Robust finance : a model randomization approach

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Mots clés : incertitude du modèle, sur-réplication, maximisation d'utilité, coût de transaction, randomisation, option américaine.

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Abstract

This PhD dissertation presents three research topics. The first two topics are related to the domain of robust finance and the last is related to a numerical method applied in risk management of insurance companies.

In the first part, we focus on the problem of super-replication duality for American options in discrete time financial models. We consider the robust framework with a family of non-dominated probability measures and the trading strategies are dynamic on the stocks and static on the options. We use two different ways to obtain the pricing-hedging duality. The first insight is that we can reformulate American options as European options on an enlarged space, and this can be seen as a weak formulation of the original problem. The second insight is that by considering a fictitious extensions of the market on which all the assets are traded dynamically, we can also re-obtain the duality. We then show that the general results apply in two important examples of the robust framework: the setup of [1] and the martingale optimal transport setup.

In the second part, we consider the problem of super-replication and utility maximization with proportional transaction cost in discrete time financial market with model uncertainty. Our key technique is to convert the original problem to a frictionless problem on an enlarged space by using a randomization technique together with the minimax theorem. For the super-replication problem, we obtain the duality results well-known in the classical dominated context. For the utility maximization problem, by suggesting a different dynamic programming argument than [2], we are able to prove the existence of the optimal strategy and the convex duality theorem in our context with transaction costs. In the frictionless framework, this alternative dynamic programming argument also allows us to generalize the main results in [2] to a weaker market condition.

In the third part, we present a numerical method based on a sparse grid approximation to compute the loss distribution of the balance sheet of an insurance company. The dynamic of the stock and the interest rate follow respectively the Black-Scholes model and Hull-White model, and the risk management model describing the evolution of the parameters is Gaussian. We compare the new numerical method with the traditional nested simulation approach and review the convergence of both methods to estimate the risk indicators under consideration. Finally, we provide numerical results showing that the sparse grid approach is extremely competitive for models with moderate dimension.

Keywords: model uncertainty, super-replication, utility maximization, transaction cost, randomization, american option.

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Résumé

Dans cette thèse, on considère trois sujets. Les deux premiers sujets sont liées avec la domaine de robuste finance et le dernier est une méthode numérique appliqué sur la gestion du risque des entreprises d'assurance.

Dans la première partie, on considère le problème de la sur-réplication des options américaines au temps discret. On considère une famille non-dominée des mesures de probabilité et les stratégies de trading sont dynamiques pour les sous-jacents et statiques pour les options. Pour obtenir la dualité de valorisation-couverture, on a deux méthodes. La première méthode est de reformuler les options américaines comme options européens dans un espace élargi. Cette méthode est liée avec la formulation faible du problème original. La deuxième méthode est de considérer un marché fictif dans lequel les stratégies pour tous les actifs sont dynamiques. Ensuite on applique le résultat général à deux exemples importants dans le contexte robust : le contexte de [1] et le contexte de transport optimal martingale.

Dans la deuxième partie, on considère le problème de sur-réplication and maximisation d'utilité au temps discret avec coût de transaction sous l'incertitude du modèle. L'idée principale est de convertir le problème original à un problème sans friction dans un espace élargi en utilisant un argument de randomisation et le théorème de minimax. Pour le problème de sur-réplication, on obtient la dualité comme dans le cas classique. Pour le problème de maximisation d'utilité, en utilisant un argument de la programmation dynamique différent à [2], on peut preuver à la fois l'existence de la stratégie optimale et le théorème de la dualité convexe. Dans le contexte sans friction, notre argument est une généralisation du [2] avec une condition de non-arbitrage plus faible.

Dans le troisième partie, on présente une méthode numérique basé sur l'approximation du sparse grid pour calculer la distribution de la perte du bilan d'un entreprise d'assurance. La dynamique du stock et le taux d'intérêt suivent le modèle de Black-Scholes et le modèle de Hull-White respectivement. On compare la nouvelle méthode numérique avec l'approche classique de la simulation et étudie la vitesse de la convergence des deux méthodes pour estimer l'indicateur du risque. On donne alors un exemple numérique pour montrer la bénéfice de l'approche de sparse grid.

Mots clés : incertitude du modèle, sur-réplication, maximisation d'utilité, coût de transaction, randomisation, option américaine.

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Chapter 1

Introduction

1.1 An Overview on robust finance

In the classical theory of financial mathematics, the financial market is modeled with a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The underlying assets is modeled by a stochastic process, denoted by $(S_t)_{t\in\mathbb{T}}$. We are in the discret time model if $\mathbb{T} = \{0, 1, \cdots, N\}$ and in the continuous time model if $\mathbb{T} = [0, T]$.

In discrete time, the First Fundamental Theorem of Asset Pricing(FTAP) asserts that the no-arbitrage condition is equivalent to the existence of equivalent martingale measure. Here the equivalent martingale measure means the probability measure on $(\Omega, \mathcal{F})$ such that the discounted value of the risky assets is a martingale. In continuous time, we have the similar theorem that the no free lunch with vanishing risk(NFLVR) condition is equivalent to the existence of equivalent local martingale measure.

In complete market, one can perfectly replicate the payoff of the contingent claims ξ by cash and the underlying assets. In this case the value $\mathbb{E}^{\mathbb{Q}}[\xi]$ provides a no-arbitrage price, where $\mathbb{Q}$ is an equivalent martingale measure. In incomplete market, where the perfect replication does not exist, we can consider the so-called super-replication price, which is the minimal initial cost such that there exists a hedging strategy to make the final value of the portfolio stay above the contingent claim under any possible scenario. If we take the supremum over the sets of martingale measures, we get the so-called pricing-hedging duality:

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi] = \inf\{x : \exists H \in \mathcal{H},\ \text{s.t.} x + V_T(H) \geqslant \xi, \mathbb{P} - \text{a.s.}\},$$

where $\mathcal{M}$ is the set of martingale measures, $\mathcal{H}$ is the set of admissible strategies, $V_T(H)$ is the P&L at T w.r.t to trading strategy H.

In recently years, especially after the 2008 financial crisis, more and more attentions are paid to the risk of model misspecification. The so-called Robust approach, is dedicated to address the model risk. Instead of a fixed model, here we consider a family of models $\mathcal{P}$, which is a set of probability measures on $(\Omega, \mathcal{F})$. We mention here two extreme cases: first, when $\mathcal{P}$ contains all Dirac measures $\{\delta_\omega, \omega \in \Omega\}$, it leads to the so-called model free pricing approach. Secondly, when $\mathcal{P} = \{\mathbb{P}\}$, it is reduced to the classical dominated case.

1.1.1 A brief litterature review of super-replication duality under model uncertainty

In the continuous time models under volatility uncertainty, the super-replication duality was first studied by Denis and Martini [3] using the capacity theory and Peng [4] using an approximation of Markovian control problems. In a series of papers [5, 6, 7], Soner, Touzi and Zhang used the super-martingale decomposition methods together with the aggregation results to prove the duality. Their approach was further extended by Nutz and Neufeld [8] which dropped the strong continuity condition and by Possamai et al.[9] which adapted to the context of Martingale optimal transport.

In [10], Nutz started to consider the super-replication problem under model uncertainty in discrete time. He established the existence of the super-replication strategy in dimension $d = 1$ and a rather abstract duality result by choosing some proper functional space to carry out the separation argument. In the paper [11], general pricing-hedging duality was shown in a pointwise setting. They gave a characterization of the subset of trajectories on which the duality gap disappears. This result was further extended in the paper of [12], to the setting of abstract model specification and generic finite set of dynamically traded options. In particular, they added the so-called prediction sets which represent the convictions on the set of models.

In their seminal paper, Bouchard and Nutz [1] proved the fundamental theorem of asset pricing in discrete time under a rather general framework. Their approach was motivated by the seminal work of Dalang-Morton-Willinger [13], and based crucially on the equivalence between the local and global no-arbitrage. In a first step, they proved the one-period result with finite-dimensional separation arguments in a local fashion. In a second step, they used the measurable selection arguments to paste the one-period result together.

More recently, the super-replication duality under model uncertainty have also been extended to the continuous time framework. In [14], the authors worked under a proper no-arbitrage condition and proved a version of fundament theorem of asset pricing and gave a representation of the superhedging price. In [15] and [16], the authors extended the framework of [12] to continuous time model by introducing the prediction sets.

1.1.2 Super-replication with information of marginals: Martingale optimal transport(MOT)

In the work of Breeden and Litzenberger [17], they have observed that one can retrieve the information of the marginal distribution of the underlyings once the option prices are available for all strikes K. Indeed, as the price of the call can be written as $c(K) = \int_{\mathbb{R}}(x-K)^+\mu(dx)$ (where c is the price of the vanilla call and μ is the market implied distribution of stock price S, and we assumed interest rate $r = 0$), we have $\mu = c''$. Thus when modeling the dynamic of the stock, we should put more constraints on the distribution of the stock in order to be consistent with the market information.

In this case, to eliminate the model risk and to stay consistent with the market information, we should consider the set of martingale measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ satisfying the marginal distribution constraint $X_T \sim^{\mathbb{Q}} \mu$, where μ is obtained from the price of the vanilla options on the market. We note $\mathcal{M}(\mu)$ the set of martingale measures satisfying the distribution constraint. Thus the upper bound of the price of the contingent claims given the market information can be written as

$$P(\mu) = \sup_{\mathbb{Q}\in\mathcal{M}(\mu)} \mathbb{E}^{\mathbb{Q}}[\xi].$$

For a hedging point of view, in the case with vanilla options liquid on the market, we are able to use the semi-static strategies to implement the super-replication. Here, "semi-static" means that we have static positions on the options and dynamic positions on the underlyings. By the Carr-Madan formula:

$$\lambda(x) = \lambda(x_i) + (x-x_i)\lambda'(x_i) + \int_{-\infty}^{x_i}(K-x)^+\lambda''(K)dK + \int_{x_i}^{+\infty}(x-K)^+\lambda''(K)dK,$$

any European option with payoff $\lambda(X_T)$ could be represented as a basket of vanilla options, whose cost is given by $\mu(\lambda) := \int_{\mathbb{R}}\lambda(x)\mu(dx)$. Thus the super-replication cost with semi-static strategy can be represented as following:

$$D(\mu) := \inf\{\mu(\lambda) : \exists(\lambda, H) \in \mathbb{L}^1 \times \mathcal{H}, \text{s.t.}\lambda(X_T) + (H\cdot X)_T \geqslant \xi_T, \mathbb{P}-\text{a.s.}, \forall\mathbb{P}\in\mathcal{P}\}.$$

As an extension of both the classical pricing-hedging duality and the classical optimal transport, we expect to obtain the following result:

$$P(\mu) = D(\mu)$$

which we shall call martingale optimal transport(MOT). We are interested in the following

questions relating the MOT:

(i) When does the duality hold?

(ii) How to characterize the optimizer of both the primal and dual problem?

(iii) How to calculate numerically the value of the primal and dual problem?

As the main subject of this thesis is on super-replication, we shall only focus on the first question here. We should also mention that before the MOT approach, the problem of super-replication with marginal information was initially studied by the approach of optimal Skorokhod embedding problem(oSEP). A basic SEP is: given a centered probability measure μ with finite first moment and a Brownian motion W, one aims to find a stopping time τ such that $W_\tau \sim \mu$ and $(W_{\tau\wedge t})_{t\geqslant 0}$ is uniformly integrable. The SEP problem has many explicitly constructive solutions and each of them optimizes a certain criterion. The optimal SEP unifies the above specific constructions and aims at finding an optimal solution of SEP w.r.t. some reward function. When the reward function is time invariant, the equivalence between the MOT and oSEP can be proved by using the Dubins-Schwarz theorem.

Typically, the duality $P(\mu) = D(\mu)$ can be proved in two steps, see for example [18]. Firstly, under the condition that Φ is bounded from above and upper semicontinuous, one can prove that

$$\mu \in \mathfrak{B}(\mathbb{R}^M) \mapsto \sup_{\mathbb{Q}\in\mathcal{M}(\mu)} \mathbb{E}^{\mathbb{Q}}\big[\Phi\big] \in \mathbb{R}$$

is concave and upper semicontinuous, where we equip $\mathfrak{B}(\mathbb{R}^M)$ with a Wasserstein kind topology. Secondly, using Fenchel–Moreau theorem, it follows that

$$\sup_{\mathbb{Q}\in\mathcal{M}(\mu)} \mathbb{E}^{\mathbb{Q}}\big[\Phi\big] \;=\; \pi^E_{\mu,0}(\Phi) \;:=\; \inf_{\lambda\in\Lambda}\Big\{\mu(\lambda) + \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\big[\Phi - \lambda\big]\Big\}.$$

The above is called the intermediate duality, where the marginal constraints have been removed. Now by applying a certain version of super-replication duality, we can finally get the desired result.

The duality can also be proved using a different approximation arguments, without passing by the intermediate duality. For more details, see Chapter 3.

1.1.3 Bouchard-Nutz discrete time framework

As it will be cited many times in the following, we first recall briefly the framework of Bouchard and Nutz [1]. Let $\Omega_0 := \{\omega_0\}$ be a singleton and Ω_1 be a Polish space. For each $t = 1, \cdots, T$, denote by $\Omega_t := \Omega_1^t$ the t-fold Cartesian product of Ω_1 and let $\mathcal{F}_t^0 := \mathcal{B}(\Omega_t)$ and $\mathcal{F}_t$ its universal completion. The measurable space $(\Omega, \mathcal{F})$ and the associated filtrations are defined by

$$\Omega := \Omega_T,\ \mathcal{F} := \mathcal{F}_T,\ \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T} \ \text{ and } \mathbb{F}^0 := (\mathcal{F}_t^0)_{0\leq t\leq T}.$$

Denoting by $\mathcal{P}_t(\omega)$ a non-empty convex set in $\mathfrak{B}(\Omega_1)$, which represents the set of all possible models for the $(t+1)$-th period, given the state $\omega \in \Omega_t$ at time $t = 0, 1\cdots, T-1$. The set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ can now be introduced as

$$\mathcal{P} := \big\{\mathbb{P} := \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \ : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot) \text{ for } t \leq T-1\big\}.$$

where $\mathbb{P}_t : \Omega_t \mapsto \mathfrak{B}(\Omega_1)$ is a probability kernel and the probability measure $\mathbb{P}$ is defined by Fubini's theorem in the sense that

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2 \cdots, \omega_T)\mathbb{P}_{T-1}(\omega_1, \cdots, \omega_{T-1}; d\omega_T) \cdots \mathbb{P}_0(d\omega_1).$$

To ensure that $\mathcal{P}$ is not empty, they gave the following measurability assumption: for each t,

$$\text{graph}(\mathcal{P}_t) := \big\{(\omega, \mathbb{P}) : \omega \in \Omega_t, \mathbb{P} \in \mathcal{P}_t(\omega)\big\} \ \subseteq \ \Omega_t \times \mathcal{P}(\Omega_1) \ \text{ is analytic.}$$

Under this framework, they gave a quasi-sure version of no-arbitrage condition and proved a corresponding First Fundamental Theorem of Asset Pricing(FTAM). Their proof is in the spirit of Dalang-Morton-Willinger [13]: first they proved the local version of the result by separation arguments of finite dimension, then used the measurable selection argument to paste the one-period result together. One of the crucial feature used in the proof is that the local version of no-arbitrage is indeed equivalent to the global version. They also considered the trading strategy with finitely many static options and obtained a non-dominated version of optional decomposition as a by-product.

In the following, we shall consider several important topics in robust finance. More precisely, in Section 1.2 we consider the super-replication of American options in discrete time financial models where some assets are traded dynamically and a family of European options are traded statically. In Section 1.3, we consider a discrete time financial market with proportional transaction cost under model uncertainty, and study a super-replication problem. The key technique is to convert the initial problem to a frictionless problem set on an enlarged space by using a randomization technique together with the minimax theorem. In Section 1.4, we study the utility maximization problem in a similar setting with the previous section.

In Section 1.5, we present a numerical method based on a sparse grid approximation to compute the loss distribution of the balance sheet of an insurance company.

1.2 Super-replication of American option under model uncertainty

1.2.1 Duality gap for the naive formulation

By far the most parts of litteratures worked with the European type of the contingent claims under model uncertainty. In the first part of the thesis, we shall consider the pricing-hedging duality for the American options in discrete time with semi-static trading strategy under model uncertainty. In the classical dominated case, if the market is complete and denoting by $(\Phi_t)_{t\in\mathbb{T}}$ the payoff of the American option, it is well-known that the price is given by the initial value of its Snell envelop.

In the non-dominated case, the superhedging cost of the American option Φ should be the minimal initial cost such that there exists a hedging strategy to dominate the payoff of the American option whatever the exercising strategy of the counterparty. In [19], we work in a general discrete time framework: let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\ldots,N}$ be a filtration, where $\mathcal{F}_0$ is trivial and $N \in \mathbb{N}$ is the time horizon. We have an adapted process $S = (S_t)_{0\leqslant t\leqslant N}$ which represents the discounted stock process and a family of random variables $(g^\lambda)_{\lambda\in\Lambda}$ which represents the static (european) options. In particular, this abstract framework will cover the following well-studied discrete time framework:

(i) The classical dominated case where $\mathcal{P} = \{\mathbb{P}\}$, i.e. there is no model uncertainty.

(ii) The Bouchard-Nutz framework with $\Omega = \Omega_1^N$. where each Ω_1 is a Polish space and the number of static options is finite.

(iii) The Martingale optimal transport(MOT) framework with infinitely many static options, with the canonical space $\Omega = \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, and $\mathcal{P}$ is the collection of all Borel probability measures on Ω.

(iv) The framework of [12] with finite static options, where Ω is a Polish space, and $\mathcal{P}$ is the collection of all Borel probability measures on Ω. In particular, the authors worked on some prediction sets representing the convictions to the set of models.

In [19], we considered the second and third cases. We denote by $\mathcal{H}$ the set of all $\mathbb{F}$-predictable processes, and by $\mathfrak{h}$ the set of real numbers with finitely many nonzero elements for the well-posedness of the static strategy. For the simplicity of presentation, we assume here the number of stocks $d = 1$, and the associated final payoff is given by

$$(H \circ S)_N + hg = \sum_{k=1}^{N} H_k \Delta S_k + \sum_{\lambda \in \Lambda} h^\lambda g^\lambda, \tag{1.2.1}$$

where $\Delta S_k = S_k - S_{k-1}$. As stated above, the natural formulation of superhedging problem should be

$$\begin{aligned}\pi_g^A(\Phi) := \inf \Big\{ x : &\exists ({}^1H, ..., {}^NH) \in \mathcal{H}^N \text{ s.t. } {}^jH_i = {}^kH_i \ \forall 1 \leqslant i \leqslant j \leqslant k \leqslant N \text{ and } h \in \mathfrak{h} \\ &\text{satisfying } x + ({}^kH \circ S)_N + hg \geqslant \Phi_k \ \forall k = 1, ..., N \ \mathcal{P}\text{-q.s.} \Big\}\end{aligned} \tag{1.2.2}$$

The above definition is consistent with the fact that we are allowed to adjust our strategy in response to an early exercise.

In spirit of the classical dominated case, a natural guess for the super-replication cost would be

$$\pi_g^A(\Phi) \stackrel{?}{=} \sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau], \tag{1.2.3}$$

where $\mathcal{T}(\mathbb{F})$ denotes the set of $\mathbb{F}$-stopping times, and $\mathcal{M}_g$ represents the set of martingale measures which are consistent with the option price. However, this duality may fail. The "numerical" reason is that the RHS in (1.2.3) may be too small since the set $\mathcal{M}_g \times \mathcal{T}(\mathbb{F})$ is too small. This phenomenon was also observed by some other work. In [20], the authors studied under the framework of Bouchard-Nutz the super-replication problem $\pi_g^A(\Phi)$ with finitely many static options, and establish the duality

$$\pi_g^A(\Phi) = \inf_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \sup_{\mathbb{Q} \in \mathcal{M}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau - hg], \tag{1.2.4}$$

under rather strong regularity conditions (see their Proposition 3.1). Due to the use of the strong stopping time, they also observed the duality gap between $\pi_g^A(\Phi)$ and $\sup_{\tau \in \mathcal{T}(\mathbb{F})} \sup_{\mathbb{Q} \in \mathcal{M}_g} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]$. In exchange, in [20] the authors studied the following subhedging problem $\sup_{\tau \in \mathcal{T}(\mathbb{F})} \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]$. In a subsequent paper [21], they extended the above superhedging duality by introducing randomized stopping time and using a minimax theorem. The new duality is somehow more complete, however, it heritates the same strong regularity condition as [20] plus some strong integrability conditions.

In [22] and [23], Hobson and Neuberger studied the same superhedging problem in a Markovian setting, where the underlying process S takes value in a discrete lattice $\mathcal{X}$. By considering

the class of weak stopping times, they then formulated both the pricing and the hedging problems as linear programming problems under linear constraints, which can be solved numerically.

In [19], we proposed to remedy (1.2.3) in two ways: the first is to use an enlarged space formulation to turn the American option into a European option, which is equivalent in spirit to the weak formulation of Hobson and Neuberger [23]. The second one is to allow the dynamical trading of the vanilla options such that the set of stopping times $\mathcal{T}$ corresponds to a larger filtration $\widehat{\mathbb{F}}$.

1.2.2 Reformulation of American options as European options

Due to the presence of the statically traded options, the pricing-hedging duality is broken and the correct formulation is to introduce an enlarged space which turns the American options into European options. The enlargement of space is based on construction of random times, previously used e.g. in [24, 25] to study the existence of random times with a given survival probability and in [26] to study a general optimal control/stopping problem, and in [18] and [27] to study the optimal Skorokhod embedding problem.

Let $\mathbb{T} := \{1, ..., N\}$ and introduce the space $\overline{\Omega} := \Omega \times \mathbb{T}$ with the canonical time $T : \overline{\Omega} \to \mathbb{T}$ given by $T(\bar{\omega}) := \theta$, where $\bar{\omega} := (\omega, \theta)$, the filtration $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_k)_{k=0,1,...,N}$ with $\overline{\mathcal{F}}_k = \mathcal{F}_k \otimes \vartheta_k$ and $\vartheta_k = \sigma(T \wedge (k+1))$, and the σ-field $\overline{\mathcal{F}} = \mathcal{F} \otimes \vartheta_N$. Let Υ be a given class of functions defined on Ω. In the following, it will correspond to the set of upper-semianalytic functions in the framework of Bouchard-Nutz and the set of upper-semicontinuous functions in the framework of Martingale optimal transport. Denote by $\overline{\mathcal{H}}$ the class of $\overline{\mathbb{F}}$-predictable processes and extend S and g^λ to $\overline{\Omega}$ naturally as $S(\bar{\omega}) = S(\omega)$ and $g^\lambda(\bar{\omega}) = g^\lambda(\omega)$ for $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$. Denote $\overline{\Upsilon}$ the class of random variables $\bar{\xi} : \overline{\Omega} \to \mathbb{R}$ such that $\bar{\xi}(\cdot, k) \in \Upsilon$ for all $k \in \mathbb{T}$ and let $\overline{\pi}^E_g(\bar{\xi})$ the superreplication cost of $\bar{\xi}$. The class of American objects $\overline{\Upsilon}$ on $\overline{\Omega}$ can be identified as European objects Υ^N on Ω via $\bar{\xi}(\bar{\omega}) = \Phi_\theta(\omega)$. Introduce the set of models and the set of martingale measures on the enlarged space: $\overline{\mathcal{P}} = \{\overline{\mathbb{P}} \in \mathfrak{P}(\overline{\Omega}) : \overline{\mathbb{P}}_{|\Omega} \in \mathcal{P}\}$, $\overline{\mathcal{M}} = \{\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega}) : \overline{\mathbb{Q}} \ll \overline{\mathcal{P}}$ and $\mathbb{E}^{\overline{\mathbb{Q}}}[\Delta S_k | \overline{\mathcal{F}}_{k-1}] = 0 \ \forall k \in \mathbb{T}\}$ and $\overline{\mathcal{M}}_g = \{\overline{\mathbb{Q}} \in \overline{\mathcal{M}} : \mathbb{E}^{\overline{\mathbb{Q}}}[g^\lambda] = 0 \ \forall \lambda \in \Lambda\}$.

The proof of the european pricing-hedging duality on the enlarged space relies crucially on the definition of concatenation on the enlarged space, which will ensure the dynamic programming property. Then we have the following first main result of [19].

Theorem 1.2.1. *For any $\Phi \in \Upsilon^N = \overline{\Upsilon}$ we have*

$$\pi^A_g(\Phi) = \overline{\pi}^E_g(\Phi) := \inf\{x : \exists\ (\overline{H}, h) \in \overline{\mathcal{H}} \times \mathfrak{h} \ \ s.t. \ \ x + (\overline{H} \circ S)_N + hg \geqslant \xi \ \overline{\mathcal{P}}\text{-}q.s.\}. \quad (1.2.5)$$

In both the framework of Bouchard-Nutz and Martingale optimal transport, we have the pricing-hedging duality

$$\pi^A_g(\Phi) = \overline{\pi}^E_g(\Phi) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]. \quad (1.2.6)$$

Remark 1.2.2. *The above $\overline{\mathcal{M}}_g$ in* (1.2.6) *is equivalent to a* weak formulation *of the RHS of* (1.2.3)*, similar to the spirit of [23]. Indeed, we define a weak stopping rule α a collection*

$$\alpha \ = \ \big(\Omega^\alpha,\ \mathcal{F}^\alpha,\ \mathbb{Q}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}^\alpha_k)_{0 \leq k \leq N},\ (S^\alpha_k)_{0 \leq k \leq N},\ (g^{\lambda,\alpha})_{\lambda \in \Lambda},\ (\Phi^\alpha_k)_{k \in \mathbb{T}},\ \tau^\alpha\big)$$

with $\big(\Omega^\alpha,\ \mathcal{F}^\alpha,\ \mathbb{Q}^\alpha, \mathbb{F}^\alpha\big)$ a filtered probability space, τ^α a $\mathbb{T}$-valued $\mathbb{F}^\alpha$-stopping time, an $\mathbb{R}^d$-valued $(\mathbb{Q}^\alpha, \mathbb{F}^\alpha)$-martingale S^α and a collection of random variables $g^{\lambda,\alpha}, \Phi^\alpha_k$, and such that there is a measurable surjective mapping $\mathtt{i}_\alpha : \Omega^\alpha \to \Omega$ with $\mathbb{Q} = \mathbb{Q}^\alpha \circ \mathtt{i}_\alpha^{-1} \in \mathcal{M}$ and $\mathtt{i}_\alpha^{-1}(\mathcal{F}_k) \subset \mathcal{F}^\alpha_k$,

$\mathtt{i}_\alpha^{-1}(\mathcal{F}) \subset \mathcal{F}^\alpha$ *and finally* $\mathcal{L}_{\mathbb{Q}^\alpha}(S^\alpha, g^\alpha, \Phi^\alpha) = \mathcal{L}_{\mathbb{Q}}(S, g, \Phi)$. *Denote by* $\mathcal{A}_g$ *the collection of all weak stopping rules* α *such that* $\mathbb{E}^{\mathbb{Q}^\alpha}\left[g^{\lambda,\alpha}\right] = 0$ *for each* $\lambda \in \Lambda$. *We have*

$$\sup_{\alpha\in\mathcal{A}_g} \mathbb{E}^{\mathbb{Q}^\alpha}\left[\Phi_{\tau^\alpha}\right] = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi].$$

Remark 1.2.3. *In the case without static options, the strong and weak formulations are indeed equivalent:*

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]. \tag{1.2.7}$$

Remark 1.2.4. *In the case with static options, we have in general*

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \geqslant \sup_{\mathbb{Q}\in\mathcal{M}_g} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau],$$

where the last inequality may be strict. We can indeed identify the subset(s) of $\overline{\mathcal{M}}_g$ *which leads to equality in place of inequality above. Introduce*

$$\begin{aligned}\underline{\mathcal{M}}_g := \big\{\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega}) :\ & \overline{\mathbb{Q}} \lll \overline{\mathcal{P}},\ \mathbb{E}^{\overline{\mathbb{Q}}}[g^\lambda] = 0,\ \lambda \in \Lambda,\ S \text{ is an } (\mathbb{F},\overline{\mathbb{Q}})\text{-martingale,}\\ & \mathbb{E}^{\overline{\mathbb{Q}}}[M_T] = \mathbb{E}^{\overline{\mathbb{Q}}}[M_0],\ \text{for all bounded } (\mathbb{F},\overline{\mathbb{Q}})\text{-martingales } M\big\},\end{aligned} \tag{1.2.8}$$

the set of measures which make S *an* $\mathbb{F}$*-martingale and* T *an* $\mathbb{F}$*-pseudo–stopping time. Then we have*

$$\sup_{\overline{\mathbb{Q}}\in\underline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\mathbb{Q}\in\mathcal{M}_g} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}\left[\Phi_\tau\right].$$

1.2.3 Transforming static options into dynamic options

The duality of (1.2.3) is violated due to the existence of statically traded options. In the previous section we suggest the introduction of an enlarged space to remedy the duality. In this section, we shall give another way to make the RHS of (1.2.3) large enough by introducing a fictitious larger market where both S and all the options $(g^\lambda)_{\lambda\in\Lambda}$ are traded dynamically.

Loosely speaking, a dynamic extension $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$ of the original market $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$ is a fictitious market where we can dynamically trade both the stock and the option, and $\widehat{\mathbb{F}} \supset \mathbb{F}$ is the filtration containing the information of both the price of the option and the stock. For more precise definition of dynamic extension, see Chapter 3.

On a dynamic extension $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}})$ of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$ we can define similarly the dynamical trading strategies $\widehat{H}$ and the superhedging costs $\widehat{\pi}^A(\Phi)$ of an American option $\widehat{\Phi} = (\widehat{\Phi}_k)_{k\leqslant N}$ on $\widehat{\Omega}$. We have the following main result:

Theorem 1.2.5. *In both the framework of Bouchard-Nutz and martingale optimal transport, for all* $\Phi \in \Upsilon^N$,

$$\pi_g^A(\Phi) = \widehat{\pi}^A(\Phi) = \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{M}}} \sup_{\hat{\tau}\in\mathcal{T}(\widehat{\mathbb{F}})} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\Phi_{\hat{\tau}}\right] = \sup_{\widehat{\mathbb{Q}}\in\overline{\widehat{\mathcal{M}}}} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\Phi\right] \geqslant \sup_{\mathbb{Q}\in\mathcal{M}_g} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}\left[\Phi_\tau\right], \tag{1.2.9}$$

where $\mathcal{M}_g$ *is the set of martingale measures on* Ω *consistent with the market information,* $\widehat{\mathcal{M}}$ *and* $\overline{\widehat{\mathcal{M}}}$ *are the set of martingale measures on* $\widehat{\Omega}$ *and the enlarged space of* $\widehat{\Omega}$.

In the context of martingale optimal transport, the dynamic extension is closely related to the measure-valued martingale. It has been used by Eldan in [28] to study the Skorokhod embedding problem and Cox and Kallblad in [29] to study the martingale optimal transport problem for Asian options. In particular, this point of view allows to obtain the dynamic programming principle with marginal constraint since the terminal constraint is transformed into the initial constraint. To illustrate the idea, consider the case with only the final marginal, i.e. the marginal dates $\mathbb{T}_0 = \{N\}$. The marginal μ is a probability measure on $\mathbb{R}$. Let $\mathfrak{P}_1(\mathbb{R})$ be the set of probability measures with finite first moment equipped with the 1-Wasserstein distance, which makes $\mathfrak{P}_1(\mathbb{R})$ a Polish space. Define the following canonical space for the measure-valued process $\widehat{\Omega} := \{\mu\} \times (\mathfrak{P}_1(\mathbb{R}))^N$ and denote $\widehat{X} = (\widehat{X}_k)_{0\leqslant k\leqslant N}$ the canonical process on $\widehat{\Omega}$. Define $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_k)_{0\leq k\leq N}$ the universal completion of the canonical filtration and denote by $\mathcal{T}(\widehat{\mathbb{F}})$ the collection of all $\widehat{\mathbb{F}}$-stopping times. For $f \in \mathcal{C}_1$, where $\mathcal{C}_1$ is the set of all continuous functions on $\mathbb{R}$ with linear growth, we denote the process of its integrals against $\widehat{X}$ as $\widehat{X}_k(f) = \int_{\mathbb{R}} f(x)\widehat{X}_k(dx)$ and $\widehat{X}_k(id) = \int_{\mathbb{R}} x\widehat{X}_k(dx)$. Define $\mathtt{i} : \widehat{\Omega} \to \Omega$ by $\mathtt{i}(\widehat{\omega}) = (\widehat{X}_0(id)(\widehat{\omega}), ..., \widehat{X}_N(id)(\widehat{\omega}))$ which is surjective and naturally extends processes on Ω to processes on $\widehat{\Omega}$. In particular the price process extends via $S_k(\widehat{\omega}) = S_k(\mathtt{i}(\widehat{\omega})) = \widehat{X}_k(id)(\widehat{\omega})$ and the statically traded options via $g^\lambda(\widehat{\omega}) = g^\lambda(\mathtt{i}(\widehat{\omega})) = \lambda(\mathtt{i}(\widehat{\omega})) - \mu(\lambda)$. Finally, we define a family of random variables $Y = (Y^\lambda)_{\lambda\in\Lambda}$:

$$Y_k^\lambda = \begin{cases} \widehat{X}_k(\lambda) - \mu(\lambda) & 0 \leqslant k \leqslant N-1 \\ g^\lambda = \lambda(\widehat{X}_N(id)) - \mu(\lambda) & k = N \end{cases}$$

The definition of measure-valued martingale measure is given as below:

Definition 1.2.6. *A probability measure $\widehat{\mathbb{Q}}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is called a measure–valued martingale measure (MVM measure) if the process $(\widehat{X}_k(f))_{0\leq k\leq N}$ is a $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingale for all $f \in \mathcal{C}_1$.*

Let us denote by

$$\widehat{\mathcal{M}}_\mu = \{\widehat{\mathbb{Q}} \in \mathfrak{B}(\widehat{\Omega}\} : \widehat{\mathbb{Q}} \text{ is MVM measure}\}.$$

For any $\mathbb{Q} \in \mathcal{M}_\mu$ we define a mapping $\mathtt{j}_\mathbb{Q} : \Omega \to \widehat{\Omega}$ by $\mathtt{j}_\mathbb{Q}(\omega) = (\mathcal{L}_\mathbb{Q}(S_N|\mathcal{F}_k)(\omega))_{k\leqslant N}$. Now we define a map $\mathtt{J} : \mathcal{M}_\mu \to \widehat{\mathcal{M}}$ by $\mathtt{J}(\mathbb{Q}) = \mathbb{Q} \circ \mathtt{j}_\mathbb{Q}^{-1}$. It can be proved that $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, Y, \mathtt{i}, J)$ is a dynamic extension of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$.

More recently, measured-valued martingales are used to study other related problems in mathematical finance. In particular, in [30], it has been used to study the martingale optimal transport problem where the cost functionals are represented by optimal stopping problems. In [31] and [32], it has been used to study the distribution constrained optimal stopping problem.

1.2.4 American options as hedging strategy

In the more recent papers of Bayraktar and Zhou [33], [34], they extended the study to the case where we can use liquid American options as static hedging strategies. The crucial assumption they introduced in their paper is the infinite divisibility of the American options, which says that one can decompose the unit American options into small pieces and trade each piece seperately. In their first paper [33], they assumed the prohibition of short-selling and considered only the subhedging duality, and proved the infinite divisibility indeed gave a correct formulation of subhedging price. While in a second paper [34], they employed our space enlargement technique to reformulate the shorted American options as European option in the enlarged space, thus solved the more complete duality allowing short-selling and super-replication. The core of analysis is

that we should distinguish between two types of American options: the first one is the long position in the hedging portfolio or the subhedging problem; the second one is the short position in the hedging portfolio or the superhedging problem. For the first one, we are the holders of the American option and can employ the optimal stopping strategy, thus our hedging strategy only need to adapt to $\mathbb{F}$; while for the second one, we should adjust our strategy in response to the American option holder, thus the hedging strategy should adapt to a large filtration $\overline{\mathbb{F}}$.

1.3 Super-replication with proportional transaction cost under model uncertainty

In the classical discrete time market with proportional transaction cost, the notion of equivalent martingale measure is replaced by the consistent price system(CPS), which consists of a couple $(\mathbb{Q}, Z)$ such that $\mathbb{Q} \ll \mathbb{P}$ and Z is a $\mathbb{Q}$-martingale which lies in the bid-ask spread.

The fundamental theorem of asset pricing is proved in different contexts with different notions of no-arbitrage conditions. Firstly, the usual no-arbitrage condition can only guarantee the existence of consistent price system when Ω is finite. In [35], the authors proposed a so-called strict no-arbitrage condition, which is equivalent to the existence of strict consistent price system when no couple of assets can be exchanged freely. In [36], the author proposed a robust no-arbitrage condition, which can guarantee the existence of strict consistent price system with less assumptions.

In the context without model uncertainty, the super-replication duality with proportional transaction cost is also well studied, see for example [37] and [38]. More general duality arised in the problem of utility maximization is then studied through the super-replication duality via its relationship to the shadow price, see [39]. We should also mention that there is work which lies beyond the scope of proportional transaction costs. In [40], the authors proved the super-replication duality with transient price impact in a continuous time financial model and established in particular the optimality of the buy-and-hold strategies. While in [41], the authors considered the problem of super-replication with fixed transaction costs as a stochastic impulse control problem with a terminal state constraint.

In the non-dominated case without transaction cost, one of the crucial features in [1] is the equivalence between the local no-arbitrage and the global no-arbitrage. This property allows one to apply a finite dimensional separation argument in the local fashion and then pass to the multi-period case by the measurable selection argument. Thus, the choice of a proper version of no-arbitrage condition is rather important. In the paper of [42], they proposed a proof using the strict no-arbitrage condition. However, as the local strict no-arbitrage is not equivalent to the global strict no-arbitrage in their sense, a rather intricate backward-forward procedure was needed to obtain the multi-period result.

An extension of the Bouchard-Nutz framework is proposed in [43]. In their paper, they used the so-called no-arbitrage of the second type(NA2($\mathcal{P}$)), proposed initially in the paper of [44]. The main advantage is that NA2($\mathcal{P}$) enjoys the equivalence between the local and global no-arbitrage.

The super-replication with transaction cost under model uncertainty was first studied by [45, 46] by first discretizing the space and then pass to the limit. Burzoni [47] gave a complementation of previous work in which pointwise super-hedging is considered. He considered a multivariate market, while all transactions should go through the cash account. By constructing a fictitious price system in which the frictionless superhedging price is the same as in the original market, he was able to prove the result with the presence of static options. In [48], the authors considered the problem of super-replication with nonlinear transaction costs.

1.3.1 The financial market with proportional transaction cost

Given $(\Omega, \mathcal{F})$ a measurable space, equipped with two filtrations $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t=0,1,\cdots,T} \subset \mathbb{F} = (\mathcal{F}_t)_{t=0,1,\cdots,T}$ for some $T \in \mathbb{N}$ and a d-dimensional measurable process $(S_t)_{0\leqslant t\leqslant T}$($\mathbb{F}^0$ is generated by S, while $\mathbb{F}$ its universal completion).

Example 1.3.1. *To give an intuition of the portfolio process with transaction cost, we consider first the case $d = 2$ with a cash account and a stock account. Because of the transaction cost, at each time t, the stock is bought at a price $\overline{S}_t$ and sold at a price $\underline{S}_t$ with $\underline{S}_t < \overline{S}_t$. We define the admissible strategy an $\mathbb{F}$-predictable process $H = (H_t)_{0\leqslant t\leqslant T+1}$ with $H_0 = H_{T+1} \equiv 0$ and denote $\Delta H_t = H_t - H_{t-1}$. Then the P&L of an admissible strategy is given by*

$$V_T(H) = -\sum_{t=1}^{T+1} (\overline{S}_{t-1}\mathbf{1}_{\Delta H_t>0} + \underline{S}_{t-1}\mathbf{1}_{\Delta H_t<0})\Delta H_t$$

Given an exotic option $\xi : \Omega \to \mathbb{R}$ and finitely many liquid options $\eta_i : \Omega \to \mathbb{R}$ with price 0, the super-replication cost of ξ using static strategy on options and dynamic strategy on stocks can be given by

$$\pi_e := \inf\{x : \exists h \in \mathbb{R}^e, H \text{ admissible}, x + \sum_{i=1}^{e} h_i\eta_i + V_T(H) \geqslant \xi, \mathcal{P} - q.s.\}$$

In a more general framework with $d \geqslant 2$, the transaction cost is modeled using the concept of solvency cone, see for example [49]. For every $t \in \{0, 1, \cdots, T\}$, $K_t : \Omega \to 2^{\mathbb{R}^d}$ is a $\mathcal{F}_t^0$-measurable random set in the sense that $\{\omega \in \Omega : K_t(\omega) \cap O \neq \emptyset\} \in \mathcal{F}_t^0$ for every closed (open) set $O \subset \mathbb{R}^d$. For each $\omega \in \Omega$, $K_t(\omega)$ is a closed convex cone containing $\mathbb{R}^d_+$, and represents the collection of positions which can be turned into non-negative ones after exchanging between assets. We denote by $K_t^* \subset \mathbb{R}^d_+$ its (nonnegative) dual cone:

$$K_t^*(\omega) \ := \ \big\{y \in \mathbb{R}^d \ : x \cdot y \geq 0 \text{ for all } x \in K_t(\omega)\big\}, \tag{1.3.10}$$

Example 1.3.2. *If π_t^{ij} is the price of asset j labeled in asset i, then the solvency cone can be represented as*

$$K_t(\omega) := \big\{x : \exists (a^{ij})_{ij} \in \mathbb{R}_+^{d\times d}, s.t. x^i + \sum_{j\neq i} a^{ji} - a^{ij}\pi_t^{ij}(\omega) \geqslant 0, i \leqslant d\big\}.$$

Remark 1.3.3. *As in [43], to obtain the result, the solvency cones are supposed to satisfy several technique conditions, such as efficient frictions and bounded frictions. For more details, see Chapter 4.*

By the concept of solvency cones, we can now give the definition of admissible strategies.

Definition 1.3.4. *We say that an $\mathbb{F}$-adapted process $\eta = (\eta_t)_{0\leq t\leq T}$ is an admissible trading strategy if*

$$\eta_t \in -K_t \quad \mathcal{P}\text{-q.s. for all } t \leq T.$$

We denote by $\mathcal{A}$ the collection of all admissible strategies.

Now the minimal super-hedging cost of the exotic option ξ using vanilla options ζ_i together

with dynamic trading strategy is given by[1]:

$$\pi_e(\xi) := \inf\Big\{y + \sum_{i=1}^{e} c_i|\ell_i| \ : y1_d + \sum_{i=1}^{e} \ell_i\zeta_i + \sum_{t=0}^{T} \eta_t - \xi \in K_T,\ \mathcal{P}\text{-q.s.}, (\eta,\ell) \in \mathcal{A}\times\mathbb{R}^e\Big\},\ (1.3.11)$$

where 1_d is the vector with all components equal to 0 but the last one that is equal to 1.

1.3.2 $NA2(\mathcal{P})$: Equivalence between local no-arbitrage and global no-arbitrage

The spirit of [1] is to pass from local to global, thus it is important to ensure that the local no-arbitrage is impossible if the global no-arbitrage holds. However, for the commonly used weak and strict no-arbitrage, it is not easy to deduce the equivalence between local version and global version. In the recent work of [43], they proposed to use the no-arbitrage of the second kind(NA2($\mathcal{P}$)), which was first introduced in [44]. $NA2(\mathcal{P})$ is defined in the following way:

Definition 1.3.5. *We say that* NA2($\mathcal{P}$) *holds if for all* $t \le T-1$ *and all* $\zeta \in L^0(\mathcal{F}_t)$,

$$\zeta \in K_{t+1}\ \ \mathcal{P}\text{-q.s.} \quad \text{implies} \quad \zeta \in K_t\ \ \mathcal{P}\text{-q.s.}$$

The following robust version of the fundamental theorem was proved in [43].

Theorem 1.3.6. *The condition* NA2($\mathcal{P}$) *is equivalent to : For all* $t \le T-1$, $\mathbb{P} \in \mathcal{P}$ *and* $Y \in L^0_{\mathcal{P}}(\mathcal{F}_t, \mathrm{int}K^*_t)$, *there exists* $\mathbb{Q} \in \mathfrak{B}(\Omega)$ *and a* $\mathbb{F}^0$*-adapted process* $(Z_s)_{s=t,\dots,T}$ *such that* $\mathbb{P} \ll \mathbb{Q}$ *and* $\mathbb{P} = \mathbb{Q}$ *on* $\mathcal{F}_t$, *and*
(i) $\mathbb{Q} \lll \mathcal{P}$
(ii) $Y = Z_t\ \ \mathbb{Q}$*-a.s.*
(iii) $Z_s \in \mathrm{int}K^*_s\ \ \mathbb{Q}$*-a.s. for* $s = t,\dots,T$
(iv) $(Z_s)_{s=t,\dots,T}$ *is a* $\mathbb{Q}$*-martingale, i.e.* $\mathbb{E}^{\mathbb{Q}}[Z_{s'}|\mathcal{F}_s] = Z_s$ *for* $t \le s \le s'$.

A couple $(\mathbb{Q}, Z)$ satisfying the conditions (i) – (iv) above for $t = 0$ is called a strictly consistent price system (SCPS). We denote $\mathcal{S}$ the collection of all SCPS, and set $\mathcal{S}_0 := \big\{(\mathbb{Q}, Z) \in \mathcal{S}$ such that $Z^d \equiv 1\big\}$. In the presence of static option, it should be defined compatible with the bid-ask spreads:

$$\mathcal{S}_e := \big\{(\mathbb{Q}, Z) \in \mathcal{S}_0\ : \mathbb{E}^{\mathbb{Q}}\big[\zeta_i \cdot Z_T\big] \in [-c_i, c_i],\ i = 1, \cdots, e\big\}.$$

The advantage of NA2($\mathcal{P}$) is that it has an equivalence between the absence of local and global arbitrage, thus the global problem can be proved by first solving the one-period problem and pasting them together.

1.3.3 Enlarged space and randomization technique

The main idea of the work is to use a randomization argument by introducing an enlarged space. We shall construct a fictitious price process X with additional randomness, and the original problem with proportional transaction cost can be transformed into a quasi-sure super-replication problem without transaction cost in the fictitious market. This randomization/enlargement technique is in fact in the same spirit of the controlled fictitious market approach of [50, 51].

[1] Here we use the convention $\sum_{i=1}^{0} = 0$

The enlarged space Let $c > 1$ be a constant bounding the transaction cost, we define $\Lambda_1 := [c^{-1}, c]^{d-1}$, $\Lambda_t := (\Lambda_1)^{t+1}$, and $\Lambda := \Lambda_T$. We then introduce the canonical process $\Theta_t(\theta) := \theta_t$, $\forall \theta = (\theta_t)_{0\le t\le T} \in \Lambda$, as well as the σ-fields $\mathcal{F}_t^\Lambda := \sigma(\Theta_s,\ s \le t)$, $t \le T$. The enlarged space can be now defined as $\overline{\Omega} := \Omega \times \Lambda$, on which we introduce an enlarged σ-field $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}_T^\Lambda$, together with two filtrations $\overline{\mathbb{F}}^0 = (\overline{\mathcal{F}}_t^0)_{0\le t\le T}$, and $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\le t\le T}$, where $\overline{\mathcal{F}}_t^0 := \mathcal{F}_t^0 \otimes \mathcal{F}_t^\Lambda$, and $\overline{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}_t^\Lambda$ for $t \le T$.

Now the randomized fictitious market can be defined via the fictitious stock price $X = (X_t)_{0\le t\le T}$:

$$X_t(\bar\omega) := \Pi_{K_t^*(\omega)}[S_t(\omega)\theta_t], \quad \text{for all } \bar\omega = (\omega, \theta) \in \overline{\Omega},\ t \le T, \tag{1.3.12}$$

where $S_t(\omega)\theta_t := (S_t^1(\omega)\theta_t^1, \cdots, S_t^{d-1}(\omega)\theta_t^{d-1}, S_t^d(\omega))$, and $\Pi_{K_t^*(\omega)}[y]$ stands for the projection of $y \in \mathbb{R}^d$ on the convex closed set $K_t^*(\omega)$.

No-arbitrage condition on the enlarged space On the enlarged space, we first introduce the set of possible models:

$$\overline{\mathcal{P}} \ := \ \big\{\overline{\mathbb{P}} \in \mathfrak{B}(\overline{\Omega}, \overline{\mathcal{F}}) \text{ such that } \overline{\mathbb{P}}|_\Omega \in \mathcal{P}\big\}.$$

The set of trading strategies can be also defined naturally on the enlarged space:

$$\overline{\mathcal{H}} := \{\text{All } \overline{\mathbb{F}}\text{-predictable processes}\}.$$

We can now define the following no-arbitrage condition on the enlarged space:

Definition 1.3.7. *We say that* $\mathrm{NA}(\overline{\mathcal{P}})$ *holds if*

$$(\overline{H} \circ X)_T \ge 0,\ \overline{\mathcal{P}}\text{-q.s.} \quad \Longrightarrow \quad (\overline{H} \circ X)_T = 0,\ \overline{\mathcal{P}}\text{-q.s.},$$

for every $\overline{H} \in \overline{\mathcal{H}}$.

By omitting a slight problem at the boundary of the bid-ask spread, the conditions $\mathrm{NA2}(\mathcal{P})$ and $\mathrm{NA}(\overline{\mathcal{P}})$ are equivalent. This allows us to apply the classical results on the enlarged space. Thus the next step is naturally to reformulate both the primal and the dual problem on the enlarged space.

Reformulation of the primal and dual on the enlarged space Denote $\overline{\mathcal{Q}}_0$ the collection of measures $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$ such that $\overline{\mathbb{Q}} \lll \overline{\mathcal{P}}$ and X is a $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale. Then we have the following reformulation result on the enlarged space for both the primal and the dual problem.

Proposition 1.3.8. (i) *For any universally measurable vector* $\xi : \Omega \to \mathbb{R}^d$*, one has*

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0} \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] \ = \ \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big].$$

(ii) *One has*

$$\pi_0(\xi) = \inf\Big\{y \in \mathbb{R} \ : y + (H \circ X)_T \ge g\ \overline{\mathcal{P}}\text{-q.s., for some } H \in \mathcal{H}\Big\}.$$

1.3.4 Main result

The main result of this section is the following super-hedging duality.

Theorem 1.3.9. *Let ξ and $(\zeta_i)_{i\leq e}$ be Borel measurable, and assume that* NA2($\mathcal{P}$) *holds true. Assume either that $e = 0$, or that $e \geq 1$ and for all $\ell \in \mathbb{R}^e$ and $\eta \in \mathcal{A}$,*

$$\sum_{i=1}^{e} \big(\ell_i\zeta_i - |\ell_i|c_i 1_d\big) + \sum_{t=0}^{T} \eta_t \in K_T \ \mathcal{P}\text{-q.s.} \quad \Longrightarrow \quad \ell = 0. \tag{1.3.13}$$

Then $\mathcal{S}_e$ is nonempty and

$$\pi_e(\xi) \ = \ \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e} \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big]. \tag{1.3.14}$$

Moreover, there exists $(\hat{\eta}, \hat{\ell}) \in \mathcal{A} \times \mathbb{R}^e$ such that

$$\pi_e(\xi)1_d + \sum_{i=1}^{e} \Big(\hat{\ell}_i\zeta_i - |\hat{\ell}_i|c_i 1_d\Big) + \sum_{t=0}^{T} \hat{\eta}_t - \xi \in K_T, \ \mathcal{P}\text{-q.s.}$$

Remark 1.3.10. (i) *The proof is done by the combination of the randomisation technique together with a minimax argument.*

(ii) *In the case $e = 1$, we can see clearly see that the condition* (1.3.13) *is a kind of no-arbitrage condition. By using the randomization techniques, $\ell_1\zeta_1 - |\ell_1|c_1\mathbf{1}_d + \sum_{t=0}^{T} \eta_t \in K_T$ $\mathcal{P}$-q.s. can be shown as equivalent to*

$$\ell_1 g_1\big(\bar{\omega}, \widehat{\theta}\big) + \Big(\sum_{t=1}^{T} H_t\Delta X_t\Big)(\bar{\omega}) \geq 0, \ \overline{\mathcal{P}}\text{-q.s. and for both } \widehat{\theta} = \pm 1,$$

where $H_t := \sum_{s=0}^{t-1} \eta_s$ and $g_1(\bar{\omega}, \pm 1) := \zeta_1 \cdot X_T \pm c_1$. For more details, see Chapter 5. The robust no-arbitrage condition on the enlarged space will lead to

$$\ell_1 g_1\big(\bar{\omega}, \widehat{\theta}\big) + \Big(\sum_{t=1}^{T} H_t\Delta X_t\Big)(\bar{\omega}) = 0, \ \overline{\mathcal{P}}\text{-q.s. and for both } \widehat{\theta} = \pm 1.$$

As $g_1(\bar{\omega}, 1) \neq g_1(\omega, -1)$ when $c_1 > 0$, one obtains $\ell_1 = 0$.

1.4 Utility maximization with proportional transaction cost under model uncertainty

1.4.1 Basic background of utility maximization problem

The problem of utility maximization is widely studied in mathematical finance. In the classical dominated market model without transaction cost, the problem of utility maximization with random endowments was extensively investigated. In particular, when dealing with the incomplete market models, the duality approach is a powerful tool which enables us to obtain the existence of the primal optimizer by solving the corresponding dual optimization problem. Typically, the dual problem is formulated on the set of equivalent (local) martingale measures (EMM), whose existence is ensured by some appropriate no arbitrage assumptions. For utilities defined on the positive real line, [52], [53] handled the random payoffs and established the bipolar relationship by choosing the appropriate closure of the dual set of EMM. For utilities defined on the whole

real line, [54], [55] and [56] attacked the problem by choosing a subset of EMM with finite general entropy on the dual side and carefully defining the class of working portfolios on the primal side.

In the dominated case with porportional transaction cost, the EMM is replaced by the Consistent Price Systems(CPS) and the duality of utility maximization is established using the connection of super-replication duality with the so-called shadow price. For the case of general càdlàg processes, [39] proved the existence of a dual optimizer as well as a shadow process while in [57] the authors considered the case of non-semimartingales. For utility maximization problems with the presence of transaction cost and random endowment, more efforts are needed, see for example [58].

In the non-dominated case, the problem of utility maximization in discrete time market without transaction cost was first studied in [59]. Similar to [60] in the dominated case, he was able to prove the existence of the optimal primal strategy by using the dynamic programming principle. In [61], the authors considered the presence of friction and a linearity type of no-arbitrage condition was introduced. [62, 63] worked in a similar context as [59], and it relaxed the bounded from above condition. More recently, [64] and [65] worked under some different contexts and directly dealt with the global problem. In [64], model uncertainty was represented by a collection of stochastic processes instead of a family of probability measures, and the authors use a Komlos-type argument to prove the existence of the optimal strategy. In [65] the authors consider a framework without domination or time-consistency, and use a functional version of Choquet's capacitability theorem and medial limits to prove the existence results and give a dual representation of the utility maximization problem.

In [2], Bartl considered the duality representation for the exponential utility preference without transaction cost, under a rather restrictive no-arbitrage condition. The main objective of [66] is therefore to study the existence of the optimal strategy, the convex duality theorem and the auxiliary dynamic programming principle for a semi-static utility maximization problem with transaction costs in a discrete time framework. In the frictionless framework, our alternative dynamic programming argument allows us to generalize the main results in [2] to a weaker market condition.

1.4.2 Robust utility maximization with transaction cost

We stay in the Bouchard-Nutz framework and the notations are the same with the previous section. To obtain a duality result, we consider the exponential preference with the utility function $U(x) := -\exp(-\gamma x)$, with $\gamma > 0$ a constant related to the risk aversion. For $e \in \mathbb{N} \cup \{0\}$, there are a finite class of $\mathcal{F}_T^0$-measurable random vectors $\zeta_i : \Omega \to \mathbb{R}^d$, $i = 1, \cdots, e$, where each ζ_i represents the payoff of some option i labeled in units of different risky assets. Let $\xi : \Omega \to \mathbb{R}^d$ represent the payoff of the random endowment, then we can consider the following utility maximization problem with random endowments and semi-static strategies:

$$V(\xi, \gamma) := \sup_{(\ell,\eta)\in\mathcal{A}_e} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[U\Big(\Big(\xi + \sum_{i=1}^{e} \big(\ell_i \zeta_i - |\ell_i| c_i 1_d \big) + \sum_{t=0}^{T} \eta_t \Big)^d \Big) \right]. \tag{1.4.15}$$

where 1_d is the vector with all components equal to 0 but the last one that is equal to 1 and $\mathcal{A}_e$ denotes the collection of all $(l, \eta) \in \mathbb{R}^e \times \mathcal{A}$ such that $\xi + \sum_{i=1}^{e} \big(\ell_i \zeta_i - |\ell_i| c_i 1_d \big) + \sum_{t=0}^{T} \eta_t \Big)^i = 0$ for $i = 1, \cdots, d-1$. This means that the last asset plays the role of a numeraire and all the other assets should be liquidated at final time T. The static options ζ_i has price 0, but induces a proportional transaction cost with rate $c_i > 0$ in the static strategy.

In order to give the dual representation for the above problem, let us introduce a robust

version of the relative entropy associated to a probability measure $\mathbb{Q}$ as

$$\mathcal{E}(\mathbb{Q},\mathcal{P}) := \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{E}(\mathbb{Q},\mathbb{P}), \quad \text{where } \mathcal{E}(\mathbb{Q},\mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right], & \text{if } \mathbb{Q}\ll\mathbb{P}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.4.16}$$

Notice that $\mathcal{S}_0$ is a subset of the collection of SCPS $(\mathbb{Q}, Z)$ defined as in section 1.3, we then define

$$\mathcal{S}_e^* := \Big\{(\mathbb{Q},Z)\in\mathcal{S}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[(\xi\cdot Z_T)_-\big]+\mathcal{E}(\mathbb{Q},\mathcal{P})<+\infty \text{ and } \mathbb{E}^{\mathbb{Q}}\big[(\zeta_i\cdot Z_T)\big]\in[-c_i,c_i],\ i=1,\cdots,e\Big\}.$$

Then the main theorem can be stated as below:

Theorem 1.4.1. *Let ξ and $(\zeta_i)_{i\le e}\colon \Omega\to\mathbb{R}^d$ be Borel measurable and assume that* NA2($\mathcal{P}$) *holds. Assume either that $e=0$, or that $e\ge 1$ and for all $\ell\in\mathbb{R}^e$ and $\eta\in\mathcal{A}$,*

$$\sum_{i=1}^{e}\big(\ell_i\zeta_i-|\ell_i|c_i1_d\big)+\sum_{t=0}^{T}\eta_t\in K_T \ \mathcal{P}\text{-q.s.} \implies \ell=0. \tag{1.4.17}$$

Then, we have

$$V(\xi,\gamma) \ = \ -\exp\Big(-\inf_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\gamma\xi\cdot Z_T\big]+\mathcal{E}(\mathbb{Q},\mathcal{P})\big\}\Big), \tag{1.4.18}$$

Moreover, the infimum over $(\ell,\eta)\in\mathcal{A}_e$ is attained by an optimal strategy $(\hat{\ell},\hat{\eta})$.

Remark 1.4.2. *Notice that up to taking logarithm on both sides and replacing $\gamma\xi$ by $-\xi$, the equality* (5.4.16) *is equivalent to*

$$\begin{aligned} &\inf_{(\ell,\eta)\in\mathcal{A}_e}\sup_{\mathbb{P}\in\mathcal{P}}\log\mathbb{E}^{\mathbb{P}}\left[\exp\left(\left(\xi-\sum_{i=1}^{e}(\ell_i\zeta_i-|\ell_i|c_i1_d)-\sum_{t=0}^{T}\eta_t\right)^d\right)\right] \\ &= \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big]-\mathcal{E}(\mathbb{Q},\mathcal{P})\big\}. \end{aligned} \tag{1.4.19}$$

1.4.3 Reformulation and the duality on the enlarged space

The technique used in this section is similarly as section 1.3. In a first step, we shall reformulate the original problem as a new problem without transaction cost on the fictitious enlarged space. Then, by proving the duality on the enlarged the space, the problem is solved. The new difficulty arised here in utility maximisation is that we need to deal with some integrability issues when doing dynamic programming.

Reformulation of the primal and dual on the enlarged space We introduce the enlarged space $\overline{\Omega}$ in the same way as the previous section, and then reformulate the utility maximization problem on the enlarged space, with X the underlying stock and $g(\bar{\omega}) := \xi(\omega)\cdot X_T(\bar{\omega})$, for all $\bar{\omega}=(\omega,\theta)\in\overline{\Omega}$ the contingent claim. Firstly, for a general utility function U, we have the following reformulation of the primal on the enlarged space.

Proposition 1.4.3. *Under certain conditions, we have*

$$V(\xi) \ = \ \sup_{H\in\mathcal{H}}\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}}\mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big].$$

For the dual representation, we will restrict ourselves to the exponential utility function $U(x) := -\exp(-\gamma x)$, for some constant $\gamma > 0$. Moreover, for simplification, we suppose here the number of static options is equal to zero. The case with the presence of static options can be considered similarly by the introduction of a further enlarged space. We define

$$\overline{\mathcal{Q}}^* := \big\{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0 \ : \mathbb{E}^{\overline{\mathbb{Q}}}\big[(\xi \cdot X_T)_-\big] + \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}) < \infty\big\}$$

Thus we have the following reformulation for the dual problem.

Proposition 1.4.4. *For any universally measurable random vector $\xi : \Omega \to \mathbb{R}^d$, one has*

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0^*} \big\{\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\big\} \ = \ \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}})\big\}.$$

Duality on the enlarged space Now it is enough to prove the duality result on the enlarged space and we have the following proposition.

Proposition 1.4.5. *Let $g := \xi \cdot X_T$ and $\mathrm{NA}(\overline{\mathcal{P}})$ hold true. Then one has*

$$\overline{V} \ := \ \inf_{H\in\mathcal{H}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + (H \circ X)_T\big)\big] \ = \ \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}})\big\}. \tag{1.4.20}$$

Moreover, the infimum of the problem $\overline{V}$ is attained by some optimal trading strategy $\widehat{H} \in \mathcal{H}$.

Example 1.4.6. *To illustate the potential difficulty in our proof, let us consider the following one-period example:*

Suppose that $T = 1$ and the dimension of assets $d = 2$. In this context, we have that $\Omega = \mathbb{R}$ and the proportional transaction cost occurs on both dates $t = 0$ and $t = 1$. Thus the buying and selling price at $t = 0$ and at $t = 1$ are respectively $S_0(1+\alpha)$, $S_0(1-\alpha)$, $S_1(1+\alpha)$ and $S_1(1-\alpha)$, with $\alpha > 0$ a constant representing the transaction cost. Our utility maximisation problem with transaction cost can be now written as

$$V_0 := \inf_{H\in\mathbb{R}} \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}[\exp(g + HS_1(1-\alpha)\mathbf{1}_{H\geqslant 0}$$
$$+HS_1(1+\alpha)\mathbf{1}_{H<0} - HS_0(1-\alpha)\mathbf{1}_{H<0} - HS_0(1+\alpha)\mathbf{1}_{H\geqslant 0})]$$

By defining the following enlarged space: $\overline{\Omega} := \mathbb{R} \times [1-\alpha, 1+\alpha]^2$, we can introduce the fictitious process as: $X_0(\theta_0) = S_0\theta_0$, $X_1(S_1, \theta_1) = S_1\theta_1$. The filtration on the enlarged space is thus $\overline{\mathcal{F}}_0 = \sigma(\theta_0)$ and $\overline{\mathcal{F}}_1 = \sigma(S_1) \otimes \sigma(\theta_0, \theta_1)$. By reformulating on the enlarged space, we can prove that both the primal and the dual become utility maximization problem without transaction cost:

$$\sup_{(\mathbb{Q},Z)\in\mathcal{Q}_\star} (\mathbb{E}^{\mathbb{Q}}[g] - \mathcal{E}(\mathbb{Q}, \mathbb{P})) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\star} (\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{P}}))$$

and

$$V_0 = \bar{V}_0 := \inf_{H\in\mathbb{R}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + H(X_1 - X_0)\big)\big].$$

We can now see the problem: in the primal formulation, H is adapted to $\mathcal{F}_0$ instead of $\overline{\mathcal{F}}_0$, thus one cannot directly apply the duality on the enlarged space,

Indeed, this problem can be overcome by a simple minimax argument:

$$\begin{aligned}\bar{V_0} &= \inf_{H\in\mathbb{R}} \sup_{\theta_0\in[1-\alpha,1+\alpha]} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g+H(X_1-X_0)\big)\big] \\ &= \sup_{\theta_0\in[1-\alpha,1+\alpha]} \inf_{H\in\mathbb{R}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g+H(X_1-X_0)\big)\big] \\ &= \sup_{\theta_0\in[1-\alpha,1+\alpha]} \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\star(\theta_0)} (\mathbb{E}^{\overline{\mathbb{Q}}}[g]-\mathcal{E}(\overline{\mathbb{Q}},\overline{\mathbb{P}})) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\star} (\mathbb{E}^{\overline{\mathbb{Q}}}[g]-\mathcal{E}(\overline{\mathbb{Q}},\overline{\mathbb{P}})),\end{aligned}$$

where $\overline{\mathcal{P}}(\theta_0)$ is the subset of $\overline{\mathcal{P}}$ such that Θ_0 is fixed to be θ_0. The minimax argument applied on the second equality of the above equation is valid as $[1-\alpha,1+\alpha]$ is compact, $H\mapsto \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g+H(X_1-X_0)\big)\big]$ is convex, and $\theta_0\mapsto \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g+H(X_1-X_0)\big)\big]$ is linear. By the arguments of dynamic programming, we can further paste the one-period result together. For more details, see Chapter 5.

1.4.4 Related results

In [66], we considere also the properties related to the utility indifference prices as a by-product of the utility maximization duality. In particuler, we re-obtain the convergence of the utility indifference price to the super-replication price when the risk-aversion coefficient tends to infinite in the non-dominated context with transaction cost.

In the dominated discrete time case, the so-called shadow price is studied by [67]. It is a fictitious frictionless market which leads to the same optimal strategy and utility as the original market with proportional transaction costs. It is also interesting to consider a potential extension of the above work in out non-dominated framework.

1.5 A sparse grid approach to balance sheet risk measurement

1.5.1 Introduction

In this part, we shall present a robust and efficient method to numerically assess risks on the balance sheet distribution of an insurance company at a given horizon which is in practice chosen to be one year.

