kappa_\alpha$ has the ω_1-approximation, an easy modification of the proof leads the same result for $\mathbb{P}^\kappa_\beta/G_\alpha$. Notice that $\mathbb{P}^\kappa_{\min(E)}$ is $\mathscr{C}$-strongly proper, and hence has the ω_1-approximation property. Suppose that the conclusion holds for every ordinal in $E \cap \alpha$. Let $\dot{f} : \mu \to 2$ be a $\mathbb{P}^\kappa_\alpha$-name forced by p to be a function that is countably approximated in V. We may assume that μ

is a cardinal in V. Suppose $\delta > \mu, \lambda$ is a sufficiently large regular cardinal. Pick a countable $M \prec V_\delta$ containing the relevant objects. Thus $N = M \cap V_\lambda$ is a standard virtual model. Let $q \leq p^N$ be a condition which decides $\dot{f} \restriction_M$ to be some function $g : M \cap \mu \to 2$ in V. Notice that q is $(M, \mathbb{P}^\kappa_\alpha)$-generic. It is enough to show that g is guessed in M since one can then apply Lemma 1.2.18 to show that $\dot{f}$ is decided in V.

Claim 5.1.23. *g is guessed in M.*

Proof. We may assume towards a contradiction that g is not guessed in M. Let $\gamma := \max(\mathrm{dom}(w_q) \cap M)$, and pick some V-generic filter $G_{\gamma+1}$ on $\mathbb{P}^\kappa_{\gamma+1}$ such that $(q \restriction_\gamma, w_q^{G_\gamma}(\gamma)) \in G_{\gamma+1}$. Let

$$\Delta = \bigcup \{P \cap \alpha : P \in \mathcal{M}_p \restriction_M \text{ is countable}\}.$$

Working in $V[G_{\gamma+1}]$, we define in $M[G_{\gamma+1}]$ an assignment $x \to (q_x, g_x)$ on $[\mu]^\omega$ as follows.

1. $q_x \in \mathbb{P}^\kappa_\alpha$.
2. $(\mathcal{M}_{q_x}, d_{q_x}) \leq (\mathcal{M}_q, d_q) \restriction_N$.
3. $M \cap \mathrm{dom}(w_q) \subseteq \mathrm{dom}(w_{q_x})$.
4. $(q_x \restriction_\gamma, w_{q_x}^{G_\gamma}(\gamma)) \in G_{\gamma+1}$.
5. $\mathrm{dom}(w_{q_x}) \cap \Delta \setminus (\gamma + 1) = \varnothing$.
6. g_x is a function with $x \subseteq \mathrm{dom}(g_x)$.
7. $q_x \Vdash \dot{f} \restriction_x = \check{g}_x \restriction_x$.

Since $x \to (q, g)$ satisfies all the properties above, such an assignment exists in $M[G_{\gamma+1}]$ by elementary. By the inductive assumption and Lemma 1.2.17, g is not guessed in $M[G_{\gamma+1}]$. On the one hand, Lemma 1.2.20 gives us a set $B \in M[G_{\gamma+1}]$ cofinal in $[\mu]^\omega$ so that for every $x \in B \cap M[G_{\gamma+1}]$, $g_x \nsubseteq g$, and hence q_x is incompatible with q. On the other hand, the conditions 1-5 above enable us to amalgamate q_x and q, this is very similar to the proof of Proposition 5.1.13, notice that in that proof the conditions 1-3 are expressed in the definition of F, and that here the condition 4 plays the same role as the genericity of $w_q(\beta)$ there. Thus for every such $x \in B \cap M[G_{\gamma+1}]$, q_x and q are compatible, and hence a contradiction. $\boxed{5.1.23}$

$\boxed{5.1.22}$

5.2 Quotients by Magidor Models

In this section we consider the virtual model based iteration and assume that for each $\alpha \in E$, $U(\alpha)$ is forced to be of size at most $\aleph_1$. Suppose p is a condition and $\alpha \in \mathrm{dom}(w_p)$, assume that M is a Magidor model in $\mathcal{M}_p$. Notice that if α is in M, then $U(\alpha)$ is in M as well. Since $U(\alpha)$ is forced to be of size $\aleph_1$, we may find a canonical $\mathbb{P}^\kappa_\alpha$-term $\dot{w}_\alpha$ in M so that

$$\mathbf{1}_{\mathbb{P}^\kappa_\alpha} \Vdash w_p(\alpha) = \dot{w}_\alpha.$$

We may assume without loss of generality that for each $\alpha \in M \cap E$ and each condition $p \in \mathbb{P}^\kappa_\lambda$ with $M \in \mathcal{M}_p$ and $\alpha \in \mathrm{dom}(w_p)$, $w_p(\alpha)$ is in M. The first lemma of this section states that in this case $\mathbb{P}^\kappa_\lambda$ is $\mathscr{U}$-strongly proper in a canonical way.

Definition 5.2.1. *Suppose p is a condition in $\mathbb{P}^\kappa_\alpha$, and $M \in \mathcal{M}_p$ is a Magidor model. Let $p \restriction_M$ be defined by $(\mathcal{M}_{p\restriction_M}, d_{p\restriction_M}) = (\mathcal{M}_p, d_p) \restriction_M$ and $w_{p\restriction_M} = w_p \restriction_M$.*

It is clear that $p \restriction_M$ belongs to $\mathbb{P}^\kappa_\alpha \cap M$.

Lemma 5.2.2. *Suppose p is a condition in $\mathbb{P}^\kappa_\alpha$, and $M \in \mathcal{M}_p$ is a Magidor model. If q is a condition in $\mathbb{P}^\kappa_\alpha \cap M$ extending $p \restriction_M$, then q is compatible with p.*

Proof. Suppose $q \in M$ is a condition extending $p \restriction_M$. We shall show that p and q are compatible. Let r be defined by letting $(\mathcal{M}_r, d_r) = (\mathcal{M}_p, d_p) \wedge (\mathcal{M}_q, d_q)$. We then define naturally w_r on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$, by letting $w_r(\gamma)$ be $w_q(\gamma)$ if γ is in M, and otherwise we let it be $w_p(\gamma)$. If $\gamma \notin M$, then we have that $\mathcal{M}_r(\gamma) = \mathcal{M}_p(\gamma)$, and that γ is not in $\mathrm{dom}(w_q)$, but if γ is in M, then γ is in $\mathrm{dom}(w_q)$, and moreover $q \restriction_\gamma$ forces that $w_q(\gamma) \leq w_p(\gamma)$. Thus $q \restriction_\gamma$ forces that $w_q(\gamma)$ is $(P[\dot{G}_\gamma], U(\gamma))$-generic for every $P \in \mathcal{M}_r$. It is now easily seen that r is a common extension of p and q. $\boxed{5.2.2}$

Remark 5.2.3. We let r in the above proof be the meet of p and q and we denote it by $p \wedge q$.

Suppose $N \in \mathscr{U}$, and $\alpha \in E$. We set

$$\mathbb{P}^\kappa_\alpha \restriction_N = \{p \in \mathbb{P}^\kappa_\alpha : N \restriction_\alpha \in \mathcal{M}_p\}.$$

and $\mathbb{P}^\kappa_{\alpha,N} = \mathbb{P}^\kappa_\alpha \cap N$. By Lemma 5.2.2, $\mathbf{1}_N = (\{N \restriction_\alpha\}, \varnothing, \varnothing)$ is $(N, \mathbb{P}^\kappa_\alpha)$-strongly generic, and hence it is clear that the mapping $p \mapsto p^{N\restriction_\alpha}$ is a complete embedding

from $\mathbb{P}^\kappa_{\alpha,N}$ to $\mathbb{P}^\kappa_\alpha \restriction_N$. Moreover, if p is a condition in $\mathbb{P}^\kappa_\alpha \restriction_N$, then there is $p \restriction_N \in \mathbb{P}^\kappa_{\alpha,N}$ such that if $q \in \mathbb{P}^\kappa_{\alpha,N}$ extends $p \restriction_M$, then p and q are compatible, and in fact the meet $p \wedge q$ exists.

Lemma 5.2.4. *Suppose $N \in \mathscr{U}$ and $\alpha \leq \eta(N)$ is in E. Let p be in $\mathbb{P}^\kappa_\lambda$ with $N \in \mathcal{M}_p$. Then $p \restriction_\alpha \restriction_N = (p \restriction_N) \restriction_\alpha$*

Proof. It is enough to show that if $M \in \mathcal{M}_p$, then $(M \restriction_N) \restriction_\alpha = M \restriction_\alpha \restriction_N$. This is of course clear from the definition of $M \restriction_N$. $\boxed{5.2.4}$

If $G_{\alpha,N}$ is a V-generic filter over $\mathbb{P}^\kappa_\alpha \cap N$, we can form the following poset

$$\mathbb{R}^N_\alpha = \mathbb{P}^\kappa_\alpha \restriction_N / G_{\alpha,N}.$$

Lemma 5.2.5 (Factorization Lemma)**.** *Suppose $N \in \mathscr{U}$, and $\alpha \leq \beta \leq \eta(N)$ are in E. Let $G_{\alpha,N}$ be a V-generic filter on $\mathbb{P}^\kappa_\alpha \cap N$. Then, in $V[G_{\alpha,N}]$, the mapping*

$$\rho : \mathbb{P}^\kappa_\beta \restriction_N / G_{\alpha,N} \to \mathbb{R}^N_\alpha \times (\mathbb{P}^\kappa_\beta \cap N)/G_{\alpha,N},$$

defined by $\rho(p) = (p \restriction_\alpha, p \restriction_N)$, is a projection.

Proof. We first observe that ρ is a well-defined function since $p \leq p \restriction_\alpha, p \restriction_N$, and that clearly ρ preserves the ordering and respects the maximal conditions. Suppose that $p \in \mathbb{P}^\kappa_\beta / G_{\alpha,N}$ and that $(r, s) \in \mathbb{P}^\kappa_\alpha \restriction_N / G_{\alpha,N} \times (\mathbb{P}^\kappa_\beta \cap N)/G_{\alpha,N}$ extends $(p \restriction_\alpha, p \restriction_N)$, thus $r \restriction_N, s \restriction_\alpha \in G_{\alpha,N}$. Fix a common extension $\bar{q} \in G_{\alpha,N}$ of them. We have that $\bar{q} \wedge r \leq \bar{q} \leq s \restriction_\alpha$ and $\bar{q} \wedge s \leq \bar{q} \leq r \restriction_N$. It follows from Proposition 5.1.9 and Lemma 5.2.2 that

$$(\bar{q} \wedge r) \wedge s = (\bar{q} \wedge s) \wedge r.$$

Set $q = (\bar{q} \wedge r) \wedge s$. We have that $q \restriction_\alpha = (\bar{q} \wedge r) \leq r$ and $q \restriction_N = (\bar{q} \wedge s) \leq s$. We have also that

$$q \restriction_\alpha \restriction_N = (q \restriction_N) \restriction_\alpha = \bar{q} \in G_{\alpha,N}.$$

Thus $q \in \mathbb{P}^\kappa_\beta / G_{\alpha,N}$, and $(q \restriction_\alpha, q \restriction_N) \leq (r, s)$. It remains to show that p and q are compatible. We define a common extension of them, say t. Let $\mathcal{M}_t = \mathcal{M}_p \cup \mathcal{M}_q$. To see that this is a condition in $\mathbb{M}^\kappa_\alpha$, pick $\delta \in E$, if $\delta \leq \alpha$, then

$$\mathcal{M}^\delta_p \subseteq \mathcal{M}^\delta_r \subseteq \mathcal{M}^\delta_q,$$

and if $\delta > \alpha$, then

$$\mathcal{M}^\delta_q \subseteq \mathcal{M}^\delta_s \subseteq \mathcal{M}^\delta_p,$$

and hence $\mathcal{M}_t^\delta$ is always an $\in_\delta$-chain. Let us define d_t naturally, namely let d_t be defined on $\mathrm{dom}(d_p) \cup \mathrm{dom}(d_q)$ as follows.

$$d_t(P) = \begin{cases} d_p(P) & \text{if } \eta(P) > \alpha \text{ and } P \notin N \\ d_q(P) & \text{otherwise} \end{cases}$$

