Fluides non newtoniens avec surface libre : modélisation et caractère bien posé

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THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITE GRENOBLE ALPES

préparée dans le cadre d'une cotutelle entre l'Université Grenoble Alpes et l'Université Libanaise

Spécialité : **Mathématiques appliquées**

Arrêté ministériel : 25 mai 2016

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préparée au sein de **laboratoire LAMA et de laboratoire de mathématiques dans les Ecoles doctorales MSTII et EDST**

Fluides Non Newtoniens: Modélisation et Caractère Bien Posé

Thèse soutenue publiquement le **25/09/2020**,
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Abstract

The work in this thesis falls in the category of comprehending dynamical systems related to thin liquid films driven by gravity; mainly shallow water models and lubrication equations. From a modeling point of view, we have derived from one hand 3-equation shallow water models for Newtonian fluids with constant viscosity that describe the evolution of the fluid's height, velocity and a new defined variable equivalent to the shear rate. Such models succeed in capturing unstable regimes and in giving satisfactory numerical results for the instability threshold and the wave speed at moderate distance from threshold. On the other hand, a lubrication equation and a shallow water model were derived for a non Newtonian fluid- known as a bi-viscous fluid- whose rheology approximates that of pseudo plastic and Bingham fluids. Concerning the analysis part, the global existence of nonnegative weak solutions for lubrication equations, such as the Derrida-Lebowitz-Speer-Spohn equation, is proved starting from nonnegative weak solutions for appropriate viscous shallow-water equations (playing with drag terms and capillarity formula). The novelty in this work is to show that the BF entropy which is introduced for lubrication equations in the context of thin films is encoded inside the BD entropy introduced for compressible Navier Stokes equations with density dependent viscosity. We also investigate the dissipative solution of Navier Stokes system of Oldroyd-B rheology, as well as the global weak solution for degenerate lake system of Bingham rheology.

Résumé

Cette thèse porte sur la compréhension de systèmes dynamiques liés aux films liquides minces entraînés par la gravité; principalement des modèles d'eau peu profonde et des équations de lubrification. D'un point de vue modélisation, nous avons obtenu d'une part des modèles à 3 équations des eaux peu profondes pour les fluides newtoniens à viscosité constante décrivant l'évolution de la hauteur du fluide, de sa vitesse et d'une nouvelle variable définie équivalente au taux de cisaillement. De tels modèles parviennent à capturer des régimes instables et à donner des résultats numériques satisfaisants pour l'étude de seuil d'instabilité et de la phase de l'onde à une distance modérée du seuil. D'autre part, une équation de lubrification et un modèle d'eau peu profonde ont été obtenu à partir d'un modèle de fluide non-Newtonien connu sous le nom de fluide bi-visqueux- dont la rhéologie se rapproche de celle des fluides pseudo-plastiques et de Bingham. Concernant la partie analyse, l'existence globale de solutions faibles non négatives pour les équations de lubrification, telles que l'équation de Derrida-Lebowitz-Speer-Spohn, partant des solutions faibles non négatives pour les équations appropriées d'eau peu profonde visqueuse (en manipulant les termes de traînée et la formule de capillarité), est montrée. La nouveauté dans ce travail est de montrer que l'entropie BF qui a été introduite pour les équations de lubrification dans le contexte des films minces est encodée dans l'entropie BD introduite pour Navier-Stokes compressible avec viscosités dépendant de la densité. On s'intéresse également aux solutions dissipatives pour des systèmes incompressible et compressible de type Oldroyd ainsi qu'aux solutions faibles pour un système dégénéré de type Bingham.

Keywords

Free surface, Shallow water, lubrication, pseudo plastic, Bingham, BD entropy, BF entropy, Oldroyd.

Dedicated to M.H.H

Acknowledgements

I would like to take the advantage of this space area to express my deep gratitude for all those who contributed, in one way or another, throughout the journey of my PhD studies and research. Many thanks as well to those who I may forgot to mention on the go...

First and foremost, my deepest gratitude is to God Almighty who paved the way for the reasons to occur in order to fulfill my studies and provided me with inner peace, determination and hope. Without these blessings, this work would not have been possible.

I would like to express my deep thanks to Prof. Raafat Talhouk, my thesis advisor, who followed my journey since my Masters degree. I highly appreciate your honesty and dedication to science and to my hometown university. I profoundly thank you for providing me with all these opportunities and introducing me to amazing research units in France.

I would like to sincerely thank my thesis supervisor Prof. Didier Bresch, who has launched my research three years ago and has been a pillar of support to me since then. I want to express my gratitude for all what you gave me, for your patience, invaluable guidance, remarkable suggestions for interesting mathematical topics for my research, repetitive encouragement and for all the opportunities you offered me, and still. Many thanks for all the words you told me in tough times and situations; they were more life lessons to me than instantaneous words of encouragement.

I'm very grateful for the Jury members, Professors: Anne-Laure Dalibard, Frédéric Lagoutière, Jean-Paul Vila, Laurent Chupin and Marguerite Gisclon for agreeing to be part of the committee. I would like to thank in particular the reporters Prof. Jean-Paul Vila and Prof. Laurent Chupin for their precious time invested in discussing and reviewing my work, especially in the hard time the world is passing through.

A great part of my thesis was a diligence of collaboration with amazing people. I would like first to thank Prof. Christian Ruyer-Quil for being such a cooperative advisor, and for your valuable time and precious discussions since my first year. I really appreciate your patience with my repeated questions and your interest to hear my ideas and suggestions. I would like also to thank Prof. Francois James for the project you proposed in CEMRACS 2019, the time you gave to extend the work after that and for all the discussions with me and my team. Thanks for helping us succeed in launching that project and for monitoring us all the way to the end. I would like as well to thank Doc.Bilal AL Taki and Prof. Jacques Saint Marie for giving me the chance to collaborate with them in my first year of PhD. I would also like to thank the former for the patience to hear my silliest questions and nagging during that period.

For the wonderful laboratory members at LAMA: lab directors, professors, researchers and crew members who were very welcoming and cooperative and never made me feel uncomfortable especially with language difficulties. My lab fellows and friends, especially those who accompanied me on a daily basis: Abir, Daniel and Clement, my amazing visitor fellows Zhanat and Gloria, many love and thanks for each one of you. You've been a wonderful family. Not to forget to express my deep thanks to the laboratory members in Lebanon at Math laboratory, especially the lab secretary Abir Mokaddem, for being a continuous source of support, always

facilitating my administrative procedures and giving me valuable advice since the beginning of this journey.

My dearest friends Darine and Hanin, thank you for being there and standing by my side whenever I needed you. Your support was a blessing to me in this journey. My dear cousin Amani, thanks for being my at-a-distance nurse and sacrificing your time to teach me how to cook during those three years, thanks for hearing my nagging and being such a patient person.

My Dad and Mom, my dear brothers, it's your prayers, faith in me, immense love and support that triggered me to keep presenting the ultimate and best performance during my years of study. My words won't express my profound and real appreciation to your sacrifices to help me achieve my ambition. My dear Grandmas, you don't know how blessed I am to have you by my side, encouraging me to chase my dreams, believing in me like no one ever did, and being very loving and supportive for my choices.

Finally, I would like to thank myself, despite how much arrogant and prosaic this might seem, but standing a moment on what I've endured and struggled during this journey, on my patience and success to overcome inner and outer challenges and reminding myself the next time in future I come across this is undoubtedly of extreme importance to me.

Contents

Motivation

"Only passions, great passions, can elevate the soul to great things"

– Denis Diderot, french philosopher

Motivation and Imagination is what we lack sometimes in life, especially in scientific life. Those two words would have been probably replaced by "passion" if we were about to read a poetry related document. For me, whatever I have had to do in life, should have been accompanied with my "passion", and that's why I'm writing this a-bit-weird introduction. My passion for math and physics was always to understand the stories behind formulas, and the most interesting story for me was that of Einstein, who made a whole acting scenario of the motion of the moon and earth in his living room to understand the phenomena that startled him, gravitation, and led him to set down his equations that ruled the universe as depicted by New York Times: "As compact and mysterious as a Viking rune, it describes space-time as a kind of sagging mattress where matter and energy, like a heavy sleeper, distort the geometry of the cosmos to produce the effect we call gravity, obliging light beams as well as marbles and falling apples to follow curved paths through space". What Einstein had besides curiosity was devotion. Motivation can trigger questions inside your brain, but devotion can create a whole new world of imagination that ignites sparks of curiosity. This lead to the prodigious relativity principle. I will try at the most accepted extent in the scientific world to explain whatever formulas or results in what follows by showing the story behind them, the way I was motivated to do things. By this far, the talk started to stray a bit, so lets go back to our main topic: Fluids.

Part I

Introduction

I dedicate this part to be an introductory part that conveys all my acquired knowledge and learning paths that I paved in the topic of modeling thin liquid films. In particular, a great concentration will be on films driven by gravity. The topic of modeling the flow of a fluid has sparked my interest since the first encounter with fluid mechanics. Alas, it is a topic not clearly demonstrated in literature. The reason of my interest in modeling is that it constitutes for me the tool that links between understanding the physics of the problem and interpreting the occurring phenomena in the flow from one hand, and from the second hand understanding the difficulties in the mathematical and numerical resolution of the problem which triggered the motivation to formulate new models in the first place. Modeling made me understand deeper a great part of the physics behind the problem and to become aware of my destined goal through the derivation process. Of course this intuition and understanding is basic for launching a research, but its indispensable as well. My thesis is for me now a corner stone for advanced learning and investigation. Besides modeling, a great part of the thesis is devoted to understand the behavior of non-Newtonian fluids from both modeling and analysis perspectives. As usual, some results in this thesis are not closed, we have a great hope and expectation in extending the studies already launched inhere to serve good mathematical results.

For convenience, I will sketch a summary of the contents of the introduction, as not all readers will be interested in it as a whole. In fact, we can split it into two chapters:

1. Chapter 1 is a basic introduction through which modeling the problem of a flow is introduced starting from the physical stand point and reaching the mathematical closure via Navier Stokes system. Then, a historical review on **hydrodynamics** and **rheology** is presented. And the final section displays a general overview on modeling. Since small pieces are crucial to complete the bigger picture, basic detailed description is provided in this part that was exposed in a way that serves the work in the thesis.

2. Chapter 2 elaborates on two main modeling theories: Shallow Water and Lubrication theories. For each topic a descriptive literature is first exhibited then followed by a kind of state of art that sheds light on the progress done in the selected theory, and after which the contribution to the thesis is presented. Respectively in this part, we deal with the following topics:

    - Derivation of 2 and 3 equation systems of shallow water type using momentum integral method and weighted residual method.
    - Derivation of a shallow water/lubrication systems for a Bi-viscous rheology and proving a formal link with Bingham and Newtonian shallow water/lubrication models.
    - Global weak solution of some lubrication equations through the link between the BD entropy for shallow water systems and the Bernis-Friedman dissipative entropy for the corresponding lubrication equation.
    - Dissipative solutions for incompressible Navier Stokes systems of Oldroyd-B type rheology.
    - Existence result for a degenerate lake system of Bingham rheology.

1

Introduction to Continuum Mechanics

1.1 Motivation of studying fluids

The story behind surveying fluids goes back to ancient people. Motion captured their sights and attention: the rhythmic percussion of soft shoreline waves, the entrancing swaying of leaves upon spring breezes, the dust caught in a sunlight ray, snowflakes swinging down the ground, Tsunamis crashing like huge beasts and washing down all sightseeing. They were skeptical about the air above them that filled the endless skies, and they feared the huge bulks of water, from oceans to lakes, springs and rivers, reverencing some of them, and describing others as the sacred paths to the divine self. But what is fear but the paving way to knowledge? It is at this level of recognition and conception, the level of doubt of fear, that one can start to wonder about the essence of such entities, and their fuzzy impact all around, and what astonishing results he can invest through them.

A scientific scrutiny synchronized with the appearance of human civilizations who chose to dwell near lakes and seas, and thus adaptively showed a practical knowledge of flows which was manifested in the design of flow systems (spears, sailing ships with oars) and particularly in hydraulic projects for irrigation systems, flood protection, water supplies (conduits, canals and wells), drainage, etc. But this pragmatic knowledge didn't just depend on observatory and applicable tools, as it was noticed that Greek people started more scientific qualitative studies and postulations, and the most remarkable ones of that era were Archimedes principles (laws of buoyancy) applied on floating and submerged bodies. The majority of contact with fluids through these eras (especially at the level of patents as is the case of the invention of the barometer by Evangelista Torricelli in 1643) was almost at the level of static fluids, which are fluids in rest state, and which also have a minimized impact compared to fluids in motion as results show until today, and from here comes the out of question importance of "flows" which have made the headlines in the research field, and still. In fact, experiments and observations continued to prosper after Archimedes by many remarkable scientists: Leonardo da Vinci (1452 – 1519) who stated the equation of conservation of mass in one-dimensional steady-state flow and experimented with waves, jets, hydraulic jumps and eddy formation and with Edme Mariotte (1620 – 1684) who built the first wind tunnel and tested models in it. The first outstanding theoretical work wasn't furnished until the 15th century when Sir Isaac Newton postulated his laws on motion and introduced the notion of linear fluids, now known as Newtonian fluids. A very little after that Daniel Bernoulli (1738) and Leonhard Euler (1757) did a pioneering, fundamental approach to write down the first set of differential equations describing the motion of fluid flow. Thanks to these equations, a colossal amount of studies and research had been conducted and a quite significant results and solutions for existing problems had been proved. Besides the distinguished approaches of both Froude and Reynold (to whom the Froude number and Reynold number correspond), it was until the 16th century that the next milestone in fluid analysis was hit by Claude-Louis Navier (1822) and George Gabriel Stokes (1842) who presented a mathematical justification of fluid flow mechanics. In particular, they wrote down the foundational axioms of fluid dynamics by exploiting Newton's laws of motion (conservation of mass and momentum) and

the first law of thermodynamics (energy conservation). These axioms are known nowadays by Navier-Stokes equations (NS) and have been the fulcrum of research in solving different problems related to fluid flow. Various remarkable scientists after that have been able to add a scientific value and progress in the chapter of studying fluids especially studies related to understanding viscosity and turbulence such as Andrey kolmogorov and Geoffrey Ingram Taylor, and not to forget as well Ludwig Prandtl who remarkably introduced the Boundary-Layer theory in 1904.

It is intrinsic at this point to remark that all the above reviewed approaches to setting fluid equations do not have a " fundamental nature", but they are rather " phenomenological equations and for this reason one cannot ask too much of them ". When this concept was realized by the scientific community due to the fact that the foundational systems presented by Euler-Bernoulli and Navier-Stokes approach, in their complex absurd formulation, couldn't answer questions using the available analysis tools, researchers started to look for condensed abbreviated models that translate the physics embodied in the preceding two systems. The main tool till today was the perturbation analysis, or asymptotic analysis, which limits the number of difficulties in the case of study and preserves at the same time its physical identity. Two main theories appeared as a result of such analysis: Shallow Water theory and Lubrication theory. This thesis will elaborate in the general context, the relation of such theories with NS system.

Nowadays, flows are a main factor in understanding most of the natural and biological phenomena, and a resolute factor in technological and industrial applications. It is of this high criticality not just for surviving, as was the motivation of ancient civilizations, but also for development and sustainability on earth. This triggered scientists and decision makers to be better equipped with theoretical understanding and capability to use experiments and numerical tools in order to increase the efficiency of the result.

In order to launch an exhaustive comprehensive study on fluids, one must narrow the targets of studying. In this context, it is good to remark that describing an entity, as a fluid, in the substantial scientific concept, can adopt several perspectives. The classifications done in this dissertation line up in a selected path that identifies with the chosen subdomain of study. Any other crucially important classifications are not included for the convenience, time and space matters. The first requisite piece of information to carry on before any study is realizing the notion behind the topic of study. As a part of matter, which is mainly split into two parts: fluids-solids, fluids consist of a number of particles with specific parametric quantities, or degrees of freedom, that gives tendency to its particles, on the contrary to solids, to reposition whenever a force is applied on it, simply we often say "to flow" preserving its macroscopic properties. Out of this definition we get three effective **elements** of study: **particles, forces, and degrees of freedom** which direct the perspective of research on fluids targeting the **outcome** element of study: the **flow**.

1.2 The relative Perspective of Motion: Macroscopic and Microscopic

At the level of particles, and according to the case of application or the phenomenon aimed for study, the fluids can be viewed either from a microscopic perspective, or from a macroscopic one, a classification better know in the mathematical description as Lagrangian or Eulerian description, respectively associated to the German scientists behind setting these two frames of study. At the microscopic level, the fluid is considered to constitute of bunch of separate individual element moving freely and independently, whereas at the macroscopic one, the fluid is regarded as a bulk entity sharing constitutional properties (mass, volume..) and exhibiting others (speed, temperature,..). Though the fundamental basis applied to derive the equations of motion in both frames are similar: Newton's laws of classical mechanics, yet there is a basic difference related to the object variable in each. A good example that illustrates this difference

is the pendulum (even though it is not a fluid case), in Eulerian frame of work, the main concentration is to determine the forces acting on the pendulum in terms of the velocity field and describe the evolution by determining the history of a specific position not in the moving pendulum but with respect to a fixed reference in space, whereas a Lagrangian study would be more concerned in determining the trajectory followed by a material point on the pendulum as it moves with time, the variables of the system become functions of the path lines of this point, and one of the approaches would be writing down a Lagrangian functional defined by a minimization of the potential and kinetic energies of the point, based on the postulate that the point will follow a path of minimum energy requirement.

1.2.1 Lagrangian Frame

In case of a fluid, this classification becomes sharp and intricate depending on the length scales compared to the size of individual particle in the fluid. If the flow is occurring in configurations where the mean free path of the particle, defined as the " average distance traveled by the particle before undergoing collision with another particle or barrier and after which it modifies its direction or energy", is crucially valuable compared to the characteristic lengths of the domain occupied by the fluid, in such case, a Lagrangian frame of reference associated to the particle is used to determine the history and prediction of its motion; the trajectory.

Definition 1.1. *In a fluid occupying initially a domain Ω_0, a fluid particle (mathematical definition) is defined by the family $(\varphi_t(x_0))_t$, where φ_t is a bijective map mapping the initial domain Ω_0 to the domain Ω_t occupied by all the particles initially present in Ω_0:*

$$\begin{aligned} \varphi : \Omega_0^q &\longrightarrow \quad \Omega_t \\ x_0 &\longmapsto \varphi_t(x_0). \end{aligned}$$

Definition 1.2. *The trajectory of a fluid particle x_0 is the map $t \mapsto X(t, t_i, x_i)$, where x_i is the position occupied by this particle at time t_i, and $X(t, t_i, x_i)$ is defined by $X_i(t) = X(t, t_i, x_i) := \varphi_t(\varphi_t^{-1}(x_i))$. $\{x_0\}$ is considered to be the frame of reference.*

In this context, the velocity of the fluid particle $\{x_i\}$ at time t is given by

$$v_{Lag}(t, X(t, t_i, x_i)) = \dot{X}(t, t_i, x_i) = \frac{d}{dt} X(t, t_i, x_i).$$

And so other variables such as acceleration, pressure, temperature, etc, are being expressed. Having determined the Lagrangian frame of reference, one can derive the Lagrangian equations of motion by determining the different forces and interactions acted upon the particle preassuming that it follows Newton's laws of classical mechanics. This would result in a system of the very general form

$$\begin{aligned} &\frac{d}{dt} X_i = M_i, \\ &\frac{d}{dt} M_i(t, X) = -\gamma M_i + F_i, \end{aligned}$$

where

$M_i = M(t, X_i)$ is the momentum of the particle,
$F_i = F(t, X_i)$ is the sum of forces on the particle due to any external potentials or interactions,
γM_i is a viscosity term.

The Lagrangian frame of work subjoined two main domains in fluid dynamics known by Kinetic Theory and Molecular Dynamics (MD). In fact, where MD is based on models similar to the above one, the models in Kinetic Theory are based on averaging techniques of the above equations in which the main state variable is a kinetic distribution $F(t, x, u)$ which is the density of the particles with velocity u found at time t in the position x. The applications in each subdomain are vast, a good example would be in biological flows as blood, where the diameter of the red blood cell, or its mean free path, is very close to the diameter of the capillaries in which it flows, thus imposing a Lagrangian frame of work.

Nevertheless, in the whole thesis we wouldn't be treating such kind of flows, but rather we will be considering Eulerian frame of work.

1.2.2 Eulerian Frame

Viewing the fluid at the macroscopic level induces the notion of "continuum" which is on the contrary position to the notion associated to the microscopic perspective of fluid being a scattering of individually separate particles. In the Eulerian description, a fixed geometric reference attached to the underlying physical space is adapted, for example, the Cartesian space $(x, y, z) \subset \mathbb{R}^3$ or polar coordinates (r, θ) in a time interval $[0, T]$. This allows to express the states of the continuum object viewed as one physically unified entity in this time-spatial domain. Furthermore, the continuum hypothesis assures that the state and motion variables of the fluid are continuous in the metric reference topology.

Definition 1.3. *The continuum hypothesis of fluid mechanics admits the following postulates:*

1. *a domain $\Omega \subset \mathbb{R}^3$ occupied by a fluid in an ambient space;*
2. *a non-negative measurable function $\rho = \rho(x)$ defined for $t \in (0, T)$, $x \in \Omega$, yielding the mass density;*
3. *a vector filed $u = u(t, x)$, $t \in (0, T)$, $x \in \Omega$, defining the velocity of the fluid;*
4. *a positive measurable function $\theta = \theta(t, x)$, $t \in (0, T)$, $x \in \Omega$, describing the distribution of temperature measured in the absolute Kelvin scale;*
5. *the thermodynamic functions: the pressure $p = p(\rho, \theta)$, the specified internal energy $e = e(\rho, \theta)$, and the specific entropy $s = s(\rho, \theta)$;*
6. *a tensor $\mathbb{T} = \{T_{i,j}\}_{i,j=1}^3$ yielding the force per unit surface that the part of a fluid on the other side of the same surface element exerts;*
7. *a vector field q giving the flux of the internal energy;*
8. *a vector field $f = f(t, x)$,$t \in (0, T)$, $x \in \Omega$, defining the distribution of a volume force acting on a fluid;*
9. *a function $\mathbb{Q} = \mathbb{Q}(t, x)$,$t \in (0, T)$, $x \in \Omega$, yielding the rate of production of internal energy*

The hypothesis guarantees the validity of the laws of mechanics and thermodynamics describing the basic state variables (ρ, u, θ). The other states are then expressed in terms of these latter variables through constitutional relations that play an important role in the classification of the third effective element of study (degrees of freedom), that we will discuss later. As these relations are related to forces and tensors in fluid, I will keep them to the second section where forces are discussed. The Eulerian frame is adapted in the whole remaining literature.

1.3 Volume and Surface Forces: A Story of Tensors

1.3.1 Body and Contact forces

Consider an element in the fluid of volume δV. We can distinguish between two kinds of forces acting on the element:

- Long-range forces (body forces): which are forces acting at a distance without having a contact between the origin of the force and the element of volume δV, such as gravity, electromagnetism, and centrifugal forces, etc. Such forces act equally on all the matter inside the element, so we can assume that the total force on the element is proportional to the size of volume of the element. If δV is centered at a position X at a certain instant t, then the total body force is given by

$$F(X,t)\delta V.$$

- Short-range forces: these are contact forces of molecular interaction origin. Their contribution is more obvious in liquids than in gases due to the fact that their values decrease rapidly with the increase in the distance separating the molecules of interaction (as it the case in gases' molecules). The forces in fact occur on the membrane of the element which is in contact with another volume element $\delta V'$. At this common membrane, two phenomena can take place: 1) the transport of momentum across it when interacting molecules are in oscillatory motion, and 2) forces between molecules on the two opposite side of the common boundary. Thus, such forces act on a surface element on the boundary of δV rather than the whole volume, and furthermore-unlike long-range forces- they don't act equally in all regions of this surface element due to the different orientations in different areas. This means that we cannot express the total contact force on our chosen element in a proportional way, or any relevant way, with respect to the volume δV, but instead, the total force will be determined locally on plane surface (in the whole closed surface of δV on which contact forces act) of area δA as the total force exerted on the fluid on one side of the element surface by the fluid on its other side. Thus, the total force will be proportional to δA, and at a position X centering δA at a certain time t, it will be given by

$$\Sigma(n,X,t)\delta A,$$

where n is the unit normal to the element. Being defined then by total force per unit area, Σ is in fact called the local stress, and a normal component of Σ in the direction of n presents a tension.

Theorem 1.1. *[6] Assume that the density field ρ, the velocity field u, and the body force density f are regular. Lets also assume that, the vector u being fixed, the function $(t,X) \to \Sigma(n,X,t)$ is continuous.*
Then, there exists a tensor-valued function $(t,X) \to \sigma(X,t)$ such that for all (t,X) and for all unit vectors u we have

$$\Sigma(n,X,t) = \sigma(X,t) \cdot n.$$

$\sigma(X,t)$ is called the stress tensor of the fluid.

Since the tensor Σ across a plane surface is represented in the Cartesian reference (Eulerian description) by a vector in $\mathbb{R}^3$ in the same direction of the corresponding contact force, then the stress tensor is represented by a matrix in $\mathcal{M}_{3\times3}$: $\sigma_{i,j}$ represents the i-th component of the force per unit area exerted across a plane surface element normal to the j-th direction Moreover,σ is symmetric in the sense that $\sigma_{ij} = \sigma_{ji}$, and these latter are decomposed into two kinds of stresses:

- normal stresses represented by the diagonal entries σ_{ii} which represent the amount of stretch or compression given by the normal component of the force applied on a plane surface element parallel to the i-th coordinate plane.
- shear stresses which represent the amount of distortion associated with the sliding of plane layers over each other represented by the six other entries σ_{ij}.

Remark 1. *Though the terminology "molecular" is associated to the surface forces described above, yet this doesn't mean that their effect is only valuable at the microscopic prospect. Nevertheless, their impact echos at the macroscopic level as well, and makes a significant difference in describing the motion of a fluid as a whole.*

1.3.2 Compressible and Incompressible Fluids

In the context of forces and tensors, it is good to remark that one of the main forces that operate on the fluid is the hydrostatic pressure. It is defined by the pressure that is exerted by a fluid at equilibrium at a given point within the fluid, due to the force of gravity. Hydrostatic pressure increases in proportion to depth measured from the surface because of the increasing weight of fluid exerting downward force from above. In fact, the range of such pressure gives rise to two types of fluids: Compressible and Incompressible fluids. In reality, all fluids are compressible, which means that they are vulnerable to change their density (or volume) upon acted external force. Nevertheless, as this variation is considered negligible compared to the absolute pressure in some cases, especially for liquids, then we assume that density is not changing upon exerted forces and motion, and the fluid is said to be incompressible.

1.4 Hydrodynamics and Rheology: a story of viscosity

The third category of classification is related to the degrees of freedom in a fluid, and by those we mean specific constitutional properties which characterize the identity of the fluid, and contributes to the variety of patterns of motion of different fluids. These degrees of freedom serve to link motion discussed in section 1.2 with forces discussed in section 1.3 , thus completing the description of the deformation. In particular, such constitutional characteristics directly impact the surface forces discussed in the previous part which in turn impacts the motion. Two primarily properties we aim at studying here correspond in fact to the way the fluid responds upon inter molecular forces in two positions: at rest, and in motion. A fluid element at rest is vulnerable to stress effect, mainly compression one. The stress tensor in this case is only formed of normal stresses (non-zero diagonal entries), whereas shear stresses are absent due to the absence of the bulk motion of the fluid. This consequently means that at rest, no resistance to motion exists by shear stresses as there is no motion supervening at the first place, and this doesn't neglect the fact that the fluid still has the property of resistance if set to motion. It is convenient to write then the stress tensor as

$$\sigma = -p\,\mathrm{Id}, \tag{1.1}$$

where p is known as the hydrostatic (also called thermodynamic) pressure coming from compression effects. The finding of viscosity and elasticity constitute in fact a part in a greater story of fluid mechanics: unveiling the enigma of a complete constitutional law for stresses. This latter need was definitely urgent for closing the kinetic description of the fluid's (or solid's) motion as will be discussed in next section (balance laws). Both viscosity and elasticity reveal the same tendency of resistance to shear motion, but it is conventional that the latter notion is used for solids whereas viscosity denotes the resistance in fluids. A rigorous scientific study of both notions synchronized starting from the 15th century, however, a primitive understanding of elasticity goes back to antiquity with the use of bows. Starting from the 15th century, especially with the work of Leonardo da Vinci on friction forces, attempts to understand and introduce

these features to mechanics harmonized in a way that some milestone findings in both states of matter were delivered by the same person, and the same community. It is at this level that we can unwrap the classification done at the level of the constitutional behaviors of fluids, mainly giving rise to two classes of fluids: Newtonian fluids which are studied in Hydrodynamics and non-Newtonian fluids studied in Rheology. What is good about this category is that one paved the way to the other: the science setup started with Newtonian fluids (laws of Hydrodynamics), but when Hydrodynamics failed in interpreting various applications and phenomena, the branch of Rheology was introduced to serve the mission. Here the story continues in chronological order:

1.4.1 Hydrodynamics: Study of Newtonian Fluids

The name's origin is merely related to Sir Isaac Newton who was the first to shed light on the viscous notion. In 1676, Robert Hook published the first quantitative concept of elasticity which showed a linear relation between forces and the extension of springs. In 1675, Edme Mariotte rediscovered the same law for fluids in France, and connected what was NOT known at that time by "stress" to the state of the fluid. But the real notion of viscosity wasn't established until 1687 by Sir Isaac Newton. The story of viscosity goes back to the experiment that he did in 1687, know as the Couette flow. In his work "Philosophie Principia Mathematica", Newton defines viscosity as

> *The resistance which arises from the lack of slipperiness of the parts of the liquid, other things being equal, is proportional to the velocity with which the parts of the liquid are separated from one another.*

Newton's new interest in viscosity was due to his attempt to disprove Descart's theory of cosmic vortices. He was wondering how spinning planets could entrain fluid bulks in a vortex motion. In the Couette flow experiment, Newton placed water between two parallel plates being apart by a distance d, one fixed on a plane surface, and the upper one moving with a speed u. He noticed that different layers of the fluid possess different speeds ranging from 0 at the first layer at the bottom to u at the top layer adjacent to the upper plate. His observation pointed out to the presence of a force that is resisting the relative motion of the layers. If layer A is moving with a constant speed u_A, then the layer just below it, say layer B with constant velocity u_B, will apply a force on A in an opposite direction to the motion of A, and thus an external force-acted by A-should be exerted on B to keep it moving with velocity u_B. Newton signified that the total of such force between layers and particles, is inversely proportional to d, but directly proportional to u. Under the above hypotheses, Newtonian fluids satisfy the following properties:

1. The viscous stress tensor $\mathbb{S}$ in a flow depends only on the strain rate tensor $D(u)$.
2. The dependence of $\mathbb{S}$ on $D(u)$ is linear, i.e a Newtonian fluid is characterized by a constant viscosity.
3. The relation linking $\mathbb{S}$ to $D(u)$ is isotropic, i.e it is invariant under a change of the orthogonal frame of reference.

Newton's fundamental finding gives rise to the notion of viscosity established through this relation

$$F = \mu A \frac{u}{d}, \tag{1.2}$$

where F is the total resistance force acted upon the fluid, μ is the viscosity coefficient and A is the total surface area on which F acts. Technically, the force due to viscous effects per unit area $\frac{F}{A}$ represents the viscous tensor $\mathbb{S}$, and $\frac{u}{d}$ is congruent to the rate of change of velocity, i.e the spatial gradient of velocity, thus we can have the following relation known as Newton's law of viscosity

$$\mathbb{S} \sim \mu \nabla u.$$

Later on, Euler, along with Bernoulli's pioneering work on kinematics of deformation, set down a rigorous system for the motion of fluid using partial differential equations. Nevertheless, Euler neglected completely Newton's laws of viscosity and these stresses, and his set of equations-till this day- govern the motion of a perfect fluid, known also for d'Alembert. The d'Alambert-Euler constitutional equation represents simply an abstract fluid at rest

$$\sigma = -p\,\mathrm{Id}\,.$$

Many mathematicians and physicist after that tried to further study elasticity and viscosity as dynamical quantities such as Coulomb and Young. At the level of elasticity, Young could elaborate a constitutional relation for solids involving shear stresses where he recognized a new concept: shear strain. As for the fluids, the constitutional relation wasn't clear by this time, but scientists could only approve comprising a viscous effect in it. In accordance with the property of the fluid of not being able to withstand any deformation tendency by applied forces without changing its volume, the pressure force acts equally in all directions, which is technically exhibited by being represented by a diagonal matrix of equal diagonal entries. Once the fluid is set to motion, shear stresses contribute again to a new resisting effect to motion, the viscous effect, which along with the hydrostatic pressure furnishes a general formulation of the stress tensor perfectly displayed by the following law

$$\sigma = \mathbb{S} - p\,\mathrm{Id}, \tag{1.3}$$

with $\mathbb{S}$ being the viscous tensor. The form and dependency of this tensor on the state variables of the system remained a debatable question till the work of Navier. On the basis of Coulomb's work, who elaborated strongly on shear stresses in fluids, Navier derived the fundamental equation of elasticity [7]. He considered a system of spherical particles among which central forces operate, and the equation developed for small displacement was given by

$$\mu(\nabla^2 u + 2\nabla\Theta) + f = 0,$$

where Θ denoted the volumetric strain of the element, μ the shear modulus (which is related to Young's modulus of elasticity) and f the external body forces.

In France 1827, Cauchy introduced the symmetric stress tensor $\mathrm{D}(u) = \frac{\nabla u + \nabla u^T}{2}$, and he set down the first complete realistic constitutional equation for solids that governed the basis of classical elasticity

$$\sigma = \Lambda\Theta\,\mathrm{Id} + 2\eta\,\mathrm{D}(u).$$

Λ and η where called *Lame*'s coefficients, Θ the volume strain given by $\operatorname{div} u$. Simultaneously, a new investigation for motion was done at the level of viscous fluids by Navier. Similar to his first attempt of considering system of spherical bodies, but in fluid, he derived the first equation of motion to account for viscosity

$$f - \nabla p = \rho\frac{\mathrm{D}\,u}{\mathrm{D}\,t} - \mu\nabla^2 u,$$

where μ being the viscosity, ρ the mass density,u the velocity and p the hydrostatic pressure. In 1845, Stokes added the last chapter in this story [8] joining the work of Cauchy and Navier, by fulfilling the notion of resistance to attempted or actual volumetric changes in a general fluid via a complete constitutional law that is still considered till today

$$\sigma = -p\,\mathrm{Id} + \lambda\operatorname{div} u\,\mathrm{Id} + 2\mu\,\mathrm{D}\,u.$$

The term in front of λ expresses volumetric dilatation; the rate of change in volume which is in essence a change of the density, while the term in front of μ corresponds to the rate of linear dilatation—a change in shape with fixed volume. We notice here the presence of two viscosity

coefficients μ and λ. Those latter are also known by *Lame*'s coefficients as called after Cauchy's work. The first viscosity is also known as the dynamic or shear viscosity. It is associated to shear deformation and it is the same one spotted in Newton's experiment. In his attempt to close the expressions of stresses, and upon the lack of measurements of λ, Stokes postulated a relation between the two viscosities so that the flow will depend on just one viscosity determined from the basis of empirical assumptions postulated in the continuum hypothesis. However this relation is not valid for all fluids, but it has been tremendously exploited in most of fluids as

$$\lambda = -\frac{2}{d}\mu$$

which means that the pure volumetric changes without shearing do not dissipate energy (see notion of bulk viscosity in the next paragraph). Stokes work was later supported by many experimental results. It is good to mention that the agreement between Stokes theory and experiment is related to boundary conditions on the wall of the capillary: no-slip boundary condition. Such law now is only guaranteed for fluids with small molecules, i,e Newtonian fluids, and under low shear stresses. In fluid dynamics, this strain tensor is decomposed into two parts in order to identify the different deformations involved and in a way to split aside the diagonal involving $\operatorname{div} u$ which expresses isotropic deformations associated with volumetric dilatation from the diagonal-free part called the deviatoric part of the deformation rate tensor and associated to the rate of linear dilatation. In $\mathcal{M}_{d\times d}$, we have

$$\mathrm{D}(u) = \frac{\operatorname{div} u}{d}\,\mathrm{Id} + (\mathrm{D}(u) - \frac{\operatorname{div} u}{d}\,\mathrm{Id}). \tag{1.4}$$

Thus, using Stokes viscosity law and the partition of the strain tensor in (1.4), we can rewrite the viscous stress as

$$\mathbb{S} = 2\mu(\mathrm{D}\,u - \frac{1}{d}\operatorname{div} u\,\mathrm{Id}). \tag{1.5}$$

Bulk viscosity. In fact, Stokes relation gives rise to a third viscosity type known as *bulk* viscosity. Bulk viscosity, denoted $\mu_B = \lambda + \frac{2}{d}\mu$, is associated to normal stresses and related to change of volume in the fluid parcel. The notion of this viscosity can be understood from this phenomenon: Supposing that we have an element fluid as a sphere, normally subjected to normal stresses from all directions being equal to the hydrostatic pressure p inside the fluid. Upon sudden work acted on the sphere, causing the normal stresses to increase in a quasi-static reversible manner, an increase in the internal energy will start to occur due to the first law of thermodynamics. This energy will be depicted in three forms of molecular motion: translation, rotation or vibration. Upon very rapid compression, a need of time-lag is noticed in order for this internal energy to re-partition into the different modes. This time is due to a resistance property in the sphere which makes it exert pressure to oppose the volumetric constraint due the exerted work of compression. This effect is treated as viscosity effect and it is what we call bulk viscosity. Due to Stokes law of viscosity, which defines the bulk viscosity as a zero quantity, most of the studies on fluids neglect this kind of resistance. It is proven to be efficient in supersonic flows (high Mach number) and multidimensional reaction flows.

1.4.2 Rheology: An Introduction To Non-Newtonian Fluids

Newton's hypothesis on the behavior of a fluid in response to deformation were accurate in plenty of study cases. However, the rise of polymers in the industrial world had spot the light on a different kind of behavior among these fluids that would refute Newton's postulates. Newton's linear constitutional law was no more valid to define the viscosity of such types of fluids-called

by analogy non Newtonian- and as a result a need to a generalized constitutional law appeared. Non Newtonian fluids are defined as those fluids whose viscous stress tensor is not linearly depending on the rate of deformation, and thus the resisting behavior they exhibit depends on the state variables of the system, for example this resistance may increase with the increase in applied stress, or decrease (i.e to say viscosity depends on rate of deformation), it may be also time dependent, or have a yield stress, and so on. Non-Newtonian fluids come in a number of different types: viscoelastic, Thixotropic, Rheopectic, Pseudoplastic, and dilatant. Our main concern in the thesis would be that of shear thinning fluids as Pseudoplastic fluids, and with viscoelastic fluids. As for the development of this branch of science, it would be surprising to know that curiosity and usage of non-Newtonian fluids predates its formal framing into a branch of science called Rheology, which was enunciated circa the twentieth century. The concept of "thinness" and "thickness" was illustrated in ancient knowledge by many applications as that of the water clock: Clepsydra, which originated in Babylon and migrated to Egypt and China, where water had been heated by those people in winter in order to keep time accurate, i.e they manipulated using heat the viscosity or "stickness" of water. Inspired by what the the Greek philosopher Heraklitus said:"Everything flows", Marcus Reiner, one of the eminent founders of Rheology, once said in reference to variable viscosity: "Everything flows if you wait long enough, even the mountains."

This non linear behavior in matter started to be studied in the beginning of 19-th century. Weber (1841) was the first to highlight new observations in silk fibers and visco-elastic effects. His experiment of a silk fiber attached to a load and then released freely to contract showed that the silk reattains its initial length, which means that the elasticity in silk fibers is not perfect, and he called this behavior visco-elastic effect in metals. He presented a law on the observed extension by experiments

$$\dot{x} = bx^{6.82}.$$

Between 1847 and 1866, Kohlrauch continued the pioneering work of his father, and established the linearity of torsional phenomenon by separating time and magnitude effects of the response through experiments. The first theoretical law towards Rheology was noted in 1867 by Maxwell who stated that "viscosity in all bodies may be described independent of hypothesis" by the equation

$$\frac{d\sigma}{dt} = E\frac{d\,\mathrm{D}(u)}{dt} - \frac{\sigma}{\tau}$$

where E is Young's modulus, and τ is a time constant. One firmly can say that Maxwell's equation is the basic formula that all constitutional models and equations for solids and non Newtonian fluids were based on. However, at the time of Maxwell, and lacking the real interpretation of this formula, he used it experimentally to calculate the viscosity $E\tau$ in gases.

Later, Oskar E Meyer in 1874 proposed that the shear stress σ and strain $\mathrm{D}(u)$ could be written in the form of

$$\sigma = G\,\mathrm{D}(u) + \eta\frac{d}{dt}\,\mathrm{D}(u)$$

which describes what is known now by Kelvin-Voigt body. Kelvin then did damping experiments on metals using the law introduced by Meyer, and Voigt later in 1889 generalized Meyer's ideas to anisotropic media. The next cornestone in Rheology of solids was set by Boltzman who is best known for contributions in Kinetic theory and entropy concepts. His additions were based on correcting Meyer's work:

- he presented the concept of "fading memory"; he assumed that the stress at time t depends not only on the strain at that time, but also on those in previous times, and that the longer the time interval from present to past, the smaller is the contribution of strain to stress. His new notion of behavior is known as "fading memory".

- He contributed in the general theory of visco-elasticity

$$\sigma_{xx} = \lambda\Theta(t) + 2G(\mathrm{D}(u))_{xx}(t) - \int_0^\infty dw\varphi_1(w)\Theta(t-w) - 2\int_0^\infty dw\varphi_2(w)e_{xx}(t-w)$$

  where φ_i are memory functions and $w = t - \tau$, with τ being a past time. Shear stresses where given in their usual form.

In 1878, Meyer criticized Boltzmann's work since it was not based on atomic hypothesis. By this time, non Newtonian behavior in fluids started to be investigated as well. The main contribution in dynamics of Rheology goes back to Poisseuille (1835) who studied blood flows, Hagen (1839) who was interested in hydraulics, and the seminal work of Couette. In 1889, Theodore Schwedoff performed the first experiment to determine viscosity variation with shear rate. He also studied yield stresses in some fluids using Maxwell's ideas. He invented a generalized Maxwell model for shear stress

$$\frac{d\sigma}{dt} = G\,\mathrm{D}(u) - \frac{1}{\tau}(\sigma - \sigma_y)$$

where σ_y is the shear yield stress. In 1903, Ladislaus Natanson of Cracow's University considered a fromulation for relaxing bodies

$$\frac{D\sigma}{Dt} = \Lambda(tr\,\mathrm{D}(u))\,\mathrm{Id} + 2G\,\mathrm{D}(u) - \frac{1}{\tau}(\sigma + p\,\mathrm{Id}).$$

$\frac{D}{Dt}$ is the usual material derivative, Λ and G moduli, and τ the relaxation time. Zaremba later corrected the work of Natanson being the first to introduce co-rotational derivatives to the dynamics of fluids, which later on proved that it is not adequate for most non Newtonian fluids in application, especially for polymeric liquids. He wrote the equation in a rotating and translating frame with the medium

$$\frac{\bar{\mathrm{D}}\sigma_{ij}}{Dt} = \frac{\mathrm{D}\,\sigma_{ij}}{Dt} + \sum_n \sigma_{in}w_{nj}$$

where $w_{ji} = \frac{\partial u_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$ being the vorticity tensor. Zaremba also replaced $\frac{p}{\lambda}$ by $\frac{tr(\sigma)}{\tau_n}$, with τ_n a relaxation time.

The next considered work in the development of a constitutional law for Rheology was done by Heinrich Hencky who is known for his work on plasticity (1929) and proposing several perspectives on frames convected with the material. He influenced later Oldroyd (1950) and Lodge (1964) to expand these ideas. He introduced a constitutional equation similar to that by Zaremba which constituted the pathway to invariant constitutional relations

$$\frac{\bar{\mathrm{D}}T_{ij}}{Dt} = \varphi_{ij} - \frac{T_{ij}}{\tau}$$

where φ_{ij} are functions of the stress and the velocity gradient, and T is the extra stress defined by $T = \sigma + p\,\mathrm{Id}$ (equivalently $\mathbb{S}$ in our previous definition).

The work after that concentrated on incompressible fluids. Weissenberg in 1931 introduced a relation between the deviatoric stress $\mathbb{T} = \sigma - \frac{tr(\sigma)}{3}\,\mathrm{Id}$ and the rate of stress tensor $\mathrm{D}(u)$

$$\mathbb{T} = F(\mathrm{D}(u)) = \eta_1\,\mathrm{D}(u) + \eta_2\,\mathrm{D}(u)^3.$$

In 1933, Hans Fromm from Berlin reconsidered Hencky's corotational derivatives, and using

tensor methods he proposed the relation known as Corotational Maxwell model

$$\frac{\bar{\mathrm{D}}\mathbb{T}}{Dt} + \frac{\mathbb{T}}{\tau} = 2G\,\mathrm{D}(u).$$

He obtained correct experimental results for steady shear flow for a Cartesian velocity distribution: $u_x = \gamma y, \quad u_y = u_z = 0$, where γ is a constant velocity gradient. He obtained the viscometric functions using which he tried to solve the capillary flow problem, and mainly he obtained

$$\sigma = \frac{\eta\gamma}{(1+\tau^2\gamma^2)}.$$

In 1945, the next cornerstone was set by Reiner by introducing the tensor methods in Rheology. The use of the deviatoric stress $\mathbb{T}$ was not adequate in his perspective (which is true in reality) as it neglects the isotropic part necessary to guarantee the momentum balance for incompressible fluids. Reiner proposed a constitutional equation for dilatant materials

$$\sigma = c_0\,\mathrm{Id} + c_1\,\mathrm{D}(u) + c_2\,\mathrm{D}(u)^2,$$

where $c_i's$ are the invariants of $\mathrm{D}(u)$. The drawback of his equation was in neglecting normal stresses.

In 1946, Fröhlic and Sack (1946) started studying dilute suspension of elastic spheres in a viscous medium, and in the same context, Oldroyd in 1950 stated that for an empirical rheological equation of states to be valid universally, convected coordinate systems should be used, and thus reviving the assumptions done by Zaremba in 1903. He suggested the following law based on experimental work

$$T + \tau\frac{\partial T}{\partial t} = 2\eta_0(\mathrm{D}(u) + \lambda_2\frac{\partial}{\partial t}\,\mathrm{D}(u)),$$

where $\eta_0, \lambda_1, \lambda_2$ are constant properties related to material's concentration. Another simpler form by Oldroyd was given by what he called liquid B of constitutional formula

$$T + \lambda_1\overset{\triangledown}{T} = 2\eta_0(\mathrm{D}(u) + \lambda_2\overset{\triangledown}{\mathrm{D}(u)}),$$

with $\triangledown$ being the upper convected time derivatives. The case of λ_2 equals zero is universally known by the Upper Convected Maxwell Model UCM, inspired from Maxwell's work. The Oldroyd and UCM models have gained a huge popularity till nowadays and benefited a vast amount of studies on viscoelastic flows especially numerically.

During 1952-1955, Rivlin and his team in the Naval Research Laboratory made some progress in the way for the constitutive relations. They assumed that in an isotropic incompressible viscoelastic fluid, the Cauchy stress depends on the deformation gradients, velocity gradients, first and second accelerations and so on. The resulting constitutive equation up to order n reads

$$T = f(A_1, A_2, ..),$$

where f is an isotropic matrix, and A_i are kinematic tensors called Rivlin-Erickson tensors. After that, invariant theory governed the research of obtaining better expression of f. A smart discovery by Rivlin in 1955 stated that for n=2 we have

$$\begin{aligned} f &= tr(A_1^2)A_1 + tr(A_1^3)A_2 + tr(A_2^2)A_1^2 + tr(A_2^3)A_2^2 + tr(A_1A_2)(A_1A_2 + A_2A_1) \\ &+ tr(A_1^2A_2)(A_1^2A_2 + A_2A_1^2) + tr(A_1A_2^2)(A_1A_2^2 + A_2^2A_1) + tr(A_1^2A_2^2)(A_1^2A_2^2 + A_2^2A_1^2) \end{aligned} \tag{1.6}$$

which is compatible with the analysis of viscometric flows known nowadays as Rivlin-Erickson fluids referring to (1.6). In 1957, Markovitz and Williamson rewrote Rivlin's formula as

$$T = \alpha_1 A_1 + \alpha_2 A + 2 + \alpha_3 A_1^2$$

with α_i being functions of $\mathrm{D}(u)^2$.

Between 1957 and 1961, Green, Rivlin nd Spencer from one side and Colman and Noll from second side lead the next crucial steps in the development of rheological theories for constituve laws. They introduced a new measure of deformation; the Cauchy Green tensor

$$T = \underset{-\infty}{\overset{t}{F}}[C(t')],$$

which means that at time t, $T(t)$ depends on the entire history of local deformation $C(t')$. Later, they approximated the functional F by a series of integrals.

During the same period, Coleman and Noll presented a simpler development to the latter's expression; they defined the measures of fading memory in terms of functional space topologies, where this allows the difficulty to be presented in the smoothness rather than the constitutional form. They called such fluids with constitutional law "simple fluids" with fading memory, and their study lead to series of approximations known as the Hierarchy equations of Coleman and Noll. The first order approximation would be Newtonian limit, and the second order leads to more complex and useful fluid in applications:

$$T = c_1 A_1 + c_2 A_2 + c_3 A_1^2$$

with c_i's are constant.

In 1960, Walters applied Maxwell's postulates by assuming that the fluid is a set of infinite number of Maxwell's elements. His constitutive relation was given by

$$T = 2 \int_{-\infty}^{t} \psi(t - t^*)\, \mathrm{D}(u)\ dt^*, \tag{1.7}$$

where

$$\psi(t - t^*) = \int_0^{\infty} \frac{N(\lambda)}{\lambda} \exp^{-\frac{t-t^*}{\lambda}}\ d\lambda,$$

and $N(\lambda)$ is a distributional function of the relaxation time. Later, Walters modified his equation by adapting convected frame as that suggested by Oldroyd. His generalized constitutive relation was made for two liquids named A' and B', however the relation for liquid B' was used expensively in literature.

We observe two main math tools which carried out the achievements done till the early 1960s: corotational derivatives and local representation of fading memory functions using integral functionals. Both tools have their drawbacks that make the hypothesis fail in some cases, as when applying corotational derivative to a shear-lead flow. Developments on the above main achievements have been carried out after 1960 mainly by remarkable scientists as J.L.White and Tokita (1967), Rivlin and Sawyers (1971), Segalman (1977), Tanner and Phan-Thien (1977), Wagner (1978) and Doi and Edwards (1986). The developments concentrated in the first place on the well definition of the integral local deformations and representations and on the constraints on the fading memory functions, also on the approximation of the integral functionals, and finally on modeling simpler constitutional relations based on the formulation principles compatible with thermodynamics that reflect microstructural postulates. The generality of the work that started

in the late 19th century and continued till 1960s was followed by a wave of modeling techniques deviating from this generality and integrating into case-applied Rheology. In conclusion, the quest to find a constitutional relation that is adequate to all general cases is not closed till today. Scientist have partitioned non-Newtonian fluids according to their behavior, which in turn resulted in plenty of approximate relations based on the work presented in the above literature. It would be noteworthy to mention some of the remarkable models presented in this context: Bingham 1922, Ostwald 1925, power law model and Herschel Bulkley 1926.

1.5 Balance Laws

Fluid dynamics stands for the examination of the interactive motion of a large number of atoms or particles in a domain in which their density distribution is so high that they can be regarded as a continuum. Thus, the continuum assumption implies that we can define a mean velocity and a mean kinetic energy of this system of particles thus specify the velocity, pressure and temperature of each particle. The dynamical behavior of a fluid is based on conservation laws. Conservation means that for a certain quantity of a fluid, its total variation is equivalent to the net amount of this quantity being transported across the boundary of the volume by the net forces (external and internal) acting on the system.

Technically, the above description is translated as follows: consider a volume element of the fluid δV of surface area dS, and let ρ and $\overrightarrow{n}$ be respectively the density of the fluid in the domain and the outward normal to dS. Then the total variation of a certain quantity M in the whole domain Ω is given by

$$\frac{\partial}{\partial t}\int_{\Omega} M \; d\Omega.$$

The transported quantity is called a flux, and it is decomposed into two parts: convective and diffusive. Usually, convective fluxes result from long range forces which act on the fluid as a bulk, whereas the diffusive property arises from internal molecular interactive forces. The convective flux is the amount of M entering the volume through the boundary with a velocity u, it is given by

$$-\oint_{\partial\Omega} M\overrightarrow{u}\cdot\overrightarrow{n}\; dS.$$

And as for the diffusive flux, it is the measure of the amount of diffusivity of this quantity per unit mass, and it is given by

$$\oint_{\partial\Omega} \kappa\rho\nabla(\frac{M}{\rho})\cdot\overrightarrow{n}\; dS.$$

Additional external sources may affect the variation of quantity M in ∂V, and thus in Ω, they are called volume and surface forces (Q_V and Q_S) depending on the manner of their work, whether it acts on the ∂V or on dS, the total of both on Ω is given by

$$\int_{\Omega} Q_V \; d\Omega + \oint_{\partial\Omega} Q_S\cdot\overrightarrow{n}\; dS.$$

Summing up we get

$$\frac{d}{dt}\int_{\Omega} M \; d\Omega + \oint_{\partial\Omega}\left(M\overrightarrow{u} - \kappa\rho\nabla(\frac{M}{\rho})\right)\cdot\overrightarrow{n}\; dS = \int_{\Omega} Q_V \; d\Omega + \oint_{\partial\Omega} Q_S\cdot\overrightarrow{n}\; dS. \tag{1.8}$$

By analogy, and in case the conserved quantity is a vector field, say $\overrightarrow{F}$, then forces are replaced by tensors, namely the convective tensor is denoted D_C and the diffusive one is denoted

F_D. Thus we get

$$\frac{d}{dt}\int_\Omega \overrightarrow{F}\ d\Omega + \oint_{\partial\Omega}\Big(F_C - F_D\Big)\cdot\overrightarrow{n}\ dS = \int_\Omega Q_V\ d\Omega + \oint_{\partial\Omega} Q_S\cdot\overrightarrow{n}\ dS. \tag{1.9}$$

All conservation equations in classical physics are derived based on the above formula.

1.5.1 Mass Conservation

It is also called continuity equation. In a closed system, the physical postulates state that matter cannot be created nor destroyed, the particles of the matter can only rearrange through displacements or chemical bonds, and thus the total mass is conserved. In the absence of external sources in a closed system, and measuring the mass quantity through its density distribution ρ, the integral formulation (1.8) becomes for mass conservation

$$\frac{d}{dt}\int_\Omega \rho\ d\Omega + \oint_{\partial\Omega}\rho\overrightarrow{u}\cdot\overrightarrow{n}\ dS = 0. \tag{1.10}$$

1.5.2 Momentum Conservation

A word on Momentum

At any interaction, a body expresses a tendency to resist motion, including decrease or increase in speed, or changing direction. This behavioral characteristic in any body of mass is called inertia. Hence inertia is a good indicator of motion fluctuation, and the interest in this fluctuation on the future of motion necessitates its measurement. This lead to introduce the notion of momentum as a quantified trait and indicator of inertia. Momentum can hold several definitions, in most literature, it is defined as mass times volume, but a more substantial definition would be the force $\times$ time, which gives the amount of force needed to contrast inertia and change the motion of the mass object in time, and this would be more logical to include it in Newton's force representative second law of motion.

In fact, inertia is a conserved quantity, thus we can apply the conservation principle stated in the beginning of this subsection for the quantity we are interested in studying its variation accordingly: momentum= mass $\times$ velocity. In fact, historical backgrounds state that Newton's second law gives an evolutionary equation on the momentum, but this is nothing but the application of the conservation principle. The total variation across Ω is given by

$$\frac{d}{dt}\int_\Omega \rho\overrightarrow{u}\ d\Omega.$$

The transport of the flux is as well defined by

$$-\oint_{\partial\Omega}\rho\overrightarrow{u}(\overrightarrow{u}\cdot\overrightarrow{n})\ dS.$$

As inertia is of no diffusive behavior, the contribution of the diffusive flux is zero. And finally, there is the external and internal forces (volume and surface forces) acting on ∂V. External forces on ∂V can include gravitational, buoyancy, Coriolis, centrifugal forces and electromagnetic

forces as well. The total of those forces per unit volumes ∂V's is given by

$$\int_{\Omega} \rho Q d\Omega.$$

On the other hand, internal forces include from one side the pressure exerted by the surrounding fluid to ∂V acting on dS, and the forces exerted due to molecular interactions of particles with the surface dS of ∂V, represented by shear and normal stresses as discussed in section 1.3. These forces are represented by the stress tensor divided into the pressure tensor and viscous tensor. The total across the surface of such force reads

$$-\oint_{\partial\Omega} p\,\mathrm{Id}\cdot\overrightarrow{n}\ dS + \oint_{\partial\omega} \mathcal{S}\cdot\overrightarrow{n}\ dS.$$

Finally, the conservation of momentum is expressed through the following integral relation

$$\frac{d}{dt}\int_{\Omega}\rho\overrightarrow{u}\ d\Omega + \oint_{\partial\Omega}\rho\overrightarrow{u}(\overrightarrow{u}\cdot\overrightarrow{n})\ dS = \int_{\Omega}\rho Q d\Omega + \oint_{\partial\omega}(\mathbb{S}-p\,\mathrm{Id})\cdot\overrightarrow{n}\ dS. \tag{1.11}$$

1.5.3 Energy Conservation

This is also known as the first law of thermodynamics. In fact, analogous with mass, energy is also a conserved quantity in nature, it can never be created or destroyed, and it can only be transformed from one shape to another (kinetic, thermal, potential, electric, etc).

The total energy in an element volume ∂V is given by the kinetic energy $\frac{1}{2}|u|^2$ and the potential (internal) energy denoted e. The total variation of the total energy per unit mass $E = \frac{1}{2}|u|^2 + e$ inside Ω summed with the corresponding convective flux reads

$$\frac{d}{dt}\int_{\Omega}\rho(\frac{1}{2}|u|^2+e)\ d\Omega + \oint_{\partial\Omega}\rho(\frac{1}{2}|u|^2+e)(\overrightarrow{u}\cdot\overrightarrow{n})\ dS.$$

On the contrary to the mass and momentum, energy is of a diffusive nature, in particular, internal energy is diffusive in the form of heat transfer (there is another form of diffusivity such as electric, electromagnetic, etc, but are beyond our scope and interest) and contributes to a diffusive flux across the boundary of Ω. This latter is demonstrated by Fourier's law of heat conduction, the total of such flux across $\partial\Omega$ reads

$$\oint_{\partial\Omega}\kappa\nabla T\cdot\overrightarrow{n}\ dS,$$

where κ is the thermal conductivity and T is the absolute temperature. Concerning the volumetric sources, those include the total of the work done by external forces per unit mass ρf_e and the time rate of heat transfer occurring in Ω-including absorption, emission or chemical reaction- represented by $\dot{q}_e$. The total of the volumetric sources is given by

$$\int_{\Omega}\rho f_e\cdot\overrightarrow{n} + \dot{q}_e\ d\Omega.$$

And finally, the surface sources should be considered, in this case they are equivalent to the time rate of work done by pressure and stresses

$$\oint_{\partial\Omega}(\sigma - \mathbf{p}\,\mathrm{Id})\overrightarrow{u}\cdot\overrightarrow{n}\ dS.$$

Summing up we get the integral form of the energy equation

$$\frac{d}{dt}\int_\Omega \rho(\frac{1}{2}|u|^2+e)\,d\Omega+\oint_{\partial\Omega}\rho(\frac{1}{2}|u|^2+e)(\overrightarrow{u}\cdot\overrightarrow{n})\,dS=\oint_{\partial\Omega}\kappa\nabla T\cdot\overrightarrow{n}\,dS+\int_\Omega \rho f_e\cdot\overrightarrow{n}+\dot{q}_e\,d\Omega + \oint_{\partial\Omega}(\sigma-\mathbf{p}\,\mathrm{Id})\overrightarrow{u}\cdot\overrightarrow{n}\,dS. \tag{1.12}$$

The exact forms of equations 1.10, 1.11 and 1.12 are given by virtue of Gauss theorem. In case the domain Ω is in a fixed frame, we can write the above integral formulas in differential forms as

$$\begin{aligned}\partial_t\rho+\nabla\cdot(\rho\overrightarrow{u})&=0,\\ \partial_t(\rho\overrightarrow{u})+\nabla\cdot(\rho\overrightarrow{u}\otimes\overrightarrow{u})&=\rho Q+\nabla\cdot(\mathbb{S}-\mathbf{p}\,\mathrm{Id}),\\ \partial_t(\rho E)+\nabla\cdot(\rho E\overrightarrow{u})&=\rho f+\nabla\cdot(\sigma-\mathbf{p}\,\mathrm{Id})\overrightarrow{u}+\kappa\Delta T.\end{aligned} \tag{1.13}$$

This is called Navier Stokes Fourier system in its complete form.

Remark 2. *In some geophysical and turbulence problems, an angular velocity contributes to the velocity of the fluid, and thus extends both the kinetic energy and the volume source Q as a centrifugal force is additionally included. The same thing applies for Coriolis forces. In such cases, an appropriate rotating frame of reference is taken to express the angular velocity, and thus the final form of Navier Stokes Fourier system will be extended. Such topics are beyond the study in this thesis, and the general form (1.13) will be adapted in our case.*

1.6 Constitutional relations

In system (1.13), the expressions of the internal energy and the stresses are not specified. It is the type of the fluid that determines the form of dependence of these quantities on the state variables (ρ, p and T) and dynamical variables (u_1, u_2 and u_3). Thus, two supplementary equation are used to identify the constitutional relations between stresses and internal energy from one side with the other variables from the second side. This closes our system as we have now 7 unknowns: ρ, u_1, u_2, u_3, p, T and E, and 7 independent equations: 5 from (1.13) and two constitutional relations. In this thesis we will be concerned with isothermal and isentropic fluids, and this would mean that no variation in the internal energy e, and hence the final energy equation is a contribution of the kinetic energy merely following what has been stated in Section 1.5.3. In such case, the kinetic energy can be derived from the system (1.13) without the need to regard the third equation in system (1.13). For reader's convenience, we will state the constitutional relation related to the internal energy though it is beyond the scope of this work.

1-Thermo-dynamical Law

Many scientists found that heat always flows spontaneously from hotter to colder regions (Carnot, Clausius and Kelvin). The direction of such process is never reversed unless external factors operate. This was the basic axiom of the second law of thermodynamics, which identifies such phenomenon with entropy. The total entropy in an isolated system is not a conserved quantity, but it can never diminish with time as the system naturally evolves toward thermodynamical equilibrium, hence the entropy should stay the same or increase. This is demonstrated

by assumptions from static physics which asserts that the term

$$\frac{1}{\theta}(\mathrm{D}\, e + p\,\mathrm{D}(\frac{1}{\rho}))$$

is a perfect gradient. Gibbs's law defines the entropy in a system $s(\rho, \theta)$ by

$$\theta\,\mathrm{D}\, s(\rho,\theta) = \mathrm{D}\, e + p\,\mathrm{D}(\frac{1}{\rho}). \tag{1.14}$$

Gibbs's law is equivalent to Maxwell relation relating the internal energy in a system with the state variables ρ and θ

$$\rho^2 \frac{\partial e}{\partial \rho} = p(\rho,\theta) - \theta \frac{\partial p}{\partial \theta}. \tag{1.15}$$

An entropy balance relation can thus be deduced from the above laws which reads

$$\partial_t(\rho s) + \mathrm{div}(\rho u s + \frac{q}{\theta}) = \frac{1}{\theta}(\mathbb{S} \colon \nabla u - \frac{q \cdot \nabla\theta}{\theta}) + \frac{\rho}{\theta}\mathbb{Q}, \tag{1.16}$$

where θ and $\mathbb{Q}$ are given in definition 1.3 by temperature distribution and internal energy production rate.

2-Constitutional Laws Related to Stress Tensor

The viscous aspect in a fluid is related to the dynamical behavior of the flow through constitutional laws. As discussed in section 1.4, these laws has passed through a vast expansion of postulates and analysis since the 19th century. In this thesis, we will be concerned in four types of fluids: Newtonian fluids, Bingham fluids, Pseudo Plastic fluids and Oldroyd fluids.

Newtonian Fluids

As discussed before, Newtonian fluids are characterized by a linear relation between the stress and the deformation. After the revolutionary work of Newton and Stokes, the Newtonian constitutional law reads

$$\mathbb{S} = 2\mu(\mathrm{D}\, u - \frac{1}{d}\,\mathrm{div}\, u\,\mathrm{Id}) \tag{1.17}$$

where μ is a given positive quantity of the fluid. The other types of fluids fall in the category of **non-Newtonian fluids**.

Non-Newtonian Fluids

1- Viscoplastic Fluids: Bingham Model

Such fluids are also known as "yield stress" fluids. For a viscoplastic fluid to flow, it should be acted upon by a deformation exceeding some yield. In particular, the shear stress should exceed a critical value for the fluid to start flowing, otherwise it acts as a solid. One of the most important models describing such fluids is the Bingham fluid. In his paper "investigation of the laws of plastic flows", Eugene C. Bingham shed the light on the fluids that can behave either like a solid or like a liquid, not just in a phenomenological observatory manner (which has been noticed by preceding scientists) but also from a theoretical physio-chemical frame. He conducted capillary flow experiments on kaolin water suspensions, and he concluded the following:

1. There exists a critical value beyond which any pressure with greater value causes the fluid to flow.

2. The speed of the flow increases proportionally with the amount by which the applied effort exceeds the critical value.

He further concluded that a constant friction term and a term depending on the velocity variation take a part in the viscous stress between adjacent slipping layers. The novelty of his findings, in addition to proposing the proportional relation between shear rate and stress for such fluids, was in giving a physical interpretation of the molecular construction and behavior of such fluids, which until today serves as the basic explanation for the family of shear thinning fluids. He presumes that a group of large molecules bond in a network which upon large deformation breaks down into smaller structures in the medium and thus the fluid will flow. Whereas upon some small deformations, such networks or bonding reform into larger structures due to the adhesion of smaller structures, and these large structures give the solid behavior of the fluid. Bingham studied simple shear cases. His primitive model reads

$$\mathbb{S} = \mathbb{S}_0 + B\frac{\partial u}{\partial z},$$

where B is the Bingham viscosity. This model can also be expressed as

$$\frac{\partial u}{\partial z} = \begin{cases} 0 & \mathbb{S} \leqslant \mathbb{S}_0, \\ B(\mathbb{S} - \mathbb{S}_0) & \mathbb{S} \geqslant \mathbb{S}_0. \end{cases} \tag{1.18}$$

In fact, such models work efficiently in simulation problems and for short range of deformation variation. Later in 1926, Herschel and Bulkley (HB) modified Bingham's model to include a wide range of shear rates with subsequent modification to account for power law variation of the additional viscous term. The HB constitutional model reads

$$\mathbb{S} = \mathbb{S}_0 + K(\frac{\partial u}{\partial z})^n,$$

with K being the HB consistency coefficient. An adequate 3D generalization was introduced by Hohenemser and Prager (1936), and later by Oldroyd (1946). A Von Mises criterion is presumed to be satisfied by the fluid, and the yielding begins when the elastic energy or distortion reaches a certain critical value. This is expressed when the second invariant of the stress tensor is equal to a critical value. The second invariant of the viscous stress tensor and the strain rate tensor read respectively

$$\mathcal{II}_s = \frac{1}{2}\sum(\mathbb{S}_{ij}\mathbb{S}_{ji} - \mathbb{S}_{kk}),$$

$$\mathcal{II}_d = \frac{1}{2}\sum((\mathrm{D}(u))_{ij}(\mathrm{D}(u))_{ji} - (\mathrm{D}(u))_{kk}).$$

The 3D model reads

$$\begin{cases} (\mathrm{D}(u))_{ij} = 0 & \text{if} \sqrt{\mathcal{II}_s} \leqslant \mathbb{S}_0, \\ \mathbb{S}_{ij} = 2B(\mathrm{D}(u))_{ij} + \mathbb{S}_0\frac{(\mathrm{D}(u))_{ij}}{\sqrt{\mathcal{II}_d}} & \text{if} \sqrt{\mathcal{II}_s} \geqslant \mathbb{S}_0. \end{cases} \tag{1.19}$$

In simple shear flows, the above model reduces to the following one

$$\mathbb{S} = \begin{cases} 2B\,\mathrm{D}(u) + \mathbb{S}_0\frac{\mathrm{D}(u)}{|\mathrm{D}(u)|} & \text{if} |\,\mathrm{D}(u)| \neq 0, \\ |\mathbb{S}| \leqslant |\mathbb{S}_0| & \text{if} |\,\mathrm{D}(u)| = 0. \end{cases} \tag{1.20}$$

For the HB model we get

$$\mathbb{S} = \begin{cases} 2K|\,\mathrm{D}(u)|^{n-1}\,\mathrm{D}(u) + \mathbb{S}_0 \frac{\mathrm{D}(u)}{|\mathrm{D}(u)|} & \text{if} |\,\mathrm{D}(u)| \neq 0, \\ |\mathbb{S}| \leqslant |\mathbb{S}_0| & \text{if} |\,\mathrm{D}(u)| = 0. \end{cases} \tag{1.21}$$

2-Visco-elastic Fluids

Such fluids demonstrate both viscous and elastic properties when exposed to a deformation. More precisely, they exhibit a resistance (apparent viscosity) to deformation and shear flow in a linear manner. If their viscosity decreases when increasing the shear rate, they are called shear-thinning fluids, whereas the counter case is known for shear-thickening. We will highlight in here two types of shear thinning fluids: Pseudo-Plastic fluid and Oldroyd-B model.

• *Pseudo-Plastid Fluids*

Pseudo plastic fluids, also known as shear thinning fluids, are the most abundant type of fluids both in nature, engineering and industrial applications. Unlike Bingham fluids, pseudo plastic fluids are not characterized by a yield value, their viscosity decreases with gradual increase in shear rate, and they behave like a perfect fluid upon very high or very low shear rates. Thus if the shear stress-shear rate relation is studied upon high deformation rate, the linear relation can be extrapolated to express a yield point thus resembling Bingham plastic fluids at such limit, and from here comes the term pseudo plastic. A valid generalization of a constitutional equation for such fluids is difficult to attain as the deformation rates differ from one fluid to another depending on several uncertain factors as the concentration, the nature of the fluid, the geometric configuration of the particles and their bonding types and finally the nature of suspensions. Thus it is more convenient to look for approximate models. Usually, the relation between the shear stress and shear rate is plotted on a $\log - \log$ coordinate system, and the data are approximated by a straight line over an interval of shear rate. This kind of approximation will be discussed in chapter 6 when a shallow water model approximating such rheologies is derived. Nevertheless, it is very common to approximate the data using curve functions. In this context, the most used model is the power law model of Ostwald de Waele. His constitutional law reads

$$\mathbb{S} = g(\mathrm{D}(u))^n, \tag{1.22}$$

and the apparent viscosity is thus defined by

$$\mu = g(\mathrm{D}(u))^{n-1},$$

where g denotes the fluid's consistency constant. For shear thinning fluids, n is an index less than 1. Experimental surveys showed that polymer liquids for example show an index between 0.3 and 0.7 taking into account the molecular weight and the concentration of the polymer. Smaller power law indices can be found in fine particle suspensions like kaolin water suspension. The smaller is n the more shear thinning is the fluid. Another model which has also gained a wide acceptance in literature is the Cross model [9]. Indeed, the power law is only valid for a short range of shear rate and cannot predict the limit of the viscosity at very low or very high ranges of shear rate. Cross presented an empirical form to describe the rheology of pseudo plastic fluids in attempt to discredit the power law model. He suggested

$$\frac{\mu - \mu_\infty}{\mu_0 - \mu_\infty} = \frac{1}{1 + g(\mathrm{D}(u))^n}.$$

Furthermore, he suggested $n = \frac{2}{3}$ as an adequate value for most of the shear thinning fluids, however, it is now considered as a good fitting constant for many fluids in applications.

- *Oldroyd-B Fluids*

As reviewed in Section 1.4.2, the Oldroyd-B model- named after J.G. Oldroyd- is a constitutive model used to describe the flow of viscoelastic fluids, especially for high-molecular-weight liquids, which include polymer melts and solutions of polymers, as well as liquids in which fine particles are suspended. This model can be regarded as an extension of the Upper Convected Maxwell model. So in ddition to classical Navier Stokes equations of mass and momentum conservation, the model add the following equation on the stress

$$\mathbb{S} + \lambda_1 \overset{\nabla}{\mathbb{S}} = 2\eta_0(\mathrm{D}(u) + \lambda_2 \overset{\nabla}{\mathrm{D}}(u))$$

where λ_1 is the relaxation time, λ_2 is the retardation time, $\overset{\nabla}{\mathbb{S}}$ is the Upper convected time derivative of stress tensor defined by

$$\overset{\nabla}{\mathbb{S}} = \frac{\partial}{\partial t}\mathbb{S} + u \cdot \nabla \mathbb{S} - ((\nabla u)^T \cdot \mathbb{S} + \mathbb{S} \cdot (\nabla u)),$$

and $\eta_0 = \eta_s + \eta_p$ is the total viscosity composed of solvent component η_s attributed to the viscous behavior in the fluid and a polymer components η_p that contributes to the elastic behavior.

1.7 Boundary conditions

Having closed the system, it is worth mentioning now that predicting the behavior of the fluid is strongly attributed to what happens on the boundary in addition to the above mentioned factors. The boundary may separate two different phases as solid and liquid or liquid and air, or same phase fluids but with different constitutions. Special important phenomena can occur on the boundary, and the most two important ones are: surface tension and boundary layers. We can distinguish between kinematic boundary conditions and dynamic ones:

1-Kinematic Equation

The kinematic assumptions on the boundary assume that this latter is a material surface for both media and thus guarantees the continuity of the velocity and temperature fields across the boundary. We will only be concerned with conditions of the velocity on the boundary. Several types of kinematic boundary conditions exist in literature, but the three main ones are given below.

No-slip condition

This is a type of wall boundary conditions which states that the fluid cannot penetrate the wall at its boundary (speaking of an impermeable boundary like the fluid-solid part of the boundary), and thus the normal component of the velocity is zero:

$$(u - u_b) \cdot n = 0,$$

where u_b is the velocity of the wall (it is zero if the wall is fixed). Sometimes even the tangential velocity is set to zero, which results in the **homogeneous Dirichlet** boundary condition:

$$u = \text{on } \partial\Omega.$$

Naveir boudary condition

This condition was proposed by Navier in 1872. It is also known as slip-with-friction boundary condition and mostly used in capillary problems and for rough or perforated surfaces. Navier assumed that there is a static layer of fluid close to the boundary allowing a fluid to slip, and the velocity of this slip (tangential velocity) is proportional to the normal vector field of the stress tensor:

$$(u - u_b) \cdot \tau + (\mathbb{S}n) \cdot \tau = 0.$$

Equation for Fluid's Height

A further Kinematic condition can be added in free surface problems. The layer of the fluid is considered to be advected with the velocity field $\overrightarrow{u}$, and this results in an evolutionary equation for the height h of the fluid: Let O be a point on $\partial\Omega$ of velocity $\overrightarrow{u}(h) = (u_1(h), u_2(h), u_3(h))$; its coordinates can thus be parametrized as $(x, y, z = h(x, y, t))$. The free surface and the boundary advection assumptions state that

$$\frac{dz}{dt} = u_3(h) \implies \frac{d}{dt}h(x, y, t) = \partial_t h + \overrightarrow{u}_{(x,y)}(h) \cdot \nabla_{(x,y)} h = u_3(h),$$

and thus we get the kinematic condition for the height h

$$\partial_t h + u_1(h)\, \partial_x h + u_2(h)\, \partial_y h = u_3(h).$$

2-Dynamic boundary conditions

The conserved properties associated with the transport fluxes yield boundary conditions which guarantee as well the continuity for example of the stress tensor and pressure.

Surface Tension

Surface tension can be observed in many phenomena, for example in the formation of air bubbles in liquid in the form of spheres or in the spreading of liquid droplets on a wetting surface. It is of molecular origin: We already know that an intermolecular interaction occurs between the particles of the fluid. Each particle is subjected to such pressure from the surrounding particles which sums up to a zero net force. However, for those in the proximity of the surface, they encounter different interaction forces from the liquid molecules compared to the forces from molecules on the other side phase particles. In particular, the surface liquid molecules exhibit a stronger attractive force (cohesion) to the inner molecules in its same medium than to the molecules on the other side of the interface. Thus, the net force is not zero, and an inward cohesive force acts on those surface particles or molecules making them contract to the inside to attain a minimum area and thus a minimum energy. This makes the interface or the boundary seems more like an elastic stretched membrane contracting into the inward of the fluid. The most interesting surface tension to our concern is that at the liquid-air interface for problems known as free surface problems. Surface tension is very important in capillarity problems and lubrication applications. Its effects appear in the continuity condition at the boundary of the extra stress tensor $\mathbb{S} - p\,\mathrm{Id}$, with h being the height of the fluid and κ being the curvature of the interface, the condition thus reads in its general formality

$$(\mathbb{S} - p\,\mathrm{Id}) \cdot \vec{\boldsymbol{n}} = (\kappa\,\mathrm{div}\,\vec{\boldsymbol{n}})\vec{\boldsymbol{n}},$$

where $\vec{\boldsymbol{n}}$ is the normal to the boundary given in terms of the gradient of the fluid's height.

1.8 Modelling : Scaling and Important Parameters

Once the full NSF system is obtained, scientists target two main questions: (1) Well posedness of the system or (2) singular limit problems in some regimes (i.e what does the field X looks or behaves like in some extreme regimes). The content in here is more concerned with the second question at a formal and theoretical basis, and investigating both starts from scaling.

From the application point of view, the full generality of Navier-Stokes-Fourier system conveys a descriptive glitch, as it describes a huge spectrum of possible motions including industrial applications of laminar and turbulent flows, motions of stars in astrophysics, climatic movements, nano-scaled biological flows and huge bulk flows of fluids as avalanches, lava, and tide waves. Even for theoretical purposes, the generality of the system couldn't help in finding analytical solutions for most of the problems even in the weak senses. In order to get a specific characterization and target individual problems precisely, the need to model the NSF system started to get increasingly pressing especially for engineering purposes. The basic intuitive attempts of simplification gave rise to incompressible NS systems. As we know all fluids are of compressible type, but since most of fluids, especially liquids, are of negligible compressibility behavior, we tend to reduce this property from the equations. And as tremendously explained in lots of books on NS systems, this is equivalent to say that the total time variation of the volume of the fluid is zero, which is expressed as having $\operatorname{div} u = 0$, which simplifies some expressions from conservation equations. Still these are very basic assumptions that one can postulate in order to carry on a simpler study.

Modeling is manifested by building in the essential balance of flow fields and canceling the insignificant and undesirable forces that probably would have negligible impact on the motion. This is done in two levels. The first one is scaling which brings out some important feature parameters to show up explicitly in the equation, and the second level is adapting an asymptotic (or perturbation) analysis tool compatible to having such forces with negligible contribution. This allows one to predict the unwanted and unnecessary terms of the equation. The solution is then expanded in terms of such significant parameters around an average mean or some explicit solution to be determined according to the case of study. Of course, such method leads to several order truncated systems according to the level of accuracy required. In this sequel, the process of how such negligible quantities are filtered off is often more important than the truncated (limit) problem itself. This analysis provides a useful tool to derive a simplified set of equations out of a general physical system which potentially achieves deeper comprehension of the problem and captures the adequate physical mechanism involved. More precisely, two consequences that happens to be motivational goals as well to adopt such analysis is that the obtained model constitute a framework and source base to identify the different flow regimes in which relevant physical effects elaborate and on which predictions and analysis tools can be applied, in addition to the ability to investigate solutions at a full scale with less analytical and numerical efforts, and both causes are unlikely to exist for the full NS system.

Scaling

Scaling analysis is the core of modeling dynamical systems and is the first attempt in such process. It is accepted as an intuitive in great amount of work. However, this intuition fails in complex problems especially nonlinear ones, and a systemic scaling method is needed to target the problems and track the correct behavior. The flow regimes and the geometric configurations can range from very small magnitudes as micro-units or even smaller to tens of thousands of kilo units. The time scale as well can exhibit a significant range that could be parts of seconds in turbulence problems and millions of years in climatic studies. As for the momentum equation for instance, it is well known that it is a balance between different forces including dissipation forces, inertia, pressure, etc. Accordingly, many applications have some forces that could be more dominant than others. For lubrication theory for example, as will be discussed later, inertia forces are often of no importance that they are neglected within the theory, and so on.

Here lies the role of scaling in shrinking the system into a one which only includes those factors having an impact on motion.

Scaling has three aspects: geometric similarity, kinetic similarity and dynamic similarity. An intrinsic property of scaling is that for two problems of different geometric or kinetic ranges, but of same prominent dynamic variation, scaling reduces both problems to one single problem. The non-dimentionalizing process applies by dividing each variable by a characteristic value or a combination of characteristic values. The geometric similarity gives a specified ratio of any length in the model to that in the actual configuration. Thus, the model resembles a 3D contracted or stretched image by the scale factors. First, we start by specifying the characteristic values of the fundamental dimensions in the system on which other variables are a combination of such dimensions. In the sequel, the fundamental dimensions are: length (coordinates), time, mass/density and temperature. We introduce for each variable X its characteristic value X_{ref}, and we introduce the new non dimensional variable $X' = \frac{X}{X_{ref}}$. To be more precise, we set characteristic values for each of those latter defined respectively by: L_{ref}, T_{ref}, ρ_{ref} and θ_{ref}, and then we define

$$x' = \frac{x}{L_{ref}}, \quad y' = \frac{y}{L_{ref}}, \quad z' = \frac{z}{H_{ref}}, \quad t' = \frac{t}{T_{ref}} \quad \text{and} \quad \theta' = \frac{\theta}{\theta_{ref}}.$$

In fact, the above similarity is applied to get laboratory flow setups to simulate the larger dimensional flow regimes. However, a kinetic and dynamic similarity must be fitted to be congruent to the real case problem. Indeed, two flows are said to be similar when the ratios of their velocities and accelerations are constant. In our case of study, we choose the kinetic characteristic value to be equal for example to that in a stationary or Nusselt flow. Thus, the above scaling should be followed by non-dimentionalizing the full set of terms in the equations by determining their characteristic values that depend on the above fundamental characteristic values. We choose reference characteristic values for each of the velocity U_{ref}, pressure p_{ref}, viscous stress tensor $\mathbb{S}_{ref}$ in addition to the other dynamic values present in the system (heat conductivity, capillarity, viscosity, etc). In the full generality of the source forces and tensors presented in system 1.13, it is almost impossible to present a general scaled model where all the parameters appear in the system. This in fact doesn't even line up with the concept of scaling from moving from a general description to a more fitting one.

After replacing the non-dimensional variables in the set of equations, different dimensionless coefficients appear infront of the terms, and here one can choose wisely one of the terms to divide the equation by its coefficient so that it becomes a factor of unity term. The resulting coefficients are called parameters of the system, and they are values which determine the effectiveness of the terms infront of them by comparing their magnitudes to unity.

The main common coefficients appearing as a result in most of the systems are the aspect ratio, Reynold number, Mach number and Froude number. We will give a brief definition of each of such numbers and shed light on their significance and scopes of applications. In what follows, we will discuss two methods for which simplified models where obtained, and therein, we will highlight the choice of scaling as well as the modeling process.

Aspect Ratio

The aspect ratio is a characteristic of problems where thin layers of fluid are considered to flow in long distances. Thus the aspect ratio, defined by the ratio of width compared to length, is supposed to be very small:

$$\varepsilon = \frac{H}{L} <<< 1.$$

This aspect ratio appears in both industrial applications and natural phenomena, such as blood flows in arteries, tear suspensions, nano-scale technological applications, pipe flows, etc. In the

sequel, we will consider velocities with negligible normal components; we define $U = U_0(1, 1, \varepsilon)$, and thus the scaling of the velocity is given by

$$(u_1', u_2', u_3') = (U_0 u_1, U_0 u_2, \varepsilon U_0 u_3).$$

Reynolds number

Reynolds number is a parameter that gives the ratio of inertial forces to viscous forces. Thus, a low Reynolds number shows high viscosity and thus a laminar type of flow, whereas once the Reynolds number becomes critically big, turbulence appears in the flow. This number is of high importance in many applications especially boundary layer problems which will be discussed in the following subsection, such as the passage of the air over the aircraft wing. The Reynolds number is given by

$$Re = \frac{\rho_{ref} U_{ref} L_{ref}}{\mu_{ref}}.$$

However, especially in non Newtonian fluids, another representation can be considered which is given by

$$Re = \varepsilon \frac{\rho_{ref} U_{ref} L_{ref}}{\mu_{ref}} = \frac{\rho_{ref} U_{ref} H_{ref}}{\mu_{ref}},$$

which certainly accounts for the strong impact of the viscosity in such fluids.

Mach number

It is a dimensionless parameter representing the ratio of fluid's speed to that of sound. According to this number, we would have two main kinds of flows depending on their speed: supersonic flows (Ma >1) and subsonic flows (Ma <1). The mach number is considered as a measure of the compressibility characteristic of a fluid, and thus it is crucial in singular problems of transition from compressible to incompressible patterns. It finds its applications mainly in aeronautics and acoustic studies where the speed of sound dominates the characteristic speed of the fluid. The Mach number is given by

$$Ma = \frac{U_{ref}}{c_0},$$

where c_0 is the speed of sound.

Froude number

Last but not least is the Froude number. It gives the ratio of the inertial forces to the external forces. It is defined by

$$Fr = \frac{U_{ref}}{\sqrt{gH}}.$$

The Froude number is a critical parameter in problems involving wave making resistance (motion of boats and objects in water), as well as for free surface problems as shallow water waves and tidal waves.

2

A Scope on the Thesis

2.1 Brief Historical Review on Modeling Gravity Driven Films

A Need To Simplify

Two main theories on which the work in this thesis relied are: Lubrication and Shallow Water theories. We will be concerned with **gravity driven flows** down inclined planes, or specifically thin film flows. This latter is a rich phenomenology that provides an adequate platform to initiate a study on nonlinear wave formations and transition to turbulence. Its worth mentioning that this class of flows bisects into two main sub fields, the first of which refers to flows with negligible inertial contributions, and the second is characterized by significant inertial impact. The former class initiates in flows exhibiting moving contact lines at low speeds which are characterized by instability to transverse perturbations encoded in fingers and rivulets formation. Wetting phenomenon is crucial at this level as well. Stability of such flows is coherently related to capillary stabilization and inclination angle. Lubrication equations of such cases are characterized by high order spatial derivatives that adds difficulty to the analytical and numerical resolution. Such case will not be our concern from the modeling point of view, however, we will show in Chapter 4 later the existence of a limit solution approximated from that of SW system for some type of lubrication equations arising from negligible inertial contributions.

On the other hand, cases where inertial terms are of great significance are basis for our study. The analysis of gravity driven flows is based on perturbation from stationary solutions. Stationary uniform parallel flows possess a trivial solution, called Nusselt solution, which is characterized by a parabolic velocity profile and a constant height. Once inclined surfaces are concerned, the smallness assumption of the flow along with steep surfaces constitute a geometric ground for instabilities. From the observational perspective, natural waves tend to move in the downstream directions once a perturbation at the inlet is produced. At the time wave amplitudes start to grow, two type of waves appear deviating from the sinusoidal shape of the natural ongoing waves: the first initiates from high frequency disturbances and forms into short wavelength waves with wide peaks leading to span wise modulations, while the second one is due to low frequency disturbances and forms into solitary structured and narrow peaked waves that are at considerably far distances apart. Those are usually preceded by capillary waves of small amplitudes called ripples. Such destabilized patterns should be avoided in many applications which require the conservancy of a uniform flow thickness. Navier Stokes system in this context doesn't provide a zoomed vision on the special force factors that would lead to instabilities, even numerical studies fail to focus on such problem from the direct treatment of the NS system as this latter conveys all the possible patterns in the flow together with no preference for the dominating patterns. Theoretical and experimental studies insure that the formation of instability in the film is a sign for the significance of inertial terms. Besides, according to analysis of Orr-Sommerfeld equation which gives the hydrostatic conditions for stability flows, it was confirmed that for long wavelength perturbations from the Nusselt solution (low flow rates), interfacial instabilities appear, and viscosity and surface tension forces become crucial as

well. Therefore, a rigorous interplay between viscous, pressure and inertial forces is necessary to initiate a realistic and correct simplification technique on the equations of motion.

Boundary Layer Theory

The first theoretical approach to fully understand such problem is to fit the study in the frame work of boundary layer theory along with the smallness assumption on the stream wise gradient of the velocity compared to the cross wise variations. However, this latter fails like the NS system in incorporating the dominating relevant physical features and focusing on transition patterns that develop non linear waves.

The boundary-layer theory is considered an intermediate level study among lubrication and shallow water theories. It is an asymptotic theory of the Navier-Stokes equations for high Reynolds numbers that still finds plenty of applicability in industry and research due to the fact that many important sub fields of fluid mechanics- as aeronautics, ship hydrodynamics, automobile aerodynamics- refer to flows at high Reynolds numbers. This theory is of critically paramount significance that one must highlight on it. In the beginning of the twentieth century, classical hydrodynamic schools failed to comprehend many phenomena rising in industry and engineering as for calculating the drag forces of bodies dragged by a fluid and interpreting the pressure loss in channels and tubes. Concurrently, the full Navier Stokes equations were known by that time, however the analysis of a rigorous solution was not presented except for very simplified cases including ideal or inviscid fluids. Such fluids don't experience a shear or a viscosity effect between there layers, and most importantly, the "no-slip" boundary condition is absent in such fluids. Where it was predicted that the failure of the theory and its discrepancy from experiments is attributed merely to neglecting viscous effects, especially neglecting the "no-slip boundary condition" (which does hold good even for very low viscosity fluids), yet, no treatment was suggested until the work of Ludwig Prandtl. It is good to mention that 100 years before, Laplace [10] elaborated on surface tension related boundary layer analysis that he is considered in many literature as the inventor of boundary layer theory. He was concerned with the study of a large drop of mercury placed on a glass plate. The shape of the drop was manifested by a partial differential equation. Of course, surface tension forces are crucial here in determining the shape of the edge. However, Laplace suggested that if the volume of the drop is large enough, surface tension effects are only crucial in a thin boundary layer of the edge, whereas in the rest inner part of the fluid these effects diminish. The boundary layer theory was fully elaborated by Prandtl in 1904 and was investigated thoroughly in the last decades by lots of mathematicians. His theory is based on a perturbation method in which he perturbed the limit solution of the inviscid fluid (high Reynolds limit) by viscosity effects. At high Reynolds number, the flow is divided into two parts: a very thin layer adjoining the body where viscous effects are of high stakes and a layer consisting of the remaining inner fluid where one can neglect viscosity effects. The full solution of the flow is then determined using the method of matched asymptotic expansions. This theory is considered one of the first theories to introduce the concept of perturbation as an approximating method to solve Navier-Stokes system. For reviews on this topic the reader may refer to [11], [12] and references therein.

Emergence to Lubrication Theory

Lubrication in fact is a part of a science and technology branch which studies close interacting surfaces including friction, lubrication and wear. For an excellent review the reader is referred to [13] and [14]. The mathematical modeling governed by lubrication theory provides vast amount and forms of one equation models which fall under the category of **Reynolds equations**.