On a filtered probability space $(\Omega,\mathcal{A},\mathbb{P},(\mathcal{F}_t)_{t\geq 0})$, the balance sheet of the company is a stochastic process summarised, at any time $t\geq 0$, by the value of the assets of the company $(A_t)_{t\geq 0}$ and the value of the liabilities $(L_t)_{t\geq 0}$. The quantity of interest is the Profit and Loss (PnL) associated to the balance sheet, which is given by

$$P_t = L_t - A_t\ , \quad t\geq 0\,.$$

Our goal is then to compute various risk indicators for the loss distribution of the balance sheet at one year namely the distribution of P_1 under the *real-world* probability measure $\mathbb{P}$, that we denote hereafter η.

More precisely, we shall measure the risk associated to η using a (law invariant) risk measure defined over the class of square integrable measures $\varrho:\mathcal{P}_2(\mathbb{R})\to\mathbb{R}$. First, we consider for ϱ the

so called Value-at-Risk ($V@R$), which is defined by the left-side quantile:

$$V@R_p(\eta) = \inf\left\{q \in \mathbb{R} \mid \eta\left((-\infty, q]\right) \geq p\right\} . \tag{1.5.21}$$

We will also work with the class of *spectral risk measures*: a spectral risk measure is defined as

$$\varrho_h(\eta) = \int_0^1 V@R_p(\eta)h(p)\mathrm{d}p , \tag{1.5.22}$$

where h is a non-decreasing probability density on $[0, 1]$.

1.5.2 Financial Model

Description of the sold product

Let us assume that a company sells put lookback option at time $t = 0$ with a payoff function G paid at the maturity $T > 0$, depending upon a one-dimensional risky asset's price S. For $\kappa \geq 1$ and $t \in \{\tau_0 = 0, \tau_1, \cdots, \tau_\kappa = T\}$, the payoff G is given by:

$$G(S_{\tau_0}, \ldots, S_{\tau_\kappa}) = \left(\max_{0\leq\ell\leq\kappa} S_{\tau_\ell}\right) - S_T. \tag{1.5.23}$$

We assume that all pricing and hedging is done with market risk-neutral measure $\mathbb{Q}$.

Market model under the risk-neutral measure

The short rate model Let $\Theta \in \mathbb{R}^3$ be a set of parameters representing some market observations. The short rate evolution is governed by the Hull & White dynamics

$$r_s^{t,\Theta} = r_t^{t,\Theta} + \int_t^s a\left(\mu_u^{t,\Theta} - r_u^{t,\Theta}\right)\mathrm{d} + b\left(B_s - B_t\right), \ s \in [t, T], \tag{1.5.24}$$

where B is a $\mathbb{Q}$-Brownian motion, a and b are real constants. The parameter $\mu^{t,\Theta}$ is calibrated using the market observations Θ, so that the model reproduces the interest rate curve observed on the market. It is given by

$$\mu_s^{t,\Theta} = f^\Theta(t, s) + \frac{1}{a}\frac{\partial f^\Theta(t, s)}{\partial s} + \frac{b^2}{2a^2}\left(1 - e^{-2a(s-t)}\right), s \in [t, T]. \tag{1.5.25}$$

Remark 1.5.1. *In this model, let $(0, \Theta_0)$ be the observation made at time 0. Consider a swap contract issued in $s = 0$, with maturity $M > 0$, rate $R > 0$, with coupons versed at each time $i \in \{1, \ldots, M\}$. Then, the price of this contract at time t is given by:*

$$SW^{t,\Theta,M,R} = \frac{P^{t,\Theta,1}}{P^{0,\Theta_0,1}} - P^{t,\Theta,M} - R\sum_{i=1}^{M} P^{t,\Theta,i}, \tag{1.5.26}$$

The stock model Given the observations Θ of the interest rate factors and the risky asset's price $x \in (0, \infty)$, the evolution of the price under the neutral-risk measure $\mathbb{Q}$ is given by

$$S_s^{t,x,\Theta} = x + \int_t^s r_u^{t,\Theta} S_u^{t,x,\Theta}\mathrm{d}u + \int_t^s \sigma S_u^{t,x,\Theta}\mathrm{d}\tilde{W}_u, s \in [t, T], \tag{1.5.27}$$

where $\sigma > 0$, $\tilde{W}$ is another $\mathbb{Q}$-Brownian motion, whose quadratic covariation with B is given by

$$\langle B, \tilde{W} \rangle_t := \rho\, t, t \in [0, T],$$

where $\rho \in [-1, 1]$.

Modeling the Balance Sheet

Liability side For any market observation $\bar{\mathcal{X}}_t := (\bar{S}_t, \bar{\Theta}_t)$, the value $L_t =: \ell(t, \bar{S}_t, \bar{\Theta}_t)$ of the liabilities has to be computed, especially at time $t = 1$ in our application. The company's liabilities are reduced to one derivative product sold at $t = 0$. In our setting, the contingent claim's price is simply given by:

$$\ell(t, x, \Theta) = \mathbb{E}^Q e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta}), \tag{1.5.28}$$

where $(r^{t,\Theta}, S^{t,x,\Theta})_{t \le s \le T}$ are the risk neutral dynamics of the short rate and stock price.

Asset side The company wants to replicate the product with payoff G. The hedging portfolio's value A is decomposed into two parts:

$$A_t = A_t^{\Delta} + A_t^{\rho}. \tag{1.5.29}$$

The process A^{Δ} is the value of the portfolio obtained to cancel the variations of the price with respect to S, while A^{ρ} is defined to deal with the variations with respect to Θ.

Δ-hedging: The value of A^{Δ} at time 1 is

$$A_1^{\Delta} = \sum_{i=0}^{n-1} \Delta(t_i, \bar{S}_{t_i}, \bar{\Theta}_{t_i}) \left(\bar{S}_{t_{i+1}} - \bar{S}_{t_i}\right) \quad \text{where} \quad \Delta(t, x, \Theta) := \frac{\partial L}{\partial x}(t, x, \Theta). \tag{1.5.30}$$

ρ-hedging: The value of A_1^{ρ} is

$$A_1^{\rho} = \sum_{i=0}^{n-1} \sum_{j=1}^{3} \rho^j(t_i, \bar{S}_{t_i}, \bar{\Theta}_{t_i}) \left(\mathrm{SW}^{t_{i+1}, \bar{\Theta}_{t_{i+1}}, T_j, R_j} - \mathrm{SW}^{t_i, \bar{\Theta}_{t_i}, T_j, R_j}\right). \tag{1.5.31}$$

The global PnL function

From the previous two sections, we conclude that the PnL of the balance sheet at time 1 can be expressed as,

$$P_1 = p_1\left((t, \bar{S}_t, \bar{\Theta}_t)_{t \in \Gamma}\right)$$

where $(\bar{S}, \bar{\Theta})$ are the market parameters (risk factors) and the PnL function $p_1 : \mathbb{R}^{\gamma} \to \mathbb{R}$, with $\gamma = 4 \times (n + 1)$, is given by

$$\ell(t_n, x_n, \Theta_n) - \sum_{i=0}^{n-1} \Delta(t_i, x_i, \Theta_i)(x_{i+1} - x_i) - \sum_{i=0}^{n-1} \sum_{j=1}^{3} \rho^j(t_i, x_i, \Theta_i) \left(\mathrm{SW}^{t_{i+1}, \Theta_{i+1}, T_j, R_j} - \mathrm{SW}^{t_i, \Theta_i, T_j, R_j}\right). \tag{1.5.32}$$

1.5.3 Estimating the risk measure

Given a risk measure ϱ and the loss distribution η of the balance sheet at one year, we estimate the quantity of interest $\varrho(\eta)$ by simulating a sample of N i.i.d random variables $(\Psi_j)_{1\leq j\leq N}$ with law η and then computing simply $\varrho(\eta^N)$ using the formulae (1.5.21), (1.5.22) with η^N instead of η. Here, η^N stands for the empirical measure associated to the Ψ_j i.e.

$$\eta^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\Psi_j} \ .$$

The loss distribution η is obtained through the following expression:

$$\eta = p_1 \sharp \nu$$

p_1 being described in (1.5.32) and ν stands for the distribution of the market parameters. Namely, ν is the law of the random variable

$$\bar{\mathcal{X}} := (\bar{S}_t, \bar{\Theta}_t)_{t\in\Gamma} \tag{1.5.33}$$

under the *real world* probability measure $\mathbb{P}$.

In order to estimate $\varrho(\eta)$ for a chosen risk measure, we need to be able to sample from η which implies two steps in our setting. First, we need to be able to sample $\bar{\mathcal{X}}$ and then we use an approximation p_1^{υ} of p_1:

- $p_1^{\mathcal{N}}$ if one chooses the *nested simulation* approach;
- $p_1^{\mathcal{S}}$ if one chooses the *sparse grid* approach.

Eventually, the estimator of $\varrho(\mu)$ is given by

$$R^{\upsilon} := \varrho(p_1^{\upsilon}\sharp\nu^N) \ , \quad \text{for } \upsilon \in \{\mathcal{N}, \mathcal{S}\} \, . \tag{1.5.34}$$

To summarise, the two numerical methods have the following steps.

Nested simulation approach

1. Outer step: Simulate the model parameters $(\mathcal{X}_j)_{j=1,\dots,N}$.
2. Inner step: Simulate $\Psi_j = p^{\mathcal{N}}(\mathcal{X}_j)$ using MC simulations. This requires to compute the option prices with Monte Carlo estimates, the interest rate derivative prices, and the various sensitivities of the price.
3. Estimate the risk measure.

Sparse grid approach

1. Fix a (sparse) grid $\mathcal{V}$ and compute the approximation $p^{\mathcal{S}}$ at each required value on the grid by an MC simulation. This involves exactly the same computations as 2. above.
2. Simulate the N model parameter samples $(\mathcal{X}_j)$ and evaluate $\Psi_j = p^{\mathcal{S}}(\mathcal{X}_j)$.
3. Estimate the risk measure.

Chapitre 2

Introduction (French)

2.1 Le domaine de robuste finance

Dans la théorie de mathématique financière, le marché financier est modélisé par un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. Le sous-jacent est modélisé par un processus stochastique noté par $(S_t)_{t\in\mathbb{T}}$. Dans le cas discret on a $\mathbb{T} = \{0, 1, \cdots, N\}$ et dans le cas continu on a $\mathbb{T} = [0, T]$.

Dans le marché au temps discret, le premier théorème de valorisation d'actif dit que la condition de non-arbitrage est équivalent à l'existence de mesure de martingale équivalente. Le mesure de martingale équivalente est défini comme l'ensemble de mesure de probabilité sur $(\Omega, \mathcal{F})$ tel que le valeur actualisé de l'actif risqué est une martingale. Dans le marché au temps continu, on a la condition de "no free lunch with vanishing risk"(NFLVR) est équivalente à l'existence de mesure martingale locale équivalente.

Dans le marché complet, on peut répliquer parfaitment le payoff des produits financiers ξ par le cash et le sous-jacent. En plus, le valeur $\mathbb{E}^{\mathbb{Q}}[\xi]$ donne un prix de non-arbitrage, où $\mathbb{Q}$ est une mesure de martingale équivalente. Dans le marché incomplet, où la réplication parfaite n'existe pas, on peut considérer le prix de la sur-réplication, qui est le coût initial minimal tel que il existe une stratégie de la couverture pour rassurer que le valeur final du portefeuille reste au-dessus de produit financier quelque soit le scénario. Si on prend le suprémum sur l'ensemble des mesures de martingale, on obtient la dualité de valorisation-couverture suivante :

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi] = \inf\{x : \exists H \in \mathcal{H},\ \text{s.t.} x + V_T(H) \geqslant \xi, \mathbb{P} - \text{a.s.}\},$$

où $\mathcal{M}$ est l'ensemble des mesures de martingale, $\mathcal{H}$ est l'ensemble des stratégies admissibles et $V_T(H)$ est le profit et perte en T en utilisant la stratégie H.

Plus récemment, notamment après la crise de 2008, on concentre de plus en plus sur le risque de mal spécification du modèle. L'approche robuste est utilisée pour traiter le risque du modèle. Au lieu d'un modèle fixé, on considère une famille de modèles $\mathcal{P}$, qui est un ensemble de mesures de probabilités sur $(\Omega, \mathcal{F})$. En particulier, ce contexte contient deux cas spécifiques : premièrement, si $\mathcal{P}$ contient toutes les mesures de Dirac $\{\delta_\omega, \omega \in \Omega\}$, c'est le cadre de "model-free". Deuxièmement, si $\mathcal{P} = \{\mathbb{P}\}$, on revient au cas classique dominé.

2.1.1 Littérature sur la sur-réplication avec l'incertitude du modèle

Dans le modèle au temps continu avec l'incertitude de la volatilité, la dualité de sur-réplication est d'abord étudiée par Denis et Martini [3] utilisant la théorie de capacité et par Peng [4] utilisant une approximation du problème de contrôl markovien. Dans [5, 6, 7], Soner, Touzi et Zhang ont utilisé la décomposition de la sur-martingale avec les résultats d'agrégation. Leur approche est ensuite généralisée par Nutz et Neufeld [8] où l'hypothèse sur la continuité est supprimée et par Possamaï et al.[9] dans le contexte de transport optimal martingale.

Pour la dualité de sur-réplication au temps discret avec l'incertitude du modèle, Nutz [10] a établi l'existence de la stratégie optimale avec la dimension $d = 1$ et une dualité relativement abstraite. Dans le papier de [11], la dualité de valorisation-couverture générale est preuvée dans un contexte dit "model free". En particulier, ils ont donné une charactérisation du sous-ensemble de trajectoires dans lequel la dualité est vraie. Le résultat est ensuite généralisé dans [12], avec une spécification de modèle abstrait et ensemble des stratégies plus générales. En particulier, ils ont ajouté les ensembles de prédiction, qui représentent la conviction des investisseurs sur le modèle.

Dans leur papier séminal, Bouchard et Nutz [1] ont preuvé le théorème fondamental de valorisation d'actif dans un cadre relativement général au temps discret. Leur approche est motivée par Dalang-Morton-Willinger [13] et basée sur l'équivalence entre le non-arbitrage local et global.

Dans une première étape, ils ont preuvé le résultat localment avec les arguments de séparation en dimensions finies. Ensuite ils ont utilisé les arguments de sélection mesurable pour coller les résultats d'une période ensemble.

Plus récemment, la dualité de sur-réplication sous l'incertitude du modéle est aussi généralisée au temps continu. Dans [14], sous une certaine condition de non-arbitrage, les auteurs ont preuvé une version de théorème fondamental de valorisation d'actif et donné une représentation du prix de la sur-réplication. Dans [15] et [16], les auteurs ont généralisé le cadre de [12] dans un contexte continu en introduisant les ensembles de prédiction.

2.1.2 La relation avec transport optimal martingale(MOT)

Dans le papier de Breeden et Litzenberger [17], ils ont observé qu'on peut obtenir l'information sur les marginaux de sous-jacent une fois les prix des options vanilles sont disponibles pour tous strikes K. En effet, comme le prix de call $c(K) = \int_{\mathbb{R}}(x-K)^+\mu(dx)$ (où μ est la distribution du prix de stock S impliquée par le marché, et on suppose en plus que le taux d'intérêt $r = 0$), on a $\mu = c''$. Donc quand on modélise la dynamique du stock, on doit mettre plus de constraints sur sa distribution pour la consistence avec l'information du marché.

Dans ce cas, pour éliminer le risque du modèle et maintenir la consistence avec l'information du marché, on doit considérer l'ensemble des mesures martingales $\mathbb{Q}$ sur $(\Omega, \mathcal{F})$ satisfaisant la contrainte sur la marginale $X_T \sim^{\mathbb{Q}} \mu$, où μ est obtenu par le prix des options vanilles sur le marché. On note $\mathcal{M}(\mu)$ l'ensemble des mesures martingales satisfaisant la contrainte sur la distribution, donc la borne supérieure du prix du produit exotique ξ étant donné l'information du marché est

$$P(\mu) = \sup_{\mathbb{Q}\in\mathcal{M}(\mu)} \mathbb{E}^{\mathbb{Q}}[\xi].$$

D'un point de vue de la couverture, s'il existe des options vanilles dans le marché, on peut utiliser la stragégie semi-statique pour implémenter la sur-réplication. Ici, on dit "semi-statique" car les positions sont statiques sur les options et dynamiques sur les sous-jacents. Par la formule Carr-Madan :

$$\lambda(x) = \lambda(x_i) + (x-x_i)\lambda'(x_i) + \int_{-\infty}^{x_i}(K-x)^+\lambda''(K)dK + \int_{x_i}^{+\infty}(x-K)^+\lambda''(K)dK,$$

toute option européen avec payoff $\lambda(X_T)$ peut être representée comme un basket des options vanilles, dont prix est donné par $\mu(\lambda) := \int_{\mathbb{R}}\lambda(x)\mu(dx)$. Donc le coût de la sur-réplication avec la stratégie semi-statique peut être représenté comme :

$$D(\mu) := \inf\{\mu(\lambda) : \exists(\lambda, H) \in \mathbb{L}^1 \times \mathcal{H}, \text{s.t.}\lambda(X_T) + (H\cdot X)_T \geqslant \xi_T, \mathbb{P}-\text{p.s.}, \forall\mathbb{P}\in\mathcal{P}\}.$$

Sous les conditions propres, on espère d'obtenir le résultat suivant, qui peut être vue comme une généralisation de la dualité classique de valorisation-couverture et de la dualité de transport optimal classique

$$P(\mu) = D(\mu)$$

On appele le problème ci-dessus transport optimal martingale(MOT). Nous sommes intéressés par les questions suivantes :

(i) Sous quelles conditions les dualités sont vraies ?

(ii) Comment charactériser l'optimisateur du problème primal et dual ?

(iii) Comment calculer numériquement le valeur du problème primal et dual ?

Comme le plupart de cette thèse est sur la sur-réplication, ici on concentre seulement sur la

première question.

Avant l'approche de MOT, le problème de la sur-réplication avec l'information de la marginale est étudié initialement par l'approche de plongement de skorokhod optimal(oSEP). Le SEP classique est : étant donné une mesure de probabilité centrée μ avec le moment d'ordre un fini et un mouvement brownian W, on cherche un temps d'arrêt τ tel que $W_\tau \sim \mu$ et $(W_{\tau\wedge t})_{t\geqslant 0}$ est uniformément intégrable. Le SEP a beaucoup de solutions construits explicitement et chacune entre eux optimise un certain critère. Le SEP optimal unifie les constructions specifiques et vise à trouver une solution optimale de SEP par rapport à une certaine fonction d'objective. Si la fonction d'objective est invariante au temps, l'équivalence entre MOT et SEP peut être obtenue à partir d'un argument classique de changement du temps et le théorème de Dubins-Schwarz.

La dualité $P(\mu) = D(\mu)$ peut être preuvée dans deux étapes, voir par exemple [18]. D'abord, sous la condition que Φ est bornée supérieurement et semi-continue supérieurement, on peut preuver que

$$\mu \in \mathfrak{B}(\mathbb{R}^M) \mapsto \sup_{\mathbb{Q}\in\mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}\big[\Phi\big] \in \mathbb{R}$$

est concave et semi-continu supérieurement, où on équippe $\mathfrak{B}(\mathbb{R}^M)$ avec une topologie de type Wasserstein. Ensuite, en utilisant le théorème Fenchel-Moreau, on a

$$\sup_{\mathbb{Q}\in\mathcal{M}_\mu} \mathbb{E}^{\mathbb{Q}}\big[\Phi\big] \;=\; \pi^E_{\mu,0}(\Phi) \;:=\; \inf_{\lambda\in\Lambda}\Big\{\mu(\lambda) + \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}\big[\Phi - \lambda\big]\Big\}.$$

La dualité ci-dessus est une dualité intermédiaire, dans laquelle les contraintes sur les marginaux sont déjà supprimées. Maintenant on peut appliquer une certaine version de la dualité de sur-réplication pour obtenir la dualité complete.

Dans la section 2.2 pour l'option américaine, on va utiliser un différent argument par l'approximation pour preuver la dualité, sans passant par la dualité intermédiaire.

2.1.3 Le cadre de Bouchard-Nutz [1]

Dans le papier de Bouchard et Nutz [1], les auteurs ont preuvé des résultats importants dans le cadre de robuste finance. On rappele ici leur contexte et les résultats principals. Soit $\Omega_0 := \{\omega_0\}$ et Ω_1 un espace polonais. Pour chaque $t = 1, \cdots, T$, on note par $\Omega_t := \Omega_1^t$ le produit cartésien t-fois de Ω_1 et $\mathcal{F}_t^0 := \mathcal{B}(\Omega_t)$ et $\mathcal{F}_t$ sa complété universelle. L'espace mesurable $(\Omega, \mathcal{F})$ et les filtrations associées sont définis par

$$\Omega := \Omega_T,\ \mathcal{F} := \mathcal{F}_T,\ \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T} \ \text{ et } \mathbb{F}^0 := (\mathcal{F}_t^0)_{0\leq t\leq T}.$$

On note $\mathcal{P}_t(\omega)$ un ensemble convex non-vide dans $\mathfrak{B}(\Omega_1)$, qui représente l'ensemble de tous les modèles possibles pour la $(t+1)$-ième période, étant donné $\omega \in \Omega_t$ au temps $t = 0, 1\cdots, T-1$. L'ensemble $\mathcal{P}$ des mesures de probabilité sur $(\Omega, \mathcal{F})$ est donné par

$$\mathcal{P} := \big\{\mathbb{P} := \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \ : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot) \text{ for } t \leq T-1\big\}.$$

où $\mathbb{P}_t : \Omega_t \mapsto \mathfrak{B}(\Omega_1)$ est un noyau de probabilité tel que la mesure de probabilité $\mathbb{P}$ est définie par le théorème de Fubini dans le sens

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2 \cdots, \omega_T)\mathbb{P}_{T-1}(\omega_1, \cdots, \omega_{T-1}; d\omega_T) \cdots \mathbb{P}_0(d\omega_1).$$

Pour garantir que l'ensemble $\mathcal{P}$ est non-vide, ils ont fait l'hypothèse de la mesurabilité suivante : pour chaque t,

$$\text{graph}(\mathcal{P}_t) := \big\{(\omega, \mathbb{P}) : \omega \in \Omega_t, \mathbb{P} \in \mathcal{P}_t(\omega)\big\} \subseteq \Omega_t \times \mathcal{P}(\Omega_1) \text{ est analytique.}$$

Dans ce contexte, ils ont donné une version quasi-sure de non-arbitrage et ont preuvé un théorème de valorisation d'actif corréspondant. Leur preuve est dans l'esprit de Dalang-Morton-Willinger [13] : d'abord on preuve une version locale de la dualité par les arguments de séparation en dimensions finies, ensuite utilise l'argument de sélection mesurable pour coller les résulatats d'une seule période ensemble. L'équivalence entre la version locale et globale de non-arbitrage est cruciale dans la preuve. Ils ont aussi considéré des positions statiques sur nombre fini des options et obtenu une version non-dominée de décomposition optionnelle comme un sous-produit.

Dans les chapitres suivants, on va considérer plusieurs sujets dans le domaine de robuste finance. Plus précisément, dans la section 2.2 on considère le problème de sur-réplication des options américaines au temps discret avec la stratégie dynamique pour les sous-jacents et la stratégie statique pour les options. Dans la section 2.3, on considére un problème de sur-réplication au temps discret avec coût de transaction proportionelle et l'incertitude du modèle. L'idée principale est de convertir le problème original à un problème sur un espace élargi sans coût de transaction en utilisant le technique de randonmisation. Dans la section 2.4, on va étudier le problème de maximisation d'utilité dans un contexte similaire avec la section précédente.

Dans la section 2.5, on considère un méthode numérique basé sur l'approximation de sparse grid pour calculer la distribution de la perte du bilan d'une entreprise d'assurance.

2.2 Sur-réplication des options américaines avec l'incertitude du modèle

2.2.1 Formulation forte : une première formulation

La plupart des littératures pour la sur-réplication étudient les options de type européenne. Dans la première part de cette thèse, on considère la dualité de valorisation-couverture pour les options de type américaine au temps discret avec la stratégie semi-statique sous l'incertitude du modèle. Dans le cas classique avec une probabilité dominée, si le marché est complet et on note $(\Phi_t)_{t\in\mathbb{T}}$ le payoff de l'option américaine, il est connu que le prix est donné par le valeur initial de sa enveloppe de Snell.

Dans le cas non-dominé, le prix de sur-réplication d'une option américaine doit être le coût minimal tel que on peut dominer le payoff pour n'importe quelle stratégie de la contrepartie. Dans [19], on travaille dans un cadre général au temps discret. Soient $(\Omega, \mathcal{F})$ un espace mesurable et $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,N}$ une filtration où $\mathcal{F}_0$ est trivial et $N \in \mathbb{N}$ est l'horizon du temps. Soient $S = (S_t)_{0\leqslant t\leqslant N}$ un processus adapté qui représente le prix actualisé de sous-jacent et $(g^\lambda)_{\lambda\in\Lambda}$ une famille de variables aléatoires qui représentent les options statiques. En particulier, le cadre abstrait ci-dessus peut incluire les cas suivants :

(i) Le cas classique dominé où $\mathcal{P} = \{\mathbb{P}\}$.

(ii) Le contexte de Bouchard-Nutz avec $\Omega = \Omega_1^N$, où Ω_1 est un espace polonais et le nombre des options statiques est fini. Pour les détails voir section 2.1.3.

(iii) Le contexte de transport optimal martingale avec nombre infini des options statiques, où l'espace canonique $\Omega = \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ et $\mathcal{P}$ est l'ensemble des measures de probabilité boréliens sur Ω.

(iv)Le contexte de [12], avec nombre fini des options statiques, où Ω est un espace polonais et $\mathcal{P}$ est l'ensemble de tous les measures de probabilité boréliens sur Ω. En particulier, ils ont travaillé

avec certains ensembles de prédiction pour ajouter leurs convictions sur l'ensemble des modèles. Dans [19], on considère le deuxième et le troisième cas. On note $\mathcal{H}$ l'ensemble de processus $\mathbb{F}$-prévisible, et $\mathfrak{h}$ l'ensemble de nombres réels avec nombre fini des éléments non nuls. On considère ici le cas la dimension $d = 1$, alors le payoff final associé avec la stratégie $(H, h) \in \mathcal{H} \times \mathfrak{h}$ est donée par

$$(H \circ S)_N + hg = \sum_{k=1}^{N} H_k \Delta S_k + \sum_{\lambda \in \Lambda} h^\lambda g^\lambda,$$

où $\Delta S_k = S_k - S_{k-1}$. Donc le prix de sur-réplication peut-être écrit comme

$$\pi_g^A(\Phi) := \inf \Big\{ x : \exists ({}^1H, ..., {}^NH) \in \mathcal{H}^N \text{ s.t. } {}^jH_i = {}^kH_i \ \forall 1 \leqslant i \leqslant j \leqslant k \leqslant N \text{ et } h \in \mathfrak{h} \\ \text{tel que } x + ({}^kH \circ S)_N + hg \geqslant \Phi_k \ \forall k = 1, ..., N \ \mathcal{P}\text{-q.s.} \Big\} \tag{2.2.1}$$

La définition ci-dessus est consistente avec le fait que on peut adjuster notre stratégie selon la stratégie de la contrepartie. Dans le cas non-dominé, on peut envisager que une formulation naturelle est le suivant :

$$\pi_g^A(\Phi) \stackrel{?}{=} \sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau], \tag{2.2.2}$$

où $\mathcal{T}(\mathbb{F})$ est l'ensemble de $\mathbb{F}$-temps d'arrêt, et $\mathcal{M}_g$ représente l'ensemble des mesures de martingale qui satisfont les contraintes sur le prix des options. Par contre, en général cette dualité n'est pas correcte. La raison "numérique" est que l'ensemble $\mathcal{M}_g \times \mathcal{T}(\mathbb{F})$ dans (2.2.2) est trop petit. Il y a aussi des résultats similaires dans les littératures. Dans [20], ils ont étudié dans le cadre de Bouchard-Nutz avec nombre fini des options statiques, et établi la dualité suivante

$$\pi_g^A(\Phi) = \inf_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \sup_{\mathbb{Q} \in \mathcal{M}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau - hg]$$

sous les conditions relativement fortes(voir leur proposition 3.1). À cause de l'usage du temps d'arrêt fort, ils ont aussi observé la différence entre $\pi_g^A(\Phi)$ et $\sup_{\tau \in \mathcal{T}(\mathbb{F})} \sup_{\mathbb{Q} \in \mathcal{M}_g} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]$. Dans [20], ils ont aussi étudié le problème de sous-réplication $\sup_{\tau \in \mathcal{T}(\mathbb{F})} \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]$. Dans un autre papier [21], ils ont généralisé la dualité de sur-réplication précédente par l'introduction de temps d'arrêt randomisé et l'usage d'un théorème de minimax. La nouvelle dualité est plus complete, mais il hérite la même condition de régularité et intégrabilité précédente.

Dans [22] and [23], Hobson et Neuberger ont étudié le même problème de sur-réplication dans un contexte Markovian, où le processus S prend le valeur dans un grid discret. En considérant la classe de temps d'arrêt faible, ils ont réformulé le problème de valorisation et couverture comme un problème de la programmation linéaire sous contraintes linéaires, qui peut être résolu numériquement.

Dans [19], on propose deux approches différentes pour obtenir une dualité dans (2.2.2). La première est de réformuler l'option américaine comme une option européenne, qui est équivalente avec l'esprit de la formulation faible de Hobson et Neuberger [23]. La deuxième est de permettre l'achat et vente dynamique des options vanilles tel que l'ensemble des temps d'arrêt correspondent à une filtration plus large $\widehat{\mathbb{F}}$.

2.2.2 Réformulation des options américaines comme options européennes

À cause de la présence des options statiques, la dualité de valorisation-couverture est violée.

La formulation correcte est d'introduire un espace élargi qui convertit options américaines aux options européennes. La technique de l'élargissement d'espace est basée sur la construction des temps aléatoires, utilisée dans [24, 25] pour étudier l'existence des temps aléatoires avec une probabilité de survie, et dans [26] pour étudier le problème général de contrôle/arrêt optimal, et dans [18] et [27] pour étudier le problème de plongement skorokhod optimal.

Soit $\mathbb{T} := \{1, ..., N\}$ et introduisons l'espace $\overline{\Omega} := \Omega \times \mathbb{T}$ avec le temps canonique $T : \overline{\Omega} \to \mathbb{T}$ donné par $T(\bar{\omega}) := \theta$, où $\bar{\omega} := (\omega, \theta)$. On introduit aussi la filtration $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_k)_{k=0,1,...,N}$ avec $\overline{\mathcal{F}}_k = \mathcal{F}_k \otimes \vartheta_k$ où $\vartheta_k = \sigma(T \wedge (k+1))$, et $\overline{\mathcal{F}} = \mathcal{F} \otimes \vartheta_N$. Soit Υ une classe de fonctions définies sur Ω. Il corréspond à l'ensemble des fonctions semi-analytiques supérieurment dans le cadre de Bouchard-Nutz et l'ensemble des fonctions semi-continue supérieurement dans le cadre de transport optimal martingale. On note $\overline{\mathcal{H}}$ la classe de processus $\overline{\mathbb{F}}$-prévisible et étend S et g^λ à $\overline{\Omega}$ naturellement par $S(\bar{\omega}) = S(\omega)$ et $g^\lambda(\bar{\omega}) = g^\lambda(\omega)$ pour $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$. On note $\overline{\Upsilon}$ la classe des variables aléatoires $\overline{\xi} : \overline{\Omega} \to \mathbb{R}$ tel que $\overline{\xi}(\cdot, k) \in \Upsilon$ pour tous $k \in \mathbb{T}$ et $\overline{\pi}^E_g(\bar{\xi})$ le prix de surréplication de $\overline{\xi}$. La classe des objets américains $\overline{\Upsilon}$ sur $\overline{\Omega}$ peuvent être identifiées comme objets européens Υ^N sur Ω par $\overline{\xi}(\bar{\omega}) = \Phi_\theta(\omega)$. On introduit l'ensemble des modèles et l'ensemble des mesures de martingale sur l'espace élargi : $\overline{\mathcal{P}} = \{\overline{\mathbb{P}} \in \mathfrak{P}(\overline{\Omega}) : \overline{\mathbb{P}}_{|\Omega} \in \mathcal{P}\}$, $\overline{\mathcal{M}} = \{\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega}) : \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}$ et $\mathbb{E}^{\overline{\mathbb{Q}}}[\Delta S_k | \overline{\mathcal{F}}_{k-1}] = 0 \ \forall k \in \mathbb{T}\}$ et $\overline{\mathcal{M}}_g = \{\overline{\mathbb{Q}} \in \overline{\mathcal{M}} : \mathbb{E}^{\overline{\mathbb{Q}}}[g^\lambda] = 0 \ \forall \lambda \in \Lambda\}$.

Pour la preuve de la dualité de valorisation-couverture dans l'espace élargi, il faut bien définir la concatenation pour rassurer la propriété de la programmation dynamique. On a alors le résultat suivant de [19] :

Theorem 2.2.1. *Pour tous $\Phi \in \Upsilon^N = \overline{\Upsilon}$ on a*

$$\pi^A_g(\Phi) = \overline{\pi}^E_g(\Phi) := \inf\{x : \exists\ (\overline{H}, h) \in \overline{\mathcal{H}} \times \mathfrak{h} \ \ s.t. \ \ x + (\overline{H} \circ S)_N + hg \geqslant \xi \ \overline{\mathcal{P}}\text{-}q.s.\}. \quad (2.2.3)$$

Dans le cadre de Bouchard-Nutz et transport optimal martingale, on a la dualité de valorisation-couverture

$$\pi^A_g(\Phi) = \overline{\pi}^E_g(\Phi) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]. \quad (2.2.4)$$

Remark 2.2.2. *$\overline{\mathcal{M}}_g$ dans* (2.2.4) *est équivalent à une* formulation faible *du RHS de* (2.2.2)*, similaire avec l'esprit du [23]. En effet, on peut définir une terme d'arrêt faible α comme*

$$\alpha \ = \ \big(\Omega^\alpha,\ \mathcal{F}^\alpha,\ \mathbb{Q}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}^\alpha_k)_{0 \leq k \leq N},\ (S^\alpha_k)_{0 \leq k \leq N},\ (g^{\lambda,\alpha})_{\lambda \in \Lambda},\ (\Phi^\alpha_k)_{k \in \mathbb{T}},\ \tau^\alpha\big)$$

avec $\big(\Omega^\alpha,\ \mathcal{F}^\alpha,\ \mathbb{Q}^\alpha, \mathbb{F}^\alpha\big)$ un espace de probabilité filtré, τ^α un $\mathbb{F}^\alpha$-temps d'arrêt à valeur dans $\mathbb{T}$, une $(\mathbb{Q}^\alpha, \mathbb{F}^\alpha)$-martingale S^α à valeur dans $\mathbb{R}^d$ et une famille de variables aléatoires $g^{\lambda,\alpha}, \Phi^\alpha_k$. En plus, il existe une application mesurable surjective $\mathtt{i}_\alpha : \Omega^\alpha \to \Omega$ avec $\mathbb{Q} = \mathbb{Q}^\alpha \circ \mathtt{i}^{-1}_\alpha \in \mathcal{M}$ et $\mathtt{i}^{-1}_\alpha(\mathcal{F}_k) \subset \mathcal{F}^\alpha_k$, $\mathtt{i}^{-1}_\alpha(\mathcal{F}) \subset \mathcal{F}^\alpha$ et finalement $\mathcal{L}_{\mathbb{Q}^\alpha}(S^\alpha, g^\alpha, \Phi^\alpha) = \mathcal{L}_{\mathbb{Q}}(S, g, \Phi)$. Si on note $\mathcal{A}_g$ l'ensemble des termes d'arrêt faible α tel que $\mathbb{E}^{\mathbb{Q}^\alpha}\big[g^{\lambda,\alpha}\big] = 0$ pour chaque $\lambda \in \Lambda$. On a

$$\sup_{\alpha \in \mathcal{A}_g} \mathbb{E}^{\mathbb{Q}^\alpha}\big[\Phi_{\tau^\alpha}\big] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]\big].$$

Remark 2.2.3. *Dans le cas sans options statiques, la formulation forte et faible sont équivalente :*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\mathbb{Q} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]. \quad (2.2.5)$$

Remark 2.2.4. *Dans le cas avec options statiques, en général on a*

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g}\mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]\geqslant\sup_{\mathbb{Q}\in\mathcal{M}_g}\sup_{\tau\in\mathcal{T}(\mathbb{F})}\mathbb{E}^{\mathbb{Q}}[\Phi_\tau],$$

et l'inégalité ci-dessus peut être stricte. En effet, on peut identifier le sous-ensemble de $\overline{\mathcal{M}}_g$ qui peut convertir l'inégalité à une égalité. Introduire

$$\begin{aligned}\underline{\mathcal{M}}_g:=\big\{\overline{\mathbb{Q}}\in\mathfrak{B}(\overline{\Omega}):\ &\overline{\mathbb{Q}}\ll\overline{\mathcal{P}},\ \mathbb{E}^{\overline{\mathbb{Q}}}[g^\lambda]=0,\ \lambda\in\Lambda,\ S\text{ est une }(\mathbb{F},\overline{\mathbb{Q}})\text{-martingale,}\\ &\mathbb{E}^{\overline{\mathbb{Q}}}[M_T]=\mathbb{E}^{\overline{\mathbb{Q}}}[M_0],\ \text{pour toutes }(\mathbb{F},\overline{\mathbb{Q}})\text{-martingales bornée }M\big\},\end{aligned}$$

l'ensemble de mesures sous lesquelle S est une $\mathbb{F}$-martingale et T est un $\mathbb{F}$-pseudo–temps d'arrêt. Alors on a

$$\sup_{\overline{\mathbb{Q}}\in\underline{\mathcal{M}}_g}\mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]=\sup_{\mathbb{Q}\in\mathcal{M}_g}\sup_{\tau\in\mathcal{T}(\mathbb{F})}\mathbb{E}^{\mathbb{Q}}\big[\Phi_\tau].$$

2.2.3 Des options statiques aux options dynamiques

La dualité de (2.2.2) est violée à cause de la présence des options statiques. Dans la section précédente on a introduit un espace élargi pour obtenir la dualité. Dans cette section on donne une autre méthode pour rendre le RHS de (2.2.2) assez grand par l'introduction d'un espace fictif plus grand où le sous-jacent S et les options $(g^\lambda)_{\lambda\in\Lambda}$ peuvent être échangé dynamiquement simultanément.

Une extension dynamique $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathcal{P}})$ du marché original $(\Omega,\mathbb{F},\mathcal{F},\mathcal{P},S,g)$ est un marché fictif où on peut acheter et venter le sous-jacent et l'option dynamiquement, et $\widehat{\mathbb{F}}\supset\mathbb{F}$ est la filtration contenant l'information du prix du sous-jacent et de l'option. Pour la définition précise de l'extension dynamique, voir Chapitre 3.

Dans une extension dynamique $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathcal{P}})$ de $(\Omega,\mathbb{F},\mathcal{F},\mathcal{P},S,g)$, on peut définir similairement la stratégie de trading $\widehat{H}$ et le coût de sur-réplication $\widehat{\pi}^A(\Phi)$ d'une option américaine $\widehat{\Phi}=(\widehat{\Phi}_k)_{k\leqslant N}$ sur $\widehat{\Omega}$. On a le résultat principal suivant :

Theorem 2.2.5. *Dans le cadre de Bouchard-Nutz et transport optimal martingale, pour tous $\Phi\in\Upsilon^N$,*

$$\pi_g^A(\Phi)=\widehat{\pi}^A(\Phi)=\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{M}}}\sup_{\hat{\tau}\in\mathcal{T}(\widehat{\mathbb{F}})}\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\Phi_{\hat{\tau}}\right]=\sup_{\widehat{\mathbb{Q}}\in\overline{\widehat{\mathcal{M}}}}\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\Phi\right]\geqslant\sup_{\mathbb{Q}\in\mathcal{M}_g}\sup_{\tau\in\mathcal{T}(\mathbb{F})}\mathbb{E}^{\mathbb{Q}}\left[\Phi_\tau\right],\tag{2.2.6}$$

où $\mathcal{M}_g$ est l'ensemble des mesures de martingale sur Ω consistent avec l'information du marché, $\widehat{\mathcal{M}}$ et $\overline{\widehat{\mathcal{M}}}$ sont l'ensemble des mesures de martingale sur $\widehat{\Omega}$ et l'espace élargi de $\widehat{\Omega}$ respectivement.

Dans le contexte de transport optimal martingale, l'extension dynamique est liée étroitement avec la martingale à valeur mesure. Elle est utilisée par Eldan dans [28] pour étudier le plongement de skorokhod et Cox et Kallblad dans [29] pour étudier le transport optimal martingale pour les options asiatiques. En particulier, ce point de vue nous permet d'obtenir le principe de la programmation dynamique avec contrainte sur les marginaux, car la contrainte terminale devient la contrainte initiale. Pour donner l'idée principale, on considère le cas avec seulement la marginale finale, i.e. la date de marginale $\mathbb{T}_0=\{N\}$. La marginale μ est une mesure de la probabilité sur $\mathbb{R}$. Soit $\mathfrak{P}_1(\mathbb{R})$ l'ensemble des mesures de probabilité avec le moment d'ordre un fini équipé avec la 1-Wasserstein distance(donc $\mathfrak{P}_1(\mathbb{R})$ est un espace polonais). On définit

l'espace canonique suivant pour le processus à valeur mesure : $\widehat{\Omega} := \{\mu\} \times (\mathfrak{P}_1(\mathbb{R}))^N$ et on note $\widehat{X} = (\widehat{X}_k)_{0\leqslant k\leqslant N}$ le processus canonique sur $\widehat{\Omega}$. On définit $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_k)_{0\leq k\leq N}$ la complété universelle de la filtration canonique et on note $\mathcal{T}(\widehat{\mathbb{F}})$ l'ensemble de tous $\widehat{\mathbb{F}}$-temps d'arrêt. Pour $f \in \mathcal{C}_1$, où $\mathcal{C}_1$ est l'ensemble des fonctions continues sur $\mathbb{R}$ à accroissance linéaire, on note les processus de l'intégral contre $\widehat{X}$ comme $\widehat{X}_k(f) = \int_{\mathbb{R}} f(x)\widehat{X}_k(dx)$ et $\widehat{X}_k(id) = \int_{\mathbb{R}} x\widehat{X}_k(dx)$. Définir $\mathtt{i} : \widehat{\Omega} \to \Omega$ par $\mathtt{i}(\widehat{\omega}) = (\widehat{X}_0(id)(\widehat{\omega}), ..., \widehat{X}_N(id)(\widehat{\omega}))$ qui est surjectif et étendre le processus sur Ω naturellement à l'espace $\widehat{\Omega}$. En particulier le processus de prix est étendu par $S_k(\widehat{\omega}) = S_k(\mathtt{i}(\widehat{\omega})) = \widehat{X}_k(id)(\widehat{\omega})$ et l'option statique est étendue par $g^\lambda(\widehat{\omega}) = g^\lambda(\mathtt{i}(\widehat{\omega})) = \lambda(\mathtt{i}(\widehat{\omega})) - \mu(\lambda)$. Définir une famille de variables aléatoires $Y = (Y^\lambda)_{\lambda\in\Lambda}$ par :

$$Y_k^\lambda = \begin{cases} \widehat{X}_k(\lambda) - \mu(\lambda) & 0 \leqslant k \leqslant N-1 \\ g^\lambda = \lambda(\widehat{X}_N(id)) - \mu(\lambda) & k = N \end{cases}$$

On peut maintenant donner la définition de la mesure de martingale à valeur mesure.

Definition 2.2.6. *Une mesure de probabilité $\widehat{\mathbb{Q}}$ sur $(\widehat{\Omega}, \widehat{\mathcal{F}})$ est dit une mesure de martingale à valeur mesure (MVM mesure) si le processus $(\widehat{X}_k(f))_{0\leq k\leq N}$ est une $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingale pour tous $f \in \mathcal{C}_1$.*

On note

$$\widehat{\mathcal{M}}_\mu := \{\widehat{\mathbb{Q}} \in \mathfrak{B}(\widehat{\Omega}\} : \widehat{\mathbb{Q}} \text{ est une MVM mesure}\}.$$

Pour tous $\mathbb{Q} \in \mathcal{M}_\mu$ on définit une application $\mathtt{j}_\mathbb{Q} : \Omega \to \widehat{\Omega}$ par $\mathtt{j}_\mathbb{Q}(\omega) = (\mathcal{L}_\mathbb{Q}(S_N|\mathcal{F}_k)(\omega))_{k\leqslant N}$. Maintenant on peut définir l'application $\mathtt{J} : \mathcal{M}_\mu \to \widehat{\mathcal{M}}$ par $\mathtt{J}(\mathbb{Q}) = \mathbb{Q} \circ \mathtt{j}_\mathbb{Q}^{-1}$ et on peut preuver que $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, Y, \mathtt{i}, J)$ est une extension dynamique de $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$.

Plus récemment, martingales à valeur mesure sont utilisées pour étudier les autres problèmes dans mathématique financière. En particulier, elle est utilisée dans [30] pour étudier le problème de transport optimal martingale où le coût est représenté par un problème de temps d'arrêt. Dans [31] et [32], elle est utilisée pour étudier le problème d'arrêt optimal avec contrainte sur la distribution.

2.2.4 Options Américaines comme stratégie de la couverture

Dans les papiers plus récent de Bayraktar and Zhou [33], [34], ils ont généralisé l'étude au cas où on peut utiliser les options américaines comme la stratégie de la couverture. L'hypothèse cruciale qu'ils ont introduit dans leur papier est la divisibilité infinie des options américaines. Autrement dit, on peut décomposer une option américaine aux petites pièces et les échange séparément. Dans leur premier papier [33], ils font l'hypothèse de l'interdiction du vente à découvert et considèrent seulement la dualité de sous-réplication, et preuvent que la divisibilité infinie donne une formulation correcte du prix de sous-réplication. Dans un deuxième papier [34], ils ont utilisé notre technique de l'élargissment d'espace pour réformuler les options américaines en position courte comme options européennes dans l'espace élargie, et ensuite résolu la dualité plus complete admettant le vente à découvert et sur-réplication. L'idée est de distinguer deux types des options américaines : la position longue dans le portefeuille de la couverture ou le problème de sous-réplication ; la position courte dans le portefeulle de la couverture ou le problème de sur-réplication. Pour le premier, comme le propriétaire de l'option américaine on peut utiliser une certaine stratégie d'exercice. En revanche, pour le deuxième, on doit ajuster sa stratégie en

réponse aux contreparties, donc la stratégie de la couverture doivent adapté à une filtration plus large $\overline{\mathbb{F}}$.

2.3 Sur-réplication avec coût de transaction proportionnelle sous l'incertitude du modèle

Nous somme dans le cadre de Bouchard-Nutz dans cette section. Dans le marché classique au temps discret avec coût de transaction proportionnelle, la notion de mesure de martingale équivalente est remplacée par le systme de pricing consistent(CPS), qui contient une couple $(\mathbb{Q}, Z)$ tel que $\mathbb{Q} \ll \mathbb{P}$ et Z est une $\mathbb{Q}$-martingale à valeur dans le spread bid-ask.

Le théorème fontamental de valorisation d'actif est preuvé dans les contextes différents avec différentes versions des conditions de non-arbitrage. D'abord, la condition de non-arbitrage usuelle peut seulement garantir l'existence de CPS si Ω est fini. Dans [35], ils ont preuvé l'existence de système de pricing consistent strict(SCPS) est équivalent à la condition de non-arbitrage stricte sous condition que tous les actifs ne peuvent pas être échangés sans frictions. Dans [36], il a proposé une condition de non-arbitrage robuste, qui peut garantir l'existence de SCPS avec moins d'hypothèse.

Dans le contexte sans l'incertitude du modèle, la dualité de la sur-réplication avec coût de transaction proportionnelle est déjà beaucoup étudiée, voir par exemple [37] et [38]. La dualité de maximisation d'utilité avec coût de transaction est aussi étudiée en utilisant la dualité de sur-réplication par sa relation avec le prix de l'ombre, voir par exemple [39]. Il y a aussi des études au delà du coût de transaction proportionnelle. Dans [40], les auteurs ont preuvé la dualité de sur-réplication avec l'impact temporel du prix dans un modèle continu et établi en particulier l'optimalité de la stratégie de non exécution. Dans [41], les auteurs ont considéré le problème de la sur-réplication avec coût de transaction fixé comme un problème de contrôl stochastique avec des impulsions sous un contraint d'état terminal.

Dans le cas non-dominé, une propriété importante dans le papier Bouchard-Nutz est l'équivalence entre la version locale et version globale de non-arbitrage. Cette propriété nous permet de utiliser l'argument de séparation en dimensions finies localement, et ensuite passe aux plusieurs périodes en utilisant l'argument de sélection mesurable. Donc il est important de choisir une version propre de la condition non-arbitrage. Plus récemment, dans le papier de Bayraktar and Zhang [42], ils ont proposé une preuve en utilisant la condition de non-arbitrage stricte. Comme le non-arbitrage local n'est pas équivalent à non-arbitrage global dans leur contexte, ils utilisent une procédure délicate de backward-forward pour obtenir le résultat multi-période.

Une généralisation de Bouchard-Nutz avec coût de transaction est proposé dans [43]. Dans ce papier, ils ont utilisé le non-arbitrage de seconde type (NA2($\mathcal{P}$)) qui est proposé par [44]. L'avantage principale de NA2($\mathcal{P}$) est l'équivalence entre de non-arbitrage local et global.

La sur-réplication avec coût de transaction est d'abord étudiée par [45, 46] en discretisant l'espace et ensuite passant à la limite. Dans [47], l'auteur considère aussi la sur-réplication pointwise. En particulier, le marché est multi-varié et toutes les transactions doivent passer par le compte cash. En construisant un systéme de valorisation fictif dans lequel le prix de sur-réplication sans friction est le même avec le coût original, il a réussi de preuver le résultat. Dans [48], les auteurs ont considéré le problème de sur-réplication avec coût de transaction non-linéaire.

2.3.1 Le marché avec coût de transaction proportionnelle

Soit $(\Omega, \mathcal{F})$ un espace mesurable, équippé avec deux filtrations $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t=0,1,\cdots,T} \subset \mathbb{F} = (\mathcal{F}_t)_{t=0,1,\cdots,T}$ pour certain $T \in \mathbb{N}$ et un processus mesurable de dimension d : $(S_t)_{0\leqslant t\leqslant T}(\mathbb{F}^0$

est généré par S, et $\mathbb{F}$ sa complété universelle).

Example 2.3.1. *Pour donner une intuition sur le processus du portefeuille avec coût de transaction, on considère ici le cas $d = 2$ avec un compte de cash et un compte de stock. À cause du coût de la transaction, le stock est acheté à prix $\overline{S}_t$ et venté à prix $\underline{S}_t$ avec $\underline{S}_t < \overline{S}_t$ à chaque date t. On définit la stratégie mesurable comme un processus $\mathbb{F}$-prévisible $H = (H_t)_{0\leqslant t\leqslant T+1}$ avec $H_0 = H_{T+1} \equiv 0$ et note $\Delta H_t = H_t - H_{t-1}$. Alors le P&L d'une stratégie admissible est donné par*

$$V_T(H) = -\sum_{t=1}^{T+1} (\overline{S}_{t-1}\mathbf{1}_{\Delta H_t>0} + \underline{S}_{t-1}\mathbf{1}_{\Delta H_t<0})\Delta H_t$$

Étant donné une option exotique $\xi : \Omega \to \mathbb{R}$ et nombre fini des options liquides $\eta_i : \Omega \to \mathbb{R}$ avec prix 0, le coût de sur-réplication minimal de ξ avec stratégie statique sur η et stratégie dynamique sur S est donné par

$$\pi_e := \inf\{x : \exists h \in \mathbb{R}^e, H \text{ admissble}, x + \sum_{i=1}^{e} h_i\eta_i + V_T(H) \geqslant \xi, \mathcal{P} - q.s.\}$$

Dans un cadre plus général avec $d \geqslant 2$, le coût de transaction est modélisé en utilisant le concept de cône de la solvabilité, voir [49]. Pour $t \in \{0, 1, \cdots, T\}$, $K_t : \Omega \to 2^{\mathbb{R}^d}$ est un ensemble aléatoire $\mathcal{F}_t^0$-mesurable dans le sens où $\{\omega \in \Omega : K_t(\omega) \cap O \neq \emptyset\} \in \mathcal{F}_t^0$ pour tous les ensembles $O \subset \mathbb{R}^d$ fermé(ouvert). Pour $\omega \in \Omega$, $K_t(\omega)$ quelconque, $K_t(\omega)$ est un cône contenant $\mathbb{R}^d_+$, et représente l'ensemble des positions qui peuvent être transformées aux positions positives(component par component) après le rebalancement entre les actifs. On note par $K_t^* \subset \mathbb{R}^d_+$ son cône dual (positive)

$$K_t^*(\omega) \ := \ \big\{y \in \mathbb{R}^d \ : x \cdot y \geq 0 \text{ for all } x \in K_t(\omega)\big\}, \tag{2.3.7}$$

Example 2.3.2. *Si π_t^{ij} est le prix de l'actif j en unité de l'actif i, alors le cône de la solvabilité peut-être représenté comme*

$$K_t(\omega) := \big\{x : \exists (a^{ij})_{ij} \in \mathbb{R}_+^{d\times d}, s.t. x^i + \sum_{j\neq i} a^{ji} - a^{ij}\pi_t^{ij}(\omega) \geqslant 0, i \leqslant d\big\}.$$

Remark 2.3.3. *Comme dans [43], pour obtenir le résultat, le cône de la solvabilité est supposé de satisfaire plusieurs conditions techniques, comme la friction efficace, friction bornée. Pour plus de détail, voir Chapitre 4.*

En utilisant le cône de la solvabilité, on peut maintenant donner la définition des stratégies admissibles.

Definition 2.3.4. *On dit que un processus $\mathbb{F}$-adapté $\eta = (\eta_t)_{0\leq t\leq T}$ est une stratégie admissible si*

$$\eta_t \in -K_t \quad \mathcal{P}\text{-q.s. pour tous } t \leq T.$$

On note par $\mathcal{A}$ la collecte des stratégies admissibles.

Maintenant le coût minimal de la sur-réplication de l'option exotique ξ utilisant options vanilles ζ_i et stratégie dynamique est donné par [1] :

$$\pi_e(\xi) := \inf\Big\{y + \sum_{i=1}^{e} c_i|\ell_i| \ : y1_d + \sum_{i=1}^{e} \ell_i\zeta_i + \sum_{t=0}^{T} \eta_t - \xi \in K_T, \ \mathcal{P}\text{-q.s.}, (\eta, \ell) \in \mathcal{A} \times \mathbb{R}^e\Big\}, \tag{2.3.8}$$

[1] Here we use the convention $\sum_{i=1}^{0} = 0$

où 1_d est le vecteur avec tous les components égal à 0 sauf le dernier égal à 1.

2.3.2 $NA2(\mathcal{P})$: Équivalence entre local and global

L'esprit de [1] est de passer de local à globale, donc il est important de garantir le non-arbitrage local est impossible si le non-arbitrage global est vrai. Par contre, pour le non-arbitrage faible et stricte, ce n'est pas claire de preuver l'équivalence entre la version locale et la version globale. Dans [43], ils ont proposé de utiliser le non-arbitrage de seconde type (NA2($\mathcal{P}$)), introduit par [44]. $NA2(\mathcal{P})$ est défini de manière suivant :

Definition 2.3.5. *On dit que* NA2($\mathcal{P}$) *est vrai si pour tous* $t \leq T-1$ *et tous* $\zeta \in L^0(\mathcal{F}_t)$,

$$\zeta \in K_{t+1} \ \ \mathcal{P}\text{-q.s.} \quad \textit{implique que} \quad \zeta \in K_t \ \ \mathcal{P}\text{-q.s.}$$

La version robuste de théorème fontamentale est preuvée dans [43].

Theorem 2.3.6. *La condition* NA2($\mathcal{P}$) *est équivalent à : Pour tous* $t \leq T-1$, $\mathbb{P} \in \mathcal{P}$ *et* $Y \in L^0_{\mathcal{P}}(\mathcal{F}_t, \mathrm{int}K^*_t)$, *il existe* $\mathbb{Q} \in \mathfrak{B}(\Omega)$ *et un processus* $\mathbb{F}^0$*-adapté* $(Z_s)_{s=t,\dots,T}$ *tel que* $\mathbb{P} \ll \mathbb{Q}$ *et* $\mathbb{P} = \mathbb{Q}$ *sur* $\mathcal{F}_t$, *et*
(i) $\mathbb{Q} \lll \mathcal{P}$
(ii) $Y = Z_t \ \ \mathbb{Q}$*-p.s.*
(iii) $Z_s \in \mathrm{int}K^*_s \ \ \mathbb{Q}$*-p.s. pour* $s = t, \dots, T$
(iv) $(Z_s)_{s=t,\dots,T}$ *est une* $\mathbb{Q}$*-martingale, i.e.* $\mathbb{E}^{\mathbb{Q}}[Z_{s'}|\mathcal{F}_s] = Z_s$ *pour* $t \leq s \leq s'$.

Une couple $(\mathbb{Q}, Z)$ satisfaisant les conditions (i) − (iv) ci-dessus pour $t = 0$ est dit un système de pricing consistent strict(SCPS). On note $\mathcal{S}$ la famille des SCPS, et note $\mathcal{S}_0 := \big\{(\mathbb{Q}, Z) \in \mathcal{S}$ tel que $Z^d \equiv 1\big\}$. Dans la présence de l'option statique , il doit être consistent avec le spread bid-ask :

$$\mathcal{S}_e := \big\{(\mathbb{Q}, Z) \in \mathcal{S}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[\zeta_i \cdot Z_T\big] \in [-c_i, c_i], \ i = 1, \cdots, e\big\}.$$

L'avantage de NA2($\mathcal{P}$) est que il y a une équivalence entre le non-arbitrage local et global. Donc pour preuver le problème global, il suffit de résoudre le problème d'une période et ensuite leur coller ensemble.

2.3.3 Espace élargi et technique de randomisation

L'idée principale est de utiliser un argument de randomisation par l'introduction d'un espace élargi. On va construire un processus de prix fictif X avec aléa supplémentaire, et le problème original avec coût de transaction proportionnelle peut être réformulé comme un problème de sur-réplication sans friction dans le marché fictif. La technique de randomisation/élargissement est en effet dans le même esprit avec l'approche de marché fictif controlé de [50, 51].

L'espace élargi Soit $c > 1$ une constante donnant la borne de la friction, on définit $\Lambda_1 := [c^{-1}, c]^{d-1}$, $\Lambda_t := (\Lambda_1)^{t+1}$, et $\Lambda := \Lambda_T$, et introduit ensuite le processus canonique $\Theta_t(\theta) := \theta_t$, $\forall \theta = (\theta_t)_{0 \leq t \leq T} \in \Lambda$, et la tribu $\mathcal{F}^\Lambda_t := \sigma(\Theta_s, \ s \leq t)$, $t \leq T$. On définit un espace élargi $\overline{\Omega} := \Omega \times \Lambda$, et une tribu $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}^\Lambda_T$, avec deux filtrations $\overline{\mathbb{F}}^0 = (\overline{\mathcal{F}}^0_t)_{0 \leq t \leq T}$, et $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0 \leq t \leq T}$ dans lesquelles $\overline{\mathcal{F}}^0_t := \mathcal{F}^0_t \otimes \mathcal{F}^\Lambda_t$, et $\overline{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}^\Lambda_t$ pour $t \leq T$.

Maintenant le marché randomisé fictif peut être défini par le prix de stock fictif $X = (X_t)_{0 \leq t \leq T}$:

$$X_t(\bar{\omega}) := \Pi_{K^*_t(\omega)}[S_t(\omega)\theta_t], \quad \text{pour tous } \bar{\omega} = (\omega, \theta) \in \overline{\Omega}, \ t \leq T, \tag{2.3.9}$$

où $S_t(\omega)\theta_t := (S_t^1(\omega)\theta_t^1, \cdots, S_t^{d-1}(\omega)\theta_t^{d-1}, S_t^d(\omega))$, et $\Pi_{K_t^*(\omega)}[y]$ représente la projection de $y \in \mathbb{R}^d$ sur l'ensemble convex fermé $K_t^*(\omega)$.

Condition non-arbitrage sur l'espace élargi Dans l'espace élargi, on introduit d'abord l'ensemble du modèle par

$$\overline{\mathcal{P}} \ := \ \big\{\overline{\mathbb{P}} \in \mathfrak{B}(\overline{\Omega}, \overline{\mathcal{F}}) \text{ tel que } \overline{\mathbb{P}}|_\Omega \in \mathcal{P}\big\}.$$

L'ensemble de la stratégie peut être aussi défini naturellement sur l'espace élargi :

$$\overline{\mathcal{H}} := \{\text{Tous processus } \overline{\mathbb{F}}\text{-prévisible}\}.$$

On peut maintenant définir la condition de non-arbitrage dans l'espace élargi comme ci-dessous :

Definition 2.3.7. *On dit que* NA($\overline{\mathcal{P}}$) *est vrai si*

$$(\overline{H} \circ X)_T \geq 0, \ \overline{\mathcal{P}}\text{-q.s.} \quad \Longrightarrow \quad (\overline{H} \circ X)_T = 0, \ \overline{\mathcal{P}}\text{-q.s.},$$

pour tous $\overline{H} \in \overline{\mathcal{H}}$.

Si on néglige un petit problème dans la borne du spread bid-ask, les conditions NA2($\mathcal{P}$) et NA($\overline{\mathcal{P}}$) sont équivalent. Elle nous permet de travailler dans un espace élargi. Donc la prochaine étape est de réformuler le problème primal et dual dans un espace élargi.

Réformulation de problème primal and dual sur l'espace élargi On note $\overline{\mathcal{Q}}_0$ la famille des mesures $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$ tel que $\overline{\mathbb{Q}} \lll \overline{\mathcal{P}}$ et X est une $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale. Alors on a la réformulation suivant sur l'espace élargi pour la formulation primal et dual :

Proposition 2.3.8. (i) *Pour tous vecteur* $\xi : \Omega \to \mathbb{R}^d$ *universellement mesurable , on a*

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0} \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] \ = \ \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big].$$

(ii) *On a*

$$\pi_0(\xi) = \inf\Big\{y \in \mathbb{R} \ : y + (H \circ X)_T \geq g \ \overline{\mathcal{P}}\text{-q.s., pour certain } H \in \mathcal{H}\Big\}.$$

2.3.4 Résultat principal

Le résultat principal de cette section est la dualité de sur-réplication suivante :.

Theorem 2.3.9. *Soient* ξ *et* $(\zeta_i)_{i\leq e}$ *borelien, et suppose que* NA2($\mathcal{P}$) *est vrai. Suppose en plus soit* $e = 0$, *soit* $e \geq 1$ *et pour tous* $\ell \in \mathbb{R}^e$ *et* $\eta \in \mathcal{A}$,

$$\sum_{i=1}^{e} \big(\ell_i\zeta_i - |\ell_i|c_i 1_d\big) + \sum_{t=0}^{T} \eta_t \in K_T \ \mathcal{P}\text{-q.s.} \quad \Longrightarrow \quad \ell = 0. \tag{2.3.10}$$

Alors $\mathcal{S}_e$ *est non-vide et*

$$\pi_e(\xi) \ = \ \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e} \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big]. \tag{2.3.11}$$

En plus, il existe $(\hat{\eta}, \hat{\ell}) \in \mathcal{A} \times \mathbb{R}^e$ *tel que*

$$\pi_e(\xi)1_d + \sum_{i=1}^{e} \Big(\hat{\ell}_i \zeta_i - |\hat{\ell}_i| c_i 1_d \Big) + \sum_{t=0}^{T} \hat{\eta}_t - \xi \in K_T, \ \mathcal{P}\text{-q.s.}$$

Remark 2.3.10. (i) *Dans la preuve, il faut utiliser une combinaison de la technique de randomisation et un argument de minimax.*

(ii) *Dans le cas* $e = 1$*, il est claire que la condition* (1.3.13) *est un type de non-arbitrage. Par les arguments de randomisation,* $\ell_1 \zeta_1 - |\ell_1| c_1 \mathbf{1}_d + \sum_{t=0}^{T} \eta_t \in K_T$ $\mathcal{P}$*-q.s. est équivalent à*

$$\ell_1 g_1\big(\bar{\omega}, \widehat{\theta}\big) + \Big(\sum_{t=1}^{T} H_t \Delta X_t \Big)(\bar{\omega}) \geq 0, \ \overline{\mathcal{P}}\text{-q.s. pour } \widehat{\theta} = \pm 1,$$

où $H_t := \sum_{s=0}^{t-1} \eta_s$ *and* $g_1(\bar{\omega}, \pm 1) := \zeta_1 \cdot X_T \pm c_1$*. Pour plus de détail, voir Chapitre 5. La condition de non-arbitrage dans l'espace élargi nous donne maintenant*

$$\ell_1 g_1\big(\bar{\omega}, \widehat{\theta}\big) + \Big(\sum_{t=1}^{T} H_t \Delta X_t \Big)(\bar{\omega}) = 0, \ \overline{\mathcal{P}}\text{-q.s. pour } \widehat{\theta} = \pm 1.$$

Comme $g_1(\bar{\omega}, 1) \neq g_1(\omega, -1)$ *quand* $c_1 > 0$*, on a* $\ell_1 = 0$*.*

2.4 Maximisation d'utilité avec coût de transaction proportionnelle sous l'incertitude du modéle

2.4.1 Le problème de maximisation d'utilité

Le problème de maximisation d'utilité est un sujet classique dans mathématique financière. Dans le modèle dominé classique, maximisation d'utilité avec coût de transaction et dotation aléatoire est déjà beaucoup étudié. En particulier, quand on considère le marché incomplet, l'approche de la dualité est un outil puissant qui nous permet d'obtenir l'existence d'optimisateur primal par résolvant le problème d'optimisation dual corréspondant. Typiquement, le problème dual est formulé sur l'ensemble de measure de martingale (locale) équivalente(EMM), dont l'existence est garantie par certaines conditions de non-arbitrage. Pour l'utilité défini sur les réels positifs, [52], [53] ont établi le théorème de bipolaire par le choix propre de l'ensemble dual de EMM. Pour l'utilité définie sur les réels, [54], [55] and [56] ont résolu le problème par définissant un sous-ensemble de EMM avec l'entropie généralisé finie pour le dual et par définissant la classe du portefeuille de travail pour le primal.

Dans le cadre dominé avec coût de transaction, le EMM est remplacé par le système de pricing consistent(CPS) et la dualité de maximisation d'utilité est établie en utilisant la connection de la dualité de sur-réplication avec le prix de l'ombre. Pour le cas général du processus càdlàg, [39] ont preuvé l'existence d'un optimisateur dual et le prix de l'ombre. Dans [57], les auteurs ont considéré le cas des processus au delà du semi-martingale. [58] a fait un peu plus d'effort pour résoudre le problème de maximisation d'utilité avec la présence de pas seulement coût de transaction mais aussi dotation aléatoire.