Every model in $\mathrm{dom}(d_t)$ is $\mathcal{M}_t$-free. This is quite similar to the proofs of Lemma 4.2.5 and Lemma 4.2.12. We sketch a proof. If $\eta(P) \leq \alpha$, then P is in $\mathcal{M}_{q\restriction\alpha}$, and thus if $Q \in \mathcal{M}_t$ is such that $P \in_{\eta(P)} Q$, then $Q \restriction_\alpha \in \mathcal{M}_{q\restriction\alpha}$. Thus $Q \restriction_\alpha$, and consequently Q is strongly active at $\eta(P)$. On the other hand if $\eta(P) > \alpha$ is in N, then P is in $\mathcal{M}_s$, and thus if $Q \in \mathcal{M}_t$ is such that $P \in_{\eta(P)} Q$, then if Q is not in N, it should be then in $\mathcal{M}_p$, but $s \leq p \restriction_N$, and hence Q should be strongly active at $\eta(P)$. Similarly one can prove that d_t satisfies $(\divideontimes)$ in Definition 4.2.3. We define w_t on $\mathrm{dom}(w_p) \cup \mathrm{dom}(w_q)$ as follows. For each $\gamma \in \mathrm{dom}(w_t)$,

$$w_t(\gamma) = \begin{cases} w_p(\gamma) & \text{if } \gamma > \alpha \text{ and } \gamma \notin N \\ w_q(\gamma) & \text{otherwise} \end{cases}$$

We observe that w_t is well-defined thanks to the definition of q. Notice that also if $\gamma < \alpha$ is in $\mathrm{dom}(w_t)$, then $\mathcal{M}_t(\gamma) = \mathcal{M}_q(\gamma)$ and $w_t(\gamma) = w_q(\gamma)$. On the other hand if $\gamma \geq \alpha$, then $\mathcal{M}_t(\gamma) = \mathcal{M}_p(\gamma)$. Thus t is a condition in $\mathbb{P}^\kappa_\beta \restriction_N$ that extends p and q. By an easy calculation we have that $t \restriction_\alpha \restriction_N = \bar{q} \in G_{\alpha,N}$. Thus t is a condition in $\mathbb{P}^\kappa_\beta \restriction_N / G_{\alpha,N}$.

5.2.5

Lemma 5.2.6. *Suppose that $N \in \mathscr{U}$ and $\alpha \leq \beta \leq \eta(N)$ are in E. Suppose $G_{\beta,N}$ is a V-generic filter on $\mathbb{P}^\kappa_\beta \cap N$. Then in $V[G_{\beta,N}]$, $\mathbb{R}^N_\alpha$ is a complete suborder of $\mathbb{R}^N_\beta$.*

Proof. Assume that $p \in \mathbb{R}^N_\beta$ and $q \leq p \restriction \alpha$ is in $\mathbb{P}^N_\alpha$. Since $q \leq p \restriction_\alpha$, we can from the meet of p and q i.e the condition $r = p \wedge q$ in $\mathbb{P}^\kappa_\beta$. Thus $r \in \mathbb{P}^\kappa_\beta \restriction_N$, and we have that $r \restriction_N = p \restriction_N \wedge q \restriction_N \in G_{\beta,N}$. 5.2.6

Final Remark

Notice that for the results in this chapter, it was enough to assume that κ is a Mahlo cardinal, of course we have then to work with κ-Mahlo models.

Chapter Six

Guessing Models

In this chapter, we first introduce certain principles in terms of guessing models and discuss their consequences. We then use our forcing constructions from §4 and §5 to prove their consistency.

6.1 Guessing Models

Recall that a transitive model R is called a *powerful* model if it is closed under taking subsets, i.e. if $x \in R$ and $y \subseteq x$ then $y \in R$. We are mainly interested in the case $R = V_\alpha$, for some ordinal α, or $R = H_\theta$, for some uncountable regular cardinal θ. For a powerful model R, a regular cardinal γ and a cardinal κ, we let

$$\mathfrak{G}_{\kappa,\gamma}(R) = \{M \in \mathcal{P}_\kappa(R) : M \prec R \text{ and } M \text{ is } \gamma\text{-guessing}\}.$$

We are interested in combinatorial properties of $\mathfrak{G}_{\kappa,\gamma}(R)$ motivated by [39] and [40].

Definition 6.1.1. *For a powerful model R, $\mathrm{GM}(\kappa, \gamma, R)$ is the statement that $\mathfrak{G}_{\kappa,\gamma}(R)$ is stationary in $\mathcal{P}_\kappa(R)$. $\mathrm{GM}(\kappa, \gamma)$ is the statement that $\mathrm{GM}(\kappa, \gamma, H_\theta)$ holds, for all sufficiently large regular θ.*

Notice that if M is γ-guessing and $\gamma \leq \gamma'$ then M is γ'-guessing. Therefore $\mathrm{GM}(\kappa, \gamma)$ implies $\mathrm{GM}(\kappa, \gamma')$ if $\gamma \leq \gamma'$. The statement $\mathrm{GM}(\kappa, \omega_1)$ is a reformulation of the principle $\mathrm{ISP}(\kappa)$ introduced by C. Weiß in [41] and further studied in [39], [40] and [42]. The equivalence between $\mathrm{GM}(\kappa, \omega_1)$ and $\mathrm{ISP}(\kappa)$ was established in

[40], where it was also proven that $\mathrm{ISP}(\omega_2)$ follows from PFA. Let us mention some consequences of this principle.

Proposition 6.1.2. *Assume* $\mathrm{GM}(\kappa, \omega_1)$. *Then*

1. *(Weiß, [41, 42]) The weak square principle* $\square(\kappa, \lambda)$[1] *fails, for every* $\lambda \geq \kappa$.
2. *(Krueger, [18]) The Singular Cardinal Hypothesis holds above* κ.

6.1.2

Though it was known that $\mathrm{GM}(\omega_2, \omega_1)$ implies the failure of CH, but Cox and Krueger [8] observed the following which in turn implies the continuum is at least $\aleph_2$.

Observation 6.1.3 (Cox–Krueger, [8])**.** *Assume* $\mathrm{GM}(\omega_2, \omega_1)$. *Then the weak Kurepa Hypothesis fails at* ω_1, *i.e every tree of size* $\aleph_1$ *has at most* $\aleph_1$ *many cofinal branches.*

6.1.3

Let us also mention that in [35] Trang showed the consistency of $\mathrm{GM}(\omega_3, \omega_2)$ assuming the existence of a supercompact cardinal. In his model the Continuum Hypothesis holds. Thus $\mathrm{GM}(\omega_2, \omega_1)$ fails by the above observation. One of the important features of $\mathrm{GM}(\omega_2, \omega_1)$ is its impact on the approachability ideal at ω_2. It was shown in [40] that under $\mathrm{GM}(\omega_2, \omega_1)$, the approachability property fails at ω_1, that is ω_2 is not in $I[\omega_2]$, however $\mathrm{GM}(\omega_2, \omega_1)$ does not imply $\mathrm{MP}(\omega_2)$ since $\mathrm{GM}(\omega_2, \omega_1)$ is consistent with $2^{\aleph_0} = \aleph_2$, but $\mathrm{MP}(\omega_2)$ implies $2^{\aleph_0} \geq \aleph_3$[2]. We shall introduce certain principles by strengthening $\mathrm{GM}(\omega_2, \omega_1)$ in order to control the approachability ideal on ω_2. To this end we first introduce a local principle which implies $\mathrm{MP}(\omega_2)$ and follows from our strong version of $\mathrm{GM}(\omega_2, \omega_1)$.

Definition 6.1.4 ($\mathrm{FS}(\kappa^+, \gamma)$)**.** *Suppose that* $\gamma \leq \kappa$ *are regular uncountable cardinals. The principle* $\mathrm{FS}(\kappa^+, \gamma)$ *asserts that, for every* $X \in H_{\kappa^{++}}$, *there is a collection* $\mathcal{G}$ *of* γ*-guessing models of cardinality* κ *all containing* X *such that* $\{M \cap \kappa^+ : M \in \mathcal{G}\}$ *is* κ*-closed and unbounded in* κ^+.

[1] $\square(\kappa, \lambda)$ asserts that there is a sequence $\langle \mathscr{C}_\alpha : \alpha \in \mathrm{Lim}(\lambda) \rangle$ such that

(a) For every limit ordinal $\alpha < \lambda$, $\mathscr{C}_\alpha$ is a nonempty collection of clubs in α, with $|\mathscr{C}_\alpha| < \kappa$.

(b) For every limit ordinal $\alpha < \lambda$, every $C \in \mathscr{C}_\alpha$ and every $\beta \in \mathrm{Lim}(C)$, $C \cap \beta \in \mathscr{C}_\beta$.

(c) There is no club $D \subseteq \lambda$ such that $D \cap \alpha \in \mathscr{C}_\alpha$, for every $\alpha \in \mathrm{Lim}(D)$.

[2] See Fact 1.1.5

Proposition 6.1.5. *Suppose that κ is a regular uncountable cardinal and* $\mathrm{FS}(\kappa^+, \kappa)$ *holds. Then* $\mathrm{MP}(\kappa^+)$ *holds.*

Proof. Suppose that $\mathrm{FS}(\kappa^+, \kappa)$ holds. Let $\bar{a} = \langle a_\xi : \xi < \kappa^+ \rangle$ be a κ^+-approaching sequence which belongs to $H_{\kappa^{++}}$. Let $\mathcal{G}$ be the family of κ-guessing models all containing $\bar{a}$ whose existence is guaranteed by $\mathrm{FS}(\kappa^+, \kappa)$. We show that $M \cap \kappa^+$ is not in $B(\bar{a})$, for any $M \in \mathcal{G}$ such that $\mathrm{cof}(M \cap \kappa^+) = \kappa$. Fix one such $M \in \mathcal{G}$. Let $\delta = M \cap \kappa^+$ and suppose that $c \subseteq \delta$ satisfies (1) and (2) of Definition 1.1.1. Let $\mu = \mathrm{otp}(c)$. Note that $\mu < \delta$, hence $\mu \in M$. Since $\bar{a} \in M$, we have that $c \cap \gamma \in M$, for all $\gamma < \delta$, and hence $c \cap Z \in M$, for all $Z \in M$ with $|Z| < \kappa$. Since M is a κ-guessing model, there must be $d \in M$ such that $c = d \cap \delta$. We may assume that $d \subseteq \kappa^+$. Then c is an initial segment of d, so if ρ is the μ-th element of d then $d \cap \rho = c$. Since $\mu, d \in M$, we have $\rho \in M$ as well, and hence $c = d \cap \rho \in M$. But then $\delta = \sup(c)$ belongs to M, a contradiction. $\boxed{6.1.5}$

$\mathrm{FS}(\omega_2, \omega_1)$ is a local principle, i.e it refers only to H_{ω_3}, therefore it cannot imply $\mathrm{GM}(\omega_2, \omega_1)$. We now formulate a principle that implies both $\mathrm{FS}(\omega_2, \omega_1)$ and $\mathrm{GM}(\omega_2, \omega_1)$. We state it for any pair of uncountable regular cardinals $\gamma \leq \kappa$.

Definition 6.1.6. *Let $\gamma \leq \kappa$ be regular uncountable cardinals. A model M of cardinality κ^+ is called* strongly γ-guessing *if it is the union of an $\in$-increasing chain $\langle M_\xi : \xi < \kappa^+ \rangle$ of γ-guessing models of cardinality κ and $M_\xi = \bigcup\{M_\eta : \eta < \xi\}$, for every ξ of cofinality κ.*

Lemma 6.1.7. *Every strongly γ-guessing model is γ-guessing.*

Proof. Suppose that M is a strongly γ-guessing model, which is witnessed by the sequence $\langle M_\xi : \xi < \kappa^+ \rangle$. Suppose that A is bounded in M, we may assume that A is bounded in each M_ξ. Since the sequence $(M_\xi : \xi < \kappa^+)$ is closed at ordinals of cofinality κ, a standard closure argument shows that the following set, modulo $S^\kappa_{\kappa^+}$, is a club in κ^+.

$$C = \{\xi < \kappa^+ : A \text{ is } \gamma\text{-approximated in } M_\xi\}$$

Thus for each $\xi \in C$, there is $A_\xi \in M_\xi$ such that $A_\xi \cap M = A \cap M_\xi$. By Fodor's lemma, there is some $\eta < \kappa^+$ and a stationary set $S \subseteq C$ such that for each $\xi \in S$, A_ξ is in M_η. On the other hand $|M_\eta| < \kappa^+$, and hence there is a stationary set $T \subseteq S$, such that for every $\xi, \xi' \in T$, $A_\xi = A_{\xi'}$. Let $A^* = A_\xi$, for some $\xi \in T$. Therefore, we obtain

$$A^* \cap M = A^* \cap (\bigcup_{\xi \in T} M_\xi) = \bigcup_{\xi \in T} (A_\xi \cap M_\xi) = \bigcup_{\xi \in T} (A \cap M_\xi) = A \cap M.$$

6.1.7

For a powerful model R and regular cardinals γ and κ, we let

$$\mathfrak{G}^+_{\kappa^{++},\gamma}(R) = \{M \in \mathcal{P}_{\kappa^{++}}(R) : M \prec R \text{ and } M \text{ is strongly } \gamma\text{-guessing}\}.$$

Definition 6.1.8. *For a powerful model R, $\mathrm{GM}^+(\kappa^{++}, \gamma, R)$ states that $\mathfrak{G}^+_{\kappa^{++},\gamma}(R)$ is stationary in $\mathcal{P}_{\kappa^{++}}(R)$. $\mathrm{GM}^+(\kappa^{++}, \gamma)$ is the statement that $\mathrm{GM}^+(\kappa^{++}, \gamma, H_\theta)$ holds, for all sufficiently large regular θ.*

We have the following immediate corollary.