Since the early 19th century, mathematicians and physicists started to show a huge interest in the phenomenon of fluids known as lubricants; flowing in narrow spaces, starting with G.A Hirn 1847, to R.H Thurston 1879 and N.P Petrov 1883. They studied the relations between friction and viscosity from one side and velocity variations from the other side. The remarkable discovery of that era was that of B. Tower in 1885 who proved experimentally the existence of hydrodynamic pressure in the film flow. His finding constitutes an experimental confirmation for the initiation of lubrication theory that started by Reynolds in 1886 when he deduced in his celebrated work [15] the famous equation known by his name for thin viscous films. Based on an asymptotic approach in terms of the film parameter or the aspect ratio ε, followed by averaging of the equations, Reynolds obtained a scalar evolutionary equation-starting from the incompressible Stokes system (and then Navier Stokes system)-describing the distribution of the pressure in the film. The Reynolds equation reads

$$\partial_t h + \partial_x(h^3\,\partial_x p) + \partial_y(h^3\,\partial_y p) = 6\mu(u_0 - u_1)\,\partial_x h + 6\mu h\,\partial_x(u_0 + u_1),$$

where h is the film's height, p is the hydrostatic pressure, μ is the viscosity and u_0 and u_1 denote respectively the first velocity component at the bottom and top of the film. We remark here that such equation is derived using the kinematic equation on the height, thus, boundary conditions manifest a clear importance not only from physical point of view but also in the derivation process. This is one of the reasons why lubrication theory conveys such various classes of one-equation models that are not alike from the structure point of view and the physical background. The other reasons sway between the adopted constitutional law and the scaling technique of the NS system, i.e to say the range of Reynolds number adopted in the light of the viscosity relation with other variables. As the physical background and correspondingly the boundary features matter, some various phenomena responsible for such lubrication equations' variety include gravity, surface tension, inclined planes, drop formation, wetting, Van Der Waals attractions, thermo-capillarity, evaporation, condensation, additional curvature due to curved solid interfaces, Hele Shaw flow (flow between near closed plates), etc. Out of these, we are most concerned with gravity driven flows of thin films down inclined planes.

The first cornerstone in such theory dates back to the pioneering work of Kapitza [16], upon which, based on both experimental and theoretical studies at that time especially the observations of Friedman and Miller [17] and Kirkbride [18], he proved that viscous thin film flows with free surfaces are more stable than laminar flows due to the propagation of waves on the surface: " laminar flows are obeyed only as an average, and refers only to the mean thickness of the layer, whereas the character of the flow differs from that of a simple laminar flow". According to Kapitza, it is the capillarity force the one responsible for the stability of undulatory flows for viscous thin films. Under the aspect ratio assumption as well as taking into account the threshold flow with respect to critical ranges of Reynolds number accounting for the transition between laminar and turbulent patterns, he was successfully able to attain a parabolic velocity profile at main order, a velocity phase and a wave amplitude under the assumption that the wavelength of the flows are much larger than the fluid's height or thickness. He also obtained a set of equations of first and second order in terms of a stream line function congruent to the height of the fluid.

Kapitza's work paved the way for the theory of one equation models of thin films enslaving the local variables of the flow to the fluctuations on the surface. Such equation is known as Lubrication equation; an evolutionary equation of the height of the flow in the form of

$$\partial_t h = G(h^n,\, \partial_x^m h),$$

where G involves various classes of algebraic forms in terms of powers of the height (n) and differentiation orders (m). The first known theory of lubrication equation which accounts for inertia, hydrostatic pressure, capillary and viscous effects including nonlinear regimes dates back to Benny in 1966 [19] who benefited from the long wave nature of the primary instability and

considered that viscous diffusion across the film is quicker than time and space evolution of the film. The perturbation method he followed was based on a film parameter $\varepsilon \sim \partial_{x,t}$ and showed that the dynamics of the film at lowest order is enslaved to its kinematics. In fact, what is known now as Benny equation is given by

$$\partial_t h + 2h^2\,\partial_x h + \varepsilon\,\partial_x\left[(\frac{16}{15}Reh^6 - \frac{2}{3}\cot\theta h^3)\,\partial_x h + \frac{1}{3}\kappa\varepsilon^2 h^3\,\partial_x^3 h\right] = 0.$$

The fact that inertial terms are included at an order εRe in the above equation makes them critical for moderate and large Reynolds numbers. This causes instability in Benny's equation leading to finite time blow up of the solution. The linear stability analysis of Benny's equation contradicts those of linearized NS equations (Kapitza and others), Sommerfeld equation or even boundary layer approximation. Many scientists tried to extend and correct the work after Benny by including higher order gradient expansions or regularization techniques [20], [21], [22] and [23], however, this theory still misses something at each time.

Satisfactory Results Via Shallow Water Models

As stated before, at critical dynamical regimes (critical steepness or Reynolds number threshold), one-equation lubrication models fail to describe the full and accurate dynamics of waves in thin films. One reason is due to restraining the internal variables (velocity and pressure) to the film thickness. This issue is relaxed in the 2-equation models, as those of shallow water type, as the velocity is treated as an independent variable. There are several main approaches in literature to model shallow water equations depending on whether we are considering inviscid fluids or viscous ones.

Inviscid Shallow Water Model

Initially, it was introduced by Saint Venant in 1871 to derive a simple 1D model for water flows [24] from Euler's system (inviscid incompressible NS systems). Saint Venant system was similar in structure to a compressible system with a pressure power law of exponent two. Another good model in this context is that derived by Zakharov-Craig-Sulem [25]. They wrote the system in terms of the fluid's height and the velocity potential defined by $\overrightarrow{u} = \nabla\varphi$ which reduces the whole system into a Bernoulli type equation. The compatible assumptions on the flow (boundary condition, irrotationality, incompressibility) lead to the full determination of the trace of φ on the boundary. Hence, the problem is reduced to a 2-scalar evolutionary equations describing the height of the fluid and the trace of the velocity potential. The well posedness of such problem was discussed in [26] and the stability was studied first by the authors themselves in [25]. For a good review on such models and applications, we refer to [27], [28] and [29].

Viscous Shallow Water Model

Such model is attained by the **method of weighted residuals** including in the simplest form the **integral method MIM** (depth averaging the momentum equation) as well as the collocation and **Galerkin methods** (mostly known as the weighted residual method **WRM**) [30]. The first viscous shallow water model was deduced by Shkadov in [31] using the MIM method. His work in fact is an extension of the theory of lubrication in order to correct the glitch that exists in the latter theory. He obtained a corrected velocity profile at main order in terms of the averaged depth velocity as well as a 2 evolutionary equation model of the fluid's height h and

the total discharge rate $q = \int_0^h u\, dz$. His model reads

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q = h - 3\frac{q}{h^2} - \frac{12}{5}\frac{q}{h}\,\partial_x q + (\frac{6}{5}\frac{q^2}{h^2} - \cot\theta h)\,\partial_x h + \kappa h\,\partial_{xxx} h. \end{cases} \tag{2.1}$$

where θ is the angle of inclination of the fluid from the horizontal and κ is the surface tension.

Integral method has also enjoyed a lot of interest in the past couple of years. Two main approaches that drove attention in the literature and gave satisfactory results are those derived by J.F Gerbeau and B. Perthame [32] and that by J.P Vila in [33] that are both revisited and justified theoretically by Bresch and Noble in [1] and [5]. For films with general topography we also mention [34]. The main difference between the two models in [32] and [33] is in the choice of scaling which was reflected on the choice of the boundary condition assumed. Where the former chose a Navier-type boundary condition (friction condition on the bottom), a no-slip boundary condition was presumed by the latter. From the modeling perspective, we will be more interested in scaling and boundary conditions investigated in [1]. It reads for the height h and the averaged depth velocity $U = \frac{q}{h} = \frac{1}{h}\int_0^h u\, dz$

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x(\frac{6}{5}hU^2 + \frac{\cos\theta}{Fr^2}h^2 - (\frac{\sin\theta Re}{Fr^2})^2\frac{h^5}{75}) - \kappa h\,\partial_x^3 h = \frac{1}{\varepsilon Re}(\frac{\sin\theta Re}{Fr^2}h - \frac{3U}{h}). \end{cases} \tag{2.2}$$

In such method, the asymptotic expansion near the Nusselt solution was used to calculate the main and first order velocity profile in the long wave expansion. The first order was used to close the system by averaging gravitational and viscous terms, and finally a consistent model at first order is obtained. However, such models even with higher order terms (viscous terms) couldn't provide nonlinear stability in critical regimes of Reynolds number. The derivation of such model will be reviewed in Chapter 3. For more details the reader can refer to their papers and to some lecture notes discussing different shallow water models in [29] and [35].

In 1998, C. Ruyer Quil and P. Manneville derived a new shallow water type model using mixed integral-collocation method. Their model is a **3 equation model**; the first two are of shallow water type in terms of the height h and the flow discharge rate q, and the third is a coupled equation for a supplementary variable measuring the departure of the wall shear stress from that imposed by the velocity profile. The velocity field was approximated by an expansion of polynomial test functions appearing in the derivation of Benny's equation. Though this model gives satisfactory numerical and experimental results for the instability threshold and the wave phase at moderate distance from threshold, finite time blow up far from the threshold still occurs. In 2000, the same authors developed another shallow water type model in [2] to get rid of the previous glitches depending on a **weighted residual method** in which they imposed a velocity profile as an expansion in terms of polynomial functions and then the weights are taken identical to the polynomials. They expanded the method so that finally they derived a 4-equation type shallow water model for the height, discharge rate q, and two other variables measuring the departure from the flat-film semi-parabolic velocity profile. Such model has been simplified into a 2-equation model in terms of the height and the discharge rate For this case, the imposed velocity profile reads $u = 3Ug_0 + \varepsilon\tilde{u}$ where $g_0 = \frac{z}{h} - \frac{z}{2h^2}$ which is the polynomial appearing at the main order Nusselt solution, and $\tilde{u} = u - 3Ug_0$ is the correction. The system then reads at first order

$$\begin{aligned} &\partial_t h + \partial_x q = 0, \\ &\partial_t q = \frac{1}{\varepsilon Re}(\frac{5}{6}\frac{Re\sin\theta}{Fr^2}h - \frac{5}{2}\frac{q}{h^2}) + \frac{9}{7}\frac{q^2}{h^2}\partial_x h - \frac{17}{7}\frac{q}{h}\partial_x q - \frac{5}{6}\frac{\cos\theta Re}{Fr^2}h\partial_x h + \frac{5}{6}\kappa h\partial_x^3 h \\ &\quad + \frac{\varepsilon}{Re}(4\frac{q}{h^2}(\partial_x h)^2 - \frac{9}{2h}\,\partial_x q\,\partial_x h - 6\frac{q}{h}\,\partial_{xx} h + \frac{9}{2}\,\partial_{xx} q). \end{aligned} \tag{2.3}$$

The first thing noticed in the above model is that it loses a conservative structure on the contrary to system (2.2). This is the main drawback of the technique which makes it lose many robust mathematical and numerical results. However, the model has given a quite satisfactory numerical and experimental results in the nonlinear regimes regarding traveling-wave and solitary-wave solutions, and the corresponding 4-equation model is considered one of the best models to describe the flow in critical regimes.

Recently in 2016, Richard and others derived a new set of 3 equation shallow water type model [3]. The model describing the evolution of film thickness, the discharge rate q and the transport behavior of the **enstrophy** φ related to the averaged velocity variance is derived using integral method with a coupling with the energy equation (the derivation is being reviewed in Chapter 3). In fact they defined φ as

$$\int_0^h (u-U)^2 \, dz = h^3\varphi,$$

and the system they obtained reads

$$\begin{cases} \partial_t h + \partial_x(hU) = 0 \\ \partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{2Fr^2}h^2) = \frac{1}{\varepsilon Re}(1+f_1)(\lambda h - \frac{3U}{h}) + \frac{f_2}{\varepsilon Re}(\varphi - \frac{\lambda^2 h^2}{45}) \\ \qquad\qquad + \frac{\kappa}{Fr^2} h\,\partial_x^3 h + \frac{\varepsilon}{Re}\,\partial_x(\frac{109}{32}h\,\partial_x U); \\ \frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) = -\frac{U}{\varepsilon Re}\frac{2f_1}{3}(\lambda h - \frac{3U}{h}) - \frac{U}{\varepsilon Re}\frac{2f_2}{3}(\varphi - \frac{\lambda^2 h^2}{45}) \\ \qquad\qquad + \frac{\varepsilon}{Re}\frac{109}{32}h(\,\partial_x U)^2 + \frac{\varepsilon}{Re}\,\partial_x(\frac{175}{48}h^3\,\partial_x\varphi). \end{cases} \tag{2.4}$$

where f_1 and f_2 are parameters arising from the choice of the asymptotic expansion. The pros of such model is in its mathematical structure that is governed by well known results of both well posedness and robust numerical schemes. Such structure encodes the three physical equations: mass, momentum and energy conservation, and it displays all the physical features through relaxation source terms, classical disperse capillary term and viscous diffusion terms. One drawback is in using the asymptotic technique to manually interchange between inertial, relaxation and viscous effects, in addition to the fact that the derivation does not involve the notion of profile in the sense that the system is not closed using assumptions on the stream wise velocity field but rather using its asymptotic expansions. It is good to remark that in the context of extended shallow water models, a 3 equation model was derived in [36] for a viscous incompressible fluid inspired by the interaction of viscous and inviscid layers in the boundary layer theory. The resulting model is formulated in terms of the depth of the fluid, depth averaged velocity and a third variable that describes the evolution of the viscous layer.

The concentration in modeling liquid films has been focused on Newtonian fluids, and it just till the recent decades that more interest in modeling shallow water systems for non-Newtonian fluids started to arise. We mention in this context the papers [37], [38], [39] and [40] where reduced systems are derived for thin films of generalized Newtonian rheology (power law), Oldroyd and Bingham Rheology, and for which we remark that the systems derived by E. Fernandez-Nieto, P. Noble and J.–P. Vila in [40] exhibit a consistent theory that has come to derive fully understandable shallow water models for both power law and Bingham fluids correcting by this all the lapses done at the level of shallow water derivation for previous work as lapses that include neglecting first order velocity profile which is necessary at the level of shallow water approximation, or neglecting corrective terms in the viscous region or in pseud-plug regions. The Bingham shallow-water model by the latter authors in [40] reads

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x(\Lambda_1 h\bar{u}^2 + \frac{\cos\theta}{2Fr^2}h^2 + \Lambda_2) = \frac{1}{\varepsilon Re}(\lambda h^* - \frac{h\bar{u}}{\Gamma_1}) + R. \end{cases} \tag{2.5}$$

where $\lambda = \frac{Re\sin\theta}{Fr^2}$. The coefficients Λ_1, Λ_2 and Γ_1 are given by

$$\begin{aligned}
\Lambda_1 =& \frac{h}{(h+(B/2\lambda))^2}\Big(\frac{6}{5}h + (\frac{9}{4}-\frac{6}{5})\frac{B}{\lambda}\Big),\\
\lambda_2 =& \Big[-\frac{\cos\theta}{Fr^2}(\frac{h_c^2}{2}-\frac{h_c h^*}{2}) + \frac{3}{4}\frac{\cos\theta}{Fr^2}h_c^2\log(\frac{h^*}{3}+\frac{h_c}{2})\\
&-\Lambda^2(\frac{3}{40}h^{*2}h_c^3 - \frac{9}{40}h_c^4h^* + \frac{27}{80}h_c^5\log(\frac{h^*}{3}+\frac{h_c}{2})) - \lambda^2(\frac{h^{*5}}{75}+\frac{h_c h^{*4}}{10}+\frac{3}{10}h_c^2h^{*3})\Big],\\
\Gamma_1 =& ((h^*)^2/3) + (Bh^*/2\lambda),
\end{aligned}$$

with B being the rescaled Bingham number, $h^* = h - \frac{B}{\sqrt{2}|\lambda|}$ (h^* being the height of the yielded part, see the scheme 2.2), $h_c = h - h^*$, and finally R is a corrective term arising from the normal stress appearing in the first order velocity expansion

$$R = \frac{\pi B^2|\,\partial_x h|}{2Re\lambda(\frac{h^*}{3}+\frac{h_c}{2})}.$$

Attempts to modify the different obtained shallow water models have been carried out extensively to include larger ranges of flows and the various physical features; the thing that lead to understand the problem thoroughly and enriched the literature especially from the application perspective and lead to some results that are satisfactory from experimental and numerical points of view to some extent. Among all what have been reviewed, we note finally that the mentioned three methods concerning viscous films-including both Newtonian and non-Newtonian fluids-are the main methods to derive shallow water systems: Integral Momentum Method **MIM** for 2-equations models, Weighted Residual Method **WRM** for 2,3and 4-equations models, and the modified MIM to derive 3-equation model using the enstrophy. The three methods will be reviewed and elaborated on in Chapter 3.

2.2 Shallow Water Theory

2.2.1 Shallow Water Theory From the Modeling Point of View

This theory is concerned with flows characterized by very long wavelength compared to the depth of the fluid, i.e to say that the aspect ratio is a very small quantity. It finds its applications in different ranges from microns (layers of conductor applied in liquid form to a printed circuit board before being baked solid) through millimeters (paint on a wall, honey spilled on a table) to kilometers (magma flow from a volcano). It is applied for free surface problems where the shape of the domain is unknown and inhere, it is considered one of the variables of the system.

One of the main techniques in deriving such simplified models for fluid flows is by using residual methods to shrink the number of unknowns in the system. This is preceded by geometric simplifications and variables transformation. The geometric simplification is manifested through the smallness assumption on the aspect ratio due to shallowness which leads to neglect the variation of the velocity in the depth direction and thus reduces the dimension of the momentum equation. In this context, many approaches appeared where they differ in the variable transformation but are all compressible type systems referred to in the literature as "shallow water" models; or simply SW models.

Since they are formulated in terms of the height of the fluid, then these models rely strongly on what happens on the boundary of the domain. This dependence is not only related to the geometric configuration of the fluid ($h(x,y,t)$) but also on the different physical features that could occur on the boundary. These features get translated and encoded in the set of equa-

tions in SW models rather than being expressed in the boundary conditions of the NS system. Thus, even the number of equations gets simplified. Indeed, various physical aspects (whether on the boundary or operating on the whole fluid) could emerge causing natural altering and adjustment of the equations such as viscosity, capillarity, Coriolis effect, drag forces, electric and electromagnetic fields, etc. This makes such models convey a wide variety of phenomenological cases including pipe systems, tide waves and numerous engineering applications. SW models constitute an affordable theoretical and numerical case of study so that even in cases where the smallness assumption (small aspect ratio) does not apply, still there is an approach to use such models by dividing the fluid into "shallow water layers" coupled together through pressure.

It is good to remark that the smallness parameter which governs the derivation of SW equations is the same used in lubrication theory that will be discussed in next subsection. One of the slight differences between both theories lies in scaling assumptions depending on the application. In particular, it is a matter of balance between viscous, pressure and inertial forces. In applications where the flow is critical (moderate flow speed and of Froude number being an order of one), a perfect balance between pressure driven forces and inertial forces takes place, and this is translated differently via scaling in cases where the speed is considerably slow corresponding to very low Froude numbers (subcritical flows) where viscous forces in such cases contribute to a pressure, and thus a balance between viscous shear forces and hydrostatic pressure should take place regardless of inertial effects (which become negligible). The latter case is applied for lubrication theory especially for viscous thin films, whereas the former case perfectly suits applications for shallow water models. This is a prescription of the motions acting on the bulk domain and applied at scaling level for NS system, however, as both theories are more concerned with what happens at the free surface, it is good to remark that from a mathematical point of view, the balance between viscous and surface tension forces at the boundary is important for attaining compatibility and consistency of the resulting system. Moreover, some view the lubrication approximation for free surfaces as the main order truncation of the perturbation analysis, whereas the shallow water approximation is of first order. This is the perspective that we will adapt in the sequel.

2.2.2 A Word On The Well Posedness Of Shallow Water Models

Lets start by a brief review on the inviscid shallow water type system neglecting viscous effects, which clearly resembles Euler system. Such system reads mainly

$$\begin{cases} \partial_t \eta + \nabla \cdot (hU) = 0, \\ \partial_t U + \varepsilon U \cdot \nabla U + \nabla \eta = 0. \end{cases} \tag{2.6}$$

where $h = 1 + \varepsilon\eta$. The initial conditions are given by

$$\eta|_{t=0} = \eta_0 \quad \text{and} \quad U|_{t=0} = U_0.$$

Such problems fall in the category of hyperbolic conservation laws. The well posedness in the infinite domain is governed by the non vanishing depth condition

$$h \geqslant h_{min} > 0,$$

and the proof relies on the theory of Friedrich symmetrizable hyperbolic systems for time scale of order $O(\frac{1}{\varepsilon})$ and initial data in $H^s(\mathbb{R}^d)$ where $s > 1 + \frac{d}{2}$ [41], [42], [43]. Solutions exhibit shock formation at this time scale and add difficulty to the problem in higher dimensions. In case the non vanishing depth condition was relaxed, a strong global solution can be obtained as in [44] based on compensated compactness method for regularized systems and after converging to the limit solution. However, this is just in 1D, whereas in 2D strong solutions are difficult to treat due to shock formation, and only results for weak solutions are available yet with no uniqueness

result. The construction of weak-entropy solutions follows that for isentropic Euler equations for compressible gases with pressure law of exponent 1 (height replacing density) from inviscid limit of viscous generalization of the SW equations [45], [46], [47]. More complications arise if the topography is not flat so that h is expressed as $h = 1 + \varepsilon\eta - b$, and the equations are transformed into

$$\begin{cases} \partial_t \eta + \nabla \cdot (hU) = 0, \\ \partial_t (hU) + \varepsilon \nabla \cdot (hU \otimes U) + \frac{1}{\varepsilon} h \nabla (\eta + b) = 0. \end{cases} \tag{2.7}$$

A strong well-posedness was proved in [48] for the strongly non linear version of the above system. Furthermore, relaxed versions of Euler system are investigated in many literature, where relaxation serves in identifying with limit problems, see for instance [49], [50]. In the strong nonlinear case, the above model converges to a lake-type system, for which we have studied in [51] an existence result for a degenerate topography b, and this will be discussed in Section 7.

The lack in a complete theory triggered more concentration on bounded domains as more theoretical and numerical results are cast in bounded frames. Furthermore, such systems have been as well solved in the context of strong solution in bounded domains. Meanwhile, in recent decades, more interest have been directed towards the viscous shallow water models; or viscous compressible models which exhibit hyperbolic-parabolic structure. Following the derivation in [32], such models read

$$\begin{cases} \partial_t h + \nabla \cdot (hU) = 0, \\ \partial_t (hU) + \nabla \cdot (hU \otimes U) = -\frac{h}{Fr^2} \nabla (h - b) + \frac{1}{Re} \nabla \cdot (h\, \mathrm{D}(U)) - We h \nabla \Delta h + D, \end{cases} \tag{2.8}$$

where D denotes here drag terms. Though the usual system given by (2.8) has not enjoyed yet a rigorous justification of well posedness in its abstract formulation in higher dimensions, it is however solvable either in 1D or with a bit modification in the density/height dependence of viscosity in the initial system. In fact, it is well known that on the contrary to incompressible systems where the difficulty lies in the existence and regularity of pressure, especially for constant viscosities or inviscid cases, the compressible systems on the other hand exhibit difficulty in defining the velocity in vanishing density regions, and so the purpose was always to control the variables in vacuum. In infinite domains, high dimension global weak solution was first established by P-L. Lions in 1990 [52] in the spirit of *a Leray* solutions (d=2,3). The author studied the following compressible model of Navier type

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) = -\nabla p(\rho) - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u, \end{cases} \tag{2.9}$$

where λ and μ are constant viscosities, the pressure follows power law dependence on the density (baratropic) mainly $p(\rho) = \rho^\alpha$ and $\alpha \geqslant \frac{3}{2}$ if d=2 and $\alpha \geqslant \frac{9}{5}$ if d=3. The solution relied on estimates from the physical energy which by virtue of the viscous term give enough regularity on the velocity field, as well as estimates on the effective flux $p - (\lambda + 2\mu) \operatorname{div} u$ which allowed passing to the limit in the pressure term in the weak formulation. Such result was extended later in [53] to include wider ranges of power laws mainly $\alpha > \frac{d}{2}$. It is good to mention here that in both results from one hand Stokes condition on the viscosities was strongly assumed; $\lambda + \frac{2\mu}{d} > 0$, and on the other hand monotonic pressure law was necessary. These methods don't apply for degenerate viscosity cases as that in (2.8). Recently in [54] and [55], it has been possible to obtain a more general result concerning the pressure state by Bresch and Jabin covering thermodynamically unstable pressure laws and some anisotropy in the viscosities. The case of density dependent viscosities was first treated by Bresch and Desjardins in [56] by introducing a new class of mathematical entropies, called BD entropy, which provides additional control on the density leading to the strong convergence in the density field due to Aubin-Simon compactness.

Consider for instance the following compressible NS system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \nabla(\lambda(\rho)\operatorname{div} u\,\mathrm{Id}) - 2\operatorname{div}(\mu(\rho)\,\mathrm{D}(u)) = 0. \end{cases} \tag{2.10}$$

The BD entropy is formulated in terms of a drift velocity and adds a new constraint on the relation between the viscosities

$$\lambda(\rho) + 2\mu(\rho) = 2\rho\mu'(\rho). \tag{2.11}$$

For reader's convenience, a derivation of the BD entropy is reviewed in the Appendix of Chapter 6. It reads

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho|v|^2 + e_{in}(\rho))\,dx + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2\,dx + \int_\Omega \mu(\rho)|A(u)|^2\,dx = 0. \tag{2.12}$$

where v is the drift velocity $v = u + \frac{2\nabla\mu(\rho)}{\rho}$, $e'_{in} = \frac{p'(\rho)}{\rho}$, and $A(u)$ is the skew-symmetric part of the velocity gradient $A(u) = \frac{\nabla u - \nabla u^T}{2}$. It is good to remark that in 1D, Stokes relation is considered a specific case of relation (2.11), yet this fails for higher dimensions. Despite the lack of physical justification of the above relation, yet a physical-mathematical compatibility was demonstrated in [57]. The authors showed that near vacuum, flows characterized by constant viscosities do not converge to a physical solution as the density approaches vacuum regions, and only if viscosity vanishes with density that mathematical-physical compatibility of the solution exists. Such degeneracy is conveyed in the above relation giving credibility to adapt it. Moreover, system (2.10) is technically solved depending on a compensated compactness argument based on the BD entropy to get strong convergence of the density field and another entropy derived by Vasseur and Mellet to govern the convergence of the non linear term $\rho u \otimes \rho$ in the weak formulation. The reader is referred to [58] for the detailed proof. In the case $\lambda(\rho) = 0$, which is the case of the viscous shallow water model (2.8), the BD entropy is enough to pass to the limit, see for instance [59].

Remark 3. *More general shallow water models, namely in [32] and [60] contradict relation (2.11) mentioned above in d=2,3. Such models still lack a proof for existence of weak solutions, and they read*

$$\begin{cases} \partial_t h + \nabla\cdot(hU) = 0, \\ \partial_t(hU) + \nabla\cdot(hU\otimes U) = \frac{h}{Fr^2}\nabla(h-b) + \frac{2}{Re}\nabla\cdot(h\,\mathrm{D}(U)) - We\,h\nabla\Delta h + \frac{2}{Re}\nabla(h\operatorname{div} u). \end{cases} \tag{2.13}$$

As for the well posedness of problem (2.8) in bounded domains, several strong existence results have been proven: existence and stability of strong classical solutions for small initial data in [61] and [62] and local existence of the Cauchy problem for any initial datum in [63] and [64].

2.2.3 Contribution to the Thesis

2.2.3.1 Refined 2 and 3 Equation Models Using WRM

The work in this subsection is a collaboration with C. Ruyer-Quil, USMB, LOCIE, Chambéry.

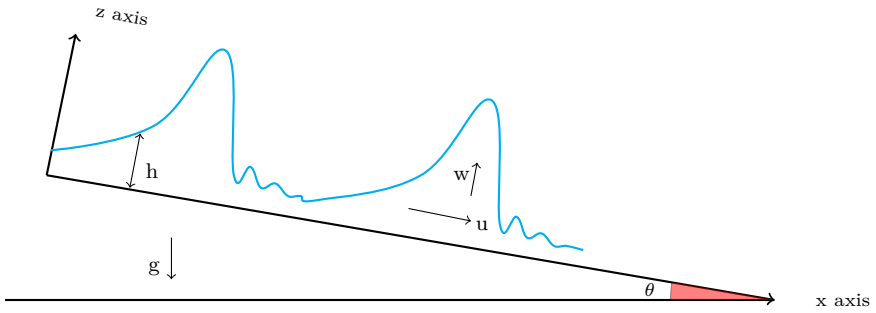


Figure 2.1: *thin liquid film moving down an inclined plane in 2D*

In this part, we will consider a free surface 2D flow of a Newtonian fluid with constant density $\rho = 1$ and constant viscosity μ where the motion is driven by gravity along the x-axis (stream wise direction) chosen such that it is inclined by an angle θ with the horizontal, and the z-axis denotes the direction perpendicular to the x-axis. The velocity field is denoted (u, w), p is the hydrostatic pressure, and $g\vec{e} = g(\sin\theta, -\cos\theta)$ is the gravitational acceleration. The non dimensional incompressible Navier Stokes system with free surface boundary conditions is given by

$$\begin{aligned} \partial_x u + \partial_z w &= 0, \\ \partial_t u + u\,\partial_x u + w\,\partial_z u &= \frac{\sin\theta}{\varepsilon Fr^2} - \frac{1}{Fr^2}\,\partial_x p + \frac{1}{\varepsilon Re}(\varepsilon^2\partial_{xx}u + \partial_{zz}u), \\ \partial_t w + u\,\partial_x w + w\,\partial_z w &= -\frac{\cos\theta}{\varepsilon^2 Fr^2} - \frac{1}{\varepsilon^2 Fr^2}\,\partial_z p + \frac{1}{\varepsilon Re}(\varepsilon^2\partial_{xx}w + \partial_{zz}w). \end{aligned} \tag{2.14}$$

The parameters ε, Re and Fr^2 denote respectively the aspect ratio, Reynolds number and Froude number. A no slip condition and the kinematic condition read

$$\begin{aligned} u = w = 0 \qquad & z = 0, \\ \partial_t h + u\,\partial_x h = w \qquad & z = h(x,t). \end{aligned} \tag{2.15}$$

The fluid is submitted to surface tension forces which yield the following boundary conditions

$$\begin{aligned} p = -\kappa\frac{\partial_{xx}h}{(1+\varepsilon^2(\partial_x h)^2)^{\frac{3}{2}}} - 2\varepsilon\frac{Fr^2}{Re}\frac{1+\varepsilon^2(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\,\partial_x u \qquad & z = h(x,t), \\ (1-\varepsilon^2 h_x^2)(\partial_z u + \varepsilon^2\partial_x w) - 4\varepsilon^2 h_x\partial_x u = 0 \qquad & z = h(x,t). \end{aligned} \tag{2.16}$$

where κ is the rescaled surface tension coefficient. And not to forget the kinematic condition

$$\partial_t h + \partial_x hu|_{z=h} = w|_{z=h}.$$

1- Generalized Weighted Residual Method

Inspired by the weighted residual method, we tried to find a generalized method which would enjoy more flexibility provided by projection and delivers a general shallow water model at main order for Newtonian fluids with constant viscosities. Indeed, such result would be very important to optimize the coefficients in the system by stressing on the weight instead on stressing on the technique or the assumptions. In our method, the chosen weight plays a role in the form of the

model obtained. So choosing a general polynomial to represent the weight, and applying the weighted residual method to system (2.14), we obtain a general model which serves to represent a general form of the Newtonian shallow water models. We chose a second order polynomial

$$f_0(x,z,t) = a\frac{z^2}{h^2} + b\frac{z}{h} + c.$$

The final general system reads

$$\begin{aligned}
&\partial_t h + \partial_x q = 0,\\
&\alpha\,\partial_t q + \partial_x\Big((\alpha + \frac{\beta}{6})\frac{q^2}{h} + \frac{\cos\theta}{2Fr^2}\gamma h^2 + (\frac{\sin\theta Re}{Fr^2})^2(\frac{\alpha}{15} - \frac{6\gamma}{75})h^5\Big) + (\alpha - \frac{5}{6}\beta)\frac{q^2}{h^2}h_x\\
&= \frac{\gamma}{\varepsilon Re}(\frac{\sin\theta Re}{Fr^2}h - \frac{3q}{h^2}) + \kappa\gamma h\,\partial_x^3 h.
\end{aligned}$$

where α, β and γ being respectively defined by

$$\gamma = \frac{1}{h}\int_0^h w\,dz, \quad \alpha = \frac{1}{q}\int_0^h u_N w \quad \text{and} \quad \beta = \frac{h}{q^2}\int_0^h u_N^2 w.$$

In fact, α, β, γ and the coefficients of the weight w_0 are related by the following linear system

$$\begin{cases}
\alpha = \frac{9}{20}a + \frac{15}{24}b + c,\\
\beta = \frac{87}{140}a + \frac{33}{40}b + \frac{6}{5}c,\\
\gamma = \frac{1}{3}a + \frac{1}{2}b + c.
\end{cases} \tag{2.17}$$

Thus, to recover either one of the two systems (2.2) or (2.3), we handle just straight forward the values for α, β and γ. In particular, for $\alpha = 1$, $\beta = \frac{6}{5}$ and $\gamma = 1$ we recover (2.2), and taking $\alpha = \frac{6}{15}$, $\beta = \frac{18}{35}$ and $\gamma = \frac{1}{3}$ we obtain $f_0 = \frac{y}{h} - \frac{y^2}{2h^2}$ and thus system (2.3). One of the advantages of the unified model is conveying a large range of shallow water models, as those satisfying conservative forms which is efficient in numerical tests especially when applying finite volume methods. Optimization techniques can be exploited for such model to get the range needed for either physical features required to study in the system. The reader is referred to [65] for a similar optimization study on several characteristics of different shallow models.

2- The Three Equation Model Using Weighted Residual Method

In this work, we have launched two approaches to derive 3-equation models.

1. The first one introduces the same variable as done in [3] and exploits as well the momentum integral method for the three equation model derived in the latter reference. Mainly in [3] they introduced a third variable: the enstrophy, which measures the departure of the velocity from its average. It is defined by

$$h^3\varphi = \int_0^h (u - U)^2\,dz,$$

where u is the velocity in the stream wise direction for a 2D flow and U is the depth

averaged velocity $U = \frac{1}{h}\int_0^h u\ dz$. The system in [3] reads

$$\begin{cases} \partial_t h + \partial_x(hU) = 0 \\ \partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{2Fr^2}h^2) = \frac{1}{\varepsilon Re}(1+f_1)(\lambda h - \frac{3U}{h}) + \frac{f_2}{\varepsilon Re}(\varphi - \frac{\lambda^2 h^2}{45}) \\ \qquad\qquad + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + \frac{\varepsilon}{Re}\,\partial_x(\frac{109}{32}h\,\partial_x U); \\ \frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) = -\frac{U}{\varepsilon Re}\frac{2f_1}{3}(\lambda h - \frac{3U}{h}) - \frac{U}{\varepsilon Re}\frac{2f_2}{3}(\varphi - \frac{\lambda^2 h^2}{45}) \\ \qquad\qquad + \frac{\varepsilon}{Re}\frac{109}{32}h(\,\partial_x U)^2 + \frac{\varepsilon}{Re}\,\partial_x(\frac{175}{48}h^3\,\partial_x\varphi). \end{cases}$$

f_1 and f_2 are parameters with some degrees of freedom due to the use of asymptotic expansions to express equivalence between different terms.

The novelty in our work however is in introducing a velocity profile related to this new variable which allows to obtain the relaxation terms automatically without manipulating other terms in the system. Also, there is difference in the way we treat the cubic term that arises from inertial terms in the energy equation $\int_0^h (u-U)^3\ dz$ between the initial and our approach. In both, the asymptotic expansion was used to deal with such term, the difference is that in the former approach this term was added to relaxation effects , however, since it descends from inertial terms, we saw that it would be more physically relevant to add this contribution to inertial part. Our adopted velocity profile reads

$$u = 3Ug_0 + \frac{1}{6\alpha\beta U}(h^2\varphi - \frac{U^2}{5})G_1 + O(\varepsilon)$$

where g_0 is the polynomial that appears in the main order velocity profile of the Nusselt solution: $g_0 = \frac{z}{h} - \frac{z^2}{2h^2}$, and G_1 is a function that is to be determined through the derivation of the system due to the constraints we have on u, and α and β are parameters related to G_1. The derived system in this context reads

$$\begin{aligned} &\partial_t h + \partial_x(hU) = 0, \\ &\partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h - \frac{3U}{h}) + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + \frac{105}{\varepsilon Re}\frac{1}{6hU}(h^2\varphi - \frac{U^2}{5}), \\ &\frac{3h^2}{4}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) = -\frac{105}{6h}\frac{1}{\varepsilon Re}(h^2\varphi - \frac{U^2}{5}). \end{aligned}$$

2. In the second approach, we used the weighted residual method (WRM) to derive 3-equation models both for 2D and 3D flows. The advantage in using WRM is that there is no need to use asymptotic expansions to close the system, and also we have an exact velocity profile that embeds a relation with the new variable to be defined, and this would eventually lead to obtain good results for the relaxation terms compatible with the eigenmodes (damping coefficients) of a perturbed viscous film. We will state the 2D result, and the 3D follows in the same manner. We started from a velocity profile in the spirit of what has been done in [2], and we defined a new variable Φ equivalent at main order to the shear rate; it reads

$$\frac{h^3\Phi^2}{5} = \int_0^h (u-U)^2\ dz.$$

Consequently we could express the velocity profile in terms of Φ in addition to a corrected term, i.e

$$u = 3Ug_0 + \frac{1}{6}(U - h\Phi)(45g_1 - 3g_0) + \varepsilon\tilde{u}_1,$$

where g_1 is the second polynomial appearing in the expansion of the velocity profile done in [2] (the first is g_0), and $\tilde{u}_1$ is the correction. This definition of the velocity profiles introduces two gauge conditions on $\tilde{u}_1$, and thus applying the weighted residual technique we obtain the following system (for 2D flow):

$$\begin{aligned}
\partial_t U = &-\frac{14\cos\theta\,\partial_x h}{15\mathrm{Fr}^2} + \frac{14}{15}\frac{\kappa}{Fr^2} h\,\partial_x^3 h - \frac{116U^2\,\partial_x h}{1287h} + \frac{1036U\,\partial_x h\Phi}{2145} - \frac{6272h\,\partial_x h\Phi^2}{6435}\\
&- \frac{4067h^2\Phi\,\partial_x\Phi}{6435} + \frac{21\Phi}{5h\mathrm{Re}\varepsilon} + \frac{476h\,\partial_x U\Phi}{2145} + \frac{322hU\,\partial_x\Phi}{1287} - \frac{532}{429}U\,\partial_x U - \frac{7U}{h^2\mathrm{Re}\varepsilon} + \frac{14\lambda}{15\mathrm{Re}\varepsilon}\\
&+ \frac{\varepsilon}{Re}\Big(-\frac{7}{12}\,\partial_x h\,\partial_x\Phi - \frac{63(\partial_x h)^2\Phi}{40h} - \frac{175}{96}\,\partial_{xx} h\Phi + \frac{109\,\partial_x h\,\partial_x U}{24h} - \frac{13(\partial_x h)^2 U}{24h^2}\\
&\qquad + \frac{113\,\partial_{xx} hU}{96h} + \frac{49}{60}h\,\partial_{xx}\Phi + \frac{83}{24}\,\partial_{xx} U\Big)
\end{aligned}$$

$$\begin{aligned}
\partial_t \Phi = &-\frac{\cos\,\partial_x h}{3\mathrm{Fr}^2 h} + \frac{\kappa}{3Fr^2}\,\partial_x^3 h + \frac{290U\,\partial_x h\Phi}{429h} - \frac{826\,\partial_x h\Phi^2}{1287} - \frac{21\Phi}{h^2\mathrm{Re}\varepsilon} + \frac{160U^2\,\partial_x h}{9009h^2}\\
&+ \frac{1070U\,\partial_x U}{3003h} - \frac{574h\Phi\,\partial_x\Phi}{1287} + \frac{70}{429}\,\partial_x U\Phi - \frac{830U\,\partial_x\Phi}{1287} + \frac{20U}{h^3\mathrm{Re}\varepsilon} + \frac{\lambda}{3h\mathrm{Re}\varepsilon}\\
&+ \frac{\varepsilon}{Re}\Big(\frac{139\,\partial_x h\,\partial_x\Phi}{24h} - \frac{15(\partial_x h)^2\Phi}{2h^2} - \frac{247\,\partial_{xx} h\Phi}{96h} + \frac{85\,\partial_x h\,\partial_x U}{6h^2} + \frac{125(\partial_x h)^2 U}{24h^3}\\
&\qquad + \frac{305\,\partial_{xx} hU}{96h^2} + \frac{10\,\partial_{xx} U}{3h} + \frac{55}{24}\,\partial_{xx}\Phi\Big).
\end{aligned}$$

However, using asymptotic expansions, we proposed to manipulate the convected part so that it meets a conservative formula that would be also good to derive an energy for the system. Hence finally we got

$$\begin{aligned}
&\partial_t h + \partial_x(hU) = 0,\\
&\partial_t(hU) + \partial_x(hU^2 + \frac{1}{5}h^3\Phi^2) = \frac{1}{\varepsilon Re}\Big(\frac{14}{15}(\lambda h - \frac{3U}{h}) + \frac{21}{5}(\Phi - \frac{U}{h})\Big)\\
&\qquad\qquad - \frac{14}{15}\Big(\frac{\cos\theta}{Fr^2}h\,\partial_x h - \frac{\kappa}{Fr^2}h\,\partial_x^3 h\Big) + \frac{\varepsilon}{Re}\mathcal{D}_1,\\
&\partial_t(h\Phi) + \partial_x(hU\Phi) - \frac{1}{7}\frac{\partial_x(h^4\Phi^3)}{h^2\Phi} = \frac{1}{\varepsilon}\Big(\frac{1}{3}(\lambda - \frac{3U}{h^2}) + 21(\frac{U}{h^2} - \Phi)\Big)\\
&\qquad\qquad - \frac{1}{3}\Big(\frac{\cos\theta}{Fr^2}\,\partial_x h - \frac{\kappa}{Fr^2}\,\partial_x^3 h\Big) + \frac{\varepsilon}{Re}\mathcal{D}_2,
\end{aligned}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are diffusion terms given by

$$\begin{aligned}
\mathcal{D}_1 = &-\frac{7}{12}h\,\partial_x h\,\partial_x\Phi - \frac{63(\partial_x h)^2\Phi}{40} - \frac{175}{96}h\Phi\,\partial_{xx} h + \frac{109\,\partial_x h\,\partial_x U}{24} - \frac{13(\partial_x h)^2 U}{24h}\\
&+ \frac{113\,\partial_{xx} hU}{96} + \frac{49}{60}h^2\,\partial_{xx}\Phi + \frac{83}{24}h\,\partial_{xx} U,\\
\mathcal{D}_2 = &\frac{139}{24}\,\partial_x h\,\partial_x\Phi - \frac{15(\partial_x h)^2\Phi}{2h} - \frac{247\,\partial_{xx} h\Phi}{96} + \frac{85\,\partial_x h\,\partial_x U}{6h} + \frac{125(\partial_x h)^2 U}{24h^2}\\
&+ \frac{305\,\partial_{xx} hU}{96h} + \frac{10\,\partial_{xx} U}{3} + \frac{55}{24}h\,\partial_{xx}\Phi.
\end{aligned}$$

2.2.3.2 Bi-viscous Shallow Water model

This work is a collaboration with D. Bresch and F.James.

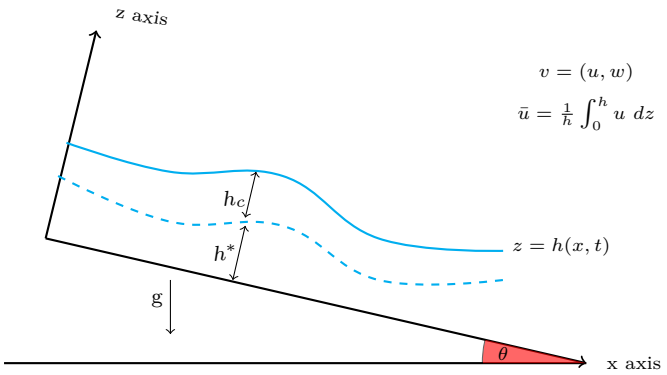
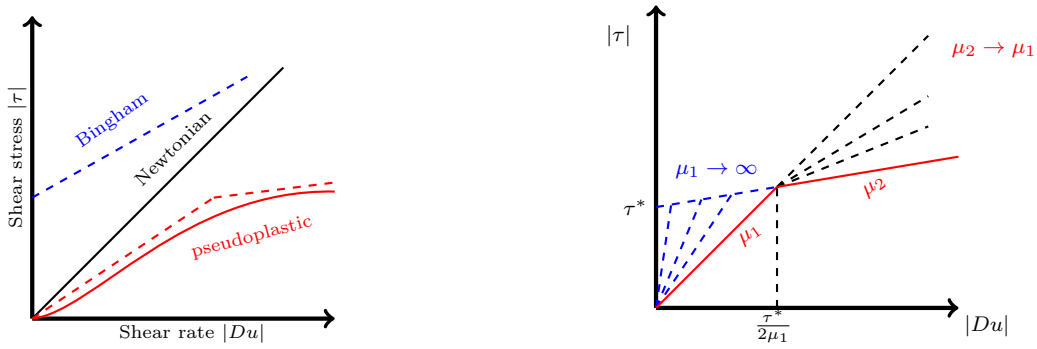


Figure 2.2: *schematic representation of the flow with the main parameters of the system: velocity vector $v = (u, w)$, gravity g, plane's inclination angle θ, fluid's height h, height of the yielded part h^*, height of the pseudo-plug part h_c, depth averaged velocity $\bar{u} = \frac{1}{h}\int_0^h u\ dz$.*

Bi-viscous Rheology: Approximation to Pseudo-Plastic, and Smoothness to Bingham

We are interested in modeling a shallow water type system for pseudo-plastic fluids. Such fluids have no yield point; their consistency curves pass through the origin. The curves are non-linear, but approach linearity at high shear rates, see Figure 2.3. If stress readings taken at high shear rates are extrapolated back to the axis, there appears to be a yield point similar to that of a Bingham plastic (hence the name pseudo plastic). Under this observation, we propose the first attempt or approach towards understanding such rheologies: approximating the relation between the shear stress and shear rate (i.e the viscosity). We approximate the corresponding function- which is a nonlinear relation decreasing with the strain rate- by a piece wise linear function as shown in figure (2.3a) (red dotted graph). Thus the approximation rheology behaves similar to a Newtonian fluid for a deformation which is less than some characteristic threshold, whereas it shows the properties of a Bingham fluid for a large deformation. We will denote by μ_1 the viscosity of the Newtonian-like behaving part, and the Bingham-like part will be characterized with a viscosity $\mu_2 < \mu_1$ (see figure below), and due to this fact that this rheology will be referred to as bi-viscous. In fact, such type of rheology was used in the lubrication-scaling context by Mei and Liu [66] where they used the bi-viscous stress type to approach the Bingham model as a limit to the bi-viscous one. We adopt a similar bi viscous constitutional law to get a shallow water type model that approximates rather both the Pseudo-Plastic and Bingham fluids, and also recover the Newtonian model as shown in the figure below.



(a) *figure 1* (b) *figure 2*

Figure 2.3: *In the first diagram the relation between the modulus of the deformation tensor $|Du|$ and the modulus of the shear rate τ for several types of rheologies is exhibited, in particular we show how a bi-viscous rheology (dashed red graph) approximates the of Pseudo-Plastic one. In the second diagram we show how the behavior of bi-viscous rheology resembles that of Newtonian and Bingham rheologies depending on the choice and range that we expect the viscosities μ_1 and μ_2 to be in.*

Departure System and Ansatz

We start from Navier Stokes system for a free surface fluid moving down an inclined plane under the effect of gravity

$$\begin{aligned} \partial_x u + \partial_z w &= 0, \\ \partial_t u + u\,\partial_x u + w\,\partial_z u &= -\frac{\partial_x p}{Fr^2} + \frac{\lambda}{\varepsilon Re} + \frac{1}{Re}\partial_x \tau_{xx} + \frac{1}{\varepsilon Re}\partial_z \tau_{xz}, \\ \partial_t w + u\,\partial_x w + w\,\partial_z w &= -\frac{\partial_z p}{\varepsilon^2 Fr^2} - \frac{\lambda \cot\theta}{\varepsilon^2 Re} + \frac{1}{\varepsilon Re}\partial_x \tau_{xz} + \frac{1}{\varepsilon^2 Re}\partial_z \tau_{zz}. \end{aligned} \tag{2.18}$$

for a velocity $v = (u, w)$, pressure p and viscous stress tensor τ. The bi-viscous rheology is defined by two viscosities μ_1 and μ_2 such that the constitutional law reads

$$\tau = \begin{cases} 2\mu_1 \,\mathrm{D}\, v & \text{if} \quad |\,\mathrm{D}\, v| \leqslant \frac{B}{2\mu_1}, \\ 2\tilde{\mu_2} Dv + (1 - \frac{\mu_2}{\mu_1}) B \dfrac{\mathrm{D}\, v}{|\,\mathrm{D}\, v|} & \text{if} \quad |\,\mathrm{D}\, v| \geqslant \frac{B}{2\mu_1}. \end{cases} \tag{2.19}$$

B is the rescaled Bingham number. The rescaled boundary conditions at $z = 0$ and $z = h$ become respectively

$$\begin{aligned} u|_{z=0} &= w|_{z=0} = 0, \\ \tau_{xz}|_{z=h} &= \frac{\varepsilon\,\partial_x h}{1 - \varepsilon^2(\,\partial_x h)^2}(\tau_{xx} - \tau_{zz})|_{z=h}, \\ p|_{z=h} &= \frac{\sin\theta}{\lambda}\frac{1}{1 - \varepsilon^2(\,\partial_x h)^2}(\tau_{zz} - \varepsilon^2(\,\partial_x h)^2 \tau_{xz})|_{z=h}. \end{aligned} \tag{2.20}$$

Due to being characterized by two viscosities, such fluid exhibit different behavior according to the intensity of deformation applied. For deformations smaller than some threshold, a part of

the fluid behave as a Newtonian fluid of apparent viscosity μ_1, and for deformations beyond this threshold, the other part of the fluid exhibits a pseudo-plug behavior similar to plastic fluids with viscosity μ_2. As a result we will denote by h^* the maximum height of the yielded part of fluid that responds as a Newtonian one, and this would be our first ansatz. The scaling approved for shallow water theory induces further ansatz on the adimentionlal numbers, namely

$$\mathrm{Fr} = O(1), \qquad \mathrm{Re} = O(1), \qquad \lambda = O(1),$$

In fact, in the light of the forgoing, we claim that we have **two small parameters** in the system, the **aspect ratio** ε and the one relating this latter to the viscosities ratio $\varepsilon\frac{\mu_1}{\mu_2}$. Our adapted perturbation analysis is made in terms of ε only. In particular we claim that if the velocity profile and the shear are denoted u and τ respectively, we assume:

$$u = u_0^{\mu_1,\mu_2} + \varepsilon u_1^{\mu_1,\mu_2} \qquad \text{and} \qquad \tau = \tau_0^{\mu_1,\mu_2} + \varepsilon \tau_1^{\mu_1,\mu_2}.$$

Of course, the main order and first order terms as shown depend on the larger viscosity μ_1. One of the goals after getting the final system is to study the formal limit of the system when μ_1 approaches infinity. At this point, we assume uniformity of the variables with respect to μ_1, and in addition neglect boundary layers by assuming that $\partial_x h$ is uniform with respect to μ_1, this would guarantee that the derivation and the perturbation analysis are made uniform in μ_1 so that the theory is not broken down. In fact, although the first order expansions of u and τ depend on the larger viscosity, yet this doesn't affect the derivation, especially for τ, due to adopting the physical condition imposed by Liu and Mei in [66] $\varepsilon\frac{\mu_1}{\mu_2} \leqslant \varepsilon \ll 1$. For the model at hand, the derivation is carried in a way that allows, in addition to what has been mentioned, to study the asymptotic limit in terms of the larger viscosity μ_1. Hence, the obtained model is valid for both regimes of μ_1: $\mu_1 \sim O(1)$ and $\mu_1 \to \infty$, where the former regime corresponds to a pseudo-plastic and Newtonian approximations, and the latter to a Bingham approximation.

Final System

We derived a consistent bi-viscous shallow-water model following shallow water approximations in the presence of a slope in the spirit of J.-P. Vila and collaborators in [40]. Our model reads

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x(\Lambda_1^P h\bar{u}^2 + \frac{\cos\theta}{2\mathrm{Fr}^2}h^2 + \Lambda_2^P) = \frac{1}{\varepsilon Re}(\lambda h^* - \frac{h\bar{u}}{\Gamma_1^P}) + R^P \end{cases} \tag{2.21}$$

where, defining $h^* = h - \frac{B}{\sqrt{2}|\lambda|}$ and $h_c = h - h^*$. The unknowns Λ_1^P, Λ_2^P, Γ_1^P and R^P are given by

$$\Lambda_1^P = h\frac{\frac{12}{\mu_1^2}(2h_c^5 - 5h_c^4 h + 5h_c^2 h^3) - \frac{15h_c}{\mu_1\mu_2}(h_c - h)^2(3h_c^2 - 2h_c h - 5h^2) + \frac{3}{\mu_2^2}(h_c - h)^4(7h_c + 8h)}{5(\frac{1}{\mu_1}(h_c^3 - 3h_c h^2) + \frac{1}{\mu_2}(3h_c h^2 - 2h^3 - h_c^3))^2}, \tag{2.22}$$

$$\Lambda_2^P = -\frac{\cos\theta}{Fr^2}\frac{h^2}{2} - \lambda^2\Big(-\frac{1}{\mu_1^2}\frac{h_c^4}{3}h^* + \frac{1}{\mu_2^2}(-\frac{h_c h^{*4}}{4} - \frac{h^{*4}}{15} - \frac{h_c^2 h^{*3}}{3}) + \frac{1}{\mu_1\mu_2}(-\frac{h_c^2 h^{*3}}{3} - \frac{h_c h^{*4}}{12} - \frac{2}{3}h_c^3 h^{*2})\Big), \tag{2.23}$$

$$\Gamma_1^P = \frac{1}{\mu_2}(\frac{h^{*2}}{3} + \frac{h^* h_c}{2}) + \frac{1}{\mu_1}(\frac{h^3}{3h^*} - \frac{hh^*}{2} + \frac{h^{*2}}{6}) \tag{2.24}$$

$$R^P = \frac{1}{Re}(\frac{1}{\Gamma_1}\int_0^h u_1\ dz) \tag{2.25}$$

with

$$\begin{aligned}\int_0^h u_1\,dz = &-\frac{Re}{\mu_1}\lambda^2\,\partial_x h\Big[-\frac{2}{15}\frac{1}{\mu_1^2}h_c^6+\frac{1}{\mu_1\mu_2}(-\frac{7}{15}h_c^5h^*-\frac{1}{6}h_c^4h^{*2})+\frac{1}{\mu_2^2}(-\frac{1}{3}h_c^4h^{*2}-\frac{1}{6}h_c^3h^{*3})\Big]\\
&-\frac{Re}{\mu_2}\lambda^2\,\partial_x h\Big[\frac{1}{\mu_1^2}(-\frac{h_c^5h^*}{3}-\frac{h_c^4h^{*2}}{6})+\frac{1}{\mu_1\mu_2}(-\frac{4}{3}h_c^4h^{*2}-3h_c^3h^{*3}-\frac{2}{3}h_c^2h^{*4}-\frac{2}{15}h_ch^{*5})\\
&+\frac{1}{\mu_2^2}(-h_c^3h^{*3}-\frac{4}{3}h_c^2h^{*4}-\frac{2}{3}h_ch^{*5}-\frac{2}{15}h^{*6})\Big]-\frac{\lambda\cot\theta}{\mu_1}\,\partial_x h\frac{h_c^3}{3}\\
&-\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(\frac{h^{*3}}{3}+h^{*2}h_c+h^*h_c^2).\end{aligned} \tag{2.26}$$

Conclusion

Using the above model, we are able from one side to give a good approximation of shallow water systems for pseudo-plastic fluids, and be consistent with the Newtonian shallow water model (2.2) when we take $\mu_1 = \mu_2$. From the second side it would help recover the Bingham shallow water model by letting one of the viscosities be very large. In fact, doing so we recover the Bingham shallow-water system (2.5) which is the same as the one derived in [40] except for the last term R which is absent in our derivation, i.e we obtain $R = 0$. Mainly, this difference arises from the boundary normal stress that is counted for in Vila and al. paper [40] and absent in our case due to the adopted continuous representation of the stress in the pseudo-plug zone especially at h^*: $\tau = 2\mu_1 Du$ and due to assuming the uniformity of $\partial_x h$ in terms of μ_1.

By this way, we get a conservative PDE with a right hand side that doesn't change sign. Note that this interesting result comes from the fact that, in our adopted rheology, the strain tensor is continuous, which means that no jump interface is present in the derivation. This is not the case for the Bingham incompressible Navier-Stokes equations. Another advantage of the bi-viscous model is conveyed in the smoothness noticed in the graduation of the depth component of the velocity gradient. This smoothness makes us avoid calculating the corrective layers calculated in different approaches, both in lubrication and shallow water approximations, in an attempt to get a consistent theory for thin layer dynamics of Bingham viscoplastic fluids as in [67] and [40]. The reason is behind the velocity profile at first order which due to the bi-viscosity characteristic encodes, in some sense, these corrective layers as shown in the calculations.

We do believe that adapting this kind of rheology is more convenient for several Bingham type fluids, since technically it helps get rid of the problem of including corrective layers in the approximation and thus directly leads us to a consistent approximation of the model, and physically it has been proven by many scientists as in [68] that Bingham fluids behave in reality as Newtonian fluids with very large viscosities at very low stress values. However, the constraint assumed concerning the uniformity of the gradient of the height may have lead us to neglect some boundary layers that may contribute to the term R that we did not recover as in [40]. This result will be discussed in details in Chapter 6.

2.3 Lubrication Theory

2.3.1 Lubrication Theory From the Modeling Point of View

Though the shortcoming that lubrication equations have had concerning instabilities, yet what they served in the study of thin films is of paramount importance. Certainly such models

are a pillar in the bearing industry; nevertheless, from a mathematical and physical points of view they were the reason to understand many phenomena related to film flows: observations of regular wave patterns in film flows down a windowpane or along guttering, the patterning of dewetting droplets, the fingering of viscous flows down a slope, thermo capillary- and surfactant-driven films, falling films and films flowing over structured surfaces, compliant and rapidly rotating substrates, evaporating films as well as those manipulated via the use of electric fields to produce nanoscale patterns, and the list goes on. Such phenomena find a wide range of application in geophysics, biophysics and engineering such as nanofluidics and microfluidics, coating flows, intensive processing, lava flows, dynamics of continental ice sheets, tear-film rupture, and surfactant replacement therapy.

Our interest in lubrication equations-as far as it is concerned with this dissertation- evolves around the modeling point of view, although a great interest exists to extend the research on the models obtained in this context and study its behavior and stability in critical regimes. Add to this that also well posedness is concerned, and a great part of the thesis is devoted to prove existence results of different lubrication models by a limit process that exploits the BD entropy in the compensated compactness technique for adequate shallow water models. This will be elaborated on in subsection 2.3.2.2 and discussed more in chapter 4.

Classically, a lubrication equation is developed from Navier Stokes system by merging to the long wave theory with proper scaling techniques based on the balance between pressure and viscous forces in the Navier-Stokes equations as discussed thoroughly before. Experiments and studies insure that the formation of instabilities in film flows synchronizes with the increased significance of the inertial terms which forms a balancing bug for the lubrication theory. However, for many cases, such theory helps encapsulate the physical features of the NS system into a more theoretically and numerically tactile scalar equation giving considerably good predictions. Another advantage of such approximation is that it is easy to attain the velocity profile (and thus the streamlines) at main order, which is a parabolic profile corresponding to Nusselt solution.

The motivation of the work in the modeling context started form the need to obtain some simplified models of complex-type fluids that correspond to different physical natural phenomena, as it is the case of granular matter flows and the lithosphere sheet flow. Such applications fall again in the context of shear thinning fluids, and as a result we will endorse the rheology of bi-viscous fluid described in pat 2.2.3.2. In this context, it is good to mention that in the treated case, we assume that the viscosity is far away from the vanishing regimes, which yields the ansatz that the corresponding Reynolds number is an order one term. This in fact will be used later on to balance the momentum equation and get a consistent system in terms of the non-dimensional aspect.

As far as well posedness is concerned, we will focus on existence results of several Newtonian lubrication equations following the method of energy estimates coupled with the BD entropy. This is in fact a result concerned with global weak solution; no more regularity is investigated.

2.3.2 Contribution to the Thesis

2.3.2.1 Bi-viscous Lubrication Equations

This work is a collaboration with F. James, M. M'baye, D. Nguyen launched in CEMRACS 2019

Bi viscous fluids:

We will Start from the same system and rheology described in part 2.2.3.2. The same ansatz as well is regarded concerning the height h^*. However, in the lubrication scaling , only main order profiles are needed, and hence, the condition suggested by Mei and Liu concerning the ratio between the viscosities and the aspect ratio (especially in critical regimes for μ_1): $\varepsilon\frac{\mu_1}{\mu_2} \ll 1$ is not needed in the lubrication derivation, since it is only significant at first order. Nevertheless, the scaling in this part is a bit different. This difference is attributed to the fact that in the

lubrication setting, the speed is considerably slow, corresponding to very low Froude numbers (subcritical flows) where viscous forces in such cases contribute to a pressure, and thus a balance between viscous shear forces and hydrostatic pressure should take place regardless of inertial effects (which become negligible). In the scaling this is attributed to claiming that Froude number is in fact very small compared to Reynolds number

$$Fr^2 = \varepsilon Re,$$

On the contrary to what have been adopted in the shallow water scaling of having

$$Fr^2 \sim Re \sim O(1).$$

The lubrication equation is derived by averaging in depth as a first step the mass equation so that we get

$$\partial_t h + \partial_x(hU) = 0.$$

Since we are looking for a one equation model that would describe the evolution of the height h, then to close the above equation it is enough to calculate the averaged depth velocity U in terms of h. This is done by depth averaging the momentum equation at main order. Finally, we would be able in the simple setup to derive a lubrication equation for bi-viscous fluids that reads

$$\partial_t h - \partial_x\Big(\partial_x h(\frac{1}{\mu_2}(\frac{h^{*3}}{3} + \frac{h_c h^{*2}}{2}) + \frac{1}{\mu_1}(h^* h_c^2 + \frac{h^{*2}h_c}{2} + \frac{h_c^3}{3}))\Big) = 0. \tag{2.27}$$

The explicit derivation and discussion concerning this model are handled in Chapter 6 where a more general initial framework is handled, as for example having a basal velocity at the bottom (which is usual in geophysical applications) and the bottom is regarded as an arbitrary topography.

2.3.2.2 Existence Results for Some Lubrication-type Equations

This part is a collaboration with D. Bresch, M. Colin, P. Noble and X. Song.

Motivation and Problem Setup

As the Reynold equation is concerned, in 1904, Sommerfeld used a change of variable and succeeded in obtaining an analytical solution for Reynolds equation for infinitely long films with $\partial_x p = 0$. However, his work and others preceding in the same spirit suffered form non physical boundary conditions, ignoring rupture formation, and obtaining negative pressure distribution near divergence zones. The lubrication equations in their generality induce numerous classes of non trivially solved pde's most of the times. Among those we are concerned with highly nonlinear parabolic pde's of general form

$$\partial_t h + \partial_x(\frac{1}{\alpha}F(h)\partial_x^3 h - \frac{1}{\alpha}D(h)\partial_x h) = 0, \tag{2.28}$$

where α is a positive constant, and the non negative functions $s \mapsto F(s)$ and $s \mapsto D(s)$ depend on the applications. Two forms of such equations are being discussed according to the choice of F and D. Due to strong non linearity, the solutions of such systems, if they exist, require high regularity on the initial data. The attempt in our approach is to stress less on the regularity by proving the existence of global weak solutions. However, due to the high non linearity, even the well posedness of the weak solution is difficult to attain in the classical sense, mainly it is not possible to pass to the limit of the weak formulation, and thus a primary additional step is required, usually it would be to figure out additional estimates. In our approach we will recast the above equation as a limit of a shallow water model that enjoys compatible physical features

as the equation at hand. We will be concerned with features as capillarity and drag terms. The corresponding shallow water model is considered as a regularized model for the equation, which, in our cases of study, enjoys a known existence result that is more likely to lead the convergence to the weak formulation of the lubrication equation rather than doing that the classical way. Thus, we are creating a singular shallow water problem that links a shallow water model to a lubrication model. The novelty in such approach in the tools used to prove such singular limit which is the link between two entropies related to the shallow water and lubrication systems.

For the general model at hand, we can thus suggest the following shallow water type model

$$\begin{aligned}&\partial_t h_\varepsilon + \partial_x(h_\varepsilon u_\varepsilon) = 0,\\ &\partial_t(h_\varepsilon u_\varepsilon) + \partial_x(h_\varepsilon u_\varepsilon^2) + \frac{1}{\varepsilon Fr^2} S(h_\varepsilon)\partial_x(h_\varepsilon) = \frac{4}{R_e}\partial_x(h_\varepsilon \partial_x u_\varepsilon) + \frac{1}{\varepsilon W_e} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha \frac{h_\varepsilon^2}{\varepsilon T(h_\varepsilon)} u_\varepsilon.\end{aligned} \tag{2.29}$$

Note that the terms on the right-hand side of the momentum equation represent respectively the viscous term, the capillarity term and the drag term. The functions $r \mapsto S(r)$ and $r \mapsto T(r)$ are two non negative functions.

Relation Between BD Entropy and Bernis-Friedman Entropy

A remarkable result in this context is the link between two types of entropies used to provide estimates for each of the models; the lubrication equation and the shallow water model. This will manifested by imposing the following relation between the general functions F, D, T and S

$$T(r) = F(r) \qquad \text{and} \qquad S(r) = \frac{rD(r)}{F(r)},$$

and equivalently we get

$$T_1 = G \qquad \text{and} \qquad S_1'' = \frac{D}{F}.$$

The Bresch-Desjardin entropy, known as BD entropy, was used by the authors in [69] to control the degenerate term in diffusive capillary models of Korteweg type by providing extra regularity on the density of the system. In the same spirit, such non-trivial entropy is necessary and indispensable to guarantee the weak existence result of (2.29). For the shallow water model at hand, the BD entropy reads

$$\begin{aligned}&\int_\Omega \frac{1}{2} h_\varepsilon(x,t) v_\varepsilon(x,t)^2 + \frac{1}{2}\frac{1}{\varepsilon W_e}(\partial_x h_\varepsilon(x,t))^2 + \frac{1}{\varepsilon Fr^2} S_1(h_\varepsilon(x,t)) + \frac{4\alpha}{\varepsilon R_e} T_1(h_\varepsilon(x,t))\, dx\\ &+ \int_0^t \int_\Omega \left(\frac{4}{R_e}\frac{1}{\varepsilon Fr^2}\frac{S(h_\varepsilon)}{h_\varepsilon}(\partial_x h_\varepsilon)^2 + \frac{4}{R_e}\frac{1}{\varepsilon W_e}(\partial_x^2 h_\varepsilon)^2 + \alpha \frac{h_\varepsilon^2 u_\varepsilon^2}{\varepsilon T(h_\varepsilon)} \right) dx\, dt\\ &= \int_\Omega \frac{1}{2}\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{1}{2}\frac{1}{\varepsilon W_e}(\partial_x h_0^\varepsilon(x))^2 + \frac{1}{\varepsilon Fr^2} S_1(h_0^\varepsilon(x)) + \frac{4\alpha}{\varepsilon R_e} T_1(h_0^\varepsilon(x))\, dx,\end{aligned} \tag{2.30}$$

where $v_\varepsilon = u_\varepsilon + \frac{4}{R_e}\partial_x(\log h_\varepsilon)$ is an augmented velocity, $S_1''(r) = \frac{S(r)}{r}$ and $T_1''(r) = \frac{1}{T(r)}$.

On the other hand, the Bernis-Friedman entropy, known as BF entropy, was also used by the authors in [70] to provide additional regularity for the existence of the weak solution of a high order non linear degenerate parabolic lubrication equation. More precisely, they studied system (2.28) with $F(h) = h^n$ where $n > 1$ and $D(h) = 0$. Their work paved the way for plenty of interesting results, for instance the general class of $F(h) = h^n$ and $D(h) = h^m$, the interested reader is referred to [71], [70], [72] and [73] and references therein. In fact, equation (2.28) can

be recast into the following model

$$\begin{aligned} \partial_t h + \partial_x(hu) &= 0, \\ hu = \frac{1}{\alpha}F(h)\partial_x^3 h - \frac{1}{\alpha}D(h)\partial_x h. & \end{aligned} \tag{2.31}$$

Thus, we have the following energy of the system

$$\int_0^t \int_\Omega \frac{\alpha h^2 u^2}{F(h)}\, dx\, dt + \int_\Omega \frac{S_1}{Fr^2} + \frac{1}{2}\frac{(\partial_x h)^2}{We} = \frac{1}{2}\int_\Omega \frac{S_1|_{t=0}}{Fr^2} + \frac{(\partial_x h)^2|_{t=0}}{We}.$$

As for the general lubrication equation we proposed (2.28), the BF entropy reads for $G''(r) = \frac{1}{F(r)}$

$$\int_\Omega G(h(x,t))\, dx + \int_0^t \int_\Omega \frac{(\partial_x^2 h)^2}{\alpha We} + \frac{1}{\alpha Fr^2}\frac{D(h)}{F(h)}(\partial_x h)^2\, dx\, dt = \int_\Omega G(h_0(x))\, dx. \tag{2.32}$$

A rigorous link between the BD entropy and the BF entropy is manifested through showing that the latter is in fact encoded in the former at main order. In fact, the BD entropy then can be rewritten as

$$\begin{aligned} &\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t)v_\varepsilon(x,t)^2 + \\ &+ \frac{4\alpha}{Re}\Big(\int_\Omega G_1(h_\varepsilon(x,t))\, dx + \int_0^t \int_\Omega \Big(\frac{1}{\alpha Fr^2}\frac{D(h_\varepsilon)}{F(h_\varepsilon)}(\partial_x h_\varepsilon)^2 + \frac{1}{\alpha W_e}(\partial_x^2 h_\varepsilon)^2\, dx\, dt\Big) \\ &+ \int_\Omega \frac{1}{2W_e}(\partial_x h_\varepsilon(x,t))^2 + \frac{1}{Fr^2}S_1(h_\varepsilon(x,t)) + \int_0^t \int_\Omega + \alpha\frac{h_\varepsilon^2 u_\varepsilon^2}{\varepsilon F(h_\varepsilon)}\Big)\, dx\, dt \\ &= \int_\Omega \frac{1}{2}\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{1}{2}\frac{1}{\varepsilon W_e}(\partial_x h_0^\varepsilon(x))^2 + \frac{1}{\varepsilon Fr^2}S_1(h_0^\varepsilon(x)) + \frac{4\alpha}{\varepsilon R_e}T_1(h_0^\varepsilon(x))\, dx, \end{aligned} \tag{2.33}$$

It was noticed that the terms on second line of the above equation resemble those in the BF entropy, and the terms on the third line are nothing but the terms of the energy of the lubrication equation. What remains is an order ε terms, hence, formally at main order, we notice that the BD entropy encodes the BF entropy coupled with the lubrication energy. This link constitutes a tool now for the next result obtained in this context.

Well-posedness of Some Lubrication Equations.

The global weak solutions of three cases of lubrication equations were investigated in this work, where the first two of which fall in the category of system (2.28):

1. In the first one, we took $F(r) = D(r) = T(r) = r^2 + r^3$ and $S(r) = r$, which corresponds to the viscous shallow water model with drag term formally derived in [4] and justified in [5].

2. In the second case, we started from a viscous compressible system with a nonlinear drag term. We took $S(r) = r^\beta$, $T(r) = r^n$, with $\beta + n \in (1,2)$. As a result we got a lubrication equation with polynomial choice of F and D: $F(r) = r^n$ and $D(r) = r^{\beta+n-1}$. Such lubrication model has been studied extensively by many scientists as Bertozzi and Pugh, see for instance [71], [72], [73] and [74].

3. A third type of lubrication equations that was also amongst the studied cases is a gener-

alized form of the Derrida-Lebowitz-Speer-Spohn equation. Such system reads

$$\partial_t h + \partial_x^2(F(h)\,\partial_x^2 G(h)) = 0. \tag{2.34}$$

We consider the following viscous compressible model with capillarity and drag terms

$$\begin{aligned}
&\partial_t h_\varepsilon + \partial_x(h_\varepsilon u_\varepsilon) = 0,\\
&\varepsilon(\partial_t(h_\varepsilon u_\varepsilon) + \partial_x(h_\varepsilon u_\varepsilon^2) + \partial_x p(h_\varepsilon)) = \varepsilon(\frac{4}{\mathrm{R_e}}\partial_x(\lambda(h_\varepsilon)\partial_x u_\varepsilon))\\
&\qquad\qquad + \frac{1}{\mathrm{W_e}} h_\varepsilon \partial_x(\sqrt{K(h_\varepsilon)}\,\partial_x^2(\int_0^{h_\varepsilon}\sqrt{K(r)}\,dr)\,) - h_\varepsilon u_\varepsilon.
\end{aligned} \tag{2.35}$$

The existence result of such system was discussed recently in [75]. In this context, a relation between the viscosity λ and and the capillarity K is manifested via a shear-type viscosity μ

$$\mu(h) = \int_0^h \frac{\lambda(r)}{2r}\,dr \quad \text{such that} \quad \mu'(h) = \frac{1}{2}\sqrt{h}\sqrt{K(h)} = \frac{\lambda(h)}{2h}. \tag{2.36}$$

This in fact helps utilize some known identities to recast the limit of the above shallow water model into the lubrication equation (2.34). To be more clear, at the formal level, system (2.35) converges to the following system

$$\begin{aligned}
&\partial_t h + \partial_x(hu) = 0,\\
&hu = \frac{1}{\mathrm{W_e}} h\partial_x(\sqrt{K(h)}\,\partial_x^2(\int_0^h \sqrt{K(r)}\,dr)),
\end{aligned} \tag{2.37}$$

which can be written as

$$\partial_t h + \partial_x\left(\frac{1}{\mathrm{W_e}} h\partial_x(\sqrt{K(h)}\,\partial_x^2(\int_0^h \sqrt{K(r)}\,dr\,))\right) = 0. \tag{2.38}$$

Thanks to relation (2.36), we can use Bohm's identity proven in [76], so that we can write

$$h\,\partial_x(\sqrt{K(h)}\,\partial_x^2(\int_0^h \sqrt{K(r)}\,dr)) = \partial_x(h\gamma'(h)\,\partial_x^2\theta(h)),$$

where $\gamma'(h) = \sqrt{hK(h)}$ and $\theta'(h) = 2\sqrt{\frac{K(h)}{h}}$. Thus $\gamma(h) = 2\mu(h)$and $\theta(h) = 2S(h)$. Substituting the identity in equation (2.38) we obtain the following equation

$$\partial_t h + \partial_x^2\Big(2h\mu'(h)\,\partial_x^2(2S)\Big) = 0, \tag{2.39}$$

which recovers equation (2.34) taking $F(h) = 2h\mu'(h)$ and $G(h) = 2S$.

For convenience, we will mention the results of the first studied case where $F(r) = D(r) = T(r) = r^2 + r^3$ and $S(r) = r$. Let's recall the definition of the weak formulation of system (2.29):

Definition 2.4. *A weak formulation of the shallow water model (2.29) is given by*

$$\int_0^\infty \int_\Omega h_\varepsilon \partial_t\psi\;dx\;dt + \int_\Omega h_0^\varepsilon \psi(\cdot,0)\;dx = -\int_0^\infty \int_\Omega h_\varepsilon u_\varepsilon \partial_x\psi\;dx\;dt, \tag{2.40}$$

and

$$\varepsilon\Big(\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon \partial_t\varphi \; dx \; dt + \int_\Omega m_0^\varepsilon \varphi(\cdot,0) \; dx + \int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon^2 \partial_x\varphi \; dx \; dt\Big)$$
$$- \frac{4\varepsilon}{R_e}\int_0^\infty\int_\Omega h_\varepsilon \partial_x u_\varepsilon \partial_x\varphi \; dx \; dt + \frac{1}{W_e}\int_0^\infty\int_\Omega \frac{1}{2}(\partial_x h_\varepsilon)^2 \, \partial_x\varphi \; dx \; dt$$
$$- \frac{1}{W_e}\int_0^\infty\int_\Omega h_\varepsilon \partial_x^2 h_\varepsilon \partial_x\varphi \; dx \; dt + \frac{1}{Fr^2}\int_0^\infty\int_\Omega h_\varepsilon^2 \partial_x\varphi \; dx \; dt - \alpha\int_0^\infty\int_\Omega \frac{h_\varepsilon^2 u_\varepsilon}{F(h_\varepsilon)}\varphi \; dx \; dt = 0, \tag{2.41}$$

for all $\psi \in \mathcal{C}_0^\infty(\Omega\times[0,\infty))$ *and* $\varphi \in \mathcal{C}_0^\infty(\Omega\times[0,\infty))$.

Furthermore, from [5] we recall

Theorem 2.2. *Let* $(h_0^\varepsilon, m_0^\varepsilon)$ *be such that* $h_0^\varepsilon > 0$ *and*

$$h_0^\varepsilon \in H^1(\Omega), \quad \varepsilon|m_0^\varepsilon|^2/h_0^\varepsilon \in L^1(\Omega), \quad \sqrt{\varepsilon}\partial_x\sqrt{h_0^\varepsilon} \in L^2(\Omega),$$
$$-(1+h_0^\varepsilon)log(\frac{h_0^\varepsilon}{1+h_0^\varepsilon}) \in L^1(\Omega),$$

then there exists a global weak solution of (2.29) *in the sense of definition* 2.4.

Our main result states

Theorem 2.3. *Given a sequence* $(h_\varepsilon, u_\varepsilon)_\varepsilon$ *a global solution of* (2.29)*, satisfying the initial conditions given in the sense of Theorem* 2.2 *then there exists a subsequence of* $(h_\varepsilon, u_\varepsilon)$ *that converges to a couple* (h,u)*, which is a solution of the weak formulation of the lubrication system satisfying the initial condition* $h|_{t=0} = h_0$*, where* h_0 *is the weak limit of* h_0^ε *in* $H^1(\Omega)$.

The same strategy applies to attain the global weak solution of the three lubrication equations discussed above. The proof of the weak limit relies on using the BD-entropy along with the energy inequality, in order to get some regularity on the weak solution and finally use compactness technique to pass to the limit in the weak formulation of the shallow water system, where the expected limit solution is found to satisfy the weak formulation of the corresponding lubrication system. Furthermore, using the regularity obtained, we can also pass to the limit form the BD entropy to the BF dissipative entropy, and thus giving finally that the obtained limit solution is a non negative weak solution of the lubrication equation. In fact, such result is not trivial in the classical sense of well posedness problems in the lubrication theory. It was one of the reasons to introduce the BF dissipative entropy by the authors in [70]. This work is further explained in Chapter 4.

2.3.2.3 Bingham Lubrication Equation Revisited

We will revisit in this part the derivation of a lubrication equation for Bingham rheology. This is to be coherent with the already stated results and to complete the presentation. Consider the 2D Navier-Stokes system

$$\begin{aligned} \partial_x u + \partial_z w &= 0, \\ \rho(\partial_t u + u\,\partial_x u + w\,\partial_z u) &= -\partial_x p + \partial_x\tau_{xx} + \partial_z\tau_{xz}, \\ \rho(\partial_t w + u\,\partial_x w + w\,\partial_z w) &= -\partial_z p + \partial_x\tau_{xz} + \partial_z\tau_{zz} - \rho g, \end{aligned} \tag{2.42}$$

where ρ is the fluid's density, $v = (u,w)$ is the velocity field, p is the pressure, h is the fluid's height, τ is the viscous stress tensor, and g the gravity constant. The above system is coupled

usually with a constitutive law relating the rheology of the fluid with its deformation behavior, i.e τ with the deformation tensor $Dv = \frac{\nabla v + \nabla^T v}{2}$. For Bingham fluids, this relation is represented by the following constitutive law

$$\tau := \begin{cases} 2\mu Dv + \tau^* \frac{Dv}{|Dv|} & Dv \neq 0, \\ |\tau| < \tau^* & Dv = 0, \end{cases} \tag{2.43}$$

where μ is the viscosity parameter and τ^* is the yield stress. The boundary conditions read (neglecting capillary effects)

$$\begin{aligned} u = w = 0 \qquad & z = 0, && \text{no-slip condition} \\ \partial_t h + u\,\partial_x h = w \qquad & z = h(x,t), && \text{kinematic condition} \\ (1 - (\partial_x h)^2)p + (1 + (\partial_x h)^2)\tau_{xx} = 0 \qquad & z = h(x,t), && \text{free-boundary condition} \\ (1 - (\partial_x h)^2)\tau_{xz} - 2\,\partial_x h \tau_{xx} = 0 \qquad & z = h(x,t). \end{aligned}$$

Setting the characteristic wavelength L and the characteristic film thickness H, we define the aspect ratio of the film $\varepsilon = \frac{H}{L}$ and we choose the following non dimensional coordinates

$$x = L\tilde{x}, \quad z = H\tilde{z}, \quad t = \frac{L}{U}\tilde{t}.$$

And thus follows the following dimentionless variables

$$h = H\tilde{h}, \quad u = U\tilde{u}, \quad w = \varepsilon U\tilde{w}, \quad p = \rho g H \tilde{\rho},$$

where U is the characteristic flow speed of the film which will be defined later following the balance of the momentum equations. As a result of the above non dimentionalizing, we set

$$D\tilde{v} = \begin{pmatrix} \varepsilon\,\partial_{\tilde{x}}\tilde{u} & \frac{1}{2}(\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\,\partial_{\tilde{x}}\tilde{w}) \\ \frac{1}{2}(\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\,\partial_{\tilde{x}}\tilde{w}) & \varepsilon\,\partial_{\tilde{z}}\tilde{w} \end{pmatrix}, \quad |D\tilde{v}| = \frac{1}{\sqrt{2}}\sqrt{(\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\,\partial_{\tilde{x}}\tilde{w})^2 + 4\varepsilon^2\,\partial_{\tilde{x}}\tilde{u}^2}.$$

Then, we can write

$$Dv = \frac{U}{H}D\tilde{v} \quad \text{and} \quad |Dv| = \frac{U}{H}|D\tilde{v}|.$$

As for the tensor in this case of rheology, we can in fact write the following

$$\tau = 2\mu Dv + \tau^*\frac{Dv}{|Dv|} = \frac{2\mu U}{H}D\tilde{v} + \tau^*\frac{D\tilde{v}}{|D\tilde{v}|} = \frac{2\mu U}{H}(D\tilde{v} + \frac{H\tau^*}{2\mu U}\frac{D\tilde{v}}{|Dv|}).$$

We define as a result the Bingham number B by

$$B = \frac{H\tau^*}{2\mu U}.$$

Hence, we can define the non dimensional stress $\tilde{\tau} = D\tilde{v} + B\frac{D\tilde{v}}{|Dv|}$ which yields as well $\tau = \frac{2\mu U}{H}\tilde{\tau}$. Dropping the tilde for simplicity we obtain the following rescaled equations

$$\begin{aligned}
\partial_x u + \partial_z w &= 0,\\
\partial_t u + u\,\partial_x u + w\,\partial_z u &= -\frac{gH}{U^2}\,\partial_x p + \frac{2\mu L}{\rho U H^2}(\varepsilon\partial_x \tau_{xx} + \partial_z \tau_{xz}),\\
\partial_t w + u\,\partial_x w + w\,\partial_z w &= -\frac{gL^2}{HU^2}\,\partial_z p + \frac{2\mu L^2}{\rho H^3 U}(\varepsilon\partial_x \tau_{xz} + \partial_z \tau_{zz}) - \frac{gL^2}{HU^2}.
\end{aligned}$$

Now using the fact that the kinematic viscosity, Froude number and Reynolds number are respectively given by

$$\nu = \frac{\mu}{\rho}, \quad Fr^2 = \frac{U^2}{gH} \quad \text{and} \quad Re = \frac{HU}{\nu},$$

we get the following model

$$\begin{aligned}
\partial_x u + \partial_z w &= 0,\\
\varepsilon Re(\,\partial_t u + u\,\partial_x u + w\,\partial_z u) &= -\frac{\varepsilon Re}{Fr^2}\,\partial_x p + 2\varepsilon\partial_x \tau_{xx} + 2\partial_z \tau_{xz},\\
\varepsilon^3 Re(\,\partial_t w + u\,\partial_x w + w\,\partial_z w) &= -\frac{-\varepsilon Re}{Fr^2}\,\partial_z p + 2\varepsilon^2\partial_x \tau_{xz} + 2\varepsilon\partial_z \tau_{zz} - \frac{\varepsilon Re}{Fr^2}.
\end{aligned} \tag{2.44}$$

As argued before, if we balance pressure and viscous terms on the right hand side, we shall assume that $\frac{-\varepsilon Re}{Fr^2} = 1$ which contributes in fact in defining the scaling of characteristic velocity by $U = \frac{gH^3}{\nu L}$. The boundary conditions become

$$\begin{aligned}
u = w &= 0 && z = 0,\\
\partial_t h + u\,\partial_x h &= w && z = h(x,t),\\
(1 - \varepsilon^2(\,\partial_x h)^2)p + 2\varepsilon(1 + \varepsilon^2(\,\partial_x h)^2)\tau_{xx} &= 0 && z = h(x,t),\\
(1 - \varepsilon^2(\,\partial_x h)^2)\tau_{xz} - 2\varepsilon\,\partial_x h\tau_{xx} &= 0 && z = h(x,t).
\end{aligned} \tag{2.45}$$

And the rheology condition in the dimensionless case becomes

$$\tau := \begin{cases} Dv + B\frac{Dv}{|Dv|} & Dv \neq 0,\\ |\tau| < B & Dv = 0. \end{cases}$$

This means explicitly

$$\begin{cases} |\tau| < B & \text{if } \partial_x u = \partial_z w = \partial_z u + \varepsilon^2\,\partial_x w = 0,\\ \tau = \left(Id + \frac{B\sqrt{2}}{|Dv|}\right)\begin{pmatrix} \varepsilon\,\partial_x u & \frac{1}{2}(\,\partial_z u + \varepsilon^2\,\partial_x w)\\ \frac{1}{2}(\,\partial_z u + \varepsilon^2\,\partial_x w & \varepsilon\,\partial_z w \end{pmatrix} & \text{if } |Dv| \neq 0, \end{cases}$$

where

$$|Dv| = \sqrt{(\,\partial_z u + \varepsilon^2\,\partial_x w)^2 + 4\varepsilon^2(\,\partial_x u)^2}.$$

As we are concerned with first order approximation of the above rescaled system, we will formally take $\varepsilon \to 0$. We obtain

$$\begin{aligned}
\partial_x u + \partial_z w &= 0,\\
\partial_x p &= 2\partial_z \tau_{xz},\\
\partial_z p &= -1,
\end{aligned}$$

The constitutional law as well as the boundary conditions become respectively

$$\tau := \begin{cases} \tau_{xx} = \tau_{zz} = 0 \quad \text{and} \quad \tau_{xz} = \frac{1}{2} u_z \left(1 + \frac{B\sqrt{2}}{|u_z|}\right) & \text{if} \quad u_z \neq 0, \\ |\tau| < B & \text{elsewhere.} \end{cases}$$

$$\begin{aligned} u = w = 0 \qquad & z = 0, \\ \partial_t h + u\,\partial_x h = w \qquad & z = h(x,t), \\ p = 0 \qquad & z = h(x,t), \\ \tau_{xz} = 0 \qquad & z = h(x,t). \end{aligned}$$

Integrating in depth the obtained equations regarding the boundary conditions, we get

$$p = h - z \qquad \text{and} \qquad \tau_{xz} = \frac{\partial_x h}{2}(z - h). \tag{2.46}$$

Now if $\partial_z u \neq 0$, from the limit value of stress tensor we have that $\tau_{xz} = \frac{1}{2}\,\partial_z u \left(1 + \frac{B\sqrt{2}}{|\partial_z u|}\right)$, then we can write

$$\frac{\partial_z u}{|\,\partial_z u|}\left(|\,\partial_z u| + \sqrt{2}B\right) = \partial_x h(z - h) \quad \text{if } \partial_z u \neq 0$$

Regard that $(z - h)$ is negative, which means that $\partial_x h$ and $\partial_z u$ have opposite signs. Apply the absolute value on both sides of the equation we get

$$|\,\partial_z u| = |\,\partial_x h|\left(h - \frac{B\sqrt{2}}{|\,\partial_x h|} - z\right) \tag{2.47}$$

Setting $Y = \max(h - \frac{B\sqrt{2}}{|\partial_x h|}, 0)$, and regarding the sign of $\partial_x h$ and $\partial_z u$ to take off the absolute value, we get that whenever $\partial_z u \neq 0$, we fall in the region $0 \leqslant z \leqslant Y$, and that

$$\partial_z u = -|\,\partial_x h| sgn(\,\partial_x h)(Y - z) = -\,\partial_x h(Y - z).$$

Integrating now between 0 and $z \leqslant Y$, and using the boundary conditions we get

$$u = -\,\partial_x h(Yz - \frac{z^2}{2}). \tag{2.48}$$

Now if $\partial_z = 0$, then u is constant function in z (so it is a function of x and t), and hence from the continuity of u and from equation (2.48), we get that for $Y \leqslant z \leqslant h(x,t)$, u is given by

$$u = -\,\partial_x h \frac{Y^2}{2} \tag{2.49}$$

As for the averaged velocity (denoted U), we get by integrating from 0 to h

$$hU = \int_0^h u\; dz = \int_0^Y -\,\partial_x h(Yz - \frac{z^2}{2})\; dz + \int_Y^h -\,\partial_x h \frac{Y^2}{2} = \frac{-\,\partial_x h Y^2}{6}[3h - Y].$$

Using the continuity equation of the height, we recover finally the following dimensionless lubrication model

$$\partial_t h - \partial_x\left(\frac{\partial_x h Y^2}{6}[3h - Y]\right) = 0.$$

We can transform it to a dimensional form so that we get

$$\partial_t h - \frac{\rho g}{\mu}\,\partial_x\left(\partial_x h\left(h - \frac{\tau^*}{\sqrt{2}\rho g|\,\partial_x h|}\right)^2\left(2h + \frac{\tau^*}{\sqrt{2}\rho g|\,\partial_x h|}\right)\right) = 0. \tag{2.50}$$

2.4 Existence Result for Degenerate Lake model for Bingham fluids

This work is a collaboration with B. Al Taki and J. Sainte Marie.

