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ANDREAS HALLBÄCK

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THÉORIE DES MODÈLES MÉTRIQUES, GROUPES POLONAIS ET DIVERSITÉS
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Abstract

We study metric model theory and Polish groups as automorphism groups of separable metric structures.

We expand upon the infinitary continuous logic treated in [12, 11, 23, 27] and give a new proof of the Omitting Types Theorem of infinitary continuous logic. We also find a new way of calculating the type distance of infinitary continuous logic. Furthermore, we show an infinitary version of the Ryll-Nardzewski Theorem.

We also study the Roelcke completion of a Polish group and give a model theoretical characterisation of locally Roelcke precompact Polish groups. We do this by showing that the Roelcke completion of a Polish group can be considered as a certain set of types in metric model theory.

Furthermore, we develop the model theory of the Urysohn metric space $\mathcal{U}$. We show that its theory $T_{\mathcal{U}}$ eliminates quantifiers, that $\mathcal{U}$ is a prime model and that any separable model of $T_{\mathcal{U}}$ is a disjoint union of isomorphic copies of $\mathcal{U}$. Moreover, we show that the isometry group of $\mathcal{U}$ is locally Roelcke precompact by applying our result above. This was already known, but our proof is new.

Finally, we study the Urysohn diversity $\mathbb{U}$. Diversities are a natural generalisation of metric spaces, where positive values are assigned not just to pairs, but to all finite subsets. We develop the model theory of $\mathbb{U}$ and show, among other things, that its automorphism group $\mathrm{Aut}(\mathbb{U})$ is locally Roelcke precompact, again by applying our result above. We also show that $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group and that it has a dense conjugacy class. Lastly, we study the automorphism group of the rational Urysohn diversity $\mathbb{U}_{\mathbb{Q}}$ and show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics – a property with many strong implications.

Keywords: Metric model theory, continuous logic, infinitary continuous logic, automorphism groups of metric structures, Polish group theory, locally Roelcke precompact groups, Urysohn metric space, diversities, Urysohn diversity, dense conjugacy class, ample generics.

Résumé

Nous étudions la théorie des modèles métriques et les groupes polonais comme groupes d'automorphismes de structures métriques séparables.

Nous développons la logique continue infinitaire traitée dans [12, 11, 23, 27] et donnons une nouvelle preuve du théorème d'omission des types pour la logique continue infinitaire. Nous trouvons également une nouvelle facon de calculer la distance de type de la logique continue infinitaire. De plus, nous montrons une version infinitaire du théorème de Ryll-Nardzewski.

Nous étudions également le complété de Roelcke d'un groupe polonais et donnons une caractérisation des groupes polonais localement Roelcke précompacts en utilisant la logique continue. Nous le faisons en montrant que le complété de Roelcke d'un groupe polonais peut être considéré comme un certain ensemble de types dans la théorie des modèles métriques.

De plus, nous développons la théorie des modèles de l'espace métrique d'Urysohn $\mathcal{U}$. Nous montrons que sa théorie $T_{\mathcal{U}}$ élimine les quantificateurs, que $\mathcal{U}$ est un modèle premier et que tout modèle séparable de $T_{\mathcal{U}}$ est une union disjointe de copies isomorphes à $\mathcal{U}$. De plus, nous montrons que le groupe d'isométries de $\mathcal{U}$ est localement Roelcke précompact en appliquant notre résultat obtenu précédemment. Bien que le résultat soit déjà connu, nous en apportons une preuve nouvelle.

Enfin, nous étudions la diversité d'Urysohn $\mathbb{U}$. Les diversités sont une généralisation naturelle des espaces métriques, les valeurs positives sont attribuées non seulement aux paires, mais à tous les sous-ensembles finis. Nous développons la théorie du modèle de $\mathbb{U}$ et montrons, entre autres, que son groupe d'automorphismes $\mathrm{Aut}(\mathbb{U})$ est localement Roelcke précompact, encore une fois en appliquant notre résultat ci-dessus. Nous montrons également que $\mathrm{Aut}(\mathbb{U})$ est un groupe polonais universel et qu'il a une classe de conjugaison dense. Enfin, nous étudions le groupe d'automorphisme de la diversité rationnelle d'Urysohn $\mathbb{U}_{\mathbb{Q}}$ et montrons que $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ a des génériques amples – une propriété riche de conséquences, certaines étant très importantes.

Mots clés : Théorie des modèles métriques, logique continue, logique continue infinitaire, groupes d'automorphismes de structures métriques séparables, théorie des groupes polonais, groupes locallement Roelcke précompacts, l'espace métrique d'Urysohn, diversités, diversité d'Urysohn, classe de conjugaison dense, génériques amples.

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Introduction

This dissertation deals with two major fields of mathematical research and the connections between them: *metric model theory* and *Polish group theory*. As it turns out, these two seemingly quite different topics are intimately connected. In fact, as we shall see in this thesis, the study of Polish groups is in some sense the same as the study of the automorphism groups of separable atomic metric structures.

Metric model theory is also known as *continuous logic*. Continuous logic was introduced in some form or another already in the 1960's, where any compact set was allowed as truth values (cf. [25]). Unfortunately, the theory never really took off, perhaps because the set-up was too general to get a satisfactory analogy between discrete model theory and the continuous counterpart. In recent years the theory has, however, experienced quite a renaissance with the works of Henson, Berenstein, Ben Yaacov, Usvyatsov and many more (cf. for instance [8, 13, 14, 15, 6, 9, 11, 30]). The focus has now been on *metric* structures, where instead of allowing a general compact set as truth values, we only allow compact intervals of the real line as possible truth values. Although we do lose some generality with this restriction, most of the structures we want to study, such as Banach spaces, Hilbert spaces, measure algebras and Polish groups, all fit well into this set-up.

The basic object on which we can define our metric structures are complete metric spaces. For technical reasons we require the metric to be bounded. This does cause a few problems, since most of the structures we are interested in are not bounded. However, in practice, this obstacle is easy to work around, and it is possible to describe even unbounded structures as metric structures without much effort. Just as in classical logic, the structures contain *predicates* and *functions* that in the metric setting must be uniformly continuous maps defined on some power of the structure. As logical *connectives* we allow any real-valued continuous function defined on products of compact intervals. Furthermore, and this is where the crucial advantage over the continuous logic of the 1960's lies, there are two natural *quantifiers*, namely the *infimum* and the *supremum*, thought of as metric versions of the existential and univer-

sal quantifiers, respectively. Equipped with these two quantifiers the analogy between metric model theory and discrete model theory becomes absolutely remarkable. Almost every single concept and result from the classical setting has a corresponding concept or result in the metric setting. For instance, there are completely natural metric versions of the Compactness Theorem, the Löwenheim-Skolem Theorem, the Omitting Types Theorem and the Ryll-Nardzewski Theorem, just to mention some. Moreover, fundamental concepts and properties generalise nicely to the metric setting. These include, for example, homogeneity, atomicity, saturation and quantifier elimination, again just to mention a few. All of these concepts can be defined using another model theoretical concept that generalises nicely to the metric setting: *Types*. In short, the *type* of a tuple of elements from a model of a given theory is the collection of everything that the logical language can tell us about this specific tuple. Naturally, such an object contains a lot of information on the tuple, so understanding the types of all tuples provides an enormous amount of information on the structure. *Homogeneity* is then defined as in the classical case: if two tuples have the same type, we can find an automorphism of the structure mapping one tuple onto the other. Usually in metric model theory, we need to allow some small error, so the structures are only *approximately* homogeneous, meaning that we can only move two tuples of the same type arbitrarily close to each other with automorphisms. However, most of the structures we will consider are *exact* homogeneous. Likewise, it is possible to define the *principal* types. These are types that must be *realised* in all models (of the theory under consideration), meaning that it is possible to find a tuple of the given type in any model. An *atomic* model is then a model that only realises principal types. *Saturation* is defined as usual: a structure is κ-saturated if it realises all types over sets of size less than κ. Finally, *quantifier elimination* means that any formula in our logical language is equivalent to a formula without quantifiers. As in the classical case, this property can also be defined in terms of saturated models and types, even though this would perhaps be somewhat artificial to do.

Type spaces in metric model theory turns out to be one of the most powerful tools available. Just as in the the discrete setting, the type spaces of metric model theory carry a nice compact topology referred to as the *logic topology*. Moreover, in metric model theory the type spaces carry a topology induced by the so-called *type distance*. This distance is a completely new phenomenon in metric model theory. In the discrete setting, the distance is always discrete. However, in the general metric set-up it provides a very flexible tool to study metric structures and their properties. In particular, the interplay between the logic topology and the type distance provides crucial new information on the structures. In this dissertation too, we will exploit the relationship between these two topologies and characterise a property of Polish groups known as *local Roelcke precompactness*. We will get back to these groups below.

Given the remarkable analogy between discrete finitary logic and finitary

metric model theory discussed above, it becomes natural to ask if the same thing is true for more general logics. In the discrete setting, one of the more successful logics is the infinitary logic denoted $\mathcal{L}_{\omega_1\omega}$. In this logic we allow conjunctions and disjunctions over countably many formulas, which gives the language more expressive power. However, the price for this power is high: the Compactness Theorem no longer holds. Nevertheless, $\mathcal{L}_{\omega_1\omega}$ provides a powerful tool in the study of structures. The most prominent example of the expressive power of this logic is probably Scott's theorem, stating that for any (classical) model, there is an $\mathcal{L}_{\omega_1\omega}$-sentence characterising it up to isomorphism (cf. e.g. [47]). In this thesis, we give an exposition of the *infinitary* continuous logic developed in [12, 11, 23, 27]. Moreover, we expand upon this logic and give a new proof of the *Omitting Types Theorem* for infinitary continuous logic. In short, this theorem states that for any non-principal type it is possible to find a model where it is not realised, i.e. the type is *omitted*. As a corollary, we obtain a version of the *Ryll-Nardzewski Theorem* for infinitary continuous logic. This theorem describes the $\aleph_0$-categorical structures, i.e. those structures whose theory has a unique separable model, in terms of the types they realise. Furthermore, we discover a new way of calculating the type distance in infinitary continuous logic. Defining such a distance is a delicate task, since the usual type distance of finitary logic depends heavily on the Compactness Theorem. Nevertheless, in [11] the authors do manage to define a type distance for infinitary types. However, the definition might seem rather artificial and complicated at first glance. Our new version of the type distance generalises directly one way to calculate the distance in finitary logic: it is the supremum over the so-called 1-Lipschitz formulas, i.e. those formulas that are interpreted as 1-Lipschitz maps. Hence, it simplifies the definition from [11] and gives perhaps a more intuitive idea of how to calculate the distance.

The other major theme of the thesis is, as mentioned, Polish group theory. Polish groups are everywhere in mathematics. Examples include the group S_∞ of all permutations of $\mathbb{N}$, the group of homeomorphisms of a compact metrisable space, the unitary group of the separable infinite dimensional Hilbert space, isometry groups of separable metric spaces and, more generally, automorphism groups of separable metric structures. In classical model theory, the group S_∞ plays a particularly important role. It turns out that the automorphism group of any countable structure can be viewed as a closed subgroup of S_∞. Conversely, given any closed subgroup of S_∞, it is possible to construct a countable structure, known as the *canonical structure*, with this subgroup as its automorphism group (cf. [4] for the details).

Given this correspondence between closed subgroups of S_∞ and automorphism groups of classical countable structures, it is natural to ask if there is a similar correspondence in the general metric framework. The answer turns out to be yes: any automorphism group of a separable metric structure is, as mentioned, a Polish group, because it is a closed subgroup of an isometry group of a complete separable metric space. Conversely, given *any* Polish group G,

it is possible to define a metric structure, known as the *canonical metric structure* associated to G, such that G is the automorphism group of this structure. Usually, this construction is attributed to Julien Melleray. However, curiously enough, he himself writes in [50] that he is not sure who the construction should be attributed to. In any case, we will use this structure to give the aforementioned characterisation of locally Roelcke precompact Polish groups. Local Roelcke precompactness is a property of the so-called *Roelcke completion* of a Polish group. This completion is the completion of a certain *uniformity* we can impose on the group. Uniformities were originally introduced in the 1930's by André Weil in order to capture the intuitive idea of relative closeness. That is, in a uniform space it makes sense to say that a is closer to x than b is to y. On general topological groups it is always possible to find uniformities inducing the group topology. Among these uniformities, there are four canonical uniformities of special interest: the *left*, *right*, *two-sided* and *Roelcke* uniformities. For Polish groups, the two-sided uniformity is always complete. Hence, only the other three can provide new information when we consider the group completions. Furthermore, by a famous theorem of Birkhoff and Kakutani, it is always possible to find a left-invariant compatible metric inducing the left uniformity. Likewise, we can always find a right-invariant compatible metric inducing the right uniformity. Moreover, it is not hard to show that *any* left-, respectively, right-invariant metric will induce the left, respectively right uniformity. The Roelcke uniformity is then defined as the *meet* of the left and the right, i.e. it is the finest uniformity contained in both of them. Using any left- or right-invariant metric, it is possible to define a metric that induces the Roelcke uniformity. The completion of the group with respect to the Roelcke uniformity is known as the *Roelcke completion*, and it is this completion that we will study by using type spaces of continuous logic.

The connection between the Roelcke completion and type spaces of continuous logic was also studied in [13]. There the authors together with C. Rosendal (cf. [54]) obtained the theorem below that characterises the so-called *Roelcke precompact* groups, i.e. those groups whose Roelcke completion is compact, in terms of their actions. Roelcke precompact groups have been studied extensively in the literature (see for instance [13, 53, 55, 62, 63, 64, 65, 68] just to mention some). One reason for this, is that their actions tend to be somewhat nicely behaved and easy to understand, as the theorem below explains. Loosely speaking, they have a somewhat compact feel to them, even though there are, of course, many Roelcke precompact groups that are nowhere near compact. The theorem from [13] (cf. also [54]) reads:

Theorem ([13, Theorem 2.4], [54])**.** *Let G be a Polish group. Then the following are equivalent:*

(i) *G is Roelcke precompact.*

(*ii*) *Whenever G acts continuously by isometries on a complete metric space X and the set of orbit closures $X /\!\!/ G$ is compact, then for all n, $X^n /\!\!/ G$ is compact, where G acts on X^n via the diagonal action.*

(*iii*) *There is a Polish metric space X and a homeomorphic group embedding $G \hookrightarrow \mathrm{Iso}(X)$ such that for the induced action of G on X, the space $X^n /\!\!/ G$ is compact for every n.*

An action such as the one in (*ii*) of the theorem above, where $X^n /\!\!/ G$ is compact for all n, is called an *approximately oligomorphic* action (compared to an *oligomorphic* action, where there is only finitely many orbits for each n). It turns out that by using the Ryll-Nardzewski Theorem for continuous logic (cf. [9, Theorem 12.10]), one gets the following description of Roelcke precompact groups:

Corollary. *Let $G = \mathrm{Aut}(\mathcal{M})$ be the automorphism group of a metric structure and suppose that $\mathcal{M} /\!\!/ G$ is compact. Then the following are equivalent:*

(*i*) *G is Roelcke precompact.*

(*ii*) $\mathrm{Th}(\mathcal{M})$ *is $\aleph_0$-categorical.*

Armed with these results, it becomes much simpler to check if a given group is Roelcke precompact. For instance, S_∞, the unitary group of a separable Hilbert space [62], the automorphism group of a standard probability space [33], the automorphism group of the random graph [13] and the isometry group of the bounded Urysohn space [65] are all Roelcke precompact, which may be seen using the result above.

Just like locally compact spaces are a natural generalisation of the compact ones, Roelcke precompact groups can be generalised to the so-called *locally Roelcke precompact* (lRpc) groups. Here we do not require the group itself to be precompact in the Roelcke uniformity, but only that there is a precompact neighbourhood of the identity. In particular, any Roelcke precompact and any locally compact group is also lRpc. However, it is not immediate that the Roelcke completion of a locally Roelcke precompact group is locally compact. Nevertheless, due to a recent theorem of Zielinski, this is fortunately the case:

Theorem (Zielinski, cf. [68])**.** *A Polish group G is locally Roelcke precompact if and only if the Roelcke completion of G is locally compact.*

Examples of locally Roelcke precompact groups that are not Roelcke precompact include the automorphism group of the regular rooted tree, the group $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Q}, <)$ of order preserving bijections of $\mathbb{Q}$ commuting with integral shifts and the group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}, <)$ of order-preserving homeomorphisms of $\mathbb{R}$ commuting with integral shifts. Furthermore, and more importantly for us, the isometry group of the Urysohn metric space and the automorphism group of

the Urysohn diversity are both locally Roelcke precompact. We will get back to diversities below.

Although lRpc groups have not received as much attention as the Roelcke precompact ones, they do posses interesting features e.g. from the viewpoint of geometric group theory. For instance, they can allow non-trivial so-called *coarse* geometry. This should be seen in opposition to the Roelcke precompact groups, because these are always coarsely bounded and hence have trivial large-scale geometry. At the same time, lRpc groups are only a slight generalisation of the Roelcke precompact ones and hence share many of the same features. Moreover, lRpc groups are always locally bounded, so their large-scale geometry is somehow still tangible. Since we will not really be concerned with these geometrical considerations in this thesis, we will not indulge ourselves and digress from the matter at hand. Instead, we refer the reader to Rosendal's manuscript [55] and Zielinski's paper [68] for an introduction to coarse geometry and more on lRpc groups. However, what we will see in this thesis is a model theoretical characterisation of lRpc groups, inspired by the one of the Roelcke precompact ones above. More specifically, we will see that it is always possible to embed the Roelcke completion isometrically into a certain type space, in such a way that the Roelcke completion is homeomorphic to the image of this embedding equipped with the logic topology τ. Since τ is a nice compact topology, it becomes much easier to verify topological properties of the Roelcke completion. In particular, the group is lRpc if and only if the Roelcke completion equipped with τ is open inside its τ-closure. It turns out that this is rather straightforward to verify for both the isometry group of the Urysohn metric space and the automorphism group of the Urysohn diversity, once you understand the model theory of these structures. Therefore, before we can make this verification we will need to develop the model theory of the Urysohn metric space $\mathcal{U}$. Recall that this space is the unique (up to isometry) universal ultrahomogeneous separable complete metric space. Here, *universal* means that it contains an isometric copy of any finite metric space. *Ultrahomogeneous* means that any isometry between finite subsets extends to a surjective isometry of the whole space. $\mathcal{U}$ was first constructed by Urysohn in the 1920's in the posthumously published paper [60] and was long almost completely forgotten. However, in the 1980's Katětov gave a new construction of $\mathcal{U}$, where he "built" a version of the Urysohn space "around" any separable metric space. Not long after, Uspenskij realised that using this construction, it was rather easy to show that the isometry group of $\mathcal{U}$ was a *universal* Polish group, i.e. that any Polish group embeds as a closed subgroup of $\mathrm{Iso}(\mathcal{U})$. This renewed the interest in $\mathcal{U}$ and its isometry group, and today new interesting results keep appearing in the literature (cf. for instance [10, 24, 48, 49, 56]).

As mentioned, we will develop the model theory of $\mathcal{U}$ in this thesis. A priori, $\mathcal{U}$ does not fit well into the set-up of metric model theory because it is an *unbounded* metric space. Therefore, most authors working in logic choose to consider the bounded version of $\mathcal{U}$. However, there are several work arounds

for the unboundedness problem, as we shall see in Chapter 5. There we will show an "almost categoricity" result concerning the theory of $\mathcal{U}$, saying that any separable model of the theory is a disjoint union of at most countably many copies of $\mathcal{U}$. From this result it will follow that the theory eliminates quantifiers and that $\mathcal{U}$ is an atomic model. Moreover, as we mentioned above, we can use our result to show that $\mathrm{Iso}(\mathcal{U})$ is lRpc. This was of course already known, but our approach is new.

We will also study the automorphism group of the Urysohn *diversity* $\mathbb{U}$. Diversities were introduced in [20] and [21] by Bryant and Tupper. Their motivation was to generalise applications of metric space theory to combinatorial optimisation and graph theory to the hypergraph setting. The idea is simple enough: a diversity assigns positive values to all finite subsets (and not just to pairs of points as for a metric). This assignment must, of course, uphold some axioms similar to those of a metric. In particular, if we only consider pairs of points we obtain a metric space. It turns out that this generalises metric spaces quite nicely. Moreover, and this is where the interest of model theory begins, it turns out that there is a unique ultrahomogeneous universal separable diversity, namely the Urysohn diversity $\mathbb{U}$. In this thesis, we shall develop the model theory of $\mathbb{U}$ along the same lines as for the Urysohn metric space. In particular, we will show the same "almost categoricity" result for $\mathbb{U}$, namely that any separable model of the theory of $\mathbb{U}$ is a disjoint union of at most countably many copies of $\mathbb{U}$. As for the metric space, this result will imply that the theory of $\mathbb{U}$ eliminates quantifiers and that $\mathbb{U}$ is an atomic structure. Moreover, it implies, using our characterisation of the lRpc groups, that the automorphism group of $\mathbb{U}$ is locally Roelcke precompact.

$\mathbb{U}$ was first constructed in [18] by a construction similar to the Katětov construction of the Urysohn metric space. As mentioned, Uspenskij exploited Katětov's construction and showed that the isometry group the Urysohn metric space is a universal Polish group. In Chapter 6, we adapt Uspenskij's proof and show that the automorphism group of $\mathbb{U}$ is universal as well.

In [19], the authors of [18] constructed $\mathbb{U}$ as a *metric Fraïssé limit* in the sense of Ben Yaacov in [8]. In this thesis, we will give yet another construction of $\mathbb{U}$ using classical *Fraïssé theory*. More precisely, we will show that the class of finite rational diversities, i.e. diversities whose map only takes rational values, is a so-called *Fraïssé class*. A Fraïssé class is a class with certain properties that ensure that there is a so-called *limit* for the class, i.e. the *Fraïssé limit*, containing all the structures of the class. Moreover, this limit is *ultrahomogeneous*, which in this more general setting means that any isomorphism between finite substructures extends to an automorphism of the whole structure. This way of approximating infinite structures by their finite substructures has proven extremely useful. Many well-known and important structures can be defined as a Fraïssé limit, e.g. the (rational) Urysohn metric space, the random graph and the ordered rationals are all Fraïssé limits of their respective classes. Moreover, as mentioned, the class of all finite rational

diversities is a Fraïssé class. Hence, the class has a limit, which we will call the *rational* Urysohn diversity and denote by $\mathbb{U}_{\mathbb{Q}}$. It is not difficult to check that the completion of $\mathbb{U}_{\mathbb{Q}}$ is (isomorphic to) $\mathbb{U}$ itself, and thus we obtain the mentioned new construction of $\mathbb{U}$. Moreover, using results of Kechris and Rosendal from [46], it is easy to show that the conjugacy action of $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ on itself admits a dense orbit. Furthermore, we will see that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$, from which it follows that the conjugacy action of $\mathrm{Aut}(\mathbb{U})$ on itself also admits a dense orbit. Having a dense conjugacy class is also known as the *topological Rokhlin property* because the well-known Rokhlin Lemma of ergodic theory implies that the automorphism group of a standard measure space (X, μ) has this property. Having a dense conjugacy class is equivalent to the following generic ergodicity property: every conjugacy invariant Baire measurable subset of the group is either meagre or comeagre (cf. [46]). Recall that a subset of a topological space is *meagre* if it is a countable union of *nowhere dense* sets, i.e. sets whose closure has empty interior. A *comeagre* set is then simply a set whose complement is meagre. A set A is *Baire measurable* if there is an open set U such that the symmetric difference between A and U is meagre. Today there are many groups that are known to admit a dense conjugacy class. Besides $\mathrm{Aut}(X, \mu)$, the unitary group of a separable infinite dimensional Hilbert space, and the homeomorphism groups of the Hilbert cube $[0, 1]^{\mathbb{N}}$ and the Cantor space $2^{\mathbb{N}}$ all share this property, just to mention a few (cf. [46]).

We conclude the thesis by showing that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has so-called *ample generics*. Ample generics is a powerful property with many strong implications. It was first introduced by Hodges, Hodkinson, Lascar and Shelah in [37] in order to study the *small index property* for automorphism groups of certain countable structures. This property says that any subgroup of index $< 2^{\aleph_0}$ must be open. In [46], Kechris and Rosendal studied ample generics further and showed that it implies several strong properties. One of these is the *automatic continuity property*. A classical result of Pettis says that any Baire measurable homomorphism from a Polish group G to a separable group must be continuous (cf. [52]). If G has ample generics, this is simply true for *any* group homomorphism. Kechris and Rosendal also show that a Polish group with ample generics has a unique Polish group topology and that it cannot be the countable union of non-open subgroups.

The thesis is organised as follows: Chapter 1 covers the basics of Polish group theory and classical descriptive set theory that we will need. In Chapter 2 we introduce metric model theory and the concepts and results we will need later in the thesis. We will also show a few smaller lemmas and propositions that will come in handy from time to time. Chapter 3 is concerned with infinitary continuous logic. Chapter 4 studies Polish groups as automorphism groups of metric structures and the Roelcke completion as a set of types. In Chapter 5 we study the Urysohn metric space as a metric structure. The last chapter, Chapter 6, studies both the rational and complete Urysohn

diversities as well as their automorphism groups.

Chapter 1

Polish Group Theory

In this first, short chapter of the thesis we have included some preliminary facts and results from descriptive set theory and Polish group theory that we will need throughout. There are no new results here and very few proofs. We invite the reader to consult e.g. Kechris' excellent book [44] for a thorough exposition of classical descriptive set theory and for instance Becker and Kechris' book [4] for more on Polish groups. Finally, Dierolf and Roelcke's [53] is a good reference for uniform spaces and group completions.

1.1 Polish spaces

The perhaps most basic object of descriptive set theory is a *Polish space*. These spaces are so named because they were first studied by a large number of well-known Polish mathematicians. Today they provide a general framework for the study of the main topics of this thesis, i.e. descriptive set theory, metric model theory and Polish group theory.

In this short section we only introduce the absolute essentials that we will need. There are many good references for Polish spaces and we invite the reader to consult e.g. [44] for a thorough exposition.

The definition of a Polish space reads as follows:

Definition 1.1.1. *A topological space X is* **separable** *if there is a countable dense subset. X is* **completely metrisable** *if there is a complete metric inducing the topology. A* **Polish** *space is a separable completely metrisable topological space.*

Polish spaces are abundant in mathematics. We have listed two well-known examples below.

Example 1.1.2. The reals $\mathbb{R}$ with their usual topology is a Polish space. So is the open unit interval $(0, 1)$ since it is homeomorphic to $\mathbb{R}$ (even though the usual metric is not complete).

Example 1.1.3 (The Cantor Space)**.** Consider the set $\mathcal{C} = 2^\omega$ consisting of all sequences in $2 = \{0, 1\}$. We endow $\{0, 1\}$ with the discrete topology and $\mathcal{C}$ with the product topology. It has as basis the sets

$$U_s = \{x \in \mathcal{C} : s \subseteq x\},$$

where $s \in 2^{<\omega}$ is a finite sequence. The space $\mathcal{C}$ is called the **Cantor space**. It is a Polish space with a complete compatible metric defined by $d(x, y) = 2^{-n-1}$ for $x \neq y$, where n is the least number such that $x(n) \neq y(n)$. The set

$$\{x \in \mathcal{C} : \exists N \ \forall n \geq N \ x(n) = 0\}$$

is a countable dense subset.

Remark 1.1.4. In the example above one could just as easily have used any countable ordinal instead of 2. The space $\mathcal{N} = \omega^\omega$ is called the **Baire space**.

The following proposition is easily verified.

Proposition 1.1.5. (*i*) *Closed subspaces of Polish spaces are Polish.*
(*ii*) *Countable products of Polish spaces are Polish. In particular, countable powers of a Polish space are again Polish.*

Recall that a $\boldsymbol{G_\delta}$ **set** is a countable intersection of open sets. We have the following important characterisation of those subspaces of a Polish space that are themselves Polish (cf. [44, Theorem 3.11]).

Theorem 1.1.6. *A subspace of a Polish space is Polish in the subspace topology if and only if it is G_δ.*

1.2 Baire category

We will from time to time encounter notions from Baire category. Therefore, we provide a short introduction to this area. The reader is again referred to [44] for a more comprehensible introduction to Baire category.

Definition 1.2.1. *Let X be a topological space. A subset $A \subseteq X$ is* **nowhere dense** *if its closure $\overline{A}$ has empty interior, i.e. $\overline{A}^\circ = \emptyset$.*

A is **meagre** *if it is a countable union of nowhere dense sets. The complement of a meagre set is called* **comeagre**.

Remark 1.2.2. Observe the following:

- A set A is nowhere dense if and only if the complement of its closure $\overline{A}^c$ is dense.
- A is meagre if and only if it is contained in a countable union of closed nowhere dense sets.

- A is comeagre if and only if it contains a countable intersection of open dense sets.

The meagre sets of a topological space can be used to define an important σ-algebra consisting of the sets with the *Baire property*.

Definition 1.2.3. *A subset $A \subseteq X$ of a topological space X has the* **Baire property** *if there is an open set U such that the symmetric difference $A \triangle U$ is meagre.*

It is not hard to see that the sets with the Baire property form a σ-algebra. Moreover, it is the smallest σ-algebra containing the open sets and the meagre sets (cf. [44, Proposition 8.22]).

Since the sets with the Baire property form a σ-algebra, we can define maps that are measurable with respect to it. These are the *Baire measurable* maps.

Definition 1.2.4. *A map $f\colon X \to Y$ between topological spaces is* **Baire measurable** *if the preimage of any open set in Y has the Baire property in X.*

It is of course evident that continuous maps are Baire measurable. In fact, most natural maps are. A general rule of thumb says that if you can write down an explicit definition of the map, then it is (probably) Baire measurable.

The crown jewel of Baire category is the *Baire Category Theorem* proved by R. Baire himself in 1899.

Theorem 1.2.5 (Baire Category Theorem, [2])**.** *Let X be a completely metrisable space. Then any countable intersection of dense open subsets of X is dense.*

Proof. Let d be a complete metric on X and let $(D_n)_{n\in\mathbb{N}}$ be a sequence of dense open subsets of X. Let U be a non-empty open set. We must show that $\bigcap D_n \cap U$ is non-empty. Since D_0 is dense and open, we can find a ball, B_0, of some radius, r_0, such that $\overline{B_0} \subseteq U \cap D_0$. Since B_0 is open and D_1 is dense and open, we can find a ball, B_1, of radius $r_1 \leq \frac{1}{2}r_0$ and such that $\overline{B_1} \subseteq B_0 \cap D_1$. Continuing in this way, we get a sequence of balls B_n of radius r_n, where $r_n \leq \frac{1}{n+1}r_0$ and such that $\overline{B_n} \subseteq B_{n-1} \cap D_n$. If we let x_n denote any point of B_n, then the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus, by completeness, (x_n) converges to some x. For any $n \in \mathbb{N}$ we must have that $x \in \overline{B_n}$ since $\overline{B_n}$ is closed, and thus $x \in \bigcap \overline{B_n}$. Furthermore, we have that $\bigcap \overline{B_n} \subseteq \overline{B_1} \subseteq U$ and for each $n \in \mathbb{N}$ we have that $\overline{B_n} \subseteq D_n$. Hence, $\bigcap \overline{B_n} \subseteq \bigcap D_n$ as well, so that $x \in \bigcap D_n \cap U$. □

Remark 1.2.6. The following are equivalent in a topological space X:

(i) A countable intersection of dense open sets in X is dense.

(*ii*) Any non-empty open set is not meagre.
(*iii*) Every comeagre set in X is dense.

A space satisfying these properties is called a **Baire space**.

The Baire Category Theorem is one of the most important and useful theorems of descriptive set theory, with a multitude of applications in many areas of mathematics. We will be using this theorem ourselves in Chapter 3.

1.3 Polish groups

One of the main objects of study in the thesis is *Polish groups*. In general, a *topological group* is a group equipped with a topology such that the group operations are continuous. Polish groups are then defined as follows:

Definition 1.3.1. *A* **Polish group** *is a topological group whose topology is separable and completely metrisable.*

Polish groups appear throughout mathematics and they have been studied for around a century. Below is a list of some examples of Polish groups.

Examples. The following groups are Polish:

(*i*) Any countable group equipped with the discrete topology.

(*ii*) $(\mathbb{R}, +)$ with the usual topology.

(*iii*) If G_n is a Polish group for each $n \in \mathbb{N}$, then $\prod G_n$ with the product topology is Polish.

(*iv*) The group of invertible $n \times n$-matrices $GL(n, \mathbb{R})$ viewed as a subspace of $\mathbb{R}^{n^2}$.

(*v*) The permutation group of $\mathbb{N}$, S_∞, equipped with the pointwise convergence topology. This group is particularly important in model theory, since its closed subgroups are exactly the automorphism groups of countable structures (cf. [4] for the details).

(*vi*) If H is a separable infinite dimensional Hilbert space, then the unitary group $U(H)$ of all bounded linear operators $T\colon H \to H$ such that $TT^* = T^*T = I$ is a Polish group.

(*vii*) If (X, d) is a separable complete metric space, then the isometry group of X, $\mathrm{Iso}(X)$, is a Polish group with the pointwise convergence topology. A compatible complete metric is given by

$$d(f, g) = \sum_{n=0}^{\infty} 2^{-n-1}\big(d_1(f(x_n), g(x_n)) + d_1(f^{-1}(x_n), g^{-1}(x_n))\big),$$

where (x_n) is a countable dense sequence in X and where

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

(*viii*) More generally, the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a separable metric structure $\mathcal{M}$ is Polish since it is a closed subgroup of the isometry group of $\mathcal{M}$.

We will get back to metric structures in the next chapter. Furthermore, we will see in Chapter 4 that *any* Polish group may be viewed as the automorphism group of a separable metric structure. Thus, the study of Polish groups can be considered as contained in the study of metric structures and their automorphism groups.

General metrisable topological groups were completely characterised by Birkhoff and Kakutani. Moreover, and this will be important to us, they showed that when the group is metrisable, it is always possible to find a *left-invariant* compatible metric, i.e. a metric d inducing the topology such that $d(gh, gf) = d(h, f)$ for all g, h and f.

Theorem 1.3.2 (Birkhoff-Kakutani)**.** *A topological group G is metrisable if and only if it is Hausdorff and has a countable neighbourhood basis at the identity. Moreover, if G is metrisable, the metric can be chosen to be left-invariant.*

Remark 1.3.3. If G is metrisable with left-invariant metric d_L, then the metric $d_R(g,h) = d_L(g^{-1}, h^{-1})$ defines a right-invariant compatible metric.

In particular, it follows that any Polish group admits a compatible left-invariant metric. However, and this is very important, the left-invariant metric need not be *complete*. For instance, S_∞ does not admit such a metric (cf. [4]). This opens up the study of completions of Polish groups with respect to for instance left-invariant metrics. We will get back to that below.

A useful theorem due to Pettis [52] tells us that Baire measurable homomorphisms between Polish groups are always continuous.

Theorem 1.3.4 (Pettis [52], cf. also [44, Theorems 9.9 and 9.10])**.** *Let G be a topological group. If $A \subseteq G$ has the Baire property and is non-meagre, then $A^{-1}A := \{gh : g \in A^{-1}, h \in A\}$ contains an open neighbourhood of the identity. It follows that if $\varphi\colon G \to H$ is a homomorphism from a Polish group G to a separable group H and φ is Baire measurable, then φ is continuous.*

Therefore, most natural homomorphism that one will encounter when working with Polish groups will be continuous. However, using the axiom of choice, it is possible to construct homomorphisms between some Polish groups that are not continuous. On the other hand, there are Polish groups G, such that

any homomorphism $\varphi\colon G \to H$ into a separable group H is continuous. This property is known as the *automatic continuity property* and we will get back to it in Chapter 6, where we provide a new example of a Polish group with this property.

Another important notion, that we will come across a few times in this thesis, is the notion of a *group action*. These are defined as follows:

Definition 1.3.5. *Let G be a group and let X be a set. An* **action** *of G on X is a map $G \times X \to X$ denoted $(g, x) \mapsto g \cdot x$, such that for all $x \in X$, $1_G \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$. We say that G* **acts** *on X and denote it by $G \curvearrowright X$.*

If G is a topological group and X is a topological space, we say that an action $G \curvearrowright X$ is **continuous** *if the action is continuous as a map $G \times X \to X$.*

For instance, if we are given a structure $\mathcal{M}$, then the automorphism group $\mathrm{Aut}(\mathcal{M})$ acts on $\mathcal{M}$ simply by $g \cdot a = g(a)$. We will therefore use these two notations interchangeably. We will also often omit the '$\cdot$' and simply write ga for $g \in \mathrm{Aut}(\mathcal{M})$ and $a \in \mathcal{M}$.

1.4 Uniform spaces

In this section, we provide a short introduction to uniform spaces and uniformities on Polish groups. We will need these concepts in Chapter 4.

A uniform space should be thought of, in terms of generality, as a structure sitting between topological spaces and metric spaces. The idea is to define the minimal structure needed in order to define uniform notions such as uniform continuity, Cauchy sequences and completeness. Thus a uniform space should be a generalisation of a metric space, since all these notions can be defined using a metric. On the other hand, uniform continuity implies continuity, so a uniform space must in particular be a topological space.

There are several equivalent ways to define a uniform space. We have chosen the definition using *entourages*, which was the definition used when André Weil first introduced the concept in the 1930's.

Definition 1.4.1. *Let X be a set. A collection $\mathcal{U}$ of subsets of $X \times X$ is a* **uniformity** *if it satisfies the following conditions:*

(i) If $U \in \mathcal{U}$, then $\Delta X = \{(x, x) : x \in X\} \subseteq U$.

(ii) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.

(iii) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.

(iv) If $U \in \mathcal{U}$, then there is $V \in \mathcal{U}$ such that $V \circ V \subseteq U$, where

$$V \circ V := \{(x, y) \in X \times X : \exists z \in X (x, z), (z, y) \in V\}.$$

(v) *If* $U \in \mathcal{U}$*, then* $U^{-1} \in \mathcal{U}$*, where* $U^{-1} := \{(x, y) : (y, x) \in U\}$.

Elements of $\mathcal{U}$ *are referred to as* **entourages**.

Given two uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X, we say that $\mathcal{U}$ is **coarser** than $\mathcal{V}$ and $\mathcal{V}$ is **finer** than $\mathcal{U}$ if $\mathcal{U} \subseteq \mathcal{V}$. If neither is contained in the other, it is natural to ask whether there is a finest, respectively coarsest, uniformity contained in, respectively containing, both $\mathcal{U}$ and $\mathcal{V}$. This is indeed the case. The coarsest uniformity containing both is called the **join** of $\mathcal{U}$ and $\mathcal{V}$ and is denoted by $\mathcal{U} \vee \mathcal{V}$. The finest uniformity contained in both is called the **meet** of $\mathcal{U}$ and $\mathcal{V}$ and is denoted $\mathcal{U} \wedge \mathcal{V}$.

A **base** for a uniformity $\mathcal{U}$ is a subset $\mathcal{B} \subseteq \mathcal{U}$ such that for all $U \in \mathcal{U}$ there is $B \in \mathcal{B}$ with $B \subseteq U$.

If we are given a uniform space $(X,\mathcal{U})$, we can define a topology on X as follows: a set $A \subseteq X$ is open if for every $x \in A$, there is $U \in \mathcal{U}$ such that $U[x] := \{y : (x, y) \in U\} \subseteq A$. We denote this topology by $\tau_{\mathcal{U}}$ and say that it is **induced** by the uniformity $\mathcal{U}$.

We can also define the notion of *uniform continuity*. If $f\colon X \to Y$ is a map between uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$, then f is **uniformly continuous** if for any $V \in \mathcal{V}$, there is $U \in \mathcal{U}$, such that for all $x, x' \in X$, if $(x, x') \in U$, then $(f(x), f(x')) \in V$.

It is also possible to define Cauchy sequences and completions for uniform spaces using the notion of a *Cauchy filter*. If $\mathscr{F}$ is a filter on the uniform space $(X,\mathcal{U})$, we say that $\mathscr{F}$ is a **Cauchy filter** if for any entourage $U \in \mathcal{U}$, there is $F \in \mathscr{F}$ such that $F \times F \subseteq U$. We say that $(X,\mathcal{U})$ is **complete** if any Cauchy filter on X converges to some $x \in X$ in the induced topology $\tau_{\mathcal{U}}$. Recall that $\mathscr{F}$ converges to x if for all neighbourhoods of x, there is an element of the filter contained in the neighbourhood. Finally, any uniform space $(X,\mathcal{U})$ has a **completion**, i.e. a complete uniform space in which X is dense. We refer the reader to [53] for a proof of this fact and for a more thorough introduction to uniform spaces.

1.4.1 Uniformities on Polish groups

On a topological group there are four canonical uniformities that all induce the group topology. These are the *left*, *right*, *two-sided* and the *Roelcke* uniformity. In this thesis, we will mainly be interested in the Roelcke uniformity (cf. Chapter 4). However, the left uniformity will also be important to us, as we will use it in several constructions. Of course we could just as easily have used the right uniformity in these constructions, but it seems to be the left one that most authors prefer.

The left uniformity of a Polish group G is generated by the entourages

$$\{(g, h) : g \in hV\},$$

where V varies over neighbourhoods of the identity. It is not hard to check that the compatibility of the group operations with the topology implies that these sets do in fact generate a uniformity that induces the group topology. The **left completion** of G, denoted $\widehat{G}_L$, is the completion of G with respect to the left uniformity.

Similarly, the right uniformity is generated by sets of the form

$$\{(g,h) : g \in Vh\},$$

where V again varies over identity neighbourhoods. The **right completion** of G is the completion with respect to this uniformity and is denoted $\widehat{G}_R$.

The two-sided uniformity is now defined as the join of the left and the right uniformity. Therefore, it is generated by intersections of entourages from these two uniformities. However, this uniformity is always complete for Polish groups, and it will therefore not play any further role for us here.

As mentioned, we are mainly interested in the **Roelcke uniformity**. It is defined as the meet of the left and right uniformities and is generated by entourages of the form

$$\{(g,h) : g \in VhV\},$$

where V varies over identity neighbourhoods. The **Roelcke completion** of G is the completion of G with respect to this uniformity and is denoted $\widehat{G}_{L\wedge R}$.

For a Polish group G, the Birkhoff-Kakutani Theorem tells us that it is always possible to find a left-invariant metric d_L compatible with the topology. Moreover, it is easy to see that any such left invariant metric induces the left uniformity on G by considering entourages of the form

$$\{(g,h) : d_L(g,h) < r\}$$

for $r > 0$. Given such a left-invariant metric d_L, we can define a metric $d_{L\wedge R}$ inducing the Roelcke uniformity on G as follows:

$$d_{L\wedge R}(g,h) = \inf_{f\in G} \max\{d_L(g,f), d_L(f^{-1},h^{-1})\}.$$

Thus, the Roelcke completion $\widehat{G}_{L\wedge R}$ is the completion of G with respect to this metric.

The Roelcke uniformity has received quite a lot of attention in the literature (see e.g. [13, 53, 55, 62, 63, 64, 65, 68]). One reason for this is that the *Roelcke precompact* groups (defined below) have particularly nice properties, as we mentioned in the introduction. In Chapter 4 we will study the Roelcke completion from the viewpoint of metric model theory. It turns out that there is a close connection between the Roelcke completion and type spaces of continuous logic. We will use this connection to study the class of the so-called *locally Roelcke precompact* groups. These groups include the Roelcke precompact ones but also many groups that are not Roelcke precompact, such as, for instance, the isometry group of the Urysohn metric space. Roelcke precompactness and local Roelcke precompactness are defined as follows:

Definition 1.4.2. *Let G be a Polish group. A subset $A \subseteq G$ is said to be* **Roelcke precompact** *if for all neighbourhoods $V \subseteq G$ of the identity, there is a finite set $F \subseteq G$ such that $A \subseteq VFV$.*

If G is a Roelcke precompact subset of itself, we say that G is **Roelcke precompact**.

If there is an open Roelcke precompact set $A \subseteq G$, we say that G is **locally Roelcke precompact**.

It is not difficult to see that a set $A \subseteq G$ is Roelcke precompact if and only if the closure of A in the Roelcke completion $\widehat{G}_{L\wedge R}$ is compact. Thus, G is Roelcke precompact if and only if $\widehat{G}_{L\wedge R}$ is compact. However, the analogous statement for locally Roelcke precompact groups is not immediate. Fortunately it is still true, as Zielinski showed in [68].

Theorem 1.4.3 ([68, Theorem 16])**.** *A Polish group G is locally Roelcke precompact if and only if the Roelcke completion of G is locally compact.*

This theorem is important for us, since we will use it in Chapter 4.

It is important to stress that properties of the Roelcke completion of a group G are not necessarily inherited by subgroups of G. In particular, if H is a subgroup of G which is Roelcke precompact *as a subset of* G, it does *not* follow that H is a Roelcke precompact group. This is because the Roelcke completion of H as a group and the closure of H inside the Roelcke completion of G need not coincide. Of course, if H is an open subgroup, then H is a Roelcke precompact group if and only if it is a Roelcke precompact subset of G, which follows from a straightforward verification of the definition above. However, in general this is false. We will study this phenomenon further in Chapter 4.

Chapter 2

Model Theory for Metric Structures

This chapter surveys the main theoretical framework of the thesis: *metric model theory* also known as *continuous logic*. Continuous logic (in some form or another) was already defined in the 1960's (cf. [25]). Back then, the authors allowed any compact Hausdorff space as the set of truth values, which perhaps was too general for a successful theory. The modern version of continuous logic focuses on the special case where only compact intervals of reals are allowed as "truth values". This has the key advantage that two natural quantifiers similar to the classical ones may be defined, namely the *supremum* and the *infimum*. This version of continuous logic has been developed over the last 20 years or so with the works of a large group of authors (cf. e.g. [8, 13, 14, 15, 6, 9, 11, 30] and the list could go on for a while). The driving force behind the development has been to define a model theory that is better suited than the classical discrete model theory for describing structures appearing naturally in other parts of mathematics, such as e.g. Banach spaces, metrisable groups, measure algebras etc. The classical model theory does not capture very well the somehow continuous nature of these structures. Continuous model theory, on the other hand, has been developed exactly to do just that. Today continuous logic has reached a point where more and more applications to other parts of mathematics have appeared. Furthermore, not only does continuous logic pose a better suited theory for these sort of continuous structures, but it is also strikingly similar to classical model theory. Most of the main results from the classical theory have a continuous counterpart. These include e.g. the Compactness Theorem, the Löwenheim-Skolem Theorem and the Omitting Types Theorem, just to mention a few (cf. [9]). This, together with the many fruitful applications, emphasises that the continuous logic we have today is not only a powerful tool, but also the "good" way to generalise classical model theory to a continuous version.

The main idea behind metric model theory is to use complete metric spaces as the foundation on which structures can be defined. In order for the Compactness Theorem to hold, we have to require the metric to be bounded which

can be a little annoying. However, usually it is not hard to work around this requirement as we will see below. We will think of the metric as the generalisation of equality in classical logic, and think of something being "close to the truth" if the distance is close to 0. Thus, 0 plays the role of truth. This might cause a bit of confusion at first, but when working with the theory it will be clear right away why this is the natural choice. On top of the metric space, just as in classical logic, we allow predicates, functions and constants with one crucial requirement: everything must be uniformly continuous. Thus, we think of these objects as being maps from the metric space into either the space itself or into the reals. The first-order language is constructed recursively just as in the classical case, using continuous map as connectives and infimum and supremum as quantifiers. This makes sense, since we think of predicates as maps into e.g. $[0, 1]$. It might seem peculiar at first, but if one considers the classical connectives as maps into the set of truth values $\{0, 1\}$, it becomes clear that the obvious generalisation to the continuous setting is to use continuous maps.

For further details on the basics of continuous logic we refer the reader to the survey [9], where everything is explained in full detail.

2.1 Languages and structures

In this first proper section of the chapter, we will develop the formalism and theoretical framework of metric model theory that we will need in the chapters to come. We begin by defining what a metric structure is, before moving on to define the formal language of metric model theory.

2.1.1 Signatures

The first thing we need is a tool that will help us control the uniform continuity of the formulas in our language.

Definition 2.1.1. *A* **modulus of uniform continuity** *is a function*

$$\delta\colon (0, \infty) \to (0, \infty).$$

For metric spaces (X, d_X) and (Y, d_Y), we say that a map $f\colon X \to Y$ **respects** *the modulus δ if for all $x, x' \in X$ and $\varepsilon > 0$, we have*

$$d_X(x, x') \leq \delta(\varepsilon) \Longrightarrow d_Y(f(x), f(x')) \leq \varepsilon.$$

In this case, f is said to be **uniformly continuous with respect to $\boldsymbol{\delta}$**.

We will often simply write '*δ is a modulus*' instead of '*δ is a modulus of uniform continuity*'.

With the moduli of uniform continuity established, we proceed to define the alphabets for our languages. These are called *metric signatures* and they

will usually be denoted by an L with various subscripts. As for all alphabets (natural and formal), L is merely a collection of symbols without meaning. It is only later that the symbols will be given meanings, when they are *interpreted* in a structure.

Definition 2.1.2. *A* **metric signature** *is a countable collection*

$$L = \{(P_i, n_i, \delta_i, I_i) : i \in \mathcal{I}\} \cup \{(F_j, m_j, \rho_j) : j \in \mathcal{J}\}$$

where for each $i \in \mathcal{I}$, $j \in \mathcal{J}$

- $n_i, m_j \in \mathbb{N}$,
- P_i *is a* **predicate symbol** *of arity n_i,*
- I_i *is a compact interval of reals, called the* **bound** *of P_i,*
- F_j *is a* **function symbol** *of arity m_j,*
- δ_i *and ρ_j are moduli of uniform continuity of arities n_i and m_j, respectively.*

Moreover, the signature contains a special binary predicate d (intended for the metric) equipped with the modulus $\delta_d(\varepsilon) = \varepsilon/2$.