Corollary 6.1.9. *Assume $\mathrm{GM}^+(\kappa^{++}, \gamma)$. Then $\mathrm{FS}(\kappa^+, \gamma)$, $\mathrm{GM}(\kappa^+, \gamma)$ and $\mathrm{GM}(\kappa^{++}, \gamma)$ hold.*

6.1.9

We now consider the indestructible version of the above principle, but at small cardinals. The indestructible version of guessing models were studied by Cox and Krueger [8]. They showed that having stationary many special ω_1-guessing models of size $\aleph_1$ follows form PFA and it is also consistent with the continuum being arbitrarily large. Let us start with the special version of $\mathrm{GM}(\omega_2, \omega_1)$[3].

Definition 6.1.10. $\mathrm{SGM}(\omega_2, \omega_1)$ *states that for every sufficiently large regular cardinal θ the following set is stationary in $\mathcal{P}_{\omega_2}(H_\theta)$*

$$\mathfrak{G}_{\omega_2,\omega_1}(H_\theta) = \{M \in \mathcal{P}_{\omega_2}(H_\theta) : M \prec H_\theta \textit{ and } M \textit{ is a special } \omega_1\textit{-guessing}\}.$$

Proposition 6.1.11 (Cox–Krueger, [8])**.** *Assume $\mathrm{SGM}(\omega_2, \omega_1)$. Then the Suslin Hypothesis holds.*

6.1.11

The principle $\mathrm{SFS}(\omega_2, \omega_1)$ is defined similarly, it states that for every $X \in H_{\omega_3}$, there is a collection $\mathcal{G}$ of special ω_1-guessing models of cardinality ω_1 all containing X such that $\{M \cap \omega_2 : M \in \mathcal{G}\}$ is ω_1-closed and unbounded in ω_2.

[3] $\mathrm{IGMP}(\omega_2)$ in Cox and Krueger's notation

Definition 6.1.12. *A model M of cardinality ω_2 is* indestructibly strongly ω_1-guessing *if it is the union of an increasing chain $\langle M_\xi : \xi < \omega_2 \rangle$ of special ω_1-guessing models of cardinality ω_1 and $M_\xi = \bigcup\{M_\eta : \eta < \xi\}$, for every ξ of cofinality ω_1.*

We now define the principle $\mathrm{SGM}^+(\omega_3, \omega_1)$.

Definition 6.1.13. *The principle $\mathrm{SGM}^+(\omega_3, \omega_1)$ states that $\mathfrak{S}_{\omega_3,\omega_1}(H_\theta)$ is stationary, for all large enough θ, where*

$$\mathfrak{S}_{\omega_3,\omega_1}(H_\theta) := \{M \in \mathcal{P}_{\omega_3}(H_\theta) : M \text{ is an indestructibly strongly } \omega_1\text{-guessing model}\}.$$

The notion of an indestructibly strongly guessing model can be thought as an indestructible version of ω_1-guessing models of size ω_2 because they remain guessing in any outer model with the same $\aleph_1$ and $\aleph_2$ as the ground model. The important feature of this principle is that to achieve its consistency we need not to specialize trees of size $\aleph_2$, but we use our machinery together with the usual specialization of trees of size and height ω_1 without cofinal branches to make sure that such models exist. We now give an interesting consequence of $\mathrm{SGM}^+(\omega_3, \omega_1)$. In his PhD thesis [1], Abraham asked if there is a forcing notion $\mathbb{P}$ in ZFC such that it does not add new reals, adds a new subset of some ordinal whose initial segments belong to the ground model and that the forcing does not collapse any cardinal. Notice that if CH holds, then $\mathrm{Add}(\omega_1, 1)$ is countably closed and $\aleph_2$-c.c while adding a new subset of ω_1. Recall that Foreman's Maximality Principle (see [12]) sates that every nontrivial forcing notion either adds a new real or collapses some cardinals. Using the forcing with initial segments of an uncountable cardinal κ ordered with the set extension, one can show that Foreman's Maximality principle violates GCH and the existence of inaccessible cardinals.

Definition 6.1.14. *For a regular cardinal κ.* Abraham's Maximality Principle *at κ^+* ($\mathrm{AMP}(\kappa^+)$) *states that if $2^\kappa < \aleph_{\kappa^+}$, then every forcing which adds a new subset of κ^+ whose initial segments are in the ground model, collapses some cardinal $\leq 2^\kappa$.*

Towards answering the above-mentioned question of Abraham, Todorčević showed in [33] that $\mathrm{AMP}(\aleph_1)$ is true if every tree of size and height $\aleph_1$ with at most $\aleph_1$ cofinal branches is weakly special. This principle studied also by Golshani and Shelah in [15], where they showed that $\mathrm{AMP}(\kappa^+)$ is consistent for every regular cardinal κ. Cox and Krueger [8] proved the following.

Proposition 6.1.15 (Cox–Krueger, [8])**.** $\mathrm{SGM}(\omega_2, \omega_1)$ *implies* $\mathrm{AMP}(\aleph_1)$.

6.1.15

We shall prove that $\mathrm{SGM}^+(\omega_3, \omega_1)$ implies $\mathrm{AMP}(\aleph_2)$, and since $\mathrm{SGM}(\omega_2, \omega_1)$ follows from $\mathrm{SGM}^+(\omega_3, \omega_1)$, we obtain the consistency of $\mathrm{AMP}(\aleph_1)$ and $\mathrm{AMP}(\aleph_2)$ simultaneously.

Theorem 6.1.16. *Suppose that $V \subseteq W$ are transitive models of* ZFC*. Assume in V, $\mathrm{SGM}^+(\omega_3, \omega_1)$ holds and $2^{\aleph_1} < \aleph_{\omega_2}$. Suppose that W has a subset of $\aleph_2^V$ which does not belong to V. Then either $\mathcal{P}^V(\omega_1) \neq \mathcal{P}^W(\omega_1)$ or some cardinal $\leq 2^{\aleph_1}$ is no longer a cardinal in W.*

Proof. Let $x \in W \setminus V$ be a subset of $\aleph_2^V$. Assume that $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$. We shall show that some cardinal $\leq 2^{\aleph_1}$ is no longer a cardinal in W. Since $\mathcal{P}^V(\omega_1) = \mathcal{P}^W(\omega_1)$, every initial segment of x belongs to V. Letting now $\mathfrak{X} = \{x \cap \gamma : \gamma < \omega_2\}$, we have that $\mathfrak{X}$ is bounded in V. Assume towards a contradiction that every cardinal $\leq 2^{\aleph_1}$ remains cardinal in W. Work in W, and let $\mu \geq \aleph_2$ be the least cardinal such that there is a set M in V of cardinality μ such that $M \cap \mathfrak{X}$ is of size $\aleph_2$. Thus $\mu \leq 2^{\aleph_1}$. We claim that $\mu = \aleph_2$. Suppose that $\mu > \aleph_2$ and M is a witness for that, then one can work in V and write M as the union of an increasing sequence $\langle M_\xi : \xi < \mathrm{cof}^V(\mu)\rangle$ of subsets of M in V whose size are less than μ. Since $\mu \leq 2^{\aleph_1} < \aleph_{\omega_2}$ and every cardinal $\leq 2^{\aleph_1}$ is a cardinal in W, $\mathrm{cof}^W(\mu) = \mathrm{cof}^V(\mu) \neq \omega_2$. Thus either μ is of cofinality at most $\aleph_1$, which then by the pigeonhole principle, there is $\xi < \mathrm{cof}(\mu)$ such that $|M_\xi \cap \mathfrak{X}| = \aleph_2$, or μ is regular, and thus there is some $\xi < \mathrm{cof}(\mu)$ such that $M \cap \mathfrak{X} \subseteq M_\xi$, but in either case we obtain a contradiction since $|M_\xi| < \mu$. Therefore, $\mu = \aleph_2$. Let M be a witness for $\mu = \aleph_2$, and let $\mathfrak{X}' = M \cap \mathfrak{X}$. Notice that $V \models |M| = \aleph_2$. Since M is in V and that V satisfies $\mathrm{SGM}^+(\omega_3, \omega_1)$, one can cover M with an indestructibly strongly ω_1-guessing model N of size $\aleph_2$. Working in W, x is countably approximated in N since if $\gamma \in N \cap \omega_2$, then there is $\gamma' > \gamma$ in N such that $x \cap \gamma' \in \mathfrak{X}' \subseteq N$, and hence $x \cap \gamma \in N$. On the other hand N is a guessing model in W by $\mathrm{SGM}^+(\omega_3, \omega_1)$ in V and that both $\aleph_1$ and $\aleph_2$ are cardinals in W. Thus x is guessed in N, but then x should be in N as $|x| \leq |N|$. Therefore, x is in V which is a contradiction.

6.1.16

The following corollaries are immediate.

Corollary 6.1.17. *Suppose that $V \subseteq W$ are transitive models of* ZFC*. Assume in V, $\mathrm{SGM}^+(\omega_3, \omega_1)$, $2^{\aleph_0} < \aleph_{\omega_1}$ and $2^{\aleph_1} < \aleph_{\omega_2}$ hold. Suppose that W has a new subset*

of $\aleph_2^V$*. Then either* W *contains a real which is not in* V *or some cardinal* $\leq 2^{\aleph_1}$ *in* V *is no longer a cardinal in* W*.*

6.1.17

Corollary 6.1.18. *Assume* $\mathrm{SGM}^+(\omega_3, \omega_1)$*. Then* $\mathrm{AMP}(\aleph_1)$ *and* $\mathrm{AMP}(\aleph_2)$ *hold.*

6.1.18

The following is a generalization of Proposition 6.1.11.