In this work, we are interested in comprehending some existence and limit problems of Bingham lake models which decode many natural as well as industrial phenomena. Mathematically, such systems can be obtained from shallow water models of Bingham fluids by passing to the limit ($\mathrm{Fr} \to 0$), where the initial height converges to a non constant function $b(x)$ depending on the space variables only. In the corresponding paper, we succeeded to prove existence result of such models, as well as its convergence to the viscous lake model. We adopt the methodology of variational inequality used in [77] to prove the existence of a weak solution of the incompressible Bingham fluid confined to a shallow basin with a varying bottom topography but with some changes in the nature of spaces used. Again recalling that we are dealing with a degenerate bathymetry $b(x)$, meaning that $b(x)$ may vanish on the boundary, we prove that the solution exists in some weighted Sobolev space where the weight is assumed to be a Muckenhoupt type. Then, we discuss the behavior of solution when the yield limit tends to zero. The results are given in theorems 2.4 and 2.5. But first, lets introduce the model and problem setup.
The Bingham lake model is given by:

$$\begin{cases} \partial_t(bu) + \operatorname{div}(bu \otimes u) - \operatorname{div}\sigma + b\nabla p &= bf, \\ \operatorname{div}(bu) &= 0. \end{cases} \tag{2.51}$$

The shear stress σ satisfies the special constitutive law of a Bingham fluid:

$$\begin{cases} \sigma = 2\mu b\,\mathrm{D}(u) + \lambda b \operatorname{div} u\,\mathbb{I} + gb\dfrac{\mathrm{D}(u)}{|\,\mathrm{D}(u)|} & \text{if } \mathrm{D}(u) \neq 0, \\ |\sigma| < gb & \text{if } \mathrm{D}(u) = 0. \end{cases} \tag{2.52}$$

Here, $u(t,x)$ denotes the velocity vector, p the pressure, $f(t,x)$ the known external force, μ and λ the Lamé viscosity coefficients, g the yield limit. We couple the system (2.51) with the so called Lions' boundary conditions given by

$$bu \cdot n = 0 \qquad (\sigma \cdot n) \cdot \tau + \kappa(x) bu \cdot \tau = 0 \quad (t,x) \in (0,T) \times \partial\Omega, \tag{2.53}$$

and with the initial data (defined in a weak sense)

$$u(t,x)|_{t=0} = u_0(x), \quad x \in \Omega. \tag{2.54}$$

We will assume in addition this condition on the bathymetry (weight $b(x)$):

Hypothesis: (**I**) We define a space function $b(x)$, locally integrable and belonging to the *Muckenhoupt* class $\mathcal{A}_q$. Generally speaking, for a weight of *Muckenhoupt* type, the definition of trace operator is well defined. More precisely, one can check that if $b \in \mathcal{A}_q$, we have $u \in W^{1,q}_b(\Omega) \hookrightarrow W^{1,q}_{loc}(\overline{\Omega})$ and hence there is a linear trace operator $\gamma_{0,b} : W^{1,q}_{loc}(\overline{\Omega}) \to L^1_{loc}(\overline{\Omega})$. Though it is well defined, yet we lack characterization of such trace regarding a general Muckenhoupt weight. That's why we restrict ourselves in what follows to a more specific weight that provides a characterization of the boundary terms. Its expression is given in a neighborhood of the boundary $V(\partial\Omega)$ by

$$b = \rho^\alpha(x),\ 0 < \alpha < 1/2, \ \ \rho(x) = \operatorname{dist}(x, \partial\Omega) \text{ for } x \in V(\partial\Omega). \tag{2.55}$$

In this situation, the definition of the trace is more accurate. The main theorem is given below

Theorem 2.4. *(Existence of weak solutions). We suppose that f and u_0 are the applied force and the initial datum given such that f lies in $L^2(0,T;L^2_b(\Omega))$ and u_0 belongs to H_b. Assume that b satisfies Hypothesis (**I**), and that $\kappa(x)$ is in $L^\infty(\Omega)$, then there exists a unique vector field u such that*

$$u \in L^2(0,T;V_b) \quad \partial_t(bu) \in L^2(0,T;V_b') \tag{2.56}$$

satisfying

$$\int_\Omega f\cdot(v-u)\,b\,dx \leqslant \int_\Omega \partial_t u\cdot(v-u)\,b\,dx+\int_\Omega u\cdot\nabla u\cdot(v-u)\,b\,dx+g\int_\Omega(|\,\mathrm{D}(v)|-|\,\mathrm{D}(u)|)\,b\,dx$$
$$+2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\,b\,dx+\lambda\int_\Omega \operatorname{div}u\operatorname{div}(v-u)\,b\,dx$$
$$+\int_{\partial\Omega}\kappa u\cdot(v-u)\,b dx \qquad \textit{for all } v\in V_b,$$

and the initial condition defined in a weak sense:

$$\Big(\int_\Omega u\cdot v\,b\,dx\Big)(0)=\int_\Omega u_0\cdot v\,b\,dx \qquad \textit{for all} \quad v\in V_b.$$

This theorem is proved in several steps:

- Firstly, we must regularize the term j induced by the Bingham's singular term.
- Then we use a Faedo-Galerkin method to transform the problem in a finite dimension problem. We obtain a priori estimate independent of the dimension.
- Thanks to compactness results we pass to the limit.
- Finally, we prove that we can pass to the limit with respect to the regularization parameter.

After proving existence result, we prove that the rigid lid approximation for the viscous lake equation can be obtained by passing to the limit in the previous variational inequality when the Bingham coefficient vanishes. The second main theorem is given by

Theorem 2.5. *Let Ω a bounded Lipschitz domain and b satisfying Hypothesis (**I**). Consider the Bingham model (2.51). We suppose that all the parameters therein are fixed except for the yield stress limit g which is assumed to be independent of other parameters and rendered to vary (consequently taken to zero). Denote by u_g the constructed solution with initial data $u|_{t=0}=u_g(0)$ proved in existence theorem 2.4. Then, if $u_g(0)$ converges to u_0 in $L^2_b(\Omega)$ as g tends to zero, then up to a sub-sequence, we have*

$$\begin{cases} u_g\to u \quad \textit{weakly in} \quad L^\infty(0,T;H_b)\cap L^2(0,T;V_b),\\ bu_g'\to bu' \quad \textit{weakly in} \quad L^2(0,T;V_b'), \end{cases} \tag{2.57}$$

where u is a weak solution of the viscous Lake system and thus

$$\|u_g(t,x)-u(t,x)\|^2_{L^2_b(\Omega)}\to 0 \quad \textit{when} \quad g \quad \textit{tends to} \quad 0,$$

and the solution u of the viscous lake equations enjoys the following property

$$u\in L^\infty(0,T;H_b)\cap L^2(0,T;V_b). \tag{2.58}$$

And finally, a numerical scheme is proposed in the final part. It is based on a finite volume approximation with Rusanov upwind fluxes. Numerical results are presented at the end of chapter 7.

2.5 Dissipative Solutions for Oldroyd-B Fluids

This work is a collaboration with D. Bresch and E.Suli.

In general, visco-elastic fluids are very difficult to treat analytically in comparison with Newtonian fluids. The reason is that there is no universal constitutional law describing the rheology that is valid for all fluids. In some cases, such relation exhibits non linearities with respect to the strain (deformation

tensor), and sometimes the fluid has a strong memory due to its elastic behavior. Additional evolution equations on the stress are in many cases added to the system of motion, and in each time the subject of visco-elasticity becomes more challenging. A review on the mathematical modeling of constitutional laws for such fluids is demonstrated in Section 1.4.2.

In this work, we will prove the existence of global dissipative solution of the incompressible Oldroyd-B system in a two-dimensional periodic setting $\Omega = \mathbb{T}^2$ namely the following equations

$$\begin{cases} \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \nabla p = (1-w)\Delta u + \dfrac{w}{\lambda_1} \operatorname{div} Z, \\ \partial_t Z + u \cdot \nabla Z - (\nabla u) Z - Z(\nabla u)^T + \dfrac{1}{\lambda_1}(Z - \mathrm{Id}) = 0, \end{cases} \tag{2.59}$$

where $w \in [0,1]$, u is the velocity field, p the pressure and Z is a 2×2 matrix linked to the elastic part of the viscous stress. The parameter $\lambda_1 > 0$ is a fixed constant. Before giving the definition of a dissipative solution for the incompressible Oldroyd system, for (u_0, Z_0) smooth enough such that $\operatorname{div} u_0 = 0$, we define $\mathcal{L}_1(u_0, Z_0)$ and $\mathcal{L}_2(u_0, Z_0)$ as follows

$$\begin{cases} \mathcal{L}_1(u_0, Z_0) = \partial_t u_0 + \mathcal{P}(u_0 \cdot \nabla u_0 - \dfrac{w}{\lambda_1} \operatorname{div} Z_0) - (1-w)\Delta u_0, \\ \mathcal{L}_2(u_0, Z_0) = \partial_t Z_0 + u_0 \cdot \nabla Z_0 - (\nabla u_0) Z_0 - Z_0 (\nabla u_0)^T + \dfrac{1}{\lambda_1}(Z_0 - \mathrm{Id}), \end{cases} \tag{2.60}$$

where $\mathcal{P}$ is the Leray-Helmholtz projector.

Definition 2.5. *A couple (u, Z) is said to be a dissipative solution of (2.59) if*

$$u \in L^2(0,T;(H^1(\Omega))^2) \cap L^\infty(0,T;(L^2(\Omega))^2) \cap \mathcal{C}_{\text{weak}}([0,T];(L^2(\Omega))^2) \quad \text{with} \quad \operatorname{div} u = 0$$

$$Z \in L^\infty(0,T;(L^2(\Omega))^{2\times 2}) \cap \mathcal{C}_{\text{weak}}([0,T];(L^2(\Omega))^{2\times 2}) \text{ is a symmetric positive definite matrix}$$

and for each smooth couple (u_0, Z_0) such that $\operatorname{div} u_0 = 0$ with Z_0 a symmetric positive definite matrix then the couple (u, Z) satisfies the following inequality

$$C\mathcal{E}(t) \leqslant \mathcal{D}_0 \exp^{\int_0^t C_3(s)\ ds} + \int_0^t \int_\Omega \exp^{\int_s^t C_3(\tau)\ d\tau} \big(2\mathcal{L}_1 \cdot (u_0 - u) + \frac{w}{\lambda_1} \mathcal{L}_2 : Z_0^{-2}(Z_0 - Z)\big) ds.$$

where $\mathcal{E}(t)$ is given by

$$\mathcal{E}(t) = \int_\Omega |u - u_0|^2 + \frac{w}{\lambda_1} |Z - Z_0|^2,$$

and $\mathcal{D}_0$ is given by

$$\mathcal{D}_0 = \int_\Omega |u^0 - u_0(0,\cdot)|^2 + \frac{w}{\lambda_1} \big(- \log\det Z^0 + \log\det Z^0(0,\cdot) + \mathbb{T}_r((Z^0(0,\cdot))^{-1}(Z^0 - Z_0(0,\cdot)))\big),$$

and C_3 is a constant depending on (u_0, Z_0) given by

$$C_3 = \frac{w^{1/2}}{\lambda_1^{1/2}} \|\nabla Z_0^{-1}\|_{L^\infty} + \frac{2w}{\lambda_1(1-w)} \|Z_0^{-1}\|_{L^\infty}^2 + \|\nabla u_0\|_{L^\infty} + \frac{w}{\lambda_1(1-w)}$$

and $C > 0$ is a constant depending on Z_0 and Ω only.

The first main result concerns the incompressible Oldroyd system and reads

Theorem 2.6. *Let (u^0, Z^0) be the initial data in $L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ such that $\operatorname{div} u^0 = 0$. If there exists a global weak solution $(u_\varepsilon, Z_\varepsilon)$ of the regularized incompressible Oldroyd system satisfying the energy*

inequality and in which the stress Z_ε is assumed to be a uniformly bounded, uniformly symmetric positive definite matrix in terms of ε, then there exists a global dissipative solution (u, Z) of System (8.1) in the sense of definition 8.10.

To prove Theorem 2.6, we start from the existence of global weak solution of the regularized system proved by Barret and Boyaval in [78] which is valid in the two dimensional periodic setting and which satisfies the energy inequality. However, we will assume further to the result proved by the latter authors that the global weak solution satisfies the energy inequality in which Z_ε is a uniformly bounded symmetric positive definite matrix in ε. It is good to remark that for the proof of theorem 2.6, we can start from any global weak solution of the regularized system on a condition that the stress satisfies being bounded and symmetric positive definite uniformly with respect to ε. In our approach, out of these global solutions we choose to start from that constructed by Barret and Boyaval in [78] under the condition that it satisfies the constraint on the stress. We first prove that such global weak solution is a dissipative solution in the sense of definition 2.5 with $\mathcal{L}_2^\varepsilon(u_0, Z_0) = \mathcal{L}_2(u_0, Z_0) - \varepsilon\Delta Z_0$. We then pass to the limit easily to get the global existence of weak solution of the Oldroyd-B system without extra regularization on Z. A reflection on this work shows that we still lack a complete strategy and method that could contribute to a weak solution of Oldroyd systems without having uniformity constraints on the stress. This explains the approaches in which FENE-P systems were considered, or where regularization of the stress was assumed to prove weak solutions. However, such relative entropy could still be powerful in case a strong solution is discussed, where the entropy would provide the weak-strong uniqueness of the solution.

Part II

Newtonian Flows with Free Surfaces

3

Derivation Of Viscous Newtonian Shallow Water Models

The work in this chapter is a collaboration with C. Ruyer-Quil, University Savoie Mont Blanc, LOCIE, Chambéry

In this work, we have tried to derive refined 3-equation shallow water models in the spirit of two approaches; the one introduced in [3] and the weighted residual technique (WRM) used in [2]. The obtained models are consistent at first order. Good numerical results are obtained for the second order model in 1D using WRM. Our derivation is done starting from the incompressible Navier Stokes system for a free surface fluid driven by gravity and flowing down an inclined plane.

The chapter is organized as follows:

-**The first section 3.1** is an introductory section where we present a small historical background of the context of simplified models derived from Navier Stokes system (NS) with free surface.

-**The second section 3.2** is where we present the primitive system on which our postulates and analysis is applied; which is the incompressible NS system for inclined thin liquid films driven by gravity, we exhibit the scaling appropriate for such physical setting, and finally we present briefly the main results of the chapter.

-**The third section 3.3** presents a review for the derivation of shallow water model for thin liquid films using the momentum integral method (MIM) presented and justified in [1].

-**The fourth section 3.4** presents a review for the derivation of shallow water model using weighted residual technique introduced in [2].

-**The fifth section 3.5** presents a review for the derivation of the 3-equation shallow water model in [3].

-**The sixth section 3.6** is a revisit for the 3-equation model using the technique provided in [3] but this time coupled with a velocity profile.

-**The seventh section 3.7** provides the derivations for a new 3-equation models using the weighted residual technique in 1D (including second order diffusion terms) and in 2D.

-**The eighth section 3.8** is the last section where we present a brief validation for the model in 1D.

3.1 Introduction

Thin liquid films driven by gravity constitute the topic of this chapter. They occupy a wide variety of phenomenological cases including pipe systems, tide waves and numerous engineering applications. A highlight on the importance and on the distinguishing of such cases is depicted in the first chapter. The motion of such films is dominated in a "juggling" manner between three different dynamical aspects: viscosity, inertia and surface tension, that, as now understood by everyone, are not clearly and easily elaborated using NS system neither analytically nor numerically. The key solution is in a modeling process that manifests a balance between flow fields and cancels out those of negligible impact on motion. The universal technique of derivation is known as residual methods which is mostly coupled with a perturbation analysis. It provides different levels of compressible models, called shallow water type models, in an increasing order of accuracy. These models are characterized by encoding the boundary conditions associated to the physical system, and thus they rely strongly on the kinematics and dynamics of the boundary. The process of how the negligible features are filtered off is often more important than the final model itself, as the process conveys the deep comprehension of the physical mechanism and helps understand how the dynamical balance leads to determine the kinetics of the flow defining by that the different possible flow regimes. More precisely, the reason of criticality of the process lies behind the success in capturing the unstable regimes. Long wavelength infinitesimal perturbations down sufficiently high slopes cause the stationary solution of the film to become unstable, which creates a train of traveling waves of high frequencies (sturated periodic waves), and low frequencies (solitary waves of one hump preceded by capillary ripples). Thus, our greatest goal is to manifest the best modeling technique to capture the instabilities in the flow.

Residual method contain in the elementary frame the momentum integral method MIM, as well as the Galerkin and collocation method [30]. The first viscous shallow water model was deduced by Shkadov in [31] using the MIM method. His work in fact is an extension of the theory of lubrication in order to correct the glitch that exists in the latter theory. Integral method has also enjoyed a lot of interest in the past couple of years. Two main approaches that drove attention in the literature and gave satisfactory results are those derived by J.F Gerbeau and B. Perthame [32] and that by J.P Vila in [33], which is also revisited and justified theoretically by Bresch and Noble in [1]. The main difference between the two models is in the boundary condition assumed. Where the former chose a Navier-type boundary condition (friction condition on the bottom), a no-slip boundary condition was presumed by the latter. In 1998, C. Ruyer Quil and P. Manneville derived a new shallow water type model using mixed integral-collocation method [79]. Their model is a 3 equation model; the first two are of shallow water type in terms of the height h and the flow discharge rate q, and the third is a coupled equation for a supplementary variable measuring the departure of the wall shear stress from that predicted by a parabolic velocity profile. The velocity field was approximated by an expansion of polynomial test functions appearing in the derivation of Benny's equation. Though this model gave satisfactory numerical and experimental results for the instability threshold and the wave phase at moderate distance from threshold, finite time blow up from the threshold still occurs. In 2000, the same authors developed another improved shallow water type model to get rid of the previous glitches depending on a Galerkin method (which will be denoted WRM) in which the test projection functions and weight functions of the approximation of the velocity profile are identical [2]. Recently in 2016, Richard and others derived a new set of 3 equation Shallow Water type model [3]. The model describes the evolution of film thickness h and discharge rate q on one hand and the transport behavior of a new variable: the enstrophy φ, related to the averaged velocity variance on the other hand. It is derived using integral method with coupled with the energy equation. The pros of such model is in its mathematical structure that is governed by well known results of both well posedness and robust numerical schemes. Such structure encode the three physical equations: mass, momentum, and energy conservation, and displays all physical features through relaxation source terms, classical disperse capillary term and viscous diffusion terms. A drawback of such model is that it does not involve the notion of profile in the sense that the system is not closed using assumptions on the stream wise velocity field, and thus this latter lacks from the study.

In this chapter, we will deal with modulated techniques to get shallow water models by joining both the integral (MIM) and the weighted residual method (WRM) and benefiting from the advantage that each method provides. In fact, such models face three main problems:

* inaccurate physical depiction of the nonlinear regimes and transition to turbulence,
* possessing numerical discretization difficulty,
* structural incompatibility between the derived system and the kinetic energy.

The first one of the above difficulties is a main problem in one equation problems, but as 2-equation problems are concerned, the problem is partially solved. Since non linear solitary waves are quite good in replicating a shallow water wave in its extreme sense, then they are a good test to check for the validity of the model in the non linear regime. Such models from MIM and WRM succeed in predicting the speed of solitary waves at critical Reynolds numbers, however they fail in obtaining an accurate amplitude of the wave. Add to this, higher order models of such techniques can always improve the results on instabilities. On the other hand, where both techniques develop consistent models at main and first order, yet they deliver different structures. The inertial terms in the MIM preserve their conservative form and this is not the case of the WRM. This contributes to difficulties both in numerical and analytical resolutions for the weighted residual models as well as structural incompatibility between the model and the corresponding energy.

In what follows, we will discuss two different approaches that would result in a three equation model of the height, the depth averaged velocity and a third variable which is either the enstrophy defined in [3] (first approach), or a variable equivalent to the shear rate (second approach). The first approach that we launch in this context is in joining the mechanism followed in [3] in comparing the averaged energy equation with the derived kinetic equation of the averaged momentum equation. However, the difference is in adopting the notion of profile- the thing neglected in [3]- which depends on both the mean velocity and the velocity variance (or enstrophy). Such dependence would include naturally the relaxation in terms of the enstrophy which was induced manually in the formal approach. The main aim of modifying the initial method is to have as a first step a velocity profile which would enhance the numerical results, and next is to obtain automatically the relaxation terms from the derivation process without the need to modify them using asymptotic expansions and this would give more physical intuition of the corresponding terms and thus the system.

The second approach is to our knowledge the first approach in such direction: we derive the 3 equation model using the weighted residual method discussed in [2] in which the residues are truncated by means of projection method which cancels them rather than truncating them by approximation techniques. The projection method follows Galerkin approach where the set of orthogonal basis in L^2 are carefully chosen such that they coincide with the weights in the variables' expansions, and this results in residues which help in obtaining the parameters of the weights of the velocity profile. Thus, such residues target two goals at once, the velocity profile and the system of equations. We carry the derivation in 1D (including diffusion terms) and in 2D. A solitary wave test is also launched for the 1D case.

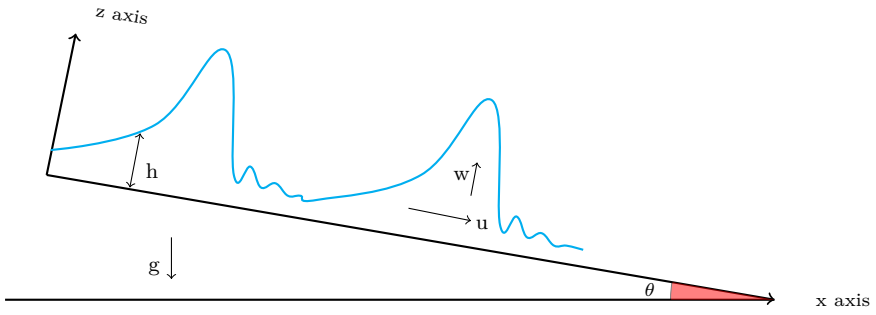


Figure 3.1: *thin liquid film moving down an inclined plane in 2D*

3.2 Preliminary System, Scaling and Main Results

3D System Setup

The 3D incompressible Navier Stokes system for a free surface Newtonian fluid with constant density $\rho = 1$ and constant viscosity μ inclined with an angle θ with the horizontal plane is given by

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p &= \rho g \, \vec{e} + \mu \Delta \mathbf{u}. \end{aligned} \tag{3.1}$$

The motion is driven along the inclined plane such that the x-axis is directed along the stream wise direction and the y-axis is along the span wise direction. The z-axis denotes the direction perpendicular to the inclined plane spanned by the x-axis and y-axis. The velocity field is denoted $\mathbf{u} = (u, v, w) = u\vec{i} + v\vec{j} + w\vec{k}$, p is the hydrostatic pressure, and $g\vec{e} = g(\sin\theta, 0, \cos\theta)$ is the gravitational acceleration. To preserve the vector representation of the system in the non dimensional case (later on), we will split the velocity in the plane velocity denoted $\mathbf{u}_X = (u, v)$ and $\mathbf{u}_Z = w$. Consequently, $\nabla_X = (\partial_x, \partial_y)$ and $\nabla_z = \partial_z$. We will consider a no-slip boundary condition on the solid substrate

$$\mathbf{u}|_{z=0} = 0 \implies u|_{z=0} = v|_{z=0} = w|_{z=0} = 0, \tag{3.2}$$

a kinematic condition of the material surface $h(x, y, z, t)$

$$w|_{z=h} = \partial_t h + \mathbf{u}_X|_{z=h} \cdot \nabla_X h = u|_{z=h}\,\partial_x h + v|_{z=h}\,\partial_y h, \tag{3.3}$$

and assuming the surface is subjected to surface tension forces, then due to continuity of the stress at the free boundary we write

$$-\,p|_{z=h}\;\vec{\boldsymbol{n}} + 2\,\mathrm{D}(\mathbf{u})|_{z=h} \cdot \vec{\boldsymbol{n}} = -(\kappa\,\mathrm{div}\,\vec{\boldsymbol{n}})\vec{\boldsymbol{n}}, \tag{3.4}$$

where κ is the surface tension, $\vec{\boldsymbol{n}}$ is the inward normal to the tangent plane of motion. $\mathrm{D}(\mathbf{u})$ is the total deformation tensor defined by $\mathrm{D}(\mathbf{u}) = \frac{\nabla u + \nabla u^T}{2}$, which is also could be written as

$$\mathrm{D}(\mathbf{u}) = \begin{pmatrix} \mathrm{D}_X(\mathbf{u}_X) & \frac{(\partial_z \mathbf{u}_X + \nabla_X w)}{2} \\ \frac{(\partial_z \mathbf{u}_X + \nabla_X w)^T}{2} & \partial_z w \end{pmatrix}.$$

Scaling

The scaling is made relative to the uniform motion of parallel plate flow, i.e to say it is related to the Nusselt solution taken such that we have the constant height H_N and uniform plane velocity $U_N = V_N$. The Reynolds number and Weber number are calculated accordingly, and the Kapitza number is related to this latter through the relation $\mathbf{K} = \kappa H_N^2$. Setting the characteristic wavelength L and the characteristic film thickness H_N, we define the aspect ratio of the film $\varepsilon = H_N/L$ and we choose the following non dimensional coordinates and variables

$$x = L\bar{x}, \qquad y = L\bar{y}, \qquad z = H_N\bar{z}, \qquad t = \frac{L}{U_N}\bar{t},$$
$$h = H_N\bar{h}, \qquad u = U_N\bar{u}, \qquad v = U_N\bar{v}, \qquad w = \varepsilon U_N\bar{w}, \quad \text{and} \quad p = \rho g H_N\bar{\rho}.$$

Consequently, we obtain the following rescaled system

$$\begin{aligned}
&\partial_{\bar{x}}\bar{u} + \partial_{\bar{y}}\bar{v} + \partial_{\bar{z}}\bar{w} = 0\\
&\frac{\rho U_N^2}{L}(\partial_{\bar{t}}\bar{u} + \bar{u}\,\partial_{\bar{x}}\bar{u} + \bar{v}\,\partial_{\bar{y}}\bar{u} + \bar{w}\,\partial_{\bar{z}}\bar{u})\\
&\qquad = \rho g\sin\theta - \frac{\rho g H_N}{L}\,\partial_{\bar{x}}\bar{p} + \frac{\mu U_N}{L^2}\,\partial_{\bar{x}\bar{x}}\bar{u} + \frac{\mu U_N}{L^2}\,\partial_{\bar{y}\bar{y}}\bar{u} + \frac{\varepsilon\mu U_N}{H_N^2}\,\partial_{\bar{z}\bar{z}}\bar{u},\\
&\frac{\rho U_N^2}{L}(\partial_{\bar{t}}\bar{v} + \bar{u}\,\partial_{\bar{x}}\bar{v} + \bar{v}\,\partial_{\bar{y}}\bar{v} + \bar{w}\,\partial_{\bar{z}}\bar{v})\\
&\qquad = -\frac{\rho g H_N}{L}\,\partial_{\bar{y}}\bar{p} + \frac{\mu U_N}{L^2}\,\partial_{\bar{x}\bar{x}}\bar{v} + \frac{\mu U_N}{L^2}\,\partial_{\bar{y}\bar{y}}\bar{v} + \frac{\varepsilon\mu U_N}{H_N^2}\,\partial_{\bar{z}\bar{z}}\bar{v},\\
&\frac{\varepsilon\rho U_N^2}{L}(\partial_{\bar{t}}\bar{w} + \bar{u}\,\partial_{\bar{x}}\bar{w} + \bar{v}\,\partial_{\bar{y}}\bar{w} + \bar{w}\,\partial_{\bar{z}}\bar{w})\\
&\qquad = -\rho g\cos\theta - \rho g\,\partial_{\bar{z}}\bar{p} + \frac{\varepsilon\mu U_N}{L^2}\,\partial_{\bar{x}\bar{x}}\bar{w} + \frac{\varepsilon\mu U_N}{L^2}\,\partial_{\bar{y}\bar{y}}\bar{w} + \frac{\varepsilon\mu U_N}{H_N^2}\,\partial_{\bar{z}\bar{z}}\bar{w}.
\end{aligned} \tag{3.5}$$

The above equations convey inertial forces, external forces (gravity) and viscous forces, so we predict to have two main parameters that would appear as a result of scaling besides the aspect ratio: Froude number and Reynolds number (as explained in the introduction of the thesis). The Froude number is supposed to appear infront of external forces whereas the Reynolds number should appear infront of inertial terms (or alternatively $\frac{1}{Re}$ appears infront of viscous terms). Dropping the bar for simplicity and recalling the Reynolds number, Weber number and Froude number as

$$Re = \frac{\mu}{\rho U_N H_N} \qquad Fr^2 = \frac{U_N^2}{gH_N} \quad \text{and} \quad W_e = \frac{\rho g H_N^2}{\kappa}, \tag{3.6}$$

we obtain

$$\begin{aligned}
&\partial_t u + u\,\partial_x u + v\,\partial_y u + w\,\partial_z u = \frac{\sin\theta}{\varepsilon Fr^2} - \frac{\partial_x p}{Fr^2} + \frac{1}{\varepsilon Re}(\partial_{zz}u + \varepsilon^2\,\partial_{xx}u + \varepsilon^2\,\partial_{yy}u),\\
&\partial_t v + u\,\partial_x v + v\,\partial_y v + w\,\partial_z v = -\frac{\partial_y p}{Fr^2} + \frac{1}{\varepsilon Re}(\partial_{zz}v + \varepsilon^2\,\partial_{xx}v + \varepsilon^2\,\partial_{yy}v),\\
&\partial_t w + u\,\partial_x w + v\,\partial_y w + w\,\partial_z w = \frac{-\cos\theta}{\varepsilon^2 Fr^2} - \frac{\partial_z p}{\varepsilon^2 Fr^2} + \frac{1}{\varepsilon Re}(\partial_{zz}w + \varepsilon\,\partial_{xx}w + \varepsilon\,\partial_{yy}w).
\end{aligned} \tag{3.7}$$

As mentioned in the beginning of this part, shallow water applications are more concerned with the balance between pressure and inertial forces, thus we expect the coefficient infront of the pressure to balance the inertial coefficient, and thus we have the ansatz that

$$\frac{1}{Fr^2} = O(1).$$

The above system can be rewritten in a vector form as

$$\begin{aligned}
&\operatorname{div}\mathbf{u} = 0,\\
&\partial_t\mathbf{u}_X + \mathbf{u}\cdot\nabla\mathbf{u}_X = \frac{\sin\theta}{\varepsilon Fr^2}\vec{e_1} - \frac{\nabla_X p}{Fr^2} + \frac{1}{\varepsilon Re}(\partial_{zz}\mathbf{u}_X + \varepsilon^2\Delta_X\mathbf{u}_X),\\
&\partial_t\mathbf{u}_z + \mathbf{u}\cdot\nabla\mathbf{u}_z = -\frac{\cos\theta}{\varepsilon^2 Fr^2} - \frac{\partial_z p}{\varepsilon^2 Fr^2} + + \frac{1}{\varepsilon Re}(\partial_{zz}\mathbf{u}_z + \varepsilon^2\Delta_X\mathbf{u}_z).
\end{aligned} \tag{3.8}$$

The rescaled boundary conditions read

$$\begin{aligned}
&u|_{z=0} = v|_{z=0} = w|_{z=0} = 0,\\
&\partial_t h + \nabla_X h \cdot \mathbf{u}_X|_{z=h} = \mathbf{u}_z|_{z=h},\\
&\partial_z \mathbf{u}_X^T|_{z=h} + \varepsilon^2 \nabla_X \mathbf{u}_z|_{z=h} = \varepsilon^2 \nabla_X h \left(\mathrm{D}_X\, \mathbf{u}_X|_{z=h} - \frac{2\nabla_z \mathbf{u}_z|_{z=h} - \varepsilon^2 \nabla_X h^T\, \mathrm{D}_X\, \mathbf{u}_X|_{z=h} \nabla_X h}{1-\varepsilon^2|\nabla_X h|^2} \right),\\
&p|_{z=h} = \frac{\varepsilon Fr^2}{Re} \left(\frac{2\nabla_z \mathbf{u}_z|_{z=h} - \varepsilon^2 \nabla_X h^T\, \mathrm{D}_X\, \mathbf{u}_X|_{z=h} \nabla_X h}{1-\varepsilon^2|\nabla_X h|^2} \right) - \frac{\varepsilon^2}{W_e} \frac{\Delta_X h}{(1+\varepsilon^2|\nabla_X h|^2)^{\frac{3}{2}}}.
\end{aligned} \tag{3.9}$$

It is good to shed light on the fact that when deriving liquid films, we count on the capillarity effects due to the crucial role they play in the energy transfer in the nonlinear regimes, and thus in the instability study in the film, and hence the secnd ansatz on the parameters in the system is to take $\frac{\varepsilon^2}{W_e} = O(1)$, we will replace it again for simplicity by κ. Thus, at main order (since in the derivation we will only need main order boundary conditions), the above boundary conditions are translated into the following

$$\begin{aligned}
&u|_{z=0} = v|_{z=0} = w|_{z=0} = 0,\\
&\partial_t h + \partial_x hu|_{z=h} + \partial_y hv|_{z=h} = w|_{z=h},\\
&p|_{z=h} = -\kappa(\partial_{xx} h + \partial_{yy} h),\\
&\partial_z u|_{z=h} = \partial_z v|_{z=h} = 0.
\end{aligned} \tag{3.10}$$

For simplicity in the following sections we will denote by $\lambda = \frac{\sin\theta Re}{Fr^1}$.

Main Results

In this work, we have launched two approaches to derive 3-equation models.

- The first one introduces the same variable as done in [3] and exploits as well the momentum integral method for the three equation model derived in the latter reference. Mainly in [3] they introduced a third variable: the enstrophy, which measures the departure of the velocity from its average. It is defined by

$$h^3\varphi = \int_0^h (u-U)^2\, dz,$$

where u is the velocity in the stream wise direction for a 2D flow and U is the depth averaged velocity $U = \frac{1}{h}\int_0^h u\, dz$. The system in [3] reads

$$\begin{cases}
\partial_t h + \partial_x(hU) = 0\\
\partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{2Fr^2}h^2) = \frac{1}{\varepsilon Re}(1+f_1)(\lambda h - \frac{3U}{h}) + \frac{f_2}{\varepsilon Re}(\varphi - \frac{\lambda^2 h^2}{45})\\
\qquad\qquad + \frac{\kappa}{Fr^2} h\, \partial_x^3 h + \frac{\varepsilon}{Re}\partial_x(\frac{109}{32} h\, \partial_x U);\\
\frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) = -\frac{U}{\varepsilon Re}\frac{2f_1}{3}(\lambda h - \frac{3U}{h}) - \frac{U}{\varepsilon Re}\frac{2f_2}{3}(\varphi - \frac{\lambda^2 h^2}{45})\\
\qquad\qquad + \frac{\varepsilon}{Re}\frac{109}{32} h(\partial_x U)^2 + \frac{\varepsilon}{Re}\partial_x(\frac{175}{48} h^3\, \partial_x \varphi).
\end{cases}$$

f_1 and f_2 are parameters with some degrees of freedom due to the use of asymptotic expansions to express equivalence between different terms.

The novelty in our work however is in introducing a velocity profile related to this new variable which allows to obtain the relaxation terms automatically without manipulating other terms in the system. Also, there is difference in the way we treat the cubic term that arises from inertial terms in the energy equation $\int_0^h (u-U)^3\, dz$ between the initial and our approach. In both, the asymptotic expansion was used to deal with such term, the difference is that in the former approach this term was added to relaxation effects , however, since it descends from inertial terms, we saw that it would be more physically relevant to add this contribution to inertial part. Our adopted

velocity profile reads

$$u = 3Ug_0 + \frac{1}{6\alpha\beta U}(h^2\varphi - \frac{U^2}{5})G_1 + O(\varepsilon)$$

where g_0 is the polynomial that appears in the main order velocity profile of the Nusselt solution: $g_0 = \frac{z}{h} - \frac{z^2}{2h^2}$, and G_1 is a function that is to be determined through the derivation of the system due to the constraints we have on u, and α and β are parameters related to G_1. The derived system in this context reads

$$\begin{aligned}
&\partial_t h + \partial_x(hU) = 0,\\
&\partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h - \frac{3U}{h}) + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + \frac{105}{\varepsilon Re}\frac{1}{6hU}(h^2\varphi - \frac{U^2}{5}),\\
&\frac{3h^2}{4}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) = -\frac{105}{6h}\frac{1}{\varepsilon Re}(h^2\varphi - \frac{U^2}{5}).
\end{aligned}$$

- In the second approach, we used the weighted residual method (WRM) to derive 3-equation models both for 2D and 3D flows. The advantage in using WRM is that there is no need to use asymptotic expansions to close the system, and also we have an exact velocity profile that embeds a relation with the new variable to be defined, and this would eventually lead to obtain good results for the relaxation terms compatible with the eigenmodes (damping coefficients) of a perturbed viscous film. We will state the 2D result, and the 3D follows in the same manner. We started from a velocity profile in the spirit of what has been done in [2], and we defined a new variable Φ equivalent at main order to the shear rate; it reads

$$\frac{h^3\Phi^2}{5} = \int_0^h (u-U)^2\, dz.$$

Consequently we could express the velocity profile in terms of Φ in addition to a corrected term, i.e

$$u = 3Ug_0 + \frac{1}{6}(U - h\Phi)(45g_1 - 3g_0) + \varepsilon\tilde{u}_1,$$

where g_1 is the second polynomial appearing in the expansion of the velocity profile done in [2] (the first is g_0), and $\tilde{u}_1$ is the correction. This definition of the velocity profiles introduces two gauge conditions on $\tilde{u}_1$, and thus applying the weighted residual technique we obtain the following system (for 2D flow):

$$\begin{aligned}
\partial_t U = &-\frac{14\cos\theta\,\partial_x h}{15\mathrm{Fr}^2} + \frac{14}{15}\frac{\kappa}{Fr^2}h\,\partial_x^3 h - \frac{116U^2\,\partial_x h}{1287h} + \frac{1036U\,\partial_x h\Phi}{2145} - \frac{6272h\,\partial_x h\Phi^2}{6435}\\
&- \frac{4067h^2\Phi\,\partial_x\Phi}{6435} + \frac{21\Phi}{5h\mathrm{Re}\varepsilon} + \frac{476h\,\partial_x U\Phi}{2145} + \frac{322hU\,\partial_x\Phi}{1287} - \frac{532}{429}U\,\partial_x U - \frac{7U}{h^2\mathrm{Re}\varepsilon} + \frac{14\lambda}{15\mathrm{Re}\varepsilon}\\
&+ \frac{\varepsilon}{Re}\Big(-\frac{7}{12}\,\partial_x h\,\partial_x\Phi - \frac{63(\partial_x h)^2\Phi}{40h} - \frac{175}{96}\,\partial_{xx}h\Phi + \frac{109\,\partial_x h\,\partial_x U}{24h} - \frac{13(\partial_x h)^2 U}{24h^2}\\
&\qquad + \frac{113\,\partial_{xx}hU}{96h} + \frac{49}{60}h\,\partial_{xx}\Phi + \frac{83}{24}\,\partial_{xx}U\Big)
\end{aligned}$$

$$\begin{aligned}
\partial_t\Phi = &-\frac{\cos\,\partial_x h}{3\mathrm{Fr}^2 h} + \frac{\kappa}{3Fr^2}\,\partial_x^3 h + \frac{290U\,\partial_x h\Phi}{429h} - \frac{826\,\partial_x h\Phi^2}{1287} - \frac{21\Phi}{h^2\mathrm{Re}\varepsilon} + \frac{160U^2\,\partial_x h}{9009h^2}\\
&+ \frac{1070U\,\partial_x U}{3003h} - \frac{574h\Phi\,\partial_x\Phi}{1287} + \frac{70}{429}\,\partial_x U\Phi - \frac{830U\,\partial_x\Phi}{1287} + \frac{20U}{h^3\mathrm{Re}\varepsilon} + \frac{\lambda}{3h\mathrm{Re}\varepsilon}\\
&+ \frac{\varepsilon}{Re}\Big(\frac{139\,\partial_x h\,\partial_x\Phi}{24h} - \frac{15(\partial_x h)^2\Phi}{2h^2} - \frac{247\,\partial_{xx}h\Phi}{96h} + \frac{85\,\partial_x h\,\partial_x U}{6h^2} + \frac{125(\partial_x h)^2 U}{24h^3}\\
&\qquad + \frac{305\,\partial_{xx}hU}{96h^2} + \frac{10\,\partial_{xx}U}{3h} + \frac{55}{24}\,\partial_{xx}\Phi\Big).
\end{aligned}$$

However, using asymptotic expansions, we proposed to manipulate the convected part so that it meets a conservative formula that would be also good to derive an energy for the system. Hence finally we got

$$\begin{aligned}
&\partial_t h + \partial_x(hU) = 0,\\
&\partial_t(hU) + \partial_x(hU^2 + \frac{1}{5}h^3\Phi^2) = \frac{1}{\varepsilon Re}\left(\frac{14}{15}(\lambda h - \frac{3U}{h}) + \frac{21}{5}(\Phi - \frac{U}{h})\right)\\
&\qquad\qquad - \frac{14}{15}\left(\frac{\cos\theta}{Fr^2}h\,\partial_x h - \frac{\kappa}{Fr^2}h\,\partial_x^3 h\right) + \frac{\varepsilon}{Re}\mathcal{D}_1,\\
&\partial_t(h\Phi) + \partial_x(hU\Phi) - \frac{1}{7}\frac{\partial_x(h^4\Phi^3)}{h^2\Phi} = \frac{1}{\varepsilon}\left(\frac{1}{3}(\lambda - \frac{3U}{h^2}) + 21(\frac{U}{h^2} - \Phi)\right)\\
&\qquad\qquad - \frac{1}{3}\left(\frac{\cos\theta}{Fr^2}\,\partial_x h - \frac{\kappa}{Fr^2}\,\partial_x^3 h\right) + \frac{\varepsilon}{Re}\mathcal{D}_2,
\end{aligned}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are diffusion terms given by

$$\begin{aligned}
\mathcal{D}_1 = &-\frac{7}{12}h\,\partial_x h\,\partial_x\Phi - \frac{63(\partial_x h)^2\Phi}{40} - \frac{175}{96}h\Phi\,\partial_{xx}h + \frac{109\,\partial_x h\,\partial_x U}{24} - \frac{13(\partial_x h)^2 U}{24h}\\
&+ \frac{113\,\partial_{xx}hU}{96} + \frac{49}{60}h^2\,\partial_{xx}\Phi + \frac{83}{24}h\,\partial_{xx}U,\\
\mathcal{D}_2 = &\frac{139}{24}\,\partial_x h\,\partial_x\Phi - \frac{15(\partial_x h)^2\Phi}{2h} - \frac{247\,\partial_{xx}h\Phi}{96} + \frac{85\,\partial_x h\,\partial_x U}{6h} + \frac{125(\partial_x h)^2 U}{24h^2}\\
&+ \frac{305\,\partial_{xx}hU}{96h} + \frac{10\,\partial_{xx}U}{3} + \frac{55}{24}h\,\partial_{xx}\Phi.
\end{aligned}$$

3.3 Overview on the Momentum Integral Method (MIM) Justified in [1]

In this part, we will give a brief overview on the derivation of a shallow water model as that reviewed and justified in [1]. However, we will do that in 2D, and thus the domain of study is $\Omega_t = \{(x,z) \in \mathbb{R}^2 : 0 < z < h(x,t)\}$. In reference to previous section, but projecting the plane (xOy)n the stream wise direction only, the non dimensional incompressible Navier Stokes system with free surface boundary conditions is given by

$$\begin{aligned}
&\partial_x u + \partial_z w = 0,\\
&\partial_t u + u\,\partial_x u + w\,\partial_z u = \frac{\sin\theta}{\varepsilon Fr^2} - \frac{1}{Fr^2}\,\partial_x p + \frac{1}{\varepsilon Re}(\varepsilon^2\partial_{xx}u + \partial_{zz}u),\\
&\partial_t w + u\,\partial_x w + w\,\partial_z w = -\frac{\cos\theta}{\varepsilon^2 Fr^2} - \frac{1}{\varepsilon^2 Fr^2}\,\partial_z p + \frac{1}{\varepsilon Re}(\varepsilon^2\partial_{xx}w + \partial_{zz}w).
\end{aligned} \tag{3.11}$$

A no slip condition and the kinematic condition read

$$\begin{aligned}
u = w = 0 \qquad & z = 0,\\
\partial_t h + u\,\partial_x h = w \qquad & z = h(x,t).
\end{aligned} \tag{3.12}$$

The fluid is submitted to surface tension forces which yield the following boundary conditions

$$\begin{aligned}
p = -\kappa\frac{\partial_{xx}h}{(1+\varepsilon^2(\partial_x h)^2)^{\frac{3}{2}}} - 2\varepsilon\frac{Fr^2}{Re}\frac{1+\varepsilon^2(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\,\partial_x u \qquad & z = h(x,t),\\
(1-\varepsilon^2 h_x^2)(\partial_z u + \varepsilon^2\partial_x w) - 4\varepsilon^2 h_x\partial_x u = 0 \qquad & z = h(x,t).
\end{aligned} \tag{3.13}$$

And not to forget the kinematic condition

$$\partial_t h + \partial_x hu|_{z=h} = w|_{z=h}.$$

1. **Asymptotic Expansion of u**

   From the first momentum equation of system (3.11), we have

$$\partial_{zz} u = -\frac{\sin\theta Re}{Fr^2} + \varepsilon \frac{Re}{Fr^2}\partial_x p + \varepsilon Re(\partial_t u + u\,\partial_x u + w\,\partial_z u) - \varepsilon^2 \partial_{xx} u.$$

   Integrating in depth first between z and h we can write

$$\begin{aligned}\partial_z u(h) - \partial_z u = &-\frac{\sin\theta Re}{Fr^2}(h-z) + \varepsilon\frac{Re}{Fr^2}\int_z^h \partial_x p\, d\zeta \\ &+ \varepsilon Re\int_z^h (\partial_t u + u\,\partial_x u + w\,\partial_z u)\, d\zeta - \varepsilon^2 \int_z^h \partial_{xx} u\, d\zeta.\end{aligned}$$

   Integrating again from 0 to z and using the boundary conditions we get

$$\begin{aligned}u = &\frac{\sin\theta Re}{Fr^2}(hz - \frac{z^2}{2}) - \varepsilon Re\int_0^z\int_z^h (\partial_t u + u\,\partial_x u + w\,\partial_z u)\, d\zeta\, dz \\ &- \varepsilon\frac{Re}{Fr^2}\int_0^z\int_z^h p\, d\zeta\, dz - \varepsilon\kappa\frac{Re}{Fr^2}\frac{\partial_x h\,\partial_{xx} h}{(1+\varepsilon^2(\partial_x h)^2)} z \\ &- \frac{2Re}{Fr^2}\varepsilon^2\frac{1+\varepsilon^2(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\partial_x h\,\partial_x u(h) + 4\varepsilon^2\frac{(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\partial_x u(h) - \varepsilon^2\partial_x w(h) \\ = &\frac{\sin\theta Re}{Fr^2}(hz - \frac{z^2}{2}) + R_u,\end{aligned}$$

   where R_u is an order $O(\varepsilon)$ term given by

$$\begin{aligned}R_u = &-\varepsilon Re\int_0^z\int_z^h (\partial_t u + u\,\partial_x u + w\,\partial_z u)\, d\zeta\, dz \\ &- \varepsilon\frac{Re}{Fr^2}\int_0^z\int_z^h p\, d\zeta\, dz - \varepsilon\kappa\frac{Re}{Fr^2}\frac{\partial_x h\,\partial_{xx} h}{(1+\varepsilon^2(\partial_x h)^2)} z + R_u^1,\end{aligned}$$

   and $R_u^1 \sim O(\varepsilon^2)$ is given by

$$R_u^1 = -\frac{2Re}{Fr^2}\varepsilon^2\frac{1+\varepsilon^2(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\partial_x h\,\partial_x u(h) + 4\varepsilon^2\frac{(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\partial_x u(h) - \varepsilon^2\partial_x w(h).$$

   It is clear that the profile of u at main order is close to the Nusselt profile (parabolic profile). We take $u_0 = \frac{\sin\theta Re}{Fr^2}(hz - \frac{z^2}{2})$.

2. **Asymptotic Expansion of Pressure**

   As for the pressure, from the second momentum equation we get

$$\partial_z p = -\cos\theta + \varepsilon\frac{Fr^2}{Re}(\partial_{zz} w + \varepsilon^2\partial_{xx} w) - \varepsilon^2 Fr^2(\partial_t w + u\,\partial_x w + w\,\partial_z w).$$

Integrating and using boundary conditions, we get

$$\begin{aligned} p &= \cos\theta(h-z) - \kappa\frac{\partial_{xx}h}{(1+\varepsilon^2(\partial_x h)^2)^{\frac{3}{2}}} - 2\varepsilon\frac{Fr^2}{Re}\frac{1+\varepsilon^2(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\,\partial_x u \\ &- \varepsilon\frac{Fr^2}{Re}\int_z^h (\partial_{zz}w + \varepsilon^2\,\partial_{xx}w)\,d\zeta + \varepsilon^2 Fr^2\int_z^h (\partial_t w_u\,\partial_x w + w\,\partial_z w)\,d\zeta \\ &= \cos\theta(h-z) - \kappa\,\partial_{xx}h + R_p, \end{aligned}$$

where $R_p \sim O(\varepsilon)$ is given by

$$\begin{aligned} R_p &= \kappa\,\partial_{xx}h\left(1 - \frac{1}{(1+\varepsilon^2(\partial_x h)^2)^{\frac{3}{2}}}\right) - 2\varepsilon\frac{Fr^2}{Re}\frac{1+\varepsilon^2(\partial_x h)^2}{1-\varepsilon^2(\partial_x h)^2}\,\partial_x u \\ &- \varepsilon\frac{Fr^2}{Re}\int_z^h (\partial_{zz}w + \varepsilon^2\,\partial_{xx}w)\,d\zeta + \varepsilon^2 Fr^2\int_z^h (\partial_t w + u\,\partial_x w + w\,\partial_z w)\,d\zeta. \end{aligned}$$

Denote $p_0 = \cos\theta(h-z) - \kappa\,\partial_{xx}h$.

3. **Asymptotix Expansion of w**

We can get this from the divergence free equation

$$\begin{aligned} w = -\int_0^z \partial_x u\,dz &= -\int_0^z \frac{\sin\theta Re}{Fr^2}(hz - \frac{z^2}{2})\,dz - \int_0^z \partial_x R_u\,dz \\ &= -\frac{\sin\theta Re}{Fr^2}\,\partial_x h\frac{z^2}{2} + R_w. \end{aligned}$$

4. **Iteration of the Asymptotic Expansions**

We define now the asymptotic expansions in the regime $\varepsilon \to 0$ of the solution $X = (u, w, p)^T$ of NS system:

$$X = X_0 + R_X, \tag{3.14}$$

where $X_0 = (u_0, w_0, p_0)$ and $R_X = (R_u, R_w, R_p)$. Setting the correct assumptions on the above problem, we can prove that the solution of the NS system is a result of a fixed point argument of a Lipschitz constant $O(\varepsilon)$ on any bounded set. Define the sequence $X_{n+1} = X_0 + R_{X_n}$. By induction, we prove that the sequence X_n is bounded and satisfies $||X_n||_\infty = O((\varepsilon)^{n+1})$. These approximations will be used to justify formally the depth averaged continuity and momentum equations, and thereafter we obtain evolutionary equations describing the **evolution of the height h and the total flow rate $q = hU = \int_0^h u$ in the regime $\varepsilon \to 0$, the so called shallow water equations.** The reader is referred to [1] for more details.

5. **Evolutionary Equation of h: Exact Equation**

Integrating the continuity equation in z, and using the kinematic condition at the boundary, we get the first equation of the model, which is an exact equation

$$\begin{aligned} \int_0^h \partial_x u + \int_0^h \partial_z w = 0 \implies \partial_x(\int_0^h u) - \partial_x hu|_{z=h} + w|_{z=h} - w|_{z=0} = 0 \\ \implies \partial_t h + \partial_x(hU) = 0. \end{aligned}$$

6. **Evolutionary Equation of hU**

Integrating the momentum equation in depth yields (neglecting terms of order ε)

$$\begin{aligned} LHS &= \int_0^h \partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{\partial_x p}{Fr^2}\,dz \\ &= \partial_t \int_0^h u\,dz - \partial_t h u(h) + \int_0^h u\,\partial_x h\,dz + \int_0^h \partial_z(wu) - u\,\partial_z w\,dz + \int_0^h \frac{\partial_x p}{Fr^2} \\ &= \partial_t(hU) - \partial_t h u(h) + 2\int_0^h u\,\partial_x u\,dz + u(h)w(h) - u(0)w(0) + \int_0^h \frac{\partial_x p}{Fr^2}\,dz \\ &= \partial_t(hU) + \partial_x(\int_0^h u^2\,dz) - \partial_x h u(h)^2 + u(h)w(h) - \partial_t h u(h) + \int_0^h \frac{\partial_x p}{Fr^2}\,dz. \end{aligned} \tag{3.15}$$

But

$$\int_0^h u^2\,dz = \int_0^h u_0^2\,dz + \int_0^h u^2 - u_0^2\,dz = (\frac{\sin\theta Re}{Fr^2})^2\frac{2}{15}h^5 + \int_0^h \underbrace{(u-u_0)}_{O(\varepsilon)}(u+u_0),$$

and since

$$\int_0^h u\,dz = \int_0^h u_0\,dz + O(\varepsilon) = \frac{\sin\theta Re}{Fr^2}\frac{h^3}{3} + O(\varepsilon),$$

then $\frac{\sin\theta Re}{Fr^2}h^2 = 3U$, and hence

$$\int_0^h u^2\,dz = \frac{6}{5}hU^2 + O(\varepsilon).$$

Also, the pressure at main order is given by

$$p = \cos\theta(h-z) - \kappa\,\partial_{xx}h.$$

Hence truncated at main order we can write

$$LHS = \partial_t(hU) + \partial_x(hU^2 + \frac{\cos\theta}{Fr^2}\frac{h^2}{2}) - \frac{\kappa}{Fr^2}h\,\partial_x^3 h. \tag{3.16}$$

As for the right hand side, we have at main order

$$RHS = \int_0^h \frac{\sin\theta}{\varepsilon Fr^2} + \frac{1}{\varepsilon Re}\partial_{zz}u\,dz = \frac{1}{\varepsilon Re}\left(\frac{\sin\theta Re}{Fr^2}h + \partial_z u(h) - \partial_z u(0)\right).$$

From boundary conditions we deduce that $\partial_z u(h) \sim O(\varepsilon^2)$, hence at main order we get

$$RHS = \frac{1}{\varepsilon Re}\left(\frac{\sin\theta Re}{Fr^2}h - \partial_z u(0)\right).$$

Thus at this stage the system reads

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x(\frac{6}{5}hU^2 + \frac{\cos\theta}{Fr^2}\frac{h^2}{2}) - \kappa h\,\partial_x^3 h = \frac{1}{\varepsilon Re}(\frac{\sin\theta Re}{Fr^2}h - \partial_z u(0)). \end{cases} \tag{3.17}$$

7. What About $\partial_z u$?

As $\partial_z u$ is a singular term, the main order perturbation of u around the Nusselt solution would cause inconsistency in the system. The settlement of this obstacle is to compute the first order

approximation of the velocity. If we assume that $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$, then we write again

$$\partial_z u(0) = \partial_z u_0(0) + \varepsilon\, \partial_z u_1(0) + O(\varepsilon^2) = \frac{\sin\theta Re}{Fr^2} h + \varepsilon\, \partial_z u_1(0) + O(\varepsilon^2).$$

Another intermediate step that is required for sake of consistency as well is calculating $\partial_z u(0)$ in terms of U, in which the added corrected term is proved as a result to achieve the consistency of the final system. We have

$$hU = \frac{\sin\theta Re}{Fr^2}\frac{h^3}{3} + \varepsilon \int_0^h u_1\, dz + O(\varepsilon^2) \implies \frac{\sin\theta Re}{Fr^2} h = \frac{3U}{h^2} - \frac{3\varepsilon}{h^2}\int_0^h u_1\, dz.$$

Hence

$$\partial_z u(0) = \frac{3U}{h^2} + \varepsilon\big(\partial_z u_1(0) - \frac{3}{h^2}\int_0^h u_1\, dz\big) + O(\varepsilon^2).$$

So it remains to compute u_1. We recall that from the momentum equation

$$\partial_{zz} u = \varepsilon Re\big(\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{\partial_x p}{Fr^2}\big) - \frac{\sin\theta Re}{Fr^2} + O(\varepsilon^2).$$

With $u(0) = 0$ and $\partial_z u(h) = O(\varepsilon^2)$, we get upon integration

$$u = \frac{\sin\theta Re}{Fr^2}(hz - \frac{z^2}{2}) - \varepsilon Re \int_0^z \int_z^h (\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{\partial_x p}{Fr^2}) + O(\varepsilon^2).$$

Since we assumed that $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$, then comparing this to the above equation we get

$$u_0 + \varepsilon u_1 = \frac{\sin\theta Re}{Fr^2}(hz - \frac{z^2}{2}) - \varepsilon Re \int_0^z \int_z^h (\partial_t u_0 + u_0\,\partial_x u_0 + w_0\,\partial_z u_0 + \frac{\partial_x p_0}{Fr^2}),$$

and thus

$$u_1 = -Re \int_0^z \int_z^h (\partial_t u_0 + u_0\,\partial_x u_0 + w_0\,\partial_z u_0 + \frac{\partial_x p_0}{Fr^2}).$$

Substituting u_0, w_0 and p_0 in the above equation and integrating we get the expression of u_1 in the long wave approximation

$$u_1 = Re(\frac{\sin\theta Re}{Fr^2})^2\,\partial_x h(\frac{h}{24}z^4 - \frac{h^2}{6}z^3 + \frac{h^4}{3}z) - \frac{\cos\theta Re}{Fr^2}\,\partial_x h(hz - \frac{z^2}{2}).$$

Therefore

$$\partial_z u_1(0) - \frac{3}{h^2}\int_0^h u_1\, dz = Re(\frac{\sin\theta Re}{Fr^2})^2 \frac{h^4}{15}\,\partial_x h = Re(\frac{\sin\theta Re}{Fr^2})^2\,\partial_x(\frac{h^5}{75}).$$

Finally, we obtain the Shallow Water model

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x(\frac{6}{5}hU^2 + \frac{\cos\theta}{Fr^2}h^2 - (\frac{\sin\theta Re}{Fr^2})^2\frac{h^5}{75}) - \kappa h\,\partial_x^3 h = \frac{1}{\varepsilon Re}(\frac{\sin\theta Re}{Fr^2}h - \frac{3U}{h}). \end{cases} \tag{3.18}$$

For details of the rigorous justification of the model (existence of solution of the Navier Stokes system and then convergence to the shallow water model) the reader is referred to [1].

3.4 Overview on the Weighted Residual Method (WRM) Explored in [2]

In 2000, C.Ruyer Quil with P. Manneville [79] launched a new approach to obtain a shallow water model equivalent to (3.18), which is obtained usually using the integral method. In the spirit of Galerkin projection method, they proved that testing the departure system (Navier Stokes system) with a suitable weight and integrating, one can obtain a consistent shallow water model that enjoys favorable traits. This averaging is a projection of the flow on a space spanned by a convenient function. The steady uniform parallel flow governed by the 2D Navier Stokes system has a trivial solution corresponding to the Nusselt solution The velocity field is given by $(u_N, w_N) = (u_N, 0)$, where u_N is given by

$$u_N(z) = \frac{\lambda}{2}(2h_N z - z^2) := \lambda h_N^2 g_0(\frac{z}{h}).$$

Their method relied on expressing the velocity field not using asymptotic expansions, but rather at exact order in terms of polynomial functions, the first of which is found later to coincide with the Nusselt solution- which is expected since the goal is to look for perturbation around the stationary uniform flow- and the rest of the polynomials express the deviation from such projection in real. Of course, the authors expressed the expansion in terms of powers of the aspect ratio ε for the flow rate is not too large due to smallness assumption on the cross stream-wise derivative ∂_x, or equivalently on the aspect ratio. Mainly, they took at first

$$u(x, z, t) = \sum_{j=0}^{N} a_j(x, t) f_j(\frac{z}{h}),$$

where the functions f_j are supposed to satisfy the boundary conditions, and they are expressed as

$$f_j(r) = r^{j+1} - \frac{j+1}{j+2} r^{j+2}.$$

Determining the coefficients of the polynomial test functions is manifested using first the definition of the assumed velocity that would impose a consistency between the definition and the gauge condition which states that $\int_0^h u \, dz = hU := q$, next inserting the velocity profile in the momentum equation, and finally solving a linear system in the coefficients.

The weighted residual method they adopted takes into account multiplying the equations by a weight w_j. Hence, if the mass and momentum conservation equations are denoted respectively $Ma(u)$ and $M(u)$, then formally we should have the following residuals

$$Ra_i(u) =< Ma(u), w_i(\frac{z}{h}) >= 0, \text{ and } R_i(u) =< M(u), w_i(\frac{z}{h}) >= 0.$$

A special feature in this approach is that the weights are chosen to be the same as the test functions as they satisfy the boundary conditions. Plugging the above velocity profile in the residues, and solving at each orders, we get then a consistent equation on $hU = q$. Furthermore, the authors proved that the velocity profile can be expanded in a new way which allows direct application of the Galerkin method to derive the same system as with f_j. They assumed that we can write

$$u(x, z, t) = \sum_{j=0}^{N} a_j(x, t) g_j(\frac{z}{h}),$$

where test functions g_j are chosen to be the same as those appearing in the long wave expansion of Benny's equation which in turn are found to fulfill the boundary conditions and also form a complete basis in L^2. And from here comes the choice of g_0 to be as that appearing in the Nusselt velocity, i.e $g_0(z) = \frac{z}{h} - \frac{z^2}{2h^2}$. In fact, to determine the coefficient a_0 starting from the above residues as a first step, a main term that requires some concern and leads to determine the latter coefficients is $\int_0^h g_0 \, \partial_{zz} u \, dz$. Upon integrating twice in depth and using the fact that g_0 is required to satisfy boundary conditions at

main order, i.e $g_0|_{z=0} = \partial_z g_0|_{z=h} = 0$, we get from one hand

$$\int_0^h \partial_{zz} u g_0 \, dz = (g_0 \, \partial_z u)|_0^h - (u \, \partial_z g_0)|_0^h - \int_0^h \partial_{zz} g_0 u \, dz = -\frac{1}{h^2}\int_0^h u \, dz = -\frac{hU}{h^2} = \frac{q}{h^2}$$

and if we proceed using the velocity profile at main order, we get

$$\int_0^h \partial_{zz} u g_0 \, dz = a_0 \int_0^h \partial_{zz} g_0 g_0 \, dz = -\frac{a_0}{3h}$$

hence we can take $a_0 = 3U = \frac{3q}{h}$. Thus, the definition of the assumed velocity profile embeds and identifies with the gauge condition $\int_0^h u \, dz = hU := q$ by using that $u(x,z,t) = 3Ug_0 + \sum_{j\geqslant 1} a_j g_j$. Of course, higher order polynomials are also being calculated. The reader can refer to [2] for further discussion and details. In a word, using the weighted residual method, a consistent system at main order is obtained without the need to calculate the first order correction of the expansion as done in the MIM method. The final system reads for the height h and the discharge rate $q = hU$

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q = \frac{1}{\varepsilon Re}(\frac{5}{6}h - \frac{5}{2}\frac{q}{h^2}) + \frac{9}{7}\frac{q^2}{h^2}\partial_x h - \frac{17}{7}\frac{q}{h}\partial_x q - \frac{5}{6}\frac{\cos\theta Re}{Fr^2} h\partial_x h + \frac{5}{6}\kappa h \partial_x^3 h. \end{cases} \tag{3.19}$$

The above system and that in (3.18) are equivalent at main order of ε. It is good to mention that the authors in [79] derived a 4 equation model of the height, the flow discharge rate, and two supplementary variables homogeneous to q measuring the corrections to the parabolic profile. The two equation model is also derived by them as a simplified one which is still consistent with the long-wave expansion and gives the correct results at critical Reynolds numbers. It is also good to mention that the 4 equation model gives certainly the best results of all models to date compared to NS system, with the only drawback being having a complicated non conservative structure leading to difficulties to get numerical solutions.

3.5 Overview On The Three Equation Model Derived in [3]

In [3], the authors derived a new asymptotic model for viscous films in 2D to capture the evolution of three variables of the system: the height, the depth averaged velocity and a new variable called enstrophy, which is congruent to the deviation from the averaged depth velocity. The authors started from a film of Newtonian incompressible fluid of free surface as depicted in figure 3.1. Again we consider the scaled system with boundary conditions (3.11),(3.12) and (3.13). For simplicity, we define $\lambda = \frac{\sin\theta Re}{Fr^2}$. The derivation was based on an integral method with perturbation analysis. The first step as in MIM is to write the momentum and energy equations as closed depth averaged equations

$$\partial_t(hU)\, dz + \partial_x(\int_0^h u^2 \, dz + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h - \tau_{xz}(0)) + \kappa h \, \partial_x^3 h + O(\varepsilon). \tag{3.20}$$

On the other hand, lets look on the energy of the NS system, the general form in correspondence to the non scaled system reads

$$\frac{1}{2}(\, \partial_t \mathbf{u}^2 + \mathrm{div}(\mathbf{u}^2 v)) = \mathrm{div}(\tau - p\,\mathrm{Id}) \cdot \mathbf{u},$$

where we recall $\mathbf{u} = (u, w)$. In the stream wise dimension, and using the scaled system (3.11) this reads

$$\frac{1}{2}(\, \partial_t u^2 + \mathrm{div}(u^2 \mathbf{u})) = (\frac{\lambda}{\varepsilon Re} - \frac{\partial_x p}{Fr^2})u + \frac{u\,\partial_{zz} u}{\varepsilon Re} + O(\varepsilon).$$

And thus integrating in depth in the stream wise direction-neglecting order $O(\varepsilon)$ terms- yields

$$\frac{1}{2}(\, \partial_t \int_0^h u^2 \, dz + \partial_x \int_0^h u^3 \, dz) = (\frac{\lambda}{\varepsilon Re} - \frac{\partial_x p}{Fr^2})hU + \frac{1}{\varepsilon Re}\int_0^h u\,\partial_{zz} u \, dz. \tag{3.21}$$

As noticed from (3.20) and (3.21), three terms determine the closed form of the so called shallow water model: $\int_0^h u^2\ dz$, $\int_0^h u^3\ dz$ and the viscous part. The novelty in this work is in calculating the deviation of the velocity from the Nusselt profile, mainly we are speaking on the term $\int_0^h u^2\ dz$. In the classical 2D MIM, the approach was to express this integral in terms of U, and the remainder would be of order ε which is neglected in the first order models, so we would write

$$\int_0^h u^2\ dz = hU^2 + \int_0^h u^2 - U^2\ dz = hU^2 + O(\varepsilon).$$

However, such systems showed sometimes inconsistency at main order. Instead, the approach in [1], which was reviewed in the previous section, was to played on the asymptotic to recover an equivalent inertial term. Mainly, from the long wave expansion we know that at main order we have

$$u = \lambda h^2(\frac{z}{h} - \frac{z^2}{2h^2}).$$

But also we know that

$$3U = \lambda h^2,$$

which implies

$$u = 3U(\frac{z}{h} - \frac{z^2}{2h^2}).$$

And now Integrating we get

$$\int_0^h u^2\ dz = \lambda^2 \frac{2}{15} h^5 + O(\varepsilon) = \frac{6}{5} hU^2 + O(\varepsilon).$$

However, in the 3D MIM in [3], a new variable is introduced at this level. So instead of comparing the squares of the velocity and its average, the square would be applied for the difference instead, giving rise to a new variable being the square of the difference, the enstrophy. In particular, by defining this latter as

$$\int_0^h (u-U)^2\ dz = h^3\varphi,$$

we write

$$\int_0^h u^2\ dz = \int_0^h (u-U+U)^2\ dz = hU^2 + \int_0^h (u-U)^2 dz = hU^2 + h^3\varphi.$$

Unsurprisingly, $h^3\varphi$ at main order contributes to $\frac{hU^2}{5}$, and thus recovers the same coefficient $\frac{6}{5}$ at the main order of the inertial terms as in [1]. Surprisingly however, the same issue appears again in deriving models at first order, since first order of the inertial terms are rarely discussed. Here lies the strength of introducing the enstrophy as it embraces all the higher order contributions of $\int_0^h u^2\ dz$, and balances consistency at any order regarding inertial terms. In the same context, the cubic term reads

$$\int_0^h u^3\ dz = \int_0^h (u-U+U)^3\ dz = hU^3 + 3h^3U\varphi + \int_0^h (u-U)^3\ dz.$$

The last integral however is expressed at long wave expansion in terms of a combination of the usual relaxation term that appears naturally in shallow water models $h - \frac{3U}{h}$ and another relaxation term relative to the enstrophy $\varphi - \frac{\lambda^2 h^2}{45}$(we say "a" combination since such integrals are expressed in terms of sums of such relaxation terms up to degrees of freedom which are formulated in terms of h, U and φ). This is also the adopted way in this approach to express the viscous term. Though this way adds some feedom in choosing the expressions of the parameters, yet this is in some perspectives considered a bug in this method as it induces an artificial relaxation term with no physical or mathematical basis. However, it still gives consistent results in the model. So, there is something missing to understand here. Having included a third variable in the depth averaged momentum and energy equations, a new equation should be derived on it in order to close the system. The approach adopted in [3] is in subtracting the kinetic

energy of the depth averaged system from the averaged work-energy equation of the NS system. Thus, this will result in a transport equation on the enstrophy with a bit complex right hand side being in fact the work of the viscous and capillarity terms. The model in [3] reads

$$\begin{cases} \partial_t h + \partial_x(hU) = 0 \\ \partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{2Fr^2}h^2) = \frac{1}{\varepsilon Re}(1+f_1)(\lambda h - \frac{3U}{h}) + \frac{f_2}{\varepsilon Re}(\varphi - \frac{\lambda^2 h^2}{45}) \\ \qquad\qquad + \frac{\kappa}{Fr^2} h\,\partial_x^3 h + \frac{\varepsilon}{Re}\,\partial_x(\frac{109}{32} h\,\partial_x U); \\ \frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) = -\frac{U}{\varepsilon Re}\frac{2f_1}{3}(\lambda h - \frac{3U}{h}) - \frac{U}{\varepsilon Re}\frac{2f_2}{3}(\varphi - \frac{\lambda^2 h^2}{45}) \\ \qquad\qquad + \frac{\varepsilon}{Re}\frac{109}{32} h(\,\partial_x U)^2 + \frac{\varepsilon}{Re}\,\partial_x(\frac{175}{48} h^3\,\partial_x\varphi). \end{cases} \tag{3.22}$$

where f_1 and f_2 are parameters arising from the choice of the asymptotic expansion.

3.6 A Revisit of the Three-Equation Approach

As mentioned before, in this section we will modify the strategy followed in [3] to derive a three-equation model by introducing a velocity profile that depends on the enstrophy. First, adapting a general representation of the velocity profile (general in terms of φ), and using long wave expansions, we will calculate the first order corrections of the different terms which will be of good use in the derivation. The momentum equation is then derived in its closed form using the enstrophy. As for the additional cubic term from the inertial part in the energy equation, it will be expressed using long wave expansion in terms of the inertial part itself and thus remains stick to it. Second, the viscous terms will be calculated using the adopted velocity profile at main order. The characterization of the velocity profile will be manifested after deriving the depth averaged momentum equation. During derivation several constraints are imposed on the profile. These constraints in addition to the boundary conditions and gauge condition will figure out the velocity profile, mainly the expression of the additional part in terms of φ. To be more clear in fact, we will adopt a velocity profile inspired by that elaborated on in [2] in the context of weighted residuals and reviewed in Section 3.4.

In the sequel we will adopt the following velocity profile

$$u = \tilde{u_0} + \varepsilon\tilde{u_1} = 3Ug_0 + \Psi(U,\varphi)G_1 + \varepsilon\tilde{u_1}. \tag{3.23}$$

for some polynomial G_1 to be determined through derivation. This choice is due to several reasons: from one hand we included the correct velocity profile at main order $3Ug_0$, which means that $\Psi \sim O(\varepsilon)$. It is intrinsic in the light of our goal discussed in previous section to take Ψ thus equivalent to the relaxation term $\varphi - \frac{\lambda h^2}{45}$ or any other equivalent formulation as $\varphi - \frac{U^2}{5h^2}$. Due to gauge condition, we know that $\int_0^h u\,dz = hU$, and thus using the above formulation we get the first constraint on Ψ and the correction $\tilde{u}_1$

$$\int_0^h \Psi(U,\varphi)G_1 + \varepsilon\tilde{u_1}\;dz = 0. \tag{3.24}$$

Moreover, the no slip boundary condition necessitate that

$$\Psi(U,\varphi)G_1(0) + \varepsilon\tilde{u_1}(0) = 0. \tag{3.25}$$

The determination of the test function G_1 will be specified according to constraints imposed on the inertial and viscous terms in the derivation below in addition to the above boundary and gauge condition constraints.

Asymptotics

In this part, we will bring the long wave expansion of u, U and φ at first order. From the momentum equation we have

$$\partial_{zz}u = \varepsilon Re\big(\,\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{\partial_x p}{Fr^2}\big) - \frac{\sin\theta Re}{Fr^2}$$

with $\lambda = \frac{\sin\theta Re}{Fr^2}$, $u(0) = 0$ and $\partial_z u(h) = O(\varepsilon^2)$, we get upon integration

$$u = \lambda(hz - \frac{z^2}{2}) - \varepsilon Re \int_0^z \int_z^h (\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{\partial_x p}{Fr^2}).$$

Assuming also in the long wave expansion that

$$u = u_0^h + \varepsilon u_1^h + O(\varepsilon^2)$$

we get

$$u_0^h = \lambda(hz - \frac{z^2}{2})$$

and

$$u_1^h = Re\lambda^2\,\partial_x h(\frac{h}{24}z^4 - \frac{h^2}{6}z^3 + \frac{h^4}{3}z) - \lambda\cot\theta\,\partial_x h(hz - \frac{z^2}{2}).$$

Thus the averaged depth velocity becomes at the long wave expansion in terms of h

$$U = \frac{1}{h}\int_0^h u\,dz = \frac{\lambda h^2}{3} + \varepsilon Re(\frac{2\lambda^2}{15}h^5\,\partial_x h - \frac{\cos\theta}{Fr^2}\frac{h^2}{3}\,\partial_x h) + O(\varepsilon^2)$$

Lets get the expansion of the averaged variance φ, in fact from [3], we have

$$\begin{aligned}\varphi &= \varphi_0^h + \varepsilon\varphi_1^h + O(\varepsilon^2)\\ &= \frac{\lambda^2 h^2}{45} + \varepsilon Re\lambda\big(\frac{2}{105}\lambda^2 h^5 - \frac{2}{45}\frac{\cos\theta}{Fr^2}h^2\big)\,\partial_x h + O(\varepsilon^2).\end{aligned}$$

We claim as well that

$$\varphi = \frac{U^2}{5h^2} + \varepsilon\tilde{\varphi} + O(\varepsilon^2)$$

with $\tilde{\varphi}$ being the first order correction of φ, thus we can write

$$\tilde{\varphi} = \frac{1}{\varepsilon}(\frac{\lambda^2 h^2}{45} - \frac{U^2}{5h^2}) + \varphi_1^h + O(\varepsilon^2).$$

But as

$$\begin{aligned}U^2 &= \frac{\lambda^2 h^4}{9} + 2\varepsilon\frac{\lambda h}{3}Reh^2(\frac{2\lambda^2}{15}h^5\,\partial_x h - \lambda\frac{\cos\theta}{Fr^2}\frac{h^2}{3}\,\partial_x h) + O(\varepsilon^2)\\ &= \frac{\lambda^2 h^4}{9} + \varepsilon\lambda Re(\frac{4\lambda^2}{45}h^7\,\partial_x h - \frac{\cos\theta}{Fr^2}\frac{2h^4}{9}\,\partial_x h) + O(\varepsilon^2)\end{aligned}$$

Thus

$$\tilde{\varphi} = \lambda^3 Re\frac{2}{1575}h^5\,\partial_x h \tag{3.26}$$

and hence

$$\varphi = \frac{U^2}{5h^2} + \varepsilon\lambda^3 Re\frac{2}{1575}h^5\,\partial_x h + O(\varepsilon^2). \tag{3.27}$$

Averaged Momentum Equation

We will start from the closed form (3.20)

$$\partial_t(hU)\,dz + \partial_x(\int_0^h u^2\,dz + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h + \int_0^h \partial_{zz}u\,dz) + \kappa h\,\partial_x^3 h + O(\frac{\varepsilon}{Re}). \tag{3.28}$$

Using the velocity profile in (3.23) we have

$$\begin{aligned}\int_0^h u^2\ dz &= 9U^2\int_0^h g_0^2\ dz + 6U\Psi\int_0^h g_0G_1\ dz + 6\varepsilon U\int_0^h g_0\tilde{u_1}\ dz + \Psi^2\int_0^h G_1^2\ dz + \varepsilon^2\int_0^h \tilde{u_1}^2\ dz\\ &= \frac{6}{5}hU^2 + 6U\Psi\int_0^h g_0G_1\ dz + 6\varepsilon U\int_0^h g_0\tilde{u_1}\ dz + O(\varepsilon^2).\end{aligned}$$

From the other side, we can write

$$\begin{aligned}\int_0^h u^2\ dz =& hU^2 + h^3\varphi\\ =& \frac{6}{5}hU^2 + h^3\varphi - \frac{hU^2}{5}\end{aligned}$$

Comparing the above two equations we get

$$\frac{6}{5}hU^2 + +6U\left(\int_0^h(\Psi G_1 + \varepsilon\tilde{u_1})g_0\ dz\right) + O(\varepsilon^2) = \frac{6}{5}hU^2 + h^3\varphi - \frac{hU^2}{5}.$$

Thus

$$6U\left(\int_0^h(\Psi G_1 + \varepsilon\tilde{u_1})g_0\ dz\right) + O(\varepsilon^2) = h^3\varphi - \frac{hU^2}{5}.$$

In fact, if we assumed that the integral on the left hand side is zero, we will get that

$$h^3\varphi - \frac{hU^2}{5} = O(\varepsilon^2),$$

which not consistent with the asymptotic derived in (3.27), as the difference on the right hand side is of order ε. This also means that one can't assume that g_0 is orthogonal to both G_1 and $\tilde{u_1}$ at the same time since this will deteriorate the consistency as explained. Hence, one can interplay with the values of the above integral, for that we can assume

$$\int_0^h(\Psi G_1 + \varepsilon\tilde{u_1})g_0\ dz = \beta\int_0^h \Psi G_1 g_0\ dz \tag{3.29}$$

for some constant β. Thus subtracting the above equations yields

$$6U\beta\Psi\int_0^h g_0G_1\ dz - (h^3\varphi - \frac{hU^2}{5}) + O(\varepsilon^2) = 0. \tag{3.30}$$

Since inertial terms are only crucial at first order (the maximum), then we get letting $\alpha h = \int_0^h g_0G_1\ dz$

$$\Psi = \frac{1}{6\alpha\beta U}(h^2\varphi - \frac{U^2}{5}) + O(\varepsilon^2). \tag{3.31}$$

It is good to mention at this level that other equivalent formulations of Ψ ar possible but will only be valid in the main order approximation unless corrections are included, for example, replacing U by its long wave approximation $\frac{h^2}{3}$ is equivalent to the above formulation at main order, but at first order the contribution relative to the first order approximation of U should be included to preserve consistency of

the system. As for the viscous term, we have in fact

$$\begin{aligned}\int_0^h \partial_{zz}u\ dz =& 3U\int_0^h \partial_{zz}g_0\ dz + \Psi\int_0^h \partial_{zz}G_1\ dz + \varepsilon\int_0^h \partial_{zz}\tilde{u_1} + O(\varepsilon^2)\\ =& -\frac{3U}{h} + \Psi\int_0^h \partial_{zz}G_1 + \varepsilon\int_0^h \partial_{zz}\tilde{u_1}\ dz + O(\varepsilon^2)\\ =& -\frac{3U}{h} + (\Psi\,\partial_z G_1(h) + \varepsilon\,\partial_z\tilde{u_1}(h)) - (\Psi\,\partial_z G_1(0) + \varepsilon\,\partial_z\tilde{u_1}(0)) + O(\varepsilon^2).\end{aligned}$$

The final averaged momentum thus reads

$$\begin{aligned}\partial_t(hU) + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) =& \frac{\Psi}{\varepsilon Re}\Big(\partial_z G_1(h) - \partial_z G_1(0)\Big) + \frac{1}{Re}\Big(\partial_z\tilde{u_1}(h) - \partial_z\tilde{u_1}(0)\Big)\\ &+ \frac{1}{\varepsilon Re}\Big(\lambda h - \frac{3U}{h}\Big) + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + O(\varepsilon).\end{aligned} \tag{3.32}$$

Hence the corresponding kinetic energy becomes

$$\begin{aligned}&\partial_t(\frac{hU^2}{2} + \frac{\cos\theta}{2Fr^2}h^2)\ dz + \partial_x(\frac{hU^3}{2} + \frac{\cos\theta}{Fr^2}h^2U) + U\,\partial_x(h^3\varphi)\\ &\qquad= \frac{\kappa}{Fr^2}hU\,\partial_x^3 h + \frac{U}{\varepsilon Re}(\lambda h - \frac{3U}{h}) + \frac{\Psi U}{\varepsilon Re}\Big(\partial_z G_1(h) - \partial_z G_1(0)\Big)\\ &\qquad\quad+ \frac{U}{Re}\Big(\partial_z\tilde{u_1}(h) - \partial_z\tilde{u_1}(0)\Big)\ dz + O(\varepsilon).\end{aligned} \tag{3.33}$$

Averaged Energy Equation

From (3.21), we have

$$\begin{aligned}&\frac{1}{2}\Big(\partial_t(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) + \partial_x(hU^3 + 3h^3U\varphi + \int_0^h (u-U)^3\ dz + \frac{\cos\theta}{Fr^2}h^2U)\Big)\\ &= \frac{\kappa}{Fr^2}hU\,\partial_x^3 h + \frac{1}{\varepsilon Re}\Big(\frac{Re\sin\theta}{Fr^2}hU + \int_0^h u\,\partial_{zz}u\ dz\Big) + O(\varepsilon).\end{aligned} \tag{3.34}$$

As for the viscous term, we have using the velocity profile

$$\begin{aligned}\int_0^h u\,\partial_{zz}u\ dz =& \int_0^h \big[3Ug_0 + \Psi G_1 + \varepsilon\tilde{u_1}\big]\big[-\frac{3U}{h} + \Psi\,\partial_{zz}G_1 + \varepsilon\,\partial_{zz}\tilde{u_1}\big]\\ =& -\frac{3U^2}{h} + 3U\Psi\int_0^h g_0\,\partial_{zz}G_1\ dz + 3\varepsilon U\int_0^h g_0\,\partial_{zz}\tilde{u_1} + \tilde{R},\end{aligned} \tag{3.35}$$

where

$$\tilde{R} = \Psi^2\int_0^h G_1\,\partial_{zz}G_1 + \varepsilon\Psi\int_0^h G_1\,\partial_{zz}\tilde{u_1} + \varepsilon\Psi\int_0^h \tilde{u_1}\,\partial_{zz}G_1\ dz + \varepsilon^2\int_0^h \tilde{u_1}\,\partial_{zz}\tilde{u_1}\ dz \sim O(\varepsilon^2) \tag{3.36}$$

Lets compute the integrals in (3.35):

- The first integral:

$$\int_0^h g_0\,\partial_{zz}G_1\;dz = \int_0^h \partial_{zz}g_0G_1\;dz + g_0(h)\,\partial_z G_1(h) - g_0(0)\,\partial_z G_1(0) - \partial_z g_0(h)G_1(h) + \partial_z g_0(0)G_1(0)$$
$$= \frac{1}{h}\Big(- \int_0^h G_1\;dz + G_1(0) + \frac{h}{2}\,\partial_z G_1(h)\Big). \tag{3.37}$$

- The second integral:

$$\int_0^h g_0\,\partial_{zz}\tilde{u_1} = \int_0^h \partial_{zz}g_0\tilde{u_1}\;dz + g_0(h)\,\partial_z\tilde{u_1}(h) - g_0(0)\,\partial_z\tilde{u_1}(0) - \partial_z g_0(h)\tilde{u_1}(h) + \partial_z g_0(0)\tilde{u_1}(0)$$
$$= \frac{1}{h}\Big(- \int_0^h \tilde{u_1}\;dz + \tilde{u_1}(0) + \frac{h}{2}\,\partial_z\tilde{u_1}(h)\Big). \tag{3.38}$$

Thus summing in (3.35) and using the constraints (3.24) and (3.25) we get

$$\int_0^h u\,\partial_{zz}u\;dz = -\frac{3U^2}{h} + \frac{3U}{2h}\Big(\Psi\,\partial_z G_1(h) + \varepsilon\,\partial_z\tilde{u_1}(h)\Big) + O(\varepsilon^2).$$

And thus the averaged energy equation reads

$$\frac{1}{2}\Big(\partial_t(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) + \partial_x(hU^3 + 3h^3U\varphi + \int_0^h (u-U)^3\;dz + \frac{\cos\theta}{Fr^2}h^2U)\Big)$$
$$= \frac{\kappa}{Fr^2}hU\,\partial_x^3 h + \frac{1}{\varepsilon Re}\Big(\frac{Re\sin\theta}{Fr^2} - \frac{3U^2}{h}\Big) + \frac{1}{\varepsilon Re}\frac{3U}{2}\Big(\Psi\,\partial_z G_1(h) + \varepsilon\,\partial_z\tilde{u_1}(h)\Big) + O(\varepsilon). \tag{3.39}$$

Transport Equation on φ

If we subtract now equations (3.39) and (3.33), we get

$$\frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) + \frac{1}{2}\,\partial_x\Big(\int_0^h (u-U)^3\;dz\Big) = \frac{1}{\varepsilon Re}\frac{U}{2}\Big(\Psi\,\partial_z G_1(h) + \varepsilon\,\partial_z\tilde{u_1}(h)\Big)$$
$$+ \frac{U}{\varepsilon Re}(\Psi\,\partial_z G_1(0) + \varepsilon\,\partial_z\tilde{u_1}(0)). \tag{3.40}$$

Summing the Constraints on G_1

We can now sum up all the conditions on G_1:

- gauge condition
$$\int_0^h u\;dz = hU \implies \int_0^h \Psi G_1 + \varepsilon\tilde{u_1} = 0.$$
- Boundary condition
$$u(0) = 0 \implies \Psi G_1(0) + \varepsilon\tilde{u_1}(0) = 0.$$
- Consistency at first order (3.29)
$$\int_0^h g_0(\Psi G_1 + \varepsilon\tilde{u_1})\;dz = \beta\Psi\int_0^h G_1 g_0.$$
- Condition $\int_0^h g_0 G_1\;dz = \alpha h$.

In order to determine G_1, we should determine $\tilde{u}_1$ at main order, and then the above constraints, we can determine the degree and then the expression of the polynomial function G_1. To do that, let's calculate $\tilde{u}_1$ at the long wave expansion.

Determing $\tilde{u_1}$ at Main Order

From the long wave expansion of φ, we already know that

$$h^2\varphi = \frac{U^2}{5} + \varepsilon h^2\tilde{\varphi} + O(\varepsilon^2) = \frac{U^2}{5} + \varepsilon\lambda^3 Re\frac{2}{1575}h^7\,\partial_x h + O(\varepsilon^2)$$

which implies that at first order

$$\Psi = \varepsilon\frac{\tilde{\varphi}}{2\alpha\beta\lambda} + O(\varepsilon^2) = \varepsilon\frac{\lambda^2 Re}{\alpha\beta}\frac{1}{1575}h^5\,\partial_x h + O(\varepsilon^2).$$

On the other hand, we can write

$$u = 3Ug_0 + \Psi G_1 + \varepsilon\tilde{u_1} == u_0^h + \varepsilon u_1^h + O(\varepsilon^2),$$

and thus we get the expression of $\tilde{u_1}$ at long wave expansion at main order

$$\tilde{u_1} = \frac{1}{\varepsilon}(u_0^h - 3Ug_0) - \frac{\tilde{\varphi}}{2\alpha\beta\lambda}G_1 + u_1^h.$$

From the expansion of U at first order, we have

$$\begin{aligned} u_0^h - 3Ug_0 &= \lambda h^2 g_0 - 3g_0\left(\frac{\lambda h^2}{3} + \varepsilon Re(\frac{2\lambda^2}{15}h^5\,\partial_x h - \frac{\cos\theta}{Fr^2}\frac{h^2}{3}\,\partial_x h)\right) \\ &= -\varepsilon Re(\frac{2\lambda^2}{5}h^5\,\partial_x h - \frac{\cos\theta}{Fr^2}h^2\,\partial_x h)g_0. \end{aligned}$$

and since

$$u_1^h = Re\lambda^2\,\partial_x h(\frac{h}{24}z^4 - \frac{h^2}{6}z^3 + \frac{h^4}{3}z) - \lambda\cot\theta\,\partial_x h(hz - \frac{z^2}{2}),$$

then

$$\begin{aligned} \tilde{u_1} &= -\frac{\tilde{\varphi}}{2\alpha\beta\lambda}G_1 + Re\lambda^2\,\partial_x h(\frac{h}{24}z^4 - \frac{h^2}{6}z^3 + \frac{h^4}{3}z - \frac{2}{5}h^5 g_0) \\ &= \lambda^2 Re\,\partial_x h\Big[-\frac{h^5}{1575\alpha\beta}G_1 + \frac{h}{24}z^4 - \frac{h^2}{6}z^3 + \frac{h^4}{3}z - \frac{2}{5}h^5 g_0\Big]. \end{aligned} \tag{3.41}$$

Determining G_1

Under the light of the above relation, we can now rewrite the conditions on G_1 and $\tilde{u_1}$.

- The gauge condition can be rewritten as

$$\begin{aligned} \int_0^h \Psi G_1 + \varepsilon\tilde{u_1}\;dz &= (\Psi - \frac{\varepsilon\tilde{\varphi}}{2\alpha\beta\lambda})\int_0^h G_1\;dz + \varepsilon Re\lambda^2\,\partial_x h\int_0^h(\frac{h}{24}z^4 - \frac{h^2}{6}z^3 + \frac{h^4}{3}z - \frac{2}{5}h^5 g_0)\;dz \\ &= (\Psi - \frac{\varepsilon\tilde{\varphi}}{2\alpha\beta\lambda})\int_0^h G_1\;dz + 0 = 0. \end{aligned}$$

Hence, since $\Psi - \frac{\varepsilon\tilde{\varphi}}{2\alpha\lambda} \sim O(\varepsilon^2)$, then this is a sufficient condition to get a condition on G_1:

$$\int_0^h G_1 \, dz = 0. \tag{3.42}$$

- The no slip condition leads to have

$$\Psi G_1(0) + \varepsilon \tilde{u_1}(0) = 0 = (\Psi - \frac{\varepsilon\tilde{\varphi}}{2\alpha\lambda}) G_1(0). \tag{3.43}$$

Therefore

$$G_1(0) = 0. \tag{3.44}$$

- Now let's calculate the relaxation term in momentum equation that read

$$\frac{\Psi}{\varepsilon Re}\Big(\partial_z G_1(h) - \partial_z G_1(0) \Big) + \frac{1}{Re}\Big(\partial_z \tilde{u_1}(h) - \partial_z \tilde{u_1}(0) \Big).$$

Then, we will stress on making the second part of the above term zero at main order, so that we take

$$\partial_z \tilde{u_1}(h) - \partial_z \tilde{u_1}(0) = 0.$$

Using the fact that at main order

$$\partial_z \tilde{u_1}(h) = -\frac{\tilde{\varphi}}{2\alpha\beta\lambda}\, \partial_z G_1(h) \quad \text{and} \quad \partial_z \tilde{u_1}(0) = -\frac{\tilde{\varphi}}{2\alpha\beta\lambda}\, \partial_z G_1(0) - Re\lambda^2\, \partial_x h \frac{h^4}{15},$$

we get a condition on G_1

$$\partial_z G_1(h) - \partial_z G_1(0) = \frac{105\alpha\beta}{h}.$$

The relaxation term now reads

$$\frac{\Psi}{\varepsilon Re}\frac{105\alpha\beta}{h}.$$

- Calculating the relaxation terms in the transport equation:

$$\begin{aligned} &\frac{1}{\varepsilon Re}\frac{U}{2}\Big(\Psi\, \partial_z G_1(h) + \varepsilon\, \partial_z \tilde{u_1}(h) \Big) + \frac{U}{\varepsilon Re}(\Psi\, \partial_z G_1(0) + \varepsilon\, \partial_z \tilde{u_1}(0)) \\ &= \frac{U\Psi}{\varepsilon Re}\left[\frac{\partial_z G_1(h)}{2} + \partial_z G_1(0) \right] + \frac{U}{Re}\left[\frac{\partial_z \tilde{u_1}(h)}{2} + \partial_z \tilde{u_1}(0) \right]. \end{aligned}$$

However

$$\begin{aligned} \frac{\partial_z \tilde{u_1}(h)}{2} + \partial_z \tilde{u_1}(0) &= -\frac{\tilde{\varphi}}{4\alpha\beta\lambda}\, \partial_z G_1(h) - \frac{\tilde{\varphi}}{2\alpha\beta\lambda}\, \partial_z G_1(0) - Re\lambda^2 \frac{h^4}{15}\, \partial_x h \\ &= \frac{\lambda^2 Re}{1575} h^4\, \partial_x h \Big(-\frac{h}{2\alpha\beta}\, \partial_z G_1(h) - \frac{h}{\alpha\beta}\, \partial_z G_1(0) - 105 \Big). \end{aligned}$$

- Condition $\int_0^h g_0 G_1 \, dz = \alpha h$.