Dans le cadre non-dominé, le problème de maximisation d'utilité en temps discret sans coût de transaction est d'abord étudié par [59]. Similaire à [60] dans le cas dominé, il est capable

de preuver l'existence de la stratégie optimale primal en utilisant le principe de la programmation dynamique. Dans [61], les auteurs ont considéré la présence de la friction et introduit une condition de non-arbitrage de type linéarité. [62, 63] ont considéré un contexte similaire à [59], et enlevé la contraint de la borne au dessus.

Plus récemment, [64] et [65] ont considéré un contexte de l'incertitude du modèle différent. Dans [64], l'incertitude du modèle est représenté par un ensemble du processus stochastique au lieu d'une famille de mesure de probabilité, et les auteurs ont preuvé l'exitence de la stratégie optimale par un argument de type Komlos. Dans [65], les auteurs ont établi une dualité de maximisation d'utilité en utilisant limite médiaire et une version fonctionnelle de théorème de capacitabilité de Choquet.

Dans [2], Bartl a considéré la représentation pour l'utilité de type exponentielle sans coût de transaction, sous une condition de non-arbitrage relativement forte. L'objectif principal de [66] est donc d'étudier l'existence de la stratégie optimale, le théorème de la dualité convexe et le principe de la programmation dynamique auxiliaire pour un problème de maximisation d'utilité avec coût de transaction au temps discret. En particulier, dans le cadre sans friction, l'argument de programmation dynamique nous permet de généraliser le résultat de [2] à une condition de non-arbitrage plus faible.

2.4.2 Maximisation d'utilité robuste avec coût de transaction

On reste dans le cadre de Bouchard-Nutz et les notations sont les même avec la section précédente. Pour obtenir un résultat de la dualité, on considère le fonction d'utilité de type exponentiel $U(x) := -\exp(-\gamma x)$, où $\gamma > 0$ un constant. Pour $e \in \mathbb{N} \cup \{0\}$, il y a une classe finie de vecteurs aléatoires $\mathcal{F}_T^0$-mesurable $\zeta_i : \Omega \to \mathbb{R}^d$, $i = 1, \cdots, e$, où chaque ζ_i représente le payoff de certaine option i noté par unité des différentes actifs risqués. Soit $\xi : \Omega \to \mathbb{R}^d$ le payoff de dotation aléatoire, on peut considérer le problème de maximisation suivant avec dotations aléatoires et stratégies semi-statiques :

$$V(\xi, \gamma) := \sup_{(\ell,\eta)\in\mathcal{A}_e} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[U\Big(\Big(\xi + \sum_{i=1}^{e} \big(\ell_i \zeta_i - |\ell_i| c_i 1_d \big) + \sum_{t=0}^{T} \eta_t \Big)^d \Big) \right]. \tag{2.4.12}$$

où 1_d est le vecteur avec tous les components égal à 0 sauf le dernier égal à 1 et $\mathcal{A}_e$ l'ensemble de tous $(l, \eta) \in \mathbb{R}^e \times \mathcal{A}$ tel que $\xi + \sum_{i=1}^{e} \big(\ell_i \zeta_i - |\ell_i| c_i 1_d \big) + \sum_{t=0}^{T} \eta_t \Big)^i = 0$ pour $i = 1, \cdots, d-1$. Autrement dit, le dernier actif joue le rôle d'un numéraire et tous les autres actifs doivent être liquidé au temps final T. Les options statiques ζ_i ont prix 0, mais ils ont un coût de transaction $c_i > 0$.

Pour donner une représentation duale au problème ci-dessus, on introduit une version robuste de l'entropie relative associé à une mesure de probabilité $\mathbb{Q}$ comme

$$\mathcal{E}(\mathbb{Q}, \mathcal{P}) := \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{E}(\mathbb{Q}, \mathbb{P}), \quad \text{où } \mathcal{E}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}} \right], & \text{si } \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{sinon.} \end{cases} \tag{2.4.13}$$

Soit $\mathcal{S}_0$ un sous-ensemble de SCPS $(\mathbb{Q}, Z)$ défini comme dans la section 2.3, on peut alors définir

$$\mathcal{S}_e^* := \Big\{ (\mathbb{Q}, Z) \in \mathcal{S}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[(\xi \cdot Z_T)_-\big] + \mathcal{E}(\mathbb{Q}, \mathcal{P}) < +\infty \text{ and } \mathbb{E}^{\mathbb{Q}}\big[(\zeta_i \cdot Z_T)\big] \in [-c_i, c_i],\ i = 1, \cdots, e \Big\}.$$

Maintenant on a le résultat principal suivant :

Theorem 2.4.1. *Soit ξ et $(\zeta_i)_{i\le e}$: $\Omega \to \mathbb{R}^d$ borelien et suppose que* NA2($\mathcal{P}$) *est vrai. Suppose en plus soit $e = 0$, soit $e \ge 1$ et pour tous $\ell \in \mathbb{R}^e$ et $\eta \in \mathcal{A}$,*

$$\sum_{i=1}^{e} \big(\ell_i\zeta_i - |\ell_i|c_i 1_d\big) + \sum_{t=0}^{T} \eta_t \in K_T \ \mathcal{P}\text{-q.s.} \quad \Longrightarrow \quad \ell = 0. \tag{2.4.14}$$

Alors, on a

$$V(\xi,\gamma) \ = \ -\exp\Big(-\inf_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\gamma\xi\cdot Z_T\big] + \mathcal{E}(\mathbb{Q},\mathcal{P})\big\}\Big), \tag{2.4.15}$$

En plus, l'infimum sur $(\ell,\eta) \in \mathcal{A}_e$ est atteint par une stratégie optimale $(\hat{\ell},\hat{\eta})$.

Remark 2.4.2. *En prenant logarithme sur les deux côtés et remplçant $\gamma\xi$ par $-\xi$, l'équalité* (2.4.15) *est équivalent à*

$$\begin{aligned} &\inf_{(\ell,\eta)\in\mathcal{A}_e}\sup_{\mathbb{P}\in\mathcal{P}}\log\mathbb{E}^{\mathbb{P}}\left[\exp\left(\left(\xi - \sum_{i=1}^{e}\left(\ell_i\zeta_i - |\ell_i|c_i1_d\right) - \sum_{t=0}^{T}\eta_t\right)^d\right)\right] \\ &= \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \mathcal{E}(\mathbb{Q},\mathcal{P})\big\}. \end{aligned} \tag{2.4.16}$$

2.4.3 Réformulation et la dualité dans l'espace élargi

La technique utilisée dans cette section est similaire avec la section 1.3. Dans une première étape, on va réformuler le problème original comme un problème sans coût de transaction dans un espace élargi. Ensuite il suffit de preuver la dualité dans l'espace élargi. La nouvelle difficulté pour la maximisation d'utilité est le problème d'intégrabilité quand on fait la programmation dynamique.

Réformulation du primal et dual dans l'espace élargi On introduit l'espace élargi $\overline{\Omega}$ comme la section précédente, et réformule le problème de maximisation d'utilité dans l'espace élargi, avec X le sous-jacent fictif et $g(\bar{\omega}) := \xi(\omega)\cdot X_T(\bar{\omega})$, pour tous $\bar{\omega} = (\omega,\theta) \in \overline{\Omega}$ le produit financier. D'abord, pour une fonction d'utilité générale, on a la réformulation suivante du primal dans l'espace élargi.

Proposition 2.4.3. *Sous certaines conditions, on a*

$$V(\xi) \ = \ \sup_{H\in\mathcal{H}}\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}}\mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g + (H\circ X)_T\Big)\Big].$$

Pour la représentation duale, on concentre sur la fonction d'utilité de type exponentielle $U(x) := -\exp(-\gamma x)$, pour certain constant $\gamma > 0$. En plus, pour la simplification, on suppose que le nombre des options statiques est égal à zéro. Le cas avec la présence des options statiques peut être considéré similairement par l'introduction d'un espace élargi plus large. On définit

$$\overline{\mathcal{Q}}^* \ := \big\{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0 \ : \mathbb{E}^{\overline{\mathbb{Q}}}\big[(\xi\cdot X_T)_-\big] + \mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}}) < \infty\big\},$$

alors on a la réformulation suivante pour le problème dual.

Proposition 2.4.4. *Pour tous vecteur aléatoire mesurable universellement $\xi : \Omega \to \mathbb{R}^d$, on a*

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0^*} \big\{\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] - \mathcal{E}(\mathbb{Q},\mathcal{P})\big\} \;=\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big] - \mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}})\big\}.$$

Dualité dans l'espace élargi Maintenant il suffit de preuver la dualité dans l'espace élargi.

Proposition 2.4.5. *Soit $g := \xi \cdot X_T$ et $\mathrm{NA}(\overline{\mathcal{P}})$ sont vrai. Alors on a*

$$\overline{V} \;:=\; \inf_{H\in\mathcal{H}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + (H\circ X)_T\big)\big] \;=\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}\big[g\big] - \mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}})\big\}. \tag{2.4.17}$$

En plus, l'infimum du problème $\overline{V}$ est atteint par certaine stratégie de trading optimal $\widehat{H} \in \mathcal{H}$.

Example 2.4.6. *Pour donner une idée sur la difficulté potentielle, considère l'exemple suivant avec le nombre de la période égal à un :*

On suppose que $T = 1$ et la dimension des actifs $d = 2$. Dans ce contexte, on a $\Omega = \mathbb{R}$ et le coût de transaction a lieu en date $t = 0$ et $t = 1$. Alors le prix d'achat et de vente au temps $t = 0$ et $t = 1$ sont $S_0(1+\alpha)$, $S_0(1-\alpha)$, $S_1(1+\alpha)$ et $S_1(1-\alpha)$ respectivement, avec $\alpha > 0$ un constant représente le coût de transaction. Notre problème de maximisation d'utilité peut-être écrit comme

$$V_0 := \inf_{H\in\mathbb{R}} \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}[\exp(g + HS_1(1-\alpha)\mathbf{1}_{H\geqslant 0}$$
$$+HS_1(1+\alpha)\mathbf{1}_{H<0} - HS_0(1-\alpha)\mathbf{1}_{H<0} - HS_0(1+\alpha)\mathbf{1}_{H\geqslant 0})]$$

On peut aussi définir l'espace élargi : $\overline{\Omega} := \mathbb{R}\times[1-\alpha,1+\alpha]^2$ et le processus fictif comme : $X_0(\theta_0) = S_0\theta_0$, $X_1(S_1,\theta_1) = S_1\theta_1$. Les filtrations sont alors $\overline{\mathcal{F}}_0 = \sigma(\theta_0)$ et $\overline{\mathcal{F}}_1 = \sigma(S_1)\otimes\sigma(\theta_0,\theta_1)$. On peut maintenant réformuler le primal et le dual comme un problème de maximisation d'utilité dans un espace élargi sans coût de transaction : $\sup_{(\mathbb{Q},Z)\in\mathcal{Q}_\star}(\mathbb{E}^{\mathbb{Q}}[g] - \mathcal{E}(\mathbb{Q},\mathbb{P})) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\star}(\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}},\overline{\mathbb{P}}))$ et

$$V_0 = \bar{V}_0 := \inf_{H\in\mathbb{R}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + H(X_1 - X_0)\big)\big].$$

Pour appliquer la dualité dans l'espace élargi, il y a un problème : dans la formulation primale, H est adapté à $\mathcal{F}_0$ au lieu de $\overline{\mathcal{F}}_0$.

En effet, le problème peut être résolu en utilisant un argument de minimax :

$$\begin{aligned}
\bar{V}_0 &= \inf_{H\in\mathbb{R}} \sup_{\theta_0\in[1-\alpha,1+\alpha]} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + H(X_1 - X_0)\big)\big] \\
&= \sup_{\theta_0\in[1-\alpha,1+\alpha]} \inf_{H\in\mathbb{R}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + H(X_1 - X_0)\big)\big] \\
&= \sup_{\theta_0\in[1-\alpha,1+\alpha]} \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\star(\theta_0)} (\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}},\overline{\mathbb{P}})) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\star} (\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}},\overline{\mathbb{P}})),
\end{aligned}$$

où $\overline{\mathcal{P}}(\theta_0)$ est le sous-ensemble de $\overline{\mathcal{P}}$ tel que Θ_0 est fixé à θ_0. L'argument de minimax est vrai pour la deuxième égalité ci-dessus car $[1-\alpha,1+\alpha]$ est compact, $H \mapsto \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + H(X_1 - X_0)\big)\big]$ est convex, et $\theta_0 \mapsto \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}(\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + H(X_1 - X_0)\big)\big]$ est linéaire. Par les arguments de la programmation dynamique, on peut maintenant coller les résultats d'une période ensemble pour obtenir le résultat global. Pour plus de détails, voir Chapitre 5.

2.4.4 Résultats associés

Dans [66], on considère aussi la propriété associé au prix d'indifférence d'utilité comme un sous-produit de la dualité de la maximisation d'utilité. En particulier, on obtient aussi la convergence du prix d'indifférence d'utilité vers le prix de sur-réplication quand le coefficient de l'aversion au risque tend vers l'infini dans le contexte non-dominé avec coût de transaction.

Dans le cas dominé au temps discret, le "prix de l'ombre" est étudié par [67]. Il est un marché fictif sans friction qui mène à la même stratégie optimale et utilisé comme le marché original avec coût de transaction proportionnelle. Il est aussi intéressant de considère sa généralisation dans le cadre non-dominé.

2.5 Gestion du risque dans le bilan par l'approche de sparse grid

2.5.1 Introduction

Dans cette partie, nous allons présenter une méthode robuste et efficace pour évaluer numériquement la distribution du risque dans le bilan pour une société de l'assurance dans un horizon donné qui est normalement un an et consistent avec la régulation du Solvency 2.

Dans un espace de probabilité filtré $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, le bilan de la société est un processus stochastique qui est composé à chaque date t par l'actif $(A_t)_{t\geq 0}$ et le passif $(L_t)_{t\geq 0}$. La quantité intéressant est le PnL(Profit and Loss en anglais) associé avec le bilan, qui est donné par

$$P_t = L_t - A_t \,, \quad t \geq 0 \,.$$

Le but est de calculer des différents indicateurs du risque pour la distribution de la perte du bilan dans un an (P_1) sous la mesure de probabilité dans le monde réelle $\mathbb{P}$. Nous le représentons par η.

Plus précisément, nous allons mesurer le risque associé avec η en utilisant une mesure de risque définie sur l'ensemble de mesures de carré intégrable $\varrho : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$. D'abord, nous considérons pour ϱ le Value-at-Risk $(V@R)$, qui est défini par

$$V@R_p(\eta) = \inf \{q \in \mathbb{R} \mid \eta((-\infty, q]) \geq p\} \,. \tag{2.5.18}$$

On va aussi travailler avec la classe de *mesures de risque spectrales* : une mesure de risque spectrale est définie par

$$\varrho_h(\eta) = \int_0^1 V@R_p(\eta) h(p) \mathrm{d}p \,, \tag{2.5.19}$$

où h est une densité de la probabilité non-décroissant sur $[0, 1]$.

2.5.2 Modèle financière

Le produit que on vente

On suppose que une société vente un put lookback au temps $t = 0$ avec la fonction de payoff G payé à la maturité $T > 0$ et écrit sur un actif risqué S. Pour $\kappa \geq 1$ et $t \in \{\tau_0 = 0, \tau_1, \cdots, \tau_\kappa = T\}$,

le payoff G est donné par

$$G(S_{\tau_0}, \dots, S_{\tau_\kappa}) = \left(\max_{0 \le \ell \le \kappa} S_{\tau_\ell} \right) - S_T. \tag{2.5.20}$$

Le modèle du marché sous mesure risque-neutre

Le modèle du taux au court terme Soit $\Theta \in \mathbb{R}^3$ l'ensemble du paramètres représentant les observations du marché. La dynamique de taux au court terme est donné par le modèle Hull & White

$$r_s^{t,\Theta} = r_t^{t,\Theta} + \int_t^s a \left(\mu_u^{t,\Theta} - r_u^{t,\Theta} \right) \mathrm{d} + b \left(B_s - B_t \right), \ s \in [t, T], \tag{2.5.21}$$

avec B un $\mathbb{Q}$-mouvement Brownien et a et b des constants réels. Le paramètre $\mu^{t,\Theta}$ est calibré en utilisant les observations du marché Θ tel que le modèle nous donne la courbe de taux d'intérêt du marché. Il est donné par

$$\mu_s^{t,\Theta} = f^\Theta(t,s) + \frac{1}{a} \frac{\partial f^\Theta(t,s)}{\partial s} + \frac{b^2}{2a^2} \left(1 - e^{-2a(s-t)} \right), s \in [t, T]. \tag{2.5.22}$$

Remark 2.5.1. *Dans ce modèle, soit $(0, \Theta_0)$ l'observation au temps 0. Le contrat de type swap émis au $s = 0$, avec la maturité $M > 0$, taux $R > 0$, et les coupons versé à chaque date $i \in \{1, \dots, M\}$. Alors le prix de ce contrat au temps t est donné par :*

$$SW^{t,\Theta,M,R} = \frac{P^{t,\Theta,1}}{P^{0,\Theta_0,1}} - P^{t,\Theta,M} - R \sum_{i=1}^{M} P^{t,\Theta,i}, \tag{2.5.23}$$

Le modèle du stock Étant donné les observations Θ les facteurs du taux d'intérêt et le prix du l'actif risqué $x \in (0, \infty)$, la dynamique du prix sous mesure de probabilité risque neutre $\mathbb{Q}$ est donné par

$$S_s^{t,x,\Theta} = x + \int_t^s r_u^{t,\Theta} S_u^{t,x,\Theta} \mathrm{d}u + \int_t^s \sigma S_u^{t,x,\Theta} \mathrm{d}\tilde{W}_u, s \in [t, T], \tag{2.5.24}$$

où $\sigma > 0$, $\tilde{W}$ est un autre $\mathbb{Q}$-mouvement Brownien, dont la covariation quadratique avec B est donné par

$$\langle B, \tilde{W} \rangle_t := \rho \, t, t \in [0, T],$$

où $\rho \in [-1, 1]$.

Le passif Pour chaque observation $\bar{\mathcal{X}}_t := (\bar{S}_t, \bar{\Theta}_t)$, le valeur $L_t =: \ell(t, \bar{S}_t, \bar{\Theta}_t)$ du passif est calculé au temps $t = 1$. Dans notre contexte, le passif de la société est simplement donné par

$$\ell(t, x, \Theta) = \mathbb{E}^Q [e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta})], \tag{2.5.25}$$

où $(r^{t,\Theta}, S^{t,x,\Theta})_{t \le s \le T}$ sont la dynamique de taux au court terme et le prix du stock sous mesure risque neutre.

Modéliser le bilan

L'actif La société veut répliquer le produit G et le portefeuille de la couverture A est décomposé en deux parties

$$A_t = A_t^{\Delta} + A_t^{\rho}. \tag{2.5.26}$$

Le processus A^{Δ} est le valeur du portefeuille qui est utilisé pour la couverture du risque associé avec S, et A^{ρ} est pour la couverture du risque associé avec Θ.

Δ-couverture : Le valeur de A^{Δ} au temps 1 est

$$A_1^{\Delta} = \sum_{i=0}^{n-1} \Delta(t_i, \bar{S}_{t_i}, \bar{\Theta}_{t_i}) \left(\bar{S}_{t_{i+1}} - \bar{S}_{t_i}\right) \quad \text{où} \quad \Delta(t,x,\Theta) := \frac{\partial L}{\partial x}(t,x,\Theta). \tag{2.5.27}$$

ρ-couverture : Le valeur de A_1^{ρ} est

$$A_1^{\rho} = \sum_{i=0}^{n-1} \sum_{j=1}^{3} \rho^j(t_i, \bar{S}_{t_i}, \bar{\Theta}_{t_i}) \left(\mathrm{SW}^{t_{i+1}, \bar{\Theta}_{t_{i+1}}, T_j, R_j} - \mathrm{SW}^{t_i, \bar{\Theta}_{t_i}, T_j, R_j}\right) . \tag{2.5.28}$$

La fonction PnL globale

Finalement, le PnL du bilan au temps 1 est donné par,

$$P_1 = p_1\left((t, \bar{S}_t, \bar{\Theta}_t)_{t\in\Gamma}\right)$$

où $(\bar{S}, \bar{\Theta})$ sont les paramètres du marché et la fonction du PnL $p_1 : \mathbb{R}^{\gamma} \to \mathbb{R}$, avec $\gamma = 4 \times (n+1)$, est donné par

$$\ell(t_n, x_n, \Theta_n) - \sum_{i=0}^{n-1} \Delta(t_i, x_i, \Theta_i)(x_{i+1} - x_i) - \sum_{i=0}^{n-1} \sum_{j=1}^{3} \rho^j(t_i, x_i, \Theta_i) \left(\mathrm{SW}^{t_{i+1}, \Theta_{i+1}, T_j, R_j} - \mathrm{SW}^{t_i, \Theta_i, T_j, R_j}\right) . \tag{2.5.29}$$

2.5.3 L'estimation de la mesure de risque

Étant donné une mesure de risque ϱ et la distribution de la perte η du bilan d'un an, on estime la quantité de taux $\varrho(\eta)$ par la simulation d'un échantillon de N variables aléatoires i.i.d. $(\Psi_j)_{1\le j\le N}$ avec loi η et le calcul de $\varrho(\eta^N)$ par (2.5.18), (2.5.19) avec η^N au lieu de η. Ici, η^N est la mesure empirique associée avec Ψ_j, i.e.

$$\eta^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{\Psi_j} .$$

La distribution de la perte η est obtenue par :

$$\eta = p_1 \sharp \nu,$$

avec p_1 donné dans (2.5.29) et ν la distribution des paramètres du marché. Autrement dit, ν est la loi des variables aléatoires

$$\bar{\mathcal{X}} := (\bar{S}_t, \bar{\Theta}_t)_{t\in\Gamma} \tag{2.5.30}$$

sous la mesure de probabilité $\mathbb{P}$.

Pour l'estimation de $\varrho(\eta)$ pour une mesure de risque donnée, les étapes sont suivantes : d'abord, il faut simuler $\bar{\mathcal{X}}$ et utiliser une approximation p_1^{υ} de p_1 :

- $p_1^{\mathcal{N}}$ si on choisit l'approche de *nested simulation* ;
- $p_1^{\mathcal{S}}$ si on choisit l'approche de *sparse grid* approach.

Ensuite, l'estimateur de $\varrho(\mu)$ est donné par

$$R^{\upsilon} := \varrho(p_1^{\upsilon}\sharp\nu^N)\,, \quad \text{pour}\, \upsilon \in \{\mathcal{N}, \mathcal{S}\}\,. \tag{2.5.31}$$

Pour conclure, les deux méthodes ont les étapes suivantes :

L'approche de nested simulation

1. Étape extérieure : Simuler les paramètres du modèle $(\mathcal{X}_j)_{j=1,\dots,N}$.
2. Étape int erieure : Simuler $\Psi_j = p^{\mathcal{N}}(\mathcal{X}_j)$ par les simulations. Il faut calculer le prix des options par Monte Carlo, le prix des produits de taux et les greques.
3. Estimer la mesure du risque.

L'approche de Sparse grid

1. Fixer un (sparse) grid $\mathcal{V}$ et calculer l'approximation $p^{\mathcal{S}}$ pour tous valeurs nécessaires dans le grid par une simulation de monte carlo.
2. Simuler les N échantillons des paramètres du modèle $(\mathcal{X}_j)$ et evaluer $\Psi_j = p^{\mathcal{S}}(\mathcal{X}_j)$.
3. Estimer la mesure du risque.

Chapitre 3

Super-replication of American option under model uncertainty

3.1 Introduction

The robust approach to pricing and hedging has been an active field of research in mathematical finance over recent years. It aims to address one of the key shortcomings of the classical approach, namely its inability to account for model misspecification risk. Accordingly, the capacity to account for model uncertainty is at the core of the robust approach. In the classical approach one postulates a fixed probability measure $\mathbb{P}$ to describe the future evolution of prices of risky assets. In contrast, in the robust approach one considers the pricing and hedging problem simultaneously under a family of probability measures, or pathwise on a set of feasible trajectories. The challenge lies in extending the arbitrage pricing theory, which is well understood in the classical setup, to the robust setting.

In the classical approach, when the reference measure $\mathbb{P}$ is fixed, the absence of arbitrage is equivalent to the existence of a martingale measure equivalent to $\mathbb{P}$, a result known as the first fundamental theorem of asset pricing, see, e.g., [68] or [69]. When the market is complete—i.e., when every contingent claim can be perfectly replicated using a self-financing trading strategy—the equivalent martingale measure $\mathbb{Q}$ is unique, and the fair price for a contingent claim is equal to the replication cost of its payoff, and may be computed as the expected value of the discounted payoff under $\mathbb{Q}$. In an incomplete market, where a perfect replication strategy does not always exist, a conservative way of pricing is to use the minimum superreplication cost of the option. Employing duality techniques, this superreplication price can be expressed as the supremum of expectations of the discounted payoff over all martingale measures equivalent to $\mathbb{P}$.

In the robust approach, in the absence of a dominating probability measure, this elegant story often becomes more involved and technical. In continuous time models under *volatility uncertainty*, analogous pricing-hedging duality results have been obtained by, among many others, [3], [5], [70], [9]. In discrete time, a general pricing-hedging duality was shown in, e.g., [1] and [11]. Importantly, in a robust setting one often wants to include additional market instruments which may be available for trading. In a setup which goes back to the seminal work of [71], one often considers dynamic trading in the underlying asset and static trading, i.e., buy and hold strategies at time zero, in some European options, often call or put options with a fixed maturity. Naturally, such additional assets constrain the set of martingale measures which may be used for pricing. General pricing-hedging duality results, in different variations of this setting, both in continuous and in discrete time, can be found in, e.g., [72], [12], [73], [74], [75], [76], [77] and we refer to the survey papers [78] and [79] for more details.

The main focus in the literature so far has been on the duality for (possibly exotic) European payoffs. However, more recently, some papers have also investigated American options. [80] studied the necessary (and, in some cases, sufficient) conditions for absence of arbitrage among American put option prices. [81] studied game options (including American options) in a non-dominated discrete time market, but in the absence of any statically traded options. [22] considered a discrete time, discrete space market in the presence of statically traded European vanilla options. He observed that the superhedging price for an American option may be strictly larger than the supremum of its expected (discounted) payoff over all stopping times and all (relevant) martingale measures. We refer to such a situation as a *duality gap*. In [22], the pricing-hedging duality was then restored by using a *weak* dual formulation. This approach was further exploited, with more general results, in [23]. [20] studied the same superhedging problem as in the setup of [1], but only considered *strong* stopping times in their dual formulation, which again, in general, leads to a duality gap. More recently, and in parallel to an earlier version of this paper, imposing suitable regularity and integrability conditions on the payoff functions, [21] were able to prove a duality result by considering *randomized* models.

Motivated by the above works, we endeavour here to understand the fundamental reasons

why pricing-hedging duality for American options holds or fails, and offer a systematic approach to mend it in the latter case. We derive two main general results which we then apply to various specific contexts, both classical and robust. Our first insight is that by considering an appropriate enlargement of the space, namely the time-space product structure, we can see an American option as a European option and recover the pricing-hedging duality, which may fail in the original formulation. This may be seen as a weak formulation of the dual (pricing) problem and leads to considering a large family of stopping times. This formulation of the dual problem is similar in spirit to [22], [23] and [21], but our approach leads to duality results in more general settings, and/or under more general conditions, see Remark 3.2.7 and Subsection 3.2.8 and also [82]. Our second main insight is that the duality gap is caused by the failure of the dynamic programming principle. To recover the duality, under the formulation with *strong* stopping times, it is necessary and sufficient to consider an enlargement which restores dynamic consistency: it is enough to consider (fictitious) extensions of the market in which all the assets are traded dynamically. As a byproduct, we find that the strategies which trade dynamically in options and the semi-static strategies described above lead to the same superhedging cost in various settings.

The first part of the paper, Section 3.2, presents the above two main insights in a very general discrete time framework which covers both classical (dominated) and robust (non-dominated) settings. In the second part of the paper, we apply our general results in the context of two important examples of the robust framework: the setup of [1] in Section 5.2, and the martingale optimal transport setup of [73] in Section 3.3. We obtain suitable pricing-hedging duality for American options in both setups. In the latter case of martingale optimal transport, there is an infinity of assets to consider and we use measure valued martingales to elegantly describe this setting. To allow for a suitable flow of narrative of our main results, technical proofs of the results in Sections 3.2,5.2 and 3.3 are postponed and presented respectively in Sections 3.4,3.5 and 3.6.

Example 3.1.1. *We conclude this introduction with a motivating example showing that the pricing-hedging duality may fail in the presence of statically traded instruments and how it may be recovered when exercise times are allowed to depend on the dynamic price processes of these instruments. This example is summarized in Figure 3.1. We consider a two period model with stock price process S given by $S_0 = S_1 = 0$ and $S_2 \in \{-2,-1,1,2\}$. The American option process Φ is defined as $\Phi_1 \equiv 1$, $\Phi_2(\{S_2 \in \{-2,2\}\}) = 0$ and $\Phi_2(\{S_2 \in \{-1,1\}\}) = 2$. The (pathwise) superhedging price of Φ, i.e., the minimal initial wealth which allows superhedging against all possible states and times by trading in the stock, can be easily computed and equals 2 (keeping 2 in cash and not trading in stock). A probability measure $\mathbb{Q}$ on the space of four possible paths is uniquely described through a choice of $q_i = \mathbb{Q}(S_2 = i) \geqslant 0$ for $i \in \{-2,-1,1,2\}$ satisfying $q_2+q_1+q_{-1}+q_{-2} = 1$. The martingale condition is equivalent to $2q_2+q_1-q_{-1}-2q_{-2} = 0$. Note that as there are only two stopping times greater than 0, namely $\tau_1 = 1$ and $\tau_2 = 2$, the market model price given as the double supremum over all stopping times τ and all martingale measures $\mathbb{Q}$ of $\mathbb{E}^{\mathbb{Q}}[\Phi_\tau]$ also equals 2 (as $\mathbb{E}^{\widetilde{\mathbb{Q}}}[\Phi_{\tau_2}] = 2$ for $\widetilde{\mathbb{Q}}$ given by $\widetilde{q}_1 = \widetilde{q}_{-1} = 1/2$ and $\widetilde{q}_2 = \widetilde{q}_{-2} = 0$) and the two prices agree.*

Suppose now that we add a European option g with a payoff $g = \mathbb{1}_{\{|S_2|=1\}} - 1/2$ and an initial price 0, which may be used as a static hedging instrument. With g and S, the superhedging price of Φ drops to 3/2 (e.g., keep 3/2 in cash and buy one option g). The presence of g also imposes a calibration constraint on the martingale measures: $q_1+q_{-1} = 1/2$. Thus, any calibrated martingale measure can be expressed by $(q_2,q_1,q_{-1},q_{-2}) = (q, 3/4-2q, 2q-1/4, 1/2-q)$ with $q \in (1/8, 3/8)$, and the market model price equals 1 (as $\mathbb{E}^{\mathbb{Q}}[\Phi_{\tau_1}] = \mathbb{E}^{\mathbb{Q}}[\Phi_{\tau_2}] = 1$ under any calibrated martingale measure). We therefore see that adding a statically traded option breaks the pricing-hedging duality.

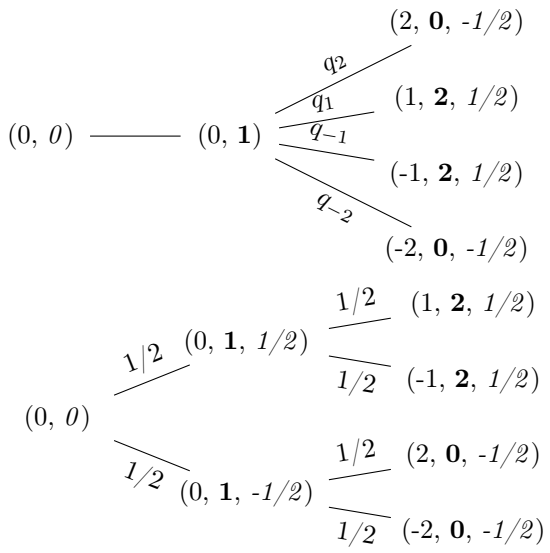

Figure 3.1: The stock prices are written in a regular font, the payoffs of the American option are in bold, and the prices of the European option g are in italic. The model without dynamic trading in g is on the left. The model with dynamic trading in g, in which the duality is recovered, is on the right.

Let us now show that the duality is recovered when we consider a fictitious market where the option g is traded dynamically and the exercise policy can depend on its current price. We model this through a process $Y = (Y_t : t = 0, 1, 2)$ given by $Y_2 = g$, $Y_1 = 1/2$ on $\{|S_2| = 1\}$, $Y_1 = -1/2$ on $\{|S_2| = 2\}$, and $Y_0 = 0$. Note that there exists a (unique) measure $\widehat{\mathbb{Q}}$, presented in Figure 1, such that both S and Y are martingales with respect to their joint natural filtration $\widehat{\mathbb{F}}$ and in particular $\widehat{\mathbb{Q}}$ is calibrated: $\mathbb{E}^{\widehat{\mathbb{Q}}}[g] = 0$. The filtration $\widehat{\mathbb{F}}$ is richer than the natural filtration of S alone and allows for an additional stopping time $\tau^* = \mathbb{1}_{\{Y_1=-1/2\}} + 2\mathbb{1}_{\{Y_1=1/2\}}$, and the duality is recovered as $\mathbb{E}^{\widehat{\mathbb{Q}}}[\Phi_{\tau^*}] = 3/2$.

3.2 Pricing-hedging duality for American options

We present in this section general results which explain when and why the pricing-hedging duality for American options holds. We work in a general discrete time setup which we now introduce. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{F} := (\mathcal{F}_k)_{k=0,1,\dots,N}$ be a filtration, where $\mathcal{F}_0$ is trivial and $N \in \mathbb{N}$ is the time horizon. We denote by $\mathfrak{P}(\Omega)$ the set of all probability measures on $(\Omega, \mathcal{F})$ and consider a subset $\mathcal{P} \subset \mathfrak{P}(\Omega)$. We say that a given property holds $\mathcal{P}$-quasi surely ($\mathcal{P}$-q.s.) if it holds $\mathbb{P}$-almost surely for every $\mathbb{P} \in \mathcal{P}$, and say that a set from $\mathcal{F}$ is $\mathcal{P}$-polar if it is a null set with respect to every $\mathbb{P} \in \mathcal{P}$. We write $\mathbb{Q} \ll\ll \mathcal{P}$ if there exists a $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{Q} \ll \mathbb{P}$. Given a random variable ξ and a sub-σ-field $\mathcal{G} \subset \mathcal{F}$, we define the conditional expectation $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{G}] := \mathbb{E}^{\mathbb{P}}[\xi^+|\mathcal{G}] - \mathbb{E}^{\mathbb{P}}[\xi^-|\mathcal{G}]$ with the convention $\infty - \infty = -\infty$, where $\xi^+ := \xi \vee 0$ and $\xi^- := -(\xi \wedge 0)$. We consider a market with no transaction costs and with financial assets, some of which are dynamically traded and some of which are only statically

traded. The former are modelled by an adapted $\mathbb{R}^d$-valued process S with $d \in \mathbb{N}$. We think of the latter as European options which are traded at time $t = 0$ and not at future times. We let $g = (g^\lambda)_{\lambda\in\Lambda}$, where Λ is a set of arbitrary cardinality, be the vector of their payoffs which are assumed to be $\mathbb{R}$-valued and $\mathcal{F}$-measurable. Up to a constant shift of the payoffs, we may assume, without loss of generality, that all options g^λ have zero initial price. All prices are expressed in units of some numeraire S^0, such as a bank account, whose price is thus normalised, so that $S^0_t \equiv 1$. We denote by $\mathcal{H}$ the set of all $\mathbb{F}$-predictable $\mathbb{R}^d$-valued processes, and by $\mathfrak{h} = \{h \in \mathbb{R}^\Lambda : \exists$ finite subset $\beta \subset \Lambda$ s.t. $h^\lambda = 0\ \forall\lambda \notin \beta\}$. A self-financing strategy trades dynamically in S and statically in finitely many of g^λ, $\lambda \in \Lambda$ and hence corresponds to a choice of $H \in \mathcal{H}$ and $h \in \mathfrak{h}$. Its associated final payoff is given by

$$(H \circ S)_N + hg = \sum_{j=1}^{d}\sum_{k=1}^{N} H^j_k \Delta S^j_k + \sum_{\lambda\in\Lambda} h^\lambda g^\lambda, \tag{3.2.1}$$

where $\Delta S^j_k = S^j_k - S^j_{k-1}$. Having defined the trading strategy, we can consider the superhedging price of an option with payoff ξ at time N, given by

$$\pi^E_g(\xi) := \inf\{x : \exists\ (H,h) \in \mathcal{H}\times\mathfrak{h}\ \text{ s.t. }\ x + (H\circ S)_N + hg \geqslant \xi\ \mathcal{P}\text{-q.s.}\}. \tag{3.2.2}$$

In particular, if $\mathcal{P} = \mathfrak{B}(\Omega)$ is the set of all probability measures on $\mathcal{F}$ and $\{\omega\} \in \mathcal{F}$ for all $\omega \in \Omega$, then the superreplication in (3.2.2) is pathwise on Ω.

To formulate a duality relationship, we need the dual elements given by rational pricing rules, or martingale measures,

$$\begin{aligned} \mathcal{M} &= \{\mathbb{Q} \in \mathfrak{B}(\Omega) : \mathbb{Q} \lll \mathcal{P} \text{ and } \mathbb{E}^{\mathbb{Q}}[\Delta S_k|\mathcal{F}_{k-1}] = 0,\ \forall k = 1,\dots,N\},\\ \mathcal{M}_g &= \{\mathbb{Q} \in \mathcal{M} : \mathbb{E}^{\mathbb{Q}}[g^\lambda] = 0,\ \forall\lambda \in \Lambda\}. \end{aligned} \tag{3.2.3}$$

Definition 3.2.1. *Let Υ be a given class of real-valued functions defined on Ω. We say that the* (European) pricing-hedging duality holds for the class Υ *if $\mathcal{M}_g \neq \emptyset$ and*

$$\pi^E_g(\xi) = \sup_{\mathbb{Q}\in\mathcal{M}_g} \mathbb{E}^{\mathbb{Q}}[\xi], \quad \xi \in \Upsilon. \tag{3.2.4}$$

Remark 3.2.2. *Note that the inequality "$\geqslant$" in* (3.2.4)*, called weak pricing-hedging duality, holds automatically from the definition of $\mathcal{M}_g$ in* (3.2.3)*.*

A number of papers, including [1] and [12], proved that the above pricing-hedging duality (3.2.4) holds under various further specifications and restrictions on Ω, $\mathbb{F}$, $\mathcal{P}$ and Υ, including in particular an appropriate no-arbitrage condition. We take the above duality for granted here and our aim is to study an analogous duality for American options. We work first in the general setup described above without specifying $\mathbb{F}$ or Υ, as our results will apply to any such further specification. Further, many abstract results in this section also extend to other setups, e.g., to trading in continuous time.

3.2.1 Superhedging of American options

An American option may be exercised at any time $k \in \mathbb{T} := \{1,\dots,N\}$ (without loss of generality we exclude exercise at time 0). It is described by its payoff function $\Phi = (\Phi_k)_{1\leq k\leq N}$, where $\Phi_k : \Omega \to \mathbb{R}$ belongs to Υ and is the payoff, delivered at time N, if the option is exercised at time k. Usually Φ_k is taken to be $\mathcal{F}_k$-measurable, but here we only assume Φ_k to be $\mathcal{F}$-measurable for

greater generality which includes, e.g., the case of a portfolio containing a mixture of American and European options. We note that when hedging our exposure to an American option, we should be allowed to adjust our strategy in response to an early exercise. As a consequence, the superhedging cost of the American option Φ using semi-static strategies is given by

$$\pi_g^A(\Phi) := \inf\Big\{x : \exists({}^1H, \dots, {}^NH) \in \mathcal{H}^N \text{ s.t. } {}^jH_i = {}^kH_i \ \forall 1 \leqslant i \leqslant j \leqslant k \leqslant N \text{ and } h \in \mathfrak{h} \\ \text{satisfying } x + ({}^kH \circ S)_N + hg \geqslant \Phi_k \ \forall k = 1, \dots, N \ \mathcal{P}\text{-q.s.}\Big\}$$

Remark 3.2.3. *We formulate the problem above with payoff delivered by the seller at maturity N irrespectively of the actual exercise time. In full generality, this is necessary because the payoff is not assumed to be known at the exercise time. However, given that we work in discounted units, if the payoff is known at the exercise time, our convention is equivalent to the one in which the payoff is delivered at its exercise time, via taking a loan, and then the seller has to be able to continue trading in such a way that her final payoff is non-negative. For this equivalence to hold it is important to allow the seller to adjust the strategy at the time of the exercise. Note that in the more classical setting when Φ_k is $\mathcal{F}_k$-measurable and there are no statically traded options, i.e., $\Lambda = \emptyset$, no-arbitrage ensures that to have a final non-negative payoff the seller has to have a non-negative wealth after delivering the payoff at the exercise time. She can then just stop trading altogether—the vector of strategies $({}^1H, \dots, {}^NH) \in \mathcal{H}^N$ above then reduces to a single trading strategy which is unwound at the exercise time.*

Classically, the pricing of an American option is recast as an optimal stopping problem and, extending (3.2.4), it would be natural to ask whether

$$\pi_g^A(\Phi) \stackrel{?}{=} \sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau] \tag{3.2.5}$$

holds, where $\mathcal{T}(\mathbb{F})$ denotes the set of $\mathbb{F}$-stopping times. However, as illustrated by the simple example in the introduction, this duality may fail. The "numerical" reason is that the right-hand side in (3.2.4) may be too small because the set $\mathcal{M}_g \times \mathcal{T}(\mathbb{F})$ is too small. Our aim here is to understand the fundamental reasons why the duality fails and hence discuss how and why the right-hand side should be modified to obtain the equality in (3.2.5).

3.2.2 An American option is a European option on an enlarged space

The first key idea of this paper offers a generic enlargement of the underlying probability space which turns all American options into European options. Depending on the particular setup, it may take more or less effort to establish (3.2.4) for the enlarged space, but this shifts the difficulty back to the better understood and well studied case of European options. Our reformulation technique—from an American to European option—can be easily extended to other contexts, such as the continuous time case. The enlargement of space is based on construction of random times, previously used, e.g., in [24, 25] to study the existence of random times with a given survival probability, in [26] to study a general optimal control/stopping problem, and in [18] and [27] to study the optimal Skorokhod embedding problem.

Recalling the notation $\mathbb{T} := \{1, \dots, N\}$, we introduce the space $\overline{\Omega} := \Omega \times \mathbb{T}$ with the canonical time $T : \overline{\Omega} \to \mathbb{T}$ given by $T(\bar{\omega}) := \theta$, where $\bar{\omega} := (\omega, \theta)$, the filtration $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_k)_{k=0,1,\dots,N}$ with $\overline{\mathcal{F}}_k = \mathcal{F}_k \otimes \vartheta_k$ and $\vartheta_k = \sigma(T \wedge (k+1))$, and the σ-field $\overline{\mathcal{F}} = \mathcal{F} \otimes \vartheta_N$. By definition, T is an $\overline{\mathbb{F}}$-stopping time. We denote by $\overline{\mathcal{H}}$ the class of $\overline{\mathbb{F}}$-predictable processes and extend naturally the definitions of S and g^λ from Ω to $\overline{\Omega}$ via $S(\bar{\omega}) = S(\omega)$ and $g^\lambda(\bar{\omega}) = g^\lambda(\omega)$ for $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}$.

We let $\overline{\Upsilon}$ be the class of random variables $\bar{\xi} : \overline{\Omega} \to \mathbb{R}$ such that $\bar{\xi}(\cdot, k) \in \Upsilon$ for all $k \in \mathbb{T}$ and we let $\overline{\pi}_g^E(\bar{\xi})$ denote the superreplication cost of $\bar{\xi}$. We may, and will, identify $\overline{\Upsilon}$ with Υ^N via $\bar{\xi}(\bar{\omega}) = \Phi_\theta(\omega)$. Finally, we introduce

$$\begin{aligned} \overline{\mathcal{P}} &= \{\overline{\mathbb{P}} \in \mathfrak{P}(\overline{\Omega}) : \overline{\mathbb{P}}_{|\Omega} \in \mathcal{P}\}, \\ \overline{\mathcal{M}} &= \{\overline{\mathbb{Q}} \in \mathfrak{P}(\overline{\Omega}) : \overline{\mathbb{Q}} \lll \overline{\mathcal{P}} \text{ and } \mathbb{E}^{\overline{\mathbb{Q}}}[\Delta S_k | \overline{\mathcal{F}}_{k-1}] = 0 \ \forall k \in \mathbb{T}\}, \\ \overline{\mathcal{M}}_g &= \{\overline{\mathbb{Q}} \in \overline{\mathcal{M}} : \mathbb{E}^{\overline{\mathbb{Q}}}[g^\lambda] = 0 \ \forall \lambda \in \Lambda\}. \end{aligned} \tag{3.2.6}$$

Theorem 3.2.4. *For any $\Phi \in \Upsilon^N = \overline{\Upsilon}$, we have*

$$\pi_g^A(\Phi) = \overline{\pi}_g^E(\Phi) := \inf\{x : \exists\ (\overline{H}, h) \in \overline{\mathcal{H}} \times \mathfrak{h} \ \ s.t. \ \ x + (\overline{H} \circ S)_N + hg \geqslant \xi \ \overline{\mathcal{P}}\text{-}q.s.\}. \tag{3.2.7}$$

In particular, if the European pricing-hedging duality on $\overline{\Omega}$ holds for Φ, then

$$\pi_g^A(\Phi) = \overline{\pi}_g^E(\Phi) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]. \tag{3.2.8}$$

Proof. First note that

$$\overline{\mathcal{H}} = \{\overline{H} = (\overline{H}(\cdot, 1), \dots, \overline{H}(\cdot, N)) \in \mathcal{H}^N : \overline{H}_i(\cdot, j) = \overline{H}_i(\cdot, k) \ \forall 1 \leqslant i \leqslant j \leqslant k \leqslant N\},$$

and hence that the dynamic strategies used for superhedging in π_g^A and in $\overline{\pi}_g^E$ are the same. The equality now follows by observing that a set $\Gamma \in \overline{\mathcal{F}}_N$ is $\overline{\mathcal{P}}$-polar if and only if its k-sections $\Gamma_k = \{\omega : (\omega, k) \in \Gamma\}$ are $\mathcal{P}$-polar for all $k \in \mathbb{T}$. Indeed, for one implication assume that $\overline{\mathbb{P}}(\Gamma) = 0$ for each $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$. For arbitrary $\mathbb{P} \in \mathcal{P}$ and $k \in \mathbb{T}$ we can define $\overline{\mathbb{P}} = \mathbb{P} \otimes \delta_k$ which belongs to $\overline{\mathcal{P}}$, and hence $\mathbb{P}(\Gamma_k) = 0$ follows.

To show the reverse implication, assume that $\mathbb{P}(\Gamma_k) = 0$ for each $\mathbb{P} \in \mathcal{P}$ and $k \in \mathbb{T}$. Observe that, for any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$,

$$\overline{\mathbb{P}}(\Gamma) = \sum_{k \in \mathbb{T}} \overline{\mathbb{P}}(\Gamma_k \times \{k\}) \leqslant \sum_{k \in \mathbb{T}} \overline{\mathbb{P}}_{|\Omega}(\Gamma_k) = 0$$

as $\overline{\mathbb{P}}_{|\Omega} \in \mathcal{P}$. This completes the proof. □

Remark 3.2.5. *If the pricing-hedging duality holds with respect to the filtration $\mathbb{F}$, then it also holds for any filtration $\mathbb{H} \subset \mathbb{F}$ such that $\mathbb{H}$ and $\mathbb{F}$ only differ up to $\mathcal{M}_g$-polar sets. Indeed, this follows from Remark 3.2.2, observing that such a change does not affect $\mathcal{M}_g$ and can only decrease the superhedging cost as one has more trading strategies available.*

Remark 3.2.6. *We note that the set $\overline{\mathcal{M}}_g$ in* (3.2.8) *is potentially much larger than the set of all pushforward measures induced by $\omega \mapsto (\omega, \tau(\omega))$ and $\mathbb{Q} \in \mathcal{M}_g$ for all $\tau \in \mathcal{T}(\mathbb{F})$. Indeed, instead of stopping times relative to $\mathbb{F}$, it allows us to consider any* random *time which can be made into a stopping time under some calibrated martingale measure. We can rephrase this as saying that $\overline{\mathcal{M}}_g$ is equivalent to a* weak formulation *of the initial problem on the right-hand side of* (3.2.5). *To make this precise, let us define a weak stopping rule α as a collection*

$$\alpha \ = \ \big(\Omega^\alpha,\ \mathcal{F}^\alpha,\ \mathbb{Q}^\alpha, \mathbb{F}^\alpha = (\mathcal{F}_k^\alpha)_{0 \leq k \leq N},\ (S_k^\alpha)_{0 \leq k \leq N},\ (g^{\lambda,\alpha})_{\lambda \in \Lambda},\ (\Phi_k^\alpha)_{k \in \mathbb{T}},\ \tau^\alpha\big)$$

with $\big(\Omega^\alpha,\ \mathcal{F}^\alpha,\ \mathbb{Q}^\alpha, \mathbb{F}^\alpha\big)$ a filtered probability space, τ^α a $\mathbb{T}$-valued $\mathbb{F}^\alpha$-stopping time, an $\mathbb{R}^d$-valued $(\mathbb{Q}^\alpha, \mathbb{F}^\alpha)$-martingale S^α and a collection of random variables $g^{\lambda,\alpha}, \Phi_k^\alpha$, and such that there is a measurable surjective mapping $\mathtt{i}_\alpha : \Omega^\alpha \to \Omega$ with $\mathbb{Q} = \mathbb{Q}^\alpha \circ \mathtt{i}_\alpha^{-1} \in \mathcal{M}$ and $\mathtt{i}_\alpha^{-1}(\mathcal{F}_k) \subset \mathcal{F}_k^\alpha$,

$\mathtt{i}_\alpha^{-1}(\mathcal{F}) \subset \mathcal{F}^\alpha$, *and finally* $\mathcal{L}_{\mathbb{Q}^\alpha}(S^\alpha, g^\alpha, \Phi^\alpha) = \mathcal{L}_{\mathbb{Q}}(S, g, \Phi)$. *Denote by* $\mathcal{A}_g$ *the collection of all weak stopping rules* α *such that* $\mathbb{E}^{\mathbb{Q}^\alpha}\big[g^{\lambda,\alpha}\big] = 0$ *for each* $\lambda \in \Lambda$. *It follows that any* $\alpha \in \mathcal{A}_g$ *induces a probability measure* $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g$ *and* $\mathbb{E}^{\mathbb{Q}^\alpha}[\Phi^\alpha_{\tau^\alpha}] = \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]$. *Reciprocally, any* $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g$, *together with the space* $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}})$ *and* (S, g, Φ), *provides a weak stopping rule in* $\mathcal{A}_g$. *As a consequence,*

$$\sup_{\alpha\in\mathcal{A}_g} \mathbb{E}^{\mathbb{Q}^\alpha}\big[\Phi_{\tau^\alpha}\big] = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi\big].$$

In summary, and similarly to a number of other contexts, see the Introduction in [83], the weak formulation (and not the strong one) offers the right framework to compute the value of the problem. In fact, the set $\overline{\mathcal{M}}_g$ *is large enough to make the problem static, or European, again. However, while it offers a solution and a corrected version of* (3.2.5), *it does not offer a fundamental insight into why* (3.2.5) *may fail and if there is a canonical "smaller" way of enlarging the objects on the right-hand side thereof to preserve the equality. These questions are addressed in the subsequent section.*

Remark 3.2.7. *[22] and [23] studied the same superhedging problem in a Markovian setting, where the underlying process* S *takes values in a discrete lattice* $\mathcal{X}$. *By considering the weak formulation (which is equivalent to our formulation, as shown in Remark 3.2.6 above), they obtain similar duality results. However, they only consider* $\Phi_k = \phi(S_k)$ *where* $\phi : \mathbb{R}^d \to \mathbb{R}$. *Then the authors show that in the optimization problem* $\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]$ *given in* (3.2.7) *one may restrict to only Markovian martingale measures. The primal and the dual problem then turn out to be linear programming problems under linear constraints, which can be solved numerically. Their arguments have also been extended to a more general context, where* S *takes values in* $\mathbb{R}_+$. *Comparing to [22] and [23], our weak formulation is very similar to theirs. However our setting is much more general and, when considering the specific setups in Sections 5.2 and 3.3, we rely on entirely different arguments to prove the duality.*

3.2.3 The loss and recovery of the dynamic programming principle and the natural duality for American options

The classical pricing of American options, on which the duality in (3.2.5) was modelled, relies on optimal stopping techniques which subsume a certain dynamic consistency, or a dynamic programming principle, as explained below. Our second key observation in this paper is that if the pricing-hedging duality (3.2.5) for American options fails it is because the introduction of static trading of European options g at time $t = 0$ destroys the dynamic programming principle. Indeed, $\pi^E_g(\xi)$ will typically be lower than the superhedging price at time $t = 0$ of the capital needed at time $t = 1$ to superhedge from thereon. To reinstate such dynamic consistency, we need to enlarge the model and consider dynamic trading in options in g. This will generate a richer filtration than $\mathbb{F}$ and one which will carry enough stopping times to obtain the correct natural duality in the spirit of (3.2.5). In particular, if $g = 0$ (or equivalently $\Lambda = \emptyset$), then (3.2.5) should hold. We now first prove this statement and then present the necessary extension when g is non-trivial.

Let Υ be a class of $\mathcal{F}$-measurable random variables such that $-\infty \in \Upsilon$, we denote $\mathcal{E}(\xi) := \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\xi]$, and suppose that there is a family of operators $\mathcal{E}_k : \Upsilon \to \Upsilon$ for $k \in \{0, \dots, N-1\}$ such that $\mathcal{E}_k(\xi)$ is $\mathcal{F}_k$-measurable for all $\xi \in \Upsilon$ and $\mathcal{E}_k(-\infty) = -\infty$. Notice that $\mathcal{F}_0$ is assumed to be trivial so that $\mathcal{E}_0(\xi)$ is deterministic. We say that the family $(\mathcal{E}_k)$ provides a dynamic programming representation of $\mathcal{E}$ if

$$\mathcal{E}(\xi) = \mathcal{E}^0(\xi) \ \ \forall \xi \in \Upsilon, \quad \text{where } \mathcal{E}^k(\xi) := \mathcal{E}_k \circ \cdots \circ \mathcal{E}_{N-1}(\xi), \ 0 \leqslant k \leqslant N-1. \tag{3.2.9}$$

The family $(\mathcal{E}_k)$ naturally extends to $(\overline{\mathcal{E}}_k)$, $0 \leqslant k \leqslant N-1$, defined for any $\Phi \in \overline{\Upsilon} = \Upsilon^N$ by

$$\begin{aligned}\overline{\mathcal{E}}_0(\Phi) &:= \mathcal{E}_0(\Phi(\cdot,1)),\\ \overline{\mathcal{E}}_k(\Phi)(\bar{\omega}) &:= \begin{cases}\mathcal{E}_k(\Phi(\cdot,\theta))(\omega) & \text{if } \theta < k,\\ \mathcal{E}_k(\Phi(\cdot,k))(\omega) \vee \mathcal{E}_k(\Phi(\cdot,k+1))(\omega) & \text{if } \theta \geqslant k,\end{cases} \quad \text{for } 1 \leqslant k \leqslant N-1. \end{aligned} \tag{3.2.10}$$

Assume that $f \vee f' \in \Upsilon$ for $f, f' \in \Upsilon$ so that $(\overline{\mathcal{E}}_k)$ maps functionals from $\overline{\Upsilon}$ to $\overline{\Upsilon}$. We introduce the following process,

$$\overline{\mathcal{E}}^k(\Phi) := \overline{\mathcal{E}}_k \circ \cdots \circ \overline{\mathcal{E}}_{N-1}(\Phi), \quad 0 \leqslant k \leqslant N-1, \tag{3.2.11}$$

which, under suitable assumptions (see Proposition 3.2.9 below) represents the $\mathcal{M}$-Snell envelope process of an American option $\Phi \in \overline{\Upsilon}$. To illustrate how the operator $\overline{\mathcal{E}}^0$ works, we develop it for the case $\mathbb{T} = \{1,2,3\}$,

$$\overline{\mathcal{E}}^0(\Phi) = \mathcal{E}_0\left(\mathcal{E}_1 \circ \mathcal{E}_2(\Phi(\cdot,1)) \bigvee \mathcal{E}_1\big(\mathcal{E}_2(\Phi(\cdot,2)) \vee \mathcal{E}_2\left(\Phi(\cdot,3)\right)\big)\right).$$

We say that the family $(\overline{\mathcal{E}}_k)$ provides a dynamic programming representation of $\overline{\mathcal{E}}(\Phi) := \sup_{\mathbb{Q}\in\overline{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[\Phi]$ if

$$\overline{\mathcal{E}}(\Phi) = \overline{\mathcal{E}}^0(\Phi), \quad \forall \Phi \in \overline{\Upsilon}. \tag{3.2.12}$$

Typically we will consider $\mathcal{E}_k$ to be a supremum over conditional expectations with respect to $\mathcal{F}_k$ (see Examples 3.2.11 and 3.2.12 below), and in such setups we automatically obtain

$$\overline{\mathcal{E}}(\Phi) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \leqslant \overline{\mathcal{E}}^0(\Phi), \quad \Phi \in \overline{\Upsilon}. \tag{3.2.13}$$

Theorem 3.2.8. *Assume that $\Lambda = \emptyset$, $\mathcal{E}_k$ satisfies* (3.2.9)*, that* (3.2.13) *holds true, and that $f \vee f' \in \Upsilon$ for all $f, f' \in \Upsilon$. Then, for all $\Phi \in \Upsilon^N = \overline{\Upsilon}$,*

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau]. \tag{3.2.14}$$

If, further, the European pricing-hedging duality holds on $\overline{\Omega}$ for the class $\overline{\Upsilon}$, then

$$\pi^A(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau].$$

The second assertion follows instantly from the first one and Theorem 3.2.4. The first assertion is reformulated and proved in Proposition 3.2.9 below, which also allows us to identify the optimal stopping time on the right-hand side of (3.2.14).

Proposition 3.2.9. *Assume that $\Lambda = \emptyset$ and $f \vee f' \in \Upsilon$ for all $f, f' \in \Upsilon$. Then the dynamic programming representation* (3.2.12) *holds if and only if* (3.2.9) *and* (3.2.13) *hold true. Moreover, under condition* (3.2.12)*, the $\mathbb{F}$-stopping time*

$$\tau^*(\omega) := \min\left\{k \geqslant 1 : \mathcal{E}^k\left(\Phi(\cdot,k)\right)(\omega) = \overline{\mathcal{E}}^k(\Phi)(\omega,k)\right\} \tag{3.2.15}$$

provides the optimal exercise policy for $\Phi \in \overline{\Upsilon}$*, in sense that*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\mathbb{Q} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau] = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi_{\tau^*}] = \overline{\mathcal{E}}^0(\Phi). \tag{3.2.16}$$

Remark 3.2.10. *The proof of Proposition 3.2.9 will be provided in Section 3.4. The results in Theorem 3.2.8 and Proposition 3.2.9 are stated on* $(\Omega, \mathcal{F})$*, where there are only finitely many dynamically traded risky assets. However, their proofs do not rely on the fact that the number of risky assets is finite, and the same results still hold true if there are infinitely many dynamically traded risky assets.*

Next, we give two examples of operators $(\mathcal{E}_k)_{k \leqslant N-1}$ satisfying (3.2.9), (3.2.13) and therefore, by Proposition 3.2.9, also (3.2.12).

Example 3.2.11. *The model-specific setting is recovered by taking* $\mathcal{P} = \{\mathbb{P}\}$*, for a fixed probability measure* $\mathbb{P}$*. Then, taking* Υ *to be the set of all* $\mathcal{F}$*-measurable random variables and*

$$\mathcal{E}_k(\xi) = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{M}_g} \mathbb{E}^{\mathbb{Q}}[\xi | \mathcal{F}_k], \tag{3.2.17}$$

where the essential supremum is taken with respect to $\mathbb{P}$*, leads to a family of operators satisfying* (3.2.9)*,* (3.2.13)*, and therefore also* (3.2.12)*. See the literature on dynamic coherent risk measures for further discussion (e.g., [84] for an overview). In particular, Theorem 3.2.8 recovers the classical superhedging theorem for American options (see, e.g., [85]).*

Example 3.2.12. *Let* (Ω, d) *be a Polish space,* $\mathcal{F}$ *the universally completed Borel* σ*-field,* $\mathcal{P}$ *a given set of probability measures on* $(\Omega, \mathcal{F})$*, and* $\mathcal{M}$ *be defined by* (3.2.3)*. We are given a filtration* $\mathbb{G} := (\mathcal{G}_k)_{k \leqslant N}$ *such that* $\mathcal{G}_0 = \{\emptyset, \Omega\}$ *and each* σ*-field* $\mathcal{G}_k$ *is countably generated. Let* $\mathcal{F}_k$ *be the universal completion of* $\mathcal{G}_k$*. Notice that* $\mathbb{E}^{\mathbb{P}}[\xi | \mathcal{G}_k] = \mathbb{E}^{\mathbb{P}}[\xi | \mathcal{F}_k]$*,* $\mathbb{P}$*-a.s. for all* $\mathcal{F}$*-measurable* ξ *and* $\mathbb{P} \in \mathcal{P}$*, and the fact that* $\mathcal{G}_k$ *is countably generated ensures the existence of a regular conditional probability of* $\mathbb{P}$ *with respect to* $\mathcal{G}_k$*.*

Assume there exists a family $(\mathcal{M}_k(\omega))_{k \leq N-1, \omega \in \Omega}$ *of sets of measures satisfying:*

- $\mathbb{Q}([\omega]_{\mathcal{G}_k}) = 1$ *for all* $\mathbb{Q} \in \mathcal{M}_k(\omega)$*, where* $[\omega]_{\mathcal{G}_k}$ *is the atom of* $\mathcal{G}_k$ *containing* ω*, i.e.,*

$$[\omega]_{\mathcal{G}_k} = \bigcap_{F \in \mathcal{G}_k : \omega \in F} F. \tag{3.2.18}$$

  Note that $[\omega]_{\mathcal{G}_k} \in \mathcal{G}_k$ *because the latter is countably generated.*

- *For every* $\mathbb{Q} \in \mathcal{M}$ *and every family of regular conditional probabilities* $(\mathbb{Q}_\omega)_{\omega \in \Omega}$ *of* $\mathbb{Q}$ *with respect to* $\mathcal{G}_k$*, one has* $\mathbb{Q}_\omega \in \mathcal{M}_k(\omega)$*, for* $\mathbb{Q}$*-a.e.* ω*.*

Define the family $(\mathcal{E}_k)_{k \leqslant N-1}$ *by*

$$\mathcal{E}_k(\xi)(\omega) = \sup_{\mathbb{Q} \in \mathcal{M}_k(\omega)} \mathbb{E}^{\mathbb{Q}}[\xi].$$

If we furthermore assume that $\mathcal{E}_k(\xi) \in \Upsilon$ *for any* $\xi \in \Upsilon$*, then the family* $(\mathcal{E}_k)_{k \leqslant N-1}$ *satisfies* (3.2.13) *(see Proposition 3.4.1). Moreover, under suitable assumptions on* $(\Omega, \mathbb{F}, \mathcal{P})$*,* $\mathcal{M}_k(\omega)$ *and* Υ*, we shall also prove that* (3.2.9) *holds for this family. This holds in particular in the setup of [1] as shown therein; see (4.12) in [1].*

Let us consider the case with statically traded options: $\Lambda \neq \emptyset$. We saw in Example 3.1.1 that this can break down dynamic consistency as the universe of traded assets differs at time $t = 0$

and times $t \geqslant 1$. To remedy this, one has to embed the market into a fictitious larger one where both S and all the options g^λ, $\lambda \in \Lambda$, are traded dynamically.

Definition 3.2.13. *Let $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}})$ be a filtered space satisfying the following properties:*

1. *There exists a surjective mapping $\mathtt{i} : \widehat{\Omega} \to \Omega$.*
2. *For each k, $\mathtt{i}^{-1}(\mathcal{F}_k) \subset \widehat{\mathcal{F}}_k$.*
3. *There exists a family of $\widehat{\mathbb{F}}$-adapted processes $Y = (Y^\lambda)_{\lambda\in\Lambda}$ such that $Y_0^\lambda = 0$ and $Y_N^\lambda(\widehat{\omega}) = g^\lambda(\mathtt{i}(\widehat{\omega}))$.*
4. *Let $S(\widehat{\omega}) := S(\mathtt{i}(\widehat{\omega}))$, $\widehat{S} := (S, Y)$, and define*

$$\widehat{\mathcal{P}} := \{\widehat{\mathbb{P}} \in \mathfrak{B}(\widehat{\Omega}) : \widehat{\mathbb{P}} \circ \mathtt{i}^{-1} \in \mathcal{P}\}, \quad \widehat{\mathcal{M}} := \{\widehat{\mathbb{Q}} \ll \widehat{\mathcal{P}} : \widehat{S} \text{ is an } (\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})\text{-martingale}\}. \quad (3.2.19)$$

There exists a mapping $\mathtt{J} : \mathcal{M}_g \to \widehat{\mathcal{M}}$ such that for $\mathbb{Q} \in \mathcal{M}_g$, $\mathtt{J}(\mathbb{Q}) \circ \mathtt{i}^{-1} = \mathbb{Q}$ and

$$\mathcal{L}_{\mathtt{J}(\mathbb{Q})}(\widehat{S}) = \mathcal{L}_{\mathbb{Q}}(S, Y^{\mathbb{Q}}),$$

where $Y^{\mathbb{Q}} = (Y^{\lambda,\mathbb{Q}})_{\lambda\in\Lambda}$ and $Y^{\lambda,\mathbb{Q}} := (\mathbb{E}^{\mathbb{Q}}[g^\lambda|\mathcal{F}_k])_{k\leqslant N}$.

The collection $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}}, Y, \mathtt{i}, \mathtt{J})$ satisfying Properties 1–4 above is called a dynamic extension of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$. In short, we shall say that $\widehat{\Omega}$ is a dynamic extension of Ω.