Proposition 6.1.19. *Suppose that* $\mathrm{SGM}(\omega_2, \omega_1)$ *holds. Then every* ω_1*-preserving forcing which is* $\aleph_1$*-proper has the* ω_1*-approximation property.*

Proof. Suppose that $\mathbb{P}$ is an ω_1-preserving forcing which is $\aleph_1$-proper. Assume that $p \in \mathbb{P}$ forces $\dot{f}$ is a countably approximated function on some ordinal γ. Let θ be a sufficiently large regular cardinal with $\gamma, \dot{f}, \mathbb{P} \in H_\theta$. Since $\mathbb{P}$ is $\aleph_1$-proper, there is an algebra F on H_θ witnessing the $\aleph_1$-properness of $\mathbb{P}$. By $\mathrm{SGM}(\omega_2, \omega_1)$, there is a special ω_1-guessing model $M \prec H_\theta$ closed under F such that $\gamma, \dot{f}, \mathbb{P} \in M$. Thus we may extend p to a $(M, \mathbb{P})$-generic condition p^M. Let G be a V-generic filter on $\mathbb{P}$ with $p^M \in G$. Thus M remains a guessing model in $V[G]$ as $\mathbb{P}$ preserves ω_1. Set $f = \dot{f}^G$. We claim that f is countably approximated in M. Let $a \in M$ be a countable subset of γ. Let D_a be the set of conditions deciding $\dot{f} \restriction_a$. Then D_a is dense below p. Thus, there is some condition q in $G \cap D_a \cap M$ since G contains a $(M, \mathbb{P})$-generic condition. By elementarity, there is some $g : a \to 2$ in M such that $q \Vdash \check{g} = \dot{f} \restriction_a$. Let $r \in G$ extends p and q. Thus $r \Vdash \dot{f} \restriction_a = \check{g}$, and hence $f \restriction_a = g \in M$ in $V[G]$. Since M is a guessing model in $V[G]$, there is $\bar{f}$ in M, and hence in V, such that $\bar{f} \restriction_M = f \restriction_M$. We now work in V, both $\dot{f}$ and $\bar{f}$ belong to M, thus by elementarity $p \Vdash \dot{f} = \bar{f}$, and hence $\mathbb{P}$ has the ω_1-approximation property. 6.1.19

Proposition 6.1.20. *Suppose that* $\mathfrak{K}$ *is a class of forcings which are* $\aleph_1$*-proper. Assume that the forcing axiom* $\mathrm{FA}_{\aleph_1}(\mathfrak{K})$ *holds. If* $\mathrm{GM}(\omega_2, \omega_1)$ *holds, then every forcing in* $\mathfrak{K}$ *has the* ω_1*-approximation property.*

Proof. The proof is more or less the same as in the last proposition thus we just sketch it. Let $\mathbb{P} \in \mathfrak{K}$. Suppose G is a V-generic filter on $\mathbb{P}$. We show that every ω_1-guessing model remains guessing in $V[G]$, and then we are done by the previous theorem. Suppose $N \in V$ is an ω_1-guessing model which is not ω_1-guessing in $V[G]$. Thus there is a function f in $V[G]$ on some ordinal $\gamma \in N$ that is countably approximated in N but it is not guessed in N. By restricting f to N, we obtain a

function of size $\aleph_1$. By the forcing axiom $\mathrm{FA}_{\aleph_1}(\mathfrak{K})$, we can interpret such a function in V, but then it is countably approximated in N, and thus should be guessed, hence a contradiction!

6.1.20

Corollary 6.1.21. *Under* MM *every stationary preserving forcing of size* $\aleph_1$ *has the* ω_1*-approximation property. The same result holds for* PFA *and proper forcings of size* $\aleph_1$.

6.1.21

It is clear that $\mathrm{SGM}^+(\omega_3, \omega_1)$ is the strongest one among the above principles. A schematic diagram expressing the relations between these principles can be found in Appendix A. In the two forthcoming sections, we shall discuss the consistency of these principles.

6.2 Consistency of $\mathrm{GM}^+(\omega_3, \omega_1)$

This section is devoted to the proof of the following theorem.

Theorem 6.2.1 ([24])**.** *Suppose that* κ *is supercompact and* $\lambda > \kappa$ *is inaccessible. Then there is a forcing notion such that in the generic extension* $\mathrm{GM}(\omega_2, \omega_1)$ *and* $\mathrm{FS}(\omega_2, \omega_1)$ *hold. If, in addition,* λ *is supercompact, then* $\mathrm{GM}^+(\omega_3, \omega_1)$ *holds as well.*

We assume that κ is supercompact and λ is inaccessible and analyze ω_1-guessing models in the the generic extension by $\mathfrak{M}^\kappa_\lambda$. Let us recall some definition from §4. For $N \in \mathscr{U}$, let $\mathbf{1}^N = (\{N\}, \varnothing)$. Recall that by Lemma 4.2.12, $\mathbf{1}^N$ is an $(N, \mathfrak{M}^\kappa_\lambda)$-strongly generic condition. Moreover, for every $q \leq \mathbf{1}^N$ and $r \leq q \restriction_N$ with $r \in N$, q and r are compatible, and the meet $q \wedge r$ exists. Let $\mathfrak{M}_N = \mathfrak{M}^\kappa_\lambda \cap N$ and let

$$\mathfrak{M}^\kappa_\lambda \restriction_N = \{q \in \mathfrak{M}^\kappa_\lambda : N \in \mathcal{M}_q\}.$$

Then the map $p \mapsto p^N$ is a complete embedding from $\mathfrak{M}_N$ to $\mathfrak{M}^\kappa_\lambda \restriction_N$. Now, fix a V-generic filter G_N over $\mathfrak{M}_N$. Suppose that G_λ is a V-generic filter over $\mathfrak{M}^\kappa_\lambda$. For $\alpha \in E$, let $G_\alpha = G_\lambda \cap \mathfrak{M}^\kappa_\alpha$.

Lemma 6.2.2. *Let* $\alpha \in E$*. Suppose that* $N \in \mathcal{M}_{G_\lambda}$ *is a Magidor model with* $\alpha \in N$*. Then* $N[G_\alpha]$ *is an* ω_1*-guessing model in* $V[G_\lambda]$*.*

Proof. Note that the projection map $N \mapsto N \restriction_\alpha$ is an isomorphism and is the identity on $\mathfrak{M}^\kappa_\alpha \cap N$. Thus, $N[G_\alpha]$ and $(N \restriction_\alpha)[G_\alpha]$ are isomorphic as well. Therefore, by replacing N with $N \restriction_\alpha$ we may assume that it is an α-model. Let $\overline{N}$ be the transitive collapse of N, and let π be the collapse map. For convenience, let us write $\bar{\kappa}$ for κ_N. Then $\overline{N} = V_{\bar{\gamma}}$, for some $\bar{\gamma}$ with $\mathrm{cof}(\bar{\gamma}) \geq \bar{\kappa}$ and $\pi(\kappa) = \bar{\kappa}$. Let $\bar{\alpha} = \pi(\alpha)$. Since $\alpha \in N$ and N is an α-model we have $\mathfrak{M}_N = \mathfrak{M}^\kappa_\lambda \cap N = \mathfrak{M}^\kappa_\alpha \cap N$. Let $\mathfrak{M}^{\bar{\kappa}}_{\bar{\alpha}} = \pi[\mathfrak{M}^\kappa_\alpha \cap N]$. Let $p \in G_\alpha$ be such that $N \in \mathcal{M}_p$. By Lemma 4.2.12 p is $(N, \mathfrak{M}^\kappa_\alpha)$-strongly generic. Hence $G_N = G_\alpha \cap N$ is V-generic over $\mathfrak{M}^\kappa_\alpha \cap N$. It follows that $G^{\bar{\kappa}}_{\bar{\alpha}} = \pi[G_N]$ is V-generic over $\mathfrak{M}^{\bar{\kappa}}_{\bar{\alpha}}$. Note that $N[G_\alpha] = N[G_N]$ and its transitive collapse $\overline{N[G_N]}$ is equal to $V_{\bar{\gamma}}[G^{\bar{\kappa}}_{\bar{\alpha}}] = V^{V[G_N]}_{\bar{\gamma}}$. Hence $N[G_N]$ is an ω_1-guessing model in $V[G_N]$. On the other hand, by Lemma 4.4.7, the quotient forcing $\mathbb{Q}_N$ is $\mathscr{C}_{\mathrm{st}}[G_N]$-strongly proper, and by Lemma 4.4.6 $\mathscr{C}_{\mathrm{st}}[G_N]$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_N]$. It follows by Proposition 1.3.6 that the pair $(V[G_N], V[G_\lambda])$ has the ω_1-approximation property. Thus, $N[G_N]$ remains an ω_1-guessing model in $V[G_\lambda]$. $\boxed{6.2.2}$

A similar argument shows the following.

Lemma 6.2.3. *Suppose that $\mu > \lambda$ and $N \prec V_\mu$ is a κ-Magidor model containing all the relevant parameters. Then $N[G_\lambda]$ is an ω_1-guessing model in $V[G_\lambda]$.*

$\boxed{6.2.3}$

Now, by Proposition 3.2.17 we have the following.

Theorem 6.2.4. *The principle* $\mathrm{GM}(\omega_2, \omega_1)$ *holds in* $V[G_\lambda]$.

$\boxed{6.2.4}$

Theorem 6.2.5. *The principle* $\mathrm{FS}(\omega_2, \omega_1)$ *holds in* $V[G_\lambda]$.

Proof. Fix $X \in H(\omega_3)^{V[G_\lambda]}$. We have to find a collection $\mathcal{G}$ of ω_1-guessing models containing X such that $\{M \cap \omega_2 : M \in \mathcal{G}\}$ is an ω_1-closed unbounded subset of ω_2. Back in V we can find $\alpha \in E$, and a canonical $\mathfrak{M}^\kappa_\lambda$-name $\dot{X}$, such that $\dot{X}[G_\alpha] = X$. Fix some $\beta \in E \setminus (\alpha + 1)$ with $\mathrm{cof}(\beta) < \kappa$. By a standard density argument, we can find a Magidor model $M \in \mathcal{M}_{G_\beta}$ such that $\alpha, \dot{X} \in M$. Suppose that $P \in \mathcal{M}_{G_\beta}$ is also a Magidor model and $M \in_\beta P$. Notice that $M \cap V_\beta \subseteq P \cap V_\beta$, so $\dot{X} \in P$, and hence $X \in P[G_\alpha]$. By Lemma 6.2.2, $P[G_\alpha]$ is an ω_1-guessing model, for all such P. Now, by Lemma 4.3.2, the set $C_\beta(G)$ is club in ω_2, and hence the family $\mathcal{G} = \{P \in \mathcal{M}_{G_\beta} \cap \mathscr{U} : M \in_\beta P\}$ is as required. $\boxed{6.2.5}$

Finally, we observe that if λ is also supercompact, then $\mathrm{GM}^+(\omega_3, \omega_1)$ holds in $V[G_\lambda]$ as well. In fact, we show that for all $\mu > \lambda$ the set of strong ω_1-guessing models is stationary in $\mathcal{P}_{\omega_3}(V_\mu[G_\lambda])$.

Lemma 6.2.6. *Suppose that $\mu > \lambda$ and $N \prec V_\mu$ is a λ-Magidor model containing all the relevant parameters. Then $N[G_\lambda]$ is a strong ω_1-guessing model.*

Proof. Since N is a λ-Magidor model, its transitive collapse $\overline{N}$ equals $V_{\bar\gamma}$, for some $\bar\gamma < \lambda$. Let $\bar\lambda = N \cap \lambda$. Note that $\mathrm{cof}(\bar\lambda) \geq \kappa$, and hence the transitive collapse $\overline{N[G_\lambda]}$ of $N[G_\lambda]$ equals $V_{\bar\gamma}[G_{\bar\lambda}]$. On the other hand, by Corollary 4.4.4, the pair $(V[G_{\bar\lambda}], V[G_\lambda])$ has the ω_1-approximation property. Therefore, $V_{\bar\gamma}[G_{\bar\lambda}]$ and hence also $N[G_\lambda]$ remains an ω_1-guessing model in $V[G_\lambda]$. To see that $V_{\bar\gamma}[G_{\bar\lambda}]$ is a strong ω_1-guessing model, fix some $\delta \in E$ with $\delta > \bar\gamma$ and $\mathrm{cof}(\delta) < \kappa$. Note that if $M \in \mathcal{M}^\delta_G$ is a Magidor model with $\bar\lambda \in M$ then by Lemma 6.2.2 $M[G_{\bar\lambda}]$ is an ω_1-guessing model. Moreover, if $M \in \mathcal{M}^\delta_{G^\kappa_\lambda}$ is a limit of such Magidor models then by Proposition 4.2.29,

$$M \cap V_{\bar\gamma} = \bigcup \{Q \cap V_\delta : Q \in_\delta M \text{ and } Q \in \mathcal{M}^\delta_{G^\kappa_\lambda}\}.$$

Hence if we let $\mathcal{G}$ be the collection of the models $(M \cap V_{\bar\gamma})[G_{\bar\lambda}]$, for Magidor models $M \in \mathcal{M}^\delta_{G^\kappa_\lambda}$ with $\bar\lambda \in M$, then $\mathcal{G}$ is an increasing $\subseteq$-chain of length ω_2 which is continuous at ω_1-limits and whose union is $V_{\bar\gamma}[G_{\bar\lambda}]$. Therefore, $V_{\bar\gamma}[G_{\bar\lambda}]$ and hence also $N[G_\lambda]$ is a strong ω_1-guessing model in $V[G_\lambda]$, as required. $\boxed{6.2.6}$

Now, by Proposition 3.2.17 and Lemma 6.2.6 we conclude the following.