Therefore, the system becomes now

$$\begin{cases} \partial_t(hU)\ dz + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h - \frac{3U}{h}) + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + \frac{\Psi}{\varepsilon Re}\frac{105\alpha\beta}{h}. \\ \frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) + \frac{1}{2}\partial_x(\int_0^h (u-U)^3\ dz) = \frac{U\Psi}{\varepsilon Re}\Big[\frac{3\,\partial_z G_1(0)}{2} + \frac{105}{2h}\alpha\beta\Big] \\ \qquad\qquad + \frac{U}{Re}\Big[-\frac{\tilde{\varphi}}{4\alpha\beta\lambda}\,\partial_z G_1(h) - \frac{\tilde{\varphi}}{2\alpha\beta\lambda}\,\partial_z G_1(0) - Re\lambda^2\frac{h^4}{15}\,\partial_x h.\Big]. \end{cases} \tag{3.45}$$

Therefore, as summed up, we have 4 conditions on G_1:

1. $\int_0^h G_1\ dz = 0$,
2. $G_1(0) = 0$,
3. $\partial_z G_1(h) - \partial_z G_1(0) = \frac{105\alpha\beta}{h}$,
4. $\alpha \neq 0$.

Since we are concerned with a parabolic velocity profile, we are looking for an even function $G_1 = P(X^2)$ where $X = \frac{z}{h} - 1$. We've got 3 equations on G_1, thus on the polynomial P, which means that P is of third order which can can take the form $P(X^2) = a + bX^2 + cX^4 + dX^6$, Hence solving we get

$$\begin{cases} 2b + 4c + 6d = 105\alpha\beta \\ a + b + c + d = 0 \\ a + \dfrac{b}{3} + \dfrac{c}{5} + \dfrac{d}{7} = 0 \\ \dfrac{a}{3} + \dfrac{b}{15} + \dfrac{c}{35} + \dfrac{d}{63} = \alpha \end{cases} \tag{3.46}$$

which implies

$$a \to \frac{315\alpha}{16} - \frac{105\beta\alpha}{16}, b \to \frac{1575\beta\alpha}{16} - \frac{2835\alpha}{16}, c \to \frac{4725\alpha}{16} - \frac{3675\beta\alpha}{16}, d \to \frac{2205\beta\alpha}{16} - \frac{2205\alpha}{16}$$

therefore

$$\begin{aligned} G_1 = {} & \frac{2205\beta\alpha z^6}{16h^6} - \frac{2205\alpha z^6}{16h^6} - \frac{6615\beta\alpha z^5}{8h^5} + \frac{6615\alpha z^5}{8h^5} + \frac{3675\beta\alpha z^4}{2h^4} - \frac{14175\alpha z^4}{8h^4} \\ & - \frac{3675\beta\alpha z^3}{2h^3} + \frac{1575\alpha z^3}{h^3} + \frac{1575\beta\alpha z^2}{2h^2} - \frac{945\alpha z^2}{2h^2} - \frac{105\beta\alpha z}{h}. \end{aligned}$$

Therefore we deduce the expression of u:

$$u = 3Ug_0 + \frac{1}{6\alpha\beta U}(h^2\varphi - \frac{U^2}{5})G_1 + \varepsilon\tilde{u_1},$$

and at first main order we can write

$$u = 3Ug_0 + \frac{1}{6\alpha\beta U}(h^2\varphi - \frac{U^2}{5})G_1 + O(\varepsilon),$$

and the final system reads

$$\begin{cases} \partial_t(hU)\ dz + \partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h - \frac{3U}{h}) + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + \frac{105}{\varepsilon Re}\frac{1}{6hU}(h^2\varphi - \frac{U^2}{5}). \\ \frac{h^2}{2}\Big(\partial_t(h\varphi) + \partial_x(hU\varphi)\Big) + \frac{1}{2}\partial_x(\int_0^h (u-U)^3\ dz) = -\frac{105}{6h}\frac{1}{\varepsilon Re}(h^2\varphi - \frac{U^2}{5}). \end{cases} \tag{3.47}$$

Calculating the Cubic Term

There is several possibilities to include the integral term in the above equation:

- express it in the long wave expansion
- express it in terms of the inertial terms in (3.40)
- express it in terms of the inertial terms in (3.39)

Intrinsically, such term arises initially from inertial terms of the transport equation, so in order not to lose its contribution to inertial effects, it would be effective to test such possibility as a first step. Using the long wave expansion we have from one side

$$\frac{1}{2}\,\partial_x\Big(\int_0^h (u-U)^3\,dz\Big) = -\,\partial_x\big(\lambda^3\frac{h^7}{945}\big) = -\frac{\lambda^3}{135}h^6\,\partial_x h.$$

And since also we can express the inertial terms as

$$\frac{h^2}{2}\big(\,\partial_t(h\varphi) + \,\partial_x(hU\varphi)\big) = -\lambda^3\frac{2}{135}h^6\,\partial_x h,$$

then we can add the term $\frac{1}{2}\,\partial_x(\int_0^h (u-U)^3\,dz)$ to the inertial terms so that the final equations read

$$\begin{cases} \partial_t h + \,\partial_x(hU) = 0, \\ \partial_t(hU) + \,\partial_x(hU^2 + h^3\varphi + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}(\lambda h - \frac{3U}{h}) + \frac{\kappa}{Fr^2}h\,\partial_x^3 h + \frac{105}{\varepsilon Re}\frac{1}{6hU}(h^2\varphi - \frac{U^2}{5}), \\ \frac{3h^2}{4}\Big(\,\partial_t(h\varphi) + \,\partial_x(hU\varphi)\Big) = -\frac{105}{6h}\frac{1}{\varepsilon Re}(h^2\varphi - \frac{U^2}{5}). \end{cases} \tag{3.48}$$

Remark 4. *It is good to remark that the dependence of the velocity profile on the parameters α and β is not obvious in the final system especially in the relaxation terms due to a simple reason that in the relaxation terms we have the term Ψ multiplied by a function of G_1, and in the time we have the definition of Ψ has $\alpha\beta$ in the denominator, the corresponding expression in terms of G_1 on the other hand is multiplied by $\alpha\beta$, and thus this compensates so that α and β do not appear in the final system.*

Conclusion

In this section, we have displayed a kind of a refinement of the approach used in [3] to derive a three equation shallow water model. The novelty is in adapting a velocity profile, though finally was not expressed at exact order but depends on the the variables φ and U rather than expressing the profile at truncated orders using the asymptotic expansion, and thus enslaving it just to h. This is very crucial to obtain more accurate streamlines for instance. The main concern of this revisiting was to preserve the physical feature of each term in the derivation. Of course, we have preserved as well the pros that the MIM for the three equation model introduced, for example as in preserving the structure of the final model which would benefit in the mathematical analysis of the system as well as in facilitating the numerical process (having a conservative form). However, at some point, the asymptotic expansions were necessary to use in order to close the system, i.e when calculating the term $\partial_x\int_0^h (u-U)^3\,dz$. From a physical perspective, this step is considered a refinement compared to what have done in the initial approach where they expressed the cubic term in terms of the relaxation terms. However, since such term arises from inertial effects, and there is no solution till the moment other than using its long wave expansion to close the system, then its preferable from our point of view to add its contribution to the inertia terms rather than adding it to contributions from other physical effects as diffusion or relaxation.

3.7 A 3-Equation Model for 2D and 3D flows Using the Weighted Residual Method

In this approach, we will use the weighted residual method reviewed in Section 3.4. The novelty of this approach is in utilizing the weighted residual method in deriving a 3 equation model of the height,

discharge rate and a new variable Φ equivalent to the shear rate. In particular, the adopted velocity profile is taken such that the new variable Φ is already embedded in it in the same manner as in Section 3.6 with the only advantage is that at the end we get an exact velocity profile using the advantage that the weighted residual method induces on the correction of the velocity through gauge conditions. We will apply the method for both 2D and then 3D flows. In 2D flow, we will include the diffusion terms (i.e second order terms) in the final system. This would enhance the ability to capture solitary waves and would reflect the instabilities in the system more accurately.

3.7.1 A 3-Equation Model for the 2D flow

Imposing a Velocity Profile

In the same spirit as proposed in [2], we will adapt the following velocity profile

$$\begin{aligned} u &= 3(U - \frac{s_1}{h})g_0 + \frac{45s_1}{h}g_1 + O(\varepsilon^2) \\ &= 3Ug_0 + \frac{s_1}{h}(45g_1 - 3g_0) + O(\varepsilon^2), \end{aligned} \tag{3.49}$$

where we recall that $U = \frac{1}{h}\int_0^h u\ dz$, and we define now a new variable Φ which expresses the deviation of the velocity profile from the its depth average

$$\frac{h^3\Phi^2}{5} = \int_0^h (u-U)^2\ dz. \tag{3.50}$$

If we expand the above relation, and using the definition of (3.49) knowing that $s_1^2 \sim O(\varepsilon^2)$ we get

$$\frac{h^3\Phi^2}{5} = \frac{h}{5}\big(U - 6\frac{s_1}{h}\big)^2 + O(\varepsilon^2),$$

and thus we can deduce a relation between Φ and s_1

$$\frac{s_1}{h} = \frac{1}{6}\big(U - h\Phi\big) + O(\varepsilon^2)$$

Hence, we can give an expression of u at main order as

$$u = 3Ug_0 + \frac{1}{6}\big(U - h\Phi\big)(45g_1 - 3g_0) + O(\varepsilon). \tag{3.51}$$

Let's define now

$$\tilde{u_0} = 3Ug_0 + \frac{1}{6}\big(U - h\Phi\big)(45g_1 - 3g_0)$$

and

$$\tilde{u_1} = u - \tilde{u_0}.$$

Hence, we adopt the following velocity profile

$$u = \tilde{u_0} + \varepsilon\tilde{u_1} = 3Ug_0 + \frac{1}{6}\big(U - h\Phi\big)(45g_1 - 3g_0) + \varepsilon\tilde{u_1}. \tag{3.52}$$

An important remark done here is concerning the above velocity profile: in fact, the above relation is an exact relation where $\varepsilon\tilde{u_1}$ contains all the corrections that would appear as a result of substituting the asymptotic relation between Φ and s_1 at main order in addition to the truncated order of the expression of u we already started with in (3.49).

Exploiting the Velocity Profile in the WRM

Using expression (3.52), the gauge condition imposes two conditions on $\tilde{u_1}$, which results in two residues

and as a result leads to the final model, in fact we have

$$R_1 =< \varepsilon\tilde{u_1}, 1 >= \int_0^h \varepsilon\tilde{u_1}\ dz = 0 \qquad R_2 =< \varepsilon\tilde{u_1}, g_0 >= \int_0^h \varepsilon\tilde{u_1} g_0\ dz = 0. \tag{3.53}$$

Remark 5. *These residues in fact are derived from the imposed definition of the velocity profile. We already have elaborated on this in the previous sections, but for clarity we will recall this briefly. The first one is simply by integrating the imposed velocity profile and using the fact that we have* $\int_0^h u\ dz = hU$*, then we get* $\int_0^h \tilde{u_1}\ dz = 0$*. In a similar manner, we compute* $\int_0^h u^2\ dz$ *using the given velocity profile, and from the second hand we calculate it using the fact that*

$$\int_0^h u^2\ dz = \int_0^h (u-U)^2\ dz + hU^2 = \frac{h^3\Phi^2}{5} + hU^2.$$

Upon comparing the two integrals, we can easily notice that

$$\int_0^h \varepsilon\tilde{u_1} g_0\ dz + O(\varepsilon^2) = O(\varepsilon^2),$$

which readily implies the second residue stating that $\int_0^h \tilde{u_1} g_0\ dz = 0$.

Using this velocity profile, and the gauge condition, we will derive the shallow water system in 1D using the weighted residual method, so first let's get the expression of the correction $\tilde{u_1}$ in order to plug it in the above residues. Recalling the momentum equation of the stream wise velocity in 2D reads

$$\partial_t u + u\,\partial_x u + w\,\partial_z u = \frac{\lambda}{\varepsilon Re} - \frac{1}{Fr^2}\,\partial_x p + \frac{1}{\varepsilon Re}(\varepsilon^2\partial_{xx}u + \partial_{zz}u),$$

we can write at main order

$$\partial_{zz}u = -\lambda + \varepsilon Re\left(\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{1}{Fr^2}\,\partial_x p\right) - \varepsilon^2\,\partial_{xx}u$$

since we know that $\partial_z u(h) = O(\varepsilon^2)$ and $u(0) = 0$ from boundary conditions, then we get

$$\begin{aligned}
\varepsilon\tilde{u_1} &= -u_0 + \lambda h^2 g_0 - \varepsilon Re\int_0^z\int_z^h \left(\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{1}{Fr^2}\,\partial_x p\right) d\zeta\ dz - \varepsilon^2\int_0^z\int_z^h \partial_{xx}u \\
&= (\lambda h^2 - 3U)g_0 - \frac{1}{6}\big(U - h\varphi\big)(45g_1 - 3g_0) \\
&- \varepsilon Re\int_0^z\int_z^h \left(\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{1}{Fr^2}\,\partial_x p\right) d\zeta\ dz - \varepsilon^2\int_0^z\int_z^h \partial_{xx}u.
\end{aligned}$$

Having obtained the expression of $\tilde{u_1}$, we can calculate the residues

$$\int_0^h \tilde{u_1}\ dz = 0 \qquad \text{and} \qquad \int_0^h \tilde{u_1} g_0\ dz = 0.$$

Two evolutionary equations on U and $h\varphi$ can be obtained reading

$$\begin{aligned}\partial_t U = &-\frac{14\cos\theta\,\partial_x h}{15\mathrm{Fr}^2}+\frac{14}{15}\frac{\kappa}{Fr^2}h\,\partial_x^3 h-\frac{116U^2\,\partial_x h}{1287h}+\frac{1036U\,\partial_x h\Phi}{2145}-\frac{6272h\,\partial_x h\Phi^2}{6435}\\&-\frac{4067h^2\Phi\,\partial_x\Phi}{6435}+\frac{21\Phi}{5h\mathrm{Re}\varepsilon}+\frac{476h\,\partial_x U\Phi}{2145}+\frac{322hU\,\partial_x\Phi}{1287}-\frac{532}{429}U\,\partial_x U-\frac{7U}{h^2\mathrm{Re}\varepsilon}+\frac{14\lambda}{15\mathrm{Re}\varepsilon}\\&+\frac{\varepsilon}{Re}\Big(-\frac{7}{12}\,\partial_x h\,\partial_x\Phi-\frac{63(\partial_x h)^2\Phi}{40h}-\frac{175}{96}\,\partial_{xx}h\Phi+\frac{109\,\partial_x h\,\partial_x U}{24h}-\frac{13(\partial_x h)^2U}{24h^2}\\&\qquad+\frac{113\,\partial_{xx}hU}{96h}+\frac{49}{60}h\,\partial_{xx}\Phi+\frac{83}{24}\,\partial_{xx}U\Big)\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}\partial_t\Phi = &-\frac{\cos\,\partial_x h}{3\mathrm{Fr}^2h}+\frac{\kappa}{3Fr^2}\,\partial_x^3 h+\frac{290U\,\partial_x h\Phi}{429h}-\frac{826\,\partial_x h\Phi^2}{1287}-\frac{21\Phi}{h^2\mathrm{Re}\varepsilon}+\frac{160U^2\,\partial_x h}{9009h^2}\\&+\frac{1070U\,\partial_x U}{3003h}-\frac{574h\Phi\,\partial_x\Phi}{1287}+\frac{70}{429}\,\partial_x U\Phi-\frac{830U\,\partial_x\Phi}{1287}+\frac{20U}{h^3\mathrm{Re}\varepsilon}+\frac{\lambda}{3h\mathrm{Re}\varepsilon}\\&+\frac{\varepsilon}{Re}\Big(\frac{139\,\partial_x h\,\partial_x\Phi}{24h}-\frac{15(\partial_x h)^2\Phi}{2h^2}-\frac{247\,\partial_{xx}h\Phi}{96h}+\frac{85\,\partial_x h\,\partial_x U}{6h^2}+\frac{125(\partial_x h)^2U}{24h^3}\\&\qquad+\frac{305\,\partial_{xx}hU}{96h^2}+\frac{10\,\partial_{xx}U}{3h}+\frac{55}{24}\,\partial_{xx}\Phi\Big).\end{aligned}\tag{3.55}$$

Perspective of the Approach

In what preceded, we have used the weighted residual method in order to get a shallow water system of the height h and the averaged depth velocity U coupled with an equation on the new variable Φ. Obtaining such system, we notice that on the left hand side, we have lost the conservative structure of the system which would lead to lose many interesting mathematical results. On the other hand, a further look on the right hand side, in particular on the relaxation terms, we notice that we obtain good results corresponding to the eigenmodes (damping coefficients) of a perturbed viscous film. Add to this of course, that we have gained an exact velocity profile for the system, and that no asymptotic expansion is needed to close the system as done in the previous revisited approache.

Asymptotic Expansions and Conservative Form

It is expected to obtain a non-conservative structure in the above system since we have used the weighted residual method. To fix this issue, one proposed approach is to express the convective part of the obtained equations (that are already the messed parts) using the asymptotic expansion of the variables at main order. We could easily prove the equivalence between the convected part in the above equations and a conservative form at main order, hence asymptotically we would recover a conservative left hand side, and we would preserve at the same time the relaxation and diffusion terms that are automatically generated by the residual technique. The system hence reads

$$\begin{aligned}&\partial_t h+\partial_x(hU)=0,\\&\partial_t(hU)+\partial_x(hU^2+\frac{1}{5}h^3\Phi^2)=\frac{1}{\varepsilon Re}\Big(\frac{14}{15}(\lambda h-\frac{3U}{h})+\frac{21}{5}(\Phi-\frac{U}{h})\Big)\\&\qquad\qquad-\frac{14}{15}\Big(\frac{\cos\theta}{Fr^2}h\,\partial_x h-\frac{\kappa}{Fr^2}h\,\partial_x^3 h\Big)+\frac{\varepsilon}{Re}\mathcal{D}_1,\\&\partial_t(h\Phi)+\partial_x(hU\Phi)-\frac{1}{7}\frac{\partial_x(h^4\Phi^3)}{h^2\Phi}=\frac{1}{\varepsilon}\Big(\frac{1}{3}(\lambda-\frac{3U}{h^2})+\frac{21}{h}(\frac{U}{h}-\Phi)\Big)\\&\qquad\qquad-\frac{1}{3}\Big(\frac{\cos\theta}{Fr^2}\,\partial_x h-\frac{\kappa}{Fr^2}\,\partial_x^3 h\Big)+\frac{\varepsilon}{Re}\mathcal{D}_2,\end{aligned}\tag{3.56}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the corresponding diffusion terms defined by

$$\begin{aligned}\mathcal{D}_1 = &-\frac{7}{12}h\,\partial_x h\,\partial_x\Phi - \frac{63(\partial_x h)^2\Phi}{40} - \frac{175}{96}h\Phi\,\partial_{xx}h + \frac{109\,\partial_x h\,\partial_x U}{24} - \frac{13(\partial_x h)^2 U}{24h}\\ &+ \frac{113\,\partial_{xx}hU}{96} + \frac{49}{60}h^2\,\partial_{xx}\Phi + \frac{83}{24}h\,\partial_{xx}U,\\ \mathcal{D}_2 = &\frac{139}{24}\,\partial_x h\,\partial_x\Phi - \frac{15(\partial_x h)^2\Phi}{2h} - \frac{247\,\partial_{xx}h\Phi}{96} + \frac{85\,\partial_x h\,\partial_x U}{6h} + \frac{125(\partial_x h)^2 U}{24h^2}\\ &+ \frac{305\,\partial_{xx}hU}{96h} + \frac{10\,\partial_{xx}U}{3} + \frac{55}{24}h\,\partial_{xx}\Phi.\end{aligned} \tag{3.57}$$

The first remark to note here is concerning the diffusion terms. Similar to what have been done to the convective part, we can manipulate the diffusion terms and rewrite them in a conservative form using asymptotic expansions, however, we have not found till the moment the correct format that would preserve the decrease of energy and give a branch of solitary wave that doesn't show blow up and is stable for all values of Reynolds number. We should also remark that we have succeeded to get a conservative form of the momentum equation on hU. Nevertheless, we notice that for the equation on $h\Phi$ we couldn't do that due to the term $-\frac{1}{7}\frac{\partial_x(h^4\Phi^3)}{h^2\Phi}$ which arises from the corresponding term $\int_0^h(u-U)^3$ in the weighted residual derivation. This choice in fact is due to two reasons, the denominator is surely inspired by the energy multiplier corresponding to Φ: $\partial_\Phi\mathcal{E} \sim h^2\Phi$, and this guarantees that we will get a dissipative kinetic energy of the final model. The second reason is concerning the term inside the derivative (in the numerator), where using asymptotic expansions we have tried several formulations to write this term in terms of h, U and Φ. None of the suggested reformulations could give an appropriate form to capture the correct solitary wave, and this was the only term that didn't show a finite blow-up in time for the solitary wave.

3.7.2 A 3-Equation Model for the 3D flow

We recall that we will work with the 3D system explained in Section 3.2 with boundary conditions (3.9). As a first step, we define the averaged velocity vector in the motion plane

$$\vec{U} = \frac{1}{h}\int_0^h \mathbf{u}_X\,dz. \tag{3.58}$$

Identifying with the velocity profile used in [2], we will propose the following velocity profile in the 3D dimension expressed in terms of the orthogonal basis $(g_i)_i$ of the L^2 space.

$$\begin{aligned}\mathbf{u}_X &= 3(\vec{U} - \frac{\vec{s_1}}{h})g_0 + \frac{45\vec{s_1}}{h}g_1 + O(\varepsilon^2)\\ &= 3\vec{U}g_0 + \frac{\vec{s_1}}{h}(45g_1 - 3g_0) + O(\varepsilon^2).\end{aligned} \tag{3.59}$$

We define now a new variable $\Phi = (\Phi_x, \Phi_y)$ which expresses the deviation of the velocity profile from the depth averaged velocity field. In fact, we consider first the tensor product of Φ with itself as a function of the depth averaged deviation of the velocity profile from the depth averaged velocity, i.e we define

$$\frac{h^3}{5}\Phi\otimes\Phi = \int_0^h(\mathbf{u}_X - \vec{U})\otimes(\mathbf{u}_X - \vec{U})\,dz.$$

By analogy with what have been done in the case of 2D flow, and since we would expect a symmetry in the behavior of the flow in the stream wise and span wise directions (which is clearly shown in the system if it is projected such that the gravity has a contribution in the span wise direction as well), then we can assume that

$$h\Phi = \vec{U} - \frac{6\vec{s_1}}{h} + O(\varepsilon^2), \tag{3.60}$$

and this would definitely assure the definition in (3.59). Thus, we have

$$\vec{s_1} = \frac{1}{6}(h\vec{U} - h^2\Phi) + O(\varepsilon^2). \tag{3.61}$$

and finally we can express $\mathbf{u}_X$ as

$$\begin{aligned} \mathbf{u}_X &= 3\vec{U}g_0 + \frac{1}{6}(\vec{U} - h\Phi)(45g_1 - 3g_0) + \varepsilon\tilde{\mathbf{u}}_X^1 \\ &:= \tilde{\mathbf{u}}_X^0 \ + \varepsilon\tilde{\mathbf{u}}_X^1. \end{aligned} \tag{3.62}$$

Using expression (3.62), the gauge condition imposes two conditions on $\tilde{\mathbf{u}}_X^1$, which results in two residues and as a result leads to the final model, in fact we have

$$\vec{R}_1 =< \varepsilon\tilde{\mathbf{u}}_X^1, 1 >= \int_0^h \varepsilon\tilde{\mathbf{u}}_X^1 \, dz = 0 \qquad \vec{R}_2 =< \varepsilon\tilde{\mathbf{u}}_X^1, g_0 >= \int_0^h \varepsilon\tilde{\mathbf{u}}_X^1 g_0 \, dz = 0. \tag{3.63}$$

From the momentum equation in the plane of motion we can write

$$\partial_t \mathbf{u}_X + \mathbf{u}\cdot\nabla\mathbf{u}_X = \lambda\vec{e_1} - \frac{\nabla_X p}{Fr^2} + \frac{1}{\varepsilon Re}(\partial_{zz}\mathbf{u}_X + \varepsilon^2\Delta_X\mathbf{u}_X),$$

and hence we can write

$$\partial_{zz}\mathbf{u}_X = -\lambda\vec{e_1} + \varepsilon Re\left(\partial_t\mathbf{u}_X + \mathbf{u}\cdot\nabla\mathbf{u}_X + \frac{\nabla_X p}{Fr^2}\right) + O(\varepsilon^2).$$

Using boundary conditions at main order, we have $\partial_z\mathbf{u}_X|_{z=h} = O(\varepsilon^2)$ and $\mathbf{u}_X|_{z=0} = 0$, then integrating twice in depth we get

$$\begin{aligned} \varepsilon\tilde{\mathbf{u}}_X^1 = (\lambda h^2\vec{e_1} - 3\vec{U})g_0 - \frac{1}{6}(h\vec{U} - h^2\Phi)(45g_1 - 3g_0) \\ - \varepsilon Re\int_0^z\int_z^h \left(\partial_t\mathbf{u}_X + \mathbf{u}\cdot\nabla\mathbf{u}_X + \frac{\nabla_X p}{Fr^2}\right) + O(\varepsilon^2). \end{aligned}$$

Having obtained the expression of $\tilde{\mathbf{u}}_X^1$, we can now apply now the weighted residual method with the new velocity profile, mainly (3.63), in which four evolutionary equations on U, V and φ are obtained,reading

$$
\begin{aligned}
&\partial_t h + \nabla_X \cdot (h\vec{U}) = 0,\\
&\partial_t(h\vec{U}) + a_1(\vec{U}\cdot\nabla_X)(h\vec{U}) - a_2\vec{U}\cdot\nabla_X h\vec{U} + a_3\,\mathrm{div}_X(h\vec{U})\vec{U}\\
&+ a_4h^3(\Phi\cdot\nabla_X)\Phi - a_5h^3\,\mathrm{div}_X\,\Phi\Phi - a_6h^2\Phi\cdot\nabla_X h\Phi - a_7h^2(\vec{U}\cdot\nabla_X)\Phi\\
&+ a_8h^2\,\mathrm{div}_X\,\Phi\vec{U} + a_9\Phi\cdot\nabla_X hh\vec{U} + a_{10}h\vec{U}\cdot\nabla_X h\Phi - a_{11}h^2(\Phi\cdot\nabla_X)\vec{U}\\
&- a_{12}h^2\,\mathrm{div}_X\,\vec{U}\Phi = -\frac{14}{15}\frac{\cos\theta}{Fr^2}h\nabla_X h + \frac{14}{15}(\Lambda h - \frac{3\vec{U}}{h}) + \frac{21}{5}(\frac{\vec{U}}{h} - h\Phi)\\
&+ \frac{14}{15}\frac{\kappa}{Fr^2}h\nabla_X(\Delta_X h),\\
&\partial_t(h\Phi) + b_1h(\vec{U}\cdot\nabla_X)\Phi + b_2\,\mathrm{div}_X\,\Phi h\vec{U} + b_3(\Phi\cdot\nabla_X h)\vec{U} + b_4(\vec{U}\cdot\nabla_X h)\Phi\\
&+ b_5h(\Phi\cdot\nabla_X)\vec{U} - b_6h\,\mathrm{div}_X\,\vec{U}\Phi + b_7\frac{\vec{U}}{h}(\vec{U}\cdot\nabla_X h) - b_8(\vec{U}\cdot\nabla_X)\vec{U} + b_9\,\mathrm{div}_X\,\vec{U}\vec{U}\\
&+ b_{10}h^2(\Phi\cdot\nabla_X)\Phi - b_{11}h^2\,\mathrm{div}_X\,\Phi\Phi + b_{12}(\Phi\cdot\nabla_X h)h\Phi\\
&+ b_{13}h(\Phi\cdot\nabla_X)U = -\frac{1}{3}\frac{\cos\theta}{Fr^2}\nabla_X h + \frac{1}{\varepsilon Re}\left(\frac{1}{3}(\Lambda - \frac{3\vec{U}}{h^2}) - 21(\frac{\Phi}{h} - \frac{\vec{U}}{h^2})\right)\\
&+ \frac{1}{3}\frac{\kappa}{Fr^2}\nabla_X(\Delta_X h).
\end{aligned}
\tag{3.64}
$$

where $\Lambda = (\lambda, 0)$ is the vector of coefficients of the projection of gravity vector on the plane of motion, and the coefficients a_i and b_i are listed in the following tables

$a_1 = \frac{1480}{1287}$	$a_2 = \frac{1480}{1287}$	$a_3 = \frac{1403}{1287}$	$a_4 = \frac{1862}{6435}$	$a_5 = \frac{-49}{143}$	$a_6 = \frac{6272}{6435}$
$a_7 = \frac{14}{117}$	$a_8 = -\frac{56}{429}$	$a_9 = -\frac{112}{429}$	$a_{10} = -\frac{476}{2145}$	$a_{11} = \frac{14}{117}$	$a_{12} = \frac{658}{6435}$

$b_1 = \frac{1160}{1287}$	$b_2 = -\frac{10}{39}$	$b_3 = -\frac{20}{39}$	$b_4 = \frac{359}{429}$	$b_5 = \frac{1160}{1287}$	$b_6 = \frac{83}{1287}$
$b_7 = -\frac{160}{9009}$	$b_8 = \frac{3050}{9009}$	$b_9 = -\frac{160}{9009}$	$b_{10} = \frac{322}{1287}$	$b_{11} = -\frac{28}{143}$	$b_{12} = \frac{826}{1287}$

Asymptotic Expansions and Conservative Form

In the same manner as in the 2D case, we can recast a conservative vectorial formulation of the above system. As for the second equation on $h\vec{U}$, using asymptotic expansions we can prove that its left hand side denoted

$$
\begin{aligned}
lhs = \;&\partial_t(h\vec{U}) + a_1(\vec{U}\cdot\nabla_X)(h\vec{U}) - a_2\vec{U}\cdot\nabla_X h\vec{U} + a_3\,\mathrm{div}_X(h\vec{U})\vec{U}\\
&+ a_4h^3(\Phi\cdot\nabla_X)\Phi - a_5h^3\,\mathrm{div}_X\,\Phi\Phi - a_6h^2\Phi\cdot\nabla_X h\Phi - a_7h^2(\vec{U}\cdot\nabla_X)\Phi\\
&+ a_8h^2\,\mathrm{div}_X\,\Phi\vec{U} + a_9\Phi\cdot\nabla_X hh\vec{U} + a_{10}h\vec{U}\cdot\nabla_X h\Phi - a_{11}h^2(\Phi\cdot\nabla_X)\vec{U}\\
&- a_{12}h^2\,\mathrm{div}_X\,\vec{U}\Phi.
\end{aligned}
$$

Now using the asymptotic expansion, we can easily prove that

$$lhs_U = \partial_t(h\vec{U}) + \mathrm{div}_X(h\vec{U}\otimes\vec{U} + \frac{h^3}{5}\Phi\otimes\Phi).$$

And hence, using the asymptotic expansion of the convected part, we can write

$$\begin{aligned}\partial_t(h\vec{U}) + \mathrm{div}_X(h\vec{U}\otimes\vec{U} + \frac{h^3}{5}\Phi\otimes\Phi)&\\ = -\frac{14}{15}\frac{\cos\theta}{Fr^2}h\nabla_X h + \frac{14}{15}(\Lambda h - \frac{3\vec{U}}{h}) + \frac{21}{5}(\frac{\vec{U}}{h} - h\Phi) + \frac{14}{15}\frac{\kappa}{Fr^2}h\nabla_X(\Delta_X h).&\end{aligned}$$

Concerning the second equation, we will denote its left hand side as

$$\begin{aligned}lhs_\Phi = \partial_t(h\Phi) + b_1 h(\vec{U}\cdot\nabla_X)\Phi + b_2\,\mathrm{div}_X\,\Phi h\vec{U} + b_3(\Phi\cdot\nabla_X h)\vec{U} + b_4(\vec{U}\cdot\nabla_X h)\Phi\\ + b_5 h(\Phi\cdot\nabla_X)\vec{U} - b_6 h\,\mathrm{div}_X\,\vec{U}\Phi + b_7\frac{\vec{U}}{h}(\vec{U}\cdot\nabla_X h) + b_8 h(\Phi\cdot\nabla)\vec{U}\\ - b_9(\vec{U}\cdot\nabla_X)\vec{U} + b_{10}\,\mathrm{div}_X\,\vec{U}\vec{U} + b_{11}h^2(\Phi\cdot\nabla_X)\Phi - b_{12}h^2\,\mathrm{div}_X\,\Phi\Phi\\ + b_{13}(\Phi\cdot\nabla_X h)h\Phi.\end{aligned}$$

Again using the asymptotic expansion, we find several formulations of the lhs, for instance we can prove that

$$lhs_\Phi = \frac{3}{2}\Big(\partial_t(h\Phi) + \frac{1}{2}\,\mathrm{div}_X(h\vec{U}\otimes\Phi + h\Phi\otimes\vec{U})\Big).$$

On the other hand, by analogy with the 1D case, we find a second formulation that seems in 2D a part of the kinetic energy rather than a function on $h\Phi$. It reads

$$lhs_\Phi = h^2\Phi\cdot\big(\partial_t(h\Phi) + \frac{1}{2}\,\mathrm{div}_X(h\vec{U}\otimes\Phi + h\Phi\otimes\vec{U})\big) - \frac{1}{7}\nabla_X\cdot(h^4|\Phi|^2\Phi).$$

Its interesting to have a full understanding in the 2D case of the such reformulation, especially when testing for solitary waves. For the moment, nothing further has been investigated, but the resolution in both 1D and 2D is still an ongoing topic of research.

3.8 Numerical validation

After obtaining a closed system of equations, a main test can identify the range of validity of the model under the variation of the parameters of the system, mainly Reynolds number and Weber number (or equivalently Kapitza number). We aim at constructing traveling wave solutions: solitary wave solutions. Solitary waves travel at a constant speed c and preserve their shape, thus in the moving frame of coordinates $\zeta = x - ct$, they remain stationary. In this frame, the partial differential equation is then transformed into an ordinary differential problem which is reciprocated as a dynamical system in the phase spanned by the variables and its derivatives, i.e the dynamical system will read

$$\frac{d}{dt}U = F(U)$$

where $U = (h, h', h", \Phi', \Phi")$. Solitary wave solutions thus correspond to homo clinic orbits that connect an equilibrium fixed point to itself. Such test allows to predict the wave shape, height and phase speed. It also leads to shed light on the behavior close to instability threshold described by the critical value of Reynolds number, especially when inertia becomes significant. Using AUTO07p, we have constructed a branch of solitary waves in a vertical plane. It is very demanding to test the solitary waves in a vertical plane that exhibits strong instabilities for all ranges of Reynolds number and exhibit as well abundance in capillary waves accumulating in the front of the wave which are usually hard to capture numerically.

In figure 3.2, the profile of each of Φ and h respectively is demonstrated. This one hump solitary wave is captured for a Kapitza number= 3400 corresponding to water. In accordance with experimental results, it is very often noticed that at the back of the wave, Φ behaves similar to the behavior of h, one reason is the gentle graduation of the height's gradient in this region, leading to similarity in behavior between the shear $\frac{U}{h}$ and h, which explains thus the similarity with Φ as this latter and $\frac{U}{h}$ are equivalent at main order. However this is not the case at the front of the wave due to steepness of the gradient of the profile and the appearance of capillary waves. This is validated as well in figure 3.3 where a comparison between the profiles of the shear and Φ is manifested showing a quite well agreement at the back of the wave and departures in the front of the wave accompanied with the appearance of capillary waves.

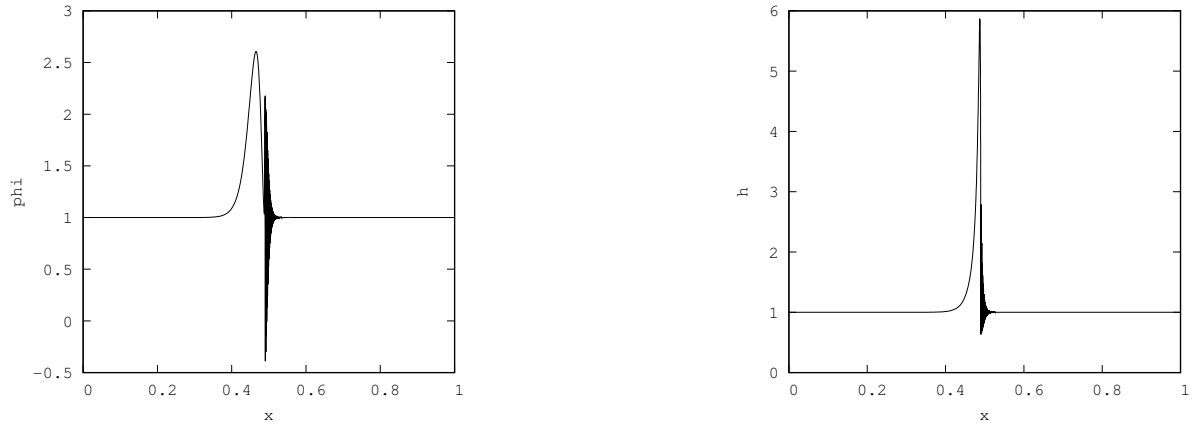


Figure 3.2: *Profiles of Φ and h respectively for a solitary wave captured at Kapitza number=3400 and Re=20.*

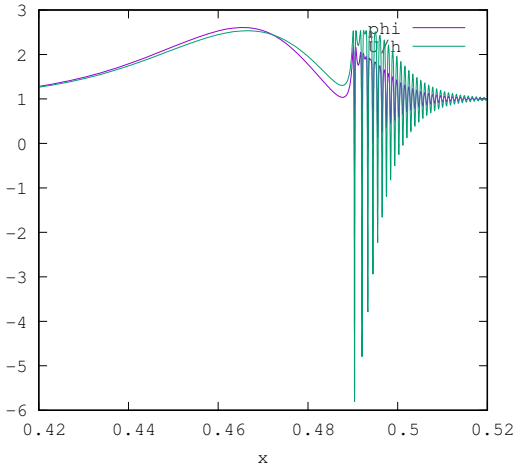


Figure 3.3: *Comparison between the shear rate and the function Φ.*

In figure 3.4, the maximum height and the phase speed are tested at different Reynolds numbers. Three regions of Reynolds number are observed that correspond to two main behaviors of the fluid, and a transition period in between. The region for low values of Reynolds number is known as the drag-gravity region, where a balance between gravity and viscous drag takes place, in this case we notice a kind of stability and slow variation in the speed phase and the maximum height. Those latter grow rapidly in the vicinity of a transition region of Reynolds number around $1.5 \sim 2$, however the solution is still captured

regardless of the instability in such region. The final region corresponds in fact to the drag-inertia region where inertial effects are dominant due to large values of Reynolds number. The interesting result in this case is in obtaining a stable state of both the maximum height and the phase speed . Thus, no blow up of the solution exists and we are still able to capture the wave in this part of the branch of the solitary wave, which is in fact absent in many other suggested models for thin liquid films.

Such numerical results are elementary and preliminary, but at the same time very important to have a basic view for the validity of the model, and to set a basis for further study on the model and on the technique from both the modeling and numerical perspectives. Thus we can simply say that this work is still in progress, and several approaches are to be examined in the future.

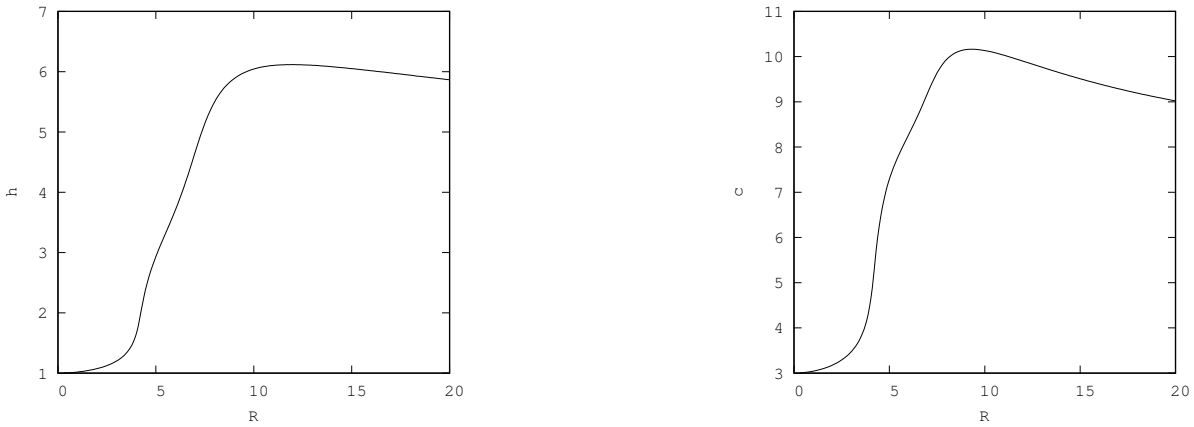


Figure 3.4: *The height and the velocity speed showing a certain stability for sufficiently high Reynolds numbers.*

4

BD Entropy and BF Dissipative Entropy

The work in this chapter is a collaboration with D. Bresch (LAMA, Universite Savoie Mont-Blanc),M. Colin (Equipe INRIA CARDAMOM), P. Noble (IMT, INSA Toulouse) and X. Song. It has been accepted in ESAIM: Proceedings and Surveys, 2020

This paper concerns the results recently announced by the authors, in *C.R. Acad. Sciences Maths* volume 357, Issue 1, 1-6 (2019), and accepted in *ESAIM 2020*, which make the link between the BD entropy introduced by D. Bresch and B. Desjardins for the viscous shallow-water equations and the Bernis-Friedman (called BF in our paper) dissipative entropy introduced to study the lubrication equations. More precisely different dissipative BF entropies are obtained from the BD entropies playing with drag terms and capillarity formula for viscous shallow water type equations. It allows in one dimension for instance to prove global existence of nonnegative weak solutions for lubrication equations starting from the global existence of nonnegative weak solutions for appropriate viscous shallow-water equations. It also allows to prove global existence of nonnegative weak solutions for fourth-order equation including the Derrida-Lebowitz-Speer-Spohn equation starting from compressible Navier-Stokes type equations.

The chapter is organized as follows:

-**In section 4.1** we introduce the general setup of the technique that relates between a shallow water model and a lubrication equation.

-**In section 4.2** we establish the weak limit between a viscous shallow water model with a drag term (derived in [4] and justified in [5]) and the coresponding lubrication equation.

-**In section 4.3** we generalize the drag term in the previous section and we establish the limit between the corresponding shallow water system and lubrication equation.

-**In section 4.4** a logarithmic fourth order lubrication equation is considered which we relate with a second order shallow water system with a general capillarity term. A special application of such case gives the weak limit of a fourth order lubrication equation known as Derrida-Lebowitz-Speer-Spohn equation.

4.1 Introduction

In a one-dimensional periodic torus domain $\Omega = \mathbb{T}$, lubrication type models with general representations of the surface tension and the second order dissipative term reads

$$\partial_t h + \partial_x\big(\frac{1}{\alpha W_e}F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2}D(h)\partial_x h\big) = 0. \tag{4.1}$$

with the initial condition

$$h|_{t=0} = h_0 \geqslant 0 \text{ where } \int_\Omega h_0 = M > 0.$$

Note that the coefficients M (total mass), α (drag term coefficient), We (Weissenberg number) and Fr (Froude number) are given positive coefficients and the nonnegative functions $s \mapsto F(s)$ and $s \mapsto D(s)$ depends on the applications. In their paper [70], Bernis and Friedman proved the existence of a weak solution for the higher order nonlinear degenerate parabolic equations with $F(h) = h^n$ and $D(h) = 0$ with $n > 1$ and suggested a new entropy inequality- referred to by BF entropy- which provides additional estimates serving for increasing the regularity of the weak solution obtained. This paper has been the starting point of plenty of important results for instance concerning the general class of $F(h) = h^n$ and $D(h) = h^m$ for some coeffficients m and n, the interested reader is referred to [71], [70], [72] and [73] and references therein. It is important to explain what is the BF entropy and how to get it formally: A regularization process is used to justify it. Let us define the functionals

$$g(s) = -\int_s^A \frac{1}{F(r)}\,dr, \qquad G(s) = -\int_s^A g(r)\,dr,$$

with A being an integer such that $A \geqslant \max|h(x,t)|$. According to Bernis and Friedman, we multiply (4.1) by $G'(h)$

$$\int_\Omega G'(h)\,\partial_t h\;dx + \int_\Omega \partial_x(F(h)\,\partial_x^3 h)G'(h)\;dx - \int_\Omega \partial_x(\frac{1}{\alpha Fr^2}D(h)\,\partial_x h)G'(h)\;dx = 0.$$

Integrating by parts the last two terms, knowing that $G'' = \frac{1}{F}$, we get

$$\frac{d}{dt}\int_\Omega G(h)\;dx + \int_\Omega \frac{(\partial_x^2 h)^2}{\alpha We} + \frac{1}{\alpha Fr^2}\frac{D(h)}{F(h)}(\partial_x h)^2\;dx = 0.$$

Finally, integrate in time, we get for all $t \in (0,T)$

$$\int_\Omega G(h(x,t))\;dx + \int_0^t\int_\Omega \frac{(\partial_x^2 h)^2}{\alpha We} + \frac{1}{\alpha Fr^2}\frac{D(h)}{F(h)}(\partial_x h)^2\;dx\;dt = \int_\Omega G(h_0(x))\;dx. \tag{4.2}$$

On the other hand, viscous shallow water type models with a general representation of the surface tension and drag term reads

$$\begin{aligned}
&\partial_t h_\varepsilon + \partial_x(h_\varepsilon u_\varepsilon) = 0,\\
&\partial_t(h_\varepsilon u_\varepsilon) + \partial_x(h_\varepsilon u_\varepsilon^2) + \frac{1}{\varepsilon Fr^2}S(h_\varepsilon)\partial_x(h_\varepsilon) = \frac{4}{R_e}\partial_x(h_\varepsilon\partial_x u_\varepsilon) + \frac{1}{\varepsilon W_e}h_\varepsilon\partial_x^3 h_\varepsilon - \alpha\frac{h_\varepsilon^2}{\varepsilon T(h_\varepsilon)}u_\varepsilon
\end{aligned} \tag{4.3}$$

with the initial conditions

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \quad (h_\varepsilon u_\varepsilon)|_{t=0} = m_0^\varepsilon$$

where α is a positive drag coefficient, R_e, W_e and Fr are respectively the adimentional Reynold, Weber and Froude numbers and ε is an non-dimensional number. Note that the terms on the right-hand side of the momentum equation represent respectively the viscous term, the capillarity term and the drag term and S(h) and T(h) are two nonnegative functions. As an indispensable step to prove global existence of a weak solution of such systems (4.3), we recall now the BD entropy inequality which was discovered by D. Bresch and B. Desjardins in solving diffusive capillary models of Korteweg type, where they showed

that such a non-trivial entropy may govern additional information on the gradient of the density leading to extra regularity on this latter as well as a control of the degenerate term near vacuum , see [69]. So, deriving the mass equation in space, multiplying by $\frac{4}{R_e}$, then summing with the momentum equation, and multiplying the sum by the augmented velocity $v_\varepsilon = u_\varepsilon + \frac{4}{R_e}\partial_x(\log h_\varepsilon)$, the authors obtained the following equation:

$$\begin{aligned}&\frac{1}{2}\int_\Omega \partial_t(h_\varepsilon v_\varepsilon^2 + \frac{1}{\varepsilon W_e}(\partial_x h_\varepsilon)^2)\,dx + \int_\Omega \frac{1}{\varepsilon Fr^2}\partial_x h_\varepsilon \frac{S(h_\varepsilon)}{h_\varepsilon}h_\varepsilon u_\varepsilon + \frac{4\alpha}{\varepsilon R_e}h_\varepsilon u_\varepsilon \frac{\partial_x h_\varepsilon}{T(h_\varepsilon)}\,dx \\ &+ \int_\Omega \frac{4}{R_e}\frac{1}{\varepsilon Fr^2}\frac{S(h_\varepsilon)}{h_\varepsilon}(\partial_x h_\varepsilon)^2 + \frac{4}{R_e}\frac{1}{\varepsilon W_e}(\partial_x^2 h_\varepsilon)^2 + \alpha\frac{h_\varepsilon^2 u_\varepsilon^2}{\varepsilon T(h_\varepsilon)}\,dx.\end{aligned} \tag{4.4}$$

Indeed, introducing the two functions T_0 and S_0 such that

$$\frac{\partial_x h_\varepsilon}{T(h_\varepsilon)} = \partial_x(T_0(h_\varepsilon)) \quad \text{and} \quad \partial_x h_\varepsilon \frac{S(h_\varepsilon)}{h_\varepsilon} = \partial_x(S_0(h_\varepsilon))$$

and using the mass equation, the second integral in (4.4) becomes

$$\int_\Omega \frac{1}{\varepsilon Fr^2}\partial_x h_\varepsilon \frac{S(h_\varepsilon)}{h_\varepsilon}h_\varepsilon u_\varepsilon + \frac{4\alpha}{\varepsilon R_e}h_\varepsilon u_\varepsilon \frac{\partial_x h_\varepsilon}{T(h_\varepsilon)}\,dx = \int_\Omega \frac{1}{\varepsilon Fr^2}\partial_t h_\varepsilon S_0(h_\varepsilon) + \frac{4\alpha}{\varepsilon R_e}\partial_t h_\varepsilon T_0(h_\varepsilon)\,dx. \tag{4.5}$$

Assuming furthermore that there exists two functions T_1 and S_1 such that

$$\partial_t h_\varepsilon S_0(h_\varepsilon) = \partial_t(S_1(h_\varepsilon)) \quad \text{and} \quad \partial_t h_\varepsilon T_0(h_\varepsilon) = \partial_t(T_1(h_\varepsilon)),$$

the second integral in (4.4) yields

$$\int_\Omega \frac{1}{\varepsilon Fr^2}\partial_x h_\varepsilon \frac{S(h_\varepsilon)}{h_\varepsilon}h_\varepsilon u_\varepsilon + \frac{4\alpha}{\varepsilon R_e}h_\varepsilon u_\varepsilon \frac{\partial_x h_\varepsilon}{T(h_\varepsilon)}\,dx = \int_\Omega \partial_t\left(\frac{1}{\varepsilon Fr^2}S_1(h_\varepsilon) + \frac{4\alpha}{\varepsilon R_e}T_1(h_\varepsilon)\right)dx. \tag{4.6}$$

Thus, the BD-entropy for (4.3) reads, for all $t \in (0,T)$

$$\begin{aligned}&\int_\Omega \frac{1}{2}h_\varepsilon(x,t)v_\varepsilon(x,t)^2 + \frac{1}{2}\frac{1}{\varepsilon W_e}(\partial_x h_\varepsilon(x,t))^2 + \frac{1}{\varepsilon Fr^2}S_1(h_\varepsilon(x,t)) + \frac{4\alpha}{\varepsilon R_e}T_1(h_\varepsilon(x,t))\,dx \\ &+ \int_0^t\int_\Omega \left(\frac{4}{R_e}\frac{1}{\varepsilon Fr^2}\frac{S(h_\varepsilon)}{h_\varepsilon}(\partial_x h_\varepsilon)^2 + \frac{4}{R_e}\frac{1}{\varepsilon W_e}(\partial_x^2 h_\varepsilon)^2 + \alpha\frac{h_\varepsilon^2 u_\varepsilon^2}{\varepsilon T(h_\varepsilon)}\right)dx\,dt \\ &= \int_\Omega \frac{1}{2}\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon} + \frac{1}{2}\frac{1}{\varepsilon W_e}(\partial_x h_0^\varepsilon(x))^2 + \frac{1}{\varepsilon Fr^2}S_1(h_0^\varepsilon(x)) + \frac{4\alpha}{\varepsilon R_e}T_1(h_0^\varepsilon(x))\,dx.\end{aligned} \tag{4.7}$$

In this paper, as indicated in [80], the main objective is to exhibit a different way to prove the existence of a non-negative global weak solution for (4.1). In fact, the authors in [70] proved that a nonnegative weak solution of the lubrication model of the form

$$\partial_t h + \partial_x(F(h)\partial_x^3 h) = 0 \text{ in } (0,T)\times\Omega, \qquad h|_{t=0} = h_0 \geqslant 0 \text{ in } \Omega \tag{4.8}$$

with $F(h) = h^n$ (with $n \geqslant 1$) satisfies the weak formulation of the above system, as well as the BF dissipative entropy given in (4.2). This has been generalized in several papers as mentionned previously. On the other hand, a weak solution of system (4.3) would satisfy its corresponding weak formulation as well as the energy and BD-entropy inequality: see for instance [69] and more recently [81], [82], [75] and the recent handbook [83]. The key tool in our method is to prove an existence result for system (4.1) by passing the limit of ε to zero in both the weak formulation of (4.3) as well as the BD-entropy. One would readily infer that, more precisely, we will prove that the BD entropy of the appropriate viscous shallow water system will implies the different BF dissipative entropy of the lubrication system at the limit playing with the drag terms and the capillarity forces. Different BF entropies are obtained depending on the choice of drag terms generalizing in some sense papers such as [84], [85].

For readers convenience, we decompose the paper in three sections. In the first section we will discuss the case $S(r) = r$ and $T(r) = F(r) = r^2 + r^3$ which corresponds to the viscous shallow water with drag term formally derived in [4] and mathematically justified in [5]. Then we discuss a viscous compressible system with a nonlinear drag term and the link with some results that we can find for instance in [71], [72], [73] and [74]. Finally we conclude with more general capillarity terms with no pressure on the viscous compressible system to derive general forms of fourth-order equation including the Derrida-Lebowitz-Speer-Spohn equation.

4.2 The limit of a viscous shallow water model formally derived in [4] and justified in [5]

In this section, we will consider the viscous shallow water model given in (4.3) regarding that $S(r) = r$ and $T(r) = F(r) = r^2 + r^3$. This choice corresponds to the model formally derived in [4] and mathematically justified in [5]. We will prove that it will be linked to the lubrication model studied for instance by A.L. Bertozzi and M. Pugh. More precisely, consider the Shallow Water system corresponding to the new drag term $\frac{h_\varepsilon^2}{F(h_\varepsilon)}u_\varepsilon$, defined in a periodic domain Ω:

$$\begin{aligned}
&\partial_t h_\varepsilon + \partial_x(h_\varepsilon u_\varepsilon) = 0,\\
&\varepsilon\left(\partial_t(h_\varepsilon u_\varepsilon) + \partial_x(h_\varepsilon u_\varepsilon^2)\right) + \frac{h_\varepsilon \partial_x(h_\varepsilon)}{\mathrm{Fr}^2} = \varepsilon\left(\frac{4}{\mathrm{R_e}}\partial_x(h_\varepsilon \partial_x u_\varepsilon)\right) + \frac{1}{\mathrm{W_e}} h_\varepsilon \partial_x^3 h_\varepsilon - \alpha\frac{u_\varepsilon}{1+h_\varepsilon}.
\end{aligned} \tag{4.9}$$

The initial conditions are given by :

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \qquad (h_\varepsilon u_\varepsilon)|_{t=0} = m_0^\varepsilon.$$

4.2.1 Formal limit

Assuming enough uniform boundedness on the different terms of the above system, and taking $\varepsilon \to 0$ in (4.9), we obtain

$$\begin{aligned}
&\partial_t h + \partial_x(hu) = 0,\\
&hu = \frac{1}{\alpha W_e}F(h)\partial_x^3 h - \frac{1}{\alpha Fr^2}F(h)\partial_x h,
\end{aligned} \tag{4.10}$$

with $F(h) = h^2 + h^3$ which can be written equivalently

$$\partial_t h + \partial_x\left(\frac{1}{\alpha W_e}F(h)\right)\partial_x^3 h - \frac{1}{\alpha Fr^2}F(h)\partial_x h = 0. \tag{4.11}$$

The mathematical justification of this model relies on an energy method with an indispensable contribution of the BD-entropy.

4.2.2 Mathematical justification

The first essential step herein is the a priori estimate which is maintained by the uniform bounds that both the energy and BD-entropy offer. The energy equation corresponding to (4.9) is obtained by multiplying the momentum equation by u_ε. Integrating then in space and using the mass equation, we obtain:

$$\frac{d}{dt}\left(\int_\Omega \varepsilon\frac{h_\varepsilon u_\varepsilon^2}{2} + \frac{h_\varepsilon^2}{2Fr^2} + \frac{(\partial_x h_\varepsilon)^2}{2W_e}\right)\,dx + \int_\Omega \frac{4\varepsilon}{R_e}h_\varepsilon(\partial_x u_\varepsilon)^2 + \alpha\frac{h_\varepsilon^2 u_\varepsilon^2}{F(h_\varepsilon)}\,dx = 0. \tag{4.12}$$

As for the BD-entropy inequality, it reads

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_\varepsilon v_\varepsilon^2\,dx+\frac{1}{\varepsilon}\left[\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{h_\varepsilon^2}{Fr^2}+\frac{(\partial_x h_\varepsilon)^2}{W_e}\,dx+\alpha\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}\,dx\right]$$
$$+\frac{4}{\varepsilon \mathrm{R_e}}\left[\int_\Omega\frac{(\partial_x h_\varepsilon)^2}{\mathrm{Fr}^2}\,dx+\int_\Omega\frac{(\partial_x^2 h_\varepsilon)^2}{\mathrm{W_e}}\,dx+\alpha\int_\Omega\frac{h_\varepsilon u_\varepsilon}{F(h_\varepsilon)}\partial_x h_\varepsilon\,dx\right]=0,$$

where $v_\varepsilon = u_\varepsilon + \frac{4\alpha}{R_e}\partial_x(\log h_\varepsilon)$.
The last term can be rewritten as:

$$\begin{aligned}\int_\Omega\frac{h_\varepsilon u_\varepsilon}{F(h_\varepsilon)}\partial_x h_\varepsilon &= \int_\Omega h_\varepsilon u_\varepsilon\frac{\partial_x h_\varepsilon}{h_\varepsilon^2+h_\varepsilon^3}\\ &=\int_\Omega h_\varepsilon u_\varepsilon\partial_x h_\varepsilon\Big(\frac{1}{h_\varepsilon^2}+\frac{1}{1+h_\varepsilon}-\frac{1}{h_\varepsilon}\Big)\\ &=\int_\Omega h_\varepsilon u_\varepsilon\partial_x\Big(\frac{-1}{h_\varepsilon}+\log(1+h_\varepsilon)-\log(h_\varepsilon)\Big)\\ &=\frac{d}{dt}\int_\Omega\underbrace{-(1+h_\varepsilon)\log\Big(\frac{h_\varepsilon}{1+h_\varepsilon}\Big)}_{>0}.\end{aligned}$$

Finally, the BD-entropy reads

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\frac{\varepsilon}{2}h_\varepsilon v_\varepsilon^2-\frac{4\alpha}{R_e}(1+h_\varepsilon)\log\Big(\frac{h_\varepsilon}{1+h_\varepsilon}\Big)+\frac{h_\varepsilon^2}{2Fr^2}+\frac{(\partial_x h_\varepsilon)^2}{2W_e}\,dx\\ &+\alpha\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}+\frac{4}{\mathrm{R_e}}\left[\int_\Omega\frac{(\partial_x h_\varepsilon)^2}{\mathrm{Fr}^2}\,dx+\frac{(\partial_{xx}h_\varepsilon)^2}{\mathrm{W_e}}\,dx\right]=0.\end{aligned}\tag{4.13}$$

Let's recall the definition of the weak formulation of system (4.9):

Definition 4.6. *A weak formulation of the shallow water model with the nonlinear drag term represented in system (4.9) is given by*

$$\int_0^\infty\int_\Omega h_\varepsilon\partial_t\psi\,dx\,dt+\int_\Omega h_0^\varepsilon\psi(\cdot,0)\,dx=-\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_x\psi\,dx\,dt,\tag{4.14}$$

and

$$\begin{aligned}&\varepsilon\Big(\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_t\varphi\,dx\,dt+\int_\Omega m_0^\varepsilon\varphi(\cdot,0)\,dx+\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon^2\partial_x\varphi\,dx\,dt\Big)\\ &-\frac{4\varepsilon}{R_e}\int_0^\infty\int_\Omega h_\varepsilon\partial_x u_\varepsilon\partial_x\varphi\,dx\,dt+\frac{1}{W_e}\int_0^\infty\int_\Omega\frac{1}{2}(\partial_x h_\varepsilon)^2\partial_x\varphi\,dx\,dt\\ &-\frac{1}{W_e}\int_0^\infty\int_\Omega h_\varepsilon\partial_x^2 h_\varepsilon\partial_x\varphi\,dx\,dt+\frac{1}{Fr^2}\int_0^\infty\int_\Omega h_\varepsilon^2\partial_x\varphi\,dx\,dt-\alpha\int_0^\infty\int_\Omega\frac{h_\varepsilon^2u_\varepsilon}{F(h_\varepsilon)}\varphi\,dx\,dt=0,\end{aligned}\tag{4.15}$$

for all $\psi\in\mathcal{C}_0^\infty(\Omega\times[0,\infty))$ and $\varphi\in\mathcal{C}_0^\infty(\Omega\times[0,\infty))$.

To proceed, we give first the definition of a global weak solution of (4.9), as well as its existence result which follows the proof in [69]

Definition 4.7. *A couple (h,u) is said to be a global weak solution of (4.9) if it satisfies the weak*

formulations (4.14)-(4.15) as well as the energy and BD-entropy inequalities respectively given by

$$\begin{aligned}&\sup_{t\in(0,T)}\left(\int_\Omega \varepsilon\frac{h_\varepsilon u_\varepsilon^2}{2}+\frac{h_\varepsilon^2}{2Fr^2}+\frac{(\partial_x h_\varepsilon)^2}{2W_e}\,dx\right)(t)+\int_0^T\int_\Omega \frac{4\varepsilon}{R_e}h_\varepsilon(\partial_x u_\varepsilon)^2+\alpha\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}\,dx\\&\leqslant\left(\int_\Omega \varepsilon\frac{m_0^\varepsilon}{2h_0^\varepsilon}+\frac{(h_0^\varepsilon)^2}{2Fr^2}+\frac{(\partial_x h_0^\varepsilon)^2}{2W_e}\right)dx.\end{aligned}\tag{4.16}$$

and

$$\begin{aligned}&\sup_{t\in(0,T)}\Big(\int_\Omega \frac{\varepsilon}{2}h_\varepsilon v_\varepsilon^2-\frac{4\alpha}{R_e}(1+h_\varepsilon)\log(\frac{h_\varepsilon}{1+h_\varepsilon})+\frac{h_\varepsilon^2}{2Fr^2}+\frac{(\partial_x h_\varepsilon)^2}{2W_e}\,dx\Big)(t)\\&+\alpha\int_0^T\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}+\frac{4}{\mathrm{R_e}}\left[\int_0^T\int_\Omega\frac{(\partial_x h_\varepsilon)^2}{\mathrm{Fr}^2}\,dx+\int_0^T\int_\Omega\frac{(\partial_{xx} h_\varepsilon)^2}{\mathrm{W_e}}\,dx\right]\\&\leqslant\int_\Omega\frac{\varepsilon}{2}(\frac{m_0^\varepsilon}{\rho_0^\varepsilon})^2-\frac{4\alpha}{R_e}(1+h_0^\varepsilon)\log(\frac{h_0^\varepsilon}{1+h_0^\varepsilon})+\frac{(h_0^\varepsilon)^2}{2Fr^2}+\frac{(\partial_x h_0^\varepsilon)^2}{2W_e}\,dx\end{aligned}\tag{4.17}$$

The existing result of global weak solutions reads

Theorem 4.7. *Let $(h_0^\varepsilon,m_0^\varepsilon)$ be such that $h_0^\varepsilon>0$ and*

$$h_0^\varepsilon\in H^1(\Omega),\quad \varepsilon|m_0^\varepsilon|^2/h_0^\varepsilon\in L^1(\Omega),\quad \sqrt{\varepsilon}\partial_x\sqrt{h_0}^\varepsilon\in L^2(\Omega),$$

$$-(1+h_0^\varepsilon)log(\frac{h_0^\varepsilon}{1+h_0^\varepsilon})\in L^1(\Omega),$$

then there exists a global weak solution of (4.9) in the sense of definition 4.6.

The main result in this part is the following result

Theorem 4.8. *Given a sequence $(h_\varepsilon,u_\varepsilon)_\varepsilon$ a global solution of (4.9), satisfying the initial conditions given in the previous theorem, in the sense of Theorem 4.7 then there exists a subsequence of $(h_\varepsilon,u_\varepsilon)$ that converges to a couple (h,u), which is a solution of the weak formulation of the lubrication system satisfying the initial condition $h|_{t=0}=h_0$, where h_0 is the weak limit of h_0^ε in $H^1(\Omega)$.*

Proof. The proof starts from integrating the energy and BD-energy inequalities in the time interval (0,t), for all t in (0,T). We obtain

$$\begin{aligned}&\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t)u_\varepsilon(x,t)^2\,dx+\frac{1}{2Fr^2}\int_\Omega h_\varepsilon(x,t)^2\,dx+\frac{1}{2W_e}\int_\Omega(\partial_x h_\varepsilon(x,t))^2\,dx\\&+\frac{4\varepsilon}{R_e}\int_0^t\int_\Omega h_\varepsilon(\partial_x u_\varepsilon)^2\,dx\,dt+\alpha\int_0^t\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}\,dx\,dt\leqslant\frac{1}{2}\int_\Omega\frac{(m_0^\varepsilon(x))^2}{h_0^\varepsilon(x)}+\frac{h_0^\varepsilon(x)^2}{Fr^2}+\frac{(\partial_x h_0^\varepsilon(x))^2}{W_e}\,dx,\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}&\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t)v_\varepsilon(x,t)^2\,dx+\frac{1}{2Fr^2}\int_\Omega h_\varepsilon(x,t)^2\,dx+\frac{1}{2W_e}\int_\Omega(\partial_x h_\varepsilon(x,t))^2\,dx-\\&\frac{4\alpha}{R_e}\int_\Omega(1+h_\varepsilon(x,t))\log(\frac{h_\varepsilon(x,t)}{1+h_\varepsilon(x,t)})\,dx+\alpha\int_0^t\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}\,dx\,dt+\frac{4}{R_e}\int_0^t\int_\Omega\frac{(\partial_x h_\varepsilon)^2}{Fr^2}+\frac{(\partial_x^2 h_\varepsilon)^2}{W_e}\,dx\,dt\\&\leqslant\frac{\varepsilon}{2}\int_\Omega\left(\frac{(m_0^\varepsilon(x))^2}{h_0^\varepsilon(x)}+2\frac{m_0^\varepsilon(x)}{h_0^\varepsilon(x)}\partial_x h_0^\varepsilon(x)+\frac{(\partial_x h_0^\varepsilon(x))^2}{h_0^\varepsilon(x)}\right)dx+\int_\Omega\frac{(h_0^\varepsilon(x))^2}{2Fr^2}\,dx+\int_\Omega\frac{(\partial_x h_0^\varepsilon(x))^2}{2W_e}\,dx\\&-\frac{4\alpha}{R_e}\int_\Omega(1+h_0^\varepsilon(x))\log(\frac{h_0^\varepsilon(x)}{1+h_0^\varepsilon(x)})\,dx.\end{aligned}\tag{4.19}$$

Using the uniform bounds on the initial data introduced in theorem 4.7, and since h_0^ε is bounded, we obtain that the right hand side of each of the above equalities is bounded uniformly in ε. Thus, the energy yields the following estimates

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \tag{4.20}$$

$$\|h_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leqslant C, \tag{4.21}$$

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\partial_x u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \tag{4.22}$$

$$\|\frac{u_\varepsilon}{\sqrt{1+h_\varepsilon}}\|_{L^2(0,T;L^2(\Omega))} \leqslant C. \tag{4.23}$$

From (4.21), and applying Rellich-Kandrachov compactness theory (since Ω is bounded and Lipschitz domain), we get that $H^1(\Omega) \subset L^\infty(\Omega)$, and thus

$$\|h_\varepsilon\|_{L^\infty(0,T,\Omega)} \leqslant \|h_\varepsilon\|_{L^\infty(0,T,H^1(\Omega))} \leqslant C. \tag{4.24}$$

A further look on the fifth term in (4.18) allows to get a uniform bound on u_ε in $L^2(0,T;L^2(\Omega))$. Indeed

$$\begin{aligned}\int_0^t\int_\Omega u_\varepsilon^2\,dx\,dt &= \int_0^t\int_\Omega \frac{\sqrt{(1+h_\varepsilon)}}{\sqrt{(1+h_\varepsilon)}}u_\varepsilon^2\,dx\,dt\\ &= \int_0^t\int_\Omega \frac{\eta}{2}(1+h_\varepsilon)u_\varepsilon^2 + \frac{1}{2\eta}\frac{u_\varepsilon^2}{1+h_\varepsilon}\,dx\,dt\\ &\leqslant \frac{\eta}{2}(1+M)\int_0^t\int_\Omega u_\varepsilon^2\,dx\,dt + C,\end{aligned}$$

where M is chosen due to Sobolev injections and the uniform boundedness of u_ε in $L^\infty(0,T;H^1(\Omega))$ such that

$$\|h_\varepsilon\|_{L^\infty(0,T,\Omega)} \leqslant M.$$

Therefore, choosing η such that $\eta \leqslant \frac{1}{1+M}$, we obtain

$$\|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leqslant C. \tag{4.25}$$

We can deduce as well that

$$\int_0^t\int_\Omega (\frac{u_\varepsilon}{1+h_\varepsilon})^2\,dx\,dt \leqslant \int_0^t\int_\Omega u_\varepsilon^2\,dx\,dt \leqslant C. \tag{4.26}$$

As for the additional estimates provided by the BD-entropy, we get

$$\sqrt{\varepsilon}\|\,\partial_x(\sqrt{h_\varepsilon})\|_{L^\infty(0,T,L^2(\Omega))} \leqslant C, \tag{4.27}$$

$$\|-(1+h_\varepsilon)\log(\frac{h_\varepsilon}{1+h_\varepsilon})\|_{L^\infty(0,T,L^1(\Omega))} \leqslant C, \tag{4.28}$$

$$\|\,\partial_x h_\varepsilon\|_{L^2(0,T,H^1(\Omega))} \leqslant C \qquad \text{and} \qquad \|\,\partial_x h_\varepsilon\|_{H^1((0,T)\times\Omega)} \leqslant C. \tag{4.29}$$

Having such bounds, on can deduce some consequences, mainly using (4.21) and (4.25), such as

$$\partial_x(h_\varepsilon u_\varepsilon) \text{ bounded uniformly in } L^2(0,T,H^{-1}(\Omega)), \tag{4.30}$$

$$h_\varepsilon u_\varepsilon, h_\varepsilon\,\partial_x^2 h_\varepsilon \text{ bounded uniformly in } L^2(0,T,L^2(\Omega)). \tag{4.31}$$

Now we utilize the various uniform bounds obtained on the different unknowns of the system to conclude some convergence results. Starting from (4.25), we can infer that there exists u in $L^2(0,T,L^2(\Omega))$ such that, up to a subsequence, u_ε converges weakly to u in $L^2(0,T,L^2(\Omega))$. Since Ω is a bounded Lipschitz domain, then applying compactness theory since $H^1(\Omega)$ is compactly injected in $L^2(\Omega)$. But having h_ε

bounded uniformly in $L^\infty(0,T,H^1(\Omega))$, and $\partial_t h_\varepsilon = -\partial_x(h_\varepsilon u_\varepsilon)$ bounded uniformly in $L^2(0,T,H^{-1})$, by Aubin-Simon's theorem we find that h_ε converges strongly to some h in $C([0,T],L^2(\Omega))$. Also, due to the uniform boundedness of $\partial_x h_\varepsilon$ in $H^1((0,T)\times\Omega)$, we get that h_ε is in $L^2(0,T,H^2(\Omega))$. But since the embedding of $H^2(\Omega)$ in $H^1(\Omega)$ is compact, again using Aubin-Simon's theorem we get that h_ε also converges strongly to h in $L^2(0,T,H^1(\Omega))$. In fact, one consequence of these latter results is that

$$h_\varepsilon(x,0) \rightharpoonup h(x,0) \quad \text{in} \quad H^1(\Omega).$$

We denote by h_0 the weak limit in $H^1(\Omega)$ of h_0^ε. Finally, passing to the limit in the weak formulation (4.14)-(4.15) of system (4.9), we obtain:

1. using uniform boundedness of $\partial_x h_\varepsilon$ in $H^1((0,T)\times\Omega)$, we get the strong convergence and thus weak convergence of h_ε to h in $L^2((0,T)\times\Omega)$. Thus we have that

$$\int_0^t\int_\Omega h_\varepsilon \partial_t\psi \; dx \; dt \underset{\varepsilon\longrightarrow 0}{\longrightarrow} \int_0^t\int_\Omega h\partial_t\psi \; dx \; dt.$$

2. h_ε converges to h weakly in $C(0,T,L^2(\Omega))$, hence

$$\int_\Omega h_0^\varepsilon\psi(.,0) \; dx \underset{\varepsilon\longrightarrow 0}{\longrightarrow} \int_\Omega h_0\psi(.,0) \; dx.$$

3. Due to strong convergence of h_ε in $L^2((0,T)\times\Omega)$, and the boundedness and weak convergence of u_ε in $L^2(0,T,L^2(\Omega))$, we get that $h_\varepsilon u_\varepsilon$ connverges weakly to hu in $L^2(0,T,L^2(\Omega))$

$$\begin{aligned}\int_0^t\int_\Omega (h_\varepsilon u_\varepsilon - hu)\partial_x\psi \; dx \; dt &= \int_0^t\int_\Omega (h_\varepsilon - h)u_\varepsilon \, \partial_x\psi \; dx \; dt + \int_0^t\int_\Omega (u_\varepsilon - u)h \, \partial_x\psi \; dx \; dt \\ &\leqslant \|\, \partial_x\psi\|_{L^\infty((0,T)\times\Omega)}\|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\|h_\varepsilon - h\|_{L^2(0,T;L^2(\Omega))} + \int_0^t\int_\Omega (u_\varepsilon - u)h \, \partial_x\psi \; dx \\ &\underset{\varepsilon\longrightarrow 0}{\longrightarrow} 0 \, .\end{aligned}$$

4. We follow now the same analysis for (4.15): the integrals multiplied by ε tend to zero upon the limit as they are uniformly bounded

$$\begin{aligned}&\varepsilon\Big(\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_t\varphi \; dx \; dt + \int_\Omega m_0^\varepsilon\varphi(\cdot,0) \; dx + \int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon^2\partial_x\varphi \; dx \; dt - \frac{4}{R_e}\int_0^\infty\int_\Omega h_\varepsilon\partial_x u_\varepsilon\partial_x\varphi \; dx \; dt\Big) \\ &\leqslant \varepsilon \max\big(\|h_\varepsilon\|_{L^\infty((0,T)\times\Omega)}, \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}, \|m_0^\varepsilon\|_{L^2(\Omega)}, \|\, \partial_t\varphi\|_{L^2(0,T;L^2(\Omega)}, \\ &\qquad \|\varphi\|_{L^\infty(0,T;L^2(\Omega)}, \|\, \partial_x\varphi\|_{L^\infty((0,T)\times\Omega)}\big) + \sqrt{\varepsilon}\|\sqrt{\varepsilon}\sqrt{h_\varepsilon}\, \partial_x u_\varepsilon\|_{L^2(0,T;L^2(\Omega)} \\ &\leqslant \varepsilon C_1 + \sqrt{\varepsilon}C_2 \underset{\varepsilon\longrightarrow 0}{\longrightarrow} 0 \, .\end{aligned}$$

5. As the for fifth and sixth terms, we use the results obtained in (4.21), (4.31), the strong convergence of h_ε to h in $C(0,T,L^2(\Omega))$ and $L^2(0,T,H^1(\Omega))$, and the weak convergence of $\partial_x^2 h_\varepsilon$ in $L^2(0,T,L^2(\Omega))$, we get

$$\begin{aligned}\int_0^t\int_\Omega h_\varepsilon\partial_x^2 h_\varepsilon\partial_x\varphi - h\partial_x^2 h\partial_x\varphi \; dx \; dt &= \int_0^t\int_\Omega (h_\varepsilon - h)\partial_x^2 h_\varepsilon\partial_x\varphi \; dx \; dt + \int_0^t\int_\Omega (\partial_x^2 h_\varepsilon - \partial_x^2 h)h \, \partial_x\varphi \; dx \; dt \\ &\leqslant \|\, \partial_x\varphi\|_{L^\infty((0,T)\times\Omega))}\|\, \partial_x^2 h_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\|h_\varepsilon - h\|_{L^2(0,T;L^2(\Omega)} \\ &+ \int_0^t\int_\Omega (\partial_x^2 h_\varepsilon - \partial_x^2 h)h \, \partial_x\varphi \; dx \; dt \\ &\underset{\varepsilon\longrightarrow 0}{\longrightarrow} 0 \, .\end{aligned}$$

As for the sixth term, we have deduced that $\partial_x h_\varepsilon$ converges strongly to $\partial_x h$ in $L^2(0,T;L^2(\Omega))$, thus we get the strong and hence weak convergence of $(\partial_x h_\varepsilon)^2$ to $(\partial_x h)^2$ in $L^1((0,T)\times\Omega)$, hence

$$\int_0^t \int_\Omega (\partial_x h_\varepsilon)^2\, \partial_x\varphi \; dx \; dt \underset{\varepsilon\longrightarrow 0}{\longrightarrow} \int_0^t \int_\Omega (\partial_x h)^2\, \partial_x\varphi \; dx \; dt.$$

6. As for the last term, we use the fact that due to strong convergence of h_ε to h in $L^2((0,T,)\times\Omega)$, and weak convergence of u_ε to u in $L^2(0,T;L^2(\Omega)$ we get

$$\begin{aligned}
&\int_0^t \int_\Omega \frac{u_\varepsilon}{1+h_\varepsilon}\varphi - \frac{u}{1+h_\varepsilon}\varphi \; dx \; dt \\
&= \int_0^t \int_\Omega u_\varepsilon \varphi(\frac{1}{1+h_\varepsilon} - \frac{1}{1+h}) \; dx \; dt + \int_0^t \int_\Omega \frac{1}{1+h}\varphi(u_\varepsilon - u) \; dx \; dt \\
&\leqslant \|\varphi\|_{L^\infty((0,T)\times\Omega))} \|u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \int_0^t \int_\Omega \frac{(h_\varepsilon - h)^2}{(1+h)(1+h_\varepsilon)} \; dx \; dt \\
&+ \int_0^t \int_\Omega \frac{1}{1+h}\varphi(u_\varepsilon - u) \; dx \; dt \\
&\leqslant C \int_0^t \int_\Omega (h_\varepsilon - h)^2 \; dx \; dt + \int_0^t \int_\Omega \frac{1}{1+h}\varphi(u_\varepsilon - u) \; dx \; dt \\
&\underset{\varepsilon\longrightarrow 0}{\longrightarrow} 0 \,.
\end{aligned}$$

where we have used the fact that $\frac{1}{1+h}\varphi \in L^2(0,T;L^2(\Omega))$.

Hence, we get that the couple (h,u) satisfies

$$\begin{aligned}
&\int_0^t \int_\Omega h\partial_t\psi \; dx \; dt + \int_\Omega h_0\psi(.,0) \; dx = -\int_0^t \int_\Omega hu\partial_x\psi \; dx \; dt, \\
&\frac{1}{W_e}\int_0^\infty \int_\Omega \frac{1}{2}(\partial_x h)^2\, \partial_x\varphi \; dx \; dt - \frac{1}{W_e}\int_0^\infty \int_\Omega h\partial_x^2 h\partial_x\varphi \; dx \; dt \\
&+ \frac{1}{Fr^2}\int_0^\infty \int_\Omega h^2\partial_x\varphi \; dx \; dt - \alpha\int_0^\infty \int_\Omega \frac{h^2 u}{F(h_\varepsilon)}\varphi \; dx \; dt = 0,
\end{aligned}$$

which is the weak formulation of the lubrication system.

4.2.3 BF-entropy information include in the limit part of the BD-entropy

In the previous subsections, we have proved that a weak limit of the shallow water system with the chosen nonlinear drag term (4.9) would converge to a couple (h,u) satisfying the weak formulation of the lubrication system (4.10). If we can now show that (h,u) satisfies as well the BF dissipative entropy, then certainly we can say that weak solution of shallow water system (4.9) converges weakly to the weak solution of the lubrication system (4.11) as defined by [70]. One significant consequence this approach shows is the implicitness of the lubrication system in the shallow water system. in particular one notices the inclusion of the BF dissipative entropy in the BD entropy which explains the convergence limit relation in between. One can naively predict that not just the entropy relations, but also other characteristics of both system are supposed to be interlaced with one another, and that under physical assumptions and bounded initial datum, the lubrication theory originates from shallow water assumptions in the weak sense as well.

The aim now, as mentioned above, is to prove the convergence of the BD entropy into the BF dissipative entropy. Lets recall first each of the BD entropy of system (4.9), and the BF dissipative entropy of the

sought for lubrication model (4.11). In fact, the BD entropy (4.17) reads

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega h_\varepsilon(x,t)v_\varepsilon(x,t)^2\ dx+\frac{1}{2Fr^2}\int_\Omega h_\varepsilon(x,t)^2\ dx+\frac{1}{2W_e}\int_\Omega(\partial_x h_\varepsilon(x,t))^2\ dx+\alpha\int_0^t\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{F(h_\varepsilon)}\ dx\ dt\\
&-\frac{4\alpha}{R_e}\int_\Omega(1+h_\varepsilon(x,t))\log(\frac{h_\varepsilon(x,t)}{1+h_\varepsilon(x,t)})\ dx+\frac{4}{R_e}\int_0^t\int_\Omega\frac{(\partial_x h_\varepsilon)^2}{Fr^2}+\frac{(\partial_x^2 h_\varepsilon)^2}{W_e}\ dx\ dt\\
&\leqslant\frac{\varepsilon}{2}\int_\Omega\left(\frac{(m_0^\varepsilon(x))^2}{h_0^\varepsilon(x)}+2\frac{m_0^\varepsilon(x)}{h_0^\varepsilon(x)}\partial_x h_0^\varepsilon(x)+\frac{(\partial_x h_0^\varepsilon(x))^2}{h_0^\varepsilon(x)}\right)dx+\int_\Omega\frac{(h_0^\varepsilon(x))^2}{2Fr^2}\ dx+\int_\Omega\frac{(\partial_x h_0^\varepsilon(x))^2}{2W_e}\ dx\\
&-\frac{4\alpha}{R_e}\int_\Omega(1+h_0^\varepsilon(x))\log(\frac{h_0^\varepsilon(x)}{1+h_0^\varepsilon(x)})\ dx.
\end{aligned}\tag{4.32}$$

As for the BF-dissipative entropy, we would notice first, comparing (4.1) and (4.11), that $D(r)=F(r)$, and hence to obtain the BF entropy we should only look after the function G_0.

$$\begin{aligned}
G_0(A)&=\int_{+\infty}^A\int_{+\infty}^s\frac{1}{F(r)}\ dr\ ds\\
&=\int_{+\infty}^A\int_{+\infty}^s\frac{1}{r^2+r^3}\ dr\ ds\\
&=\int_{+\infty}^A\int_{+\infty}^s\frac{1}{r^2}+\frac{1}{1+r}-\frac{1}{r}\ dr\ ds\\
&=-(1+A)\log(\frac{A}{1+A}).
\end{aligned}$$

Hence, substituting in the general form of the BF-entropy (4.2), we get

$$\begin{aligned}
&\int_\Omega-(1+h(x,t))\log(\frac{h(x,t)}{1+h(x,t)})\ dx+\int_0^t\int_\Omega\frac{(\partial_x^2 h)^2}{\alpha We}+\frac{(\partial_x h)^2}{\alpha Fr^2}\ dx\ dt\\
&\leqslant-\int_\Omega(1+h_0(x))\log(\frac{h_0(x)}{1+h_0(x)})\ dx.
\end{aligned}\tag{4.33}$$

Multiplying by $\frac{4\alpha}{R_e}$ yields

$$\begin{aligned}
&\frac{4}{R_e}\left[-\alpha\int_\Omega(1+h(x,t))\log(\frac{h(x,t)}{1+h(x,t)})\ dx+\int_0^t\int_\Omega\frac{(\partial_x^2 h)^2}{We}+\frac{(\partial_x h)^2}{Fr^2}\ dx\ dt\right]\\
&\leqslant-\frac{4\alpha}{R_e}\int_\Omega(1+h_0(x))\log(\frac{h_0(x)}{1+h_0(x)})\ dx.
\end{aligned}\tag{4.34}$$

Due to the weak convergence of the systems and the lack of higher orders of convergence on most of the terms in the BD-entropy, the convergence between the entropies will be exclusively a convergence of inequalities. Hence, using the weakly lower semi continuity property of the L^2 norm and the weak convergence of the terms stated in section 4.2.2, we get the convergence of the energy inequality (4.12) to the following inequality

$$\frac{1}{2}\int_\Omega\frac{h^2}{Fr^2}+\frac{(\partial_x h)^2}{W_e}dx+\alpha\int_0^t\int_\Omega\frac{h^2u^2}{F(h)}\ dx\ dt\leqslant C.\tag{4.35}$$

As for the Bd-entropy, we obtain the convergence to the following inequality

$$\begin{aligned}&\frac{1}{2}\int_\Omega \frac{h^2}{Fr^2}+\frac{(\partial_x h)^2}{W_e}dx-\frac{4\alpha}{R_e}(1+h(x,t))\log(\frac{h(x,t)}{1+h(x,t)})\ dx\\&+\alpha\int_0^t\int_\Omega \frac{h^2u^2}{F(h)}+\frac{4}{Re}\int_0^t\int_\Omega \frac{(\partial_x^2 h)^2}{\alpha We}+\frac{(\partial_x h)^2}{\alpha Fr^2}\ dx\ dt\leqslant C.\end{aligned}\tag{4.36}$$

Coupling the last inequality with the energy inequality (4.35) yields

$$-\alpha\int_\Omega (1+h(x,t))\log(\frac{h(x,t)}{1+h(x,t)})\ dx+\int_0^t\int_\Omega \frac{(\partial_x^2 h)^2}{We}+\frac{(\partial_x h)^2}{Fr^2}\ dx\ dt\leqslant C,\tag{4.37}$$

which obviously resembles the so called BF-entropy inequality that can be noticed from (4.33).

4.3 The limit of a viscous compressible system with a general drag term

Following the footsteps of the previous section, but considering a viscous compressible model with a general drag term given by $\alpha\, h^{2-n}u$ and general pressure term $p(s)=s^{\beta+1}/\big((\beta+1)\mathrm{Fr}^2\big)$ namely

$$\begin{aligned}&\partial_t h_\varepsilon+\partial_x(h_\varepsilon u_\varepsilon)=0,\\&\varepsilon\big(\partial_t(h_\varepsilon u_\varepsilon)+\partial_x(h_\varepsilon u_\varepsilon^2)\big)+\frac{1}{Fr^2}h_\varepsilon^\beta\partial_x(h_\varepsilon)=\varepsilon\big(\frac{4}{R_e}\partial_x(h_\varepsilon\partial_x u_\varepsilon)\big)+\frac{1}{W_e}h_\varepsilon\partial_x^3 h_\varepsilon-\alpha\frac{h_\varepsilon^2u_\varepsilon}{h_\varepsilon^n},\end{aligned}\tag{4.38}$$

where $\beta+n\in(1,2)$. Such system would converge in the formal sense to the following fourth order lubrication approximation that has been studied in several papers [71]

$$\begin{aligned}&\partial_t h+\partial_x(hu)=0,\\&hu=\frac{1}{\alpha W_e}h^n\partial_x^3 h-\frac{1}{\alpha Fr^2}h^{\beta+n-1}\partial_x h.\end{aligned}\tag{4.39}$$

Hence, compared to (4.1), we get that $F(h)=h^n$ and $D(h)=h^{\beta+n-1}$. We will consider in the sequel that $m=\beta+n$. In [71] for instance, the authors considered the above lubrication model with the same choice of F and D. They proved the existence of a global in time nonnegative weak solution starting from nonnegative datum for all $n>0$, and $1<m<2$. In particular, The most critical case is the most significantly physical one when $n=3$ (moving contact line in a thin film). In this case, a distributional solution is proven to exist, where it becomes a strong positive solution in the infinite time limit. The BF entropy corresponding to the latter system is given by

$$\int_\Omega G_0(h(x,T))\ dx+\int_0^T\int_\Omega \frac{(\partial_x^2 h)^2}{\alpha We}+\frac{1}{\alpha Fr^2}h^{m-n-1}(\partial_x h)^2\ dx\ dt=\int_\Omega G_0(h_0(x))\ dx.\tag{4.40}$$

A further look on G_0 allows us distinguish 3 cases:

1. $n\neq 1$ and $n\neq 2$

$$\begin{aligned}G_0(A)&=\int_{C_1}^A\int_{C_2}^s\frac{1}{F(r)}\ dr\ ds\\&=\int_{C_1}^A\int_{C_2}^s\frac{1}{r^n}\ dr\ ds\\&=\int_{C_1}^A\frac{1}{1-n}s^{1-n}\ ds\\&=\frac{1}{(1-n)(2-n)}A^{2-n}.\end{aligned}$$

$(C_i)_{i=1,2}$ are chosen such that: $C_1 = C_2 = +\infty$ if $n > 2$, and $C_1 = 0$, $C_2 = +\infty$ when $1 < n < 2$, $C_1 = C_2 = 0$ when $n < 1$.

2. $n = 1$

$$\begin{aligned} G_0(A) &= \int_e^A \int_1^s \frac{1}{F(r)}\, dr\, ds \\ &= \int_e^A \int_1^s \frac{1}{r}\, dr\, ds \\ &= \int_e^A \log(s)\, ds \\ &= A\log(A) - A. \end{aligned}$$

3. $n = 2$

$$\begin{aligned} G_0(A) &= \int_1^A \int_{+\infty}^s \frac{1}{F(r)}\, dr\, ds \\ &= \int_1^A \int_{+\infty}^s \frac{1}{r^2}\, dr\, ds \\ &= \int_1^A \frac{-1}{s}\, ds \\ &= -\log(A). \end{aligned}$$

The third case $n = 2$ is studied thoroughly in [80], therefore we wouldn't consider it at the moment. In the rest of this part will consider n is not equal to 1 nor to 2.

4.3.1 Ansatz between n and m on the physical basis

Consider the system (4.39) that we call PLM (corresponding to porous-lubrication model). In [71] the authors studied system PLM for the case where $n > 0$ and $1 < m < 2$. Their paper in fact is an outcome of two previous papers [86] and [72]; one studying the porous medium problem describing the flow of a gas in a porous medium given by system (4.41), and the second is the study of the lubrication approximation model of surface tension dominated motion given by system (4.42):

$$(PM) \qquad \partial_t h + \partial_x^2(h^m) = 0, \tag{4.41}$$

$$(LM) \qquad \partial_t h + \partial_x(h^n\, \partial_x^3 h) = 0. \tag{4.42}$$

Physically such systems model droplets spreading on a solid interface. Consider just a wetting case, the liquid is characterized by a positive spreading constant and thus the drops tend to wet the surface totally. In this case, intermolecular forces near the interface are important to account for in the dynamics of the system. Technically, this can be represented by a cut-off of the disjoint pressure at the molecular length scale which is interpreted as a second order term, or Van Der Waals term or porous medium term in the mass conservation evolution equation. This in fact may explain the convergence at the physical basis of the Shallow water model into the lubrication model as for instance the case of thin films at the nanoscale or droplets array spreading. In such cases, the effect of capillarity dominating the motion, and the spreading manner which resembles the manner that a liquid spreads in a porous medium explain that the motion would emanate from a drag and pressure forces in (4.38) as well as a capillarity effect contribution.
It is good to remark that in (4.41), the existence of solution as well as regularity is proven for a more general choice of m, but in accordance with LM in (4.42), only for $1 < m < 2$ that PM possess a source type solution that touches down with a zero slope which coincides with a similar manner to the solution of LM. An important discussion here is the competition as a function of n and m between the porous medium term which is represented by a second order derivative term in the evolution equation, and the capillarity effect represented by the fourth order term. For $n > 0$ and $1 < m < 2$, an asymptotic analysis via leading order of transport waves at the edge of the support of the solution is applied and predicts a kind of power law behavior of the solution at the edge. The results are clearly manifested in [71], and

the reader can refer to table 1 in the paper for a brief summary of the competition with respect to the relation between n and m.