Predicates will also be referred to as *relations* and 0-ary function symbols will be referred to as *constant* symbols.

Remark 2.1.3. The countability assumption on the signatures is in general not necessary. However, this assumption ensures that the logic topology on the type spaces defined below is metrisable, which will be convenient for some of our arguments. Moreover, we will only consider countable signatures in this thesis, so this is no real restriction for us.

2.1.2 Metric structures

With the signatures defined, we are now ready to explain what a metric structure is exactly. They are defined as follows:

Definition 2.1.4. *Let $L = \{(P_i, n_i, \delta_i, I_i) : i \in \mathcal{I}\} \cup \{(F_j, m_j, \rho_j) : j \in \mathcal{J}\}$ be a metric signature. A* **metric L-structure** *$\mathcal{M}$ consists of the following data:*

- *A bounded complete metric space $(M, d^{\mathcal{M}})$. M will be referred to as the* **universe** *of $\mathcal{M}$.*
- *A map $F_j^{\mathcal{M}} \colon \mathcal{M}^{m_j} \to \mathcal{M}$ respecting ρ_j for each $j \in \mathcal{J}$.*
- *A function $P_i^{\mathcal{M}} \colon \mathcal{M}^{n_i} \to I_i$ respecting δ_i for each $i \in \mathcal{I}$.*

For a symbol $S \in L$ the map $S^{\mathcal{M}}$ is the **interpretation** *of S in $\mathcal{M}$. The special symbol d is always interpreted as the metric $d^{\mathcal{M}}$.*

Metric L-structures will also be referred to as *L-models* or simply *structures* or *models*. To ease the notation, we will not discern between the structure $\mathcal{M}$ and the underlying metric space M. This will not cause any confusion and it will be perfectly clear from context whether we consider the structure or the metric space.

The boundedness condition on the metric is, as mentioned, a bit of a nuisance. However, the boundedness is important for the theory to work smoothly and to make sure that the type spaces (defined below) are compact. Fortunately, there are ways to work around this requirement. One way is to consider *many-sorted structures*, where we allow several bounded complete metric spaces as part of the structure together with maps between the sorts. However, many-sorted structure are a little cumbersome to work with, so when dealing with unbounded spaces we have chosen another approach explained below.

The work-around we will implement is the following: if we want to consider an unbounded space as a metric structure, we will add countably many predicates to the language that will encode the unbounded distance up to some bound. In particular, this approach works out nicely when considering the unbounded Urysohn metric space and diversity in Chapters 5 and 6 below.

Before we move on, let us provide a few examples of metric structures to indicate how versatile the concept is.

Examples. The following can be considered as metric structures.

(*i*) *Complete metric spaces.* If (M, d_M) is a bounded complete metric space, we can consider it as a structure in the empty signature. If d_M is not bounded, we can add predicates to the language as described above.

(*ii*) *Classical first-order structures.* As the metric we simply use the discrete metric. Predicates and functions are clearly uniformly continuous. Thus, metric model theory generalises the classical discrete model theory.

(*iii*) *Hilbert spaces over* $\mathbb{R}$. They can be considered as many-sorted structures or, equivalently, we can restrict ourselves to the unit ball. The signature is $L_{\text{HILB}} = \{0, +, -, \langle\cdot,\cdot\rangle, (m_\lambda)_{\lambda\in\mathbb{Q}}\}$. With some care, one can also consider Hilbert spaces over $\mathbb{C}$ as metric structures by considering the real and imaginary parts separately.

(*iv*) *Measure algebras.* Suppose (X, μ) is a probability space and let MALG_μ denote the corresponding measure algebra, i.e. the measurable sets where A and B are identified if the symmetric difference $A \triangle B$ has measure 0. The map $d_\mu(A, B) = \mu(A \triangle B)$ defines a metric on MALG_μ and therefore MALG_μ is a metric structure in the signature $L_{\text{BOOL}} = \{0, 1, \cap, \cup, \mu\}$.

Just as in classical logic, we can augment a given signature with constants coming from some given structure.

Definition 2.1.5. *Let L be a signature and let $\mathcal{M}$ be an L-structure. Given a countable subset $A \subseteq \mathcal{M}$ we denote by $L(A)$ the signature obtained by augmenting L with constant symbols c_a for each element $a \in A$. $\mathcal{M}$ is naturally an $L(A)$-structure simply by setting $c_a^{\mathcal{M}} = a$.*

We can also talk about *embeddings* and *isomorphisms* of structures.

Definition 2.1.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures. An* **embedding** *of $\mathcal{M}$ into $\mathcal{N}$ is a map $\iota\colon \mathcal{M} \to \mathcal{N}$ that commutes with the interpretations of all predicate and function symbols of L. That is, if $F \in L$ is an n-ary function symbol and P a k-ary predicate symbol, we have*

$$\iota\big(F^{\mathcal{M}}(a_0, \dots, a_{n-1})\big) = F^{\mathcal{N}}\big(\iota(a_0), \dots, \iota(a_{n-1})\big)$$

and

$$P^{\mathcal{M}}\big(b_0, \dots, b_{k-1}\big) = P^{\mathcal{N}}\big(\iota(b_0), \dots, \iota(b_{k-1})\big)$$

for any $a_i, b_j \in \mathcal{M}$. In particular, ι must be an isometry and is therefore always injective. If there is an embedding $\iota\colon \mathcal{M} \to \mathcal{N}$, we write $\mathcal{M} \hookrightarrow \mathcal{N}$ and say that $\mathcal{M}$ **embeds** *into $\mathcal{N}$.*

If $\mathcal{M} \subseteq \mathcal{N}$ as sets and the inclusion map is an embedding, we say that $\mathcal{M}$ is a **substructure** *of $\mathcal{N}$ and we write $\mathcal{M} \subsetneq \mathcal{N}$.*

An **isomorphism** *is a surjective embedding and we write $\mathcal{M} \cong \mathcal{N}$ when there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.*

An isomorphism from $\mathcal{M}$ to itself will be called an **automorphism**.

Automorphisms will play a significant role for us. Let us therefore fix some notation concerning them.

Definition 2.1.7. *Let $\mathcal{M}$ be an L-structure. The* **automorphism group** *of $\mathcal{M}$, denoted $\mathrm{Aut}(\mathcal{M})$, is the group of all automorphisms of $\mathcal{M}$.*

Since the metric is always included in the signature, automorphisms are in particular isometries. Thus, $\mathrm{Aut}(\mathcal{M})$ is a subgroup of $\mathrm{Iso}(\mathcal{M})$. Moreover, it is easy to check that $\mathrm{Aut}(\mathcal{M})$ is a closed subgroup in the pointwise convergence topology. Hence, $\mathrm{Aut}(\mathcal{M})$ is a Polish group if $\mathcal{M}$ is separable. In fact, as we shall see in Chapter 4, any Polish group is (isomorphic to) the automorphism group of some separable metric structure.

2.1.3 Syntax

We will now define the *language* of a given signature consisting of the expressions that we will be able to assign meaning to. Just like for natural languages, only certain combinations of letters make sense as words, and we are only allowed to combine words in certain ways depending on the grammar of the language. As in classical model theory, the language is built recursively creating *terms* and *formulas* that we can combine according to certain rules using the

symbols of the signature and create new formulas. One of the key differences between continuous and classical logic is that we allow uniformly continuous functions as our connectives instead of the classical first-order connectives. This will allow us to introduce methods and terminology from functional analysis, because the collection of all formulas may be viewed as a Banach algebra of real valued continuous functions on a compact Hausdorff space.

Fix a set of *variables* $\mathcal{V}$. Since we will only need countably many variables, we can assume that $\mathcal{V}$ is countable, but this is not essential. $\mathcal{V}$ should formally be considered as a part of any metric signature. *Terms* are then defined as follows:

Definition 2.1.8. *Let L be a metric signature. The class of* **L-terms** *is recursively defined as follows:*

- *All variables of $\mathcal{V}$ are L-terms.*
- *If $t_0, \ldots, t_{n-1}$ are L-terms and $F \in L$ is an n-ary function symbol, then $F(t_0, \ldots, t_{n-1})$ is an L-term.*

Notice that the definition above includes constants as 0-ary function symbols. If $\bar{v} = (v_0, \ldots, v_{n-1})$ are the variables used to define some term t, we will write $t(\bar{v})$ to indicate this and say that t **depends** on $\bar{v}$.

We can now use the terms to define the simplest meaningful expressions, namely the *atomic formulas*.

Definition 2.1.9. *Let L be a signature. An* **atomic formula** *is an expression of the form $P(t_0, \ldots, t_{n-1})$, where P is an n-ary predicate symbol from L and $t_0, \ldots, t_{n-1}$ are terms. The* **bound** *for $P(t_0, \ldots, t_{n-1})$ is the bound for P (determined by L).*

The bound for an atomic formula is used in the definition below to define a bound for all formulas. Formulas are defined recursively starting from the atomic formulas using uniformly continuous functions defined on the products of the bounds. Note, however, that since these functions are defined on compact spaces, it is enough to assume them to be merely continuous.

Definition 2.1.10. *Let L be a signature. The class of* **L-formulas** *is defined recursively as follows:*

- *Every atomic formula is a formula.*
- *If $\varphi_0, \ldots, \varphi_{n-1}$ are formulas with bounds I_i and $f \colon \prod_i I_i \to \mathbb{R}$ is a continuous function, then $f(\varphi_0, \ldots, \varphi_{n-1})$ is a formula with bound $\mathrm{Im}(f)$. Such functions f are referred to as* **connectives**.
- *If φ is a formula with bound I and v is a variable, then $\sup_v \varphi$ and $\inf_v \varphi$ are formulas with bound I. $\sup$ and $\inf$ are called* **quantifiers**.

The collection of all L-formulas is denoted $\mathcal{L}(L)$ or simply $\mathcal{L}$.

The notions of *free* and *bound* variables are defined as in the classical case: an occurrence of a variable v in a formula φ is **bound** if it occurs within the scope of a quantifier, i.e. if φ contains a formula of the form $\inf_v \psi$ or $\sup_v \psi$ and the occurerrence of v in question is in ψ. If v occurs outside the scope of any quantifier it is called **free**. A **sentence** is a formula with no free variables and a collection of sentences is called a **theory**. As usual, a formula is **quantifier free** if it is built without using the quantifiers. We will use the notation $\varphi(v_0, \ldots, v_{n-1})$ to denote that the formula φ has free variables among $v_0, \ldots, v_{n-1}$, and we will say that φ **depends** on $v_0, \ldots, v_{n-1}$. Note that a formula only depends on finitely many free variables.

Like in classical logic, we can define the *complexity* of a term or a formula. This will come in handy, since many arguments and definitions are done by induction or recursion on the complexity of formulas.

Definition 2.1.11. *The* **complexity** *or* **rank** *of a term t, denoted* $\mathrm{rk}(t)$*, is defined recursively as follows:*

- *If t is a variable, then* $\mathrm{rk}(t) = 0$.
- *If t is $F(t_1, \ldots, t_m)$, then* $\mathrm{rk}(t) = \max_i \mathrm{rk}(t_i) + 1$.

We can then define the complexity of a formula recursively as follows:

Definition 2.1.12. *The* **complexity** *or* **rank** *of a formula $\varphi \in \mathcal{L}(L)$, denoted* $\mathrm{rk}(\varphi)$*, is defined recursively as follows:*

(*i*) *If φ is an atomic formula, then* $\mathrm{rk}(\varphi) = 0$.
(*ii*) *If φ is $f(\varphi_1, \ldots, \varphi_n)$ for formulas φ_j and a connective f, then*

$$\mathrm{rk}(\varphi) = \max_i \mathrm{rk}(\varphi_i) + 1.$$

(*iii*) *If φ is $\inf_v \psi$ or $\sup_v \psi$, then* $\mathrm{rk}(\varphi) = \mathrm{rk}(\psi) + 1$.

2.1.4 Semantics

The above establishes the *syntax* of our language. The *semantics* are provided by structures. Any term $t(v_0, \ldots, v_{n-1})$ is naturally interpreted in a structure $\mathcal{M}$ as a map $t^{\mathcal{M}} \colon \mathcal{M}^n \to \mathcal{M}$. Likewise, any formula $\varphi(v_0, \ldots, v_{n-1})$ is naturally interpreted in $\mathcal{M}$ as a map $\varphi^{\mathcal{M}} \colon \mathcal{M}^n \to I_\varphi$, i.e. to the bound of φ. Moreover, these maps will respect moduli of uniform continuity that are determined by the signature and therefore independent of the structure. Hence, no matter where we interpret a formula or term, it will always respect *the same* modulus.

The formal definitions of the interpretations of terms and formulas are given recursively as follows:

Definition 2.1.13. *Let L be a signature and let $\mathcal{M}$ be an L-structure. Let moreover $t(\bar{v})$ be an L-term depending on the variables $\bar{v} = (v_0, \ldots, v_{n-1})$. The map $t^{\mathcal{M}} \colon \mathcal{M}^n \to \mathcal{M}$ is defined recursively on the complexity of t as follows:*

- *If t is the variable v_i, then $t^{\mathcal{M}}(\bar{a}) := a_i$ for any $\bar{a} \in \mathcal{M}^n$.*
- *If t is the term $F(t_1(\bar{v}), \ldots, t_m(\bar{v}))$ for a function symbol F, then*

$$t^{\mathcal{M}}(\bar{a}) := F^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_m^{\mathcal{M}}(\bar{a}))$$

  for any $\bar{a} \in \mathcal{M}^n$.

The interpretations of formulas are then defined as follows:

Definition 2.1.14. *Let L be a signature and let $\mathcal{M}$ be an L-structure. Let moreover $\varphi(\bar{v})$ be a formula with free variables among $\bar{v} = (v_0, \ldots, v_{n-1})$. The map $\varphi^{\mathcal{M}} \colon \mathcal{M}^n \to I_\varphi$ is defined by recursion on the complexity of φ as follows:*

- *If $\varphi(\bar{v})$ is the atomic formula $P(t_1(\bar{v}), \ldots, t_m(\bar{v}))$ for a predicate P and terms t_i, then*

$$\varphi^{\mathcal{M}}(\bar{a}) := P^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_m^{\mathcal{M}}(\bar{a}))$$

  for all $\bar{a} \in \mathcal{M}^n$.

- *If $\varphi(\bar{v})$ is the formula $f(\varphi_1(\bar{v}), \ldots, \varphi_m(\bar{v}))$ for a connective f and formulas φ_i, then*

$$\varphi^{\mathcal{M}}(\bar{a}) := f(\varphi_1^{\mathcal{M}}(\bar{a}), \ldots, \varphi_m^{\mathcal{M}}(\bar{a})).$$

- *If $\varphi(\bar{v})$ is the formula $\sup_x \psi(\bar{v}, x)$ or $\inf_x \psi(\bar{v}, x)$, then*

$$\varphi^{\mathcal{M}}(\bar{a}) := \sup_{b \in \mathcal{M}} \psi^{\mathcal{M}}(\bar{a}, b) \quad \text{or} \quad \varphi^{\mathcal{M}}(\bar{a}) := \inf_{b \in \mathcal{M}} \psi^{\mathcal{M}}(\bar{a}, b),$$

  respectively.

As we have mentioned, the interpretation of any term or formula is uniformly continuous. Moreover, the interpretation will respect a *syntactically* defined modulus of uniform continuity, i.e. a modulus that only depends on the signature L and not the structures. The proof of this is a simple induction that we leave for the reader.

Proposition 2.1.15 ([9, Theorem 3.5])**.** *Let L be a signature. For any L-term $t(\bar{v})$ and any L-formula $\varphi(\bar{v})$ there are moduli of uniform continuity δ_t and δ_φ, such that for any L-structure $\mathcal{M}$, the map $t^{\mathcal{M}}$ will respect δ_t and the map $\varphi^{\mathcal{M}}$ will respect δ_φ.*

Of course, strictly speaking, we should specify which metric we put on finite powers of some structure $\mathcal{M}$ before we can talk about moduli of uniform continuity. The usual convention is to choose the maximum distance, so that is what we will do, but of course any metric inducing the product uniform structure would work as well. We will also need to fix metrics on countably infinite powers of structures. In particular, we will need metrics on $\mathcal{M}^\omega$ and $\mathcal{M}^{\omega\oplus\omega}$ (i.e. pairs of infinite sequences) for reasons that will become clear in Chapter 4. As we shall see there, the type space $S_{\omega\oplus\omega}(\mathrm{Th}(\mathcal{M}))$ (defined below) contains the Roelcke completion of $\mathrm{Aut}(\mathcal{M})$. Since this type space uses the metric on $\mathcal{M}^{\omega\oplus\omega}$, we need to be specific about how $d^{\mathcal{M}}_{\omega\oplus\omega}$ is defined. Again, we can choose any metric inducing the product uniformity, but to make things explicit and work smoothly we have to make *some* choice. The metrics are defined below.

Definition 2.1.16. *Let $\mathcal{M}$ be an L-structure with metric $d^{\mathcal{M}}$. The metric $d^{\mathcal{M}}_n$ on $\mathcal{M}^n$ for $n \in \mathbb{N}$ is defined to be the maximum*

$$d^{\mathcal{M}}_n(\bar{a}, \bar{b}) = \max\{d^{\mathcal{M}}(a_i, b_i) : i < n\}.$$

On $\mathcal{M}^\omega$, we define the metric $d^{\mathcal{M}}_\omega$ to be the sum

$$d^{\mathcal{M}}_\omega(\bar{a}, \bar{b}) = \sum_{i\in\mathbb{N}} 2^{-i-1} d^{\mathcal{M}}(a_i, b_i). \tag{2.1}$$

Finally, on $\mathcal{M}^{\omega\oplus\omega} = \{(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in \mathcal{M}^\omega\}$, we define the metric $d^{\mathcal{M}}_{\omega\oplus\omega}$ to be the following:

$$d^{\mathcal{M}}_{\omega\oplus\omega}((\bar{a}, \bar{a}'), (\bar{b}, \bar{b}')) = \max\{d^{\mathcal{M}}_\omega(\bar{a}, \bar{b}), d^{\mathcal{M}}_\omega(\bar{a}', \bar{b}')\}. \tag{2.2}$$

We will for the most part omit the n, ω and $\omega \oplus \omega$ subscripts to make the notation more readable.

Observe that the metric $d^{\mathcal{M}}_n$ can be defined using the L-formula

$$\varphi(\bar{x}, \bar{y}) = \max\{d(x_i, y_i) : i < n\},$$

where max is considered as a connective defined on I^n_d. However, $d^{\mathcal{M}}_\omega$ and $d^{\mathcal{M}}_{\omega\oplus\omega}$ cannot directly be defined using a formula, since they depend on infinitely many variables. Fortunately, they can be approximated uniformly by formulas, and hence we will consider them as formulas as well. We will explain this in more details in a subsequent subsection.

2.2 Basic model theoretical concepts

In this subsection, we define several basic concepts of metric model theory that we will need throughout the thesis. These concepts are metric generalisations of concepts from classical first-order logic.

The first concept we define is *satisfaction* of formulas in a structure.

Definition 2.2.1. *Let L be a signature and let $\mathcal{M}$ be an L-structure. We say that $\mathcal{M}$* **satisfies** *the formula $\varphi(\bar{v})$ with the tuple $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ if $\varphi^{\mathcal{M}}(\bar{a}) = 0$. This is denoted $\mathcal{M} \vDash \varphi(\bar{a})$.*

In particular, if we are given a sentence σ, then $\mathcal{M}$ satisfies σ if $\sigma^{\mathcal{M}} = 0$. We will denote this by $\mathcal{M} \vDash \sigma$.

The notions of *satisfiable* and *complete* theories are then defined as follows:

Definition 2.2.2. *We say that a theory T is* **satisfiable** *if there is a structure $\mathcal{M}$ satisfying all sentences of T. In this case, we say that $\mathcal{M}$ is a* **model** *of T and write $\mathcal{M} \vDash T$.*

If $\mathcal{M}$ is a structure, then the **theory of $\mathcal{M}$** *is the collection of all sentences that are satisfied by $\mathcal{M}$. The theory of $\mathcal{M}$ is denoted by $\mathrm{Th}(\mathcal{M})$.*

A theory T is said to be **complete** *if there is a structure $\mathcal{M}$ such that $T = \mathrm{Th}(\mathcal{M})$.*

We can also define the concept of elementarity as follows:

Definition 2.2.3. *Let L be a signature and let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures. We say that $\mathcal{M}$ and $\mathcal{N}$ are* **elementarily equivalent**, *denoted $\mathcal{M} \equiv \mathcal{N}$, if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.*

If $\mathcal{M} \subsetneq \mathcal{N}$, i.e. if $\mathcal{M}$ is a substructure of $\mathcal{N}$, and for all formulas $\varphi(\bar{x})$ and all tuples $\bar{a}$ of elements of $\mathcal{M}$, we have $\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(\bar{a})$, we say that $\mathcal{M}$ is an **elementary substructure** *of $\mathcal{N}$ and that $\mathcal{N}$ is an* **elementary extension** *of $\mathcal{M}$. We denote this by $\mathcal{M} \preceq \mathcal{N}$.*

An **elementary embedding** *of $\mathcal{M}$ into $\mathcal{N}$ is a map $\Phi\colon \mathcal{M} \to \mathcal{N}$ such that for all formulas $\varphi(\bar{x})$ and all tuples $\bar{a}$ of elements of $\mathcal{M}$, we have*

$$\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(\Phi(\bar{a})).$$

If there is such a map $\mathcal{M} \to \mathcal{N}$, we say that $\mathcal{M}$ **embeds elementarily** *into $\mathcal{N}$ and write $\mathcal{M} \preccurlyeq \mathcal{N}$.*

Note that an isomorphism is automatically an elementary embedding (by induction on complexities), so there is no need to define an "elementary isomorphism".

Another well-known notion from the classical case that has a metric counterpart is the notion of *quantifier elimination*. As it is so often the case in the metric setting, we need to introduce some ε's in the appropriate places in order to generalise the classical notion.

Definition 2.2.4. *A theory T admits* **quantifier elimination** *if for any formula $\varphi(\bar{x})$ and any $\varepsilon > 0$, there is a quantifier free formula $\psi(\bar{x})$ such that*

$$\sup\{|\varphi^{\mathcal{M}}(\bar{a}) - \psi^{\mathcal{M}}(\bar{a})| : \mathcal{M} \vDash T, \bar{a} \in \mathcal{M}^{|\bar{x}|}\} < \varepsilon.$$

In the next section we will see an equivalent formulation of quantifier elimination that can be easier to verify. However, it uses so-called *saturated* models and *quantifier free types* and these concepts have not yet been introduced. We invite the impatient reader to confer Proposition 2.4.5 below.

The last basic concept of this subsection, which we will need in Chapters 5 and 6, is the concept of a *prime* model.

Definition 2.2.5. *A model $\mathcal{M}$ of a theory T is a* **prime** *model if it is elementarily embeddable into any other model of T.*

Prime models are of course interesting in their own right. However, we are mainly interested in them because they are examples of so-called *atomic* models. In fact, a separable model is prime if and only if it is atomic. Atomic models models will play an important role for us in Chapter 4, and they will be defined in Section 2.5.

2.3 The Banach algebras of formulas

In this section we introduce notions from functional analysis that will be helpful when dealing with formulas and, more importantly, uniform limits of formulas.

The collection of all formulas $\mathcal{L}(L)$ generated by a signature L carries the structure of a commutative unital algebra in a natural way. If we fix some enumeration $\bar{v}_\omega = (v_0, \ldots)$ of the variables of $\mathcal{V}$ (recall we assumed $\mathcal{V}$ to be countable), then we may consider the formulas of $\mathcal{L}(L)$ as uniformly continuous maps $\mathcal{M}^\omega \to \mathbb{R}$ for any L-structure $\mathcal{M}$. Since multiplication and addition are continuous maps, they are connectives, and thus we can add and multiply formulas and multiply with real scalars as well. Therefore, $\mathcal{L}(L)$ becomes an algebra. Moreover, given any complete theory T, we can define a natural semi-norm on $\mathcal{L}(L)$ as follows:

Definition 2.3.1. *Let L be a metric signature and let $\mathcal{L}(L)$ denote the formulas it generates. Let moreover T be a complete theory. Given $\varphi(\bar{v}_\omega) \in \mathcal{L}(L)$, we define the map $\|\cdot\|_T \colon \mathcal{L}(L) \to [0, \infty)$ by*

$$\|\varphi(\bar{v}_\omega)\|_T = \sup\{|\varphi^{\mathcal{M}}(\bar{a})| : \mathcal{M} \vDash T \ \&\ \bar{a} \in \mathcal{M}^\omega\}. \tag{2.3}$$

We will omit the variables from time to time and simply write $\|\varphi\|_T$ for a formula φ. If the theory is empty, or if it is clear from context which theory we are considering, we will just write $\|\varphi\|$.

Since any formula has bounded image, it follows that $\|\cdot\|_T$ only attains finite values. Moreover, it is easy to see that this map defines a semi-norm. However, there are several formulas such that $\|\varphi\|_T = 0$, so $\|\cdot\|_T$ is not an actual norm. Therefore, we make the following definition:

Definition 2.3.2. *Two formulas $\varphi, \psi \in \mathcal{L}(L)$ are said to be* **logically equivalent with respect to T** *if $\|\varphi - \psi\|_T = 0$.*

It is easy to see that logical equivalence is in fact an equivalence relation on $\mathcal{L}(L)$. If we let $\mathcal{I}_T$ denote the equivalence class of the zero-map $\mathcal{M}^\omega \to \mathbb{R}$, then $\mathcal{I}_T$ is an ideal in $\mathcal{L}(L)$. Hence, we obtain a normed commutative unital algebra by considering the quotient $\mathcal{L}(L)/\mathcal{I}_T$ with the norm

$$\|\varphi^\bullet\|_T := \|\varphi\|_T, \tag{2.4}$$

where $\varphi^\bullet$ denotes the class of φ in $\mathcal{L}(L)/\mathcal{I}_T$.

Definition 2.3.3. *The completion of $\mathcal{L}(L)/\mathcal{I}_T$ with respect to the norm $\|\cdot\|_T$ is called* **the Banach algebra of L-formulas over T** *and is denoted $\mathcal{F}(T)$. Elements of $\mathcal{F}(T)$ are called* **L-definable predicates** *or simply* **formulas**. *We denote $\mathcal{F}(\emptyset)$ simply by $\mathcal{F}$.*

For the most part, we only consider formulas up to logical equivalence. Since two equivalent formulas always define the same map on any structure, we will still write φ even if we are actually considering the class $\varphi^\bullet$.

We can now show that the metrics $d_\omega^{\mathcal{M}}$ and $d_{\omega\oplus\omega}^{\mathcal{M}}$ defined above, for a structure $\mathcal{M}$, are definable predicates.

Proposition 2.3.4. *Given a signature L, there are L-definable predicates d_ω and $d_{\omega\oplus\omega}$ in $\mathcal{F}$ such that for any L-structure $\mathcal{M}$, the maps $d_\omega^{\mathcal{M}}$ and $d_{\omega\oplus\omega}^{\mathcal{M}}$ defined in Definition 2.1.16 above are exactly equal to the interpretations of d_ω and $d_{\omega\oplus\omega}$ in $\mathcal{M}$.*

Proof. The expressions

$$\varphi_n(\bar{x}, \bar{y}) = \sum_{i=0}^{n} 2^{-i-1} d(x_i, y_i)$$

for each $n \in \mathbb{N}$ are formulas and converge in $\|\cdot\|$ to the formula d_ω we are looking for. It is clear that the interpretation of d_ω in any $\mathcal{M}$ is exactly $d_\omega^{\mathcal{M}}$.

Similarly, the expressions

$$\psi_n((\bar{x}, \bar{x}'), (\bar{y}, \bar{y}')) = \max\left\{\sum_{i=0}^{n} 2^{-i-1} d(x_i, y_i), \sum_{i=0}^{n} 2^{-i-1} d(x_i', y_i')\right\}$$

are formulas and converge in $\|\cdot\|$ to the formula we are looking for. □

We observe that by [1, Theorem 2.3], $\mathcal{F}$ is isometrically isomorphic to the continuous real-valued functions on some compact Hausdorff space S. As we will see in the next section, this space and similar spaces may be viewed as the type spaces of metric model theory. In some sense, this is a better definition of types than the usual one, since it generalises naturally to the infinitary case that we will consider in Chapter 3. However, there is no need to cause too much confusion in this chapter, so we will stick to the classical definition of types in the next section.

It is easy to see that the formulas with free variables among some fixed set $I \subseteq \mathcal{V}$ is a subalgebra of $\mathcal{L}(L)$. Thus, it follows that everything we have defined so far can be done for this subalgebra. The corresponding Banach algebra will be denoted $\mathcal{F}_I(T)$ or $\mathcal{F}_I$ when $T = \emptyset$. Elements of $\mathcal{F}_I(T)$ are called **I-ary definable predicates** or **I-ary formulas**.

Similarly, it is easy to see that the quantifier free formulas with free variables among I constitute a subalgebra of $\mathcal{L}(L)$. The corresponding Banach algebra is denoted $Q\mathcal{F}_I(T)$. The subalgebra of all quantifier free formulas is denoted $Q\mathcal{F}(T)$. Observe that T eliminates quantifiers if and only if $Q\mathcal{F}(T) = \mathcal{F}(T)$.

2.4 Type spaces

In this section we will define one of the absolute key concepts of the thesis: *types*. The type of a tuple of elements from a structure tells us exactly what properties are true for the given tuple. Thus, understanding the collection of types is the same as understanding the structure and its model theory. Just as in the classical first-order setting, metric types are simply defined as the set of formulas satisfied by the given tuple. Furthermore, the set of types carries a topology, the *logic topology*, that turns out to be compact thanks to the Compactness Theorem – again just as in the classical setting. However, unlike the classical setting, the metric type spaces carry another topology induced by the *type distance*. In the classical case, this distance is always discrete, but in the general metric setting the distance turns out to provide crucial new information about the types. In fact, one of the key virtues of metric model theory is this type distance, and understanding its interplay with the logic topology will be one of our major tools. Types are defined as follows:

Definition 2.4.1. *Let $\bar{v}$ be a tuple of distinct variables (finite or infinite) and let T be a complete L-theory. A* **$\bar{v}$-type over T** *is a collection $p(\bar{v})$ of L-formulas with free variables among $\bar{v}$, such that there is an L-structure $\mathcal{M} \vDash T$ and some tuple $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ for which we have*

$$p(\bar{v}) = \{\varphi(\bar{v}) : \varphi^{\mathcal{M}}(\bar{a}) = 0\}.$$

We say that $\bar{a}$ **realises** *$p(\bar{v})$ and write $\bar{a} \vDash p(\bar{v})$. We also say that $p(\bar{v})$ is* **the type of $\bar{a}$ over T** *and write $p(\bar{v}) = \mathrm{tp}(\bar{a})$.*

The collection of all $\bar{v}$-types over T is denoted $S_{\bar{v}}(T)$.

The set of $\bar{v}$-types realised by a fixed structure $\mathcal{M} \vDash T$ is denoted $S_{\bar{v}}^{\mathcal{M}}(T)$, i.e.

$$S_{\bar{v}}^{\mathcal{M}}(T) = \{p(\bar{v}) \in S_{\bar{v}}(T) : \exists \bar{a} \in \mathcal{M}^{|\bar{v}|}\ \bar{a} \vDash p(\bar{v})\}.$$

A **quantifier free $\bar{v}$-type over T** *is a collection of the form*

$$\{\varphi(\bar{v}) \in Q\mathcal{F}_{\bar{v}}(T) : \exists \mathcal{M} \vDash T\ \exists \bar{a} \in \mathcal{M}^{|\bar{v}|}\ \varphi^{\mathcal{M}}(\bar{a}) = 0\}.$$

This set is denoted $\mathrm{tp}_{qf}(\bar{a})$ *and we say that* $\mathrm{tp}_{qf}(\bar{a})$ *is the* **quantifier free type of $\bar{a}$ over T**. *The collection of quantifier free types is denoted* $QS_{\bar{v}}(T)$.

For the most part, we will not really care about the specific variables in a given type. Therefore, we will usually omit the variables and simply write 'p' instead of '$p(\bar{v})$' and refer to p as a '*type*' instead of '$\bar{v}$*-type over* T'. Likewise, we will write $S_n(T)$ for the collection of types with free variables among some n-tuple of variables and refer to these types as 'n*-types*' or simply '*types*'. All of this terminology is completely standard and should not cause any confusion.

Note that we allow infinitely many free variables as well. Two important cases for us using infinitely many variables are $S_\omega(T)$ and $S_{\omega\oplus\omega}(T)$, where we allow the formulas to have free variables among, respectively, one and two countably infinite tuples of free variables.

Another important observation to make is that the type of a tuple depends on the structure it comes from. That is, if $\bar{a}$ is a tuple of elements from a structure $\mathcal{M}$, which is contained in some other structure $\mathcal{N}$, it is not necessarily true that the type of $\bar{a}$ in $\mathcal{M}$ is the same as the type of $\bar{a}$ in $\mathcal{N}$. However, it is easy to see that if $\mathcal{M} \preceq \mathcal{N}$, i.e. if $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, then the type of $\bar{a}$ in $\mathcal{M}$ is the same as the type of $\bar{a}$ in $\mathcal{N}$. We should therefore actually write $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ to emphasise that we are considering $\bar{a}$ as a tuple in $\mathcal{M}$. However, it will usually be clear from context where a given tuple comes from, so our notation should not cause any confusion.

Next, we observe that there are natural *projection maps* between the various type spaces. We define them here explicitly for future reference.

Definition 2.4.2. *Let T be a complete theory and suppose $m < \alpha \leq \omega$. Define a map $\pi^\alpha_m \colon S_\alpha(T) \to S_m(T)$ by $\pi^\alpha_m(p)(\varphi) = p(\varphi)$.*

Define moreover maps $\pi^\omega, \pi_\omega \colon S_{\omega\oplus\omega}(T) \to S_\omega(T)$ by

$$\pi^\omega(p)(\varphi(\bar{x})) = p(\varphi(\bar{x}, \bar{y})) \quad \textit{and} \quad \pi_\omega(p)(\varphi(\bar{y})) = p(\varphi(\bar{x}, \bar{y})).$$

That is, $\pi^\omega(p)$ is the type p restricted to formulas with free variables among the first infinite tuple $\bar{x}$, and $\pi_\omega(p)$ is the type p restricted to formulas with the second infinite tuple $\bar{y}$ as free variables.

These maps will be referred to as the **projection maps** *between the type spaces.*

As always the super- and subscripts will be omitted whenever there is no chance of confusion.

Once the logic topology and type distance are defined (cf. Definitions 2.4.7 and 2.4.10), it will we straightforward to verify that these projections maps are continuous with respect to both topologies.

Like in classical model theory, we can also talk about types over some (countable) subset $A \subseteq \mathcal{M} \vDash T$ for some given structure $\mathcal{M}$. This is simply defined using the signature $L(A)$, where we have added constant symbols for

each element of A, and the theory T_A of the $L(A)$-structure $\mathcal{M}_A$ based on $\mathcal{M}$. We will denote the type of a tuple $\bar{b}$ over T_A in $\mathcal{M}_A$ by $\mathrm{tp}(\bar{b} \mid A)$ to emphasise that we have augmented our language. As in the classical case, *saturated* structures are structures that realise *all* types over any subset of certain cardinalities. Since we have restricted ourselves to countable signatures, we can only define ω-saturation. However, it is of course possible to define saturation for any cardinal. Fortunately, we will only need ω-saturation. This is defined as follows:

Definition 2.4.3. *A model $\mathcal{M}$ of a theory T is* **ω-saturated** *if for all finite subsets $A \subseteq \mathcal{M}$ and all $n \in \mathbb{N}$, we have $S_n^{\mathcal{M}_A}(T_A) = S_n(T_A)$, i.e. $\mathcal{M}$ realises all n-types over A.*

Using this notion of saturation, we can now give an equivalent formulation of quantifier elimination, as promised earlier. The equivalent condition is the so-called *back-and-forth property* defined as follows:

Definition 2.4.4 ([15, Definition 4.15])**.** *A theory T has the* **back-and-forth property** *if for all $n \in \mathbb{N}$ and any two ω-saturated models of T, $\mathcal{M}$ and $\mathcal{N}$, and any two tuples $\bar{a} \in \mathcal{M}^n$, $\bar{b} \in \mathcal{N}^n$, and any singleton $c \in \mathcal{M}$, it holds that if $\mathrm{tp}_{qf}(\bar{a}) = \mathrm{tp}_{qf}(\bar{b})$, then there is $c' \in \mathcal{N}$ such that $\mathrm{tp}_{qf}(\bar{a}, c) = \mathrm{tp}_{qf}(\bar{b}, c')$.*

We then have:

Proposition 2.4.5 ([15, Theorem 4.16])**.** *The following are equivalent for a theory T:*

(*i*) *T admits quantifier elimination.*
(*ii*) *T has the back-and-forth property.*

In Chapter 5 and 6 we will use this proposition to show that the theories of the Urysohn metric space and diversity eliminate quantifiers.

Any type $p(\bar{v})$ defines a map from the Banach algebra $\mathcal{F}_{\bar{v}}(T)$ of $\bar{v}$-ary formulas, defined in the previous section, to the reals, by letting $p(\varphi)$ be the value $\varphi^{\mathcal{M}}(\bar{a})$ for a realisation of p in a structure $\mathcal{M} \vDash T$. In fact, this map is a multiplicative linear functional on $\mathcal{F}_{\bar{v}}(T)$.

Proposition 2.4.6. *Let $p(\bar{v}) \in S_{\bar{v}}(T)$ be realised by $\bar{a}$ in some model $\mathcal{M}$ of T. Then the map $p\colon \mathcal{F}_{\bar{v}}(T) \to \mathbb{R}$ defined by*

$$p(\varphi(\bar{v})) := \varphi^{\mathcal{M}}(\bar{a})$$

is a multiplicative linear functional on $\mathcal{F}_{\bar{v}}(T)$.

Moreover, there is a one-to-one correspondence between multiplicative linear functionals on $\mathcal{F}_{\bar{v}}(T)$ and types in $S_{\bar{v}}(T)$. We could therefore just as well define types in this way. Indeed, this turns out to be the good way to generalise types to infinitary metric model theory, as we will discover in Chapter 3. However,

we have chosen to stick to the classical definition in this chapter to not confuse the reader unnecessarily. It is, however, useful in metric model theory to think of types as functionals instead of collections of formulas.

2.4.1 The logic topology

In classical first-order logic, the set $S_{\bar{v}}(T)$ is endowed with a topology called the *logic topology*. As mentioned earlier, in metric model theory we have not only one, but *two* important topologies on $S_{\bar{v}}(T)$. The first topology is the natural generalisation of the topology from the classical setting. It is therefore also called the *logic topology*.

Definition 2.4.7. *The* **logic topology** $\tau_{\bar{v}}$ *on* $S_{\bar{v}}(T)$ *is the topology generated by the sets*

$$[\varphi < r] = \{p \in S_{\bar{v}}(T) : p(\varphi) < r\}$$

for a formula $\varphi(\bar{v})$ *and real number* $r > 0$.

We will also denote this topology simply by $\tau_{\bar{v}}$ or τ. It is not hard to see that τ is Hausdorff. Furthermore, if we view a type as a multiplicative linear functional, we see that τ is the topology of pointwise convergence on formulas. Moreover, by the Compactness Theorem for metric structures (cf. [9, Corollary 5.12]), it follows that τ is compact. In fact, the compactness of $(S_I(T), \tau)$ for any set of variables I is just a restatement of this theorem.

Proposition 2.4.8. *Let* L *be a signature and let* T *be a complete* L*-theory. For any set of variables* $I \subseteq \mathcal{V}$*, the type space* $(S_I(T), \tau)$ *is a compact Hausdorff space.*

Proof. Apply [9, Corollary 5.12]. □

Another easy fact is that the projection maps between the various type spaces defined above are continuous with respect to the logic topologies. We state it here as a lemma for future reference. The proof is left for the reader.

Lemma 2.4.9. *All the projections* π^n_m*,* π^ω_m*,* π_ω *and* π^ω *from Definition 2.4.2 above are* τ*-continuous.*

2.4.2 The type distance

The other topology on the type spaces is induced by the *type distance* which is defined below. This distance is a new phenomenon in metric model theory compared to the classical discrete setting. In the classical case, the type distance is always discrete because the structures are viewed as discrete metric spaces. However, in the general metric setting the type distance provides interesting new information on the structures we are dealing with.

Definition 2.4.10. *Let I be an ordered tuple of variables from $\mathcal{V}$ and let T be a complete theory. We define the* **type distance** *∂_I on $S_I(T)$ by*

$$\partial_I(p,q) = \inf\{d_I^{\mathcal{M}}(\bar{a},\bar{b}) : \mathcal{M} \vDash T \ \&\ \bar{a},\bar{b} \in \mathcal{M}^I \ \&\ \bar{a} \vDash p \ \&\ \bar{b} \vDash q\}.$$

Of course, the definition above only makes sense once we have chosen some metric on powers of a given structure $\mathcal{M}$. As long as I is countable, this can always be done. The important cases for us will be when I is either finite, a countable tuple or a pair of countable tuples. In all these cases, we have already specified the metrics on the corresponding powers of $\mathcal{M}$ in Definition 2.1.16 above.

That ∂ is in fact a metric can be shown (once again) using the Compactness Theorem.

Proposition 2.4.11 ([9, Section 8])**.** *∂ is a metric.*

Proof. It is easy to see that ∂ is a pseudo-metric. If p and q are two types, we can use the Compactness Theorem to construct a model realising the type distance $\partial(p,q)$. In particular, if $\partial(p,q) = 0$, we must have $p = q$. □

Another easy fact is that ∂ induces a finer topology than τ. We will denote this topology by τ_∂. Moreover, ∂ is *lower semi-continuous* with respect to τ. Recall that this means that the sets

$$\{(p,q) : \partial(p,q) \leq r\}$$

are τ-closed for any $r > 0$. Finally, ∂ is complete, which essentially follows from the Compactness Theorem (or the compactness of τ to be precise). We collect these easy facts in the following proposition for future reference.

Proposition 2.4.12 ([9, Propositions 8.7 and 8.8], [15, Section 4.3] and [6, Lemma 1.9])**.** *The type distance is a complete lower semi-continuous metric inducing a finer topology than τ.*

In fact, because of these properties, the study of type spaces has motivated the study of so-called *topometric spaces* in [7] and [6]. These spaces are defined as follows:

Definition 2.4.13. *A* **topometric** *space is a triplet (X, τ_X, d_X), where X is a set, τ_X is a topology on X and d_X is a lower semi-continuous metric inducing a finer topology than τ_X.*

In Chapter 3 we will need some of the general results concerning topometric spaces, but for now we content ourselves with the definition. It is, however, important to mention here that we will follow the usual convention when dealing with topometric spaces, i.e. all words and concepts coming from topology (open, closed, compact etc) will refer to the topology and all words and concepts from metric space theory (balls, distances etc) will refer to the metric.

We end the section with an easy observation, namely that the projection maps mentioned earlier are uniformly continuous with respect to ∂. The proof is again left for the reader.

Lemma 2.4.14. *All the projection maps mentioned earlier are ∂-uniformly continuous.*

2.5 Atomic & homogeneous structures

Two important concepts of classical model theory, that generalise nicely to the metric setting, are *atomic* and *homogeneous* structures. In this section, we study these concepts and show that an atomic structure *knows* the type distance, i.e. that the infimum defining the type distance between two types realised by the structure may be taken over realisations *in the structure itself* instead of over all models of the given theory. The concept of knowledge of ∂ will be important to us in Chapter 4, when we study the Roelcke completion of a Polish group as a set of types.

Homogeneity of metric structures is defined as follows:

Definition 2.5.1. *A metric structure $\mathcal{M}$ is* **approximately homogeneous** *or simply* **homogeneous** *if for any $n \in \mathbb{N}$ and any two n-tuples $\bar{a}, \bar{b} \in \mathcal{M}^n$ of the same type, we have*

$$\overline{\mathrm{Aut}(\mathcal{M}) \cdot \bar{a}} = \overline{\mathrm{Aut}(\mathcal{M}) \cdot \bar{b}}.$$

$\mathcal{M}$ is said to be **exact homogeneous** *if for any $n \in \mathbb{N}$ and $\bar{a}, \bar{b} \in \mathcal{M}^n$ of the same type, we have*

$$\bar{b} \in \mathrm{Aut}(\mathcal{M}) \cdot \bar{a}.$$

If the above holds for all tuples of the same quantifier free type, we say that $\mathcal{M}$ is **approximately ultrahomogeneous***, respectively* **exact ultrahomogenous***.*

An *atomic* model only realises those types that it absolutely must realise. These types are known as the *principal* or *isolated* types. Since we have too many topologies around and "isolated" could be misunderstood, we prefer the notion "principal". Principality and atomicity are defined as follows:

Definition 2.5.2. *A type $p \in S_I(T)$ is* **principal** *if the following holds:*

$$\forall r > 0 \ \exists \varphi \ \exists s > 0 \ p \in [\varphi < s] \subseteq B(p, r).$$

In other words, p belongs to the τ-interior of any ∂-ball around p.

A model $\mathcal{M}$ of T is **atomic** *if all types realised by $\mathcal{M}$ are principal.*

As mentioned, and just as in the classical setting, principal types are realised everywhere.

Theorem 2.5.3 (Omitting Types Theorem, [9, Theorem 12.6])**.** *Let T be a complete theory in a countable signature and let $p \in S_n(T)$. Then the following are equivalent:*

(*i*) *p is principal.*
(*ii*) *p is realised in every model of T.*

In the next section we will show a version of this theorem for infinitary continuous logic.

As a corollary to the Omitting Types Theorem, we can show, as promised earlier, that prime models are atomic.

Proposition 2.5.4. *Let T be a complete theory. Then any prime model $\mathcal{M}$ of T is atomic.*

Proof. Suppose $\mathcal{M}$ is a prime model of T that is not atomic. Then there is a type p realised in $\mathcal{M}$ which is non-principal. Therefore, by the Omitting Types Theorem, there is a model $\mathcal{N}$ of T omitting p. But since $\mathcal{M}$ is prime, it embeds elementarily into $\mathcal{N}$. Therefore, any realisation of p in $\mathcal{M}$ is mapped to a realisation of p in $\mathcal{N}$ under this embedding. Thus, $\mathcal{M}$ must be atomic. □

In fact, the other direction is also true: if $\mathcal{M}$ is a separable atomic model, then $\mathcal{M}$ is prime. However, we will not use this implication. We refer the reader to [14, Proposition 1.18] for a proof.

Another key property of separable atomic models is that they are always homogeneous.

Proposition 2.5.5. *Suppose $\mathcal{M}$ is a separable atomic metric structure. Then $\mathcal{M}$ is approximately homogeneous.*

Proof. This follows from a standard back-and-forth argument. See for instance [43, Propositions 6.8 and 6.9]. □

Another useful lemma that will come in handy later on is the following. It says that in order to check principality of a type, it is enough to check that any ball around it has non-empty τ-interior.

Lemma 2.5.6 ([9, Proposition 12.5])**.** *Let T be a complete theory and let $n \in \mathbb{N}$. Then $p \in S_n(T)$ is principal if and only if for all $r > 0$, the ball $B_\partial(p, r)$ has non-empty τ-interior.*

In order to check whether or not a type is principal, it is moreover enough to check the inclusion of the τ-open neighbourhood in the ∂-ball for a τ-dense subset. More precisely we have the following useful lemma:

Lemma 2.5.7. *Let T be a complete theory and let $p \in S_I(T)$. Suppose that $D \subseteq S_I(T)$ is a τ-dense subset. Then p is principal if and only if for all $r > 0$ there is a formula φ and an $s > 0$ such that $p(\varphi) < s$ and*

$$[\varphi < s] \cap D \subseteq B_\partial(p, r).$$

Proof. If p is principal, it is clear that the condition above is satisfied, so we must show the other direction. Let therefore $r > 0$ be given and find φ and $s > 0$ such that $p(\varphi) < s$ and

$$[\varphi < s] \cap D \subseteq B_\partial(p, r/2).$$

We claim that $[\varphi < s] \subseteq B(p, r)$. To see this, let $q \in [\varphi < s]$. By density of D, we find a sequence (q_n) of types in $[\varphi < s] \cap D$ converging in τ to q (recall that our signatures are countable, so τ is metrisable). Therefore, $\partial(p, q_n) < r/2$ so (p, q_n) is a sequence converging to (p, q) in τ inside the set

$$\{(p_1, p_2) : \partial(p_1, p_2) \leq r/2\},$$

which is τ-closed by lower semi-continuity of ∂. Hence, $\partial(p, q) \leq r/2 < r$, which was what we wanted. □

As it turns out, the set of types realised in a model of T is always τ-dense. Thus, to show principality of some type, we can restrict ourselves to the types realised in a model. This will prove very convenient for us in many cases.

Proposition 2.5.8. *For any model $\mathcal{M}$ of the complete theory T and any countable I, the set $S_I^{\mathcal{M}}(T)$ of I-types realised in $\mathcal{M}$ is τ-dense in $S_I(T)$.*

Proof. Let $[\varphi < r]$ be a non-empty basic τ-open set. Take some q such that $q(\varphi) < r$. Then the sentence $\inf_{\bar{x}} \max\{0, \varphi(\bar{x}) - q(\varphi)\}$ must be in T, so it is satisfied by $\mathcal{M}$. Hence, there is some $\bar{a} \in \mathcal{M}$ such that $\varphi^{\mathcal{M}}(\bar{a}) < r$ and therefore $\mathrm{tp}(\bar{a}) \in [\varphi < r]$ as we wanted. □

2.5.1 KNOWLEDGE OF ∂

The following property will be important for us when we study the Roelcke completion as a set of types in Chapter 4. It is a very natural property saying that a given structure $\mathcal{M}$ contains "enough" realisations of the types it realises to calculate the distance between them. In lack of a better term for it, we have chosen to say that $\mathcal{M}$ *knows* the distance. Since it is such a basic property, it might already be defined in the literature, but we have not (yet) stumbled upon it.