Theorem 6.2.7. *Suppose that $\kappa < \lambda$ are supercompact cardinals. Let G_λ be a V-generic filter over $\mathfrak{M}^\kappa_\lambda$. Then in $V[G_\lambda]$ the principle $\mathrm{GM}^+(\omega_3, \omega_1)$ holds.*

$\boxed{6.2.7}$

Remark 6.2.8. The ω_1-guessing models obtained above in $V[G_\lambda]$ are IC-models. To see this, suppose that N is a κ-Magidor model in some $\mu > \kappa$. Thus its transitive collapse $\overline{N}$ is $V_{\bar\gamma}$, for some $\bar\gamma < \kappa$. By Theorem 4.2.30, we can over $V_{\bar\gamma}$ using a continuous increasing chain of countable models in $\mathcal{M}_{G_\lambda}$ containing $V_{\bar\gamma}$. Hence in $V[G_\lambda]$, $V_{\bar\gamma}$ and consequently N is an IC-model. Therefore, whenever $N[G_\lambda]$ or $N[G_\alpha]$ for $\alpha \in E$ make sense, we can ensure that they are IC-models in $V[G_\lambda]$.

6.3 Consistency of $\mathrm{SGM}^+(\omega_3, \omega_1)$

In this section, we provide a proof for the following consistency result.

Theorem 6.3.1 ([25]). *Suppose that κ is supercompact and $\lambda > \kappa$ is inaccessible. Then there is a forcing notion such that in the generic extension* $\mathrm{SGM}(\omega_2, \omega_1)$ *and* $\mathrm{SFS}(\omega_2, \omega_1)$ *hold. If, in addition, λ is supercompact, then* $\mathrm{SGM}^+(\omega_3, \omega_1)$ *holds as well.*

In order to prove the above theorem, we apply our iteration from §4, but as in the previous section we need to analyze the quotients of the iteration by Magidor models. Thus our main goal in this section is to show that the evaluation of a Magidor model in the generic extension by G_α is a guessing model, however we will no longer have the $\mathscr{C}$-strong properness of the quotients. It would be better to fix our iteration and prove certain lemmata about it before diving into the details of the proof. Thus fix a supercompact cardinal κ and an inaccessible $\lambda > \kappa$. Our aim is to iterate the Baumgartner forcing for the specialization of trees of size and height ω_1 with no cofinal branches, along the virtual model based iteration from §5. Notice that the Baumgartner forcing is c.c.c and hence we do not need to require genericity in the third clause in the definition of the iteration, however we ignore this point. As before, we shall consider a suitable structure $\mathfrak{A} = (V_\lambda, \in, \kappa, U)$, where U is a predicate. We may first use an auxiliary function $u : \lambda \to \lambda \times \lambda$ to enumerate all the possible names, in V_λ, for trees of height and size ω_1 without cofinal branches. Thus consider the suitable structure $\mathfrak{A}^* = (V_\lambda, \in, \kappa, u)$, where $u : \lambda \to \lambda \times \lambda$ is an onto map satisfying $\beta \leq \alpha$ whenever $u(\alpha) = (\beta, \gamma)$. We then define a predicate U on the structure $\mathfrak{A}^* = (V_\lambda, \in, \kappa, u)$ as follows. Let $U(x) = \varnothing$ if x is not an ordinal. For an ordinal $\alpha < \lambda$ if $u(\alpha) = (\beta, \gamma)$ with $\alpha, \beta \in E$, we let $U(\alpha)$ be a $\mathbb{P}^\kappa_\alpha$-name for the Baumgartner forcing $\mathbb{B}(\dot{T})$ where $\dot{T}$ is the γ-th (in a prescribed enumeration) $\mathbb{P}^\kappa_\beta$-name for a tree of size and height ω_1 without cofinal branches, and otherwise let $U(\alpha) = \varnothing$. We then work with the suitable structure $\mathfrak{A} = (V_\lambda, \in, \kappa, U)$. Thus let us consider, for the rest of this chapter, the virtual model based iteration $\langle \mathbb{P}^\kappa_\alpha : \alpha \in E \cup \{\lambda\} \rangle$ obtained from $\mathfrak{A}$. Recall that if $N \in \mathscr{U}$ and $G_{\alpha,N}$ is a V-generic filter over $\mathbb{P}^\kappa_\alpha \cap N$, we can then form the following quotient

$$\mathbb{R}^N_\alpha = \mathbb{P}^\kappa_\alpha \restriction_N / G_{\alpha,N}.$$

Definition 6.3.2. *Let $G_{\alpha,N}$ be a V-generic filter over $\mathbb{P}^\kappa_\alpha \cap N$. Let $\mathscr{C}_{\mathrm{st}}[G_{\alpha,N}]$ denote the set of all $M \in \mathscr{C}_{\mathrm{st}}$ such that $\alpha, N \in M$ and $(N \wedge M) \restriction_\alpha \in \mathcal{M}_{G_{\alpha,N}}$.*

The following is analogous to Lemma 4.4.6 and can be proved similarly.

Lemma 6.3.3. *$\mathscr{C}_{\mathrm{st}}[G_{\alpha,N}]$ is stationary in $\mathcal{P}_{\omega_1}(V_\lambda)$ in the model $V[G_{\alpha,N}]$.*

6.3.3

Lemma 6.3.4. *Assume that N is a Magidor model. Suppose that $p \in \mathbb{R}^N_\alpha$. If $M \in \mathscr{C}$ such that $\alpha, p \in M$ and $(N \wedge M) \restriction_\alpha \in \mathcal{M}_{G_{\alpha,N}}$. Then there is a condition $p^M \leq p$ in $\mathbb{R}^N_\alpha$ such that $M \restriction_\alpha \in \mathcal{M}_{p^M}$.*

Proof. We define p^M in the same way as it is in $\mathbb{P}^\kappa_\alpha$, except that we let $w_{p^M} = w_p$. Since each $U(\gamma)$ is forced to be a c.c.c forcing, p^M is a condition in $\mathbb{P}^\kappa_\alpha$. On the other hand as in Lemma 4.4.7, $(\mathcal{M}_p, d_p)^M \restriction_N = ((\mathcal{M}_p, d_p) \restriction_N)^{N \wedge M}$ and since $N \wedge M$ is in $\mathcal{M}_{G_{\alpha,N}}$, we have that $((\mathcal{M}_{p\restriction_N}, d_{p\restriction_N})^{N\wedge M}, \varnothing) \in G_{\alpha,N}$. Therefore, we should have that $p^M \restriction_N = (\mathcal{M}^{N\wedge M}_{p\restriction_N}, d^{N\wedge M}_{p\restriction_N}, w_p \restriction_N) \in G_{\alpha,N}$. It is now clear that p^M is in $\mathbb{R}^N_\alpha$. 6.3.4

Lemma 6.3.5. *Assume that $N \in \mathscr{U}$. Suppose that $p \in \mathbb{R}^N_\alpha$. If M is a countable virtual model such that $\alpha, p, N \in M$ and $(N \wedge M) \restriction_\alpha \in \mathcal{M}_{G_{\alpha,N}}$. Then $p^{M\restriction\alpha}$ is $(M[G_{\alpha,N}], \mathbb{R}^N_\alpha)$-generic.*

Proof. Though the condition p^M defined here is not obtained in the same way as in Proposition 5.1.13, but the same proof as in Proposition 5.1.13 works here, and thus p^M is $(M, \mathbb{P}^\kappa_\alpha \restriction_N)$-generic. Therefore $p \restriction_N$ is $(M, \mathbb{P}^\kappa_\alpha \cap N)$-generic, and forces that p is $(M[\dot{G}_{\alpha,N}], \mathbb{R}^N_\alpha)$-generic. Since $p \restriction_N$ is in $G_{\alpha,N}$, p is $(M[G_{\alpha,N}], \mathbb{R}^N_\alpha)$-generic.

6.3.5

We now state our key lemma.

Lemma 6.3.6. *Suppose that $N \in \mathscr{U}$ and that $\gamma \in a(N)$. Let $G_{\gamma,N}$ be a V-generic filter on $\mathbb{P}^\kappa_\gamma \cap N$. Then in $V[G_{\gamma,N}]$, $\mathbb{R}^N_\gamma$ has the ω_1-approximation property.*

Proof. We prove by induction that for every $\beta \leq \gamma$, $\mathbb{R}^N_\beta$ has the ω_1-approximation property over $V[G_{\gamma,N}]$, where in the definition of $\mathbb{R}^N_\beta$, we use $G_{\gamma,N} \cap \mathbb{P}^\kappa_\beta$. Thus fix β, and assume that the conclusion of the lemma holds for every ordinal in $E \cap \beta$. Suppose that $\dot{f}$ is an $\mathbb{R}^N_\beta$-term forced by some condition p to be a function on some ordinal μ that is countably approximated in $V[G_{\gamma,N}]$. One can use Lemma 6.3.3 to pick a countable model M elementary in V_δ with $\delta > \lambda$ such that $E, \beta, \gamma, N, p, \mu$ and $\dot{f}$ belong to M such that $(N \wedge (M \cap V_\lambda)) \restriction_\gamma$ is in $\mathcal{M}_{G_{\gamma,N}}$. Set $M_\gamma = (M \cap V_\lambda) \restriction_\gamma$ and $M_\beta = M_\gamma \restriction_\beta$. We may then use Lemma 6.3.4 to extend p to a condition p^{M_β} in $\mathbb{R}^N_\beta$ so that $M_\beta \in \mathcal{M}_{p^{M_\beta}}$. Since p forces $\dot{f}$ is countably approximated in $V[G_{\gamma,N}]$, there is a condition $q \leq p^{M_\beta}$ which decides the values of $\dot{f} \restriction_M$, thus there is a function $g : M \cap \mu \to 2$ such that $q \Vdash \check{g} = \dot{f} \restriction_M$. Moreover, by Lemma 5.2.6 and

Lemma 6.3.5, p^{M_β} is $(M_\gamma[G_{\gamma,N}], \mathbb{R}^N_\beta)$-generic. Therefore, it is $(M[G_{\gamma,N}], \mathbb{R}^N_\beta)$-generic. By Lemma 1.2.18, we are done if g is guessed in $M[G_{\gamma,N}]$. Thus we may assume towards a contradiction that g is not guessed in $M[G_{\gamma,N}]$.

Let $\alpha = \max(\mathrm{dom}(w_q) \cap M)$. Let G^N_α be a $V[G_{\gamma,N}]$-generic filter on $\mathbb{R}^N_\alpha$ containing $q \restriction_\alpha$, where again we use $G_{\alpha,N} = G_{\gamma,N} \cap \mathbb{P}^\kappa_\alpha$ in order to form the quotient $\mathbb{R}^N_\alpha$. Lemma 5.2.5 together with Lemma 6.3.5 imply that $q \restriction_\alpha$ is $(M[G_{\gamma,N}], \mathbb{R}^N_\alpha)$-generic, on the other hand by the inductive hypothesis, $\mathbb{R}^N_\alpha$ has the ω_1-approximation property over $V[G_{\gamma,N}]$, thus by Lemma 1.2.20 the function g is not yet guessed in $M[G_{\gamma,N}][G^N_\alpha]$. Since G^N_α is a $V[G_{\gamma,N}]$-generic filter on $\mathbb{R}^N_\alpha$, the factorization lemma implies that $V[G_{\gamma,N}][G^N_\alpha] = V[G^N_\alpha][G_{\gamma,N}]$.

Claim 6.3.7. *The pair $(V[G^N_\alpha], V[G^N_\alpha][G_{\gamma,N}])$ has the ω_1-approximation property.*

Proof. Let G^N_γ be a $V[G^N_\alpha][G_{\gamma,N}]$-generic filter on $\mathbb{R}^N_\gamma$. Notice that both G^N_α and G^N_γ are also V-generic filters on $\mathbb{P}^\kappa_\gamma$ and $\mathbb{P}^\kappa_\alpha$, respectively. Now, Proposition 5.1.22 implies that the pair $(V[G^N_\alpha], V[G^N_\gamma])$ has the ω_1-approximation property, and hence the pair $(V[G^N_\alpha], V[G^N_\alpha][G_{\gamma,N}])$ has the ω_1-approximation property. $\boxed{6.3.7}$

Notice that if $\alpha \in N$, then $U(\alpha)$ is already interpreted by $G_{\gamma,N}$, and moreover there is a generic filter on $U^{G_{\gamma,N}}(\alpha)$ in $V[G_{\gamma,N}]$. Thus in this case conditions in $U(\alpha)$ cannot prevent us from amalgamating the conditions in $\mathbb{R}^N_\beta$ i.e if p' and q' are conditions in $\mathbb{R}^N_\beta$ such that $\alpha \in \mathrm{dom}(w_{p'}) \cap \mathrm{dom}(w_{q'})$, we do know that $w^{G_{\gamma,N}}_{p'}(\alpha)$ and $w^{G_{\gamma,N}}_{q'}(\alpha)$ are compatible. Let $\mathbb{Q}$ be the the interpretation of $U(\alpha)$ under G^N_α if α is not in N, and otherwise let it be the trivial forcing, and let also, for every $p' \in \mathbb{R}^N_\beta$, $z_{p'} = w^{G^N_\alpha}_{p'}(\alpha)$ if $\alpha \notin N$, and otherwise let it be an $\mathbb{R}^N_\alpha$-name in N for the maximal condition of the trivial forcing. These conventions are necessary to avoid giving two almost identical arguments.