4.3.2 Mathematical justification

Consider the energy and BD entropy inequalities of system (4.38)

$$\begin{aligned}&\frac{1}{2}\left(\int_\Omega \varepsilon h_\varepsilon(x,t)u_\varepsilon^2(x,t)+\frac{1}{Fr^2}\frac{h_\varepsilon(x,t)^{\beta+1}}{\beta(\beta+1)}+\frac{(\partial_x h_\varepsilon(x,t))^2}{W_e}\right)\,dx+\int_0^t\int_\Omega \frac{4\varepsilon}{R_e}h_\varepsilon(\partial_x u_\varepsilon)^2+\alpha\frac{h_\varepsilon^2u_\varepsilon^2}{h_\varepsilon^n}\,dx\,dt\\&\leqslant\frac{1}{2}\int_\Omega \varepsilon\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon}+\frac{1}{Fr^2}\frac{(h_0^\varepsilon)^{\beta+1}}{\beta(\beta+1)}+\frac{(\partial_x h_0^\varepsilon)^2}{W_e}\,dx.\end{aligned}\tag{4.43}$$

and

$$\begin{aligned}&\frac{1}{2}\int_\Omega \varepsilon h_\varepsilon(x,t)v_\varepsilon(x,t)^2+\frac{1}{F_r^2}\frac{h_\varepsilon(x,t)^{\beta+1}}{\beta(\beta+1)}+\frac{(\partial_x h_\varepsilon(x,t))^2}{W_e}\,dx+\frac{4\alpha}{Re}\int_\Omega\frac{1}{(1-n)(2-n)}h_\varepsilon(x,t)^{2-n}\,dx\\&+\frac{4}{Re}\Big[\int_0^t\int_\Omega\frac{1}{Fr^2}h_\varepsilon^{\beta-1}(\partial_x h_\varepsilon)^2\,dx+\frac{1}{W_e}(\partial_x^2 h_\varepsilon)^2\,dx\,dt\Big]+\alpha\int_0^t\int_\Omega\frac{h_\varepsilon^2u_\varepsilon^2}{h_\varepsilon^n}\,dx\,dt\\&\leqslant\frac{1}{2}\int_\Omega\Big(\varepsilon\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon}+2\frac{m_0^\varepsilon}{h_0^\varepsilon}\partial_x h_0^\varepsilon(x)+\frac{(\partial_x h_0^\varepsilon(x))^2}{h_0^\varepsilon(x)}\Big)\,dx+\frac{1}{2}\int_\Omega\Big(\frac{1}{Fr^2}\frac{(h_0^\varepsilon)^{\beta+1}}{\beta(\beta+1)}+\frac{(\partial_x h_0^\varepsilon)^2}{W_e}\Big)\,dx\\&+\frac{4\alpha}{Re}\int_\Omega\frac{1}{(1-n)(2-n)}(h_0^\varepsilon)^{2-n}(x)\,dx.\end{aligned}\tag{4.44}$$

Definition 4.8. *A weak solution of system (4.38) is a couple (h,u) satsifying the following integral equations as well as the energy and BD-entropy inequality given in (4.43) and (4.44)*

$$\int_0^\infty\int_\Omega h_\varepsilon\partial_t\psi\,dx\,dt+\int_\Omega h_0^\varepsilon\psi(\cdot,0)\,dx=-\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_x\psi\,dx\,dt,\tag{4.45}$$

and

$$\begin{aligned}&\varepsilon\Big(\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_t\varphi\,dx\,dt+\int_\Omega m_0^\varepsilon\varphi(\cdot,0)\,dx+\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon^2\partial_x\varphi\,dx\,dt\Big)\\&-\frac{4\varepsilon}{R_e}\int_0^\infty\int_\Omega h_\varepsilon\partial_x u_\varepsilon\partial_x\varphi\,dx\,dt+\frac{1}{W_e}\int_0^\infty\int_\Omega\frac{1}{2}(\partial_x h_\varepsilon)^2\partial_x\varphi\,dx\,dt\\&-\frac{1}{W_e}\int_0^\infty\int_\Omega h_\varepsilon\partial_x^2 h_\varepsilon\partial_x\varphi\,dx\,dt+\frac{1}{Fr^2}\int_0^\infty\int_\Omega\frac{1}{\beta+1}h_\varepsilon^{\beta+1}\partial_x\varphi\,dx\,dt-\alpha\int_0^\infty\int_\Omega\frac{h_\varepsilon^2u_\varepsilon}{h_\varepsilon^n}\varphi\,dx\,dt=0,\end{aligned}\tag{4.46}$$

for all $\psi\in\mathcal{C}_0^\infty(\Omega\times[0,\infty))$ and $\varphi\in\mathcal{C}_0^\infty(\Omega\times[0,\infty))$.

We can prove the global existence of weak solutions following the recent papers by [81], [82], [75]

Theorem 4.9. *Let $(h_0^\varepsilon,m_0^\varepsilon)$ be such that $h_0^\varepsilon\geqslant 0$ and*

$$\varepsilon\frac{(m_0^\varepsilon)^2}{h_0^\varepsilon}\in L^1(\Omega),\qquad h_0^\varepsilon\in H^1(\Omega),$$

$$\sqrt{\varepsilon}\partial_x(\sqrt{h_0^\varepsilon})\in L^2(\Omega),\qquad (h_0^\varepsilon)^{2-n}\in L^1(\Omega).$$

Then there exists a global weak solution of (4.38) in the sense of Definition 4.8.

Theorem 4.10. *Suppose there exits a sequence $(h_\varepsilon,u_\varepsilon)$ a weak solution of system (4.38) in the sense of definition 4.8 given by Theorem 4.9, then, up to a subsequence, this couple converges in a weak sense to the weak solution (h,u) of the lubrication model given in system (4.39).*

Proof. We will sketch the proof briefly since it follows that of theorem 4.8.
From (4.43) and (4.44), we can deduce the following uniform bounds accounting for initial data satisfying the bounds in theorem 4.9

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C,\qquad \|h_\varepsilon^{\frac{\beta+1}{2}}\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C,\qquad \|\,\partial_x h_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C,$$

$$\sqrt{\varepsilon}\|\sqrt{h_\varepsilon}\,\partial_x u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\leqslant C,\qquad \|h_\varepsilon^{\frac{2-n}{2}}u_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\leqslant C,\qquad \sqrt{\varepsilon}\|\sqrt{h_\varepsilon}v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C,$$

$$\|h_\varepsilon^{\frac{2-n}{2}}\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C,\qquad \|\,\partial_x(h_\varepsilon^{\frac{\beta+1}{2}})\|_{L^2(0,T;L^2(\Omega))}\leqslant C,\qquad \|\,\partial_x^2 h_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\leqslant C.$$

Regarding that h_ε is in $L^\infty(0,T;L^1(\Omega))$ from the conservation of mas (mass equation), we can infer the following using Poincare's inequality

$$\begin{aligned}\|h_\varepsilon\|_{L^2(\Omega)}&\leqslant\|h_\varepsilon-\bar{h_\varepsilon}\|_{L^2(\Omega)}+\|\bar{h_\varepsilon}\|_{L^2(\Omega)}\\&\leqslant\|\,\partial_x h_\varepsilon\|_{L^2(\Omega)}+C\|h_\varepsilon\|_{L^1(\Omega)}\\&\leqslant C,\end{aligned}$$

where $\bar{h_\varepsilon}=\frac{1}{|\Omega|}\int_\Omega h_\varepsilon\,dx$. Hence

$$h_\varepsilon\in L^\infty(0,T;H^1(\Omega)).$$

Due to compactness we get as well

$$h_\varepsilon\in L^\infty((0,T)\times\Omega).$$

Again, by compactness results we find that h_ε converges strongly to some h in $C([0,T],L^2(\Omega))$. Thus we get

$$h_\varepsilon(x,0)\rightharpoonup h(x,0)\quad\text{ in }\quad L^2(\Omega).$$

We denote by h_0 the weak limit in $L^2(\Omega)$ of h_0^ε. Due to uniform boundedness of $\partial_x^2 h_\varepsilon$ in $L^2(0,T;L^2(\Omega))$ we get that h_ε is bounded uniformly in $L^2(0,T;H^2(\Omega))$, and using Aubin's compactness theorem, we deduce that h_ε would converge strongly to h in $L^2(0,T;H^1(\Omega))$.
Now, we are concerned with the convergence of the velocity field in (4.38). In fact, we can argue here for two situations: the first is when $n>2$, and the second is when $0<n<2$. In the first case, the proof follows exactly the steps in the proof of theorem 4.9 so that we get u_ε converges weakly to some function u in $L^2(0,T;L^2(\Omega))$. For $0<n<2$, we will prove first the convergence of the momentum following a similar approach to that introduced by the authors in [69] using the presence of the drag term mainly using the uniform bounds from energy and BD-entropy inequalities.
The term $h_\varepsilon u_\varepsilon$ can be rewritten as

$$h_\varepsilon u_\varepsilon=h_\varepsilon^{\frac{n}{2}}h_\varepsilon^{\frac{2-n}{2}}u_\varepsilon.$$

From the energy estimates, we have $h_\varepsilon^{\frac{2-n}{2}}u_\varepsilon$ is uniformly bounded in $L^2((0,T)\times\Omega)$ and thus it converges weakly in such space to a function that we will denote ξ. Now since $n>0$, then $h_\varepsilon^{\frac{n}{2}}$ is bounded in $L^\infty((0,T)\times\Omega)$, which implies that the momentum is bounded uniformly in $L^2((0,T)\times\Omega)$. Thus this insures also that $h_\varepsilon u_\varepsilon$ converges weakly to some function in $L^2((0,T)\times\Omega)$. Besides, for ε large enough, we may assume that on the set $\{(x,t)\in(0,\infty)\times\Omega/h(x,t)=0\}$ h_ε is taken equal to zero, and thus ξ is also equals to zero on such set. Now we can define the function u such that

$$u=\begin{cases}\frac{\xi}{h^{\frac{2-n}{2}}} & h\neq 0,\\ 0 & \text{elsewhere.}\end{cases}$$

We claim now that the momentum $h_\varepsilon u_\varepsilon$ converges weakly in $L^2((0,T)\times\Omega)$ to hu. In fact, we have

$$\int_0^t\int_\Omega(h_\varepsilon u_\varepsilon-hu)\psi\;dx\;dt=\int_0^t\int_\Omega h^{\frac{n}{2}}(h_\varepsilon^{\frac{2-n}{2}}u_\varepsilon-h^{\frac{2-n}{2}}u)\psi\;dx\;dt+\int_0^t\int_\Omega h_\varepsilon^{\frac{2-n}{2}}u_\varepsilon(h_\varepsilon^{\frac{n}{2}}-h^{\frac{n}{2}})\psi\;dx\;dt.$$

As for the first term on the right hand side, it converges to zero due to weak convergence of the drag term in $L^2((0,T)\times\Omega)$. The second term converges to zero as well using almost everywhere convergence

of h_ε to h and a LDCT approach.
Remark here that u is not defined uniquely on the set $\{(x,t) \in (0,\infty)\times\Omega / h(x,t) = 0\}$.
Once obtained enough convergence results, we are able now to pass to the limit in the weak formulations (4.45) and (4.46) following the same analysis as in the proof of theorem 4.8. As for the weak formulation of the mass equation, we use the strong convergence of h_ε to h in $L^2(0,T;L^2(\Omega))$ as well as the weak convergence of u_ε to u in $L^2(0,T;L^2(\Omega))$ in case $n > 2$, or the weak convergence of the momentum in $L^2((0,T)\times\Omega)$ in case $0 < n < 2$. Concerning the terms multiplied by ε in the second integral equation of the weak formulation, the uniform boundedness in ε of such terms allows them to converge to zero as ε tends to zero (using the estimates obtained at the beginning of the proof). As for the fifth and sixth terms which come from the capillarity term in system (4.38), the convergence comes as a result of the strong convergence of h_ε to h in $L^2(0,T;H^1(\Omega))$ and the weak convergence $\partial_x^2 h_\varepsilon$ to $\partial_x^2 h$ in $L^2(0,T;L^2(\Omega))$. Using LDCT and the fact that the strong convergence of h_ε implies almost everywhere convergence up to a subsequence, we can prove the convergence of the seventh term as well. Finally, for the last term corresponding to the drag term, it follows from its weak convergence to $\xi = h^{\frac{2-n}{2}}u$ in $L^2((0,T)\times\Omega)$ for $0 < n < 2$ (otherwise its convergence follows as in the proof of theorem 4.8). Therefore, we get that the weak limit couple (h,u) is in fact a weak solution of the lubrication system (4.39) since it satisfies its corresponding weak formulation:

$$\int_0^\infty \int_\Omega h\partial_t\psi \; dx \; dt + \int_\Omega h_0\psi(\cdot,0) \; dx = -\int_0^\infty \int_\Omega hu\partial_x\psi \; dx \; dt, \tag{4.47}$$

$$\begin{aligned}&\frac{1}{We}\int_0^\infty \int_\Omega \frac{1}{2}(\partial_x h)^2\partial_x\varphi \; dx \; dt - \frac{1}{We}\int_0^\infty \int_\Omega h\partial_x^2 h_\varepsilon \partial_x\varphi \; dx \; dt \\ &+ \frac{1}{Fr^2}\int_0^\infty \int_\Omega \frac{1}{\beta+1}h^{\beta+1}\partial_x\varphi \; dx \; dt - \alpha\int_0^\infty \int_\Omega \frac{h^2u}{h^n}\varphi \; dx \; dt = 0.\end{aligned} \tag{4.48}$$

□

4.3.3 Convergence of the BD-entropy

Using the uniform bounds of the initial datum, we recall that the energy and BD-entropy inequalities of system (4.9) is given by

$$\begin{aligned}&\frac{1}{2}\left(\int_\Omega \varepsilon h_\varepsilon(x,t)u_\varepsilon^2(x,t) + \frac{1}{Fr^2}\frac{h_\varepsilon(x,t)^{\beta+1}}{\beta(\beta+1)} + \frac{(\partial_x h_\varepsilon(x,t))^2}{We}\right) \; dx + \int_0^t \int_\Omega \frac{4\varepsilon}{Re}h_\varepsilon(\partial_x u_\varepsilon)^2 \\ &+ \alpha\frac{h_\varepsilon^2 u_\varepsilon^2}{h_\varepsilon^n} \; dx \; dt \leqslant C.\end{aligned} \tag{4.49}$$

$$\begin{aligned}&\frac{1}{2}\int_\Omega \varepsilon h_\varepsilon(x,t)v_\varepsilon(x,t)^2 + \frac{1}{F_r^2}\frac{h_\varepsilon(x,t)^{\beta+1}}{\beta(\beta+1)} + \frac{(\partial_x h_\varepsilon(x,t))^2}{We} \; dx + \frac{4\alpha}{Re}\int_\Omega \frac{1}{(1-n)(2-n)}h_\varepsilon(x,t)^{2-n} \; dx \\ &+ \frac{4}{Re}\Big[\int_0^t \int_\Omega \frac{1}{Fr^2}h_\varepsilon^{\beta-1}(\partial_x h_\varepsilon)^2 \; dx + \frac{1}{We}(\partial_x^2 h_\varepsilon)^2 \; dx \; dt\Big] + \alpha\int_0^t \int_\Omega \frac{h_\varepsilon^2 u_\varepsilon^2}{h_\varepsilon^n} \; dx \; dt \leqslant C.\end{aligned} \tag{4.50}$$

Using the weakly lower semi continuity property of the L^2 norm and the weak convergence of the different terms in the above two inequalities, we can deduce the following

$$\frac{1}{2}\left(\int_\Omega \frac{1}{Fr^2}\frac{h(x,t)^{\beta+1}}{\beta(\beta+1)} + \frac{(\partial_x h(x,t))^2}{We}\right) \; dx + \alpha\int_0^t \int_\Omega \frac{h^2u^2}{h^n} \; dx \; dt \leqslant C. \tag{4.51}$$

$$\begin{aligned}&\frac{1}{2}\int_\Omega \frac{1}{F_r^2}\frac{h(x,t)^{\beta+1}}{\beta(\beta+1)} + \frac{(\partial_x h(x,t))^2}{We} \; dx + \frac{4\alpha}{Re}\int_\Omega \frac{1}{(1-n)(2-n)}h(x,t)^{2-n} \; dx \\ &+ \frac{4}{Re}\Big[\int_0^t \int_\Omega \frac{1}{Fr^2}h^{\beta-1}(\partial_x h)^2 \; dx + \frac{1}{We}(\partial_x^2 h)^2 \; dx \; dt\Big] + \alpha\int_0^t \int_\Omega \frac{h^2u^2}{h^n} \; dx \; dt \leqslant C.\end{aligned} \tag{4.52}$$

Coupling the above two inequalities together yields

$$\frac{4\alpha}{Re}\int_\Omega \frac{1}{(1-n)(2-n)}h(x,t)^{2-n}\,dx + \frac{4}{Re}\int_0^t\int_\Omega \frac{1}{Fr^2}h^{\beta-1}(\partial_x h)^2\,dx + \frac{1}{W_e}(\partial_x^2 h)^2\,dx\,dt \leqslant C, \tag{4.53}$$

which coincides with the BF-dissipative entropy of system (4.39) given by inequality (4.40). Once again we can say that the BD entropy of shallow water system converges in a weak sense to the BF-dissipative entropy of the corresponding lubrication system.

4.4 A more general framework

Consider the following logarithmic fourth-order equation:

$$\partial_t h + \partial_x^2(F(h)\,\partial_x^2 G(h)) = 0 \tag{4.54}$$

On the other hand we consider the following viscous compressible model with capillarity and drag term (neglecting pressure) given by

$$\begin{aligned}
&\partial_t h_\varepsilon + \partial_x(h_\varepsilon u_\varepsilon) = 0,\\
&\varepsilon\big(\partial_t(h_\varepsilon u_\varepsilon) + \partial_x(h_\varepsilon u_\varepsilon^2) + \partial_x p(h_\varepsilon)\big) = \varepsilon\big(\frac{4}{\mathrm{R_e}}\partial_x(\lambda(h_\varepsilon)\partial_x u_\varepsilon)\big)\\
&\qquad\qquad + \frac{1}{\mathrm{W_e}}h_\varepsilon\partial_x\big(\sqrt{K(h_\varepsilon)}\,\partial_x^2\big(\int_0^{h_\varepsilon}\sqrt{K(r)}\,dr\big)\big) - h_\varepsilon u_\varepsilon.
\end{aligned} \tag{4.55}$$

where $p(s) = as^\gamma$ with $a > 0$ and $\gamma > 1$ two fixed parameter, Re and We respectively the Reynolds and Weissenberg coefficients and with initial data

$$h_\varepsilon|_{t=0} = h_0^\varepsilon, \qquad h_\varepsilon u_\varepsilon|_{t=0} = m_0^\varepsilon.$$

The viscosity function λ is supposed to embed a shear-type viscosity $\mu(h) = \int_0^h \frac{\lambda(r)}{2r}\,dr$. In fact, a link indispensably relating between the viscosity μ and the capillarity K (to be able to follow existence result recently obtained in [75]) is given by

$$\mu'(h) = \frac{1}{2}\sqrt{h}\sqrt{K(h)}.$$

Define now

$$S(h) = \int_0^h \frac{\sqrt{K(r)}}{\sqrt{r}}\,dr, \qquad Z(h) = \int_0^h \frac{\mu'(r)\sqrt{\mu(r)}}{r}\,dr. \tag{4.56}$$

Furthemore, we will assume in the sequel that μ satisfies the following hypothesis:

$$\mu \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)\cap\mathcal{C}^2(\mathbb{R}_+^*;\mathbb{R}). \tag{4.57}$$

There exists a_1, a_2 and some constant C such that

$$0 < \frac{2}{3} < a_1 < a_2 < 4, \tag{4.58}$$

$$0 < \frac{h\mu'(h)}{a_2} \leqslant \mu(h) \leqslant \frac{h\mu'(h)}{a_1} \qquad \forall h > 0, \tag{4.59}$$

and

$$\Big|\frac{h\mu''(h)}{\mu'(h)}\Big| \leqslant C < +\infty. \tag{4.60}$$

At the formal level, system (4.55) converges to the following system

$$\begin{aligned} &\partial_t h + \partial_x(hu) = 0,\\ &hu = \frac{1}{\mathrm{W_e}} h\partial_x\big(\sqrt{K(h)}\,\partial_x^2\big(\int_0^h \sqrt{K(r)}\,dr\big)\big), \end{aligned} \tag{4.61}$$

which can be written as

$$\partial_t h + \partial_x\left(\frac{1}{\mathrm{W_e}} h\partial_x\big(\sqrt{K(h)}\,\partial_x^2\big(\int_0^h \sqrt{K(r)}\,dr\,\big)\big)\right) = 0. \tag{4.62}$$

Using Bohm's identity proven in [76] we have

$$h\,\partial_x\big(\sqrt{K(h)}\,\partial_x^2\big(\int_0^h \sqrt{K(r)}\,dr\big)\big) = \partial_x\big(h\gamma'(h)\,\partial_x^2\theta(h)\big)$$

where $\gamma'(h) = \sqrt{hK(h)}$ and $\theta'(h) = 2\sqrt{\frac{K(h)}{h}}$. In accordance with our definition in (4.56), we get that $\gamma(h) = 2\mu(h)$, and $\theta(h) = 2S(h)$. Thus substituting the identity in equation (4.62) we obtain the following

$$\partial_t h + \partial_x^2\left(2h\mu'(h)\,\partial_x^2(2S)\right) = 0. \tag{4.63}$$

Which recovers equation (4.54) taking $F(h) = 2h\mu'(h)$ and $G(h) = 2S$. In the sequel, we will define $\Psi'(h) = \sqrt{K(h)}$ for simplicity. The global existence of weak solutions for the compressible system with capillarity and drag terms follows quite similar lines than for the approximate system in [75]. We will not rewrite the proof because it is not really the objective of the paper which concerns the link between Equation 4.54 and System 4.55 with their corresponding initial data.

4.4.1 Limit problem

Following the same framework based on the energy estimations as in the previous section, we will start from a global weak solution (whence nonnegative) of the shallow water model (4.55). The energy and BD-entropy of the corresponding system are given respectively by

$$\begin{aligned} &\sup_{t\in(0,T)}\Big[\frac{1}{2}\big(\int_\Omega \varepsilon h_\varepsilon u_\varepsilon^2 + (\,\partial_x\Psi(h))^2\,dx\big)(t) + \int_\Omega \varepsilon\Pi(h_\varepsilon)\,dx\Big] + \int_0^T\int_\Omega h_\varepsilon u_\varepsilon^2 + \varepsilon 2h_\varepsilon\mu'(h_\varepsilon)(\,\partial_x u_\varepsilon)^2\,dx\\ &\leqslant \frac{1}{2}\int_\Omega \varepsilon\frac{m_0^\varepsilon}{h_0^\varepsilon} + (\,\partial_x\Psi(h_0^\varepsilon))^2\,dx + \int_\Omega \varepsilon\Pi(h_0^\varepsilon)\,dx, \end{aligned} \tag{4.64}$$

$$\begin{aligned} &\sup_{t\in(0,T)}\Big[\frac{1}{2}\frac{d}{dt}\int_\Omega \varepsilon h_\varepsilon v_\varepsilon^2 + M(h_\varepsilon) + (\,\partial_x\Psi(h))^2\,dx + \int_\Omega \varepsilon\Pi(h_\varepsilon)\,dx\Big] + \int_\Omega h_\varepsilon u_\varepsilon^2 + h_\varepsilon\mu'(h_\varepsilon)(\,\partial_x^2(2S))^2\,dx\\ &+ 2\varepsilon\int_0^T\int_\Omega p'(h_\varepsilon)S'(h_\varepsilon)|\partial_x h_\varepsilon|^2 \leqslant \frac{1}{2}\int_\Omega \varepsilon h_0^\varepsilon|v_0^\varepsilon|^2 + (\,\partial_x\Psi(h_0^\varepsilon))^2\,dx + \int_\Omega \varepsilon\Pi(h_0^\varepsilon)\,dx, \end{aligned} \tag{4.65}$$

with $\Pi(s) = s\int_a^s p(\tau)/\tau^2\,d\tau$ where $v_\varepsilon = u_\varepsilon + \partial_x(2S)$. The new term $m(h_\varepsilon)$ is a drag descent term, as we multiply by $\partial_x(2s)$ for the sake of BD-entropy, we come across the following term $\int_\Omega h_\varepsilon u_\varepsilon\,\partial_x(2s)\,dx$ which can be written as follows:

$$\begin{aligned} \int_\Omega h_\varepsilon u_\varepsilon\,\partial_x(2s)\,dx &= \int_\Omega h_\varepsilon u_\varepsilon\,\partial_x(4\int_1^r \frac{\mu'}{r}\,dr)\,dx\\ &= \int_\Omega \partial_t h_\varepsilon(4\int_1^r \frac{\mu'}{r}\,dr)\,dx. \end{aligned}$$

Thus we define $M(h) = \int_1^h \int_1^r 4\frac{\mu'}{s}\, ds\, dr$. But due to monotonicity of μ over $(0,\infty)$ ($\mu' > 0$), we can assume that $\mu'(\zeta) > \lim_{h\to 0} \mu'(h) = C$ and $C < \infty$. Hence we have that:

$$M(h) \geqslant C \int_1^h \int_1^r \frac{1}{s}\, ds\, dr = C(h\log h - h).$$

Hence we recover the following entropy:

$$\begin{aligned} &\frac{1}{2}\int_\Omega \varepsilon h_\varepsilon v_\varepsilon^2 + (h_\varepsilon \log h_\varepsilon - h_\varepsilon)_+ + (\partial_x \Psi)^2\, dx + \int_0^T \int_\Omega h_\varepsilon u_\varepsilon^2 + h_\varepsilon \mu'(h_\varepsilon)(\partial_x^2(2S))^2\, dx\, dt \\ &= \frac{1}{2}\int_\Omega \varepsilon h_\varepsilon^0 (v_\varepsilon^0)^2 + (h_\varepsilon^0 \log h_\varepsilon^0 - h_\varepsilon^0)_+ + (\partial_x \Psi_0)^2\, dx - \int_\Omega (h_\varepsilon \log h_\varepsilon - h_\varepsilon)_-\, dx \leqslant C\, dx, \end{aligned} \tag{4.66}$$

where we used the fact that $-\int_\Omega (h_\varepsilon \log h_\varepsilon - h_\varepsilon)_-$ is a positive quantity that does not vanish whenever h_ε belongs to the set $\{x < e\}$ and is bounded by 1. Thus we can write

$$-\int_\Omega h_\varepsilon (\log h_\varepsilon - 1)_-\, dx < C.$$

Furthermore, using the logarithmic Sobolev inequality proved in [75], we can obtain some additional estimates from the bounded term $h_\varepsilon \mu'(h_\varepsilon)(\partial_x^2(2S))^2$ in $L^2(0,T;\Omega)$. We recall the following lemma:

Lemma 4.1. *We suppose that μ satisfies the assumptions (4.57)–(4.60), then for all $h > 0$, and $h \in L^2(0,T;H^2(\Omega))$, there exists some constant α such that we have the following estimate*

$$\int_0^T \int_\Omega (\partial_x^2 Z(h))^2\, dx\, dt + \alpha \int_0^T \int_\Omega \frac{h^2}{\mu(h)^3}(\partial_x h)^4\, dx\, dt \leqslant \frac{C}{\alpha}\int_0^T \int_\Omega \mu(h)(\partial_x^2 S(h))^2\, dx. \tag{4.67}$$

Finally, we can obtain the following entropy

$$\int_\Omega \frac{\varepsilon}{2} h_\varepsilon v_\varepsilon^2 + h_\varepsilon(\log h_\varepsilon - 1)_+ + (\partial_x \Psi)^2\, dx + \int_0^T \int_\Omega h_\varepsilon u_\varepsilon^2 + (\partial_x^2 Z(h_\varepsilon))^2\, dx\, dt \leqslant C. \tag{4.68}$$

The estimations obtained from (4.64) and (4.68):

$$\begin{gathered} \sqrt{\varepsilon}\|\sqrt{h_\varepsilon} u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \qquad \|\partial_x \Psi\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \\ \sqrt{\varepsilon}\|\sqrt{h_\varepsilon \mu'}\partial_x u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \qquad \|\sqrt{h_\varepsilon} u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \\ \|\partial_x Z(h_\varepsilon)\|_{L^2(0,T;H^1(\Omega))} \leqslant C, \qquad \sqrt{\varepsilon}\|\sqrt{h_\varepsilon} v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \\ \|h_\varepsilon(\log h_\varepsilon - 1)_+\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C. \end{gathered}$$

1-Uniform boundedness of $\mu(h_\varepsilon)$. Using mass conservation equation, we can write a transport equation on $\mu(h_\varepsilon)$:

$$\partial_t(\mu(h_\varepsilon)) + \partial_x(h_\varepsilon \mu'(h_\varepsilon) u_\varepsilon) - h_\varepsilon \mu''(h_\varepsilon) u_\varepsilon\, \partial_x h_\varepsilon = 0.$$

Integrating in space and using the fact that $h_\varepsilon \mu''(h_\varepsilon) \leqslant C\mu'(h_\varepsilon)$ we get

$$\frac{d}{dt}\int_\Omega \mu(h_\varepsilon)\, dx \leqslant \int_\Omega \Big|\frac{\partial_x \mu}{\sqrt{h_\varepsilon}}\sqrt{h_\varepsilon} u_\varepsilon\Big|\, dx.$$

Remark that $\frac{\partial_x \mu(h_\varepsilon)}{\sqrt{h_\varepsilon}} = \partial_x \Psi(h_\varepsilon)$, and since $\partial_x \Psi(h_\varepsilon)$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$, and $\sqrt{h_\varepsilon} u_\varepsilon$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$, then we get that $\mu(h_\varepsilon)$ is bounded in $L^\infty(0,T;L^1(\Omega))$.

Concerning $\partial_x \mu(h_\varepsilon)$ we can write:

$$\partial_x \mu(h_\varepsilon) = \frac{\partial_x \mu(h_\varepsilon)}{\sqrt{h_\varepsilon}} \sqrt{h_\varepsilon} = 2\sqrt{h_\varepsilon}\, \partial_x \Psi(h_\varepsilon).$$

Using mass conservation, $\sqrt{h_\varepsilon}$ is in $L^\infty(0,T;L^2(\Omega))$, and so is $\partial_x \Psi(h_\varepsilon)$, which yields that $\partial_x \mu(h_\varepsilon)$ is in $L^\infty(0,T;L^1(\Omega))$. Therefore $\mu(h_\varepsilon)$ is in $L^\infty(0,T;W^{1,1}(\Omega))$, and eventually in $L^\infty(0,T;\Omega)$, as $W^{1,1}(\Omega)$ is continuously injected in $L^\infty(\Omega)$. Lets get a further look on $\frac{d}{dt}\int_\Omega M(h_\varepsilon)$, in fact we can write

$$\begin{aligned}
\int_\Omega \frac{d}{dt} = \int_\Omega M(h_\varepsilon)\, dx = \int_\Omega u_\varepsilon\, \partial_x \mu(h_\varepsilon)\, dx &= -\int_\Omega \mu(h_\varepsilon)\, \partial_x u_\varepsilon\, dx \\
&= -\int_\Omega \frac{\mu}{h_\varepsilon^2} h_\varepsilon^2\, \partial_x u_\varepsilon `dx \\
&= \int_\Omega h_\varepsilon \frac{\mu}{h_\varepsilon^2}(u_\varepsilon\, \partial_x h_\varepsilon + \partial_t h_\varepsilon)\, dx \\
&= \int_\Omega h_\varepsilon u_\varepsilon\, \partial_x (\int_a^{h_\varepsilon} \frac{\mu(s)}{s^2}\, ds) + h_\varepsilon \frac{\mu}{h_\varepsilon^2}\, \partial_t h_\varepsilon\, dx \\
&= \int_\Omega (\int_a^{h_\varepsilon} \frac{\mu(s)}{s^2}\, ds)\, \partial_t h_\varepsilon + h_\varepsilon\, \partial_t (\int_0^{h_\varepsilon} \frac{\mu(s)}{s^2}\, ds)\, dx \\
&= \frac{d}{dt}\int_\Omega h_\varepsilon (\int_a^{h_\varepsilon} \frac{\mu(s)}{s^2}\, ds)\, dx.
\end{aligned}$$

And thus from (4.65), we get that $h_\varepsilon(\int_a^{h_\varepsilon} \frac{\mu(s)}{s^2}\, ds$ is uniformly bounded in $L^\infty(0,T;L^1(\Omega))$. Again using assumptions on μ in (4.57)-(4.60), we get that $h_\varepsilon(\int_a^{h_\varepsilon} \frac{\mu'(s)}{s}\, ds$ is bounded in $L^\infty(0,T;L^1(\Omega))$, and thus we get

$$\|\mu\|_{L^\infty(0,T;L^1(\Omega))} \leqslant \|\int_a^{h_\varepsilon} \mu'(s)\, ds\|_{L^\infty(0,T;L^1(\Omega))} \leqslant \|h_\varepsilon(\int_a^{h_\varepsilon} \frac{\mu'(s)}{s}\|_{L^\infty(0,T;L^1(\Omega))} < \infty.$$

Remark 6. *Using the assumptions (4.57)-(4.60), we can deduce that*

$$h_\varepsilon^{c_1} \leqslant \mu(h_\varepsilon) \leqslant h_\varepsilon^{c_2},$$

and so we can deduce the uniform boundedness of h_ε in $(0,T;\Omega)$.

2-Information on $\partial_t \mu(h_\varepsilon)$. From above we have

$$\partial_t(\mu(h_\varepsilon)) = -\partial_x(h_\varepsilon \mu'(h_\varepsilon) u_\varepsilon) + h_\varepsilon \mu''(h_\varepsilon) u_\varepsilon\, \partial_x h_\varepsilon.$$

The second term is proven to be bounded in $L^\infty(0,T;L^1(\Omega))$. As for the first term, we have in fact $|h_\varepsilon \mu'(h_\varepsilon) u_\varepsilon| \leqslant C|\mu(h_\varepsilon) u_\varepsilon|$, hence lets check the boundedness of μu_ε:

$$\mu(h_\varepsilon) u_\varepsilon = \frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}} \sqrt{h_\varepsilon} u_\varepsilon.$$

Now for $h_\varepsilon \leqslant 1$, we have $h_\varepsilon^{a_1 - \frac{1}{2}} \leqslant \frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}} \leqslant h_\varepsilon^{a_2 - \frac{1}{2}}$, which is equivalent to have $h_\varepsilon^{\frac{1}{6}} \leqslant \frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}} \leqslant h_\varepsilon^{\frac{7}{2}} \leqslant 1$. And if $h_\varepsilon \geqslant 1$, then $\frac{\mu(h_\varepsilon)}{\sqrt{he}} \leqslant \mu(h_\varepsilon)$. Hence, in all cases, $\frac{\mu}{\sqrt{h_\varepsilon}}$ is in $L^\infty(0,T;L^1(\Omega))$. Regard that

$$\partial_x(\frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}}) = \frac{\partial_x \mu(h_\varepsilon)}{\sqrt{h_\varepsilon}} + \mu(h_\varepsilon)\, \partial_x \frac{1}{\sqrt{h_\varepsilon}} = 2\, \partial_x \Psi(h_\varepsilon) + \mu(h_\varepsilon)\, \partial_x \frac{1}{\sqrt{h_\varepsilon}}.$$

Hence

$$|\,\partial_x(\frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}})| \leqslant 2(|\,\partial_x\Psi(h_\varepsilon)| + |h_\varepsilon\mu'(h_\varepsilon)\frac{\partial_x h_\varepsilon}{h_\varepsilon^{\frac{3}{2}}}| \leqslant C(|\,\partial_x\Psi(h_\varepsilon)|).$$

Thus $\partial_x(\frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}})$ is bounded in $L^\infty(0,T;L^2(\Omega)) \subset L^\infty(0,T;L^1(\Omega))$, and therefore $\frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}}$ is bounded in $L^\infty(0,T;W^{1,1}(\Omega))$. Now returning to $\mu(h_\varepsilon)u_\varepsilon = \frac{\mu(h_\varepsilon)}{\sqrt{h_\varepsilon}}\sqrt{h_\varepsilon}u_\varepsilon$, we deduce that $\mu(h_\varepsilon)u_\varepsilon$ is bounded in $L^2(0,T;L^2(\Omega))$, and so is $h_\varepsilon\mu'(h_\varepsilon)u_\varepsilon$. Hence, $\partial_x(h_\varepsilon\mu'(h_\varepsilon)u_\varepsilon)$ belongs to $L^2(0,T;W^{-1,1}(\Omega))$. and so we can deduce the uniform boundedness of $\partial_t\mu(h_\varepsilon)$ in $L^2(0,T;W^{-1,1}(\Omega))$.

Convergence of μ. As a result of the previous two parts, we got the following

$$\mu \text{ is uniformly bounded in } L^\infty(0,T;W^{1,1}(\Omega)),$$

$$\partial_t\mu \text{ is uniformly bounded in } L^2(0,T;W^{-1,1}(\Omega)),$$

$$W^{1,1}(\Omega) \subset\subset L^2(\Omega) \subset W^{-1,1}(\Omega).$$

Applying the compactness result of Aubin-Simon, we get the strong convergence of μ in $C(0,T,L^2(\Omega))$.

Convergence of h_ε. Now we claim that h_ε converges almost everywhere to some h such that the limit of $\mu(h_\varepsilon)$ is $\mu(h)$. This claim in fact is guaranteed using the fact that μ is continuous monotone function and thus invertible. On the other hand, we have from mass equation that $\partial_t h_\varepsilon = -\,\partial_x(\sqrt{h_\varepsilon}\sqrt{h_\varepsilon}u_\varepsilon)$ is in $L^2(0,T;L^2(\Omega))$ since $\sqrt{h_\varepsilon}$ is bounded , and the bound on $\sqrt{h_\varepsilon}u_\varepsilon$ follows from the estimations. Now regard that if $h_\varepsilon \leqslant 1$, then h_ε is in $L^\infty(0,T;L^p(\Omega))$ for all $p>0$. If $h_\varepsilon \geqslant 1$, then from the assumptions on μ we have

$$h_\varepsilon^{\frac{2}{3}} \leqslant \mu(h_\varepsilon) \leqslant h_\varepsilon^4, \quad \text{and} \quad c_1 h_\varepsilon\mu'(h_\varepsilon) \leqslant \mu(h_\varepsilon) \leqslant c_2 h_\varepsilon\mu'(h_\varepsilon).$$

Thus we get

$$0 \leqslant \frac{1}{h_\varepsilon} \leqslant \frac{c\mu'(h_\varepsilon)\sqrt{\mu(h_\varepsilon)}}{h_\varepsilon} = Z'(h_\varepsilon),$$

which implies

$$|\,\partial_x \log h_\varepsilon| = |\frac{\partial_x h_\varepsilon}{h_\varepsilon}| \leqslant |\,\partial_x Z(h_\varepsilon)| \in L^2(0,T;L^2(\Omega)).$$

But as $\partial_x h_\varepsilon = h_\varepsilon\,\partial_x \log h_\varepsilon$ and h is bounded locally in time and space, then we deduce that $\partial_x h_\varepsilon \in L^2(0,T;L^2(\Omega))$. Hence, using Poincare Wirtinger inequality we deduce that h_ε belongs to $L^2(0,T;H^1(\Omega))$. Finally applying compactness result, we get that h_ε will converges strongly to some h in $L^2(0,T;L^2(\Omega))$.

Passage to the limit in the weak formulation. A weak formulation of the shallow water model (4.55) reads

$$\int_0^\infty\int_\Omega h_\varepsilon\partial_t\psi\;dx\;dt + \int_\Omega h_0^\varepsilon\psi(h_0^\varepsilon)\;dx = -\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_x\psi\;dx\;dt, \tag{4.69}$$

and

$$\begin{aligned}\varepsilon\Big(\int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\partial_t\varphi\;dx\;dt + \int_\Omega m_0^\varepsilon\varphi(\cdot,0)\;dx + \int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon^2\partial_x\varphi\;dx\;dt \\ - \frac{8}{R_e}\int_0^\infty\int_\Omega h_\varepsilon\mu'(h_\varepsilon)\partial_x u_\varepsilon\partial_x\varphi\;dx\;dt + \int_0^\infty\int_\Omega p(h_\varepsilon)\partial_x\varphi\;dx\;dt\Big) \\ - \frac{1}{W_e}\int_0^\infty\int_\Omega h_\varepsilon\,\partial_x(\Psi'\,\partial_x^2\Psi)\varphi\;dx\;dt - \int_0^\infty\int_\Omega h_\varepsilon u_\varepsilon\varphi\;dx\;dt = 0,\end{aligned} \tag{4.70}$$

for all $\psi \in \mathcal{C}_0^\infty(\Omega\times[0,\infty))$ and $\varphi \in \mathcal{C}_0^\infty(\Omega\times[0,\infty))$.

The obtained convergence results allow us to pass to the limit in the weak formulations. A dominated Lebesgue convergence theorem as well as the strong convergence of h_ε and the weak convergence of the momentum in $L^2(0,T;L^2(\Omega))$ are used to pass to limit in all terms of the weak formulations except for the capillarity term. Concerning this latter, we will use the estimations on $Z(h)$ by showing the relation

between Ψ and Z through the following identity:

$$h\,\partial_x(\Psi'\,\partial_{xx}\Psi) = \partial_x(\frac{4h_\varepsilon\mu'}{\sqrt{\mu}}\,\partial_{xx}Z(h))) - \partial_{xx}(\kappa(h)\,\partial_x Z(h)) + \partial_x(\kappa(h)\,\partial_{xx}Z(h)),$$

where $\kappa(h)$ is a functional of h given by

$$\kappa(h) = \int_0^h \frac{2s\mu'^2}{\mu\sqrt{\mu}}\,ds. \tag{4.71}$$

Now we can write the fifth term in the weak formulation as

$$\begin{aligned}
&-\frac{1}{W_e}\int_0^\infty\int_\Omega h_\varepsilon\,\partial_x(\Psi'\,\partial_x^2\Psi)\varphi\,dx\,dt \\
&=\frac{1}{W_e}\int_0^\infty\int_\Omega \big(\frac{4h_\varepsilon\mu'}{\sqrt{\mu}}+\kappa(h_\varepsilon)\big)\,\partial_{xx}Z(h_\varepsilon)\,\partial_x\varphi+\kappa(h_\varepsilon)\,\partial_x Z(h_\varepsilon)\,\partial_{xx}\varphi\,dx\,dt.
\end{aligned}$$

Lets have a look on the terms $\frac{4h_\varepsilon\mu'}{\sqrt{\mu}}$ and $\kappa(h_\varepsilon)$. In fact, both terms are positive quantities that can be treated in the same manner as $\sqrt{\mu}$. More precisely, using the assumptions (4.57)-(4.60) we've got

$$|\frac{4h_\varepsilon\mu'(h_\varepsilon)}{\sqrt{\mu}}| \leqslant C|\sqrt{\mu}|.$$

But $\sqrt{\mu}$ belongs to $L^\infty(0,T;(\Omega)\cap L^\infty(0,T;L^2(\Omega))$. Thus, we infer the weak convergence of $\frac{4h_\varepsilon\mu'}{\sqrt{\mu}}\,\partial_{xx}Z(h_\varepsilon)$. The same applies to the remaining terms in (4.4.1) since

$$\kappa(h) = \int_0^h \frac{2s\mu'(s)^2}{\mu(s)\sqrt{\mu(s)}}\,ds \leqslant C\int_0^h \frac{\mu^2(s)}{\sqrt{\mu(s)}}\,ds = \sqrt{h_\varepsilon}.$$

Thus we obtain the following weak formulation:

$$\int_0^\infty\int_\Omega h\partial_t\Psi(h)\,dx\,dt + \int_\Omega h_0\Psi(h_0)\,dx = -\int_0^\infty\int_\Omega hu\partial_x\Psi(h)\,dx\,dt, \tag{4.72}$$

$$\frac{1}{W_e}\int_0^\infty\int_\Omega \big(\frac{4h\mu'(h)}{\sqrt{\mu(h)}}+\kappa(h)\big)\,\partial_{xx}Z(h)\,\partial_x\varphi+\kappa(h)\,\partial_x Z(h)\,\partial_{xx}\varphi\,dx\,dt - \int_0^\infty\int_\Omega hu\varphi\,dx\,dt = 0, \tag{4.73}$$

which is the corresponding weak formulation of (4.63).

Remark. Note that the case $F(h) = h$ and $G(h) = \log h$ corresponds to the equation derived by Derrida, Lebowitz, Speer and Spohn describing several applications as in the fluctuation of a stationary non-equilibrium interface, and in zero-temperature zero-field approximation of quantum models for semi conductors. Solving such equation (in 1D and 2D) has been a great interest of mathematician and several results were obtained. As for the 1 dimensional case. a local in time classical solution was obtained in [87], and a nonnegative global weak solution was also proved to exist under several assumptions (Dirichlet-Neuman boundary condition, smallness of initial data,etc), see for instance [88]. Jungel and al. later in [89] proved existence of nonnegative global weak solution under a weaker condition on the initial data (measurability), and boundedness of the corresponding Lyapunov functional at time $t = 0$: $\int u_0 - \log u_0 < \infty$. Their proof was based on regularizing the equation, using a semi-discrete system (in time), and compactness results to obtain convergence. Our method provides an other approach making the link between different models.

4.5 Appendix

Derivation of BD Entropy

For completeness, we will give a brief derivation of the BD-entropy starting from the 3D Navier Stokes system (2.10) stated in the introduction system. First we derive an equation on $\mu(\rho)$, so multiplying the mass equation by $\mu'(\rho)$ we get

$$\partial_t \mu(\rho) + \operatorname{div}(\mu(\rho)u) + (\rho\mu'(\rho) - \mu)\operatorname{div} u = 0.$$

Using the relation introduced above we get

$$\partial_t \mu(\rho) + \operatorname{div}(\mu(\rho)u) + \frac{\lambda(\rho)}{2}\operatorname{div} u = 0.$$

Taking the gradient of the above equation

$$\partial_t(\nabla\mu(\rho)) + \nabla\operatorname{div}(\mu(\rho)u) + \nabla(\frac{\lambda(\rho)}{2}\operatorname{div} u) = 0.$$

But notice that

$$\begin{aligned}
\nabla\operatorname{div}(\mu(\rho)u) &= \nabla(\mu(\rho)\operatorname{div} u + \nabla\mu(\rho)\cdot u)\\
&= \operatorname{div} u\nabla\mu(\rho) + \mu(\rho)\nabla\operatorname{div} u + \nabla\nabla\mu(\rho)\cdot u + \nabla\mu(\rho)\cdot\nabla u\\
&= \operatorname{div}(u\otimes\nabla\mu(\rho)) - \nabla\nabla\mu(\rho)\cdot u + \mu\nabla\operatorname{div} u + \nabla\nabla\mu(\rho)\cdot u + \nabla\mu(\rho)\cdot\nabla u\\
&= \operatorname{div}(u\otimes\nabla\mu(\rho)) + \mu\nabla\operatorname{div} u + \nabla\mu(\rho)\cdot\nabla u.
\end{aligned}$$

Keeping the last two terms above wouldn't help get the desired estimation, and so a simple trick would be by substituting ∇u in terms of its symmetric and skew-symmetric parts $\mathrm{D}(u)$ and $A(u)$, we get

$$\begin{aligned}
\nabla\mu(\rho)\cdot\nabla u &= \nabla\mu(\rho)\cdot(\mathrm{D}(u) - A(u))\\
&= \operatorname{div}(\mu(\rho)\mathrm{D}(u)) - \operatorname{div}(\mu(\rho)A(u)) - \mu\operatorname{div}\nabla u^T.
\end{aligned}$$

By simple computations, we find that

$$\nabla\operatorname{div} u - \operatorname{div}\nabla u^T = 0.$$

And finally the equation becomes (multiplied by 2)

$$\partial_t(2\nabla\mu(\rho)) + 2\operatorname{div}(u\otimes\nabla\mu(\rho)) + 2\operatorname{div}(\mu(\rho)\mathrm{D}(u)) - 2\operatorname{div}(\mu(\rho)A(u)) + \nabla(\lambda(\rho)\operatorname{div} u) = 0.$$

Summed to the momentum equation yields

$$\partial_t(\rho(u + \frac{2\nabla\mu(\rho)}{\rho})) + \operatorname{div}(\rho(u + \frac{2\nabla\mu(\rho)}{\rho})\otimes u) + \nabla p - 2\operatorname{div}(\mu(\rho)A(u)) = 0.$$

Defining the drift velocity $v = u + \frac{2\nabla\mu(\rho)}{\rho}$, and multiplying by v^T to the left of the above equation, we get

$$\frac{1}{2}\partial_t(\rho|v|^2) + \frac{1}{2}\operatorname{div}(\rho|v|^2 v) + v^T\nabla p - 2v^T\operatorname{div}(\mu(\rho)A(u)) = 0. \tag{4.74}$$

As for the last term we have in fact

$$\begin{aligned}
-2v^T\operatorname{div}(\mu(\rho)A(u)) &= -2\operatorname{div}(\mu(\rho)A(u)^T\cdot v) + 2\mu(\rho)A(u) : \nabla v\\
&= -2\operatorname{div}(\mu(\rho)A(u)^T\cdot v) + 2\mu(\rho)A(u) : \nabla u + 4\mu(\rho)A(u) : \nabla(\frac{\nabla\mu(\rho)}{\rho})\\
&= -2\operatorname{div}(\mu(\rho)A(u)^T\cdot v) + \mu(\rho)|A(u)|^2 + 4\mu(\rho)A(u) : \nabla(\frac{\nabla\mu(\rho)}{\rho}).
\end{aligned}$$

One can check that

$$\mu(\rho)A(u) : \nabla(\frac{\nabla\mu(\rho)}{\rho}) = 0.$$

Therefore the equation reads

$$\frac{1}{2}\,\partial_t(\rho|v|^2) + \frac{1}{2}\,\mathrm{div}(\rho|v|^2 v) + v^T\nabla p - 2\,\mathrm{div}(\mu(\rho)A(u)^T \cdot v) + \mu(\rho)|A(u)|^2 = 0. \tag{4.75}$$

Integrating in space assuming periodic boundary conditions yields

$$\frac{1}{2}\,\partial_t \int_\Omega (\rho|v|^2)\ dx + \int_\Omega \nabla p \cdot v\ dx + \int_\Omega \mu(\rho)|A(u)|^2\ dx = 0. \tag{4.76}$$

But

$$\begin{aligned}\int_\Omega \nabla p \cdot v\ dx &= \int_\Omega \nabla\psi \cdot \rho u\ dx + \int_\Omega \nabla\psi \cdot \nabla\mu(\rho)\ dx \\ &= \frac{d}{dt}\int_\Omega e_{in}(\rho) + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2\ dx,\end{aligned}$$

with $\psi'(\rho) = \frac{p'(\rho)}{\rho}$ and $e'_{in}(\rho) = \psi$. Finally the BD entropy reads

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho|v|^2 + e_{in}(\rho))\ dx + \int_\Omega \frac{p'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2\ dx + \int_\Omega \mu(\rho)|A(u)|^2\ dx = 0. \tag{4.77}$$

Part III

Bi-viscous Rheology: Lubrication and Shallow Water Equations

5

A Lubrication Equation for a Simplified Model of Shear-Thinning Fluid

This work started in Marseille Juillet 2019, in a CEMRACS project: PYROCLAST. It is a collaboration between F. James from Institut Denis Poisson, Universite d'Orleans, and the PhD sudents D. Nguyen (LAMA, Universite Paris-Est Marne-la-Vallee) and M. M'Baye (Laboratoire Jean Leray, Universite de Nantes). It has been submitted to ESAIM: Proceedings and Surveys. The link of the paper on HAL is given below:
https://hal.archives-ouvertes.fr/hal-02570086.

A lubrication equation is obtained for a simplified shear-thinning fluid. The simplified rheology consists of a piece wise linear stress tensor, resulting in a two-viscosity model. This can be interpreted as a modified Bingham fluid, which can be recovered in a specific limit. The lubrication equation is obtained in two steps. First two scalings are performed on the incompressible Navier-Stokes equations, namely the long-wave scaling and the slow motion scaling. Second, the resulting equations are averaged along the vertical direction. Numerical illustations are provided, bringing to light the different possible behaviours.

This chapter is divided into sections as following:

- **The first section** 5.1 is where we introduce a brief state of art, the motivation and the procedure of the work.

-**The second section** 5.2 exhibits the departure system with boundary conditions, the corresponding rheology and the choice of scaling for lubrication setting.

-**The third section** 5.3 is where the derivation of the lubrication equation is carried on using depth averaging the system stated in Section 5.2 and the main order profile in terms of smallness parameters sponsored by scaling.

-**The fourth section** 5.4 contains numerical tests of the final model, in which several behaviors of the bi-viscous fluid are illustrated.

5.1 Introduction

The lubrication equation is quite a classical simplification of the incompressible Navier-Stokes system. It is obtained for thin films of fluid, when viscous effects balance the pressure force. This occurs for instance for thin films of oil, hence the name of the equation. The study of this approximation goes back to Reynolds in 1886 [15]. Several scalings are involved to obtain this model. First the aspect ratio between the thickness of the film and the characteristic length of the substrate must be small, say δ. Simultaneously, the time scale has to be of order $1/\delta$. This is the so-called long wave regime, and is classically used in the shallow-water approximation. The lubrication equation requires another assumption of balance between the viscous effects and the pressure effects, which amounts to neglect all kinematic effects. This simplified flow is known as the Stokes flow. The lubrication equation itself is then obtained by integration over the fluid thickness.

We are interested here in the lubrication model for a class of non Newtonian fluids. Several fluids are known to depart from the usual Newtonian rheology, where the deviatoric stress tensor is a linear function of the strain rate tensor, thus defining the dynamical viscosity of the fluid. The lubrication equation for Newtonian fluids has been studied for instance by Huppert [90]. Non Newtonian fluids arise in several applications in engineering, biology, geophysics... In particular, viscoplastic or pseudoplastic fluids are involved in various geological problems, for instance lava flows, mudslides and avalanches. We refer to [91] for a review on the subject. A model which is widely used is the so-called Bingham-plastic model. This model involves a yield stress, namely a threshold on strain rate: for values of the strain rate above this threshold the fluid behaves like a viscous fluid, for values below, it looks like a solid. This can be thought of as an infinite viscosity fluid. We refer to the papers by Liu and Mei [92] and Balmforth et al. [93] for the study of such fluids in the lubrication approximation. Both papers contain also a complete bibliography. Liu and Mei also introduced in [66] a perturbed Bingham model, which is actually a two viscosities model, with a high viscosity for small deformations. When this viscosity goes to ∞ the Bingham model is recovered, thus giving a fluid mechanics interpretation of this solid behaviour.

This is precisely the two viscosities model we investigate here. First we describe the mathematical model we use, namely the incompressible Navier-Sokes equations in a time-dependant domain, since we consider a free-boundary problem. In particular we explain in some details all the scalings involved. Next, we turn to the lubrication equation itself, which is a one-dimensional equation, obtained by averaging the previous ones along the thickness. Finally we provide a few numerical illustrations based on a finite volume scheme.

5.2 Mathematical model

In this section we set up the model. The starting point is the incompressible Navier-Stokes system. We limit ourselves in this paper to the two-dimensional case, thus aiming at a one-dimensional lubrication equation. Similar computations can be performed in three space dimensions. The domain we consider is Ω_t defined by $f_b(x) < z < \varphi(t,x)$, for $t > 0$ and $x \in (-\infty, +\infty)$, where f_b is given topography, and φ is a free surface. The notations we use are gathered in Figure 5.1.

The incompressible Navier-Stokes equations are

$$\partial_x u + \partial_z w = 0, \tag{5.1}$$

$$\partial_t u + u\partial_x u + v\partial_z u = -\frac{1}{\rho}\partial_x p + \frac{1}{\rho}\big(\partial_x \tau_{xx} + \partial_z \tau_{xz}\big), \tag{5.2}$$

$$\partial_t w + u\partial_x w + w\partial_z w = -g - \frac{1}{\rho}\partial_z p + \frac{1}{\rho}\big(\partial_x \tau_{zx} + \partial_z \tau_{zz}\big), \tag{5.3}$$

where ρ is the density of the fluid, $U = (u, w)$ is the velocity field, and the stress tensor σ is written as the sum of a volumetric stress tensor, involving the pressure p, and a deviatoric stress tensor τ:

$$\sigma = -p\mathbb{I}_2 + \tau, \qquad \tau = \begin{pmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{pmatrix}.$$

where $\mathbb{I}_2$ is the identity matrix in dimension 2. The density ρ is assumed to be constant here, and the tensor σ will be defined in Section 5.2.1 below.

Boundary conditions are:

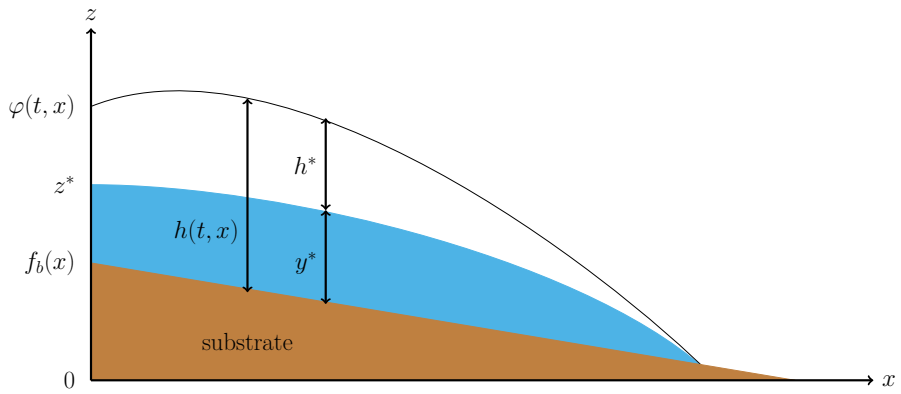


Figure 5.1: *Notations for the two viscosities fluid: φ is the free surface; f_b is the topography of the substrate; z^* is the ordinate which separates "small deformations" (white zone) from "large deformations" (green zone), see Section 5.3 below. We introduce the thicknesses $h = \varphi - f_b$, $h^* = \varphi - z^*$, $h_* = h - h^*$.*

$z = \varphi$: fluid-atmosphere interface. We have continuity of the stress tensor at the free surface, together with a kinematic boundary condition. Since the atmosphere can be viewed as an ideal fluid, the stress tensor can be taken equal to zero above φ. Hence we get

$$\sigma \cdot n|_\varphi = (-p\mathbb{I}_2 + \tau) \cdot n|_\varphi = 0, \qquad \partial_t \varphi + u_\varphi \partial_x \varphi = w_\varphi. \tag{5.4}$$

$z = f_b$: interface between the fluid and the substrate, which is fixed. This is a material interface, on which we have the no-slip boundary condition

$$u|_{f_b} = u_b, \quad w|_{f_b} = w_b. \tag{5.5}$$

Here (u_b, w_b) is the so-called basal velocity. Often in fluid mechanics the basal velocity is zero, but for geophysical applications it can actually be the driving force, and thus depend on (t, x).

5.2.1 Rheology

For a fluid, the deviatoric stress tensor τ is usually a function of the strain rate tensor

$$\dot\varepsilon = \begin{pmatrix} \dot\varepsilon_{xx} & \dot\varepsilon_{xz} \\ \dot\varepsilon_{zx} & \dot\varepsilon_{zz} \end{pmatrix} = \frac{1}{2}(\nabla U + \nabla U^T) = \frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_x w + \partial_z u \\ \partial_x w + \partial_z u & 2\partial_z w \end{pmatrix}. \tag{5.6}$$

A Newtonian fluid is characterized by a linear relation, defining the viscosity of the fluid, which is assumed here to be isotropic and constant. Therefore we introduce the dynamical viscosity coefficient μ, and define the Newtonian stress tensor by

$$\tau_N = 2\mu\dot\varepsilon = 2\rho\nu\dot\varepsilon,$$

where $\nu = \mu/\rho$ is the kinematic viscosity.

Fluids that do not follow this kind of constitutive law are non-Newtonian. In the general case, the material invariance principle implies that the stress tensor depends only on the similarity invariants of the strain rate tensor, in particular the coefficients of its characteristic polynomial. In dimension 2 there are only two such coefficients $\dot\varepsilon_I$ and $\dot\varepsilon_{II}$. Namely $\dot\varepsilon_I$ is the trace of the matrix and $\dot\varepsilon_{II}$ its determinant. For an incompressible fluid, the trace is zero, and moreover we have

$$\dot\varepsilon_{II} = \dot\varepsilon_{xx}\dot\varepsilon_{zz} - \dot\varepsilon_{zx}\dot\varepsilon_{xz} = \partial_x u \partial_z w - \frac{1}{4}(\partial_x w + \partial_z u)^2 = -\big((\partial_x u)^2 + \frac{1}{4}(\partial_x w + \partial_z u)^2\big).$$

This allows to define the strain rate $\dot{\gamma}$ as

$$\dot{\gamma} = 2\sqrt{-\dot{\varepsilon}_{II}} = 2\sqrt{(\partial_x u)^2 + \frac{1}{4}(\partial_x w + \partial_z u)^2}. \tag{5.7}$$

In a similar way we can check that the Frobenius norm of $\dot{\varepsilon}$, that is $\|\dot{\varepsilon}\|^2 = \sum_{i,j}(\varepsilon_{ij})^2$, satisfies

$$\|\dot{\varepsilon}\|^2 = \dot{\gamma}^2/2. \tag{5.8}$$

A very sketchy illustration of the possible behaviours of non-Newtonian fluids is given in Figure 5.2. We will be mostly interested in this work in the so-called pseudoplastic case, that is the red curve in Figure 5.2, for which experimental evidence can be given, see [94]. This kind of models are also used in geophysics, see [95], [96], [97]. We wish to give a simplified model for this pseudo-plastic fluid, that

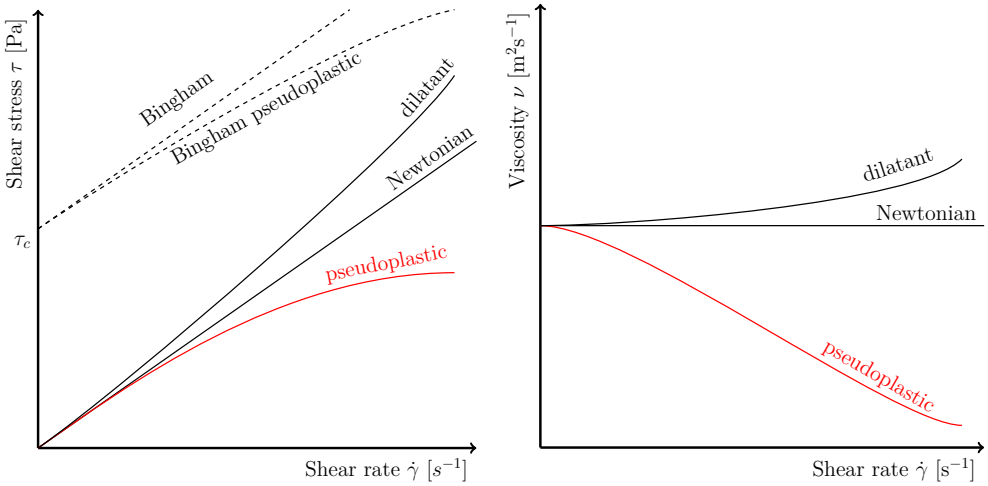


Figure 5.2: *Qualitative behaviour of various types of fluids. Left: stress vs shear stress – Right: apparent viscosity vs shear stress. The Bingham type fluids can be viewed as enjoying infinite apparent viscosity below the threshold τ_c.*

allows to handle explicit computations. The main feature of this kind of fluids is a nonlinear viscosity, decreasing with the strain rate. Mimicking the Bingham model, which is based on a threshold on the shear stress, we consider a model with a threshold on the strain rate: the viscosity is equal to some large μ_B for small deformations, that is $\dot{\gamma} < \gamma_c$, where $\gamma_c > 0$ is a given constant, and to another value μ for large deformations, $\dot{\gamma} > \gamma_c$. Such models were introduced by Liu and Mei [66], and the limit case $\nu_B \to \infty$, which leads to a Bingham fluid, is studied in [92] and [93]. Notice that using (5.8) the threshold γ_c on $\dot{\gamma}$ can be replaced by a threshold $\gamma_c' = \gamma_c/\sqrt{2}$ on $\|\dot{\varepsilon}\|$.

A multidimensional formulation for these simplified pseudo-plastic fluids is therefore

$$\tau_{PP} = \begin{cases} 2\rho\nu_B\dot{\varepsilon} & \text{if } \|\dot{\varepsilon}\| \leqslant \gamma_c' \\ 2\rho\nu\dot{\varepsilon} + 2\rho(\nu_B - \nu)\gamma_c' \dfrac{\dot{\varepsilon}}{\|\dot{\varepsilon}\|} & \text{if } \|\dot{\varepsilon}\| > \gamma_c' \end{cases} \tag{5.9}$$

A particular limit case is $\nu_B \to \infty$, which leads to a Bingham type fluid. To view this, it is convenient to define the following quantities (see Figure 5.3 below for an illustration in 1 dimension)

$$\tau_c = \nu_B\gamma_c', \qquad \tau_* = (\nu_B - \nu)\gamma_c' = (1 - \nu/\nu_B)\tau_c, \tag{5.10}$$

so that definition (5.9) can be rewritten

$$\tau_{PP} = \begin{cases} 2\rho\nu_B\dot{\varepsilon} & \text{if } \|\dot{\varepsilon}\| \leqslant \tau_c/\nu_B \\ 2\rho\nu\dot{\varepsilon} + 2\rho(1-\nu/\nu_B)\tau_c\dfrac{\dot{\varepsilon}}{\|\dot{\varepsilon}\|} & \text{if } \|\dot{\varepsilon}\| > \tau_c/\nu_B \end{cases} \tag{5.11}$$

It is clear on this formulation that the relevant limit is $\nu_B \to +\infty$, together with $\gamma_c' \to 0$, keeping $\nu_B\gamma_c' = \tau_c$. In doing so, we recover the classical Bingham stress tensor, with threshold τ_c:

$$\tau_{\text{Bing}} = \begin{cases} \text{any } \tau \text{ s.t. } \|\tau\| \leqslant \tau_c & \text{if } \dot{\varepsilon} = 0 \\ 2\rho\nu\dot{\varepsilon} + 2\rho\tau_c\dfrac{\dot{\varepsilon}}{\|\dot{\varepsilon}\|} & \text{if } \|\dot{\varepsilon}\| > 0 \end{cases}$$

Finally, notice that in the pseudo-plastic (or shear thinning) context, we consider $0 < \nu < \nu_B$, but similar computations can be performed in any case.

It is convenient for the scalings below to rewrite expression (5.11) using an equivalent kinematic viscosity ν_{eq}, which satisfies $\nu \leqslant \nu_{eq} \leqslant \nu_B$:

$$\tau_{PP} = 2\rho\nu_{eq}\dot{\varepsilon}, \qquad \text{where } \nu_{eq} = \begin{cases} \nu_B & \text{if } \|\dot{\varepsilon}\| \leqslant \gamma_c' \\ \nu + (1-\nu/\nu_B)\dfrac{\tau_c}{\|\dot{\varepsilon}\|} & \text{if } \|\dot{\varepsilon}\| > \gamma_c' \end{cases}. \tag{5.12}$$

5.2.2 Scalings

We introduce now the scaling laws, namely thin layer, or more precisely long wave approximation, and slow motion, in order to finally obtain the lubrication model. This kind of scalings is already present e.g. in [93] in the context of a visco-plastic fluid. Hence we propose the following family of scalings: we introduce a first set of characteristic scales, namely dimensions ℓ_0 and h_0, characteristic velocities u_0 and v_0, and a characteristic time t_0. The quantities ℓ_0 and u_0 correspond to the horizontal direction, h_0 and v_0 to the vertical one. The aspect ratio $\delta = h_0/\ell_0$ will be an important parameter, assumed to be small in the thin layer case. Dimensionless variables are then defined by

$$\begin{aligned} x &= \ell_0\bar{x}, \quad z = h_0\bar{z}, \quad t = t_0\bar{t} \\ u &= u_0\bar{u}, \quad w = w_0\bar{w}. \end{aligned}$$

First, we rewrite the incompressibility equation (5.1) in the rescaled variables. We obtain

$$\frac{u_0}{\ell_0}\partial_{\bar{x}}\bar{u} + \frac{v_0}{h_0}\partial_{\bar{z}}\bar{w} = 0,$$

and following the least degeneracy principle [98], this implies $u_0/\ell_0 = w_0/h_0$, or equivalently $\ell_0/h_0 = u_0/w_0$. Thus $w_0/u_0 = \delta$, so that in the thin layer approximation w_0 is also small compared to u_0.

We turn now to the kinematic part of the equation. Using $u_0/\ell_0 = w_0/h_0$, we readily obtain

$$\partial_t u + u\partial_x u + w\partial_z u = \frac{u_0}{t_0}\partial_{\bar{t}}\bar{u} + \frac{u_0w_0}{h_0}\bar{u}\partial_{\bar{x}}\bar{u} + \frac{u_0w_0}{h_0}\bar{w}\partial_{\bar{z}}\bar{u}.$$

Once again we apply the least degeneracy principle and obtain $t_0 = \ell_0/u_0 = h_0/w_0$, or, as expected, $u_0 = \ell_0/t_0$ and $w_0 = h_0/t_0$. We proceed in the same way for the momentum equation in v and finally obtain

$$\partial_t u + u\partial_x u + w\partial_z u = \frac{u_0w_0}{h_0}\left(\partial_{\bar{t}}\bar{u} + \bar{u}\partial_{\bar{x}}\bar{u} + \bar{w}\partial_{\bar{z}}\bar{u}\right) = \delta\frac{u_0^2}{h_0}\left(\partial_{\bar{t}}\bar{u} + \bar{u}\partial_{\bar{x}}\bar{u} + \bar{w}\partial_{\bar{z}}\bar{u}\right), \tag{5.13}$$

$$\partial_t w + u\partial_x w + w\partial_z w = \frac{u_0w_0}{\ell_0}\left(\partial_{\bar{t}}\bar{w} + \bar{u}\partial_{\bar{x}}\bar{w} + \bar{v}\partial_{\bar{z}}\bar{w}\right) = \delta^2\frac{u_0^2}{h_0}\left(\partial_{\bar{t}}\bar{w} + \bar{u}\partial_{\bar{x}}\bar{w} + \bar{w}\partial_{\bar{z}}\bar{w}\right), \tag{5.14}$$

where we have emphasized the aspect factor $\delta = h_0/\ell_0 = w_0/u_0$.

Following Balmforth [93], we rescale the pressure and the stress tensor by

$$p = \rho g h_0 \bar{p}, \qquad \tau = \rho \nu \frac{u_0}{h_0} \bar{\tau}, \tag{5.15}$$

where we recall that ν is the kinematic viscosity for *large deformations*. We can write now the rescaled version of the Navier-Stokes momentum equations (5.2) and (5.3):

$$\delta \frac{u_0^2}{h_0} \left(\partial_{\bar{t}} \bar{u} + \bar{u}\partial_{\bar{x}} \bar{u} + \bar{w}\partial_{\bar{z}} \bar{u}\right) = -\delta g \partial_{\bar{x}} \bar{p} + \nu \frac{u_0}{h_0^2} \left(\delta \partial_{\bar{x}} \bar{\tau}_{xx} + \partial_{\bar{z}} \bar{\tau}_{xz}\right), \tag{5.16}$$

$$\delta^2 \frac{u_0^2}{h_0} \left(\partial_{\bar{t}} \bar{w} + \bar{u}\partial_{\bar{x}} \bar{w} + \bar{w}\partial_{\bar{z}} \bar{w}\right) = -g \partial_{\bar{z}} \bar{p} - g + \nu \frac{u_0}{h_0^2} \left(\delta \partial_{\bar{x}} \bar{\tau}_{xz} + \partial_{\bar{z}} \bar{\tau}_{zz}\right). \tag{5.17}$$

At this stage, we introduce two classical dimensionless quantities, namely the Froude and Reynolds numbers, defined from the characteristic *horizontal* velocity u_0 and the *vertical* extension h_0

$$\frac{1}{Fr^2} = \frac{g h_0}{u_0^2}, \qquad \frac{1}{Re} = \frac{\nu}{u_0 h_0}. \tag{5.18}$$

We divide the previous two equations by u_0^2/h_0, and noticing that $\tau = \rho Re u_0^2 \bar{\tau}$, we obtain

$$\delta \left(\partial_{\bar{t}} \bar{u} + \bar{u}\partial_{\bar{x}} \bar{u} + \bar{w}\partial_{\bar{z}} \bar{u}\right) = -\frac{\delta}{Fr^2} \partial_{\bar{x}} \bar{p} + \frac{1}{Re} \left(\delta \partial_{\bar{x}} \bar{\tau}_{xx} + \partial_{\bar{z}} \bar{\tau}_{xz}\right), \tag{5.19}$$

$$\delta^2 \left(\partial_{\bar{t}} \bar{w} + \bar{u}\partial_{\bar{x}} \bar{w} + \bar{w}\partial_{\bar{z}} \bar{w}\right) = -\frac{1}{Fr^2} \partial_{\bar{z}} \bar{p} - \frac{1}{Fr^2} + \frac{1}{Re} \left(\delta \partial_{\bar{x}} \bar{\tau}_{xz} + \partial_{\bar{z}} \bar{\tau}_{zz}\right). \tag{5.20}$$

The idea now is to send δ to zero, thus implementing the thin layer assumption, but in a regime where the Reynolds number Re is kept of order 1, together with a balance between viscosity and gravity forces. Therefore we set

$$Fr^2 = \delta Re, \qquad Re = O(1). \tag{5.21}$$

This readily gives

$$u_0 = \delta \frac{g h_0^2}{\nu} = \frac{g h_0^3}{\ell_0 \nu}, \tag{5.22}$$

the latter being the scaling proposed in [93]. It introduces another characteristic velocity, namely $u_0' = (g h_0^2)/\nu$. The latter equality shows that this is indeed a slow motion scaling, thus we meet the initial requirement.

Inserting (5.21) in equations (5.19) and (5.20), and keeping only the dominant terms of order δ^{-1} gives first the dimensionless Stokes equation

$$-\partial_{\bar{x}} \bar{p} + \partial_{\bar{z}} \bar{\tau}_{xz} = 0, \tag{5.23}$$

then the dimensionless hydrostatic relation for the pressure

$$\partial_{\bar{z}} \bar{p} = -1. \tag{5.24}$$

Now we compute $\bar{\tau}$ from (5.12). We start by rewriting $\dot{\varepsilon}$ in rescaled variables

$$\dot{\varepsilon} = \frac{1}{2} \begin{pmatrix} 2\frac{u_0}{\ell_0}\partial_{\bar{x}} \bar{u} & \frac{w_0}{\ell_0}\partial_{\bar{x}} \bar{w} + \frac{u_0}{h_0}\partial_{\bar{z}} \bar{u} \\ \frac{w_0}{\ell_0}\partial_{\bar{x}} \bar{w} + \frac{u_0}{h_0}\partial_{\bar{z}} \bar{u} & 2\frac{w_0}{h_0}\partial_{\bar{z}} \bar{w} \end{pmatrix} = \frac{1}{2}\frac{u_0}{h_0} \begin{pmatrix} 2\delta\partial_{\bar{x}} \bar{u} & \delta^2\partial_{\bar{x}} \bar{w} + \partial_{\bar{z}} \bar{u} \\ \delta^2\partial_{\bar{x}} \bar{w} + \partial_{\bar{z}} \bar{u} & 2\delta\partial_{\bar{z}} \bar{w} \end{pmatrix}. \tag{5.25}$$

From this we easily deduce

$$\bar{\tau} = \frac{\nu_{eq}}{\nu} \begin{pmatrix} 2\delta\partial_{\bar{x}} \bar{u} & \delta^2\partial_{\bar{x}} \bar{w} + \partial_{\bar{z}} \bar{u} \\ \delta^2\partial_{\bar{x}} \bar{w} + \partial_{\bar{z}} \bar{u} & 2\delta\partial_{\bar{z}} \bar{w} \end{pmatrix}. \tag{5.26}$$

We define a dimensionless equivalent viscosity by $\bar{\nu}_{eq} = \nu_{eq}/\nu$, and rewrite equation (5.23)

$$\partial_{\bar{z}}\big(\bar{\nu}_{eq}\partial_z \bar{u}\big) = \partial_{\bar{x}}\bar{p}. \tag{5.27}$$

We turn now to the expression of $\bar{\nu}_{eq}$. We first notice that, using (5.25)

$$\|\dot{\varepsilon}\| = \frac{u_0}{h_0}\sqrt{2}\sqrt{\delta^2(\partial_{\bar{x}}\bar{u})^2 + \frac{1}{4}(\delta^2\partial_{\bar{x}}\bar{w} + \partial_{\bar{z}}\bar{u})^2} \underset{\delta\to 0}{\longrightarrow} \frac{1}{\sqrt{2}}\frac{u_0}{h_0}|\partial_{\bar{z}}\bar{u}|. \tag{5.28}$$

Hence the condition $\|\dot{\varepsilon}\| > \gamma_c'$ leads us to define a dimensionless threshold $\bar{\gamma}_c = (\sqrt{2}h_0/u_0)\gamma_c' = (h_0/u_0)\gamma_c$, so that the condition $\|\dot{\varepsilon}\| > \gamma_c'$ becomes $|\partial_{\bar{z}}\bar{u}| > \bar{\gamma}_c$, and $\bar{\nu}_{eq}$ becomes

$$\bar{\nu}_{eq} = \frac{\nu_{eq}}{\nu} = \begin{cases} \dfrac{\nu_B}{\nu} & \text{if } |\partial_{\bar{z}}\bar{u}| \leqslant \bar{\gamma}_c \\ 1 + (1-\nu/\nu_B)\dfrac{\tau_c}{\nu}\dfrac{h_0}{u_0}\dfrac{\sqrt{2}}{|\partial_{\bar{z}}\bar{u}|} & \text{if } |\partial_{\bar{z}}\bar{u}| > \bar{\gamma}_c \end{cases} \tag{5.29}$$

We introduce a dimensionless viscosity $\bar{\nu}_B$ and a dimensionless yield stress B by setting

$$\bar{\nu}_B = \frac{\nu_B}{\nu} \geqslant 1, \qquad B = \frac{\sqrt{2}\tau_c h_0}{\nu u_0}, \tag{5.30}$$

so that the expression of $\bar{\tau}_{xz}$, which is the only part of the deviatoric stress tensor remaining in the equations, becomes (see Figure 5.3)

$$\bar{\tau}_{xz} = \begin{cases} \bar{\nu}_B\partial_{\bar{z}}\bar{u} & \text{if } |\partial_{\bar{z}}\bar{u}| \leqslant \bar{\gamma}_c \\ \partial_{\bar{z}}\bar{u} + (1-1/\bar{\nu}_B)B\dfrac{\partial_{\bar{z}}\bar{u}}{|\partial_{\bar{z}}\bar{u}|} & \text{if } |\partial_{\bar{z}}\bar{u}| > \bar{\gamma}_c \end{cases} \tag{5.31}$$

This is the model proposed by Liu and Mei in [66].

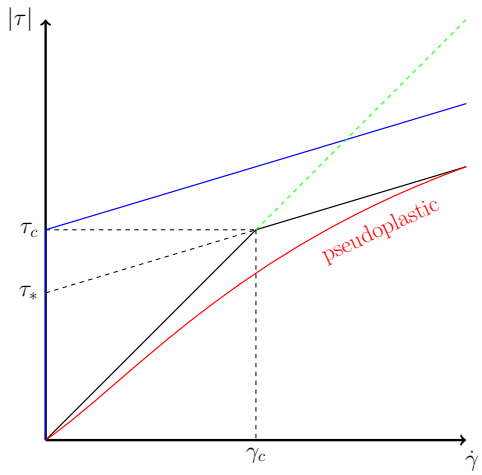


Figure 5.3: *Simplified shear-thinning model. We consider a piecewise linear approximation (in black) of the "theoretical" pseudoplastic law (in red). Parameters τ_* and τ_c are defined by (5.10). The blue curve is the Bingham limit: $\nu_B \to +\infty$, $\gamma_c \to 0$ with $\gamma_c\nu_B = \tau_c$. The green dashed line is the pure Newtonian limit $\nu_B \to \nu$.*

As concerns the boundary conditions, we notice that the no-slip and kinematic boundary conditions remain unchanged by the scaling. In contrast, the continuity of the stress tensor across the free surface

φ is greatly simplified. Recalling that $\varphi = h_0\bar{\varphi}$, we indeed obtain

$$\sigma \cdot n = \begin{pmatrix} -p\partial_x\varphi + \tau_{xx}\partial_x\varphi - \tau_{xz} \\ p + \tau_{xz}\partial_x\varphi - \tau_{zz} \end{pmatrix} = \rho \begin{pmatrix} -\delta\bar{p}gh_0\partial_{\bar{x}}\bar{\varphi} + \delta\nu\dfrac{u_0}{h_0}\bar{\tau}_{xx}\partial_{\bar{x}}\bar{\varphi} - \nu\dfrac{u_0}{h_0}\bar{\tau}_{xz} \\ \bar{p}gh_0 + \delta\nu\dfrac{u_0}{h_0}\bar{\tau}_{xz}\partial_{\bar{x}}\bar{\varphi} - \nu\dfrac{u_0}{h_0}\bar{\tau}_{zz} \end{pmatrix}.$$

Now making use of (5.26), we obtain

$$\sigma \cdot n = \rho \begin{pmatrix} \delta\big((\delta\nu_{eq}\dfrac{u_0}{h_0}\partial_{\bar{x}}\bar{u} - gh_0\bar{p})\partial_{\bar{x}}\bar{\varphi} - \delta\nu_{eq}\dfrac{u_0}{2h_0}\partial_{\bar{z}}\bar{w}\big) - \nu_{eq}\dfrac{u_0}{2h_0}\partial_{\bar{z}}\bar{u} \\ \\ \delta\nu_{eq}\dfrac{u_0}{2h_0}\big(\delta^2\partial_{\bar{x}}\bar{w} + \partial_{\bar{z}}\bar{u}\partial_{\bar{x}}\bar{\varphi} - \partial_{\bar{z}}\bar{w}\big) + gh_0\bar{p} \end{pmatrix}.$$

Letting δ go to zero implies therefore that (5.4) becomes

$$\bar{p}|_{\bar{\varphi}} = 0, \qquad \partial_{\bar{z}}\bar{u}_{\bar{\varphi}} = 0, \tag{5.32}$$

In other words, we recover separately the continuity of $\partial_z u$ and the continuity of the pressure.

Remark 7. *It is worth to notice here that we assume implicitly in this paper that ν_{eq} is bounded. However, if we wish to consider $\nu_B \to +\infty$, we should take care of the product $\delta\nu_{eq}$ that appears at several places in the equations. Thus another scaling arises, namely $\delta\nu_B$ should go to zero when δ goes to zero. This was pointed out by Liu and Mei [66].*

5.3 Lubrication equation

The so-called lubrication equation is obtained by integrating equations (5.1) along the vertical direction. The long wave and slow motion assumptions imply that we obtain a single nonlinear equation on the depth φ. Similar computations were performed by Liu and Mei [66], for a two-viscosity model, in order to justify the Bingham case, which corresponds to $\nu_B \to \infty$ in our context. For Bingham fluids, we refer to Liu and Mei [92], and more recently to Balmforth [93]. The final equation is obtained through three steps we present in detail now.

We recall the equations we obtained in the preceding section, dropping the bars for clarity. First we have the hydrostatic relation

$$\partial_z p = -1, \quad f_b \leqslant z \leqslant \varphi, \tag{5.33}$$

next, the dimensionless Stokes equation (5.23)

$$\partial_z \tau_{xz} = \partial_x p, \quad f_b \leqslant z \leqslant \varphi, \tag{5.34}$$

where τ_{xz} is the dimensionless deviatoric stress tensor defined by (5.31).

These equations are coupled with the following boundary conditions (in these relations, t and x are hidden parameters):

- on the free surface $z = \varphi$
$$p(\varphi) = 0, \quad \partial_z u(\varphi) = 0, \tag{5.35}$$
- on $z = f_b$
$$u(f_b) = u_b, \quad w(f_b) = w_b. \tag{5.36}$$

Concerning first the pressure, using the boundary condition on the free surface we obtain the usual hydrostatic approximation

$$p(z) = \varphi - z, \qquad f_b \leqslant z \leqslant \varphi. \tag{5.37}$$

The averaged equation we look for is obtained by integrating in z the incompressibility equation, or mass conservation,

$$\partial_x u + \partial_z w = 0.$$

This is quite classical, see e.g. [93] in the same slow motion context, or [99] for shallow water approximation. We obtain

$$w(t,x,\varphi) = w(t,x,f_b) - \int_{f_b}^{\varphi} \partial_x u(z)\,dz \tag{5.38}$$

$$= w(t,x,f_b) - \partial_x\left(\int_{f_b}^{\varphi} u(z)\,dz\right) + u(t,x,\varphi)\partial_x\varphi - u(t,x,f_b)\partial_x f_b. \tag{5.39}$$

The kinematic boundary condition on $z=\varphi$ leads to $w(t,x,\varphi) - u(t,x,\varphi)\partial_x\varphi = \partial_t\varphi = \partial_t h$, and for $z=f_b$, we make use of the no-slip boundary condition (5.36), to obtain the following averaged equation

$$\partial_t h + \partial_x\left(\int_{f_b}^{\varphi} u(z)\,dz\right) = w_b. \tag{5.40}$$

The flux $\displaystyle\int_{f_b}^{\varphi} u(z)\,dz$ can be computed explicitly as a function of φ, by integrating twice equation (5.34).

The first step towards the computation of the flux is to obtain the vertical velocity profile. The general structure of this profile is as follows. We have $\tau_{xz} = F(\partial_z u)$, where F is a continuous, one-to-one, increasing function, with $F(0)=0$, see (5.31) and Figure 5.3. From (5.34) and (5.37) we are led to solve $\partial_z\big(F(\partial_z u)\big) = \partial_x\varphi$. Since $F(\partial_z u) = 0$ for $z=\varphi$ (or equivalently $\partial_z u = 0$) we get $F(\partial_z u) = \partial_x\varphi(z-\varphi)$, so that $F(\partial_z u)$ is monotone (increasing if $\partial_x\varphi \geqslant 0$, decreasing if not). Because F is increasing, $\partial_z u$ is monotone as well, in particular, since $\partial_z u = 0$ for $z=\varphi$, its sign remains constant. Therefore $|\partial_z u|$ is decreasing in z (increasing with depth).

The threshold in formula (5.31) eventually splits the fluid in two layers. Let z^* be defined by $|\partial_z u(z^*)| = \gamma_c$. Provided $z^* \in]f_b,\varphi[$ (see below for precise formulas), we have a "small deformation" region, that is $|\partial_z u(z)| < \gamma_c$, for $z \in]z^*,\varphi[$, because $|\partial_z u|$ is decreasing from 0 for increasing depth. Similarly for $z \in]f_b, z^*[$ we have $|\partial_z u(z)| > \gamma_c$, so that finally, according to (5.31), the velocity is ruled by the system of equations

$$\begin{aligned} \nu_B \partial_{zz} u &= \partial_x\varphi, \qquad & z^* \leqslant z \leqslant \varphi, \\ \partial_{zz} u &= \partial_x\varphi, \qquad & f_b \leqslant z \leqslant z^*, \end{aligned}$$

where for the second equation we have used that $\partial_z u$ has a constant sign. These equations are complemented with the boundary conditions

$$\partial_z u = 0, \quad z=\varphi\,; \qquad u = 0, \quad z = f_b.$$

Notice that the curve $z=z^*$ is not a physical interface, yet we have continuity of the stress tensor, or equivalently here continuity of $\partial_z u$.

Now the computations are quite easy. We integrate once the first equation between φ and z^*, to obtain

$$\partial_z u = \frac{1}{\nu_B}\partial_x\varphi(z-\varphi).$$

This leads to

$$\dot\gamma = |\partial_z u| = \frac{1}{\nu_B}|\partial_x\varphi|(\varphi - z),$$

so that the value of z^* and the thickness h^* of this layer are given by

$$z^* = \max\left(\varphi - \frac{B}{|\partial_x\varphi|}, f_b\right), \qquad h^* = \varphi - z^* = \min\left(\frac{B}{|\partial_x\varphi|}, h\right). \tag{5.41}$$

These definitions ensure that $z^* \geqslant f_b$ and $h^* \leqslant h$, and are valid for $\partial_x\varphi = 0$ with the convention $B/0 = \infty$. Notice that z^* can be equal to f_b for weak slopes (small $\partial_x\varphi$), or small depths (small h). Conversely, $z^* \to \varphi$ when $|\partial_x\varphi|$ goes to ∞.

Integrating once again between z^* and φ, we obtain the velocity profile for $\varphi \geqslant z \geqslant z^*$:

$$u(z) = \frac{1}{2\nu_B}\partial_x\varphi(\varphi - z)^2 + K,$$

where the constant K will be determined later. Notice for further use that by construction

$$\partial_z u(z^*) = \frac{1}{\nu_B}\partial_x\varphi(z^* - \varphi) = \gamma_c. \tag{5.42}$$

We turn now to the lower layer, $z^* \geqslant z \geqslant f_b$. The fluid here has dimensionless viscosity 1, and we use the boundary conditions (5.42) for $z = z^*$, and no slip (5.36) at $z = f_b$. First we get, using (5.42),

$$\partial_z u = \partial_x\varphi(z - z^*) - \frac{1}{\nu_B}\partial_x\varphi h^*,$$

next, integrating once again between f_b and z^*,

$$u = \frac{1}{2}\partial_x\varphi(z^* - z)^2 - \frac{1}{\nu_B}\partial_x\varphi h^* z + L,$$

where L is computed using (5.36), leading to

$$L = u_b - \frac{1}{2}\partial_x\varphi(z^* - f_b)^2 + \frac{1}{\nu_B}\partial_x\varphi h^* f_b,$$

so that

$$u = \frac{1}{2}\partial_x\varphi\left((z^* - z)^2 - (z^* - f_b)^2\right) - \frac{1}{\nu_B}\partial_x\varphi h^*(z - f_b) + u_b, \tag{5.43}$$

Finally, we use the continuity of the velocity at $z = z^*$ to obtain the constant K:

$$\frac{1}{2\nu_B}\partial_x\varphi(\varphi - z^*)^2 + K = -\frac{1}{2}\partial_x\varphi(z^* - f_b)^2 - \frac{1}{\nu_B}\partial_x\varphi h^*(z^* - f_b) + u_b.$$

The velocity profile is therefore given by

$$u(z) = \begin{cases} \dfrac{\partial_x\varphi}{2}\left((z^* - z)^2 - (z^* - f_b)^2\right) - \dfrac{\partial_x\varphi}{\nu_B}h^*(z - f_b) + u_b, & f_b \leqslant z \leqslant z^* \\ \dfrac{\partial_x\varphi}{2\nu_B}\left((\varphi - z)^2 - (\varphi - z^*)^2\right) - \dfrac{\partial_x\varphi}{2}(z^* - f_b)^2 & \\ \qquad - \dfrac{\partial_x\varphi}{\nu_B}h^*(z^* - f_b) + u_b, & z^* \leqslant z \leqslant \varphi \end{cases} \tag{5.44}$$

Notice that for $\nu_B = 1$, easy computations show that the profile is the same in the two layers, namely $u = \dfrac{\partial_x\varphi}{2}(z - f_b)(z - f_b - 2h) + u_b$, which is as expected the usual parabolic profile for a Newtonian fluid.