Remark 3.2.14. *A measure $\mathbb{Q} \in \mathcal{M}_g$ is an admissible pricing measure under which the time-k prices for European options g^λ are given by $Y_k^{\lambda,\mathbb{Q}}$. Property 4 in the above definition says that any such $\mathbb{Q} \in \mathcal{M}_g$ can be lifted to a measure $\mathtt{J}(\mathbb{Q}) \in \widehat{\mathcal{M}}$ which preserves the joint distribution of the stock and option prices. In general, we do not expect the reverse to be true and we may have $\mathtt{J}(\mathcal{M}_g) \subsetneq \widehat{\mathcal{M}}$. More precisely, $\widehat{\mathcal{M}}$ may offer scope for a richer description and dynamics so that the mapping $\widehat{\mathcal{M}} \ni \widehat{\mathbb{Q}} \mapsto \widehat{\mathbb{Q}} \circ \mathtt{i}^{-1} \in \mathcal{M}_g$ is surjective but typically not injective.*

Example 3.2.15. *In practice, the map $\mathtt{J}$ in a dynamic extension is often built from a family of mappings from Ω to $\widehat{\Omega}$. Assume that, for each $\mathbb{Q} \in \mathcal{M}_g$, one has a mapping $\mathtt{j}_\mathbb{Q} : \Omega \to \widehat{\Omega}$ such that $\mathtt{i} \circ \mathtt{j}_\mathbb{Q} = id$ and $\mathcal{L}_{\mathbb{Q}\circ\mathtt{j}_\mathbb{Q}^{-1}}(Y) = \mathcal{L}_\mathbb{Q}(Y^\mathbb{Q})$. Let $\mathtt{J}(\mathbb{Q}) := \mathbb{Q} \circ \mathtt{j}_\mathbb{Q}^{-1}$. Then $\mathtt{J} : \mathcal{M}_g \to \widehat{\mathcal{M}}$ satisfies Property 4 of Definition 3.2.13.*

Let us illustrate this with an example of a dynamic extension of Ω in the case of finitely many statically traded options, i.e., $\Lambda = \{1, \dots, e\}$ for some $e \in \mathbb{N}$. Consider the space $\widehat{\Omega} = \Omega \times \mathbb{R}^{(N-1)\times e}$. An element $\widehat{\omega}$ of $\widehat{\Omega}$ can be written as $\widehat{\omega} = (\omega, y)$ where $y = (y^1, \dots, y^e) \in \mathbb{R}^{(N-1)\times e}$ with $y^i = (y^i_1, \dots, y^i_{N-1})$. Define a mapping $\mathtt{i} : \widehat{\Omega} \to \Omega$ by $\mathtt{i}(\widehat{\omega}) = \omega$ which is clearly surjective. We also introduce the process Y as $Y_k(\widehat{\omega}) = y_k = (y^1_k, \dots, y^e_k)$ for $k \in \{1, \dots, N-1\}$, $Y_0(\widehat{\omega}) = 0$ and $Y_N(\widehat{\omega}) = g(\widehat{\omega}) = g(\omega)$. Let $\mathcal{Y}_k := \sigma(Y_n : n \leqslant k)$ and $\widehat{\mathcal{F}}_k$ be the universal completion of $\mathcal{F}_k \otimes \mathcal{Y}_k$, so that one obtains the filtration $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_k)_{k=0,1,\dots,N}$. In this context, we define $\mathtt{j}_\mathbb{Q}(\omega) := (\omega, (y_k^{\mathbb{Q},i})_{i\leqslant e, k\leqslant N-1})$, where $y_k^{\mathbb{Q},i}$ is a version of $\mathbb{E}^\mathbb{Q}[g^i|\mathcal{F}_k](\omega)$ for each $\mathbb{Q} \in \mathcal{M}_g$. Then it is clear that $\mathtt{i} \circ \mathtt{j}_\mathbb{Q} = id$ and $\mathcal{L}_{\mathbb{Q}\circ\mathtt{j}_\mathbb{Q}^{-1}}(Y) = \mathcal{L}_\mathbb{Q}(Y^\mathbb{Q})$. This implies that $\mathtt{J} : \mathbb{Q} \mapsto \mathbb{Q} \circ \mathtt{j}_\mathbb{Q}^{-1}$ satisfies Property 4 of Definition 3.2.13. In line with Remark 3.2.14, the inverse of $\mathtt{J}$, given by $\widehat{\mathcal{M}} \ni \widehat{\mathbb{Q}} \mapsto \widehat{\mathbb{Q}} \circ \mathtt{i}^{-1} = \widehat{\mathbb{Q}}_{|\Omega} \in \mathcal{M}_g$ is surjective.

We consider a class of functions $\widehat{\Upsilon}$ on $\widehat{\Omega}$ and assume that $\Upsilon \subset \widehat{\Upsilon}$ in the sense that for $\xi \in \Upsilon$, $\xi(\widehat{\omega}) := \xi(\mathtt{i}(\widehat{\omega}))$ belongs to $\widehat{\Upsilon}$. The relationship between $\widehat{\mathcal{M}}$ and $\mathcal{M}_g$ observed in Remark 3.2.14

then yields

$$\sup_{\mathbb{Q}\in\widehat{\mathcal{M}}}\mathbb{E}^{\mathbb{Q}}[\xi]\geqslant\sup_{\mathbb{Q}\in\mathcal{M}_g}\mathbb{E}^{\mathbb{Q}}[\xi]\quad\text{for any }\xi\in\Upsilon.$$

We can apply the enlargement construction introduced in Section 3.2.2 to the space $\widehat{\Omega}$ which leads to the set of martingale measures $\overline{\widehat{\mathcal{M}}}$ on $\widehat{\Omega}$, and the above inequality extends to

$$\sup_{\mathbb{Q}\in\overline{\widehat{\mathcal{M}}}}\mathbb{E}^{\mathbb{Q}}[\Phi]\geqslant\sup_{\mathbb{Q}\in\overline{\mathcal{M}}_g}\mathbb{E}^{\mathbb{Q}}[\Phi]\quad\text{for any }\Phi\in\overline{\Upsilon}. \tag{3.2.20}$$

We now consider a dynamic extension $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathcal{P}},Y,\mathtt{i},\mathtt{J})$ of $(\Omega,\mathbb{F},\mathcal{F},\mathcal{P},S,g)$ as a fictitious market in which we can trade dynamically in $\widehat{S}=(S,Y)$ using the class of trading strategies $\widehat{H}$, which are the $\widehat{\mathbb{F}}$-predictable $\mathbb{R}^{\widehat{\Lambda}}$-valued processes which have only finitely many non-zero coordinates where $\widehat{\Lambda}=\{(i,s):i\in\{1,\dots,d\}\}\cup\{(\lambda,y):\lambda\in\Lambda\}$, i.e.,

$$\begin{aligned}\widehat{\mathcal{H}}=\big\{\widehat{H}=&(\widehat{H}^{\widehat{\lambda}}_k:\widehat{\lambda}\in\widehat{\Lambda})_{k\leqslant N}:\ \widehat{\mathbb{F}}\text{-predictable }\mathbb{R}^{\widehat{\Lambda}}\text{-valued process s.t.}\\&\exists\text{ finite subset }\widehat{\Lambda}_0\subset\widehat{\Lambda}\text{ s.t. }\widehat{H}^{\widehat{\lambda}}_k=0,\ \forall k,\ \forall\widehat{\lambda}\notin\widehat{\Lambda}_0\big\}.\end{aligned}$$

A self-financing strategy corresponds to a choice of $\widehat{H}\in\widehat{\mathcal{H}}$ and yields a final payoff of

$$(\widehat{H}\circ\widehat{S})_N=\sum_{j=1}^{d}\sum_{k=1}^{N}\widehat{H}^{(j,s)}_k\Delta S^j_k+\sum_{\lambda\in\Lambda}\sum_{k=1}^{N}\widehat{H}^{(\lambda,y)}_k\Delta Y^\lambda_k. \tag{3.2.21}$$

Note that our choice of trading strategies ensures that the sums are finite. The supehedging costs of a European option $\widehat{\xi}$ and an American option $\widehat{\Phi}=(\widehat{\Phi}_k)_{k\leqslant N}$ on $\widehat{\Omega}$ are given by

$$\begin{aligned}\widehat{\pi}^E(\widehat{\xi})&=\inf\big\{x:\exists\widehat{H}\in\widehat{\mathcal{H}}\text{ s.t. }x+(\widehat{H}\circ\widehat{S})_N\geqslant\widehat{\xi},\ \widehat{\mathcal{P}}\text{-q.s.}\big\},\\\widehat{\pi}^A(\widehat{\Phi})&=\inf\big\{x:\exists({}^1\widehat{H},\dots,{}^N\widehat{H})\in\widehat{\mathcal{H}}^N\text{ s.t. }{}^j\widehat{H}_i={}^k\widehat{H}_i\ \forall\,1\leqslant i\leqslant j\leqslant k\leqslant N\\&\qquad\qquad\text{and s.t. }x+({}^k\widehat{H}\circ\widehat{S})_N\geqslant\widehat{\Phi}_k,\ \forall k=1,\dots,N,\ \widehat{\mathcal{P}}\text{-q.s.}\big\}.\end{aligned} \tag{3.2.22}$$

Remark 3.2.16. *Clearly $\widehat{\mathbb{F}}$ is much richer than $\mathbb{F}$ as it captures not only the evolution of prices of S but also of all the vanilla options. The inequality $\widehat{\pi}^A(\Phi)\leqslant\pi^A_g(\Phi)$ holds trivially as a buy-and-hold strategy is a special case of a dynamic trading strategy and $\mathcal{P}=\widehat{\mathcal{P}}\circ\mathtt{i}^{-1}$.*

The following result shows that if the pricing-hedging duality holds then the superhedging prices in the fictitious dynamic extension market are the same as in the original market. This will apply to the setups we consider in Sections 5.2 and 3.3 below.

Proposition 3.2.17. *Let $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathcal{P}},Y,\mathtt{i},\mathtt{J})$ be a dynamic extension of $(\Omega,\mathbb{F},\mathcal{F},\mathcal{P},S,g)$ with its superhedging prices given by* (3.2.22).
(a) Assume that the European pricing-hedging duality holds for the class Υ on Ω. Then

$$\pi^E_g(\xi)=\widehat{\pi}^E(\xi),\quad\xi\in\Upsilon.$$

(b) Assume that the European pricing-hedging duality holds for the class Υ^N on $\overline{\Omega}$. Then

$$\pi^A_g(\Phi)=\widehat{\pi}^A(\Phi),\quad\Phi\in\Upsilon^N.$$

Proof. Note that $\pi_g^A \geqslant \widehat{\pi}^A$ holds by Remark 3.2.16. Using (3.2.7) twice we obtain

$$\overline{\pi}_g^E(\Phi) = \pi_g^A(\Phi) \geqslant \widehat{\pi}^A(\Phi) = \overline{\widehat{\pi}}^E(\Phi) \geqslant \sup_{\widehat{\mathbb{Q}} \in \overline{\widehat{\mathcal{M}}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\Phi] \geqslant \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi], \tag{3.2.23}$$

where the penultimate inequality always holds by Remark 3.2.2 and the last inequality follows by (3.2.20). The assumed pricing-hedging duality on $\overline{\Omega}$ implies that we have equalities throughout. The proof of (a) is analogous but simpler. □

Remark 3.2.18. *The above result may at first seem surprising. The dynamic extension introduces many new dynamically traded assets, yet the superhedging prices remain the same. The intuition behind this is that under pricing-hedging duality, the cheapest superhedge is a perfect hedge (or very nearly so) under some (worst case) model. Our dynamic extensions do not introduce any constraints on the prices of options g and hence do not restrict the set of martingale measures. The worst case model will remain an admissible model and for this model the additional traded assets make no difference. They could however make a difference in many other (specific) models. If we considered a restricted version of dynamic trading in which we make further assumptions about the price dynamics of vanilla options, then this could imply that $\mathtt{J} \circ \mathtt{i}^{-1}$ is not surjective and the superhedging prices might strictly decrease. Such a setup is studied in [86] where the authors consider restrictions on the levels of implied volatility through time.*

Let $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}}, Y, \mathtt{i}, \mathtt{J})$ be a dynamic extension of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$ and $\widehat{\mathcal{E}}_k$ be a family of operators on the space $\widehat{\Upsilon}$ of functionals on $\widehat{\Omega}$. One can define the corresponding extended operators $\widehat{\mathcal{E}}^k$ as well as $\overline{\widehat{\mathcal{E}}}^k$ as in (3.2.9) and (3.2.10). We can then apply Theorem 3.2.8 and Proposition 3.2.17 to obtain the following result,

Corollary 3.2.19. *Let $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}}, Y, \mathtt{i}, \mathtt{J})$ be a dynamic extension of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$ with operators $\widehat{\mathcal{E}}_k : \widehat{\Upsilon} \to \widehat{\Upsilon}$ and the corresponding extended operators $\widehat{\mathcal{E}}^k$ as well as $\overline{\widehat{\mathcal{E}}}^k$ satisfying* (3.2.9) *and* (3.2.13)*. Assume that the European pricing-hedging duality holds for the class Υ^N on $\overline{\Omega}$, and $f \vee f' \in \Upsilon$ for all $f, f' \in \Upsilon$. Then for all $\Phi \in \Upsilon^N$,*

$$\pi_g^A(\Phi) = \widehat{\pi}^A(\Phi) = \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \sup_{\hat{\tau} \in \mathcal{T}(\widehat{\mathbb{F}})} \mathbb{E}^{\widehat{\mathbb{Q}}}[\Phi_{\hat{\tau}}] = \sup_{\widehat{\mathbb{Q}} \in \overline{\widehat{\mathcal{M}}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\Phi] = \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\Phi_{\hat{\tau}^*}], \tag{3.2.24}$$

where

$$\hat{\tau}^* := \min\left\{ k \geqslant 1 : \widehat{\mathcal{E}}^k\left(\Phi(\cdot, k)\right)(\omega) = \overline{\widehat{\mathcal{E}}}^k(\Phi)(\omega, k) \right\}.$$

Remark 3.2.20. *In Section 5.2, in the context of [1], we will adopt the dynamic extension introduced in Example 3.2.15, and show that it admits a family of operators $\widehat{\mathcal{E}}_k$ to which we can apply Corollary 3.2.19.*

Remark 3.2.21. *We believe that Corollary 3.2.19 describes a canonical, and in some sense* minimal*, solution to the pricing-hedging duality of American option, when compared to addition of all consistent random times, as discussed in Remark 3.2.6. A dynamic extension $\widehat{\Omega}$ is crucial to establish the DPP, which in turn allows one to define the optimal stopping time $\hat{\tau}^*$.*

Remark 3.2.22. *Let us consider the two period ($N = 2$) example of [82]; see Figure 2. For simplicity, we introduce only one statically traded option g with payoff $\mathbb{1}_{\{S_2=4\}}$ at time $t = 2$ and price $2/5$ at time $t = 0$. This already destroys the pricing-hedging duality for the American option Φ. In [82], the duality is recovered by considering a (calibrated) mixture of martingale measures.*

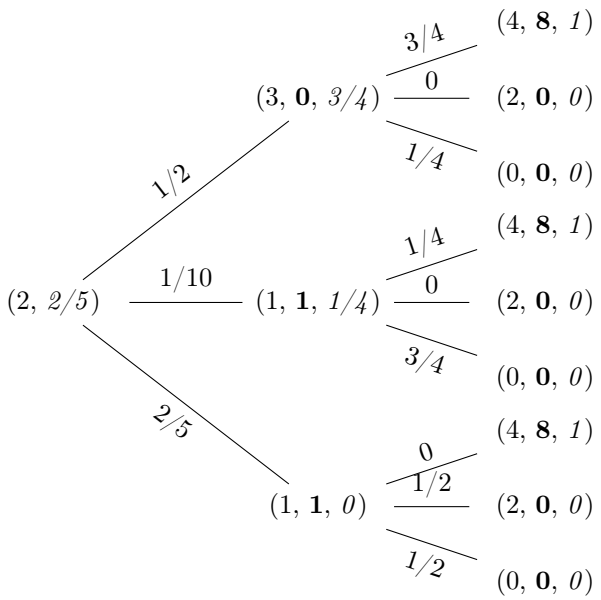

Figure 3.2: The model on $\widehat{\Omega}$ which corresponds to the mixture model in [82] attaining the superhedging price. Prices of the stock are written in regular font, payoffs of the American option in bold and prices of European option in italic.

It is insightful to observe that their mixture model is nothing else but a martingale measure for an augmented setup with dynamic trading in g which, following Corollary 3.2.19, restores the dynamic programming principle and the pricing-hedging duality for American options. To show this, let Y denote the price process of the option g, so that $Y_0 = 2/5$ and $Y_2 = g$. Figure 2 illustrates a martingale measure $\mathbb{Q}$ along with the intermediate prices Y_1 such that the processes S and Y are martingales. With $\tau = \mathbb{1}_{\{S_1=1,Y_1=0\}} + 2\mathbb{1}_{\{S_1=1,Y_1=1/4\}\cup\{S_1=3\}}$ we find $\mathbb{E}^{\mathbb{Q}}[\Phi_\tau] = 18/5$, which is the superhedging price, and the duality is recovered.

3.2.4 Pseudo-stopping times

In this subsection we study the connection of our problem to pseudo-stopping times in the filtration $\mathbb{F}$ which form a bigger class than $\mathbb{F}$-stopping times. We refer the reader to [87], [88] and [89] for an introduction to pseudo-stopping times.

It follows from Theorem 3.2.4 that in general we expect to see

$$\pi_g^A(\Phi) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \geqslant \sup_{\mathbb{Q}\in\mathcal{M}_g} \sup_{\tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau],$$

where the inequality may be strict. We showed above that this is linked with the necessity to use random times beyond $\tau \in \mathcal{T}(\mathbb{F})$. To conclude our general results, we explore this property from another angle and identify the subset(s) of $\overline{\mathcal{M}}_g$ which lead to equality in the place of the

inequality above. We introduce

$$\underline{\mathcal{M}}_g := \big\{\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega}) : \ \overline{\mathbb{Q}} \lll \overline{\mathcal{P}},\ \mathbb{E}^{\overline{\mathbb{Q}}}[g^\lambda] = 0,\ \lambda \in \Lambda,\ S \text{ is an } (\mathbb{F}, \overline{\mathbb{Q}})\text{-martingale},$$
$$\mathbb{E}^{\overline{\mathbb{Q}}}[M_T] = \mathbb{E}^{\overline{\mathbb{Q}}}[M_0],\ \text{for all bounded } (\mathbb{F}, \overline{\mathbb{Q}})\text{-martingales } M\big\}, \quad (3.2.25)$$

as the set of measures which make S an $\mathbb{F}$-martingale and T an $\mathbb{F}$-pseudo-stopping time. These are natural because the martingale part of the Snell envelope can be stopped at the pseudo-stopping time with null expectation.

Proposition 3.2.23. *Assume that $\mathcal{M}_g \neq \emptyset$. Then*

$$\sup_{\overline{\mathbb{Q}} \in \underline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}\big[\Phi_\tau\big]. \quad (3.2.26)$$

Proof. Let $\overline{\mathbb{Q}} \in \underline{\mathcal{M}}_g$ such that $\mathbb{E}^{\overline{\mathbb{Q}}}[|g^\lambda|] < \infty$ and $\mathbb{E}^{\overline{\mathbb{Q}}}[|\Phi_k|] < \infty$ for all $\lambda \in \Lambda$ and $k = 1, \dots, N$. We next consider the optimal stopping problem $\sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi_\tau\big]$. Define its Snell envelope $(Z_k)_{0 \le k \le N}$ by

$$Z_k := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}(\mathbb{F}), \tau \ge k} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi_\tau \big| \mathcal{F}_k\big],$$

which is an $(\mathbb{F}, \overline{\mathbb{Q}})$-supermartingale. Its Doob–Meyer decomposition is given by

$$Z_k = Z_0 + M_k - A_k, \quad \text{where } A = (A_k)_{0 \le k \le N} \text{ is an } \mathbb{F}\text{-predictable increasing process},$$

and $A_0 = M_0 = 0$. It follows that

$$\mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] \ \le\ \mathbb{E}^{\overline{\mathbb{Q}}}[Z_T] \ \le\ Z_0 + \mathbb{E}^{\overline{\mathbb{Q}}}[M_T] \ =\ Z_0. \quad (3.2.27)$$

We deduce that $\sup_{\overline{\mathbb{Q}} \in \underline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \le \sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}\big[\Phi_\tau\big]$. Then (3.2.26) holds as every stopping time $\tau \in \mathcal{T}(\mathbb{F})$ is a pseudo-stopping time and hence the inverse inequality is trivial. □

Remark 3.2.24. *The above allows us to see that it is not enough to use randomized stopping times to recover the equality in* (3.2.5)*. Such a time corresponds to an $\mathbb{F}$-adapted increasing process V with $V_0 = 0$ and $V_N = 1$. It may be seen as a distribution over all possible stopping times, in our setup a distribution η on $\mathbb{T}$ such that $\eta(\{k\}) := \Delta V_k = V_k - V_{k-1}$ for each $k \in \mathbb{T}$. For any pseudo-stopping time τ, the dual optional projection of the process $\mathbb{1}_{[\![\tau, N]\!]}$ is a randomized stopping time. Conversely, for a given V, if we take a uniformly distributed random variable Θ independent of V, possibly enlarging the probability space, then $\tau := \inf\{t : V_t \geqslant \Theta\}$ is $\mathbb{F}$-pseudo-stopping time which generates V. Let $\mathcal{R}$ be the set of such randomized stopping times. Then, from Proposition 3.2.23 and the definition of the dual optional projection,*

$$\sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}\big[\Phi_\tau\big] = \sup_{\mathbb{Q} \in \mathcal{M}_g} \sup_{V \in \mathcal{R}} \mathbb{E}^{\mathbb{Q}}\left[\sum_k \Phi_k \Delta V_k\right].$$

Remark 3.2.25. *[88] showed that under a progressive enlargement with pseudo-stopping time τ, all martingales from the smaller filtration stopped at τ remain martingales in the larger filtration. One can relate this to a more restrictive situation, when all martingales from the smaller filtration remain martingales in the bigger filtration, which is called the immersion property in the context of filtration enlargement. Clearly each random time satisfying the immersion property is a pseudo-stopping time. Thus, keeping the equality* (3.2.26) *true, the pseudo-stopping time property in*

the definition of $\underline{\mathcal{M}}_g$ above can be replaced by a stronger condition characterizing the immersion property,

$$\overline{\mathbb{Q}}[T > k|\mathcal{F}_n] = \overline{\mathbb{Q}}[T > k|\mathcal{F}_k], \quad \textit{for all } 0 \le k \le n \le N. \tag{3.2.28}$$

See Section 3.1.2 of [90] for the discrete time context of progressive enlargement of filtration and [91] for connections between pseudo-stopping times, the immersion property and projections.

3.2.5 A detailed study of the non-dominated setup of [1]

In this section we work in the non-dominated setup introduced in [1] which is a special case of Example 3.2.12. We let $\Omega_0 = \{\omega_0\}$ be a singleton and Ω_1 be a Polish space. For each $k \in \{1, \dots, N\}$, we define $\Omega_k := \{\omega_0\} \times \Omega_1^k$ as the k-fold Cartesian product. For each k, we denote $\mathcal{G}_k := \mathcal{B}(\Omega_k)$ and by $\mathcal{F}_k$ its universal completion. In particular, $\mathcal{G}_0$ and $\mathcal{F}_0$ are trivial, and $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{G}_k] = \mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_k]$ for all $\xi \in \mathcal{F}_N$ and every probability measure $\mathbb{P}$ on $(\Omega_N, \mathcal{F}_N)$. We shall often see $\mathcal{G}_k$ and $\mathcal{F}_k$ as sub-σ-fields of $\mathcal{F}_N$, and hence obtain two filtrations $\mathbb{G} = (\mathcal{G}_k)_{0\le k\le N}$ and $\mathbb{F} = (\mathcal{F}_k)_{0\le k\le N}$ on Ω. Denote

$$\Omega := \Omega_N, \quad \mathcal{G} := \mathcal{G}_N \ \text{ and } \ \mathcal{F} := \mathcal{F}_N.$$

Recall that a subset of a Polish space Ω is analytic if it is the image of a Borel subset of another Polish space under a Borel measurable mapping. We take Υ to be the class of upper semianalytic functions $f : \Omega \to \overline{\mathbb{R}} := [-\infty, \infty]$, i.e., such that $\{\omega \in \Omega \ : f(\omega) > c\}$ is analytic for all $c \in \mathbb{R}$.

The price process S is a $\mathbb{G}$-adapted $\mathbb{R}^d$-valued process and the collection of options $g = (g^1, \dots, g^e)$ is a $\mathcal{G}$-measurable $\mathbb{R}^e$-valued vector for $e \in \mathbb{N}$ (thus $\Lambda = \{1, \dots, e\}$).

Let $k \in \{0, \dots, N-1\}$ and $\omega \in \Omega_k$. We are given a nonempty convex set $\mathcal{P}_k(\omega) \subseteq \mathfrak{B}(\Omega_1)$ of probability measures, which represents the set of all possible models for the $(k+1)$-th period, given state ω at times $0, 1, \dots, k$. We assume that for each k,

$$\text{graph}(\mathcal{P}_k) := \{(\omega, \mathbb{P}) : \omega \in \Omega_k, \mathbb{P} \in \mathcal{P}_k(\omega)\} \ \subseteq \ \Omega_k \times \mathfrak{B}(\Omega_1) \ \text{ is analytic.} \tag{3.2.29}$$

Given a universally measurable kernel $\mathbb{P}_k : \Omega_k \to \mathfrak{B}(\Omega_1)$ for each $k \in \{0, 1, \dots, N-1\}$, we define a probability measure $\mathbb{P} = \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{N-1}$ on Ω by

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2, \dots, \omega_N) \mathbb{P}_{N-1}(\omega_1, \dots, \omega_{N-1}; d\omega_N) \cdots \mathbb{P}_0(d\omega_1).$$

We can then introduce the set $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$ of possible models for the multi-period market up to time N by

$$\mathcal{P} := \big\{\mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{N-1} \ : \mathbb{P}_k(\cdot) \in \mathcal{P}_k(\cdot), k = 0, 1, \dots, N-1\big\}. \tag{3.2.30}$$

Notice that the condition (5.2.2) ensures that $\mathcal{P}_k$ always has a universally measurable selector: $\mathbb{P}_k : \Omega_k \to \mathfrak{B}(\Omega_1)$ such that $\mathbb{P}_k(\omega) \in \mathcal{P}_k(\omega)$ for all $\omega \in \Omega_k$. Then the set $\mathcal{P}$ defined in (5.2.1) is nonempty. We also denote

$$\mathcal{M}_{k,k+1}(\omega) := \{\mathbb{Q} \in \mathfrak{P}(\Omega_1) : \mathbb{Q} \lll \mathcal{P}_k(\omega) \text{ and } \ \mathbb{E}^{\delta_\omega \otimes_k \mathbb{Q}}[\Delta S_{k+1}] = 0\}, \tag{3.2.31}$$

where $\delta_\omega \otimes_k \mathbb{Q} := \delta_{(\omega_1,\dots,\omega_k)} \otimes \mathbb{Q}$ is a Borel probability measure on $\Omega_{k+1} := \Omega_k \times \Omega_1$, and

$$\mathcal{M}_k(\omega) := \big\{\delta_\omega \otimes_k \mathbb{Q}_k \otimes \cdots \otimes \mathbb{Q}_{N-1} \ : \mathbb{Q}_i(\cdot) \in \mathcal{M}_{i,i+1}(\cdot), i = k, \dots, N-1\big\}. \tag{3.2.32}$$

The following notion of no-arbitrage NA($\mathcal{P}$) has been introduced in [1]: NA($\mathcal{P}$) holds if for all $(H,h) \in \mathcal{H} \times \mathbb{R}^e$,

$$(H \circ S)_N + hg \geqslant 0 \quad \mathcal{P}\text{-q.s.} \implies (H \circ S)_N + hg = 0 \quad \mathcal{P}\text{-q.s.}$$

Analogously, we will say that NA($\overline{\mathcal{P}}$) holds if for all $(\overline{H},h) \in \overline{\mathcal{H}} \times \mathbb{R}^e$,

$$(\overline{H} \circ S)_N + hg \geqslant 0 \quad \overline{\mathcal{P}}\text{-q.s.} \implies (\overline{H} \circ S)_N + hg = 0 \quad \overline{\mathcal{P}}\text{-q.s.} \tag{3.2.33}$$

Recall also that $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ have been defined in (3.2.3) and (3.2.6). As established in [1], the condition NA($\mathcal{P}$) is equivalent to the statement that $\mathcal{P}$ and $\mathcal{M}_g$ have the same polar sets. The following lemma extends this result to $\overline{\Omega}$.

Lemma 3.2.26. NA($\mathcal{P}$) $\iff$ NA($\overline{\mathcal{P}}$) $\iff$ *[$\overline{\mathcal{P}}$ and $\overline{\mathcal{M}}_g$ have the same polar sets.]*

Proof. The two conditions NA($\mathcal{P}$) and NA($\overline{\mathcal{P}}$) are equivalent by the same arguments as in proving (3.2.7). It is enough to show that $\overline{\mathcal{P}}$ and $\overline{\mathcal{M}}_g$ have the same polar sets if and only if $\mathcal{P}$ and $\mathcal{M}_g$ have the same polar sets. This boils down to proving that a set $\Gamma \in \overline{\Omega}$ is an $\overline{\mathcal{M}}_g$ polar set if and only if the k-section $\Gamma_k = \{\omega : (\omega,k) \in \Gamma\}$ is an $\mathcal{M}_g$ polar set for each $k \in \mathbb{T}$, which can be shown similarly to the analogous statement involving $\mathcal{P}$ and $\overline{\mathcal{P}}$ established in the proof of Theorem 3.2.4. □

3.2.6 Duality on the enlarged space $\overline{\Omega}$

Our first main result is the following duality under the no-arbitrage condition (3.2.33).

Theorem 3.2.27. *Let* NA($\overline{\mathcal{P}}$) *hold. Then the set* $\overline{\mathcal{M}}_g$ *is nonempty, and, for any upper semianalytic* $\Phi : \overline{\Omega} \to \mathbb{R}$*, one has*

$$\overline{\pi}_g^E(\Phi) \quad = \quad \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi], \tag{3.2.34}$$

and in particular the pricing-hedging duality (3.2.8) *holds. Moreover, there exists* $(\overline{H},h) \in \overline{\mathcal{H}} \times \mathbb{R}^e$ *such that*

$$\overline{\pi}_g^E(\Phi) + (\overline{H} \circ S)_N + hg \geq \Phi, \quad \overline{\mathcal{P}}\text{-q.s.}$$

The proof is postponed to Section 3.5 and uses the following lemma. Let us work with the operators $\mathcal{E}_k$ introduced in Example 3.2.12 with $\mathcal{M}_k(\omega)$ defined as in (3.2.32). Observe that

$$\mathcal{E}_k \circ \cdots \circ \mathcal{E}_{N-1}(\xi)(\omega) = \mathcal{E}_{k,k+1} \circ \cdots \circ \mathcal{E}_{N-1,N}(\xi)(\omega), \quad \xi \in \Upsilon,$$

where $\mathcal{E}_{k,k+1}(\xi)(\omega) = \sup_{\mathcal{M}_{k,k+1}(\omega)} \mathbb{E}^{\mathbb{Q}}[\xi]$. By Proposition 3.2.9, (4.12) in [1] and using that the maximum of upper semianalytic functions is still upper semianalytic, we conclude the following.

Lemma 3.2.28. *Consider the case* $e = 0$*, i.e.,* $\Lambda = \emptyset$*. Let* $\Phi \in \overline{\Upsilon}$*. Then* $\overline{\mathcal{E}}_k(\Phi)$ *in* (3.2.10) *is also upper semianalytic and*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \ = \ \overline{\mathcal{E}}^0(\Phi) := \overline{\mathcal{E}}_0 \circ \cdots \circ \overline{\mathcal{E}}_{N-1}(\Phi).$$

3.2.7 Dynamic programming principle on $\widehat{\Omega}$

We consider a dynamic extension $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathcal{P}}, Y, \mathtt{i}, \mathtt{J})$ of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$ as defined in Example 3.2.15, where $\widehat{\Omega} = \Omega \times \mathbb{R}^{(N-1)\times e}$ has the same product structure as Ω, and $\widehat{\mathcal{P}}$ is defined by (3.2.19). Moreover, $\widehat{\mathcal{G}}_k := \mathcal{G}_k \otimes \sigma(Y_i, i \le k)$ is countably generated and $\widehat{\mathcal{F}}_k$ is the universal completion of $\widehat{\mathcal{G}}_k$. For $k = 0, \dots, N-1$ and $\widehat{\omega} = (\omega, y) \in \widehat{\Omega}$, we define

$$\widehat{\mathcal{P}}_k(\omega, y) := \big\{\widehat{\mathbb{P}} : \ \widehat{\mathbb{P}}|_{\Omega_1} \in \mathcal{P}_k(\omega)\big\}.$$

By the definition of $\mathcal{P}$ in (5.2.1), it is clear that $\widehat{\mathcal{P}}$ has the same product structure, i.e.,

$$\widehat{\mathcal{P}} = \big\{\widehat{\mathbb{P}}_0 \otimes \cdots \otimes \widehat{\mathbb{P}}_{N-1} \ : \widehat{\mathbb{P}}_k(\cdot) \in \widehat{\mathcal{P}}_k(\cdot), \ k = 0, \dots, N-1\big\}.$$

Further, recalling the definitions of $\mathcal{M}_{k,k+1}(\omega)$ and $\mathcal{M}_k(\omega)$ in (3.2.31)-(3.2.32), we define

$$\widehat{\mathcal{M}}_{k,k+1}(\omega, y) \ := \ \big\{\widehat{\mathbb{Q}} \in \mathfrak{B}(\Omega_1 \times \mathbb{R}^e) \ : \widehat{\mathbb{Q}}|_{\Omega_1} \in \mathcal{M}_{k,k+1}(\omega), \ \mathbb{E}^{\widehat{\mathbb{Q}}}\big[(\Delta S_{k+1}, \Delta Y_{k+1})\big] = 0\big\},$$

and

$$\widehat{\mathcal{M}}_k(\widehat{\omega}) := \big\{\delta_{\widehat{\omega}} \otimes_k \widehat{\mathbb{Q}}_k \otimes \cdots \otimes \widehat{\mathbb{Q}}_{N-1} \ : \widehat{\mathbb{Q}}_i(\cdot) \in \widehat{\mathcal{M}}_{i,i+1}(\cdot), i = k, \dots, N-1\big\},$$

with $\Delta S_{k+1} := S_{k+1} - S_k$ and $\Delta Y_{k+1} := Y_{k+1} - Y_k$. Similarly to the definition of $\mathcal{E}_k$ in Example 3.2.12 with $\mathcal{M}_k(\omega)$, we define a family of operators $(\widehat{\mathcal{E}}_k)$ on functionals on $\widehat{\Omega}$ using $\widehat{\mathcal{M}}_k(\widehat{\omega})$, and show in the following theorem that it provides a dynamic programming representation of $\sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\,\cdot\,]$, where $\widehat{\mathcal{M}}$ is defined in Property 4 of Definition 3.2.13. The proof of Theorem 3.2.29 is left to Section 3.5.3.

Theorem 3.2.29. *Let $\widehat{\xi} : \widehat{\Omega} \to \mathbb{R}$ be an upper semianalytic functional. Then $\widehat{\mathcal{E}}_k(\widehat{\xi})$ is also upper semianalytic and*

$$\sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}] \ = \ \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_0} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}] \ = \ \widehat{\mathcal{E}}^0(\widehat{\xi}) \ := \ \widehat{\mathcal{E}}_0 \circ \cdots \circ \widehat{\mathcal{E}}_{N-1}(\widehat{\xi}). \tag{3.2.35}$$

Remark 3.2.30. (i) *The above result is in fact a classical dynamic programming principle result studied in [92] and [93]. The only crucial step is to prove that the graph set $[[\widehat{\mathcal{M}}_{k,k+1}]] := \{(\widehat{\omega}, \mathbb{Q}) \ : \mathbb{Q} \in \mathcal{M}_{k,k+1}(\widehat{\omega})\}$ is analytic.*

(ii) *Assume that* NA($\mathcal{P}$) *holds. It then follows by Lemma 3.2.26 and Theorem 3.2.27 that the pricing-hedging duality on $\overline{\Omega}$ in* (3.2.34) *holds. Further, by defining $\widehat{\mathcal{E}}^k$ and $\overline{\widehat{\mathcal{E}}}^k$ with $\widehat{\mathcal{E}}_k$ as in* (3.2.9) *and* (3.2.11)*, one has that* (3.2.9) *holds for $\widehat{\mathcal{E}}^k$ from Theorem 3.2.29, and moreover that* (3.2.13) *holds for $\overline{\widehat{\mathcal{E}}}^k$ as it is a special case of Example 3.2.12. It then follows by Corollary 3.2.19 that* (3.2.24) *holds in this framework.*

3.2.8 Comparison with [20] and [21]

In [20] the authors considered the same superhedging problem $\pi_g^A(\Phi)$ with the finite set $\Lambda = \{1, \dots, e\}$, and established the duality

$$\pi_g^A(\Phi) = \inf_{h \in \mathbb{R}^e} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \sup_{\mathbb{Q} \in \mathcal{M}_0} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau - hg], \tag{3.2.36}$$

under some regularity conditions (see Proposition 3.1 in [20]). Our duality in Theorem 3.2.27 is more general and more complete, and moreover, together with Lemma 3.2.28, it induces the above duality (3.2.36). In exchange, [20] also studied another subhedging problem $\sup_{\tau\in\mathcal{T}(\mathbb{F})}\inf_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}^{\mathbb{Q}}[\Phi_\tau]$ which we do not consider here.

More recently, [21] consider the "randomized" stopping times, and obtain a more complete duality for $\pi_g^A(\Phi)$. The dual formulations in [21] and in our results are more or less in the same spirit (as in [22] and [23]). Nevertheless, the duality in [21] is established under strong integrability conditions and an abstract condition which is checked under regularity conditions (see their Assumption 2.1 and Remark 2.1). In particular, when $\mathcal{P}$ is the class of all probability measures on Ω, the integrability condition in their Assumption 2.1 is equivalent to saying that Φ_k and g^i are all uniformly bounded. In our paper, we only assume that g^i are Borel measurable, Φ_k are upper semianalytic and all are $\mathbb{R}$-valued.

Technically, [21] use the duality results in [1] together with a minimax theorem to prove their results. Our first main result consists of introducing an enlarged canonical space (together with an enlarged canonical filtration) to reformulate the main problem as a superhedging problem for European options. Then by adapting the arguments in [1], we establish our duality under general conditions as in [1]. Moreover, we do not assume that Φ_k is $\mathcal{F}_k$-measurable, which enables us to study the superhedging problem for a portfolio containing an American option and some European options. Finally, our setting enables us to apply an approximation argument to study a new class of martingale optimal transport problems and to obtain a Kantorovich duality as in Section 3.3.

3.3 A martingale (optimal) transport setup

In this section we study the duality for American options in a martingale optimal transport setup, with canonical space $\Omega := \{s_0\}\times\mathbb{R}^{d\times N}$ for some $s_0\in\mathbb{R}^d$, the canonical process S on Ω and $\mathcal{P} := \mathfrak{B}(\Omega)$. Then, with $\overline{\Omega} := \Omega\times\mathbb{T}$, we have $\overline{\mathcal{P}} = \mathfrak{P}(\overline{\Omega})$. We assume that the statically traded options on the market are all vanilla options and are arbitrage-free (see [94] and [95]), and numerous enough such that one can recover the marginal distribution of the underlying process S at some maturity times $\mathbb{T}_0 = \{t_1,\dots,t_M\}\subseteq\mathbb{T}$, where $t_M = N$. More precisely, we are given a vector $\mu = (\mu_1,\dots,\mu_M)$ of marginal distributions. We write $\mu(f) := (\int f(x)\mu_1(dx),\dots,\int f(x)\mu_M(dx))$ and we assume that $\mu(|\cdot|)<\infty$ and

$$\mu_i(f)\le\mu_j(f)\ \text{ for all } i\le j,\ i,j\leqslant M,\ \text{and any convex function } f:\mathbb{R}^d\to\mathbb{R}. \tag{3.3.37}$$

The condition (3.3.37) ensures the existence of a calibrated martingale measure, i.e., that the following sets are nonempty

$$\begin{aligned}\mathcal{M}_\mu &:= \big\{\mathbb{Q}\in\mathfrak{B}(\Omega)\ :\mathcal{L}_{\mathbb{Q}}(S_{t_i})=\mu_i,\ i\leqslant M,\ \text{and } S \text{ is a } (\mathbb{Q},\mathbb{F})\text{-martingale}\big\},\\ \overline{\mathcal{M}}_\mu &:= \big\{\overline{\mathbb{Q}}\in\mathfrak{B}(\overline{\Omega})\ :\mathcal{L}_{\overline{\mathbb{Q}}}(S_{t_i})=\mu_i,\ i\leqslant M,\ \text{and } S \text{ is a } (\overline{\mathbb{Q}},\overline{\mathbb{F}})\text{-martingale}\big\}.\end{aligned}$$

Let Λ_0 be the class of all Lipschitz functions $\lambda:\mathbb{R}^d\to\mathbb{R}$, and denote $\Lambda := \Lambda_0^M$. The statically traded options $g = (g^\lambda)_{\lambda\in\Lambda}$ are given by $g^\lambda(\omega) := \lambda(\omega) - \mu(\lambda)$ where $\lambda(\omega) := \sum_{i=1}^M\lambda_i(\omega_{t_i})$ and $\mu(\lambda) := \sum_{i=1}^M\mu_{t_i}(\lambda_i)$. Recall that $\mathcal{M}_g = \mathcal{M}_\mu$. As Λ is a linear space, the superhedging cost of the American option Φ using semi-static strategies $\pi_g^A(\Phi)$ defined in Subsection 3.2.1 can be

rewritten as

$$\pi_g^A(\Phi) = \pi_\mu^A(\Phi) := \inf\{\mu(\lambda) : \exists({}^1H, \dots, {}^NH) \in \mathcal{H}^N \text{ s.t. } {}^jH_i = {}^kH_i \ \forall 1 \leqslant i \leqslant j \leqslant k \leqslant N \\ \text{and } \lambda \in \Lambda \text{ satisfying } \lambda(\omega) + ({}^kH \circ S)_N(\omega) \geqslant \Phi_k(\omega) \text{ for all } k \in \mathbb{T},\ \omega \in \Omega\}.$$

Similarly, we denote by $\overline{\pi}_\mu^E(\Phi)$ the corresponding superhedging cost for a European option with payoff Φ defined on $\overline{\Omega}$, and one has $\pi_\mu^A(\Phi) = \overline{\pi}_\mu^E(\Phi)$ by Theorem 3.2.4.

Example 3.3.1. *We shall construct an example similar to Example 3.1.1 to highlight that we may have a strict inequality in* (3.2.5)*. Consider the case* $N = 2$, $\mathbb{T}_0 = \mathbb{T} = \{1, 2\}$, $\mu_1 = \delta_{\{0\}}$ *and* $\mu_2 = \frac{1}{4}\big(\delta_{\{-2\}} + \delta_{\{-1\}} + \delta_{\{1\}} + \delta_{\{2\}}\big)$*. Let* $\Phi_1(\{S_1 = 0\}) = 1$, $\Phi_2(\{|S_2| = 1\}) = 2$ *and* $\Phi_2(\{|S_2| = 2\}) = 0$*. Then* $\mathcal{M}_\mu$ *contains only one probability measure* $\mathbb{Q}$*, and by direct computation, one has*

$$\mathbb{E}^{\mathbb{Q}}\big[\Phi_\tau\big] = 1, \quad \textit{for all } \ \tau \in \mathcal{T}(\mathbb{F}).$$

Let us now construct a martingale measure $\overline{\mathbb{Q}}_0$ *by*

$$\overline{\mathbb{Q}}_0(d\omega, d\theta) \ := \ \frac{1}{4}\delta_{\{1\}}(d\theta) \otimes \big(\delta_{(0,1)} + \delta_{(0,-1)}\big)(d\omega) \ + \ \frac{1}{4}\delta_{\{2\}}(d\theta) \otimes \big(\delta_{(0,2)} + \delta_{(0,-2)}\big)(d\omega).$$

Then one can check that $\overline{\mathbb{Q}}_0 \in \overline{\mathcal{M}}_\mu$*, and it follows that*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \geq \mathbb{E}^{\overline{\mathbb{Q}}_0}[\Phi] \ = \ \frac{3}{2} \ > \ 1 \ = \ \sup_{\mathbb{Q} \in \mathcal{M}_\mu} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}\big[\Phi_\tau\big].$$

The superhedging price of Φ *is equal to* $3/2$*, as one can consider a superhedging strategy consisting of holding* $3/2$ *in cash and one option* g *from Example 3.1.1. In a similar way as in Example 3.1.1, the duality may be recovered by allowing dynamic trading options.*

3.3.1 Duality on the enlarged space $\overline{\Omega}$

The following theorem shows the duality for $\overline{\Omega}$. Its proof is postponed to Section 3.6.

Theorem 3.3.2. *Suppose that* $\Phi : \overline{\Omega} \to \mathbb{R}$ *is bounded from above and upper semicontinuous. Then there exists an optimal martingale measure* $\overline{\mathbb{Q}}^* \in \overline{\mathcal{M}}_\mu$ *and the pricing-hedging duality holds,*

$$\mathbb{E}^{\overline{\mathbb{Q}}^*}\big[\Phi\big] = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] = \overline{\pi}_\mu^E(\Phi),$$

and in particular (3.2.8) *holds.*

Remark 3.3.3. *Note that in the above formulation each* μ_i *is an element of* $\mathfrak{B}(\mathbb{R}^d)$*. Instead one could take* μ_i *to be an element of* $(\mathfrak{B}(\mathbb{R}))^d$*, and the same statements with analogous proofs would still hold. This alternative formulation has a more transparent financial interpretation as it corresponds only to marginal laws of terminal values of each stock price as opposed to the full distribution; see also [96] for a related discussion.*

3.3.2 Dynamic programming principle on $\widehat{\Omega}$

[28] and [29] studied the Skorokhod embedding and martingale optimal transport problems in continuous time using measure-valued martingales. This point of view enables one to obtain

the dynamic programming principle with marginal constraint because the terminal constraint is transformed into the initial constraint. We adopt this perspective which proves to be very useful.

As before we work with the set of marginal times $\mathbb{T}_0 = \{t_1, \dots, t_M\} \subset \{1, \dots, N\}$ such that $t_M = N$, and a vector of marginal measures $\mu = (\mu_1, \dots, \mu_M)$ satisfying (3.3.37), where each μ_i is a probability measure on $\mathbb{R}^d$. We let $\mathfrak{P}_1(\mathbb{R}^d) = \{\eta \in \mathfrak{P}(\mathbb{R}^d) : \eta(|\cdot|) < \infty\}$ be the set of probability measures with finite first moment which we equip with the 1-Wasserstein distance, i.e., $\eta_n \to \eta_0$ if and only if

$$\int_{\mathbb{R}^d} f(x)\eta_n(dx) \to \int_{\mathbb{R}^d} f(x)\eta_0(dx), \quad \forall f \in \mathcal{C}_1,$$

where $\mathcal{C}_1$ denotes the set of all continuous functions on $\mathbb{R}^d$ with at most linear growth, which makes $\mathfrak{P}_1(\mathbb{R}^d)$ a Polish space. Continuing with the construction from Example 3.2.15, $\widehat{\Omega}$ has to be an infinite dimensional space, and it is convenient to parametrize it as the canonical space for the measure-valued processes

$$\widehat{\Omega} := \{\mu\} \times (\mathfrak{P}_1(\mathbb{R}^d))^{M \times N}$$

and denote by $\widehat{X} = (\widehat{X}_k^1, \dots, \widehat{X}_k^M)_{0 \leqslant k \leqslant N}$ the canonical process on $\widehat{\Omega}$. Let $\widehat{\mathbb{G}} = (\widehat{\mathcal{G}}_k)_{0 \leq k \leq N}$ be the canonical filtration and $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_k)_{0 \leq k \leq N}$ its universal completion. Denote by $\mathcal{T}(\widehat{\mathbb{F}})$ the collection of all $\widehat{\mathbb{F}}$-stopping times. For $f \in \mathcal{C}_1$ we denote the process of its integrals against $\widehat{X}$ as

$$\widehat{X}_k(f) = (\widehat{X}_k^1(f), \dots, \widehat{X}_k^M(f)), \text{ where } \widehat{X}_k^i(f) := \int_{\mathbb{R}^d} f(x)\widehat{X}_k^i(dx) \text{ and}$$
$$\widehat{X}_k(id) = (\widehat{X}_k^1(id), \dots, \widehat{X}_k^M(id)), \text{ where } \widehat{X}_k^i(id) = \int_{\mathbb{R}^d} x\widehat{X}_k^i(dx).$$

Define $\mathtt{i} : \widehat{\Omega} \to \Omega$ by $\mathtt{i}(\widehat{\omega}) = (\widehat{X}_0^M(id)(\widehat{\omega}), \dots, \widehat{X}_N^M(id)(\widehat{\omega}))$ which is surjective and naturally extends processes on Ω to processes on $\widehat{\Omega}$. In particular the price process extends via $S_k(\widehat{\omega}) = S_k(\mathtt{i}(\widehat{\omega})) = \widehat{X}_k^M(id)(\widehat{\omega})$ and the statically traded options via $g^\lambda(\widehat{\omega}) = g^\lambda(\mathtt{i}(\widehat{\omega})) = \lambda(\mathtt{i}(\widehat{\omega})) - \mu(\lambda)$. Define a family of processes $Y = (Y^\lambda)_{\lambda \in \Lambda}$ by $Y^\lambda = \sum_{i=1}^M Y^{\lambda_i}$, where

$$Y_k^{\lambda_i} = \begin{cases} \widehat{X}_k^i(\lambda_i) - \mu_i(\lambda_i) & 0 \leqslant k \leqslant t_i - 1, \\ g^{\lambda_i} = \lambda_i(\widehat{X}_{t_i}^i(id)) - \mu_i(\lambda_i) & t_i \leqslant k \leqslant N. \end{cases}$$

Note that $Y_0^{\lambda_i} = 0$.

For any $\mathbb{Q} \in \mathcal{M}_\mu$ we define a mapping $\mathtt{j}_\mathbb{Q} : \Omega \to \widehat{\Omega}$ by $\mathtt{j}_\mathbb{Q}(\omega) = (\mathcal{L}_\mathbb{Q}(S_{t_i}|\mathcal{F}_k)(\omega))_{i \leqslant M, k \leqslant N}$. As in Example 3.2.15, the map $\mathtt{J} : \mathcal{M}_\mu \to \widehat{\mathcal{M}}$ defined by $\mathtt{J}(\mathbb{Q}) = \mathbb{Q} \circ \mathtt{j}_\mathbb{Q}^{-1}$ satisfies Property 4 of Definition 3.2.13.

Definition 3.3.4. (i) *A probability measure $\widehat{\mathbb{Q}}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ is called a measure-valued martingale measure (MVM measure) if the process $(\widehat{X}_k(f))_{0 \leq k \leq N}$ is a $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingale for all $f \in \mathcal{C}_1$.*
(ii) *A MVM measure $\widehat{\mathbb{Q}}$ is terminating if for $i \in \{1, \dots, M\}$,*

$$\widehat{X}_{t_i}^i \in \Delta := \{\eta \in \mathfrak{P}(\mathbb{R}^d) : \eta = \delta_x, \ x \in \mathbb{R}^d\} \quad \widehat{\mathbb{Q}}\text{-a.s.}$$

(iii) *A MVM measure $\widehat{\mathbb{Q}}$ is consistent if $S_k = \widehat{X}_k^i(id)$ for $k \leqslant t_i$ and $i \in \{1, \dots, M\}$, $\widehat{\mathbb{Q}}$-a.s.*

Let us denote by

$$\widehat{\mathcal{M}}_\mu = \{\widehat{\mathbb{Q}} \in \mathfrak{B}(\widehat{\Omega}) : \widehat{\mathbb{Q}} \text{ is a terminating, consistent, MVM measure}\}.$$

The following lemma shows that the marginal distribution of S at t_i is equal to μ_i, $\widehat{\mathcal{M}}_\mu$-q.s., and hence $\widehat{\mathbb{Q}} \circ \mathtt{i}^{-1} \in \mathcal{M}_\mu$ for any $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_\mu$.

Lemma 3.3.5. *For a measure $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_\mu$ the following holds:*
(i) $\mathcal{L}_{\widehat{\mathbb{Q}}}(S_{t_i}|\widehat{\mathcal{F}}_k) = \widehat{X}^i_k$ $\widehat{\mathbb{Q}}$*-a.s. for* $k \leqslant t_i$, *and in particular* $\mathcal{L}_{\widehat{\mathbb{Q}}}(S_{t_i}) = \mu_i$.
(ii) *For* $k \leqslant t_j \leqslant t_i$, $\widehat{X}^j_k \preceq \widehat{X}^i_k$ $\widehat{\mathbb{Q}}$*-a.s., i.e., for any convex function* f,

$$\int_{\mathbb{R}^d} f(x)\widehat{X}^j_k(dx) \leqslant \int_{\mathbb{R}^d} f(x)\widehat{X}^i_k(dx) \quad \widehat{\mathbb{Q}}\text{-a.s.}$$

Proof. (i) Let $A \subset \mathbb{R}^d$ and recall that $S_k = \widehat{X}^i_k(id)$ $\widehat{\mathbb{Q}}$-a.s. Then we have

$$\int_{\mathbb{R}^d} \mathbb{1}_A(x)\mathcal{L}_{\widehat{\mathbb{Q}}}\left(\widehat{X}^i_{t_i}(id)\middle|\widehat{\mathcal{F}}_k\right)(dx) = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\mathbb{1}_{\{\widehat{X}^i_{t_i}(id)\in A\}}\middle|\widehat{\mathcal{F}}_k\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\widehat{X}^i_{t_i}(\mathbb{1}_A)\middle|\widehat{\mathcal{F}}_k\right] = \widehat{X}^i_k(\mathbb{1}_A),$$

where the second equality holds as $\widehat{\mathbb{Q}}$ is terminating and the third one follows by the definition of the MVM measure in Definition 3.3.4 as well as Remark 2.2 of [29]. Hence the first assertion is proven.
(ii) Let $j \leqslant i$, $k \leqslant t_j$ and f be a convex function. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)\widehat{X}^i_k(dx) &= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[f\big(\widehat{X}^i_{t_i}(id)\big)|\widehat{\mathcal{F}}_k\right] \\
&\geqslant \mathbb{E}^{\widehat{\mathbb{Q}}}\left[f\big(\mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{X}^i_{t_i}(id)|\widehat{\mathcal{F}}_{t_j}]\big)|\widehat{\mathcal{F}}_k\right] \\
&= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[f\big(\widehat{X}^i_{t_j}(id)\big)|\widehat{\mathcal{F}}_k\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[f\big(\widehat{X}^j_{t_j}(id)\big)|\widehat{\mathcal{F}}_k\right] \\
&= \int_{\mathbb{R}^d} f(x)\widehat{X}^j_k(dx),
\end{aligned}$$

where the first and the last equalities follow by (a), the penultimate is due to the consistency of $\widehat{\mathbb{Q}}$, and the inequality follows by conditional Jensen's inequality. □

Recall the set $\widehat{\mathcal{M}}$ of martingale measures in Definition 3.2.13. The following lemma shows how to build the map $\mathtt{J}$ and that $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, Y, \mathtt{i}, J)$ is a dynamic extension of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$.

Lemma 3.3.6. (i) *Under any $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_\mu$, the processes S and Y^λ, for $\lambda \in \Lambda$, are $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingales. In particular, one has $\widehat{\mathcal{M}}_\mu \subset \widehat{\mathcal{M}}$.*
(ii) *For $\mathbb{Q} \in \mathcal{M}_\mu$, let $\mathtt{J}(\mathbb{Q})$ be the distribution of the measure-valued process $\eta = (\eta^1_k, \dots, \eta^M_k)_{k\leqslant N}$, where $\eta^i_k = \mathcal{L}_{\mathbb{Q}}(S_{t_i}|\mathcal{F}_k)$. Then $\mathtt{J} : \mathcal{M}_\mu \to \widehat{\mathcal{M}}$ and $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, Y, \mathtt{i}, J)$ is a dynamic extension of $(\Omega, \mathbb{F}, \mathcal{F}, \mathcal{P}, S, g)$.*

Proof. (i) The process $S = \widehat{X}^M(id)$ is a $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingale as $\widehat{\mathbb{Q}}$ is an MVM measure. To prove that Y^λ is a $(\widehat{\mathbb{Q}}, \widehat{\mathbb{F}})$-martingale for any $\lambda \in \Lambda$, it is enough to show that for any $i \leqslant M$ and $\lambda \in \Lambda_0$

one has $\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\lambda(\widehat{X}^M_{t_i}(id))|\widehat{\mathcal{F}}_k\right] = \widehat{X}^i_k(\lambda)$ for any $k < t_i$. The latter holds because

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\lambda(\widehat{X}^M_{t_i}(id))|\widehat{\mathcal{F}}_k\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\lambda(\widehat{X}^i_{t_i}(id))|\widehat{\mathcal{F}}_k\right] = \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\widehat{X}^i_{t_i}(\lambda)|\widehat{\mathcal{F}}_k\right] = \widehat{X}^i_k(\lambda),$$

where the first equality follows by consistency of $\widehat{\mathbb{Q}}$, the second holds as $\widehat{\mathbb{Q}}$ is terminating, and the last one holds because $\widehat{\mathbb{Q}}$ is an MVM measure.

(ii) We have $\eta^i_{t_i} = \mathcal{L}_{\widehat{\mathbb{Q}}}(S_{t_i}|\mathcal{F}_{t_i}) = \delta_{S_{t_i}} \in \Delta$, and it follows that $\mathtt{J}(\mathbb{Q}) \in \widehat{\mathcal{M}}_\mu \subset \widehat{\mathcal{M}}$. □

For $\widehat{\omega} \in \widehat{\Omega}$, we define the set $[\widehat{\omega}]_{\widehat{\mathcal{G}}_k}$ as in (3.2.18), and denote by $\widehat{\mathcal{M}}^k_\mu(\widehat{\omega})$ the following set of measures

$$\widehat{\mathcal{M}}^k_\mu(\widehat{\omega}) := \Big\{\widehat{\mathbb{Q}} \in \mathfrak{B}(\widehat{\Omega}) : \widehat{\mathbb{Q}} \text{ is terminating and consistent,}$$
$$\widehat{\mathbb{Q}}([\widehat{\omega}]_{\widehat{\mathcal{G}}_k}) = 1, \text{ and } (\widehat{X}_l)_{k\leqslant l\leqslant N} \text{ is a } (\widehat{\mathbb{Q}},\widehat{\mathbb{F}})\text{-MVM}\Big\}.$$

Let us define a family of operators $\widehat{\mathcal{E}}_k$, etc., as in Example 3.2.12,

$$\widehat{\mathcal{E}}_k(\widehat{\xi})(\widehat{\omega}) = \sup_{\mathbb{Q}\in\widehat{\mathcal{M}}^k_\mu(\widehat{\omega})} \mathbb{E}^{\mathbb{Q}}[\widehat{\xi}], \quad \widehat{\xi} \in \widehat{\Upsilon},$$

and then the extension $\overline{\widehat{\mathcal{E}}}_k$ as well as $\overline{\widehat{\mathcal{E}}}^0$ on the enlarged space as in Section 3.2.2. We then have the following theorem.

Theorem 3.3.7. *For all upper semianalytic functionals $\widehat{\xi} : \widehat{\Omega} \to \mathbb{R}$, $\overline{\widehat{\mathcal{E}}}_k(\widehat{\xi})$ is also upper semianalytic and*

$$\sup_{\tilde{\mathbb{Q}}\in\overline{\widehat{\mathcal{M}}}_\mu} \mathbb{E}^{\tilde{\mathbb{Q}}}[\widehat{\xi}] = \overline{\widehat{\mathcal{E}}}^0(\widehat{\xi}). \tag{3.3.38}$$

In particular the pricing-hedging duality (3.2.24) *holds in this martingale optimal transport context for all functionals $\Phi : \overline{\Omega} \to \mathbb{R}^N$ which are upper semicontinuous and bounded from above.*

Proof. Notice that the pricing-hedging duality on $\overline{\Omega}$ holds by Theorem 3.3.2. Then by Corollary 3.2.19, it is enough to establish the dynamic programming principle on $\widehat{\Omega}$ to prove the pricing-hedging duality (3.2.24). Using exactly the same arguments as in (4.12) of [1], to establish the dynamic programming principle on $\widehat{\Omega}$, it is enough to argue that $\widehat{\mathcal{M}}_\mu$ is such that

$$\{(\widehat{\omega}, \widehat{\mathbb{Q}}) : \widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}^k_\mu(\widehat{\omega})\} \text{ is analytic.}$$

To prove the above analyticity property, we first observe that

$$\widehat{\mathcal{E}}_k \circ \cdots \circ \widehat{\mathcal{E}}_{N-1}(\widehat{\xi})(\widehat{\omega}) = \widehat{\mathcal{E}}_{k,k+1} \circ \cdots \circ \widehat{\mathcal{E}}_{N-1,N}(\widehat{\xi})(\widehat{\omega}), \quad \widehat{\xi} \in \widehat{\Upsilon},$$

where $\mathcal{E}_{k,k+1}(\widehat{\xi})(\omega) = \sup_{\mathcal{M}^{k,k+1}_\mu(\omega)} \mathbb{E}^{\mathbb{Q}}[\widehat{\xi}]$ and

$$\widehat{\mathcal{M}}^{k,k+1}_\mu(\widehat{\omega}) := \Big\{\widehat{\mathbb{Q}} \in \mathfrak{B}(\widehat{\Omega}) : \widehat{\mathbb{Q}} \text{ is terminating and consistent,}$$
$$\widehat{\mathbb{Q}}([\widehat{\omega}]_{\widehat{\mathcal{G}}_k}) = 1, \text{ and } \widehat{\omega}_k(f) = \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{X}_{k+1}(f)], \ \forall f \in \mathcal{C}_1\Big\}.$$

Next, let $\mathcal{C}_1^0$ denote a countable dense subset of $\mathcal{C}_1$ under the uniform convergence topology. Then it is clear that for each $k \in \mathbb{T}$,

$$\begin{aligned}\left\{(\widehat{\omega},\widehat{\mathbb{Q}}) \in \widehat{\Omega}\times\mathfrak{B}(\widehat{\Omega}):\ \widehat{\mathbb{Q}}\in\widehat{\mathcal{M}}_\mu^{k,k+1}(\widehat{\omega})\right\} = \big\{(\widehat{\omega},\widehat{\mathbb{Q}})\in\widehat{\Omega}\times\mathfrak{B}(\widehat{\Omega})\ :\widehat{\mathbb{Q}}([\widehat{\omega}]_{\widehat{\mathcal{G}}_k})=1\\ \widehat{\mathbb{Q}} \text{ is terminating and consistent, and } \ \widehat{\omega}_k(f)=\mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{X}_{k+1}(f)],\ \ \forall f\in\mathcal{C}_1^0\big\}\end{aligned}$$

is a Borel set. □

3.4 Proofs for Section 3.2

We recall that Section 3.2 stays in a context with an abstract space $(\Omega,\mathcal{F})$ equipped with an underlying process S and a family $\mathcal{P}$ of probability measures. $\mathcal{M}$ denotes the collection of all measures $\mathbb{Q}$ dominated by some $\mathbb{P}\in\mathcal{P}$ and such that S is a $\mathbb{Q}$-martingale, and $\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}^{\mathbb{Q}}[\,\cdot\,]$ admits a dynamic programming representation by $\mathcal{E}_k$ (see (3.2.9)), from which one defines the family of operators $\widehat{\mathcal{E}}_k$ in (3.2.10). A first enlarged space $\overline{\Omega}:=\Omega\times\{1,\dots,N\}$ is introduced in Section 3.2.2 to reduce an American option to a European option, and a dynamic extension $\widehat{\Omega}$ of Ω is defined in Definition 3.2.13 to introduce a fictitious market allowing dynamic trading of options.

Proof of Proposition 3.2.9. First we prove that (3.2.12) implies (3.2.9). For a given ξ on Ω let us define Φ on $\overline{\Omega}$ by

$$\begin{aligned}\Phi((\omega,k)) &= -\infty \ \text{ if } k\in\{1,\dots,N-1\},\\ \Phi((\omega,N)) &= \xi(\omega).\end{aligned}$$

The definition of Φ combined with (3.2.12) implies that

$$\overline{\mathcal{E}}_0\circ\overline{\mathcal{E}}_1\circ\cdots\circ\overline{\mathcal{E}}_{N-1}(\xi)=\mathcal{E}_0\circ\mathcal{E}_1\circ\cdots\circ\mathcal{E}_{N-1}(\xi).$$

Moreover, one has that

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}}\mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]=\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}^{\mathbb{Q}}[\Phi_N]=\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}^{\mathbb{Q}}[\xi],$$

because for a measure $\overline{\mathbb{Q}}\in\overline{\mathcal{M}}$ such that $\overline{\mathbb{Q}}(\Omega\times\{1,\dots,N-1\})>0$, the expected value drops to $-\infty$.