Claim 6.3.8. *$\mathbb{Q}$ has the ω_1-approximation property over $V[G_{\gamma,N}][G^N_\alpha]$, and z_q is $(M[G_{\gamma,N}][G^N_\alpha], \mathbb{Q})$-generic.*

Proof. It is enough to assume that $\alpha \notin N$ and that $\mathbb{Q}$ is nontrivial, thus it is the Baumgartner forcing for the specialization of a tree T of size and height ω_1 without branches in $V[G^N_\alpha]$. Claim 6.3.7 implies that in $V[G^N_\alpha][G_{\gamma,N}] = V[G_{\gamma,N}][G^N_\alpha]$, T is still a tree of size and height ω_1 which has no cofinal branches. Thus $\mathbb{Q}$ is, in $V[G_{\gamma,N}][G^N_\alpha]$, a c.c.c forcing with the ω_1-approximation property. Therefore, z_q is also $(M[G_{\gamma,N}][G^N_\alpha], \mathbb{Q})$-generic. $\boxed{6.3.8}$

Let H be a $V[G_{\gamma,N}][G^N_\alpha]$-generic filter on $\mathbb{Q}$ containing z_q. Therefore, by Claim 6.3.8 and Lemma 1.2.17 the function g is not yet guessed in $M^* := M[G_{\gamma,N}][G^N_\alpha][H]$. We consider an assignment $x \to (q_x, g_x)$ on $[\mu]^\omega$ in M^* such that:

1. $q_x \in \mathbb{P}^\kappa_\beta \restriction_N$.
2. $(\mathcal{M}_{q_x}, d_{q_x}) \leq (\mathcal{M}_p, d_p) \restriction_M$ in $\mathfrak{M}^\kappa_\beta$.
3. $M \cap \mathrm{dom}(w_q) \subseteq \mathrm{dom}(w_{q_x})$.
4. $q_x \restriction_\alpha \in G^N_\alpha$.
5. $q_x \restriction_N \in G_{\beta,N}$.
6. $z_{q_x} \in H$.
7. $x \subseteq \mathrm{dom}(g_x)$ and $q_x \Vdash g_x \restriction_x = \dot{f} \restriction_x$.

Such an assignment exists in M^* as $x \mapsto (q, g)$ witnesses it. Thus by Lemma 1.2.20, there is a set $B \in M^*$ cofinal in $[\mu]^\omega$ such that for every $x \in B \cap M^*$, $g_x \nsubseteq g$, but then it implies that for such x, q_x and q are incompatible. We shall show that for every $x \in B \cap M^*$, q_x and q are compatible. Fix such an x. Then $q_x \in M^*$ by Lemma 5.2.5 and Lemma 6.3.5, and hence it is in M. Let $r \in G^N_\alpha$ be a common extension of $q_x \restriction_\alpha$ and $q \restriction_\alpha$, and let also $s \in G_{\beta,N}$ be a common extension of $q_x \restriction_N$ and $q \restriction_N$. Let us also put $\hat{q}_0 = q_x \wedge q$ where the amalgamation is computed in $\mathbb{P}^\kappa_\beta \restriction_N$. Set

$$d = \{\xi \in \mathrm{dom}(w_{q_0}) : \alpha \leq \xi \notin N\}.$$

Let $\hat{q} = (\mathcal{M}_{\hat{q}_0}, d_{\hat{q}_0}, w_{\hat{q}_0} \restriction_d)$. Notice that $\hat{q} \in \mathbb{P}^\kappa_\beta \restriction_N$, and that $(r, s) \leq (\hat{q} \restriction_\alpha, \hat{q} \restriction_N)$. We now apply Lemma 5.2.5 to (r, s) and $\hat{q}$ to find some condition $t \in \mathbb{P}^\kappa_\beta \restriction_N / G_{\alpha,N}$ as in the proof of Lemma 5.2.5 with $t \leq r, s, \hat{q}$ such that $t \restriction_N \in G_{\beta,N}$. Notice that $t \leq q_x, q$ in $\mathbb{P}^\kappa_\beta \restriction_N$, but t is in $\mathbb{R}^N_\beta$, and hence q and q_x are compatible in $\mathbb{R}^N_\beta$. $\boxed{6.3.6}$

Corollary 6.3.9. *Suppose that $\alpha \in E$. Let $N \in \mathscr{U}$ be such that $\alpha \in N \cap a(N)$. Assume that G_α is a V-generic filter on $\mathbb{P}^\kappa_\alpha$ such that $N \restriction_\alpha \in \mathcal{M}_{G_\alpha}$. Then the pair $(V[G_\alpha \cap N], V[G_\alpha])$ has the ω_1-approximation property.*

Proof. Since N is active at α, we have that $\alpha = \max(a(N) \cap (\alpha + 1))$. Thus, the previous lemma implies that $\mathbb{R}^N_\alpha$ has the ω_1-approximation property, and hence $(V[G_\alpha \cap N], V[G_\alpha])$ has the ω_1-approximation property. $\boxed{6.3.9}$

Lemma 6.3.10. *Suppose that $\mu > \lambda$. Let $N \prec V_\mu$ be a κ-Magidor model containing the relevant objects. Suppose that G_λ is a V-generic filter on $\mathbb{P}^\kappa_\lambda$. Then $N[G_\lambda]$ is an ω_1-guessing model in $V[G_\lambda]$.*

Proof. Let $\eta = \sup(N \cap \lambda)$. Then $\mathbb{P}^\kappa_\lambda \cap N = \mathbb{P}^\kappa_\eta \cap N = \bigcup_{\alpha \in N \cap \eta} \mathbb{P}^\kappa_\alpha \cap N$. Let π be the collapse map of N, and let $V_\gamma = \pi[N]$, $\bar{\lambda} = \pi(\lambda)$ and $\bar{\kappa} = \pi(\kappa)$. Thus

$$\pi[\mathbb{P}^\kappa_\lambda \cap N] = \bigcup_{\alpha \in N \cap \eta} \pi[\mathbb{P}^\kappa_\alpha \cap N].$$

On the other hand $\pi[\mathbb{P}^\kappa_\alpha \cap N] = \mathbb{P}^{\bar{\kappa}}_{\pi(\alpha)}$, and hence since $\mathrm{cof}(\eta) \geq \bar{\kappa}$, we have that $\pi[\mathbb{P}^\kappa_\lambda \cap N] = \bigcup_{\alpha < \bar{\lambda}} \mathbb{P}^{\bar{\kappa}}_\alpha = \mathbb{P}^{\bar{\kappa}}_{\bar{\lambda}}$. Therefore,

$$\overline{N[G_\lambda]} = V_{\bar{\gamma}}[G^{\bar{\kappa}}_{\bar{\lambda}}] = V^{V[G_\eta]}_{\bar{\gamma}}.$$

Since $N \cap V_\lambda$ is active at η, Lemma 6.3.6 implies that $(V[G_\eta \cap N], V[G_\eta])$ has the ω_1-approximation, on the other hand $(V[G_\eta], V[G_\lambda])$ has the ω_1-approximation property by Proposition 5.1.22, and hence $N[G_\lambda]$ is an ω_1-guessing model in $V[G_\lambda]$.

$\boxed{6.3.10}$

If λ is supercompact, one can then use a similar argument as above in Lemma 6.3.10 to conclude the following lemma.

Lemma 6.3.11. *Assume that λ is a supercompact cardinal. Suppose that $\mu > \lambda$ and $N \prec V_\mu$ is a λ-Magidor model containing all the relevant objects. Then $N[G_\lambda]$ is an ω_1-guessing model.*

$\boxed{6.3.11}$

Proof of Theorem 6.3.1

Let G_λ be a V-generic filter on $\mathbb{P}^\kappa_\lambda$. We first show that as in the last section, $\mathrm{GM}(\omega_2, \omega_1)$ and $\mathrm{FS}(\omega_2, \omega_1)$ hold in $V[G_\lambda]$, and moreover if λ is supercompact, then $\mathrm{GM}(\omega_3, \omega_1)$ is also true in $V[G_\lambda]$. In fact the way we obtain many ω_1-guessing models to witness the above principles is that if N is a κ-Magidor model in some $\mu > \kappa$ containing the relevant objects, such that $N \cap V_\lambda \in \mathcal{M}_{G_\lambda}$, then for every $\alpha \in E \cup \{\lambda\}$, $N[G_\alpha]$ is an ω_1-guessing model in $V[G_\lambda]$, Notice that by Remark 6.2.8, the models we obtain in this way are IC-models. As it is easily seen the sequence witnessing that some model M is strongly ω_1-guessing models consist of ω_1-guessing models which are IC.

Claim 6.3.12. $V[G_\lambda] \models \mathrm{FS}(\omega_2, \omega_1)$.

Proof. If $N \in \mathcal{M}_{G^\kappa_\lambda}$ is a Magidor model and α is in $a(N)$. Then, $N[G_\alpha]$ is an ω_1-guessing model in $V[G_\lambda \cap N]$, and hence by Corollary 6.3.9, it remains an ω_1-guessing model in $V[G_\lambda]$. A proof quite similar to the one in Theorem 6.2.5 shows that $\mathrm{FS}(\omega_2, \omega_1)$ holds in $V[G_\lambda]$ $\boxed{6.3.12}$

Claim 6.3.13. $V[G_\lambda] \models \mathrm{GM}(\omega_2, \omega_1)$.

Proof. This follows easily form Proposition 3.2.17 and Corollary 6.3.9. $\boxed{6.3.13}$

Claim 6.3.14. *Suppose that λ is supercompact. Then $V[G_\lambda] \models \mathrm{GM}^+(\omega_3, \omega_1)$.*

Proof. Suppose that $\mu > \lambda$ is an ordinal. Suppose that $\dot{F}$ is a $\mathbb{P}^\kappa_\lambda$-name for an algebra on V_μ, then by Proposition 3.2.17, there is a λ-Magidor model N containing all relevant information so that $\dot{F} \in N$. Lemma 6.3.10 implies that $N[G_\lambda]$ is an ω_1-guessing model in $V[G_\lambda]$. A similar proof as in Lemma 6.2.6 shows that it is in fact a strongly ω_1-guessing model. $\boxed{6.3.14}$

By the properties of the iteration from §5, we do know that $\mathbb{P}^\kappa_\lambda$ is λ-c.c, it preserves ω_1 and κ, and makes $\kappa = \aleph_2$ and $\lambda = \aleph_3$. To conclude the theorem, it is now enough to prove the following.