On the other hand, letting $\nu_B \to +\infty$, and $\gamma_c \to 0$, keeping $\nu_B\gamma_c = \tau_c$, we recover formally the Bingham fluid velocity, as in Balmforth [93]:

$$u(z) = \begin{cases} -\dfrac{\partial_x\varphi}{2}\left((h_*)^2 - (h_* - (z - f_b))^2\right) + u_b, & f_b \leqslant z \leqslant z^* \\ -\dfrac{\partial_x\varphi}{2}(h_*)^2 + u_b, & z^* \leqslant z \leqslant \varphi \end{cases}$$

where we have set $h_* = z^* - f_b = h - h^*$.

It is now straightforward to obtain the flux in (5.40), since

$$\int_{f_b}^{\varphi} u(z)\,dz = \int_{f_b}^{z^*} u(z)\,dz + \int_{z^*}^{\varphi} u(z)\,dz.$$

We have on the one hand

$$\int_{f_b}^{z^*} u(z)\,dz = -\frac{\partial_x\varphi}{3}(h_*)^3 - \frac{\partial_x\varphi}{2\nu_B}h^*(h_*)^2 + u_b h_*,$$

on the other hand

$$\int_{z^*}^{\varphi} u(z)\,dz = -\frac{\partial_x\varphi}{3\nu_B}(h^*)^3 - \frac{\partial_x\varphi}{2}(h_*)^2 h^* - \frac{\partial_x\varphi}{\nu_B}h_*(h^*)^2 + u_b h^*.$$

Therefore the flux we are looking for is given by

$$\int_{f_b}^{\varphi} u(z)\,dz = -\frac{\partial_x\varphi}{3}\left((h_*)^3 + \frac{3}{2}\Big(\frac{1}{\nu_B}+1\Big)(h_*)^2h^* + \frac{3}{\nu_B}h_*(h^*)^2 + \frac{1}{\nu_B}(h^*)^3\right) + u_b h. \tag{5.45}$$

It is easy once again to check on this formula that we recover the usual cubic flux $-\frac{\partial_x\varphi}{3}h^3$ for the Newtonian fluid $\nu_B = 1$. On the other hand the limit case $\nu_B \to \infty$ gives back Balmforth's formula

$$\int_{f_b}^{\varphi} u(z)\,dz = -\frac{\partial_x\varphi}{6}(h_*)^2(h_* - 3h).$$

Inserting (5.45) in the conservation equation (5.40) leads to the following advection-diffusion equation:

$$\partial_t h + \partial_x(u_b h) = v_b + \partial_x\left(D(h,\partial_x h)\partial_x(h+f_b)\right), \tag{5.46}$$

where

$$D(h,\partial_x h) = \frac{1}{3}\left((h_*)^3 + \frac{3}{2}\Big(\frac{1}{\nu_B}+1\Big)(h_*)^2h^* + \frac{3}{\nu_B}h_*(h^*)^2 + \frac{1}{\nu_B}(h^*)^3\right) \tag{5.47}$$

and we recall the definitions of h^* from (5.41), and h_*

$$h^* = \min\left(\frac{B}{|\partial_x\varphi|}, h\right), \qquad h_* = h - h^* = z^* - f_b. \tag{5.48}$$

Notice that $0 < D(h,\partial_x h) \leqslant h^3/(3\nu_B)$.

5.4 Numerical illustrations

We turn now to numerical examples to illustrate the behaviour of the two-viscosity fluid. The point here is not to give an accurate specific scheme, which is an interesting perspective since the diffusion term may degenerate, but is beyond the scope of this work. We merely apply here a simple finite volume strategy. The infinite space domain is replaced by some finite computational domain $[a,b]$. Since we do not want to cope with boundary conditions here, we merely impose a free flux on the boundaries, which is compatible with the examples we choose. Positive time and space steps Δt and Δx being given, we introduce the usual notations $t^n = \Delta t$, $n \geqslant 0$, and $x_j = j\Delta x$, $0 \leqslant j \leqslant J$, where $J = (b-a)/\Delta x$. An approximation of the depth h is sought for in the form

$$h_j^{n+1} = h_k^n - \frac{\Delta t}{\Delta x}(F_{j+1/2}^n - F_{j-1/2}^n) + \frac{\Delta t}{\Delta x}(G_{j+1/2}^n - G_{j-1/2}^n),$$

where $F_{j+1/2}^n$ is the numerical advection flux, and $G_{j+1/2}^n$ the numerical diffusion flux, both computed at interface $x_{j+1/2}$. In the following we denote u_j^n the discretized basal velocity, and f_j the discrete

topography, which are both given functions.

The advection flux is merely an upwind flux

$$F^n_{j+1/2} = \begin{cases} h^n_j(u^n_j + u^n_{j+1})/2 & \text{if } u^n_j + u^n_{j+1} \geqslant 0 \\ h^n_{j+1}(u^n_j + u^n_{j+1})/2 & \text{if } u^n_j + u^n_{j+1} \leqslant 0 \end{cases}$$

For the diffusive flux, we write $G^n_{j+1/2} = D^n_{j+1/2}K^n_{j+1/2}$, where $K^n_{j+1/2}$ is the approximate value of the slope $\partial_x\varphi$

$$K^n_{j+1/2} = \frac{h^n_{j+1} - h^n_j}{\Delta x} + \frac{f_{j+1} - f_j}{\Delta x},$$

and $D^n_{j+1/2}$ is a discretization of (5.47). To obtain it we need to compute h^* and h_* at the interface. Accordingly to (5.48), we put

$$(h^*)^n_{j+1/2} = \begin{cases} \dfrac{B}{|K^n_{j+1/2}|} & \text{if } B < \dfrac{h^n_{j+1} + h^n_j}{2}|K^n_{j+1/2}| \\ \dfrac{h^n_{j+1} + h^n_j}{2} & \text{if not} \end{cases}$$

and $(h_*)^n_{j+1/2} = (h^n_{j+1} + h^n_j)/2 - ((h^*)^n_{j+1/2}$, so that $D^n_{j+1/2}$ is given by

$$D^n_{j+1/2} = \frac{1}{3}\Bigg(\big((h_*)^n_{j+1/2}\big)^3 + \frac{3}{2}\Big(\frac{1}{\nu_B} + 1\Big)\big((h_*)^n_{j+1/2}\big)^2(h^*)^n_{j+1/2} + \frac{3}{\nu_B}(h_*)^n_{j+1/2}\big((h^*)^n_{j+1/2}\big)^2 + \big((h^*)^n_{j+1/2}\big)^3\Bigg),$$

The time step Δt is actually updated at each time step using the CFL condition

$$\frac{\Delta t^n}{\Delta x^2} = \frac{\sigma}{2D^n}, \qquad \text{with } D^n = \max_j D^n_{j+1/2}, \quad \text{where } \sigma < 1.$$

The following simulations have been performed with $J = 200$ cells in the interval $[-1, 1]$, together with $\sigma = 0.9$. All figures are gathered at the end of the paper.

The first set of simulations concerns the collapse of a square-shaped stack on a horizontal flat bottom: $h^0(x) = 1$ for $x \in]-1/3, 1/3[$, 0 elsewhere, with zero basal velocity ($u_b = v_b = 0$). We first propose a comparison between the two viscosities model and the high viscosity and low viscosity models. The small deformation viscosity is $\nu_B = 100$ (recall that $\nu = 1$), and the yield stress is 0.1 in Figure 5.4, and 0.5 in Figure 5.5. These figures are complemented by Figure 5.6 where we display for four values of the yield stress B a timelapse of the evolution of both the total thickness of the fluid h (plain lines) and the thickness of the low velocity layer (dashed lines).

For $B = 0.1$, the fluid clearly behaves similarly as the low viscosity fluid in the early stages, then eventually it slows down, when the low viscosity layer tends to disappear, see Figure 5.6, top left. With a yield stress $B = 0.5$, the two viscosities model stays inbetween the other two, as expected, faster that the high viscosity model, slower than the low viscosity one, see Figure 5.5. However, one can check that the front hardly moves between $t = 10$ and $t = 50$, indicating that the fluid tends to behave as the high velocity one. This is made more explicit in Figure 5.6, top right, where for $t = 10$ and $t = 50$ the low viscosity layer is very small. In general, the thickness h_* decreases with time, faster when B is larger. It is hardly observable for $t = 50$ when $B = 2.5$, indicating that the fluid is almost completely driven by the high viscosity.

Using the same initial data, we check the convergence of the two viscosities model towards the Bingham fluid when ν_B goes to ∞. We take a yield stress $B = 1.25$, and $\nu_B = 10, 100, 1000$. As expected, the behaviour becomes close to the Bingham fluid, yet it departs from it for larger times, see Figure 5.7. This somehow justifies a posteriori that we have taken into account the scaling $\delta\nu_B \to 0$, see Remark 7.

We turn now to a different context, closer to the situation in geophysics. The flow here is no longer purely gravity driven, it is actually dragged along by a non zero basal velocity. The idea here is that our

pseudo-plastic fluid is a very crude model of some planetary lithosphere, below which lies the mantle. The basal velocity is the upper trace of convection currents in the mantle, which are supposed to be the main drivers of plate tectonics. The initial thickness is constant equal to 1, and we use two basal velocities $U_b(x) = (u_b(x), 0)$, where

$$u_b(x) = -\sin(2\pi x)/10 \cdot \mathbb{1}_{]-0.5,0.5[}(x), \qquad u_b(x) = \sin(2\pi x)/10 \cdot \mathbb{1}_{]-0.5,0.5[}(x). \tag{5.49}$$

These velocities crudely correspond respectively to the vertical motion of a magma bubble, which generates local perturbations of the velocity. The first one corresponds to some bubble lift, with negative velocity on the left and positive on the right. It generates some kind of a valley surrounded by mountains, see Figure 5.8. Conversely, the descent of a bubble reverses the velocities, and produces a mountain surrounded with valleys, Figure 5.9. We notice in both cases that the small viscosity model has very little influence on the time evolution, and that the two viscosity model leads to rather sharp angles in the thickness.

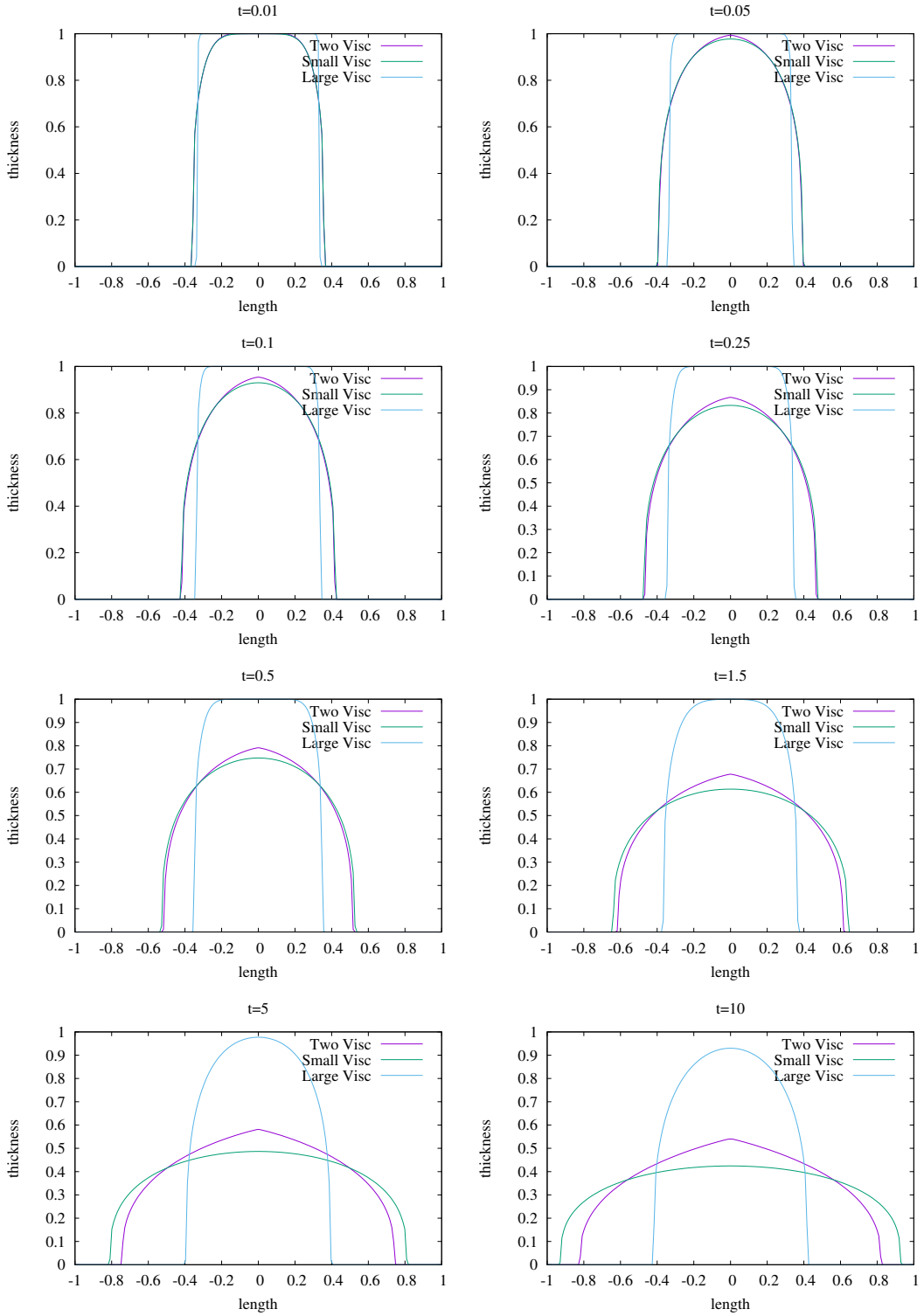


Figure 5.4: *Comparison between the three models, time evolution. Yield stress* $B = 0.1$, $\nu_B = 100$

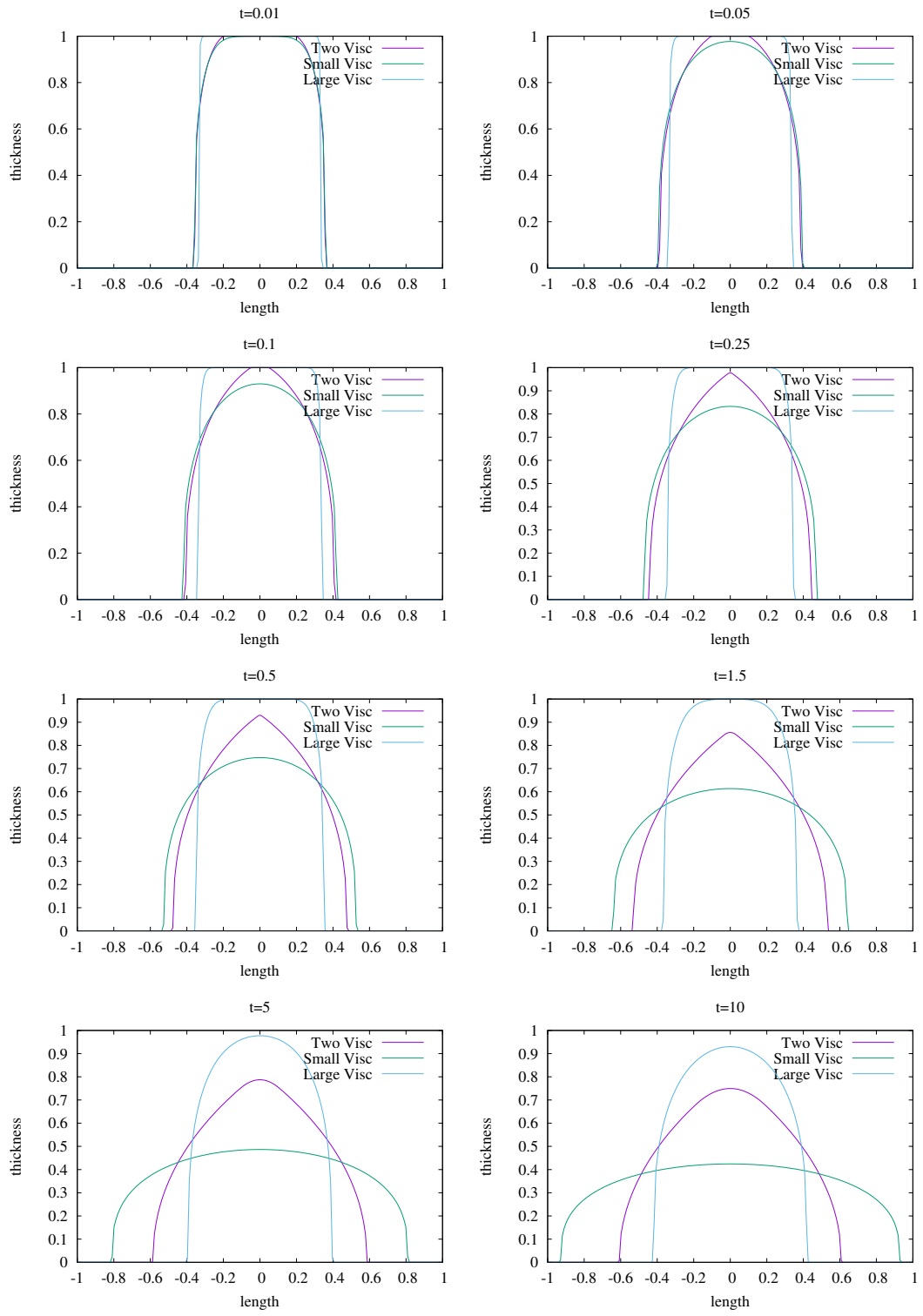


Figure 5.5: *Comparison between the three models, time evolution. Yield stress* $B = 0.5$, $\nu_B = 100$

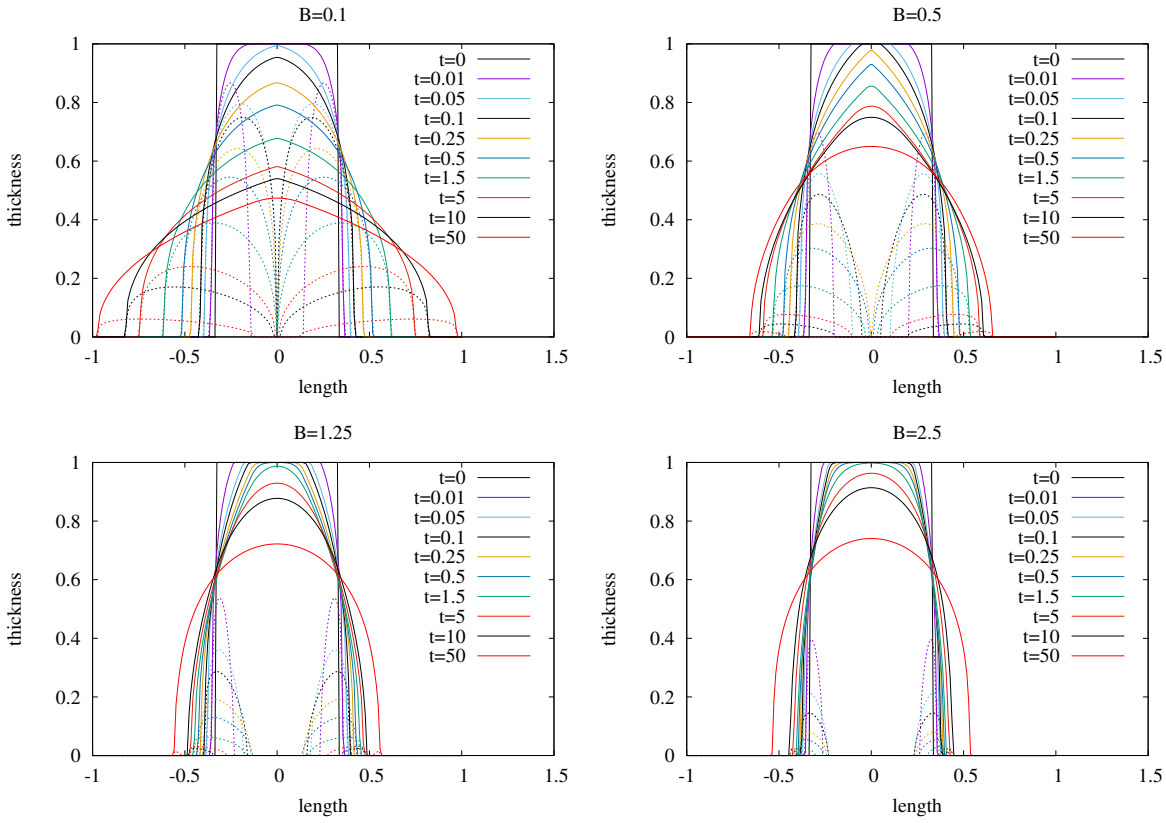


Figure 5.6: *Influence of the yield stress B. $\nu_B = 100$. Color code in pictures - plain lines: total thickness h, dashed lines: small viscosity zone thicness h_*.*

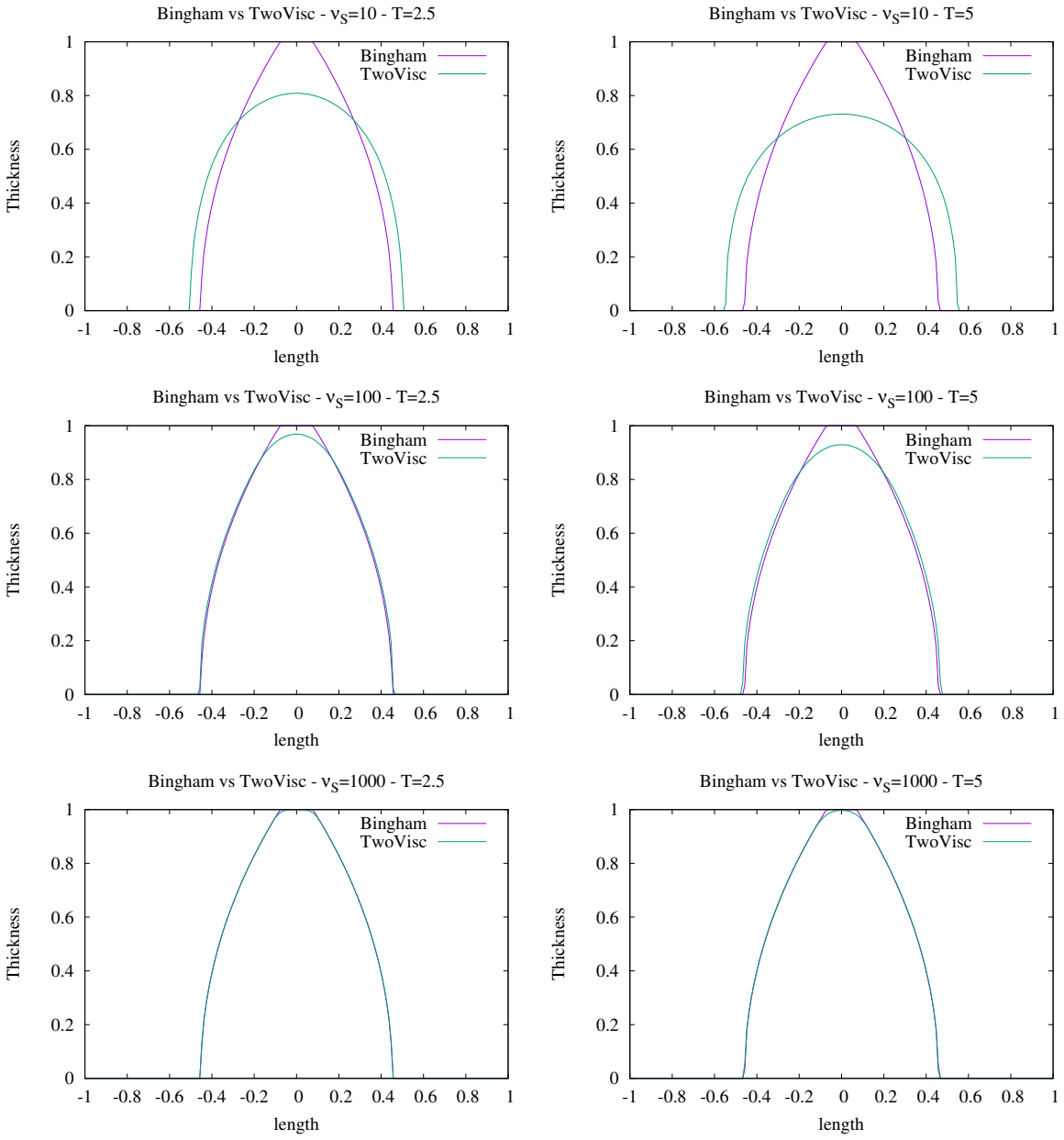


Figure 5.7: *Convergence towards the Bingham model. Yield stress* $B = 1.25$. *From top to bottom:* $\nu_B = 10, 100, 1000$.

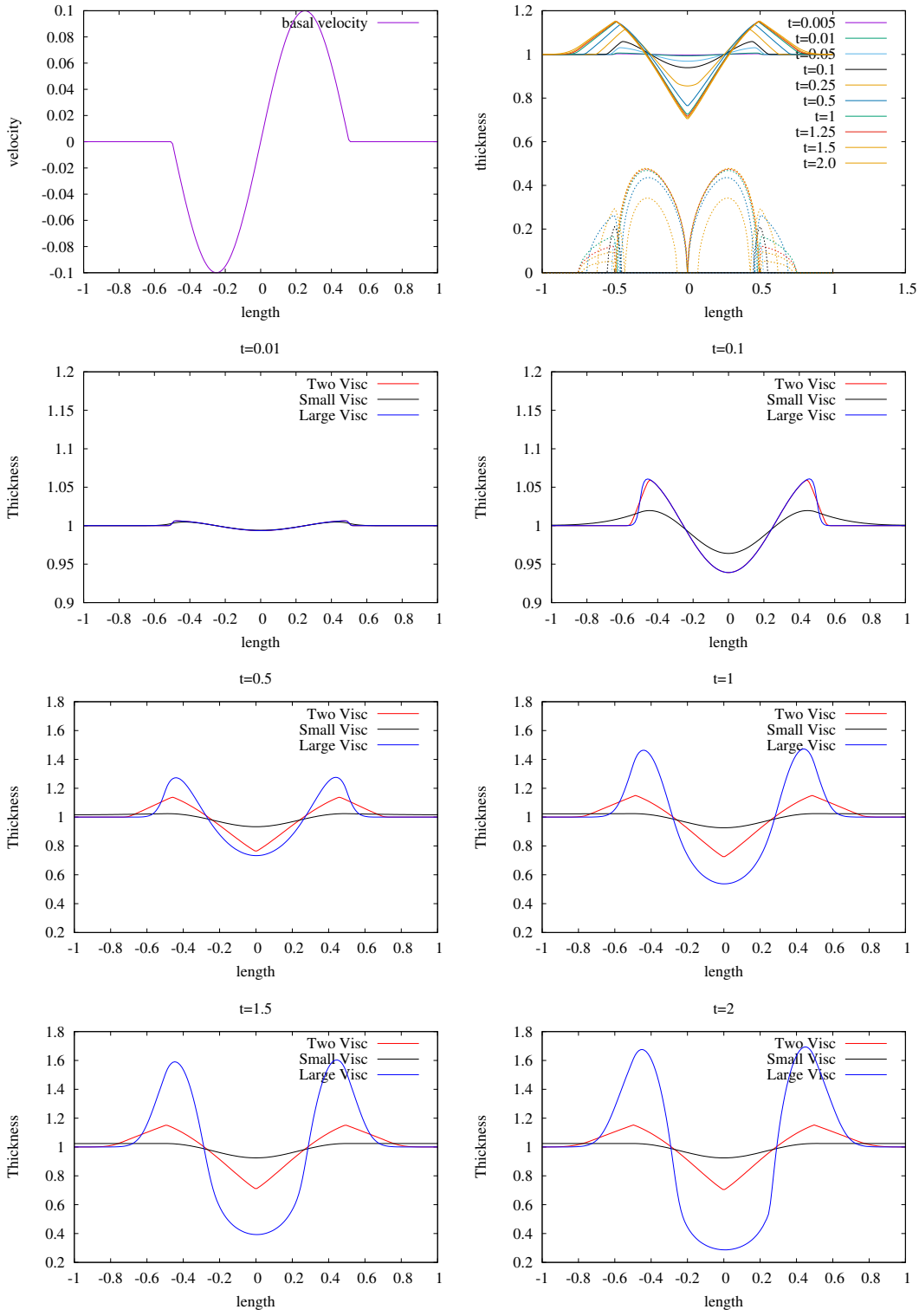


Figure 5.8: *Lift of a magma bubble - Yield stress* $B = 1.25$ *-* $\nu_B = 100$ *-Top left: basal velocity - Top right: Timelapse of thickness* h *and low viscosity layer* h_* *- Next 6 pictures: time evolution of the three models*

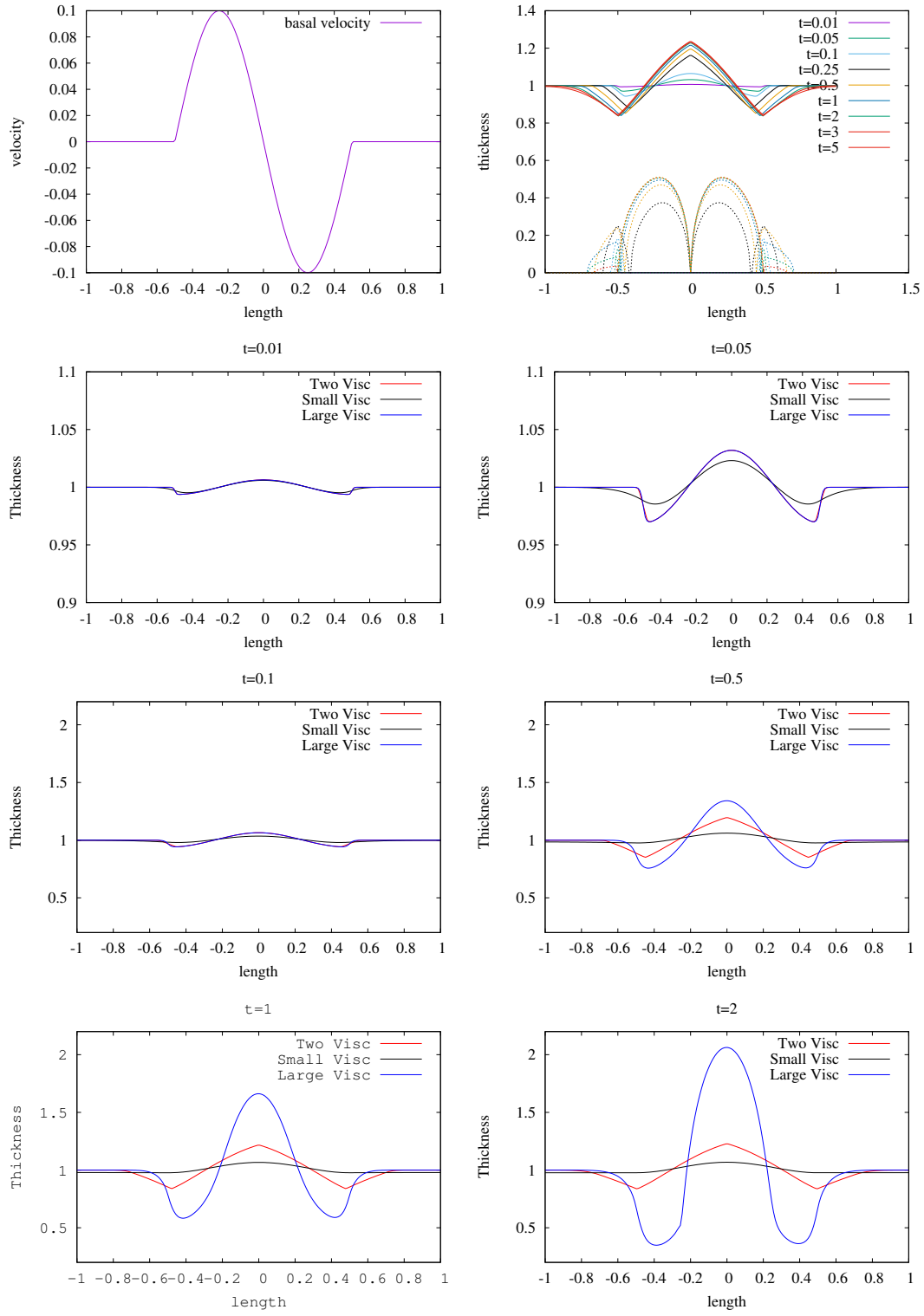


Figure 5.9: *Descent of a magma bubble - Yield stress* $B = 1.25$ *-* $\nu_B = 100$ *- Top left: basal velocity - Top right: Timelapse of thickness* h *and low viscosity layer* h_* *- Next 6 pictures: time evolution of the three models*

6

Bi-viscous Shallow Water Model

The work in this chapter is a collaboration work still in progress with the thesis supervisor D. Bresch (LAMA, Universite Savoie Mont Blanc) and F. James (Institut Denis Poisson, Universite d'Orleans).

This chapter is concerned with the derivation of a shallow water type model for a rheology that approximates that of pseudo-plastic models flowing down an inclined plane under the effect of gravity. The carried on derivation is formal, and the obtained model conveys a consistent thin layer theory. Being characterized with two viscosities referring to two regions: yielded region and pseudo-plug region (following the terminologies by N.J. Balmforth and R.V. Craster [*J. of Non-Newtonian Fluid Mech.* 1999]), the limit of the obtained model reattains a kind of shallow water model for a Bingham rheology once we are in large viscosity limit in the pseudo-plug zone under some uniformity costraints on the variables, and it reattains the classical Newtonian shallow water model mathematically justified by D. Bresch and P. Noble [*Methods Application analysis* 2010] when the viscosities become equal. In the large limit regime, the condition induced by K.F. Liu and C.C. Mei [*J. Fluid Mech.* 1989] concerning the relation between the aspect ratio ε, and the ratio of viscosities-stating that $\varepsilon\mu_1/\mu_2 << 1$- is conserved in this work. Besides giving a good approximation for the complex model of pseudo-plastic fluids, the bi-viscous model helps attain a one shot thin layer theory for thin sheet spreading of Bingham fluids avoiding the need to correct the viscosity profiles and include corrective layers throughout the approximations. To present a simplified presentation of the derivation, our calculus is done under some restrictive assumptions on the Froude and Reynolds numbers and uniformity of the variables through the derivation, yet it could be done with more general assumptions.

The chapter is organized as follows:

-**The first section 6.1** is an introductory section that sheds light on the history of the bi-viscous rheology.

-**The second section 6.2** where we present the departure system and boundary conditions and exhibit our main result in the work which is the final bi-viscous shallow water model (6.9). Then we compare the limit of the bi-viscous model when the viscosity μ_1 approaches large values, and compare the final result with that obtained in [40]. Similar comparison now is made when the two viscosities are made equal, and so the expected model would be the classical shallow water model for Newtonian fluids proved rigorously in [1].

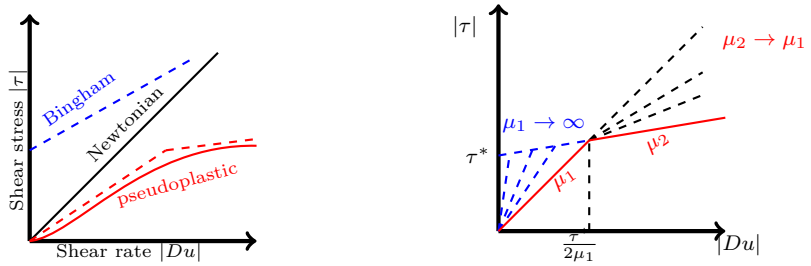
-**The third section 6.3** whose main goal is to show the derivation leading to the final bi-viscous shallow water model stated in previous section, using the the depth averaging of the mass and momentum equations. This is done in subsection 6.3.1, however the system can't be closed at this point as the theory still misses the first order shear and velocity profile. For that, we calculate the shear at first order in subsection 6.3.2, and this paves the way to calculate the velocity profile as well in the same subsection.

-**The fourth section 6.4** is a summary of the work which highlights on the novelty and main details and concludes the advantage of the final derived shallow model.

6.1 Introduction

Explaining the weird and unexpected behavior of pseudo-plastic fluids has been a topic of research for decades. A pseudo-plastic fluid is a non-Newtonian fluid exhibiting shear thinning behavior, meaning that its viscosity decreases gradually with the increase of shear rate. Thus, once the force is applied, the shear created allows such material to thin and spread as for a normal flow, and once the force stops, the fluid reattains its high viscosity. Pseudo-plastic fluids occupy a wide range of presence in different natural and industrial phenomena, and a good example would be a biological fluid called Synovial fluid which is found in the joints and helps reduce shocks. Also, from daily life usage, we have paints, ketchup, blood and molten polymers, all being examples of a pseudo-plastic fluid. The main distinguishable difference in this latter type is its stress which depends non linearly on the corresponding viscosity, and this in turns plays a crucial role in modifying the flow dynamics of the fluid.

The motivation of this work started form the need to obtain some simplified models of such complex-type fluids and to get a better vision of their behavior mechanically and try relating them to other types of non-Newtonian fluids- such as the Bingham one- by interpreting their physical characteristics as pressure, velocity, run out time, etc. For that, the first attempt is to get an approximation of the relation between the shear stress and shear rate (i.e the viscosity). We approximate the corresponding function- which is a nonlinear relation decreasing with the strain rate- by a piece wise linear function as shown in figure (6.1a) (red dotted graph). Thus the approximation rheology behaves similar to a Newtonian fluid for a deformation which is less than some characteristic threshold, whereas it shows the properties of a Bingham fluid for a large deformation. We will denote by μ_1 the viscosity of the Newtonian-like behaving part, and the Bingham-like part will be characterized with a viscosity $\mu_2 < \mu_1$ (see figure below).



(a) *fig1* (b) *fig2*

Figure 6.1: *Bi-viscous rheology as an approximation to Pseudo-Plastic rheology: relation with Newtonian and Bingham rheology.*

The inevitable result of obtaining a shallow water model for a bi-viscous fluid-beside its own importance in different biological and physical applications- is the limit of such model when the higher viscosity becomes too large. As shown in figure 6.1b, as the viscosity μ_1 becomes very large, the relation between the shear stress and shear rate becomes similar to that of Bingham fluids which results in parallel in modulations in the shallow water model. In fact, Bingham fluids have gained a great interest in research among other non-Newtonian fluids, and many characteristics of such fluids are still till now a topic of discussion and argumentation, starting from the consistence theory for thin layer dynamics or long wave approximations, to the reality of the yield threshold, is it a fake one or not? And last but not least is the mechanism of dealing with the stress in the plug (or pseudo-plug) zone. The literature of all previous work is presented in several papers, but it is good to shed light here on is the degradation of such problems in both lubrication and shallow water theory.

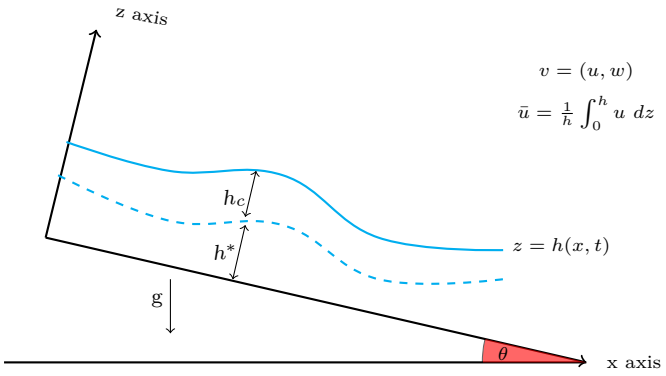


Figure 6.2: *schematic representation of the flow with the main parameters of the system: velocity vector $v = (u, w)$, gravity g, plane's inclination angle θ, fluid's height h, height of the yielded part h^*, height of the pseudo-plug part h_c, depth averaged velocity $\bar{u} = \frac{1}{h}\int_0^h u\ dz$.*

Lubrication type systems. As for lubrication systems, the literature is vast with such kind of models for non-Newtonian fluids and especially for Bingham fluids. Deriving Bingham lubrication models started with many people, for instance with Stanmore and al (1981), then Mei and Liu (1989), Tichy (1992) and Piau (1996). Later on, Balmforth and Craster [67] made a significant step in this field correcting all the previous works by providing a coherent, self-consistent thin layer theory for the thin layer dynamics of Bingham fluids. They proved that the previously admitted yield threshold doesn't really exist, thus a Bingham fluid is no more a fully plastic fluid, and no true plugs occur. The terminology was replaced by pseudo-plug zones. Their attempt relied on calculating corrective terms in the yielded region, and this resulted for them in a consistent parabolic profile, and thus they proved that the lubrication theory for Bingham fluids is robust. The usual asymptotic expansions neglected first order normal stress terms. This used to breakdown the theory, and this was their main correction to count on. However, they neglected the correcting terms in the pseud-plug zones. Whereas this would not affect the lubrication approximation, yet, it would break down the shallow water theory. But before this, another attempt by Mei and Liu, which was different in spirit, came also to correct the inconsistency presented in the literature of Bingham lubrication models. This attempt used the bi-viscous stress type, and approached the Bingham model as a limit to the bi-viscous one. As stated by Balmforth in [67], " One way to surmount the problem of consistency is to use bi-viscous fluids to derive the thin-layer equations and thereby avoid the Bingham model. This replaces the plug by a slowly yielding flow, and then the inconsistency magically disappears.... In other words, an arbitrarily small perturbation of the Bingham model appears to lead to a consistent theory". Such derivation has been also recently revisited in [100] where an equation describing the evolution of the surface of a bi-viscous fluid is derived under the theory of lubrication, and several numerical tests are carried out demonstrating the different possible flow behaviors of the fluid. This work is being discussed in details in Chapter 5 and has been already submitted in [100].

Shallow-water type systems. Shallow water systems for Bingham fluid were not that abundant in research records, but a recent crucial paper by E. Fernandez-Nieto, P. Noble and J.–P. Vila [40] has come to derive fully understandable shallow water models for both power law and Bingham fluids correcting by this all the lapses done at the level of shallow water derivation for previous work as lapses that include neglecting first order velocity profile which is necessary at the level of shallow water approximation, or neglecting corrective terms in the viscous region or in pseud-plug regions. The Bingham shallow-water model by the latter authors in [40] reads

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x(\Lambda_1 h\bar{u}^2 + \frac{\cos\theta}{2Fr^2}h^2 + \Lambda_2) = \frac{1}{\varepsilon Re}\big(\lambda h^* - \frac{h\bar{u}}{\Gamma_1}\big) + R. \end{cases} \tag{6.1}$$

In the above model, Fr^2 and Re denote the Froude and Reynolds numbers respectively, $\lambda = \frac{Re\sin\theta}{Fr^2}$, and ε is the aspect ratio that expresses the smallness of the length compared to the width of the flow. The

coefficients Λ_1, Λ_2 and Γ_1 are given by

$$\begin{aligned}
\Lambda_1 =& \frac{h}{(h+(B/2\lambda))^2}\Big(\frac{6}{5}h + (\frac{9}{4}-\frac{6}{5})\frac{B}{\lambda}\Big),\\
\lambda_2 =& \Big[-\frac{\cos\theta}{Fr^2}(\frac{h_c^2}{2}-\frac{h_c h^*}{2}) + \frac{3}{4}\frac{\cos\theta}{Fr^2}h_c^2\log(\frac{h^*}{3}+\frac{h_c}{2})\\
& -\Lambda^2(\frac{3}{40}h^{*2}h_c^3 - \frac{9}{40}h_c^4 h^* + \frac{27}{80}h_c^5\log(\frac{h^*}{3}+\frac{h_c}{2})) - \lambda^2(\frac{h^{*5}}{75}+\frac{h_c h^{*4}}{10}+\frac{3}{10}h_c^2 h^{*3})\Big],\\
\Gamma_1 =& ((h^*)^2/3) + (Bh^*/2\lambda),
\end{aligned}$$

with B the rescaled Bingham number, $h^* = h - \frac{B}{\sqrt{2}|\lambda|}$ (h^* being the height of the yielded part, see the scheme 6.2), $h_c = h - h^*$, and finally R is a corrective term arising from the normal stress appearing in the first order velocity expansion

$$R = \frac{\pi B^2 |\,\partial_x h|}{2Re\lambda(\frac{h^*}{3}+\frac{h_c}{2})}.$$

The goal of this paper is to derive a consistent bi-viscous shallow-water model (6.9)–(6.14) which from one side can give a good approximate description of shallow water systems for pseudo-plastic fluids, and from the second side it would help recover the Bingham shallow water model by letting one of the viscosities be very large. In fact, doing so we recover the Bingham shallow-water system (6.1) which is the same as the one derived in [40] except for the last term R which is absent in our derivation, i.e we obtain $R = 0$. The absence of the remainder can be attributed to several reasons, the continuity of the stress assumed, the limit process, in particular the uniformity assumed on the variables (mainly gradient of height) with respect to ε and μ_1 and thus neglecting any boundary layers that may occur upon limit. Under the uniformity assumptions we assumed-as a first step- we get a conservative PDE with a right hand side that doesn't change sign. Note that this interesting result comes from the fact that, in our adopted rheology, the strain tensor is continuous, which means that no jump interface is present in the derivation. This is not the case for the Bingham incompressible Navier-Stokes equations. We do believe that adapting this kind of rheology is more convenient for several Bingham type fluids, since technically it helps get rid of the problem of including corrective layers in the approximation and thus directly leads us to a consistent approximation of the model, and physically it has been proven by many scientists as in [68] that Bingham fluids behave in reality as Newtonian fluids with very large viscosities at very low stress values. Add to this, the model obtained is consistent with the Newtonian shallow water model justified in [1] when we take $\mu_1 = \mu_2$.

Starting from Navier Stokes system for a free surface fluid moving down an inclined plane under the effect of gravity, we will follow shallow water approximations in the presence of a slope in the spirit of J.-P. Vila and collaborators (see [*J. of Non-Newtonian Fluid Mec* 2010] and [*Methods Application Analysis* 2010]) in order to get a new model describing the dynamics of such fluid. Such models are simplified models describing the evolution of the **fluid's height h** and the **depth averaged velocity** $\bar{u}$ by merging to the long wave theory with proper scaling techniques. Our derivation is based on a depth averaging technique of the mass and momentum equations accompanied with a perturbation analysis on the shear and velocity profile to close the system. In fact, we claim that we have **two small parameters** in the system, the **aspect ratio** ε and the one relating this latter to the viscosities ratio $\varepsilon\frac{\mu_1}{\mu_2}$. Our adapted perturbation analysis is made in terms of ε only. In particular we claim that if the velocity profile and the shear are denoted u and τ respectively, we assume:

$$u = u_0^{\mu_1,\mu_2} + \varepsilon u_1^{\mu_1,\mu_2} \qquad \text{and} \qquad \tau = \tau_0^{\mu_1,\mu_2} + \varepsilon\tau_1^{\mu_1,\mu_2}.$$

Of course, the main order and first order terms as shown depend on the larger viscosity μ_1. One of the goals after getting the final system is to study the formal limit of the system when μ_1 approaches infinity. At this point, we assume uniformity of the variables with respect to μ_1, and in addition neglect boundary layers by assuming that $\partial_x h$ is uniform with respect to μ_1, this would guarantee that the derivation and the perturbation analysis are made uniform in μ_1 so that the theory is not broken down. In fact, although the first order expansions of u and τ depend on the larger viscosity, yet this doesn't affect the derivation, especially for τ, due to adapting the physical condition 6.8 imposed by Liu and Mei in [66] $\varepsilon\frac{\mu_1}{\mu_2} \leqslant \varepsilon \ll 1$.

The theory of shallow water helps reduce the number of unknowns the system possesses, and preserves

at the same time the physical features of the system by expressing them, especially those on the boundary, throughout the derivation. Another advantage of such approximation is that it is easy to attain the velocity profile at main order, which is a parabolic profile corresponding usually to Nusselt solution. For the model at hand, this theory is carried in a way that allows, in addition to what has been mentioned, to study the asymptotic limit in terms of the larger viscosity μ_1. Hence, the obtained model is valid for both regimes of μ_1: $\mu_1 \sim O(1)$ and $\mu_1 \to \infty$, where the former regime corresponds to a pseudo-plastic and Newtonian approximations, and the latter to a Bingham approximation.

6.2 Starting model, scaling choices and main result

We will start from the two-dimensional incompressible Navier-Stokes system

$$\begin{aligned} \partial_x u + \partial_z w &= 0, \\ \rho(\partial_t u + u\,\partial_x u + w\,\partial_z u) &= -\partial_x p + \rho g \sin\theta + \partial_x \tau_{xx} + \partial_z \tau_{xz}, \\ \rho(\partial_t w + u\,\partial_x w + w\,\partial_z w) &= -\partial_z p - \rho g \cos\theta + \partial_x \tau_{xz} + \partial_z \tau_{zz}, \end{aligned} \tag{6.2}$$

where ρ is the fluid's density, $\vec{v} = (u, w)$ is the velocity field, p is the pressure, τ is the viscous stress tensor, g is the gravity constant and θ the angle of inclination of the fluid that will be assumed fixed and strictly positive. The above system is coupled usually with a constitutive law relating the rheology of the fluid with its deformation behavior, i.e τ with the deformation tensor $Dv = \frac{\nabla v + \nabla^T v}{2}$. In the sequel, we will consider the rheology defined by the following stress already elaborated on in Chapter 5 (see also [100])

$$\tau := \begin{cases} 2\mu_1 Dv & |Dv| \leqslant \dfrac{\tau^*}{2\mu_1}, \\ 2\mu_2 Dv + (1 - \dfrac{\mu_2}{\mu_1})\tau^* \dfrac{Dv}{|Dv|} & |Dv| \geqslant \dfrac{\tau^*}{2\mu_1}. \end{cases} \tag{6.3}$$

As noticed, the considered stress is continuous. As for the boundary conditions, we will consider a no slip boundary condition at the flat bottom $z = 0$ and continuity of the normal stress at the free surface $z = h$, namely

$$\begin{aligned} u|_{z=0} &= w|_{z=0} = 0, \\ \tau_{xz}|_{z=h} &= \frac{\partial_x h}{1 - (\partial_x h)^2}(\tau_{xx} - \tau_{zz})|_{z=h}, \\ p|_{z=h} &= \frac{1}{1 - (\partial_x h)^2}(\tau_{zz} - (\partial_x h)^2 \tau_{xz})|_{z=h}. \end{aligned} \tag{6.4}$$

In addition to the kinematic condition describing the free boundary, which is given by

$$\partial_t h + (u|_{z=h})\,\partial_x h = w|_{z=h}.$$

The previous equations modelize an incompressible pseudo-plastic fluid with free surface falling along a plane with an angle of slope equal to θ.

6.2.1 Adimensionalized System and Boundary Conditions

The rescaling of the system in this case follows the same lines as in the shallow water case for Newtonian fluids with slight yet crucial modifications. As noticed, the stress τ is a continuous function of Dv at $\tau^*/2\mu_1$, which means that our scaling should preserve the continuity of the stress. Another important thing to mention is that **the rescaling of u will be taken by default to depend on a unitary viscosity μ which will also serve in rescaling the viscosity. This would free the scaling from the varying viscosities in the fluid and presents those latter as free parameters in the system, which will help later the comparison with other similar models by modifying the viscosities without affecting the scaling**. Setting the characteristic wavelength L and the characteristic film thickness H, we define the aspect ratio of the film $\varepsilon = H/L$ and we choose the

following non dimensional coordinates

$$x = L\tilde{x}, \quad z = H\tilde{z}, \quad t = \frac{L}{U}\tilde{t},$$

and thus follows the following dimentionless variables

$$h = H\tilde{h}, \quad u = U\tilde{u}, \quad w = \varepsilon U\tilde{w}, \quad p = \rho g H\tilde{\rho}, \quad \mu_i = \mu\tilde{\mu_i},$$

where μ is a unit viscosity and U is the characteristic flow speed of the film which will be defined later following the balance of the momentum equations. As a result of the above non dimentionalizing, we set

$$D\tilde{v} = \begin{pmatrix} \varepsilon\,\partial_{\tilde{x}}\tilde{u} & \frac{1}{2}(\,\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\,\partial_{\tilde{x}}\tilde{w}) \\ \frac{1}{2}(\,\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\,\partial_{\tilde{x}}\tilde{w}) & \varepsilon\,\partial_{\tilde{z}}\tilde{w} \end{pmatrix}, \quad |D\tilde{v}| = \frac{1}{\sqrt{2}}\sqrt{(\,\partial_{\tilde{z}}\tilde{u} + \varepsilon^2\,\partial_{\tilde{x}}\tilde{w})^2 + 4\varepsilon^2(\,\partial_{\tilde{x}}\tilde{u})^2}.$$

Then,

$$Dv = \frac{U}{H}D\tilde{v}, \qquad |Dv| = \frac{U}{H}|D\tilde{v}|,$$

and

$$\tau = \frac{\mu U}{H}\begin{cases} 2\tilde{\mu_1}D\tilde{v} & \text{if} \quad |D\tilde{v}| \leqslant \frac{B}{2\tilde{\mu_1}}, \\ 2\tilde{\mu_2}D\tilde{v} + (1 - \frac{\tilde{\mu_2}}{\tilde{\mu_1}})B\dfrac{D\tilde{v}}{|D\tilde{v}|} & \text{if} \quad |D\tilde{v}| \geqslant \frac{B}{2\tilde{\mu_1}}, \end{cases}$$

where $B = H\tau^*/U$. Thus we define

$$\tilde{\tau} = \begin{cases} 2\tilde{\mu_1}D\tilde{v} & \text{if} \quad |D\tilde{v}| \leqslant \frac{B}{2\tilde{\mu_1}}, \\ 2\tilde{\mu_2}D\tilde{v} + (1 - \frac{\tilde{\mu_2}}{\tilde{\mu_1}})B\dfrac{D\tilde{v}}{|D\tilde{v}|} & \text{if} \quad |D\tilde{v}| \geqslant \frac{B}{2\tilde{\mu_1}}. \end{cases} \tag{6.5}$$

Hence we got the relation between τ and $\tilde{\tau}$ as $\tau = \frac{\mu U}{H}\tilde{\tau}$. In fact, this rheology can be considered more refined than that of Bingham case, as we can get the explicit relation between stress and deformation in the region $|D\tilde{v}| \leqslant \frac{B}{2\tilde{\mu_1}}$. Thus, we are optimistic at this level to obtain a refined study of the Bingham rheology (at the level of calculating the corrective layers) by regarding them as limits of pseudo plastic fluids when μ_1 tends to infinity. Now using the fact that the kinematic viscosity, Froude number and Reynold number are respectively given by

$$\nu = \frac{\mu}{\rho}, \qquad \mathrm{Fr}^2 = \frac{U^2}{gH} \quad \text{and} \qquad \mathrm{Re} = \frac{HU}{\nu}$$

and recalling that λ is defined by

$$\lambda = \frac{Re\sin\theta}{Fr^2},$$

we get the following rescaled Navier Stokes system (dropping the tilde)

$$\begin{aligned} \partial_x u + \partial_z w &= 0, \\ \partial_t u + u\,\partial_x u + w\,\partial_z u &= -\frac{\partial_x p}{Fr^2} + \frac{\lambda}{\varepsilon Re} + \frac{1}{Re}\partial_x\tau_{xx} + \frac{1}{\varepsilon Re}\partial_z\tau_{xz}, \\ \partial_t w + u\,\partial_x w + w\,\partial_z w &= -\frac{\partial_z p}{\varepsilon^2 Fr^2} - \frac{\lambda\cot\theta}{\varepsilon^2 Re} + \frac{1}{\varepsilon Re}\partial_x\tau_{xz} + \frac{1}{\varepsilon^2 Re}\partial_z\tau_{zz}. \end{aligned} \tag{6.6}$$

The rescaled boundary conditions at $z = 0$ and $z = h$ become respectively

$$\begin{aligned} &u|_{z=0} = w|_{z=0} = 0, \\ &\tau_{xz}|_{z=h} = \frac{\varepsilon\,\partial_x h}{1-\varepsilon^2(\,\partial_x h)^2}\big(\tau_{xx}-\tau_{zz}\big)|_{z=h}, \\ &p|_{z=h} = \frac{\sin\theta}{\lambda}\frac{1}{1-\varepsilon^2(\,\partial_x h)^2}(\tau_{zz}-\varepsilon^2(\,\partial_x h)^2\tau_{xz})|_{z=h}. \end{aligned} \tag{6.7}$$

In fact, understanding boundary conditions triggers a priori expecting the behavior of the fluid and the physical phenomenon that would result in such case. To be more specific, the fluid in our case will have some (interface/separation height) h^* that will divide the region where the behavior of the high viscosity dominates for low deformations (pseudo plug region: upper layer), and the yielded region where low viscosity behavior dominates (lower layer) for high deformations. Thus, we need information on the velocity, stress and pressure at this imaginary/fake interface. By analogy with τ, we conclude that h^* should occur also when $|Dv| = \frac{B}{2\mu_1}$. Assuming the continuity of the stress and velocity at $|Dv| = \frac{B}{2\mu_1}$, we suggest imposed boundary conditions at the interface. We will assume the continuity of velocity, pressure and stress at h^*, thus the boundary conditions will be imposed from the second layer. Now by returning to regime of higher viscosity which corresponds to the upper layer of the fluid being the affected part by small deformation: $|Dv| \leqslant \frac{B}{2\mu_1}$, we are assuming that higher viscosity regime occurs in the region where $h^* < z < h$ which is a compatible assumption to have a stability (higher viscosity up, lower viscosity down).

6.2.2 Scaling and expansion

For simplicity, in this paper, we assume the following hypothesis on the adimensional numbers

$$\begin{gathered} \mathrm{Fr} = O(1), \qquad \mathrm{Re} = O(1), \qquad \lambda = O(1), \\ \frac{\varepsilon\mu_1}{\mu_2} \ll 1 \end{gathered} \tag{6.8}$$

In conclusion, we are dealing with two assumptions : the smallness of the aspect ratio ε, and smallness of the ratio in (6.8) which represents a comparison between the viscosity ratio to the aspect ratio of the theory. These two assumptions are present in the work of Liu and Mei concerning the lubrication theory [66]. It is important to remark that we consider inhere some special hypothesis on the Froude and Reynolds dimensionless numbers to simplify the calculus. It is, of course, possible to consider more general setting as in [40] and [101], but it's not the objective of the study to be as general as possible to cover different types of flows. The main objective in our paper is to show that if we consider a piece wise description of a pseudo-plastic flow, we can get a shallow-water type system in the plastic regime with a better structure than the one derived from Bingham system with free surface, as done for instance in [40].

Remark 8. *From the relation between λ, Re and Fr^2 we can deduce the characteristic fluid speed up to a constant*

$$U = \frac{gH^2\sin\theta}{\nu\lambda}.$$

6.2.3 Main result – Shallow Water type systems

In this paper, we formally derive the following shallow-water type system from the two-viscosities fluid system with free boundary

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x(\Lambda_1^P h\bar{u}^2 + \frac{\cos\theta}{2\mathrm{Fr}^2}h^2 + \Lambda_2^P) = \frac{1}{\varepsilon Re}(\lambda h^* - \frac{h\bar{u}}{\Gamma_1^P}) + R^P \end{cases} \tag{6.9}$$

where, defining $h^* = h - \frac{B}{\sqrt{2}|\lambda|}$ and $h_c = h - h^*$. The unknowns Λ_1^P, Λ_2^P, Γ_1^P and R^P are given by

$$\Lambda_1^P = h\frac{\frac{12}{\mu_1^2}(2h_c^5 - 5h_c^4 h + 5h_c^2 h^3) - \frac{15h_c}{\mu_1\mu_2}(h_c - h)^2(3h_c^2 - 2h_c h - 5h^2) + \frac{3}{\mu_2^2}(h_c - h)^4(7h_c + 8h)}{5\big(\frac{1}{\mu_1}(h_c^3 - 3h_c h^2) + \frac{1}{\mu_2}(3h_c h^2 - 2h^3 - h_c^3)\big)^2}, \tag{6.10}$$

$$\Lambda_2^P = -\frac{\cos\theta}{Fr^2}\frac{h^2}{2} - \lambda^2\bigg(-\frac{1}{\mu_1^2}\frac{h_c^4}{3}h^* + \frac{1}{\mu_2^2}\big(-\frac{h_c h^{*4}}{4} - \frac{h^{*4}}{15} - \frac{h_c^2 h^{*3}}{3}\big) + \frac{1}{\mu_1\mu_2}\big(-\frac{h_c^2 h^{*3}}{3} - \frac{h_c h^{*4}}{12} - \frac{2}{3}h_c^3 h^{*2}\big)\bigg), \tag{6.11}$$

$$\Gamma_1^P = \frac{1}{\mu_2}\big(\frac{h^{*2}}{3} + \frac{h^* h_c}{2}\big) + \frac{1}{\mu_1}\big(\frac{h^3}{3h^*} - \frac{hh^*}{2} + \frac{h^{*2}}{6}\big) \tag{6.12}$$

$$R^P = \frac{1}{Re}\big(\frac{1}{\Gamma_1}\int_0^h u_1\, dz\big) \tag{6.13}$$

with

$$\begin{aligned}\int_0^h u_1\, dz = &-\frac{Re}{\mu_1}\lambda^2\,\partial_x h\bigg[-\frac{2}{15}\frac{1}{\mu_1^2}h_c^6 + \frac{1}{\mu_1\mu_2}\big(-\frac{7}{15}h_c^5 h^* - \frac{1}{6}h_c^4 h^{*2}\big) + \frac{1}{\mu_2^2}\big(-\frac{1}{3}h_c^4 h^{*2} - \frac{1}{6}h_c^3 h^{*3}\big)\bigg]\\
&-\frac{Re}{\mu_2}\lambda^2\,\partial_x h\bigg[\frac{1}{\mu_1^2}\big(-\frac{h_c^5 h^*}{3} - \frac{h_c^4 h^{*2}}{6}\big) + \frac{1}{\mu_1\mu_2}\big(-\frac{4}{3}h_c^4 h^{*2} - 3h_c^3 h^{*3} - \frac{2}{3}h_c^2 h^{*4} - \frac{2}{15}h_c h^{*5}\big)\\
&+\frac{1}{\mu_2^2}\big(-h_c^3 h^{*3} - \frac{4}{3}h_c^2 h^{*4} - \frac{2}{3}h_c h^{*5} - \frac{2}{15}h^{*6}\big)\bigg] - \frac{\lambda\cot\theta}{\mu_1}\,\partial_x h\frac{h_c^3}{3}\\
&-\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h\big(\frac{h^{*3}}{3} + h^{*2}h_c + h^* h_c^2\big).\end{aligned} \tag{6.14}$$

6.2.3.1 Comparison with Bingham model (6.1)

The derivation leading to the final shallow water system will be uniform with respect to μ_1 under the scaling assumptions, especially if this latter will be made to vary very large, and for that, we would be able at this level to obtain formally the limit shallow water model for Bingham rheology, simply by letting μ_1 tends to infinity (or $\frac{1}{\mu_1} = \eta \to 0$) and testing the different coefficients in the system. For simplicity also, we will set μ_2 equal to 1, and the sup-indices B to Bingham case.

- To begin with, we will start by Λ_1^P, we obtain as a limit

$$\Lambda_1^B = h\frac{(h_c - h)^4(7h_c + 8h)}{5\big((3h_c h^2 - 2h^3 - h_c^3)\big)^2} = \frac{3}{5}h\frac{7h_c + 8h}{(h_c + 2h)^2} = \frac{h}{(h + \frac{h_c}{2})^2}\big(\frac{6}{5}h + \frac{21}{20}h_c\big).$$

  which is the same expression as that figured out in [40].

- It is easy to check for Γ_1^P, we get the limit

$$\Gamma_1^B = \frac{h^{*2}}{3} + \frac{h^* h_c}{2}$$

  coinciding again with the corresponding one for Bingham case in [40].

- Now, Lets compute Λ_2^B, in fact both the limits of Λ_2^P and the limit of R^P contribute to Λ_2^B, as for the limit of Λ_2^P, which we will denote $\Lambda_2^{B,1}$, we get

$$\Lambda_2^{B,1} = -\frac{\cos\theta}{Fr^2}\frac{h^2}{2} + \lambda^2\bigg(\frac{h_c h^{*4}}{4} + \frac{h^{*5}}{15} + \frac{h_c^2 h^{*3}}{3}\bigg).$$

- At this point, lets get a simpler expression of R^P, as it contributes to Λ_2^B in this limit as we will prove later. Denoting $R^B = -\partial_x \Lambda_2^{B,2}$ by the limit of R^P we get

$$
\begin{aligned}
R^B &= \frac{1}{Re}\Big(\frac{1}{\Gamma_1^B}\int_0^h u_1^B\, dz\Big)\\
&= -\frac{\cos\theta}{Fr^2}\,\partial_x h\frac{\frac{h^{*2}}{3}+h^*h_c+h_c^2}{\frac{h^*}{3}+\frac{h_c}{2}} + \lambda^2\,\partial_x h\frac{h_c^3h^{*2}+\frac{4}{3}h_c^2h^{*3}+\frac{2}{3}h_ch^{*4}+\frac{2}{15}h^{*5}}{\frac{h^*}{3}+\frac{h_c}{2}}\\
&= -\frac{3\cos\theta}{Fr^2}\,\partial_x h\Big[\frac{h^*}{3}+\frac{h_c}{2}+\frac{1}{12}\frac{h_c^2}{\frac{h^*}{3}+\frac{h_c}{2}}\Big] + \lambda^2\,\partial_x h\Big[\frac{2h^{*4}}{5}+\frac{7}{5}h^{*3}h_c+\frac{19}{10}h_c^2h^{*2}+\frac{3}{20}h^*h_c^3-\frac{9}{40}h_c^4+\frac{9}{80}\frac{h_c^5}{\frac{h^*}{3}+\frac{h_c}{2}}\\
&= -\frac{c}{Fr^2}\,\partial_x\Big[\frac{h^{*2}}{2}+\frac{3}{2}h_ch^*\Big] - \frac{3}{4}\frac{c}{Fr^2}h_c^2\,\partial_x\Big[\log(\frac{h^*}{3}+\frac{h_c}{2})\Big]\\
&\quad + \lambda^2\,\partial_x\Big[\frac{2h^{*5}}{25}+\frac{7}{20}h^{*4}h_c+\frac{19}{30}h_c^2h^{*3}+\frac{3}{40}h^{*2}h_c^3-\frac{9}{40}h_c^4h^*+\frac{27}{80}h_c^5-\log(\frac{h^*}{3}+\frac{h_c}{2})\Big]\\
&:= -\partial_x\Lambda_2^{B,2}.
\end{aligned}
$$

Thus adding $-R_B$ to $\partial_x\Lambda_2^{B,1}$, we obtain

$$
\begin{aligned}
-R^B + \partial_x\Lambda_2^{B,1} &= \partial_x\Lambda_2^{B,1} + \partial_x\Lambda_2^{B,2}\\
&= \partial_x\Big[-\frac{\cos\theta}{Fr^2}(\frac{h_c^2}{2}-\frac{h_ch^*}{2})+\frac{3}{4}\frac{\cos\theta}{Fr^2}h_c^2\log(\frac{h^*}{3}+\frac{h_c}{2})\\
&\quad -\lambda^2(\frac{3}{40}h^{*2}h_c^3-\frac{9}{40}h_c^4h^*+\frac{27}{80}h_c^5\log(\frac{h^*}{3}+\frac{h_c}{2}))-\lambda^2(\frac{h^{*5}}{75}+\frac{h_ch^{*4}}{10}+\frac{3}{10}h_c^2h^{*3})\Big].
\end{aligned}
$$

We define Λ_2^B (up to a constant) such that

$$
\begin{aligned}
\partial_x\Lambda_2^B &= \partial_x\Lambda_2^{B,1} + \partial_x\Lambda_2^{B,2}\\
&= \partial_x\Big[-\frac{\cos\theta}{Fr^2}(\frac{h_c^2}{2}-\frac{h_ch^*}{2})+\frac{3}{4}\frac{\cos\theta}{Fr^2}h_c^2\log(\frac{h^*}{3}+\frac{h_c}{2})\\
&\quad -\lambda^2(\frac{3}{40}h^{*2}h_c^3-\frac{9}{40}h_c^4h^*+\frac{27}{80}h_c^5\log(\frac{h^*}{3}+\frac{h_c}{2}))-\lambda^2(\frac{h^{*5}}{75}+\frac{h_ch^{*4}}{10}+\frac{3}{10}h_c^2h^{*3})\Big].
\end{aligned}
\tag{6.15}
$$

Hence we recover the same system than in [40] except that we don't have the term R ($R=0$ in our case) and thus the system is conservative. Note that as mentioned before, the absence of R may be attributed to the uniformity assumption with respect to μ_1 on $\partial_x h$ that otherwise may lead to assuming a boundary layer and thus engaging the remainder R in the derivation.

6.2.3.2 Comparison with Newtonian Model

On the contrary to the previous case, and referring back to figure 6.1b, both μ_1 and μ_2 at this point will be equal and $O(1)$. The Bingham number B is set to zero (and so is h_c). In particular choosing $\mu_1 = \mu_2 = 1$ (without loss of generality) and returning to the obtained final system, we get

$$
\begin{cases}
\partial_t h + \partial_x(h\bar{u}) = 0,\\
\partial_t(h\bar{u}) + \partial_x(\Lambda_1^N h\bar{u}^2 + \frac{\cos\theta}{2Fr^2}h^2 + \Lambda_2^N) = \frac{1}{\varepsilon Re}\big(\lambda h^* - \frac{h\bar{u}}{\Gamma_1^N}\big) + R^N.
\end{cases}
\tag{6.16}
$$

where the coefficients-indexed by N to refer to Newtonian- simplify as follows

- $\Lambda_1^N = \frac{6}{5}$,
- $\Gamma_1^N = \frac{h^2}{3}$,

- $\Lambda_2^N = -\frac{\cos\theta}{Fr^2}\frac{h^2}{2} + \lambda^2\frac{h^6}{15}$,
- $R^N = -\frac{\cos\theta}{Fr^2}\,\partial_x(\frac{h^2}{2}) + \lambda^2\,\partial_x(\frac{2h^5}{25})$.

Collecting similar terms together, the system may be rewritten

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x(\frac{6}{5}h\bar{u}^2 + \frac{\cos\theta}{2Fr^2}h^2 - \lambda^2\frac{h^5}{75}) = \frac{1}{\varepsilon Re}(\lambda h - \frac{3\bar{u}}{h}). \end{cases} \tag{6.17}$$

This is the shallow-water equation for newtonian flow that has been mathematically justified in [1].

6.3 Shallow-water equation derivation.

As usual if we integrate the divergence free condition with respect to the vertical coordinate from the bottom to the free surface and if we use the boundary conditions, we get

$$\partial_t h + \partial_x(h\bar{u}) = 0 \qquad \text{where} \qquad \bar{u} = \frac{1}{h}\int_0^h u(t,x,z)\,dz. \tag{6.18}$$

In order to get a closed system with respect to $(h,\bar{u})$, we need a supplementary equation governing $\bar{u}$, we integrate the horizontal component of the momentum equation in the vertical variable from the bottom to the free surface h. The horizontal component of the momentum equation reads

$$\partial_t u + u\,\partial_x u + w\,\partial_z u = -\frac{\partial_x p}{Fr^2} + \frac{\lambda}{\varepsilon Re} + \frac{1}{Re}\partial_x\tau_{xx} + \frac{1}{\varepsilon Re}\partial_z\tau_{xz}.$$

Let us first deal with the mean of the left-hand side given by

$$LHS = \int_0^h \partial_t u + u\,\partial_x u + w\,\partial_z u.$$

We have

$$\begin{aligned} \int_0^h \partial_t u + u\,\partial_x u + w\,\partial_z u\;dz &= \big[\int_0^h \partial_t u\;dz + \int_0^h u\,\partial_x u\;dz + \int_0^h w\,\partial_z u\;dz\big] \\ &= \partial_t(\int_0^h u\;dz) - \partial_t hu(h) + \int_0^h u\,\partial_x u\;dz - \int_0^h \partial_z wu\;dz + u(h)w(h) \\ &= \partial_t(h\bar{u}) - \partial_t hu(h) + 2\int_0^h u\,\partial_x u\;dz + u(h)w(h) \\ &= \partial_t(h\bar{u}) + \partial_x(\int_0^h u^2\;dz) - \partial_t hu(h) - \partial_x hu(h)^2 + u(h)w(h). \end{aligned}$$

Notice that from the kinetic boundary condition we have

$$-u(h)[\,\partial_t h + \partial_x hu(h) - w(h)] = 0.$$

Hence

$$LHS = \partial_t(h\bar{u}) + \partial_x(\int_0^h u^2\;dz).$$

To close the expression above in terms of $\bar{u}$, we will prove that

$$\int_0^h u^2\;dz = \Lambda_1^P h(\bar{u})^2 + O(\varepsilon) \tag{6.19}$$

for some specific Λ_1^P which depends on h, $|\lambda|$ and B. To do so, we will use the perturbation analysis (in terms of ε) on the velcity profile discussed in Subsection 6.3.1 where we assumed that the velocity profile may be approximated as follows $u = u_0^{\mu_1,\mu_2} + O(\varepsilon)$ and that the integral of $\int_0^h (u_0^{\mu_1,\mu_2})^2$ with respect to z may be calculated explicity in terms of $(\int_0^h u_0)^2\, dz$.

As for the right-hand side of the horizontal momentum equation. we have that

$$\begin{aligned} RHS &= \int_0^h -\frac{\partial_x p}{Fr^2} dz + \int_0^h \frac{\partial_x \tau_{xx}}{Re}\, dz + \frac{1}{\varepsilon Re}\int_0^h (\lambda + \partial_z \tau_{xz})\, dz \\ &= -\partial_x\big[\int_0^h \frac{p}{Fr^2}\, dz\big] + \partial_x h \frac{p(h)}{Fr^2} + \partial_x\big[\int_0^h \frac{\tau_{xx}}{Re}\, dz\big] - \partial_x h \frac{\tau_{xx}(h)}{Re} + \frac{1}{\varepsilon Re}\big[\lambda h + \tau_{xz}(h) - \tau_{xz}(0)\big] \\ &= \partial_x\big[\int_0^h \frac{\tau_{xx}}{Re} - \frac{p}{Fr^2}\, dz\big] + + \frac{1}{\varepsilon Re}\big[\lambda h + \tau_{xz}(h) - \tau_{xz}(0)\big] + \partial_x h\big[h\frac{p(h)}{Fr^2} - \frac{\tau_{xx}(h)}{Re}\big]. \end{aligned}$$

Using the boundary conditions, we have

$$\frac{p(h)}{Fr^2} = -\frac{\tau_{xx}(h)}{Re} + O(\varepsilon^2),$$

$$\tau_{xz}(h) = 2\varepsilon\tau_{xx}(h) + O(\varepsilon^2).$$

However, if τ_{xx} and τ_{xz} depend- in the worst scenario possible- on μ_1, then we can write

$$\frac{p(h)}{Fr^2} = -\frac{\tau_{xx}(h)}{Re} + O(\varepsilon^2 + \varepsilon^2\frac{\mu_1}{\mu_2}),$$

$$\tau_{xz}(h) = 2\varepsilon\tau_{xx}(h) + O(\varepsilon^2 + \varepsilon^2\frac{\mu_1}{\mu_2}).$$

This is in fact is assured when applying the perturbation analysis an calculating the main and first order of the shear (the same thing applies for the velocity profile). We will check later that the pressure at main order satisfies the usual hydrostatic approximation namely

$$p = \cos\theta(h - z) + O(\varepsilon + \frac{\varepsilon\mu_1}{\mu_2}), \tag{6.20}$$

and that

$$\frac{\tau_{xx}}{Re} = O(\varepsilon + \frac{\varepsilon\mu_1}{\mu_2}).$$

Thus

$$\frac{\tau_{xz}(h)}{\varepsilon Re} = O(\varepsilon + \varepsilon\frac{\mu_1}{\mu_2}).$$

The right hand side is finally rendered

$$RHS = -\frac{\cos\theta}{Fr^2} h\,\partial_x h + \frac{1}{\varepsilon Re}\big[\lambda h - \tau_{xz}(0)\big] + O(\varepsilon + \varepsilon\frac{\mu_1}{\mu_2}).$$

Thus, in conclusion, we get the following averaged horizontal momentum equation

$$\partial_t(h\bar{u}) + \partial_x(\int_0^h u^2\, dz + \frac{\cos\theta}{Fr^2}h^2) = \frac{1}{\varepsilon Re}\big[\lambda h - \tau_{xz}(0)\big] + O(\varepsilon + \varepsilon\frac{\mu_1}{\mu_2}). \tag{6.21}$$

The above system of course is not closed, for that we will use a perturbation analysis to close it. This is done through the following subsections where we will calculate the main and first order expansions of the shear and the velocity profile to be able to express $\int_0^h u^2\, dz$ in terms of $h(\bar{u})^2$ (Subsection 6.3.1) and $\tau_{xz}(0)$ at first order- since it is a singular term- in terms of h, $\partial_x h$, $\bar{u}$, $|\lambda|$ and B (Subsection 6.3.2). By that we get a shallow-water-type system which we can play with by letting μ_1 and μ_2 vary to recover the

regimes of a shallow-water system for Bingham flow or a shallow-water system for Newtonian flow.

6.3.1 Main order profile and hydrostatic pressure constraint.

Let us calculate the main order profile of the velocity and the strain tensor τ. We will show that there is no jump on these quantities contrarily to the calculations made in [40] starting with the incompressible Bingham system. We will deal with the formal limit in two different regimes depending on the threshold of deformation, which will interpret also the physical behavior of the fluid in each regime.

Asymptotic expansions of the pressure, stress tensor and velocity profile. Following the classical perturbation done the velocity profile in terms of the aspect ratio, we will adapt the following ansatz

$$u = u_0^{\mu_1,\mu_2} + \varepsilon u_1^{\mu_1,\mu_2} + O(\varepsilon^2) \qquad \tau = \tau_0^{\mu_1,\mu_2} + \varepsilon \tau_1^{\mu_1,\mu_2} + O(\varepsilon^2 + \frac{\varepsilon^2 \mu_1}{\mu_2})$$

$$p = p_0^{\mu_1,\mu_2} + O(\varepsilon + \frac{\varepsilon \mu_1}{\mu_2}). \tag{6.22}$$

The case $|Dv| \leqslant B/2\mu_1$. Note that at the free surface, this constraint is satisfied at the main order due to the stress free condition at the surface. Using the continuity of the velocity, we therefore define h^* as the minimal depth where the flow may exhibit Newtonian behavior (this value will be calculated later-on). More precisely, for all z such that $h^* \leqslant z \leqslant h$, we fall in Newtonian-like behavior of the stress and the stress in this case is given by

$$\tau = 2\mu_1 Dv.$$

Formally speaking, and under the above ansatz (asymptotic expansion of u, τ_{xz} and p), we will have the following main order PDE system to obtain the main order approximation of the velocity profile (dropping the indices μ_i from the expansion for simplicity)

$$\begin{aligned} &\partial_x u_0 + \partial_z w_0 = 0, \\ &\partial_z \tau_{xz}^0 + \lambda = 0, \\ &\partial_z p_0 = -\cos\theta + \frac{Fr^2}{Re} \partial_z \tau_{zz}^0, \end{aligned}$$

as well as the boundary conditions using the fact that the value of u_0 is imposed at h^* from the lower velocity profile

$$u_0|_{z=h^*} = u^*.$$

Remark that in the proceeding calculations h^* and u^* will be explicitly defined in term of $h, B, |\lambda|, \mu_1$ and μ_2 as we will see by imposing the continuity of the velocity. It will be noticed as well that there is no jump of the stress obtained after all. Integrating now for p_0 and τ_{xz}^0 using the boundary conditions, we get that

$$\tau_{xz}^0 = \tau_{xz}^0(h) + \lambda(h - z),$$

and

$$p_0 = \cos\theta(h - z) + p_0(h) + \frac{Fr^2}{Re}(\tau_{zz}^0 - \tau_{zz}^0(h)).$$

From boundary conditions, we can write again

$$p_0(h) = \frac{Fr^2}{Re} \tau_{zz}^0(h),$$

$$\tau_{xz}^0(h) = 0,$$

and so we get

$$\tau^0_{xz} = \tau^0_{xz}(h) + \lambda(h-z) \qquad \text{and} \qquad p_0 = \cos\theta(h-z) + \frac{Fr^2}{Re}\tau^0_{zz}.$$

But, from the definition of τ in this case, i.e in the case $|Dv| \leqslant B/2\mu_1$ we know that

$$\tau_{xz} = \mu_1(\,\partial_z u + \varepsilon^2\,\partial_x w) \qquad \text{and} \qquad \tau_{zz} = -\tau_{xx} = -2\varepsilon\mu_1\,\partial_x u.$$

This implies using condition 6.8 that

$$\tau^0_{xz} = \mu_1\,\partial_z u_0 \qquad \text{and} \qquad \tau^0_{zz} = 0,$$

and as a result we get

$$\tau^0_{xz} = \lambda(h-z) \qquad \text{and} \qquad p_0 = \cos\theta(h-z).$$

Therefore

$$\partial_z u_0 = \frac{\lambda}{\mu_1}(h-z).$$

Hence integrating again between h^* and z, for all $h^* \leqslant z \leqslant h$, we get

$$u_0 - u^* = \frac{\lambda}{\mu_1}(h^* z + h_c z - \frac{z^2}{2} - \frac{h^{*2}}{2} - h_c h^*),$$

where we have defined $h_c = h - h^*$. Plugging the value of $|\,\partial_z u|$ in the bound relation at main order yields

$$\frac{|\,\partial_z u_0|}{\sqrt{2}} \leqslant \frac{B}{2\mu_1},$$

and therefore using the expression of u, we get that

$$z \geqslant h - \frac{B}{\sqrt{2}|\lambda|}.$$

This implies that $h^* = \max(0, h - \frac{B}{\sqrt{2}|\lambda|})$ and thus $h_c = \frac{B}{\sqrt{2}|\lambda|}$.

The case $|Dv| \geqslant B/2\mu_1$. The limit problem in this case will be given by

$$\begin{aligned}
&\partial_x u_0 + \partial_z w_0 = 0,\\
&\partial_z \tau^0_{xz} + \lambda = 0,\\
&\partial_z p_0 = -\cos\theta + \frac{Fr^2}{Re}\,\partial_z \tau^0_{zz}.
\end{aligned}$$

But now we have to take care of the boundary conditions. The assumed region now should fall in $[0, h^*]$, hence from the continuity assumption of the solution, and using the main order approximations from the previous region, we will assume the following boundary conditions

$$\begin{aligned}
u_0 &= 0 \qquad && z = 0,\\
p_0 &= \cos\theta h_c \qquad && z = h^*,\\
\tau^0_{xz} &= \lambda(h - h^*) \qquad && z = h^*.
\end{aligned}$$

In this regime, τ is given by

$$\tau = 2\mu_2\,\mathrm{D}\,v + B(1 - \frac{\mu_2}{\mu_1})\frac{\mathrm{D}\,v}{|\,\mathrm{D}\,v|},$$

so that

$$\tau_{zz} = 2\mu_2 \varepsilon\, \partial_z w + B(1 - \frac{\mu_2}{\mu_1}) \frac{\varepsilon\, \partial_z w}{|\,\mathrm{D}\, v|} \sim O(\varepsilon).$$

So at main order we get

$$\tau^0_{zz} = 0.$$

Again, upon integration between $z \leqslant h^*$ and h^*, we get that

$$p_0 = \cos\theta(h - z) \qquad \text{and} \qquad \tau^0_{xz} = \lambda(h - z).$$

On the other hand, we know that τ^0_{xz} is defined by

$$\tau^0_{xz} = 2\mu_2\, \partial_z u_0 + (1 - \frac{\mu_2}{\mu_1}) \frac{B}{\sqrt{2}} \frac{\partial_z u_0}{|\,\partial_z u_0|}. \tag{6.23}$$

Thus at main order we can write

$$2\mu_2\, \partial_z u_0 + (1 - \frac{\mu_2}{\mu_1}) \frac{B}{\sqrt{2}} \frac{\partial_z u_0}{|\,\partial_z u_0|} = \lambda(h - z) \tag{6.24}$$

and hence

$$\begin{aligned} \partial_z u_0 &= \frac{\lambda}{\mu_2}(h - (1 - \frac{\mu_2}{\mu_1}) \frac{B}{\sqrt{2}|\lambda|} - z) \\ &= \frac{\lambda}{\mu_2}(h^* - z) + \frac{\lambda}{\mu_1} h_c. \end{aligned}$$

Integrating between 0 and z, we get finally that for $0 \leqslant z \leqslant h^*$, the velocity at main order is given by

$$u_0 = \frac{\lambda}{\mu_2}(h^* z - \frac{z^2}{2}) + \frac{\lambda}{\mu_1} h_c z. \tag{6.25}$$

As a consequence, we can get now the value of the velocity at h^*, due to the continuity property of the velocity field, thus

$$u^* = \frac{\lambda}{\mu_2} \frac{h^{*2}}{2} + \frac{\lambda}{\mu_1} h_c h^*.$$

To sum up, at main order we recovered the hydrostatic pressure

$$p_0 = \cos\theta(h - z). \tag{6.26}$$

and the following main order velocity profile

$$u_0 = \begin{cases} \frac{\lambda}{\mu_2}(h^* z - \frac{z^2}{2}) + \frac{\lambda}{\mu_1} h_c z & 0 \leqslant z \leqslant h^*, \\ \frac{\lambda}{\mu_2} \frac{h^{*2}}{2} + \frac{\lambda}{\mu_1}(h^* z + h_c z - \frac{z^2}{2} - \frac{h^{*2}}{2}) & h^* \leqslant z \leqslant h. \end{cases} \tag{6.27}$$

Remark 9. *It is good to remark here that the type of stress in the bi-viscous case creates a smooth enough velocity profile at main order. In particular, the stress at main order is itself continuous, and this blows up the theory of lubrication/shallow water attempts in the non-Newtonian case which provoked the calculation of corrective layers of orders of powers of ε in the vicinity of the fake threshold h^*. This corrective layer came as a solution of the discontinuity of $\partial_z u$ at main order between the pseudo plug zone and the strained zone, and thus, the degradation into a layer of order ε of $\partial_z u$ was a must to attain*

consistent results with the long wave theory. However, in our case, the continuity breaks down this need, so the procedure will follow the same general lines of the derivation in Newtonian case ([1]), yet the kind of stress and the model requires slight differences in details.

Proof of (6.19). At this level, we are able to calculate the depth averaged velocity at main order $\bar{u}_0 = \int_0^h u_0\, dz$ (or equivalently the discharge rate $q = h\bar{u}$), and thus we can calculate $\int_0^h u_0^2\, dz$. We get

$$\begin{aligned} h\bar{u}^0 = q^0 &= \int_0^h u_0\, dz \\ &= \int_0^{h^*} \frac{\lambda}{\mu_2}(h^* z - \frac{z^2}{2}) + \frac{\lambda}{\mu_1} h_c z\, dz + \int_{h^*}^h \frac{\lambda}{\mu_2}\frac{h^{*2}}{2} + \frac{\lambda}{\mu_1}(h^* z + h_c z - \frac{z^2}{2} - \frac{h^{*2}}{2})\, dz \\ &= \frac{\lambda}{\mu_2}(\frac{hh^{*2}}{2} - \frac{h^{*3}}{6}) + \frac{\lambda}{\mu_1}(\frac{h^3}{3} - \frac{hh^{*2}}{2} + \frac{h^{*3}}{6}). \end{aligned} \tag{6.28}$$

Now using the above expression and the main order profile, we can show first by simple but tedious calculations that

$$\begin{aligned} \int_0^h u_0^2\, dz = \frac{\lambda^2}{\mu_1^2}\left(\frac{h_c^2 h^3}{3} - \frac{h h_c^4}{3} + \frac{2}{15}h_c^5\right) + \frac{\lambda^2}{\mu_1\mu_2}(h-h_c)^2\left(\frac{5}{12}h_c h^2 + \frac{h h_c^2}{6} - \frac{h_c^2}{4}\right) \\ + \frac{\lambda^2}{\mu_1^2}\left(\frac{2}{15}h_c^5 + \frac{h^3 h_c^2}{3} - \frac{h h_c^4}{3}\right). \end{aligned}$$

And

$$(h\bar{u}_0)^2 = \left(\frac{\lambda}{\mu_2}(\frac{hh^{*2}}{2} - \frac{h^{*3}}{6}) + \frac{\lambda}{\mu_1}(\frac{h^3}{3} - \frac{hh^{*2}}{2} + \frac{h^{*3}}{6})\right)^2.$$

Hence we can rewrite

$$\begin{aligned} \int_0^h u_0^2\, dz &= h\frac{\int_0^h u_0^2\, dz}{(h\bar{u}_0)^2} h(\bar{u}_0)^2 \\ &= h\frac{\int_0^h u_0^2\, dz}{(h\bar{u}_0)^2}\big(h(\bar{u})^2 + O(\varepsilon)\big). \end{aligned}$$

Hence, we define Λ_1^P (which is already stated in Identity 6.19) by

$$\begin{aligned} \Lambda_1^P &= h\frac{\int_0^h u_0^2\, dz}{(h\bar{u}_0)^2} \\ &= h\frac{\frac{12}{\mu_1^2}(2h_c^5 - 5h_c^4 h + 5h_c^2 h^3) - \frac{15h_c}{\mu_1\mu_2}(h_c-h)^2(3h_c^2 - 2h_c h - 5h^2) + \frac{3}{\mu_2^2}(h_c-h)^4(7h_c+8h)}{5\big(\frac{1}{\mu_1}(h_c^3 - 3h_c h^2) + \frac{1}{\mu_2}(3h_c h^2 - 2h^3 - h_c^3)\big)^2}. \end{aligned} \tag{6.29}$$

6.3.2 Calculating $\tau_{xz}(0)$

Now to close the system, there remains to calculate $\tau_{xz}(0)$. Since this term is a singular term, we will have to expand our asymptotic expansion of τ up to first order as shwon in (6.3.1), so that we rewrite

$\tau_{xz}(0)$ as

$$\begin{aligned}\tau_{xz}(0) &= \tau^0_{xz}(0) + \varepsilon\tau^1_{xz}(0) + O(\varepsilon^2 + \frac{\varepsilon^2\mu_1}{\mu_2}) \\ &= \lambda h + \varepsilon\tau^1_{xz}(0) + O(\varepsilon^2 + \frac{\varepsilon^2\mu_1}{\mu_2}) \\ &= \lambda h^* + \lambda h_c + \varepsilon\tau^1_{xz}(0) + O(\varepsilon^2 + \frac{\varepsilon^2\mu_1}{\mu_2}) \\ &= \frac{h\bar{u}_0}{\Gamma^P_1} + \lambda h_c + \varepsilon\tau^1_{xz}(0) + O(\varepsilon^2 + \frac{\varepsilon^2\mu_1}{\mu_2}),\end{aligned}$$

where we define

$$\Gamma^P_1 = \frac{h\bar{u}_0}{\lambda h^*} = \frac{1}{\mu_2}(\frac{hh^*}{2} - \frac{h^{*2}}{6}) + \frac{1}{\mu_1}(\frac{h^3}{3h^*} - \frac{hh^*}{2} + \frac{h^{*2}}{6}). \tag{6.30}$$

Finally, noting that $h\bar{u}_0 = h\bar{u} - \varepsilon\int_0^h u_1\, dz + O(\varepsilon^2)$, we get

$$\tau_{xz}(0) = \frac{1}{\Gamma^P_1}h\bar{u} + \lambda h_c + \varepsilon(\tau^1_{xz}(0) - \frac{1}{\Gamma^P_1}\int_0^h u_1\, dz) + O(\varepsilon^2 + \frac{\varepsilon^2\mu_1}{\mu_2}). \tag{6.31}$$

Therefore, to close the system, we have to compute $\tau^1_{xz}(0)$ and $\int_0^h u_1\, dz$ at main orders. For that, we will calculate first τ^1_{xz} at any z in $[0, h]$, which will be used to bring both $\tau^1_{xz}(0)$ and $\int_0^h u_1\, dz$.