Now let us prove that (3.2.9) and (3.2.13) imply (3.2.12). Define an $\mathbb{F}$-stopping time τ^* by

$$\begin{aligned}\tau^*(\omega) &:= \min\left\{k\geqslant 1:\mathcal{E}^k\left(\Phi(\cdot,k)\right)(\omega)=\overline{\mathcal{E}}^k(\Phi)(\omega,k)\right\} \qquad (3.4.39)\\ &= \min\left\{k\geqslant 1:\mathcal{E}^k\left(\mathcal{E}^{k+1}\left(\Phi(\cdot,k)\right)\right)(\omega)\geqslant\mathcal{E}_k\left(\overline{\mathcal{E}}^{k+1}(\Phi)(\cdot,k+1)\right)(\omega)\right\}.\end{aligned}$$

Note that on $\{k<\tau^*\}$ one has

$$\mathcal{E}^k\left(\Phi(\cdot,k)\right)(\omega)<\overline{\mathcal{E}}^k(\Phi)(\omega,k)=\mathcal{E}_k\left(\overline{\mathcal{E}}^{k+1}(\Phi)(\cdot,k+1)\right)(\omega). \qquad (3.4.40)$$

Then

$$\begin{aligned}
&\overline{\mathcal{E}}_0 \circ \cdots \circ \overline{\mathcal{E}}_{N-1}(\Phi) \\
&\quad = \mathcal{E}_0 \left(\mathbb{1}_{\{\tau^*=1\}} \mathcal{E}_1 \left(\overline{\mathcal{E}}^2(\Phi)(\cdot,1) \right) + \mathbb{1}_{\{\tau^*>1\}} \mathcal{E}_1 \left(\overline{\mathcal{E}}^2(\Phi)(\cdot,2) \right) \right) \\
&\quad = \dots \\
&\quad = \mathcal{E}_0 \circ \mathcal{E}_1 \circ \cdots \circ \mathcal{E}_{N-1}(\Phi_{\tau^*}) \\
&\quad = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi_{\tau^*}],
\end{aligned}$$

where the last equality follows from the DPP on Ω, (3.2.9). Note as well that

$$\overline{\mathcal{E}}_0 \circ \cdots \circ \overline{\mathcal{E}}_{N-1}(\Phi) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi_{\tau^*}] \leqslant \sup_{\mathbb{Q} \in \mathcal{M}} \sup_{\tau \in \mathcal{T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}}[\Phi_\tau] \leqslant \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]. \tag{3.4.41}$$

Combining this with (3.2.13), we conclude the proof. □

Proposition 3.4.1. *The family $(\overline{\mathcal{E}}_k)$ given in Example 3.2.12, with $\mathcal{M}_k(\omega)$ defined by* (3.2.32)*, satisfies* (3.2.13)*.*

Proof. In the context of Example 3.2.12, the family $(\overline{\mathcal{E}}_k)$ take the following form,

$$\overline{\mathcal{E}}_0(\Phi) := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi(\cdot,1)], \tag{3.4.42}$$

$$\overline{\mathcal{E}}_k(\Phi)(\bar{\omega}) := \begin{cases} \sup_{\mathbb{Q} \in \mathcal{M}_k(\omega)} \mathbb{E}^{\mathbb{Q}}[\Phi(\cdot,\theta)] & \text{if } \theta < k, \\ \sup_{\mathbb{Q} \in \mathcal{M}_k(\omega)} \mathbb{E}^{\mathbb{Q}}[\Phi(\cdot,k)] \vee \sup_{\mathbb{Q} \in \mathcal{M}_k(\omega)} \mathbb{E}^{\mathbb{Q}}[\Phi(\cdot,k+1)] & \text{if } \theta \geqslant k. \end{cases}$$

To see that (3.2.13) holds, it is insightful to rewrite $\overline{\mathcal{E}}^0$ in a slightly different way, as $\widetilde{\mathcal{E}}^0$ below. Recall that $\mathcal{F}_k$ is the universal completion of $\mathcal{G}_k$, where the latter is countably generated. Let

$$\overline{\mathcal{G}}_k^- := \mathcal{G}_k \otimes \sigma(T \wedge k) \subset \overline{\mathcal{G}}_k := \mathcal{G}_k \otimes \sigma(T \wedge (k+1)) \subset \mathcal{F}_k \otimes \sigma(T \wedge (k+1)) =: \overline{\mathcal{F}}_k,$$

$$\overline{\mathcal{M}}_k^-(\bar{\omega}) := \{\overline{\mathbb{Q}} \lll \overline{\mathcal{P}} : \ \overline{\mathbb{Q}} \left[[\bar{\omega}]_{\overline{\mathcal{G}}_k^-} \right] = 1 \text{ and } \mathbb{E}^{\overline{\mathbb{Q}}}[\Delta S_n | \overline{\mathcal{F}}_{n-1}] = 0 \ \forall n \in \{k+1, \dots, N\}\},$$

where $[\bar{\omega}]_{\overline{\mathcal{G}}_k^-}$ is defined as in (3.2.18). Next, for $\Phi \in \overline{\Upsilon}$, let us introduce the operators

$$\widetilde{\mathcal{E}}_0(\Phi) := \sup_{\mathbb{Q} \in \overline{\mathcal{M}}} \mathbb{E}^{\mathbb{Q}}[\Phi(\cdot,1)], \qquad \widetilde{\mathcal{E}}_k(\Phi)(\bar{\omega}) := \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_k^-(\bar{\omega})} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi], \ k \leqslant N-1.$$

Denote $\overline{\mathcal{E}}^k(\cdot) := \overline{\mathcal{E}}_k \circ \cdots \circ \overline{\mathcal{E}}_{N-1}(\cdot)$ and $\widetilde{\mathcal{E}}^k(\cdot) := \widetilde{\mathcal{E}}_k \circ \cdots \circ \widetilde{\mathcal{E}}_{N-1}(\cdot)$. We claim that

$$\overline{\mathcal{E}}^k(\Phi)(\bar{\omega}) = \widetilde{\mathcal{E}}^k(\Phi)(\bar{\omega}), \ 0 \leqslant k < N, \ \Phi \in \overline{\Upsilon}. \tag{3.4.43}$$

Note that the regular conditional probabilities of any $\mathbb{Q} \in \overline{\mathcal{M}}$ with respect to $\overline{\mathcal{G}}_k^-$, denoted $\mathbb{Q}_{\bar{\omega}}$, satisfy $\mathbb{Q}\left[\{\bar{\omega} : \mathbb{Q}_{\bar{\omega}} \in \overline{\mathcal{M}}_k^-(\bar{\omega})\}\right] = 1$ and one has $\mathbb{E}^{\mathbb{Q}}[\Phi | \overline{\mathcal{F}}_k^-] \leqslant \widetilde{\mathcal{E}}_k(\Phi)$, $\mathbb{Q}$-a.s., which implies (3.2.13) by the tower property of conditional expectations and the definition of $\widetilde{\mathcal{E}}^0$.

It is then enough to prove the claim (3.4.43). Note that, for $\bar{\omega} = (\omega, \theta)$ with $\theta \leqslant k-1$, a measure $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_k^-(\bar{\omega})$ satisfies $\overline{\mathbb{Q}}_{|\Omega} \in \mathcal{M}_k(\omega)$ and $\overline{\mathbb{Q}}(\Omega \times \{\theta\}) = 1$; and a measure $\mathbb{Q} \in \mathcal{M}_k(\omega)$

satisfies $\mathbb{Q}\otimes\delta_\theta\in\overline{\mathcal{M}}_k^-(\bar\omega)$. It is thus clear that, in this case, $\overline{\mathcal{E}}^k(f)(\bar\omega)=\widetilde{\mathcal{E}}^k(f)(\bar\omega)$.

As a second step, for $\bar\omega=(\omega,\theta)$ with $\theta\geqslant k$, we show that $\overline{\mathcal{E}}_k(f)(\bar\omega)\leqslant\widetilde{\mathcal{E}}_k(f)(\bar\omega)$. Take any $\mathbb{Q}\in\mathcal{M}_k(\omega)$. Then, for $n\in\{k,\dots,N\}$, $\mathbb{Q}\otimes\delta_n\in\overline{\mathcal{M}}_k^-(\bar\omega)$ and $\mathbb{Q}\otimes\delta_n(\Omega\times\{n\})=1$. Hence it follows that $\overline{\mathcal{E}}_k(f)(\bar\omega)\leqslant\widetilde{\mathcal{E}}_k(f)(\bar\omega)$.

In a final step, we show that, for $\bar\omega=(\omega,\theta)$ with $\theta\geqslant k$, $\overline{\mathcal{E}}_k(f)(\bar\omega)\geqslant\widetilde{\mathcal{E}}_k(f)(\bar\omega)$ holds. Let us start with $k=N-1$. Take any $\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_{N-1}^-(\bar\omega)$ and consider its regular conditional probability with respect to $\overline{\mathcal{G}}_{N-1}$ (the atom $\{\omega\}\times\{N-1,N\}$ is divided into atoms $\{\omega\}\times\{N-1\}$ and $\{\omega\}\times\{N\}$) denoted by $\overline{\mathbb{Q}}_N$ and $\overline{\mathbb{Q}}_{N-1}$. Then, clearly, $\overline{\mathbb{Q}}_{N|\Omega}$ and $\overline{\mathbb{Q}}_{N-1|\Omega}$ belong to $\mathcal{M}^{N-1}(\omega)$, and $\overline{\mathbb{Q}}_N(\{\omega\}\times\{N\})=1$ and $\overline{\mathbb{Q}}_{N-1}(\{\omega\}\times\{N-1\})=1$. Thus, it follows that $\overline{\mathcal{E}}_{N-1}(f)(\bar\omega)\geqslant\widetilde{\mathcal{E}}_{N-1}(f)(\bar\omega)$.

Finally, to complete the proof, we need to show that $\overline{\mathcal{E}}^{k+1}(f)(\bar\omega)=\widetilde{\mathcal{E}}^{k+1}(f)(\bar\omega)$ implies $\overline{\mathcal{E}}^k(f)(\bar\omega)\geqslant\widetilde{\mathcal{E}}^k(f)(\bar\omega)$ for $\bar\omega=(\omega,\theta)$ with $\theta\geqslant k$. First note that $\overline{\mathcal{E}}^{k+1}(f)(\bar\omega)=\widetilde{\mathcal{E}}^{k+1}(f)(\bar\omega)$ is constant on $\theta\in\{k,\dots,N\}$, i.e.,

$$\overline{\mathcal{E}}^{k+1}(f)((\omega,\theta_1))=\overline{\mathcal{E}}^{k+1}(f)((\omega,\theta_2))\quad\text{for all }\omega\in\Omega\text{ and }\theta_1,\theta_2\in\{k,\dots,N\}.\tag{3.4.44}$$

Take any $\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_k^-(\bar\omega)$ and consider its regular conditional probability with respect to ϑ_N (the atom $\{\omega\}\times\{k,\dots,N\}$ is divided into atoms $\{\omega\}\times\{n\}$ for $n=k,\dots,N$) denoted by $\overline{\mathbb{Q}}_n$ for $n=k,\dots,N$. Then, clearly, $\overline{\mathbb{Q}}_{n|\Omega}\in\mathcal{M}_k(\omega)$ and $\overline{\mathbb{Q}}_n([\omega]_k\times\{n\})=1$ where $[\omega]_k$ denotes an atom of $\mathcal{G}_k$ which contains ω. Thus, combining with (3.4.44), it follows that $\overline{\mathcal{E}}_k(f)(\bar\omega)\geqslant\widetilde{\mathcal{E}}_k(f)(\bar\omega)$. □

3.5 Proofs for Section 3.3

We now recall the context of Section 5.2, where $\Omega_0:=\{\omega_0\}$ is a singleton, Ω_1 is a nonempty Polish space and $\Omega:=\Omega_0\times\Omega_1^N$. For technical reasons, we introduce a Ω_1-valued canonical process $X=(X_k)_{0\leq k\leq N}$ on the enlarged space $\overline{\Omega}$ by $X_k(\bar\omega):=\omega_k$ for all $\bar\omega:=(\omega,\theta)\in\overline{\Omega}$, and an enlarged filtration $\overline{\mathbb{G}}=(\overline{\mathcal{G}}_k)_{0\leq k\leq N}$ by

$$\overline{\mathcal{G}}_0:=\{\emptyset,\overline{\Omega}\}\quad\text{and}\quad\overline{\mathcal{G}}_k:=\sigma\big\{X_i,\ \{T\leq i\},\ i=1,\dots,k\big\},$$

and the universally completed filtration $\overline{\mathbb{F}}=(\overline{\mathcal{F}}_k)_{0\leq k\leq N}$ by defining $\overline{\mathcal{F}}_k$ as the universal completion of $\overline{\mathcal{G}}_k$. It follows that the random time $T:\overline{\Omega}\to\mathbb{T}$ is a $\overline{\mathbb{G}}$-stopping time. We also define a restricted space $\overline{\Omega}_k$, for every $k=1,\dots,N$,

$$\overline{\Omega}_k\ :=\ \Omega_k\times\{1,\dots,k\}\ =\ \Omega_1^k\times\{1,\dots,k\}.$$

Lemma 3.5.1. *Let $\overline{\mathbb{P}}\in\overline{\mathcal{P}}$ be a probability measure on $(\overline{\Omega},\overline{\mathcal{G}}_N)$, and $(\overline{\mathbb{P}}_{\bar\omega})_{\bar\omega\in\overline{\Omega}}$ be a family of regular conditional probability distributions of $\overline{\mathbb{P}}$ with respect to $\overline{\mathcal{G}}_k$. Then for every $k\in\{0,1,\dots,N-1\}$, one has $\overline{\mathbb{P}}_{\bar\omega}\circ X_{k+1}^{-1}\in\mathcal{P}_k(\omega)$ for $\overline{\mathbb{P}}$-a.e. $\bar\omega=(\omega,\theta)\in\overline{\Omega}$.*

Let us introduce the following set of measures

$$\begin{aligned}\overline{\mathcal{M}}_g^{loc}\ :=\ \{\overline{\mathbb{Q}}\ :\overline{\mathbb{Q}}\ll\overline{\mathcal{P}},\ \mathbb{E}^{\overline{\mathbb{Q}}}[g^i]=0,\ i\in\{1,\dots,e\}\\ \text{and } S\text{ is an }(\overline{\mathbb{F}},\overline{\mathbb{Q}})\text{-local martingale}\}.\end{aligned}$$

Lemma 3.5.2. *Let Φ be upper semianalytic and $\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g^{loc}$. Then for any $x\in\mathbb{R}$ and $(\overline{H},h)\in\overline{\mathcal{H}}\times\mathbb{R}^e$ such that $x+(\overline{H}\circ S)_N(\bar\omega)+hg(\omega)\geq\Phi(\bar\omega)$, $\overline{\mathbb{Q}}$-a.s., one has $\mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big]\leq x$.*

Proof. The proof follows by exactly the same arguments as in Lemma A.2 of [1], using the discrete time local martingale characterization in their Lemma A.1. □

By Lemma 3.5.2, we easily obtain the weak duality for all upper semianalytic Φ

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \leq \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g^{loc}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \leq \overline{\pi}_g^E(\Phi). \tag{3.5.45}$$

The following lemma shows that, for a fixed Φ, we can restrict to martingale measures satisfying a further integrability constraint.

Lemma 3.5.3. *Let Φ be upper semianalytic, $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_0^{loc}$ and $\varphi : \Omega \to [1,\infty)$ be such that $|\Phi(\omega,k)| \leq \varphi(\omega)$ for all $\bar{\omega} = (\omega,k) \in \overline{\Omega}$. Then $\overline{\mathcal{M}}^{\varphi,\overline{\mathbb{Q}}} \neq \emptyset$ and*

$$\mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \;\leq\; \sup_{\overline{\mathbb{Q}}'\in\overline{\mathcal{M}}^{\varphi,\overline{\mathbb{Q}}}} \mathbb{E}^{\overline{\mathbb{Q}}'}[\Phi],$$

where

$$\overline{\mathcal{M}}^{\varphi,\overline{\mathbb{Q}}} := \{\overline{\mathbb{Q}}' \sim \overline{\mathbb{Q}} \;: \mathbb{E}^{\overline{\mathbb{Q}}'}[\varphi] < \infty, \text{ and } S \text{ is an } (\overline{\mathbb{F}},\overline{\mathbb{Q}}')\text{-martingale}\}.$$

Proof. First, by Lemma A.3 of [1], there exists a probability measure $\overline{\mathbb{P}}_*$, equivalent to $\overline{\mathbb{Q}}$ on $(\overline{\Omega},\overline{\mathcal{F}}_N)$, such that $\mathbb{E}^{\overline{\mathbb{P}}_*}[\varphi] < \infty$. On the filtered probability space $(\overline{\Omega},\overline{\mathcal{F}}_N,\overline{\mathbb{F}},\overline{\mathbb{P}}_*)$, one defines $\overline{\mathcal{M}}_*^{loc}$ as the collection of all probability measures $\overline{\mathbb{Q}}' \sim \overline{\mathbb{Q}} \sim \overline{\mathbb{P}}_*$ under which S is an $\overline{\mathbb{F}}$-local martingale. Denote

$$\pi_0^{E,\overline{\mathbb{Q}}}(\Phi) := \inf\left\{x : \exists \overline{H} \in \overline{\mathcal{H}} \text{ s.t. } x + (\overline{H}\circ S)_N \geqslant \Phi, \overline{\mathbb{Q}}\text{-a.s.}\right\},$$

then by the classical arguments for the dominated discrete time market (such as [97] and [98], see also Lemma A.3 of [1]), one can easily obtain the inequality

$$\mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \;\leq\; \sup_{\overline{\mathbb{Q}}'\in\overline{\mathcal{M}}_*^{loc}} \mathbb{E}^{\overline{\mathbb{Q}}'}[\Phi] \;\leq\; \overline{\pi}_0^{E,\overline{\mathbb{Q}}}(\Phi) \;\leq\; \sup_{\overline{\mathbb{Q}}'\in\overline{\mathcal{M}}^{\varphi,\overline{\mathbb{Q}}}} \mathbb{E}^{\overline{\mathbb{Q}}'}[\Phi],$$

which concludes the proof. □

Using Theorem 2.2 of [1], which is stated for a general abstract space $(\Omega,\mathcal{F})$, one directly obtains a closeness result for the set of all payoffs which can be superreplicated from initial capital $x = 0$, in our context. Let us denote by $\overline{\mathcal{L}}_+^0$ the set of all positive random variables on $\overline{\Omega}$, and define

$$\overline{\mathcal{C}} := \left\{(\overline{H}\circ S)_N + hg \;: \overline{H} \in \overline{\mathcal{H}}, h \in \mathbb{R}^e\right\} - \overline{\mathcal{L}}_+^0.$$

Lemma 3.5.4 (Theorem 2.2 of [1])**.** *Let Φ be upper semianalytic and assume that $\mathrm{NA}(\overline{\mathcal{P}})$ holds. Then the set $\overline{\mathcal{C}}$ is closed in the following sense. Whenever $(W^n)_{n\geq 1} \subset \overline{\mathcal{C}}$ and W is a random variable such that $W^n \to W$, $\overline{\mathcal{P}}$-q.s., then $W \in \overline{\mathcal{C}}$.*

3.5.1 Proof of Theorem 3.2.27: the case $e = 0$, equivalently $\Lambda = \emptyset$

For each $1 \leq i \leq j \leq N$, we introduce a map from Ω_j to Ω_i (resp. $\overline{\Omega}_j$ to $\overline{\Omega}_i$) by

$$[\omega]_i := (\omega_1,\ldots,\omega_i), \text{ for all } \omega \in \Omega_j \text{ (resp.}[\bar{\omega}]_i \;:=\; ([\omega]_i, \theta\wedge i), \text{ for all } \bar{\omega} = (\omega,\theta) \in \overline{\Omega}_j).$$

Note that $\overline{\mathcal{F}}_k^-$ is the smallest σ-field on $\overline{\Omega}$ generated by $[\,\cdot\,]_k : \overline{\Omega} \to \overline{\Omega}_k$; equivalently, an $\overline{\mathcal{F}}_k^-$-measurable random variable f defined on $\overline{\Omega}$ can be identified as a Borel measurable function on $\overline{\Omega}_k$. The process S is naturally defined on the restricted spaces Ω_k and $\overline{\Omega}_k$.

We next recall the notion of local no-arbitrage condition $\mathrm{NA}(\mathcal{P}_k(\omega))$ introduced at the beginning of Section 4.2 in [1]. Given a fixed $\omega \in \Omega_k$, we can consider $\Delta S_{k+1}(\omega,\cdot) := S_{k+1}(\omega,\cdot) - S_k(\omega)$ as a random variable on Ω_1, which determines a one-period market on $(\Omega_1, \mathcal{B}(\Omega_1))$ endowed with a class $\mathcal{P}_k(\omega)$ of probability measures. Then $\mathrm{NA}(\mathcal{P}_k(\omega))$ denotes the corresponding no-arbitrage condition in this one-period market, i.e., $\mathrm{NA}(\mathcal{P}_k(\omega))$ holds if for all $H \in \mathbb{R}^d$,

$$H\Delta S_{k+1}(\omega,\cdot) \geqslant 0 \quad \mathcal{P}_k(\omega)\text{-q.s.} \implies H\Delta S_{k+1}(\omega,\cdot) = 0 \quad \mathcal{P}_k(\omega)\text{-q.s.}$$

Lemma 3.5.5. *In the context of Section 5.2, let $f : \overline{\Omega}_{k+1} \to \overline{\mathbb{R}}$ be upper semianalytic. Then $\overline{\mathcal{E}}_k(f) : \overline{\Omega}_k \to \overline{\mathbb{R}}$ is still upper semianalytic. Moreover, there exist two universally measurable functions $(y_1, y_2) : \overline{\Omega}_k \to \mathbb{R}^d \times \mathbb{R}^d$ such that*

$$\begin{aligned} \overline{\mathcal{E}}_k(f)(\bar\omega) \;+\; y_1(\bar\omega)\Delta S_{k+1}(\omega,\cdot) \;&\geq\; f(\omega,\cdot,\theta) \quad \mathcal{P}_k(\omega)\text{-q.s.}, \\ \overline{\mathcal{E}}_k(f)(\bar\omega) \;+\; y_2(\bar\omega)\Delta S_{k+1}(\omega,\cdot) \;&\geq\; f(\omega,\cdot,k+1) \quad \mathcal{P}_k(\omega)\text{-q.s.}, \end{aligned}$$

for all $\bar\omega = (\omega,\theta) \in \overline{\Omega}_k$ such that $\mathrm{NA}(\mathcal{P}_k(\omega))$ holds and $f(\omega,\cdot,\theta) > -\infty$, $\mathcal{P}_k(\omega)$-q.s. $f(\omega,\cdot,k+1) > -\infty$, $\mathcal{P}_k(\omega)$-q.s.

Proof. Notice that $f_1 \vee f_2$ is upper semianalytic whenever f_1 and f_2 are both upper semianalytic. Using the definition of $\overline{\mathcal{E}}_k$, the above lemma follows by applying Lemma 4.10 of [1] for every fixed θ. □

Proof of Theorem 3.2.27 (the case $e = 0$). First, one has the weak duality as in (3.5.45)

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] \;\leq\; \overline{\pi}_g^E(\Phi).$$

Next, for the inverse inequality, we can assume, without loss of generality, that Φ is bounded from above. Indeed, by Lemma 3.5.4, one has $\lim_{n\to\infty} \overline{\pi}_g^E(\Phi \wedge n) = \overline{\pi}_g^E(\Phi)$ (see also the proof of Theorem 3.4 of [1]). Besides, the approximation $\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \lim_{n\to\infty} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi \wedge n]$ is an easy consequence of the monotone convergence theorem.

When Φ is bounded from above, by Lemma 3.2.28, it is enough to prove that there is some $\overline{H} \in \overline{\mathcal{H}}$ such that

$$\overline{\mathcal{E}}^0[\Phi] \;+\; (\overline{H} \circ S)_N \;\geq\; \Phi \quad \overline{\mathcal{P}}\text{-q.s.} \tag{3.5.46}$$

In view of Lemma 3.5.3, we know that $\overline{\mathcal{E}}^k(\Phi)(\bar\omega) > -\infty$ for all $\bar\omega \in \overline{\Omega}_k$. Further, by Lemma 3.5.5, there exist two universally measurable functions $(y_1^k, y_2^k) : \overline{\Omega}_k \to \mathbb{R}^d \times \mathbb{R}^d$ such that

$$\begin{aligned} y_1^k(\bar\omega)\Delta S_{k+1}(\omega,\cdot) &\geq \overline{\mathcal{E}}^{k+1}(\Phi)(\omega,\cdot,\theta) - \overline{\mathcal{E}}^k(\Phi)(\bar\omega) \quad \mathcal{P}_k(\omega)\text{-q.s.}, \\ y_2^k(\bar\omega)\Delta S_{k+1}(\omega,\cdot) &\geq \overline{\mathcal{E}}^{k+1}(\Phi)(\omega,\cdot,k+1) - \overline{\mathcal{E}}^k(\Phi)(\bar\omega) \quad \mathcal{P}_k(\omega)\text{-q.s.}, \end{aligned}$$

for all $\bar\omega = (\omega,\theta) \in \overline{\Omega}_k$ such that $\mathrm{NA}(\mathcal{P}_k(\omega))$ holds.

As $N_k := \{\omega_k : \mathrm{NA}(\mathcal{P}_k(\omega)) \text{ fails}\}$ is $\mathcal{P}$-polar by Theorem 4.5 of [1], it follows that, with

$\overline{H}_{k+1}(\bar{\omega}) := y_1^k([\bar{\omega}]_k)\mathbf{1}_{\{\theta\leq k\}} + y_1^k([\bar{\omega}]_k)\mathbf{1}_{\{\theta>k\}}$, one has

$$\sum_{k=0}^{N-1} \overline{H}_{k+1}\Delta S_{k+1} \;\geq\; \sum_{k=0}^{N-1} \big(\overline{\mathcal{E}}^{k+1}(\Phi) - \overline{\mathcal{E}}^{k}(\Phi)\big) \;=\; \Phi - \overline{\mathcal{E}}(\Phi), \quad \mathcal{P}\text{-q.s.}$$

To conclude, it is enough to notice that the above $\overline{H}$ is an optimal dual strategy for the case of Φ being bounded from above. The existence of the optimal dual strategy for general Φ is then a consequence of Lemma 3.5.4. □

3.5.2 Proof of Theorem 3.2.27: the case $e \geq 1$, equivalently $\Lambda \neq \emptyset$

We will adapt the arguments in Section 5 of [1] to prove Theorem 3.2.27 in the context with finitely many options $e \geq 1$.

For technical reasons, we introduce

$$\varphi(\omega,\theta) := 1 + |g^1(\omega)| + \cdots + |g^e(\omega)| + \max_{1\leq k\leq N} |\Phi_k(\omega)|,$$

which depends only on ω, and

$$\overline{\mathcal{M}}_g^{\varphi} := \{\overline{\mathbb{Q}} \in \overline{\mathcal{M}} \;: \mathbb{E}^{\overline{\mathbb{Q}}}[\varphi] < \infty \text{ and } \mathbb{E}^{\overline{\mathbb{Q}}}[g^i] = 0 \text{ for } i = 1,\ldots,e\}. \tag{3.5.47}$$

Moreover, in view of Lemma 3.5.3, one has

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi] = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g^{\varphi}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi].$$

Proof of Theorem 3.2.27 (the case $e \geq 1$). The existence of some $\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g$ is an easy consequence of Theorem 5.1 of [1] under NA($\mathcal{P}$). Moreover, similarly to [1], there exists an optimal dual strategy by Lemma 3.5.4. Let us now focus on the duality results.

First, the duality $\overline{\pi}_g^E(\Phi) = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_g} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]$ in (3.2.34) has already been proved for the case $e = 0$. We will use an inductive argument. Suppose that the duality (3.2.34) holds true for the case with $e \geqslant 0$, We aim to prove the duality with $e+1$ options

$$\overline{\pi}_{(g,f)}^E(\Phi) \;=\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_{(g,f)}^{\varphi}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi],$$

where the additional option has a Borel measurable payoff function $f \equiv g^{e+1}$ such that $|f| \leqslant \varphi$, and has an initial price $f_0 = 0$. By the weak duality in (3.5.45) and Lemma 3.5.3, the "$\geq$" side of the inequality holds true, so we will focus on the "$\leq$" side of the inequality, i.e.,

$$\overline{\pi}_{(g,f)}^E(\Phi) \;\leq\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_{(f,g)}^{\varphi}} \mathbb{E}^{\overline{\mathbb{Q}}}[\Phi]. \tag{3.5.48}$$

If f is replicable by some semi-static strategy with underlying S and options $(g^1,\ldots,g^e)$ in the sense that $\exists \overline{H} \in \overline{\mathcal{H}}, h \in \mathbb{R}^e$, such that $f = (\overline{H}\circ S)_N + hg$, $\overline{\mathcal{P}}$-q.s. (or equivalently, $\exists H \in \mathcal{H}, h \in \mathbb{R}^e$, such that $f = (H \circ S)_N + hg, \mathcal{P}$-q.s.), then the problem is reduced to the case with e options and the result is trivial. Let us assume that f is not replicable, and we claim that there exists a

sequence $(\overline{\mathbb{Q}}_n)_{n\geq 1} \subset \overline{\mathcal{M}}_g^{\varphi}$ such that

$$\mathbb{E}^{\overline{\mathbb{Q}}_n}[f] \longrightarrow f_0 \quad \text{and} \quad \mathbb{E}^{\overline{\mathbb{Q}}_n}[\Phi] \longrightarrow \overline{\pi}^E_{(g,f)}(\Phi), \quad \text{as} \quad n \longrightarrow \infty. \tag{3.5.49}$$

Next, denote by $\overline{\pi}^E_g(f)$ the minimum superhedging cost of the European option f using S and $(g^1, \dots, g^e)$, i.e.,

$$\pi^E_g(f) = \overline{\pi}^E_g(f) = \inf\{x : \exists \overline{H} \in \overline{\mathcal{H}}, h \in \mathbb{R}^e, \text{ s.t. } x + (\overline{H} \circ S)_N + hg \geq f, \ \ \overline{\mathcal{P}}\text{-q.s.}\}.$$

As f is not replicable, by the second fundamental theorem in Theorem 5.1.(c) of [1], we have that $\overline{\mathbb{Q}} \mapsto \mathbb{E}^{\overline{\mathbb{Q}}}[f]$ is not constant on $\overline{\mathcal{M}}_g^{\varphi}$. Then, under the no-arbitrage condition, one has $0 = f_0 < \overline{\pi}^E_g(f)$. It follows that $0 = f_0 < \overline{\pi}^E_g(f) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g^{\varphi}} \mathbb{E}^{\overline{\mathbb{Q}}}[f]$. Thus there exists some $\overline{\mathbb{Q}}_+ \in \overline{\mathcal{M}}_g^{\varphi}$, such that $0 < \mathbb{E}^{\overline{\mathbb{Q}}_+}[f] < \overline{\pi}^E_g(f)$. With the same argument on $-f$, we can find another $\overline{\mathbb{Q}}_- \in \overline{\mathcal{M}}_g^{\varphi}$ such that

$$-\overline{\pi}^E_g(-f) < \mathbb{E}^{\overline{\mathbb{Q}}_-}[f] < f_0 < \mathbb{E}^{\overline{\mathbb{Q}}_+}[f] < \overline{\pi}^E_g(f).$$

Then one can choose an appropriate sequence of weights $(\lambda^n_-, \lambda^n_0, \lambda^n_+) \in \mathbb{R}^3_+$, such that $\lambda^n_- + \lambda^n_0 + \lambda^n_+ = 1$, $\lambda^n_\pm \to 0$,

$$\overline{\mathbb{Q}}'_n := \lambda^n_- \overline{\mathbb{Q}}_- + \lambda^n_0 \overline{\mathbb{Q}}_n + \lambda^n_+ \overline{\mathbb{Q}}_+ \in \overline{\mathcal{M}}_g, \quad \text{and} \quad \mathbb{E}^{\overline{\mathbb{Q}}'_n}[f] = f_0 = 0,$$

i.e., $\overline{\mathbb{Q}}'_n \in \overline{\mathcal{M}}^{\varphi}_{(g,f)}$. Moreover, as $\lambda^n_\pm \to 0$, it follows that $\mathbb{E}^{\overline{\mathbb{Q}}'_n}[\Phi] \to \overline{\pi}^E_{(g,f)}(\Phi)$, and we hence have the inequality (3.5.48).

It is enough to prove the claim (3.5.49), for which we suppose without loss of generality that $\overline{\pi}^E_{(g,f)}(\Phi) = 0$. Assuming that (3.5.49) fails, one then has

$$0 \notin \overline{\{\mathbb{E}^{\overline{\mathbb{Q}}}[(f, \Phi)] \ : \overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g^{\varphi}\}} \subseteq \mathbb{R}^2.$$

By the convexity of the above set and the separation argument, there exists $(y, z) \in \mathbb{R}^2$ with $|(y, z)| = 1$, such that

$$0 > \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_g^{\varphi}} \mathbb{E}^{\overline{\mathbb{Q}}}[yf + z\Phi] = \overline{\pi}^E_g(yf + z\Phi) \geq \overline{\pi}^E_{(g,f)}(z\Phi). \tag{3.5.50}$$

The strict inequality $\overline{\pi}^E_{(g,f)}(z\Phi) < 0$ implies that $z \neq 0$. Now, if $z > 0$, then we have $\overline{\pi}^E_{(g,f)}(\Phi) < 0$, which contradicts $\overline{\pi}^E_{(f,g)}(\Phi) = 0$. If $z < 0$, then by (3.5.50) one has $0 > \mathbb{E}^{\overline{\mathbb{Q}}'}[yf + z\Phi] = \mathbb{E}^{\overline{\mathbb{Q}}'}[z\Phi]$ for some $\overline{\mathbb{Q}}' \in \overline{\mathcal{M}}_{(g,f)} \subseteq \overline{\mathcal{M}}_g$, as $\overline{\mathcal{M}}_{(f,g)}$ is nonempty under the NA($\overline{\mathcal{P}}$) assumption in the case of $e + 1$ options. Then in the case $z < 0$, one has $\mathbb{E}^{\overline{\mathbb{Q}}'}[\Phi] > 0 = \overline{\pi}^E_{(f,g)}(\Phi)$, which contradicts the weak duality result (3.5.45), and we hence conclude the proof of the duality. □

3.5.3 Proof of Theorem 3.2.29.

(i) For the first equality in (3.2.35), we first notice that for every $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}$ and a regular conditional probability measure $(\widehat{\mathbb{Q}}_{\widehat{\omega}})$ of $\widehat{\mathbb{Q}}$ knowing $\widehat{\mathcal{G}}_k$, one has $\widehat{\mathbb{Q}}_{\widehat{\omega}} \in \widehat{\mathcal{M}}_k(\widehat{\omega})$. It then follows that $\sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}] \leq \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_0} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}]$. Next, let $\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_0$. By its definition, one has $\widehat{\mathbb{Q}} \ll\ll \widehat{\mathcal{P}}$ and (S, Y) is a generalised martingale and hence a local martingale under $\widehat{\mathbb{Q}}$ (see, e.g., Lemma A.1 of [1]). Using Lemma A.3 of [1], there is a $\widehat{\mathbb{Q}}' \sim \widehat{\mathbb{Q}}$ such that $\mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}] \leq \mathbb{E}^{\widehat{\mathbb{Q}}'}[\widehat{\xi}]$, and (S, Y) is a

$\widehat{\mathbb{Q}}'$-martingale. As $\widehat{\mathbb{Q}}' \sim \widehat{\mathbb{Q}} \lll \widehat{\mathcal{M}}$, then $\widehat{\mathbb{Q}}' \in \widehat{\mathcal{M}}$ and hence $\mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}] \leq \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}]$. Then it follows that $\sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_0} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}] \leq \sup_{\widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}} \mathbb{E}^{\widehat{\mathbb{Q}}}[\widehat{\xi}]$, and one obtains the first equality of (3.2.35).

(ii) Next, we claim that the graph set

$$[[\widehat{\mathcal{M}}_{k,k+1}]] := \big\{(\widehat{\omega}, \widehat{\mathbb{Q}}) \ : \widehat{\mathbb{Q}} \in \widehat{\mathcal{M}}_{k,k+1}(\widehat{\omega})\big\} \tag{3.5.51}$$

is analytic. Then using the (analytic) measurable projection theorem (see, e.g., Proposition 4.47 of [92]), $\widehat{\mathcal{E}}_k(\widehat{\xi})$ is upper semianalytic whenever $\widehat{\xi}$ is upper semianalytic. Further, the second equality in (3.2.35) is just a classical dynamic programming principle result, which follows by the measurable selection arguments as in [93] or [92].

It is enough to prove the claim (3.5.51), for which we notice that

$$[[\widehat{\mathcal{M}}_{k,k+1}]] \ = \ \big\{(\omega, y, \widehat{\mathbb{Q}}) \ : (\omega, \widehat{\mathbb{Q}}|_{\Omega_1}) \in [[\mathcal{M}_{k,k+1}]], \ \mathbb{E}^{\widehat{\mathbb{Q}}}[(\Delta S_{k+1}, \Delta Y_{k+1})] = 0\big\},$$

where the graph $[[\mathcal{M}_{k,k+1}]]$ is analytic by Lemma 4.8 of [1]. As $\widehat{\mathbb{Q}} \mapsto \mathbb{E}^{\widehat{\mathbb{Q}}}[(\Delta S_{k+1}, \Delta Y_{k+1})]$ is Borel measurable, $(\widehat{\omega}, \widehat{\mathbb{Q}}) \mapsto (\omega, \widehat{\mathbb{Q}}|_{\Omega_1})$ is continuous and hence is also Borel measurable. Then, by Proposition 7.40 of [92], $[[\widehat{\mathcal{M}}_{k,k+1}]]$ is also analytic. □

3.6 Proofs for Section 3.4

We finally complete here the proof of Theorem 3.3.2, which concerns the pricing-hedging duality in the martingale optimal transport problem setup. Recall that, in this setup, $\overline{\Omega} := \Omega \times \{1, \dots, N\}$, with $\Omega := \mathbb{R}^{d \times N}$, and $\mathcal{P}$ the collection of all Borel probability measures on Ω.

A first idea of how to prove Theorem 3.3.2 could be the following two step argument as in [18]. Firstly, under the condition that Φ is bounded from above and upper semicontinuous, one could prove that

$$\mathfrak{B}((\mathbb{R}^d)^M) \ni \mu \longmapsto \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] \in \mathbb{R}$$

is concave and upper semicontinuous, where we equip $\mathfrak{B}((\mathbb{R}^d)^M)$ with a Wasserstein type topology. Secondly, using the Fenchel–Moreau theorem, it follows that

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] \ = \ \overline{\pi}^E_{\mu,0}(\Phi) \ := \ \inf_{\lambda \in \Lambda} \Big\{\mu(\lambda) + \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi - \lambda\big]\Big\}. \tag{3.6.52}$$

Solving the maximization problem (3.6.52), by using Theorem 3.2.27, concludes the proof of Theorem 3.3.2.

However, in the following we will provide another proof based on an approximation argument. For simplicity, we suppose that $\mathbb{T}_0 = \{N\}$, where the same arguments work for more general $\mathbb{T}_0$. In preparation, let us provide a technical lemma. In the context of the martingale optimal transport problem, we introduce a sequence of payoff functions $(g^i)_{i \geq 1}$ by

$$g^i(\omega) := f^i(\omega_N) - c^i \quad \text{with} \ \ c^i := \int_{\mathbb{R}^d} f^i(x)\mu(dx),$$

where $f^i : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz and $(f^i)_{i \geq 1}$ is dense in the space of all Lipschitz functions on $\mathbb{R}^d$ under the uniform convergence topology, and moreover, it contains all functions of the form

$(x_j - K_n)^+$, $(-K_n - x_j)^+$ for $j = 1, \dots, d$ and $n \geq 1$, where $(K_n)_{n\geq 1} \subset \mathbb{R}$ is a sequence such that $K_n \to \infty$. Notice that μ has finite first order and hence the c^i are all finite constants.

Next, let us introduce an approximate dual problem by

$$\pi^A_{\mu,m}(\Phi) := \inf\Big\{x \ : \exists(\overline{H}, h) \in \overline{\mathcal{H}} \times \mathbb{R}^m \text{ s.t. for all } k \in \mathbb{T},\ \omega \in \Omega, \\ x + \sum_{i=1}^m h^i g^i(\omega_N) + (\overline{H}(\cdot, k) \circ S)_N(\omega) \geq \Phi_k(\omega)\Big\}.$$

Similarly,

$$\overline{\mathcal{M}}_{\mu,m} := \big\{\overline{\mathbb{Q}} \in \overline{\mathcal{M}} \ : \mathbb{E}^{\overline{\mathbb{Q}}}[g^i] = 0 \ \text{ for } i = 1, \dots, m\big\},$$

and

$$P_{\mu,m} := \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{M}}_{\mu,m}} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big].$$

Lemma 3.6.1. *Let $(\overline{\mathbb{Q}}_m)_{m\geqslant 1} \subset \overline{\mathcal{M}}$ be a sequence of martingale measures such that $\overline{\mathbb{Q}}_m \in \overline{\mathcal{M}}_{\mu,m}$ for each $m \geq 1$. Then,*

(i) *$(\overline{\mathbb{Q}}_m)_{m\geqslant 1}$ is relatively compact under the weak convergence topology.*

(ii) *The sequence $(S_N, \overline{\mathbb{Q}}_m)_{m\geq 1}$ is uniformly integrable, and any accumulation point of $(\overline{\mathbb{Q}}_m)_{m\geqslant 1}$ belongs to $\mathcal{M}_\mu$.*

Proof. (i) Without loss of generality, we assume that $f_1(x) = \sum_{i=1}^d |x_i|$ so that

$$\sup_{m\geq 1} \mathbb{E}^{\overline{\mathbb{Q}}_m}\left[\sum_{i=1}^d |S^i_N|\right] < \int_{\mathbb{R}^d} \sum_{i=1}^d |x_i| \mu(dx) < \infty.$$

Let us first prove the relative compactness of $(\overline{\mathbb{Q}}_m)_{m\geqslant 1}$. By the Prokhorov theorem, it is enough to find, for every $\varepsilon > 0$, a compact set $D_\varepsilon \subset \mathbb{R}^d$ such that $\overline{\mathbb{Q}}_m[S_k \notin D_\varepsilon] \leq \varepsilon$ for all $k = 1, \dots, N$. It is then enough to find, for every $\varepsilon > 0$, a constant $K_\varepsilon > 0$ such that $\overline{\mathbb{Q}}_m\big[|S^i_k| \geq K_\varepsilon\big] \leq \varepsilon$ for all $i = 1, \dots, d$ and $k = 1, \dots, N$. Next, by the martingale property, one has $\mathbb{E}^{\overline{\mathbb{Q}}_m}[|S^i_k|] \leq \mathbb{E}^{\overline{\mathbb{Q}}_m}[|S^i_N|]$. Then for every $\varepsilon > 0$, one can choose $K_\varepsilon > 0$ such that $\sup_{m\geq 1} \mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\sum_{i=1}^d |S^i_N|\big] \leq K_\varepsilon \varepsilon$. It follows that $\overline{\mathbb{Q}}_m\big[|S^i_k| \geq K_\varepsilon\big] \leq \frac{\mathbb{E}^{\overline{\mathbb{Q}}_m}[|S^i_k|]}{K_\varepsilon} \leq \varepsilon$, and hence that $(\overline{\mathbb{Q}}_m)_{m\geq 1}$ is relatively compact.

(ii) To see that the sequence $(S_N, \overline{\mathbb{Q}}_m)_{m\geq 1}$ is uniformly integrable, it is enough to notice that $|x_i|\mathbf{1}_{|x_i|\geq 2K_n} \leq 2(|x_i| - K_n)\mathbf{1}_{|x_i|\geq K_n}$, where the latter is a payoff function contained in the sequence $(f_k)_{k\geq 1}$.

(iii) Let $\overline{\mathbb{Q}}_0$ be an accumulation point of $(\overline{\mathbb{Q}}_m)_{m\geqslant 1}$, then one has $\mathbb{E}^{\overline{\mathbb{Q}}_0}[f^i(S_N)] = \mu(f^i) := \int_{\mathbb{R}^d} f^i(x)\mu(dx)$ for all $i \geq 1$. As $(f^i)_{i\geq 1}$ is supposed to be dense in the space of all Lipschtiz functions on $\mathbb{R}^d$ under the uniformly convergence topology, by dominated convergence, it follows that $\mathbb{E}^{\overline{\mathbb{Q}}_0}[f(S_N)] = \mu(f) := \int_{\mathbb{R}^d} f(x)\mu(dx)$ for all Lipschitz functions f. Therefore, one has $\overline{\mathbb{Q}}_0 \circ S_N^{-1} = \mu$.

(iv) To conclude the proof, it is enough to show that the martingale property is preserved for the limiting measure $\overline{\mathbb{Q}}_0$. By extracting a subsequence, we assume that $\overline{\mathbb{Q}}_m \to \overline{\mathbb{Q}}_0$ weakly.

First, for any $k = 1, \dots, N$, $i = 1, \dots, d$ and $K > 0$, one has

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{Q}}_0}[|S^i_k| \wedge K] &= \lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{Q}}_m}[|S^i_k| \wedge K] \leq \limsup_{m\to\infty} \mathbb{E}^{\overline{\mathbb{Q}}_m}[|S^i_k|] \\ &\leq \limsup_{m\to\infty} \mathbb{E}^{\overline{\mathbb{Q}}_m}[|S^i_N|] = \int_{\mathbb{R}^d} |x^i| \mu(dx) < \infty.\end{aligned}$$

Let $K \to \infty$, it follows by the monotone convergence theorem that $\mathbb{E}^{\overline{\mathbb{Q}}_0}[|S^i_k|] < \infty$.

Next, we prove that for all $1 \leqslant k_1 < k_2 \leqslant N$, and any bounded continuous function $\varphi : (\mathbb{R}^d)^{k_1} \times \mathbb{T} \to \mathbb{R}$, one has

$$\mathbb{E}^{\overline{\mathbb{Q}}_0}\big[\varphi\big(S_1, \dots, S_{k_1}, T \wedge (k_1+1)\big)(S_{k_2} - S_{k_1})\big] \;=\; 0. \tag{3.6.53}$$

Let $K > 0$, and $\chi_K : \mathbb{R}^d \to \mathbb{R}^d$ a continuous function, uniformly bounded by K, satisfying $\chi_K(x) = x$ when $\|x\| \leq K$, and $\chi_K(x) = 0$ when $\|x\| \geq K+1$. Then for every $m = 0$ or $m \geq 1$, one has

$$\begin{aligned}\big|\mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\varphi(S,T)(S_{k_2} - S_{k_1})\big]\big| \;\leq\; &\big|\mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\varphi(S,T)\big(\chi_K(S_{k_2}) - \chi_K(S_{k_1})\big)\big]\big| \\ &+ |\varphi|_\infty \mathbb{E}^{\overline{\mathbb{Q}}_m}\big[|S_{k_2}|\mathbf{1}_{|S_{k_2}|\geqslant K} + |S_{k_1}|\mathbf{1}_{|S_{k_1}|\geqslant K}\big],\end{aligned} \tag{3.6.54}$$

where we simplify $\varphi(S_1, \dots, S_{k_1}, T \wedge (k_1+1))$ to $\varphi(S,T)$.

For every $\varepsilon > 0$, by uniformly integrability of $(S_N, \overline{\mathbb{Q}}_m)_{m\geq 1}$, there is $K_\varepsilon > 0$ such that

$$|\varphi|_\infty \mathbb{E}^{\overline{\mathbb{Q}}_m}\big[|S_{k_2}|\mathbf{1}_{|S_{k_2}|\geqslant K_\varepsilon} + |S_{k_1}|\mathbf{1}_{|S_{k_1}|\geqslant K_\varepsilon}\big] \;\leq\; \varepsilon, \quad \text{for all } m = 0, 1, \dots \tag{3.6.55}$$

Moreover, for $m \geq 1$, $\overline{\mathbb{Q}}_m$ is a martingale measure, so that $\mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\varphi(S,T)(S_{k_2} - S_{k_1})\big] = 0$ and hence $\big|\mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\varphi(S,T)\big(\chi_K(S_{k_2}) - \chi_K(S_{k_1})\big)\big]\big| \leq \varepsilon$. Then, by taking the limit as $m \to \infty$, it follows that

$$\big|\mathbb{E}^{\overline{\mathbb{Q}}_0}\big[\varphi(S,T)\big(\chi_K(S_{k_2}) - \chi_K(S_{k_1})\big)\big]\big| \;\leq\; \varepsilon. \tag{3.6.56}$$

Combining (3.6.54), (3.6.55) and (3.6.56), and by the arbitrariness of $\varepsilon > 0$, it follows that (3.6.53) holds true for all bounded continuous functions φ.

Recall that $\overline{\mathcal{F}}_{k_1} = \sigma\big(S_1, \dots, S_{k_1}, T \wedge (k_1+1)\big)$, as defined at the beginning of Section 3.2.2, and observe that $\overline{\mathcal{F}}_{k_1} = \sigma(\mathcal{K})$, where

$$\mathcal{K} := \{\xi := \varphi\big(S_1, \dots, S_{k_1}, T \wedge (k_1+1)\big) \;: \varphi \text{ is bounded and continuous}\}.$$

It follows that $\mathcal{K}$ is included in the vector space of all bounded $\overline{\mathcal{F}}_{k_1}$-measurable random variables ξ for which $\mathbb{E}^{\overline{\mathbb{Q}}_0}[\xi(S_{k_2} - S_{k_1})] = 0$. An application of the monotone class theorem (see, e.g., Theorem I.8 of [99]) yields

$$\mathbb{E}^{\overline{\mathbb{Q}}_0}[\xi(S_{k_2} - S_{k_1})] = 0, \quad \text{for all bounded } \overline{\mathcal{F}}_{k_1}\text{-measurable random variables } \xi,$$

which is equivalent to S being a $\overline{\mathbb{Q}}_0$-martingale, and concludes the proof. □

Proof of Theorem 3.3.2. We notice that by Theorem 3.2.27,

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_{\mu,m}} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] = \pi^A_{\mu,m}(\Phi) \geq \pi^A_\mu(\Phi).$$

Let $(\overline{\mathbb{Q}}_m)_{m\geq 1}$ be a sequence of probability measures such that $\overline{\mathbb{Q}}_m \in \overline{\mathcal{M}}_{\mu,m}$ for each $m \geq 1$ and

$$\limsup_{m\to\infty} \mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\Phi_T(S)\big] \;=\; \limsup_{m\to\infty} \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_{\mu,m}} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big].$$

It follows by Lemma 3.6.1 that there exists some $\overline{\mathbb{Q}}_0 \in \overline{\mathcal{M}}_\mu$ and a subsequence such that $\overline{\mathbb{Q}}_{m_k} \to \overline{\mathbb{Q}}_0$ under the weak convergence topology. Using upper semicontinuity of Φ and by Fatou's lemma, it follows that $\mathbb{E}^{\overline{\mathbb{Q}}_0}\big[\Phi_T(S)\big] \geq \limsup_{m\to\infty} \mathbb{E}^{\overline{\mathbb{Q}}_m}\big[\Phi_T(S)\big]$. This leads to the inequality

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_\mu} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] \;\geq\; \mathbb{E}^{\overline{\mathbb{Q}}_0}\big[\Phi\big] \;\geq\; \limsup_{m\to\infty} \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{M}}_{\mu,m}} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\Phi\big] \;=\; \limsup_{m\to\infty} \pi^A_{\mu,m}(\Phi) \;\geq\; \pi^A_\mu(\Phi),$$

and we hence conclude the proof by the weak duality (3.5.45). □

Chapter 4

Super-replication with proportional transaction cost under model uncertainty

4.1 Introduction

Discrete time financial markets have been widely studied, and are now well understood. In the frictionless setting, the Fundamental Theorem of Asset Pricing and the traditional dual formulation of the set of contingent claims that can be super-hedged from a zero initial endowment are proved by first showing that this latter set is closed in probability and by then using a Hahn-Banach separation argument in $\mathbb{L}^1$, as in the celebrated Kreps–Yan theorem [100], see e.g. [68, 69, 101]. In the presence of proportional transaction costs, serveral notions of no-arbitrage properties can be considered, but they all aim at obtaining a similar closure property, so that Hahn-Banach separation arguments can still be applied, see [35, 44, 36] and [49] for a survey monograph.

In the context of model uncertainty, the market is defined with respect to a family $\mathcal{P}$ of (typically singular) probability measures. Closure properties can still be proved, in the quasi-sure sense, but no satisfactory $\mathbb{L}^p$-type duality argument can be used, because of the lack of a reference (dominating) probability measure. In the frictionless context, [1] suggested to use a one period argument *à la* Dalang–Morton–Willinger [13], and then to appeal to measurable selection techniques to paste the periods together. This is unfortunatly not possible (in general) when proportional transaction costs are present: local no-arbitrage is not equivalent to global no-arbitrage. Still, a quasi-sure versions of the classical *weak* and *strict* no-arbitrage conditions could be characterized in [42]. This requires the use of an intricate forward-backward construction. A quite similar construction was later used in Burzoni [47] for a version of the no-model independent arbitrage condition based on the robust no-arbitrage property of Schachermayer [36] .

Bouchard and Nutz [43] proposed to follow a simpler route and to use a quasi-sure version of the only no-abitrage condition for which local no-arbitrage and global no-arbitrage are equivalent. This notion was first suggested by [44] under the name of *no-arbitrage of the second kind*, or *no-sure gains in liquidation value*. Because of the equivalence between absence of local and absence of global arbitrage, they could use the same one period based arguments as in [1] to provide a quasi-sure version of the Fundamental Theorem of Asset Pricing in this context.

Unfortunately, they were no able to come up with an equaly easy proof of the super-hedging duality that seems to require a global argument. The difficulty comes from the fact that a portfolio is described by a vector valued process. At time t, one needs to define a vector position allowing to super-hedge the required time $t+1$-position. In frictionless markets, the time $t+1$-position reduces to a scalar valued random variable, the time $t+1$-value of the super-hedging price. Its computation can be done backward, by iterating on the time periods. In models with proportional transaction costs, there is an infinity of possible positions at time $t+1$, that are all minimal in the sense that none of them is dominated by another one, and that are enough to build up a super-hedging strategy. A backward induction does not tell which one is consistent with the global super-hedging strategy.

Burzoni [47] was able to solve this issue by constructing a fictitious price system in which the frictionless superhedging price is the same as in the original market. In his context, the super-hedging has to hold pointwise on the so-called *efficient support of the family of consistent price systems*. Also the market is multivariate, all transactions goes through the cash account (no direct exchanges between assets). It complements the work of Dolinsky and Soner [45] in which pointwise super-hedging is considered, and a static position on an arbitrary European option can be initially taken.

In this paper, we suggest to use a very simple randomization argument to tackle this problem in a general multivariate setting, under the quasi-sure no-arbitrage of second type condition of [43]. In particular, we allow for direct exchanges between the risky assets, which is not possible in [47, 45]. Technically, we add additional randomness to our initial probability space to construct a *fictitious* price process X that is consistent with the original bid-ask bounds. This additional

randomness controls the positions of prices within these bounds. We then consider the problem of quasi-sure super-hedging in the fictionless market with price process X and show that it matches with the super-hedging price in the original market with proportional transaction costs. This essentially follows from a minimax argument in the one period setting, that can be iterated in a backward way. Then, it suffices to apply the super-hedging duality of Bouchard and Nutz [1], and to project back all involved quantities on the original probability space.

Note that this randomization/enlargement technique is in fact in the same spirit of the *controlled fictitious market* approach of [50, 51], used in a dominated Markovian continuous time setting.

The rest of the paper is organized as follows. In Section 5.2, we first describe our discrete time market with proportional transaction costs, and introduce our randomization approach. We then specialize to the probabilistic setting suggested by Bouchard and Nutz [43] and link their quasi-sure no-arbitrage condition of second kind to a quasi-sure no-arbitrage condition set on our randomized frictionless market. In Section 4.3, we consider the super-replication problem and prove the duality, by using our randomization technique.

Notations. Given a measurable space $(\Omega, \mathcal{F})$, we denote by $\mathfrak{B}(\Omega, \mathcal{F})$ the set of all probability measures on $(\Omega, \mathcal{F})$. If Ω is a topological space, $\mathcal{B}(\Omega)$ denotes its Borel σ-field and we abbreviate the notation $\mathfrak{B}(\Omega) := \mathfrak{B}(\Omega, \mathcal{B}(\Omega))$. If Ω is a Polish space, a subset $A \subseteq \Omega$ is analytic if it is the image of a Borel subset of another Polish space under a Borel measurable mapping. A function $f : \Omega \to \overline{\mathbb{R}} := [-\infty, \infty]$ is upper semianalytic if $\{\omega \in \Omega \ : f(\omega) > a\}$ is analytic for all $a \in \mathbb{R}$. Given a probability measure $\mathbb{P} \in \mathfrak{B}(\Omega)$ and a measurable function $f : \Omega \to \overline{\mathbb{R}}$, we define the expectation

$$\mathbb{E}^{\mathbb{P}}[f] \ := \ \mathbb{E}^{\mathbb{P}}[f^+] - \mathbb{E}^{\mathbb{P}}[f^-], \quad \text{with the convention } \infty - \infty = -\infty.$$

For a family $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$ of probability measures, a subset $A \subset \Omega$ is called $\mathcal{P}$-polar if $A \subset A'$ for some universally measurable set A' satisfying $\mathbb{P}[A'] = 0$ for all $\mathbb{P} \in \mathcal{P}$, and a property is said to hold $\mathcal{P}$-quasi surely or $\mathcal{P}$-q.s if it holds true outside a $\mathcal{P}$-polar set. For $\mathbb{Q} \in \mathfrak{B}(\Omega)$, we write $\mathbb{Q} \lll \mathcal{P}$ if there exists $\mathbb{P}' \in \mathcal{P}$ such that $\mathbb{Q} \ll \mathbb{P}'$. Given a sigma algebra $\mathcal{G}$, we denote by $L^0(\mathcal{G})$ the collection of $\mathbb{R}^d$-valued random variable that are $\mathcal{G}$-measurable, d being given by the context. If we are given a measurable random set A and a family of probability measures $\mathcal{P}$, we denote by $L^0_{\mathcal{P}}(\mathcal{G}, A)$ the collection of $\mathcal{G}$-measurable random variables taking values in A $\mathcal{P}$-q.s.

4.2 A randomization approach for market with proportional transaction cost

We first introduce an abstract discrete-time market with proportional transaction cost, and show how to reduce to a fictitious market without transaction cost by using a randomization technique. Then, we specialize to the setting of Bouchard and Nutz [43] and discuss in particular how their quasi-sure version of the second kind no-arbitrage condition can be related to a no-arbitrage condition set on our enlarged fictitious market.

4.2.1 The financial market with proportional transaction cost

Let $(\Omega, \mathcal{F})$ be a measurable space, equipped with two filtrations $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t=0,1,\cdots,T} \subset \mathbb{F} = (\mathcal{F}_t)_{t=0,1,\cdots,T}$ for some $T \in \mathbb{N}$ (later on the first one will be the raw filtration, while the second one will be its universal completion). We fix a family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$, which

represents the model uncertainty. In particular, when $\mathcal{P}$ is a singleton, it reduces to the classical dominated market model framework.

Following [43], we specify our financial market with proportional transaction cost in terms of random cones. Let $d \geq 2$, for every $t \in \{0, 1, \cdots, T\}$, $K_t : \Omega \to 2^{\mathbb{R}^d}$ is a $\mathcal{F}_t^0$-measurable random set in the sense that $\{\omega \in \Omega : K_t(\omega) \cap O \neq \emptyset\} \in \mathcal{F}_t^0$ for every closed (open) set $O \subset \mathbb{R}^d$. Here, for each $\omega \in \Omega$, $K_t(\omega)$ is a closed convex cone containing $\mathbb{R}^d_+$, called the solvency cone at time t. It represents the collection of positions, labelled in units of different d financial assets, that can be turned into non-negative ones (component by component) by performing immediately exchanges between the assets. We denote by $K_t^* \subset \mathbb{R}^d_+$ its (nonnegative) dual cone:

$$K_t^*(\omega) \ := \ \big\{y \in \mathbb{R}^d \ : x \cdot y \geq 0 \text{ for all } x \in K_t(\omega)\big\}, \tag{4.2.1}$$

where $x \cdot y := \sum_{i=1}^d x^i y^i$ is the inner product on $\mathbb{R}^d$. For later use, let us also introduce

$$K_t^{*,0}(\omega) \ := \ \big\{y = (y^1, \cdots, y^d) \in K_t^*(\omega), \ y^d = 1\big\}.$$

As in [43], we assume the following conditions throughout the paper:

Assumption 4.2.1. *$K_t^* \cap \partial\mathbb{R}^d_+ = \{0\}$ and $\text{int}K_t^*(\omega) \neq \emptyset$ for every $\omega \in \Omega$ and $t \leq T$.*

It follows from the above assumption and [43, Lemma A.1] that K_t^*, $K_t^{*,0}$ and $\text{int}K_t^*$ are all $\mathcal{F}_t^0$-measurable. Moreover, there is a $\mathbb{F}^0$-adapted process S satisfying

$$S_t(\omega) \in K_t^{*,0}(\omega) \cap \text{int}K_t^*(\omega) \ \text{ for every } \omega \in \Omega, \, t \leq T. \tag{4.2.2}$$

We also assume that transaction costs are bounded and uniformly strictly positive. This is formulated in terms of S above.

Assumption 4.2.2. *There is some constant $c > 1$ such that*

$$c^{-1} S_t^i(\omega) \leq y^i \leq c S_t^i(\omega), \ \text{for every } i \leq d-1 \text{ and } y \in K_t^{*,0}(\omega).$$

Example 4.2.3. *Let us consider a market with one risky asset with mid price $S_t^1 > 0$ and one risk-free asset $S_t^2 \equiv 1$. Here $d = 2$. Because of a proportional transaction cost parametrized by $c \geq 1$, the bid price of the risky asset is given by $c^{-1}S_t^1$ and the ask price is cS_t^1. Then*

$$K_t(\omega) := \{x \in \mathbb{R}^2 : x^1 c^{-1} S_t^1(\omega) \mathbf{1}_{\{x^1 \geq 0\}} + x^1 c S_t^1(\omega) \mathbf{1}_{\{x^1 < 0\}} + x^2 \geq 0\},$$

$$K_t^*(\omega) = \big\{(y^1, y^2) \in \mathbb{R}^2_+ : y^1 \in \big[y^2 c^{-1} S_t^1(\omega), y^2 c S_t^1(\omega)\big]\big\},$$

and

$$K_t^{*,0}(\omega) = \big\{(y^1, 1) \in \mathbb{R}^2_+ : y^1 \in \big[c^{-1} S_t^1(\omega), c S_t^1(\omega)\big]\big\}.$$

Although there is, in the above example, a risk-free asset $S_t^2 \equiv 1$ which serves as a numéraire, this is not required in general. We refer to [43] for an example with d risky assets. See also the monograph [49].

Let us now turn to the definition of admissible trading strategies.

Definition 4.2.4. *We say that an $\mathbb{F}$-adapted process $\eta = (\eta_t)_{0 \leq t \leq T}$ is an admissible trading strategy if*

$$\eta_t \in -K_t \quad \mathcal{P}\text{-q.s. for all } t \leq T.$$

We denote by $\mathcal{A}$ the collection of all admissible strategies.

The constraint $\eta_t \in -K_t$ means that $0 - \eta_t \in K_t$, i.e., starting at t with 0, one can perform immediate transfers to reach the position η_t. Then, given $\eta \in \mathcal{A}$, the corresponding wealth process associated to a zero initial endowment at time 0 is $\big(\sum_{s=0}^{t} \eta_s\big)_{t\leq T}$.

Example 4.2.5. *In the context of Example 4.2.3, $\eta \in \mathcal{A}$ if and only if*

$$\eta_t^1 c^{-1} S_t^1 \mathbf{1}_{\{\eta_t^1 \leq 0\}} + \eta_t^1 c S_t^1 \mathbf{1}_{\{\eta_t^1 > 0\}} + \eta_t^2 \leq 0 \quad \mathcal{P}\text{-q.s. for all } t \leq T.$$

4.2.2 The randomization approach

As explained in the introduction, we aim at considering a frictionless market set on an enlarged probability space, that is equivalent (in a certain sense) to our original market. This will be used later on to apply results that are already known in the frictionless setting.

Let us therefore first introduce an enlarged space. Let $c > 1$ be the constant in Assumption 4.2.2, we define $\Lambda_1 := [c^{-1}, c]^{d-1}$, $\Lambda_t := (\Lambda_1)^{t+1}$, and $\Lambda := \Lambda_T$, and then introduce the canonical process $\Theta_t(\theta) := \theta_t$, $\forall \theta = (\theta_t)_{0\leq t\leq T} \in \Lambda$, as well as the σ-fields $\mathcal{F}_t^\Lambda := \sigma(\Theta_s,\ s \leq t)$, $t \leq T$. We next introduce an enlarged space $\overline{\Omega} := \Omega \times \Lambda$, an enlarged σ-field $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}_T^\Lambda$, together with three filtrations $\overline{\mathbb{F}}^0 = (\overline{\mathcal{F}}_t^0)_{0\leq t\leq T}$, $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{0\leq t\leq T}$ and $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\leq t\leq T}$ in which $\overline{\mathcal{F}}_t^0 := \mathcal{F}_t^0 \otimes \mathcal{F}_t^\Lambda$, $\tilde{\mathcal{F}}_t := \mathcal{F}_t \otimes \{\emptyset, \Lambda\}$ and $\overline{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}_t^\Lambda$ for $t \leq T$.

Then, we define our randomized fictitious market by letting the fictitious stock price $X = (X_t)_{0\leq t\leq T}$ be defined by

$$X_t(\bar{\omega}) := \Pi_{K_t^{*,0}(\omega)}[S_t(\omega)\theta_t], \quad \text{for all } \bar{\omega} = (\omega, \theta) \in \overline{\Omega},\ t \leq T, \tag{4.2.3}$$

where $S_t(\omega)\theta_t := (S_t^1(\omega)\theta_t^1, \cdots, S_t^{d-1}(\omega)\theta_t^{d-1}, S_t^d(\omega))$, and $\Pi_{K_t^{*,0}(\omega)}[y]$ stands for the projection of $y \in \mathbb{R}^d$ on the convex closed set $K_t^{*,0}(\omega)$. Recall that $S_t \in K_t^{*,0}$ for $t \leq T$. Finally, we introduce

$$\overline{\mathcal{P}} \ := \ \big\{\overline{\mathbb{P}} \in \mathfrak{B}(\overline{\Omega}, \overline{\mathcal{F}}) \text{ such that } \overline{\mathbb{P}}|_\Omega \in \mathcal{P}\big\}.$$

Lemma 4.2.6. *The process X is $\overline{\mathbb{F}}^0$-adapted.*

Proof. Notice that $K_t^{*,0}$ is $\mathcal{F}_t^0$-measurable. By the Castaing representation of $K_t^{*,0}$ (see e.g. [43, Lemma A.1]), there exists a sequence of $\mathcal{F}_t^0$-measurable functions $(\psi_n)_{n\geq 1}$ such that $K_t^{*,0}(\omega) = \overline{\{\psi_n(\omega), n \geq 1\}}$ for all $\omega \in \Omega$.