Definition 2.5.9. *We say that a metric structure $\mathcal{M}$* **knows** *the type distance ∂_I if for any $p, q \in S_I^{\mathcal{M}}(T)$, we have*

$$\partial_I(p, q) = \inf\{d_I^{\mathcal{M}}(\bar{a}, \bar{b}) : \bar{a}, \bar{b} \in \mathcal{M}^I \ \&\ \bar{a} \vDash p \ \&\ \bar{b} \vDash q\}.$$

The main example of a model that knows ∂_n for each $n \in \mathbb{N}$ is an atomic model.

Proposition 2.5.10. *Any atomic model $\mathcal{M}$ of T knows ∂_n for all $n \in \mathbb{N}$.*

Proof. Let $p, q \in S_n^{\mathcal{M}}(T)$ and let $\bar{a} \vDash p$ and $\bar{b} \vDash q$ in $\mathcal{M}$. Let moreover $\varepsilon > 0$ be given and set $\partial(p, q) =: s$. We will find $\bar{b}_0$ realising q such that $d^{\mathcal{M}}(\bar{a}, \bar{b}_0) \leq s+\varepsilon$ as required. To do that, we will construct a Cauchy sequence $(\bar{b}_i)$ such that $\mathrm{tp}(\bar{b}_i) \to q$ in ∂ and such that $d^{\mathcal{M}}(\bar{a}, \bar{b}_i) \leq s + \varepsilon$. It follows that $\bar{b}_i$ converges to a realisation of q close enough to $\bar{a}$.

First we find a formula φ_1 and r_1 such that

$$q \in [\varphi_1 < r_1] \subseteq B(q, \varepsilon/3).$$

By considering the formula $\frac{\varepsilon}{3r_1}\varphi_1$, we may moreover assume that r_1 is actually equal to $\frac{\varepsilon}{3}$. To find the first element of our Cauchy sequence, we note that

$$p(\inf_{\bar{y}} \max\{d(\bar{x}, \bar{y}) \dot{-} s, |\varphi_1(\bar{y})|\}) < \varepsilon/3,$$

where $\dot{-}$ denotes the map $\mathbb{R}^2 \to \mathbb{R}$ given by

$$s \dot{-} t = \begin{cases} s - t & \text{if } s \geq t \\ 0 & \text{otherwise.} \end{cases}$$

The inequality essentially follows from the Compactness Theorem, since it allows us to construct a model realising the type distance between p and q. It follows that we can find $\bar{b}_1$ in $\mathcal{M}^n$ such that $d^{\mathcal{M}}(\bar{a}, \bar{b}_1) < s + \frac{\varepsilon}{3}$ and such that $|\varphi_1(\bar{b}_1)| < \frac{\varepsilon}{3}$. Therefore, $\partial(q, \mathrm{tp}(\bar{b}_1)) < \frac{\varepsilon}{3}$ by our choice of φ_1. Set $q_1 := \mathrm{tp}(\bar{b}_1)$.

To find the second element of our Cauchy sequence, we let φ_2 be a formula such that

$$q \in [\varphi_2 < \varepsilon/3^2] \subseteq B(q, \varepsilon/3^2).$$

Again, by the Compactness Theorem, it follows that

$$q_1(\inf_{\bar{y}} \max\{d(\bar{x}, \bar{y}) \dot{-} \varepsilon/3, |\varphi_2(\bar{y})|\}) < \varepsilon/3^2.$$

Therefore, we can find $\bar{b}_2 \in \mathcal{M}^n$ such that

$$d^{\mathcal{M}}(\bar{b}_1, \bar{b}_2) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3^2}$$

and $|\varphi_2(\bar{b}_2)| < \frac{\varepsilon}{3^2}$. It follows that $\partial(q, \mathrm{tp}(\bar{b}_2)) < \frac{\varepsilon}{3^2}$.

Continuing in this fashion, we obtain a sequence $(\bar{b}_i)$ of elements in $\mathcal{M}^n$ such that

$$d^{\mathcal{M}}(\bar{b}_i, \bar{b}_{i+1}) < \varepsilon/3^i + \varepsilon/3^{i+1}$$

and such that $\partial(q, \mathrm{tp}(\bar{b}_i)) < \frac{\varepsilon}{3^i}$. It follows that $(\bar{b}_i)$ is a Cauchy sequence converging to a realisation $\bar{b}_\omega$ of q. Moreover, $d^{\mathcal{M}}(\bar{a}, \bar{b}_i) \leq s + \varepsilon$ for all i and thus $d^{\mathcal{M}}(\bar{a}, \bar{b}_\omega) \leq s + \varepsilon$, which was what we wanted. □

By definition, if $\mathcal{M}$ is atomic, all types in $S_n^{\mathcal{M}}(T)$ are principal. It turns out that the same holds for the types in $S_\omega^{\mathcal{M}}(T)$ and $S_{\omega\oplus\omega}^{\mathcal{M}}(T)$.

Lemma 2.5.11. *Let $\mathcal{M}$ be an atomic metric structure with theory T. Then all types in $S_\omega^{\mathcal{M}}(T)$ and $S_{\omega\oplus\omega}^{\mathcal{M}}(T)$ are principal.*

Proof. We will only consider types in $S_\omega(T)$, since the argument for types in $S_{\omega\oplus\omega}^{\mathcal{M}}(T)$ is completely analogous.

Let $p \in S_\omega(T)$ be a type realised in $\mathcal{M}$ and let $r > 0$ be given. To show principality of p, it is enough, by Lemma 2.5.7, to find a τ-open set U such that

$$p \in U \cap S_\omega^{\mathcal{M}}(T) \subseteq B(p, r).$$

Now, for each n, let $\pi_n \colon S_\omega(T) \to S_n(T)$ denote the τ-continuous projection map. Let $N \in \mathbb{N}$ be such that $\sum_{i>N} 2^{-i} < r/2$. By atomicity of $\mathcal{M}$, we may find a formula φ and $r' > 0$ such that

$$\pi_N(p) \in [\varphi < r'] \subseteq B(\pi_N(p), r/2).$$

Set $U := \pi_N^{-1}([\varphi < r'])$. Then of course $p \in U$ and for any $q \in U$ realised in $\mathcal{M}$, we have that $\partial(\pi_N(p), \pi_N(q)) < r/2$. Moreover, by Proposition 2.5.10, $\mathcal{M}$ knows ∂_n for all n. Hence, we can find realisations $\bar{a}_N \vDash \pi_N(p)$ and $\bar{b}_N \vDash \pi_N(q)$ in $\mathcal{M}$ such that $d^{\mathcal{M}}(\bar{a}_N, \bar{b}_N) < r/2$. Since q and p are realised in $\mathcal{M}$, we can, by homogeneity of $\mathcal{M}$, find realisations $\bar{a} \vDash p$ and $\bar{b} \vDash q$, such that for all $i \leq N$, $d^{\mathcal{M}}(a_i, b_i) < r/2$. Since $\sum_{i>N} 2^{-i} < r/2$, it follows that $d^{\mathcal{M}}(\bar{a}, \bar{b}) < r$ and therefore $\partial(p, q) < r$ as well. □

By the same argument as in Proposition 2.5.10, this implies that any atomic structure knows ∂_ω and $\partial_{\omega\oplus\omega}$.

Lemma 2.5.12. *Let $\mathcal{M}$ be an atomic metric structure. Then $\mathcal{M}$ knows ∂_ω and $\partial_{\omega\oplus\omega}$.*

Proof. The proof from Proposition 2.5.10 carries over by Lemma 2.5.11. □

We will need these results in the chapters to come.

2.5.2 THE EXACT HOMOGENEOUS CASE

In this subsection we will focus on the *exact* homogeneous structures. It turns out that atomicity of an exact homogeneous structure is preserved when we add constant symbols to the language. This will follow from the well-known result below due to Effros in [28]. First recall that an action $G \curvearrowright X$ is **transitive** if for all $x \in X$ we have $G \cdot x = X$.

Theorem 2.5.13 (Effros' Theorem, cf. [66, Theorem 1.1])**.** *Suppose $G \curvearrowright X$ is a transitive continuous action of a Polish group G on a separable metrisable space X. Then the following are equivalent:*

- *X is Polish.*

- *For every $x \in X$ and every 1_G-neighbourhood $U \subseteq G$ the set*

$$U \cdot x := \{y \in X : \exists g \in U \ y = g \cdot x\}$$

is a neighbourhood of x.

We now have:

Proposition 2.5.14. *Let $\mathcal{M}$ be a separable atomic metric structure which is exact homogeneous and let $\bar{c} \in \mathcal{M}^n$ be a tuple of elements. Then $\mathcal{M}_{\bar{c}}$ is also exact homogeneous and atomic.*

Proof. Exact homogeneity of $\mathcal{M}_{\bar{c}}$ is clear.

Let $\bar{c}$ be a tuple of elements of $\mathcal{M}$ and let $p \in S_n(T_{\bar{c}})$ be a type over $\bar{c}$ realised in $\mathcal{M}_{\bar{c}}$, say by $\bar{a}$, and let $r > 0$ be given. Set

$$U := \{g \in G : d(\bar{a}, g\bar{a}) < r/2\},$$

where $G = \mathrm{Aut}(\mathcal{M})$. Then U is a symmetric open neighbourhood of the identity. Since $\mathcal{M}$ is exact homogeneous, all orbits are closed and hence Polish. Therefore, by Effros' theorem, the set $U \cdot \bar{c}$ is open inside $G \cdot \bar{c}$. Thus, we find $\delta > 0$ such that $B(\bar{c}, \delta) \cap G \cdot \bar{c} \subseteq U \cdot \bar{c}$. Set $\delta' := \min\{\delta, r/2\}$ and find a formula $\varphi(\bar{x}, \bar{y})$ such that

$$\mathrm{tp}(\bar{a}, \bar{c}) \in [\varphi < \delta'] \subseteq B(\mathrm{tp}(\bar{a}, \bar{c}), \delta').$$

We now claim that $p \in [\varphi(\bar{x}, \bar{c}) < \delta'] \subseteq B(p, r)$, showing that p is principal. To see this, it is enough, by Lemma 2.5.7, to show this inclusion for types realised in $\mathcal{M}_{\bar{c}}$. Let therefore q be a type over $\bar{c}$ with $q(\varphi) < \delta'$ realised in $\mathcal{M}_{\bar{c}}$, say by $\bar{b}$. Then $\mathrm{tp}(\bar{b}, \bar{c}) \in B(\mathrm{tp}(\bar{a}, \bar{c}), \delta')$, so since $\mathcal{M}$ knows the type distance and is homogeneous, there is $g \in G$ such that $d(g \cdot (\bar{b}, \bar{c}), (\bar{a}, \bar{c})) < \delta'$. In particular, $g\bar{c} \in B(\bar{c}, \delta)$, so $g\bar{c} = h\bar{c}$ for some $h \in U$. Thus, $h^{-1}g \in \mathrm{Stab}(\bar{c}) = \mathrm{Aut}(\mathcal{M}_{\bar{c}})$. Moreover, $h^{-1}g \cdot \bar{b} \vDash \mathrm{tp}(\bar{b} \mid \bar{c}) = q$ and

$$d(h^{-1}g \cdot \bar{b}, \bar{a}) \leq d(h^{-1}g \cdot \bar{b}, h^{-1}\bar{a}) + d(h^{-1}\bar{a}, \bar{a}) < \delta' + r/2 \leq r.$$

It follows that $\partial(q, p) < r$ as we wanted.

We conclude that $p \in [\varphi(\bar{x}, \bar{c}) < \delta'] \subseteq B_{\partial_{\bar{c}}}(p, r)$. Since $r > 0$ was arbitrary, we conclude that p is principal and therefore that $\mathcal{M}_{\bar{c}}$ is atomic. □

This proposition will be important for us in Chapter 4, when we study stabiliser subgroups of locally Roelcke precompact automorphism groups of exact homogeneous structures.

Chapter 3

Infinitary Metric Model Theory

In this chapter, we will develop a version of *infinitary* model theory for metric structures and prove generalisations of several results from *finitary* metric model theory, i.e. the model theory developed in the previous chapter. Most notably, we prove a version of the Omitting Types Theorem for infinitary metric model theory (Theorem 3.4.2 below). This result is originally due to Eagle [27], but our proof here is new. The proof uses the Baire Category Theorem, like Eagle's original proof, but our proof is directly inspired by a proof of the Omitting Types Theorem from classical logic. Therefore, we hope it will be easier to follow than Eagle's proof. Moreover, the precise statement of Theorem 3.4.2, which was suggested by Itaï Ben Yaacov, is slightly more general than Eagle's version. As a corollary to Theorem 3.4.2, we obtain an infinitary version of the Ryll-Nardzewski Theorem describing the $\aleph_0$-categorical structures using type spaces. Moreover, we obtain a new useful description of the type distance on infinitary type spaces.

As in the classical case, the main idea is to allow conjunctions and disjunctions over countably infinite sets of formulas, thus gaining expressive power of the language. However, just as in the classical case, the price we pay is the loss of the Compactness Theorem, which has been an essential tool for us so far. Therefore, we must be more careful and creative when we are building the theory and proving results.

Most of the concepts that we have defined so far in Chapter 2 are defined exactly in the same way for infinitary logic. However, there are a few minor differences, mainly due to technical reasons. The main difference comes in the recursive definition of formulas. First, to make some arguments work more smoothly, we want our set of connectives to be countable. Therefore, we restrict ourselves to use polynomials with rational coefficients as connectives. This does not make that much of a difference, since it does not change the definable predicates by the Weierstraß Approximation Theorem. Secondly, and more importantly, we allow the supremum and infimum over *countably* many formulas respecting the same modulus of continuity and bound. These

connectives correspond to countable conjunctions and disjunctions in the classical case and are hence called *infinitary* connectives. By requiring the formulas to respect the same modulus we ensure that the interpretations of the formulas are uniformly continuous, as they should be.

3.1 INFINITARY LANGUAGES

In this section we will explain how the infinitary language is defined. The basic definitions are the same as in Chapter 2, so we will only explain the differences and adjustments one needs to do, to set up the infinitary language properly. As in Chapter 2 there are two parts: syntax and semantics. We begin with the syntax.

3.1.1 INFINITARY SYNTAX

Signatures and structures are defined in the same way as for finitary metric model theory. In particular, we stress that a signature is still assumed to be countable. Terms and atomic formulas are also defined in the same way, but general formulas are defined slightly differently. We therefore make the following revision of Definition 2.1.10 describing the recursive construction of formulas.

Definition 3.1.1. *Let L be a signature. The class of* **infinitary L-formulas** *is denoted $\mathcal{L}_{\omega_1\omega}(L)$ and is defined recursively as follows:*

- *Atomic formulas are in $\mathcal{L}_{\omega_1\omega}(L)$.*
- *If $\varphi_0, \ldots, \varphi_n$ are formulas in $\mathcal{L}_{\omega_1\omega}(L)$ with bounds I_i and if $f\colon \prod_i I_i \to \mathbb{R}$ is any polynomial with rational coefficients, then $f(\varphi_0, \ldots, \varphi_n)$ is a formula in $\mathcal{L}_{\omega_1\omega}(L)$.*
- *If φ is a formula in $\mathcal{L}_{\omega_1\omega}(L)$, then $\sup_v \varphi$ and $\inf_v \varphi$ are formulas in $\mathcal{L}_{\omega_1\omega}(L)$ too for any variable v.*
- *If $\{\varphi_n : n \in \mathbb{N}\}$ is a countable family of formulas in $\mathcal{L}_{\omega_1\omega}(L)$ with the same bound I and respecting the same modulus δ, then $\bigwedge_n \varphi_n$ and $\bigvee_n \varphi_n$ are formulas in $\mathcal{L}_{\omega_1\omega}(L)$.*

When the signature is insignificant we will allow ourselves to write $\mathcal{L}_{\omega_1\omega}$ instead of $\mathcal{L}_{\omega_1\omega}(L)$.

$\bigvee_i$ and $\bigwedge_i$ are, as mentioned, referred to as *the infinitary connectives* and the other connectives are referred to as *finitary*. The *finitary* formulas are the formulas we get if we refrain from using the infinitary connectives. The collection of all finitary formulas is denoted $\mathcal{L}_{\omega\omega}(L)$ or simply $\mathcal{L}_{\omega\omega}$.

Sentences and free variables are defined as in the finitary case. By $\varphi(\bar{v})$ we denote that the formula φ has free variables among $\bar{v}$. Note that we allow a formula to have infinitely many free variables. However, any formula may be approximated uniformly by formulas with finitely many free variables (cf. Proposition 3.3.1), so this does not change much compared to the finitary case.

We will also need to extend the notion of complexity of formulas to incorporate the infinitary formulas. The definition is a straightforward extension of Definition 2.1.12 above.

Definition 3.1.2 (Addendum to Definition 2.1.12)**.** *Let $\{\varphi_n : n \in \mathbb{N}\}$ be a countable collection of formulas respecting the same modulus and bound. The* **complexity** *or* **rank** *of the formulas $\bigwedge_n \varphi_n$ and $\bigvee_n \varphi_n$ is defined to be $\sup_n \mathrm{rk}(\varphi_n) + 1$. As before, the complexity of any formula φ is denoted $\mathrm{rk}(\varphi)$.*

Observe that in the infinitary case, the complexity of a formula may be infinite and will, by definition, always be a successor ordinal (or 0).

As mentioned above, we will give a new proof of the Omitting Types Theorem below. In the classical infinitary setting, this theorem is stated only for countable *fragments* of the language. Hence, we will require the same thing here, although we could just as well allow separable fragments. The countability assumption does, however, simplify some of the arguments. Moreover, we lose no real generality, since the definable predicates do not change.

Definition 3.1.3. *A* **fragment** *of $\mathcal{L}_{\omega_1\omega}(L)$ is a countable subset $\mathscr{F} \subseteq \mathcal{L}_{\omega_1\omega}(L)$ that contains $\mathcal{L}_{\omega\omega}(L)$ and that is closed under the following operations,*

(*i*) *Subformulas,*
(*ii*) *Substitution of terms for variables, i.e. if $\varphi(\bar{t}) \in \mathscr{F}$ and $\bar{v}$ is a tuple of variables, then $\varphi(\bar{v}) \in \mathscr{F}$,*
(*iii*) *The finitary continuous connectives of Definition 3.1.1*

Moreover, any $\varphi \in \mathscr{F}$ is required to have only finitely many free variables.

The collection of those formulas of a fragment $\mathscr{F}$ that have free variables among some fixed set of variables $I \subseteq \mathcal{V}$ will be denoted $\mathscr{F}_I$.

Remark 3.1.4. Note that since we assume the fragments to be countable, we tacitly have to assume that the signature is countable as well. Otherwise, if we were working with an uncountable signature, there would be no fragments at all.

Remark 3.1.5. Recall that $\mathcal{L}_{\omega\omega}$ is assumed to be countable since we only use a countable set of connectives. Therefore we can ask a fragment to be both countable and contain $\mathcal{L}_{\omega\omega}$.

If we want to relativise a notion to some fragment $\mathscr{F}$, we will add the pre-fix "$\mathscr{F}$-" to the name of the notion. Thus, for example, an *$\mathscr{F}$-theory* is a collection of sentences from the fragment $\mathscr{F}$. Likewise, we make the following definition of a *complete* $\mathscr{F}$-theory.

Definition 3.1.6. *An $\mathscr{F}$-theory T is* **complete** *if there is a model $\mathcal{M}$ such that T is the $\mathscr{F}$-theory of $\mathcal{M}$. That is, T is all the $\mathscr{F}$-sentences that are satisfied by $\mathcal{M}$. T is* **satisfiable** *if there is a structure $\mathcal{M}$ satisfying all sentences of T.*

Similarly, the notion of categoricity can be made with respect to a fragment.

Definition 3.1.7. *Let $\mathscr{F}$ be a fragment and let T be an $\mathscr{F}$-theory. We say that T is* **$\mathscr{F}$-$\aleph_0$-categorical** *if any two separable models of T are isomorphic.*

As usual, we will try to avoid the cumbersome "$\mathscr{F}$"-notation as often as possible.

3.1.2 INFINITARY SEMANTICS

The interpretation of an $\mathcal{L}_{\omega_1\omega}$-formula in a structure is defined recursively as in the finitary case. The only thing we need to define here is therefore how the infinitary connectives are interpreted.

Definition 3.1.8 (Addendum to Definition 2.1.14)**.** *Let $\mathcal{M}$ be a structure and let $\{\varphi_n : n \in \mathbb{N}\}$ be a countable collection of formulas with the same modulus δ and the same bound I. The* **interpretations** *of the infinitary formulas $\bigwedge_n \varphi_n$ and $\bigvee_n \varphi_n$ are defined as the maps*

$$\left(\bigwedge\nolimits_n \varphi_n\right)^{\mathcal{M}}, \left(\bigvee\nolimits_n \varphi_n\right)^{\mathcal{M}} \colon \mathcal{M}^\omega \to I$$

given by

$$\left(\bigwedge\nolimits_n \varphi_n\right)^{\mathcal{M}}(\bar{a}) = \inf_n \varphi_n^{\mathcal{M}}(\bar{a}) \quad \textit{and} \quad \left(\bigvee\nolimits_n \varphi_n\right)^{\mathcal{M}}(\bar{a}) = \sup_n \varphi_n^{\mathcal{M}}(\bar{a}).$$

With the interpretations of the infinitary formulas established, we can now show that they always give uniformly continuous maps on the structures in which they are interpreted.

Proposition 3.1.9. *If $\{\varphi_n : n \in \mathbb{N}\}$ is a collection of formulas with the same modulus δ and bound I, then there is a syntactically defined modulus δ_0 such that for any structure $\mathcal{M}$, $(\bigwedge_n \varphi_n)^{\mathcal{M}}$ will respect δ_0 and I. Similarly, there is a syntactically determined modulus and bound for $\bigvee_n \varphi_n$. It follows that all $\mathcal{L}_{\omega_1\omega}$-formulas can be assigned a syntactically determined modulus of uniform continuity.*

Proof. It is clear that the interpretations of the infinitary formulas must respect the (compact) bound I. Moreover, the argument for the two kinds of infinitary formulas is the same, so we will only provide it for $\bigwedge_n \varphi_n$. Furthermore, the conclusion follows from a straightforward argument using induction on the complexity of formulas.

We claim that $\delta_0(\varepsilon) := \delta(\varepsilon/3)$ is a modulus for $(\bigwedge_n \varphi_n)^{\mathcal{M}}$ for any structure $\mathcal{M}$. To see this, suppose $\varepsilon > 0$ and that $\bar{a}$ and $\bar{b}$ are sequences from $\mathcal{M}$ such that $d(\bar{a}, \bar{b}) \leq \delta_0(\varepsilon)$. Then we can find some $N \in \mathbb{N}$ such that

$$|\inf_n \varphi_n^{\mathcal{M}}(\bar{a}) - \varphi_N^{\mathcal{M}}(\bar{a})|, |\inf_n \varphi_n^{\mathcal{M}}(\bar{b}) - \varphi_N^{\mathcal{M}}(\bar{b})| < \varepsilon/3.$$

It follows that

$$|\inf_n \varphi^{\mathcal{M}}(\bar{a}) - \inf_n \varphi_n^{\mathcal{M}}(\bar{b})| \leq |\varphi_N^{\mathcal{M}}(\bar{a}) - \varphi_N^{\mathcal{M}}(\bar{b})| + 2\varepsilon/3 < \varepsilon,$$

which was what we wanted. □

3.2 Elementary substructures

We move on to show generalisations of two classical theorems of model theory. These are the downwards Löwenheim-Skolem Theorem and the Tarski-Vaught Test. In order to state them properly, we need to generalise some of the concepts of finitary metric model theory to the infinitary case. What this means more precisely is that we need to make some of the definitions relative to a fragment.

First of all, *embeddings*, *substructures*, *isomorphisms* and *automorphisms* are all defined as in the finitary case (cf. Definition 2.1.6). The notion of elementarity is then defined with respect to some fragment. Suppose therefore that $\mathscr{F}$ is a fragment of $\mathcal{L}_{\omega_1\omega}(L)$.

Definition 3.2.1. *We say that two structures $\mathcal{M}$ and $\mathcal{N}$ are* **$\mathscr{F}$-elementarily equivalent** *if they have the same $\mathscr{F}$-theory, i.e. if it holds that for any sentence $\sigma \in \mathscr{F}$, we have $\sigma^{\mathcal{M}} = \sigma^{\mathcal{N}}$. We denote this by $\mathcal{M} \equiv_{\mathscr{F}} \mathcal{N}$.*

If $\mathcal{M} \subsetneq \mathcal{N}$ and it holds that for all formulas $\varphi \in \mathscr{F}$ we have

$$\varphi^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{N}}(\bar{a})$$

for all tuples $\bar{a}$ of elements of $\mathcal{M}$, we say that $\mathcal{M}$ is an **$\mathscr{F}$-elementary substructure** *of $\mathcal{N}$ and we write $\mathcal{M} \preceq_{\mathscr{F}} \mathcal{N}$. In this case, we will also say that $\mathcal{N}$ is an* **$\mathscr{F}$-elementary extension** *of $\mathcal{M}$.*

We will mostly omit the $\mathscr{F}$ sub- and prescripts in the definition above, because we always tacitly assume we are given some fixed fragment.

We now have the following version of the Tarski-Vaught Test for infinitary metric model theory. The proof of it is very similar to the proof in the finitary case, and the reader might want to confer [9, Proposition 4.5].

Proposition 3.2.2 (Tarski-Vaught Test)**.** *Let L be a metric signature and let $\mathscr{F}$ be a fragment of $\mathcal{L}_{\omega_1\omega}(L)$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are L-structures such that $\mathcal{M} \subsetneq \mathcal{N}$. Then the following are equivalent:*

(*i*) $\mathcal{M} \preceq_{\mathscr{F}} \mathcal{N}$

(*ii*) *For all $n \in \mathbb{N}$ and all n-ary formulas $\varphi(\bar{x}) \in \mathscr{F}$ and all $\bar{a} \in \mathcal{M}^n$ we have*

$$\inf\{\varphi^{\mathcal{N}}(\bar{a}_{\restriction a_i=b}) : b \in \mathcal{N}\} = \inf\{\varphi^{\mathcal{N}}(\bar{a}_{\restriction a_i=b}) : b \in \mathcal{M}\}$$

for each $i \leq n$, where $\bar{a}_{\restriction a_i=b}$ denotes the tuple obtained by replacing a_i with b in $\bar{a}$.

Proof. If $\mathcal{M} \preceq \mathcal{N}$, then (*ii*) must hold since the equality holds for *any* formula of $\mathcal{F}$.

Conversely, suppose (*ii*) holds. We proceed by induction on the complexity of formulas. If φ is an atomic formula then the interpretations of φ in the two models must coincide on $\mathcal{M}$, since by assumption $\mathcal{M} \subsetneq \mathcal{N}$.

Suppose now that for all formulas $\varphi \in \mathscr{F}$ with $\mathrm{rk}(\varphi) \leq \alpha$ we have

$$\varphi^{\mathcal{N}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}).$$

Let ψ be a formula of rank $\alpha + 1$.

If ψ is $f(\psi_0, \ldots, \psi_n)$ for some formulas ψ_i and connective f, it follows by the induction hypothesis that $\psi^{\mathcal{N}}(\bar{a}) = \psi^{\mathcal{M}}(\bar{a})$.

If ψ is $\bigwedge_n \psi_n$, then each ψ_n has rank less than or equal to α so we see that

$$\psi^{\mathcal{N}}(\bar{a}) = \inf_n \psi_n^{\mathcal{N}}(\bar{a}) = \inf_n \psi_n^{\mathcal{M}}(\bar{a}) = \psi^{\mathcal{M}}(\bar{a}).$$

Similarly, the interpretations must coincide if ψ is $\bigvee_n \psi_n$.

Finally, if ψ is $\inf_{x_i} \varphi(\bar{x})$, then by (*ii*) we directly get that $\psi^{\mathcal{N}}(\bar{a}) = \psi^{\mathcal{M}}(\bar{a})$ for any tuple of elements from $\mathcal{M}$.

We conclude that $\mathcal{M} \preceq_{\mathscr{F}} \mathcal{N}$, which was what we wanted. □

Remark 3.2.3. Note that any closed *set* $M \subseteq \mathcal{N}$ satisfying the second condition in the proposition above is automatically a substructure, because given any function symbol F and a tuple $\bar{a}$ from M, we have

$$\inf_{c \in M} d^{\mathcal{N}}(F^{\mathcal{N}}(\bar{a}), c) = 0.$$

Thus, $F^{\mathcal{N}}(\bar{a}) \in M$.

With the Tarski-Vaught Test at our disposal, we can now prove a version of the downwards Löwenheim-Skolem Theorem for infinitary metric model theory. We will only use the separable case, so that is all we will prove here. The proof is more or less the same as in [9, Proposition 7.3], although we have included a few more details.

Theorem 3.2.4 (Downward Löwenheim-Skolem Theorem)**.** *Let L be a signature and let $\mathscr{F}$ be a fragment. Suppose $\mathcal{M}$ is an L-structure and that $A \subseteq \mathcal{M}$ is a separable subset of $\mathcal{M}$. Then there exists a separable $\mathscr{F}$-elementary substructure $\mathcal{N} \preceq \mathcal{M}$ such that $A \subseteq \mathcal{N}$.*

Proof. Let $A_0 \subseteq A$ be a countable dense subset. For $\varphi \in \mathscr{F}$, $\varepsilon \in \mathbb{Q}_{\geq 0}$, $i \in \mathbb{N}$ and $\bar{a}$ a finite tuple of elements of A_0, let

$$C_{\varphi,\varepsilon,\bar{a},i} = \{c \in \mathcal{M} : \varphi^{\mathcal{M}}(\bar{a}_{\restriction a_i=c}) \leq \varepsilon\}.$$

Let B_0 consist of exactly one element of $C_{\varphi,\varepsilon,\bar{a},i}$ for each tuple $(\varphi, \varepsilon, \bar{a}, i)$ such that $C_{\varphi,\varepsilon,\bar{a},i}$ is non-empty. Let $B_1 = A_0 \cup B_0$.

We now recursively iterate the above procedure to obtain $B_n \supseteq B_{n-1}$ such that B_n contains one element of each non-empty set $C_{\varphi,\varepsilon,\bar{b},i}$, for finite tuples $\bar{b} \in B_{n-1}^{<\omega}$. Let $\mathcal{N}$ denote the closure of $B_\omega := \bigcup_{n\in\mathbb{N}} B_n \subseteq \mathcal{M}$. Note that since B_ω is countable, $\mathcal{N}$ is separable.

We now claim that $\mathcal{N}$ satisfies the Tarski-Vaught Test. Let therefore $\varepsilon > 0$, $\bar{a} \in \mathcal{N}^n$ and $\varphi(x_0, \ldots, x_{n-1}) \in \mathscr{F}$ be given. For each $i \leq n-1$ we set

$$r_i := \inf_{c\in\mathcal{M}} \{\varphi^{\mathcal{M}}(\bar{a}_{\restriction a_i=c})\} + \varepsilon/2$$

By the uniform continuity of $\varphi^{\mathcal{M}}$, we can find a tuple $\bar{b} \in (B_\omega)^n$ such that for all $c \in \mathcal{M}$ and $i \leq n-1$ we have

$$|\varphi^{\mathcal{M}}(\bar{a}_{\restriction a_i=c}) - \varphi^{\mathcal{M}}(\bar{b}_{\restriction b_i=c})| < \varepsilon/2.$$

Find $m \in \mathbb{N}$ such that $\bar{b} \in (B_m)^n$. Then for any $i \leq n-1$, B_{m+1} contains an element c_i of $C_{\varphi,r_i,\bar{b},i}$. Therefore, $\varphi^{\mathcal{M}}(\bar{b}_{\restriction b_i=c_i}) \leq r_i$ and thus it follows that

$$\begin{aligned}\inf_{c\in\mathcal{N}} \{\varphi^{\mathcal{M}}(\bar{a}_{\restriction a_i=c})\} &\leq \varphi^{\mathcal{M}}(\bar{a}_{\restriction a_i=c_i}) \\ &\leq \varphi^{\mathcal{M}}(\bar{b}_{\restriction b_i=c_i}) + \varepsilon/2 \\ &\leq r_i + \varepsilon/2.\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we conclude that $\mathcal{N}$ satisfies the Tarski-Vaught Test. It follows that $\mathcal{N}$ is in fact an L-structure and moreover $A \subseteq \mathcal{N} \preceq \mathcal{M}$, which was what we wanted. □

3.3 Type spaces in infinitary logic

We will now define types and type spaces in infinitary metric model theory. As hinted to a few times, the definition uses the Banach algebras of definable predicates for infinitary metric model theory, so we begin by explaining exactly how they are defined in the infinitary setting. We should, however, first mention that the definition of types presented here is taken from [11].

3.3.1 Banach algebras of infinitary formulas

The construction of the Banach algebras of infinitary formulas or *infinitary predicates* is more or less identical to what we did in the previous chapter, but we do need to fix some notation.

The semi-norm on the commutative algebra (over $\mathbb{Q}$) of $\mathcal{L}_{\omega_1\omega}$-formulas is defined as in Definition 2.3.1 above. *Logical equivalence* is then defined as in Definition 2.3.2. The associated Banach algebra is denoted $\mathcal{F}_{\omega_1\omega}$ or, if we consider equivalence with respect to a theory T, $\mathcal{F}_{\omega_1\omega}(T)$. We will still write $\varphi^\bullet$ if we want to emphasise that we are considering the class of the formula φ. Elements of $\mathcal{F}_{\omega_1\omega}(T)$ are called **$\mathcal{L}_{\omega_1\omega}$-definable prediates** or, by an abuse of language, simply **formulas**.

If we make this construction starting with a given fragment $\mathscr{F}$, the resulting algebra is denoted $\mathcal{F}_{\mathscr{F}}(T)$ or, if T is empty, simply $\mathcal{F}_{\mathscr{F}}$.

Similarly, if we restrict ourselves to formulas with free variables among some set $I \subseteq \mathcal{V}$, we obtain Banach algebras denoted $\mathcal{F}_{\omega_1\omega,I}(T)$ and $\mathcal{F}_{\mathscr{F}_I}(T)$. To simplify our notation, these algebras will also be denoted $\mathcal{F}_I(T)$ whenever it is clear whether we consider a fragment or the full algebra.

We recall that since all these algebras are commutative Banach algebras over $\mathbb{R}$, it follows from [1, Theorem 2.3] that they are isometrically isomorphic to the space of continuous functions on some compact Hausdorff space. It is these spaces that we will use as our type spaces in infinitary metric model theory (cf. Definition 3.3.2 below). However, before moving on, we have the following convenient proposition, saying that the formulas with finitely many free variables are dense in $\mathcal{L}_{\omega_1\omega}(L)$ and therefore give us the same Banach algebras.

Proposition 3.3.1. *Let L be a signature and T a complete theory. The formulas with finitely many free variables are dense inside $\mathcal{L}_{\omega_1\omega}(L)$ with respect to the semi-norm $\|\cdot\|_T$. Hence, they give rise to the same Banach algebra over $\mathbb{R}$.*

Proof. As we mentioned, the proof is done by transfinite induction on the complexity of formulas. We will show that any formula may be approximated in $\|\cdot\|_T$ by formulas with finitely many free variables.

Any formula of complexity 0 must be atomic and hence it only contains finitely many free variables by definition.

Suppose now we can approximate any formula of complexity less than or equal to the ordinal α with a formula of finitely many free variables respecting the same modulus of continuity. Let φ be a formula with $\mathrm{rk}(\varphi) = \alpha+1$ and let $\varepsilon > 0$ be given. We must find a formula φ_0 with finitely many free variables that respects the same modulus as φ and such that $\|\varphi - \varphi_0\| \leq \varepsilon$.

If φ is $f(\varphi_1, \ldots, \varphi_n)$ for a polynomial f, then it simply follows from the (uniform) continuity of f and our induction hypothesis that we may find φ_0

with finitely many free variables such that $\|\varphi - \varphi_0\| \leq \varepsilon$. Moreover it is easy to check that φ_0 can be chosen to respect the same modulus as φ.

If φ is $\sup_{x_i} \psi(\bar{x})$, then by the induction hypothesis, we find ψ_0 with finitely many free variables that respects the modulus of ψ and such that $\|\psi - \psi_0\| \leq \varepsilon$. This means that

$$\sup\{|\psi^{\mathcal{M}}(\bar{a}) - \psi_0^{\mathcal{M}}(\bar{a} \restriction n)| : \mathcal{M} \text{ is a model, } \bar{a} \in \mathcal{M}^\omega\} \leq \varepsilon,$$

where $\bar{a} \restriction n := (a_0, \ldots, a_{n-1})$. We claim that this implies

$$\sup\{|\sup_{b \in \mathcal{M}} \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b}) - \sup_{c \in \mathcal{M}} \psi_0^{\mathcal{M}}((\bar{a} \restriction n)_{\restriction a_i = c})| : \mathcal{M},\ \bar{a} \in \mathcal{M}^\omega\} \leq \varepsilon, \tag{3.1}$$

where $\bar{a}_{\restriction a_i = b}$ denotes the sequence $\bar{a}$ with b on the i'th coordinate. To see this, let $\mathcal{M}$ be a model and $\bar{a} \in \mathcal{M}^\omega$. We want to show that

$$|\sup_{b \in \mathcal{M}} \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b}) - \sup_{c \in \mathcal{M}} \psi_0^{\mathcal{M}}((\bar{a} \restriction n)_{\restriction a_i = c})| \leq \varepsilon.$$

We may suppose that one of the supremums is greater than the other, say

$$\sup_{b \in \mathcal{M}} \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b}) \geq \sup_{c \in \mathcal{M}} \psi_0^{\mathcal{M}}((\bar{a} \restriction n)_{\restriction a_i = c}).$$

For any $\varepsilon' > 0$ we may find b_0 such that $\sup_b \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b}) \leq \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b_0}) + \varepsilon'$. Therefore, we obtain

$$\begin{aligned}
&|\sup_{b \in \mathcal{M}} \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b}) - \sup_{c \in \mathcal{M}} \psi_0^{\mathcal{M}}((\bar{a} \restriction n)_{\restriction a_i = c})| \\
&\qquad \leq \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b_0}) + \varepsilon' - \sup_{c \in \mathcal{M}} \psi_0^{\mathcal{M}}((\bar{a} \restriction n)_{\restriction a_i = c}) \\
&\qquad \leq \psi^{\mathcal{M}}(\bar{a}_{\restriction a_i = b_0}) + \varepsilon' - \psi_0^{\mathcal{M}}((\bar{a} \restriction n)_{\restriction a_i = b_0}) \\
&\qquad \leq \sup_{\bar{b} \in \mathcal{M}^\omega} \{|\psi^{\mathcal{M}}(\bar{b}) - \psi_0^{\mathcal{M}}(\bar{b} \restriction n)|\} + \varepsilon' \\
&\qquad \leq \varepsilon + \varepsilon'.
\end{aligned}$$

Since ε' was arbitrary, we conclude the inequality holds for $\varepsilon' = 0$ as well. Since $\mathcal{M}$ and $\bar{a} \in \mathcal{M}^\omega$ were arbitrary, we conclude that (3.1) holds. Finally we note that φ respects the same modulus as ψ. Hence $\varphi_0 := \sup_{x_i} \psi_0$ is a formula with finitely many free variables that respects the same modulus as φ and such that $\|\varphi - \varphi_0\| \leq \varepsilon$ as we wanted.

If φ is $\inf_{x_i} \psi(\bar{x})$, we use that φ is logically equivalent to $\sup_{x_i} -\psi(\bar{x})$ and make the same calculations as above.

Assume next that φ is $\bigwedge_n \varphi_n(\bar{x})$, where all φ_n respect the same modulus δ and bound. Since each φ_n has rank less than α, we may find φ_n^0 with finitely many free variables respecting the same modulus as φ_n and such that $\|\varphi_n - \varphi_n^0\|_T < \varepsilon$. We wish to form their infinitary conjunction to obtain the formula we are looking for. However, the number of free variables they depend

on may be unbounded. Therefore, we find some $N \in \mathbb{N}$ such that for any model $\mathcal{M} \vDash T$ and any two sequences $\bar{a}, \bar{b} \in \mathcal{M}^\omega$ we have

$$d_\omega^{\mathcal{M}}(\bar{a}, \bar{b}) \leq \sum_{i=0}^{N} 2^{-i-1} d^{\mathcal{M}}(a_i, b_i) + \delta(\varepsilon), \tag{3.2}$$

where $d_\omega^{\mathcal{M}}$ is the metric on $\mathcal{M}^\omega$ defined in Definition 2.1.16. Let now ψ_n be the formula $\inf_{x_i, i>N} \varphi_n^0(\bar{x})$. Then each ψ_n has at most the first $N+1$ variables as free variables. Moreover, they respect the same modulus δ as the φ_n's. Furthermore, for any $\mathcal{M} \vDash T$ and any $\bar{a} \in \mathcal{M}^\omega$ we have that

$$|(\varphi_n^0)^{\mathcal{M}}(\bar{a}) - \psi_n^{\mathcal{M}}(\bar{a})| \leq \varepsilon$$

because of (3.2) above. Therefore, we are allowed to form the formula $\bigwedge_n \psi_n$. Denote this formula by ψ. Since all the ψ_n's respect δ it follows that φ and ψ respect the same modulus as well. Furthermore, we claim that $\|\varphi - \psi\|_T \leq 3\varepsilon$, which of course is enough for what we want. Let therefore $\mathcal{M} \vDash T$ and let $\bar{a} \in \mathcal{M}^\omega$ be given. Then we can find an $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
|\varphi^{\mathcal{M}}(\bar{a}) - \psi^{\mathcal{M}}(\bar{a})| &= |\inf_n \varphi_n^{\mathcal{M}}(\bar{a}) - \inf_m \psi_m^{\mathcal{M}}(\bar{a})| \\
&\leq |\varphi_{n_0}^{\mathcal{M}}(\bar{a}) - \psi_{n_0}^{\mathcal{M}}(\bar{a})| + \varepsilon \\
&\leq |\varphi_{n_0}^{\mathcal{M}}(\bar{a}) - (\varphi_{n_0}^0)^{\mathcal{M}}(\bar{a})| + |(\varphi_{n_0}^0)^{\mathcal{M}}(\bar{a}) - \psi_{n_0}^{\mathcal{M}}(\bar{a})| + \varepsilon \\
&\leq \varepsilon + \varepsilon + \varepsilon,
\end{aligned}$$

which was what we wanted.

Finally, a similar argument works if φ is $\bigvee \varphi_n$.

All in all, we conclude that the formulas with finitely many free variables are dense in $\mathcal{L}_{\omega_1\omega}(L)$. □

3.3.2 INFINITARY TYPES

We are now ready to define the types of our infinitary metric model theory. Again, we would like to stress that this defintion is taken from [11]. We assume throughout that we are given a fragment $\mathscr{F}$, so that everything below is done with respect to this fragment.

Definition 3.3.2 ([11, Section 7]). *Let T be a complete $\mathscr{F}$-theory and let $\bar{v}$ be a tuple of distinct variables. A* **$\bar{v}$-type over T** *(or simply a* **type***) is a multiplicative bounded linear functional on $\mathcal{F}_{\mathscr{F},\bar{v}}(T)$ preserving the unit. We equip the set of types with the pointwise convergence topology it inherits as a subspace of the dual space. This topology is denoted $\tau_{\bar{v}}$ or simply τ and is called* **the logic topology**. *We will denote the resulting space by $\widehat{S_{\bar{v}}(T)}$.*

Again, the specific variables will most of the time not be important. Hence, we will often use the notation $\widehat{S_n(T)}$ for the type space associated to some n-tuple of variables. By $\widehat{S_\omega(T)}$, we denote the type space associated to some infinite tuple of variables.

Remark 3.3.3. We note that if we want to augment our signature L with new constant symbols from a given subset of a structure $\mathcal{M}$, we may define types with respect to this augmented language just as in the finitary case. In this light, the types defined above may be considered as types over the empty set. However, since it will not be necessary for us to consider types over given sets in this chapter, we will stick to the types defined above and not indulge ourselves in such generalisations.

Since $\widehat{S_n(T)}$ is a closed subset of the unit ball of the dual space, it is a compact space by Alaoglu's Theorem (cf. [41, Theorem 1.6.5]). Furthermore, we note that τ is generated by the sets

$$[\varphi < r] := \{p \in \widehat{S_n(T)} : p(\varphi) < r\}.$$

This is similar to the finitary setting described in Chapter 2.

The somewhat artificial $\widehat{}$-notation in the definition of the type space is used because we want to denote the subset of the so-called *realisable types* by $S_{\bar{v}}(T)$. These types are defined in the usual way. That is, a type $p \in \widehat{S_{\bar{v}}(T)}$ is **realisable** if there is a structure $\mathcal{M} \vDash T$ and a tuple $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ such for any $\varphi \in \mathcal{F}_{\bar{v}}(T)$ we have $p(\varphi) = \varphi^{\mathcal{M}}(\bar{a})$. In this case, we say that $\bar{a}$ **realises** the type p and write $\bar{a} \vDash p$. As usual, the type of a given tuple $\bar{a}$ is denoted $\mathrm{tp}(\bar{a})$. As mentioned, the subspace of all realisable types in a given type space $\widehat{S_{\bar{v}}(T)}$ will be denoted by $S_{\bar{v}}(T)$ or just $S_n(T)$. We leave it as an exercise for the reader to show that there are types that cannot be realised anywhere, so that in general we have $S_n(T) \subsetneq \widehat{S_n(T)}$. This essentially follows from the failure of the Compactness Theorem in infinitary logic. However, we do have the following proposition:

Proposition 3.3.4. *The space of realisable types $S_{\bar{v}}(T)$ is τ-dense in $\widehat{S_{\bar{v}}(T)}$.*

Proof. Let $\varphi(\bar{v})$ be a formula and let $r > 0$. We must show that there is a realisable type in $[\varphi < r]$ whenever this set is non-empty. Suppose not and that no type in $[\varphi < r]$ is realised. We claim that then the set is empty. If not, there is some p such that $p(\varphi) < r$. Moreover, since no type in $[\varphi < r]$ is realised, we must have that for any model $\mathcal{M} \vDash T$ and any tuple $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ we have $\varphi^{\mathcal{M}}(\bar{a}) \geq r$ (otherwise such a tuple $\bar{a}$ would realise a type in $[\varphi < r]$). Let $\varepsilon > 0$ be such that $p(\varphi) < r - \varepsilon$ and consider the formula $\psi = \varphi - (r - \varepsilon)$. Since p is multiplicative, $p(\psi) - \psi = p(\varphi) - \varphi$ is not invertible in $\mathcal{F}_{\bar{v}}(T)$. However, for any structure $\mathcal{M}$ and any $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ we have

$$(p(\varphi) - \varphi)^{\mathcal{M}}(\bar{a}) = p(\varphi) - \varphi^{\mathcal{M}}(\bar{a}) < r - \varepsilon - r = -\varepsilon < 0.$$

This implies that $p(\psi) - \psi$ *is* invertible – a contradiction. $\square$

We will see below that $S_{\bar{v}}(T)$ is, in fact, a Polish space, so in particular it is a G_δ subset of $\widehat{S_{\bar{v}}(T)}$.

The Gelfand transform from functional analysis allows us to identify $\mathcal{F}_{\bar{v}}(T)$ with the commutative Banach algebra over $\mathbb{R}$ $C(\widehat{S_{\bar{v}}(T)})$ of continuous real-valued functions on the type space equipped with the uniform norm. Recall that the Gelfand transform Γ is defined by $\Gamma(\varphi)(p) = p(\varphi)$. Since we are working with real valued Banach algebras, it is not a priori the case that Γ is an isometric isomorphism. However, in our case this is not hard to verify.

Proposition 3.3.5. *Let $\bar{v}$ be any tuple of distinct variables (finite or infinite) and let T be a complete $\mathscr{F}$-theory. Then the Gelfand transform*

$$\Gamma\colon \mathcal{F}_{\bar{v}}(T) \to C(\widehat{S_{\bar{v}}(T)})$$

is an isometric isomorphism of Banach algebras.

Proof. It is clear that Γ is a homomorphism. To see that Γ preserves the norm, note that

$$\|\Gamma(\varphi)\| := \sup\{|p(\varphi)| : p \in \widehat{S_{\bar{v}}(T)}\} = \sup\{|p(\varphi)| : p \in S_{\bar{v}}(T)\},$$

because the realisable types are dense in $\widehat{S_{\bar{v}}(T)}$ with respect to τ. Moreover,

$$\sup\{|p(\varphi)| : p \in S_{\bar{v}}(T)\} = \sup\{|\varphi^{\mathcal{M}}(\bar{a})| : \mathcal{M} \vDash T, \bar{a} \in \mathcal{M}^{|\bar{v}|}\} = \|\varphi\|.$$

Thus, Γ is an isometry and therefore injective as well. Furthermore, the image is closed, contains the constant maps and separates points, so by the Stone-Weierstraß Theorem (cf. e.g. [26, Theorem 16, IV.6.15, p. 272]), Γ is surjective as well. $\square$

In the light of this proposition, we will simply identify $\mathcal{F}_{\bar{v}}(T)$ and $C(\widehat{S_{\bar{v}}(T)})$. Hence, we will write $\varphi(p)$ instead of $\Gamma(\varphi)(p)$ for a formula φ and type p.