Calculating τ^1_{xz}. Truncating the expansions at order $O(\varepsilon^2 + \frac{\varepsilon^2\mu_1}{\mu_2})$, we can write then

$$\partial_z\tau_{xz} = \partial_z\tau^0_{xz} + \varepsilon\,\partial_z\tau^1_{xz} = -\lambda + \varepsilon Re\big[\partial_t u + u\,\partial_x u + w\,\partial_z u + \frac{1}{Fr^2}\,\partial_x p\big] - \varepsilon\,\partial_x\tau_{xx} \tag{6.32}$$

which radically tells that

$$\partial_z\tau^0_{xz} = -\lambda \qquad \text{and} \qquad \partial_z\tau^1_{xz} = Re[\,\partial_t u_0 + u_0\,\partial_x u_0 + w\,\partial_z u_0 + \frac{1}{Fr^2}\,\partial_x p_0] - \partial_x\tau^0_{xx}. \tag{6.33}$$

Remark 10. *The expression of τ^1_{xx} in (6.33) depends on τ_{xx} at main order. A further look on τ_{xx}, we notice that from the definition of the stress we have*

- *in the region $[h^*, h]$, we have*
$$\tau_{xx} = 2\varepsilon\mu_1\,\partial_x u,$$
and using the main order expansion of u calculated in Subsection 6.3.1, we have
$$\partial_x u_0 = \frac{\lambda}{\mu_1}[z - h^*]\,\partial_x h + \frac{\lambda}{\mu_2}h^*\,\partial_x h.$$
Thus, we get
$$\tau_{xx} = 2\varepsilon\lambda[z - h^*]\,\partial_x h + \lambda\frac{2\varepsilon\mu_1}{\mu_2}h^*\,\partial_x h.$$
Hence, using the condition 6.8, we get that the latter term is sufficiently small $O(\varepsilon + \frac{\varepsilon\mu_1}{\mu_2})$.

- *Similarly in $[0, h^*]$, we have*
$$\tau_{xx} = 2\varepsilon\mu_2\,\partial_x u + B(1 - \frac{\mu_2}{\mu_2})\frac{\varepsilon\,\partial_x u}{|(\mathrm{D}\,v)|},$$

and since $\partial_x u$ at main order in this region is of order 1

$$\partial_x u_0 = \frac{\lambda}{\mu_2}\,\partial_x h z,$$

thus, we get in $[0, h^*]$ that τ^0_{xx} is also very small $(O(\varepsilon))$.

Having stressed on this remark, we can rewrite now (6.33) as

$$\partial_z \tau^0_{xz} = -\lambda \qquad \text{and} \qquad \partial_z \tau^1_{xz} = Re[\,\partial_t u_0 + u_0\,\partial_x u_0 + w_0\,\partial_z u_0 + \frac{1}{Fr^2}\,\partial_x p_0]. \tag{6.34}$$

First, we will bring τ^1_{xz} in the upper part at main order. From boundary conditions we have

$$\tau_{xz}(h) = 2\varepsilon\,\partial_x h \tau_{xx}(h) + O(\varepsilon^2 + \varepsilon^2\frac{\mu_1}{\mu_2}),$$

and since $\tau_{xx}(h) \sim O(\varepsilon + \frac{\varepsilon\mu_1}{\mu_2})$ then

$$\tau^1_{xz}(h) = 2\mu_1\,\partial_x h \tau_{xx}(h) \sim O(\varepsilon + \frac{\varepsilon\mu_1}{\mu_2}),$$

and thus integrating (6.33) between $z \geqslant h^*$ and h we get (regarding only main order terms)

$$\tau^1_{xz} = -Re\int_z^h \partial_t u_0 + u_0\,\partial_x u_0 + w_0\,\partial_z u_0\; dz - \lambda\cot\theta\,\partial_x h(h-z). \tag{6.35}$$

Let

$$\chi_0 = \int_z^h \partial_t u_0 + u_0\,\partial_x u_0 + w_0\,\partial_z u_0.$$

Recall that we already know the expression of u at main order

$$u_0 = \begin{cases} \frac{\lambda}{\mu_2}(h^* z - \frac{z^2}{2}) + \frac{\lambda}{\mu_1} h_c z & 0 \leqslant z \leqslant h^*, \\ \frac{\lambda}{\mu_2}\frac{h^{*2}}{2} + \frac{\lambda}{\mu_1}(h^* z + h_c z - \frac{z^2}{2} - \frac{h^{*2}}{2}) & h^* \leqslant z \leqslant h. \end{cases} \tag{6.36}$$

As for w_0, we will use the incompressibility condition at main order $\partial_z w_0 + \partial_x u_0 = 0$, which implies

$$w_0 = \begin{cases} -\frac{\lambda}{\mu_2}\,\partial_x h\frac{z^2}{2} & 0 \leqslant z \leqslant h^*, \\ -\frac{\lambda}{\mu_2}\,\partial_x h(h^* z - \frac{h^{*2}}{2}) - \frac{\lambda}{\mu_1}(\frac{z^2}{2} - h^* z + \frac{h^{*2}}{2})\,\partial_x h & h^* \leqslant z \leqslant h. \end{cases}$$

As a result, we can get χ_0 at main order for all z such that $h^* \leqslant z \leqslant h$

$$\begin{aligned}
\chi_0 &= \int_z^h \partial_t u_0 + u_0\,\partial_x u_0 + w_0\,\partial_z u_0 \\
&= \lambda^2\,\partial_x h\Big(\frac{1}{\mu_1^2}\big(-\frac{h_c^4}{3} + \frac{h_c^2}{2}z^2 - \frac{h_c}{6}z^3 - h_c^2 h^* z + \frac{h_c h^*}{2}z^2 + \frac{h_c^2 h^{*2}}{2} - \frac{h_c h^{*2}}{2}z + \frac{h_c h^{*3}}{6}\big) \\
&+ \frac{1}{\mu_1\mu_2}\big(-\frac{4}{3}h_c^3 h^* + h_c^2 h^* z + \frac{h_c h^*}{2}z^2 - \frac{h^*}{6}z^3 - \frac{3h_c^2 h^{*2}}{2} - \frac{h_c h^{*2}}{2}z + \frac{h^{*2}}{2}z^2 - \frac{h^{*3}}{2}z + \frac{h^{*4}}{6}\big) \\
&+ \frac{1}{\mu_2^2}\big(-h_c^2 h^{*2} + h_c h^{*2} z - \frac{3h_c h^{*3}}{2} + \frac{h^{*3}}{2}z - \frac{h^{*4}}{2}\big)\Big).
\end{aligned}$$

Thus, for $h^* \leqslant z \leqslant h$, we obtain

$$\tau^1_{xz} = -Re\chi_0 - \lambda\cot\theta\,\partial_x h(h-z). \tag{6.37}$$

As for the lower part where $0 \leqslant z \leqslant h^*$, using as well the relation (6.34), but now integrating between

h^* and z, we get

$$\tau^1_{xz} = \tau^1_{xz}(h^*) - \lambda \cot\theta\, \partial_x h (h^* - z) - Re \int_z^{h^*} \partial_t u_0 + u_0\, \partial_x u_0 + w_0\, \partial_z u_0 \; dz.$$

Define

$$\chi_1 = \int_z^{h^*} \partial_t u_0 + u_0\, \partial_x u_0 + w_0\, \partial_z u_0 \; dz.$$

At main order, we get

$$\chi_1 = \lambda^2\, \partial_x h \Big(\frac{1}{\mu_2^2} \big(\frac{h_c h^*}{2} z^2 - \frac{h^*}{6} z^3 + \frac{h^{*2}}{2} z^2 - \frac{h_c h^{*3}}{2} - \frac{h^{*4}}{3} \big) + \frac{1}{\mu_1 \mu_2} \big(\frac{h_c^2}{2} z^2 - \frac{h_c}{6} z^3 + \frac{h_c h^*}{2} z^2 - \frac{h_c^2 h^{*2}}{2} - \frac{h_c h^{*3}}{3} \big) \Big). \tag{6.38}$$

Furthermore, from the expression of τ^1_{xz} in the upper region, and due to assumed continuity of the stress we deduce

$$\begin{aligned} \tau^1_{xz}(h^*) &= -Re \chi_0(h^*) - \lambda \cot\theta\, \partial_x h h_c \\ &= -Re\lambda^2\, \partial_x h \Big(-\frac{1}{\mu_1^2} \frac{h_c^4}{3} + \frac{1}{\mu_1 \mu_1} \big(-\frac{4}{3} h_c^3 h^* - \frac{1}{2} h_c^2 h^{*2} \big) + \frac{1}{\mu_2^2} \big(- h_c^2 h^{*2} - \frac{1}{2} h_c h^{*3} \big) \Big) \\ &\quad - \lambda \cot\theta\, \partial_x h h_c. \end{aligned} \tag{6.39}$$

And thus

$$\begin{aligned} \tau^1_{xz} &= -\lambda \cot\theta\, \partial_x h (h^* - z) - \lambda \cot\theta\, \partial_x h h_c - Re(\chi_0(h^*) + \chi_1) \\ &= -\lambda \cot\theta\, \partial_x h (h - z) - Re\lambda^2\, \partial_x h \Big(-\frac{1}{\mu_1^2} \frac{h_c^4}{3} + \frac{1}{\mu_2^2} \big(\frac{h_c h^*}{2} z^2 - \frac{h^*}{6} z^3 + \frac{h^{*2}}{2} z^2 - h_c h^{*3} - \frac{h^{*4}}{3} - h_c^2 h^{*2} \big) \\ &\quad + \frac{1}{\mu_1 \mu_2} \big(\frac{h_c^2}{2} z^2 - \frac{h_c}{6} z^3 + \frac{h_c h^*}{2} z^2 - h_c^2 h^{*2} - \frac{h_c h^{*3}}{3} - \frac{4}{3} h_c^3 h^* \big) \Big). \end{aligned} \tag{6.40}$$

We define again for simplicity

$$\begin{aligned} \chi_2 = \lambda^2\, \partial_x h \Big(&-\frac{1}{\mu_1^2} \frac{h_c^4}{3} + \frac{1}{\mu_2^2} \big(\frac{h_c h^*}{2} z^2 - \frac{h^*}{6} z^3 + \frac{h^{*2}}{2} z^2 - h_c h^{*3} - \frac{h^{*4}}{3} - h_c^2 h^{*2} \big) \\ &+ \frac{1}{\mu_1 \mu_2} \big(\frac{h_c^2}{2} z^2 - \frac{h_c}{6} z^3 + \frac{h_c h^*}{2} z^2 - h_c^2 h^{*2} - \frac{h_c h^{*3}}{3} - \frac{4}{3} h_c^3 h^* \big) \Big). \end{aligned} \tag{6.41}$$

Hence, for $0 \leqslant z \leqslant h^*$

$$\tau^1_{xz} = -\lambda \cot\theta\, \partial_x h (h - z) - Re \chi_2. \tag{6.42}$$

Calculating $\tau^1_{xz}(0)$. To obtain $\tau^1_{xz}(0)$ one can simply substitute 0 for z in (6.42) and get the result.

$$\begin{aligned} \tau^1_{xz}(0) &= -\lambda \cot\theta h\, \partial_x h - Re \chi_2(0) \\ &= -\lambda \cot\theta h\, \partial_x h \\ &\quad - Re\lambda^2\, \partial_x h \Big(-\frac{1}{\mu_1^2} \frac{h_c^4}{3} + \frac{1}{\mu_2^2} \big(- h_c h^{*3} - \frac{h^{*4}}{3} - h_c^2 h^{*2} \big) + \frac{1}{\mu_1 \mu_2} \big(- h_c^2 h^{*2} - \frac{h_c h^{*3}}{3} - \frac{4}{3} h_c^3 h^* \big) \Big) \\ &= Re\, \partial_x \Big(-\frac{\cos\theta}{Fr^2} \frac{h^2}{2} - \lambda^2 \Big(-\frac{1}{\mu_1^2} \frac{h_c^4}{3} h^* + \frac{1}{\mu_2^2} \big(-\frac{h_c h^{*4}}{4} - \frac{h^{*4}}{15} - \frac{h_c^2 h^{*3}}{3} \big) \\ &\quad + \frac{1}{\mu_1 \mu_2} \big(-\frac{h_c^2 h^{*3}}{3} - \frac{h_c h^{*4}}{12} - \frac{2}{3} h_c^3 h^{*2} \big) \Big) \Big). \end{aligned}$$

At this point, we define Λ_2^P such that

$$\tau_{xz}^1(0) := Re\,\partial_x\Lambda_2^P, \tag{6.43}$$

and

$$\Lambda_2^P = -\frac{\cos\theta}{Fr^2}\frac{h^2}{2} - \lambda^2\Big(-\frac{1}{\mu_1^2}\frac{h_c^4}{3}h^* + \frac{1}{\mu_2^2}\big(-\frac{h_c h^{*4}}{4} - \frac{h^{*4}}{15} - \frac{h_c^2 h^{*3}}{3}\big) + \frac{1}{\mu_1\mu_2}\big(-\frac{h_c^2 h^{*3}}{3} - \frac{h_c h^{*4}}{12} - \frac{2}{3}h_c^3 h^{*2}\big)\Big). \tag{6.44}$$

Calculating $\int_0^h u_1\ dz$. We recall that we truncated the expansions at order $O(\varepsilon^2 + \varepsilon^2\frac{\mu_1}{\mu_2})$ so that we write $\tau_{xz} = \lambda(h-z) + \varepsilon\tau_{xz}^1$. But also, τ_{xz} takes another formula depending on the region as described by the rheology in (6.5). Starting from the upper part, we have from one side (up to order $O(\varepsilon^2 + \varepsilon^2\frac{\mu_1}{\mu_2})$)

$$\begin{aligned}\tau_{xz} = \mu_1(\,\partial_z u + \varepsilon^2\,\partial_x w) &= \mu_1\,\partial_z u_0 + \varepsilon\mu_1\,\partial_z u_1 + \varepsilon^2\mu_1\,\partial_z u_2 + \varepsilon^2\mu_1\,\partial_x w_0\\ &= \mu_1\,\partial_z u_0 + \varepsilon(\mu_1\,\partial_z u_1 + \varepsilon\mu_1\,\partial_z u_2 + \varepsilon\mu_1\,\partial_x w_0)\\ &= \tau_{xz}^0 + \varepsilon\tau_{xz}^1\end{aligned}$$

and from the second side from previous subsection

$$\tau_{xz}^1 = -Re\chi_0 - \lambda\cot\theta\,\partial_x h(h-z).$$

Hence, by analogy we deduce that

$$\begin{aligned}&\tau_{xz}^1 = \mu_1\,\partial_z u_1 + \varepsilon\mu_1\,\partial_z u_2 + \varepsilon\mu_1\,\partial_x w_0 = -Re\chi_0 - \lambda\cot\theta\,\partial_x h(h-z)\\ \implies\ &\partial_z u_1 = -\frac{Re}{\mu_1}\chi_0 - \frac{\lambda\cot\theta\,\partial_x h}{\mu_1}(h-z) - \varepsilon\,\partial_x w_0 - \varepsilon\,\partial_z u_2\\ \implies\ &\partial_z u_1 = -\frac{Re}{\mu_1}\chi_0 - \frac{\lambda\cot\theta}{\mu_1}\,\partial_x h(h-z) - \varepsilon\Big[\,\partial_x w_0 + \,\partial_z u_2\Big].\end{aligned} \tag{6.45}$$

But we are concerned with the expression of u_1 at main order, so we neglect the terms of order ε. Integrating between h^* and z for all $h^* \leqslant z \leqslant h$, we obtain at main order

$$\int_{h^*}^z \partial_z u_1\ dz = u_1 - u_1(h^*) = -\frac{Re}{\mu_1}\int_{h^*}^z \chi_0\ dz - \frac{\lambda\cot\theta}{\mu_1}\,\partial_x h(hz - \frac{z^2}{2} - \frac{h^{*2}}{2} - h_c h^*)$$

$$\implies u_1 = u_1(h^*) - \frac{\lambda\cot\theta}{\mu_1}\,\partial_x h(hz - \frac{z^2}{2} - \frac{h^{*2}}{2} - h_c h^*) - \frac{Re}{\mu_1}\int_{h^*}^z \chi_0\ dz.$$

And

$$\begin{aligned}
&\int_{h^*}^{z} \chi_0 \, dz \\
&= \lambda^2 \, \partial_x h \Big(\frac{1}{\mu_1^2} \big(-\frac{h_c^4}{3} z + \frac{h_c^2}{6} z^3 - \frac{h_c}{24} z^4 + \frac{h_c^4 h^*}{3} - \frac{h_c^2 h^*}{2} z^2 + \frac{h_c h^*}{6} z^3 + \frac{h_c^2 h^{*2}}{2} z - \frac{h_c h^{*2}}{4} z^2 \\
&\qquad - \frac{h_c^2 h^{*3}}{6} + \frac{h_c h^{*3}}{6} z - \frac{h_c h^{*4}}{24} \big) \\
&+ \frac{1}{\mu_1 \mu_2} \big(-\frac{4}{3} h_c^3 h^* z + \frac{h_c^2 h^*}{2} z^2 + \frac{h_c h^*}{6} z^3 - \frac{h^*}{24} z^4 + \frac{4}{3} h_c^3 h^{*2} - \frac{3}{2} h_c^2 h^{*2} z - \frac{h_c h^{*2}}{4} z^2 \\
&\qquad + \frac{h^{*2}}{6} z^3 + h_c^2 h^{*3} - \frac{h^{*3}}{4} z^2 + \frac{h_c h^{*4}}{12} + \frac{h^{*4}}{6} z - \frac{h^{*5}}{24} \big) \\
&+ \frac{1}{\mu_2^2} \big(-h_c^2 h^{*2} z + \frac{h_c h^{*2}}{2} z^2 + h_c^2 h^{*3} - \frac{3}{2} h_c h^{*3} z + \frac{h^{*3}}{4} z^2 + h_c h^{*4} - \frac{h^{*4}}{2} z + \frac{h^{*5}}{4} \big) \Big).
\end{aligned}$$

Finally, we integrate u_1 in the region $[h^*, h]$, we get

$$\int_{h^*}^{h} u_1 \, dz = u_1(h^*) h_c - \frac{\lambda \cot\theta}{\mu_1} \, \partial_x h \frac{h_c^3}{3} - \frac{Re}{\mu_1} I_0, \tag{6.46}$$

where

$$I_0 = \int_{h^*}^{h} \int_{h^*}^{z} \chi_0(y) \, dy \, dz.$$

After calculation it may be formulated as follows

$$I_0 = \lambda^2 \, \partial_x h \Big(-\frac{2}{15} \frac{1}{\mu_1^2} h_c^6 + \frac{1}{\mu_1 \mu_2} (-\frac{7}{15} h_c^5 h^* - \frac{1}{6} h_c^4 h^{*2}) + \frac{1}{\mu_2^2} (-\frac{1}{3} h_c^4 h^{*2} - \frac{1}{6} h_c^3 h^{*3}) \Big). \tag{6.47}$$

Now it remains to calculate $u_1$and consequently $\int_0^{h^*} u_1 \, dz$ in the lower part $0 \leqslant z \leqslant h^*$, we have in fact

$$\tau = 2\mu_2 Dv + (1 - \frac{\mu_2}{\mu_1}) B \frac{Dv}{|Dv|}. \tag{6.48}$$

To compute the right hand side at first order, we should compute $\frac{1}{|Dv|}$ at first order as a first step. Using Taylor's expansion we can write

$$\begin{aligned}
\frac{1}{|Dv|} &= \frac{\sqrt{2}}{\sqrt{(\partial_z u)^2 + 2\varepsilon^2 \, \partial_z u \, \partial_x w + 4\varepsilon^2 (\partial_x u)^2 + \varepsilon^4 (\partial_x w)^2}} \\
&= \frac{\sqrt{2}}{|\partial_z u|} \frac{1}{\sqrt{1 + \varepsilon^2 (\frac{2 \partial_x w}{\partial_z u} + 4 \frac{(\partial_x u)^2}{(\partial_z u)^2}) + \varepsilon^4 \frac{(\partial_x w)^2}{(\partial_z u^2)}}} \\
&= \frac{\sqrt{2}}{|\partial_z u|} (1 + O(\varepsilon^2)) \\
&= \frac{\sqrt{2}}{\sqrt{(\partial_z u_0)^2 + 2\varepsilon \, \partial_z u_0 \, \partial_z u_1 + \varepsilon^2 (\partial_z u_1)^2}} \\
&= \frac{\sqrt{2}}{|\partial_z u_0|} (1 - \varepsilon \frac{\partial_z u_1}{\partial_z u_0} + O(\varepsilon^2)).
\end{aligned}$$

Hence, from the expression of τ in (6.48), and the expansion of Dv, we get

$$\begin{aligned}\tau_{xz} &= \mu_2\,\partial_z u + (1-\frac{\mu_2}{\mu_1})\frac{B}{2}\,\partial_z u\frac{\sqrt{2}}{|\,\partial_z u_0|}(1-\varepsilon\frac{\partial_z u_1}{\partial_z u_0}) + O(\varepsilon^2)\\ &= \mu_2\,\partial_z u_0 + (1-\frac{\mu_2}{\mu_1})\frac{B}{\sqrt{2}}\frac{\partial_z u_0}{|\,\partial_z u_0|} + \varepsilon\mu_2\,\partial_z u_1 + O(\varepsilon^2).\end{aligned} \tag{6.49}$$

Thus, by analogy of $\tau_{xz} = \tau^0_{xz} + \varepsilon\tau^1_{xz} + O(\varepsilon^2)$, we deduce that

$$\tau^1_{xz} = \mu_2\,\partial_z u_1.$$

Recall that for $0 \leqslant z \leqslant h^*$ the expression of τ^1_{xz} is given by

$$\tau^1_{xz} = -\lambda\cot\theta\,\partial_x h(h-z) - Re\chi_2.$$

which implies that

$$\partial_z u_1 = \frac{1}{\mu_2}\tau^1_{xz} = -\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(h-z) - \frac{Re}{\mu_2}\chi_2,$$

and since $u(0)=0$ implies that $u_1(0)=0$, then

$$u_1 = -\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(hz-\frac{z^2}{2}) - \frac{Re}{\mu_2}\chi_3,$$

where

$$\begin{aligned}\chi_3 &= \int_0^z \chi_2\;dz\\ &= \lambda^2\,\partial_x h\big[\frac{1}{\mu_2^2}(\frac{h_c h^*}{6}z^3 - \frac{h^*}{24}z^4 - h_c^2 h^{*2}z + \frac{h^{*2}}{6}z^3 - h_c h^{*3}z - \frac{h^{*4}}{3}z)\\ &+ \frac{1}{\mu_1\mu_2}(\frac{h_c^2}{6}z^3 - \frac{h_c}{24}z^4 - \frac{4}{3}h_c^3 h^* z + \frac{h_c h^*}{6}z^3 - h_c^2 h^{*2}z - \frac{h_c h^{*3}}{3}z) - \frac{1}{\mu_1^2}\frac{h_c^4}{3}z\big].\end{aligned}$$

And thus we deduce that

$$\int_0^{h^*} u_1\;dz = -\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(\frac{h^{*3}}{3} + \frac{h^{*2}h_c}{2}) - \frac{Re}{\mu_2}I_2, \tag{6.50}$$

where

$$\begin{aligned}I_2 &= \int_0^{h^*}\chi_3\;dz\\ &= \lambda^2\,\partial_x h\Big[-\frac{1}{\mu_1^2}\frac{h_c^4 h^{*2}}{6} + \frac{1}{\mu_1\mu_2}(-\frac{2}{3}h_c^3 h^{*3} - \frac{11}{24}h_c^2 h^{*4} - \frac{2}{15}h_c h^{*5}) + \frac{1}{\mu_2^2}(-\frac{h_c^2 h^{*4}}{2} - \frac{11}{24}h_c h^{*5} - \frac{2}{15}h^{*6})\Big].\end{aligned} \tag{6.51}$$

Using the continuity of the velocity profile, we also deduce that

$$\begin{aligned} u_1(h^*) &= -\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(h_c h^* + \frac{h^{*2}}{2}) - \frac{Re}{\mu_2}\chi_3(h^*) \\ &= -\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(h_c h^* + \frac{h^{*2}}{2}) - \frac{Re}{\mu_2}\chi_3(h^*) \\ &- \frac{Re}{\mu_2}\lambda^2\,\partial_x h\big[-\frac{1}{\mu_1^2}\frac{h_c^4 h^*}{3} + \frac{1}{\mu_1\mu_2}(-\frac{4}{3}h_c^3 h^{*2} - \frac{5}{6}h_c^2 h^{*3} - \frac{5}{24}h_c h^{*4}) \\ &+ \frac{1}{\mu_2^2}(-h_c^2 h^{*3} - \frac{5}{6}h_c h^{*4} - \frac{5}{24}h^{*5})\big]. \end{aligned} \tag{6.52}$$

We will denote by I_1

$$I_1 = \lambda^2 h_c\,\partial_x h\big[-\frac{1}{\mu_1^2}\frac{h_c^4 h^*}{3} + \frac{1}{\mu_1\mu_2}(-\frac{4}{3}h_c^3 h^{*2} - \frac{5}{6}h_c^2 h^{*3} - \frac{5}{24}h_c h^{*4}) + \frac{1}{\mu_2^2}(-h_c^2 h^{*3} - \frac{5}{6}h_c h^{*4} - \frac{5}{24}h^{*5})\big]. \tag{6.53}$$

Hence, $\int_{h^*}^{h} u_1\,dz$ becomes

$$\int_{h^*}^{h} u_1\,dz = -\frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(\frac{h_c h^{*2}}{2} + h_c^2 h^*) - \frac{\lambda\cot\theta}{\mu_1}\,\partial_x h\frac{h_c^3}{3} - \frac{Re}{\mu_1}I_0 - \frac{Re}{\mu_2}I_1. \tag{6.54}$$

Finally, summing (6.54) and (6.50), we get

$$\begin{aligned} \int_0^h u_1\,dz &= \int_0^{h^*} u_1\,dz + \int_{h^*}^h u_1\,dz \\ &= -\frac{Re}{\mu_1}I_0 - \frac{Re}{\mu_2}I_3 - \frac{\lambda\cot\theta}{\mu_1}\,\partial_x h\frac{h_c^3}{3} - \frac{\lambda\cot\theta}{\mu_2}\,\partial_x h(\frac{h^{*3}}{3} + h^{*2}h_c + h^* h_c^2). \end{aligned} \tag{6.55}$$

where I_0 is given by (6.47) and

$$\begin{aligned} I_3 &= I_1 + I_2 \\ &= \lambda^2\,\partial_x h\Big[\frac{1}{\mu_1^2}(-\frac{h_c^5 h^*}{3} - \frac{h_c^4 h^{*2}}{6}) + \frac{1}{\mu_1\mu_2}(-\frac{4}{3}h_c^4 h^{*2} - 3h_c^3 h^{*3} - \frac{2}{3}h_c^2 h^{*4} - \frac{2}{15}h_c h^{*5}) \\ &\quad + \frac{1}{\mu_2^2}(-h_c^3 h^{*3} - \frac{4}{3}h_c^2 h^{*4} - \frac{2}{3}h_c h^{*5} - \frac{2}{15}h^{*6})\Big] \end{aligned} \tag{6.56}$$

due to the expressions of I_1 and I_2 respectively given by (6.53) and (6.51).

Plugging together the expressions (6.19) with (6.29), (6.31), (6.43) with (6.44) and (6.55) in (6.18)–(6.21), we get the desired bi-viscous shallow-water system.

Remark 11. *An important remark to stress on is the uniformity of the perturbation analysis with respect to μ_1. In particular, the controversial term is the shear in the pseudo plug regime $h^* \leqslant z \leqslant h$. We have in fact*

$$\tau_0 = 2\mu_1(\mathrm{D}\,v)^0,$$

however, in this region as well, we have

$$\begin{aligned} \tau_{xx}^0 &= 2\varepsilon\mu_1\,\partial_x u_0 = 2\varepsilon\lambda\,\partial_x h(z-h^*) + \frac{2\varepsilon\mu_1}{\mu_2}\lambda h^*\,\partial_x h \sim O(\varepsilon + \frac{\varepsilon\mu_1}{\mu_2}), \\ \tau_{xz}^0 &= \mu_1(\,\partial_z u_0) = \lambda(h-z) \sim O(1). \end{aligned}$$

And thus, in both regimes of μ_1, i.e whether μ_1 is compared to μ_2 or μ_1 is taken very large, this doesn't cause breakdown in the perturbation analysis and in the derivation process as long as we assume the uniformity of $\partial_x h$ in terms of μ_1.

6.4 Conclusion

In what proceeded, a shallow water model was derived starting from a free surface problem of an incompressible fluid defined by a bi-viscous rheology. The first attempt was to free the scaling from the variable viscosities and thus obtain free parameters in the rescaled departure system. One advantage of this is the ability to modulate the final system using these parameters in order to recover the already presented shallow water systems in the literature: Bingham model (μ_1 tends to infinity), and Newtonian model ($\mu_1 = \mu_2 =$ constant), see figure 6.1b to see the effect of modulating the viscosity on the type of stress defining the rheology. We were able to recover an exact Newtonian model as that figured out in literature for Newtonian case [1]. However, there was a different term obtained in the Bingham limit. Mainly, this difference arises from the boundary normal stress that is counted for in Vila and al. paper [40], and absent in our case due to the adapted continuous representation of the stress in the pseudo-plug zone, especially at h^*: $\tau = 2\mu_1 Du$. Finally we lack the correction term in [40] due to the difference in calculating the first order normal stress that contributes to the first order velocity profile, mainly by assuming the uniformity of $\partial_x h$ in terms of μ_1. One advantage of the bi-viscous model is conveyed in the smoothness noticed in the graduation of depth component of velocity gradient. This smoothness makes us avoid calculating the corrective layers calculated in different approaches, both in lubrication and shallow water approximations, in an attempt to get a consistent theory for thin layer dynamics of Bingham viscoplastic fluid, as in [67] and [40]. The reason is behind the velocity profile at first order, which due to the bi-viscosity characteristic encodes, in some sense, these corrective layers as shown in the calculations.

Part IV

Degenerate Lake System for Bingham Fluids

7

On The Rigid-Lid Approximation of Shallow Water Bingham Model

The work in this chapter is a collaboration with B. Al Taki (Laboratoire Jacques Louis Lions, Sorbonne Universite) and J. Sainte Marie (Inria, Paris). It is published in *Discrete Contin. Dyn. Syst. Ser. B 2020*. The link of the paper on HAL is given below:
https://hal.inria.fr/hal-02052055/file/Bing-K-J.pdf.

This work discusses the well posedness of an initial value problem describing the motion of a Bingham fluid in a basin with a degenerate bottom topography. A physical interpretation of such motion is discussed. The system governing such motion is obtained from the Shallow Water-Bingham models in the regime where the Froude number degenerates, i.e taking the limit of such equations as the Froude number tends to zero. Since we are considering equations with degenerate coefficients, then we shall work with weighted Sobolev spaces in order to establish the existence of a weak solution. In order to overcome the difficulty of the discontinuity in Bingham's constitutive law, we follow a similar approach to that introduced in [G. Duvaut and J.-L. Lions, Springer-Verlag, 1976]. We study also the behavior of this solution when the yield limit vanishes. Finally, a numerical scheme for the system in 1D is furnished.

The chapter is organized as follows:

-**The first section** 7.1 is an introductory section where we present a small historical background of the problem of incompressible Bingham systems.

-**The second section** 7.2 introduces the spaces that we prove the solution in and some preliminary results related to the functional analysis of the problem.

-**The third section** 7.3 highlights the equivalence between the initial system and the variational inequality introduced in it.

-**The fourth section** 7.4 is the section of main results where we exhibit the existence of the solution via a Galerkin method.

-**The fifth section** 7.5 is where we study the behavior of the solution when the Bingham number tends to zero.

-**The sixth section** 7.6 discusses a numerical scheme that is implemented in 1D verifying our theoretical results.

7.1 Introduction

Bingham fluids constitute a crucial topic of study on which many applicable researches are conducted. As an interesting example we suggest the snow that appears in some important natural phenomena such as avalanches. Technically speaking, the avalanche dynamics can be described through different perspectives, such as the center of mass consideration [102] and the density one. In addition, many approaches consider the avalanche to be a deformable body whereas others describe it as a granular material. Not only the dynamics of snow, but also its constitutive behavior can be described according to different points of views such as Newtonian fluid and Bingham fluid. In many literatures, snow is considered to be a non-Newtonian fluid. One of the reasons is that Newtonian fluids adapt rapidly to deform themselves where they reach a negligible depth in an unbounded space, on the other hand, snow will pile in such a case and thus will rest with a finite depth. This can be explained by the fact that upon being in rest state, snow achieves a yield value with a non zero shear stress, and thus a threshold stress value should be exceeded to start deformation. Another reason is the experimental results of the analysis of the avalanche's velocity profile along its depth, which reveals the viscosity dependence of the shear [103] [104]. These two reasons triggered many authors in the literature to treat snow as a Bingham fluid which is characterized mainly by a non constant stress that depends on viscosity and that may differ in the same body of mass. Hence, one would notice some portions flowing while others moving in a bulk motion as solids. More precisely, at low shear stress, Bingham fluids possess high viscosity, and thus they behave as a shear thinning fluid at low or zero speed. As the shear stress reaches a certain limit-denoted yield stress- one notices a sudden drop in the viscosity, while above the yield stress, snow behaves like a low viscosity liquid.

The need to use reduced models arises from the difficulty that the three dimensional Cauchy momentum equations induce in the analytical and numerical studies, especially if we consider a fee boundary case. In this sequel, the reduced model adopted is the Shallow water-Bingham model that is derived using depth integration of the Navier-Stokes-Bingham system with free surface, taking into account that the horizontal length scale is much greater than the vertical one. Since there is no well posedness result on the initial system and the approximated one, then any mathematical justification of this procedure is still as far as we know an interesting open problem due to the wide range of applicability especially in numerical applications. For instance, in [105], [106], the authors proposed several models of a Shallow water type system for Bingham fluid. The resulting system is given by

$$\begin{cases} \partial_t h + \operatorname{div}(hv) = 0 \\ \partial_t (hv) + \operatorname{div}(hv \otimes v) - \operatorname{div}(\sigma) + \frac{1}{2\,\mathrm{Fr}^2} \nabla h^2 = hf, \end{cases} \tag{7.1}$$

where $h(t,x)$ is the water height, $v(t,x)$ is the horizontal velocity. We denote by $hv \otimes v$ the matrix with component $hv_i v_j$, Fr is the Froude number and σ is the stress tensor given by

$$\begin{cases} \sigma = 2\mu h\, \mathrm{D}(v) + \lambda h \operatorname{div} v\, \mathbb{I} + gh \dfrac{\mathrm{D}(v)}{|\,\mathrm{D}(v)|} & \text{if } \mathrm{D}(v) \neq 0, \\ |\sigma| < gb & \text{if } \mathrm{D}(v) = 0, \end{cases} \tag{7.2}$$

where $\mathrm{D}(v)$ represents the symmetric part of the velocity gradient given by

$$\mathrm{D}(v) = \frac{1}{2}(\nabla v + \nabla^t v).$$

As mentioned above, the Shallow water Bingham model is still for today an open problem. We remark that, in general, the usual strategy of the existence theory is based on two steps: the first one consists of finding u_n, a solution of a suitably chosen approximating problem (a Galerkin

approximation for example). The second one amounts to establish uniform estimates in proper spaces and one can then obtains u, a solution to System (7.3)-(7.4), as a "weak" limit of a suitably chosen subsequence of u_n. The central difficulty is the passage to the limit in the nonlinear terms and in the discontinuous constitutive law. Actually, the ideas developed in studying the well posedness issue of Shallow water equations (or compressible Navier-Stokes equations) for Newtonian fluids (see [107], [108], [109], [110]) don't work for non Newtonian ones. As a forward step in this approach, we study here the rigid lid approximation of such system, that we shall call Bingham lake equations. Bingham lake equations characterizing lake equations decode many natural as well as industrial phenomena. The most interesting example is that of mixed flows of Bingham type fluids (such as petroleum) in closed supply pipes. Two types of flows are exhibited inhere characterizing a transition phase from free surface (i.e when only part of the section of pipe is filled and the pressure is being known: atmospheric pressure), to pressurized flow (i.e when the section of the pipe is full, and the pressure is an unknown). The authors in [111] have studied such kind of flows occurring for Newtonian fluids thoroughly. They presented a new model called PFS model, based on coupling the free surface part equations derived from the incompressible Navier Stokes or Euler systems, and the pressurized part equations derived from the Compressible Euler equations. A finite volume discretization have been studied in [111], and a kinetic formulation of such models is presented in [112].

Mathematically, this system can be obtained from (7.1)-(7.2) system by passing to the limit (Fr $\to$ 0), where the initial height converges to a non constant function $b(x)$ depending on the space variables only. The obtained model is given by:

$$\begin{cases} \partial_t(bu) + \operatorname{div}(bu \otimes u) - \operatorname{div}\sigma + b\nabla p &= bf, \\ \operatorname{div}(bu) &= 0. \end{cases} \tag{7.3}$$

The shear stress σ satisfies the special constitutive law of a Bingham fluid (see [113]) :

$$\begin{cases} \sigma = 2\mu b\,\mathrm{D}(u) + \lambda b \operatorname{div} u\,\mathbb{I} + gb\dfrac{\mathrm{D}(u)}{|\,\mathrm{D}(u)|} & \text{if } \mathrm{D}(u) \neq 0, \\ |\sigma| < gb & \text{if } \mathrm{D}(u) = 0. \end{cases} \tag{7.4}$$

Here, $u(t,x)$ denotes the velocity vector, p the pressure, $f(t,x)$ the known external force, μ and λ the Lamé viscosity coefficients, g the yield limit. We couple the system (7.3) with the so called Lions' boundary conditions given by

$$bu \cdot n = 0 \qquad (\sigma \cdot n) \cdot \tau + \kappa(x) bu \cdot \tau = 0 \quad (t,x) \in (0,T) \times \partial\Omega, \tag{7.5}$$

and with the initial data (defined in a weak sence, see Theorem 7.12)

$$u(t,x)|_{t=0} = u_0(x), \quad x \in \Omega. \tag{7.6}$$

In (7.5), n and τ denote respectively the unit normal and tangential vectors to the boundary, whereas $\kappa(x)$ is the curvature of $\partial\Omega$. Lions' boundary conditions, the particular case of Navier boundary condition that were first used by Navier in 1872, regard that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress.

Since the singular term $\mathrm{D}(u)/|\,\mathrm{D}(u)|$ is not always defined, numerical and mathematical obstacles appear, which forced many authors to develop new formulations of the problem in order to tackle the difficulty. One of the approaches used was introduced in [114] for an unsteady flow of an incompressible fluid having its Cauchy stress given implicitly and relating both the symmetric part of the velocity gradient $\mathrm{D}(u)$ and the deviatoric part of the Cauchy stress. The authors regard the stress tensor σ as a new variable (along with the density and the velocity

vector), and this results in a dissipation rate in the form of a Young function depending on $\mathrm{D}(u)$ and σ^t. The second approach, that we will adopt in the sequel, was initiated by G. Duvaut and J. L. Lions [77] who replaced the original system by a variational inequality to get rid of the singularity.

In this paper, we will first adopt the methodology of variational inequality used in [77] to prove the existence of a weak solution of the incompressible Bingham fluid confined to a shallow basin with a varying bottom topography but with some changes in the nature of spaces used. Again recalling that we are dealing with a degenerate bathymetry $b(x)$, meaning that $b(x)$ may vanish on the boundary, we will prove that the solution exists in some weighted Sobolev space where the weight is assumed to be a Muckenhoupt type. For more details about these spaces, we refer the reader to [115]. Then, we will discuss the behavior of solution when g tends to zero. It is good to mention that the vanishing viscosity limit (μ tends to zero) of Bingham fluids is studied by J.-L. Lions [116] in a bounded open set of $\mathbb{R}^2$ without bottom topography. However, an improvement of weak regularity on the solution is required in order to achieve the convergence on the nonlinear term. Yet, as long as we can't improve the regularity of our solution due to the degeneracy of our equations, then we are not able to study such limit in our case.

7.2 Functional spaces

Before we start the analysis, and since we are going to work with weighted Sobolev spaces, let us first give a brief definition of some of the spaces and preliminaries that we shall need in the sequel, especially the constraints on the weight b and the domain Ω.

Domain: (**I**) For an integer m, $1 \leq m \leq 2$, we set $Q_m = (0,1)^m$. We assume that there exists a bi-Lipschitz mapping

$$B : Q_2 \to \Omega, \quad \text{such that} \quad B(\overline{Q_1}) = \partial\Omega.$$

Weight: (**II**) We define a space function $b(x)$, locally integrable and belonging to the *Muckenhoupt* class $\mathcal{A}_q$. Generally speaking, for a weight of *Muckenhoupt* type, the definition of trace operator is well defined. More precisely, one can check that if $b \in \mathcal{A}_q$, we have $u \in W^{1,q}_b(\Omega) \hookrightarrow W^{1,q}_{loc}(\overline{\Omega})$ and hence there is a linear trace operator $\gamma_{0,b} : W^{1,q}_{loc}(\overline{\Omega}) \to L^1_{loc}(\overline{\Omega})$. Though it is well defined, yet we lack characterization of such trace regarding a general Muckenhoupt weight. That's why we restrict ourselves in what follows to a more specific weight that provides a characterization of the boundary terms. Its expression is given in a neighborhood of the boundary $V(\partial\Omega)$ by

$$b = \rho^\alpha(x),\ 0 < \alpha < 1/2, \quad \rho(x) = \mathrm{dist}(x, \partial\Omega) \text{ for } x \in V(\partial\Omega). \tag{7.7}$$

In this situation, the definition of the trace is more accurate. For more details about examples of weights satisfying Muckenhoupt condition, we refer the reader to [117], [118], [119].

Weighted Sobolev spaces: As mentioned in the introduction, we will prove our solution

in Weighted Sobolev space V_b. We introduce the following weighted spaces:

$$
\begin{aligned}
&D_b(\Omega) = \{\varphi \in C_0^\infty(\Omega); \operatorname{div}(b\varphi) = 0 \text{ in } \Omega, \;\; b\varphi \cdot n = 0 \;\text{ on }\; \partial\Omega\},\\
&L_b^q(\Omega) = \{\varphi\,; \int_\Omega |\varphi|^q\, b\, dx \;< \infty\},\\
&(L_b^q(\Omega))' = L_{b'}^{q'}(\Omega) \quad \text{with} \quad \frac{1}{q} + \frac{1}{q'} = 1 \text{ and } b' = b^{-\frac{1}{q-1}},\\
&H_b(\Omega) = \{\varphi; \varphi \in L_b^2(\Omega), \operatorname{div}(b\varphi) = 0,\; b\varphi \cdot n = 0 \text{ on } \partial\Omega\},\\
&V_b(\Omega) = \{\varphi; \varphi \in H_b^1(\Omega), \operatorname{div}(b\varphi) = 0,\; b\varphi \cdot n = 0 \text{ on } \partial\Omega\}.
\end{aligned}
$$

For more details about *Muckenhoupt* classes and Weighted Sobolev spaces we refer the reader to [115] and [120].

Trace: As mentioned above in (7.7), we will adopt a special choice of Muckenhopt type weights that provide a good characterization of the trace. In fact, this result is proved by A. Nekvinda via the following theorem.

Theorem 7.11. *([Theorem 2.8, [121]]). Suppose that Hypothesis (**I**) holds. Then for $b = \rho^\alpha(x)$, $-1 < \alpha < q-1$, there exists a unique bounded linear operator*

$$T_b^{1,q}(\partial\Omega) : W_b^{1,q}(\Omega) \to W^{1-\frac{1+\alpha}{q},q}(\partial\Omega),$$

such that

$$T_b^{1,q}(\partial\Omega)(u) = u_{|\partial\Omega}.$$

As a consequence of this theorem, we remark that if $f \in H^1_{\rho^\alpha}(\Omega)$, $0 < \alpha < 1$ the trace of f is well defined and belong to $L^2(\partial\Omega) \hookrightarrow L_b^2(\partial\Omega)$. This helps us define the boundary integrals coming from the Navier boundary conditions.

Throughout this paper, we assume that the domain Ω satisfies Hypothesis (**I**), and b satisfies Hypothesis (**II**), and any other restrictions would be specified .

Notation Remark: Let us fix some notations which will be used throughout the sequel:

- $\mathrm{D}(u) : \mathrm{D}(v) = \sum_{i,j=1}^{2} \mathrm{D}_{i,j}(u)\, \mathrm{D}_{i,j}(v)$.
- $\nabla b \otimes u := (\partial_j b\, u_i)_{1\le i,j\le 2}$.
- We say that u is a b-divergence free or u satisfies the b-incompressibility condition if $\operatorname{div}(bu) = 0$.

Remark 12. *We should remark that in the sequel of the study, we will need to have $q = \frac{3}{2}$, i.e the weight b should belong to $A_{\frac{3}{2}}$. In fact, this choice is for the sake of bounding the nonlinear term $(u\cdot\nabla)u$ in lemma 7.2. Notice that in studying the Stokes problem, we just need to take $q = 2$ which is necessary to have a weighted version of Poincaré (or Korn's inequality). The reader can refer to lemma 4.1 in [120] for a detailed explanation of such choice.*

Without loss of generality, C will denote a generic constant.

7.3 Variational Inequality

Following Duvaut and Lions in [77], we will derive in this section a variational inequality and prove (at least formally) that the resolution of this variational inequality is equivalent to solve the problem (7.3)-(7.4) in a weak sense (solution of variational problem). Indeed, let us establish first the variational formulation of system (7.3)-(7.4). For this sake, we consider a set V of functions $v : \Omega \to \mathbb{R}^2$ of enough regularity and such that $\operatorname{div}(bv) = 0$ in Ω and $bv \cdot n = 0$ in $\partial\Omega$. Then, we multiply (7.3) by $(v - u)$ and integrate in space. We get

$$\int_\Omega \partial_t u \cdot (v-u)\, b\, dx + \int_\Omega \underbrace{\operatorname{div}(bu \otimes u)}_{\operatorname{div}(bu)\cdot u + bu\cdot\nabla u} \cdot (v-u)\, dx - \int_\Omega \operatorname{div}\sigma \cdot (v-u)\, dx + \int_\Omega \nabla p \cdot (v-u)\, b\, dx = \int_\Omega f \cdot (v-u)\, b\, dx.$$

Accounting for the Navier boundary conditions and the relation $\operatorname{div}(bv) = 0$, we can write

$$\int_\Omega \partial_t u \cdot (v-u)\, b\, dx + \int_\Omega (u\cdot\nabla)u \cdot (v-u)\, b\, dx + \int_\Omega \sigma : \mathrm{D}(v-u)\, dx - \int_{\partial\Omega} \sigma\cdot n\cdot(v-u)\, ds - \int_\Omega p \underbrace{\operatorname{div}(bv-bu)}_{=0}\, dx + \int_{\partial\Omega} p \underbrace{(bv-bu)\cdot n}_{=0}\, ds = \int_\Omega f\cdot(v-u)\, b\, dx.$$

Thus

$$\int_\Omega \partial_t u \cdot (v-u)\, b\, dx + \int_\Omega (u\cdot\nabla)u\cdot(v-u)\, b\, dx + 2\mu\int_\Omega \mathrm{D}(u) : \mathrm{D}(v-u)\, b\, dx + \lambda\int_\Omega \operatorname{div} u \operatorname{div}(v-u)\, b\, dx + \int_\Omega g\frac{\mathrm{D}(u)}{|\,\mathrm{D}(u)|} : \mathrm{D}(v-u)\, b\, dx + \int_{\partial\Omega} \kappa u\cdot(v-u)\, b\, ds = \int_\Omega f\cdot(v-u)\, b\, dx. \tag{7.8}$$

In equation (7.8), the ratio $\frac{\mathrm{D}(u)}{|\,\mathrm{D}(u)|}$ is not always defined. The physical understanding of this term indicates that it should be interpreted when $\mathrm{D}(u) = 0$ as "any trace-free symmetric matrix with norm less or equal to one". In the mathematical language such quantity is called "multivalued". This fundamental difficulty makes the approximation of the problem (7.8) already a complex challenge , and has motivated a large literature, see [77], [114] and references therein. Below, we follow the procedure in [77] to derive a "formal" equivalent form of the variational formulation (7.8). Indeed, regarding the Cauchy-Schwartz inequality

$$\mathrm{D}(f) : \mathrm{D}(g) = \sum_{i,j} \mathrm{D}_{i,j}(f)\,\mathrm{D}_{i,j}(g) \le |\,\mathrm{D}(f)||\,\mathrm{D}(g)|,$$

as $\mathrm{D}(u) \neq 0$, we have

$$\begin{aligned}\int_\Omega \frac{\mathrm{D}(u) : \mathrm{D}(v-u)}{|\,\mathrm{D}(u)|}\, b\, dx &= \int_\Omega \left[\frac{1}{|\,\mathrm{D}(u)|}\,\mathrm{D}(u) : \mathrm{D}(v) - \frac{1}{|\,\mathrm{D}(u)|}\,\mathrm{D}(u) : \mathrm{D}(u)\right] b\, dx \\ &\leqslant \int_\Omega \left[\frac{1}{|\,\mathrm{D}(u)|}|\,\mathrm{D}(u)||\,\mathrm{D}(v)| - \frac{1}{|\,\mathrm{D}(u)|}|\,\mathrm{D}(u)|^2\right] b\, dx \\ &\leqslant \int_\Omega \left[|\,\mathrm{D}(v)| - |\,\mathrm{D}(u)|\right] b\, dx.\end{aligned}$$

Hence we get

$$\int_\Omega f\cdot(v-u)\,b\,dx \leqslant \int_\Omega \partial_t u\cdot(v-u)\,b\,dx+\int_\Omega u\cdot\nabla u\cdot(v-u)\,b\,dx+2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\,b\,dx$$
$$+\lambda\int_\Omega \operatorname{div}u\operatorname{div}(v-u)\,b\,dx+g\int_\Omega(|\,\mathrm{D}(v)|-|\,\mathrm{D}(u)|)\,b\,dx+\int_{\partial\Omega}\kappa u\cdot(v-u)\,b\,ds \qquad \text{for all } v\in V. \tag{7.9}$$

By now we have proved that if (u,p) is a formal solution of (7.3), then it would satisfy (7.9). Reciprocally, assuming that u is a solution of the variational inequality (7.9) such that $\mathrm{D}(u)\neq 0$ a.e. in Ω. Letting $\theta w=v-u$ in the previous inequality, one gets

$$\int_\Omega f\cdot(\theta w)\,b\,dx \leqslant \int_\Omega \partial_t u\cdot(\theta w)\,b\,dx+\int_\Omega (u\cdot\nabla)u\cdot\theta w\,b\,dx+2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(\theta w)\,b\,dx$$
$$+\lambda\int_\Omega \operatorname{div}u\operatorname{div}(\theta w)\,b\,dx+g\int_\Omega(|\,\mathrm{D}(u+\theta w)|-|\,\mathrm{D}(u)|)\,b\,dx+\int_{\partial\Omega}\kappa u\cdot(\theta w)\,b\,ds \qquad \text{for all } w\in V.$$

Now, as we divide by θ we get:

$$\int_\Omega f\cdot w\,b\,dx \leqslant \int_\Omega \partial_t u\cdot w\,b\,dx+\int_\Omega u\cdot\nabla u\cdot w\,b\,dx+2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(w)\,b\,dx$$
$$+\lambda\int_\Omega \operatorname{div}u\operatorname{div}w\,b\,dx+\underbrace{g\int_\Omega\frac{1}{|\,\mathrm{D}(u)|}(\mathrm{D}(u):\mathrm{D}(w)}_{\mathbf{S}}\,b\,dx+\int_{\partial\Omega}\kappa u\cdot w\,b\,ds. \tag{7.10}$$

Concerning the term S, we have in fact:

$$\mathbf{S}=\lim_{\theta\to 0}\frac{g}{\theta}\int_\Omega(|\,\mathrm{D}(u+\theta w)|-|\,\mathrm{D}(u)|)\,b\,dx. \tag{7.11}$$

On the other hand, we have

$$|\,\mathrm{D}(u+\theta w)|^2-|\,\mathrm{D}(u)|^2=\sum_{i,j}(\mathrm{D}_{i,j}(u+\theta w))^2-\sum_{i,j}(\mathrm{D}_{i,j}(u))^2$$
$$=\sum_{i,j}[(\mathrm{D}_{i,j}(u))^2+2\theta\,\mathrm{D}_{i,j}(u)\,\mathrm{D}_{i,j}(w)+\theta^2(\mathrm{D}_{i,j}(w))^2-(\mathrm{D}_{i,j}(u))^2].$$

Hence

$$\mathbf{S}=\lim_{\theta\to 0}\Big(\frac{g}{\theta}\int_\Omega\frac{|\,\mathrm{D}(u+\theta w)|^2-|\,\mathrm{D}(u)|^2}{|\,\mathrm{D}(u+\theta w)|+|\,\mathrm{D}(u)|}\,b\,dx\Big)$$
$$=g\int_\Omega\lim_{\theta\to 0}\Big(\frac{1}{|\,\mathrm{D}(u)|+|\,\mathrm{D}(u+\theta w)|}\Big)\times\lim_{\theta\to 0}\Big(\sum_{i,j}\frac{2\theta\,\mathrm{D}_{i,j}(u)\,\mathrm{D}_{i,j}(w)\,b\,dx+\theta^2\,\mathrm{D}_{i,j}(w)}{\theta}\Big)\,b\,dx$$
$$=g\int_\Omega\frac{1}{|\,\mathrm{D}(u)|}\,\mathrm{D}(u):\mathrm{D}(w)\,b\,dx.$$

By changing w into $-w$, we find the opposite inequality of (7.10), and so we get

$$\int_\Omega \partial_t(bu)\cdot w\,dx + \int_\Omega bu\cdot\nabla u\cdot w\,dx + 2\mu\int_\Omega b\,\mathrm{D}(u):\mathrm{D}(w)\,dx$$
$$+\lambda\int_\Omega b\,\mathrm{div}\,u\,\mathrm{div}\,w\,dx + g\int_\Omega \frac{b}{|\,\mathrm{D}(u)|}\,\mathrm{D}(u):\mathrm{D}(w)\,dx + \int_{\partial\Omega}\kappa u\cdot w\,b\,ds = \int_\Omega bf\cdot w\,dx.$$

Thus, we establish formally the equivalence between (7.9) and (7.3).

Let us now precise our definition of weak solution of System (7.3)-(7.4).

Definition 7.9. *We say that u is a solution of system (7.3)-(7.4) equipped with boundary and initial conditions given in (7.5)-(7.6) if u satisfies the following regularity*

$$u\in L^\infty(0,T;H_b)\cap L^2(0,T;V_b),$$
$$\partial_t(bu)\in L^2(0,T;V_b'),$$

and the following variational inequality holds

$$\int_\Omega f\cdot(v-u)\,b\,dx \leqslant \int_\Omega \partial_t u\cdot w\;b\,dx + \int_\Omega u\cdot\nabla u\cdot(v-u)\,b\,dx + 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\,b\,dx$$
$$+\lambda\int_\Omega \mathrm{div}\,u\,\mathrm{div}(v-u)\,b\,dx + g\int_\Omega(|\,\mathrm{D}(v)|-|\,\mathrm{D}(u)|)\,b\,dx + \int_{\partial\Omega}\kappa u\cdot(v-u)\,b\,ds \qquad \text{for all } v\in V_b. \tag{7.12}$$

Remark 13. *The well definition of the right hand side will be discussed in the sequel as we take u and $v\in V_b$ (mainly in lemma 7.2). It is good to mention here that for left hand side term in (7.12), one should pay attention for the regularity of bf, hence assuming the least regularity possible, i.e in the dual of V_b, we can replace the integral form by the dual representation: $\langle bf, v-u\rangle_{V_b',V_b}$. Though, this in fact doesn't affect the computations. Without loss of generality, we will assume in the sequel that bf is in $(L^2_b(\Omega))' = L^2_{b^{-1}}(\Omega)$, or equivalently f in $L^2_b(\Omega)$.*

7.4 Main Results

We state in this section the existence result of problem (7.12). The presence of b in the diffusion operator and the "b-incompressibility condition" make the weighted Sobolev spaces the ambient ones to prove existence within. In what follows, we introduce a priori estimates concerning the non linear term in lemma 7.2, and which will later on serve the well definition of the integral forms. Next, we exhibit in theorem 7.12 the existence and uniqueness results. We will rely in the proof on a Galerkin approximation technique. Several operators will be used in the latter method for which different properties will be given in lemma 7.2.

Lemma 7.2. *Let Ω be an open bounded Lipschitz domain of $\mathbb{R}^2$, and b satsifying Hypothesis (**II**). Then, for $u\in L^\infty(0,T;H_b)\cap L^2(0,T;V_b)$ and $v\in V_b$, we have:*

$$u\in L^2(0,T;L^6_b(\Omega)),$$
$$(u\cdot\nabla)u\in L^1(0,T;L^{3/2}_b(\Omega)),$$
$$(u\cdot\nabla)u\cdot v\in L^1(0,T;L^1_b(\Omega)),$$

and there exists a positive real number C independent of u and v such that

$$||u||_{L^2(0,T;L^6_b(\Omega))} \leq C||\nabla u||_{L^2(0,T;L^2_b(\Omega))},$$
$$\Big|\int_0^T \int_\Omega (u\cdot\nabla)u\, b\, dx\Big| \leq C\,||\nabla u||^2_{L^2(0,T;L^2_b(\Omega))},$$
$$\Big|\int_0^T \int_\Omega (u\cdot\nabla)u\cdot v\, b\, dx\Big| \leq C\,||\nabla u||^2_{L^2(0,T;L^2_b(\Omega))}||\nabla v||_{L^2_b(\Omega)}.$$

We define the following linear operators A and B such that

$$Au \in L^2(0,T;V_b') \quad \textit{and} \quad Bu \in L^2(0,T;V_b'),$$

where A and B are given by

$$\langle Au, v\rangle_{V_b',V_b} := a(u,v) = 2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v)\, b\, dx + \lambda\int_\Omega \operatorname{div} u \operatorname{div} v\, b\, dx + \int_{\partial\Omega} \kappa u\cdot v\, b\, ds,$$

$$\langle Bu, v\rangle_{V_b',V_b} := b(u,u,v) = \int_\Omega (u\cdot\nabla)u\cdot v\, b\, dx,$$

and b satisfies $b(u,u,u) = 0$.

Proof. Here we just want to prove that $Au \in L^2(0,T;V_b')$ and $Bu \in L^2(0,T;V_b')$. The other properties are proved in [120]. Indeed, For $u \in L^\infty(0,T;H_b)\cap L^2(0,T;V_b)$ and $v \in V_b$, we estimate using Hölder's inequality

$$\begin{aligned}|\langle Au, v\rangle_{V_b',V_b}| &= \Big|2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v)\, b\, dx + \lambda\int_\Omega \operatorname{div} u \operatorname{div} v\, b\, dx, + \int_{\partial\Omega}\kappa u\cdot v\, b\, ds\Big| \\ &\leq C\|u\|_{V_b}\|v\|_{V_b}.\end{aligned}$$

Since $u \in L^2(0,T;V_b)$, thus by duality we get

$$Au \in L^2(0,T;V_b').$$

As for the operator B, we estimate the nonlinear term using Hölder and Gagliardo-Nirenberg inequalities as follows

$$\begin{aligned}|\langle Bu, v\rangle_{V_b',V_b}| &= \Big|\int_\Omega (u\cdot\nabla)u\cdot v\, b\, dx\Big| = \Big|-\int_\Omega (u\cdot\nabla)v\cdot u\, b\, dx\Big| \\ &\leq C\|u\|^2_{L^4_b(\Omega)}\|\nabla v\|_{L^2_b(\Omega)} \\ &\leq C\|u\|_{L^2_b(\Omega)}\|\nabla u\|_{L^2_b(\Omega)}\|\nabla v\|_{L^2_b(\Omega)} \\ &\leq C\|u\|_{H_b}\|u\|_{V_b}\|v\|_{V_b}.\end{aligned}$$

Since $\|u\|_{H_b}\|u\|_{V_b} \in L^2(0,T)$, thus by duality we get

$$Bu \in L^2(0,T;V_b').$$

□

Remark 14. *Notice that, in [Proposition 3.1, [120]] , the first author proved a weighted version of Korn's inequality. However, some constraints on the weight b and the domain Ω were needed.*

More precisely, the author proved that if we exclude that case when the domain Ω is a disc, b is radial and not identically zero on the boundary, then we have

$$\|\mathrm{D}(u)\|_{L^2_b(\Omega)} \geqslant \|u\|_{H^1_b} \qquad \textit{for all } u \textit{ in } V_b.$$

Nevertheless, he showed also that these assumptions are not required when studying the evolution problem. That is why in our case we won't suppose such assumptions. For more details, the reader is referred to [120].

The main result of this paper is given below.

Theorem 7.12. *(Existence of weak solutions). We suppose that f and u_0 are the applied force and the initial datum given such that f lies in $L^2(0,T;L^2_b(\Omega))$ and u_0 belongs to H_b. Assume that b satisfies Hypothesis (**II**), and that $\kappa(x)$ is in $L^\infty(\Omega)$, then there exists a unique vector field u such that*

$$u \in L^2(0,T;V_b) \quad \partial_t(bu) \in L^2(0,T;V_b') \tag{7.13}$$

satisfying

$$\int_\Omega f\cdot(v-u)\,b\,dx \leqslant \int_\Omega \partial_t u\cdot(v-u)\,b\,dx + \int_\Omega u\cdot\nabla u\cdot(v-u)\,b\,dx + g\int_\Omega(|\,\mathrm{D}(v)| - |\,\mathrm{D}(u)|)\,b\,dx$$
$$+2\mu\int_\Omega \mathrm{D}(u):\mathrm{D}(v-u)\,b\,dx + \lambda\int_\Omega \operatorname{div} u\operatorname{div}(v-u)\,b\,dx + \int_{\partial\Omega}\kappa u\cdot(v-u)\,b dx \qquad \textit{for all } v\in V_b,$$

and the initial condition is defined in a weak sense:

$$\Big(\int_\Omega u\cdot v\,b\,dx\Big)(0) = \int_\Omega u_0\cdot v\,b\,dx \qquad \textit{for all} \quad v\in V_b.$$

Proof. In the same spirit of Lions and Duvaut's approach, our proof will be composed of fourth main steps. First, let us denote by the operator j the integral form:

$$j(\psi) = g\int_\Omega |\,\mathrm{D}(\psi)|\,b\,dx.$$

1. **Step1: Regularizing j**

   In attempt to approximate the problem, we start first by regularizing the operator j which is derived from Bingham's singular term.

   Let $\varepsilon\in(0,1)$. We approximate j by a differentiable functional

   $$j_\varepsilon(\psi) = \frac{g}{1+\varepsilon}\int_\Omega(|\,\mathrm{D}(\psi)|)^{1+\varepsilon}\,b\,dx.$$

   In fact, $j_\varepsilon(\psi)$ is well defined since as we deal with a bounded domain and the fact that $\mathrm{D}(\psi)$ is in $L^2_b(\Omega)$, we infer that $j_\varepsilon(\psi)<\infty$. The Gateaux differential of $j_\varepsilon(\cdot)$ along v is given by

   $$D_v(j_\varepsilon(w)) := \lim_{\tau\to0}\frac{j_\varepsilon(w+\tau v)-j_\varepsilon(w)}{\tau} = \frac{g}{1+\varepsilon}\int_\Omega D_v(|\,\mathrm{D}(w)|^{1+\varepsilon})\,b\,dx.$$

Then

$$\begin{aligned} D_v(|\,\mathrm{D}(w)|^{1+\varepsilon}) &= \mathrm{D}_v((|\,\mathrm{D}(w)|^2)^{\frac{1+\varepsilon}{2}}) \\ &= \frac{1+\varepsilon}{2}(|\,\mathrm{D}(w)|^2)^{\frac{\varepsilon-1}{2}} D_v(|\,\mathrm{D}(w)|^2) \\ &= \frac{1+\varepsilon}{2}|\,\mathrm{D}(w)|^{\varepsilon-1} \times 2\,\mathrm{D}(w) : \mathrm{D}(v) \qquad \text{from (7.11).} \end{aligned}$$

Therefore

$$D_v(j_\varepsilon(w)) = g\int_\Omega |\,\mathrm{D}(w)|^{\varepsilon-1}\,\mathrm{D}(w) : \mathrm{D}(v)\, b\, dx \quad < +\infty. \tag{7.14}$$

In fact, we claim that $|\,\mathrm{D}(w)|^{\varepsilon-1}\,\mathrm{D}(w)$ belong to $L^2(0,T;H_b)$

$$\left|\int_0^t\int_\Omega |\,\mathrm{D}(w)|^{\varepsilon-1} \times \mathrm{D}(w)\, b\, dx\, dt\right| \leqslant \int_0^t\int_\Omega |\,\mathrm{D}(w)|^{\varepsilon}\, b\, dx\, dt < \infty.$$

We adopt the notation in [77]

$$(j'_\varepsilon(w), v) = g\int_\Omega |\,\mathrm{D}(w)|^{\varepsilon-1}\,\mathrm{D}(w) : \mathrm{D}(v)\, b\, dx. \tag{7.15}$$

It is good to mention here that $j'_\varepsilon(w) \in L^2(0,T,V_b')$ for all w in V_b. More precisely

$$\begin{aligned} \left|\int_0^T (j'_\varepsilon(w), v)\, dt\right| &= \left|\int_0^t\int_\Omega g|\,\mathrm{D}(w)|^{\varepsilon-1}\,\mathrm{D}(w) : \mathrm{D}(v)\, b\, dx\, dt\right| \\ &\leqslant C\int_0^t \|(\mathrm{D}(w))^\varepsilon\|_{L^2_b(\Omega)}\|\mathrm{D}(v)\|_{L^2_b(\Omega)} \\ &\leqslant C\|w\|_{L^2(0,T;V_b)}\|v\|_{L^2(0,T:V_b)} \\ &\leqslant C. \end{aligned}$$

We have the following property

Lemma 7.3. *If w_ε converges weakly to w in $L^2(0,T;V_b)$, then*

$$\int_0^t j(w) \leqslant \liminf_\varepsilon \int_0^t j_\varepsilon(w_\varepsilon).$$

Proof. By Hölder's inequality, we have first

$$\begin{aligned}\int_0^t j(w_\varepsilon)\,dt &= \int_0^t \int_\Omega |\,\mathrm{D}(w_\varepsilon)|\,b\,dx\,dt \\ &\leqslant \int_0^t \Big[\Big(\int_\Omega |\,\mathrm{D}(w_\varepsilon)|^{1+\varepsilon}\ b\ dx\Big)^{\frac{1}{1+\varepsilon}} \Big(\int_\Omega b\,dx\Big)^{\frac{\varepsilon}{1+\varepsilon}}\Big]dt \\ &\leqslant C\int_0^t \Big(\int_\Omega |\,\mathrm{D}(w_\varepsilon)|^{1+\varepsilon}\ b\ dx\Big)^{\frac{1}{1+\varepsilon}}\ dt \\ &\leqslant C\int_0^t \Big(\int_\Omega |\,\mathrm{D}(w_\varepsilon)|\ b\ dx\Big)^{\frac{1}{1+\varepsilon}}\ dt \\ &\leqslant C\Big(\int_0^t j_\varepsilon(w_\varepsilon)\,dt\Big)^{\frac{1}{1+\varepsilon}}.\end{aligned}$$

Thus

$$\Big(\int_0^t j(w_\varepsilon)\,dt\Big)^{1+\varepsilon} \leqslant C\int_0^t j_\varepsilon(w_\varepsilon)\,dt.$$

As $v \longrightarrow j(v)$ is lower semi continuous for the weak topology of $L^2(0,T;V_b)$ (from the definition of weak convergence), then

$$\int_0^t j(w)\,dt \leqslant \liminf_\varepsilon \int_0^t j(w_\varepsilon)\,dt.$$

Therefore, combining the above two inequalities we get our result

$$\int_0^t j(w) \leqslant \liminf_\varepsilon \int_0^t j_\varepsilon(w_\varepsilon).$$

□

We render the problem into a new approximated one corresponding to j_ε such that:

$$\begin{aligned}&\int_\Omega \partial_t u_\varepsilon \cdot v\,b\,dx + \int_\Omega (u_\varepsilon\cdot\nabla)u_\varepsilon\cdot v\,b\,dx + 2\mu\int_\Omega \mathrm{D}(u)_\varepsilon : \mathrm{D}(v)\,b\,dx \\ &\quad + \lambda\int_\Omega \operatorname{div} u_\varepsilon\cdot\operatorname{div} v\,b\,dx + \int_{\partial\Omega}\kappa(x)u_\varepsilon\cdot v\,b\,ds + (j'_\varepsilon(u_\varepsilon),v) = \int_\Omega f\cdot(v-u)\,b\,dx \qquad \forall v\in V_b,\end{aligned} \tag{7.16}$$

where the representation $(j_\varepsilon(\cdot),\cdot)$ stands for (7.15).

2. **Step 2: Seeking for a sequence (u_m) via Galerkin approximation**

Consider the canonical isomorphism $\wedge : V_b \longrightarrow V_b'$ with $\{w_1,...,w_m,..\}$ being the set of unit eigenfunctions of the operator $\wedge$, Define now the space $V_m = \mathrm{Span}\{w_1,\dots w_m\}$, where $\{w_1,\dots,w_m\}$ is a free and total family in H_b (it is permissible to choose the set as that since V_b is separable). In fact this isomorphism is proven for weighted Sobolev spaces by *Fröhlich* in bounded domains, check for instance [122]. Upon projecting our system on V_m, and according to the Cauchy theory in a finite space, we can construct a solution $u_m := u_{\varepsilon m}$ in V_m for the approximated variational inequality (7.16)

$$\int_\Omega \partial_t u_m \cdot w_j \, b \, dx + \int_\Omega (u_m \cdot \nabla u_m) \cdot w_j \, b \, dx + 2\mu \int_\Omega \mathrm{D}(u_m) : \mathrm{D}(w_j) \, b \, dx + \lambda \int_\Omega \operatorname{div} u_m \operatorname{div} w_j \, b \, dx$$
$$+ \int_{\partial\Omega} \kappa(x) u_m \cdot w_j \, b \, ds + (j'_\varepsilon(u_m), w_j) = \int_\Omega f \cdot w_j \, b \, dx, \qquad 1 \leqslant j \leqslant m, \tag{7.17}$$

with $u_m(0)$ being the projection of u_0 on V_m, and u_m being defined in a space interval $[0, t_m]$. Since $u_m = \sum_{i=0}^{m} (u_m, w_i)_{V_b} w_i$, we multiply (7.17) by $(u_m, w_j)_{V_b}$ and sum over j, we get:

$$\int_\Omega \partial_t u_m \cdot u_m \, b \, dx + \int_\Omega (u_m \cdot \nabla) u_m \cdot u_m \, b \, dx + 2\mu \int_\Omega |\, \mathrm{D}(u_m)|^2 \, b \, dx + \lambda \int_\Omega |\operatorname{div} u_m|^2 \, b \, dx$$
$$+ \int_{\partial\Omega} \kappa(x) |u_m|^2 b \, ds + (j'_\varepsilon(u_m), u_m(t)) \, dx = \int_\Omega f \cdot u_m \, b \, dx. \tag{7.18}$$

From Lemma 7.2, we deduce that the second term is equal to zero, and since

$$(j'_\varepsilon(u), u) = \int_\Omega g |\mathrm{D}(u)|^{\varepsilon - 1} \, \mathrm{D}(u) : \mathrm{D}(u) \, b \, dx = \int_\Omega |\mathrm{D}(u)|^{\varepsilon + 1} \, dx \geqslant 0.$$

Notice that using Hölder's and Young's inequalities, we can estimate the term on the right hand of equation (7.18) as follows

$$\begin{aligned} \Big| \int_\Omega f \cdot u_m \, b \, dx \Big| &\leq \|f\|_{L^2_b(\Omega)} \|u_m\|_{L^2_b(\Omega)} \\ &\leq \eta \|f\|^2_{L^2_b(\Omega)} + C(\eta) \|u_m\|^2_{V_b}, \end{aligned}$$

for some arbitrary $\eta > 0$. Now, after choosing η sufficiently small, we can deduce using weighted Korn's inequality from Remark 14 that

$$\frac{d}{dt} \|u_m\|^2_{L^2_b(\Omega)} + \|u_m\|^2_{H^1_b(\Omega)} \leqslant C \|f\|^2_{L^2_b(\Omega)}.$$

Hence, we deduce

$$u_m \in L^\infty(0, T; H_b) \cap L^2(0, T; V_b), \tag{7.19}$$

and u_m remains in a bounded set of $L^\infty(0, T; H_b)$ and $L^2(0, T; V_b)$ uniformly with respect to m. The next step is to prove $(bu_m)' \in L^2(0, T; V'_b)$ uniformly, i.e for all m, $(bu_m)'$ remains in a bounded set of $L^2(0, T; V'_b)$. Introduce the projection operator $P_m : V_b \longrightarrow V_m$. Since Au_m, Bu_m and $j'_\varepsilon(u_m)$ belong to $L^2(0, T; V'_b)$, then, we can write from (7.17)

$$\int_\Omega \partial_t (bu_m) \cdot w_j \, dx = \langle bf - Au_m - Bu_m - j'_\varepsilon(u_m), w_j \rangle_{V'_b, V_b} \qquad \forall 1 \leqslant j \leqslant m.$$

Since we have $P_m(\, \partial_t (bu_m)) = \partial_t (bu_m)$, then we can write

$$\partial_t (bu_m) = P_m (bf - Au_m - Bu_m - j'_\varepsilon(u_m)).$$

Therefore, we get

$$\begin{aligned} b\partial_t u_m &= P_m(bf - Au_m - Bu_m - j'_\varepsilon(u_m)) \quad \text{in } D'(0,T,V_m), \\ &:= P_m(k_m). \end{aligned}$$

Since k_m is bounded in $L^2(0,T;V_b')$, then what remains to show is that

$$\|P_m k_m\|_{V_b'} \leqslant \|k_m\|_{V_b'}.$$

Since $\wedge$ is an isomorphism from V_b to V_b', then we can deduce that $\lambda^{\frac{1}{2}} w_j$ constitute an orthogonal basis of V_b' for the norm $\|X\|_{V_b'} = \|\wedge^{-1} X\|_{V_b}$. Thus, we get that

$$\begin{aligned} \|P_m k_m\|_{V_b'} &= \sum_{j=1}^{m} (k_m, \lambda^{\frac{1}{2}} w_j)_{L_b^2} \\ &\leqslant c \sum_{j=1}^{+\infty} (k_m, \lambda^{\frac{1}{2}} w_j)_{L_b^2} \\ &\leqslant c \|k_m\|_{V_b'}. \end{aligned}$$

Finally, we have our aimed result

$$bu'_m \in L^2(0,T;V_b'). \tag{7.20}$$

As a result of (7.19) and (7.20), we can deduce that up to a sequence (denoted again u_m), we have

$$\begin{aligned} u_m &\overset{*}{\rightharpoonup} u_\varepsilon \quad \text{in } L^\infty(0,T;H_b), \\ u_m &\rightharpoonup u_\varepsilon \quad \text{in } L^2(0,T;V_b), \\ bu'_m &\rightharpoonup bu_\varepsilon \quad \text{in } L^2(0,T;V_b'). \end{aligned} \tag{7.21}$$

Now, in order to use compactness result we must establish a fractional estimate in time. Indeed, following [120], we can proved that[1]

$$||\tau_h u_m - u_m||_{L^2(0,T-h;L_b^2(\Omega))} \leq c h^{\frac{1}{4}},$$

where the we denote by $\tau_h u_m(t) = u_m(t+h)$. Thus we are able to use the compactness result and deduce that $u_m \to u_\varepsilon$ strongly in $L^2(0,T;H_b)$.

3. **Step 3. Passage to the limit in m:**
The concern now is in the convergence of terms in (7.17). We will prove the convergence of the operators as $m \to \infty$.

- **Convergence of $j'_\varepsilon(u_m)$**
$j'_\varepsilon(u_m)$ is bounded uniformly in $L^2(0,T;V_b')$, hence

$$j'_\varepsilon(u_m) \rightharpoonup \xi \quad \text{in } L^2(0,T;V_b').$$

- **Convergence of the linear operator A**

[1] Inequality can be proved following the same idea used in [Proposition 4.1. in [120]]

As for the operator A given by

$$\langle Au_m, v\rangle_{V_b',V_b} = \int_\Omega 2\mu\, \mathrm{D}(u_m) : \mathrm{D}(v) + \lambda\, \mathrm{div}\, u_m\, \mathrm{div}\, v\, b\, dx + \int_{\partial\Omega} \kappa u_m \cdot v\, b\, ds,$$

we have in fact that $u_m \rightharpoonup u_\varepsilon$ in $L^2(0,T;V_b)$, then $\partial_i u_j \rightharpoonup \partial_i (u_\varepsilon)_j$ in $L^2(0,T;L^2_b(\Omega))$, and hence

$$\int_0^T \int_\Omega \mathrm{D}(u_m) : \mathrm{D}(v)\, b\, dx\, dt \to \int_0^T \int_\Omega \mathrm{D}(u_\varepsilon) : \mathrm{D}(v)\, b\, dx\, dt,$$

and

$$\int_0^T \int_\Omega \mathrm{div}\, u_m\, \mathrm{div}\, v\, b\, dx\, dt \to \int_0^T \int_\Omega \mathrm{div}\, u_\varepsilon\, \mathrm{div}\, v\, b\, dx\, dt.$$

Before we treat the convergence on the boundary term, let us prove the following lemma.

Lemma 7.4. *Let u and v be two sufficiently smooth vectors such that $bu \cdot n = 0$ and $bv \cdot n = 0$, then*

$$(v \cdot \nabla(bu)) \cdot n = -\kappa bu \cdot v. \tag{7.22}$$

Proof. Since we have $bu \cdot n = 0$ on $\partial\Omega$, then

$$\begin{aligned} 0 = \frac{\partial}{\partial\tau}(bu \cdot n) &= \frac{\partial(bu)}{\partial\tau} \cdot n + bu \cdot \frac{\partial n}{\partial\tau} \\ &= (\tau \cdot \nabla(bu)) \cdot n + \kappa bu \cdot \tau. \end{aligned}$$

Recall that κ is the curvature of $\partial\Omega$ where we have

$$\frac{\partial n}{\partial\tau} := \frac{dn}{ds} = \kappa u \cdot \tau.$$

But v is parallel to τ, so (7.22) follows by linearity. □

Now back to prove the convergence of the boundary term, we notice first using (7.22) and the following identity

$$\nabla(bu) : \nabla^t v = \partial_i(bu_j)\, \partial_j v_i = \partial_j(v_i\, \partial_i(bu_j)) = \mathrm{div}(v \cdot \nabla(bu))$$

that we can write

$$\begin{aligned} \int_{\partial\Omega} \kappa(u_m - u_\varepsilon) \cdot v\, b\, ds &= -\int_{\partial\Omega} (v \cdot \nabla(b(u_m - u_\varepsilon))) \cdot n\, ds \\ &= -\int_\Omega \mathrm{div}(v \cdot \nabla(b(u_m - u_\varepsilon)))\, dx \\ &= -\int_\Omega \nabla^t v : \nabla(b(u_m - u_\varepsilon))\, dx \\ &= -\int_\Omega \nabla^t v : \nabla(u_m - u_\varepsilon)\, b\, dx + \int_\Omega \nabla^t v : \frac{\nabla b}{b} \otimes (u_m - u_\varepsilon)\, b\, dx. \end{aligned}$$

We easily remark that the first term on the right hand side of the above equality converges to zero because of the weak convergence of u_m to u_ε in V_b. For the second

term, we need to use a Hardy's type inequality to show that

$$\frac{\nabla b}{b} \otimes (u_m - u_\varepsilon) \quad \text{is uniformaly bounded in} \quad L^2_b(\Omega).$$

Indeed for $b = \rho^\alpha$, we have $\nabla b/b \sim 1/\rho$, then (see [123])

$$\begin{aligned}\int_\Omega \Big|\frac{\nabla b}{b} \otimes (u_m - u_\varepsilon)\Big|^2 b\, dx &= \int_\Omega \Big|\frac{1}{\rho}(u_m - u_\varepsilon)\Big|^2 b\, dx \\ &\leq C \int_\Omega |\nabla(u_m - u_\varepsilon)|^2 b\, dx.\end{aligned}$$

Thus we get the convergence of the boundary term.

- **Convergence of the trilinear term B**
  For the trilinear term, we remark that

$$b(u_m, u_m, v) = -b(u_m, v, u_m).$$

Using the fact that $H^1_b(\Omega) \hookrightarrow L^6_b(\Omega) \hookrightarrow L^4_b(\Omega)$ (since Ω is bounded, and b $\in A_{\frac{3}{2}}$ [124]), and due to the boundedness and strong convergence results proved for u_m, we obtain

$$\begin{aligned}&|\int_0^T \big(\int_\Omega (u_m \cdot \nabla)v \cdot u_m\, b\, dx - \int_\Omega (u_\varepsilon \cdot \nabla)v \cdot u_\varepsilon\, b\, dx\big)\, dt| \\ &= |\int_0^T \big(\int_\Omega (u_m - u_\varepsilon) \cdot \nabla v \cdot u_m \ + \int_\Omega u_\varepsilon \cdot \nabla v \cdot (u_m - u_\varepsilon)\, b\, dx\big)\, dt| \\ &\leqslant \int_0^T \|u_m\|_{L^4_b} \|\nabla v\|_{L^2_b} \|u_m - u_\varepsilon\|_{L^4_b}\, dt + \int_0^T \|u_\varepsilon\|_{L^4_b} \|\nabla v\|_{L^2_b} \|u_m - u_\varepsilon\|_{L^4_b}\, dt \\ &\leqslant c \int_0^T \|\nabla v\|_{H_b} \big(\|u_m\|_{V_b} + \|u_\varepsilon\|_{V_b}\big) \big(\|\nabla u_m\|_{L^2_b} + \|\nabla u_\varepsilon\|_{L^2_b}\big)^{\frac{1}{2}} \|u_m - u_\varepsilon\|^{\frac{1}{2}}_{L^2_b}\, dt \\ &\leqslant c \|u_m - u_\varepsilon\|^{\frac{1}{2}}_{L^2(0,T,L^2_b)} \underset{m\to 0}{\longrightarrow} 0\,.\end{aligned}$$

- **Convergence of the integral containing u'_m**
  Since we have $bu'_m \rightharpoonup bu'_\varepsilon$ in $L^2(0, T, V'_b)$, thus

$$\int_0^T \int_\Omega \partial_t u_m \cdot v\, b\, dx\, dt \longrightarrow \int_0^T \int_\Omega \partial_t u_\varepsilon \cdot v\, b\, dx\, dt \qquad \forall v \in V_b.$$

- **Convergence of the initial condition**
  The space V_b can be viewed as

$$V_b = \overline{D_b(\Omega)}^{||.||_{H^1_b(\Omega)}}$$

Now, we choose the orthonormal base of V_b such that for all j, we have $w_j \in D_b(\Omega)$. In this case,

$$\int_\Omega (u_m \cdot \nabla) u_m \cdot w_j\, b\, dx \quad \text{is bounded in} \quad L^2(0, T)$$

since $(bu_m \cdot \nabla)u_m$ is bounded in $L^2(0,T;(L^1(\Omega))^2)$. Moreover we have

$$a(u_m, w_j) + b(u_m, u_m, w_j) \in L^2(0,T).$$

Thus Equation (7.17) shows that

$$\partial_t \langle u_m, w_j \rangle_b \quad \text{is bounded in} \quad L^2(0,T), \tag{7.23}$$

and hence $\int_\Omega u_m.w_j\, b\, dx$ is bounded in $H^1(0,T)$. On the other hand, we know that $H^1(0,T)$ is compact in $\mathcal{C}_u(0,T)$ (see Theorem III.2.34 in [125]) which yields

$$\int_\Omega u_m \cdot w_j\, b\, dx \to \int_\Omega u_\varepsilon \cdot w_j\, b\, dx \ \text{ in } \ \mathcal{C}_u(0,T).$$

The above convergence holds also in $L^2(0,T)$ (due to strong convergence of u_m in $L^2(0,T;H_b)$). In particular

$$\Big(\int_\Omega u_m \cdot w_j\, b\, dx\Big)(0) \to \Big(\int_\Omega u_\varepsilon \cdot w_j\, b\, dx\Big)(0).$$

Notice that, by definition, we have $u_m(0) = P_m(u_0)$, so we can write

$$\int_\Omega (u_m(0,x) - P_m(u_0(x))) \cdot w_j(x)\, b\, dx = 0, \tag{7.24}$$

Passing to the limit in Equation (7.24) yields to

$$\Big(\int_\Omega u_\varepsilon \cdot w_j\, b\, dx\Big)(0) = \int_\Omega u_0 \cdot w_j\, b\, dx. \tag{7.25}$$

On other hand, by virtue of Equation (7.16), we have for all $v \in V_b$,

$$\partial_t \int_\Omega u_\varepsilon \cdot v\, b\, dx = \partial_t \langle u_\varepsilon, v \rangle_b \in L^1(0,T)$$

which yields that

$$\langle u_\varepsilon, v \rangle_b \in C([0,T]).$$

Since $u_\varepsilon \in L^\infty(0,T;H_b)$ and V_b is dense in H_b, we infer that (see Lemma 1.4 in [126]).

$$u_\varepsilon \in C([0,T];H_b - \text{weak}) \text{ and that } u_\varepsilon(0) = u_0 \text{ in } C([0,T];H_b - \text{weak}).$$

It remains to show that $\xi = j'_\varepsilon(u_\varepsilon)$. By now we have gotten

$$\int_0^T \int_\Omega u'_\varepsilon \cdot v\, b\, dx\, dt + \int_0^T \langle Bu_\varepsilon, v \rangle_{V_b', V_b}\, dt + \int_0^T \langle Au_\varepsilon, v \rangle_{V_b', V_b}\, dt + \int_0^T (\xi, v)\, dt = \int_0^T \int_\Omega f \cdot v\, b\, dx\, dt. \tag{7.26}$$

We choose a function φ such that φ in $L^2(0,T,V_b)$, $b\varphi'$ in $L^2(0,T,V_b')$ then introduce the following quantity

$$X_m = \int_0^T \langle bu'_m - b\varphi', u_m - \varphi \rangle_{V_b', V_b}\, dt + \int_0^T (j'_\varepsilon(u_m) - j'_\varepsilon(\varphi), u_m - \varphi)\, dt + \int_0^T \langle Au_m - A\varphi, u_m - \varphi \rangle_{V_b', V_b}\, dt.$$

One can show obviously that X_m is positive. For instance, we can rewrite X_m such that

$$\begin{aligned} X_m = \int_0^T & (j_\varepsilon'(u_m), u_m) + \langle Au_m, u_m\rangle_{V_b',V_b} + \langle bu_m', u_m\rangle_{V_b',V_b} \\ & - (j_\varepsilon'(u_m), \varphi) - (j_\varepsilon'(\varphi), u_m - \varphi) \\ & - \langle Au_m, \varphi\rangle_{V_b',V_b} - \langle A\varphi, u_m - \varphi\rangle_{V_b',V_b} \\ & - \langle bu_m', \varphi\rangle_{V_b',V_b} - \langle b\varphi', u_m - \varphi\rangle_{V_b',V_b}\, dt. \end{aligned}$$

Substituting (7.17) in the above expression

$$\begin{aligned} X_m = \int_0^T & \langle bf, u_m\rangle_{V_b',V_b} - (j_\varepsilon'(u_m), \varphi) - (j_\varepsilon'(\varphi), u_m - \varphi) \\ & - \langle Au_m, \varphi\rangle_{V_b',V_b} - \langle A\varphi, u_m - \varphi\rangle_{V_b',V_b} - \langle bu_m', \varphi\rangle_{V_b',V_b} - \langle b\varphi', u_m - \varphi\rangle_{V_b',V_b}\, dt. \end{aligned}$$

Then

$$\begin{aligned} X_m \underset{m\to 0}{\longrightarrow} \int_0^T & \langle bf, u_\varepsilon\rangle_{V_b',V_b} - (\xi, \varphi) - (j_\varepsilon'(\varphi), u_\varepsilon - \varphi) \\ & - \langle Au_\varepsilon, \varphi\rangle_{V_b',V_b} - \langle A\varphi, u_\varepsilon - \varphi\rangle - \langle bu_\varepsilon', \varphi\rangle_{V_b',V_b} - \langle b\varphi', u_\varepsilon - \varphi\rangle_{V_b',V_b}\, dt \\ & := X_\varepsilon. \end{aligned}$$

Again, take $v = u_\varepsilon$ in (7.26), and then substitute this latter in X_ε, we get

$$\begin{aligned} X_\varepsilon = \int_0^T & \Big[\langle bu_\varepsilon', u_\varepsilon\rangle_{V_b',V_b} + \langle Au_\varepsilon, u_\varepsilon\rangle_{V_b',V_b} + \langle \xi, u_\varepsilon\rangle_{V_b',V_b} - \langle \xi, \varphi\rangle_{V_b',V_b} - (j_\varepsilon'(\varphi), u_\varepsilon - \varphi) \\ & - \langle Au_\varepsilon, \varphi\rangle_{V_b',V_b} - \langle A\varphi, u_\varepsilon - \varphi\rangle_{V_b',V_b} - \langle bu_\varepsilon', \varphi\rangle_{V_b',V_b} - \langle b\varphi', u_\varepsilon - \varphi\rangle_{V_b',V_b}\Big]\, dt \\ = \int_0^T & \Big[\langle bu_\varepsilon' - b\varphi', u_\varepsilon - \varphi\rangle_{V_b',V_b} + \langle Au_\varepsilon - A\varphi, u_\varepsilon - \varphi\rangle_{V_b',V_b} + (\xi - j_\varepsilon'(\varphi), u_\varepsilon - \varphi)\Big]\, dt. \end{aligned}$$

As $X_m \geqslant 0$, then $X_\varepsilon \geqslant 0$. Take now $\varphi = u_\varepsilon - \theta\psi$ with $\theta \in \mathbb{R}$ in the expression of X_ε. This yields

$$X_\varepsilon = \int_0^T \langle b\theta\psi,', \theta\psi\rangle_{V_b',V_b} + \langle A(\theta\psi), \theta\psi\rangle_{V_b',V_b} + (\xi - j_\varepsilon'(u_\varepsilon - \theta\psi), \theta\psi)\, dt \geqslant 0.$$

Dividing by θ we get

$$\int_0^T \theta\langle b\psi,', \psi\rangle_{V_b',V_b} + \theta\langle A(\psi), \psi\rangle_{V_b',V_b} + (\xi - j_\varepsilon'(u_\varepsilon - \theta\psi), \psi)\, dt \geqslant 0.$$

Take $\theta \to 0$, we obtain

$$\int_0^T (\xi - j_\varepsilon'(u_\varepsilon), \psi)\, dt \geqslant 0.$$

This is true for all ψ, hence:

$$\xi = j_\varepsilon'(u_\varepsilon).$$

Therefore in this step we have proved the existence of a sequence $(u_\varepsilon)_\varepsilon$ such that:

$$\begin{cases} u_\varepsilon \text{ in, } L^\infty(0,T;H_b)\cap L^2(0,T;V_b), \\ \\ bu'_\varepsilon \text{ in } L^2(0,T;V_b'), \\ \\ \int_0^T \langle bu'_\varepsilon, v\rangle_{V_b',V_b} + \langle Bu_\varepsilon, v\rangle_{V_b',V_b} + \langle Au_\varepsilon, v\rangle_{V_b',V_b} + (j'_\varepsilon(u_\varepsilon), v) = \int_0^T \langle bf, v\rangle_{V_b',V_b} \\ \forall v \in L^2(0,T;V_b). \end{cases} \tag{7.27}$$

4. **Step 4: Passage to the limit in ε.** Since the bounds obtained in (7.27) are uniformly bounded in ε in the desired spaces, we conclude that there exists a subsequence of u_ε, that is still denoted the same, such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \text{ in } L^2(0,T;V_b), \\ u_\varepsilon &\overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H_b), \\ bu'_\varepsilon &\rightharpoonup bu' \text{ in } L^2(0,T;V_b'). \end{aligned}$$

Again, by compactness theory we get $u_\varepsilon \to u$ strongly in $L^2(0,T;H_b)$. We introduce the following quantity

$$\begin{aligned} Z_\varepsilon = \int_0^T &\langle bu'_\varepsilon, v-u_\varepsilon\rangle_{V_b',V_b} + \langle Au_\varepsilon, v-u_\varepsilon\rangle_{V_b',V_b} + \langle Bu_\varepsilon, v-u_\varepsilon\rangle_{V_b',V_b} \\ &+ j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) - \langle bf, v-u_\varepsilon\rangle_{V_b',V_b}\, dt. \end{aligned}$$

Taking $v = u_\varepsilon$ in (7.27) (which is allowed since $u_\varepsilon \in L^2(0,T;V_b)$), and then substituting in Z_ε, we get

$$Z_\varepsilon = \int_0^T j_\varepsilon(v) - j_\varepsilon(u_\varepsilon) - (j'_\varepsilon(u_\varepsilon), v-u_\varepsilon)\, dt.$$

In fact, since j_ε is coercive, then we get $Z_\varepsilon \geqslant 0$. Hence

$$\begin{aligned} &\int_0^T \Big[\langle bu'_\varepsilon, v\rangle_{V_b',V_b} + \langle Au_\varepsilon, v\rangle_{V_b',V_b} + \langle Bu_\varepsilon, v\rangle_{V_b',V_b} + j_\varepsilon(v) - \langle bf, v-u_\varepsilon\rangle_{V_b',V_b}\Big]\, dt \\ &\geqslant \int_0^T \Big[\langle bu'_\varepsilon, u_\varepsilon\rangle_{V_b',V_b} + \langle Au_\varepsilon, u_\varepsilon\rangle_{V_b',V_b} + \underbrace{\langle Bu_\varepsilon, u_\varepsilon\rangle_{V_b',V_b}}_{0} + j_\varepsilon(u_\varepsilon)\Big]\, dt. \end{aligned}$$

Now as we have $u_\varepsilon \to u$ in $L^2(0,T;H_b)$, then there exists a subsequence such that $u_\varepsilon(T) \to u(T)$ in H_b. Therefore:

$$\begin{aligned} &\liminf_{\varepsilon\to 0}\Big[\int_0^T \langle bu'_\varepsilon, v\rangle_{V_b',V_b} + \langle Au_\varepsilon, v\rangle_{V_b',V_b} + \langle Bu_\varepsilon, v\rangle_{V_b',V_b} + j_\varepsilon(v) - \langle bf, v-u_\varepsilon\rangle_{V_b',V_b}\, dt\Big] \\ &\geqslant \frac{1}{2}\liminf_{\varepsilon\to 0}\|u_\varepsilon(T)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \liminf_{\varepsilon\to 0}\int_0^T \langle Au_\varepsilon, u_\varepsilon\rangle_{V_b',V_b}\, dt + \liminf_{\varepsilon\to 0}\int_0^T j_\varepsilon(u_\varepsilon)\, dt. \end{aligned}$$

Due to the fact that all terms on the right hand side are lower semi continuous (norms

and linear continuous maps are lower semi continuous), and using lemma 7.3, we obtain

$$\begin{aligned}&\int_0^T \langle bu', v\rangle_{V_b',V_b} + \langle Au, v\rangle_{V_b',V_b} + \langle Bu, v\rangle_{V_b',V_b} + j(v) - \langle bf, v-u\rangle_{V_b',V_b}\\ &\geqslant \frac{1}{2}\|u(T)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \int_0^T \langle Au, u\rangle_{V_b',V_b}\,dt + \int_0^T j(u)\,dt.\end{aligned}$$

Finally, we get

$$\begin{aligned}&\int_0^T \Big(\langle bu', v-u\rangle_{V_b',V_b} + \langle Au, v-u\rangle_{V_b',V_b} + \langle Bu, v-u\rangle_{V_b',V_b} + j(v) - j(u)\\ &- \langle bf, v-u\rangle_{V_b',V_b}\Big)\,dt \geqslant 0 \quad \forall v \in L^2(0,T;V_b).\end{aligned} \tag{7.28}$$

The last step is to prove that the above inequality not only holds in integral form but also almost everywhere in $(0,T)$, in particular, we have to prove

$$\begin{aligned}&\langle bu'(t), v-u(t)\rangle_{V_b',V_b} + \langle Au(t), v-u(t)\rangle_{V_b',V_b} + \langle Bu(t), v-u(t)\rangle_{V_b',V_b} + j(v) - j(u(t))\\ &\quad\geqslant \langle bf, v-u(t)\rangle_{V_b',V_b}\,dt, \quad t\in(0,T), \quad v\in L^2(0,T;V_b).\end{aligned} \tag{7.29}$$

Proving this in fact allows us to prove the existence of solution for system (7.12). Let us fix some test function $v \in V_b$, and $t_0 \in (0,T)$. Define

$$\Theta_j = \left(t_0 - \frac{1}{j}, t_0 + \frac{1}{j}\right) \subset (0,T) \qquad j \text{ large enough.}$$

We introduce

$$\bar{v} = \begin{cases} v & t \in \Theta_j \\ u(t) & t \in [0,T]\backslash\Theta_j. \end{cases}$$

Hence taking $v = \bar{v}$, inequality (7.28) becomes

$$\begin{aligned}&\underbrace{\int_{\Theta_j} \langle bu' + Au + Bu - bf, v\rangle_{V_b',V_b}\,dt}_{X}\\ &+ \underbrace{\int_{\Theta_j}\Big[j(v) - j(u) - \langle bu' + Au - bf, u\rangle_{V_b',V_b}\Big]\,dt}_{Y} \geqslant 0.\end{aligned}$$

Using the Lebesgue theorem in the differentiation of set functions, we have

$$X \underset{j\to 0}{\longrightarrow} \langle bu'(t_0) + Au(t_0) + Bu(t_0) - bf, v\rangle_{V_b',V_b},$$

and

$$Y \underset{j\to 0}{\longrightarrow} \langle bu'(t_0) + Au(t_0) - bf, u(t_0)\rangle_{V_b',V_b} + j(u(t_0)) - j(v).$$

But t_0 is arbitrary in (0,T), thus we get

$$\begin{aligned}&\langle bu'(t), v-u(t)\rangle_{V_b',V_b} + \langle Au(t), v-u(t)\rangle_{V_b',V_b} + \langle Bu(t), v-u(t)\rangle_{V_b',V_b}\\ &\quad + j(v) - j(u(t)) \geqslant \langle bf, v-u(t)\rangle_{V_b',V_b},\end{aligned} \tag{7.30}$$

for all $t \in (0,T)$, for all $v \in L^2(0,T;V_b)$. By this we end the proof of existence.