Next, let us define a sequence of random variables $(X_t^n(\bar{\omega}))_{n\geq 1}$ by $X_t^1(\bar{\omega}) := \psi_1(\omega)$ and $X_t^{n+1}(\bar{\omega}) := X_t^n(\bar{\omega})\mathbf{1}_{A_n} + \psi_{n+1}(\omega)\mathbf{1}_{A_n^c}$, where $A_n := \{\|X_t^n - S_t(\omega)\theta_t\| < \|\psi_{n+1} - S_t(\omega)\theta_t\|\}$. It is clear that X_t^n are all $\overline{\mathcal{F}}_t^0$-measurable. Then, it is enough to notice that $X_t = \lim_{n\to\infty} X_t^n$ to conclude that X_t is $\overline{\mathcal{F}}_t^0$-measurable. □

Remark 4.2.7. *We shall use several times the following important property related to the structure of $\overline{\mathcal{P}}$. Let $Y : (\omega, \theta) \in \overline{\Omega} \mapsto Y(\omega, \theta) \in \mathbb{R}$ be a random variable. Then, the following are equivalent:*

(i) $Y \geq 0$ $\overline{\mathcal{P}}$*-q.s*

(ii) $Y(\cdot, \theta) \geq 0$ *for all* $\theta \in \Lambda$ $\mathcal{P}$*-q.s.*

The fact that (ii) *implies* (i) *is clear. As for the reverse implication, we observe that if $\mathbb{P} \in \mathcal{P}$ is such that $\mathbb{P}[\inf_{\theta\in\Lambda} Y(\cdot, \theta) < -c] > 0$ for some $c > 0$, then one can find a Borel map $\omega \in \Omega \mapsto \theta(\omega) \in \Lambda$ such that $\mathbb{P}[Y(\cdot, \theta(\cdot)) < -c/2] > 0$, see [92, Lemma 7.27 and Proposition 7.49]. Since*

$\mathbb{P} \otimes \delta_{\theta(\cdot)} \in \overline{\mathcal{P}}$, *this contradicts* (i).
Moreover, if $\theta \mapsto Y(\cdot, \theta)$ *is upper semicontinuous, then* (i), (ii) *are equivalent to*

(iii) $Y(\cdot, \theta) \geq 0$ $\mathcal{P}$*-q.s. for all* $\theta \in \Lambda$.

If one keeps the filtration $\mathbb{F}$ (or equivalently $\tilde{\mathbb{F}}$) to define admissible (or self-financing) trading strategies on this fictitious financial market, then they coincide with the admissible strategies in the sense of Definition 5.2.3 above. More precisely, we have the following.

Theorem 4.2.8. (i) *Fix* $t \leq T$ *and* $\zeta_t \in L^0(\mathcal{F}_t)$. *Then*

$$\zeta_t \in -K_t \ \mathcal{P}\text{-q.s.} \quad \text{if and only if} \quad \zeta_t \cdot X_t \leq 0 \ \overline{\mathcal{P}}\text{-q.s.}$$

(ii) *Consequently, an* $\mathbb{F}$*-adapted process* η *is an admissible strategy in the sense of Definition 5.2.3 if and only if* $\eta_t \cdot X_t \leq 0$ $\overline{\mathcal{P}}$*-q.s. for all* $t \leq T$.

Proof. The assertion (ii) is an immediate consequence of (i). To show that (i) holds, let us first note that, by definition,

$$x \in -K_t(\omega) \iff x \cdot y \leq 0 \ \forall y \in K_t^{*,0}(\omega). \tag{4.2.4}$$

(a) First assume that $\zeta_t \in -K_t, \mathcal{P}$-q.s. and fix $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$ together with $\mathbb{P} := \overline{\mathbb{P}}|_\Omega$. Then, $\zeta_t(\omega) \in -K_t(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Recalling (4.2.4), this implies that $\zeta_t(\omega) \cdot X_t(\omega, \theta) \leq 0$, for every $\theta \in \Lambda$ and $\mathbb{P}$-a.e. $\omega \in \Omega$. Hence, $\zeta_t \cdot X_t \leq 0$ $\overline{\mathbb{P}}$-a.s. by Remark 4.2.7. By arbitrariness of $\overline{\mathbb{P}}$, the later holds $\overline{\mathcal{P}}$-q.s.

(b) We now assume that $\zeta_t \cdot X_t \leq 0$ $\overline{\mathcal{P}}$-q.s. Let $\mathbb{P} \in \mathcal{P}$, then for every $\theta \in \Lambda$, one has $\mathbb{P} \otimes \delta_\theta \in \overline{\mathcal{P}}$. Let $\Lambda_\circ \subset \Lambda$ be a countable dense subset, it follows that, for $\mathbb{P}$-a.e. ω and every $y \in \left\{ \prod_{K_t^{*,0}(\omega)} [S_t(\omega)\theta_t] \ : \theta \in \Lambda_\circ \right\}$, one has $\zeta_t(\omega) \cdot y \leq 0$. By continuity of the inner product and Assumption 4.2.2, we then have $\zeta_t \cdot y \leq 0$ for all $y \in K_t^{*,0}$ $\mathbb{P}$-a.s. The measure $\mathbb{P} \in \mathcal{P}$ being arbitrary chosen, we then deduce from (4.2.4) that $\zeta_t \in -K_t, \mathcal{P}$-q.s. □

4.2.3 Equivalence of the no-arbitrage conditions under the framework of Bouchard & Nutz [43]

For the newly introduced fictitious market under model uncertainty, a first issue is to formulate a no-arbitrage condition. Let us now specialize to the framework of Bouchard & Nutz [43]. In this probabilistic framework, we show that the quasi-sure no-arbitrage condition of second kind used in [43] is equivalent to the quasi-sure no-arbitrage condition of [1] on the frictionless market defined on $\overline{\Omega}$ with stock price process X and $\overline{\mathbb{F}}$-predictable strategies.

No-arbitrage condition under Bouchard & Nutz's [43] framework We first recall the framework of Bouchard & Nutz [43]. Let $\Omega_0 = \{\omega_0\}$ be a singleton and Ω_1 be a Polish space. For each $t \in \{1, \cdots, T\}$, we denote by $\Omega_t := \Omega_0 \times \Omega_1^t$, where Ω_1^t denotes the t-fold Cartesian product of Ω_1, we set $\mathcal{F}_t^0 := \mathcal{B}(\Omega_t)$ and let $\mathcal{F}_t$ be its universal completion. In particular, the σ-field $\mathcal{F}_0^0$ and $\mathcal{F}_0$ are trivial. From now on,

$$\Omega := \Omega_T, \ \ \mathcal{F} := \mathcal{F}_T, \ \ \mathbb{F}^0 := (\mathcal{F}_t^0)_{0 \leq t \leq T}, \ \ \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}.$$

Given $t \in \{0, \cdots, T-1\}$ and $\omega \in \Omega_t$, we are given a non-empty convex set $\mathcal{P}_t(\omega) \subseteq \mathfrak{B}(\Omega_1)$, which represents the set of possible models for the $(t+1)$-th period, given state ω at time t. We

assume that for each t

$$\text{graph}(\mathcal{P}_t) := \{(\omega, \mathbb{P}) : \omega \in \Omega_t, \mathbb{P} \in \mathcal{P}_t(\omega)\} \subseteq \Omega_t \times \mathcal{P}(\Omega_1) \text{ is analytic.} \tag{4.2.5}$$

Given probability kernels $\mathbb{P}_t : \Omega_t \mapsto \mathfrak{B}(\Omega_1)$, for each $t \leq T-1$, we define a probability measure $\mathbb{P}$ on Ω by Fubini's theorem:

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2 \cdots, \omega_T) \mathbb{P}_{T-1}(\omega_1, \cdots, \omega_{T-1}; d\omega_T) \cdots \mathbb{P}_0(d\omega_1)$$

We can then introduce the set $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$ of possible models for the multi-period market up to time T:

$$\mathcal{P} := \big\{\mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \ : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot) \text{ for } t \leq T-1\big\}. \tag{4.2.6}$$

Notice that the condition (5.2.2) ensures that $\mathcal{P}_t$ admits a universally measurable selector: $\mathbb{P}_t : \Omega_t \to \mathcal{P}(\Omega_1)$ such that $\mathbb{P}_t(\omega) \in \mathcal{P}_t(\omega)$ for all $\omega \in \Omega_t$. Then the set $\mathcal{P}$ defined in (5.2.1) is nonempty.

Let us now recall the no-arbitrage condition used in [43].

Definition 4.2.9. *We say that* NA2($\mathcal{P}$) *holds if for all* $t \leq T-1$ *and all* $\zeta \in L^0(\mathcal{F}_t)$,

$$\zeta \in K_{t+1} \ \ \mathcal{P}\text{-q.s.} \quad \textit{implies} \quad \zeta \in K_t \ \ \mathcal{P}\text{-q.s.}$$

The following robust version of the fundamental theorem has been proved in [43].

Theorem 4.2.10. *The condition* NA2($\mathcal{P}$) *is equivalent to : For all* $t \leq T-1$, $\mathbb{P} \in \mathcal{P}$ *and* $Y \in L^0_{\mathcal{P}}(\mathcal{F}_t, \text{int}K_t^*)$, *there exists* $\mathbb{Q} \in \mathfrak{B}(\Omega)$ *and a* $\mathbb{F}^0$*-adapted process* $(Z_s)_{s=t,\ldots,T}$ *such that* $\mathbb{P} \ll \mathbb{Q}$ *and* $\mathbb{P} = \mathbb{Q}$ *on* $\mathcal{F}_t$, *and*
(i) $\mathbb{Q} \lll \mathcal{P}$
(ii) $Y = Z_t \ \ \mathbb{Q}$*-a.s.*
(iii) $Z_s \in \text{int}K_s^* \ \ \mathbb{Q}$*-a.s. for* $s = t, \ldots, T$
(iv) $(Z_s)_{s=t,\ldots,T}$ *is a* $\mathbb{Q}$*-martingale, i.e.* $\mathbb{E}^{\mathbb{Q}}[Z_{s'}|\mathcal{F}_s] = Z_s$ *for* $t \leq s \leq s'$.

A couple $(\mathbb{Q}, Z)$ satisfying the conditions (i) $-$ (iv) above for $t = 0$ is called a strictly consistent price system (SCPS). For later use, let $\mathcal{S}$ denote the collection of all SCPS, and set

$$\mathcal{S}_0 := \big\{(\mathbb{Q}, Z) \in \mathcal{S} \text{ such that } Z^d \equiv 1\big\}. \tag{4.2.7}$$

For later use, we also recall the notion of NA2(t, ω) for each $t \leq T$ and $\omega \in \Omega_t$: we say NA2(t, ω) holds true if

$$\zeta \in K_{t+1}(\omega, \cdot) \ \mathcal{P}_t(\omega)\text{-q.s.} \quad \text{implies} \ \ \zeta \in K_t(\omega), \quad \text{for all } \zeta \in \mathbb{R}^d. \tag{4.2.8}$$

The following result is proved in [43, Lemma 3.6].

Lemma 4.2.11. *The set* $N_t := \{\omega : \text{NA2}(t, \omega) \textit{ fails}\}$ *is universally measurable. Moreover,* N_t *is a* $\mathcal{P}$*-polar set if* NA2($\mathcal{P}$) *holds.*

No-arbitrage condition on the enlarged space We next consider the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}})$ and define a subset of probability measures $\overline{\mathcal{P}}_{\text{int}} \subset \overline{\mathcal{P}}$, in order to introduce the quasi-sure no-

arbitrage condition of [1] w.r.t. the price process X and the set of strategies

$$\overline{\mathcal{H}} := \{\text{All } \overline{\mathbb{F}}\text{-predictable processes}\}.$$

Given $t \leq T$, we denote $\overline{\Omega}_0 := \Omega_0 \times \Lambda_1$ and $\overline{\Omega}_t := \overline{\Omega}_0 \times (\Omega_1 \times \Lambda_1)^t$, so that one has $\overline{\Omega} := \Omega \times \Lambda = \overline{\Omega}_T$. We write $(\omega, \theta) \in \overline{\Omega}$ in the form $\omega = (\omega_0, \dots, \omega_T)$ and $\theta = (\theta_0, \dots, \theta_T)$. For $t \leq T$, we use the notations $\omega^t := (\omega_0, \dots, \omega_t)$, $\theta^t = (\theta_0, \dots, \theta^t)$ and $\bar{\omega}^t = (\omega^t, \theta^t)$. We now introduce the subset $\overline{\mathcal{P}}_{\text{int}} \subset \overline{\mathcal{P}} := \{\overline{\mathbb{P}} \in \mathfrak{B}(\overline{\Omega}) \ : \overline{\mathbb{P}}|_\Omega \in \mathcal{P}\}$ defined as follows:

- For $t = 0, 1, \cdots, T-1$ and $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}_t$, define $\overline{\mathcal{P}}_t(\bar{\omega}) := \big\{\overline{\mathbb{P}} \in \mathfrak{B}(\Omega_1 \times \Lambda_1) \ : \overline{\mathbb{P}}|_{\Omega_1} \in \mathcal{P}_t(\omega)\big\}$, and

$$\overline{\mathcal{P}}_t^{\text{int}}(\bar{\omega}) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_t(\bar{\omega}) \ : (\delta_{\bar{\omega}^t} \otimes \overline{\mathbb{P}})[X_{t+1} \in \text{int}K^*_{t+1}] = 1\big\}, \tag{4.2.9}$$

  where $\delta_{\bar{\omega}^t} \otimes \overline{\mathbb{P}}$ is a probability measure on $\overline{\Omega}_{t+1} = \overline{\Omega}_t \times (\Omega_1 \times \Lambda_1)$ and X_{t+1} (defined in (5.2.6)) is considered as a random variable defined on $\overline{\Omega}_{t+1}$.

- Let $\overline{\mathcal{P}}_\emptyset^{\text{int}}$ be the collection of all probability measures $\overline{\mathbb{P}}$ on $\overline{\Omega}_0$ such that $\overline{\mathbb{P}}[X_0 \in \text{int}K^*_0] = 1$, and define

$$\overline{\mathcal{P}}_{\text{int}} \ := \ \big\{\overline{\mathbb{P}}_\emptyset \otimes \overline{\mathbb{P}}_0 \otimes \cdots \otimes \overline{\mathbb{P}}_{T-1} \ : \overline{\mathbb{P}}_\emptyset \in \overline{\mathcal{P}}_\emptyset^{\text{int}} \text{ and } \overline{\mathbb{P}}_t(\cdot) \in \overline{\mathcal{P}}_t^{\text{int}}(\cdot) \text{ for } t \leq T-1\big\},$$

  where $\overline{\mathbb{P}}_t(\cdot)$ is a universally measurable selector of $\overline{\mathcal{P}}_t^{\text{int}}(\cdot)$, whose existence is ensured by Lemma 4.2.13 below.

By a slight abuse of notations, we shall later write $\overline{\mathcal{P}}_t(\bar{\omega})$ for $\overline{\mathcal{P}}_t(\bar{\omega}^t)$ when $\bar{\omega} \in \overline{\Omega}$. The same convention will be used for $\overline{\mathcal{P}}_t^{\text{int}}(\bar{\omega})$, etc.

Remark 4.2.12. *s Note that the equivalence observed in Remark 4.2.7 still holds for the above construction. In particular, let $Y : (\omega, \theta) \in \overline{\Omega} \mapsto Y(\omega, \theta)$ be a random variable. Then, the following are equivalent:*

(i) $Y \geq 0$ $\overline{\mathcal{P}}_{\text{int}}$*-q.s*

(ii) $Y(\omega, \theta) \geq 0$ *for all* $\theta \in \Lambda^{\text{int}}(\omega) := \{\theta' : S_t(\omega)\theta'_t \in \text{int}K^*_t(\omega), \ \forall t \leq T\}$*, for all* ω *outside a* $\mathcal{P}$*-polar set.*

Lemma 4.2.13. *For every $0 \leq t \leq T-1$, the set*

$$\text{graph}\big(\overline{\mathcal{P}}_t^{\text{int}}\big) := \big\{(\bar{\omega}, \overline{\mathbb{P}}) : \bar{\omega} \in \overline{\Omega}_t, \overline{\mathbb{P}} \in \overline{\mathcal{P}}_t^{\text{int}}(\bar{\omega})\big\} \text{ is analytic.}$$

Proof. We only consider the case $t \geq 1$, the proof for the case $t = 0$ is an obvious modification.

(i) Since $\text{graph}(\mathcal{P}_t)$ is an analytic set, there is some Polish space E and a Borel set $A \subset \Omega_t \times \mathfrak{B}(\Omega_1) \times E$ such that $\text{graph}(\mathcal{P}_t)$ is the projection set of A on $\Omega_t \times \mathfrak{B}(\Omega_1)$, i.e. $\text{graph}\big(\mathcal{P}_t\big) = \Pi_{\Omega_t \times \mathfrak{B}(\Omega_1)}\big[A\big]$. Let us define

$$\overline{A} \ := \ \big\{\big(\bar{\omega}, \overline{\mathbb{P}}, e\big) \in \overline{\Omega}_t \times \mathfrak{B}(\Omega_1 \times \Lambda_1) \times E \ : \big(\omega, \overline{\mathbb{P}}|_{\Omega_1}, e\big) \in A\big\},$$

which is a Borel set in $\overline{\Omega}_t \times \mathfrak{B}(\Omega_1 \times \Lambda_1) \times E$ since $\bar{\omega} = (\omega, \theta) \mapsto \omega$ and $\overline{\mathbb{P}} \mapsto \overline{\mathbb{P}}|_{\Omega_1}$ are Borel. Then $\text{graph}(\overline{\mathcal{P}}_t) = \Pi_{\overline{\Omega}_t \times \mathfrak{B}(\Omega_1 \times \Lambda_1)}\big[\ \overline{A}\ \big]$ is an analytic set.

(ii) Let $t \geq 0$ and $\gamma : \overline{\Omega}_{t+1} \to \mathbb{R}$ be a bounded measurable function, then $(\omega^t, \theta^t, \overline{\mathbb{P}}) \mapsto \mathbb{E}^{\overline{\mathbb{P}}}[\gamma(\omega^t, \theta^t, \cdot)]$ is a Borel measurable map from $\overline{\Omega}_t \times \mathfrak{B}(\Omega_1 \times \Lambda_1)$ to $\mathbb{R}$, see e.g. **[NutzVanHandel.12]**. Next, by [43, Lemma A.1], we know that

$$B := \big\{(\omega^{t+1}, \theta^{t+1}, x) \in \overline{\Omega}_{t+1} \times \mathbb{R}^d \ : x \in \mathrm{int}K^*_{t+1}(\omega^{t+1})\big\}$$

is Borel measurable. Then

$$\begin{aligned}(\omega^t, \theta^t, \overline{\mathbb{P}}) \quad &\mapsto \quad \big(\delta_{(\omega^t, \theta^t)} \otimes \overline{\mathbb{P}}\big)\big[X_{t+1} \in \mathrm{int}K^*_{t+1}\big] \\ &= \int_{\Omega_1 \times \Lambda_1} \mathbf{1}_{\{(\omega^t, \omega_{t+1}, \theta^t, \theta_{t+1}, X_{t+1}(\omega^t, \cdot)) \in B\}} d\overline{\mathbb{P}}((\omega_{t+1}, \theta_{t+1}))\end{aligned}$$

is also Borel measurable. It follows that

$$C := \Big\{\big(\bar{\omega}^t, \overline{\mathbb{P}}\big) \in \overline{\Omega}_t \times \mathfrak{B}(\Omega_1 \times \Lambda_1) \ : \big(\delta_{\bar{\omega}^t} \otimes \overline{\mathbb{P}}\big)\big[X_{t+1} \in \mathrm{int}K^*_{t+1}\big] = 1\Big\}$$

is a Borel set, and hence $\mathrm{graph}\big(\overline{\mathcal{P}}^{\mathrm{int}}_t\big) = C \cap \mathrm{graph}(\overline{\mathcal{P}}_t)$ is an analytic set. □

Definition 4.2.14. *We say that* $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *holds if*

$$(\overline{H} \circ X)_T \geq 0, \ \overline{\mathcal{P}}_{\mathrm{int}}\text{-}q.s. \quad \Longrightarrow \quad (\overline{H} \circ X)_T = 0, \ \overline{\mathcal{P}}_{\mathrm{int}}\text{-}q.s.,$$

for every $\overline{H} \in \overline{\mathcal{H}}$.

Using [1] and Lemma 4.2.13 above, it follows that the following fundamental theorem of asset pricing holds.

Theorem 4.2.15. *The condition* $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *is equivalent to : For all* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\mathrm{int}}$*, there exists* $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$ *such that* $\overline{\mathbb{P}} \ll \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\mathrm{int}}$ *and* X *is a* $(\overline{\mathbb{Q}}, \overline{\mathbb{F}})$*-martingale.*

Hereafter, we denote by $\overline{\mathcal{Q}}_0$ the collection of measures $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$ such that $\overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\mathrm{int}}$ and X is a $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale.

The main result of this section says that the two no-arbitrage conditions defined above are equivalent.

Proposition 4.2.16. *The conditions* $\mathrm{NA2}(\mathcal{P})$ *and* $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *are equivalent.*

Proof. (i) Let us first suppose that $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ holds. Assume that $\mathrm{NA2}(\mathcal{P})$ does not hold. Then, for some $t \leq T-1$, there is $\zeta \in L^0(\mathcal{F}_t)$ such that $\zeta \in K_{t+1}$ $\mathcal{P}$-q.s. and $\mathbb{P}[A] > 0$ for some $\mathbb{P} \in \mathcal{P}$ and

$$A \ := \ \big\{\omega \ : \zeta(\omega) \notin K_t(\omega)\big\} \ = \ \big\{\omega \ : \exists y \in K^{*,0}_t(\omega) \cap \mathrm{int}K^*_t(\omega) \text{ s.t. } y \cdot \zeta(\omega) < 0\big\}.$$

Notice that $\mathrm{graph}(K^{*,0}_t \cap \mathrm{int}K^*_t)$ is Borel measurable. Hence,

$$B := \big\{(\omega, \delta) \in \Omega_t \times \Lambda_1 \ : S_t(\omega)\delta \in K^{*,0}_t(\omega) \cap \mathrm{int}K^*_t(\omega) \text{ and } (S_t(\omega)\delta) \cdot \zeta(\omega) < 0\big\}$$

is also Borel measurable. Then $A = \Pi_{\Omega_t}[B]$ is analytic. Using Jankov-von Neumann's measurable selection theorem (see e.g. [43, Lemma A.2]), there is a universally measurable $\rho_t : A \to \Lambda_1$ such that $(\omega, \rho_t(\omega)) \in B$ and hence $S_t(\omega)\rho_t(\omega) \in K^{*,0}_t(\omega) \cap \mathrm{int}K^*_t(\omega)$ and $(S_t(\omega)\rho_t(\omega)) \cdot \zeta(\omega) < 0$, for all $\omega \in A$. We then extend ρ_t on Ω_t by setting $\rho_t = 1$ on A^c. Then, $Y_t(\omega) := S_t(\omega)\rho_t(\omega) \in K^{*,0}_t(\omega) \cap \mathrm{int}K^*_t(\omega)$ for all $\omega \in \Omega_t$, and $A \subset \{Y_t \cdot \zeta < 0\}$. Set $\bar{\zeta} := \zeta \mathbf{1}_{\{\zeta \cdot X_t \leq 0\}}$, so that

$\bar{\zeta}\cdot\big(X_{t+1}-X_t\big)\geq 0$ $\overline{\mathcal{P}}_{\rm int}$-q.s. However, using Assumption 4.2.2, (5.2.4) and the fact that S does not depend on $\theta\in\Lambda$, and $A\in\mathcal{F}$, it follows that there is some $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}$ such that $\overline{\mathbb{P}}[(X_t,X_{t+1})=(Y_t,S_{t+1})]=1$ and $\overline{\mathbb{P}}[\bar{\zeta}\cdot(X_{t+1}-X_t)>0]\geq\mathbb{P}[A]>0$. This contradicts NA($\overline{\mathcal{P}}_{\rm int}$).

(ii) Conversely, assume that NA2($\mathcal{P}$) holds, we aim at proving that NA($\overline{\mathcal{P}}_{\rm int}$) holds. In view of Theorem 4.2.15, it is enough to prove that, for every $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}$, there is a $\overline{\mathbb{Q}}\ll\!\ll\overline{\mathcal{P}}_{\rm int}$ such that $\overline{\mathbb{P}}\ll\overline{\mathbb{Q}}$ and X is a $(\overline{\mathbb{F}},\overline{\mathbb{Q}})$-martingale.

Fix $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}$, then, by the definition of $\overline{\mathcal{P}}_{\rm int}$, one has the representation:

$$\overline{\mathbb{P}}\ :=\ \overline{\mathbb{P}}_\emptyset\otimes(\mathbb{P}_0\otimes q_0)\otimes\cdots\otimes(\mathbb{P}_{T-1}\otimes q_{T-1}),$$

where $\mathbb{P}_t:\overline{\Omega}_t\to\mathcal{P}(\Omega_1)$ and $q_t\big(d\theta_{t+1}|\bar{\omega}^t,\omega_{t+1}\big):\overline{\Omega}_t\times\Omega_1\to\mathcal{P}(\Lambda_1)$ are all Borel kernels such that $\mathbb{P}_t=(\mathbb{P}_t(\cdot|\bar{\omega}^t))_{\bar{\omega}^t\in\overline{\Omega}_t}$ satisfies $\mathbb{P}_t(\cdot|\bar{\omega}^t)\in\mathcal{P}(t,\omega^t)$ and the support of $q_t(\cdot|\bar{\omega}^t,\omega_{t+1})$ is contained in $\Lambda^{\rm int}_{t+1}(\omega^{t+1}):=\{\theta_{t+1}\in\Lambda_1\ :S_{t+1}(\omega^{t+1})\theta_{t+1}\in{\rm int}K^*_{t+1}(\omega^{t+1})\}$.

We next construct another kernel q'_t in order to define a martingale measure $\overline{\mathbb{Q}}$ dominating $\overline{\mathbb{P}}$. For every $t\leq T-1$, we consider a Borel kernel $(\mathbb{P}_t(\cdot|\bar{\omega}^t))_{\bar{\omega}^t\in\overline{\Omega}_t}$. First, we observe that $\{\omega:{\rm NA2}(t,\omega)\ \text{fails}\}\times\Lambda$ is $\mathcal{P}$-polar by Lemma 4.2.11. Then given $\bar{\omega}^t=(\omega^t,\theta^t)$ such that ω^t is outside this polar set, we can consider a 1 period market on $\{t,t+1\}$, with measure uncertainty $\mathcal{P}_t(\omega^t)$ on Ω_1. It follows from [43, Proposition 3.1] that given $\bar{\omega}^t$, and therefore $X_t(\bar{\omega}^t)$ and $\mathbb{P}_t(\cdot|\bar{\omega}^t)$, there exists $(Z_{t+1}(\bar{\omega}^t,\cdot),\mathbb{Q}_t(\cdot|\bar{\omega}^t),\mathbb{P}'_t(\cdot|\bar{\omega}^t))$ such that $\mathbb{P}'_t(\cdot|\bar{\omega}^t)\in\mathcal{P}_t(\omega^t)$, $Z_{t+1}(\bar{\omega}^t,\omega_{t+1})\in({\rm int}K^*_{t+1}\cap K^{*,0}_{t+1})(\omega^t,\omega_{t+1})$ for all $\omega_{t+1}\in\Omega_1$, and

$$\mathbb{P}_t(\cdot|\bar{\omega}^t)\ll\mathbb{Q}_t(\cdot|\bar{\omega}^t)\ll\mathbb{P}'_t(\cdot|\bar{\omega}^t),\ \ \text{and}\ \ \mathbb{E}^{\mathbb{Q}_t(\cdot|\bar{\omega}^t)}[Z_{t+1}]=S_t(\omega^t)\theta_t=X_t(\bar{\omega}^t).\tag{4.2.10}$$

Moreover, by [43, Lemma 3.8] together with a measurable selection argument (see e.g. [43, Lemma A.2]), we can ensure that

$$(\bar{\omega}^t,\omega_{t+1})\in\overline{\Omega}_t\times\Omega_1\ \mapsto\ \big(Z_{t+1}(\bar{\omega}^t,\omega_{t+1}),\mathbb{Q}_t(\cdot|\bar{\omega}^t),\mathbb{P}'_t(\cdot|\bar{\omega}^t)\big)$$

is measurable w.r.t. the universally completed Borel σ-field on $\overline{\Omega}_t\times\Omega_1$.

We next consider the family $\big(q_t(\cdot|\bar{\omega}^t,\omega_{t+1})\big)_{\bar{\omega}^t\in\overline{\Omega}_t,\omega_{t+1}\in\Omega_1}$. By [1, Lemma 4.7], the set of all $(\bar{\omega}^t,\omega_{t+1},\alpha,q'(d\theta_{t+1}))$ satisfying

$$q'[\Lambda^{\rm int}_{t+1}(\omega^{t+1})]=1,\ \ q'\gg q_t(\cdot|\bar{\omega}^t,\omega_{t+1})\ \ \text{and}\ \mathbb{E}^{q'}\big[X_{t+1}(\omega^{t+1},\cdot)\big]=\alpha,$$

is a Borel set. It is not difficult to see that this set is non-empty. Then, by a standard measurable selection argument, there is a universally measurable family $q'_t(\cdot|\bar{\omega}^t,\omega_{t+1})$ with support in $\Lambda^{\rm int}_{t+1}(\omega^{t+1})$ such that

$$q_t(\cdot|\bar{\omega}^t,\omega_{t+1})\ll q'_t(\cdot|\bar{\omega}^t,\omega_{t+1})\quad\text{and}\ \ \mathbb{E}^{q'_t(\cdot|\bar{\omega}^t,\omega_{t+1})}[X_{t+1}(\omega^{t+1},\cdot)]=Z_{t+1}(\bar{\omega}^t,\omega_{t+1}).$$

Let us finally define

$$\overline{\mathbb{Q}}:=\overline{\mathbb{P}}_\emptyset\otimes\big(\mathbb{Q}_0\otimes q'_0\big)\otimes\cdots\otimes\big(\mathbb{Q}_{T-1}\otimes q'_{T-1}\big)$$

and

$$\overline{\mathbb{P}}':=\overline{\mathbb{P}}_\emptyset\otimes\big(\mathbb{P}'_0\otimes q'_0\big)\otimes\cdots\otimes\big(\mathbb{P}'_{T-1}\otimes q'_{T-1}\big).$$

Then it is easy to check that

$$\overline{\mathbb{P}} \ll \overline{\mathbb{Q}} \ll \overline{\mathbb{P}}', \ \ \overline{\mathbb{P}}' \in \overline{\mathcal{P}}, \ \ \overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0,$$

and we hence conclude the proof. □

Remark 4.2.17. *Let us define* $\Lambda_0^{\rm int}(\omega_0) := \{\theta_0 \in \Lambda_1 : S_0(\omega_0)\theta_0 \in {\rm int}K_0^*\}$*, and for each* $\theta_0 \in \Lambda_0^{\rm int}(\omega_0)$*,*

$$\overline{\mathcal{P}}_0^{{\rm int},\delta}(\theta_0) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int} \ : \overline{\mathbb{P}}[\Theta_0 = \theta_0] = 1\big\}, \quad \overline{\mathcal{Q}}_0^{\delta}(\theta_0) := \big\{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0 \ : \overline{\mathbb{Q}}[\Theta_0 = \theta_0] = 1\big\}.$$

Define ${\rm NA}(\overline{\mathcal{P}}_0^{{\rm int},\delta}(\theta_0))$ *as* ${\rm NA}(\overline{\mathcal{P}}_{\rm int})$ *with* $\overline{\mathcal{P}}_0^{{\rm int},\delta}(\theta_0)$ *in place of* $\overline{\mathcal{P}}_{\rm int}$*. Then,* ${\rm NA}(\overline{\mathcal{P}}_{\rm int})$ *implies that* ${\rm NA}(\overline{\mathcal{P}}_0^{{\rm int},\delta}(\theta_0))$ *holds for every* $\theta_0 \in \Lambda_0^{\rm int}(\omega_0)$*. Indeed, assume that* ${\rm NA}(\overline{\mathcal{P}}_{\rm int})$ *holds. Then, Theorem 4.2.15 applied to* $\overline{\mathcal{P}}_{\rm int}$ *implies that, for any* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$ *such that* $\overline{\mathbb{P}}[\Theta_0 = \theta_0] = 1$*, one can find* $\overline{\mathbb{P}} \ll \overline{\mathbb{Q}}(\theta_0)$ *such that* X *is a* $\overline{\mathbb{Q}}(\theta_0)$*-martingale.*

4.3 A robust pricing-hedging duality result

We now concentrate on the super-replication problem under the framework of [43]. Given $e \in \mathbb{N} \cup \{0\}$, we are given a random vector $\xi : \Omega \to \mathbb{R}^d$ as well as $\zeta_i : \Omega \to \mathbb{R}^d$ for $i = 1, \cdots, e$ such that $\zeta_i \not\equiv 0$. The random vectors ξ and ζ_i represents the final payoffs, in number of units of each risky assets, of respectively an exotic option and vanilla options. We assume that the bid and ask prices of each vanilla option ζ_i are respectively $-c_i$ and c_i for some constant $c_i \geq 0$, this symmetry is without loss of generality. Then, the minimal super-hedging cost of the exotic option ξ using vanilla options ζ_i together with dynamic trading strategy is given by[1]:

$$\pi_e(\xi) := \inf\Big\{y + \sum_{i=1}^e c_i|\ell_i| \ : y1_d + \sum_{i=1}^e \ell_i\zeta_i + \sum_{t=0}^T \eta_t - \xi \in K_T, \ \mathcal{P}\text{-q.s.}, (\eta,\ell) \in \mathcal{A} \times \mathbb{R}^e\Big\}, \quad (4.3.11)$$

where 1_d is the vector will all components equal to 0 but the last one that is equal to 1. Let us introduce the subset of the set of SCPS (recall (5.2.12)) that are compatible with the bid-ask speads of the vanilla options used for static hedging:

$$\mathcal{S}_e := \big\{(\mathbb{Q}, Z) \in \mathcal{S}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[\zeta_i \cdot Z_T\big] \in [-c_i, c_i], \ i = 1, \cdots, e\big\}.$$

Then, we have the following super-hedging duality.

Theorem 4.3.1. *Let* ξ *and* $(\zeta_i)_{i\leq e}$ *be Borel measurable, and assume that* ${\rm NA2}(\mathcal{P})$ *holds true. Assume either that* $e = 0$*, or that* $e \geq 1$ *and for all* $\ell \in \mathbb{R}^e$ *and* $\eta \in \mathcal{A}$*,*

$$\sum_{i=1}^e \big(\ell_i\zeta_i - |\ell_i|c_i1_d\big) + \sum_{t=0}^T \eta_t \in K_T \ \mathcal{P}\text{-q.s.} \implies \ell = 0. \quad (4.3.12)$$

Then $\mathcal{S}_e$ *is nonempty and*

$$\pi_e(\xi) \ = \ \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e} \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big]. \quad (4.3.13)$$

[1]Here we use the convention $\sum_{i=1}^0 = 0$

Moreover, there exists $(\hat{\eta}, \hat{\ell}) \in \mathcal{A} \times \mathbb{R}^e$ *such that*

$$\pi_e(\xi)1_d + \sum_{i=1}^{e} \Big(\hat{\ell}_i \zeta_i - |\hat{\ell}_i| c_i 1_d \Big) + \sum_{t=0}^{T} \hat{\eta}_t - \xi \in K_T, \ \mathcal{P}\text{-q.s.}$$

The proof is provided in the subsequent sections. We start with the case $e = 0$.

4.3.1 Proof of Theorem 4.3.1: case $e = 0$

4.3.2 Reformulation of the super-hedging problem

Before providing the proof of Theorem 4.3.1, we first reformulate the optimization problem (4.3.13) and the super-hedging problem (4.3.11), under proportional transaction cost, in terms of the fictitious market defined on our enlarged space.

We start with the pricing problem. Let us define

$$\overline{\mathcal{Q}}_0 := \big\{ \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\rm int} \ : X \text{ is a } (\overline{\mathbb{F}}, \overline{\mathbb{Q}})\text{-martingale s.t. } X_t \in {\rm int} K_t^* \ \forall t \le T \ \overline{\mathbb{Q}}\text{-a.s.} \big\}$$

and

$$\overline{\mathcal{Q}}_0^{\rm loc} := \big\{ \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\rm int} \ : X \text{ is a } (\overline{\mathbb{F}}, \overline{\mathbb{Q}})\text{-local martingale s.t. } X_t \in {\rm int} K_t^* \ \forall t \le T \ \overline{\mathbb{Q}}\text{-a.s.} \big\}.$$

Proposition 4.3.2. *For any universally measurable vector* $\xi : \Omega \to \mathbb{R}^d$*, one has*

$$\sup_{(\mathbb{Q}, Z) \in \mathcal{S}_0} \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] \ = \ \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big].$$

Proof. (i) First, let $(\mathbb{Q}, Z) \in \mathcal{S}_0$ be a consistent price system, where $\mathbb{Q} \ll \mathbb{P}$ for some $\mathbb{P} \in \mathcal{P}$ and Z is a $\mathbb{Q}$-martingale s.t. $Z_t \in {\rm int} K_t^*$, $\mathbb{Q}$-a.s. for $t \le T$. Then, for every $t \le T$, there is a $\mathcal{F}_t^0$-measurable map $\rho_t : \Omega_t \to \Lambda_1$ such that $X_t(\omega, \rho_t(\omega)) = S_t(\omega)\rho_t(\omega) = Z_t(\omega)$ for $\mathbb{Q}$-a.e. $\omega \in \Omega$. Let us define $\overline{\mathbb{P}}$ and $\overline{\mathbb{Q}}$ by

$$\mathbb{E}^{\overline{\mathbb{P}}}[f(\omega, \theta)] := \mathbb{E}^{\mathbb{P}}\big[f\big((\omega_t, \rho_t(\omega))_{t \le T}\big)\big] \text{ and } \mathbb{E}^{\overline{\mathbb{Q}}}[f(\omega, \theta)] := \mathbb{E}^{\mathbb{Q}}\big[f\big((\omega_t, \rho_t(\omega))_{t \le T}\big)\big],$$

for all (bounded) $\overline{\mathcal{F}}_T$-measurable $f : \overline{\Omega} \to \mathbb{R}$. Then, it is clear that $\overline{\mathbb{Q}} \ll \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$. Further, given a bounded $\overline{\mathcal{F}}_t$-measurable r.v. Y_t, it follows from the definition of $\overline{\mathbb{Q}}$ that

$$\begin{aligned} \mathbb{E}^{\overline{\mathbb{Q}}}[(X_{t+1} - X_t)Y_t] \ &= \ \mathbb{E}^{\mathbb{Q}}\big[\big(X_{t+1}(\omega, \rho_{t+1}(\omega)) - X_t(\omega, \rho_t(\omega))\big)Y_t(\omega, \rho_t(\omega))\big] \\ &= \ \mathbb{E}^{\mathbb{Q}}[(Z_{t+1}(\omega) - Z_t(\omega))Y_t(\omega, \rho_t(\omega))\big] \ = \ 0, \end{aligned}$$

since Z is a $\mathbb{Q}$-martingale. It follows that X is a $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale and hence $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0$. Moreover, one has $\mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big] = \mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big]$.

Conversely, given $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0$, let us define $\mathbb{Q} := \overline{\mathbb{Q}}|_\Omega$ and $Z_t := \mathbb{E}^{\overline{\mathbb{Q}}}[X_t|\mathcal{F}_t]$ for $t \le T$. Since $\overline{\mathbb{Q}} \ll \overline{\mathbb{P}}$ for some $\overline{\mathbb{P}} \in \overline{\mathcal{P}}$, then $\mathbb{Q} \ll \mathbb{P} := \overline{\mathbb{P}}|_\Omega \in \mathcal{P}$. Moreover, the fact that X is a $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale implies that Z is $(\mathbb{F}, \mathbb{Q})$-martingale. Then, $(\mathbb{Q}, Z)$ is a strictly consistent price system, and $\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] = \mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big]$. □

We next reformulate the super-hedging problem (4.3.11) on the enlarged space. Let us define

$$g(\bar{\omega}) := \xi(\omega) \cdot X_T(\bar{\omega}), \text{ for all } \bar{\omega} = (\omega, \theta) \in \overline{\Omega},$$

as the contingent claim. Denote by $\mathcal{H}$ the collection of all $\tilde{\mathbb{F}}$-predictable processes and let $(H \circ X)_t := \sum_{s=1}^t H_s \cdot (X_s - X_{s-1})$, $t \le T$, be the wealth process associated to $H \in \mathcal{H}$.

Proposition 4.3.3. *One has*

$$\begin{aligned}\pi_0(\xi) &= \inf\Big\{y \in \mathbb{R} \ : y + (H \circ X)_T \ge g \ \overline{\mathcal{P}}\text{-q.s., for some } H \in \mathcal{H}\Big\} \\ &= \inf\Big\{y \in \mathbb{R} \ : y + (H \circ X)_T \ge g \ \overline{\mathcal{P}}_{\rm int}\text{-q.s., for some } H \in \mathcal{H}\Big\}.\end{aligned}$$

Proof. For ease of notations, we write $\Delta X_t := X_t - X_{t-1}$.
(i) Let $(y, \eta) \in \mathbb{R} \times \mathcal{A}$ be such that and $y1_d + \sum_{t=0}^T \eta_t - \xi \in K_T$ $\mathcal{P}$-q.s. Define the $\tilde{\mathbb{F}}$-predictable process H by $H_t := \sum_{s=1}^t \Delta H_s$ with $\Delta H_t := \eta_{t-1}$, for $t = 1, \cdots, T$. By exactly the same arguments as in part (i) of the proof of Theorem 4.2.8, this is equivalent to

$$\begin{aligned}0 &\le \Big(y1_d + \sum_{t=0}^T \eta_t - \xi\Big) \cdot X_T \\ &= y + \sum_{t=0}^T \eta_t \cdot (X_T - X_t) + \sum_{t=0}^T \eta_t \cdot X_t - g \\ &= y + \sum_{t=1}^T H_t \cdot \Delta X_t + \sum_{t=0}^T \eta_t \cdot X_t - g \quad \overline{\mathcal{P}}\text{-q.s.}, \end{aligned} \tag{4.3.14}$$

where the last equivalence follows by direct computation using that $X_t^d \equiv 1$. Since $\eta_t \cdot X_t \le 0$ $\overline{\mathcal{P}}$-q.s., by Theorem 4.2.8, we deduce that $y + (H \circ X)_T \ge g$ $\overline{\mathcal{P}}$-q.s. This shows that

$$\pi_0(\xi) \ \ge \ \inf\big\{y \in \mathbb{R} \ : y + (H \circ X)_T \ge g \ \overline{\mathcal{P}}\text{-q.s. for some } H \in \mathcal{H}\big\}.$$

(ii) We next prove the converse inequality. Let $(y, H) \in \mathbb{R} \times \mathcal{H}$ be such that $y + (H \circ X)_T \ge g$ $\overline{\mathcal{P}}$-q.s. We use the convention $H_0 = 0$. Set $\eta_t^i := \Delta H_{t+1}^i$ for all $i = 1, \cdots, d-1$ and $t \le T-1$, and $\eta_T := 0$. We next define η_t^d for $t = 0, \cdots, T-1$ by

$$\eta_t^d(\omega) := \min_{\theta \in \Lambda} m_t^d(\omega, \theta) \ \text{ with } m_t^d(\bar\omega) := -\sum_{i=1}^{d-1} \eta_t^i(\omega) X_t^i(\bar\omega), \tag{4.3.15}$$

for all $\bar\omega = (\omega, \theta) \in \overline{\Omega}$. Notice that $m_t^d(\omega, \theta)$ is bounded continuous in θ, then, η_t^d is $\mathcal{F}_t$-measurable, by the Measurable Maximum Theorem (see e.g. Theorem 18.19 of [102]). Using its construction, one has $\eta \in \mathcal{A}$. Moreover, it follows from the choice of (y, H) and the fact that $\mathbb{P} \times \delta_\theta \in \overline{\mathcal{P}}$ for all $\mathbb{P} \in \mathcal{P}$ that

$$\begin{aligned}0 &\le \inf_{\theta \in \Lambda} \big(y + (H \circ X)_T - g\big)(\cdot, \theta) \\ &= \inf_{\theta \in \Lambda} \Big(y + \sum_{t=0}^T \eta_t \cdot (X_T - X_t) - \xi \cdot X_T\Big)(\cdot, \theta) \\ &= \inf_{\theta \in \Lambda} \Big(\big(y1_d + \sum_{t=0}^T \eta_t - \xi\big) \cdot X_T - \sum_{t=0}^{T-1} \eta_t \cdot X_t\Big)(\cdot, \theta) \ \mathcal{P}\text{-q.s.}, \end{aligned} \tag{4.3.16}$$

recall that $\eta_T = 0$ by its construction above (4.3.15). We now use the fact that each X_t depends

on θ only through θ_t to obtain

$$0 \;\leq\; \inf_{\theta\in\Lambda}\big\{(y1_d+\sum_{t=0}^{T}\eta_t-\xi)\cdot X_T\big\}(\cdot,\theta)-\sum_{t=0}^{T-1}\sup_{\theta\in\Lambda}\big\{\eta_t\cdot X_t\big\}(\cdot,\theta).$$

Further, by the definition of η^d in (4.3.15), it follows that

$$\sup_{\theta\in\Lambda}\{\eta_t\cdot X_t\}(\omega,\theta) \;=\; \eta_t^d(\omega)+\sup_{\theta\in\Lambda}\sum_{i=1}^{d-1}\eta_t^i(\omega)X_t^i(\omega,\theta) \;=\; 0.$$

Therefore,

$$0 \;\leq\; \inf_{\theta\in\Lambda}\big\{(y1_d+\sum_{t=0}^{T}\eta_t-\xi)\cdot X_T\big\}(\cdot,\theta)\ \mathcal{P}\text{-q.s.}$$

The latter is equivalent to $y1_d+\sum_{t=0}^{T}\eta_t-\xi\in K_T$, $\mathcal{P}$-q.s. This shows that

$$\pi_0(\xi) \;\leq\; \inf\big\{y\in\mathbb{R}\ : y+(H\circ X)_T\geq g\ \overline{\mathcal{P}}\text{-q.s., for some } H\in\mathcal{H}\big\}.$$

(iii) Let us now prove that

$$\pi_0(\xi)=\inf\big\{y\in\mathbb{R}\ : y+(H\circ X)_T\geq g\ \overline{\mathcal{P}}_{\rm int}\text{-q.s., for some } H\in\mathcal{H}\big\}.$$

Since $\overline{\mathcal{P}}_{\rm int}\subset\overline{\mathcal{P}}$, one inequality follows from (i) – (ii) above. As for the converse one, let $(y,H)\in\mathbb{R}\times\mathcal{H}$ be such that $y+(H\circ X)_T\geq g$ $\overline{\mathcal{P}}_{\rm int}$-q.s. and define η as in (4.3.15). Observe that the right-hand side term of (4.3.16) is equal to

$$\inf_{\theta\in\Lambda^{\rm int}(\cdot)}\big(y+(H\circ X)_T-g\big)(\cdot,\theta)$$

$\mathcal{P}$-q.s., in which, for $\omega\in\Omega$, $\Lambda^{\rm int}(\omega)$ is defined as the collection of $\theta\in\Lambda$ such that $S_t(\omega)\theta_t\in{\rm int}K_t^*(\omega)$ for all $t\leq T$.

Next, to each $\theta\in\Lambda$, we associate the probability kernels

$$q_s^\theta:\omega\in\Omega\mapsto q_s^\theta(\cdot|\omega):=\delta_{\theta_s}\mathbf{1}_{A_s^\theta(\omega)}+\delta_{\mathbf{1}}\mathbf{1}_{(A_s^\theta(\omega))^c}\in\mathfrak{B}(\Lambda_1),\ s\leq T, \tag{4.3.17}$$

where $\mathbf{1}$ is the vector of $\mathbb{R}^d$ with all entries equal to 1, $A_s^\theta(\omega):=\emptyset$ for $s\neq t$ and $A_t^\theta(\omega):=\{S_t(\omega)\theta_t\in{\rm int}K_t^*(\omega)\}$. It follows that $\mathbb{P}\otimes(q_0^\theta\otimes q_1^\theta\otimes\cdots\otimes q_T^\theta)\in\overline{\mathcal{P}}_{\rm int}$ for every $\mathbb{P}\in\mathcal{P}$. Then it suffices to argue as in (ii) above to obtain that

$$0\leq\inf_{\theta\in\Lambda^{\rm int}(\cdot)}\big(y+(H\circ X)_T-g\big)(\cdot,\theta)=\inf_{\theta\in\Lambda}\big(y+(H\circ X)_T-g\big)(\cdot,\theta)\ \mathcal{P}\text{-q.s.}$$

which implies that

$$0\leq\inf_{\theta\in\Lambda}\big\{(y1_d+\sum_{t=0}^{T}\eta_t-\xi)\cdot X_T\big\}(\cdot,\theta)\ \mathcal{P}\text{-q.s.},$$

and we hence conclude as in step (ii). □

Remark 4.3.4. *Notice that the proof of the first equality in Proposition 5.2.6 does not depend on any special structure conditions on Ω as in the framework of [43]. In other words, it holds*

still true for an abstract space $(\Omega, \mathcal{F})$ with an arbitrary family of probability measures $\mathcal{P}$.

Remark 4.3.5. *Let us observe that the reformulations in Proposition 5.2.6 on the enlarged space do not exactly correspond to standard quasi-sure super-hedging problem. Indeed, we still restrict the class of strategies to $\tilde{\mathbb{F}}$-predictable processes, as opposed to $\overline{\mathbb{F}}$-predictable processes. The fact that the formulation with these two different filtrations are equivalent will be proved by using a minimax argument in the next section.*

4.3.3 Proof of Theorem 4.3.1, case $e = 0$

In view of Propositions 4.3.2 and 5.2.6, Theorem 4.3.1 will be proved if one can show that, with $g := \xi \cdot X_T$:

$$\inf\Big\{y \in \mathbb{R} \ : y + (H \circ X)_T \geq g \ \overline{\mathcal{P}}_{\rm int}\text{-q.s., for some } H \in \mathcal{H}\Big\} \ = \ \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g\big].$$

Let us start with a weak duality result, which is an immediate consequence of [1, Lemmas A.2 and A.3].

Lemma 4.3.6. *For any universally measurable variables $g : \overline{\Omega} \to \mathbb{R}$, one has*

$$\begin{aligned} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g\big] \ &= \ \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0^{\rm loc}} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g\big] \\ \leq \ \inf\Big\{y \in \mathbb{R} \ &: y + (H \circ X)_T \geq g \ \overline{\mathcal{P}}_{\rm int}\text{-}q.s., \text{ for some } H \in \mathcal{H}\Big\}. \end{aligned}$$

We prove the converse inequality in the rest of this section. Let us proceed by induction, by first considering the one period case $T = 1$. Recall that $\Lambda_0^{\rm int}(\omega_0)$, $\overline{\mathcal{P}}_0^{{\rm int},\delta}(\theta_0)$ and $\overline{\mathcal{Q}}_0^{\delta}(\theta_0)$ are defined in Remark 4.2.17.

Lemma 4.3.7. *Let $e = 0$, $T = 1$ and $g_1 : \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ be upper semi-analytic and such that $(\omega, \theta_0, \theta_1) \in \Omega \times \Lambda_1 \times \Lambda_1 \to g_1(\omega, \theta_0, \theta_1)$ depends only on (ω, θ_1). Assume that $\mathrm{NA}(\overline{\mathcal{P}}_{\rm int})$ holds true. Then,*

$$\begin{aligned} \sup_{\theta_0 \in \Lambda_0^{\rm int}(\omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0^{\delta}(\theta_0)} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_1\big] &= \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_1\big] \\ &= \inf\big\{y \in \mathbb{R} \ : y + (H \circ X)_T \geq g_1, \ \overline{\mathcal{P}}_{\rm int}\text{-}q.s., \ H \in \mathcal{H}\big\} \\ &> -\infty. \end{aligned} \tag{4.3.18}$$

Proof. First, notice that $\mathcal{H} = \mathbb{R}^d$ when $T = 1$, and that $(g_1, X_1)(\omega, \theta_0, \theta_1)$ are independent of θ_0. Then, for all $\theta_0 \in \Lambda_0^{\rm int}(\omega_0)$,

$$\big\{\overline{\mathbb{P}} \circ (g_1, X_1)^{-1} : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_0^{{\rm int},\delta}(\theta_0)\big\} \ = \ \big\{\overline{\mathbb{P}} \circ (g_1, X_1)^{-1} : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_0^{{\rm int},\delta}(\mathbf{1})\big\},$$

where $\mathbf{1}$ represents the vector of $\mathbb{R}^d$ with all entries equal to 1. Then

$$\begin{aligned}
&\inf\big\{y \ : y + (H \circ X)_1 \geq g_1, \ \overline{\mathcal{P}}_{\text{int}}\text{-q.s.}, \ H \in \mathcal{H}\big\} \\
&= \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda_0^{\text{int}}(\omega_0)} \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}_0^{\text{int},\delta}(\theta_0)} \mathbb{E}^{\overline{\mathbb{P}}}[g_1 - h_1 \cdot (X_1 - S_0\theta_0)] \\
&= \sup_{\theta_0 \in \Lambda_0^{\text{int}}(\omega_0)} \inf_{h_1 \in \mathbb{R}^d} \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}_0^{\text{int},\delta}(\theta_0)} \mathbb{E}^{\overline{\mathbb{P}}}[g_1 - h_1 \cdot (X_1 - S_0\theta_0)] \\
&= \sup_{\theta_0 \in \Lambda_0^{\text{int}}(\omega_0)} \inf\big\{y \ : y + H_1 \cdot (X_1 - X_0) \geq g_1, \ \overline{\mathcal{P}}_0^{\text{int},\delta}(\theta_0)\text{-q.s.}, \ H \in \mathcal{H}\big\}. \qquad (4.3.19)
\end{aligned}$$

In the above, the second equality follows by the minimax theorem since

$$(\theta_0, h_1) \mapsto \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}_0^{\text{int},\delta}(\theta_0)} \mathbb{E}^{\overline{\mathbb{P}}}[g_1 - h_1 \cdot (X_1 - S_0\theta_0)] = \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}_0^{\text{int},\delta}(\mathbf{1})} \mathbb{E}^{\overline{\mathbb{P}}}[g_1 - h_1 \cdot (X_1 - S_0\theta_0)]$$

is convex in h_1 and linear in θ_0, as $(g_1, X_1)(\omega, \theta_0, \theta_1)$ is independent of θ_0. Moreover, the infimum over h_1 is concave and therefore lower semicontinuous (in particular, one can replace $\Lambda_0^{\text{int}}(\omega_0)$ by its closure, that is a compact set, in all the above terms). Observe that $\overline{\mathcal{Q}}_0^{\delta}(\theta_0)$ is nonempty for every $\theta_0 \in \Lambda_0^{\text{int}}(\omega_0)$, recall Remark 4.2.17 and Theorem 4.2.15. Then, by the duality result in [1, Theorem 3.4], the right-hand side of (4.3.19) has a finite negative part and is equal to

$$\sup_{\theta_0 \in \Lambda_0^{\text{int}}(\omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0^{\delta}(\theta_0)} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_1\big] \ = \ \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_1\big],$$

where the last equality follows from the fact that $\mathcal{Q}_0^{\delta}(\theta_0) \subset \mathcal{Q}_0$ and that every probability measure $\overline{\mathbb{Q}}$ in $\overline{\mathcal{Q}}_0$ can be disintegrated into a combinaison of elements in $(\mathcal{Q}_0^{\delta}(\theta_0))_{\theta_0 \in \Lambda_0^{\text{int}}(\omega_0)}$. □

We now prepare for the general case $T \geq 1$, which is based on a dynamic programming argument. We extend the definitions of $\Lambda_0^{\text{int}}(\omega_0)$, $\overline{\mathcal{P}}_0^{\text{int},\delta}(\theta_0)$ and $\overline{\mathcal{Q}}_0^{\delta}(\theta_0)$, see Remark 4.2.17, to an arbitrary initial time t and initial path $\bar{\omega}^t$. For $t \geq 1$ and $\bar{\omega} = \bar{\omega}^t = (\omega^t, \theta^t) \in \overline{\Omega}_t$, we firt recall the definition of $\Lambda_t^{\text{int}}(\omega^t)$ that was already used in the proof of Proposition 5.2.11:

$$\Lambda_t^{\text{int}}(\omega^t) \ := \ \{\theta_t \in \Lambda_1 \ : S_t(\omega^t)\theta_t \in \text{int}K_t^*(\omega^t)\} \ \subset \ \Lambda_1.$$

Next, recall that $\overline{\mathcal{P}}_t^{\text{int}}(\bar{\omega}) \subset \mathfrak{B}(\Omega_1 \times \Lambda_1)$ is defined in (5.2.7). We define

$$\overline{\mathcal{P}}_t^{\text{int},\delta}(\bar{\omega}) := \big\{\delta_{\bar{\omega}^t} \otimes \overline{\mathbb{P}}_{t+1} \ : \overline{\mathbb{P}}_{t+1} \in \overline{\mathcal{P}}_t^{\text{int}}(\bar{\omega})\big\}$$

and

$$\overline{\mathcal{Q}}_t^{\delta}(\bar{\omega}) \ := \ \big\{\overline{\mathbb{Q}}_{t+1} \lll \overline{\mathcal{P}}_t^{\text{int},\delta}(\bar{\omega}) \ : \mathbb{E}^{\overline{\mathbb{Q}}_{t+1}}[X_{t+1} - X_t] = 0\big\}.$$

as well as

$$\widetilde{\mathcal{P}}_t^{\text{int},\delta}(\omega) := \big\{(\delta_{\omega^t} \times \mu(d\theta^t)) \otimes \overline{\mathbb{P}}_{t+1} \ : \overline{\mathbb{P}}_{t+1} \in \overline{\mathcal{P}}_t^{\text{int}}(\bar{\omega}), \ \mu \in \mathfrak{B}\big(\Lambda_0^{\text{int}}(\omega_0) \times \cdots \times \Lambda_t^{\text{int}}(\omega^t)\big)\big\},$$

a version of $\overline{\mathcal{P}}_t^{\text{int},\delta}$ in which θ^t is not fixed anymore. Let $g_{t+1} : \overline{\Omega}_{t+1} \to \mathbb{R} \cup \{\infty\}$ be an upper semi-analytic functional and be such that $g_{t+1}(\omega^{t+1}, \theta_0, \cdots, \theta_{t+1})$ depends only on $(\omega^{t+1}, \theta_{t+1})$.

We define

$$g_t(\bar\omega^t) \;:=\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_t^\delta(\bar\omega)} \mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}]. \tag{4.3.20}$$

Remark 4.3.8. *Let $\bar\omega=(\omega,\theta)$ and $\bar\omega'=(\omega',\theta')$ be such that $\omega^t=(\omega')^t$ and $\theta_t=\theta'_t$. Then, it follows from the definition of $\overline{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega)$ and $\overline{\mathcal{Q}}_t^\delta(\bar\omega)$ that*

$$\{\overline{\mathbb{Q}}\circ(g_{t+1},X_t,X_{t+1})^{-1}\;:\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_t^\delta(\bar\omega)\}=\{\overline{\mathbb{Q}}\circ(g_{t+1},X_t,X_{t+1})^{-1}\;:\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_t^\delta(\bar\omega')\}.$$

Hence, $g_t(\bar\omega^t)$ depends only on (ω^t,θ_t) for $\bar\omega^t=(\omega^t,\theta_0,\cdots,\theta_t)$.

We then define

$$g'_t(\omega^t,h_t):=\sup_{\theta_t\in\Lambda_t^{\mathrm{int}}(\omega^t)}\big\{g_t(\omega^t,\theta_t)-h_t\cdot S_t(\omega)\theta_t\big\},\;h_t\in\mathbb{R}^d.$$

Remark 4.3.9. (i) *For a fixed $\bar\omega\in\overline{\Omega}_t$, we define* $\mathrm{NA}(\overline{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega))$ *by*

$$h\cdot(X_{t+1}-X_t)\ge 0\;\overline{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega)\text{-}q.s.\quad\Longrightarrow\quad h\cdot(X_{t+1}-X_t)=0\;\overline{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega)\text{-}q.s.,$$

for every $h\in\mathbb{R}^d$. Then, it follows from [1, Theorem 4.5] and Lemma 4.2.13 that $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *implies that* $\mathrm{NA}(\overline{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega))$ *holds for all $\bar\omega\in\overline{\Omega}$ outside a $\overline{\mathcal{P}}_{\mathrm{int}}$-polar set.*

(ii) *Now, for a fixed $\omega\in\Omega_t$, let us define* $\mathrm{NA}(\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega))$ *by*

$$h(X_t)\cdot(X_{t+1}-X_t)\ge 0\;\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega)\text{-}q.s.\;\Longrightarrow\;h(X_t)\cdot(X_{t+1}-X_t)=0\;\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\bar\omega)\text{-}q.s.,$$

for every universally measurable functions $h:\mathbb{R}^d\to\mathbb{R}^d$. Then by applying Proposition 5.2.11 with $\mathcal{P}(t,\omega)$ in place of $\mathcal{P}$, one obtains that $\mathrm{NA2}(t,\omega)$ *defined in* (5.5.29) *is equivalent to* $\mathrm{NA}(\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega))$.

(iii) *It follows from* (ii) *and Lemma 4.2.11 that* $\mathrm{NA2}(t,\omega)$ *or equivalently* $\mathrm{NA}(\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega))$ *holds for all ω outside a $\mathcal{P}$-polar set N, whenever* $\mathrm{NA2}(\mathcal{P})$ *holds. The later is equivalent to* $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *by Proposition 5.2.11. Therefore, when* $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *holds, there exists a $\overline{\mathcal{P}}_{\mathrm{int}}$-polar $\overline{N}:=N\times\Lambda$, such that for all $\bar\omega=(\omega,\theta)\notin\overline{N}$,* $\mathrm{NA}(\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega))$ *holds. Moreover, by similar arguments as in Remark 4.2.17,* $\mathrm{NA}(\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega))$ *implies* $\mathrm{NA}(\overline{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega,\theta))$ *for all $\theta\in\Lambda$.*

Lemma 4.3.10. *Assume that* $\mathrm{NA}(\overline{\mathcal{P}}_{\mathrm{int}})$ *holds. Then, both g'_t and g_t are upper semi-analytic. Moreover, there is a universally measurable map $h_{t+1}:\Omega_t\times\mathbb{R}^d\to\mathbb{R}^d$ and a $\mathcal{P}$-polar set N such that, for every $(\omega,h_t)\in N^c\times\mathbb{R}^d$, one has $g'_t(\omega^t,h_t)>-\infty$ and*

$$g'_t(\omega^t,h_t)+h_t\cdot X_t+h_{t+1}(\omega^t,h_t)\cdot(X_{t+1}-X_t)\;\ge\;g_{t+1}\;\;\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega)\text{-}q.s.$$

Proof. The proof follows from the same measurable selection arguments as in [1, Lemma 4.10]. We provide a sketch of proof for completeness. Let us define

$$\pi'_t(\omega^t,h_t):=\inf\big\{y\in\mathbb{R}:y+h_t\cdot X_t+h_{t+1}\cdot(X_{t+1}-X_t)\ge g_{t+1}\;\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega)\text{-q.s.},\;h_{t+1}\in\mathbb{R}^d\big\}.$$

Notice that $(X_{t+1},g_{t+1})(\omega^{t+1},\theta_0,\cdots,\theta_t,\theta_{t+1})$ are independent of $(\theta_0,\cdots,\theta_t)$. Using the same minimax theorem argument as in the proof of Lemma 4.3.7, one obtains that

$$\pi'_t(\omega^t,h_t)=g'_t(\omega^t,h_t)\;>\;-\infty\quad\text{if }\mathrm{NA}(\widetilde{\mathcal{P}}_t^{\mathrm{int},\delta}(\omega))\text{ holds true.}$$

In view of (iii) in Remark 5.5.6, this is true outside a $\mathcal{P}$-polar set N.

Further, g_{t+1} is assumed to be upper semi-analytic, X_t is Borel measurable, the graph of $\overline{\mathcal{Q}}_t^\delta$ is analytic by [1, Lemma 4.8], and the graph of $\text{int}K_t^*$ is Borel. It thus follows from a measurable selection argument (see e.g. [92, Propositions 7.26, 7.48]) that the maps $\bar{\omega}^t \mapsto g_t(\bar{\omega}^t)$ and $(\omega^t, h_t) \mapsto g_t'(\omega^t, h_t)$ are both upper semi-analytic.

We now define $\hat{g}_t := g_t' \mathbf{1}_{\mathbb{R}}(g_t')$, which is universally measurable, i.e. in $\mathcal{U}(\Omega_t \times \mathbb{R}^d)$, and consider the random set:

$$\Psi(\omega^t, h_t) := \left\{ y \in \mathbb{R}^d : \gamma_y(\cdot, h_t) \leq 0 \ \widetilde{\mathcal{P}}_t^{\text{int},\delta}(\omega)\text{-q.s.} \right\}$$

where

$$\gamma_y(\cdot, h_t) := g_{t+1} - \hat{g}_t(\cdot, h_t) - y \cdot \left(X_{t+1} - X_t \right) - h_t X_t.$$

It is enough to show that $\{\Psi \neq \emptyset\}$ is universally measurable and that Ψ admits a universally measurable selector $h_{t+1}(\cdot)$ on $\{\Psi \neq \emptyset\}$. It is not hard to see that[2] $\gamma_y \in \text{USA}[\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d \times \overline{\Omega}_1)]$. Note that, given a probability measure $\hat{\mathbb{P}}$, $\gamma_y(\cdot, h_t) \leqslant 0$ $\hat{\mathbb{P}}$-a.s. iff $\mathbb{E}^{\tilde{\mathbb{P}}}[\gamma_y(\cdot, h_t)] \leqslant 0$ for all $\tilde{\mathbb{P}} \ll \hat{\mathbb{P}}$. By an application of [1, Lemma 3.2], it is further equivalent to have the above for all $\tilde{\mathbb{P}} \ll \hat{\mathbb{P}}$ satisfying $\mathbb{E}^{\tilde{\mathbb{P}}}[|X_{t+1} - X_t|] < \infty$ and $\mathbb{E}^{\tilde{\mathbb{P}}}[|X_t|] < \infty$. Therefore, we introduce the random set

$$\widetilde{\mathcal{P}}_t^{\text{int},\delta'}(\omega) := \{\tilde{\mathbb{P}} \lll \widetilde{\mathcal{P}}_t^{\text{int},\delta}(\omega) \ : \mathbb{E}^{\tilde{\mathbb{P}}}[|X_{t+1} - X_t|] + \mathbb{E}^{\tilde{\mathbb{P}}}[|X_t|] < \infty\}.$$

By the same arguments in the proof of [1, Lemma 4.8], one can prove that $\widetilde{\mathcal{P}}_t^{\text{int},\delta'}$ has an analytic graph. Define now

$$\Gamma_y(\omega^t, h_t) := \sup_{\tilde{\mathbb{P}} \in \widetilde{\mathcal{P}}_t^{\text{int},\delta'}(\omega^t)} \mathbb{E}^{\tilde{\mathbb{P}}}[\gamma_y(\cdot, h_t)],$$

so that $\gamma_y(\cdot, h_t) \leqslant 0$ $\widetilde{\mathcal{P}}_t^{\text{int},\delta}(\omega)$-q.s. iff $\Gamma_y(\omega^t, h_t) \leqslant 0$. We now show that $(\omega^t, h_t) \mapsto \Gamma_y(\omega^t, h_t)$ is universally measurable. Indeed, the first term in the difference

$$\mathbb{E}^{\tilde{\mathbb{P}}}[g_{t+1}(\omega^t, \cdot)] - \hat{g}_t(\omega^t, h_t) - y \cdot \mathbb{E}^{\tilde{\mathbb{P}}}[X_{t+1}(\omega^t, \cdot) - X_t(\omega^t, \cdot)] - h_t \cdot \mathbb{E}^{\tilde{\mathbb{P}}}[X_t(\omega^t, \cdot)]$$

is a upper semianalytic function of $(\omega^t, \tilde{\mathbb{P}})$. The second term is universally measurable. The third and the fourth terms are Borel. As a result, $(\omega^t, h_t, \tilde{\mathbb{P}}) \mapsto \mathbb{E}^{\tilde{\mathbb{P}}}[\gamma_y(\omega^t, \cdot, h_t)]$ is in $\text{USA}[\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathfrak{B}(\mathbb{R}^d \times \overline{\Omega}_1))]$. Thus, by the Projection Theorem in the form of [1, Lemma 4.11],

$$\begin{aligned}\{\Gamma_y > c\} &= \text{proj}_{\Omega_t \times \mathbb{R}^d}\{(\omega^t, h_t, \tilde{\mathbb{P}}) : (\omega^t, \tilde{\mathbb{P}}) \in \text{graph}(\widetilde{\mathcal{P}}_t^{\text{int},\delta'}),\ \mathbb{E}^{\tilde{\mathbb{P}}}[\gamma_y(\omega^t, \cdot, h_t)] > c\} \\ &\in \mathcal{U}(\Omega_t \times \mathbb{R}^d)\end{aligned}$$

for all $c \in \mathbb{R}$. This means that $(\omega^t, h_t) \mapsto \Gamma_y(\omega^t, h_t)$ is universally measurable for any fixed y.