3.3.3 Connection to the usual definition

We will now explain exactly how the definition of types presented here and the usual definition relates. In finitary logic, the usual definition of a type is, that it is a set of formulas of the form

$$\{\varphi \in \mathcal{F}_{\bar{v}}(T) : \mathcal{M} \vDash \varphi(\bar{a})\} = \{\varphi : \varphi^{\mathcal{M}}(\bar{a}) = 0\}$$

for a given model $\mathcal{M}$ of T and a fixed tuple $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ (cf. Definition 2.4.1 from the previous chapter). It is not hard to see that a set of this form is a maximal ideal in $\mathcal{F}_{\bar{v}}(T)$. From the theory of Gelfand duality, we know that maximal ideals correspond to kernels of bounded multiplicative linear

functionals, and this is the connection to our definition of types. Given a set of the form above, the tuple $\bar{a}$ gives us, of course, a multiplicative linear functional with $\{\varphi : \varphi^{\mathcal{M}}(\bar{a}) = 0\}$ as its kernel. Conversely, given a multiplicative linear functional p, we may, by the Compactness Theorem, find some realisation of $\bar{a} \vDash p$. Hence, the correspondence that maps sets of the form above to the multiplicative functional it defines, is a bijective correspondence between types and maximal ideals of $\mathcal{F}_{\bar{v}}(T)$.

Unfortunately, in the infinitary case we are working with, the Compactness Theorem fails. It follows that there are types that cannot be realised anywhere. However, the correspondence between maximal ideals in $\mathcal{F}_{\bar{v}}(T)$ and elements of $\widehat{S_{\bar{v}}(T)}$ still holds. This follows essentially from the Gelfand-Mazur Theorem for algebras over $\mathbb{R}$. The theorem states that any real normed division algebra, i.e. an algebra where every element except 0 is invertible, is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or the quarternions $\mathbb{H}$ (cf. for instance [16, §14 Theorem 7]). Using this theorem, we have the following proposition:

Proposition 3.3.6. *There is a bijection between maximal ideals of $\mathcal{F}_{\bar{v}}(T)$ and the type space.*

Proof. It is clear that the map taking a type to its kernel is an injective map into the set of maximal ideals. We need to show this map is surjective.

Let therefore $I \leq \mathcal{F}_{\bar{v}}(T)$ be a maximal ideal. Then $\mathcal{F}_{\bar{v}}(T)/I$ is a normed division algebra over the reals. By the Gelfand-Mazur Theorem, it follows that this quotient is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $\mathcal{F}_{\bar{v}}(T)/I$ is isomorphic to either $\mathbb{C}$ or $\mathbb{H}$, we find some $\varphi \in \mathcal{F}_{\bar{v}}(T)$ and $\psi \in I$ such that $\varphi^2 - \psi = -1$. Thus, given a model $\mathcal{M}$ and a tuple $\bar{a} \in \mathcal{M}^{|\bar{v}|}$, we have

$$\psi^{\mathcal{M}}(\bar{a}) = (\varphi^{\mathcal{M}}(\bar{a}))^2 + 1 \geq 1 > 0.$$

Therefore, ψ is invertible, so $I = \mathcal{F}_{\bar{v}}(T)$. This is a contradiction, since a maximal ideal must be proper. We conclude that $\mathcal{F}_{\bar{v}}(T)/I \cong \mathbb{R}$.

We now define a type $p \in \widehat{S_{\bar{v}}(T)}$ by letting $p(\varphi)$ be the real number corresponding to $\varphi + I$ under the isomorphism obtained above. It is clear that the kernel of p is exactly I, so p is mapped to I under our correspondence. Hence, this correspondence is surjective as well. □

Following these considerations, it is therefore natural to consider types as multiplicative linear functionals instead of certain sets of formulas – also in finitary metric model theory. The key advantage of this approach is that it generalises nicely to other more general logics where the Compactness Theorem fails, such as the $\mathcal{L}_{\omega_1\omega}$-logic we are working with here.

3.3.4 The infinitary type distance

We move on to define a distance on the spaces of types. We will need this for the formulation of the Omitting Types Theorem. Recall that in finitary

continuous logic we define the distance between two types to be

$$\partial_{FIN}(p,q) = \inf\{d^{\mathcal{M}}(\bar{a},\bar{b}) : \mathcal{M} \vDash T,\ \bar{a},\bar{b} \in \mathcal{M}^n,\ \bar{a} \vDash p \text{ and } \bar{b} \vDash q\}.$$

Unfortunately, this definition does not work in infinitary fragments, since one of the types might not be realisable or there might not be one model realising both types, due to the failure of the Compactness Theorem. Nevertheless, we can still define a type distance. Recall that in the definition below $\dot{-}$ denotes the map $\mathbb{R}^2 \to \mathbb{R}$ given by

$$s \dot{-} t = \begin{cases} s - t & \text{if } s \geq t \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.3.7 ([11, Section 7])**.** *Let $\mathscr{F}$ be a fragment and $\bar{v}$ be a tuple of variables. Let moreover T be a complete $\mathscr{F}$-theory. On $\widehat{S_{\bar{v}}(T)}$ we define the* **type distance** *$\partial_{\mathscr{F}_{\bar{v}}}$ by*

$$\partial_{\mathscr{F}_{\bar{v}}}(p,q) \leq s \iff \forall \varphi \in \mathscr{F}_{\bar{v}}\ q\Big(\inf_{\bar{y}} \big\{ \max\{(d(\bar{v},\bar{y}) \dot{-} s), |\varphi(\bar{y}) - p(\varphi)|\}\big\}\Big) = 0.$$

The $\mathscr{F}_{\bar{v}}$-subscript will mostly be omitted, since we will only be dealing with one fixed fragment. Note also that the formulas above are (by our usual abuse of notation) considered as elements of the algebra $\mathcal{F}_{\bar{v}}(T)$, and hence it makes sense to use any continuous function and not just polynomials as connectives. Thus, the absolute value and $\dot{-}$ make sense (alternatively they could simply be allowed as connectives without changing our countability assumption). We would also like to note here that if two types $p, q \in S_{\bar{v}}(T)$ are realised in the same model by $\bar{a}, \bar{b} \in \mathcal{M}^{|\bar{v}|}$, respectively, then $\partial(p,q) \leq d^{\mathcal{M}}(\bar{a},\bar{b})$ just as in the finitary setting. To see this, we simply observe that for any formula φ, if we let s be equal to $d^{\mathcal{M}}(\bar{a},\bar{b})$ in the above definition, the infimum is obviously 0 when we evaluate in $\bar{b}$ (by plugging in $\bar{a}$ in the place of $\bar{y}$). It should also be noted that if our given fragment $\mathscr{F}$ is just $\mathcal{L}_{\omega\omega}$, then by the Compactness Theorem, ∂ is equal to the usual type distance.

The above definition is, as noted, taken from [11]. However, the authors there only define ∂ on the space of realisable types. We claim that ∂ is in fact a metric on all of $\widehat{S_{\bar{v}}(T)}$. Moreover, we claim that $(\widehat{S_{\bar{v}}(T)}, \tau, \partial)$ is a topometric space (cf. Definition 2.4.13).

Proposition 3.3.8. *$(\widehat{S_{\bar{v}}(T)}, \tau, \partial)$ is a topometric space for any tuple of variables $\bar{v}$.*

Proof. We will first check that ∂ is in fact a metric. Below we will denote $\inf_{\bar{y}} \max\{d(\bar{v},\bar{y}) \dot{-} r, |\varphi(\bar{y}) - p(\varphi)|\}$ by $\psi_{p,r,\varphi}(\bar{v})$ for a type p, $r \in \mathbb{R}$ and formula φ. As usual, the distance symbol d is interpreted according to which power of $\mathcal{M}$ we are working with, as defined in Definition 2.1.16 above. We begin by noting that for any realisable type q_0, we have $q_0(\psi_{p,r,\varphi}) \geq 0$, since we take

the infimum over positive numbers. Therefore, by τ-density of the realisable types, we have that $q(\psi_{p,r,\varphi}) \geq 0$ for all types q.

We will now verify that ∂ is a metric.

$\underline{\partial(p,q) \geq 0 \text{ for all } p,q.}$

Suppose not, so that $\partial(p,q) \leq r < 0$ for some types p and q. Then $q(\psi_{p,r,\varphi}) = 0$ for every φ. Fix some φ and let $0 < \varepsilon < |r|$. By τ-density, we can find a realisable type q_0 such that

$$|q_0(\psi_{p,r,\varphi}) - q(\psi_{p,r,\varphi})| = q_0(\psi_{p,r,\varphi}) < \varepsilon.$$

Suppose $\bar{a} \in \mathcal{M}^{|\bar{v}|}$ realises q_0. Then

$$q_0(\psi_{p,r,\varphi}) = \inf_{\bar{b} \in \mathcal{M}^n} \max\{d^{\mathcal{M}}(\bar{a},\bar{b}) \dot{-} r, |\varphi^{\mathcal{M}}(\bar{b}) - p(\varphi)|\} < \varepsilon,$$

so we may find some $\bar{b} \in \mathcal{M}^n$ such that

$$|r| \leq d^{\mathcal{M}}(\bar{a},\bar{b}) + |r| = d^{\mathcal{M}}(\bar{a},\bar{b}) \dot{-} r < \varepsilon < |r|,$$

which is a contradiction.

$\underline{\partial(p,q) = 0 \iff p = q.}$

First we show $\partial(p,p) = 0$. To do this, we must show $p(\psi_{p,0,\varphi}) = 0$ for every φ. Thus, we let φ be a formula and $\varepsilon > 0$. By τ-density of the realisable types, we may find some $p_0 \in S_{\bar{v}}(T)$ such that both $|p_0(\varphi) - p(\varphi)| < \varepsilon/2$ and $|p_0(\psi_{p,0,\varphi}) - p(\psi_{p,0,\varphi})| < \varepsilon/2$. Suppose $\bar{a} \vDash p_0$ in $\mathcal{M}$. Then

$$0 \leq p_0(\psi_{p,0,\varphi}) \leq |\varphi^{\mathcal{M}}(\bar{a}) - p(\varphi)| < \varepsilon/2.$$

Therefore,

$$p(\psi_{p,0,\varphi}) < p_0(\psi_{p,0,\varphi}) + \varepsilon/2 < \varepsilon$$

and since this holds for any $\varepsilon > 0$, we must have $p(\psi_{p,0,\varphi}) = 0$.

Conversely, suppose $\partial(p,q) = 0$. We show that for all $\bar{v}$-ary formulas φ and $\varepsilon > 0$ we have $|p(\varphi) - q(\varphi)| < \varepsilon$, which of course implies $p = q$.

Let therefore φ and ε be given. Since φ respects its modulus of continuity, we can find a $\delta > 0$ such that for any model $\mathcal{M} \vDash T$ and $\bar{a}, \bar{b} \in \mathcal{M}^{|\bar{v}|}$ we have that if $d^{\mathcal{M}}(\bar{a},\bar{b}) < \delta$, then $|\varphi^{\mathcal{M}}(\bar{a}) - \varphi^{\mathcal{M}}(\bar{b})| < \varepsilon/3$. Next, we let q_0 be a realisable type such that $q_0(\psi_{p,0,\varphi}) < \min\{\varepsilon/3, \delta\} =: \varepsilon'$ and $|q_0(\varphi) - q(\varphi)| < \varepsilon'$. Let $\bar{a} \vDash q_0$ in $\mathcal{M} \vDash T$. Then we find $\bar{b} \in \mathcal{M}^{|\bar{v}|}$ such that $d^{\mathcal{M}}(\bar{a},\bar{b}) < \varepsilon'$ and $|\varphi^{\mathcal{M}}(\bar{b}) - p(\varphi)| < \varepsilon'$. We now have that

$$|p(\varphi) - q(\varphi)| \leq |p(\varphi) - \varphi^{\mathcal{M}}(\bar{b})| + |\varphi^{\mathcal{M}}(\bar{b}) - q_0(\varphi)| + |q_0(\varphi) - q(\varphi)| < \varepsilon.$$

We conclude that $p = q$.

Symmetry.

Suppose towards a contradiction that symmetry fails for some p, q so that $\partial(p,q) \leq s$ but $\partial(q,p) > s$. This means that for some φ, $p(\psi_{q,s,\varphi}) =: r > 0$. We may find $q_0 \in S_{\bar{v}}(T)$ such that $|q_0(\varphi) - q(\varphi)| < r/2$ and

$$q_0(\inf_{\bar{y}} \max\{d(\bar{v},\bar{y}) \dot{-} s, |\psi_{q,s,\varphi}(\bar{y}) - p(\psi_{q,s,\varphi})|\}) < r/2.$$

Let $\bar{a} \vDash q_0$ in $\mathcal{M}$. Then we may find $\bar{b} \in \mathcal{M}^{|\bar{v}|}$ such that $d(\bar{a},\bar{b}) \dot{-} s < r/2$ and $|\psi^{\mathcal{M}}_{q,s,\varphi}(\bar{b}) - p(\psi_{q,s,\varphi})| < r/2$. The latter implies

$$\psi^{\mathcal{M}}_{q,s,\varphi}(\bar{b}) > p(\psi_{q,s,\varphi}) - r/2 = r/2.$$

On the other hand, we have

$$\psi^{\mathcal{M}}_{q,s,\varphi}(\bar{b}) \leq \max\{d^{\mathcal{M}}(\bar{a},\bar{b}) \dot{-} s, |\varphi^{\mathcal{M}}(\bar{a}) - q(\varphi)|\} < r/2,$$

which is a contradiction.

Triangle inequality.

The argument is very similar to the other arguments. Since we must show something for each formula φ, we will approximate a given type on finitely many formulas arbitrarily with a realisable type. Since the triangle inequality is true for the realisable ones, it will hold for the non-realisable types as well.

The details are as follows: suppose that $\partial(p_1,p_2) \leq s$ and $\partial(p_2,p_3) \leq r$. We must show that $\partial(p_1,p_3) \leq s + r$. Let φ be a $\bar{v}$-ary formula. Then we know that

$$p_2(\inf_{\bar{y}} \max\{d(\bar{v},\bar{y}) \dot{-} s, |\varphi(\bar{y}) - p_1(\varphi)|\}) = 0.$$

This implies that

$$p_3\big(\inf_{\bar{y}} \max\big\{d(\bar{v},\bar{y}) \dot{-} r, \inf_{\bar{z}} \max\{d(\bar{z},\bar{y}) \dot{-} s, |\varphi(\bar{z}) - p_1(\varphi)|\}\big\}\big) = 0 \tag{3.3}$$

by using the definition of $\partial(p_2,p_3) \leq r$ with the formula

$$\inf_{\bar{z}} \max\{d(\bar{z},\bar{y}) \dot{-} s, |\varphi(\bar{z}) - p_1(\varphi)|\},$$

which is mapped to 0 by p_2. Of course, strictly speaking, the type p_2 is only defined for formulas with free variables among $\bar{v}$, but we may naturally view it as defined on formulas with free variables among $\bar{y}$ as well. Hence, we may apply p_2 to the formula above simply by setting $p_2(\psi(\bar{y})) := p_2(\psi(\bar{v}))$. We now claim that (3.3) implies

$$p_3\big(\inf_{\bar{z}} \max\{d(\bar{v},\bar{z}) \dot{-} (s+r), |\varphi(\bar{z}) - p_1(\varphi)|\}\big) = 0,$$

why it follows that $\partial(p_1, p_3) \leq s + r$. We will show the claim, but first, to ease the notation a bit, put

$$\Psi_1(\bar{v}) := \inf_{\bar{y}} \big\{ \max \big\{ d(\bar{v}, \bar{y}) \dot{-} r, \inf_{\bar{z}} \max\{d(\bar{z}, \bar{y}) \dot{-} s, |\varphi(\bar{z}) - p_1(\varphi)|\} \big\} \big\}$$

and

$$\Psi_2(\bar{v}) := \inf_{\bar{z}} \big\{ \max\{d(\bar{v}, \bar{z}) \dot{-} (s + r), |\varphi(\bar{z}) - p_1(\varphi)|\} \big\}.$$

To show the claim, let $\varepsilon > 0$ and let p_3^0 be a realisable type such that

$$|p_3(\Psi_1) - p_3^0(\Psi_1)| = p_3^0(\Psi_1) < \varepsilon$$

and

$$|p_3(\Psi_2) - p_3^0(\Psi_2)| < \varepsilon.$$

Suppose that $\bar{a} \vDash p_3^0$ in $\mathcal{M} \vDash T$. Then we may find $\bar{b}, \bar{c} \in \mathcal{M}^{|\bar{v}|}$ such that

$$\begin{aligned} d^{\mathcal{M}}(\bar{a}, \bar{b}) &< r + \varepsilon, \\ d^{\mathcal{M}}(\bar{b}, \bar{c}) &< s + \varepsilon \text{ and} \\ |\varphi^{\mathcal{M}}(\bar{c}) - p_1(\varphi)| &< \varepsilon. \end{aligned}$$

Therefore,

$$d^{\mathcal{M}}(\bar{a}, \bar{c}) \dot{-} (r + s) < 2\varepsilon,$$

so $p_3^0(\Psi_2) < 2\varepsilon$. We conclude that $p_3(\Psi_2) < 3\varepsilon$. Since ε was arbitrary and since $p_3(\Psi_2) \geq 0$, we must have $p_3(\Psi_2) = 0$. Since this holds for any $\bar{v}$-ary formula φ, we conclude that $\partial(p_1, p_3) \leq s + r$.

All in all we conclude that ∂ is a metric on $\widehat{S_{\bar{v}}(T)}$.

We move on to show that ∂ is lower semi-continuous and that it refines the topology.

<u>Lower semi-continuity.</u>

By definition, we must show that

$$\{(p, q) \in \widehat{S_{\bar{v}}(T)}^2 : \partial(p, q) \leq r\}$$

is τ-closed for any $r \in \mathbb{R}$. Of course, this set is empty for $r < 0$ and it is the diagonal for $r = 0$. Therefore, the set is closed in both cases.

Let $r > 0$. Let $(p_n, q_n) \to (p, q)$ pointwise with $\partial(p_n, q_n) \leq r$. We must show that $\partial(p, q) \leq r$ as well. Thus, we must show

$$q(\inf_{\bar{y}} \max\{d(\bar{v}, \bar{y}) \dot{-} r, |\varphi(\bar{y}) - p(\varphi)|\}) = 0$$

for all $\bar{v}$-ary formulas φ. Let therefore φ be a $\bar{v}$-ary formula and put

$$\Psi(\bar{v}) := \inf_{\bar{y}} \max\{d(\bar{v}, \bar{y}) \dot{-} r, |\varphi(\bar{y}) - p(\varphi)|\}$$

and

$$\Psi_n(\bar{v}) := \inf_{\bar{y}} \max\{d(\bar{v}, \bar{y}) \dot{-} r, |\varphi(\bar{y}) - p_n(\varphi)|\}.$$

We must show that $q(\Psi) = 0$. First, we note that $q(\Psi) \geq 0$ always holds because this inequality must hold for a realisable type and hence it must hold for all types by τ-density. To show equality, we will show $q(\Psi) < \varepsilon$ for all $\varepsilon > 0$.

Since $p_n(\varphi) \to p(\varphi)$, we have $\Psi_n \to \Psi$. Moreover, $q_n(\Psi) \to q(\Psi)$ since $q_n \to q$ in τ. Thus, we may find some N such that $\|\Psi - \Psi_N\| < \varepsilon/2$ and $|q_N(\Psi) - q(\Psi)| < \varepsilon/2$. Then

$$q_N(\Psi) = |q_N(\Psi) - q_N(\Psi_N)| \leq \|\Psi - \Psi_N\| < \varepsilon/2,$$

and hence $q(\Psi) < \varepsilon$ as required.

<u>Refinement.</u>

Let $[\varphi < r]$ be a basic τ-open set and let $p \in [\varphi < r]$. Let $\varepsilon > 0$ be such that $p(\varphi) + \varepsilon < r$ and find $\delta > 0$ such that for any $\mathcal{M} \vDash T$ and $\bar{a}, \bar{b} \in \mathcal{M}^{|\bar{v}|}$ with $d^{\mathcal{M}}(\bar{a}, \bar{b}) < \delta$ we have $|\varphi^{\mathcal{M}}(\bar{a}) - \varphi^{\mathcal{M}}(\bar{b})| < \varepsilon/3$. Let $\varepsilon' := \min\{\varepsilon/3, \delta/2\}$. We claim that $B_\partial(p, \varepsilon') \subseteq [\varphi < r]$.

Thus, we let $q \in B_\partial(p, \varepsilon')$ and must show $q(\varphi) < r$. We may find a realisable type q_0 such that

$$q_0(\inf_{\bar{y}} \max\{d(\bar{v}, \bar{y}) \dot{-} \varepsilon', |\varphi(\bar{y}) - p(\varphi)|\}) < \varepsilon'$$

and

$$|q_0(\varphi) - q(\varphi)| < \varepsilon'.$$

Let $\bar{a} \vDash q_0$ in some model $\mathcal{M} \vDash T$. Then we may find $\bar{b} \in \mathcal{M}$ such that

$$\max\{d^{\mathcal{M}}(\bar{a}, \bar{b}) \dot{-} \varepsilon', |\varphi^{\mathcal{M}}(\bar{b}) - p(\varphi)|\} < \varepsilon'.$$

We now have

$$\begin{aligned} |p(\varphi) - q(\varphi)| &\leq |q(\varphi) - q_0(\varphi)| + |q_0(\varphi) - \varphi^{\mathcal{M}}(\bar{b})| + |p(\varphi) - \varphi^{\mathcal{M}}(\bar{b})| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

where $|p(\varphi) - \varphi^{\mathcal{M}}(\bar{b})| < \varepsilon/3$ follows from $d^{\mathcal{M}}(\bar{a}, \bar{b}) \dot{-} \varepsilon' < \varepsilon'$, since this implies $d^{\mathcal{M}}(\bar{a}, \bar{b}) < \delta$. Thus, $q(\varphi) < p(\varphi) + \varepsilon < r$. We conclude that $B_\partial(p, \varepsilon')$ is contained in $[\varphi < r]$. Therefore, $[\varphi < r]$ is ∂-open.

All in all we conclude that ∂ is a lower semi-continuous metric on $\widehat{S_{\bar{v}}(T)}$ refining τ and hence that $\widehat{S_{\bar{v}}(T)}$ is a topometric space. $\square$

The topology that the metric ∂ induces on $\widehat{S_{\bar{v}}(T)}$ will be denoted τ_∂.

As a corollary to the proposition above, we will obtain another way to compute the distance between two types. In order to explain this properly, we will need to introduce some terminology. The first definition deals with so-called *normal* topometric spaces.

Definition 3.3.9. *We say that a topometric space (X, τ_X, d_X) is* **normal** *if the following two conditions are satisfied:*

(*i*) *Any two τ_X-closed subsets $F, G \subseteq X$ with positive distance $d_X(F, G) > 0$ may be separated by disjoint τ_X-open subsets.*
(*ii*) *For any τ_X-closed set F and any $r > 0$, the closed r-ball around F, $\overline{B}(F, r) := \{x : d_X(x, F) \leq r\}$, is τ_X-closed.*

If (ii) above holds, X is said to have **closed metric neighbourhoods**.

The following lemma follows easily from [6, Lemma 1.8]

Lemma 3.3.10 (Essentially [6, Lemma 1.8])**.** *Let (X, τ, ∂) be a topometric space. If τ is compact, then (X, τ, ∂) is normal.*

As a direct corollary, we obtain the following:

Corollary 3.3.11. *$\widehat{S_{\bar{v}}(T)}$ is a normal topometric space for any $\bar{v}$.*

Proof. Since we have already shown that $(\widehat{S_{\bar{v}}(T)}, \tau, \partial)$ is a topometric space, the corollary follows directly from the lemma above. □

This allows us to exploit some of the results of [7] for normal topometric spaces. In particular, we will use the following topometric version of Urysohn's lemma.

Theorem 3.3.12 (Urysohn's Lemma, [7, Theorem 1.6])**.** *Let (X, τ, ∂) be a normal topometric space and suppose $F, G \subseteq X$ are τ-closed with $d(F, G) > 0$. Then for any $r \in]0, d(F, G)[$ there is a ∂-1-Lipschitz τ-continuous function $f \colon X \to [0, r]$ such that f is 0 on F and r on G.*

We will use this result to show that the type distance can be computed as a supremum over the so-called 1-*Lip formulas*. These are defined below. First, recall that the algebra of formulas $\mathcal{F}_{\bar{v}}(T)$ may be identified with the real valued continuous functions on $(\widehat{S_{\bar{v}}(T)}, \tau)$. Hence, we may use the two notations $\varphi(p)$ and $p(\varphi)$ interchangeably. Both notations will be convenient below.

Definition 3.3.13. *A formula $\varphi \in \mathcal{F}_{\bar{v}}(T)$ is* **k-Lip** *for $k > 0$ if for any model $\mathcal{M} \vDash T$, we have*

$$|\varphi^{\mathcal{M}}(\bar{a}) - \varphi^{\mathcal{M}}(\bar{b})| \leq k d^{\mathcal{M}}(\bar{a}, \bar{b})$$

for all $\bar{a}, \bar{b} \in \mathcal{M}^{|\bar{v}|}$. That is, φ is always interpreted as a k-Lipschitz map.

We now have the following convenient description of the type distance:

Theorem 3.3.14. *The type distance $\partial_{\mathscr{F}_{\bar{v}}}$ on $\widehat{S_{\bar{v}}(T)}$ can be computed as the supremum over* 1*-Lip formulas. I.e.,*

$$\partial_{\mathscr{F}_{\bar{v}}}(p,q) = \sup_{\varphi}\{|p(\varphi) - q(\varphi)| : \varphi \textit{ is a 1-Lip formula in } \mathcal{F}_{\bar{v}}(T)\} \tag{3.4}$$

Proof. Suppose first that $\partial(p,q) \leq s$. We must show that the right-hand side in (3.4) above is less than s as well.

Let φ be a 1-Lip formula and $\varepsilon > 0$. We will show that $|\varphi(p)-\varphi(q)| \leq s+\varepsilon$, which is sufficient to obtain the desired inequality.

Let $q_0 \in S_{\bar{v}}(T)$ be a realisable type such that

$$q_0(\inf_{\bar{y}} \max\{d(\bar{v},\bar{y}) \dot{-} s, |\varphi(p) - \varphi(\bar{y})|\}) < \varepsilon/3$$

and such that $|q_0(\varphi) - q(\varphi)| < \varepsilon/3$. Suppose $\bar{a} \vDash q_0$ in some model $\mathcal{M} \vDash T$ and let $\bar{b} \in \mathcal{M}^{|\bar{v}|}$ be such that both $d^{\mathcal{M}}(\bar{a},\bar{b}) \dot{-} s < \varepsilon/3$ and $|\varphi(p) - \varphi^{\mathcal{M}}(\bar{b})| < \varepsilon/3$. This gives us

$$\begin{aligned}
|\varphi(p) - \varphi(q)| &\leq |\varphi(p) - \varphi^{\mathcal{M}}(\bar{b})| + |\varphi^{\mathcal{M}}(\bar{b}) - \varphi^{\mathcal{M}}(\bar{a})| + |\varphi^{\mathcal{M}}(\bar{a}) - \varphi(q)| \\
&< d(\bar{a},\bar{b}) + 2\varepsilon/3 \\
&< s + \varepsilon,
\end{aligned}$$

which was what we wanted.

The other inequality follows from Urysohn's lemma for topometric spaces stated above:

Let $\partial(p,q) > \varepsilon > 0$. Since $(\widehat{S_{\bar{v}}(T)}, \tau, \partial)$ is a normal topometric space, Urysohn's Lemma above (with $\{p\}$ and $\{q\}$ as the two closed sets) gives us a ∂-1-Lipschitz function $\varphi_0 \colon \widehat{S_{\bar{v}}(T)} \to \mathbb{R}$ such that

$$\varphi_0(p) = 0 \quad \text{and} \quad \varphi_0(q) = \partial(p,q) - \varepsilon.$$

If we consider φ_0 as an element of $\mathcal{F}_{\bar{v}}(T)$, then φ_0 is also a 1-Lip formula, since given $\mathcal{M} \vDash T$ and $\bar{a}, \bar{b} \in \mathcal{M}^{|\bar{v}|}$, we have

$$|\varphi_0^{\mathcal{M}}(\bar{a}) - \varphi_0^{\mathcal{M}}(\bar{b})| = |\varphi_0(\mathrm{tp}(\bar{a})) - \varphi_0(\mathrm{tp}(\bar{b}))| \leq \partial(\mathrm{tp}(\bar{a}), \mathrm{tp}(\bar{b})) \leq d^{\mathcal{M}}(\bar{a},\bar{b}),$$

where the last inequality always holds for two types that are realised in the same model $\mathcal{M}$. It now follows that

$$\sup\{|\varphi(p) - \varphi(q)| : \varphi \text{ is 1-Lip}\} + \varepsilon \geq |\varphi_0(p) - \varphi_0(q)| + \varepsilon = \partial(p,q).$$

Letting ε tend to 0, we conclude that

$$\sup\{|\varphi(p) - \varphi(q)| : \varphi \text{ is a 1-Lip formula}\} \geq \partial(p,q)$$

as well, which was what we wanted. $\square$

3.3.5 Principality of infinitary types

The notion of principality also generalises to the infinitary setting. However, since not all types are realised, the matter is a little more delicate. As we are dealing with a space equipped with two different topologies, i.e. the logic topology τ and the metric topology τ_∂, it will be convenient to introduce some terminology on how the topologies interact. The first definition is a very general one concerning arbitrary topological spaces.

Definition 3.3.15. *Let X be a space equipped with two topologies τ_1 and τ_2. We say that the topologies* **coincide** *at a given point $x \in X$ if for any $U \in \tau_1$ containing x, there is $V \in \tau_2$ such that $x \in V \subseteq U$, and vice versa.*

The next definition defines the *principal types* for infinitary metric model theory, i.e. those types where the two topologies interact nicely. In order for the topologies around principal types to behave as in the finitary case, we will restrict ourselves to the subspace topologies on the realisable type space.

Definition 3.3.16. *Let T be an $\mathscr{F}$-theory and let $\bar{v}$ be a tuple of variables. A type $p \in \widehat{S_{\bar{v}}(T)}$ is* **principal** *if the following holds:*

$$\forall r > 0\ \exists \varphi \in \mathscr{F}\ \exists s > 0 : \emptyset \neq [\varphi < s] \cap S_{\bar{v}}(T) \subseteq B_\partial(p, r) \cap S_{\bar{v}}(T).$$

In other words, any ball around p has non-empty τ-interior in the subspace topologies.

Notice that it is not a priori the case that a principal type is realised, since we do not assume that the type p is contained in any of the two intersections above. However, it turns out that principal types are always realised in any model of the theory T (cf. Lemma 3.3.19 below).

Given the definition of principality above, another natural concept arises, namely those types where the above holds not just for the subspace topologies but for the full topologies. We have chosen to call this property *compatibility* of ∂.

Definition 3.3.17. *Let T be a complete $\mathscr{F}$-theory and let $\bar{v}$ be a finite tuple of variables. We say that ∂ is* **compatible** *at a type $p \in \widehat{S_{\bar{v}}(T)}$ if the following hold:*

$$\forall r > 0\ \exists \varphi \in \mathscr{F}\ \exists s > 0 : \emptyset \neq [\varphi < s] \subseteq B_\partial(p, r).$$

Observe that there are some obvious implications between the three concepts defined above: if τ and τ_∂ coincides at a type p, then ∂ is compatible at p, and if ∂ is compatible at p, then p is principal. Moreover, in [11, Lemma 7.4] the authors show that our definition of principality and their definition are equivalent for realisable types. We state the lemma below for convenience.

Lemma 3.3.18 ([11, Lemma 7.4])**.** *Let $p \in S_{\bar{v}}(T)$. Then the following are equivalent:*

(*i*) *p is principal.*
(*ii*) *$\tau \restriction S_{\bar{v}}(T)$ and $\tau_\partial \restriction S_{\bar{v}}(T)$ coincide at p.*

As mentioned, we have that a principal type is always realisable. Moreover, it is realised in all models of the given theory. Hence, if τ and τ_∂ coincide or if ∂ is compatible at p, then p is realisable.

Lemma 3.3.19. *Let T be a complete $\mathscr{F}$-theory and let $p \in \widehat{S_n(T)}$ be a principal type. Then p is realised in all models of T.*

Moreover, if $q \in S_n(T)$ is any type realised in a model $\mathcal{M} \vDash T$ by some $\bar{b} \in \mathcal{M}^n$, we may for any $\varepsilon > 0$ find a realisation $\bar{a} \vDash p$ in $\mathcal{M}$ such that $d^{\mathcal{M}}(\bar{a}, \bar{b}) \leq \partial(p, q) + \varepsilon$.

Proof. To simplify the notation in the proof below we will assume that all sets considered are intersected with the set of realisable types $S_n(T)$.

Let p be a principal type in $\widehat{S_n(T)}$. The first thing we will show is that for any $r, s > 0$ and formula φ such that $[\varphi < s] \subseteq B(p, r)$, we can find a formula φ_1 such that $p(\varphi_1) = 0$ and $[\varphi_1 < r] \subseteq B(p, r)$. For this, we suppose $r > 0$ is given. Then we may find $s > 0$ and a formula φ such that $[\varphi < s] \subseteq B(p, r/3)$. Pick any $p_0 \in [\varphi < s]$ and set

$$\varphi_1(\bar{x}) := \inf_{\bar{y}} \max\{d(\bar{x}, \bar{y}) \dot{-} r/3, |\varphi(\bar{y}) - p_0(\varphi)|\}.$$

Then $p(\varphi_1) = 0$ because $\partial(p_0, p) < r/3$. Set

$$s' := \min\{s - p_0(\varphi), r/3\}.$$

We now claim that $[\varphi_1 < s'] \subseteq B(p, r)$. To see this, suppose $q(\varphi_1) < s'$ and that $\bar{a} \vDash q$ in some $\mathcal{M} \vDash T$. Then we can find some tuple $\bar{b}$ in $\mathcal{M}$ such that $d^{\mathcal{M}}(\bar{a}, \bar{b}) < s' + r/3$ and $|\varphi^{\mathcal{M}}(\bar{b}) - p_0(\varphi)| < s'$. It follows that $\varphi^{\mathcal{M}}(\bar{b}) < p_0(\varphi) + s' \leq s$ and therefore that $\partial(\mathrm{tp}(\bar{b}), p) < r/3$. Moreover,

$$\partial(\mathrm{tp}(\bar{b}), q) \leq d^{\mathcal{M}}(\bar{a}, \bar{b}) < r/3 + s' \leq 2r/3.$$

Therefore, we get that

$$\partial(p, q) \leq \partial(p, \mathrm{tp}(\bar{b})) + \partial(\mathrm{tp}(\bar{b}), q) < r$$

proving our claim. If we let φ_1' be the formula $(r/s')\varphi_1$, we get that

$$[\varphi_1' < r] \subseteq B(p, r)$$

and that $p(\varphi_1') = 0$ as we wanted.

With this observation at our disposal, we can now show the lemma. Let q be a realisable type realised by some $\bar{b}$ in some $\mathcal{M} \vDash T$ and let $\varepsilon > 0$. Set

$s := \partial(p,q)$ and find some φ_1 such that $p(\varphi_1) = 0$ and $[\varphi_1 < \varepsilon/3] \subseteq B(p, \varepsilon/3)$. Then

$$q(\inf_{\bar{y}} \max\{d(\bar{x},\bar{y}) \dot{-} s, |\varphi_1(\bar{y})|\}) = 0,$$

and therefore we can find some tuple $\bar{a}_1$ in $\mathcal{M}$ such that $d^{\mathcal{M}}(\bar{b},\bar{a}_1) \leq s + \varepsilon/3$ and $|\varphi_1^{\mathcal{M}}(\bar{a}_1)| < \varepsilon/3$. Hence, $\partial(\mathrm{tp}(\bar{a}_1), p) < \varepsilon/3$.

Next, we find some φ_2 with $p(\varphi_2) = 0$ and $[\varphi_2 < \varepsilon/3^2] \subseteq B(p, \varepsilon/3^2)$. Since $\partial(\mathrm{tp}(\bar{a}_1), p) < \varepsilon/3$, it follows that

$$\inf_{\bar{c}} \max\{d^{\mathcal{M}}(\bar{a}_1,\bar{c}) \dot{-} \varepsilon/3, |\varphi_2(\bar{c})|\} = 0.$$

Therefore, we can find some $\bar{a}_2$ in $\mathcal{M}$ such that $d^{\mathcal{M}}(\bar{a}_1,\bar{a}_2) < \varepsilon/3 + \varepsilon/3^2$ and $|\varphi_2^{\mathcal{M}}(\bar{a}_2)| < \varepsilon/3^2$. Hence, $\partial(\mathrm{tp}(\bar{a}_2), p) < \varepsilon/3^2$ as well.

Continuing in this fashion, we obtain a sequence $(\bar{a}_n)$ such that

$$d^{\mathcal{M}}(\bar{a}_{n-1},\bar{a}_n) < \frac{\varepsilon}{3^{n-1}} + \frac{\varepsilon}{3^n}$$

and $\partial(\mathrm{tp}(\bar{a}_n), p) < \frac{\varepsilon}{3^n}$. The latter implies, of course, that $\mathrm{tp}(\bar{a}_n)$ converges to p with respect to ∂. Moreover, the former implies that $(\bar{a}_n)$ is a Cauchy sequence because

$$\begin{aligned} d^{\mathcal{M}}(\bar{a}_n,\bar{a}_{n+k}) &\leq \varepsilon\Big(\sum_{i=0}^{k-1} \frac{1}{3^{n+i}} + \frac{1}{3^{n+i+1}}\Big) \\ &= 2\varepsilon\frac{3^k - 1}{3^{k+n}} \\ &\leq 2\varepsilon\frac{1}{3^n} \to 0 \quad \text{when } n \to \infty. \end{aligned}$$

Thus, by completeness of $\mathcal{M}$, we must have that $(\bar{a}_n)$ converges to some $\bar{a}$. Moreover, this $\bar{a}$ must realise p by continuity of the language. Thus, p is realised in $\mathcal{M}$, and since $\mathcal{M}$ was arbitrary, p is realised in any model of T.

Finally, for the 'moreover'-part, we have that

$$\begin{aligned} d^{\mathcal{M}}(\bar{b},\bar{a}_n) &\leq s + \frac{\varepsilon}{3} + \sum_{i=1}^{n-1} \frac{\varepsilon}{3^i} + \frac{\varepsilon}{3^{i+1}} \\ &= s + \varepsilon\Big(1 - \frac{2}{3^n}\Big) \leq s + \varepsilon. \end{aligned}$$

It follows that $d(\bar{a},\bar{b}) \leq s + \varepsilon$ as well, which was what we wanted. □

3.4 Omitting types

In this section we will give a new proof of the Omitting Types Theorem for infinitary metric model theory. This theorem was originally proved by Eagle

in [27]. However, the precise formulation of the theorem below is slightly more general than that of Eagle's. This formulation for finitary continuous logic is originally due to Itaï Ben Yaacov.

The first thing we need to do, is to define exactly what we mean by "omitting". As usual, $\mathscr{F}$ denotes a fragment.

Definition 3.4.1. *Let T be a complete $\mathscr{F}$-theory and suppose $\mathcal{M} \vDash T$. Let $\bar{v}$ be a finite tuple of variables and let $p \in S_{\bar{v}}(T)$ be a type. We say that $\mathcal{M}$* **omits** *p if there is no realisation of p in $\mathcal{M}$.*

We now have:

Theorem 3.4.2 (Omitting Types, cf. [27, Theorem A])**.** *Let T be a complete $\mathscr{F}$-theory and suppose that for each $n \in \mathbb{N}$, $X_n \subseteq S_n(T)$ is a τ-meagre and ∂-open set. Then there is a separable model $\mathcal{M} \vDash T$ omitting all types in $\bigcup X_n$.*

Before proceeding to the proof, we have a few remarks.

Remark 3.4.3. Recall that we have assumed our signatures and fragments to be countable. The countability of our signatures is essential for the theorem, as it may fail for uncountable signatures.

Remark 3.4.4. Note that if $X \subseteq S_n(T)$ is τ-meagre and ∂-open, it follows that no $p \in X$ can be principal. Hence, this theorem shows, in particular, that if p is non-principal, then there is a model omitting it.

The proof of the theorem depends on two lemmas. To state the first of them, we need to define a convenient subspace of $S_\omega(T)$. This is the space of the types whose realisations are actually models of T. We say that such a type *enumerates a model*. Below, we will denote the closure of the set of all elements of a sequence $\bar{a}$ by $\mathrm{cl}(\bar{a})$.

Definition 3.4.5. *We say that a realisable type $p \in S_\omega(T)$* **enumerates a model of T** *if for all realisations $\bar{a} \vDash p$ in $\mathcal{M} \vDash T$, we have that $\mathrm{cl}(\bar{a}) \preceq \mathcal{M}$. We denote the space of the types that enumerate a model of T by $S_\omega^{\mathrm{en}}(T)$. Given $p \in S_\omega^{\mathrm{en}}(T)$ and $\bar{a} \vDash p$ in $\mathcal{M} \vDash T$, we let $\mathcal{M}_{\bar{a}}$ denote the corresponding elementary substructure with universe $\mathrm{cl}(\bar{a})$.*

It turns out that for p to enumerate a model, it is enough that there is *some* realisation $\bar{a} \vDash p$ in a model $\mathcal{M} \vDash T$ such that $\mathcal{M}_{\bar{a}} \vDash T$ *and* such that $\bar{a}$ realises p *in* $\mathcal{M}_{\bar{a}}$. This is the content of the lemma below.

Lemma 3.4.6. *$p \in S_\omega(T)$ enumerates a model of T if and only if there is some $\bar{a} \vDash p$ in a model $\mathcal{M}$ of T such that $\mathcal{M}_{\bar{a}} \vDash T$ and $\bar{a}$ realises p in $\mathcal{M}_{\bar{a}}$.*

Proof. One direction is obvious: if $p \in S_\omega(T)$ enumerates a model, then there is of course a realisation $\bar{a}$ such that $\mathcal{M}_{\bar{a}} \vDash T$ and $\bar{a}$ realises p inside $\mathcal{M}_{\bar{a}}$.

Suppose conversely that $\bar{a} \vDash p$ in a model $\mathcal{M} \vDash T$ such that $\mathcal{M}_{\bar{a}} \vDash T$ and $\bar{a}$ realises p inside $\mathcal{M}_{\bar{a}}$. Let $\bar{b} \vDash p$ be another realisation of p inside some model $\mathcal{N}$ of T. Then the map $\Phi\colon a_i \mapsto b_i$ extends to an isometry $\mathcal{M}_{\bar{a}} \to \mathcal{N}$ whose image is $\mathrm{cl}(\bar{b})$. Moreover, Φ is an elementary embedding, since for any formula $\varphi(\bar{x})$ in our fragment, we have

$$\varphi^{\mathcal{M}_{\bar{a}}}(\bar{a}_I) = p(\varphi(\bar{x})) = \varphi^{\mathcal{N}}(\bar{b}_I),$$

where $\bar{a}_I$ denotes the tuple $(a_i : i \in I)$. By uniform continuity of formulas, it follows that the above equalities hold for any tuple of elements $\bar{c} \in \mathcal{M}_{\bar{a}}$. By the Tarski-Vaught Test, it follows that $\mathrm{cl}(\bar{b})$ is in fact the universe of a model $\mathcal{M}_{\bar{b}}$ of T and, moreover, that $\mathcal{M}_{\bar{b}} \preceq \mathcal{N}$, which was what we wanted. □

The first main lemma that we will use to prove Theorem 3.4.2 reads:

Lemma 3.4.7. *$(S^{\mathrm{en}}_{\omega}(T), \tau)$ is a Polish space.*

We postpone the proof of Lemma 3.4.7 until after the proof of Theorem 3.4.2. Instead, we move on to state the second lemma that we will need. In order to do that, we need to introduce some projection maps $S^{\mathrm{en}}_{\omega}(T) \to S_n(T)$. We will also need to be specific about which variables we are using to construct these spaces, so let us suppose that we use the infinite tuple $\bar{x} = (x_0, \ldots)$ to construct the space $S_\omega(T)$, and that we use the first n variables of $\bar{x}$ to construct $S_n(T)$.

For each $n \in \mathbb{N}$ and each $s \in \mathbb{N}^n$ we define a map

$$\pi_s\colon S^{\mathrm{en}}_{\omega}(T) \to S_n(T)$$

by

$$\pi_s(p)(\varphi(x_0, \ldots, x_{n-1})) = p(\varphi(x_{s_0}, \ldots, x_{s_{n-1}})).$$

The second lemma now says the following:

Lemma 3.4.8. *For each $n \in \mathbb{N}$ and each injective $s \in \mathbb{N}^n$, i.e. $s_i \neq s_j$ when $i \neq j$, the map π_s is an open continuous surjection with respect to τ.*

Remark 3.4.9. Note that this implies that $S_n(T)$ (and hence any $S_{\bar{v}}(T)$ for a finite tuple $\bar{v}$) is a G_δ set, since the continuous open image of a Polish space is Polish (cf. [29, Exercise 5.5.8 (d)]).

Assuming these two lemmas, we can prove Theorem 3.4.2.

Proof of Theorem 3.4.2. For each $n \in \mathbb{N}$ we let $X_n \subseteq S_n(T)$ be a τ-meagre and ∂-open set. Then, by Lemma 3.4.8, $X := \bigcup_n \bigcup_{s \in \mathbb{N}^n} \pi_s^{-1}(X_n)$ is a meagre subset of $S^{\mathrm{en}}_{\omega}(T)$, since π_s is an open and continuous map (cf. [44, Exercise 8.45]). Hence, by the Baire Category Theorem, we find some $p \in S^{\mathrm{en}}_{\omega}(T) \setminus X$. Let $\bar{a}$ be a sequence realising p in some model $\mathcal{M} \vDash T$. Then we claim $\mathcal{M}_{\bar{a}}$ is a separable model omitting all types in $\bigcup_n X_n$. To see this, we first note that

no tuple of elements from $\bar{a}$ can realise a type in any X_n, since $\pi_s(p) \notin X_n$ for all $s \in \mathbb{N}^n$ and $n \in \mathbb{N}$. Moreover, if for some tuple $\bar{b} \in \mathcal{M}_{\bar{a}}^n$ we had that $\mathrm{tp}(\bar{b}) \in X_n$, then we could find some ∂-ball around $\mathrm{tp}(\bar{b})$ contained in X_n, say $B_\partial(\mathrm{tp}(\bar{b}), \varepsilon) \subseteq X_n$. Since the elements of $\bar{a}$ are dense in $\mathcal{M}_{\bar{a}}$ by definition, we could find some $s \in \mathbb{N}^n$ such that $\bar{a}_s = (a_{s_0}, \ldots, a_{s_{n-1}})$ would be less than ε from $\bar{b}$. Thus, $\partial(\mathrm{tp}(\bar{a}_s), \mathrm{tp}(\bar{b})) < \varepsilon$ as well, which of course would imply that $\pi_s(p) \in X_n$, and this is not possible. Hence, we conclude that no tuple of elements from $\mathcal{M}_{\bar{a}}$ realises a type in any X_n. □

It remains of course to prove the two lemmas.

Proof of lemma 3.4.7. We will prove that $S_\omega^{\mathrm{en}}(T)$ is a G_δ subset of $\widehat{S_\omega(T)}$. For each formula $\varphi(\bar{x}) \in \mathscr{F}_{\bar{x}}$ and $\varepsilon > 0$ define

$$U_{\varphi,\varepsilon,i} = \{p \in \widehat{S_\omega(T)} : \inf_{j \in \mathbb{N}} p(\varphi(\bar{x}_{\restriction i = x_j})) < p(\inf_{x_i} \varphi(\bar{x})) + \varepsilon\},$$

where $\bar{x}_{\restriction i = x_j}$ denotes the tuple $\bar{x}$ but with x_j on the i'th coordinate. $U_{\varphi,\varepsilon,i}$ is open since it is equal to

$$\bigcup_j [\varphi(\bar{x}_{\restriction i = x_j}) - \inf_{x_i} \varphi(\bar{x}) < \varepsilon].$$

Moreover, these sets are dense. To show this, we must show that the intersection $U_{\varphi,\varepsilon,i} \cap [\psi < r]$ is non-empty for any formula ψ. Recall that all formulas in our fragment are assumed to have finitely many free variables. Thus, suppose ψ has free variables among $x_0, \ldots, x_{m-1}$ and that φ has free variables among $x_0, \ldots, x_{n-1}$. We can without loss of generality assume $m \geq n$, since we can always consider ψ as having free variables among arbitrarily large (finite) sets of variables. Take some type $p_0 \in [\psi < r]$. Since we have shown that the realisable types are τ-dense, we may assume p_0 is realised, say by $\bar{a} \in \mathcal{M}^\omega$. Let

$$s := p_0(\inf_{x_i} \varphi(\bar{x})) = \inf_{c \in \mathcal{M}} \varphi^{\mathcal{M}}(\bar{a}_{\restriction i = c}).$$

Then for some $b \in \mathcal{M}$ we must have $\varphi^{\mathcal{M}}(\bar{a}_{\restriction i = b}) < s + \varepsilon$. Consider the type $p = \mathrm{tp}(\bar{a}_{\restriction m = b})$. Our choice of b implies that

$$p(\varphi(\bar{x}_{\restriction i = x_m})) = \varphi^{\mathcal{M}}(\bar{a}_{\restriction i = b}) < s + \varepsilon.$$

Therefore, we have

$$\inf_j p(\varphi(\bar{x}_{\restriction i = x_j})) \leq p(\varphi(\bar{x}_{\restriction i = x_m})) < p_0(\inf_{x_i} \varphi(\bar{x})) + \varepsilon = p(\inf_{x_i} \varphi(\bar{x})) + \varepsilon.$$

In other words, $p \in U_{\varphi,\varepsilon,i}$. Moreover, as we made sure not to change the first m coordinates, we still have $p(\psi) < r$. We conclude that $U_{\varphi,\varepsilon,i}$ is dense.