Uniqueness of solution. let $\mathrm{u}_1, \mathrm{u}_2$ be two solutions of the variational inequality (7.30). We take $v = \mathrm{u}_2(t)$ (resp $v = \mathrm{u}_1(t)$) as a test function in the variational inequality satisfied by u_1 (resp. u_2). Let $U = \mathrm{u}_1 - \mathrm{u}_2$. Now, if we add both inequalities we get

$$\begin{aligned}&\int_\Omega \partial_t U \cdot U\, b\, dx + 2\mu \int_\Omega |\,\mathrm{D}(U)|^2\, b\, dx + \lambda \int_\Omega |\,\mathrm{div}(U)|^2\, b\, dx\\ &\quad - \int_\Omega (\mathrm{u}_1\cdot\nabla)\,\mathrm{u}_1\cdot U\, b\, dx + \int_\Omega (\mathrm{u}_2\cdot\nabla)\,\mathrm{u}_2\cdot U\, b\, dx \leq 0.\end{aligned}$$

Therefore, using Remark 14, and the fact that

$$\int_\Omega (u\cdot\nabla)v\cdot v\, b\, dx = 0 \quad \text{for all} \quad u, v \quad \text{in} \ V_b,$$

we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |U|^2\, b\, dx + 2C_1\mu\|U\|_{V_b}^2 &\leq \int_\Omega (\mathrm{u}_1\cdot\nabla)\,\mathrm{u}_1\cdot U\, b\, dx - \int_\Omega (\mathrm{u}_2\cdot\nabla)\,\mathrm{u}_2\cdot U\, b\, dx\\ &= \int_\Omega (\mathrm{u}_1\cdot\nabla)U\cdot U\, b\, dx + \int_\Omega (\mathrm{u}_1\cdot\nabla)\,\mathrm{u}_2\cdot U\, b\, dx - \int_\Omega (\mathrm{u}_2\cdot\nabla)\,\mathrm{u}_2\cdot U\, b\, dx\\ &= \int_\Omega (U\cdot\nabla)\,\mathrm{u}_2\cdot U\, b\, dx,\end{aligned} \tag{7.31}$$

where C_1 is the constant resulting from Korn's inequality. We estimate the term on the right hand side of Inequality (7.31) using Weighted Gagliardo-Nirenberg inequality and Young's inequality as follows

$$\begin{aligned}\left|\int_\Omega (U\cdot\nabla)\cdot \mathrm{u}_2\cdot U\, b\, dx\right| &\leqslant \|U\|^2_{L^4_b(\Omega)}\|\nabla u_2\|_{L^2_b(\Omega)}\\ &\leqslant C\|\nabla u_2\|_{L^2_b(\Omega)}\|U\|_{V_b}\|U\|_{L^2_b(\Omega)}\\ &\leqslant C\|\nabla u_2\|_{L^2_b(\Omega)}\Big(\frac{1}{2C_1\mu}\|U\|^2_{L^2_b(\Omega)} + \frac{C_1\mu}{2}\|U\|^2_{V_b}\Big).\end{aligned} \tag{7.32}$$

From Inequality (7.31) and estimate (7.32), we deduce that

$$\frac{1}{2}\frac{d}{dt}\|U\|^2_{L^2_b(\Omega)} \leq C\|\nabla u_2\|_{L^2_b(\Omega)}\|U\|^2_{L^2_b(\Omega)}.$$

which readily ensures that $U = 0$, whence $\mathrm{u}_1 - \mathrm{u}_2 = 0$ by applying a Gronwall's type inequality. The proof of uniqueness of solution is finished.

7.5 Newtonian fluids as a limit of Non-Newtonian fluids

Viscous lake systems for Newtonian fluids are asymptotically derived systems according to two successive approximations that are characterized by the smallness of non dimensional parameters. The first is the rigid lid approximation assuming that the typical deviation of the top of the fluid's surface from the mean level is much smaller than the typical depth. This smallness can be regarded also as a Froude number due to the dynamics of the physical case at hand. The second approximation is the shallow water approximation characterizing the ansatz of the smallness of the typical depth compared to the typical horizontal length, see for instance [127]. Nevertheless, in this section, we will prove the existence of a weak solution of the viscous lake system by passing to the zero limit of g (yield stress limit) in the variational inequality satisfied by the weak solution of Bingham system (7.3). Roughly speaking, one can naively conclude that the fluid will behave as a viscous liquid once g vanishes, yet for the mathematical justification we need to set on a rigorous proof. So, we will prove the following theorem.

Theorem 7.13. *Let Ω a bounded Lipschitz domain and b satisfying Hypothesis (**II**). Consider the Bingham model introduced in section 7.3. We suppose that all the parameters therein are fixed except for the the yield stress limit g which is assumed to be independent of other parameters and rendered to vary (consequently taken to zero). Denote by u_g the constructed solution of system (7.12) with initial data $u|_{t=0} = u_g(0)$ proved in section 7.4. Then, if $u_g(0)$ converges to u_0 in $L^2_b(\Omega)$ as g tends to zero, then up to a sub-sequence, we have*

$$\begin{cases} u_g \to u \quad \text{weakly in} \quad L^\infty(0,T;H_b) \cap L^2(0,T;V_b), \\ bu'_g \to bu' \quad \text{weakly in} \quad L^2(0,T;V'_b), \end{cases} \tag{7.33}$$

where u is a weak solution of the viscous Lake system and thus

$$\|u_g(t,x) - u(t,x)\|^2_{L^2_b(\Omega)} \to 0 \quad \text{when} \quad g \quad \text{tends to} \quad 0,$$

and the solution u of the viscous lake equations enjoys the following property

$$u \in L^\infty(0,T;H_b) \cap L^2(0,T;V_b).$$

Proof. Let's recall the definition of the weak solution of viscous lake equations: $u \in L^2(0,T,V_b) \cap L^\infty(0,T,H_b)$ is said to be a weak solution of the viscous lake equations if it satisfies

$$\langle u', v\rangle_{L^2_b} + b(u,u,v) + a(u,v) = \langle bf, v\rangle_{V'_b,V_b} \ \forall v \in V_b$$

From (7.12), we find that the weak solution u_g satisfies

$$\frac{1}{2}\frac{d}{dt}\|u_g\|^2_{L^2_b(\Omega)} + \mu\|u_g\|^2_{V_b} + \lambda\|\operatorname{div} u\|^2_{L^2_b(\Omega)} \leqslant C.$$

Integrating in time, we get

$$\frac{1}{2}\|u_g(t)\|^2_{L^2_b(\Omega)} + \mu\int_0^t \|u_g\|^2_{V_b}\,ds + \lambda\int_0^t \|\operatorname{div} u\|^2_{L^2_b(\Omega)}\,ds \leqslant \frac{1}{2}\|u_g(0)\|^2_{L^2_b(\Omega)} + C \leqslant C.$$

Thus, u_g remains in a bounded set with respect to g in $L^\infty(0,T;H_b) \cap L^2(0,T;V_b)$. In addition, from previous section, we have

$$bu'_{\varepsilon g} \rightharpoonup bu'_g \ \in L^2(0,T;V'_b),$$

so, bu'_g is also bounded in $L^2(0,T;V'_b)$ (the bound is uniform in g). Hence, there exists $u \in L^2(0,T;V_b) \cap L^\infty(0,T;H_b)$ such that

$$u_g \rightharpoonup u \in L^2(0,T;V_b),$$
$$u_g \overset{*}{\rightharpoonup} u \in L^\infty(0,T;H_b),$$
$$bu'_g \rightharpoonup bu' \in L^2(0,T;V'_b).$$

Following the same compactness technique used in previous section, we get $u_g \longrightarrow u \in L^2(0,T;H_b)$, and the convergence of the corresponding operators follows as in section 7.4. Hence u_g satisfies

$$\langle bu'_g, v-u_g\rangle_{V'_b,V_b} + a(u_g, v-u_g) + b(u_g,u_g,v-u_g) + j(v) - j(u_g) \geqslant \langle bf, v-u_g\rangle_{V'_b,V_b}.$$

Consequently

$$\langle bu'_g, v\rangle_{V'_b,V_b} + a(u_g,v) + b(u_g,u_g,v) + j(v) - j(u_g) - \langle bf, v-u_g\rangle_{V'_b,V_b} \geqslant \frac{1}{2}\frac{d}{dt}\langle u_g,u_g\rangle_{L^2_b(\Omega)} + a(u_g,u_g).$$

Integrating in time gives

$$\begin{aligned}&\int_0^t \langle bu'_g, v\rangle_{V'_b,V_b} + a(u_g,v) + b(u_g,u_g,v) + j(v) - j(u_g) - \langle bf, v-u_g\rangle_{V'_b,V_b}\, dt \\ &\quad\geqslant \frac{1}{2}\|u_g(t)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_g(0)\|^2_{L^2_b(\Omega)} + \int_0^t a(u_g,u_g)\, dt.\end{aligned}$$

Using the fact that RHS is lower semi continuous , and since $|\,\mathrm{D}(u_g)|$ and $|\,\mathrm{D}(v)|$ are bounded in $L^1(0,T;H_b)$, then

$$\lim_{g\to 0} j(v) = \lim_{g\to 0}\int_\Omega g|\,\mathrm{D}(v)|\, b\, dx = 0,$$

and

$$\lim_{g\to 0} j(u_g) = \lim_{g\to 0}\int_\Omega g|\,\mathrm{D}(u_g)|\, b\, dx = 0.$$

Therefore, for all t in $(0,T)$, we have

$$\begin{aligned}&\liminf_{g\to 0}\left[\int_0^t \langle bu'_g, v\rangle_{V'_b,V_b} + a(u_g,v) + b(u_g,u_g,v) - \langle bf, v-u_g\rangle_{V'_b,V_b}\, dt\right] \\ &\quad\geqslant \liminf_{g\to 0}\left[\frac{1}{2}\|u_g(t)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_g(0)\|^2_{L^2_b(\Omega)} + \int_0^t a(u_g,u_g)\, dt\right].\end{aligned} \tag{7.34}$$

Due to the convergence of the LHS terms and the last term in the RHS of (7.34) (following the same strategy as in the proof of theorem (7.12) in section 7.4), and due to the lower semi-continuity of the norm operator for the topology of $L^2_b(\Omega)$, we obtain

$$\begin{aligned}&\int_0^t \langle bu', v\rangle_{V'_b,V_b} + a(u,v) + b(u,u,v) - \langle bf, v-u\rangle_{V'_b,V_b}\, dt \\ &\quad\geqslant \frac{1}{2}\|u(t)\|^2_{L^2_b(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2_b(\Omega)} + \int_0^t a(u,u)\, dt \\ &\quad= \frac{1}{2}\int_0^t \frac{d}{dt}\langle u,u\rangle_{L^2_b(\Omega)}\, dt + \int_0^t a(u,u)\, dt.\end{aligned}$$

Thus, for all $v \in L^2(0,T;V_b)$, and for all t in $(0,T)$, we get

$$\int_0^t \langle bu', v-u\rangle_{V_b',V_b} + a(u,v-u) + b(u,u,v-u)\,dt \geqslant \int_0^t \langle bf, v-u\rangle_{V_b',V_b}\,dt.$$

Again, using Lebesgue theory in the differentiation of set functions, we obtain for all $v \in L^2(0,T;V_b)$

$$\langle bu', v-u\rangle_{V_b',V_b} + a(u,v-u) + b(u,u,v-u) \geqslant \langle bf, v-u\rangle_{V_b',V_b} \quad \text{a.e in } [0,T]. \tag{7.35}$$

If we suppose that $\varphi = \pm(u-v) \in L^2(0,T;V_b)$, then substituting v in (7.35) yields

$$\langle bu', \varphi\rangle_{V_b',V_b} + a(u,\varphi) + b(u,u,\varphi) = \langle bf, \varphi\rangle_{V_b',V_b} \quad \text{a.e in } [0,T],$$

which means that u satisfies the weak formulation of the viscous Lake system. Thus we end the proof.

7.6 Numerical Scheme

In this section, we propose in the one dimensional case a numerical scheme for the approximation of the studied model i.e. the system (7.3)-(7.4).

7.6.1 Semi-discrete scheme

The system (7.3)-(7.4) can be rewritten as follows

$$\begin{aligned} \partial_t X + \partial_x F(X) - b\,\partial_x p &= bf, \\ \partial_x X &= 0, \end{aligned}$$

where

$$X = (bu), \quad F(X) = bu^2.$$

Notice that in the 1d case, the definition of σ given by (7.4) reduces to

$$\sigma := \begin{cases} 2\mu b\,\partial_x u + gb\dfrac{\partial_x u}{|\partial_x u|} & \text{if } \partial_x u \neq 0, \\ |\sigma| < bg, & \text{if } \partial_x u = 0. \end{cases} \tag{7.36}$$

For the time discretisation, we denote $t^n = \sum_{k\leq n} \Delta t^k$ where the time steps Δt^k will be precised later though a CFL condition. Following [128], we use an operator splitting technique resulting in a two step scheme

$$\frac{X^{n+1/2} - X^n}{\Delta t^n} = -\partial_x F(X^n) + bf^n, \tag{7.37}$$

$$\frac{X^{n+1} - X^{n+1/2}}{\Delta t^n} = b\,\partial_x p^{n+1}, \tag{7.38}$$

where the quantity X^{n+1} satisfies the divergence free constraint

$$\partial_x(X^{n+1}) = 0. \tag{7.39}$$

The system (7.37)-(7.39) has to be completed with suitable boundary conditions that will be

precised later, see paragraph 7.6.4.

More precisely, the prediction step (7.37) consists in the resolution of advection diffusion equation i.e.

$$X^{n+1/4} = X^n - \Delta t^n \partial_x F(X^n) + \Delta t^n \, \partial_x \Big(\mu b \, \partial_x \big(\frac{X^{n+1/4}}{b} \big) \Big), \tag{7.40}$$

$$X^{n+1/2} = X^{n+1/4} + \Delta t^n \partial_x (b \tilde{\sigma}^{n+1/4}) + \Delta t^n b f^n, \tag{7.41}$$

where

$$\tilde{\sigma}^{n+1/4} = \begin{cases} g \frac{\partial_x u^{n+1/4}}{|\partial_x u^{n+1/4}|} & \text{if } \partial_x u^{n+1/4} \neq 0, \\ |\tilde{\sigma}^{n+1/4}| \leq g & \text{else.} \end{cases}$$

Notice that the definition of $\tilde{\sigma}^{n+1/4}$ will be precised hereafter. Notice also that in order to avoid a too restrictive CFL condition we propose an implicit discretisation of the linear viscosity term.

Concerning the correction step (7.38)-(7.39), inserting (7.39) into (7.38) gives the elliptic equation governing the pressure p^{n+1} under the form

$$\partial_x \big(b \, \partial_x p^{n+1} \big) = - \frac{1}{\Delta t^n} \, \partial_x X^{n+1/2}. \tag{7.42}$$

Thus, the numerical approximation of (7.3)-(7.4) consists in the numerical resolution of equations (7.40)-(7.41), together with (7.42) and (7.38).

7.6.2 Discrete scheme

To approximate the solution $(X, p)^T$ of the system (7.3)-(7.4), we use a combined finite volume/finite element framework. We assume that the computational domain is discretized with I nodes x_i, $i = 1, \ldots, I$. We denote C_i the cell $(x_{i-1/2}, x_{i+1/2})$ of length $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ with $x_{i+1/2} = (x_i + x_{i+1})/2$. We denote $X_i^n = (b_i u_i^n)$ with

$$X_i^n \approx \frac{1}{\Delta x_i} \int_{C_i} X(x, t^n) dx,$$

the approximate solution at time t^n on the cell C_i. The pressure p is discretized on a staggered grid (in fact the dual mesh if we consider the 2d case)

$$p_{i+1/2}^n \approx \frac{1}{\Delta x_{i+1/2}} \int_{x_i}^{x_{i+1}} p(x, t^n) dx,$$

with $\Delta x_{i+1/2} = x_{i+1} - x_i$.

Hence, the discrete (in space and time) version of equations (7.40)-(7.41),(7.42) is given by

$$X_i^{n+1/4} = X_i^n - \frac{\Delta t^n}{\Delta x_i}(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n) + \mu\frac{\Delta t^n}{\Delta x_i^2}\Big(\frac{b_{i+1/2}}{b_{i+1}}X_{i+1}^{n+1/4} - \frac{b_{i+1/2}+b_{i-1/2}}{b_i}X_i^{n+1/4} + \frac{b_{i-1/2}}{b_{i-1}}X_{i-1}^{n+1/4}\Big), \tag{7.43}$$

$$X_i^{n+1/2} = X_i^{n+1/4} + \frac{\Delta t^n}{\Delta x_i}\left(b_{i+1/2}(\tilde{\sigma}_{i+1/2}^{n+1/4} + \tau_{i+1/2-}^n) - b_{i-1/2}(\tilde{\sigma}_{i-1/2}^{n+1/4} + \tau_{i-1/2+}^n)\right), \tag{7.44}$$

$$X_i^{n+1} = X_i^{n+1/2} + \frac{\Delta t^n}{\Delta x_i}b_i(p_{i+1/2}^{n+1} - p_{i-1/2}^{n+1}), \tag{7.45}$$

$$b_{i+1}p_{i+3/2}^{n+1} - (b_{i+1}+b_i)p_{i+1/2}^{n+1} + b_i p_{i-1/2}^{n+1} = -\frac{\Delta x_i}{\Delta t^n}(X_{i+1}^{n+1/2} - X_i^{n+1/2}), \tag{7.46}$$

with

$$b_{i+1/2} = \min\{b_i, b_{i+1}\}, \tag{7.47}$$

and $\mathcal{F}_{i+1/2}^n$ is a numerical flux accounting for the advection term bu^2 and any classical flux e.g. full upwind, Rusanov, etc can be used. In the simulation results given at the end of this section, an upwind formula has been used and having the form

$$\mathcal{F}_{i+1/2}^n = u_{i+1/2}^n\left(\frac{b_{i+1/2}}{b_i}X_i^n\mathbf{1}_{u_{i+1/2}^n\geq 0} + \frac{b_{i+1/2}}{b_{i+1}}X_{i+1}^n\mathbf{1}_{u_{i+1/2}^n\leq 0}\right), \tag{7.48}$$

with $u_{i+1/2}^n = (u_{i+1}^n + u_i^n)/2$. The quantities $\tau_{i+1/2-}^{n+1/4}$ and $\tau_{i-1/2+}^{n+1/4}$ correspond to discretisation using the apparent topography technique (see [129]) of the source term f and we have

$$\tau_{i+1/2-}^n = (x_{i+1/2} - x_i)f_{i+1/2}^n, \qquad \tau_{i-1/2+}^n = (x_i - x_{i-1/2})f_{i-1/2}^n.$$

It remains to define the quantities $\tilde{\sigma}_{i+1/2}^{n+1/4}$ in equation (7.44) and we use the strategy proposed by Bouchut [129, paragraph 4.12.1]. The definition of σ given by (7.36) has to be understood as multivalued: σ can be any value in $[-bg, bg]$ when $\partial_x u = 0$. When $\partial_x u = 0$, equations (7.40)-(7.41) become

$$X^{n+1/4} = X^n,$$
$$X^{n+1/2} = X^n + \Delta t^n\partial_x(b\tilde{\sigma}^{n+1/4} + b\tau^n),$$

with $\partial_x(b\tau^n) = f^n$ and we define

$$\tilde{\sigma}_{i+1/2}^{n+1/4} = -\mathrm{proj}_g\left(\tau_{i+1/2-}^n + \frac{1}{\Delta t^n}\Big(\frac{X_{i+1}^{n+1/4}}{b_{i+1}} - \frac{X_i^{n+1/4}}{b_i}\Big)\right), \tag{7.49}$$

where

$$\mathrm{proj}_g(X) = \begin{cases} X & \text{if } |X| \leq g \\ g\frac{X}{|X|} & \text{if } |X| > g \end{cases}$$

The formula (7.49) is consistent with the definition (7.36) because $|\tilde{\sigma}_{i+1/2}^{n+1/4}| \leq g$ and if $\frac{X_{i+1}^{n+1/4}}{b_{i+1}} \neq$

$\frac{X_i^{n+1/4}}{b_i}$ then for Δt^n small enough the quantity

$$\frac{1}{\Delta t^n}\Big(\frac{X_{i+1}^{n+1/4}}{b_{i+1}} - \frac{X_i^{n+1/4}}{b_i}\Big),$$

will dominate the other giving

$$\tilde{\sigma}_{i+1/2}^{n+1/4} \approx g\mathrm{sign}\Big(\frac{X_{i+1}^{n+1/4}}{b_{i+1}} - \frac{X_i^{n+1/4}}{b_i}\Big).$$

7.6.3 When $b \to 0$

Thanks to the definition of $b_{i\pm1/2}$ given by (7.47), equations (7.43)-(7.45) well behave when b_i tends to zero. More precisely, when $b_i = 0$, Equations (7.43)-(7.45) reduce to $X_i^{n+1} = X_i^{n+1/2} = X_i^{n+1/4} = X_i^n = 0$. And we adopt the modified version of (7.46) under the form

$$b_{i+1}p_{i+3/2}^{n+1} - (b_{i+1}^{\varepsilon} + b_i^{\varepsilon})p_{i+1/2}^{n+1} + b_i p_{i-1/2}^{n+1} = -\frac{\Delta x_i}{\Delta t^n}(X_{i+1}^{n+1/2} - X_i^{n+1/2}), \tag{7.50}$$

with $b_i^{\varepsilon} = \max\{b_i, \varepsilon\}$ and $0 < \varepsilon \ll 1$.

7.6.4 Boundary conditions

Boundary conditions have to be defined for Eqs. (7.43) and (7.46) at both side of the domain, and we have to face two difficulties

- equation (7.43) contains a hyperbolic part and a parabolic part whereas equation (7.46) is an elliptic equation.
- The boundary conditions applied to (7.46) have to be consistent with those applied to (7.43).

The proposed solution has been adapted from [130] but notice that other solutions can be investigated since the coupling of the boundary conditions between a hyperbolic step and a parabolic/elliptic step is far from being obvious.

Let us consider the boundary at the entry of the domain i.e. at abscissa $x_{1/2}$. We assume the inflow is prescribed typically $b_0 u_0^n = q_{in}(t^n)$ where $q_{in}(t^n)$ is a given quantity then the definition (7.48) can be used to define $\mathcal{F}_{1/2}^n$ under the form

$$\mathcal{F}_{1/2}^n = u_{1/2}^n \left(q_{in}(t^n)\mathbf{1}_{u_{1/2}^n \geq 0} + X_1^n \mathbf{1}_{u_{1/2}^n \leq 0}\right),$$

with $u_{1/2}^n = (u_1^n + q_{in}(t^n)/b_1)/2$ and assuming $b_0 = b_1 > 0$. For the parabolic part of equation (7.43) we use Dirichlet boundary conditions defined by $q_{in}(t^n)$.

Since the inflow $q_{in}(t^n)$ is prescribed, it is natural to assume Neumann boundary at the entry for p in equation (7.46) i.e. $\partial_x p|_0(t^{n+1}) = 0$.

Now we consider the boundary at the exit of the domain i.e. at abscissa $x_{I+1/2}$. Assuming Neumann boundary conditions, we define

$$\mathcal{F}_{I+1/2}^n = u_I^n X_I^n,$$

and $\partial_x X|_{I+1/2}(t^{n+1/4}) = 0$ with $b_{I+1} = b_I$ for the parabolic part. In this context, it is convenient to assume Dirichlet boundary at the exit for p in equation (7.46) i.e. $p|_{I+1}(t^{n+1}) = 0$.

For a more complete justification of the choices for the boundary conditions, the reader can refer to [130].

7.6.5 Simulation results

We present now some simulations results for the model (7.3)-(7.4) with the numerical scheme (7.43)-(7.46) where we have chosen the source term f

$$f = g\,\partial_x b,$$

mimicking the effects of the slope over the fluid rheology.

We consider a fluid domain defined by $x \in [0, x_{max}]$ with $x_{max} = 20$ meters and $I = 500$ nodes and a domain profile (see Fig. 7.1) defined by

$$b(x) = 1 - \frac{1}{2}\tanh\Big(x - \frac{x_{max}}{2}\Big) + 0.3e^{-(x-15)^2} - \frac{1}{2}e^{-(x-6)^2/2},$$

the fluid is initially at rest i.e.

$$u_i^0 = 0, \qquad \forall i \in I.$$

As mentioned in paragraph 7.6.4, the inflow is prescribed at the entry $x = 0$ with

$$q_{in}(t) = 2 + \sin(2\pi t/T),$$

the simulations are carried out over the time interval $(0, T)$ with $T = 20$ seconds.

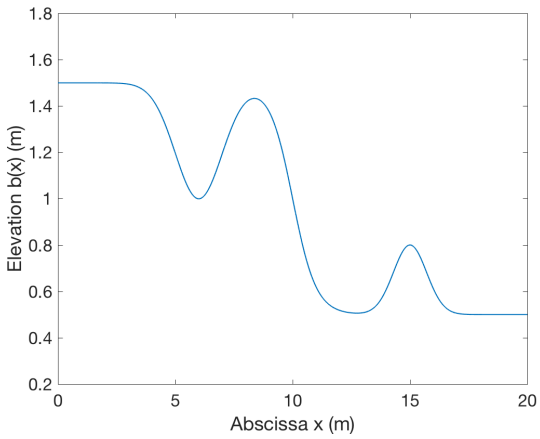
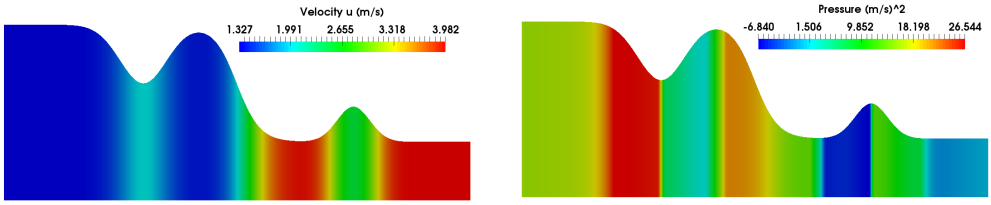


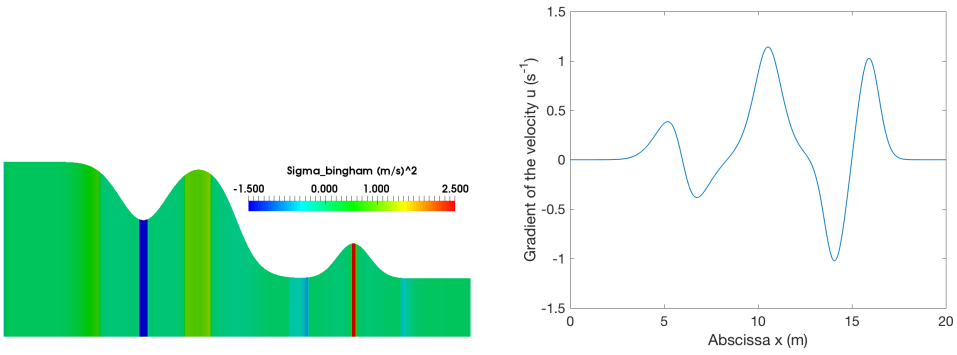
Figure 7.1: *Chosen profile for* $b(x)$.

The simulated velocity profile at time $t = T/2$ is depicted over Fig. 7.2-*(a)* whereas the variations of the simulated pressure is given over Fig. 7.2-*(b)*. The variations of $\tilde{\sigma}$ appear over Fig.7.3-*(b)*. Near the boundaries, the gradient of the velocity is very small $\partial_x u \approx 0$, see Fig. 7.3-*(b)*.



(a) *(b)*

Figure 7.2: *(a) variations of the velocity u and (b) variations of the pressure p in the fluid domain at time $t = T/2$.*



(a) *(b)*

Figure 7.3: *(a) variations of the quantity $\tilde{\sigma}$ and (b) variations of $\partial_x u$ at time $t = T/2$.*

Part V

Dissipative Solutions for Oldroyd Systems

8

Dissipative Solution for Oldroyd Systems

This work is a collaboration with D. Bresch and E.Suli, and it is still on the go.

This chapter is concerned with the existence of dissipative solutions for the incompressible Oldroyd system starting with the global existence of weak solutions for similar systems with some physical extra diffusion in the two dimensional periodic setting $\Omega = \mathbb{T}^2$. The main part is to modulate in an appropriate way the free energy originally derived in the incompressible setting by Hu and Lelièvre and extended to a new compressible visco-elastic system.

The chapter is organized as follows:

-**The first section** 8.1 is an introductory section for the problem at hand where we introduce the system to be studied.

-**The second section** 8.2 is where we recall the definition of dissipative solution for the Oldroyd B system and present the main result.

-**The third section** 8.3 is the section containing the proof of the main result. We first recall the regularized system and present why the global weak solution of such regularized system is a dissipative solution based on the modulation of the free energy. Then we prove global existence of dissipative solution of the Oldroyd-B system without diffusion passing to the limit from such global dissipative solution of the regularized Oldroyd-B system.

-**The fourth section** 8.4 is an appendix where we recall a brief system set-up, how to get the free energy for the regularized system and some convexity properties of various functionals.

8.1 Introduction

Existence of solutions for incompressible systems related to visco-elastic fluids of Oldroyd-B type has been the subject of several approaches during this last decade. As usual in nonlinear analysis related to fluid mechanics, an important step is to prove that systems are well-posed in a sense to be defined: dissipative solutions, global weak solutions à la Leray, intermediate solutions à la Hoff, strong solution in critical spaces, strong solutions in Sobolev spaces and classical solutions. Despite this amount of work that will not be detailed here, only some papers and for special cases (co-rotational flow or scalar case) have been able to prove global existence of weak solutions: see [131] for the incompressible Oldroyd-B type systems. In opposition, let us remark that there exists several interesting approaches which show global existence of weak solutions of the incompressible (physically regularized) Oldroyd-B system (see [78]).When we have in possession global existence of solution for systems with diffusion, it is a good starting point to prove global existence of dissipative solutions for the corresponding systems without diffusion (see for instance [52] for the incompressible Euler system, [132] and [133] for the compressible Euler system, [134] for the compressible Euler-Korteweg system and [135] for an other topic). Remark that for more simple incompressible visco-elastic systems, there exists some recent results related to the existence of dissipative solutions, see for instance [136]. It is also important to note that the concept of relative entropy and therefore dissipative solutions has been for instance strongly used to design appropriate numerical schemes for compressible Navier-Stokes equations and could be interesting to extend for non-Newtonian flows.

In this paper, we will prove the existence of global dissipative solution of the incompressible Oldroyd-B system in a two-dimensional periodic setting $\Omega = \mathbb{T}^2$ namely the following equations

$$\begin{cases} \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \nabla p = (1-w)\Delta u + \dfrac{w}{\lambda_1} \operatorname{div} Z, \\ \partial_t Z + u \cdot \nabla Z - (\nabla u) Z - Z(\nabla u)^T + \dfrac{1}{\lambda_1}(Z - \mathrm{Id}) = 0 \end{cases} \tag{8.1}$$

where $w \in [0,1]$, u is the velocity field, p the pressure and Z is a 2×2 matrix linked to the elastic part of the viscous stress. The parameter $\lambda_1 > 0$ is a fixed constant. We explain in the Appendix the choice of the adapted system starting from the stress equation suggested by Oldroyd. In fact, the work in this chapter is still in progress and has triggered several critical cases to discuss. We will present in this chapter a kind of an "if-Theorem" which relied strongly on the assumed constraint: uniform boundedness, symmetrical and positive definite characters of the stress in the solution of the regularized system in terms of epsilon, as we will see in the sequel.

8.2 Main Results.

Recall that we consider a two-dimensional periodic domain $\Omega = \mathbb{T}^2$. Before giving the definition of a dissipative solution for the incompressible Oldroyd system, for (u_0, Z_0) smooth enough such that $\operatorname{div} u_0 = 0$, we define $\mathcal{L}_1(u_0, Z_0)$ and $\mathcal{L}_2(u_0, Z_0)$ as follows

$$\begin{cases} \mathcal{L}_1(u_0, Z_0) = \partial_t u_0 + \mathcal{P}(u_0 \cdot \nabla u_0 - \dfrac{w}{\lambda_1} \operatorname{div} Z_0) - (1-w)\Delta u_0 \\ \mathcal{L}_2(u_0, Z_0) = \partial_t Z_0 + u_0 \cdot \nabla Z_0 - (\nabla u_0) Z_0 - Z_0(\nabla u_0)^T + \dfrac{1}{\lambda_1}(Z_0 - \mathrm{Id}). \end{cases} \tag{8.2}$$

where $\mathcal{P}$ is the Leray-Helmholtz projector.

Definition 8.10. *A couple* (u, Z) *is said to be a dissipative solution of* (8.1) *if*

$$u \in L^2(0,T;(H^1(\Omega))^2) \cap L^\infty(0,T;(L^2(\Omega))^2) \cap \mathcal{C}_{\text{weak}}([0,T];(L^2(\Omega))^2) \quad \textit{with} \quad \text{div} u = 0$$

$$Z \in L^\infty(0,T;(L^2(\Omega))^{2\times 2}) \cap \mathcal{C}_{\text{weak}}([0,T];(L^2(\Omega))^{2\times 2}) \textit{ is a symmetric positive definite matrix}$$

and for each smooth couple (u_0, Z_0) *such that* $\text{div} u_0 = 0$ *with* Z_0 *a symmetric positive definite matrix then the couple* (u, Z) *satisfies the following inequality*

$$C\mathcal{E}(t) \leqslant \mathcal{D}_0 \exp^{\int_0^t C_3(s)\ ds} + \int_0^t \int_\Omega \exp^{\int_s^t C_3(\tau)\ d\tau}(2\mathcal{L}_1 \cdot (u_0 - u) + \frac{w}{\lambda_1}\mathcal{L}_2 : Z_0^{-2}(Z_0 - Z))ds.$$

where $\mathcal{E}(t)$ *is given by*

$$\mathcal{E}(t) = \int_\Omega |u - u_0|^2 + \frac{w}{\lambda_1}|Z - Z_0|^2,$$

and $\mathcal{D}_0$ *is given by*

$$\mathcal{D}_0 = \int_\Omega |u^0 - u_0(0,\cdot)|^2 + \frac{w}{\lambda_1}(-\log\det Z^0 + \log\det Z^0(0,\cdot) + \mathbb{T}_r((Z^0(0,\cdot))^{-1}(Z^0 - Z_0(0,\cdot)))),$$

and C_3 *is a constant depending on* (u_0, Z_0) *given by*

$$C_3 = \frac{w^{1/2}}{\lambda_1^{1/2}}\|\nabla Z_0^{-1}\|_{L^\infty} + \frac{2w}{\lambda_1(1-w)}\|Z_0^{-1}\|_{L^\infty}^2 + \|\nabla u_0\|_{L^\infty} + \frac{w}{\lambda_1(1-w)}$$

and $C > 0$ *is a constant depending on* Z_0 *and* Ω *only.*

The first main result concerns the incompressible Oldroyd system and reads

Theorem 8.14. *Let* (u^0, Z^0) *be the initial data in* $L^2(\Omega)^2 \times L^2(\Omega)^{2\times 2}$ *such that* $\text{div}\, u^0 = 0$ *. If there exists a global weak solution* $(u_\varepsilon, Z_\varepsilon)$ *of the regularized incompressible Oldroyd system* (8.3) *satisfying the energy inequality and in which the stress* Z_ε *is assumed to be a uniformly bounded, uniformly symmetric positive definite matrix in terms of* ε*, then there exists a global dissipative solution* (u, Z) *of System* (8.1) *in the sense of definition* 8.10*.*

Remark:

1. if $w = 0$, we find the same definition of dissipative solution than the one introduced by P.–L. Lions in [52].
2. Note that if there exists a strong solution (u_0, Z_0) of the incompressible Oldroyd System which is sufficiently regular that means satisfying at least the regularity mentioned previously, then the dissipative solution is this strong solution because in that case $\mathcal{L}_1 = \mathcal{L}_2 = 0$.
3. Remark that to be able to define the right-hand side in the definition, it is easy to check the needed regularity.

To prove Theorem 8.14, we start from the existence of global weak solution of the regularized system proved by Barret and Boyaval in [78] which is valid in the two dimensional periodic setting and which satisfies the energy inequality. However, we will assume further to the result proved by the latter authors that the global weak solution satisfies the energy inequality in which Z_ε

is a uniformly bounded symmetric positive definite matrix in ε. It is good to remark that for the proof of theorem 8.14, we can start from any global weak solution of the regularized system on a condition that the stress satisfies being bounded and symmetric positive definite uniformly with respect to ε. In our approach, out of these global solutions we choose to start from that constructed by Barret and Boyaval in [78] under the condition that it satisfies the constraint on the stress. We first prove that such global weak solution is a dissipative solution in the sense of definition 8.10 with $\mathcal{L}_2^\varepsilon(u_0, Z_0) = \mathcal{L}_2(u_0, Z_0) - \varepsilon\Delta Z_0$. We then pass to the limit easily to get the global existence of weak solution of the Olroyd-B system without extra regularization on Z. This is a standard procedure and interested readers may consult the book by P.-L. Lions in [52] for explanation around incompressible Euler equations and incompressible Navier-Stokes equations.

Remark 15. *In the FENE-P system, the stress tensor is bounded by a constant tensor* $b\,\mathrm{Id}$ *(see for instance [137]). Thus, we can apply what have been done to the FENE-P system to show that the global weak solution constructed by N. Masmoudi is a global dissipative solution.*

8.3 Dissipative solution for the Incompressible Oldroyd system

Due to lack of estimations to control the stress tensor, studying the well posedness of model (8.1) in the weak sense is far from being possible at the moment. Hence, our goal is to prove the existence of weaker forms of solutions- called dissipative solutions- in the same manner done by P.L Lions to prove the existence of a dissipative solution for the Euler equation by regularizing it by a diffusive term, and utilizing the result on the weak existence of the solution of the Navier Stokes system to prove the existence of a solution of the regularized system. In the same manner, and in a periodic spatial domain $\Omega = \mathbb{T}^2$, we will add a very small diffusive term to the stress equation, and we will consider the following regularized system

$$\begin{cases} \operatorname{div} u_\varepsilon = 0, \\ \partial_t u_\varepsilon + u_\varepsilon \cdot \nabla u_\varepsilon + \nabla \tilde{p_\varepsilon} = (1-w)\Delta u_\varepsilon + \dfrac{w}{\lambda_1} \operatorname{div} Z_\varepsilon, \\ \partial_t Z_\varepsilon + u_\varepsilon \cdot \nabla Z_\varepsilon - (\nabla u_\varepsilon) Z_\varepsilon - Z_\varepsilon (\nabla u_\varepsilon)^T + \dfrac{1}{\lambda_1}(Z_\varepsilon - \mathrm{Id}) - \varepsilon \Delta Z_\varepsilon = 0, \end{cases} \tag{8.3}$$

with initial data

$$u_\varepsilon(.,0) = u_\varepsilon^0, \qquad Z_\varepsilon(.,0) = Z_\varepsilon^0.$$

The existence of a global weak solution of such regularized Oldroyd-B model is obtained in [78] in bounded domain with appropriate boundary conditions using a finite element scheme satisfying a free energy bound that we present in the appendix. The solution $(u_\varepsilon, Z_\varepsilon)$ is found such that u_ε is in $L^\infty(0,T;L^2) \cap L^2(0,T;H_0^1) \cap W^{1,4/\theta}(0,T;H^{-1})$ and Z_ε is in $L^\infty(0,T;[L^2]_{SPD}) \cap L^2(0,T;[H^1]_{SPD})$, where the subscript SPD denotes the space of symmetric positive definite matrices. We obtain the following estimates

$$\sup_{t\in(0,T)} \|u_\varepsilon\|_{L^2(\Omega)}^2 + \int_0^T \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \, dt \leqslant C,$$

$$\sup_{t\in(0,T)} \|Z_\varepsilon\|_{L^2(\Omega)}^2 + \varepsilon \int_0^T \|\nabla Z_\varepsilon\|_{L^2(\Omega)}^2 \, dt \leqslant C(\frac{1}{\varepsilon}, T),$$

and the global weak solution satisfies the free energy estimate which is formally derived in the Appendix and is given by

$$\begin{aligned}&\int u_\varepsilon^2+\frac{w}{\lambda_1}(\mathbb{T}_r(Z_\varepsilon)-\log det(Z_\varepsilon)-d)+\frac{w}{\lambda_1^2}\int_0^t\int_\Omega \mathbb{T}_r(Z_\varepsilon+Z_\varepsilon^{-1}-2\,\mathrm{Id})\\&+2(1-w)\int_0^t\int_\Omega \nabla u_\varepsilon^2-\varepsilon\int_0^t\int_\Omega \nabla Z_\varepsilon :: \nabla Z_\varepsilon^{-1}\\&\leqslant \int_\Omega |u_\varepsilon^0|^2+\frac{w}{\lambda_1}(\mathbb{T}_r(Z_\varepsilon^0)-\log det(Z_\varepsilon^0)-d).\end{aligned}\tag{8.4}$$

Note that the same result of existence can be obtained for a domain with periodic boundary conditions $\Omega=\mathbb{T}^2$, and this is what we will adopt in the sequel. As indicated before, in addition to the results obtained in [78] on the global weak solution assumed,we will assume as well that the stress in the weak solution is uniformly bounded and symmetric positive definite in terms of ε. This is necessary later on to prove the convexity of a function in terms of the stress (F) so that it can give us a bound in L^2 of $Z_\varepsilon - Z_0$. Besides, we define the following system satisfied by a smooth enough couple (u_0, Z_0) such that

$$\begin{cases}\operatorname{div} u_0=0,\\ \partial_t u_0+\mathcal{P}(u_0\cdot\nabla u_0-\dfrac{w}{\lambda_1}\operatorname{div} Z_0)-(1-w)\Delta u_0=\mathcal{L}_1(u_0,Z_0),\\ \partial_t Z_0+u_0\cdot\nabla Z_0-(\nabla u_0)Z_0-Z_0(\nabla u_0)^T+\dfrac{1}{\lambda_1}(Z_0-\mathrm{Id})-\varepsilon\Delta Z_0=\mathcal{L}_2^\varepsilon(u_0,Z_0),\end{cases}\tag{8.5}$$

with $\mathcal{L}_2^\varepsilon(u_0,Z_0)=\mathcal{L}_2(u_0,Z_0)-\varepsilon\Delta Z_0$ where $\mathcal{L}_1(u_0,Z_0)$ and $\mathcal{L}_2(u_0,Z_0)$ are defined as in definition (8.2). The above system satisfied by (u_0,Z_0) is supplemented with smooth enough initial data (u_0^0,Z_0^0). From the free energy in (8.4), we will define a modulated free energy using (u_0,Z_0) as test functions, and show that a global weak solution of the regularized system is a dissipative solution. Then we will pass to the limit with respect to the diffusion term in the stress equation to get the result mentioned in Section 8.2.

8.3.1 Modulated free energy

In this part, we will show how to carry the linearization of the nonlinear parts using first order expansion, and then we will carry on the derivation of the modulated energy. For a symmetric positive definite matrix X, we define

$$F(X):=-\log\det X+\mathbb{T}_r(X)-d,$$

$$H(X):=\mathbb{T}_r(X^{-1})+\mathbb{T}_r(X)-2d,$$

$$G(X)=\Delta X:(X^{-1}-\mathrm{Id}).$$

We prove in the appendix the convexity of such terms. The free energy of system (8.3) reads

$$\begin{aligned}&\int_\Omega u_\varepsilon^2+\frac{w}{\lambda_1}F(Z_\varepsilon)+\frac{w}{\lambda_1^2}\int_0^t\int_\Omega H(Z_\varepsilon)+2(1-w)\int_0^t\int_\Omega\nabla u_\varepsilon^2\\&+\frac{\varepsilon w}{\lambda_1}\int_0^t<\Delta Z_\varepsilon,Z_\varepsilon^{-1}>_{H^{-1}(\Omega),H^1(\Omega)}\leqslant\int_\Omega|u_\varepsilon^0|^2+\frac{w}{\lambda_1}F(Z_\varepsilon^0).\end{aligned}$$

The above inequality holds, specifically in the case of non-smooth enough solutions, due to the fact that the term infront of ε is equivalent in the distributional sense to G which is proven in

the smooth case to be positive. The same justification is used later when we want to use the convexity of G, and thus the positiveness of its second order expansion.

Linearized Quantities

The first order expansion of a function $\mathcal{K}$ near X_0 is defined by

$$\mathcal{K}_1(X|X_0) = \mathcal{K}(X) - \mathcal{K}(X_0) - D\mathcal{K}(X_0)(X - X_0). \tag{8.6}$$

Lets look at the third term in the expansions of F, H and G for symmetric definite matrices. In fact, we will use the following relation

$$\log\det X = \mathbb{T}_r(\log X).$$

Thus F now reads

$$F(X) = -\mathbb{T}_r(\log X) + \mathbb{T}_r(X),$$

and the derivative thus reads

$$DF(X_0)(X - X_0) = -\mathbb{T}_r(X_0^{-1}(X - X_0)) + \mathbb{T}_r(X - X_0).$$

Also we have

$$DH(X_0)(X - X_0) = \mathbb{T}_r(X_0^{-2}(X - X_0)) + \mathbb{T}_r(X - X_0).$$

Similarly for G we get

$$DG(X_0)(X - X_0) = \Delta X_0(X^{-1} - X_0^{-1}) + \Delta(X - X_0)X_0^{-1}.$$

Thus, (8.6) for F, G and H becomes at first order of expansion

$$F_1(X|X_0) = -\log\det X + \log\det X_0 + \mathbb{T}_r(X_0^{-1}(X - X_0)), \tag{8.7}$$

$$H_1(X|X_0) = \mathbb{T}_r(X^{-1}) - \mathbb{T}_r(X_0^{-1}) - \mathbb{T}_r(X_0^{-2}(X - X_0)), \tag{8.8}$$

$$G_1(X|X_0) = \Delta X : X^{-1} - \Delta X_0 : X_0^{-1} - \Delta X : X_0^{-1} + \Delta X_0 : X_0^{-1}XX_0^{-1}. \tag{8.9}$$

Since we don't guarantee enough regularity on X, we will use the equivalence of G_1 in the duality space so that we still guarantee the positivity of the terms that we will obtain in the derivation. In fact, in the smooth case we can write

$$\begin{aligned} 0 \leqslant \int_\Omega G_1(X|X_0) =& < \Delta X, X^{-1} >_{H^{-1}(\Omega),H^1(\Omega)} - \int_\Omega \Delta X_0 : X_0^{-1} \\ & - < \Delta X, X_0^{-1} >_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega \Delta X_0 : X_0^{-1}XX_0^{-1}. \end{aligned}$$

Thus, we denote

$$\begin{aligned} G_2(X|X_0) =& < \Delta X, X^{-1} >_{H^{-1}(\Omega),H^1(\Omega)} - \int_\Omega \Delta X_0 : X_0^{-1} \\ & - < \Delta X, X_0^{-1} >_{H^{-1}(\Omega),H^1(\Omega)} + \int_\Omega \Delta X_0 : X_0^{-1}XX_0^{-1} \geqslant 0. \end{aligned}$$

We define now the following modulated energy

$$\mathcal{D}_\varepsilon(t) = \int_\Omega |u_\varepsilon - u_0|^2 + \frac{w}{\lambda_1}F_1(Z_\varepsilon|Z_0),$$

and the following quantity

$$\mathcal{E}_\varepsilon(t) := \int_\Omega |u_\varepsilon - u_0|^2 + \frac{w}{\lambda_1}\int_\Omega |Z_\varepsilon - Z_0|^2.$$

Due to the convexity of F, H and G, we claim that the couples (u_0, Z_0) and $(u_\varepsilon, Z_\varepsilon)$ satisfy the following relation

$$C\mathcal{E}_\varepsilon(t) \leqslant \mathcal{D}_\varepsilon(t)$$

where C depends only on Z_0 and Ω. This suffices to apply the appendix to the unknown $Z_0^{-1}Z$.

Derivation of the modulated free energy

In order for the weak solution of system (8.3) to fulfill the definition of the dissipative solution, it needs to satisfy the relative entropy in definition 8.10. Starting from the free energy satisfied by the global weak solution $(u_\varepsilon, Z_\varepsilon)$, we have

$$\begin{aligned}
&\int_\Omega |u_\varepsilon|^2 + \frac{w}{\lambda_1}(\mathbb{T}_r(Z_\varepsilon) - \log det(Z_\varepsilon) - d) + \frac{w}{\lambda_1^2}\int_0^t\int_\Omega \mathbb{T}_r(Z_\varepsilon + Z_\varepsilon^{-1} - 2\,\mathrm{Id}) \\
&+ 2(1-w)\int_0^t\int_\Omega |\nabla u_\varepsilon|^2 + \frac{\varepsilon w}{\lambda_1}\int_0^t < \Delta Z_\varepsilon, Z_\varepsilon^{-1} >_{H^{-1}(\Omega),H^1(\Omega)} \\
&\leqslant \int_\Omega |u_\varepsilon^0|^2 + \frac{w}{\lambda_1}(\mathbb{T}_r(Z_\varepsilon^0) - \log det(Z_\varepsilon^0) - d).
\end{aligned} \tag{8.10}$$

We will use the smooth couple (u_0, Z_0) as test functions. We will proceed as following: first multiply the momentum equation of system (8.3) by u_0, we get

$$\begin{aligned}
\int_\Omega |u_0|^2 + 2(1-w)\int_0^t\int_0^t |\nabla u_0|^2 + \frac{2w}{\lambda_1}\int_0^t\int_\Omega Z_0 : \nabla u_0 \\
= \int_0^t\int_\Omega 2\mathcal{L}_1 \cdot u_0 + \int_\Omega |u_0^0|^2.
\end{aligned} \tag{8.11}$$

And testing the stress equation on Z_0 with $\frac{w}{\lambda_1}(Z_0^{-1} - \mathrm{Id})$ we further get

$$\begin{aligned}
&\frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r Z_0 - \log\det Z_0 + \frac{w}{\lambda_1^2}\int_0^t\int_\Omega (\mathbb{T}_r(Z_0 + Z_0^{-1}) - 2d) \\
&- \frac{2w}{\lambda_1}\int_0^t\int_\Omega Z_0 : \nabla u_0 + \frac{\varepsilon w}{\lambda_1}\int_0^t\int_\Omega \Delta Z_0 : Z_0^{-1} \\
&= -\frac{w}{\lambda_1}\int_0^t\int_\Omega \mathcal{L}_2 : (Z_0^{-1} - \mathrm{Id}) + \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r Z_0^0 - \log\det Z_0^0.
\end{aligned} \tag{8.12}$$

Multiplying the momentum equation of (8.3) by $-2u_0$ and using the equation satisfied by u_0, we further get

$$\begin{aligned}
&-2\int_\Omega u_\varepsilon \cdot u_0 - 4(1-w)\int_0^t\int_\Omega \nabla u_\varepsilon : \nabla u_0 = \frac{2w}{\lambda_1}\int_0^t\int_\Omega Z_\varepsilon : \nabla u_0 + Z_0 : \nabla u_\varepsilon \\
&+ 2\int_0^t\int_\Omega u_0 \cdot \nabla u_0 \cdot u_\varepsilon - u_\varepsilon \cdot \nabla u_0 \cdot u_\varepsilon - 2\int_0^t\int_\Omega \mathcal{L}_1 \cdot u_\varepsilon - 2\int_\Omega u_\varepsilon^0 \cdot u_0^0.
\end{aligned} \tag{8.13}$$

The above relation together with (8.10) and (8.11) provide the first part of the relative entropy related to the velocity, i.e $\int_\Omega |u_\varepsilon - u_0|^2$. Moreover, the relative entropy should provide a bound on $Z_\varepsilon - Z_0$. Since the corresponding terms of the stress are nonlinear, this needs a one more step to obtain the bound on the difference. A convex function of the determinant and the trace of the stress is thus used to deduce the required bound, as derived in the free energy section. Now multiplying the stress equation in (8.3) by Z_0^{-1} and using the equation satisfied by Z_0 and taking the trace of the result we obtain

$$\begin{aligned}
&\frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r(Z_0^{-1}Z_\varepsilon) + \frac{w}{\lambda_1^2}\int_0^t\int_\Omega \mathbb{T}_r(Z_0^{-2}Z_\varepsilon - Z_0^{-1}) \\
&= -\frac{w}{\lambda_1}\int_0^t\int_\Omega \mathcal{L}_2^\varepsilon : Z_0^{-2}Z_\varepsilon + \frac{\varepsilon w}{\lambda_1}\int_0^t \big(-\int_\Omega \Delta Z_0 Z_0^{-1}Z_\varepsilon Z_0^{-1} + < \Delta Z_\varepsilon, Z_0^{-1} >_{H^{-1(\Omega)},H^1(\Omega)} \big) \\
&+ \frac{w}{\lambda_1}\int_0^t\int_\Omega \mathbb{T}_r\big(- Z_0^{-1}u_\varepsilon\cdot\nabla Z_\varepsilon + Z_0^{-1}\nabla u_\varepsilon Z_\varepsilon + Z_0^{-1}Z_\varepsilon\nabla u_\varepsilon^T \\
&\qquad + Z_0^{-1}u_0\cdot\nabla Z_0 Z_0^{-1}Z_\varepsilon - Z_0^{-1}\nabla u_0 Z_\varepsilon - \nabla u_0^T Z_0^{-1}Z_\varepsilon\big) + \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r((Z_0^0)^{-1}Z_\varepsilon^0).
\end{aligned} \tag{8.14}$$

Finally, adding relations (8.10) + (8.11) − (8.12) +(8.13) +(8.14) we get

$$\begin{aligned}
&\int_\Omega |u_\varepsilon - u_0|^2 + \frac{w}{\lambda_1}(F_1 + \mathbb{T}_r(Z_\varepsilon - Z_0)) + 2(1-w)\int_0^t\int_\Omega |\nabla u_\varepsilon - \nabla u_0|^2 \\
&+ \frac{w}{\lambda_1^2}\int_0^t\int_\Omega (H_1 + \mathbb{T}_r(Z_\varepsilon - Z_0)) + \frac{\varepsilon w}{\lambda_1}\int_0^t G_2 \\
&\leqslant \frac{2w}{\lambda_1}\int_0^t\int_\Omega Z_0 : \nabla u_\varepsilon - 2Z_0 : \nabla u_0 + Z_\varepsilon : \nabla u_0 + 2\int_0^t\int_\Omega (u_0 - u_\varepsilon)\cdot\nabla u_0\cdot(u_\varepsilon - u_0) \\
&+ \frac{w}{\lambda_1}\int_0^t\int_\Omega \mathcal{L}_2^\varepsilon : (Z_0^{-1} - \mathrm{Id} - Z_0^{-2}Z_\varepsilon) + 2\int_0^t\int_\Omega \mathcal{L}_1\cdot(u_0 - u_\varepsilon) \\
&+ \frac{w}{\lambda_1}\int_0^t\int_\Omega \mathbb{T}_r\big(- Z_0^{-1}u_\varepsilon\cdot\nabla Z_\varepsilon + Z_0^{-1}u_0\cdot\nabla Z_0 Z_0^{-1}Z_\varepsilon + Z_0^{-1}\nabla u_\varepsilon Z_\varepsilon \\
&\qquad - Z_0^{-1}\nabla u_0 Z_\varepsilon + Z_0^{-1}Z_\varepsilon\nabla u_\varepsilon^T - \nabla u_0^T Z_0^{-1}Z_\varepsilon\big) \\
&+ \int_\Omega |u_\varepsilon^0 - u_0^0|^2 + \frac{w}{\lambda_1}(F_1^0 + \mathbb{T}_r(Z_\varepsilon^0 - Z_0^0)).
\end{aligned} \tag{8.15}$$

Lemma 8.5. *Using the stress equations in (8.3) and (8.5), we easily get*

$$\begin{aligned}
\frac{d}{dt}\frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r(Z_\varepsilon - Z_0) + \frac{w}{\lambda_1^2}\int_\Omega \mathbb{T}_r(Z_\varepsilon - Z_0) \\
= \frac{2w}{\lambda_1}\int_\Omega Z_\varepsilon : \nabla u_\varepsilon - Z_0 : \nabla u_0 - \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r\mathcal{L}_2.
\end{aligned} \tag{8.16}$$

Proof. The proof is quite simple and direct, by taking the trace of the stress equations of systems (8.3) and (8.5), and then subtracting them we get the above equality. □

Finally substituting the expression in the above lemma (integrated in time) in inequality

(8.15), we get

$$\begin{aligned}
&\int_\Omega |u_\varepsilon - u_0|^2 + \frac{w}{\lambda_1} F_1 + 2(1-w)\int_0^t \int_\Omega |\nabla u_\varepsilon - \nabla u_0|^2 + \frac{w}{\lambda_1^2}\int_0^t \int_\Omega H_1 + \frac{\varepsilon w}{\lambda_1}\int_0^t G_2 \\
&\leqslant \frac{2w}{\lambda_1}\int_0^t \int_\Omega Z_0 : \nabla u_\varepsilon - Z_0 : \nabla u_0 + Z_\varepsilon : \nabla u_0 - Z_\varepsilon : \nabla u_\varepsilon \\
&+ 2\int_0^t \int_\Omega (u_0 - u_\varepsilon)\cdot \nabla u_0 \cdot (u_\varepsilon - u_0) \\
&+ 2\int_0^t \int_\Omega \mathcal{L}_1 \cdot (u_0 - u_\varepsilon) + \frac{w}{\lambda_1}\int_0^t \int_\Omega \mathcal{L}_2 : (Z_0^{-1} - Z_0^{-2} Z_\varepsilon) \\
&+ \frac{w}{\lambda_1}\int_0^t \int_\Omega \mathbb{T}_r\big(- Z_0^{-1} u_\varepsilon \cdot \nabla Z_\varepsilon + Z_0^{-1} u_0 \cdot \nabla Z_0 Z_0^{-1} Z_\varepsilon \\
&\qquad + Z_0^{-1}\nabla u_\varepsilon Z_\varepsilon - Z_0^{-1}\nabla u_0 Z_\varepsilon + Z_0^{-1} Z_\varepsilon \nabla u_\varepsilon^T - Z_0^{-1} Z_\varepsilon \nabla u_0^T\big) \\
&+ \int_\Omega |u_\varepsilon^0 - u_0^0|^2 + \frac{w}{\lambda_1} F_1^0.
\end{aligned} \tag{8.17}$$

In order to be able to apply Gronwall's lemma, we need to bound all the excess terms on the right hand side of (8.17). We split them into several remainders:

$$R_1 = \frac{2w}{\lambda_1}\int_\Omega Z_0 : \nabla u_\varepsilon - Z_0 : \nabla u_0 + Z_\varepsilon : \nabla u_0 - Z_\varepsilon : \nabla u_\varepsilon, \tag{8.18}$$

$$R_2 = 2\int_\Omega (u_0 - u_\varepsilon)\cdot \nabla u_0 \cdot (u_\varepsilon - u_0), \tag{8.19}$$

$$R_3 = 2\int_\Omega \mathcal{L}_1 \cdot (u_0 - u_\varepsilon) + \frac{w}{\lambda_1}\int_\Omega \mathcal{L}_2 : (Z_0^{-1} - Z_0^{-2} Z_\varepsilon), \tag{8.20}$$

$$\begin{aligned}
R_4 = \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r\big(&- Z_0^{-1} u_\varepsilon \cdot \nabla Z_\varepsilon + Z_0^{-1} u_0 \cdot \nabla Z_0 Z_0^{-1} Z_\varepsilon \\
&+ Z_0^{-1}\nabla u_\varepsilon Z_\varepsilon - Z_0^{-1} Z_\varepsilon \nabla u_0 + Z_0^{-1} Z_\varepsilon \nabla u_\varepsilon^T - Z_0^{-1} Z_\varepsilon \nabla u_0^T\big).
\end{aligned} \tag{8.21}$$

First, we establish the following lemmas:

Lemma 8.6. *We have*

$$R_1 + R_2 \leqslant C\max\Big(\frac{2w}{\lambda_1(1-w)}, 2\|\nabla u_0\|_{L^\infty}\Big)\Big(\frac{w}{\lambda_1}\|Z_0 - Z_\varepsilon\|_{L^2}^2 + \|u_\varepsilon - u_0\|_{L^2}^2\Big) + \frac{(1-w)}{16}\|\nabla(u_\varepsilon - u_0)\|_{L^2}^2.$$

Proof. • As for R_1, we can write for small η using Young's inequality

$$\begin{aligned}
R_1 &= \frac{2w}{\lambda_1}\int_\Omega (Z_0 - Z_\varepsilon) : \nabla(u_\varepsilon - u_0) \\
&\leqslant \frac{w}{\lambda_1 \eta}\int_\Omega |Z_0 - Z_\varepsilon|^2 + \frac{\eta w}{\lambda_1}\int_\Omega |\nabla u_\varepsilon - u_0|^2,
\end{aligned}$$

choosing $\eta = \frac{\lambda_1(1-w)}{16w}$ so that the second part of the right hand side can be absorbed in

the left hand side of (8.17), we get

$$R_1 \leqslant C\frac{w}{\lambda_1(1-w)}\int_\Omega \frac{w}{\lambda_1}|Z_0 - Z_\varepsilon|^2 + \frac{(1-w)}{16}\int_\Omega |\nabla u_\varepsilon - u_0|^2.$$

- As for R_2, we have in fact

$$\begin{aligned} R_2 &= -2\int_\Omega (u_\varepsilon - u_0)\cdot\nabla u_0\cdot(u_\varepsilon - u_0) \\ &\leqslant 2\|\nabla u_0\|_{L^\infty}\|u_\varepsilon - u_0\|_{L^2}^2. \end{aligned}$$

□

The term R_3 will be treated later on. Finally, we consider the last term R_4:

Lemma 8.7.

$$R_4 \leqslant \frac{(1-w)}{2}\|\nabla(u_\varepsilon - u_0)\|_{L^2}^2 + C_2\Big(\frac{w}{\lambda_1}\|Z_\varepsilon - Z_0\|_{L^2}^2 + \|u_\varepsilon - u_0\|_{L^2}^2\Big)$$

where

$$C_2 = \frac{w^{1/2}}{\lambda_1^{1/2}}\|\nabla Z_0^{-1}\|_{L^\infty} + \frac{2w}{\lambda_1(1-w)}\|Z_0^{-1}\|_{L^\infty}^2.$$

Proof.

$$\begin{aligned} R_4 &= \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r\big(- Z_0^{-1}u_\varepsilon\cdot\nabla Z_\varepsilon + Z_0^{-1}Z_\varepsilon u_0\cdot\nabla Z_0 Z_0^{-1}Z_\varepsilon + Z_0^{-1}\nabla u_\varepsilon Z_\varepsilon - Z_0^{-1}\nabla u_0 Z_\varepsilon \\ &\quad + Z_0^{-1}Z_\varepsilon\nabla u_\varepsilon^T - Z_0^{-1}Z_\varepsilon\nabla u_0^T\big) \\ &= \underbrace{\frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r(-Z_0^{-1}u_\varepsilon\cdot\nabla Z_\varepsilon + Z_0^{-1}u_0\cdot\nabla Z_0 Z_0^{-1}Z_\varepsilon)}_{R_{4,1}} + \underbrace{\frac{2w}{\lambda_1}\int_\Omega Z_0^{-1}Z_\varepsilon : \nabla(u_\varepsilon - u_0)}_{R_{4,2}}. \end{aligned}$$

- The first part can be processed easily

$$\begin{aligned} R_{4,1} &= \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r(-Z_0^{-1}u_\varepsilon\cdot\nabla Z_\varepsilon + Z_0^{-1}u_0\cdot\nabla Z_0 Z_0^{-1}Z_\varepsilon) \\ &= \frac{w}{\lambda_1}\int_\Omega -\mathbb{T}_r\Big(Z_0^{-1}(u_\varepsilon - u_0)\cdot\nabla Z_\varepsilon\Big) - \mathbb{T}_r\Big(Z_0^{-1}u_0\cdot\nabla Z_\varepsilon - Z_0^{-1}u_0\cdot\nabla Z_\varepsilon Z_0^{-1}Z_\varepsilon\Big). \end{aligned}$$

Using the fact that

$$\int_\Omega \mathbb{T}_r(Au\cdot\nabla B) = -\int_\Omega \mathbb{T}_r(B^T u\cdot\nabla A),$$

we get that by integration by parts the last two terms in the above integral cancel each

other, thus

$$\begin{aligned} R_{4,1} &= \frac{w}{\lambda_1}\int_\Omega -\mathbb{T}_r\Big(Z_0^{-1}(u_\varepsilon-u_0)\cdot\nabla Z_\varepsilon\Big)\\ &= \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r\Big(Z_\varepsilon(u_\varepsilon-u_0)\cdot\nabla Z_0^{-1}\Big)\\ &= \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r\Big((Z_\varepsilon-Z_0)(u_\varepsilon-u_0)\cdot\nabla Z_0^{-1}\Big)+\mathbb{T}_r\Big(Z_0^{-1}(u_\varepsilon-u_0)\cdot\nabla Z_0\Big). \end{aligned}$$

The last term again is equal to zero due to incompressibility condition, more precisely

$$\int_\Omega \mathbb{T}_r\Big(Z_0^{-1}(u_\varepsilon-u_0)\cdot\nabla Z_0\Big)=\int_\Omega (u_\varepsilon-u_0)\cdot\nabla\mathbb{T}_r(\log Z_0)=-\int_\Omega \operatorname{div}(u_\varepsilon-u_0)\mathbb{T}_r(\log Z_0)=0.$$

Therefore, the first part of R_4 can be bounded as following

$$\begin{aligned} R_{4,1} &= \frac{w}{\lambda_1}\int_\Omega \mathbb{T}_r\Big((Z_\varepsilon-Z_0)(u_\varepsilon-u_0)\cdot\nabla Z_0^{-1}\Big)\\ &\leqslant \frac{w^{1/2}}{\lambda_1^{1/2}}\|\nabla Z_0^{-1}\|_{L^\infty}\Big(\frac{w}{\lambda_1}\|Z_\varepsilon-Z_0\|_{L^2}^2+\|u_\varepsilon-u_0\|_{L^2}^2\Big). \end{aligned}$$

- And for the second part, again due to incompressibility condition we get

$$\begin{aligned} R_{4,2} &= \frac{2w}{\lambda_1}\int_\Omega Z_0^{-1}Z_\varepsilon:\nabla(u_\varepsilon-u_0)\\ &= \frac{2w}{\lambda_1}\int_\Omega Z_0^{-1}(Z_\varepsilon-Z_0):\nabla(u_\varepsilon-u_0)+\underbrace{\mathrm{Id}:\nabla(u_\varepsilon-u_0)}_{=\operatorname{div}(u_\varepsilon-u_0)=0}\\ &\quad 2\|Z_0^{-1}\|_{L^\infty(\Omega)}\frac{w^{\frac12}}{\lambda_1^{\frac12}}\Big(\int_\Omega\frac{w}{\lambda_1}|Z_\varepsilon-Z_0|^2\Big)^{\frac12}\Big(\int_\Omega|\nabla u_\varepsilon-u_0|^2\Big)^{\frac12}\\ &\leqslant \|Z_0^{-1}\|_{L^\infty(\Omega)}^2\frac{w}{\lambda_1\eta}\int_\Omega\frac{w}{\lambda_1}|Z_\varepsilon-Z_0|^2+\frac{\eta}{2}\int_\Omega|\nabla u_\varepsilon-u_0|^2. \end{aligned}$$

Choosing $\eta=\frac{(1-w)}{2}$ the second term on the right hand side can be absorbed in the left hand side of (8.17), and so R_4 is bounded such that

$$R_4\leqslant C_2\Big(\frac{w}{\lambda_1}\|Z_\varepsilon-Z_0\|_{L^2}^2+\|u_\varepsilon-u_0\|_{L^2}^2\Big)+\frac{(1-w)}{4}\|\nabla(u_\varepsilon-u_0)\|_{L^2}^2$$

where

$$C_2=\frac{w^{1/2}}{\lambda_1^{1/2}}\|\nabla Z_0^{-1}\|_{L^\infty}+\frac{2w}{\lambda_1(1-w)}\|Z_0^{-1}\|_{L^\infty}^2.$$

□

Having all the necessary bounds, therefore we can write in (8.17)

$$\begin{aligned}&\int_\Omega |u_\varepsilon - u_0|^2 + \frac{w}{\lambda_1}F_1 + \int_0^t \int_\Omega \frac{w}{\lambda_1^2}H^1 + \frac{\varepsilon w}{\lambda_1}\int_0^t G_2 \\ &\leqslant \int_\Omega |u_\varepsilon - u_0|^2(0) + \frac{w}{\lambda_1}F_1(0) + \int_0^t \int_\Omega 2\mathcal{L}_1 \cdot (u_0 - u_\varepsilon) + \frac{w}{\lambda_1}\mathcal{L}_2 : Z_0^{-2}(Z_0 - Z_\varepsilon) \\ &+ \int_0^t C_3(s) \int_\Omega \frac{w}{\lambda_1}|Z_0 - Z_\varepsilon|^2 + |u_\varepsilon - u_0|^2 \, ds.\end{aligned} \tag{8.22}$$

and C_3 is a function of time depending on the norms of u_0 and Z_0. In particular, C_3 reads

$$\begin{aligned}C_3 &= C_1 + C_2 \\ &= \frac{w^{1/2}}{\lambda_1^{1/2}}\|\nabla Z_0^{-1}\|_{L^\infty} + \frac{2w}{\lambda_1(1-w)}\|Z_0^{-1}\|_{L^\infty}^2 + \|\nabla u_0\|_{L^\infty} + \frac{w}{\lambda_1(1-w)}.\end{aligned}$$

Recalling that we have the control

$$C\int_\Omega |Z_\varepsilon - Z_0|^2 \leqslant \int_\Omega F_1 \tag{8.23}$$

with a uniform constant C which does not not depend on ε, w and λ_1 we write

$$\begin{aligned}\min(1, C)\mathcal{E}_\varepsilon(t) \leqslant \mathcal{D}_\varepsilon(0) + \int_0^t C_3(s)\mathcal{E}_\varepsilon(s) \, ds \\ + \int_0^t \int_\Omega 2\mathcal{L}_1 \cdot (u_0 - u_\varepsilon) + \frac{w}{\lambda_1}\mathcal{L}_2^\varepsilon : Z_0^{-2}(Z_0 - Z_\varepsilon).\end{aligned} \tag{8.24}$$

Applying Gronwall Lemma and then passing to the limit with respect to epsilon, we end the proof.

Remark 16. *A reflection on this work shows that we still lack a complete strategy and method that could contribute to a weak solution of Oldroyd systems without having uniformity constraints on the stress. This explains the approaches in which FENE-P systems were considered, or where regularization of the stress was assumed to prove weak solutions. However, such relative entropy could still be powerful in case a strong solution is discussed, where the entropy would provide the weak-strong uniqueness of the solution.*

8.4 Appendix

8.4.1 System Setup

Oldroyd in 1950 stated that for an empirical rheological equation of states to be valid universally, convected coordinate systems should be used. He suggested the following law based on experimental work

$$S + \tau\frac{\partial S}{\partial t} = 2\eta_0(D(u) + \lambda_2\frac{\partial}{\partial t}D(u)),$$

where $\eta_0, \lambda_1, \lambda_2$ are constant properties related to material's concentration. Another simpler form by Oldroyd was given by what he called liquid B of constitutional formula

$$S + \lambda_1\overset{\triangledown}{S} = 2\eta_0(D(u) + \lambda_2\overset{\triangledown}{D}(u)), \tag{8.25}$$

with $\triangledown$ being the upper convected time derivative, known as well as corrotational or frame-invariant derivative defined by

$$\begin{aligned}\overset{\triangledown}{S} &= \frac{d}{dt}S - S(\nabla^T u) - \nabla u S \\ &= \partial_t S + u\cdot\nabla S - (\nabla^T u)S - S\nabla u.\end{aligned}$$

λ_1 is called relaxation time, and $\lambda_2 < \lambda_1$ is the retraction time. The case of λ_2 equals zero is universally known by the Upper Convected Maxwell Model UCM, inspired from Maxwell's work. The Oldroyd and UCM models have gained a huge popularity till nowadays and benefited a vast amount of studies on visco elastic flows especially numerically.

Concerning the extra stress of such fluid, there is two approaches in this context, letting p be the hydrostatic pressure, we define the augmented pressure $\tilde{p}$

$$\tilde{p} = p - \frac{1}{d}\mathbb{T}_r S, \tag{8.26}$$

and we define the deviatoric viscous stress by

$$S_D = S - \frac{1}{d}\mathbb{T}_r(S)\,\mathrm{Id}\,. \tag{8.27}$$

Since we are in an incompressible case, following the discussion in [138], the extra stress tensor can be expressed as either one of the following relations

$$\sigma = -\tilde{p}\,\mathrm{Id} + S_D = -p\,\mathrm{Id} + S. \tag{8.28}$$

Recall that the incompressible Navier Stokes system (with constant density) reads

$$\begin{cases}\operatorname{div} u = 0,\\ \partial_t u + u\cdot\nabla u = \operatorname{div}(\sigma) \iff \partial_t u + u\,\partial_x u + \nabla\tilde{p} = \operatorname{div}(S_D),\end{cases} \tag{8.29}$$

The aim is to calculate $\operatorname{div}(S_D)$. Due to the viscous and plastic behaviors of Oldroyd-B models, we will divide the fluid into two parts : a Newtonian part (index n) and a plastic part (index p). As a result, the viscosity η_0 is divided into two

$$\eta_0 = \eta_n + \eta_p, \quad \text{such that} \quad \eta = \frac{\eta_0\lambda_2}{\lambda_1} = constant.$$

As a result, the viscous stress S_D can be expressed as

$$\begin{aligned}S_D = S - \frac{1}{d}\mathbb{T}_r(S)\,\mathrm{Id} &= S_n + S_p - \frac{1}{d}\mathbb{T}_r(S_n+S_p)\,\mathrm{Id}\\ &= 2\eta_n D(u) + S_p - \frac{1}{d}\mathbb{T}_r(S_n+S_p)\,\mathrm{Id},\end{aligned}$$

and hence

$$\operatorname{div}(S_D) = 2\eta_n\operatorname{div}(D(u)) + \operatorname{div}(S_p) - \frac{1}{d}\mathbb{T}_r(\operatorname{div}(S_n+S_p))\,\mathrm{Id}\,. \tag{8.30}$$

However due to incompressibility condition

$$\mathbb{T}_r(S_n) = \mathbb{T}_r(2\eta_n D(u)) = 2\eta_n\mathbb{T}_r(D(u)) = 2\eta_n\operatorname{div}(u) = 0.$$

Therefore

$$\operatorname{div}(S_D) = 2\eta_n \operatorname{div}(D(u)) + \operatorname{div}(S_p) - \frac{1}{3}\operatorname{div}\mathbb{T}_r(S_p). \tag{8.31}$$

Moreover, for Oldroyd B-type, the isotropic part of S_p is neglected, i.e $\mathbb{T}_r(S_p) = 0$, thus the momentum equation becomes

$$\partial_t u + u \cdot \nabla u + \nabla \tilde{p} - 2\eta_n \operatorname{div}(D(u)) - \operatorname{div}(S_p) = 0.$$

Recalling the equation on S and substituting $S_n = 2\eta_n D(u)$, we can write

$$\begin{aligned} S + \lambda_1 \overset{\triangledown}{S} &= 2\eta_0 (D(u) + \lambda_2 \overset{\triangledown}{D}(u)) \\ \Longrightarrow S_p + \lambda_1 \overset{\triangledown}{S}_p &= 2\eta_p D(u). \end{aligned} \tag{8.32}$$

Considering the following change of variable

$$S_p = \frac{\eta_p}{\lambda_1}(Z - \mathrm{Id}) \Leftrightarrow Z = \frac{\lambda_1}{\eta_p} S_p + \mathrm{Id}\,. \tag{8.33}$$

Taking $\eta_n = 1 - w$ and $\eta_p = w$, the adopted model in terms of (u, Z) thus reads

$$\begin{cases} \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \nabla \tilde{p} = (1-w)\Delta u + \dfrac{w}{\lambda_1} \operatorname{div} Z, \\ \partial_t Z + u \cdot \nabla Z - (\nabla u) Z - Z(\nabla u)^T + \dfrac{1}{\lambda_1}(Z - \mathrm{Id}) = 0. \end{cases} \tag{8.34}$$

Assuming this system, we can drop the tilde for simplicity.

8.4.2 Free Energy of the Regularized System: A Priori Estimates

Recall that for two matrices A, B and X in $\mathbb{R}^{d\times d}$, we have the following relations

$$A : B = \mathbb{T}_r(A^T B) = \mathbb{T}_r(B^T A) \tag{8.35}$$

$$\partial_t \mathbb{T}_r(f(A)) = \mathbb{T}_r(\,\partial_t A f'(A)) = \,\partial_t A : f'(A) \tag{8.36}$$

$$\partial(det(X)) = det(X)\mathbb{T}_r(X^{-1}\,\partial X) \tag{8.37}$$

Using the above relations, first we formally multiply the momentum equation of (8.3) by u_ε and the stress equation by $Z_\varepsilon^{-1} - \mathrm{Id}$, then we integrate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_\varepsilon^2 + (1-w)\int_\Omega \nabla u_\varepsilon^2 + \frac{w}{\lambda_1}\int_\Omega Z_\varepsilon : \nabla u_\varepsilon = 0. \tag{8.38}$$

Second, for the stress equation, we multiply it by $Z_\varepsilon^{-1} - \mathrm{Id}$, using the above relations,and so we get

$$\begin{aligned}
\partial_t Z_\varepsilon : (Z_\varepsilon^{-1} - \mathrm{Id}) &= \mathbb{T}_r(\,\partial_t Z_\varepsilon Z_\varepsilon^{-1} - \,\partial_t Z_\varepsilon) = \,\partial_t(\log det(Z_\varepsilon)) - \,\partial_t \mathbb{T}_r(Z_\varepsilon),\\
\frac{1}{\lambda_1}(Z_\varepsilon - \mathrm{Id}) : (Z_\varepsilon^{-1} - \mathrm{Id}) &= \frac{1}{\lambda_1}\mathbb{T}_r(2\,\mathrm{Id} - Z_\varepsilon - Z_\varepsilon^{-1}),\\
(\nabla u_\varepsilon Z_\varepsilon + Z_\varepsilon(\nabla u_\varepsilon)^T) : (Z_\varepsilon^{-1} - \mathrm{Id}) &= \mathbb{T}_r((Z_\varepsilon(\nabla u_\varepsilon)^T \nabla u_\varepsilon Z_\varepsilon)(Z_\varepsilon^{-1} - \mathrm{Id})\\
&= \mathbb{T}_r(\nabla u_\varepsilon) + \mathbb{T}_r(Z_\varepsilon(\nabla u_\varepsilon)^T Z_\varepsilon^{-1}) - \mathbb{T}_r(\nabla u_\varepsilon Z_\varepsilon + Z_\varepsilon(\nabla u_\varepsilon)^T)\\
&= \underbrace{\operatorname{div} u_\varepsilon}_{=0} + \underbrace{\mathbb{T}_r(Z_\varepsilon(\nabla u_\varepsilon)^T Z_\varepsilon^{-1})}_{\mathbb{T}_r(\nabla u_\varepsilon^T)=0} - 2D(u_\varepsilon) : Z_\varepsilon\\
&= -2D(u_\varepsilon) : Z_\varepsilon.
\end{aligned}$$

Upon integrating in space the nonlinear term $\mathbb{T}_r(u_\varepsilon \cdot \nabla Z_\varepsilon(Z_\varepsilon^{-1} - \mathrm{Id}))$ is rendered zero due to the incompressibility condition. More precisely we have

$$\begin{aligned}
\int_\Omega u_\varepsilon \cdot \nabla Z_\varepsilon : Z_\varepsilon^{-1} &= \int_\Omega \Big(\operatorname{div}(u_\varepsilon Z_\varepsilon) - \underbrace{(\operatorname{div} u_\varepsilon)}_{=0} Z_\varepsilon\Big) : Z_\varepsilon^{-1}\\
&= \int_\Omega \mathbb{T}_r(u_\varepsilon \cdot \nabla Z_\varepsilon Z_\varepsilon^{-1}) = u_\varepsilon \cdot \nabla \mathbb{T}_r(\log Z_\varepsilon)\\
&= -\int_\Omega \mathbb{T}_r(\log Z_\varepsilon) \operatorname{div} u_\varepsilon = 0
\end{aligned}$$

We get the following equation

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^2 + \frac{w}{\lambda_1}(\mathbb{T}_r(Z_\varepsilon) - \log det(Z_\varepsilon) - d) + \frac{w}{\lambda_1^2}\int_\Omega \mathbb{T}_r(Z_\varepsilon + Z_\varepsilon^{-1} - 2\,\mathrm{Id})\\
+ 2(1-w)\int_\Omega \nabla u_\varepsilon^2 + \frac{\varepsilon w}{\lambda_1}\int_\Omega \Delta Z_\varepsilon : (Z_\varepsilon^{-1} - \mathrm{Id}) = 0.
\end{aligned} \tag{8.39}$$

Remark 17. *The quantity $u_\varepsilon^2 + \frac{w}{\lambda_1}(\mathbb{T}_r(Z_\varepsilon) - \log det(Z_\varepsilon) - d)$ is dissipative. This is in fact due to the following lemma which guarantees the positiveness of the required terms in equation (8.39), and which renders this latter a free energy equation of the regularized system (8.3).*

Lemma 8.8. *Let X be a symmetric positive definite matrix Then we have*

$$\mathbb{T}_r(X) - \log det(X) \geqslant d, \tag{8.40}$$

$$\mathbb{T}_r(X) + \mathbb{T}_r(X^{-1}) \geqslant 2d, \tag{8.41}$$

$$\int \Delta X : (X^{-1} - \mathrm{Id}) = \sum_j \int \mathbb{T}_r\Big((\,\partial_{x_j} X X^{-1})^2\Big) \geqslant \frac{1}{d}\int |\nabla \mathbb{T}_r(\log X)|^2 \geqslant 0. \tag{8.42}$$

Moreover, the functions F, H and G defined by

$$F(X) = \mathbb{T}_r(X) - \log det(X) - d,$$

$$H(X) = \mathbb{T}_r(X) + \mathbb{T}_r(X^{-1}) - 2d,$$

$$G(H) = \Delta X : (X^{-1} - \mathrm{Id})$$

are convex functions and attain a global minimum at $X = \mathrm{Id}$.

Proof. Since X is positive definite, then it possesses d-eigenvalues which are all positive $0 \leqslant \lambda_1 \leqslant ... \leqslant \lambda_d$. Moreover the function $\mathbb{T}_r X - \log\det X - d$ can be rewritten as

$$\begin{aligned}\mathbb{T}_r X - \log\det X - d &= \sum \lambda_i - \log \prod \lambda_i - d \\ &= \sum (\lambda_i - \log\lambda_i - 1).\end{aligned}$$

In $\mathbb{R}_+$, the function $x - \log x - 1$ is a positive function and admits a global minimum at 1. By analogy we deduce that $\mathbb{T}_r(X) - \log det(X) - d \geqslant 0$ and admits a global minimum at $X = \mathrm{Id}$. Similarly, we show that $\mathbb{T}_r(X) + \mathbb{T}_r(X^{-1}) - 2d$ is a positive function and admits a global minimum at Id.
Concerning the convexity of F, it follows from the convexity of the $-\log$ function and the linearity of $\mathbb{T}_r$. Similarly for the inverse function for positive definite matrices, we easily prove that H as well is convex.
The third inequality is true due to the fact that X is a symmetric positive definite matrix, in fact we can write

$$\begin{aligned}\int_\Omega \nabla X :: X^{-1}\nabla X X^{-1} &= \sum_k \int_\Omega (\nabla X)_k : (X^{-1}\nabla X X^{-1})_k \\ &= \sum_k \int_\Omega \mathbb{T}_r(\,\partial_{x_k} X : (X^{-1}\,\partial_{x_k} X X^{-1})) \\ &= \sum_k \int_\Omega \mathbb{T}_r\Big((\,\partial_{x_k} X)^2 X^{-2}\Big) \geqslant 0.\end{aligned}$$

The proof of the inequality bound is found in [139]. The convexity of G follows from the fact that

$$\begin{aligned}\int G(X) &= \sum_k \int_\Omega \mathbb{T}_r\Big((\,\partial_{x_k} X)^2 X^{-2}\Big) \\ &= \sum_k \int_\Omega \mathbb{T}_r\Big(\,\partial_{x_k}(\log X)\Big)^2\end{aligned}$$

which is a combination of the composition of convex ones, which means that G itself is convex. □

Using the above lemma and the fact that $F \leqslant H$, the decay of the free energy clearly gives the following a priori estimates

$$u_\varepsilon \quad \text{u.b in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$

$$Z_\varepsilon - \mathrm{Id} \ \text{ u.b in } L^\infty(0,T;L^2) \cap L^2(0,T;L^2(\Omega)).$$

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