Claim 6.3.15. *In $V[G_\lambda]$ every tree of size and height $\aleph_1$ without cofinal branches is special, and hence every ω_1-guessing model of size $\aleph_1$ is special.*

Proof. Let T be a tree of size and height $\aleph_1$ without cofinal branches in $V[G_\lambda]$. There exists some $\alpha \in E$ and some $\mathbb{P}^\kappa_\alpha$-name $\dot{T}$ such that $\dot{T}^{G_\lambda} = T$. We may assume that $\dot{T}$ is forced by the maximal condition to be a tree of size and height $\aleph_1$ without cofinal branches. By definition of u and the fact that E is a club in λ, there exists some $\beta \geq \alpha$ in E such that $\dot{T}$ is the δ-th $\mathbb{P}^\kappa_\gamma$-name where $u(\beta) = (\gamma, \delta)$ with $\gamma \geq \alpha$. Then, the maximal condition of $\mathbb{P}^\kappa_{\beta+1}$ forces that $\dot{T}$ is a special tree. $\boxed{6.3.15}$

Appendix One

A Schematic Diagram of the Principles

$$
\begin{tikzcd}
 & & & & 2^{\aleph_0} \geq \aleph_3 \\
 & \neg\square(\omega_2, \lambda) & \mathrm{TP}(\aleph_2) & \mathrm{SCH} & \mathrm{MP}(\omega_2) \arrow[u] \\
\mathrm{AMP}(\aleph_1) & \mathrm{SH} & \mathrm{GM}(\omega_2, \omega_1) \arrow[ul] \arrow[u] \arrow[ur] & \mathrm{TP}(\aleph_3) & \mathrm{FS}(\omega_2, \omega_1) \arrow[u] \\
\mathrm{AMP}(\aleph_2) & \mathrm{SGM}(\omega_2, \omega_1) \arrow[ul] \arrow[u] \arrow[ur] & & \mathrm{GM}^+(\omega_3, \omega_1) \arrow[ul] \arrow[u] \arrow[ur] & \mathrm{SFS}(\omega_2, \omega_1) \arrow[u] \\
 & & \mathrm{SGM}^+(\omega_3, \omega_1) \arrow[ul] \arrow[ull] \arrow[ur] \arrow[urr] & &
\end{tikzcd}
$$

All the arrows are logical implications. The symbols MP, SH, SCH, AMP and TP stand, respectively, for the Mitchell Property, the Suslin Hypothesis, the Singular Cardinal Hypothesis, Abraham's Maximality Principle and the Tree Property.

Appendix Two

MRP **with Finite Conditions**

The Mapping Reflection Principle (MRP) was discovered and shown to be a consequence of PFA by Moore [26]. It is a strong reflection principle which decides the value of the continuum to be $\aleph_2$, it implies SCH and the failure of the square principle. Moore used a forcing with countable conditions to show that MRP follows from PFA. In this appendix, we give a proper forcing with finite conditions which forces an instance of MRP.

Definition B.0.1 (Ellentuck topology)**.** *Let X be an uncountable set. The Ellentuck topology on $[X]^{\leq\omega}$ is the topology generated by the following sets as basic open sets.*

$$[a, A] \coloneqq \{x \subseteq X : a \subseteq x \subseteq A\}, \text{ where } a \text{ is finite and } A \subseteq X \text{ is countable .}$$

Definition B.0.2. *Suppose M and X are sets. A set $\Sigma \subseteq [X]^{\leq\omega}$ is called M-stationary if for every algebra $F \in M$ over X, there is some countable set $A \in M \cap \Sigma$ closed under F.*

Definition B.0.3. *A function Σ is called open and stationary mapping if there are a regular cardinal $\theta = \theta_\Sigma$ and an uncountable set $X = X_\Sigma$ such that:*

1. $\mathrm{dom}(\Sigma)$ *is the collection of countable elementary submodels of H_θ containing X.*

2. *For each $M \in \mathrm{dom}(\Sigma)$, $\Sigma(M)$ is M-stationary, and also open in $[X]^{\leq\omega}$ with respect to the Ellentuck topology.*

Definition B.0.4. *We say that an open mapping Σ as above reflects if there is a continuous $\in$-chain $\langle M_\xi : \xi < \omega_1\rangle$ of models in $\mathrm{dom}(\Sigma)$ such that for every $\xi < \omega_1$,*

there is $\zeta < \xi$ so that for every $\eta \in \xi \setminus \zeta$, $M_\eta \cap X \in \Sigma(M_\xi)$. The sequence $\langle M_\xi : \xi < \omega_1 \rangle$ is called a reflecting sequence.

Definition B.0.5 (Moore, [26])**.** *The Mapping Reflection Principle (*MRP*) states that every open stationary mapping reflects.*

Suppose Σ is an open stationary mapping. Let $X = X_\Sigma$ and $\theta = \theta_\Sigma$.

Definition B.0.6. *We let $\mathbb{P}_\Sigma$ consist of triples $p = (\mathcal{M}_p, d_p, f_p)$ satisfying the following.*

1. *$\mathcal{M}_p$ is a finite $\in$-chain of models in* $\mathrm{dom}(\Sigma)$.
2. *$d_p : \mathcal{M}_p \to [H_\theta]^{<\omega}$ is a function such that if $P \in M$, then $d_p(P) \in M$.*
3. *f_p is a regressive function with finite values on $\mathcal{M}_p$, i.e for every $M \in \mathcal{M}_p$, $f_p(M) \in [M]^{<\omega}$, such that whenever $M \in N$ are $\mathcal{M}_p$ and $f_p(N) \in M$, then $M \cap X \in \Sigma(N)$.*

We equip $\mathbb{P}_\Sigma$ with the following ordering. Let $p \leq q$ if and only if

1. *$\mathcal{M}_q \subseteq \mathcal{M}_p$.*
2. *For each $M \in \mathcal{M}_q$, $d_q(M) \subseteq d_p(M)$.*
3. *$f_q \subseteq f_p$.*

Convention: $\varnothing$ is in $\mathcal{M}_p$, for every $p \in \mathcal{M}_p$, and $f_p(\varnothing)$ is undefined.

Proposition B.0.7. *Suppose p is a condition in $\mathbb{P}_\Sigma$. Let M be an element of* $\mathrm{dom}(\Sigma)$ *containing p. Then there is a condition $p^M \leq p$ such that $M \in \mathcal{M}_{p^M}$.*

Proof. We let p^M be defined as follows. Let $\mathcal{M}_{p^M}$ be just $\mathcal{M}_p \cup \{M\}$, extend d_p as a function by letting $d_{p^M}(M) = \varnothing$, and also extend f_p as a function by letting $r_{p^M}(M)$ be some finite set in $M \setminus \mathcal{M}_p$. It is easily seen p^M is a condition which extends p. B.0.7

Proposition B.0.8. *Suppose $\theta^* > \theta$ is a sufficiently large regular cardinal. Assume $M^* \prec H_{\theta^*}$ is countable and contains $\Sigma, X, \mathcal{P}(\theta)$. Let $M := H_\theta \cap M^*$. Suppose $p \in \mathbb{P}_\Sigma$ is such that $M \in \mathcal{M}_p$. Then p is $(M^*, \mathbb{P}_\Sigma)$-generic.*

Proof. Let $p \restriction M = (\mathcal{M}_p \cap M, d_p \restriction_M, f_p \restriction_M)$. It is clear that $p \restriction M$ is a condition belonging to M. Set

$$\mathfrak{X} = \{(P, Q) \in \mathcal{M}_p \times \mathcal{M}_p : P \subseteq M, M \in Q \text{ and } f_p(Q) \in P \in Q\}.$$

If $(P, Q) \in \mathfrak{X}$, then there is a finite set b_P^Q in P, and hence in M, such that

$$[b_P^Q, P \cap X] \subseteq \Sigma(Q).$$

Fix such sets. Set $b_P = \bigcup\{b_P^Q : (P, Q) \in \mathfrak{X}\}$ and $B = \{b_P : P \in \mathcal{M}_p\}$. Let D be a dense subset of $\mathbb{P}_\Sigma$ in M^*. We pick some regular cardinal μ with $\theta < \mu < \theta^*$ and consider the following set which is easily verified that it contains a club of $[X]^\omega$ and belongs to M^*.

$$E = \{R \cap X : \{B, X, D, p \restriction M, \Sigma, \theta\} \subseteq R \prec H_\mu \text{ is countable }\}.$$

Since $X \in H_\theta$, every algebra on X belongs to H_θ, and hence $\Sigma(M)$ is M^*-stationary. Now by M^*-stationarity of $\Sigma(M)$, one can find $A \in E \cap M^* \cap \Sigma(M)$. Since $M \prec H_{\theta^*}$, there is $R \in M^*$ with $\{B, X, D, p \restriction M, \Sigma, \theta\} \subseteq R$ such that $A = R \cap X$. Now using the openness of $\Sigma(M)$ there exists a finite set $b_R \subseteq A$ such that the interval $[b_R, A]$ is included in $\Sigma(M)$. We need to extend $p \restriction M$ to a condition in M so that its further extensions in M do not violate the third condition of Definition B.0.6. To this end, let p^* be the same as $p \restriction M$ except about d_{p^*}, where we let it be defined as follows.

$$\text{For each } P \in \mathcal{M}_p \cap M,\ d_{p^*}(P) = d_p(P) \cup b_{P^+},$$

where P^+ is the next model of P in $\mathcal{M}_p \cap M$, and we let the next model of the largest one be defined artificially as R. It is clear that p^* is a condition belonging to R since $p \restriction M, B, b_R \in H_\theta \cap R$. Now there exists by elementarity a condition q extending p^* such that $q \in R \cap D$, but then $q \in M$ since $R \in M^*$. We claim that q is compatible with p. Put $\mathcal{M}_r = \mathcal{M}_q \cup \mathcal{M}_p$. Let also d_r be defined on $\mathcal{M}_r$ as follows $d_r(P) = d_q(P)$ if $P \in M$, and $d_r(P) = d_p(P)$ otherwise. It is easy to see that $\mathcal{M}_r$ is an $\in$-chain, and d_r decorates it. Since $\mathcal{M}_p \cap M \subseteq \mathcal{M}_q$, we simply put $f_r = f_p \cup f_q$. What remains to be shown is that whenever $P \in Q$ are in $\mathcal{M}_r$ with $f_r(Q) \in P$, we would have that $P \cap X \in \Sigma(Q)$. It is easily seen the only situation that can potentially worry us is when $Q \notin M$, and $P \in \mathcal{M}_q \setminus \mathcal{M}_p$, but in this case there is $S \in \mathcal{M}_p \cap M$ such that $S \in P \in S^+$, which implies $b_{S^+}^Q \subseteq b_{S^+} \in P$, and

hence $P \cap X \in [b^Q_{S^+}, S^+ \cap X] \subseteq \Sigma(Q)$. This concludes the proof.

B.0.8

Corollary B.0.9. $\mathbb{P}_\Sigma$ *is proper.*

Proof. It is clear from Proposition B.0.7 and Proposition B.0.8. B.0.9

If G is a V-generic filter on $\mathbb{P}_\Sigma$, we then let

$$\mathcal{M}_G = \{M : \exists p \in G \text{ such that } M \in \mathcal{M}_p\}.$$

Lemma B.0.10. *Suppose G is a V-generic filter on $\mathbb{P}_\Sigma$. Then $\mathcal{M}_G$ is a continuous $\in$-chain.*

Proof. We show that if $(\mathcal{M}_n)_n$ is a sequence in $\mathcal{M}_G$, then $\bigcup_{n<\omega} M_n$ is also in $\mathcal{M}_G$. This is equivalent to saying that if $M \in \mathcal{M}_G$ is not the minimal member and is such that for every $P \in \mathcal{M}_G$ below M, there is a model in $\mathcal{M}_G$ between P and M. Then M is the union of models below M in $\mathcal{M}_G$. Thus suppose countable model $M \prec H_\theta$ is forced, by a condition p, to be in $\mathcal{M}_G$ with the above property. Without loss of generality, we may assume $\mathcal{M}_p$ contains M and some model below M in $\mathcal{M}_G$ as well. Now let $x \in M$. If $q \leq p$ is an arbitrary condition, then one can extend q to a condition q_x such that $x \in d_{q_x}(Q_x)$ where Q_x is the largest model below M in $\mathcal{M}_{q_x}$, It then implies that any extension of q_x which has a model above Q_x should contain x. This is possible as there is some model below M in $\mathcal{M}_q$. This shows that the set of conditions such that x belongs to some model below M is dense below p. Thus for every $x \in M$, p forces that there is a model below M in $\mathcal{M}_G$ containing x. Therefore, M is the union of its predecessors in $\mathcal{M}_G$ whenever G is a $\mathbb{P}_\Sigma$-generic filter containing p.

B.0.10

Let f_G be defined on $\mathcal{M}_G$ by letting $f_G(M) = f_p(M)$ for some, or equivalently all, $p \in G$ with $M \in \mathcal{M}_p$.

Theorem B.0.11. $\mathbb{P}_\Sigma$ *adds a reflecting sequence for* Σ.