Next, for each $\varepsilon > 0$ and each formula $\Phi \in \mathscr{F}_{\bar{x}}$ of the form $\bigwedge \varphi_k$, we let $V_{\Phi,\varepsilon}$ be the set

$$V_{\Phi,\varepsilon} = \{p \in \widehat{S_\omega(T)} : \inf_k p(\varphi_k) < p(\Phi) + \varepsilon\}.$$

It is clear that $V_{\Phi,\varepsilon}$ is an open set for each Φ and ε. Moreover, $V_{\Phi,\varepsilon}$ contains all realisable types and is therefore dense.

We will prove that $S^{\mathrm{en}}_\omega(T)$ is the intersection of all the $U_{\varphi,\varepsilon,i}$'s and all the $V_{\Phi,\varepsilon}$'s. The idea is, roughly, that the sets $V_{\Phi,\varepsilon}$ will make sure any type in the intersection will be realisable, and the sets $U_{\varphi,\varepsilon,i}$ will make sure that the type enumerates a model. The latter follows from the Tarski-Vaught Test (cf. Proposition 3.2.2). What we must show is therefore that

$$S^{\mathrm{en}}_\omega(T) = \bigcap_{\substack{n,i\in\mathbb{N}\\ \varphi\in\mathscr{F}_{\bar{x}}}} U_{\varphi,\frac{1}{n},i} \cap \bigcap_{\substack{n\in\mathbb{N}\\ \Phi\in\mathscr{F}_{\bar{x}}}} V_{\Phi,\frac{1}{n}}. \tag{3.5}$$

By the Tarski-Vaught Test (cf. Proposition 3.2.2), $S^{\mathrm{en}}_\omega(T)$ is included in the intersection on the right-hand side. Therefore, we only need to show the other inclusion.

Let p be a type from the set on the right-hand side of (3.5). Now, since the Compactness Theorem holds for finitary logic (cf. [9, Theorem 5.8]), we may find a model $\mathcal{M} \vDash T \cap \mathcal{L}_{\omega\omega}$ where p restricted to the finitary fragment $\mathcal{L}_{\omega\omega}$ is realised by some $\bar{a} \in \mathcal{M}^\omega$, i.e. for any finitary formula φ, we have $p(\varphi) = \varphi^{\mathcal{M}}(\bar{a})$. We claim that $M_0 := \mathrm{cl}(\bar{a})$ is a model of T and that $\bar{a}$ realises p in both this model and in $\mathcal{M}$. From Lemma 3.4.6 it then follows that p enumerates a model of T.

We begin by showing that $\bar{a}$ does in fact realise all of p in $\mathcal{M}$. To show this, we proceed by transfinite induction on the complexity of formulas.

If $\varphi \in \mathscr{F}_{\bar{x}}$ is atomic, then φ is from the finitary part, so by our choice of $\bar{a}$, $p(\varphi) = \varphi^{\mathcal{M}}(\bar{a})$.

Suppose now that $p(\psi) = \psi^{\mathcal{M}}(\bar{a})$ for every $\psi \in \mathscr{F}_{\bar{x}}$ of rank less than the ordinal α and let $\varphi \in \mathscr{F}_{\bar{x}}$ have rank $\alpha + 1$.

If φ is $f(\psi_1, \ldots, \psi_n)$ for a connective f, then since p is a multiplicative linear functional, we have

$$p(\varphi) = f(p(\psi_1), \ldots, p(\psi_n)).$$

Since ψ_i has rank less than α, and since $\mathscr{F}_{\bar{x}}$ is closed under subformulas, we have $p(\psi) = \psi_i^{\mathcal{M}}(\bar{a})$. Thus, $p(\varphi) = \varphi^{\mathcal{M}}(\bar{a})$ as well.

If φ is $\inf_{x_i} \psi(\bar{x})$, then since $p \in \bigcap_n U_{\psi,\frac{1}{n},i}$, we have

$$p(\inf_{x_i} \psi(\bar{x})) = \inf_{j\in\mathbb{N}} p(\psi(\bar{x}_{\restriction i=x_j})).$$

By our induction hypothesis, we have $p(\psi(\bar{x}_{\restriction i=x_j})) = \psi^{\mathcal{M}}(\bar{a}_{\restriction i=a_j})$, and so we obtain

$$p(\inf_{x_i} \psi(\bar{x})) = \inf_{j\in\mathbb{N}} \psi^{\mathcal{M}_{\bar{a}}}(\bar{a}_{\restriction i=a_j}) = \inf_{b\in\mathcal{M}} \psi^{\mathcal{M}}(\bar{a}_{\restriction i=b})$$

as required.

Finally, we assume that φ is of the form $\bigwedge \varphi_k$, where the complexity of each φ_k is less than or equal to α. Since $p \in \bigcap_n V_{\varphi,\frac{1}{n}}$, we have

$$p(\bigwedge \varphi_k) = \inf_k p(\varphi_k) = \inf_k \varphi_k^{\mathcal{M}}(\bar{a}) = \varphi^{\mathcal{M}}(\bar{a}).$$

We conclude that p is realised by $\bar{a}$ in $\mathcal{M}$.

We proceed to show that M_0 satisfies condition (ii) of the Tarski-Vaught Test, so that it is the universe of a structure. Let therefore $\varphi(x_0, \ldots, x_{m-1})$ be a formula from our fragment $\mathscr{F}$ and let $\bar{b} = (b_0, \ldots, b_{m-2})$ be a tuple of elements from M_0. It is enough to show that

$$\inf\{\varphi^{\mathcal{M}}(\bar{b}, c) : c \in \mathcal{M}\} = \inf\{\varphi^{\mathcal{M}}(\bar{b}, c) : c \in M_0\}.$$

Clearly, the left hand side is less than the right hand side. Therefore we proceed to show the other inequality. For this, we let $\varepsilon > 0$ be given. For each element b_i of the tuple $\bar{b}$ we may find, by the uniform continuity of formulas, a_{n_i} such that the following holds:

$$\begin{aligned}\inf_{c \in M_0} \varphi^{\mathcal{M}}(\bar{b}, c) &\leq \inf_{c \in M_0} \varphi^{\mathcal{M}}((a_{n_i}), c) + \varepsilon \\ &\leq \inf_{c \in \mathcal{M}} \varphi^{\mathcal{M}}((a_{n_i}), c) + 2\varepsilon \\ &\leq \inf_{c \in \mathcal{M}} \varphi^{\mathcal{M}}(\bar{b}, c) + 3\varepsilon.\end{aligned}$$

Here we used that $p \in U_{\varphi,\varepsilon,|\bar{b}|+1}$ and that $\bar{a} \vDash p$ in $\mathcal{M}$ in the second inequality. Since $\varepsilon > 0$ was arbitrary, we conclude that the equality above is satisfied. It follows that M_0 is in fact the universe of a model $\mathcal{M}_{\bar{a}}$ of T. Moreover, by the Tarski-Vaught Test, $\mathcal{M}_{\bar{a}} \preceq \mathcal{M}$.

By Lemma 3.4.6, we conclude that $p \in S_\omega^{\mathrm{en}}(T)$, which was what we wanted. □

Finally, we will prove the second lemma.

Proof of lemma 3.4.8. Let $s \in \mathbb{N}^n$ be a finite injective sequence. We begin by showing continuity of π_s. Let $\varphi(x_0, \ldots, x_{n-1}) \in \mathcal{F}_n(T)$ and $r > 0$. We want to show $\pi_s^{-1}([\varphi < r])$ is open. But this is clear as

$$\begin{aligned}\pi_s^{-1}([\varphi(x_0, \ldots, x_{n-1}) < r]) &= \{p : p(\varphi(x_{s_0}, \ldots, x_{s_{n-1}})) < r\} \\ &= [\varphi(x_{s_0}, \ldots, x_{s_{n-1}}) < r].\end{aligned}$$

Next we show surjectivity. Let $p \in S_n(T)$ and let $\bar{a} \vDash p$ in $\mathcal{M} \vDash T$. By the downwards Löwenheim-Skolem Theorem (cf. Theorem 3.2.4), we find a separable substructure $\mathcal{M}_0 \preceq \mathcal{M}$ containing all elements of $\bar{a}$. If we let $\tilde{p}$ be the type realised by a dense sequence of $\mathcal{M}_0$ with the elements of $\bar{a}$ on the coordinates given by s, we see that $\tilde{p}$ enumerates a model and that $\pi_s(\tilde{p}) = p$.

Finally, we show that the map is open. By Proposition 3.3.1, it is enough to show that $\pi_s([\varphi < r])$ is open for each formula $\varphi(x_0, \ldots, x_{m-1})$ with finitely many variables and $r > 0$. We may assume $m > \max\{s_i : i < n\}$ by simply considering φ as a formula with free variables among the variables $x_0, \ldots, x_k$ for k sufficiently large. We claim that

$$\pi_s([\varphi < r]) = \{p \in S_n(T) : p(\psi) < r\}, \tag{3.6}$$

where ψ is defined by

$$\psi(x_{s_0}, \ldots, x_{s_{n-1}}) := \inf\{\varphi(x_0, \ldots, x_{m-1}) : \{x_i : i \notin s\}\}.$$

To see why (3.6) holds, let $\pi_s(p) \in \pi_s([\varphi < r])$ and let $\bar{x}_k$ denote the first k variables of $\bar{x}$ for each $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}\pi_s(p)(\psi(\bar{x}_n)) &= p(\psi(x_{s_0}, \ldots, x_{s_{n-1}})) \\ &= p(\inf\{\varphi(\bar{x}_m) : \{x_i : i \notin s\}\}) \\ &\leq p(\varphi(x_0, \ldots, x_{m-1})) < r.\end{aligned}$$

Thus, $\pi_s(p)$ is in the set on the right-hand side of (3.6).

Conversely, if p_0 is a type such that $p_0(\psi) < r$, we take a realisation $\bar{a} \vDash p_0$. Then we find elements $\bar{b}$ witnessing the infimum $\inf \varphi < r$. Finally, we use the Löwenheim-Skolem Theorem to find a separable model containing the elements of $\bar{a}$ and $\bar{b}$, which gives us a type p enumerating a model such that $\pi_s(p) = p_0$ and such that $p(\varphi) < r$. We conclude that the map π_s is in fact open as we wanted. □

3.5 The infinitary Ryll-Nardzewski Theorem

As a corollary to Theorem 3.4.2, we obtain a version of the Ryll-Nardzewski Theorem for infinitary metric model theory (cf. [9, Theorem 12.10] for a continuous finitary version of this theorem).

Theorem 3.5.1. *Let $\mathscr{F}$ be a fragment and let T be a complete $\mathscr{F}$-theory. Then the following are equivalent:*

(i) T is $\mathscr{F}$-$\aleph_0$-categorical.
(ii) All $p \in S_n(T)$ are principal for all n.
(iii) $\tau \restriction S_n(T) = \tau_\partial \restriction S_n(T)$ for all n.

Proof. $(i) \Longrightarrow (ii)$.

We will show the contrapositive. Suppose $p \in S_n(T)$ is not principal. Then there is $r > 0$ such that the ∂-ball $B(p, r)$ around p has empty τ-interior. Therefore, $B(p, r) \cap S_n(T)$ is a ∂-open and τ-meagre subset of the realisable

types. Hence, by the Omitting Types Theorem, there is a separable model $\mathcal{M} \vDash T$ omitting $B(p,r) \cap S_n(T)$ (of course it will omit the whole ball $B(p,r)$, but we do not need that here). However, p is realisable, so there is some separable model $\mathcal{N} \vDash T$ realising p. It follows that $\mathcal{M}$ and $\mathcal{N}$ cannot be isomorphic since they don't realise the same types. Therefore, T cannot be $\aleph_0$-categorical.

$(ii) \iff (iii)$.

First we note that if the two subspace topologies agree, then every realisable p is of course principal, so (iii) implies (ii).

We proceed to show the other implication. From Lemma 3.3.18 above we know that $\tau \restriction S_n(T)$ coincides with $\tau_\partial \restriction S_n(T)$ at every realisable $p \in S_n(T)$. Thus, what we must show is that the two subspace topologies actually are the same. Let therefore p and $r > 0$ be given and suppose that $q \in B(p,r) \cap S_n(T)$. Then we find some $s > 0$ such that $B(q,s) \subseteq B(p,r)$. By the above, the two topologies coincide at q, so we find some φ such that

$$q \in [\varphi < s] \cap S_n(T) \subseteq B(q,s) \cap S_n(T) \subseteq B(p,r) \cap S_n(T).$$

We conclude that $B(p,r) \cap S_n(T)$ is $\tau \restriction S_n(T)$-open. Since τ_δ refines τ we conclude that the two subspace topologies are the same.

$(ii) \implies (i)$.

Suppose all types are principal. We will show that T is $\aleph_0$-categorical. Thus, let $\mathcal{M}$ and $\mathcal{N}$ be separable models of T. We will construct an isomorphism between the two models by a back and forth argument more or less identical to the one in the proof of Fact 1.5 in [14]

Let $\{a_i : i \in \mathbb{N}\} =: M_0 \subseteq \mathcal{M}$ and $\{b_i : i \in \mathbb{N}\} =: N_0 \subseteq \mathcal{N}$ be dense countable subsets. We will recursively define elementary maps, $f_i \colon A_i \to \mathcal{N}$ and $g_i \colon B_i \to \mathcal{M}$, defined on finite subsets, A_i and B_i, of $\mathcal{M}$ and $\mathcal{N}$ respectively, such that the following holds:

- $A_{i+1} = \{a_0, \dots, a_i\} \cup A_i \cup g_i(B_i)$.
- $B_{i+1} = \{b_0, \dots, b_i\} \cup B_i \cup f_{i+1}(A_{i+1})$.
- For all $a \in A_i$ we have $d(a, g_i f_i(a)) \leq 2^{-i}$.
- For all $b \in B_i$ we have $d(b, f_{i+1} g_i(b)) \leq 2^{-i}$.

Let us first note why this will be enough. For any $a \in A_i$, we have that

$$\begin{aligned} d(f_{i+1}(a), f_i(a)) &\leq d(f_{i+1}(a), f_{i+1} g_i f_i(a)) + d(f_{i+1} g_i f_i(a), f_i(a)) \\ &\leq d(a, g_i f_i(a)) + 2^{-i} \\ &\leq 2^{-i+1}. \end{aligned}$$

Thus, the maps f_i converge to a map $f\colon \bigcup A_i \to \mathcal{N}$. Moreover, f must be elementary, since if we let $\bar{c} = (a_{i_0}, \ldots, a_{i_{n-1}})$, we have

$$\varphi^{\mathcal{N}}(f(\bar{c})) = \lim_i \varphi^{\mathcal{N}}(f_i(\bar{c})) = \varphi^{\mathcal{M}}(\bar{c}),$$

where we have used the continuity of the language. Since f is in particular an isometry, we may extend it to a map $\bar{f}\colon \bar{A} = \mathcal{M} \to \mathcal{N}$. Again by the continuity of the language, $\bar{f}$ is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

Similarly, we find an elementary embedding $\bar{g}\colon \mathcal{N} \to \mathcal{M}$ using the g_i's. Moreover we see that $\bar{g}$ must be the inverse of $\bar{f}$, since for any $a \in M_0$, we have that

$$d(a, \bar{g}\bar{f}(a)) \leq d(a, g_{j+1}f_j(a)) + d(g_{j+1}f_j(a), \bar{g}f_j(a)) + d(\bar{g}f_j(a), \bar{g}\bar{f}(a)),$$

and we see that each of these values tend to 0 as $j \to \infty$. Thus, $\bar{g}\bar{f}$ is the identity on M_0 and therefore also on $\mathcal{M}$. Similarly, we see that $\bar{f}\bar{g}$ is the identity on $\mathcal{N}$ as well.

It remains of course to construct the sequences (f_i) and (g_i). We begin by letting $f_0 = g_0 = \emptyset$ and $A_0 = B_0 = \emptyset$. Then, since T is complete and $\mathcal{M}, \mathcal{N} \vDash T$, we have $\mathcal{M} \equiv \mathcal{N}$ so f_0 and g_0 are elementary.

Suppose now that f_i and A_i are given. Then B_i is also defined by the requirement above. Let $A_i = (a_0^i, \ldots, a_n^i)$ and $B_i = (f_i(a_0^i), \ldots, f_i(a_n^i), \ldots)$ be enumerations as tuples of these two sets. Let $p = \mathrm{tp}(B_i)$ and let $\varphi(x_0, \ldots, x_m)$ be a formula such that $[\varphi < 2^{-i}] \subseteq B(p, 2^{-i})$ (in the subspace topologies) and such that $p(\varphi) = 0$. This can be done since all types are principal, so the subspace topologies agree by $(ii) \Leftrightarrow (iii)$.

Now, $\psi = \inf_{x_{n+1},\ldots,x_m} |\varphi(\bar{x})|$ is mapped to 0 by p. Moreover, by elementarity, $\mathrm{tp}(A_i)(\psi) = 0$ as well. Hence, we may find a tuple A' of elements of $\mathcal{M}$ such that $\mathrm{tp}(A_i, A') \in B(p, 2^{-i})$. Then by Lemma 3.3.19, we can find a realisation B' of p in $\mathcal{M}$ such that $d((A_i, A'), B') \leq 2^{-i}$. We can now define g_i by mapping B_i to this tuple B'. Since $\mathrm{tp}(B_i) = \mathrm{tp}(B')$, this defines an elementary map. Moreover, since $d((A_i, A'), B') \leq 2^{-i}$, the third condition above is also satisfied.

We construct f_{i+1} from g_i in an analogous way. By our comments above, we conclude $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. □

Observe that in the argument of $(ii) \implies (i)$ we only used that all types realised by $\mathcal{M}$ and $\mathcal{N}$ are principal. Thus, we have actually shown the following proposition:

Proposition 3.5.2. *Any two separable atomic models of the complete theory T are isomorphic.*

As a corollary to the theorem above, we obtain that for $\aleph_0$-categorical T, the realisable type space is complete with respect to ∂.

Corollary 3.5.3. *Let T be a complete theory. Then the space of principal n-types is complete with respect to the type distance for each $n \in \mathbb{N}$. In particular, if T is an $\aleph_0$-categorical complete theory, then $S_n(T)$ is closed with respect to the type distance ∂ for each $n \in \mathbb{N}$.*

Proof. Let (p_k) be a sequence of principal types converging to some p. We may assume that $\partial(p_k, p_{k+1}) < 2^{-k}$ by simply passing to a subsequence.

Take $\bar{a}_1 \vDash p_1$ in some $\mathcal{M} \vDash T$. Then, by Lemma 3.3.19, we find $\bar{a}_2 \vDash p_2$ in $\mathcal{M}$ such that $d^{\mathcal{M}}(\bar{a}_1, \bar{a}_2) \leq 2^{-1}$. Since p_3 is principal, we can use the lemma again to obtain a realisation $\bar{a}_3 \vDash p_3$ with $d^{\mathcal{M}}(\bar{a}_2, \bar{a}_3) \leq 2^{-2}$.

Continuing in this fashion, we find realisations $\bar{a}_k \vDash p_k$ in $\mathcal{M}$ such that $(\bar{a}_k)_{k\in\omega}$ is Cauchy. Therefore, this sequence converges to some $\bar{a}$ in $\mathcal{M}^n$. Moreover, this $\bar{a}$ must realise p, since for any formula $\varphi(\bar{x})$, we have

$$\varphi^{\mathcal{M}}(\bar{a}_k) \to \varphi^{\mathcal{M}}(\bar{a})$$

by uniform continuity of the language. At the same time

$$\varphi^{\mathcal{M}}(\bar{a}_k) = p_k(\varphi) \to p(\varphi),$$

since convergence in ∂ implies pointwise convergence.

The 'in particular'-part of the corollary follows by Theorem 3.5.1 above, since if T is $\aleph_0$-categorical, it follows that all realisable types are principal. $\square$

We do not know if the above result holds for more general theories. Thus, we finish the chapter with the following open problem:

Question 3.5.4. Is the space of realisable types closed with respect to the type distance for any complete theory T?

Chapter 4

Polish Groups & Metric Model Theory

There is a very close connection between Polish groups and automorphism groups of separable metric structures. In fact, these are just two sides of the same coin: a group is Polish if and only if it is (isomorphic to) the automorphism group of a separable metric structure. We have already mentioned in Chapter 1 that isometry groups of Polish spaces equipped with the pointwise convergence topology are Polish. It follows that automorphism groups of complete separable metric structures are Polish as well, since they are closed subgroups of the isometry group of the metric space the structure is based on. In the first section of this chapter, we explain the other direction of this equivalence by constructing the so-called *canonical metric structure* $\mathcal{M}_G$ associated to a Polish group G.

We are going to use the canonical metric structure to study the Roelcke completion of a Polish group as a certain set of types. Recall from Chapter 1 that this completion is the completion of the Roelcke uniformity, which is the meet of the left and right uniformities. If d_L is a left-invariant compatible metric on the Polish group G, then the metric given by

$$d_\wedge(g,h) = \inf_f \max\{d_L(g,f), d_L(f^{-1},h^{-1})\}$$

induces the Roelcke uniformity. Hence, the Roelcke completion is the completion of G with respect to $d_\wedge$. Note moreover that this works for any left-invariant metric.

The connection between the Roelcke completion and type spaces of continuous logic was studied in [13]. In that paper, the authors show that there is an intimate connection between $\aleph_0$-categorical metric structures and Roelcke precompact groups, i.e. the groups with compact Roelcke completion. Using

the Ryll-Nardzewski Theorem of continuous logic, they obtain the following theorem:

Theorem 4.0.1 ([13]). *Let $G = \mathrm{Aut}(\mathcal{M})$ be the automorphism group of a separable metric structure and suppose that the space of orbit closures $\mathcal{M} /\!\!/ G$ is compact. Then the following are equivalent:*

(*i*) *G is Roelcke precompact.*
(*ii*) $\mathrm{Th}(\mathcal{M})$ *is $\aleph_0$-categorical.*

Of course, the assumption that G is the automorphism group of a separable metric structure will turn out to be no real restriction, since the canonical metric structure will allow us to view any Polish group in this way. Thus, this theorem is really a characterisation of when a Polish group in general is Roelcke precompact.

Using this result, it is easy to show that many well-known metric structures have Roelcke precompact automorphism groups. For instance, S_∞ is Roelcke precompact because it is the automorphism group of the countably infinite set (which is $\aleph_0$-categorical of course). Similarly, the automorphism group of the dense linear order $(\mathbb{Q}, <)$ is Roelcke precompact because its theory is $\aleph_0$-categorical. Likewise, the unitary group of a separable Hilbert space [62], the automorphism group of a standard probability space [33], the isometry group of the bounded Urysohn space [65] and the automorphism group of the random graph are all Roelcke precompact, which may be seen using this theorem.

The key observation made in [13] that connects type spaces and Roelcke completions, is, that for an $\aleph_0$-categorical structure it is possible to embed the Roelcke completion of the group isometrically into the type space $S_{\omega\oplus\omega}(T)$. In the present chapter, we will show that under weaker assumptions than $\aleph_0$-categoricity it is still possible to make this embedding (cf. Proposition 4.2.2 below). In particular, one can view the Roelcke completion of *any* Polish group G as a subspace of the type space $S_{\omega\oplus\omega}(\mathrm{Th}(\mathcal{M}_G))$, where $\mathcal{M}_G$ is the canonical metric structure associated to G. We will use this embedding to study a property of the Roelcke completion known as *local Roelcke precompactness* introduced in Chapter 1.

Local Roelcke precompactness is a natural generalisation of Roelcke precompactness. Instead of having a compact Roelcke completion, locally Roelcke precompact (lRpc) groups have a precompact identity neighbourhood, i.e. a neighbourhood whose closure in the Roelcke completion is compact. Due to a recent theorem of Zielinski from [68], the lRpc groups are, as expected, the groups with a locally compact Roelcke completion. Using this, and inspired by the results on the Roelcke precompact groups mentioned above, we will obtain a characterisation of the lRpc Polish groups in terms of type spaces of continuous logic (cf. Theorem 4.2.5). Of course, since we do not have compactness of the Roelcke completion, our analysis of the connection to metric

model theory must and will be much more delicate and careful than in the Roelcke precompact case.

We will also study what happens in the exact homogeneous case. Recall that a metric structure is *exact* homogeneous if whenever two tuples have the same type, we can find an automorphism of the structure mapping one tuple to the other. In general, homogeneity for metric structures usually only means *approximate* homogeneity, meaning that we can move two tuples of the same type arbitrarily close to each other with automorphisms. However, many well-known metric structures are actually exact homogeneous, and hence it is not an unreasonable assumption to impose on the structure even in the general metric setting. For instance, the two most important metric structures for us, the Urysohn metric space and diversity, are both exact homogeneous.

As mentioned in the introduction, the automorphism group of a discrete structure can be viewed as a closed subgroup of the permutation group S_∞. These groups are exactly the so-called *non-archimedean* groups, i.e. the Polish groups having a basis at the identity consisting of clopen subgroups. It follows that the automorphism group of a discrete structure is locally Roelcke precompact if and only if it contains a Roelcke precompact open *subgroup*. However, some caution should be taken here, since in general it is not true that a subgroup which is Roelcke precompact as a *subset* is also Roelcke precompact as a group. On the other hand, if the subgroup is open, it follows directly from the definition that the subgroup is Roelcke precompact as a group if and only if it is Roelcke precompact as a subset.

As usual, things are more complicated when we pass to the general metric setting. However, if the structure is exact homogeneous, we do get one direction of the above equivalence for locally Roelcke precompact groups: if the automorphism group of an exact homogeneous structure is locally Roelcke precompact, then there is a finite tuple of elements from the structure such that the stabiliser of this tuple is Roelcke precompact. Furthermore, any stabiliser subgroup is a locally Roelcke precompact group. We will investigate the exact homogeneous case in the last section of the chapter.

4.1 The canonical metric structure

In this section, we will go through the construction of the canonical metric structure associated to a Polish group. Moreover, we will observe that this structure is in fact atomic, which we believe to be a new result.

The construction goes as follows: given a Polish group G, denote its left completion by $\widehat{G}_L$ with left-invariant metric d_L. G acts naturally on $\widehat{G}_L$ by

$$g \cdot x = g \cdot \lim_n g_n := \lim_n g g_n$$

for $g \in G$, $x \in \widehat{G}_L$ and a sequence (g_n) in G converging to x with respect to d_L. This is well-defined by left-invariance of d_L. Moreover, this action is

continuous, since d_L is compatible with the group topology. Note that this implies that G acts continuously on any countable power of $\widehat{G}_L$ simply by the diagonal action. We want to use $\widehat{G}_L$ as the universe of a metric structure $\mathcal{M}_G$ such that G is the automorphism group of $\mathcal{M}_G$. In order to do that, for each $n \in \mathbb{N}$ and n-tuple $\bar{a}$ of elements from some dense countable subset of $\widehat{G}_L$, we let $P_{\bar{a}}$ be an n-ary predicate. Since the metric d_L may be assumed to be bounded by 1, we let $[0,1]$ be the bound for each $P_{\bar{a}}$. Moreover, the modulus of continuity will just be the identity so that the predicates become 1-Lipschitz maps when interpreted. We interpret $P_{\bar{a}}$ in $\mathcal{M}_G$ by setting

$$P_{\bar{a}}^{\mathcal{M}_G}(\bar{b}) := d_L(\overline{G \cdot \bar{a}}, \overline{G \cdot \bar{b}}),$$

where $d_L(\overline{G \cdot \bar{a}}, \overline{G \cdot \bar{b}})$ denotes the infimum

$$\inf\{d_L(\bar{a}', \bar{b}') : \bar{a}' \in \overline{G \cdot \bar{a}}, \bar{b}' \in \overline{G \cdot \bar{b}}\}.$$

Then it is easy to verify that each $P_{\bar{a}}^{\mathcal{M}_G}$ respects its modulus and bound, and thus $\mathcal{M}_G = (\widehat{G}_L, \{P_{\bar{a}}^{\mathcal{M}_G}\})$ is a metric structure. We now have the following proposition, originally due to Melleray (cf. [50, Theorem 6]), saying that $\mathcal{M}_G$ is ultrahomogeneous and has G as its automorphism group. Moreover, $\mathcal{M}_G$ is atomic which is, as mentioned, a new result. Therefore we only include the proof of this fact.

Proposition 4.1.1 ([50, Theorem 6])**.** *$\mathcal{M}_G$ is an approximately ultrahomogeneous separable atomic metric structure with* $\mathrm{Aut}(\mathcal{M}_G) = G$.

Proof. As mentioned, we will only show atomicity, since that is the only new result. Let therefore $p \in S_n(T)$ be a type realised in $\mathcal{M}_G$ and let $r > 0$ be given. Suppose first that there is a predicate $P_{\bar{a}}$ for a tuple $\bar{a}$ realising p. We then claim that

$$[P_{\bar{a}} < r] \cap S_n^{\mathcal{M}_G}(T) \subseteq B(p, r),$$

which by Lemma 2.5.7 is enough to show principality of p. Suppose therefore $q \in [P_{\bar{a}} < r]$ is a type realised by $\mathcal{M}_G$, say by $\bar{b}$. Then $d_L(\overline{G \cdot \bar{a}}, \overline{G \cdot \bar{b}}) < r$, so we find $g, h \in G$ such that $d_L(g\bar{a}, h\bar{b}) < r$ as well. Since $g\bar{a} \vDash p$ and $h\bar{b} \vDash q$, it follows that $\partial(p, q) \leq d_L(g\bar{a}, h\bar{b}) < r$ as well. Hence, $q \in B(p, r)$ as we wanted.

Suppose now that p is realised by an arbitrary tuple $\bar{a}$ of elements of $\widehat{G}_L$. By density, we find a predicate $P_{\bar{a}_0}$ for a tuple $\bar{a}_0$ such that $d_L(\bar{a}, \bar{a}_0) < r$. It follows that $\mathrm{tp}(\bar{a}_0) = q \in B(p, r)$, and hence that there is some $r' > 0$ such that $B(q, r') \subseteq B(p, r)$. Since q is principal by the above, we find a formula φ and an $s > 0$ such that $[\varphi < s] \subseteq B(q, r')$. Hence, $B(p, r)$ has non-empty τ-interior and therefore, by Lemma 2.5.6, we get that p is principal. We conclude that $\mathcal{M}_G$ is atomic. □

The structure $\mathcal{M}_G$ will prove to be extremely useful to us. In particular, the atomicity of $\mathcal{M}_G$ is crucial for the proof of Theorem 4.2.5 below, because it implies knowledge of the type distance.

4.2 The Roelcke completion & model theory

In this section, we will study the Roelcke completion $\widehat{G}_{L\wedge R}$ of a Polish group G as a set of types. By what we have done in the previous section, we can always assume that G is given as the automorphism group of a separable atomic metric structure, namely the structure $\mathcal{M}_G$. This construction encourages us to study the connection between $\widehat{G}_{L\wedge R}$ and metric model theory further. The connection has, as mentioned, already been studied in [13], where the authors show that for $\aleph_0$-categorical structures, one can embed $\widehat{G}_{L\wedge R}$ into $S_{\omega\oplus\omega}(\mathrm{Th}(\mathcal{M}))$, where $\mathcal{M}$ is a structure with $G = \mathrm{Aut}(\mathcal{M})$. Below we show that for a homogeneous model which knows $\partial_{\omega\oplus\omega}$ it is still possible to make this embedding. Since $\mathcal{M}_G$ is atomic, it does indeed know the type distance, and thus this embedding allows us to give a characterisation of local Roelcke precompactness in Theorem 4.2.5. We begin with a simple but useful observation connecting Roelcke precompact subsets and type spaces.

Proposition 4.2.1. *Let $\mathcal{M}$ be a separable approximately homogeneous metric structure that knows ∂_n for each $n \in \mathbb{N}$. Then $A \subseteq \mathrm{Aut}(\mathcal{M})$ is Roelcke precompact if and only if for any $n \in \mathbb{N}$ and any finite n-tuple $\bar{b} \in \mathcal{M}^n$, the set $\{\mathrm{tp}(\bar{b}, g\bar{b}) : g \in A\} \subseteq S_{n\oplus n}(\mathrm{Th}(\mathcal{M}))$ is ∂-totally bounded.*

Proof. Suppose that A is Roelcke precompact and let $\varepsilon > 0$ and $\bar{b} \in \mathcal{M}^n$ be given. Set $U := \{g \in \mathrm{Aut}(\mathcal{M}) : d^{\mathcal{M}}(\bar{b}, g\bar{b}) < \varepsilon\}$. Then U is an open neighbourhood of the identity, so we can find a finite set $F \subseteq \mathrm{Aut}(\mathcal{M})$ such that $A \subseteq UFU$. We claim that for any $g \in A$, the type $\mathrm{tp}(\bar{b}, g\bar{b})$ is within ε of some type in the set $\{\mathrm{tp}(\bar{b}, f\bar{b}) : f \in F\}$. To see this, let $g \in A$. Then $g = ufu'$ for some $u, u' \in U$ and $f \in F$. Moreover, we have that

$$\mathrm{tp}(\bar{b}, ufu'\bar{b}) = \mathrm{tp}(u^{-1}\bar{b}, fu'\bar{b}).$$

Therefore, we have

$$\begin{aligned}\partial(\mathrm{tp}(\bar{b}, ufu'\bar{b}), \mathrm{tp}(\bar{b}, f\bar{b})) &\leq d^{\mathcal{M}}((u^{-1}\bar{b}, fu'\bar{b}), (\bar{b}, f\bar{b})) \\ &= \max\{d^{\mathcal{M}}(u^{-1}\bar{b}, \bar{b}), d^{\mathcal{M}}(fu'\bar{b}, f\bar{b})\} \\ &< \varepsilon\end{aligned}$$

as we claimed. Thus, $\{\mathrm{tp}(\bar{b}, g\bar{b}) : g \in A\}$ is totally bounded.

Conversely, suppose $\{\mathrm{tp}(\bar{b}, g\bar{b}) : g \in A\}$ is totally bounded for any finite tuple $\bar{b}$ and let U be a neighbourhood of the identity. Without loss of generality, let us assume that U is of the form $U_{\bar{b},r} := \{g : d(g\bar{b}, \bar{b}) < r\}$ for some finite tuple $\bar{b}$ and $r > 0$. Let $\{p_i\}$ be a finite set of types such that the balls of radius r centered at p_i cover $\{\mathrm{tp}(\bar{b}, g\bar{b}) : g \in A\}$. We can assume without loss of generality that $p_i = \mathrm{tp}(\bar{b}, f_i\bar{b})$ for some $f_i \in A$. We now claim that $A \subseteq U\{f_i\}U$. To see this, suppose $g \in A$ and find some f_i such that $\partial(\mathrm{tp}(\bar{b}, g\bar{b}), \mathrm{tp}(\bar{b}, f_i\bar{b})) < r$. By homogeneity and knowledge of ∂, we find some $h \in \mathrm{Aut}(\mathcal{M})$ such that

$d(h \cdot (\bar{b}, f_i\bar{b}), (\bar{b}, g\bar{b})) < r$. It follows that $h \in U$. Moreover, $f_i^{-1}h^{-1}g \in U$ as well. Thus, $g \in hf_iU \subseteq U\{f_i\}U$ as claimed. $\square$

Next, we will explain how to embed the Roelcke completion of a Polish group isometrically into the type space $S_{\omega\oplus\omega}(T)$. To do this, it will be convenient for us to fix, once and for all, a metric inducing the Roelcke uniformity. Since we are only dealing with automorphism groups of separable metric structures, suppose that $G = \mathrm{Aut}(\mathcal{M})$ is such a group and fix a dense sequence $\xi \in \mathcal{M}^\omega$, i.e. a sequence whose elements form a dense subset of $\mathcal{M}$. We can then define a left-invariant metric d_L on G by

$$d_L(g,h) = d_\omega^{\mathcal{M}}(g \cdot \xi, h \cdot \xi) = \sum_{i\in\omega} 2^{-i-1} d^{\mathcal{M}}(g\xi_i, h\xi_i),$$

where G acts on $\mathcal{M}^\omega$ via the diagonal action. It is easy to check that d_L is in fact left-invariant. Thus, we obtain a metric $d_{L\wedge R}$ inducing the Roelcke completion by setting

$$d_{L\wedge R}(g,h) = \inf_{f\in G} \max\{d_L(g,f), d_L(f^{-1},h^{-1})\}.$$

We now have:

Proposition 4.2.2. *Suppose $\mathcal{M}$ is a separable approximately homogeneous metric structure that knows the type distance $\partial_{\omega\oplus\omega}$. Let $G := \mathrm{Aut}(\mathcal{M})$ and let $\xi \in \mathcal{M}^\omega$ be a dense sequence. Then the map*

$$\iota\colon g \mapsto \mathrm{tp}(\xi, g\xi)$$

embeds $(G, d_{L\wedge R})$ isometrically into the type space $(S_{\omega\oplus\omega}(T), \partial_{\omega\oplus\omega})$. It follows that we can identify $\widehat{G}_{L\wedge R}$ with the $\partial_{\omega\oplus\omega}$-closure of the image of this embedding.

Proof. We have by definition that

$$d_{L\wedge R}(g,h) = \inf_{f\in G} \max\{d_L(g,f), d_L(f^{-1},h^{-1})\},$$

where $d_L(g,h) := d_\omega^{\mathcal{M}}(g\cdot\xi, h\cdot\xi)$. On the other hand, we have

$$\begin{aligned}
\partial_{\omega\oplus\omega}(\iota(g),\iota(h)) &= \partial(\mathrm{tp}(\xi,g\xi), \mathrm{tp}(\xi,h\xi)) \\
&= \inf_{f\in G} d_{\omega\oplus\omega}^{\mathcal{M}}(f^{-1}(\xi,g\xi), (h^{-1}\xi,\xi)) \\
&= \inf_{f\in G} \max\{d_\omega^{\mathcal{M}}(f^{-1}\xi, h^{-1}\xi), d_\omega^{\mathcal{M}}(g\xi, f\xi)\} \\
&= d_{L\wedge R}(g,h),
\end{aligned}$$

where we used homogeneity and knowledge of $\partial_{\omega\oplus\omega}$ in the second equality. $\square$

We will therefore from now on identify $\widehat{G}_{L\wedge R}$ with the ∂-closure of the image of the map ι from the lemma above whenever we are working in a homogeneous model that knows ∂. However, from time to time it will be important for us to be able to distinguish the Roelcke completion and the closure of the image of ι, since we may work in models that do not a priori know the type distance. Therefore, we will denote the ∂-closure of the image of ι by $S(G)$, but the reader is encouraged to think of $S(G)$ simply as $\widehat{G}_{L\wedge R}$. Since the construction of $S(G)$ depends on the dense sequence ξ, we will from now on tacitly assume that we have fixed such a sequence in any separable structure under consideration.

Since $S(G)$ is a subset of a type space, it inherits the logic topology τ in addition to the τ_∂-topology. By $\overline{S(G)}$, we will denote the τ-closure of $S(G)$. Note that $\overline{S(G)}$ is compact, being a closed subset of the compact space $S_{\omega\oplus\omega}(T)$. We wish to exploit the compactness of $(\overline{S(G)},\tau)$ to show local compactness of $(S(G),\partial)$, and hence that G is locally Roelcke precompact. Thus, if we can make sure that $(S(G),\tau)$ is homeomorphic to $(S(G),\partial)$, it follows that G is locally Roelcke precompact if and only if $S(G)$ is τ-open in $\overline{S(G)}$. Fortunately for us, it turns out that $(S(G),\partial)$ and $(S(G),\tau)$ are homeomorphic whenever G is the automorphism group of a separable atomic structure. Moreover, as we have seen above, we are in the fortunate situation that any Polish group *is* the automorphism group of such a structure. Thus, we obtain our characterisation of locally Roelcke precompact groups in Theorem 4.2.5 below.

However, before we can prove Theorem 4.2.5, we will need to do some work. The first thing we need is the following description of $S(G)$. Recall from Chapter 2 that π^ω and π_ω denote the projection maps from $S_{\omega\oplus\omega}(T)$ onto, respectively, the first and second ω variables.

Proposition 4.2.3. *Let $\mathcal{M}$ be a separable approximately homogeneous metric structure that knows $\partial_{\omega\oplus\omega}$ and let G denote* $\mathrm{Aut}(\mathcal{M})$. *Then*

$$S(G) = \{p \in \overline{S(G)} : \mathcal{M} \text{ realises } p\}.$$

Proof. Suppose first that $p \in \overline{S(G)}$ and that $\mathcal{M}$ realises p. Then, by continuity of the projections, we have that $\pi^\omega(p) = \pi_\omega(p) = \mathrm{tp}(\xi)$, where $\xi \in \mathcal{M}^\omega$ is our fixed dense sequence. Furthermore, we find a realisation $(\bar{a},\bar{b})$ of p in $\mathcal{M}$. We must show that we can approximate p in ∂ with types of the form $\mathrm{tp}(g\xi, h\xi)$ for $g,h \in G$. Let therefore $\varepsilon > 0$ be given, and find $n \in \mathbb{N}$ such that $\sum_{i\geq n} 2^{-i} < \varepsilon/2$. Since $\mathrm{tp}(\bar{a}\restriction n) = \mathrm{tp}(\bar{b}\restriction n) = \mathrm{tp}(\xi\restriction n)$, we get by homogeneity of $\mathcal{M}$ two automorphisms g and h such that

$$d^{\mathcal{M}}(g\xi\restriction n, a\restriction n), d^{\mathcal{M}}(h\xi\restriction n, \bar{b}\restriction n) < \varepsilon/2.$$

It follows that $\partial(\mathrm{tp}(g\xi, h\xi), p) < \varepsilon$ as we wanted. Thus, $p \in S(G)$.

Conversely, if $p \in S(G)$, we must show that we can find a realisation in $\mathcal{M}$. For this, we need $\mathcal{M}$ to know $\partial_{\omega\oplus\omega}$. First, we find a sequence (p_n) of types of

the form $\mathrm{tp}(\xi, g_n\xi)$ converging in ∂ to p. By passing to a subsequence, we may suppose that $\partial(p_n, p_{n+1}) < 2^{-n}$. Since $\mathcal{M}$ knows ∂, we can find realisations $(\bar{a}_1, \bar{b}_1) \vDash p_1$ and $(\bar{a}_2, \bar{b}_2) \vDash p_2$ such that

$$d^{\mathcal{M}}((\bar{a}_1, \bar{b}_1), (\bar{a}_2, \bar{b}_2)) < 2^{-1}.$$

Similarly, we can find $(\bar{a}_2', \bar{b}_2') \vDash p_2$ and $(\bar{a}_3', \bar{b}_3') \vDash p_3$ such that

$$d^{\mathcal{M}}((\bar{a}_2', \bar{b}_2'), (\bar{a}_3', \bar{b}_3')) < 2^{-2}.$$

By homogeneity, we can find some $g \in G$ such that

$$d^{\mathcal{M}}((\bar{a}_2, \bar{b}_2), g(\bar{a}_3', \bar{b}_3')) < 2^{-2}$$

as well. In this way, we construct a Cauchy sequence $((\bar{a}_n, \bar{b}_n))_n$, where $(\bar{a}_n, \bar{b}_n)$ realises p_n. By completeness, this sequence must converge to some realisation $(\bar{a}, \bar{b})$ of p, which was what we wanted. □

For separable atomic structures, we have the following convenient description of $\overline{S(G)}$.

Proposition 4.2.4. *Let $\mathcal{M}$ be a separable atomic structure with theory T and let $\xi \in \mathcal{M}^\omega$ be a dense sequence. Then*

$$\overline{S(G)} = \{p \in S_{\omega\oplus\omega}(T) : \pi_\omega(p) = \pi^\omega(p) = \mathrm{tp}(\xi)\}. \tag{4.1}$$

Proof. Denote the set on the right-hand side of (4.1) above by A. By continuity of the projections, A is τ-closed. Moreover, G (or rather $\iota(G)$) is contained in A. Hence, it is enough to show that G is τ-dense in A. Let therefore $p \in A$ and suppose $p \in [\varphi < r]$ for a formula $\varphi(\bar{x}, \bar{y})$ with finitely many free variables $\bar{x}$ and $\bar{y}$ and $r > 0$. We can assume that $p(\varphi) = 0$. Let δ be the modulus of continuity for φ. By atomicity, we find a formula $\psi(\bar{y})$ such that $\psi^{\mathcal{M}}(\xi \restriction |\bar{y}|) < r/2$ and

$$[\psi < r/2] \subseteq B(\mathrm{tp}(\xi \restriction |\bar{y}|), \delta(r/2)).$$

Then

$$p(\inf_{\bar{z}} \max\{\varphi(\bar{x}, \bar{z}), \psi(\bar{z})\}) < r/2.$$

Since $p \in A$, we have that

$$p(\inf_{\bar{z}} \max\{\varphi(\bar{x}, \bar{z}), \psi(\bar{z})\}) = \inf_{\bar{c} \in \mathcal{M}} \max\{\varphi^{\mathcal{M}}(\xi \restriction |\bar{x}|, \bar{c}), \psi^{\mathcal{M}}(\bar{c})\}.$$

Therefore, we can find some tuple $\bar{b}$ of elements from $\mathcal{M}$ such that

$$\max\{\varphi^{\mathcal{M}}(\xi \restriction |\bar{x}|, \bar{b}), \psi^{\mathcal{M}}(\bar{b})\} < r/2.$$

It follows that $\partial(\mathrm{tp}(\xi \restriction |\bar{y}|), \mathrm{tp}(\bar{b})) < \delta(r/2)$. By atomicity, $\mathcal{M}$ is homogeneous and knows ∂. Hence, we find $g \in \mathrm{Aut}(\mathcal{M})$ such that $d^{\mathcal{M}}(\bar{b}, g(\xi \restriction |\bar{y}|)) < \delta(r/2)$. It follows that

$$|\varphi^{\mathcal{M}}(\xi \restriction |\bar{x}|, g(\xi \restriction |\bar{y}|)) - \varphi^{\mathcal{M}}(\xi \restriction |\bar{x}|, \bar{b})| < r/2,$$

which implies that

$$\varphi^{\mathcal{M}}(\xi \restriction |\bar{x}|, g(\xi \restriction \bar{y})) < r/2 + \varphi^{\mathcal{M}}(\xi \restriction |\bar{x}|, \bar{b}) < r.$$

Hence, $\mathrm{tp}(\xi, g\xi) \in [\varphi < r]$. We conclude that G is τ-dense in A. □

We can now state and prove our main result of this chapter.

Theorem 4.2.5. *Let $G = \mathrm{Aut}(\mathcal{M})$ be the automorphism group of a separable atomic metric structure $\mathcal{M}$. Then $\widehat{G}_{L\wedge R}$ is homeomorphic to $(S(G), \tau)$.*

Moreover, the following are equivalent for any Polish group G:

(*i*) *G is locally Roelcke precompact.*
(*ii*) *There is a separable atomic metric structure $\mathcal{M}$ with $G = \mathrm{Aut}(\mathcal{M})$ and such that $(S(G), \tau)$ is open in $(\overline{S(G)}, \tau)$.*

Proof. Let $G = \mathrm{Aut}(\mathcal{M})$ be given. Since $\mathcal{M}$ is atomic, we have that it knows $\partial_{\omega\oplus\omega}$ (cf. Lemma 2.5.12). Therefore, from Proposition 4.2.2, it follows that we can identify the Roelcke completion of G with $(S(G), \partial_{\omega\oplus\omega})$. By Proposition 4.2.3, we know that $S(G) \subseteq S^{\mathcal{M}}_{\omega\oplus\omega}(T)$. Furthermore, it follows from Lemma 2.5.11 that all types of $S(G)$ are principal, so τ_∂ and τ agree on $S(G)$. Hence, $\widehat{G}_{L\wedge R}$ is homeomorphic to $(S(G), \tau)$.

For the 'moreover'-part, suppose that G is any locally Roelcke precompact Polish group. Let $\mathcal{M}_G$ be the canonical metric structure associated to G. Then $\mathcal{M}_G$ is a separable atomic metric structure such that $G = \mathrm{Aut}(\mathcal{M}_G)$. By Zielinski's theorem (cf. Theorem 1.4.3), it follows that $\widehat{G}_{L\wedge R}$, and hence also $(S(G), \partial)$, are both locally compact. Since $(S(G), \partial)$ is homeomorphic to $(S(G), \tau)$, it follows that $(S(G), \tau)$ is locally compact. Since $(\overline{S(G)}, \tau)$ is compact, it follows that $S(G)$ is τ-open in $\overline{S(G)}$.

For the other direction, assume we have a separable atomic metric structure $\mathcal{M}$ such that $G = \mathrm{Aut}(\mathcal{M})$ with $(S(G), \tau)$ open in $(\overline{S(G)}, \tau)$. Then $(S(G), \tau)$ is locally compact and homeomorphic to $(S(G), \partial)$ which may be identified with $\widehat{G}_{L\wedge R}$. Thus, $(S(G), \partial)$ is locally compact, and therefore by Zielinski's theorem, G is locally Roelcke precompact. □

4.3 THE EXACT HOMOGENEOUS CASE

The automorphism group of a countable homogeneous (classical) first-order structure is locally Roelcke precompact if and only if it contains a Roelcke precompact open subgroup. This is because, as mentioned above, automorphism

groups of such structures are exactly the closed subgroups of the permutation group of $\mathbb{N}$, S_∞. In the general setting of metric model theory, one of the these implications still holds for *exact* homogeneous structures as the theorem below explains. In particular, this applies to the Urysohn metric space and the Urysohn diversity, since both of these structures are exact homogeneous. We will study these structures further in Chapters 5 and 6 below,

Theorem 4.3.1. *Let $\mathcal{M}$ be an exact homogeneous separable atomic structure with theory T and automorphism group G. Let $\bar{c}$ be any finite tuple of elements of $\mathcal{M}$ and let $G_{\bar{c}}$ denote the stabiliser of $\bar{c}$. Then the Roelcke completion of $G_{\bar{c}}$ is homeomorphic to the closure of $G_{\bar{c}}$ in the Roelcke completion of G.*

In particular, if G is locally Roelcke precompact, then $G_{\bar{c}}$ is locally Roelcke precompact for all $\bar{c}$. Moreover, there is a tuple $\bar{a}$ such that $G_{\bar{a}}$ is Roelcke precompact.