On the other hand, given $(\omega^t, h_t) \in \Omega_t \times \mathbb{R}^d$ and $m \geqslant 1$, the function $y \mapsto \Gamma_y(\omega^t, h_t) \wedge m$ is lower semicontinuous as the supremum over $\widetilde{\mathcal{P}}_t^{\text{int},\delta'}$ of a family of continuous functions. By [1, Lemma 4.12], $(\omega^t, h_t, y) \mapsto \Gamma_y(\omega^t, h_t) \wedge m$ is $\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable as well. As a result,

$$\begin{aligned}\text{graph}(\Psi) \ = \ \{(\omega^t, h_t, y) : \Gamma_y(\omega^t, h_t) \leqslant 0\} \ &\in \ \mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d) \\ &\subset \text{A}[\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)],\end{aligned}$$

[2] $\text{USA}[\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d \times \overline{\Omega}_1)]$ denotes the convex cone generated by both the upper semianalytic maps on $(\Omega_t \times \mathbb{R}^d) \times (\mathbb{R}^d \times \overline{\Omega}_1)$ and the $\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d \times \overline{\Omega}_1)$-measurable functions.

where $\mathrm{A}[\mathcal{U}(\Omega_t\times\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)]$ stands for the collection analytic sets of $\mathcal{U}(\Omega_t\times\mathbb{R}^d)\otimes\mathcal{B}(\mathbb{R}^d)$. Finally, [1, Lemma 4.11] yields that $\{\Psi\neq\emptyset\}\in\mathcal{U}(\Omega_t\times\mathbb{R}^d)$ and that Ψ admits a $\mathcal{U}(\Omega_t\times\mathbb{R}^d)$-measurable selector on $\{\Psi\neq\emptyset\}$. □

Proof of Theorem 4.3.1 (case $e=0$). The existence of the optimal super-hedging strategy will be proved in Lemma 4.3.11 below for the general case $e\geq 1$. By Propositions 4.3.2 and 5.2.6 and Lemma 4.3.6, it is enough to prove

$$\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_0^{\rm loc}}\mathbb{E}^{\overline{\mathbb{Q}}}[g]\geq\inf\Big\{y\in\mathbb{R}\ :y+(H\circ X)_T\geq g\ \overline{\mathcal{P}}_{\rm int}\text{-q.s., for some } H\in\mathcal{H}\Big\},\tag{4.3.21}$$

for $g:=\xi\cdot X_T$. We use an induction argument. Recall that (4.3.21) is already proved for the case $T=1$, this is the content of Lemma 4.3.7. Assume that (4.3.21) holds true for $T=t$ and let us prove that it also holds true for the case $T=t+1$.

Given an upper semianalytic random variable $g_{t+1}:=\overline{\Omega}_{t+1}\to\mathbb{R}\cup\{\infty\}$ such that $g_{t+1}(\omega^{t+1},\theta_0,\cdots,\theta_{t+1})$ depends only on $(\omega^{t+1},\theta_{t+1})$. we define g_t by (5.5.30), and denote

$$\pi_0^t(g_t)\ :=\ \inf\big\{y\ :y+(H\circ X)_t\geq g_t\ \ \overline{\mathcal{P}}_{\rm int}\text{-q.s.},\ H\in\mathcal{H}\big\}.$$

Fix $(y,H)\in\mathbb{R}\times\mathcal{H}$ such that $y+(H\circ X)_t\geq g_t$, $\overline{\mathcal{P}}_{\rm int}$-q.s.. Then, $y+(H\circ X)_{t-1}-H_t\cdot X_{t-1}\geq g_t-H_t\cdot X_t$ $\overline{\mathcal{P}}_{\rm int}$-q.s. and therefore

$$y+(H\circ X)_{t-1}-H_t\cdot X_{t-1}\geq g'_t(\cdot,H_t)\ \overline{\mathcal{P}}_{\rm int}\text{-q.s.}$$

Hence, if we define H' by $H'_s:=H_s$ for $s\leq t$ and $H'_{t+1}(\omega^t):=h_{t+1}(\omega^t,H_t(\omega^{t-1}))$, with h_{t+1} as in Lemma 4.3.10, we obtain

$$y+(H'\circ X)_{t+1}\geq g_{t+1}\ \ \overline{\mathcal{P}}_{\rm int}\text{-q.s.}$$

Hence,

$$\pi_0^{t+1}(g_{t+1})\ \leq\ \pi_0^t(g_t)\ =\ \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_0^{\rm loc}}\mathbb{E}^{\overline{\mathbb{Q}}}[g_t]\ \leq\ \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_0^{\rm loc}}\mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}],$$

where the last inequality follows from a classical concatenation argument. This is in fact (4.3.21) for the case $T=t+1$, and we hence conclude the proof. □

4.3.4 Proof of Theorem 4.3.1: case $e\geq 1$

To take into account the transaction costs generated by the trading of the static options $(\zeta_i,i=1,\cdots,e)$, we introduce a further enlarged space:

$$\widehat{\Lambda}:=\prod_{i=1}^e[-c_i,c_i],\ \ \widehat{\Omega}:=\overline{\Omega}\times\widehat{\Lambda},\ \ \widehat{\mathcal{F}}_t:=\overline{\mathcal{F}}_t\otimes\mathcal{B}\big(\widehat{\Lambda}\big),\ \ \widehat{\mathcal{P}}_{\rm int}:=\big\{\widehat{\mathbb{P}}\in\mathfrak{B}(\widehat{\Omega})\ :\widehat{\mathbb{P}}|_{\overline{\Omega}}\in\overline{\mathcal{P}}_{\rm int}\big\},$$

and define

$$\hat{f}_i:\widehat{\Omega}\ \longrightarrow\ \mathbb{R},\quad \hat{f}_i(\widehat{\omega})=\zeta_i(\omega)\cdot X_T(\bar{\omega})-\hat{\theta}_i\ \text{ for all }\widehat{\omega}=(\bar{\omega},\hat{\theta})=(\omega,\theta,\hat{\theta})\in\widehat{\Omega}.$$

The process $(X_t)_{0\le t\le T}$ and the random variable $g := \xi \cdot X_T$ defined on $\overline{\Omega}$ can be naturally extended on $\widehat{\Omega}$. We can then consider the super-hedging problem on $\widehat{\Omega}$:

$$\hat{\pi}_e(g) \;:=\; \inf\Big\{y \;: y + \sum_{i=1}^{e} \ell_i \hat{f}_i + (H \circ X)_T \ge g,\ \widehat{\mathcal{P}}_{\rm int}\text{-q.s.},\ \ell \in \mathbb{R}^e,\ H \in \mathcal{H}\Big\}.$$

Let us also introduce

$$\widehat{\mathcal{Q}}_e \;:=\; \big\{\widehat{\mathbb{Q}} \in \mathfrak{B}(\widehat{\Omega}) \;: \widehat{\mathbb{Q}} \lll \widehat{\mathcal{P}}_{\rm int},\ X \text{ is } (\widehat{\mathbb{F}}, \widehat{\mathbb{Q}})\text{-martingale},\ \mathbb{E}^{\widehat{\mathbb{Q}}}[\hat{f}_i] = 0,\ i = 1, \cdots, e\big\},$$

and

$$\widehat{\mathcal{Q}}_e^{\varphi} := \big\{\widehat{\mathbb{Q}} \in \widehat{\mathcal{Q}}_e \;: \mathbb{E}^{\widehat{\mathbb{Q}}}[\varphi] < \infty\big\}, \quad \text{for } \varphi := 1 + |g| + \sum_{i=1}^{e} |\hat{f}_i|.$$

Lemma 4.3.11. *Let* NA2($\mathcal{P}$) *hold. Assume further that and* (5.4.15) *holds true for all* $\ell \in \mathbb{R}^e$ *and* $\eta \in \mathcal{A}$. *Then:*
(a) There exist $\hat{\ell} \in \mathbb{R}^e$ *and a* $\mathbb{F}$*-predictable process* $\widehat{H}$ *such that*

$$\hat{\pi}_e(g) + \sum_{i=1}^{e} \hat{\ell}_i \hat{f}_i + (\widehat{H} \circ X)_T \ge g,\ \widehat{\mathcal{P}}_{\rm int}\text{-}q.s. \tag{4.3.22}$$

(b) Consequently, there exists $(\hat{\eta}, \hat{\ell}) \in \mathcal{A} \times \mathbb{R}^e$ *such that*

$$\pi_e(\xi) 1_d + \sum_{i=1}^{e} \Big(\hat{\ell}_i \zeta_i - |\hat{\ell}_i| c_i 1_d\Big) + \sum_{t=0}^{T} \hat{\eta}_t - \xi \in K_T,\ \mathcal{P}\text{-}q.s. \tag{4.3.23}$$

Proof. (a) It suffices to show that the collection of claims that can be super-hedged from 0 is closed for the $\widehat{\mathcal{P}}_{\rm int}$-q.s. convergence. Note the results in [1, Section 2] are given in a general abstract context, where the underlying asset is not assumed to be adapted to the filtration of the strategy. Then, [1, Theorem 2.3] implies our claim in the case $e = 0$ (recall that NA2($\mathcal{P}$) implies NA($\overline{\mathcal{P}}_{\rm int}$)). Assume now that it holds for $e - 1 \ge 0$ and let us deduce that it holds for e as well.

Let $(g^n)_{n\ge1} \subset \mathbb{L}^0$ be such that $g^n \to g$ $\widehat{\mathcal{P}}_{\rm int}$-q.s., and $\pi_e(g^n) \le 0$ for $n \ge 1$. Let $(\hat{\ell}^n)_n \subset \mathbb{R}^e$ and let $(\widehat{H}^n)_{n\ge1}$ be a sequence of $\mathbb{F}$-predictable processes such that $\sum_{i=1}^{e} \hat{\ell}_i^n \hat{f}_i + (\widehat{H}^n \circ X)_T \ge g^n$ $\widehat{\mathcal{P}}_{\rm int}$-q.s. If $(\hat{\ell}_e^n)_{n\ge1}$ is bounded, then one can assume that it converges to some $\hat{\ell}_e \in \mathbb{R}$. Hence, [1, Theorem 2.3] implies that one can find $\hat{\ell} \in \mathbb{R}^{e-1}$ and a $\mathbb{F}$-predictable process $\widehat{H}$ such that $\sum_{i=1}^{e-1} \hat{\ell}_i \hat{f}_i + (\widehat{H} \circ X)_T \ge g - \hat{\ell}_e \hat{f}_e$ $\widehat{\mathcal{P}}_{\rm int}$-q.s.

If $(\hat{\ell}_e^n)_{n\ge1}$ is not bounded, then one can assume that $|\hat{\ell}_e^n| \to \infty$, so that $(g^n - \hat{\ell}_e^n \hat{f}_e)/(1+|\hat{\ell}_e^n|) \to -\chi \hat{f}_e$ $\widehat{\mathcal{P}}_{\rm int}$-q.s. for some $\chi \in \{-1, 1\}$ and [1, Theorem 2.3] implies that one can find $\hat{\ell} \in \mathbb{R}^{e-1}$ and a $\mathbb{F}$-predictable process $\widehat{H}$ such that $\sum_{i=1}^{e-1} \hat{\ell}_i \hat{f}_i + (\widehat{H} \circ X)_T \ge -\chi \hat{f}_e$ $\widehat{\mathcal{P}}_{\rm int}$-q.s. By similar arguments as in the proof of Proposition 5.2.6, this implies that $\chi \hat{f}_e - |\chi| c_e 1_d + \sum_{i=1}^{e-1} (\hat{\ell}_i \zeta_i - |\hat{\ell}_i| c_i 1_d) + \sum_{t=0}^{T} \eta_t \in K_T$ $\mathcal{P}$-q.s. for some $\eta \in \mathcal{A}$. Then, $\chi = 0$ by (5.4.15), a contradiction.

(b) Finally, by the same arguments as in the proof of Proposition 5.2.6, one can show $\pi_e(\xi) = \hat{\pi}_e(g)$ for $g := \xi \cdot X_T$. Moreover, using the construction (4.3.15), one can obtain explicitly $(\hat{\eta}, \hat{\ell})$ satisfying (4.3.23) from $(\widehat{H}, \hat{\ell})$ satisfying (4.3.22). □

Proof of Theorem 4.3.1 (case $e \ge 1$). The existence of a super-hedging strategy has been proved in Lemma 4.3.11 above. Moreover, it is easy to adapt the arguments of Propositions 4.3.2

and 5.2.6 to obtain

$$\pi_e(\xi) = \hat{\pi}_e(g) \text{ and } \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e} \mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] = \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e} \mathbb{E}^{\widehat{\mathbb{Q}}}\big[g\big], \quad \text{for } g := \xi\cdot X_T.$$

Remember that, by [1, Lemma A.3], one has $\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e^{\varphi}} \mathbb{E}^{\widehat{\mathbb{Q}}}[g] = \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e} \mathbb{E}^{\widehat{\mathbb{Q}}}[g]$. Hence, it is enough to prove that

$$\hat{\pi}_e(g) \;=\; \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e^{\varphi}} \mathbb{E}^{\widehat{\mathbb{Q}}}[g]. \tag{4.3.24}$$

Note that we have already proved (5.5.36) for the case $e = 0$ in Section 5.5.2, although the formulations are slightly different (the additional randomness induced by $\widehat{\Lambda}$ obviously does not play any role when $e = 0$). We argue by induction as in the proof of [1, Theorem 5.1]. Let us assume that (5.5.36) holds for $e-1 \geq 0$ and then prove it for e.

First, it follows from (5.4.15) that we can not find $\eta \in \mathcal{A}$ and $(\ell_i)_{1\leq i\leq e} \neq 0$ such that $\sum_{i=1}^{e}(\ell_i\zeta_i - |\ell_i|c_i 1_d) + \sum_{t=0}^{T}\eta_t \in K_T$ $\mathcal{P}$-q.s. By the same arguments as in the proof of Proposition 5.2.6, there is no $H \in \mathcal{H}$, $\ell_1, \cdots, \ell_{e-1}$ and $\ell_e \in \{-1, 1\}$ such that $\sum_{i=1}^{e-1}\ell_i\hat{f}_i + (H\circ X)_T \geq -\ell_e\hat{f}_e$, $\widehat{\mathcal{P}}_{\text{int}}$-q.s. It follows that $\hat{\pi}_{e-1}(\hat{f}_e), \hat{\pi}_{e-1}(-\hat{f}_e) > 0$, which, by Lemma 4.3.11 and our induction hypothesis, implies that there is $\widehat{\mathbb{Q}}_-, \widehat{\mathbb{Q}}_+ \in \widehat{\mathcal{Q}}_{e-1}^{\varphi}$ such that

$$-\hat{\pi}_{e-1}(-\hat{f}_e) \;<\; \mathbb{E}^{\widehat{\mathbb{Q}}_-}[\hat{f}_e] \;<\; 0 \;<\; \mathbb{E}^{\widehat{\mathbb{Q}}_+}[\hat{f}_e] \;<\; \hat{\pi}_{e-1}(\hat{f}_e). \tag{4.3.25}$$

We now claim that

$$\text{there exists a sequence } \big(\widehat{\mathbb{Q}}_n\big)_{n\geq 1} \subset \widehat{\mathcal{Q}}_{e-1}^{\varphi} \text{ s.t. } \mathbb{E}^{\widehat{\mathbb{Q}}_n}[\hat{f}_e] \to 0,\ \mathbb{E}^{\widehat{\mathbb{Q}}_n}[g] \to \hat{\pi}_e(g). \tag{4.3.26}$$

Indeed, if the above fails, then

$$(0, \hat{\pi}_e(g)) \notin \overline{\big\{\mathbb{E}^{\widehat{\mathbb{Q}}}[(\hat{f}_e, g)] : \widehat{\mathbb{Q}} \in \widehat{\mathcal{Q}}_{e-1}^{\varphi}\big\}} \subset \mathbb{R}^2,$$

and one obtains a contradiction by following line by line the same arguments in the end of the proof of [1, Theorem 5.1]. In view of (5.5.40) and (4.3.26), we can find $(\lambda_n^-, \lambda_n, \lambda_n^+) \in [0, 1]$ such that $\lambda_n^- + \lambda_n + \lambda_n^+ = 1$, $(\lambda_n^-, \lambda_n^+) \to 0$, and

$$\widehat{\mathbb{Q}}'_n := \lambda_n^-\widehat{\mathbb{Q}}_- + \lambda_n\widehat{\mathbb{Q}}_n + \lambda_n^+\widehat{\mathbb{Q}}_+ \in \widehat{\mathcal{Q}}_{e-1}^{\varphi} \text{ satisfies } \mathbb{E}^{\widehat{\mathbb{Q}}'_n}[f_e] = 0.$$

In particular, one has $\widehat{\mathbb{Q}}'_n \in \widehat{\mathcal{Q}}_e^{\varphi}$ and hence $\widehat{\mathcal{Q}}_e^{\varphi}$ is nonempty, which implies that $\mathcal{S}_e$ is nonempty by the projection argument in Proposition 4.3.2.

Moreover, since $\mathbb{E}^{\widehat{\mathbb{Q}}'_n}[g] \to \hat{\pi}_e(g)$, this shows that

$$\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e^{\varphi}} \mathbb{E}^{\widehat{\mathbb{Q}}}[g] \;\geq\; \hat{\pi}_e(g).$$

To conclude, it is enough to notice that the reverse inequality is the classical weak duality which can be easily obtained from [1, Lemmas A.1 and A.2]. □

Chapter 5

Utility maximization with proportional transaction cost under model uncertainty

5.1 Introduction

The optimal investment via utility maximization has always been one of the fundamental problems in quantitative finance. In particular, the optimal semi-static portfolio among risky assets and liquid options and the associated utility indifference pricing of unhedgeable illiquid contingent claims have attracted a lot of research interests recently. In the classical dominated market model, the so-called utility maximization with random endowments was extensively investigated, see among [60], [52], [53], [54], [55] and [56]. In particular, the duality approach has been proposed and developed as a powerful tool to deal with general incomplete market models. Without knowing the specific underlying model structures, the convex duality relationship enables one to obtain the existence of the primal optimizer by solving the corresponding dual optimization problem first. Typically, the dual problem is formulated on the set of equivalent (local) martingale measures (EMM), whose existence is ensured by some appropriate no arbitrage assumptions. Depending on the domain of the utility function, different techniques are involved in order to obtain some convex duality results, see for example [104], [103], [106]. For utilities defined on the positive real line, to handle the random payoffs and to establish the bipolar relationship, the appropriate closure of the dual set of EMM plays the key role, see [52] and [53] for instance. On the other hand, for utilities defined on the whole real line, a subset of EMM with finite general entropy is usually chosen to define the dual problem while the appropriate definition of working portfolios turns out to be critical to guarantee and relate the primal and dual optimizers, see [54], [55], [56], [105], [107] and the references therein.

Because of the growing complexity of real financial markets, the aforementioned optimization problems have been actively extended mainly in two directions. The first fruitful extension incorporates the practical trading frictions, namely transaction costs, into decision making and the resulting wealth process. As transaction costs will generically break the (local) martingale property of the self-financing wealth process under EMM, the dual pricing kernel is not expected to be the same as in the frictionless counterpart. Instead, the no-arbitrage condition is closely related to the existence of a pair of dual elements named the consistent price system (CPS), see [108]. Briefly speaking, the first component of CPS is a process evolving inside the bid-ask spread, while the second component is an equivalent probability measure under which the first component becomes a martingale. However, similar to the case in the frictionless model, for utility maximization with random endowments, the set of CPS can only serve as the first step to formulate the naive dual problem. More efforts are demanded to deal with the random payoffs from options, see some related work in [109], [58], [110] and [111].

The second compelling extension in the literature is to take into account the model uncertainty, for instance the volatility uncertainty, by starting with a set of possibly mutually singular probability measures. Namely, different probability measures describe the believes of different investors on the market. In the discrete time framework, the no-arbitrage condition and the fundamental theorem in robust finance have been essentially studied in [72, 1, 11, 12], etc. for frictionless markets, and in [45, 42, 43, 47, 112] for market with transaction costs. Analogous to the dominated case, the pricing-hedging duality can usually be obtained by studying the super-hedging problem under some appropriate no-arbitrage conditions. The non-dominated robust utility maximization in the discrete time frictionless market was first examined by [59], where the dynamic programming principle plays the major role to derive the existence of the optimal primal strategy without passing to the dual problem, see some further extensions in [61, 62, 63]. In a context where the model uncertainty is represented by a collection of stochastic processes, [64] proved the existence of the optimal strategy for the utility function defined either over the positive or over the whole real line. However, whether the convex duality holds remained open

in these pioneer work of utility maximization. Recently, [2] established the duality representation for the exponential utility preference in the frictionless model under some restrictive no arbitrage conditions, which motivates us to reconsider the validity of duality theorem in this paper with proportional transaction costs under weaker market conditions using some distinctive arguments. We also note a recent paper [65], in which the authors proved a robust utility maximization duality using medial limits and a functional version of Choquet's capacitability theorem.

The main objective of this paper is therefore to study the existence of the optimal strategy, the convex duality theorem and the auxiliary dynamic programming principle for a semi-static utility maximization problem with transaction costs in a discrete time framework. To be precise, we envision an investor who chooses the optimal semi-static portfolio in stocks and liquid options with an extra random endowment for the case of exponential utility preference and meanwhile each trading incurs proportional transaction fees. The core idea of our analysis is to reduce the complexity of transaction costs significantly by employing the randomization method as in [113]. Consequently, the unpleasant mathematical obstacles caused by trading fees can be hidden in an enlarged space with additional randomness and some techniques in the literature of robust hedging and utility maximization in frictionless models can be modified and adopted. It is worth noting that by applying the randomization approach in [113] but with a different and more involved definition of family of probability measures on the enlarged space, [114] recently established a super-replication duality with transaction cost under a weaker no-arbitrage condition.

Our main contributions are the following. First, we develop a distinctive dynamic programming argument comparing to [2] in a frictionless market. This allows us to overcome a measurability difficulty in [2] and hence generalize their main results (duality and existence) under a weaker market condition. This generalization is presented in Appendix. Secondly, we generalize the randomization technique in [113] in this utility maximization problem, which relies essentially on a minimax argument to resolve a filtration enlargement problem. While the corresponding convex/concave property is quite natural for the super-replication problem in [113], it is much less obvious for the utility maximization problem and we use a log transformation technique in this exponential utility maximization problem. Finally, to manifest the value of the duality representation, we also investigate an application to utility indifference pricing. Several fundamental properties of indifference prices including the asymptotic convergence of indifference prices to the superhedging price and some continuity results with respect to random endowments are confirmed in the robust setting with transaction costs.

The rest of the paper is organized as follows. In Section 5.2, we introduce the market model with transaction costs, and show how to reformulate the robust utility maximization problem on a frictionless market on an enlarged space using the randomization method. In Section 5.3, we restrict to the case of the exponential utility preference. A convex duality theorem and the existence of the optimal trading strategy are first obtained in the presence of both model uncertainty and transaction costs. As an application, several properties of the utility indifference prices are concluded. Section 5.5 mainly provides the proof of the duality result using a dynamic programming argument.

Notation. Given a measurable space $(\Omega, \mathcal{F})$, we denote by $\mathfrak{B}(\Omega, \mathcal{F})$ the set of all probability measures on $(\Omega, \mathcal{F})$. For a topological space Ω, $\mathcal{B}(\Omega)$ denotes its Borel σ-field with the abbreviate notation $\mathfrak{B}(\Omega) := \mathfrak{B}(\Omega, \mathcal{B}(\Omega))$. For a Polish space Ω, a subset $A \subseteq \Omega$ is called analytic if it is the image of a Borel subset of another Polish space under a Borel measurable mapping. A function $f : \Omega \to \overline{\mathbb{R}} := [-\infty, \infty]$ is upper semianalytic if $\{\omega \in \Omega \ : f(\omega) > a\}$ is analytic for all $a \in \mathbb{R}$.

Given a probability measure $\mathbb{P} \in \mathfrak{B}(\Omega)$ and a measurable function $f : \Omega \to \overline{\mathbb{R}}$, we define the expectation

$$\mathbb{E}^{\mathbb{P}}[f] \ := \ \mathbb{E}^{\mathbb{P}}[f^+] - \mathbb{E}^{\mathbb{P}}[f^-], \quad \text{with the convention } \infty - \infty = -\infty.$$

For a family $\mathcal{P} \subseteq \mathfrak{B}(\Omega)$ of probability measures, a subset $A \subset \Omega$ is called $\mathcal{P}$-polar if $A \subset A'$ for some universally measurable set A' satisfying $\mathbb{P}[A'] = 0$ for all $\mathbb{P} \in \mathcal{P}$, and a property is said to hold $\mathcal{P}$-quasi surely or $\mathcal{P}$-q.s if it holds true outside a $\mathcal{P}$-polar set. For $\mathbb{Q} \in \mathfrak{B}(\Omega)$, we write $\mathbb{Q} \lll \mathcal{P}$ if there exists $\mathbb{P}' \in \mathcal{P}$ such that $\mathbb{Q} \ll \mathbb{P}'$. Given a sigma algebra $\mathcal{G}$, we denote by $L^0(\mathcal{G})$ the collection of $\mathbb{R}^d$-valued random variable that are $\mathcal{G}$-measurable, d being given by the context.

5.2 Market model and Problem Formulation

We first introduce a financial market with proportional transaction costs in a multivariate setting under model uncertainty. A utility maximization problem is formulated afterwards and we then reformulate the problem further in a frictionless market setting on an enlarged space. Although the reformulation technique can be used for a more general framework, we will stay essentially in the context of Bouchard and Nutz [1, 43].

5.2.1 Market model and preliminaries

A product space with a set of probability measures Let $\Omega_0 := \{\omega_0\}$ be a singleton and Ω_1 be a Polish space. For each $t = 1, \cdots, T$, we denote by $\Omega_t := \Omega_1^t$ the t-fold Cartesian product of Ω_1 and let $\mathcal{F}_t^0 := \mathcal{B}(\Omega_t)$ and $\mathcal{F}_t$ its universal completion. In particular, $\mathcal{F}_0$ is trivial. We define the filtered measurable space $(\Omega, \mathcal{F})$ by

$$\Omega := \Omega_T, \ \mathcal{F} := \mathcal{F}_T, \ \mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T} \ \text{ and } \mathbb{F}^0 := (\mathcal{F}_t^0)_{0 \le t \le T}.$$

Let us then introduce a set $\mathcal{P}$ of probability measures on $(\Omega, \mathcal{F})$ by

$$\mathcal{P} := \big\{ \mathbb{P} := \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \ : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot) \text{ for } t \le T-1 \big\}. \tag{5.2.1}$$

In the definition above, $\mathbb{P}_t : \Omega_t \mapsto \mathfrak{B}(\Omega_1)$ are probability kernels such that the probability measure $\mathbb{P}$ is defined by Fubini's theorem in the sense that

$$\mathbb{P}(A) := \int_{\Omega_1} \cdots \int_{\Omega_1} \mathbf{1}_A(\omega_1, \omega_2 \cdots, \omega_T) \mathbb{P}_{T-1}(\omega_1, \cdots, \omega_{T-1}; d\omega_T) \cdots \mathbb{P}_0(d\omega_1),$$

and $\mathcal{P}_t(\omega)$ is a non-empty convex set in $\mathfrak{B}(\Omega_1)$, which represents the set of all possible models for the $(t+1)$-th period, given the state $\omega \in \Omega_t$ at time $t = 0, 1 \cdots, T-1$. As in the literature, we assume that, for each t,

$$[[\mathcal{P}_t]] := \big\{ (\omega, \mathbb{P}) : \omega \in \Omega_t, \mathbb{P} \in \mathcal{P}_t(\omega) \big\} \ \subseteq \ \Omega_t \times \mathcal{P}(\Omega_1) \ \text{ is analytic.} \tag{5.2.2}$$

This ensures in particular that $\mathcal{P}$ in (5.2.1) is nonempty.

A financial market with proportional transaction cost The financial market with proportional transaction cost is formulated in terms of random cones. Let $d \ge 2$, for every $t \in \{0, 1, \cdots, T\}$, $K_t : \Omega \to 2^{\mathbb{R}^d}$ is a $\mathcal{F}_t^0$-measurable random set in the sense that $\{\omega \in \Omega : K_t(\omega) \cap O \neq \emptyset\} \in \mathcal{F}_t^0$ for every closed (open) set $O \subset \mathbb{R}^d$. Here, for each $\omega \in \Omega$, $K_t(\omega)$ is a

closed convex cone containing $\mathbb{R}^d_+$, called the solvency cone at time t. It represents the collection of positions, labelled in units of different d financial assets, that can be turned into non-negative ones (component by component) by performing immediately exchanges between the assets. We denote by $K_t^* \subset \mathbb{R}^d_+$ its (nonnegative) dual cone:

$$K_t^*(\omega) \ := \ \big\{y \in \mathbb{R}^d \ : x \cdot y \geq 0 \text{ for all } x \in K_t(\omega)\big\}, \tag{5.2.3}$$

where $x \cdot y := \sum_{i=1}^d x^i y^i$ is the inner product on $\mathbb{R}^d$. For later use, let us also introduce

$$K_t^{*,0}(\omega) \ := \ \big\{y = (y^1, \cdots, y^d) \in K_t^*(\omega), \ y^d = 1\big\}.$$

As in [43], we assume the following conditions throughout the paper:

Assumption 5.2.1. *$K_t^* \cap \partial\mathbb{R}^d_+ = \{0\}$ and $\text{int}K_t^*(\omega) \neq \emptyset$ for every $\omega \in \Omega$ and $t \leq T$.*

It follows from the above assumption and [43, Lemma A.1] that K_t^*, $K_t^{*,0}$ and $\text{int}K_t^*$ are all $\mathcal{F}_t^0$-measurable. Moreover, there is a $\mathbb{F}^0$-adapted process S satisfying

$$S_t(\omega) \in K_t^{*,0}(\omega) \cap \text{int}K_t^*(\omega) \ \text{ for every } \omega \in \Omega, \, t \leq T. \tag{5.2.4}$$

We also assume that transaction costs are bounded and uniformly strictly positive. This is formulated in terms of S above.

Assumption 5.2.2. *There is some constant $c > 1$ such that*

$$c^{-1} S_t^i(\omega) \leq y^i \leq c S_t^i(\omega), \ \textit{for every } i \leq d-1 \textit{ and } y \in K_t^{*,0}(\omega).$$

Finally, we define the collection of admissible strategies as follows.

Definition 5.2.3. *We say that an $\mathbb{F}$-adapted process $\eta = (\eta_t)_{0 \leq t \leq T}$ is an admissible trading strategy if*

$$\eta_t \in -K_t \quad \mathcal{P}\text{-q.s. for all } t \leq T.$$

We denote by $\mathcal{A}$ the collection of all admissible strategies.

The constraint $\eta_t \in -K_t$ means that $0 - \eta_t \in K_t$, i.e., starting at t with 0, one can perform immediate transfers to reach the position η_t. Then, given $\eta \in \mathcal{A}$, the corresponding wealth process associated to a zero initial endowment at time 0 is $\big(\sum_{s=0}^t \eta_s\big)_{t \leq T}$. We can refer to [113, 43] for concrete examples. See also the monograph [49].

5.2.2 A utility maximization problem and its reformulation

Let $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a non-decreasing concave utility function. We are interested in the following robust utility maximization problem with random endowments:

$$V(\xi) \ := \ \sup_{\eta \in \mathcal{A}_0} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[U\Big(\Big(\xi + \sum_{t=0}^T \eta_t\Big)^d\Big)\Big], \tag{5.2.5}$$

where $\mathcal{A}_0$ denotes the collection of all $\eta \in \mathcal{A}$ such that $(\xi + \sum_{t=0}^T \eta_t)^i = 0$ for $i = 1, \cdots, d-1$.

Remark 5.2.4. *Note that (5.2.5) is a numéraire based utility maximization problem, and the d-th asset plays the role of the numéraire. For an admissible strategy in $\mathcal{A}_0$, it is required to liquidate the position of all other assets for $i = 1, \cdots, d-1$ at the terminal time T.*

The mixture of model uncertainty, transaction costs and random endowments can bring a lot of new mathematical challenges. Our paramount remedy to reduce the complexity is to reformulate it on a fictitious market without transaction cost. In particular, this allows us to use some well known results and techniques in the existing literature.

A frictionless market on the enlarged space Given the constant $c > 1$ in Assumption 5.2.2, we define $\Lambda_1 := [c^{-1}, c]^{d-1}$ and $\Lambda := (\Lambda_1)^{T+1}$, and then introduce the canonical process $\Theta_t(\theta) := \theta_t$, $\forall \theta = (\theta_t)_{t\leq T} \in \Lambda$, as well as the σ-fields $\mathcal{F}_t^\Lambda := \sigma(\Theta_s,\ s \leq t)$, $t \leq T$. We next introduce an enlarged space $\overline{\Omega} := \Omega \times \Lambda$, an enlarged σ-field $\overline{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}_T^\Lambda$, together with three filtrations $\overline{\mathbb{F}}^0 = (\overline{\mathcal{F}}_t^0)_{0\leq t\leq T}$, $\tilde{\overline{\mathbb{F}}} = (\tilde{\overline{\mathcal{F}}}_t)_{0\leq t\leq T}$ and $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\leq t\leq T}$ in which $\overline{\mathcal{F}}_t^0 := \mathcal{F}_t^0 \otimes \mathcal{F}_t^\Lambda$, $\tilde{\overline{\mathcal{F}}}_t := \mathcal{F}_t \otimes \{\emptyset, \Lambda\}$ and $\overline{\mathcal{F}}_t := \mathcal{F}_t \otimes \mathcal{F}_t^\Lambda$ for $t \leq T$.

Next, let us introduce our randomized market model with the fictitious underlying stock $X = (X_t)_{0\leq t\leq T}$ defined by

$$X_t(\bar{\omega}) := \Pi_{K_t^{*,0}(\omega)}[S_t(\omega)\theta_t], \quad \text{for all } \bar{\omega} = (\omega, \theta) \in \overline{\Omega},\ t \leq T, \tag{5.2.6}$$

where $S_t(\omega)\theta_t := (S_t^1(\omega)\theta_t^1, \cdots, S_t^{d-1}(\omega)\theta_t^{d-1}, S_t^d(\omega))$, and $\Pi_{K_t^{*,0}(\omega)}[y]$ stands for the projection of $y \in \mathbb{R}^d$ on the convex closed set $K_t^{*,0}(\omega)$. It is worth noting that $S_t \in K_t^{*,0}$ for $t \leq T$ and that X is $\overline{\mathbb{F}}^0$-adapted by Lemma 2.6 of [113].

We then define two sets of strategy processes by

$$\mathcal{H} := \{\text{All } \tilde{\overline{\mathbb{F}}}\text{-predictable processes}\} \ \text{ and } \ \overline{\mathcal{H}} := \{\text{All } \overline{\mathbb{F}}\text{-predictable processes}\}.$$

Notice that $\tilde{\overline{\mathcal{F}}}_t := \mathcal{F}_t \otimes \{\emptyset, \Lambda\}$, and hence a $\tilde{\overline{\mathbb{F}}}$-predictable process can be identified to be a $\mathbb{F}$-predictable process. Given a strategy $H \in \overline{\mathcal{H}}$, the resulting wealth process is given by $(H \circ X)_t := \sum_{s=1}^t H_s \cdot (X_s - X_{s-1})$, $t \leq T$.

Finally, let us introduce some sets of probability measures on the enlarged space $(\overline{\Omega}, \overline{\mathcal{F}})$. Let

$$\overline{\mathcal{P}} \ := \ \big\{\overline{\mathbb{P}} \in \mathfrak{B}(\overline{\Omega}, \overline{\mathcal{F}}) \text{ such that } \overline{\mathbb{P}}|_\Omega \in \mathcal{P}\big\}.$$

We next introduce a subset $\overline{\mathcal{P}}_{\rm int} \subset \overline{\mathcal{P}}$ as follows. Recall that $\overline{\Omega}$ has a product structure as Ω. More precisely, for a fixed $t \leqslant T$, let $\overline{\Omega}_0 := \Omega_0 \times \Lambda_1$, $\overline{\Omega}_t := \overline{\Omega}_0 \times (\Omega_1 \times \Lambda_1)^t$ and $\overline{\Omega} := \Omega \times \Lambda = \overline{\Omega}_T$. For $(\omega = (\omega_0, \cdots, \omega_T), \theta = (\theta_0, \cdots, \theta_T)) \in \overline{\Omega}$ and $t \leqslant T$, we denote $\omega^t := (\omega_0, \cdots, \omega_t)$, $\theta^t := (\theta_0, \cdots, \theta_t)$ and $\bar{\omega}^t := (\omega^t, \theta^t)$.

- For $t = 0, 1, \cdots, T-1$ and $\bar{\omega} = (\omega, \theta) \in \overline{\Omega}_t$, we define $\overline{\mathcal{P}}(t, \bar{\omega}) := \big\{\overline{\mathbb{P}} \in \mathfrak{B}(\Omega_1 \times \Lambda_1) \ : \overline{\mathbb{P}}|_{\Omega_1} \in \mathcal{P}_t(\omega)\big\}$, and

$$\overline{\mathcal{P}}_{\rm int}(t, \bar{\omega}) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}(t, \bar{\omega}) \ : \delta_{\bar{\omega}} \otimes \overline{\mathbb{P}}[X_{t+1} \in {\rm int}K_{t+1}^*] = 1\big\}, \tag{5.2.7}$$

  where $\delta_{\bar{\omega}} \otimes \overline{\mathbb{P}}$ is a probability measure on $\overline{\Omega}_{t+1} = \overline{\Omega}_t \times (\Omega_1 \times \Lambda_1)$ and X_{t+1} (defined in (5.2.6)) is considered as a random variable defined on $\overline{\Omega}_{t+1}$.

- Let $\overline{\mathcal{P}}_{{\rm int},\emptyset}$ be the collection of all probability measures $\overline{\mathbb{P}}$ on $\overline{\Omega}_0$ such that $\overline{\mathbb{P}}[X_0 \in {\rm int}K_0^*] = 1$. We define

$$\overline{\mathcal{P}}_{\rm int} \ := \ \big\{\overline{\mathbb{P}}_\emptyset \otimes \overline{\mathbb{P}}_0 \otimes \cdots \otimes \overline{\mathbb{P}}_{T-1} \ : \overline{\mathbb{P}}_\emptyset \in \overline{\mathcal{P}}_{{\rm int},\emptyset} \text{ and } \overline{\mathbb{P}}_t(\cdot) \in \overline{\mathcal{P}}_{\rm int}(t, \cdot) \text{ for } t \leq T-1\big\},$$

  where $\overline{\mathbb{P}}_t(\cdot)$ is a universally measurable selector of $\overline{\mathcal{P}}_{\rm int}(t, \cdot)$.

Remark 5.2.5. *Assume that the analyticity condition* (5.2.2) *for* $[[\mathcal{P}_t]]$ *holds, Lemma 2.13 of [113] asserts that*

$$\big[\big[\overline{\mathcal{P}}_{\rm int}(t)\big]\big] := \big\{(\bar\omega,\overline{\mathbb{P}}) : \bar\omega \in \overline{\Omega}_t, \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}(t,\bar\omega)\big\} \text{ is also analytic,}$$

which in particular ensures that $\overline{\mathcal{P}}_{\rm int}$ *is nonempty.*

Reformulation on the enlarged space We now reformulate the utility maximization (5.2.5) on the enlarged space $\overline{\Omega}$ using the underlying stock X. Let us set

$$g(\bar\omega) := \xi(\omega)\cdot X_T(\bar\omega), \text{ for all } \bar\omega = (\omega,\theta) \in \overline{\Omega},$$

as the contingent claim.

Proposition 5.2.6. *Suppose that Assumptions 5.2.1 and 5.2.2 hold, then*

$$V(\xi) \;=\; \sup_{H\in\mathcal{H}} \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big] \;=\; \sup_{H\in\mathcal{H}} \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big].$$

Proof. To simplify the notation, let us write $\Delta X_t := X_t - X_{t-1}$. We shall follow closely the arguments in Proposition 3.3 of [113].

Step 1: Fix $\eta \in \mathcal{A}_0$ and define the $\tilde{\mathbb{F}}$-predictable process H by $H_t := \sum_{s=1}^t \Delta H_s$ with $\Delta H_t := \eta_{t-1}$ for $t = 1,\cdots,T$. By rearranging all terms, we have

$$\left(\xi+\sum_{t=0}^T \eta_t\right)^d = \left(\xi+\sum_{t=0}^T \eta_t\right)\cdot X_T = \sum_{t=1}^T H_t\cdot\Delta X_t + \sum_{t=0}^T \eta_t\cdot X_t + g \;\le\; \sum_{t=1}^T H_t\cdot\Delta X_t + g,$$

where the last inequality follows by the fact that $\eta_t \in -K_t$ and hence $\eta_t\cdot X_t \le 0$. As U is non-decreasing, it follows that

$$\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[U\Big(\Big(\xi+\sum_{t=0}^T \eta_t\Big)^d\Big)\Big] \;\le\; \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big],$$

which yields that

$$V(\xi) \;\le\; \sup_{H\in\mathcal{H}} \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big].$$

By the same argument using $\overline{\mathcal{P}}_{\rm int}$ to replace $\overline{\mathcal{P}}$, we can similarly obtain the inequality

$$V(\xi) \;\le\; \sup_{H\in\mathcal{H}} \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big].$$

Step 2: To prove the reverse inequality, we fix $H \in \mathcal{H}$. Define $\eta = (\eta_t)_{0\le t\le T}$ by $\eta_t^i := \Delta H_{t+1}^i$, $t \le T-1$ and $\eta_T^i := -\xi^i - \sum_{s=0}^{T-1}\eta_s^i$ for $i \le d-1$, and

$$\eta_t^d(\omega) := \inf_{\theta\in\Lambda} m_t^d(\omega,\theta) \text{ with } m_t^d(\bar\omega) := -\sum_{i=1}^{d-1}\eta_t^i(\omega)X_t^i(\bar\omega),\ t\le T. \tag{5.2.8}$$

for all $\bar\omega = (\omega,\theta) \in \overline{\Omega}$. As $m_t^d(\omega,\theta)$ is bounded and continuous in θ, η_t^d is $\mathcal{F}_t$-measurable by the

Measurable Maximum Theorem (see e.g. Theorem 18.19 of [102]). From the construction, we know $\eta \in \mathcal{A}_0$. Thus we have

$$\begin{aligned}\inf_{\theta\in\Lambda}\big((H\circ X)_T+g\big)(\cdot,\theta) &= \inf_{\theta\in\Lambda}\Big\{\Big(\sum_{t=0}^{T}\eta_t+\xi\Big)\cdot X_T-\sum_{t=0}^{T-1}\eta_t\cdot X_t\Big\}(\cdot,\theta)\\ &= \inf_{\theta\in\Lambda}\Big\{\Big(\sum_{t=0}^{T}\eta_t+\xi\Big)\cdot X_T\Big\}(\cdot,\theta)-\sum_{t=0}^{T-1}\sup_{\theta\in\Lambda}\Big\{\eta_t\cdot X_t\Big\}(\cdot,\theta).\\ &= \inf_{\theta\in\Lambda}\Big\{\Big(\sum_{t=0}^{T}\eta_t+\xi\Big)\cdot X_T\Big\}(\cdot,\theta) \;=\; \Big(\xi+\sum_{t=0}^{T}\eta_t\Big)^d, \qquad (5.2.9)\end{aligned}$$

where in the second equality we exchange the the infimum and the summation, because each X_t depends on θ only through θ_t for $t=0,\cdots,T$. Let $\varepsilon>0$, we can use a measurable selection argument (see e.g. Proposition 7.50 of [92]) to choose a universally measurable map $\omega\in\Omega\mapsto\theta_\varepsilon(\omega)\in\Lambda$ such that, for all $\omega\in\Omega$, one has

$$U\Big(\big(H\circ X\big)_T\big(\omega,\theta_\varepsilon(\omega)\big)+g\big(\omega,\theta_\varepsilon(\omega)\big)\Big) \;\le\; \inf_{\theta\in\Lambda}U\Big(\big(H\circ X\big)_T(\omega,\theta)+g(\omega,\theta)\Big)+\varepsilon,$$

where the r.h.s. term is a universally measurable random variable defined on Ω. Then given $\mathbb{P}\in\mathcal{P}$, one defines $\overline{\mathbb{P}}_\varepsilon:=\mathbb{P}\circ(\omega,\theta_\varepsilon(\omega))^{-1}\in\overline{\mathcal{P}}$ and obtains

$$\mathbb{E}^{\overline{\mathbb{P}}_\varepsilon}\Big[U\big((H\circ X)_T+g\big)\Big] \;\le\; \mathbb{E}^{\mathbb{P}}\Big[\inf_{\theta\in\Lambda}U\Big(\big(H\circ X\big)_T(\cdot,\theta)+g(\cdot,\theta)\Big)\Big] \;+\; \varepsilon.$$

By arbitrariness of $\varepsilon>0$ and the fact that $\overline{\mathbb{P}}_\varepsilon\in\overline{\mathcal{P}}$, it follows that

$$\begin{aligned}\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}}\mathbb{E}^{\overline{\mathbb{P}}}\Big[U\big((H\circ X)_T+g\big)\Big] &\le \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\Big[\inf_{\theta\in\Lambda}U\Big((H\circ X)_T(\cdot,\theta)+g(\cdot,\theta)\Big)\Big] \qquad (5.2.10)\\ &= \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\Big[U\Big(\inf_{\theta\in\Lambda}\big[(H\circ X)_T(\cdot,\theta)+g(\cdot,\theta)\big]\Big)\Big]\\ &= \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\Big[U\Big(\Big(\xi+\sum_{t=0}^{T}\eta_t\Big)^d\Big)\Big].\end{aligned}$$

This leads to

$$\sup_{H\in\mathcal{H}}\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}}\mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g+(H\circ X)_T\Big)\Big] \;\le\; V(\xi),$$

and hence we have the desired equality.

Step 3: For the case with $\overline{\mathcal{P}}_{\rm int}$ in place of $\overline{\mathcal{P}}$, it is enough to notice as in *Step 2* that

$$\inf_{\theta\in\Lambda_{\rm int}(\cdot)}\big[(H\circ X)_T(\cdot,\theta)+g(\cdot,\theta)\big]=\Big(\xi+\sum_{t=0}^{T}\eta_t\Big)^d,$$

where $\Lambda_{\rm int}(\omega)$ is defined as the collection of $\theta\in\Lambda$ such that $S_t(\omega)\theta_t\in{\rm int}K_t^*(\omega)$.

Next, for each $\theta\in\Lambda$, we define $A_t^\theta(\omega):=\emptyset$ for $s\neq t$ and $A_t^\theta(\omega):=\{S_t(\omega)\theta_t\in{\rm int}K_t^*(\omega)\}$. Note that $\omega\mapsto{\rm int}K_t^*(\omega)$ is $\mathcal{F}_t^0$-measurable. Then $\big\{(\omega,y)\in\Omega_t\times\mathbb{R}\;:S_t(\omega)\theta_t=y$ and $y\in{\rm int}K_t^*(\omega)\big\}$ is a Borel set and hence $\omega\mapsto 1_{A_t^\theta(\omega)}$ is a universally measurable map. We then define

the universally measurable probability kernels by

$$q_t^\theta : \omega \in \Omega \mapsto q_t^\theta(\cdot|\omega) := \delta_{\theta_t} 1_{A_t^\theta(\omega)} + \delta_{\mathbf{1}} 1_{(A_t^\theta(\omega))^c} \in \mathfrak{B}(\Lambda_1),\ t \le T, \tag{5.2.11}$$

where $\mathbf{1}$ is the vector of $\mathbb{R}^d$ with all entries equal to 1, and $\mathfrak{B}(\Lambda_1)$ denotes the collection of all Borel probability measures on Λ_1.

It follows that $\mathbb{P} \otimes (q_0^\theta \otimes q_1^\theta \otimes \cdots \otimes q_T^\theta) \in \overline{\mathcal{P}}_{\rm int}$ for every $\mathbb{P} \in \mathcal{P}$. Then it suffices to argue as in *Step 2* above to obtain that

$$\begin{aligned} \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}} \mathbb{E}^{\overline{\mathbb{P}}}\Big[U\Big(g + (H \circ X)_T\Big)\Big] &\le \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[U\Big(\inf_{\theta \in \Lambda_{\rm int}(\cdot)} \Big((H \circ X)_T(\cdot,\theta) + g(\cdot,\theta)\Big)\Big)\right] \\ &= \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[U\Big(\Big(\xi + \sum_{t=0}^T \eta_t\Big)^d\Big)\Big], \end{aligned}$$

and we hence conclude as in *Step 2*. □

5.2.3 The robust no-arbitrage condition of Bouchard and Nutz

To conclude, we will discuss the no-arbitrage condition on Ω and its link to that on the enlarged space $\overline{\Omega}$.

Definition 5.2.7. (i) *We say the robust no-arbitrage condition of second kind* NA2($\mathcal{P}$) *on* Ω *holds true if for all* $t \le T-1$ *and all* $\xi \in L^0(\mathcal{F}_t)$,

$$\xi \in K_{t+1}\ \ \mathcal{P}\text{-q.s.} \quad \textit{implies} \quad \xi \in K_t\ \ \mathcal{P}\text{-q.s.}$$

(ii) *Let* $(\mathbb{Q}, Z)$ *be a couple where* $\mathbb{Q} \in \mathfrak{B}(\Omega)$ *and* $Z = (Z_t)_{t=0,\cdots,T}$ *an adapted process,* $(\mathbb{Q}, Z)$ *is called a strict consistent price system (SCPS) if* $\mathbb{Q} \lll \mathcal{P}$, $Z_t \in {\rm int}K_t^*$ $\mathbb{Q}$*-a.s. for all* $t = 0, \cdots, T$ *and* Z *is a* $\mathbb{Q}$*-martingale.*

We denote by $\mathcal{S}$ the collection of all SCPS, and also denote the subset

$$\mathcal{S}_0 := \big\{(\mathbb{Q}, Z) \in \mathcal{S} \text{ such that } Z^d \equiv 1\big\}. \tag{5.2.12}$$

Remark 5.2.8. *As stated in the fundamental theorem of asset pricing proved in [43] (see also [42, 115]), the no-arbitrage condition* NA2($\mathcal{P}$) *is equivalent to: for all* $t \le T-1$, $\mathbb{P} \in \mathcal{P}$ *and* $\mathcal{F}_t$*-random variable* Y *taking value in* ${\rm int}K_t^*$*, there exists a SCPS* $(\mathbb{Q}, Z)$ *such that* $\mathbb{P} \ll \mathbb{Q}$, $\mathbb{P} = \mathbb{Q}$ *on* $\mathcal{F}_t$ *and* $Y = Z_t$ $\mathbb{P}$*-a.s..*

On the enlarged space $\overline{\Omega}$, we also follow [1] to introduce a notion of the robust no-arbitrage condition.

Definition 5.2.9. *We say that the robust no-arbitrage condition* NA($\overline{\mathcal{P}}_{\rm int}$) *on* $\overline{\Omega}$ *holds true if, for every* $H \in \overline{\mathcal{H}}$,

$$(H \circ X)_T \ge 0,\ \overline{\mathcal{P}}_{\rm int}\text{-q.s.} \quad \Longrightarrow \quad (H \circ X)_T = 0,\ \overline{\mathcal{P}}_{\rm int}\text{-q.s.}$$

Remark 5.2.10. *The fundamental theorem of asset pricing in [1] proves that the condition* NA($\overline{\mathcal{P}}_{\rm int}$) *(resp.* NA($\overline{\mathcal{P}}$) *) is equivalent to : for all* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$ *(resp.* $\overline{\mathcal{P}}$ *), there exists* $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$ *such that* $\overline{\mathbb{P}} \ll \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\rm int}$ *(resp.* $\overline{\mathcal{P}}$ *) and* X *is an* $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$*-martingale.*

Hereafter, we denote by $\overline{\mathcal{Q}}_0$ the collection of measures $\overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega})$ such that $\overline{\mathbb{Q}} \ll\ll \overline{\mathcal{P}}_{\rm int}$ and X is an $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale. The above two no-arbitrage conditions on Ω and on $\overline{\Omega}$ are related by Proposition 2.16 of [113], that we recall as below.

Proposition 5.2.11. *The condition* NA2($\mathcal{P}$) *on* Ω *is equivalent to the condition* NA($\overline{\mathcal{P}}_{\rm int}$) *on* $\overline{\Omega}$.

5.3 Exponential utility maximization

Starting from this section, we will restrict ourselves to the case of the exponential utility function, i.e.,

$$U(x) := -\exp(-\gamma x), \quad \text{for some constant } \gamma > 0,$$

and provide a detailed study on the corresponding utility maximization problem.

We will consider a general context, where one is allowed to trade some liquid options statically at the initial time whose payoffs would also contribute to the terminal wealth. Namely, for $e \in \mathbb{N} \cup \{0\}$, there are a finite class of $\mathcal{F}_T^0$-measurable random vectors $\zeta_i : \Omega \to \mathbb{R}^d$, $i = 1, \cdots, e$, where each ζ_i represents the payoff of some option i labeled in units of d risky assets. Let $\xi : \Omega \to \mathbb{R}^d$ represent the payoff of the random endowment, then our maximization problem is given by.

$$V(\xi, \gamma) \ := \ \sup_{(\ell,\eta)\in\mathcal{A}_e} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[U\Big(\Big(\xi + \sum_{i=1}^{e} \big(\ell_i \zeta_i - |\ell_i| c_i \mathbf{1}_d \big) + \sum_{t=0}^{T} \eta_t \Big)^d \Big) \right]. \tag{5.3.13}$$

where $\mathbf{1}_d$ is the vector with all components equal to 0 but the last one that is equal to 1 and $\mathcal{A}_e$ denotes the collection of all $(l, \eta) \in \mathbb{R}^e \times \mathcal{A}$ such that $\xi + \sum_{i=1}^{e} \big(\ell_i \zeta_i - |\ell_i| c_i \mathbf{1}_d \big) + \sum_{t=0}^{T} \eta_t \Big)^i = 0$ for $i = 1, \cdots, d-1$. In above, we write γ in $V(\xi, \gamma)$ to emphasize the dependence of value in parameter γ in the utility function U. Also, each static option ζ_i has price 0, but the static trading induces the proportional transaction cost with rate $c_i > 0$.

5.4 The convex duality result

In the robust frictionless setting, the same exponential utility maximization problem has been studied by Bartl [2], in which a convex duality theorem has been established. Here, we apply and generalize their results in our context with transaction costs under weaker market conditions.

Let us introduce a robust version of the relative entropy associated to a probability measure $\mathbb{Q}$ as

$$\mathcal{E}(\mathbb{Q}, \mathcal{P}) := \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{E}(\mathbb{Q}, \mathbb{P}), \quad \text{where } \mathcal{E}(\mathbb{Q}, \mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}} \right], & \text{if } \mathbb{Q} \ll \mathbb{P}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.4.14}$$

Note that $\mathcal{S}_0$ is a subset of the collection of SCPS $(\mathbb{Q}, Z)$ defined in (5.2.12), we then define

$$\mathcal{S}_e^* := \Big\{ (\mathbb{Q}, Z) \in \mathcal{S}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[(\xi \cdot Z_T)_- \big] + \mathcal{E}(\mathbb{Q}, \mathcal{P}) < +\infty \text{ and } \mathbb{E}^{\mathbb{Q}}\big[\zeta_i \cdot Z_T \big] \in [-c_i, c_i],\ i = 1, \cdots, e \Big\}.$$

Theorem 5.4.1. *Let ξ and $(\zeta_i)_{i\leq e}\colon \Omega \to \mathbb{R}^d$ be Borel measurable and assume that* NA2($\mathcal{P}$) *holds. Assume either that $e = 0$, or that $e \geq 1$ and for all $\ell \in \mathbb{R}^e$ and $\eta \in \mathcal{A}$,*

$$\sum_{i=1}^{e} \big(\ell_i\zeta_i - |\ell_i|c_i\mathbf{1}_d\big) + \sum_{t=0}^{T} \eta_t \in K_T \ \mathcal{P}\text{-q.s.} \quad \Longrightarrow \quad \ell = 0. \tag{5.4.15}$$

Then, we have

$$V(\xi,\gamma) \ = \ -\exp\Big(-\inf_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\gamma\xi\cdot Z_T\big] + \mathcal{E}(\mathbb{Q},\mathcal{P})\big\}\Big), \tag{5.4.16}$$

Moreover, the infimum over $(\ell,\eta) \in \mathcal{A}_e$ is attained by an optimal strategy $(\hat{\ell},\hat{\eta})$.

Remark 5.4.2. *Note that up to taking logarithm on both sides and replacing $\gamma\xi$ by $-\xi$, the equality* (5.4.16) *is equivalent to*

$$\begin{aligned} &\inf_{(\ell,\eta)\in\mathcal{A}_e}\sup_{\mathbb{P}\in\mathcal{P}}\log\mathbb{E}^{\mathbb{P}}\left[\exp\left(\left(\xi - \sum_{i=1}^{e}\left(\ell_i\zeta_i - |\ell_i|c_i\mathbf{1}_d\right) - \sum_{t=0}^{T}\eta_t\right)^d\right)\right] \\ &= \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \mathcal{E}(\mathbb{Q},\mathcal{P})\big\}. \end{aligned} \tag{5.4.17}$$

Remark 5.4.3. *When $e \geq 1$, ζ_i is considered as statically traded options and $c_i > 0$ is the corresponding proportional transaction cost, then the condition* (5.4.15) *should be understood as a kind of robust no-arbitrage condition as defined in [1]. For simplicity, let us consider the case $e = 1$. By following arguments in Proposition 3.3 of [113], $\ell_1\zeta_1 - |\ell_1|c_1\mathbf{1}_d + \sum_{t=0}^{T}\eta_t \in K_T$ $\mathcal{P}$-q.s. can be shown as equivalent to*

$$\ell_1 g_1\big(\bar{\omega},\widehat{\theta}\big) + \Big(\sum_{t=1}^{T} H_t\Delta X_t\Big)(\bar{\omega}) \geq 0, \ \overline{\mathcal{P}}\text{-q.s. and for both } \widehat{\theta} = \pm 1,$$

where $H_t := \sum_{s=0}^{t-1}\eta_s$ and $g_1(\bar{\omega},\pm 1) := \zeta_1\cdot X_T \pm c_1$. The robust no-arbitrage condition in Definition 5.2.9 will lead to

$$\ell_1 g_1\big(\bar{\omega},\widehat{\theta}\big) + \Big(\sum_{t=1}^{T} H_t\Delta X_t\Big)(\bar{\omega}) = 0, \ \overline{\mathcal{P}}\text{-q.s. and for both } \widehat{\theta} = \pm 1.$$

As $g_1(\bar{\omega},1) \neq g_1(\omega,-1)$ when $c_1 > 0$, one obtains $\ell_1 = 0$.

Remark 5.4.4. (i) *The existence of optimal trading strategy $(\hat{\ell},\hat{\eta})$ in Theorem 5.4.1 is an auxiliary result in the proof of duality* (5.4.16) *in our context with exponential utility function $U(x) := -\exp(-\gamma x)$. Both duality and the existence of optimal strategy rely crucially on the minimax argument (Lemma 5.5.10) which uses the affine feature of the exponential utility.*

(ii) *In the robust context and for general utility functions (with or without transaction cost), different results on the existence of the optimal strategy have been obtained in the literature. Nutz [59] seems to be the first to introduce this discrete time robust utility maximization problem and obtains the existence result for general utility functions bounded from above and defined on the positive real line. Blanchard and Carassus [62] were able to relax the boundedness condition to some integrability condition. Neufeld and Sikic [61] study the robust utility maximization problem*

with friction and obtain some existence result under a linear type of no-arbitrage condition. Rasonyi and Meireles-Rodrigues [64] use a Komlós-type argument to prove the existence of the optimal strategy. Bartl et al. [65] study similar problem by the medial limit argument.

(iii) *After the completion of our paper, Bayraktar and Burzoni [114] provided a generalization of the randomization approach in [113] and proved a pricing-hedging duality under a weaker no-arbitrage condition than the* NA2($\mathcal{P}$) *condition. Their generalized randomization approach should also allow to study the above utility maximization problem under the weak no-arbitrage condition.*

5.4.1 Properties of utility indifference prices

It is well known that the superhedging price is too high in practice. As an alternative way, the utility-based indifference price has been actively studied, in which the investor's risk aversion is inherently incorporated. This section presents an application of the convex duality relationship (5.4.17) for the exponential utility maximization and provides some interesting features of indifference prices in the presence of both proportional transaction costs and model uncertainty. Generally speaking, the indifference pricing in our setting can be generated by semi-static trading strategies on risky assets and liquid options.

In the robust framework, similar to Theorem 2.4 of [2] in the frictionless model, the duality representation (5.4.17) can help us to derive that the asymptotic indifference prices converge to the superhedging price as the risk aversion $\gamma \to \infty$ regardless of the transaction costs. To see this, let us first recall the superhedging price defined by

$$\begin{aligned}\pi(\xi) &:= \inf\left\{ y + \sum_{i=1}^{e} c_i|\ell_i| : y\mathbf{1}_d + \sum_{i=1}^{e} \ell_i\zeta_i + \sum_{t=0}^{T} \eta_t - \xi \in K_T, \mathcal{P}-q.s., (\ell,\eta) \in \mathcal{A}_e \right\} \\ &= \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e} \mathbb{E}^{\mathbb{Q}}[\xi \cdot Z_T],\end{aligned}$$

where the equality follows from Theorem 3.1 of [113] with

$$\mathcal{S}_e := \Big\{ (\mathbb{Q}, Z) \in \mathcal{S}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[(\zeta_i \cdot Z_T)\big] \in [-c_i, c_i],\ i = 1, \cdots, e \Big\}.$$

The indifference price $\pi_\gamma(\xi) \in \mathbb{R}$ of derivative option ξ is, one the other hand, defined by equation

$$V(0\mathbf{1}_d,\ \gamma) \ = \ V\big(\pi_\gamma(\xi)\mathbf{1}_d - \xi,\ \gamma\big), \tag{5.4.18}$$

where $V(\cdot)$ is defined by (5.3.13). Plugging the expression of $V(\cdot)$ into (5.4.18), and recall that $U(x) = -e^{-\gamma x}$, we obtain

$$\begin{aligned}\exp(-\gamma\pi_\gamma(\xi)) \quad \times \quad & \sup_{(\ell,\eta)\in\mathcal{A}_e} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[-\exp\Big(-\gamma\Big(-\xi + \sum_{i=1}^{e} (\ell_i\zeta_i - |\ell_i|c_i\mathbf{1}_d) + \sum_{t=0}^{T} \eta_t \Big)^d \Big)\Big] \\ = \quad & \sup_{(\ell,\eta)\in\mathcal{A}_e} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Big[-\exp\Big(-\gamma\Big(\sum_{i=1}^{e} (\ell_i\zeta_i - |\ell_i|c_i\mathbf{1}_d) + \sum_{t=0}^{T} \eta_t \Big)^d \Big)\Big]. \end{aligned} \tag{5.4.19}$$

By the duality representation (5.4.16), we finally have that

$$\pi_\gamma(\xi) = \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\} - \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}. \tag{5.4.20}$$

The formula (5.4.20) yields directly the next few properties of the utility indifference price.

Lemma 5.4.5. *The following basic properties hold:*

(i) $\pi_\gamma(\xi)$ *does not depend on the initial wealth* x_0.

(ii) $\pi_\gamma(\xi)$ *is increasing in* γ *(monotonicity in* γ*).*

(iii) $\pi_\gamma(\beta\xi) = \beta\pi_{\beta\gamma}(\xi)$ *for any* $\beta\in(0,1]$ *(volume scaling).*

(iv) $\pi_\gamma(\xi + c) = c + \pi_\gamma(\xi)$ *for* $c\in\mathbb{R}$ *(translation invariance).*

(v) $\pi_\gamma(\alpha\xi_1 + (1-\alpha)\xi_2) \leqslant \alpha\pi_\gamma(\xi_1) + (1-\alpha)\pi_\gamma(\xi_2)$ *(convexity).*

(vi) $\pi_\gamma(\xi_1) \leqslant \pi_\gamma(\xi_2)$ *if* $\xi_1 \leqslant \xi_2$ *(monotonicity).*

The next result shows the risk-averse asymptotics on the utility indifference prices. Similar results can also be found in [116, 63, 2].

Proposition 5.4.6. *In the robust setting of Theorem 5.4.1 with proportional transaction costs, we have*

$$\pi(\xi) = \lim_{\gamma\to\infty}\pi_\gamma(\xi). \tag{5.4.21}$$

We postpone the proof of the above result to Section 5.5.4, as it demands some notations and results given afterwards.

Remark 5.4.7. *Observing the scaling property in item (iii) of Lemma* 5.4.5*, the limit* (5.4.21) *can be rewritten as* $\lim_{\beta\to\infty}\frac{1}{\beta}\pi_\gamma(\beta\xi) = \pi(\xi)$*, in which the term* $\frac{1}{\beta}\pi_\gamma(\beta\xi)$ *can be understood as the price per unit for a given amount volume* β *of the contingent claim* ξ*.*

Furthermore, with increasing risk aversion, the convex duality result (5.4.17) also yields that the optimal hedging strategies under the exponential utility preference converge to the superhedging counterpart in the following sense.