Proof. Let G be a V-generic filter on $\mathbb{P}_\Sigma$. By the above lemma, $\mathcal{M}_G$ is a continuous $\in$-chain of models. Let $\langle M_\xi : \xi < \omega_1 \rangle$ be the natural enumeration of $\mathcal{M}_G$, i.e for each $\xi < \omega_1$, $M_\xi \in M_{\xi+1}$. We claim that this is a reflecting sequence for Σ. If $\xi < \omega_1$ is a limit ordinal, then $f_G(M_\xi)$ belongs to M_ξ, and thus by the continuity,

there is $\zeta < \xi$ so that $f_G(M_\xi) \in M_\zeta$, and thus for each $\eta \in \xi \setminus \zeta$, $f_G(M_\xi) \in M_\eta$. It is enough to pick $q \in G$ such that $M_\xi, M_\eta \in \mathcal{M}_q$, and hence $M_\eta \cap X \in \Sigma(M_\xi)$.

B.0.11

Appendix Three

Open Problems

In this appendix, we state some questions related to the research line of this thesis.

Our fist problem concerns the concept of structured $\aleph_{1.5}$-c.c-ness.

Problem C.0.1. *Does every $\aleph_{1.5}$-c.c forcing have the structured $\aleph_{1.5}$-chain condition? Is $\mathrm{MA}^{s-1.5}_{\aleph_2}$ equivalent to $\mathrm{MA}^{1.5}_{\aleph_2}$?*

The second problem concerns the failure of $\mho_2$ (cf Definition 1.4.7). It is not clear whether it has an answer under $\mathrm{MA}^{s-1.5}_{\aleph_2}$.

Problem C.0.2. *Is the failure of $\mho_2$ consistent with $2^{\aleph_0} > \aleph_2$?*

The failure of $\mho_2$ follows from MRP for sets of size $\aleph_1$, say $\mathrm{MRP}(\aleph_1)$, which states that every open stationary mapping Σ with $|X_\Sigma| = \aleph_1$ reflects (cf Definition B.0.5). The next natural question is the following.

Problem C.0.3. *Is $\mathrm{MRP}(\aleph_1)$ consistent with $2^{\aleph_0} > \aleph_2$?*

We notice that in contrast to the failure of $\mho_2$, it is hard, or even impossible, to imagine that one can force every instance of $\mathrm{MRP}(\aleph_1)$ using a forcing of size $\aleph_1$. The last relevant question is about proper forcings of size $\aleph_1$. Let $\mathfrak{K}_{\mathrm{proper},\aleph_1}$ denote the class of all proper forcings of size $\aleph_1$.

Problem C.0.4. *Is $\mathrm{FA}_{\aleph_1}(\mathfrak{K}_{\mathrm{proper},\aleph_1})$ consistent with $2^{\aleph_0} > \aleph_2$?*

We can be more ambitious and ask the following.

Problem C.0.5. *Is $\mathrm{FA}_{\aleph_2}(\mathfrak{K}_{\mathrm{proper},\aleph_1})$ consistent with $2^{\aleph_0} > \aleph_2$?*

Now we turn to a subject with more combinatorial flavor. A natural question related to the approachability ideal is the following. Recall that $\mathrm{MP}(\kappa^+)$ stands for the Mitchell Property at κ (cf Definition 1.1.4).

Problem C.0.6. *Is it consistent to have* $\mathrm{MP}(\omega_2)$ *and* $\mathrm{MP}(\omega_3)$ *simultaneity?*

I am almost sure there is no *problem-meter*, thus let us be more ambitious again and ask another question.

Problem C.0.7. *Is it consistent to have* $\mathrm{MP}(\omega_{\mathrm{n}})$*, for every natural number* $n \geq 2$ *simultaneity?*

For a natural number n, one can easily define $\mathrm{GM}(\omega_{\mathrm{n+1}}, \omega_{\mathrm{n}})$, $\mathrm{GM}^+(\omega_{\mathrm{n+2}}, \omega_1)$ and $\mathrm{SGM}^+(\omega_{\mathrm{n+2}}, \omega_1)$, and then study their consistencies. See the first section of §5 for the relevant definitions. Let us first define $\mathrm{SGM}^+(\omega_{\mathrm{n+2}}, \omega_1)$ (cf Definition 6.1.6).

Definition C.0.8. *A model of size* $\aleph_{n+1}$*, for* $n \geq 2$*, is an indestructibly strongly* ω_1*-guessing if it is the union of a sequence* $\langle M_\xi : \xi < \omega_{n+1} \rangle$ *of indestructibly strongly* ω_1*-guessing models of size* ω_n *such that for each* ξ *of cofinality* ω_n*, we have that* $M_\xi = \bigcup_{\eta<\xi} M_\eta$*.*

Set

$$\mathfrak{S}_{\omega_{n+2},\omega_1}(H_\theta) := \{M \in \mathcal{P}_{\omega_{n+2}}(H_\theta) : M \text{ is an indestructibly strongly } \omega_1\text{-guessing model}\}.$$

Then we let naturally, for every $n \geq 1$, $\mathrm{SGM}^+(\omega_{\mathrm{n+2}}, \omega_1)$ assert that for all sufficiently large regular cardinal θ, $\mathfrak{S}_{\omega_{n+2},\omega_1}(H_\theta)$ is stationary in $\mathcal{P}_{\omega_{n+2}}(H_\theta)$. I am sure no one[1] reads these lines, so let us state a natural conjecture.

Conjecture C.0.9. *It is consistent to have* $\mathrm{SGM}^+(\omega_{\mathrm{n+2}}, \omega_1)$ *simultaneously, for every* $n \geq 1$*.*

If V is a transitive model of

$$T \equiv \mathrm{ZFC} + \forall \mathrm{n}\ (\mathrm{SGM}^+(\omega_{\mathrm{n+2}}, \omega_1) + 2^{\aleph_{\mathrm{n}}} < \aleph_{\omega_{\mathrm{n}}}),$$

and $W \supseteq V$ is transitive model of ZFC which contains a new subset of ω_n, for some $n \in \omega$, then either W has a real which does not belong to V, or some V-cardinal $\leq \sup_{n\in\omega}(2^{\aleph_n})$ is no longer a cardinal in W. In particular, if $\aleph_\omega$ is strong limit in V and $\aleph_n^V = \aleph_n^W$ for every $n \in \omega$, then $\mathbb{R}^V \neq \mathbb{R}^W$ (cf Theorem 6.1.16). Notice that also $\mathrm{SGM}^+(\omega_{\mathrm{n+2}}, \omega_1)$ does imply $\mathrm{AMP}(\aleph_{\mathrm{n+1}})$, $\mathrm{MP}(\omega_{\mathrm{n+1}})$ and $\mathrm{TP}(\aleph_{\mathrm{n+1}})$, for every $n \geq 1$, and possibly many other things.

[1] Bravo! you are an exception!

Bibliography

[1] Uri Abraham. *The isomorphism types of Aronszajn trees and the forcing without the generalized continuum hypothesiss.* 1979. Thesis (Ph.D.)–Jerusalem.

[2] Uri Abraham. Aronszajn trees on $\aleph_2$ and $\aleph_3$. *Ann. Pure Appl. Logic*, 24(3):213–230, 1983.

[3] David Asperó and Miguel Angel Mota. Forcing consequences of PFA together with the continuum large. *Trans. Amer. Math. Soc.*, 367(9):6103–6129, 2015.

[4] David Asperó and Miguel Angel Mota. A generalization of Martin's axiom. *Israel J. Math.*, 210(1):193–231, 2015.

[5] James Baumgartner, Jerome I. Malitz, and William N. Reinhardt. Embedding trees in the rationals. *Proc. Nat. Acad. Sci. U.S.A.*, 67:1748–1753, 1970.

[6] James E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[7] Sean Cox and John Krueger. Quotients of strongly proper forcings and guessing models. *The Journal of Symbolic Logic*, 81(1):264–283, 2016.

[8] Sean Cox and John Krueger. Indestructible guessing models and the continuum. *Fund. Math.*, 239(3):221–258, 2017.

[9] James Cummings and Matthew Foreman. The tree property. *Adv. Math.*, 133(1):1–32, 1998.

[10] Laura Fontanella. Strong tree properties for two successive cardinals. *Arch. Math. Logic*, 51(5-6):601–620, 2012.

[11] Laura Fontanella. Strong tree properties for small cardinals. *J. Symbolic Logic*, 78(1):317–333, 2013.

[12] Matthew Foreman, Menachem Magidor, and Saharon Shelah. $0^\sharp$ and some forcing principles. *J. Symbolic Logic*, 51(1):39–46, 1986.

[13] Sy-David Friedman. *Forcing with finite conditions*, pages 285–295. Trends Math. Birkhäuser, Basel, 2006.

[14] Thomas Gilton and Itay Neeman. Side conditions and iteration theorems. *https://www.math.ucla.edu/ ineeman/scit.pdf/.*

[15] Mohammad Golshani and Saharon Shelah. Specializing trees and answer to a question of williams. *Journal of Mathematical Logic, to appear.*

[16] Joel David Hamkins. Extensions with the approximation and cover properties have no new large cardinals. *Fundamenta Mathematicae*, 180(3):257–277, 2003.

[17] Bernhard König. Forcing indestructibility of set-theoretic axioms. *J. Symbolic Logic*, 72(1):349–360, 2007.

[18] John Krueger. Guessing models imply the singular cardinal hypothesis. *Proc. Amer. Math. Soc.*, 147(12):5427–5434, 2019.

[19] Menachem Magidor. On the role of supercompact and extendible cardinals in logic. *Israel Journal of Mathematics*, 10(2):147–157, Jun 1971.

[20] Donald A. Martin and Robert M. Solovay. Internal Cohen extensions. *Ann. Math. Logic*, 2(2):143–178, 1970.

[21] William J. Mitchell. A weak variation of Shelah's $I[\omega_2]$. *Journal of Symbolic Logic*, 69(1):94–100, 2004.

[22] William J. Mitchell. Adding closed unbounded subsets of ω_2 with finite forcing. *Notre Dame J. Formal Logic*, 46(3):357–371, 07 2005.

[23] William J. Mitchell. $I[\omega_2]$ can be the nonstationary ideal on $\mathrm{Cof}(\omega_1)$. *Transactions of the American Mathematical Society*, 361(2):561–601, 2009.

[24] Rahman Mohammadpour and Boban Velickovic. Guessing models and the approachability ideal. *Journal of Mathematical Logic, to appear.*

[25] Rahman Mohammadpour and Boban Veličković. On special guessing models. In preperation.

[26] Justin Tatch Moore. Set mapping reflection. *J. Math. Log.*, 5(1):87–97, 2005.

[27] Itay Neeman. Forcing with sequences of models of two types. *Notre Dame J. Formal Logic*, 55(2):265–298, 2014.

[28] Saharon Shelah. On successors of singular cardinals. In *Logic Colloquium '78 (Mons, 1978)*, volume 97 of *Stud. Logic Foundations Math.*, pages 357–380. North-Holland, Amsterdam-New York, 1979.

[29] Saharon Shelah. Reflecting stationary sets and successors of singular cardinals. *Archive for Mathematical Logic*, 31(1):25–53, Jan 1991.

[30] Saharon Shelah. Advances in cardinal arithmetic. In *Finite and infinite combinatorics in sets and logic (Banff, AB, 1991)*, volume 411 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 355–383. Kluwer Acad. Publ., Dordrecht, 1993.

[31] Saharon Shelah. *Proper and improper forcing.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[32] Robert M. Solovay and Stanley Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Ann. of Math. (2)*, 94:201–245, 1971.

[33] Stevo Todorčević. Some combinatorial properties of trees. *Bull. London Math. Soc.*, 14(3):213–217, 1982.

[34] Stevo Todorčević. *Partition problems in topology*, volume 84 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989.

[35] Nam Trang. PFA and guessing models. *Israel Journal of Mathematics*, 215(2):607–667, Sep 2016.

[36] Spencer Unger. A model of Cummings and Foreman revisited. *Ann. Pure Appl. Logic*, 165(12):1813–1831, 2014.

[37] Boban Velickovic. Iteration of semiproper forcing revisited, 2014.

[38] Boban Veličković. Notes on proper and semi-proper forcing. *Lecture Notes*, 2015.

[39] Matteo Viale. Guessing models and generalized Laver diamond. *Ann. Pure Appl. Logic*, 163(11):1660–1678, 2012.

[40] Matteo Viale and Christoph Weiß. On the consistency strength of the proper forcing axiom. *Adv. Math.*, 228(5):2672–2687, 2011.

[41] Christoph Weiß. *Subtle and Ineffable tree properties.* PhD thesis, Ludwig Maximilians Universität München, January 2010.

[42] Christoph Weiß. The combinatorial essence of supercompactness. *Ann. Pure Appl. Logic*, 163(11):1710–1717, 2012.