Proof. The proof uses the correspondence with type spaces developed above.

Let ξ be a dense sequence in $\mathcal{M}$ and let $\bar{c} \in \mathcal{M}^n$ be any finite tuple. Then $\bar{c}\xi$, i.e. $\bar{c}$ concatenated with ξ, is also a dense sequence. Since $\mathcal{M}$ is atomic, it knows $\partial_{\omega\oplus\omega}$. Therefore, by Proposition 4.2.2, we can view the Roelcke completion of G as the ∂-closure of the set

$$\{\mathrm{tp}(\bar{c}\xi, g(\bar{c}\xi)) : g \in G\}.$$

As before, we denote this closure by $S(G)$. Since we will need to know exactly which variables we are using for our types, we will consider $S(G)$ as a subset of the space $S_{\bar{x}\oplus\bar{y}}(T)$, i.e. the types using the two infinite tuples of free variables $\bar{x} = (x_0, \ldots)$ and $\bar{y} = (y_0, \ldots)$. We can now consider the closure of $G_{\bar{c}}$ in the Roelcke completion of G as the ∂-closure of the set

$$\{\mathrm{tp}(\bar{c}\xi, \bar{c}g(\xi)) : g \in G_{\bar{c}}\}.$$

Denote this closure by $S(G_{\bar{c}})$. By the same argument as in Proposition 4.2.3 and using exact homogeneity, it follows that

$$S(G_{\bar{c}}) = \{p \in S(G) : \exists \bar{a}, \bar{b} \in \mathcal{M}^\omega (\bar{c}\bar{a}, \bar{c}\bar{b}) \vDash p\}.$$

Let now $\mathcal{M}_{\bar{c}}$ be the structure $\mathcal{M}$ with the elements of $\bar{c}$ as constant symbols and with automorphism group $G_{\bar{c}}$ and theory $T_{\bar{c}}$. By Proposition 2.5.14, the structure $\mathcal{M}_{\bar{c}}$ is atomic and exact homogeneous. Moreover, ξ is a dense sequence in $\mathcal{M}_{\bar{c}}$, so we can identify the Roelcke completion of $G_{\bar{c}}$ with the ∂-closure of the following set of types over $\bar{c}$:

$$\{\mathrm{tp}(\xi, g\xi \mid \bar{c}) : g \in G_{\bar{c}}\}.$$

Denote this closure by $S_{\bar{c}}(G_{\bar{c}})$. To make our notation easier, we will view this set as a subset of the type space $S_{\bar{x}'\oplus\bar{y}'}(T_{\bar{c}})$, where we use the infinite tuples of variables $\bar{x}' = (x_n, \ldots)$ and $\bar{y}' = (y_n, \ldots)$.

The claim is now that $(S_{\bar{c}}(G_{\bar{c}}), \partial)$ is homeomorphic to $(S(G_{\bar{c}}), \partial)$. To show this, we first note that any type $p \in S_{\bar{c}}(G_{\bar{c}})$ is realised in $\mathcal{M}_{\bar{c}}$ by Proposition 4.2.3. Therefore, p is of the form $\mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c})$ for two sequences $\bar{a}, \bar{b} \in \mathcal{M}^\omega$ with $\mathrm{tp}(\bar{a} \mid \bar{c}) = \mathrm{tp}(\bar{b} \mid \bar{c}) = \mathrm{tp}(\xi \mid \bar{c})$.

Define a map $\lambda \colon S_{\bar{c}}(G_{\bar{c}}) \to S_{\bar{x} \oplus \bar{y}}(T)$ by

$$\lambda(\mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c})) = \mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b}).$$

First of all, we should argue why this map is well-defined. Suppose therefore $(\bar{a}, \bar{b})$ and $(\bar{a}', \bar{b}')$ realise the type $p \in S_{\bar{c}}(G_{\bar{c}})$ over $\bar{c}$. We must show that $\mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b}) = \mathrm{tp}(\bar{c}\bar{a}', \bar{c}\bar{b}')$. Let therefore $\varphi(\bar{x}, \bar{y})$ be a formula with free variables among $\bar{x}$ and $\bar{y}$. By $\varphi(\bar{c}\bar{x}', \bar{c}\bar{y}')$, we denote the formula with parameters from $\bar{c}$, where each occurrence of x_i and y_i in $\varphi(\bar{x}, \bar{y})$ is replaced with c_i for $i < n$. Then

$$\begin{aligned}
\mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b})(\varphi(\bar{x}, \bar{y})) = \varphi^{\mathcal{M}}(\bar{c}\bar{a}, \bar{c}\bar{b}) \quad &= \varphi^{\mathcal{M}_{\bar{c}}}(\bar{c}^{\mathcal{M}_{\bar{c}}}\bar{a}, \bar{c}^{\mathcal{M}_{\bar{c}}}\bar{b}) \\
= p(\varphi(\bar{c}\bar{x}', \bar{c}\bar{y}')) &= \varphi^{\mathcal{M}_{\bar{c}}}(\bar{c}^{\mathcal{M}_{\bar{c}}}\bar{a}', \bar{c}^{\mathcal{M}_{\bar{c}}}\bar{b}') \\
= \varphi^{\mathcal{M}}(\bar{c}\bar{a}', \bar{c}\bar{b}') \quad &= \mathrm{tp}(\bar{c}\bar{a}', \bar{c}\bar{b}')(\varphi(\bar{x}, \bar{y})).
\end{aligned}$$

It follows that λ is well-defined.

We now claim that λ is a ∂-∂-homeomorphism with image equal to $S(G_{\bar{c}})$. First we argue why the image of λ must be $S(G_{\bar{c}})$. Let therefore $p \in S_{\bar{c}}(G_{\bar{c}})$, and suppose $p = \mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c})$. Then $\lambda(p) = \mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b})$, and we must therefore argue why

$$\mathrm{tp}(\bar{c}\bar{a}) = \mathrm{tp}(\bar{c}\bar{b}) = \mathrm{tp}(\bar{c}\xi)$$

so that $\lambda(p) \in S(G)$ (cf. Propositions 4.2.3 and 4.2.4). Let $\varphi(\bar{x})$ be a formula. Replace each occurrence of x_i in φ by c_i for $i < n$. Denote the resulting formula by $\varphi(\bar{c}\bar{x}')$. We know that $\mathrm{tp}(\bar{a} \mid \bar{c}) = \mathrm{tp}(\xi \mid \bar{c})$, since $p \in S_{\bar{c}}(G_{\bar{c}})$. Hence,

$$\begin{aligned}
\mathrm{tp}(\bar{c}\bar{a})(\varphi(\bar{x})) &= \mathrm{tp}(\bar{a} \mid \bar{c})(\varphi(\bar{c}\bar{x}')) \\
&= \mathrm{tp}(\xi \mid \bar{c})(\varphi(\bar{c}\bar{x}')) \\
&= \mathrm{tp}(\bar{c}\xi)(\varphi(\bar{x})).
\end{aligned}$$

A similar calculation shows that $\mathrm{tp}(\bar{c}\bar{b}) = \mathrm{tp}(\bar{c}\xi)$. Thus, $\lambda(p) \in S(G_{\bar{c}})$.

To show the other inclusion, suppose $q \in S(G_{\bar{c}})$. Then there are sequences $\bar{a}$ and $\bar{b}$ in $\mathcal{M}$ such that $q = \mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b})$ with $\mathrm{tp}(\bar{c}\bar{a}) = \mathrm{tp}(\bar{c}\bar{b}) = \mathrm{tp}(\bar{c}\xi)$. If we let p be the type $\mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c})$, it is easy to check that $p \in S_{\bar{c}}(G_{\bar{c}})$, and clearly $\lambda(p) = q$. We conclude that $\mathrm{Im}(\lambda) = S(G_{\bar{c}})$.

Next we show that λ is ∂-∂-continuous. In fact, λ is a ∂-∂-1-Lipschitz map. To see this, let $p, q \in S_{\bar{c}}(G_{\bar{c}})$ be given and let $\varepsilon > 0$. By Proposition 4.2.3, it follows that both types are realised in $\mathcal{M}_{\bar{c}}$. By Proposition 2.5.14, $\mathcal{M}_{\bar{c}}$ is atomic. Therefore, by Lemma 2.5.12, we can find two realisations, $(\bar{a}_p, \bar{b}_p) \vDash p$

and $(\bar{a}_q, \bar{b}_q) \vDash q$, such that $d((\bar{a}_p, \bar{b}_p), (\bar{a}_q, \bar{b}_q)) \leq \partial(p,q) + \varepsilon$. By definition of λ, we have that $(\bar{c}\bar{a}_p, \bar{c}\bar{b}_p) \vDash \lambda(p)$ and $(\bar{c}\bar{a}_q, \bar{c}\bar{b}_q) \vDash \lambda(q)$. It follows that

$$\begin{aligned}\partial(\lambda(p), \lambda(q)) &\leq d^{\mathcal{M}}((\bar{c}\bar{a}_p, \bar{c}\bar{b}_p), (\bar{c}\bar{a}_q, \bar{c}\bar{b}_q)) \\ &\leq d^{\mathcal{M}_{\bar{c}}}((\bar{a}_p, \bar{b}_p), (\bar{a}_q, \bar{b}_q)) \\ &\leq \partial(p,q) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\partial(\lambda(p), \lambda(q)) \leq \partial(p,q)$ as we claimed. In particular, λ is ∂-∂-continuous.

We proceed to show that λ is injective with continuous inverse. For this, let $\gamma\colon S(G_{\bar{c}}) \to S_{\bar{x}'\oplus\bar{y}'}(T_{\bar{c}})$ be the map given by

$$\gamma(\mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b})) = \mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c}).$$

First of all, we must argue why this is well-defined. Let therefore $(\bar{c}\bar{a}, \bar{c}\bar{b})$ and $(\bar{c}\bar{a}', \bar{c}\bar{b}')$ both realise $p \in S(G_{\bar{c}})$, and let $\varphi(\bar{x}', \bar{y}')$ be a formula with parameters from $\bar{c}$. Let φ' be the formula φ but where all occurrences of the constant symbol c_i is replaced by the free variable x_i, for $i < n$. Then $\varphi'(\bar{x}, \bar{y}')$ is an formula with free variables among $(\bar{x}, \bar{y}')$. In particular, the free variables are among $(\bar{x}, \bar{y})$, and thus we can apply p to φ' and obtain:

$$\begin{aligned}\mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c})(\varphi(\bar{x}', \bar{y}')) &= \varphi^{\mathcal{M}_{\bar{c}}}(\bar{a}, \bar{b}) &&= \varphi'^{\mathcal{M}}(\bar{c}\bar{a}, \bar{b}) \\ &= p(\varphi'(\bar{x}, \bar{y}')) &&= \varphi'^{\mathcal{M}}(\bar{c}\bar{a}', \bar{b}') \\ &= \varphi^{\mathcal{M}_{\bar{c}}}(\bar{a}', \bar{b}') &&= \mathrm{tp}(\bar{a}', \bar{b}' \mid \bar{c})(\varphi(\bar{x}', \bar{y}')).\end{aligned}$$

We conclude that $\mathrm{tp}(\bar{a}, \bar{b} \mid \bar{c}) = \mathrm{tp}(\bar{a}', \bar{b}' \mid \bar{c})$, rendering γ well-defined. Furthermore, clearly $\gamma(\lambda(p)) = p$ and $\lambda(\gamma(p)) = p$, so λ is injective. All we are missing now is therefore that γ is ∂-∂-continuous.

To show this, we show continuity at each $p \in S(G_{\bar{c}})$. Let therefore $\varepsilon > 0$ and $p = \mathrm{tp}(\bar{c}\bar{a}, \bar{c}\bar{b})$ be given. We will produce a $\delta > 0$ such that if $\partial(p,q) < \delta$, then $\partial(\gamma(p), \gamma(q)) < \varepsilon$.

First we find $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} 2^{-i-1}\mathrm{diam}(\mathcal{M}) < \varepsilon/3$. Let $U \subseteq G$ be the basic open set

$$U = \{g \in G : d^{\mathcal{M}}(g(\bar{a}_N, \bar{b}_N), (\bar{a}_N, \bar{b}_N)) < \varepsilon/3\},$$

where $\bar{a}_N$ and $\bar{b}_N$ denote the tuples consisting of the N first elements of $\bar{a}$ and $\bar{b}$, respectively. By exact homogeneity and Effros' Theorem (Theorem 2.5.13 above), we get a $\delta' > 0$ such that

$$B((\bar{c}, \bar{c}), \delta') \cap G \cdot (\bar{c}, \bar{c}) \subseteq U \cdot (\bar{c}, \bar{c}).$$

Set $\delta := \min\{\varepsilon/3, \delta'\}/2^n$, where we recall that $n = |\bar{c}|$ is the length of $\bar{c}$. Suppose now that $\partial(p,q) < \delta$. We will show $\partial(\gamma(p), \gamma(q)) < \varepsilon$. By atomicity of $\mathcal{M}$, we find a realisation $(\bar{c}_0\bar{a}', \bar{c}_1\bar{b}') \vDash q$ such that

$$d^{\mathcal{M}}((\bar{c}\bar{a}, \bar{c}\bar{b}), (\bar{c}_0\bar{a}', \bar{c}_1\bar{b}')) < \delta.$$

Since $q \in S(G_{\bar{c}})$, it follows (by τ-continuity of projections between type spaces) that $\mathrm{tp}(\bar{c}_0, \bar{c}_1) = \mathrm{tp}(\bar{c}, \bar{c})$. By exact homogeneity, we find $h \in G$ such that $h \cdot (\bar{c}, \bar{c}) = (\bar{c}_0, \bar{c}_1)$. In particular, this means $\bar{c}_0 = \bar{c}_1$, so we let $\bar{c}'$ denote this tuple. Moreover, we have that

$$\begin{aligned} d^{\mathcal{M}}((\bar{c}, \bar{c}), (\bar{c}', \bar{c}')) &= d^{\mathcal{M}}(\bar{c}, h \cdot \bar{c}) \\ &\leq 2^n \sum_{i=0}^{n-1} 2^{-i-1} d^{\mathcal{M}}(c_i, c_i') \\ &\leq 2^n d^{\mathcal{M}}((\bar{c}\bar{a}, \bar{c}\bar{b}), (\bar{c}'\bar{a}', \bar{c}'\bar{b}')) \\ &< 2^n \delta \leq \delta'. \end{aligned}$$

It follows that $h \cdot (\bar{c}, \bar{c}) \in B((\bar{c}, \bar{c}), \delta') \cap G \cdot (\bar{c}, \bar{c})$. Therefore, there is $h' \in U$ such that $h \cdot (\bar{c}, \bar{c}) = h' \cdot (\bar{c}, \bar{c})$. We now have that

$$d^{\mathcal{M}}(h'(\bar{a}, \bar{b}), (\bar{a}, \bar{b})) \leq \sum_{i=0}^{N-1} 2^{-i-1} d^{\mathcal{M}}(h'(\bar{a}_N, \bar{b}_N), (\bar{a}_N, \bar{b}_N)) + \varepsilon/3 \leq 2\varepsilon/3.$$

Therefore, we have that $h'^{-1}(\bar{c}'\bar{a}', \bar{c}'\bar{b}') = (\bar{c}\, h'^{-1}\bar{a}', \bar{c}\, h'^{-1}\bar{b}') \vDash q$ and that

$$\begin{aligned} \partial(\gamma(p), \gamma(q)) &\leq d^{\mathcal{M}_{\bar{c}}}((\bar{a}, \bar{b}), (h'^{-1}\bar{a}', h'^{-1}\bar{b}')) \\ &= d^{\mathcal{M}_{\bar{c}}}(h'(\bar{a}, \bar{b}), (\bar{a}', \bar{b}')) \\ &\leq d^{\mathcal{M}_{\bar{c}}}(h'(\bar{a}, \bar{b}), (\bar{a}, \bar{b})) + d^{\mathcal{M}_{\bar{c}}}((\bar{a}, \bar{b}), (\bar{a}', \bar{b}')) \\ &< 2\varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

where we have used, that even though h is not (necessarily) an $\mathcal{M}_{\bar{c}}$ automorphism, it is still an $\mathcal{M}_{\bar{c}}$-*isometry*. We conclude that γ is ∂-∂-continuous, and so λ is a homeomorphism between $S_{\bar{c}}(G_{\bar{c}})$ and $S(G_{\bar{c}})$.

We can now show the 'in particular'- and 'moreover'-parts of the theorem: suppose that G is locally Roelcke precompact. Then by Zielinski's theorem, $S(G)$ is ∂-locally compact. It follows that for any finite tuple $\bar{c}$, the ∂-closed set $S(G_{\bar{c}}) \subseteq S(G)$ is also ∂-locally compact. Since $S(G_{\bar{c}})$ is homeomorphic to the Roelcke completion of $G_{\bar{c}}$ by the above, it follows, again by Zielinski's result, that $G_{\bar{c}}$ is locally Roelcke precompact.

Moreover, by definition of locally Roelcke precompactness, G contains a Roelcke precompact identity neighbourhood. Such a neighbourhood must contain a set of the form

$$\{g : d^{\mathcal{M}}(g(\bar{a}), \bar{a}) < r\}$$

for some finite tuple of elements $\bar{a}$. It follows that the stabiliser of $\bar{a}$, $G_{\bar{a}}$, is a Roelcke precompact *subset* of G. By the above, the closure of $G_{\bar{a}}$ inside $S(G)$ is homeomorphic to the Roelcke completion of $G_{\bar{a}}$ (as a group). Therefore, $G_{\bar{a}}$ is a Roelcke precompact *group* as well. $\square$

Chapter 5

The Urysohn Metric Space

In the 1920's, in the posthumously published paper [60], P. S. Urysohn constructed a remarkable metric space that we today have named after him: the Urysohn metric space $\mathcal{U}$. It is remarkable because it not only contains an isometric copy of any separable metric space (i.e. it is *universal*), but it is the unique such space that is also *ultrahomogeneous*, i.e. any isometry between finite subsets extends to an isometry of the whole space. Around the same time, Banach and Mazur constructed another much more famous universal Polish space, $C([0,1])$ cf. [3], and perhaps that is why $\mathcal{U}$ did not catch a lot of attention at the time. In fact, it was almost forgotten for 60 years (!) before Katětov in the 1980's gave a new construction of $\mathcal{U}$ in [42], where he used the so-called *Katětov functions* to build a version of $\mathcal{U}$ around any separable metric space. These functions correspond to one-point extensions of the metric space. Katětov showed that $\mathcal{U}$ is the unique complete separable metric space with the so-called *extension property* for metric spaces (cf. Definition 5.1.3 below), meaning that any one-point extension of a finite metric space is realised in $\mathcal{U}$.

When it comes to (metric) model theory, it is not just the universality of $\mathcal{U}$ that is important. It is this property *together* with ultrahomogeneity that makes $\mathcal{U}$ such an interesting space from the viewpoint of model theory. However, $\mathcal{U}$ itself does not fit well into the setup of metric model theory because it is unbounded. That is probably why most authors content themselves with studying the Urysohn sphere $\mathcal{U}_1$ from a model theoretical point of view. Recall that $\mathcal{U}_1$ is the unique ultrahomogeneous universal separable complete metric space with distances bounded by 1. Viewed as a metric structure, this space is $\aleph_0$-categorical, and this simplifies the model theoretical study of $\mathcal{U}_1$ tremendously. In the case of the unbounded space $\mathcal{U}$, things are not quite as easy. The goal for this chapter is therefore to develop the metric model theory of $\mathcal{U}$ with its unbounded metric. Our main result is a sort of almost-categoricity theorem for the theory of $\mathcal{U}$, $T_{\mathcal{U}}$. We show that any model of $T_{\mathcal{U}}$ is a disjoint union of structures, each of which is similar to $\mathcal{U}$ itself (cf. Theorem 5.3.3). In

particular, it follows that any separable model is a countable disjoint union of copies of $\mathcal{U}$. Hence, there are only countably many separable models of $T_{\mathcal{U}}$. Using this theorem, we obtain a number of corollaries concerning the model theory of $\mathcal{U}$ and its automorphism group: $T_{\mathcal{U}}$ eliminates quantifiers, $\mathcal{U}$ is a prime model of $T_{\mathcal{U}}$ and $\mathrm{Iso}(\mathcal{U})$ is locally Roelcke precompact. This latter result was already known, but our proof here is new, since it uses Theorem 4.2.5 from the previous chapter.

We begin with a short section containing the preliminaries we need concerning the Urysohn space. Then we explain how to view even unbounded metric spaces as metric structures before finally proving the mentioned categoricity-like theorem for $T_{\mathcal{U}}$ together with all its corollaries.

5.1 Preliminaries on $\mathcal{U}$

By now, there is a vast body of literature on $\mathcal{U}$, and we will not attempt to cover it at all. Instead, we will only include what we will be using later on. As a matter of fact, we are not going to need many results concerning the Urysohn space, so this section will be quite short. The interested reader is referred to the excellent exposition of the theory of the Urysohn space in [49].

Basically, all we need are the two most fundamental results concerning $\mathcal{U}$, together with the building blocks of Katětov's construction. These building blocks are known as *Katětov maps* and the *extension property*. Katětov maps correspond, as mentioned, to one-point extensions of a given metric space, and the extension property says that any such extension is already present in the space. This property turns out to characterise the Urysohn space, since any two separable complete metric spaces with the extension property are isometric and hence isometric to $\mathcal{U}$. Katětov maps are defined as follows:

Definition 5.1.1. *Let (X, d) be a metric space. A map $f\colon X \to \mathbb{R}$ is a* **Katětov map** *if for all $a, b \in X$, it holds that*

$$|f(a) - f(b)| \leq d(a, b) \leq f(a) + f(b).$$

The set of all Katětov maps on X is denoted by $E(X)$.

If we are given a finite tuple $\bar{a} \in X^n$ and a tuple $\bar{r} \in \mathbb{R}^n$, we say that $\bar{r}$ is a **Katětov assignment** *(with respect to d) if the map $a_i \mapsto r_i$ is a Katětov map.*

Remark 5.1.2. It is worth mentioning here that one may view the elements of $E(X)$ as the quantifier free types over X, if we consider X as a metric structure or as a subset of some larger metric structure.

Observe that if (X, d) is a metric space, y is some new element (possibly already in X) and $f \in E(X)$, then the map d', defined by $d'(x, y) = f(x)$ and

$d'(x, x') = d(x, x')$ for $x, x' \in X$, defines a metric on $X \cup \{y\}$ extending d. Thus, each $f \in E(X)$ corresponds to a one-point extension of X.

The extension property is now defined as follows:

Definition 5.1.3. *Let (X, d) be a metric space. Then X has the* **approximate extension property** *if the following holds for all $n \in \mathbb{N}$:*

$$\forall \bar{a} = (a_i) \in X^n \, \forall f \in E(\bar{a}) \, \forall \varepsilon > 0 \, \exists b \in X \, \forall a_i \, |d(a_i, b) - f(a_i)| < \varepsilon.$$

If the b above can be chosen such that $d(a_i, b) = f(a_i)$ for each i, then (X, d) is said to have the **extension property**.

It turns out that it is possible to axiomatise the approximate extension property in first-order continuous logic, as we will see below.

We finish the section by stating the two fundamental results concerning $\mathcal{U}$ that we will need.

Theorem 5.1.4 ([49, Theorem 3.4])**.** *Any complete metric space with the approximate extension property has the actual extension property. In particular, $\mathcal{U}$ has the extension property.*

Theorem 5.1.5 ([60],[49, Theorems 3.2, 3.3 and 3.4])**.** *Any complete separable metric space with the approximate extension property is isometric to the Urysohn space.*

5.2 THE THEORY OF METRIC SPACES

In this section, we explain exactly how we can view unbounded metric spaces as metric structures. In general, there are several ways to get around this obstacle. In our case, one easy and simple way around it is to add countably many predicates to our language that are meant to "encode" the unbounded distance truncated at some finite value. Of course, this does complicate things slightly. For instance, the theory of $\mathcal{U}$, $T_{\mathcal{U}}$, will no longer be $\aleph_0$-categorical as it is the case for the Urysohn sphere. This follows (one is tempted to say "as usual") by the Compactness Theorem, since using that, we can construct models consisting of several copies of $\mathcal{U}$ with "infinite distance" between the copies. Fortunately, it turns out that these are the only models of $T_{\mathcal{U}}$ (cf. Theorem 5.3.3 below). This will be the content of the next section, but first things first. We begin by defining the signature we will use and the theory of metric spaces in this signature.

For each natural number $n \geq 1$, we let P_n be a binary predicate symbol. The bound for P_n is $[0, n]$ and the modulus of uniform continuity for each P_n is the map $r \mapsto \min\{r, 1\}/2$. Denote the resulting signature by $L_{\mathcal{U}}$. To keep our notation a little simpler, we will identify the metric symbol d with P_1.

The $L_{\mathcal{U}}$*-theory of metric spaces* is denoted by T_{MET} and is defined as follows:

Definition 5.2.1. *For each $n \geq 1$ and variables x, y, z define the following sentences:*

- $\text{SYM}_n := \sup_{x,y} |P_n(x,y) - P_n(y,x)|$,
- $\text{REF}_n := \sup_x P_n(x,x)$,
- $\text{TRI}_n := \sup_{x,y,z} P_n(x,y) \dot{-} (P_n(x,z) + P_n(z,y))$,
- $\text{INCR}_n := \sup_{x,y} P_n(x,y) \dot{-} P_{n+1}(x,y)$,
- $\text{STAB}_n := \sup_{x,y} \min\{n - P_n(x,y), |P_n(x,y) - P_{n+1}(x,y)|\}$.

Denote the collection of all these sentences by T_{MET}.

We note that the first three sentences above are simply the axioms of a metric – hence their names. The last two ensure that interpretations of the P_n's behave in the way we want them to: INCR_n says that the P_n's are increasing in n, and STAB_n says that they *stabilise*, i.e. if $P_n(a,b) < n$ at some n, then $P_m(a,b) = P_n(a,b)$ for all $m \geq n$.

It is easy to check that any metric space $(\mathcal{M}, d^{\mathcal{M}})$ (bounded or not) is naturally an $L_{\mathcal{U}}$-structure by interpreting P_n as the map $P_n^{\mathcal{M}} \colon \mathcal{M}^2 \to \mathbb{R}$ given by

$$P_n^{\mathcal{M}}(a,b) = \min\{d^{\mathcal{M}}(a,b), n\}.$$

Moreover, it is easy to check that $\mathcal{M} \vDash T_{\text{MET}}$. Furthermore, conversely, if $\mathcal{M} \vDash T_{\text{MET}}$, then we can define a map $\rho_{\mathcal{M}}$ that is almost a metric on $\mathcal{M}$. We write "almost" because we do need to allow the value ∞ in the range of $\rho_{\mathcal{M}}$. However, if $\rho_{\mathcal{M}}$ never attains this value, then $\rho_{\mathcal{M}}$ is actually a metric on $\mathcal{M}$ that may or may not be bounded. Thus, T_{MET} can be viewed as an axiomatisation of the theory of generalised metric spaces, where the metric is allowed to attain the value ∞. To explain these things precisely, we first need to define an equivalence relation on $\mathcal{M}$ using the predicates.

Definition 5.2.2. *Let $\mathcal{M} \vDash T_{\text{MET}}$. Define a relation $E_{\mathcal{M}}$ on $\mathcal{M}$ by*

$$aE_{\mathcal{M}}b \iff \exists n \; P_n^{\mathcal{M}}(a,b) < n.$$

We verify that $E_{\mathcal{M}}$ is an equivalence relation below. Moreover, each class is actually clopen.

Proposition 5.2.3. *$E_{\mathcal{M}}$ is an equivalence relation whose classes are clopen subsets of $\mathcal{M}$.*

Proof. Reflexivity and symmetry are both clear. Transitivity follows easily as well, since if $aE_{\mathcal{M}}b$ and $bE_{\mathcal{M}}c$, we find some $n, m \in \mathbb{N}$ witnessing this and let $N = m + n$. Then

$$\begin{aligned} P_N^{\mathcal{M}}(a,c) &\leq P_N^{\mathcal{M}}(a,b) + P_N^{\mathcal{M}}(b,c) \\ &= P_n(a,b) + P_m(a,b) \\ &< n + m = N, \end{aligned}$$

where we used TRI_N, STAB_n and STAB_m.

Openness of the classes follows as each predicate $P_n^{\mathcal{M}}$ is continuous and the class of any $a \in \mathcal{M}$ is given by

$$[a]_{E_{\mathcal{M}}} = \bigcup_n \{b \in \mathcal{M} : P_n^{\mathcal{M}}(a,b) < n\}.$$

Thus, $[a]_{E_{\mathcal{M}}}$ is open and therefore also closed, since the complement is the union of all other classes. □

The map $\rho_{\mathcal{M}}$ is now defined as follows:

Definition 5.2.4. *Let $\mathcal{M} \vDash T_{\text{MET}}$. We define a map $\rho_{\mathcal{M}} \colon \mathcal{M} \times \mathcal{M} \to [0,\infty]$ by*

$$\rho_{\mathcal{M}}(a,b) = \begin{cases} \sup_n P_n^{\mathcal{M}}(a,b) & \text{if } aE_{\mathcal{M}}b \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

It is easy to check that since $\mathcal{M} \vDash T_{\text{MET}}$, $\rho_{\mathcal{M}}$ restricted to any $E_{\mathcal{M}}$-class is actually a metric. Moreover, each class is complete with respect to $\rho_{\mathcal{M}}$, since it is complete with respect to $d^{\mathcal{M}}$. Thus, any model of T_{MET} is a disjoint union of $\rho_{\mathcal{M}}$ complete metric spaces. In the next section, we will show that if $\mathcal{M} \vDash T_{\mathcal{U}}$, then each $E_{\mathcal{M}}$-class has the extension property with respect to the metric $\rho_{\mathcal{M}}$. It follows that if $\mathcal{M} \vDash T_{\mathcal{U}}$ is separable, then $\mathcal{M}$ is a disjoint union of at most countably many Urysohn spaces. From this it will follow that $T_{\mathcal{U}}$ eliminates quantifiers, that $\mathcal{U}$ is a prime model and that $\text{Aut}(\mathcal{U}) = \text{Iso}(\mathcal{U})$ is locally Roelcke precompact.

5.3 Model theory of $\mathcal{U}$

In this section we will develop the model theory of $\mathcal{U}$ using the constructions from the previous section. The first thing we need to do is to explain exactly how we view $\mathcal{U}$ as an $L_{\mathcal{U}}$-structure.

For each $n \geq 1$, we let $P_n^{\mathcal{U}}$ be defined by

$$P_n^{\mathcal{U}}(a,b) = \min\{\rho(a,b), n\},$$

where ρ denotes the usual unbounded distance on $\mathcal{U}$. As the metric on our structure we just truncate at 1, i.e.:

$$d^{\mathcal{U}}(a,b) = P_1^{\mathcal{U}}(a,b) = \min\{\rho(a,b), 1\}.$$

It is not hard to check that each $P_n^{\mathcal{U}}$ does indeed respect the prescribed modulus. We will denote the metric structure obtained in this way by $(\mathcal{U}, \{P_n^{\mathcal{U}}\})$ or simply $\mathcal{U}$ when it is clear from context that we consider the Urysohn space as an $L_{\mathcal{U}}$-structure. As mentioned, the theory of $\mathcal{U}$ is denoted by $T_{\mathcal{U}}$.

The next thing we want to do is to axiomatise the approximate extension property. As explained in the first section of the chapter, the extension property characterises $\mathcal{U}$. Therefore, once we have axiomatised this property, it will follow that any model $\mathcal{M}$ of $T_{\mathcal{U}}$ will have the extension property. Moreover, each $E_{\mathcal{M}}$ class will also have this property, and so it will follow that $\mathcal{M}$ is a disjoint union of structures each having the extension property. The sentences that describe the approximate extension property are somewhat overwhelming at first glance, so we will make a few remarks below the definition to explain what they are saying exactly.

Definition 5.3.1. *For each $n \geq 2$, $\bar{r} \in \mathbb{R}^n$ and $m \in \mathbb{N}$ and a finite tuple of variables $\bar{x} = (x_1, \ldots, x_n)$ define a sentence $\sigma_{\bar{r},m}$ by*

$$\sigma_{\bar{r},m} := \sup_{\bar{x}} \min \left\{ \begin{array}{ll} \min\limits_{i,j}\{r_i + r_j \dot{-} P_m(x_i, x_j)\}, & \min\limits_{i \neq j}\{P_m(x_i, x_j) \dot{-} |r_i - r_j|\}, \\ \min\limits_{i,j}\{m \dot{-} P_m(x_i, x_j)\}, & \min\limits_{i}\{m \dot{-} r_i\}, \\ \inf\limits_{y} \max\limits_{i} |P_m(x_i, y) - r_i| & \end{array} \right\}.$$

We denote the collection of all these sentences by T_{EXT}.

In fact, we will show below that T_{EXT} together with T_{MET} *axiomatise* $T_{\mathcal{U}}$, meaning that $\mathcal{M} \vDash T_{\mathcal{U}}$ if and only if $\mathcal{M} \vDash T_{\text{EXT}} \cup T_{\text{MET}}$. However, before moving on, a few remarks on what the sentences defined above are saying are in order.

The four minimums will, if they all *fail* to be 0, express that for all distinct i, j the following hold:

(i) $r_i + r_j > P_m(x_i, x_j)$,
(ii) $P_m(x_i, x_j) > |r_i - r_j|$,
(iii) $P_m(x_i, x_j) < m$,
(iv) $r_i < m$.

The last two items above just say that m is big enough to describe the distances between a given finite tuple of elements. The first two items say that $\bar{r}$ describes a Katětov assignment with respect to the metric P_m. Thus, if items (i) through (iv) above hold, the last part of the sentence $\sigma_{\bar{r},m}$ will make sure that we can extend a finite tuple by an element with the distance prescribed by $\bar{r}$ to the elements of the tuple. We hope that these remarks will help the reader understand the sentences of T_{EXT} better.

To show that T_{EXT} together with T_{MET} axiomatise $T_{\mathcal{U}}$, we must of course verify that $\mathcal{U}$ satisfies T_{EXT}.

Lemma 5.3.2. *The Urysohn space $\mathcal{U}$ satisfies T_{EXT}, i.e. for any $\sigma \in T_{\text{EXT}}$ we have $\sigma^{\mathcal{U}} = 0$.*

Proof. Let $\bar{a} = (a_1, \ldots, a_n) \in \mathcal{U}^n$ and let $\bar{r} \in \mathbb{R}^n$, $m \in \mathbb{N}$. We may assume that all the four minimums of $\sigma^{\mathcal{U}}_{\bar{r},m}$ fail to be 0. Thus, items (i) through (iv) above hold for $\bar{a}$, i.e. we have

(i) $r_i + r_j > P^{\mathcal{U}}_m(a_i, a_j)$,
(ii) $P^{\mathcal{U}}_m(a_i, a_j) > |r_i - r_j|$,
(iii) $P^{\mathcal{U}}_m(a_i, a_j) < m$,
(iv) $r_i < m$.

Therefore, since $\mathcal{U}$ has the extension property, we can for any $\varepsilon > 0$ find some $b \in \mathcal{U}$ such that $\max_i |P_m(a_i, b) - r_i| = \max_i |\rho(a_i, b) - r_i| < \varepsilon$. In other words, the infimum in $\sigma^{\mathcal{U}}_{\bar{r},m}$ is 0. Hence, $\sigma^{\mathcal{U}}_{\bar{r},m} = 0$ for all $\bar{r}$ and m. □

Next we show that all models of $T_{\text{EXT}} \cup T_{\text{MET}}$ are disjoint unions of complete metric spaces with the extension property. In particular, a separable model of $T_{\text{EXT}} \cup T_{\text{MET}}$ is a disjoint union of Urysohn spaces.

Theorem 5.3.3. *Let $\mathcal{M}$ be a model of $T := T_{\text{MET}} \cup T_{\text{EXT}}$, and let $E_{\mathcal{M}}$ and $\rho_{\mathcal{M}}$ be defined as in Definitions 5.2.2 and 5.2.4 above. Then each $E_{\mathcal{M}}$-class has the extension property with respect to $\rho_{\mathcal{M}}$. In particular, if $\mathcal{M}$ is separable, $\mathcal{M}$ is a countable disjoint union of $L_{\mathcal{U}}$-structures each of which is isomorphic to $(\mathcal{U}, \{P^{\mathcal{U}}_n\})$.*

Proof. Let $\mathcal{M} \vDash T$ be given. To show that each $E_{\mathcal{M}}$-class has the extension property, we will show that each class satisfies the approximate extension property. It follows by Theorem 5.1.4 above that each class actually has the extension property, since each class is complete.

Let C be an $E_{\mathcal{M}}$-class and let $\bar{a} = (a_1, \ldots, a_n)$ be a tuple of elements from C. Suppose that we are given a Katětov map f on $(\bar{a}, \rho_{\mathcal{M}})$. Then we let $\bar{r} = (f(a_i))_i$ and $m \in \mathbb{N}$ be large enough so that $P^{\mathcal{M}}_m(a_i, a_j) < m$ for all i, j and so that m is a lot larger than $\max r_i$ ($2 \max r_i < m$ will suffice).

Let now $\varepsilon > 0$ be given. Without loss of generality we may assume $\varepsilon < 1$. Since f is Katětov, we have that

$$|r_i - r_j| \leq P^{\mathcal{M}}_m(a_i, a_j) \leq r_i + r_j.$$

By a slight perturbation of $\bar{r}$, we can obtain an $\bar{r}'$ such that for all distinct i, j we have

$$|r'_i - r'_j| < P^{\mathcal{M}}_m(a_i, a_j) < r'_i + r'_j,$$

and so that for any $b \in \mathcal{M}$, we have that if $\max_i |P^{\mathcal{M}}_m(a_i, b) - r'_i| < \varepsilon/2$, then $\max_i |P^{\mathcal{M}}_m(a_i, b) - r_i| < \varepsilon$. Since the inequalities are strict, items (i) through (iv) hold for $\bar{a}$, i.e.

(i) $r'_i + r'_j > P^{\mathcal{M}}_m(a_i, a_j)$,
(ii) $P^{\mathcal{M}}_m(a_i, a_j) > |r'_i - r'_j|$,
(iii) $P^{\mathcal{M}}_m(a_i, a_j) < m$,

(*iv*) $r'_i < m$.

Hence, the infimum of $\sigma^{\mathcal{M}}_{\bar{r}',m}$ must be 0, so we find b in $\mathcal{M}$ such that

$$\max_i |P^{\mathcal{M}}_m(a_i, b) - r'_i| < \varepsilon/2.$$

By the choice of $\bar{r}'$, it follows that $\max_i |P^{\mathcal{M}}_m(a_i, b) - r_i| < \varepsilon$. Finally, since $\varepsilon < 1$ and since m is a lot larger than $\max r_i$, it follows that $b \in C$. In other words, we see that C has the approximate extension property. Since each class is complete, it follows from Theorem 5.1.4 that all classes have the actual extension property.

For the 'in particular'-part of the proposition, assume that $\mathcal{M}$ is separable. Then, since each $E_{\mathcal{M}}$-class is open, it follows that there can be at most countably many classes. Moreover, any $E_{\mathcal{M}}$-class C equipped with the metric $\rho_{\mathcal{M}}$ is separable, since any $d^{\mathcal{M}}$-dense subset of C is $\rho_{\mathcal{M}}$-dense as well. It follows from Theorem 5.1.5 that $(C, \rho_{\mathcal{M}})$ is isometric to $(\mathcal{U}, \rho)$, which implies that $(C, \{P^{\mathcal{M}}_n\})$ is isomorphic to $(\mathcal{U}, \{P^{\mathcal{U}}_n\})$. Thus, $\mathcal{M}$ is a countable disjoint union of Urysohn spaces. □

As a corollary, we can prove that $T_{\text{MET}} \cup T_{\text{EXT}}$ admits quantifier elimination.

Corollary 5.3.4. *$T := T_{\text{MET}} \cup T_{\text{EXT}}$ admits quantifier elimination.*

Proof. We show that T has the back-and-forth property. Thus, we let $\mathcal{M}$ and $\mathcal{N}$ be ω-saturated models of T and suppose two tuples $\bar{a} \in \mathcal{M}^k$ and $\bar{b} \in \mathcal{N}^k$ have the same quantifier free type. We must show that for any $c \in \mathcal{M}$, we can find some $c' \in \mathcal{N}$ such that the quantifier free type of $(\bar{a}, c)$ is the same as that of $(\bar{b}, c')$.

Suppose first that c has infinite distance to all elements of $\bar{a}$, i.e. that for all $n \in \mathbb{N}$ and $i < k$ we have $P^{\mathcal{M}}_n(a_i, c) = n$. Let $(\mathcal{N}, \bar{b})$ denote the model with the elements of $\bar{b}$ named as constant symbols and let $T_{\bar{b}}$ denote the theory of this model. Then, by the Compactness Theorem and Theorem 5.3.3, there is some type $p \in S_1(T_{\bar{b}})$ such that $p(P_n(x, b_i)) = n$ for all $i \leq k$ and $n \in \mathbb{N}$. The existence of such a type follows, since the collection of these formulas is finitely satisfiable because we can use the extension property to find elements arbitrarily far from each other in the metric $\rho_{\mathcal{N}}$ from Definition 5.2.4 above. The ω-saturation of $\mathcal{N}$ implies that there is some c' realising this type p. It follows that $(\bar{b}, c')$ has the same quantifier free type as $(\bar{a}, c)$, since they agree on each $P_n(x_i, x_j)$.

The other case is similar: if c has finite distance to some a_i of $\bar{a}$, i.e. if there is some n such that $P^{\mathcal{M}}_n(a_i, c) < n$, then $a_i E_{\mathcal{M}} c$. Since the $E_{\mathcal{N}}$-class of b_i has the extension property for $\rho_{\mathcal{N}}$ and since the quantifier free types of $\bar{a}$ and $\bar{b}$ are the same, it follows that we may find the c' in $\mathcal{N}$ that we need. We conclude that T eliminates quantifiers by Proposition 2.4.5 above. □

As another corollary, we get that $\mathcal{U}$ is a prime model of $T_{\text{EXT}} \cup T_{\text{MET}}$.

Corollary 5.3.5. *$\mathcal{U}$ is a prime model of $T := T_{\text{MET}} \cup T_{\text{EXT}}$.*

Proof. Let $\mathcal{M}$ be a model of T. Then, by the Löwenheim-Skolem Theorem, we can find a separable substructure $\mathcal{M}_0 \preceq \mathcal{M}$. By Theorem 5.3.3, it follows that $\mathcal{M}_0$ is a disjoint union of isomorphic copies of $\mathcal{U}$. We may therefore assume that $\mathcal{U} \subseteq \mathcal{M}_0$. Since T admits quantifier elimination, it follows that this inclusion is an elementary embedding. □

We can now show that $T_{\text{MET}} \cup T_{\text{EXT}}$ is in fact an axiomatisation of the theory of $\mathcal{U}$. Thus, all results we have shown for $T_{\text{MET}} \cup T_{\text{EXT}}$ also hold for $T_{\mathcal{U}}$.

Corollary 5.3.6. *For any $L_{\mathcal{U}}$-structure $\mathcal{M}$ we have that $\mathcal{M} \vDash T_{\mathcal{U}}$ if and only if $\mathcal{M} \vDash T := T_{\text{MET}} \cup T_{\text{EXT}}$. Thus, all results for T above also hold for $T_{\mathcal{U}}$.*

Proof. If $\mathcal{M} \vDash T_{\mathcal{U}}$, then clearly $\mathcal{M} \vDash T_{\text{MET}} \cup T_{\text{EXT}}$. Suppose therefore that $\mathcal{M}$ is a model of $T_{\text{MET}} \cup T_{\text{EXT}}$. Then $\mathcal{U} \preccurlyeq \mathcal{M}$ by primeness and so $\mathcal{M} \vDash T_{\mathcal{U}}$. The conclusions follow immediately. □

Finally, we can show that $\mathcal{U}$ satisfies the conditions of Theorem 4.2.5, and hence that $\text{Iso}(\mathcal{U})$ is locally Roelcke precompact. Of course, as we have mentioned, this is not a new result, but the proof and our approach are new.

Corollary 5.3.7. $\text{Iso}(\mathcal{U})$ *is locally Roelcke precompact.*

Proof. Let $G := \text{Iso}(\mathcal{U}) = \text{Aut}(\mathcal{U})$. We begin by noting that $\mathcal{U}$ is homogeneous simply by the definition of the Urysohn space. Hence, $(\mathcal{U}, \{P_n^{\mathcal{U}}\})$ is a homogeneous metric structure. From Proposition 5.3.6, we know that $\mathcal{U}$ is prime and therefore atomic. Thus, $\mathcal{U}$ knows $\partial_{\omega\oplus\omega}$ by Proposition 2.5.12. All that is left to show is therefore that $S(G)$ is τ-open inside $\overline{S(G)}$.

We claim that $S(G) = \overline{S(G)} \setminus \{p_\infty\}$, where $p_\infty \in \overline{S(G)}$ is the type defined by $p_\infty(P_n(x_i, y_j)) = n$ for all i, j, n. First of all, by the Compactness Theorem and by quantifier elimination, there is a unique such type. Next, we note that by Proposition 4.2.3, $p_\infty \notin S(G)$, since clearly $\mathcal{U}$ cannot realise p_∞. Secondly, if $p \in \overline{S(G)}$ is not p_∞, we claim it is in $S(G)$. By Proposition 4.2.3, what we must show is that $\mathcal{U} \vDash p$. To see this, we observe that if $p \neq p_\infty$, we have that for all i, j, there is some n such that $p(P_n(x_i, y_j)) < n$. Therefore, if $\mathcal{M} \vDash T_{\mathcal{U}}$ is a separable model realising p, it follows by Proposition 5.3.6 that $\mathcal{M} = \bigsqcup \mathcal{U}_i$, where each $\mathcal{U}_i$ is an $E_{\mathcal{M}}$-class isomorphic as an $L_{\mathcal{U}}$-structure to $\mathcal{U}$. If $(\bar{a}, \bar{b}) \vDash p$ in $\mathcal{M}$, it follows that all elements of $\bar{a}$ and $\bar{b}$ are $E_{\mathcal{M}}$-equivalent. Hence, they come from the same $\mathcal{U}_{i_0}$. Moreover, by quantifier elimination, it follows that $(\bar{a}, \bar{b})$ is a realisation in the structure $\mathcal{U}_{i_0}$. Therefore, p is realised by the Urysohn space, so $p \in S(G)$.

We conclude that $S(G) = \overline{S(G)} \setminus \{p_\infty\}$, which is a τ-open subset of $\overline{S(G)}$. Theorem 4.2.5 therefore implies that G is locally Roelcke precompact. □

We conclude the chapter with another corollary of our analysis above saying that the *stabilisers* of any $a \in \mathcal{U}$, i.e. the group of those $g \in \mathrm{Aut}(\mathcal{U})$ such that $g(a) = a$, is a Roelcke precompact group.

Corollary 5.3.8. *Let $a \in \mathcal{U}$ be given and let $\mathcal{U}_a$ denote the $L(a)$-structure based on $\mathcal{U}$. Then $\mathrm{Aut}(\mathcal{U}_a)$ is Roelcke precompact.*

Proof. By Proposition 2.5.14, $\mathcal{U}_a$ is exact homogeneous and atomic. Therefore, we can identify the Roelcke completion of $G_a := \mathrm{Aut}(\mathcal{U}_a)$ with the ∂-closure of $\{\mathrm{tp}(\xi, g\xi \mid a) : g \in G_a\}$. Denote this closure by $S_a(G_a)$. By Theorem 4.3.1, $(S_a(G_a), \partial)$ is homeomorphic to the ∂-closure of $\{\mathrm{tp}(a\xi, ag\xi) : g \in G_a\}$. Denote this closure by $S(G_a)$. By Theorem 4.2.5, $(S(G_a), \partial)$ is homeomorphic to $(S(G_a), \tau)$, where τ denotes the logic topology as usual. We claim that $(S(G_a), \tau)$ is closed, hence compact, rendering $(S_a(G_a), \partial)$ compact as well.

To show this, we show that all types in the τ-closure $\overline{S(G_a)}$ are realised in $\mathcal{U}$. It follows by Proposition 4.2.3 that $\overline{S(G_a)} = S(G_a)$. Let therefore $p \in \overline{S(G_a)}$ be given. Then find some separable model $\mathcal{M} \vDash T_{\mathcal{U}}$ realising p with $(\bar{b}_1, \bar{b}_2)$. By Theorem 5.3.3, $\mathcal{M}$ is a disjoint union of structures isomorphic to $\mathcal{U}$. By Proposition 4.2.4, we have that $\mathrm{tp}(\bar{b}_1) = \mathrm{tp}(\bar{b}_2) = \mathrm{tp}(a\xi)$. It follows that for all $i, j \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that

$$P_n^{\mathcal{M}}(b_1^i, b_2^j) \leq P_n(b_1^i, a) + P_n(b_2^j, a) < n.$$

Therefore, $\bar{b}_1$ and $\bar{b}_2$ belong to the same copy of $\mathcal{U}$, and so p is realised in $\mathcal{U}$. $\square$

Chapter 6

The Urysohn Diversity

Diversities were introduced by Bryant and Tupper in [20] and further developed in [21] in order to generalise applications of metric space theory to combinatorial optimisation and graph theory to the hypergraph setting. The idea is very simple: instead of only assigning real numbers to pairs of elements, a diversity assigns a real number to every finite subset of the space. This turns out to generalise metric spaces quite nicely, and in [20] and [21] the authors prove diversity versions of a number of results concerning or using metric spaces. The term *diversity* comes from a special example of a diversity that appears in *phylogenetics* and *ecological* diversities demonstrating the broad variety of applications of diversities and of mathematics in general of course.

The interest in diversities from a model theoretical point of view began with the paper [18], where the authors construct a diversity analogue of the Urysohn metric space by adapting Katětov's construction to the diversity setting. They call the resulting structure $\mathbb{U}$ the *Urysohn diversity*. The existence of such a universal object among diversities gives rise to a plethora of questions concerning it, both from the side of model theory, but also from the side of Polish group theory, since the automorphism group of $\mathbb{U}$ is virtually unstudied.

In this chapter, we will develop the model theory of the Urysohn diversity along the lines of what we did for the Urysohn metric space. This study was briefly started in [19], where the authors of [18] construct $\mathbb{U}$ as a metric Fraïssé limit in the sense of Ben-Yaacov in [8]. Hence, they have to explain how to view $\mathbb{U}$ as a metric structure. In this chapter, we will further develop the model theory of $\mathbb{U}$. We will show that $\mathbb{U}$ is a prime model and that its theory eliminates quantifiers. This will allow us to apply Theorem 4.2.5 and show that the automorphism group of $\mathbb{U}$ is locally Roelcke precompact. These results follow from an almost-categoricity result similar to Theorem 5.3.3, saying that any separable model of the theory of $\mathbb{U}$ is a disjoint union of isomorphic copies of $\mathbb{U}$.

The rest of the results of this chapter are to appear in the paper [34]. The first of these results is that $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group. We will show

this by using the Katětov construction of $\mathbb{U}$ done in [18] and adapt Uspenskij's proof of this fact for the isometry group of the Urysohn metric space. We have therefore, for the convenience of the reader, included the main ingredients from the construction done in [18].

We will also construct the *rational* Urysohn diversity $\mathbb{U}_{\mathbb{Q}}$ by showing that the class of finite rational diversities is a *Fraïssé class* (in the classical sense). Here, *rational* simply means that the diversity map only attains rational values. It follows that this class has a *Fraïssé limit*, which establishes the existence of $\mathbb{U}_{\mathbb{Q}}$. Moreover, we will show that the completion of $\mathbb{U}_{\mathbb{Q}}$ is (isomorphic to) the Urysohn diversity, thus providing another proof of the existence of $\mathbb{U}$. For the convenience of the reader, we have included a brief introduction to Fraïssé theory.

With the existence of $\mathbb{U}_{\mathbb{Q}}$ established, it is easy to show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has a *dense conjugacy class* by applying a theorem of Kechris and Rosendal from [46]. Furthermore, we will show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$, from which it follows immediately that $\mathrm{Aut}(\mathbb{U})$ also has a dense conjugacy class.

In the last section of the chapter (and the thesis), we will show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has *ample generics*. Ample generics is a property with many strong implications such as the *automatic continuity property*, the *small index property*, having a unique Polish group topology and the fact that the group cannot be the union of countably many non-open subgroups. All of these notions will be explained below. Moreover, it actually implies that the group has a dense conjugacy class. Nevertheless, the proof mentioned above is so easy that we have chosen to include it in any case. Ample generics of $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ follows from an extension theorem for diversities inspired by a result of Solecki in [57] and another theorem of Kechris and Rosendal from [46].

6.1 Introduction to diversties

In this section, we have collected a few preliminaires on diversities. For further study, we refer the reader to [18, 19, 20, 21].

Let us begin with the defintion of a diversity.

Definition 6.1.1. *A* **diversity** *is a set X equipped with a map δ, the* **diversity map***, defined on the finite subsets of X to $\mathbb{R}$ such that for all finite $A, B, C \subseteq X$ we have*

(D1) $\delta(A) \geq 0$ *and* $\delta(A) = 0$ *if and only if* $|A| \leq 1$.

(D2) If $B \neq \emptyset$*, then* $\delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cup C)$.

As an abuse of language, we will follow [18] and [19] and from time to time refer to the diversity map as a diversity as well. Hopefully this confusion of names will not cause confusion for the reader.

The following observation is useful and is easy to verify:

Lemma 6.1.2. *(D2) holds for δ if and only if the following two conditions hold:*

(D2') **Monotonicity**, *i.e.* $\delta(A) \leq \delta(B)$ *if* $A \subseteq B$.
(D2") **Connected sublinearity**, *i.e.* $\delta(A \cup B) \leq \delta(A) + \delta(B)$ *if* $A \cap B \neq \emptyset$.

Another general observation to make is that any diversity is automatically a metric space, since the map $d(a, b) = \delta(\{a, b\})$ defines a metric. We refer to this metric as the **induced metric**. A diversity is **complete**, respectively **separable**, if the induced metric is complete, respectively separable. A bijective map $f\colon X \to Y$ between two diversities that preserves the values of the diversity map will be called an **isoversity**. If $Y = X$, we will call f an **autoversity** or simply an automorphism of X if the structure f preserves is clear from context. The group of all autoversities of a diversity X is denoted by $\mathrm{Aut}(X)$. It is, as usual, equipped with the pointwise convergence topology with which it becomes a topological group.

A very important observation to make is that for each $n \in \mathbb{N}$, the diversity map induces a uniformly continuous map δ^n on X^n given by

$$\delta^n(x_0, \ldots, x_{n-1}) = \delta(\{x_0, \ldots, x_{n-1}\}).$$

We will use this fact several times, so we state it here for future reference and the convenience of the reader. The proof can be found both in [18] and in [19]. Note that this implies that $\mathrm{Aut}(X)$ is a closed subgroup of $\mathrm{Iso}(X)$. It follows that $\mathrm{Aut}(X)$ is a Polish group whenever X is a complete and separable diversity.

Lemma 6.1.3 ([19, Lemma 21])**.** *Let (X, δ) be a diversity and for $n \in \mathbb{N}$ let δ^n denote the map on X^n that δ induces. Then δ^n is 1-Lipschitz in each argument. It follows that for all $\bar{x}, \bar{y} \in X^n$ we have*

$$|\delta^n(\bar{x}) - \delta^n(\bar{y})| \leq \sum \delta(x_i, y_i).$$

In particular, δ^n is uniformly continuous.

In fact, to ease our notation, we will hardly discern between the maps δ^n and the diversity map itself. Hence, we will from time to time write $\delta(\bar{a})$ for an ordered tuple $\bar{a} = (a_0, \ldots, a_n)$ instead of writing $\delta(\{a_0, \ldots, a_n\})$. It will always be perfectly clear from the context what is meant and therefore this causes no confusion.

Given a metric space, there are several ways we can use the metric to define a diversity map. Below we have included some such examples. We have also included a few other examples and the interested reader is referred to the two papers [20] and [21] for a substantial list of further examples.

Examples. The following are examples of diversities:

(*i*) *Diameter diversity.* Given a metric space (M, d), then

$$\delta_{\text{DIAM}}(A) = \max_{a,b \in A} d(a, b)$$

defines a diversity map on M called the *diameter diversity.*

(*ii*) *Steiner diversity.* Let (M, d) be a metric space. Given finite $A \subseteq M$, we let $\delta_S(A)$ be the minimal length of an undirected tree connecting A. Here, the length of a tree is the sum of all $d(a, b)$ such that $\{a, b\}$ is an edge in the tree. (M, δ_S) is called the *Steiner diversity* on M because such a minimal tree is known as a *Steiner tree.*

(*iii*) *Hypergraph Steiner diversity.* More generally, if (X, E, w) is a connected hypergraph equipped with a weight function $w\colon E \to \mathbb{R}_{\geq 0}$, we let $\delta_H(A)$ be the minimal weight of a connected subhypergraph $E' \subseteq E$ covering A. (X, δ_H) is called the *hypergraph Steiner diversity.* This diversity served as inspiration for a crucial step in Theorem 6.5.6 below.

(*iv*) *Smallest enclosing ball diversity.* Let (M, d) be a metric space. For finite $A \subseteq M$, we let $\delta(A)$ be the minimal diameter of a ball containing A.

(*v*) *Truncated diversity.* Let (X, δ) be a diversity and $k \in \mathbb{N}$. The *k-truncated diversity* on X is the diversity (X, δ_k), where $\delta_k(A)$ is the maximum $\delta(B)$ for $B \subseteq A$ with $|B| \leq k$. Note that δ_2 is just the diameter diversity.

The diameter and Steiner diversities are important examples, since it turns out that they give lower and upper bounds for the values of any diversity.

Theorem 6.1.4 ([21, Theorem 4])**.** *Let (X, δ) be a diversity and let δ_{DIAM} and δ_S denote the diameter and the Steiner diversities, respectively, that the induced metric gives rise to. Then for any finite $A \subseteq X$ we have*

$$\delta_{\text{DIAM}}(A) \leq \delta(A) \leq \delta_S(A).$$

Intuitively speaking, the diameter gives us a lower bound because it only takes into account two elements of any finite set, i.e. the two of maximal distance. Similarly, the Steiner diversity takes all elements into account since the tree must connect all the elements of a given finite subset. Therefore it gives us an upper bound. For a detailed proof, we refer the reader to [21].

It is natural to ask whether or not there are intermediate diversity maps. In [18, Proposition 23] the authors show that an easy consequence of the universality of $\mathbb{U}$ is that it is neither a diameter diversity nor a Steiner diversity. Hence, $\mathbb{U}$ is an example of an intermediate diversity. Another good class of examples is linear combinations of a diameter and a Steiner diversity using positive coefficients.

As mentioned, the construction of the Urysohn diversity in [18] follows Katětov's construction of the ditto metric space. Since we shall use this construction to prove that $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group, we need to introduce the diversity analogue of Katětov functions. These are the so-called *admissible* maps. First, we let $[X]^{<\omega}$ denote the finite subsets of X. Admissible maps are then defined as follows:

Definition 6.1.5. *Let (X, δ_X) be a diversity. A map $f\colon [X]^{<\omega} \to \mathbb{R}$ is* **admissible** *if the following hold:*

(*i*) $f(\emptyset) = 0$,
(*ii*) $f(A) \geq \delta_X(A)$ *for every* A,
(*iii*) $f(A \cup C) + \delta_X(B \cup C) \geq f(A \cup B)$ *for all* A, B, C *with* $C \neq \emptyset$,
(*iv*) $f(A) + f(B) \geq f(A \cup B)$.

The set of all admissible maps on (X, δ_X) is denoted $E(X)$.

Remark 6.1.6. Note that if we view the diversity X as a metric structure, the admissible maps are just the quantifier free 1-types over X. This is similar to the Katětov maps on a metric space (cf. Remark 5.1.2).

The reason why these maps are called admissible is because they define diversity extensions as the lemma below tells us.

Lemma 6.1.7 ([18, Lemma 2])**.** *Let (X, δ) be a diversity and $f\colon [X]^{<\omega} \to \mathbb{R}$ be a map. Then $f \in E(X)$ if and only if for some y, the map $\widehat{\delta}\colon [X \cup \{y\}]^{<\omega} \to \mathbb{R}$ given by*

$$\widehat{\delta}(A) = \delta(A), \quad \widehat{\delta}(A \cup \{y\}) = f(A)$$

for $A \subseteq X$ finite, defines a diversity map on $X \cup \{y\}$.

Similarly to the metric setting, we can define a diversity map on the set of admissible maps. This is defined as follows:

Definition 6.1.8 ([18, Page 5])**.** *Let (X, δ) be a diversity. On $[E(X)]^{<\omega}$ we define a map $\widehat{\delta}$ by*

$$\widehat{\delta}(\{f_1, \ldots, f_n\}) = \max_{j \leq k} \sup \Big\{ f_j\big(\bigcup_{i \neq j} A_i\big) - \sum_{i \neq j} f_i(A_i) : A_i \subseteq X \text{ finite} \Big\}$$

whenever $n \geq 2$ and $\widehat{\delta}(f) = \widehat{\delta}(\emptyset) = 0$.

Observe that

$$\widehat{\delta}(f_1, f_2) = \sup_{B \text{ finite}} |f_1(B) - f_2(B)|.$$

Moreover, as the notation suggests, $\widehat{\delta}$ is a diversity map on $E(X)$.

Theorem 6.1.9 ([18, Theorem 3]). *Let (X, δ) be a diversity. Then $(E(X), \widehat{\delta})$ is a diversity and (X, δ) embeds into $(E(X), \widehat{\delta})$ via the map $x \mapsto \kappa_x$, where $\kappa_x(A) = \delta(A \cup \{x\})$.*

Unfortunately, just like in the metric setting, $E(X)$ need not be separable even if X is. Therefore, we need to restrict ourselves to a subspace of $E(X)$ to maintain separability. This is the subspace of the *finitely supported admissible maps.* These are defined as follows:

Definition 6.1.10. *Let (X, δ) be a diversity and let $S \subseteq X$ be any subset. If $f \in E(S)$, then we define the extension of f to X by*

$$f_S^X(A) = \inf\Big\{f(B) + \sum_{b \in B} \delta(A_b \cup \{b\}) : B \subseteq S \text{ finite}, \bigcup_{b \in B} A_b = A\Big\},$$

where $A \subseteq X$ is finite. We say that S is the **support** *of f_S^X.*

The set of all **finitely supported admissible maps** *on X is denoted $E(X, \omega)$, i.e. $E(X, \omega)$ is the set of all those $h \in E(X)$ such that for some finite $S \subseteq X$ and some $f \in E(S)$, we have $h = f_S^X$.*

Of course, one needs to check that the extension map f_S^X is in fact admissible. We refer the reader to [18, Lemma 6] for the details. It is easy to check that κ_x is supported on $\{x\}$ for any $x \in X$, and hence that X embeds into $E(X, \omega)$. Therefore, $(E(X, \omega), \widehat{\delta})$ is a diversity extension of X. Moreover, $E(X, \omega)$ is separable.

Theorem 6.1.11 ([18, Theorem 9]). *Let (X, δ) be a separable diversity. Then $(E(X, \omega), \widehat{\delta})$ is a separable diversity as well.*

One can then iterate this construction and obtain a *Katětov tower* on a given separable diversity X consisting of a sequence of separable diversities (X_n, δ_n), where $X_n = E(X_{n-1}, \omega)$ embeds into X_{n+1}. The union X_ω of all of these diversities turns out to have an extension property similar to the extension property for metric spaces. Furthermore, it turns out that this property characterises the Urysohn diversity – just like for the metric space. The extension property for diversities is defined as follows:

Definition 6.1.12. *A diversity (X, δ_X) has the* **approximate extension property** *if for any finite subset $F \subseteq X$, any admissible map f defined on F and any $\varepsilon > 0$, there is $x \in X$ such that $|f(A) - \delta_X(A \cup \{x\})| \leq \varepsilon$ for $A \subseteq F$.*

If the above holds for $\varepsilon = 0$, (X, δ_X) has the **extension property**.

Just like in the metric setting, it turns out that complete diversities with the approximate extension property actually have the extension property. In fact, we have:

Proposition 6.1.13 ([18, Lemmas 16 and 17])**.** *Suppose (X, δ_X) is a separable diversity with the approximate extension property. Then its completion has the extension property.*

Furthermore, as mentioned, this property characterises the Urysohn diversity, meaning that any two *Polish* diversities, i.e. with complete and separable induced metrics, that have the extension property are isomorphic. This is one of the main results of [18].

Theorem 6.1.14 ([18, Theorems 14 and 22])**.** *Any two Polish diversities both having the extension property are isomorphic. In particular, any Polish diversity with the extension property is isomorphic to the Urysohn diversity.*

Of course, we invite the reader to study [18] for the details. These were all the preliminaries we will need on diversities.

6.2 Model theory of the Urysohn diversity

As we have mentioned, Theorem 4.2.5 applies to the Urysohn diversity. In fact, as we shall see in this section, the model theory of the Urysohn diversity is very similar to the one of the Urysohn metric space. However, we do need to make some adjustment to the arguments and set things up properly, since we are dealing with a different structure.

In order for us to study the model theory of the Urysohn diversity, we need to explain how to describe the unbounded diversity. In [19], the authors work in a continuous logic where unbounded predicates are allowed (cf. [5] for more details on this version of continuous logic). However, our set-up is slightly more restrictive and does not allow this. On the other hand, just like for the Urysohn metric space, we can amend this simply by adding more predicates to the language. Another easily fixed problem, which is addressed in [19] as well, is that a diversity map does not depend on a fixed number of variables. Therefore, for every $m \geq 2$ and $n \geq 1$, we will introduce an m-ary predicate to describe the diversity map restricted to subsets of size m and truncated at n.

With the above considerations in mind, we let $D_{n,m}$ be an m-ary predicate symbol with bound $[0, n]$ and modulus of continuity $\Delta_{n,m}(\varepsilon) = \min\{\varepsilon, 1\}/m$. Let $L_{\mathcal{D}}$ denote the first order metric signature containing these predicates. We now turn the Urysohn diversity into an $L_{\mathcal{D}}$-structure by interpreting each predicate as the map $D^{\mathbb{U}}_{n,m} \colon \mathbb{U}^m \to \mathbb{R}$ given by

$$D^{\mathbb{U}}_{n,m}(\bar{a}) = \min\{\delta_{\mathbb{U}}(\bar{a}), n\}.$$

As the metric we choose

$$d^{\mathbb{U}}(a, b) = D_{1,2}(a, b) = \min\{\delta_{\mathbb{U}}(a, b), 1\}.$$

The first thing we should do is to verify that these maps actually respect the prescribed moduli. This follows from a straightforward application of Lemma 6.1.3, but for the convenience of the reader, we provide the details in the lemma below.

Lemma 6.2.1. *For each $n \geq 1$ and $m \geq 2$ the interpretation $D^{\mathbb{U}}_{n,m}$ respects the modulus $\Delta_{n,m}$.*

Proof. We must show that if

$$d^{\mathbb{U}}_m(\bar{a}, \bar{b}) = \max_i d^{\mathbb{U}}(a_i, b_i) < \Delta_{n,m}(\varepsilon),$$

then

$$|D^{\mathbb{U}}_{n,m}(\bar{a}) - D^{\mathbb{U}}_{n,m}(\bar{b})| < \varepsilon$$

for two m-tuples $\bar{a} = (a_0, \ldots, a_{m-1})$ and $\bar{b} = (b_0, \ldots, b_{m-1})$.

It is clear that if both $\delta_{\mathbb{U}}(\bar{a}), \delta_{\mathbb{U}}(\bar{b}) \geq n$, the difference above is 0, so the inequality is true. Next, we note that since $d^{\mathbb{U}}_m(\bar{a}, \bar{b}) < \Delta_{n,m}(\varepsilon) < 1$, we have $d^{\mathbb{U}}(a_i, b_i) = \delta_{\mathbb{U}}(a_i, b_i)$. Therefore, by applying Lemma 6.1.3, we have that

$$|D^{\mathbb{U}}_{n,m}(\bar{a}) - D^{\mathbb{U}}_{n,m}(\bar{b})| \leq |\delta_{\mathbb{U}}(\bar{a}) - \delta_{\mathbb{U}}(\bar{b})| \leq \sum_i \delta_{\mathbb{U}}(a_i, b_i) < m\Delta_{n,m}(\varepsilon) \leq \varepsilon.$$

We conclude that $D^{\mathbb{U}}_{n,m}$ respects $\Delta_{n,m}$. □

Thus, $\mathbb{U}$ equipped with this structure is a metric $L_{\mathcal{D}}$-structure. Let $T_{\mathbb{U}}$ denote the theory of $(\mathbb{U}, \{D^{\mathbb{U}}_{n,m}\})$.

In [19, Section 5], there is a list of useful properties that the predicates $D_{n,m}$ must have in all models of $T_{\mathbb{U}}$. We have included it here for the convenience of the reader. Suppose therefore that $\mathcal{M} \vDash T_{\mathbb{U}}$. Then we have the following:

(*i*) $D^{\mathcal{M}}_{n,1}(a) = 0$ for all $a \in \mathcal{M}$.
(*ii*) $D^{\mathcal{M}}_{n,m}$ is permutation invariant.
(*iii*) $D^{\mathcal{M}}_{n,m+2}(a_0, \ldots, a_{m-1}, a_m, a_m) = D^{\mathcal{M}}_{n,m+1}(a_0, \ldots, a_m)$ for $a_i \in \mathcal{M}$.
(*iv*) $D^{\mathcal{M}}_{n,m+1}(\bar{a}, b) \geq D^{\mathcal{M}}_{n,m}(\bar{a})$ for $\bar{a} \in \mathcal{M}^m$ and $b \in \mathcal{M}$.
(*v*) For all tuples $\bar{a}$, $\bar{b}$ and $\bar{c}$ (not necessarily of the same lengths) with $|\bar{b}| > 0$ we have

$$D^{\mathcal{M}}_{n,|\bar{a}|+|\bar{c}|}(\bar{a}, \bar{c}) \leq D^{\mathcal{M}}_{n,|\bar{a}|+|\bar{b}|}(\bar{a}, \bar{b}) + D^{\mathcal{M}}_{n,|\bar{b}|+|\bar{c}|}(\bar{b}, \bar{c}).$$

(*vi*) The predicates *stabilise*, i.e. if $D^{\mathcal{M}}_{n,m}(\bar{a}) < n$, then for all $k \geq n$ we have $D^{\mathcal{M}}_{k,m}(\bar{a}) = D^{\mathcal{M}}_{n,m}(\bar{a})$

In light of item (*iii*), we will write $D^{\mathcal{M}}_{n,m}(\bar{a})$ also for tuples of length less than m, since we can just repeat the last element until we have a tuple of length m. This will simplify our notation a bit.

As in the previous chapter concerning the Urysohn metric space, we will exploit Theorem 6.1.14 to show that any model of the theory of $\mathbb{U}$ is a disjoint union of models having the extension property. In particular, a separable model of $T_{\mathbb{U}}$ will be a disjoint union of copies of $\mathbb{U}$. In order for us to state and prove this, we need to introduce some notation and make some definitions that are similar to the ones we made for the Urysohn metric space.

Definition 6.2.2. *Given a model $\mathcal{M} \vDash T_{\mathbb{U}}$ define a relation $E_{\mathcal{M}}$ on $\mathcal{M}$ by*

$$aE_{\mathcal{M}}b \iff \exists n\ D^{\mathcal{M}}_{n,2}(a,b) < n.$$

Lemma 6.2.3. *For any $\mathcal{M} \vDash T_{\mathbb{U}}$, $E_{\mathcal{M}}$ is an equivalence relation with clopen classes.*

Proof. Symmetry and reflexivity are both clear. If $aE_{\mathcal{M}}b$ and $bE_{\mathcal{M}}c$, then we find n, m witnessing this. If we let $N = n + m$, we see that

$$D^{\mathcal{M}}_{N,2}(a,c) \leq D^{\mathcal{M}}_{n,2}(a,b) + D^{\mathcal{M}}_{m,2}(b,c) < n + m = N.$$

Thus, $E_{\mathcal{M}}$ is transitive as well.

Openness of the classes follows from continuity of $D^{\mathcal{M}}_{n,2}$, since given any $a \in \mathcal{M}$, the class of a is given by

$$[a]_{E_{\mathcal{M}}} = \bigcup_n \{b : D^{\mathcal{M}}_{n,2}(a,b) < n\}.$$

Thus, each class is also closed, since the complement is the union of all other classes. □

Using the predicates, we can equip any model of $T_{\mathbb{U}}$ with a "diversity" map if we allow the value infinity. Recall that $[\mathcal{M}]^{<\omega}$ denotes the set of all finite subsets of $\mathcal{M}$.

Definition 6.2.4. *Let $\mathcal{M}$ be a model of $T_{\mathbb{U}}$. Define a map*

$$\delta_{\mathcal{M}} \colon [\mathcal{M}]^{<\omega} \to [0, \infty]$$

by

$$\delta_{\mathcal{M}}(\{a_0, \ldots, a_{m-1}\}) := \begin{cases} \sup_n D^{\mathcal{M}}_{n,m}(\{a_0, \ldots, a_{m-1}\}) & \text{if } \forall i < m\ a_0 E_{\mathcal{M}} a_i \\ \infty & \text{otherwise.} \end{cases}$$

Note that if we restrict $\delta_{\mathcal{M}}$ to any $E_{\mathcal{M}}$-class, we obtain an actual diversity map because $\mathcal{M} \vDash T_{\mathbb{U}}$. Moreover, it is not hard to see that $\delta_{\mathcal{M}}$ induces a complete metric on each $E_{\mathcal{M}}$-class. Thus, $\mathcal{M}$ is a disjoint union of complete $\delta_{\mathcal{M}}$-diversities. Furthermore, we obtain the following theorem:

Theorem 6.2.5. *Let $\mathcal{M}$ be a model of $T_{\mathbb{U}}$ and let $E_{\mathcal{M}}$ and $\delta_{\mathcal{M}}$ be defined as above. Then each $E_{\mathcal{M}}$-class has the extension property with respect to $\delta_{\mathcal{M}}$. In particular, if $\mathcal{M}$ is separable, $\mathcal{M}$ is a disjoint union of $L_{\mathcal{D}}$-structures isomorphic to $\mathbb{U}$.*

Proof. Let $\mathcal{M} \vDash T_{\mathbb{U}}$ be given. We will show that each $E_{\mathcal{M}}$-class has the approximate extension property with respect to $\delta_{\mathcal{M}}$. It follows from Proposition 6.1.13 that each class has the actual extension property.

To do this, we will axiomatise the approximate extension property for diversities in continuous logic. For $n \geq 1$, $m \geq 2$ and $\bar{r} \in \mathbb{R}^{\mathbb{P}(m)}$ (where $\mathbb{P}(m)$ denotes the power set of m) such that $\bar{r}(\emptyset) = 0$, we define a sentence $\sigma^{\bar{r}}_{n,m}$ by

$$\sup_{\bar{x}} \min \left\{ \begin{array}{l} \min\limits_{I \subseteq m}\{\bar{r}(I) \dot{-} D_{n,m}(\bar{x}_I)\}, \quad \min\limits_{I,J \subseteq m}\{\bar{r}(I) + \bar{r}(J) \dot{-} \bar{r}(I \cup J) : I, J \neq \emptyset\} \\ \min\limits_{I,J,K \subseteq m}\big\{\bar{r}(I \cup J) + D_{n,m}(\bar{x}_{I \cup K}) \dot{-} \bar{r}(J \cup K) : I \neq \emptyset \ \& \ |I \cup K| \geq 2\big\} \\ n - D_{n,m}(\bar{x}), \quad \min\limits_{I \subseteq m}\{n \dot{-} \bar{r}(I)\}, \quad \min\limits_{i \in m, J \subseteq m}\{\bar{r}(\{i\} \cup J) - \bar{r}(J)\} \\ \inf\limits_{y} \max\limits_{I \subseteq m} |D_{n,m+1}(\bar{x}_I, y) - \bar{r}(I)| \end{array} \right\}$$

Here, for a set $I \subseteq m$, $\bar{x}_I$ denotes the tuple $(x_i : i \in I)$. The formulas within the first minimum of the sentence above will be referred to as the *subformulas* of the sentence.

We now claim, first of all, that since $\mathbb{U}$ has the extension property, it will satisfy every $\sigma^{\bar{r}}_{n,m}$. To see this, let $n \geq 1$, $m \geq 2$ and $\bar{r} \in \mathbb{R}^{\mathbb{P}(m)}$ with $\bar{r}(\emptyset) = 0$ be given. We must show that $\mathbb{U} \vDash \sigma^{\bar{r}}_{n,m} = 0$, i.e. we must show that for every tuple $\bar{a} \in \mathbb{U}^m$, at least one of the subformulas of $\sigma^{\bar{r}}_{n,m}$ is 0.

Of course, we can suppose that all subformulas except for the infimum are non-zero. Then what we must show is that the infimum is 0. Define a map f on $\mathbb{P}(\bar{a})$ by $f(\bar{a}_I) := \bar{r}(I)$. We now claim that f is admissible for the usual diversity map $\delta_{\mathbb{U}}$ on $\mathbb{U}$. To see this, we verify that conditions (i)-(iv) of Definition 6.1.5 hold. First, we note that $\delta_{\mathbb{U}}(\bar{a}_I) = D^{\mathbb{U}}_{n,m}(\bar{a}_I)$ because $n > D^{\mathbb{U}}_{n,m}(\bar{a})$. Furthermore, we have that

(i) $f(\emptyset) = \bar{r}(\emptyset) = 0$.

(ii) $f(\bar{a}_I) = \bar{r}(I) > \delta_{\mathbb{U}}(\bar{a}_I)$ since the first subformula is non-zero.

(iii) For all $I, J, K \subseteq m$ such that $I \neq \emptyset$ and $|I \cup K| \geq 2$ we have

$$f(\bar{a}_{I \cup J}) + \delta_{\mathbb{U}}(\bar{a}_{I \cup K}) > \bar{r}(J \cup K) = f(\bar{a}_{J \cup K}),$$

since the third subformula is non-zero. Furthermore, if $|I \cup K| = 1$ and $K = I$, then clearly

$$f(\bar{a}_{I \cup J}) + \delta_{\mathbb{U}}(\bar{a}_{I \cup K}) = f(\bar{a}_{I \cup J}) = f(\bar{a}_{J \cup K}).$$

If $|I| = 1$ and $K = \emptyset$, then by the sixth subformula, we get

$$f(\bar{a}_{I\cup J}) + \delta_{\mathbb{U}}(\bar{a}_{I\cup K}) = f(\bar{a}_{I\cup J}) \geq f(\bar{a}_J) = f(\bar{a}_{J\cup K}).$$

We conclude that (iii) of Definition 6.1.5 holds.

(iv) For all non-empty $I, J \subseteq m$, we have

$$f(\bar{a}_I) + f(\bar{a}_J) = \bar{r}(I) + \bar{r}(J) > \bar{r}(I \cup J) = f(\bar{a}_{I\cup J}),$$

since the second subformula is non-zero. Moreover, if one of I or J is empty, then trivially we have $f(\bar{a}_I) + f(\bar{a}_J) \geq f(\bar{a}_{I\cup J})$ as well. Thus, (iv) of Definition 6.1.5 holds.

It follows that f is admissible. Therefore, since $\mathbb{U}$ has the extension property, we find $b \in \mathbb{U}$ such that for all $I \subseteq m$, we have $\delta_{\mathbb{U}}(\bar{a}_I \cup \{b\}) = f(\bar{a}_I)$. Since $\min_I\{n \dot{-} \bar{r}(I)\} > 0$ by assumption, it follows that $f(\bar{a}_I) < n$, and hence

$$\inf_{c\in\mathbb{U}} \max_I |D^{\mathbb{U}}_{n,m+1}(\bar{a}_I, c) - \bar{r}(I)| = 0,$$

so $\mathbb{U} \vDash \sigma^{\bar{r}}_{n,m} = 0$.

Therefore, if $\mathcal{M} \vDash T_{\mathbb{U}}$, then in particular $\mathcal{M} \vDash \sigma^{\bar{r}}_{n,m}$ for all n, m and $\bar{r} \in \mathbb{R}^{\mathbb{P}(m)}$ with $\bar{r}(\emptyset) = 0$. We claim that this implies that each $E_{\mathcal{M}}$-class has the approximate extension property with respect to $\delta_{\mathcal{M}}$. Let therefore $(a_0, \ldots, a_{m-1}) =: \bar{a}$ be a tuple of $E_{\mathcal{M}}$-equivalent elements of $\mathcal{M}$ and let $\varepsilon > 0$ be given. Let moreover f be an admissible map on $\mathbb{P}(\bar{a})$. Choose n large enough so that for every $I \subseteq m$, we have $D^{\mathcal{M}}_{n,m}(\bar{a}_I) < n$ and $f(\bar{a}_I) < n$. Then $\delta_{\mathcal{M}}(\bar{a}_I) = D^{\mathcal{M}}_{n,m}(\bar{a}_I)$ for all $I \subseteq m$. It follows that

(i) $f(\emptyset) = 0$,
(ii) $f(\bar{a}_I) \geq D^{\mathcal{M}}_{n,m}(\bar{a}_I)$ for all $I \subseteq m$,
(iii) $f(\bar{a}_{I\cup J}) + D^{\mathcal{M}}_{n,m}(\bar{a}_{I\cup K}) \geq f(\bar{a}_{J\cup K})$ for all $I, J, K \subseteq m$ with $I \neq \emptyset$,
(iv) $f(\bar{a}_I) + f(\bar{a}_J) \geq f(\bar{a}_{I\cup J})$ for all $I, J \subseteq m$.

Let now $\bar{r} \in \mathbb{R}^{\mathbb{P}(m)}$ be the image of f, i.e. $\bar{r}(I) = f(\bar{a}_I)$. By a slight perturbation of $\bar{r}$, we can obtain an $\bar{r}' \in \mathbb{R}^{\mathbb{P}(m)}$ such that for all $I \subseteq m$, if

$$|D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, b) - \bar{r}'(I)| < \varepsilon/2,$$

then $|D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, b) - \bar{r}(I)| < \varepsilon$, and moreover such that

- $\bar{r}'(\emptyset) = 0$,
- $\bar{r}'(I) > D^{\mathcal{M}}_{n,m}(\bar{a}_I)$ for all $I \subseteq m$,
- $\bar{r}'(I \cup J) + D^{\mathcal{M}}_{n,m}(\bar{a}_{I\cup K}) > \bar{r}'(J \cup K)$ for all $I, J, K \subseteq m$ such that $I \neq \emptyset$ and $|I \cup K| \geq 2$,
- $\bar{r}'(\{i\} \cup J) > \bar{r}'(J)$ for all $J \subseteq m$ and $i \notin J$,
- $\bar{r}'(I) + \bar{r}'(J) > \bar{r}'(I \cup J)$ for $I, J \subseteq m$ non-empty,

- $\bar{r}'(I) < n$ for all $I \subseteq m$.

Since $\mathcal{M} \vDash \sigma^{\bar{r}'}_{n,m}$ and all subformulas of this sentence except the infimum are non-zero, we must have that $\inf_{c\in\mathcal{M}} \max_I |D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, c) - \bar{r}'(I)| = 0$. Therefore, we find some $b \in \mathcal{M}$ such that for all I, we have

$$|D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, b) - \bar{r}'(I)| < \varepsilon/2.$$

By our choice of $\bar{r}'$, it follows that $|D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, b) - \bar{r}(I)| < \varepsilon$. Moreover, for ε small enough, it follows that $D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, b) < n$ as well and hence $D^{\mathcal{M}}_{n,m+1}(\bar{a}_I, b) = \delta_{\mathcal{M}}(\bar{a}_I \cup \{b\})$. Therefore, $([a]_{E_{\mathcal{M}}}, \delta_{\mathcal{M}})$ has the extension property, which was what we wanted.

It now follows that if $\mathcal{M}$ is a separable model of $T_{\mathbb{U}}$, it must be a countable disjoint union of clopen $E_{\mathcal{M}}$-classes having the extension property for $\delta_{\mathcal{M}}$. Moreover, it is not hard to see that each class is $\delta_{\mathcal{M}}$-separable and $\delta_{\mathcal{M}}$-complete. Therefore, it follows that each class equipped with $\delta_{\mathcal{M}}$ as its diversity map is isomorphic as a diversity to $\mathbb{U}$. Moreover, it is easy to check that this implies that each class is $L_{\mathcal{D}}$-isomorphic to $(\mathbb{U}, \{D^{\mathbb{U}}_{n,m}\})$ as claimed. □

Remark 6.2.6. It can be shown that the collection of all the sentences $\sigma^{\bar{r}}_{n,m}$ together with the axioms for diversity maps is an axiomatisation of the theory of $T_{\mathbb{U}}$. This is completely similar to the case of the Urysohn metric space. We leave the details to be worked out by the reader.

As a corollary to the theorem above, we get that the theory of $\mathbb{U}$ eliminates quantifiers and that $\mathbb{U}$ is a prime model.

Corollary 6.2.7. *$T_{\mathbb{U}}$ eliminates quantifiers.*

Proof. We show that $T_{\mathbb{U}}$ has the back-and-forth property (cf. Definition 2.4.4). Let therefore $\mathcal{M}$ and $\mathcal{N}$ be two ω-saturated models of $T_{\mathbb{U}}$ and let $\bar{a}$ and $\bar{b}$ be two tuples with the same quantifier free type from $\mathcal{M}$ and $\mathcal{N}$, respectively, and let $c \in \mathcal{M}$.

Like for the Urysohn metric space, there are two cases: either c is not $E_{\mathcal{M}}$ related to any a_i in $\bar{a}$ or there is some a_i in the $E_{\mathcal{M}}$-class of c. In the first case, using ω-saturation of $\mathcal{N}$, we find a $c' \in \mathcal{N}$ not $E_{\mathcal{N}}$-related to any b_i. Hence, $(\bar{a}, c)$ and $(\bar{b}, c')$ have the same quantifier free type. In the other case, we use Theorem 6.2.5 above and find, by the extension property, some c' in the $E_{\mathcal{N}}$-class of b_i such that $(\bar{b}, c')$ has the same quantifier free type as $(\bar{a}, c)$. □

It now follows easily that $\mathbb{U}$ is a prime model.

Corollary 6.2.8. *$\mathbb{U}$ is a prime model. In particular, $\mathbb{U}$ is atomic.*

Proof. The proof is similar to the one for the Urysohn metric space: by Theorem 6.2.5 we find an isomorphic copy of $\mathbb{U}$ inside any model of $T_{\mathbb{U}}$, and the inclusion is elementary by quantifier elimination. □

We can now show, using Theorem 4.2.5, that Aut($\mathbb{U}$) is locally Roelcke precompact.

Corollary 6.2.9. *$G := \mathrm{Aut}(\mathbb{U}, \{D^{\mathbb{U}}_{n,m}\})$ is locally Roelcke precompact.*

Proof. We will verify that $\mathbb{U}$ satisfies the conditions of Theorem 4.2.5. Let therefore $S(G)$ be the embedding of the Roelcke completion of G in the type space $S_{\omega\oplus\omega}(T_{\mathbb{U}})$.

First of all, simply by definition (or rather construction), $\mathbb{U}$ is separable and complete. Moreover, it is easy to check that the automorphism group of the *diversity* is the same as the automorphism group of the metric structure. Hence, $\mathbb{U}$ is homogeneous simply by construction.

Since $\mathbb{U}$ is atomic, it follows that it knows the type distance. Thus, what remains to be seen is that $S(G)$ is τ-open inside $\overline{S(G)}$.

We claim that $S(G) = \overline{S(G)} \setminus \{p_\infty\}$ where $p_\infty(D_{n,m}(\bar{x}, \bar{y})) = n$ for all n and m. To see this, first note that there is a unique such type by quantifier elimination. Moreover, p_∞ is not in $S(G)$ by Proposition 4.2.3, since it cannot be realised in $\mathbb{U}$. Conversely, if p is any type different from p_∞, then there are variables $\bar{x}$ and $\bar{y}$ and an $n \in \mathbb{N}$ such that $p(D_{n,m}(\bar{x}, \bar{y})) < n$, where $m = |\bar{x}| + |\bar{y}|$. By the properties of the predicates, it follows that $p(D_{n,2}(x_i, y_j)) < n$ for any i and j. Let now $(\bar{a}, \bar{b}) \vDash p$ in some separable model $\mathcal{M}$ of $T_{\mathbb{U}}$. Then, by Theorem 6.2.5, it follows that $\mathcal{M}$ is a disjoint union of isomorphic copies of $\mathbb{U}$. Moreover, since $p(D_{n,2}(x_i, y_j)) < n$, it follows that $\bar{a}$ and $\bar{b}$ belong to the same $E_{\mathcal{M}}$-class, i.e. they belong to the same copy of $\mathbb{U}$. By quantifier elimination it follows that $\mathbb{U} \vDash p$. Thus, $p \in S(G)$, which was what we wanted. □

We finish the section by observing that, by repeating the proof of Corollary 5.3.8, we can show that the stabiliser of any element $a \in \mathbb{U}$ is Roelcke precompact.

Corollary 6.2.10. *The stabiliser of any $a \in \mathbb{U}$ is Roelcke precompact.*

Proof. Repeat the proof of Corollary 5.3.8 (almost) verbatim. □

6.3 Universality of Aut($\mathbb{U}$)

In this section, we will show that Aut($\mathbb{U}$) is a universal Polish group. We will do this by adapting Uspenskij's proof of this fact for the isometry group of the Urysohn metric space in [61]. In order to do that, we will need the Katětov construction introduced earlier in Section 6.1.

Let us begin by defining exactly what we mean by *universality*.

Definition 6.3.1. *A Polish group G is a* **universal Polish group** *if any Polish group H is isomorphic (as a topological group) to a subgroup of G, denoted $H \hookrightarrow G$.*

We will also say that H *embeds (as a topological group)* into G whenever H is isomorphic to a subgroup of G. Note that this is the same as saying that there is a map $\Phi\colon H \to G$ such that Φ is a continuous algebraic injective homomorphism with continuous inverse.

The strategy to show that $\mathrm{Aut}(\mathbb{U})$ is universal is the following: any Polish group G can be embedded into the automorphism group of a separable diversity (X, δ_X). Denote the diversity Katětov tower on X by X_ω. Then $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(X_\omega)$, which in turn embeds into $\mathrm{Aut}(\mathbb{U})$ because the completion of X_ω is isomorphic to $\mathbb{U}$. Moreover, these embeddings are all continuous (by Pettis' theorem) with continuous inverses. Below we elaborate each of these steps. First we need a lemma:

Lemma 6.3.2. *Let (X, δ_X) be a separable diversity and let $X_1 := E(X, \omega)$ denote the diversity of admissible maps on X with finite support. Then* $\mathrm{Aut}(X)$ *embeds as a topological group into* $\mathrm{Aut}(X_1)$.

Proof. Let $\Phi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(X_1)$ be the map defined by $\Phi(g)(f_S^X) = f'^X_{g(S)}$, where $f'(g(A)) = f(A)$ for $A \subseteq S$. It is straightforward to check that $\Phi(g)$ is a bijection of X_1 extending g. Moreover, we note that

$$\Phi(g)(f_S^X)(A) = f_S^X(g^{-1}A) \tag{6.1}$$

for any finite $A \subseteq X$. Using this, it is straightforward to verify that $\Phi(g)$ is an automorphism of X_1 and that Φ is injective. Furthermore, continuity of Φ follows either from Pettis' theorem (cf. [52]) or simply by a direct argument using (6.1). Finally, continuity of the inverse of Φ can be seen as follows:

Suppose $\Phi(g_n) \to \Phi(g)$ and let $x \in X$ be given. We must show that $g_n(x) \to g(x)$. For this, let $\kappa_x \in X_1$ denote the image of x under the embedding of X into X_1. Then $\Phi(g_n)(\kappa_x) \to \Phi(g)(\kappa_x)$ which means that

$$\sup_{B \text{ finite}} |\Phi(g_n)(\kappa_x)(B) - \Phi(g)(\kappa_x)(B)| \to 0.$$

In particular, this is true for $B = \{gx\}$. Therefore, we have

$$\begin{aligned} |\Phi(g_n)(\kappa_x)(\{gx\}) - \Phi(g)(\kappa_x)(\{gx\})| &= |\delta(\{g_n^{-1}gx, x\}) - \delta(\{g^{-1}gx, x\})| \\ &= \delta(\{gx, g_n x\}) \to 0, \end{aligned}$$

where we have used that $\Phi(h)(\kappa_x) = \kappa_{hx}$ for any $h \in \mathrm{Aut}(X)$, which easily follows from (6.1) above. We conclude that $g_n \to g$ in $\mathrm{Aut}(X)$. □

With this lemma established, we can show that $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group.

Theorem 6.3.3. $\mathrm{Aut}(\mathbb{U})$ *is a universal Polish group.*

Proof. First, any Polish group G can be embedded into the isometry group of its left completion $(\widehat{G}_L, d_L)$ where d_L is a left-invariant metric. We turn $\widehat{G}_L$ into a diversity by using the diameter diversity δ_L associated to d_L, i.e. $\delta_L(A)$ is the diameter of A. Then $\mathrm{Aut}(\widehat{G}_L, \delta_L)$ is still just $\mathrm{Iso}(\widehat{G}_L, d_L)$ so G embeds into this group.

Given any separable diversity X, we let X_1 denote $E(X, \omega)$ and for any $n \in \mathbb{N}$ we let X_n denote $E(X_{n-1}, \omega)$. By X_ω we denote the union $\bigcup X_n$. In the lemma above we saw that $\mathrm{Aut}(X_i)$ embeds into $\mathrm{Aut}(X_{i+1})$ for every i. Hence we obtain a chain of embeddings

$$\mathrm{Aut}(X) \xhookrightarrow{\varphi_0} \mathrm{Aut}(X_1) \xhookrightarrow{\varphi_1} \mathrm{Aut}(X_2) \xhookrightarrow{\varphi_2} \dots,$$

where $\varphi_i(g)$ extends $g \in \mathrm{Aut}(X_i)$. It follows that we obtain a map

$$\Phi\colon\ \mathrm{Aut}(X) \to \mathrm{Aut}(X_\omega),$$

where $\Phi(g) \in \mathrm{Aut}(X_\omega)$ is defined by

$$\Phi(g)(a) = \Phi_n(g)(a),$$

where Φ_n denotes $\varphi_{n-1} \circ \dots \circ \varphi_0$ for $n \in \mathbb{N}$ such that $a \in X_n$ for $a \in X_\omega$. Note that this is well defined since if $a \in X_m$ as well for $m < n$ we must have $\Phi_m(g)(a) = \Phi_n(g)(a)$ because

$$\begin{aligned}\Phi_n(g)(a) &= (\varphi_{n-1} \circ \dots \circ \varphi_0)(g)(a)\\ &= (\varphi_{n-1} \circ \dots \circ \varphi_m \circ \Phi_m)(g)(a)\\ &= \Phi_m(g)(a).\end{aligned}$$

It is easy to check that Φ is a topological group embedding and therefore it follows that $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(X_\omega)$.

Finally, by [18, Theorem 19], the completion of X_ω is isomorphic to $\mathbb{U}$. Therefore, it follows from uniform continuity of δ (cf. Lemma 6.1.3) that $\mathrm{Aut}(X_\omega)$ embeds into $\mathrm{Aut}(\mathbb{U})$.

In conclusion, we have seen that given any Polish group G, we can embed G into $\mathrm{Aut}(G_L, \delta_L)$, which in turn may be embedded into $\mathrm{Aut}(\mathbb{U})$ using the construction above. Hence, $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group, which was what we wanted. □

6.4 A DENSE CONJUGACY CLASS IN Aut(𝕌)

In this section, we will show that $\mathrm{Aut}(\mathbb{U})$ has a *dense conjugacy class* (cf. Corollary 6.4.17 below). We will do this by first showing that the automorphism group of the *rational* Urysohn diversity $\mathbb{U}_\mathbb{Q}$ has a dense conjugacy class and then show that this group embeds densely into $\mathrm{Aut}(\mathbb{U})$ (cf. Corollary 6.4.12 and Theorem 6.4.16). Here, *rational* simply means that the diversity map only

attains rational values. Of course in order to do that, the first thing we must show is that $\mathbb{U}_{\mathbb{Q}}$ exists. To do that, we will need to introduce some classical Fraïssé theory before defining $\mathbb{U}_{\mathbb{Q}}$ as the *Fraïssé limit* of the *Fraïssé class* of finite diversities with rational values, denoted by $\mathscr{D}$. We will therefore recall the basics of Fraïssé theory in the first subsection, where we also provide a useful *free amalgamation* of diversities (cf. Definition 6.4.7). This amalgamation is a generalisation of the free amalgamation of metric spaces, and we will use it several times below. With these preliminaries done, it is straightforward to verify that $\mathscr{D}$ is a Fraïssé class and therefore that $\mathbb{U}_{\mathbb{Q}}$ exists. Moreover, we will show that $\mathbb{U}_{\mathbb{Q}}$ satisfies the approximate extension property of Definition 6.1.12 above. It follows that the completion of $\mathbb{U}_{\mathbb{Q}}$ is isomorphic to $\mathbb{U}$. Thus, we obtain another way of constructing $\mathbb{U}$.

With the existence of $\mathbb{U}_{\mathbb{Q}}$ established, it is straightforward to show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has a dense conjugacy class by applying a result of Kechris and Rosendal from [46] (cf. Theorem 6.4.11 below). This result characterises exactly when the automorphism group of a Fraïssé limit has such a conjugacy class. Afterwards, we show that $\mathbb{U}$ and $\mathbb{U}_{\mathbb{Q}}$ have a property referred to as *propinquity* in [68]. From this it follows that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$. Therefore, $\mathrm{Aut}(\mathbb{U})$ has a dense conjugacy class as well (Theorem 6.4.16 and Corollary 6.4.17).

6.4.1 Fraïssé theory

Recall that given two structures A and B we denote by $A \hookrightarrow B$ that A embeds into B. An embedding between classical first-order structures is defined just as in Definition 2.1.6: it is an injective map commuting with the interpretations of the symbols of the signature. Fraïssé classes in relational signatures are then defined as follows:

Definition 6.4.1. *Let L be a countable relational signature for a (classical) first-order language and let $\mathcal{K}$ be a class of finite L-structures. Then $\mathcal{K}$ is a* **Fraïssé class** *if it has the following properties:*

(*i*) *(HP) $\mathcal{K}$ is* **hereditary**, *i.e. if $B \in \mathcal{K}$ and $A \hookrightarrow B$, then $A \in \mathcal{K}$.*

(*ii*) *(JEP) $\mathcal{K}$ has the* **joint embedding property**, *i.e. if $A, B \in \mathcal{K}$, then there is some $C \in \mathcal{K}$ such that $A, B \hookrightarrow C$.*

(*iii*) *(AP) $\mathcal{K}$ has the* **amalgamation property**, *i.e. if $A, B, C \in \mathcal{K}$ and $f\colon A \to B$ and $g\colon A \to C$ are embeddings, then there is $D \in \mathcal{K}$ and embeddings $h_B\colon B \to D$ and $h_C\colon C \to D$ such that $h_B \circ f = h_C \circ g$. In*

diagram form:

$$\begin{tikzcd}
 & \forall B \arrow[dr, dashed, "\exists h_B"] & \\
A \arrow[ur, "\forall f"] \arrow[dr, "\forall g"'] & \circlearrowleft & \exists D \\
 & \forall C \arrow[ur, dashed, "\exists h_C"'] &
\end{tikzcd}$$

We call such a structure D an **amalgam of B and C over A**.

(*iv*) *$\mathcal{K}$ contains countably many structures (up to isomorphism), and contains structures of arbitrarily large (finite) cardinality.*

Remark 6.4.2. Note that if the class $\mathcal{K}$ above contains the empty structure, then AP implies JEP.

The main reason for studying Fraïssé classes is that any Fraïssé class $\mathcal{K}$ has a so-called *Fraïssé limit* $\mathbb{K}$ which is *universal* and *ultrahomogeneous*. Ultrahomogeinity was already defined previously: any isomorphism between finite substructures extends to an automorphism. Universality in this case means that the class of all finite structures that embeds into $\mathbb{K}$ is $\mathcal{K}$. This class is the so-called *age* of $\mathbb{K}$ and is denoted $\mathrm{Age}(\mathbb{K})$. Fraïssé's theorem now reads:

Theorem 6.4.3 (Fraïssé, [32, 31], cf. also [36, Theorem 7.1.2])**.** *Let L be a countable relational signature and let $\mathcal{K}$ be a Fraïssé class of L-structures. Then there exists a unique (up to isomorphism) countable structure $\mathbb{K}$ satisfying:*

(*i*) *$\mathbb{K}$ is ultrahomogeneous.*
(*ii*) $\mathrm{Age}(\mathbb{K}) = \mathcal{K}$.

The structure $\mathbb{K}$ in the theorem above is called the **Fraïssé limit** of $\mathcal{K}$. Using this theorem, we will show that there is a universal ultrahomogeneous countable rational diversity. First we need a couple of definitions and an amalgamation lemma to make it simpler for us to verify the AP for the class of finite rational diversities.

Definition 6.4.4. *Let Y be a set and let $X \subseteq Y$. A* **connected cover** *of X is a collection $\{E_i\}$ of subsets of Y such that $X \subseteq \bigcup E_i$ and such that the* **intersection graph** *$\mathcal{G}$ defined on $\{E_i\}$ by $E_i \mathcal{G} E_j \iff E_i \cap E_j \neq \emptyset$ is connected.*

Remark 6.4.5. If (Y, δ) is a diversity and $X \subseteq Y$ is finite, then for any finite connected cover $\{E_i\}$ of X with each E_i finite we have that $\delta(X) \leq \sum \delta(E_i)$. This inequality is the main reason why we are interested in connected covers.

With this terminology established, we can define a *free* amalgamation of two diversities sharing a common sub-diversity. This is a diversity version of the free amalgamation of metric spaces. Recall first that if A and B are

structures, then $A \subsetneq B$ denotes that $A \subseteq B$ and that the inclusion is an embedding.

Definition 6.4.6. *Let (A, δ_A), (B, δ_B) and (C, δ_C) be non-empty finite diversities such that $A = B \cap C$ and such that $A \subsetneq B, C$. The* **free amalgam of B and C over A** *is the diversity (D, δ_D) where, $D = B \cup C$ and where, for $X \subseteq D$, $\delta_D(X)$ is given by the minimum over sums $\sum_i \delta(E_i)$ for a connected cover $\{E_i : i \leq n\}$ of X such that for each i, either $E_i \subseteq B$ or $E_i \subseteq C$.*

Note that if X has elements from both B and C, the definition of $\delta_D(X)$ requires the connected cover to include elements from A. Hence, if we restrict δ_D to pairs, we obtain the usual free amalgamation of metric spaces, i.e.

$$\delta_D(\{b, c\}) = \min_{a \in A}\{\delta_B(\{b, a\}) + \delta_D(\{a, c\})\}$$

for $b \in B$ and $c \in C$.

Of course, it is not necessarily evident that δ_D above defines a diversity map and that both (B, δ_B) and (C, δ_C) embeds into (D, δ_D). Let us therefore verify this.

Lemma 6.4.7. *δ_D defined above is a diversity map on $B \cup C$ extending both δ_B and δ_C. It follows that (D, δ_D) is an amalgam of B and C over A.*

Proof. First we show δ_D agrees with δ_B and δ_C on B and C, respectively. Suppose therefore $X \subseteq B$ (the other case is similar). Then $\{X\}$ is a connected cover of X so $\delta_D(X) \leq \delta_B(X)$. To show equality, let $\{E_i\}$ be a connected cover of X. We can assume $E_i \subseteq B$. By monotonicity of δ_B, we have

$$\delta_B(X) \leq \delta_B(\bigcup E_i).$$

By connectivity, we have

$$\delta_B(\bigcup E_i) \leq \sum \delta_B(E_i).$$

We conclude that $\delta_B(X) \leq \delta_D(X)$ as well, so in fact $\delta_D(X) = \delta_B(X)$. In particular, $\delta_D(X) = 0$ if $|X| \leq 1$.

Next, we show monotonicity. Let therefore $X \subseteq Y \subseteq B \cup C$. Then any connected cover of Y whose elements are contained in either B or C must also cover X. Hence, $\delta_D(X) \leq \delta_D(Y)$.

Lastly we show connected sublinearity. Suppose therefore that $X \cap Y \neq \emptyset$. Let $\{E_i\}$ and $\{F_j\}$ be connected covers realising $\delta_D(X)$ and $\delta_D(Y)$, respectively. Then, since $X$and Y intersect, we have that $\{E_i, F_j\}$ is a connected cover of $X \cup Y$ whose elements are either contained in B or C. Hence, we must have

$$\delta_D(X \cup Y) \leq \sum \delta_*(E_i) + \sum \delta_*(F_j) = \delta_D(X) + \delta_D(Y),$$

where δ_* denotes either δ_B or δ_C depending on which diversity E_i and F_j are contained in. Thus, (D, δ_D) is a diversity such that $B, C \subsetneq D$, which was what we wanted. □

Note that if the diversities A, B and C above are all rational, then the amalgam D will also be a rational diversity. It follows that the class of finite rational diversities, denoted $\mathscr{D}$, has the AP, and hence that this class is a Fraïssé class.

Proposition 6.4.8. *$\mathscr{D}$ is a Fraïssé class with limit $\mathbb{U}_{\mathbb{Q}}$. Moreover, the completion of $\mathbb{U}_{\mathbb{Q}}$ is (isomorphic to) the Urysohn diversity.*

Proof. We first note that there are clearly rational diversities of arbitrarily large finite cardinality. Moreover, up to isomorphism, there are only countably many possible finite rational diversities. Hence, $\mathscr{D}$ satisfies property (iv) above. We verify that $\mathscr{D}$ has the three properties HP, JEP and AP.

HP is clearly satisfied, since if $B \in \mathscr{D}$ and $A \xhookrightarrow{\sim} B$, then A must be a finite rational diversity so $A \in \mathscr{D}$.

JEP follows from AP, which we show below, since clearly $\emptyset \in \mathscr{D}$.

AP follows from Lemma 6.4.7 above. To see this, suppose we are given $A, B, C \in \mathscr{D}$ with $A \xhookrightarrow{\sim} B, C$ via embeddings f_B and f_C. Then we let $D = B \cup_A C$ be the union of B and C where we identify $f_B(A)$ with $f_C(A)$ while leaving $B \setminus f_B(A)$ and $C \setminus f_C(A)$ disjoint. Identifying A with its image inside D, we now have that $A = B \cap C$ and therefore Definition 6.4.6 applies. Thus, we obtain an amalgam (D, δ_D) of B and C over A.

We conclude that $\mathscr{D}$ is a Fraïssé class and hence that it has a Fraïssé limit $\mathbb{U}_{\mathbb{Q}}$.

The 'moreover'-part follows since $\mathbb{U}_{\mathbb{Q}}$ has the approximate extension property: if $F \subseteq \mathbb{U}_{\mathbb{Q}}$ is finite, $f \in E(F)$ is admissible and $\varepsilon > 0$, we can find an admissible map f' with rational values such that $|f'(A) - f(A)| < \varepsilon$. Then f' defines a rational diversity on $F \cup \{z\}$ for some new element z. By universality and ultrahomogeneity of $\mathbb{U}_{\mathbb{Q}}$, we find $x \in \mathbb{U}_{\mathbb{Q}}$ such that for all $A \subseteq F$ we have

$$|\delta(A \cup \{x\}) - f(A)| = |f'(A) - f(A)| < \varepsilon.$$

It now follows from Proposition 6.1.13 above that the completion of $\mathbb{U}_{\mathbb{Q}}$ has the extension property. Moreover, from Theorem 6.1.14 it follows that this completion is isomorphic to $\mathbb{U}$ as claimed. □

6.4.2 A dense conjugacy class

With the existence of $\mathbb{U}_{\mathbb{Q}}$ established, we set out to show that Aut($\mathbb{U}_{\mathbb{Q}}$) has a *dense conjugacy class*. First recall that the *conjugacy action* of a group on itself is given by $g \cdot h := ghg^{-1}$. Having a dense conjugacy class is then defined as follows:

Definition 6.4.9. *A Polish group G is said to have a* **dense conjugacy class** *if there is some element of G whose orbit under the conjugacy action of G on itself is dense.*

In [46], Kechris and Rosendal characterise when the automorphism group of a Fraïssé limit of a class $\mathcal{K}$ has a dense conjugacy class. They do this in terms of the JEP, not for $\mathcal{K}$ itself, but for the class of all $\mathcal{K}$*-systems* defined as follows:

Definition 6.4.10. *Let $\mathcal{K}$ be a Fraïssé class. A* **$\mathcal{K}$-system** *consists of a structure A in $\mathcal{K}$ together with a substructure $A_0 \subsetneq A$ and a partial automorphism $f\colon A_0 \to A$. Such a system is denoted $\mathcal{A} = (A, (f, A_0))$. The class of all $\mathcal{K}$-systems is denoted $\mathcal{K}_p$.*

An **embedding** *of a $\mathcal{K}$-system $\mathcal{A} = (A, (f, A_0))$ into another $\mathcal{K}$-system $\mathcal{B} = (B, (g, B_0))$ is a map $\Phi\colon A \to B$ that embeds A into B, A_0 into B_0 and $f(A_0)$ into $g(B_0)$ such that $\Phi \circ f \subseteq g \circ \Phi$. In diagram form:*

$$\begin{tikzcd} A_0 \arrow[r, "\Phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B_0 \arrow[d, "g"] \\ f(A_0) \arrow[r, "\Phi"'] & g(B_0) \end{tikzcd}$$

Kechris and Rosendal then obtain the following characterisation of having a dense conjugacy class:

Theorem 6.4.11 ([46, Theorem 2.1])**.** *Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbb{K}$. Then the following are equivalent:*

(*i*) *There is a dense conjugacy class in* $\mathrm{Aut}(\mathbb{K})$.
(*ii*) *$\mathcal{K}_p$ has the JEP.*

As an immediate corollary to this, we obtain that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ has a dense conjugacy class. Recall that $\sqcup$ denotes disjoint union.

Corollary 6.4.12. *$\mathscr{D}_p$ has the JEP. Hence,* $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ *has a dense conjugacy class.*

Proof. Let $\mathcal{A} = (A, (f, A_0))$ and $\mathcal{B} = (B, (g, B_0))$ be $\mathscr{D}$-systems. Then we define a system $\mathcal{C} = (C, (h, C_0))$ by setting $C = A \sqcup B$, $C_0 = A_0 \sqcup B_0$ and $h = f \cup g$, and where the diversity map δ_C is defined to be δ_A on A, δ_B on B and on subsets with elements from both A and B, δ_C is constant, equal to some $N > \delta_A(A), \delta_B(B)$. It is easy to check that $\mathcal{C}$ is in $\mathcal{K}_p$ and that both $\mathcal{A}$ and $\mathcal{B}$ embeds into $\mathcal{C}$. $\square$

We now wish to show the same thing for the automorphism group of the full Urysohn diversity. In order to do that, we will show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$. This will follow from a homogeneity-like property that the rational and complete Urysohn diversities and metric spaces all share. In [68], the author refers to this property for metric spaces as *pair propinquity*. To emphasise that we are working with diversities, we will call this property *diversity propinquity*. In short, the property says that if two finite subspaces are close to being isomorphic, then we can find an isomorphic copy of one space close to the other space. It is defined as follows:

Definition 6.4.13. *Let (X, δ_X) be a diversity and let $\bar{a} = (a_i)$ and $\bar{b} = (b_i)$ be two tuples of length $n \in \mathbb{N}$ of the same cardinality. For $\varepsilon > 0$ we say that $\bar{a}$ and $\bar{b}$ are* **ε-isomorphic** *if we have*

$$|\delta_X(\bar{a}_J) - \delta_X(\bar{b}_J)| < \varepsilon$$

for all $J \subseteq n$, where $\bar{b}_J := (b_j)_{j \in J}$.

Definition 6.4.14. *Let (X, δ_X) be a diversity. We say that (X, δ_X) has* **diversity propinquity** *if for all $\varepsilon > 0$, there is an $\varepsilon' > 0$ such that for all ε'-isomorphic finite tuples $\bar{a}$ and $\bar{b}$ in X, there is $\bar{a}'$ isomorphic to $\bar{a}$ and pointwise within ε of $\bar{b}$, i.e. $\max_i \delta_X(a'_i, b_i) < \varepsilon$.*

We now have the following lemma, the proof of which is modelled on the proof of the corresponding fact for the Urysohn metric space in [54, Lemma 6.5].

Lemma 6.4.15. *$\mathbb{U}$ and $\mathbb{U}_{\mathbb{Q}}$ both have diversity propinquity. Moreover, the ε' of the definition may simply be chosen to be the given ε.*

Proof. The proof for the two diversities is the same. In the rational case all one needs to check is that the diversity maps defined below are rational. However, since we are dealing with finite sets, this is easily verified.

Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. The first thing we need is to introduce some notation for dealing with the various diversities one may assign to an n-tuple. Thus, let $D_{\bar{x}}$ be the set of all *diversity assignments* to the n-tuple $\bar{x} = (x_0, \ldots, x_{n-1})$. That is, if we denote $\{x_i : i \in I\}$ by $\bar{x}_I$, then $D_{\bar{x}}$ is the set of those maps on the power set of $\bar{x}$, $\bar{r} \colon \mathbb{P}(\bar{x}) \to \mathbb{R}$ (or into $\mathbb{Q}$ for the rational case), such that

(*i*) $\bar{r}(\bar{x}_I) = 0$ if and only if $|I| \leq 1$,

(*ii*) For all I_1, I_2 and all $I \neq \emptyset$, we have $\bar{r}(\bar{x}_{I_1} \cup \bar{x}_{I_2}) \leq \bar{r}(\bar{x}_{I_1} \cup \bar{x}_I) + \bar{r}(\bar{x}_I \cup \bar{x}_{I_2})$.

Of course, any $\bar{r} \in D_{\bar{x}}$ corresponds to an element of $\mathbb{R}^{2^n}$ that we will also denote by $\bar{r}$. Thus, we will use the notation $\bar{r}(I)$ for $\bar{r}(\bar{x}_I)$. This will be convenient below.

Let now d_∞ denote the maximum metric on $D_{\bar{x}}$, i.e.

$$d_\infty(\bar{r}, \bar{r}') = \sup_{I \subseteq n} \{|\bar{r}(I) - \bar{r}'(I)|\}.$$

Next, we define another metric on $D_{\bar{x}}$ that measures how close together we can embed two diversities with n elements into a third diversity. To define this metric, let $\bar{y}$ be another n-tuple of elements disjoint from $\bar{x}$. Then define d_1 to be the metric given by

$$d_1(\bar{r}_1, \bar{r}_2) = \inf_{\bar{r}} \max_{i \leq n} \{\bar{r}(x_i, y_i)\} : \bar{r} \in D_{\bar{x} \cup \bar{y}}, \bar{r} \restriction \bar{x} = \bar{r}_1, \bar{r} \restriction \bar{y} = \bar{r}_2\},$$

where $\bar{r}_1, \bar{r}_2 \in D_{\bar{x}}$ are two different diversity assignments. If $\bar{r}_1 = \bar{r}_2$, we set $d_1(\bar{r}_1, \bar{r}_1) = 0$. Of course, here $\bar{r} \restriction \bar{y} = \bar{r}_2$ means that the diversity assignment on $\bar{y}$ given by $\bar{r}_2$ (i.e. $\bar{y}_I \mapsto \bar{r}_2(\bar{x}_I)$) is equal to $\bar{r} \restriction \bar{y}$. That d_1 is in fact a metric follows from [18, Proposition 10]. We now claim that $d_1(\bar{r}_1, \bar{r}_2) \leq d_\infty(\bar{r}_1, \bar{r}_2)$. Moreover, we claim that this will imply the proposition, but let us do one thing at a time.

Let therefore $\bar{r}_1, \bar{r}_2 \in D_{\bar{x}}$ be two different diversity assignments and set $c = d_\infty(\bar{r}_1, \bar{r}_2)$. We need to define some $\bar{r} \in D_{\bar{x} \cup \bar{y}}$ such that $\bar{r} \restriction \bar{x} = \bar{r}_1$, $\bar{r} \restriction \bar{y} = \bar{r}_2$ and such that $\max \bar{r}(x_i, y_i) \leq c$. In order to define such an $\bar{r}$ we need to introduce some notation. Given a subset $s = \{y_{i_1}, \ldots, y_{i_k}\} \subseteq \bar{y}$, we denote the corresponding set $\{x_{i_1}, \ldots x_{i_k}\} \subseteq \bar{x}$ by s'. A collection of subsets $\{E_i\}$ of $\bar{x}$ or $\bar{y}$ is said to be *connected* if the intersection graph on $\{E_i\}$ forms a connected graph. Let now $\bar{r}$ be the diversity assignment where for each $s \subseteq \bar{x} \cup \bar{y}$, $\bar{r}(s)$ is defined to be the minimum over sums of the form

$$\sum_i \bar{r}_1(E_i) + \sum_j \bar{r}_2(F'_j) + c/2,$$

where

- $E_i \subseteq \bar{x}$,
- $F_j \subseteq \bar{y}$,
- $\{E_i, F'_j\}$ is connected,
- $s \cap \bar{x} \subseteq \bigcup E_i$,
- $s \cap \bar{y} \subseteq \bigcup F_j$.

Let us argue why $\bar{r}$ is a diversity assignment. If $s_1 \subseteq s_2$, then any collection satisfying the properties of the minimum above for s_2 will also satisfy the properties for s_1. Hence, $\bar{r}(s_1) \leq \bar{r}(s_2)$. If $s_1 \cap s_2 \neq \emptyset$, we let $\{E^1_i\}, \{F^1_j\}$ realise $\bar{r}(s_1)$ and $\{E^2_l\}, \{F^2_k\}$ realise $\bar{r}(s_2)$. Then it is easy to check that $\{E^1_i, E^2_l\}, \{F^1_j, F^2_k\}$ satisfiy the properties of the minimum for $s_1 \cup s_2$. Therefore, $\bar{r}(s_1 \cup s_2) \leq \bar{r}(s_1) + \bar{r}(s_2)$ as required. We conclude that $\bar{r}$ is in fact a diversity assignment. Moreover, we see that $\sup_i \bar{r}(x_i, y_i) = c/2$, since the singletons $\{x_i\}$ and $\{y_i\}$ satisfy the properties of the minimum. This shows that $d_1(\bar{r}_1, \bar{r}_2) \leq c/2 < d_\infty(r_1, r_2)$ as we claimed.

It now follows that both $\mathbb{U}$ and $\mathbb{U}_\mathbb{Q}$ have diversity propinquity. Since the argument for both diversities is the same, we only provide it for $\mathbb{U}$. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then we claim that ε works as the ε' of Definition 6.4.14

above. To see this, let $\bar{a}$ and $\bar{b}$ be n-tuples of elements of $\mathbb{U}$ and suppose $\sup_{I\subseteq n} |\delta(\bar{a}_I) - \delta(\bar{b}_I)| < \varepsilon$. Let $\bar{r}_{\bar{a}}$ and $\bar{r}_{\bar{b}}$ be the diversity assignments corresponding to $\bar{a}$ and $\bar{b}$. Then $d_\infty(\bar{r}_{\bar{a}}, \bar{r}_{\bar{b}}) < \varepsilon$ and so $d_1(\bar{r}_{\bar{a}}, \bar{r}_{\bar{b}}) < \varepsilon$ as well. Therefore, we find a diversity assignment $\bar{r}$ on $\bar{a} \cup \bar{b}$ such that restricted to $\bar{a}$ we get $\bar{r}_{\bar{a}}$ and restricted to $\bar{b}$ we get $\bar{r}_{\bar{b}}$ and such that $\sup_i \bar{r}(a_i, b_i) < \varepsilon$. By universality of $\mathbb{U}$, we find $\bar{a}', \bar{b}' \in \mathbb{U}^n$ isomorphic as diversities to $\bar{a}$ and $\bar{b}$, respectively, such that $\sup \delta_{\mathbb{U}}(a'_i, b'_i) < \varepsilon$. By ultrahomogeneity, we find an automorphism g of $\mathbb{U}$ such that $\delta_{\mathbb{U}}(a_i, g \cdot b_i) < \varepsilon$, which was what we wanted. □

We can now show that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$.

Theorem 6.4.16. $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ *continuously embeds as a dense subgroup into* $\mathrm{Aut}(\mathbb{U})$.

Proof. Recall that $\mathbb{U}_\mathbb{Q}$ is dense in $\mathbb{U}$ by Proposition 6.4.8. Furthermore, since the diversity map defines uniformly continuous maps on finite powers of $\mathbb{U}$ (cf. Lemma 6.1.3 above), it follows that any $g \in \mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ uniquely extends to an autoversity of $\mathbb{U}$. Thus, $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ embeds into $\mathrm{Aut}(\mathbb{U})$. Moreover, this embedding must be continuous by Pettis' theorem (cf. [52] or Theorem 1.3.4).

We move on to show that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ is dense in $\mathrm{Aut}(\mathbb{U})$. Recall that the topology on $\mathrm{Aut}(\mathbb{U})$ is the pointwise convergence topology for which a basis at the identity is given by sets of the form

$$U_{\bar{a},r} := \{g \in \mathrm{Aut}(\mathbb{U}) : \delta(g(\bar{a}), \bar{a}) < r\}$$

for a tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of $\mathbb{U}$ and some $r > 0$. In each of these sets, we must find an autoversity extending a rational autoversity. Let therefore $U_{\bar{a},r}$ be given and let $g \in U_{\bar{a},r}$. Set $\varepsilon := r - \max_i \delta(g(a_i), a_i) > 0$ and find a tuple $\bar{x}$ of n elements of $\mathbb{U}_\mathbb{Q}$ with $\max_i \delta(a_i, x_i) < \varepsilon/4$. Let moreover $\bar{y}$ be an n-tuple of elements of $\mathbb{U}_\mathbb{Q}$ such that $\max_i \delta(y_i, g(x_i)) < \varepsilon/(4n)$. Note that $g(x_i)$ is not necessarily in $\mathbb{U}_\mathbb{Q}$ – hence this approximation. By Lemma 6.1.3, it follows that $(\bar{y}, \delta)$ is $\varepsilon/4$-isomorphic to $(\bar{x}, \delta)$ and therefore, by propinquity and ultrahomogeneity of $\mathbb{U}_\mathbb{Q}$, we find an autoversity g_0 of $\mathbb{U}_\mathbb{Q}$ such that $\max_i \delta(g_0(x_i), y_i) < \varepsilon/4$. We claim that the extension of g_0 to $\mathbb{U}$ is in $U_{\bar{a},r}$. Let therefore $\tilde{g}_0$ denote this extension. We have

$$\begin{aligned}
\delta(a_i, \tilde{g}_0(a_i)) &\leq \delta(a_i, g(a_i)) + \delta(g(a_i), g(x_i)) + \delta(g(x_i), y_i) + \delta(y_i, \tilde{g}_0(x_i)) \\
&\quad + \delta(\tilde{g}_0(x_i), \tilde{g}_0(a_i)) \\
&< r - \varepsilon + \varepsilon/4 + \varepsilon/(4n) + \varepsilon/4 + \varepsilon/4 \\
&\leq r.
\end{aligned}$$

We conclude that $\tilde{g}_0 \in U_{\bar{a},r}$ and hence that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ is a dense subgroup of $\mathrm{Aut}(\mathbb{U})$. □

As an immediate corollary, we obtain that $\mathrm{Aut}(\mathbb{U})$ has a dense conjugacy class.

Corollary 6.4.17. $\mathrm{Aut}(\mathbb{U})$ *has a dense conjugacy class.*

Proof. This follows easily since $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has a dense conjugacy class and is densely embedded into $\mathrm{Aut}(\mathbb{U})$. □

6.5 Ample generics of $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$

In this section, we will show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has *ample generics*. Ample generics is a powerful property introduced by Hodges, Hodkinson, Lascar and Shelah in [37] in order to study the *small index property* for automorphism groups of certain countable structures. This property was also studied by Ivanov in [40], where he describes when $\aleph_0$-categorical classical structures have automorphisms with comeagre conjugacy classes. Later, Kechris and Rosendal studied ample generics further in [46], where they gave equivalent conditions for when the automorphism group of a Fraïssé limit has ample generics. These conditions are the *joint embedding property* and the *weak amalgamation property* of the so-called *n-systems* in the given Fraïssé class. They also show that ample generics have very strong implications such as the *automatic continuity property*, *uncountable cofinality of non-open subgroups*, unique Polish group topology and the *small index property*. All of these concepts will of course be explained below.

Ample generics of $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ follows from an extension result inspired by Solecki's result in [57]. This will allow us to extend any partial isoversity of a finite diversity to a full autoversity of some larger finite diversity containing the original one.

We begin our endeavour by defining the notion of ample generics.

Definition 6.5.1. *A Polish group G has* **ample generics** *if for each $n \in \mathbb{N}$, there is a comeagre orbit for the diagonal conjugacy action of G on G^n defined by*

$$g \cdot (g_1, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

Before explaining the consequences of ample generics mentioned above, let us mention a few examples of groups that are known to have ample generics.

Examples. The following groups have ample generics:

(i) The automorphism group of the random graph, [38], cf. also [37].
(ii) The automorphism group of the free group on countably many generators, [22].
(iii) The group of measure preserving homeomorphisms of the Cantor space, [46].
(iv) The automorphism group of $\mathbb{N}^{<\omega}$ seen as the infinitely splitting regular rooted tree, [46].
(v) The isometry group of the rational Urysohn space, [57].

In [46], where these examples are taken from, Kechris and Rosendal show, as mentioned, a number of interesting consequences of ample generics. We have collected the most important ones in the theorem below.

Theorem 6.5.2 (Kechris-Rosendal [46])**.** *Let G be a Polish group with ample generics. Then G has the following properties:*

(1) **Automatic continuity property**, *i.e. any homomorphism from G to a separable group is continuous.*
(2) **Small index property**, *i.e. any subgroup of G of index $< 2^{\aleph_0}$ is open.*
(3) G cannot be the union of countably many non-open subgroups.
(4) G has a unique Polish group topology.

Another important result from [46] is a characterisation of when the automorphism group of a Fraïssé limit has ample generics in terms of the JEP and a weak form of the AP. This weaker form of amalgamation is, naturally enough, called the *weak amalgamation property* (or *WAP* for short) and is defined as follows:

Definition 6.5.3. *Let $\mathcal{K}$ be a class of finite structures. Then $\mathcal{K}$ has the* **weak amalgamation property** *(WAP) if for any $A_0 \in \mathcal{K}$, there is $A \in \mathcal{K}$ and an embedding $f_0 \colon A_0 \to A$ such that whenever $g_B \colon A \to B$ and $g_C \colon A \to C$ are embeddings into $B, C \in \mathcal{K}$, there is $D \in \mathcal{K}$ and embeddings $h_B \colon B \to D$ and $h_C \colon C \to D$ such that $h_B \circ g_B \circ f_0 = h_C \circ g_C \circ f_0$. In diagram form:*

$$\begin{tikzcd}
 & & \forall B \arrow[dr, dashed, "\exists h_B"] & \\
A_0 \arrow[r, dashed, "\exists f_0"] & \exists A \arrow[ur, "\forall g_B"] \arrow[dr, "\forall g_C"'] \arrow[rr, phantom, "\circlearrowleft"] & & \exists D \\
 & & \forall C \arrow[ur, dashed, "\exists h_C"'] &
\end{tikzcd}$$

However, it is not the Fraïssé class itself that must have the WAP and the JEP in order for the automorphism group to have ample generics, but the class of so-called *n-systems* for $n \geq 1$. This is the class of finite structures A, together with n substructures of A and n partial automorphisms of A defined on these substructures. The exact definition is as follows:

Definition 6.5.4. *Let $\mathcal{K}$ be a Fraïssé class and let $n \geq 1$ be given. An* **n-system in $\mathcal{K}$** *consists of a structure A in $\mathcal{K}$ together with n substructures $A_1, \ldots, A_n \subsetneq A$ and n embeddings $f_1 \colon A_1 \to A, \ldots, f_n \colon A_n \to A$. We denote such a system by $\mathcal{A} = (A, (f_i, A_i)_{i \leq n})$. The class of all n-systems in $\mathcal{K}$ is denoted $\mathcal{K}_p^n$.*

An **embedding** *of an n-system $\mathcal{A} = (A, (f_i, A_i))$ into another n-system $\mathcal{B} = (B, (g_i, B_i))$ is a map $\Phi \colon A \to B$ that embeds A into B, A_i into B_i and*

$f_i(A_i)$ into $g_i(B_i)$ such that $\Phi \circ f_i \subseteq g_i \circ \Phi$ for each $i \leq n$. In diagram form, for each $i \leq n$:

$$\begin{tikzcd} A_i \arrow[r, "\Phi"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & B_i \arrow[d, "g_i"] \\ f_i(A_i) \arrow[r, "\Phi"'] & g_i(B_i) \end{tikzcd}$$

Note that since we have defined embeddings between n-systems, we can talk about the WAP and the JEP for the class $\mathcal{K}_p^n$. In [46], Kechris and Rosendal show that these two properties for $\mathcal{K}_p^n$ actually characterise ample generics.

Theorem 6.5.5 ([46, Theorem 6.2])**.** *Let $\mathcal{K}$ be a Fraïssé class and let $\mathbb{K}$ denote its limit. Then the following are equivalent:*

(*i*) $\mathrm{Aut}(\mathbb{K})$ *has ample generics.*
(*ii*) *For all $n \geq 1$, $\mathcal{K}_p^n$ has the JEP and the WAP.*

Using this theorem, we can show that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ has ample generics. The proof uses the following extension result inspired by [57, Theorem 2.1]. Recall that an *isoversity* is an isomorphism of diversities. An isoversity of a diversity to itself is called an *autoversity*. A *partial* isoversity of a diversity A is simply an isoversity $f\colon A_0 \to B$ defined on a subset $A_0 \subseteq A$ into some diversity B. A *partial* autoversity of A is a partial isoversity into A. We now have the following theorem:

Theorem 6.5.6. *Let (A, δ_A) be a finite diversity. Then there is a finite diversity (B, δ_B) containing A as a subdiversity and such that any partial autoversity of A extends to a full autoversity of B.*

The proof of this theorem uses a theorem due to Herwig and Lascar from [35]. For the convenience of the reader, we have included it here. However, before we can state it properly, we need to make a few definitions.

Definition 6.5.7. *A class of structures $\mathcal{K}$ has the* **extension property for partial automorphisms** *(EPPA for short) if for any finite $A \in \mathcal{K}$ and any $B \in \mathcal{K}$ such that $A \subsetneq B$ and any partial automorphisms $g_1, \ldots, g_n$ of A that extend to automorphisms of B, there is a finite $C \in \mathcal{K}$ and automorphisms $h_1, \ldots, h_n$ of C such that h_i extends g_i.*

Loosely speaking, this property says that if it is possible to extend partial automorphisms to full automorphisms of *some* structure, then there is a *finite* structure where the partial automorphisms also extend.

Definition 6.5.8. *Let L be a finite relational classical signature. A map between two L-structures $h\colon A \to B$ is a* **weak homomorphism** *if for all predicates $R \in L$ and all tuples $\bar{a}$ of elements from A such that $A \vDash R(\bar{a})$, we have $B \vDash R(h(\bar{a}))$.*

Definition 6.5.9. *Let L be a finite relational classical signature and let $\mathcal{T}$ be a set of L-structures. An L-structure A is said to be* **$\mathcal{T}$-free** *if there is no structure $T \in \mathcal{T}$ and weak homomorphism $h\colon T \to A$.*

Herwig and Lascar's theorem then reads:

Theorem 6.5.10 ([35, Theorem 3.2])**.** *Let L be a finite classical signature and $\mathcal{T}$ a finite set of finite L-structures. Then the class of $\mathcal{T}$-free L-structures has the EPPA.*

We are now ready to prove Theorem 6.5.6.

Proof of Theorem 6.5.6. We can of course assume without loss of generality that $|A| \geq 2$. Let D be the set

$$D := \{(\delta_A(X), |X|) : X \subseteq A\} \setminus \{(0,1),(0,0)\}.$$

That is, D is all pairs of the non-zero values of δ_A together with the size of the set the value comes from. For each $(r,n) \in D$, we let $R_{(r,n)}$ be an n-ary relation symbol and let L be the (finite) relational language consisting of these symbols. We call a tuple of elements of D, $\alpha = ((r_0,n_0),\dots,(r_k,n_k))$, a *configuration* if we have that

$$\sum_{i=1}^{k} r_i < r_0 \quad \text{and} \quad 1 + \sum_{i=1}^{k}(n_i - 1) \geq n_0.$$

Given a configuration $\alpha = ((r_i,n_i))$ let $Y_0, Y_1, \dots, Y_k$ be sets such that

(*i*) $Y_0 \subseteq \bigcup_{i=1}^{k} Y_i$,

(*ii*) $|Y_i| = n_i$,

(*iii*) The intersection graph on $\{Y_1,\dots,Y_k\}$ is connected.

We call such a family of sets $\{Y_i : 0 \leq i \leq k\}$ an α-family. Note that since α is a configuration it is always possible to find at least one α-family. Moreover, we note that there are only finitely many α-families.

Given a configuration $\alpha = ((r_i,n_i) : 0 \leq i \leq k)$ and an α-family $\beta = \{Y_i\}$, we define an L-structure $\mathcal{M}_{\alpha,\beta}$ with universe $\bigcup Y_i$ by declaring that the only relations satisfied by $\mathcal{M}_{\alpha,\beta}$ are the following:

$$\mathcal{M}_{\alpha,\beta} \vDash R_{(r_i,n_i)}(\sigma(Y_i))$$

for any permutation σ of the elements of Y_i (considered here as an ordered tuple and not just a set). The permutations merely ensure that the relations are symmetric and do not really serve any other purpose. Let $\mathcal{T}$ denote the family of all $\mathcal{M}_{\alpha,\beta}$ for all configurations α and all α-families β. Note that $\mathcal{T}$ is finite.

Any diversity (X, δ_X) is naturally also an L-structure by letting

$$X \vDash R_{(r,n)}(Y) \Longleftrightarrow \delta_X(Y) = r \text{ and } |Y| = n$$

for any finite subset/tuple Y of elements of X, meaning that we are considering Y as a subset on the right-hand side and as an ordered tuple on the left-hand side above. Note, however, that the order we choose on Y is not important. Observe that any partial autoversity of X is also a partial *automorphism* of X as an L-structure.

Suppose now we are given a configuration $\alpha = ((r_i, n_i) : 0 \leq i \leq k)$ and an α-family $\beta = \{Y_0, \ldots, Y_k\}$. We then claim that X is $\mathcal{T}$-free, i.e. that there are no weak homomorphisms $h\colon \mathcal{M}_{\alpha,\beta} \to X$. To see this, suppose h is such a map. Then since β is an α-family, we have that

$$h(Y_0) \subseteq \bigcup_{i=1}^{k} h(Y_i)$$

and so, by the monotonicity of the diversity map,

$$\delta_X(h(Y_0)) \leq \delta_X(\bigcup_{i=1}^{k} h(Y_i)).$$

Since the intersection graph on $\{Y_i : 1 \leq i \leq k\}$ is connected, it follows that the intersection graph on the images $\{h(Y_i) : 1 \leq i \leq k\}$ is connected too. Therefore, we can find Y_{i_0} such that the intersection graph on the family $\{h(Y_i) : i \neq i_0\}$ remains connected (this is always possible for finite connected graphs). Hence, by connected sublinearity of the diversity map, it follows that

$$\delta_X(\bigcup_{i=1}^{k} h(Y_i)) \leq \delta_X(h(Y_{i_0})) + \delta_X(\bigcup_{i \neq i_0} h(Y_i)).$$

By induction, we obtain that

$$\delta_X(\bigcup_{i=1}^{k} h(Y_i)) \leq \sum_{i=1}^{k} \delta_X(h(Y_i)).$$

However, since h is a weak homomorphism and β is an α-family, we have that $X \vDash R_{(r_i,n_i)}(h(Y_i))$. Hence, $\delta_X(h(Y_i)) = r_i$ for each i, and so we have that

$$r_0 = \delta_X(h(Y_0)) \leq \sum_{i=1}^{k} \delta_X(h(Y_i)) = \sum_{i=1}^{k} r_i < r_0,$$

which is of course a contradiction.

Next, by universality of $(\mathbb{U}, \delta_{\mathbb{U}})$, we can embed (A, δ_A) into $(\mathbb{U}, \delta_{\mathbb{U}})$. By ultrahomogeneity, we can extend each partial autoversity of A to an autoversity of $\mathbb{U}$. Note that since D includes all values of δ_A, any partial L-automorphism of A is a partial autoversity of A (and vice versa of course). Hence, we can extend any partial L-automorphism of A to a full L-automorphism of $\mathbb{U}$ viewed as an L-structure. By Theorem 6.5.10 above, we can find a finite $\mathcal{T}$-free L-structure C containing A as a substructure such that each partial L-automorphism of A extends to an automorphism of C. Given a partial automorphism g of A, we will denote its extension to C by $\tilde{g}$. By convention, we will assume that the empty map is extended to the identity map.

A sequence of subsets $e_1, \ldots, e_k \subseteq C$ is called a *connection* if the intersection graph on $\{e_i\}$ is connected and if there are $(r_1, n_1), \ldots, (r_n, n_k) \in D$ such that for each $i \leq k$

(*i*) $|e_i| = n_i$,

(*ii*) $C \vDash R_{(r_i, n_i)}(\sigma(e_i))$ for any permutation σ of e_i considered as an ordered tuple.

Given $c, c' \in C$, we say that they are *connected* if there is a connection $e_1, \ldots, e_k$ such that $c \in e_1$ and $c' \in e_k$. Let $B \subseteq C$ be those $b \in C$ that are connected to some $a \in A$. Note that any $b \in B$ is connected to all $a \in A$, since if b is connected to $a' \in A$ via the connection $e_1, \ldots, e_k$, then $\{a, a'\}, e_1, \ldots, e_k$ is a connection between a and b. Moreover, clearly $A \subseteq B$, since given $a \in A$, we pick $a' \in A \setminus \{a\}$ (remember that we have assumed $|A| \geq 2$) and see that $\{a, a'\}$ is a connection between a and a'.

Given a partial automorphism g of A, we claim that $\tilde{g}(B) = B$. To show this, it is enough to show that $\tilde{g}(B) \subseteq B$, since we are dealing with finite sets. If g is the empty map, then we extend it to the identity and there is nothing to show. If not, pick a in the domain of g and let $b \in B$. Then, as noted above, we can find a connection between a and b. Let $e_1, \ldots, e_k$ denote such a connection. Since $\tilde{g}$ is an automorphism, it follows that $\tilde{g}(e_1), \ldots, \tilde{g}(e_k)$ is a connection between $\tilde{g}(a) = g(a) \in A$ and $\tilde{g}(b)$, because clearly the intersection graph on $\{\tilde{g}(e_i) : 1 \leq i \leq k\}$ is connected and $\tilde{g}$ preserves the relations. We conclude that $\tilde{g}(b) \in B$ as we claimed.

Define now a diversity δ_B on B by letting $\delta_B(X)$ be 0 if $|X| \leq 1$ and otherwise letting it be the minimum over all sums $\sum_{i=1}^{k} r_i$, where for some connection $e_1, \ldots, e_k$ with $e_i \subseteq B$, we have $C \vDash R_{(r_i, |e_i|)}(\sigma(e_i))$ for any permutation of e_i considered as a tuple and where $X \subseteq \bigcup e_i$. Note that since $X \subseteq B$, each element of X is connected to the *same* element of $a \in A$. Hence, the collection of all these connections, one for each $x \in X$, forms a connection containing X. Therefore, this minimum is not over the empty set and so δ_B is well-defined.

We must argue why δ_B is a diversity map, i.e. we must show that for each $X, Y, Z \subseteq B$ with $Z \neq \emptyset$ we have

$$\delta_B(X \cup Y) \leq \delta_B(X \cup Z) + \delta_B(Z \cup Y).$$

Let $\{e_i\}$ and $\{r_i\}$ realise $\delta_B(X \cup Z)$ and let $\{f_j\}$ and $\{s_j\}$ realise $\delta_B(Z \cup Y)$. Then since $Z \subseteq X \cup Z \subseteq \bigcup e_i$ and $Z \subseteq Z \cup Y \subseteq \bigcup f_j$, it follows that the intersection graph on $\{e_i\} \cup \{f_j\}$ is connected. Hence, $\{e_i\} \cup \{f_j\}$ forms a connection. Moreover, this connection covers $X \cup Y$. Therefore, we have that

$$\delta_B(X \cup Y) \leq \sum r_i + \sum s_i = \delta_B(X \cup Z) + \delta_B(Z \cup Y)$$

as we wanted. Moreover, if g is a partial autoversity of A, it follows that the extension $\tilde{g}$ and its inverse $\tilde{g}^{-1}$ maps connections to connections. Therefore, we must have that $\tilde{g}\colon B \to B$ is an autoversity with respect to δ_B.

Finally, we must show that δ_B extends δ_A. First of all, it is clear that we must have $\delta_B(X) \leq \delta_A(X)$ for all $X \subseteq A$, since $\{X\}$ is itself a connection covering X as $C \vDash R_{(\delta_A(X),|X|)}(X)$.

Suppose towards a contradiction that we have $\delta_B(X) < \delta_A(X)$. Then let $e_1, \ldots, e_k$ be a connection with corresponding values $r_1, \ldots, r_k$ witnessing this, i.e. $\sum r_i < \delta_A(X)$. It follows that

$$(\delta_A(X), |X|), (r_1, |e_1|), \ldots, (r_k, |e_k|)$$

is a configuration because $X \subseteq \bigcup e_i$, so

$$\begin{aligned} |X| &\leq |e_1| + |e_2 \setminus e_1| + \ldots + |e_k \setminus (\bigcup_{i=1}^{k-1} e_i)| \\ &\leq 1 + \sum (|e_i| - 1), \end{aligned}$$

where the second inequality follows since the first sum counts each element of $\bigcup e_i$ exactly once and the second sum counts each element at least once given that the intersection graph on $\{e_i\}$ is connected. If we denote this configuration by α, then $\{e_i, X : 1 \leq i \leq k\}$ is an α-family β. Therefore, $\mathcal{M}_{\alpha,\beta}$ is in $\mathcal{T}$ and the identity map on $\mathcal{M}_{\alpha,\beta}$ is a weak homomorphism into C. This contradicts that C is $\mathcal{T}$-free. We conclude that $\delta_B(X) = \delta_A(X)$.

All in all, we have extended each partial autoversity of A to an autoversity of (B, δ_B), and this diversity contains (A, δ_A) as a subdiversity. This was what we wanted. □

We are now ready to prove that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ has ample generics.

Theorem 6.5.11. $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ *has ample generics.*

Proof. We show that for each $n \in \mathbb{N}$, the class $\mathscr{D}_p^n$ of n-systems in $\mathscr{D}$ has the WAP. Since it clearly has the JEP, it follows from Kechris and Rosendal's Theorem 6.5.5 above that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics.

Let therefore $\mathcal{A} = (A, (f_i, A_i))$ be an n-system in $\mathscr{D}_p^n$. By the extension theorem above, we find a rational diversity B containing A, where the partial isoversities of A extend to autoversities of B. Let $\tilde{f}_i$ denote the extension of f_i to B, and let $\mathcal{B}$ denote the resulting n-system in $\mathscr{D}_p^n$. Suppose now that we are given n-systems $\mathcal{C}_1 = (C_1, (g_1^i, C_1^i))$ and $\mathcal{C}_2 = (C_2, (g_2^i, C_2^i))$ and embeddings $\Phi_j \colon \mathcal{B} \to \mathcal{C}_j$, $j = 1, 2$. We need to construct an amalgam of $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathcal{B}$. To do that, we apply the extension theorem to both C_1 and C_2 and get $\widetilde{C}_1$ and $\widetilde{C}_2$, where the partial isoversities g_1^i and g_2^i extend to full autoversities $\tilde{g}_1^i$ and $\tilde{g}_2^i$ of $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively. Denote the resulting n-systems by $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$. As usual, we can assume that $B = \widetilde{C}_1 \cap \widetilde{C}_2$. Therefore, we can construct the free amalgam D of $\widetilde{C}_1$ and $\widetilde{C}_2$ over B. Moreover, we can define an n-system using D by letting h_i be $\tilde{g}_1^i \cup \tilde{g}_2^i$, which is an autoversity of D because $\tilde{g}_1^i$ and $\tilde{g}_2^i$ must agree on B. Denote the resulting n-system by $\mathcal{D}$. In diagram form for the n-systems:

$$\begin{tikzcd}
 & & \mathcal{C}_1 \arrow[r] & \widetilde{\mathcal{C}}_1 \arrow[dr] & \\
\mathcal{A} \arrow[r] & \mathcal{B} \arrow[ur] \arrow[dr] & \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{D} \\
 & & \mathcal{C}_2 \arrow[r] & \widetilde{\mathcal{C}}_2 \arrow[ur] &
\end{tikzcd}$$

and in diagram form for $j = 1, 2$ and each i:

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\tilde{f}_i"'] \arrow[dr, phantom, "\circlearrowleft"] & C_j^i \arrow[r] \arrow[d, "g_j^i"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{C}_j \arrow[r] \arrow[d, "\tilde{g}_j^i"'] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[d, "h_i"] \\
f_i(A_i) \arrow[r] & B \arrow[r] & g_j^i(C_j^i) \arrow[r] & \widetilde{C}_j \arrow[r] & D
\end{tikzcd}$$

It is easy to check that $\mathcal{D}$ is an amalgam of $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathcal{B}$. We conclude that $\mathscr{D}_p^n$ has the WAP and hence that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics. $\square$

$*$

Appendix A

Open problems

We present here a list of open problems related to our work. The first was already mentioned in Chapter 3.

Question A.1. Is the set of realisable types in infinitary continuous logic always complete with respect to the type distance?

In Chapter 3, we showed that this was true for $\aleph_0$-categorical theories, and we conjecture that it is true in more generality.

In [68], the author claims to show that the group of affine isometries of the Gurarij space is locally Roelcke precompact. However, there is a gap in the proof, since the author seems to assume that this space is homogeneous as a metric space, which it is not. Therefore, the following remains an open question:

Question A.2. Is the group of affine isometries of the Gurarij space locally Roelcke precompact?

Since the Gurarij space is approximately homogeneous as a metric *structure*, our approach to local Roelcke precompactness might be better suited for this problem.

Diversities were defined only a few years ago and so there are many natural questions concerning them. We list some of them here.

Question A.3. Is the Urysohn diversity $\mathbb{U}$ compactly homogenous? That is, can every isoversity defined on a compact subset of $\mathbb{U}$ be extended to all of $\mathbb{U}$?

This is true for the Urysohn metric space. The authors of [18] claim that the usual proof for the metric space goes through for the diversity as well. Unfortunately, this turned out to be wrong, because the usual proof uses that we only measure distances between pairs of points when we exploit the triangle inequality to do the appropriate approximations. In the general diversity setting we have to add an ε to this approximation every time we add an element to our

finite subset. Hence, we need some upper bound on the sizes of the subsets we are measuring. Since we measure arbitrarily large finite subsets such a bound does not exist and therefore the argument fails. We invite the reader to study the proof of compact homogeneity for the Urysohn metric space in the section *Compact Homogeneity* of [49] for more details. If compact homogeneity were to fail for the Urysohn diversity, it would be the first example of a property true for the Urysohn metric space but false for the Urysohn diversity.

Another interesting question related to what we have done so far is the following:

Question A.4. Does the automorphism group of the Urysohn diversity have the automatic continuity property? What about other consequences of ample generics?

Again, this is true for the Urysohn metric space, which was shown by Sabok in [56] and later also by Malicki in [48] by similar methods. However, we discovered that there was a small gap in both proofs. Fortunately, Sabok has recently communicated to us a possible fix for this small gap in both papers. It would be interesting to see if the new proofs can be adapted to the diversity case as well.

We would also like to study simplicity of the automorphism group of the Urysohn diversity along the lines of what Tent and Ziegler did for the Urysohn metric space in [58] and [59]. In [58], the authors show that the isometry group of the bounded Urysohn space is simple. In the diversity case the first thing to do is simply to show that the bounded Urysohn diversity exists. An easy way to do that is by using Fraïssé theory as we did in Chapter 6 above. With the existence established, we can then ask the following:

Question A.5. Is the automorphism group of the bounded Urysohn diversity simple?

Lastly, we would like to investigate the question of extreme amenability. Pestov showed in [51] that the isometry group of the Urysohn space is extremely amenable. It was recently brought to our attention by Matěj Konečný, that using results from [39] it should be possible to show that the automorphism group of the Urysohn diversity is extremely amenable. However, we have not yet worked out all the details, so we would like to end this appendix by posing it as a question.

Question A.6. Is the automorphism group of the Urysohn diversity extremely amenable?

∗

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