Proposition 5.4.8. *We have that*

$$\lim_{\gamma\to\infty}\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\left[\left(\pi(\xi)\mathbf{1}_d + \sum_{i=1}^{e}\left(\ell_i^{\star,\gamma}\zeta_i - |\ell_i^{\star,\gamma}|c_i\mathbf{1}_d\right) + \sum_{t=0}^{T}\eta_t^{\star,\gamma} - \xi\right)^-\right] = 0,$$

where $(\ell^{\star,\gamma}, \eta^{\star,\gamma})$ *is an optimal semi-static strategy to the problem* (5.4.17) *under the risk aversion level* γ*.*

Proof. Let us set $\Gamma_\gamma := \pi(\xi)\mathbf{1}_d + \sum_{i=1}^{e}\left(\ell_i^{\star,\gamma}\zeta_i - |\ell_i^{\star,\gamma}|c_i\mathbf{1}_d\right) + \sum_{t=0}^{T}\eta_t^{\star,\gamma} - \xi$ and it follows by (5.4.17) that

$$\sup_{\mathbb{P}\in\mathcal{P}}\log\mathbb{E}^{\mathbb{P}}[e^{-\gamma\Gamma_\gamma}] = \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{\gamma\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \gamma\pi(\xi) - \mathcal{E}(\mathbb{Q},\mathcal{P})\right\}. \tag{5.4.22}$$

If $\pi(\xi) = +\infty$, it is clear that $\sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}[e^{-\gamma\Gamma_\gamma}] = -\infty$. Otherwise, if $\pi(\xi) < +\infty$, it follows by item (ii) of Lemma 5.4.5 that $\pi_\gamma(\xi)$ is increasing in γ and moreover $\pi_\gamma(\xi) \leqslant \pi(\xi)$. Therefore, it yields that $\sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}[e^{-\gamma\Gamma_\gamma}] \leqslant 0$ and hence $\mathbb{E}^{\mathbb{P}}[e^{-\gamma\Gamma_\gamma}] \leqslant 1$ uniformly for all $\mathbb{P} \in \mathcal{P}$. By Jensen's inequality, we have

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\Gamma_\gamma^-] \leqslant \frac{1}{\gamma} \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}[e^{\gamma\Gamma_\gamma^-}] \ \leqslant \ \frac{1}{\gamma} \sup_{\mathbb{P}\in\mathcal{P}} \log(1 + \mathbb{E}^{\mathbb{P}}[e^{-\gamma\Gamma_\gamma}]),$$

which completes the proof. □

Remark 5.4.9. *Similar results have been obtained in Corollary* 5.1 *and Theorem* 5.2 *of [54] in the classical dominated frictionless market model. Thanks to the convex duality* (5.4.17)*, this paper makes nontrivial extension of the asymptotic convergence on risk aversion level to the setting with both proportional transaction costs and model uncertainty.*

Again, based on the convex duality representation obtained in the enlarged space, the continuity property and Fatou property of the indifference prices can be shown in the following sense.

Proposition 5.4.10. (i) *If* $(\xi_n)_{n\in\mathbb{N}}$ *is a sequence of option payoffs such that*

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} \mathbb{E}^{\mathbb{Q}}[(\xi_n - \xi)\cdot Z_T] \to 0 \quad \textit{and} \quad \inf_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} \mathbb{E}^{\mathbb{Q}}[(\xi_n - \xi)\cdot Z_T] \to 0. \tag{5.4.23}$$

then $\pi_\gamma(\xi_n) \to \pi_\gamma(\xi)$ *for any* $\gamma > 0$.

(ii) *For* $\xi_n \geqslant 0$*, we have*

$$\pi_\gamma(\liminf_n \xi_n) \leqslant \liminf_n \pi_\gamma(\xi_n). \tag{5.4.24}$$

(iii) *If* $(\xi_n)_{n\in\mathbb{N}}$ *is a sequence of option payoffs such that* $\xi_n \nearrow \xi$*,* $\mathbb{P}$*-a.s., then* $\pi_\gamma(\xi_n) \nearrow \pi_\gamma(\xi)$.

Proof. (i) Recall that $\pi_\gamma(\xi) = \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} \left\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\} - \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} \left\{-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}$ in (5.4.20), we can obtain that

$$\begin{aligned} |\pi_\gamma(\xi_n) - \pi_\gamma(\xi)| &= \Big| \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} \left\{\mathbb{E}^{\mathbb{Q}}\big[\xi_n\cdot Z_T\big] - \frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\} - \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} \left\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\} \Big| \\ &\leqslant \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*} |\mathbb{E}^{\mathbb{Q}}[(\xi_n - \xi)\cdot Z_T]|. \end{aligned}$$

The continuity $\pi_\gamma(\xi_n) \to \pi_\gamma(\xi)$ follows directly by (5.4.23).

(ii) The Fatou property can be derived by observing that

$$\begin{aligned}\pi_\gamma(\liminf_n \xi_n) &= \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{\mathbb{E}^{\mathbb{Q}}\big[\liminf_n \xi_n\cdot Z_T\big]-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}-\sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}\\ &\leqslant \sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{\liminf_n \mathbb{E}^{\mathbb{Q}}\big[\xi_n\cdot Z_T\big]-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}-\sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}\\ &\leqslant \liminf_n\left(\sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{\mathbb{E}^{\mathbb{Q}}\big[\xi_n\cdot Z_T\big]-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}-\sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\left\{-\frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P})\right\}\right)\\ &= \liminf_n \pi_\gamma(\xi_n).\end{aligned}$$

(iii) By the Fatou property from part (ii) and item (vi) of Lemma 5.4.5, we have

$$\pi_\gamma(\xi)\geqslant\liminf_n\pi_\gamma(\xi_n)\geqslant\pi_\gamma(\xi),$$

which completes the proof. □

5.5 Proof of main results

This section provides the technical arguments to establish the convex duality (5.4.17) and we shall first work in the fictitious frictionless market on the enlarged space. All three results, namely the convex duality theorem, the dynamic programming principle and the existence of the optimal portfolio will be confirmed. Translating the transaction costs into additional randomness on the enlarged space in both primal and dual problems plays a crucial role to develop some key equivalences.

5.5.1 Reformulation of the dual problem

As a first step to reduce the complexity of the proof, the standard dual problem based on CPS in the model with transaction costs will be reformulated on the enlarged dual space. Define

$$\overline{\mathcal{Q}}^* := \big\{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_0 \ : \mathbb{E}^{\overline{\mathbb{Q}}}\big[(\xi\cdot X_T)_-\big]+\mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}}_{\rm int})<\infty\big\},$$

where $\mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}}_{\rm int})$ is defined exactly as $\mathcal{E}(\mathbb{Q},\mathcal{P})$ in (5.4.14). For any universally measurable random variable $\varphi:\overline{\Omega}\to\mathbb{R}_+$, we further define

$$\overline{\mathcal{Q}}^*_\varphi := \big\{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^* \ : \mathbb{E}^{\overline{\mathbb{Q}}}[\varphi]<\infty\big\} \text{ and } \overline{\mathcal{Q}}^*_\varphi(0,\theta_0) := \big\{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*_\varphi \ : \overline{\mathbb{Q}}[\Theta_0=\theta_0]=1\big\}.$$

The function φ will be chosen depending on the context, it allows to control the integrability of some extra random variables when one considers the subsets of $\overline{\mathcal{Q}}^*$ and also in some iteration arguments.

Lemma 5.5.1. *For any universally measurable random vector $\xi:\Omega\to\mathbb{R}^d$, one has*

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big]-\mathcal{E}(\mathbb{Q},\mathcal{P})\big\} = \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*}\big\{\mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi\cdot X_T\big]-\mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}}_{\rm int})\big\}.$$

Proof. First, for a given $(\mathbb{Q}, Z) \in \mathcal{S}_0^*$, we associate the probability kernel:

$$q^Z : \omega \in \Omega \mapsto q^Z(\cdot|\omega) := \delta_{(Z/S)(\omega)} \in \mathfrak{B}(\Lambda),$$

and define $\overline{\mathbb{Q}} := \mathbb{Q} \otimes q^Z$. The construction implies that $\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] = \mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big]$ and that $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*$. Moreover, for every $\mathbb{P} \in \mathcal{P}$, one can similarly define $\overline{\mathbb{P}} := \mathbb{P} \otimes q^Z \in \overline{\mathcal{P}}_{\rm int}$. If $\mathbb{Q} \ll \mathbb{P}$, one has $\overline{\mathbb{Q}} \ll \overline{\mathbb{P}}$ and $d\mathbb{Q}/d\mathbb{P} = d\overline{\mathbb{Q}}/d\overline{\mathbb{P}}$, $\overline{\mathbb{P}}$-a.s. If $\mathbb{Q} \ll \mathbb{P}$ is not true, then $\mathcal{E}(\mathbb{Q}, \mathbb{P}) = \infty$ by definition. This implies that $\mathcal{E}(\mathbb{Q}, \mathcal{P}) \geq \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})$. Therefore,

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0^*} \big\{\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\big\} \;\leq\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})\big\}.$$

Conversely, let us fix $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*$, and define $\mathbb{Q} := \overline{\mathbb{Q}}|_\Omega$ and $Z_t := \mathbb{E}^{\overline{\mathbb{Q}}}\big[X_t\big|\mathcal{F}_t\big]$ for $t \leqslant T$. As $\overline{\mathbb{Q}} \ll \overline{\mathbb{P}}$ for some $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$, then $\mathbb{Q} \ll \mathbb{P} := \overline{\mathbb{P}}|_\Omega \in \mathcal{P}$. Moreover, the fact that X is an $(\overline{\mathbb{F}}, \overline{\mathbb{Q}})$-martingale implies that Z is an $(\mathbb{F}, \mathbb{Q})$-martingale. Then, $(\mathbb{Q}, Z) \in \mathcal{S}_0^*$ and $\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] = \mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big]$. Now as $d\mathbb{Q}/d\mathbb{P} = \mathbb{E}^{\overline{\mathbb{P}}}[d\overline{\mathbb{Q}}/d\overline{\mathbb{P}}|\mathcal{F}_T]$ and $x \mapsto x\log(x)$ is convex on $\mathbb{R}_+$, we have $\mathcal{E}(\mathbb{Q}, \mathbb{P}) \leq \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{P}})$ by Jensen's inequality. It follows that

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_0^*} \big\{\mathbb{E}^{\mathbb{Q}}\big[\xi \cdot Z_T\big] - \mathcal{E}(\mathbb{Q}, \mathcal{P})\big\} \;\geq\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}\big[\xi \cdot X_T\big] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})\big\},$$

and we hence conclude the proof. □

5.5.2 Proof of Theorem 5.4.1(Case $e = 0$)

In view of Lemma 5.5.1 and Proposition 5.2.6, one can first establish the duality result of the utility maximization problem on the enlarged space $\overline{\Omega}$, in order to prove Theorem 5.4.1.

Proposition 5.5.2. *Let $g := \xi \cdot X_T$ and $\mathrm{NA}(\overline{\mathcal{P}}_{\rm int})$ hold true. Then for any universally measurable random variable $\varphi : \overline{\Omega} \to \mathbb{R}_+$, one has*

$$\begin{aligned} \overline{V} \;:=\; \inf_{H\in\mathcal{H}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + (H \circ X)_T\big)\big] \;&=\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})\big\} \quad (5.5.25) \\ &=\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*_\varphi} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})\big\}. \end{aligned}$$

Moreover, the infimum of the problem $\overline{V}$ is attained by some optimal trading strategy $\widehat{H} \in \mathcal{H}$.

Remark 5.5.3. *The above duality result is similar to that in [2], but differs substantially with theirs in the following two points:*

(i) *In our current work, we have relaxed the strong one-period no-arbitrage condition for all $\omega_t \in \Omega_t$ assumed in [2]. Indeed, the strong no-arbitrage condition is needed in [2] because their duality and dynamic programming are mixed with each other. More precisely, with the notations in [2, Section 4], they need the relation "$\mathcal{E}_t(\omega, x) = \mathcal{D}_t(\omega) + x$" to hold for all t and $\omega \in \Omega_t$ to guarantee the measurability of $\mathcal{E}_t$ through $\mathcal{D}_t$ (see in particular their equation (21) and their Proof of Lemma 4.6). In Appendix 5.6.3, we shall give more details on this point.*

(ii) *It is worth noting that the reformulations in Proposition 5.2.6 on the enlarged space do not exactly correspond to standard quasi-sure utility maximization problem. Indeed, we still restrict the class of strategies to $\widetilde{\mathbb{F}}$-predictable processes, as opposed to $\overline{\mathbb{F}}$-predictable processes. The fact*

that the formulation with these two different filtrations are equivalent will be proved by using a minimax argument.

Proof of Theorem 5.4.1 (case $e = 0$) First, using Lemma 5.5.1 and Proposition 5.2.6, the duality (5.4.17) can be deduced immediately from (5.5.25) in Proposition 5.5.2. Moreover, given the optimal trading strategy $\widehat{H} \in \mathcal{H}$ in Proposition 5.5.2, we can construct $\hat{\eta}$ by (5.2.8) and show its optimality by almost the same arguments as in Step 2 of Proposition 5.2.6 (ii). □

In the rest of Section 5.5.2, we will provide the proof of Proposition 5.5.2 in several steps.

The weak duality As in the classical results, one can easily obtain a weak duality result.

Lemma 5.5.4. *For any universally measurable function $g : \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$, one has*

$$\inf_{H\in\mathcal{H}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + (H \circ X)_T\big)\big] \;\geq\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})\big\}.$$

Proof. Using the result in the [2, Proof of Theorem 4.1 - dynamic programming principle], one knows that for any $H \in \mathcal{H}$, $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$ and $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*$, one has

$$\log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + (H \circ X)_T\big)\big] \;\geq\; \mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{P}}).$$

(Note that $\mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathbb{P}}) = \infty$ if $\overline{\mathbb{Q}}$ is not dominated by $\overline{\mathbb{P}}$.) Therefore it is enough to take supremum over $\overline{\mathbb{Q}}$ (and $\overline{\mathbb{P}}$) and then take infimum over $H \in \mathcal{H}$ to obtain the two weak duality results in the claim. □

We can next turn to (and for the duality, it suffices to) prove that

$$\inf_{H\in\mathcal{H}} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g + (H \circ X)_T\big)\big] \;\leq\; \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*_\varphi} \big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})\big\}, \tag{5.5.26}$$

for any universally measurable random variable $\varphi : \overline{\Omega} \to [0, \infty)$.

The one-period case $T = 1$ Let us first consider the one-period case $T = 1$. Define

$$\Lambda_{\rm int}(0, \omega_0) := \{\theta_0 \in \Lambda_1 : S_0(\omega_0)\theta_0 \in {\rm int}K_0^*\},$$

and for each $\theta_0 \in \Lambda_{\rm int}(0, \omega_0)$,

$$\overline{\mathcal{P}}^{\delta}_{\rm int}(0, \theta_0) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int} \;: \overline{\mathbb{P}}[\Theta_0 = \theta_0] = 1\big\}.$$

Define ${\rm NA}(\overline{\mathcal{P}}^{\delta}_{\rm int}(0, \theta_0))$ as ${\rm NA}(\overline{\mathcal{P}}_{\rm int})$ in Definition 5.2.9 with $\overline{\mathcal{P}}^{\delta}_{\rm int}(0, \theta_0)$ in place of $\overline{\mathcal{P}}_{\rm int}$. Then, ${\rm NA}(\overline{\mathcal{P}}_{\rm int})$ implies that ${\rm NA}(\overline{\mathcal{P}}^{\delta}_{\rm int}(0, \theta_0))$ holds for every $\theta_0 \in \Lambda_{\rm int}(0, \omega_0)$.

Lemma 5.5.5. *Let $T = 1$, and $g_1 : \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ be upper semi-analytic and also $(\omega, \theta_0, \theta_1) \in \Omega \times \Lambda_1 \times \Lambda_1 \to g_1(\omega, \theta_0, \theta_1)$ depend only on (ω, θ_1). Assume that ${\rm NA}(\overline{\mathcal{P}}_{\rm int})$ holds. Then, for $g = g_1$, the inequality (5.5.26) holds for any universally measurable random variable $\varphi : \overline{\Omega} \to [0, \infty)$ and both terms are not equal to $-\infty$. Moreover, there exists an optimal solution $\widehat{H} \in \mathcal{H}$ for the infimum problem at the left hand side. In consequence, Proposition 5.5.2 holds true for the case $T = 1$.*

Proof. Step 1: Although the context is slightly different, we can still follow the same arguments line by line in step (b) of the proof of Theorem 3.1 and Lemma 3.2 of [2] to obtain the existence of the optimal strategy $\widehat{H}$ (see also the proof of Theorem 2.2 of [59]), where the key argument is to show that $h \mapsto \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g + h(X_1 - X_0))\big]$ is lower-semicontinuous.

Step 2: We then turn to prove the duality. First, notice that $\mathcal{H} = \mathbb{R}^d$ when $T = 1$, and that $(g_1, X_1)(\omega, \theta_0, \theta_1)$ is independent of θ_0. Then, for all $\theta_0 \in \Lambda_{\rm int}(0, \omega_0)$,

$$\big\{\overline{\mathbb{P}} \circ (g_1, X_1)^{-1} : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}(0, \theta_0)\big\} \;=\; \big\{\overline{\mathbb{P}} \circ (g_1, X_1)^{-1} : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}(0, \mathbf{1})\big\},$$

where $\mathbf{1}$ represents the vector of $\mathbb{R}^d$ with all entries equal to 1. Thanks to the standard concatenation argument, it is clear that

$$\overline{V} \;=\; \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)} \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}^{\delta}_{\rm int}(0,\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0\theta_0)\big].$$

Define the function

$$\alpha(h_1, \theta_0) \;:=\; \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}^{\delta}_{\rm int}(0,\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0\theta_0)\big].$$

We first observe that

$$\theta_0 \;\mapsto\; \alpha(h_1, \theta_0) = \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}^{\delta}_{\rm int}(0,\mathbf{1})} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0\theta_0)] \text{ is affine.}$$

We claim further that $h_1 \mapsto \alpha(h_1, \theta_0)$ is convex. Indeed, for any (universally measurable) random variables Y_1 and Y_2, it follows by Cauchy-Schwarz inequality that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big((Y_1 + Y_2)/2\big)\big] \;\leqslant\; \Big(\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(Y_1)\big]\Big)^{\frac{1}{2}} \Big(\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(Y_2)\big]\Big)^{\frac{1}{2}}.$$

By taking logarithm on both sides, one has

$$\log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big((Y_1 + Y_2)/2\big)\big] \;\leqslant\; \frac{1}{2}\Big(\log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(Y_1)\big] + \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(Y_2)\big]\Big),$$

from which one observes that

$$h_1 \;\mapsto\; \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0\theta_0)\big] \text{ is convex.}$$

As the pointwise supremum of an arbitrary family of convex functions is still convex, it follows that $h_1 \mapsto \alpha(h_1, \theta_0)$ is convex.

This allows us to use minimax theorem to deduce that

$$\begin{aligned}\overline{V} \;&=\; \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)} \alpha(h_1, \theta_0) = \inf_{h_1 \in \mathbb{R}^d} \sup_{\theta_0 \in \Lambda(0,\omega_0)} \alpha(h_1, \theta_0) = \sup_{\theta_0 \in \Lambda(0,\omega_0)} \inf_{h_1 \in \mathbb{R}^d} \alpha(h_1, \theta_0) \\ &=\; \sup_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)} \inf_{h_1 \in \mathbb{R}^d} \sup_{\overline{\mathbb{P}} \lll \overline{\mathcal{P}}^{\delta}_{\rm int}(0,\theta_0)} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g_1 + h_1 \cdot X_1 - h_1 \cdot S_0\theta_0)\big].\end{aligned}$$

In the above argument, $\Lambda(0, \omega_0)$ denotes the closure of $\Lambda_{\rm int}(0, \omega_0)$, and we can replace $\Lambda_{\rm int}(0, \omega_0)$ by $\Lambda(0, \omega_0)$ since $\theta_0 \mapsto \alpha(h_1, \theta_0)$ is affine, and $\theta_0 \mapsto \inf_{h_1 \in \mathbb{R}^d} \alpha(h_1, \theta_0)$ is concave and hence lower

semicontinuous. Using the one period duality result in [2, Theorem 3.1], we obtain

$$\overline{V} = \sup_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_\varphi(0,\theta_0)} \Big\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^\delta_{\rm int}(0,\theta_0)\big) \Big\}.$$

Step 3: To conclude the proof, it is enough to prove that

$$\sup_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_\varphi(0,\theta_0)} \Big\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^\delta_{\rm int}(0,\theta_0)\big) \Big\} \;\geq\; \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_\varphi} \big\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}) \big\}, \quad (5.5.27)$$

as the reverse inequality is trivial by the fact that $\overline{\mathcal{Q}}^*_\varphi(0,\theta_0) \subset \overline{\mathcal{Q}}^*_\varphi$ and that $\mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^\delta_{\rm int}(0,\theta_0)\big) = \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int})$ for all $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_\varphi(0,\theta_0)$. Let $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_\varphi$ and denote by $(\overline{\mathbb{Q}}_{\theta_0})_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)}$ a family of r.c.p.d. of $\overline{\mathbb{Q}}$ knowing θ_0, then by [2, Lemma 4.4], we have

$$\begin{aligned} \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_1\big] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}) &= \mathbb{E}^{\overline{\mathbb{Q}}}\Big[\mathbb{E}^{\overline{\mathbb{Q}}_{\theta_0}}\big[g_1\big] - \mathcal{E}\big(\overline{\mathbb{Q}}_{\theta_0}, \overline{\mathcal{P}}^\delta_{\rm int}(0,\theta_0)\big)\Big] - \mathcal{E}\big(\overline{\mathbb{Q}} \circ \theta_0^{-1}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_0}\big) \\ &\leq \sup_{\theta_0 \in \Lambda_{\rm int}(0,\omega_0)} \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_\varphi(0,\theta_0)} \Big\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_1] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^\delta_{\rm int}(0,\theta_0)\big) \Big\}. \end{aligned}$$

Taking the supremum over $\overline{\mathbb{Q}}$ in $\overline{\mathcal{Q}}^*_\varphi$, we verify (5.5.27). □

The multi-period case: measurable selection of the dynamic strategy Let us extend the above definitions of $\Lambda_{\rm int}(0,\omega_0)$, $\overline{\mathcal{P}}^\delta_{\rm int}(0,\theta_0)$ and $\overline{\mathcal{Q}}^*_\varphi(0,\theta_0)$ to an arbitrary initial time t and initial path $\bar{\omega}^t$. For $t \geq 1$ and $\bar{\omega} = \bar{\omega}^t = (\omega^t, \theta^t) \in \overline{\Omega}_t$, let us first recall the definition of $\Lambda_{\rm int}(t,\omega^t)$:

$$\Lambda_{\rm int}(t,\omega^t) \;:=\; \{\theta_t \in \Lambda_1 \;: S_t(\omega^t)\theta_t \in {\rm int}K^*_t(\omega^t)\} \;\subset\; \Lambda_1.$$

Note that $\overline{\mathcal{P}}_{\rm int}(t,\bar{\omega}) \subset \mathfrak{B}(\Omega_1 \times \Lambda_1)$ is defined in (5.2.7). We introduce

$$\overline{\mathcal{P}}^\delta_{\rm int}(t,\bar{\omega}) := \big\{\delta_{\bar{\omega}^t} \otimes \overline{\mathbb{P}}_{t+1} \;: \overline{\mathbb{P}}_{t+1} \in \overline{\mathcal{P}}_{\rm int}(t,\bar{\omega})\big\}, \quad (5.5.28)$$

and

$$\widetilde{\mathcal{P}}^\delta_{\rm int}(t,\omega) := \big\{(\delta_{\omega^t} \times \mu(d\theta^t)) \otimes \overline{\mathbb{P}}_{t+1} \;: \overline{\mathbb{P}}_{t+1} \in \overline{\mathcal{P}}_{\rm int}(t,\bar{\omega}),\; \mu \in \mathfrak{B}\big(\Lambda_{\rm int}(0,\omega_0) \times \cdots \times \Lambda_{\rm int}(t,\omega^t)\big)\big\},$$

where the latter consists in a version of $\overline{\mathcal{P}}^\delta_{\rm int}(t,\bar{\omega})$ in which θ^t is not fixed anymore.

Remark 5.5.6. (i) *For a fixed $\omega \in \Omega_t$, let us define* NA$(\widetilde{\mathcal{P}}^\delta_{\rm int}(t,\omega))$ *by*

$$h(X_t)\cdot(X_{t+1} - X_t) \geq 0,\; \widetilde{\mathcal{P}}^\delta_{\rm int}(t,\omega)\text{-}q.s. \implies h(X_t)\cdot(X_{t+1} - X_t) = 0,\; \widetilde{\mathcal{P}}^\delta_{\rm int}(t,\omega)\text{-}q.s.,$$

for every universally measurable function $h : \mathbb{R}^d \to \mathbb{R}^d$. By applying Proposition 5.2.11 with $\mathcal{P}(t,\omega)$ in place of $\mathcal{P}$, one obtains that NA2(t,ω) *defined in* (5.5.29) *is equivalent to* NA$(\widetilde{\mathcal{P}}^\delta_{\rm int}(t,\omega))$.

(ii) *We recall that for each $t \leq T$ and $\omega \in \Omega_t$, the condition* NA2(t,ω) *is satisfied if*

$$\zeta \in K_{t+1}(\omega,\cdot),\; \mathcal{P}_t(\omega)\text{-}q.s. \quad \textit{implies} \;\; \zeta \in K_t(\omega), \quad \textit{for all } \zeta \in \mathbb{R}^d. \quad (5.5.29)$$

Then by [43, Lemma 3.6], the set $N_t := \{\omega :$ NA2(t,ω) *fails$\}$ is universally measurable. Moreover, N_t is a $\mathcal{P}$-polar set if* NA2$(\mathcal{P})$ *holds.*

(iii) *It follows from* (i) *and* (ii) that NA2(t,ω) or equivalently $\widetilde{\mathcal{P}}^{\delta}_{\rm int}(t,\omega)$ holds for all ω outside a $\mathcal{P}$-polar set N, whenever NA2$(\mathcal{P})$ holds. The latter is equivalent to NA$(\overline{\mathcal{P}}_{\rm int})$ by Proposition 5.2.11. Therefore, if NA$(\overline{\mathcal{P}}_{\rm int})$ holds, there exists a $\overline{\mathcal{P}}_{\rm int}$-polar $\overline{N} := N \times \Lambda$, such that for all $\bar{\omega} = (\omega, \theta) \notin \overline{N}$, NA$(\widetilde{\mathcal{P}}^{\delta}_{\rm int}(t,\omega))$ holds.

(iv) Finally, for a fixed $\bar{\omega} \in \overline{\Omega}_t$, we define NA$(\overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega}))$ by

$$h \cdot (X_{t+1} - X_t) \geq 0,\ \overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega})\text{-q.s.} \quad \Longrightarrow \quad h \cdot (X_{t+1} - X_t) = 0,\ \overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega})\text{-q.s.},$$

for every $h \in \mathbb{R}^d$. Then, NA$(\widetilde{\mathcal{P}}^{\delta}_{\rm int}(t,\omega))$ implies NA$(\overline{\mathcal{P}}^{\delta}_{\rm int}(t,\omega,\theta))$ for all $\theta \in \Lambda$ (see also Remark 3.9 of [113]).

Let us fix a functional $g_{t+1} : \overline{\Omega}_{t+1} \to \mathbb{R} \cup \{\infty\}$ which is upper semi-analytic and such that $g_{t+1}(\omega^{t+1}, \theta_0, \cdots, \theta_{t+1})$ depends only on $(\omega^{t+1}, \theta_{t+1})$. Then for any universally measurable random variable $Y_{t+1} : \overline{\Omega}_{t+1} \to \mathbb{R}_+$, we introduce

$$\overline{\mathcal{Q}}^{*}_{Y_{t+1}}(t,\bar{\omega}) := \left\{ \overline{\mathbb{Q}} \in \mathfrak{B}(\overline{\Omega}_{t+1}) : \begin{array}{l} \overline{\mathbb{Q}} \ll \overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega}),\ \mathbb{E}^{\overline{\mathbb{Q}}}[X_{t+1} - X_t] = 0,\ \ \mathbb{E}^{\overline{\mathbb{Q}}}[Y_{t+1}] < \infty, \\ \mathbb{E}^{\overline{\mathbb{Q}}}\big[g^{-}_{t+1} + |X_{t+1} - X_t|\big] + \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega})\big) < \infty \end{array} \right\},$$

and by setting $Y_{t+1} \equiv 0$, we define

$$g_t(\bar{\omega}^t) := \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^{*}_0(t,\bar{\omega}^t)} \Big\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega}^t)\big) \Big\}, \ \text{ for all } \bar{\omega}^t \in \overline{\Omega}_t. \tag{5.5.30}$$

Remark 5.5.7. *Let $\bar{\omega} = (\omega, \theta)$ and $\bar{\omega}' = (\omega', \theta')$ be such that $\omega^t = (\omega')^t$ and $\theta_t = \theta'_t$. Then, it follows from the definition of $\overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega})$ and $\overline{\mathcal{Q}}^{*}_{Y_{t+1}}(t,\bar{\omega})$ that*

$$\Big\{ \overline{\mathbb{Q}} \circ (g_{t+1}, X_t, X_{t+1})^{-1} \ : \overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^{*}_{Y_{t+1}}(t,\bar{\omega}) \Big\} = \Big\{ \overline{\mathbb{Q}} \circ (g_{t+1}, X_t, X_{t+1})^{-1} \ : \overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^{*}_{Y_{t+1}}(t,\bar{\omega}') \Big\}.$$

Since $g_{t+1}(\omega^{t+1}, \theta_0, \cdots, \theta_{t+1})$ is assumed to be independent of $(\theta_0, \cdots, \theta_t)$, then it is clear that $g_t(\bar{\omega}^t)$ depends only on (ω^t, θ_t) for $\bar{\omega}^t = (\omega^t, \theta_0, \cdots, \theta_t)$.

The above remark allows us to define

$$g'_t(\omega^t, h) := \sup_{\theta_t \in \Lambda_{\rm int}(t,\omega^t)} \big\{ g_t(\omega^t, \theta_t) + h \cdot S_t(\omega)\theta_t \big\}, \ \ \forall\ (\omega^t, h) \in \Omega_t \times \mathbb{R}^d. \tag{5.5.31}$$

Remark 5.5.8. *From Remark 5.5.6,* NA$(\overline{\mathcal{P}}_{\rm int})$ *implies that* NA$(\overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega}))$ *holds for $\overline{\mathbb{P}}$-a.e. $\bar{\omega} \in \overline{\Omega}$ under any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$. We can in fact apply Theorem 3.1 of [2] to obtain that*

$$g_t(\bar{\omega}) = \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^{*}_{Y_{t+1}}(t,\bar{\omega})} \Big\{ \mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^{\delta}_{\rm int}(t,\bar{\omega})\big) \Big\},\ \overline{\mathcal{P}}_{\rm int}\text{-q.s.},$$

for all universally measurable random variables $Y_{t+1} : \overline{\Omega}_{t+1} \to \mathbb{R}_+$.

Lemma 5.5.9. *For every t, the graph set*

$$\Big[\Big[\overline{\mathcal{Q}}^{*}_0(t)\Big]\Big] := \Big\{ (\bar{\omega}, \overline{\mathbb{Q}}) : \bar{\omega} \in \overline{\Omega}_t, \overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^{*}_0(t,\bar{\omega}) \Big\} \ \textit{is analytic.}$$

Proof. We follow the arguments in Lemma 4.5 of [2] and Lemma 4.8 of [1]. First, as $g_{t+1} \wedge 0 + |X_{t+1} - X_t|$ is upper semi-analytic, an application of Proposition 7.48 of [92] shows that $(\bar{\omega}, \overline{\mathbb{Q}}) \mapsto \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_{t+1} \wedge 0 - |X_{t+1} - X_t|\big]$ is upper semi-analytic.

Furthermore, from the definition of $\overline{\mathcal{P}}^{\delta}_{\text{int}}(t)$ in (5.5.28), one observes that the graph set $\big[\big[\overline{\mathcal{P}}^{\delta}_{\text{int}}(t)\big]\big]$ is analytic, as $\big[\big[\overline{\mathcal{P}}_{\text{int}}(t)\big]\big]$ is analytic (see Remark 5.2.5). Then using the Borel measurability of the relative entropy (Lemma 4.2 of [2]), one obtains from a measurable selection argument (see e.g. Proposition 7.47 of [92]) that $(\bar{\omega}, \overline{\mathbb{Q}}) \mapsto -\mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega})\big)$ is upper semi-analytic.

It follows that

$$A := \Big\{(\bar{\omega}, \overline{\mathbb{Q}}) : \mathbb{E}^{\overline{\mathbb{Q}}}\left[g_{t+1} \wedge 0 - |X_{t+1} - X_t|\right] - \mathcal{E}\left(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega})\right) > -\infty\Big\}$$

is an analytic set. By Lemma 4.8 of [1], we know

$$B(\bar{\omega}) := \{(\overline{\mathbb{Q}}, \overline{\mathbb{P}}) \in \mathfrak{B}(\overline{\Omega}_{t+1}) \times \mathfrak{B}(\overline{\Omega}_{t+1}) : \overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega}),\ \mathbb{E}^{\overline{\mathbb{Q}}}[X_{t+1} - X_t] = 0,\ \overline{\mathbb{Q}} \ll \overline{\mathbb{P}}\}$$

has an analytic graph. Notice that the set

$$C := \big\{\big(\bar{\omega}, \overline{\mathbb{Q}}\big) : \overline{\mathbb{Q}} \lll \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega}),\ \mathbb{E}^{\overline{\mathbb{Q}}}[X_{t+1} - X_t] = 0\big\}$$

is the image of the graph set $\big[\big[B\big]\big]$ under canonical projection $\overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1}) \times \mathfrak{B}(\overline{\Omega}_{t+1}) \mapsto \overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1})$ and thus analytic. Finally, it is shown that

$$\Big[\Big[\overline{\mathcal{Q}}^{*}_{0}(t)\Big]\Big] = A \cap C$$

is analytic. □

Lemma 5.5.10. *Assume that* $\text{NA}(\overline{\mathcal{P}}_{\text{int}})$ *holds true. Then both* g_t *and* g'_t *are upper semi-analytic, and there is a universally measurable map* $h_{t+1} : \Omega_t \times \mathbb{R}^d \to \mathbb{R}^d$ *together with a* $\mathcal{P}$*-polar set* N *such that, for every* $(\omega, h_t) \in N^c \times \mathbb{R}^d$*, one has*

$$g'_t(\omega^t, h_t) = \sup_{\theta_t \in \Lambda_{\text{int}}(t, \omega^t)} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\big(g_{t+1} + h_{t+1}(\omega^t, h_t)(X_{t+1} - X_t) + h_t X_t\big)\Big] > -\infty.$$

Proof. The proof follows the track of measurable selection arguments as in Lemma 3.7 of [59] with some modifications for our setting. Let us denote, for all $\omega^t \in \Omega_t$ and $h_t \in \mathbb{R}^d$,

$$V'_t(\omega^t, h_t) := \inf_{h \in \mathbb{R}^d} \sup_{\theta_t \in \Lambda_{\text{int}}(t, \omega^t)} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\big(g_{t+1} + h(X_{t+1} - X_t) + h_t X_t\big)\Big].$$

By Remark 5.5.7, we can employ the same minimax theorem argument as in Lemma 5.5.5 above and obtain that

$$V'_t(\omega^t, h_t) = g'_t(\omega^t, h_t) \ > \ -\infty, \quad \text{if } \text{NA}(\widetilde{\mathcal{P}}^{\delta}_{\text{int}}(t, \omega)) \text{ holds true.}$$

In view of (iii) in Remark 5.5.6, this holds true outside a $\mathcal{P}$-polar set N.

Further, let us denote by $\mathcal{U}(\Omega_t \times \mathbb{R}^d)$ the universal σ-field on $\Omega_t \times \mathbb{R}^d$. Notice that g_{t+1} is assumed to be upper semi-analytic, the graph set $\big[\big[\overline{\mathcal{Q}}^{*}_{0}(t)\big]\big]$ is analytic by Lemma 5.5.9, $(\bar{\omega}^t, \overline{\mathbb{Q}}) \in \overline{\Omega}_t \times \mathfrak{B}(\overline{\Omega}_{t+1}) \mapsto \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}^{\delta}_{\text{int}}(t, \bar{\omega}^t)\big)$ is lower semi-analytic by [2, Lemma 4.2] and [92,

Proposition 7.47]. It follows from the measurable selection argument (see e.g. [92, Propositions 7.26, 7.47, 7.48]) that the maps $\bar{\omega}^t \mapsto g_t(\bar{\omega}^t)$ is upper semi-analytic. As $\big[\big[\mathrm{int}K_t^*\big]\big]$ is Borel and hence $\big[\big[\Lambda_{\mathrm{int}}(t,\cdot)\big]\big]$ is also Borel, it follows from [92, Proposition 7.47] that $(\omega^t, h_t) \mapsto g_t'(\omega^t, h_t)$ is upper semi-analytic, hence belongs to $\mathcal{U}(\Omega_t \times \mathbb{R}^d)$.

Next, we claim that the function

$$\phi(\omega^t, h_t, h) := \sup_{\theta_t \in \Lambda_{\mathrm{int}}(t,\omega^t)} \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\delta}_{\mathrm{int}}(t,\bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\big(g_{t+1} + h(X_{t+1} - X_t) + h_t X_t\big)\Big].$$

is $\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. To see this, we first fix h and h_t. Then from the same argument as above, as $\Big[\Big[\overline{\mathcal{P}}^{\delta}_{\mathrm{int}}(t)\Big]\Big]$ is analytic and by [92, Propositions 7.26, 7.47, 7.48], we have that

$$(\omega^t, \theta_t) \mapsto \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\delta}_{\mathrm{int}}(t,\bar{\omega})} \log \mathbb{E}^{\overline{\mathbb{P}}}[\exp\big(g_{t+1} + h(X_{t+1} - X_t) + h_t X_t\big)]$$

is upper semi-analytic. Again, $\big[\big[\mathrm{int}K_t^*\big]\big]$ is Borel implies that $\big[\big[\Lambda_{\mathrm{int}}(t,\cdot)\big]\big]$ is also Borel, by [92, Proposition 7.47], we have that $\omega^t \mapsto \phi(\omega^t, h_t, h)$ is upper semi-analytic. On the other hand, for fixed ω^t, it follows by an application of Fatou's lemma (as [2, Lemma 4.6]) that $(h, h_t) \mapsto \phi(\omega^t, h_t, h)$ is lower semi-continuous. Moreover, as $(h, h_t) \mapsto \phi(\omega^t, h_t, h)$ is convex, by [62, Lemma 4.5], we have that ϕ is indeed $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and thus belongs to $\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$.

Let us consider the random set

$$\Phi(\omega^t, h_t) := \big\{h \in \mathbb{R}^d : \phi(\omega^t, h_t, h) = g_t'(\omega^t, h_t)\big\}.$$

By the previous arguments, we have that $[[\Phi]]$ is in $\mathcal{U}(\Omega_t \times \mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$. Thus Φ admits an $\mathcal{U}(\Omega_t \times \mathbb{R}^d)$-measurable selector h_{t+1} on the universally measurable set $\Phi(\omega^t, h_t) \neq \emptyset$; cf. the corollary and scholium of [117, Theorem 5.5]. Moreover, Lemma 5.5.5 and Remark 5.5.6 imply that $\Phi(\omega^t, h_t) \neq \emptyset$ holds true outside a $\mathcal{P}$-polar set N, it yields that h_{t+1} solves the infimum $\mathcal{P}$-q.s. □

The multi-period case: the final proof We provide a last technical lemma and then finish the proof of Proposition 5.5.2. Recall that $g_{t+1} := \overline{\Omega}_{t+1} \to \mathbb{R} \cup \{\infty\}$ is a given upper semi-analytic functional, such that $g_{t+1}(\omega^{t+1}, \theta_0, \cdots, \theta_{t+1})$ depends only on $(\omega^{t+1}, \theta_{t+1})$, and g_t is defined in (5.5.30). Given a universally measurable random variable $Y_t : \overline{\Omega}_t \to \mathbb{R}_+$, we define

$$\overline{\mathcal{Q}}^*_{Y_t,t} := \{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}_0|_{\overline{\Omega}_t} : \mathbb{E}^{\overline{\mathbb{Q}}}\big[g_t^-\big] + \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\mathrm{int}}|_{\overline{\Omega}_t}\big) < +\infty,\ \mathbb{E}^{\overline{\mathbb{Q}}}[Y_t] < +\infty\}.$$

Lemma 5.5.11. *Let $t+1 \leq T$, then for any universally measurable random variable $Y_{t+1} : \overline{\Omega}_{t+1} \to \mathbb{R}_+$ and $\varepsilon > 0$, there is a universally measurable random variable $Y_t^{\varepsilon} : \overline{\Omega}_t \to \mathbb{R}_+$ such that*

$$\sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{Y_t^{\varepsilon},t}} \Big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g_t] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\mathrm{int}}|_{\overline{\Omega}_t}\big)\Big\} \leq \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{Y_{t+1},t+1}} \Big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\mathrm{int}}|_{\overline{\Omega}_{t+1}}\big)\Big\} + \varepsilon. \quad (5.5.32)$$

Proof. (i) In view of Corollary 5.6.7, we can assume w.l.o.g. that $Y_{t+1} \equiv 0$. Then Lemma 5.5.9 and a measurable selection argument (see e.g. Proposition 7.50 of [92]) guarantees that there exists a universally measurable kernel $\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot) : \overline{\Omega}_t \to \mathfrak{B}(\overline{\Omega}_1)$ such that $\delta_{\bar{\omega}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\bar{\omega}) \in \overline{\mathcal{Q}}_0^*(t, \bar{\omega})$ for

all $\bar{\omega} \in \overline{\Omega}_t$, and

$$g_t(\bar{\omega}) \ \leq \ \mathbb{E}^{\delta_{\bar{\omega}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\bar{\omega})}[g_{t+1}] - \mathcal{E}\big(\overline{\mathbb{Q}}_t^{\varepsilon}(\bar{\omega}), \overline{\mathcal{P}}_{\rm int}(t, \bar{\omega})\big) + \varepsilon.$$

(ii) Let us define Y_t^{ε} by

$$Y_t^{\varepsilon}(\cdot) := \mathbb{E}^{\delta_\cdot \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)}\big[g_{t+1}^- + |X_{t+1} - X_t|\big] + \mathcal{E}\big(\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot), \overline{\mathcal{P}}_{\rm int}(t, \cdot)\big).$$

By the definition of $\overline{\mathcal{Q}}_0^*(t, \cdot)$ and [2, Lemma 4.2], Y_t^{ε} is $\mathbb{R}_+$-valued and universally measurable.

Then for any $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{Y_t^{\varepsilon}, t}$, one has

$$\begin{aligned}
& \mathbb{E}^{\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)}\big[g_{t+1}^- + |X_{t+1} - X_t|\big] + \mathcal{E}(\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot), \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_{t+1}}) \\
= \ & \mathbb{E}^{\overline{\mathbb{Q}}}\Big[\mathbb{E}^{\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)}[g_{t+1}^- + |X_{t+1} - X_t|]\Big] + \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}) + \mathbb{E}^{\overline{\mathbb{Q}}}\Big[\mathcal{E}(\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot), \overline{\mathcal{P}}_{\rm int}(t, \cdot))\Big] \\
\leq \ & \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}) + \mathbb{E}^{\overline{\mathbb{Q}}}[Y_t^{\varepsilon}] < +\infty,
\end{aligned}$$

where the first equality follows from Lemma 4.4 of [2]. Further, $\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)$ is a martingale measure on $\overline{\Omega}_{t+1}$ by the martingale property of $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)$. Finally, because $\overline{\mathbb{Q}} \lll \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}$ and $\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot) \lll \overline{\mathcal{P}}_{\rm int}(t, \cdot)$, it follows that $\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot) \lll \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_{t+1}}$. This implies that $\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot) \in \overline{\mathcal{Q}}^*_{0,t+1}$.

Thus for any $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{Y_t^{\varepsilon}, t}$, one has

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{Q}}}[g_t] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}) \ &\leqslant \ \mathbb{E}^{\overline{\mathbb{Q}}}\Big[\mathbb{E}^{\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)}[g_{t+1}] - \mathcal{E}\big(\overline{\mathbb{Q}}_t^{\varepsilon}(\cdot), \overline{\mathcal{P}}_{\rm int}(t, \cdot)\big) + \varepsilon\Big] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}) \\
&= \ \mathbb{E}^{\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot)}[g_{t+1}] - \mathcal{E}(\overline{\mathbb{Q}} \otimes \overline{\mathbb{Q}}_t^{\varepsilon}(\cdot), \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_{t+1}}) + \varepsilon \\
&\leqslant \ \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{0,t+1}} \Big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}] - \mathcal{E}(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_{t+1}})\Big\} + \varepsilon.
\end{aligned}$$

We hence conclude the proof as $\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{Y_t^{\varepsilon}, t}$ is arbitrary. □

Proof of Proposition 5.5.2. We will use an induction argument. First, Proposition 5.5.2 in case $T = 1$ is already proved in Lemma 5.5.5. Next, assume that Proposition 5.5.2 holds true for the case $T = t$, we then consider the case $T = t + 1$.

In the case $T = t + 1$, let us denote $g_{t+1} := g := \xi \cdot X_{t+1}$. It is clear that g_{t+1} is a Borel random variable and $g_{t+1}(\omega^{t+1}, \theta_0, \cdots, \theta_{t+1})$ depends only on $(\omega^{t+1}, \theta_{t+1})$. Let g_t be defined by (5.5.30). Since Proposition 5.5.2 is assumed to hold true for the case $T = t$, it follows that there is $\hat{H} = (\hat{H}_1, \cdots, \hat{H}_t) \in \mathcal{H}_t$ such that, for any universally measurable random variable $Y_t : \overline{\Omega}_t \to \mathbb{R}_+$, one has

$$\sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}} \log \mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g_t + (\hat{H} \circ X)_t)\big] \ = \ \sup_{\overline{\mathbb{Q}} \in \overline{\mathcal{Q}}^*_{Y_t, t}} \Big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g_t] - \mathcal{E}\big(\overline{\mathbb{Q}}, \overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}\big)\Big\}. \tag{5.5.33}$$

Then with the function h_{t+1} defined in Lemma 5.5.10, we define

$$\hat{H}_{t+1}(\omega^t) \ := \ h_{t+1}(\omega^t, \hat{H}_t(\omega^{t-1})). \tag{5.5.34}$$

Further, for any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{\rm int}$, one has the representation $\overline{\mathbb{P}} = \overline{\mathbb{P}}_0 \otimes \cdots \otimes \overline{\mathbb{P}}_t$, where $\overline{\mathbb{P}}_s(\cdot)$ is

measurable kernel in $\overline{\mathcal{P}}^{\delta}_{\rm int}(s,\cdot)$. It follows by direct computation that

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\Big(g_{t+1}+(\hat{H}\circ X)_{t+1}\Big)\Big] &= \mathbb{E}^{\overline{\mathbb{P}}_0\otimes\cdots\otimes\overline{\mathbb{P}}_{t-1}}\Big[\exp\Big(\log\mathbb{E}^{\overline{\mathbb{P}}_t}\Big[\exp\big(g_{t+1}+(\hat{H}\circ X)_{t+1}\big)\Big]\Big)\Big]\\ &\le \mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\Big(\sup_{\overline{\mathbb{P}}'\in\overline{\mathcal{P}}^{\delta}_{\rm int}(t,\cdot)}\log\mathbb{E}^{\overline{\mathbb{P}}'}\Big[\exp\big(g_{t+1}+(\hat{H}\circ X)_{t+1}\big)\Big]\Big)\Big]\\ &\le \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g'_t(\cdot,\hat{H}_t)+(\hat{H}\circ X)_{t-1}-\hat{H}_tX_{t-1}\big)\big],\end{aligned}$$

where the last inequality follows by the definition of $\hat{H}_{t+1}$ in (5.5.34) and Lemma 5.5.10. Taking the supremum over $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}$, it follows from the definition of g'_t in (5.5.31) together with a dynamic programming argument that

$$\begin{aligned}\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\Big(g_{t+1}+(\hat{H}\circ X)_{t+1}\Big)\Big] &\le \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g_t+\hat{H}_tX_t+(\hat{H}\circ X)_{t-1}-\hat{H}_tX_{t-1}\big)\big]\\ &= \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(g_t+(\hat{H}\circ X)_t\big)\big]. \qquad (5.5.35)\end{aligned}$$

Then for any universally measurable random variable $\varphi:\overline{\Omega}\to\mathbb{R}_+$, we set $Y_{t+1}:=\varphi$ and use sequentially Lemma 5.5.11, (5.5.33), (5.5.35), to obtain

$$\begin{aligned}\sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*_{\varphi}}\Big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g_{t+1}]-\mathcal{E}(\overline{\mathbb{Q}},\overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_{t+1}})\Big\} &\ge \sup_{\overline{\mathbb{Q}}\in\overline{\mathcal{Q}}^*_{Y_t,t}}\Big\{\mathbb{E}^{\overline{\mathbb{Q}}}[g_t]-\mathcal{E}\big(\overline{\mathbb{Q}},\overline{\mathcal{P}}_{\rm int}|_{\overline{\Omega}_t}\big)\Big\}\\ &= \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp(g_t+(\hat{H}\circ X)_t)\big]\\ &\ge \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\overline{\mathbb{P}}}\Big[\exp\Big(g_{t+1}+(\hat{H}\circ X)_{t+1}\Big)\Big]\\ &\ge \inf_{H\in\mathcal{H}}\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\overline{\mathbb{P}}}\left[\exp\left(g_{t+1}+(H\circ X)_{t+1}\right)\right].\end{aligned}$$

Because the reverse inequality is the weak duality in Lemma 5.5.4, we obtain the equality everywhere in the above formula, which is the duality result (5.5.25) for the case $T=t+1$. In particular, $(\hat{H}_1,\cdots,\hat{H}_t,\hat{H}_{t+1})$ is the optimal trading strategy for the case $T=t+1$. □

5.5.3 Proof of Theorem 5.4.1(Case $e\geqslant 1$)

In this section, we are interested in the utility maximization problem with semi-static strategy. To take into account of the transaction costs caused by trading the static options $(\zeta_i, i=1,\cdots,e)$, we work in the framework of [113] and introduce a further enlarged space by

$$\widehat{\Lambda}:=\prod_{i=1}^{e}[-c_i,c_i],\quad \widehat{\Omega}:=\overline{\Omega}\times\widehat{\Lambda},\quad \widehat{\mathcal{F}}_t:=\overline{\mathcal{F}}_t\otimes\mathcal{B}\big(\widehat{\Lambda}\big),\quad \widehat{\mathcal{P}}_{\rm int}:=\Big\{\widehat{\mathbb{P}}\in\mathfrak{B}(\widehat{\Omega})\ :\widehat{\mathbb{P}}|_{\overline{\Omega}}\in\overline{\mathcal{P}}_{\rm int}\Big\},$$

and define

$$\hat{f}_i:\widehat{\Omega}\longrightarrow\mathbb{R},\quad \hat{f}_i(\widehat{\omega})=\zeta_i(\omega)\cdot X_T(\bar{\omega})-\hat{\theta}_i\ \text{ for all }\widehat{\omega}=(\bar{\omega},\hat{\theta})=(\omega,\theta,\hat{\theta})\in\widehat{\Omega}.$$

The process $(X_t)_{0\le t\le T}$ and the random variable $g:=\xi\cdot X_T$ defined on $\overline{\Omega}$ can be naturally extended on $\widehat{\Omega}$.

We can then consider the exponential utility maximization problem on $\widehat{\Omega}$:

$$\inf_{(H,\ell)\in\mathcal{H}\times\mathbb{R}^e}\sup_{\widehat{\mathbb{P}}\in\widehat{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\widehat{\mathbb{P}}}\left[\exp\left(g+\sum_{i=1}^e\ell_i\hat{f}_i+(H\circ X)_T\right)\right].$$

Let us also introduce

$$\widehat{\mathcal{Q}}_e^*\ :=\ \left\{\widehat{\mathbb{Q}}\in\mathfrak{B}(\widehat{\Omega})\ :\ \begin{array}{l}\widehat{\mathbb{Q}}\ll\widehat{\mathcal{P}}_{\rm int},\ X\text{ is }(\widehat{\mathbb{F}},\widehat{\mathbb{Q}})\text{-martingale},\ \mathbb{E}^{\widehat{\mathbb{Q}}}[\hat{f}_i]=0,\ i=1,\cdots,e,\\ \mathbb{E}^{\widehat{\mathbb{Q}}}\big[(\xi\cdot X_T)_-\big]+\mathcal{E}(\widehat{\mathbb{Q}},\widehat{\mathcal{P}}_{\rm int})<+\infty\end{array}\right\},$$

and

$$\widehat{\mathcal{Q}}_{e,\varphi}^*:=\{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e^*:\mathbb{E}^{\widehat{\mathbb{Q}}}[\varphi]<+\infty\},\quad\text{for all }\varphi:\widehat{\Omega}\to\mathbb{R}_+.$$

It is easy to employ similar arguments for Lemma 5.5.1 and Proposition 5.2.6 to obtain

$$\begin{aligned}&\inf_{(\ell,\eta)\in\mathcal{A}_e}\sup_{\mathbb{P}\in\mathcal{P}}\log\mathbb{E}^{\mathbb{P}}\left[\exp\left(\left(\xi-\sum_{i=1}^e(\ell_i\zeta_i-|\ell_i|c_i\mathbf{1}_d)-\sum_{t=0}^T\eta_t\right)^d\right)\right]\\ =\ &\inf_{(H,\ell)\in\mathcal{H}\times\mathbb{R}^e}\sup_{\widehat{\mathbb{P}}\in\widehat{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\widehat{\mathbb{P}}}\left[\exp\big(g+\sum_{i=1}^e\ell_i\hat{f}_i+(H\circ X)_T\big)\right],\end{aligned}$$

and

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}_e^*}\big\{\mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big]-\mathcal{E}(\mathbb{Q},\mathcal{P})\big\}=\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_e^*}\big\{\mathbb{E}^{\widehat{\mathbb{Q}}}\big[g\big]-\mathcal{E}(\widehat{\mathbb{Q}},\widehat{\mathcal{P}}_{\rm int})\big\},$$

with $g:=\xi\cdot X_T$. Hence, to conclude the proof of Theorem 5.4.1(case $e\geqslant 1$), it is sufficient to prove that, for any universally measurable $\varphi:\widehat{\Omega}\to\mathbb{R}_+$, one has

$$\inf_{(H,\ell)\in\mathcal{H}\times\mathbb{R}^e}\sup_{\widehat{\mathbb{P}}\in\widehat{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\widehat{\mathbb{P}}}\left[\exp\left(g+\sum_{i=1}^e\ell_i\hat{f}_i+(H\circ X)_T\right)\right]=\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}_{e,\varphi}^*}\Big\{\mathbb{E}^{\widehat{\mathbb{Q}}}\big[g\big]-\mathcal{E}(\widehat{\mathbb{Q}},\widehat{\mathcal{P}}_{\rm int})\Big\}\quad(5.5.36)$$

Let us first provide a useful lemma.

Lemma 5.5.12. *Let $g:\overline{\Omega}\to\mathbb{R}$ be upper semi-analytic, and assume that* NA2($\mathcal{P}$) *holds. Assume either that $e=0$, or that $e\geq 1$ and for all $\ell\in\mathbb{R}^e$ and $\eta\in\mathcal{A}$,* (5.4.15) *holds. Then, for all $\varphi:\widehat{\Omega}\to\mathbb{R}_+$, one has*

$$\inf\Big\{y\in\mathbb{R}\ :y+\sum_{i=1}^e\ell_i\hat{f}_i+(H\circ X)_T\geq g,\ \widehat{\mathcal{P}}_{\rm int}\text{-}q.s.,l\in\mathbb{R}^e,H\in\mathcal{H}\Big\}\ =\ \sup_{\overline{\mathbb{Q}}\in\widehat{\mathcal{Q}}_{e,\varphi}^*}\mathbb{E}^{\overline{\mathbb{Q}}}\big[g\big].\quad(5.5.37)$$

Proof. By Proposition 5.2.11, NA2($\mathcal{P}$) implies NA($\overline{\mathcal{P}}_{\rm int}$). For the case $e=0$, as observed by [2, Lemma 3.5], Lemma 3.3 of [1] has indeed proved the following stronger version of fundamental lemma with $T=1$:

$$0\in\mathrm{ri}\{\ \mathbb{E}^{\overline{\mathbb{Q}}}[\Delta X],\ \overline{\mathbb{Q}}\in\overline{\mathcal{Q}}_\varphi^*\}.\quad(5.5.38)$$

Using (5.5.38), we can proceed as [1, Lemma 3.5, 3.6, Theorem 4.1] to prove (5.5.37) in the case without options($e=0$).

For the case $e\geqslant 1$, we can argue by induction. Suppose the super-replication theorem with

$e-1$ options holds with $g=\hat{f}_e$:

$$\begin{aligned}\hat{\pi}_{e-1}(g) &:= \inf\Big\{y \ : y+\sum_{i=1}^{e-1}\ell_i\hat{f}_i+(H\circ X)_T\geq g,\ \widehat{\mathcal{P}}_{\rm int}\text{-q.s.},\ \ell\in\mathbb{R}^{e-1},\ H\in\mathcal{H}\Big\}\\ &= \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}}\mathbb{E}^{\widehat{\mathbb{Q}}}[g],\end{aligned}\tag{5.5.39}$$

and we shall pass to e. By the no arbitrage condition (5.4.15), there is no $H\in\mathcal{H}$, $\ell_1,\cdots,\ell_{e-1}$ and $\ell_e\in\{-1,1\}$ such that $\sum_{i=1}^{e-1}\ell_i\hat{f}_i+(H\circ X)_T\geq-\ell_e\hat{f}_e$, $\widehat{\mathcal{P}}_{\rm int}$-q.s. It follows that $\hat{\pi}_{e-1}(\hat{f}_e),\hat{\pi}_{e-1}(-\hat{f}_e)>0$, which, by [1, Lemma 3.12] and (5.5.39), implies that there is $\widehat{\mathbb{Q}}_-,\widehat{\mathbb{Q}}_+\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}$ such that

$$-\hat{\pi}_{e-1}(-\hat{f}_e)\ <\ \mathbb{E}^{\widehat{\mathbb{Q}}_-}[\hat{f}_e]\ <\ 0\ <\ \mathbb{E}^{\widehat{\mathbb{Q}}_+}[\hat{f}_e]\ <\ \hat{\pi}_{e-1}(\hat{f}_e).\tag{5.5.40}$$

In particular, we have

$$0\in\mathrm{ri}\{\ \mathbb{E}^{\widehat{\mathbb{Q}}}[\hat{f}_e],\ \widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}\}.\tag{5.5.41}$$

Then we can argue line by line as [113, Proof of Theorem 3.1(case $e\geqslant1$)] to prove that

$$\text{there exists a sequence }\big(\widehat{\mathbb{Q}}_n\big)_{n\geq1}\subset\widehat{\mathcal{Q}}^*_{e,\varphi}\text{ s.t. }\mathbb{E}^{\widehat{\mathbb{Q}}_n}[g]\to\hat{\pi}_e(g),$$

which shows that

$$\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e,\varphi}}\mathbb{E}^{\widehat{\mathbb{Q}}}[g]\ \geq\ \hat{\pi}_e(g).$$

To conclude, it is enough to notice that the reverse inequality is the classical weak duality which can be easily obtained from [1, Lemmas A.1 and A.2]. □

Proof of Theorem 5.4.1 (case $e\geqslant1$). Notice that (5.5.36) has been proved for the case $e=0$ in Section 5.5.2, although the formulations are slightly different. The proof is still based on the induction as in the proof of [2, Theorem 2.2]. Let us assume that (5.5.36) holds for $e-1\geq0$ and then prove it for the case of e. Define

$$J:\widehat{\mathcal{Q}}^*_{e-1,\varphi}\times\mathbb{R}\to\mathbb{R},\quad(\widehat{\mathbb{Q}},\beta)\mapsto\mathbb{E}^{\widehat{\mathbb{Q}}}[g]+\beta\mathbb{E}^{\widehat{\mathbb{Q}}}[\hat{f}_e]-H(\widehat{\mathbb{Q}},\widehat{\mathcal{P}}_{\rm int}).$$

Clearly, J is concave in the first argument and convex in the second argument. By (5.5.41), J satisfies the compactness-type condition (14) in [2], thus we can apply the minimax theorem. Using the induction hypothesis and the same arguments as in [2], we have

$$\begin{aligned}&\inf_{(H,\ell)\in\mathcal{H}\times\mathbb{R}^e}\sup_{\widehat{\mathbb{P}}\in\widehat{\mathcal{P}}_{\rm int}}\log\mathbb{E}^{\widehat{\mathbb{P}}}\left[\exp\left(g+\sum_{i=1}^{e}\ell_i\hat{f}_i+(H\circ X)_T\right)\right]\\ =\ &\inf_{\beta\in\mathbb{R}}\min_{(H,\ell)\in\mathcal{H}\times\mathbb{R}^{e-1}}\sup_{\widehat{\mathbb{P}}\in\widehat{\mathcal{P}}_{\rm int}}\mathbb{E}^{\widehat{\mathbb{P}}}\left[\exp\left(g+\sum_{i=1}^{e-1}\ell_i\hat{f}_i+\beta\hat{f}_e+(H\circ X)_T\right)\right]\\ =\ &\inf_{\beta\in\mathbb{R}}\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}}J(\widehat{\mathbb{Q}},\beta)\\ =\ &\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}}\inf_{\beta\in\mathbb{R}}J(\widehat{\mathbb{Q}},\beta)=\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e,\varphi}}\big(\mathbb{E}^{\widehat{\mathbb{Q}}}[g]-H(\widehat{\mathbb{Q}},\widehat{\mathcal{P}}_{\rm int})\big).\end{aligned}\tag{5.5.42}$$

The duality holds as a consequence. Moreover, from (15) of [2], $\forall c < \inf_{\beta\in\mathbb{R}} \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}} J(\widehat{\mathbb{Q}},\beta)$, $\exists n,$ s.t. for all β satisfying $|\beta| > n, \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}} J(\widehat{\mathbb{Q}},\beta) > c$. Thus (5.5.42) can be rewritten as

$$\inf_{|\beta|\le n} \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e-1,\varphi}} J(\widehat{\mathbb{Q}},\beta).$$

Now the lower-continuity of $\beta \mapsto \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e,\varphi}} J(\widehat{\mathbb{Q}},\beta)$ implies the existence of some $\hat{\beta}$ such that

$$\sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e,\varphi}} J(\widehat{\mathbb{Q}},\hat{\beta}) = \inf_{\beta\in\mathbb{R}} \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_{e,\varphi}} J(\widehat{\mathbb{Q}},\beta).$$

Combining $\hat{\beta}$ with the optimal strategy with $e-1$ options $(\hat{H},\hat{\ell}^\star)$, we deduce the existence of an optimal strategy for e options, namely $(\hat{H},\hat{\ell}) := (\hat{H},(\hat{\ell}^\star,\hat{\beta}))$. Using the construction (5.2.8), one can obtain $(\hat{\eta},\hat{\ell})$ explicitly attaining the infimum in (5.4.17) from $(\hat{H},\hat{\ell})$ which is constructed already in previous steps. □

5.5.4 Proof of Proposition 5.4.6

Using the expression in (5.4.20), one has

$$\lim_{\gamma\to\infty} \pi_\gamma(\xi) \;=\; \lim_{\gamma\to\infty} \sup_{(\mathbb{Q},Z)\in\mathcal{S}^*_e} \left\{ \mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P}) \right\},$$

where the r.h.s. is increasing in γ. Replacing the limit by supremum, and then interchanging the order of two supremums, we have

$$\lim_{\gamma\to\infty} \pi_\gamma(\xi) \;=\; \sup_{(\mathbb{Q},Z)\in\mathcal{S}^*_e} \sup_{\gamma} \left\{ \mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] - \frac{1}{\gamma}\mathcal{E}(\mathbb{Q},\mathcal{P}) \right\} \;=\; \sup_{(\mathbb{Q},Z)\in\mathcal{S}^*_e} \mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big].$$

By similar arguments as in Section 3.2 of [113], we can reformulate the problem at the r.h.s. on the enlarged space $\widehat{\Omega}$ and then use Lemma 5.5.12 to obtain that

$$\sup_{(\mathbb{Q},Z)\in\mathcal{S}^*_e} \mathbb{E}^{\mathbb{Q}}\big[\xi\cdot Z_T\big] \;=\; \sup_{\widehat{\mathbb{Q}}\in\widehat{\mathcal{Q}}^*_e} \mathbb{E}^{\widehat{\mathbb{Q}}}\big[\xi\cdot X_T\big] \;=\; \pi(\xi).$$

This concludes the proof. □

5.6 Appendix : Exponential utility maximization duality without transaction cost

In this appendix, we shall present an auxiliary result on the exponential utility maximization problem without transaction cost by applying the same procedure as in the proof of Theorem 5.4.1. This allows to extend the main results in Bartl [2] without a restrictive ω-wise no-arbitrage condition. Moreover, an auxiliary result in the dominated case consists a key ingredient in the proof of our main result in Theorem 5.4.1 with transaction cost (in particular in Lemma 5.5.11).

Throughout this appendix, we stay in the context of Section 5.2.1, where $\Omega := \Omega_1^T$ is a (product) Polish space with the raw canonical filtration $\mathbb{F}^0 = (\mathcal{F}^0_t)_{0\le t\le T}$ and the universally

completed filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ and $\mathcal{F} := \mathcal{F}_T$. The space $(\Omega, \mathcal{F})$ is equipped with a family of (possibly) non-dominated probability measures $\mathcal{P}$ defined by (5.2.1) with a given family of classes of probability measures $\mathcal{P}_t(\omega)$ on Ω_1, that is,

$$\mathcal{P} := \big\{\mathbb{P} := \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1} \ : \mathbb{P}_t(\cdot) \in \mathcal{P}_t(\cdot) \text{ for } t \le T-1\big\},$$

which satisfies the measurability condition (5.2.2). We take the $\mathbb{F}^0$-adapted process $(S_t)_{0\le t\le T}$ in (5.2.4) and let it represent the discounted stock price, which can be traded without any transaction cost. Finally, by a slight abuse of language, we denote $g : \Omega \to \mathbb{R}$ an upper semi-analytic random variable representing the payoff of a derivative option, and let

$$\mathcal{H} \ := \ \{\text{All } \mathbb{F}\text{-predictable processes}\}$$

represent the set of all admissible trading strategies, and denote $(H\circ S)_T := \sum_{t=1}^T H_t\cdot(S_{t+1}-S_t)$. Following Bouchard and Nutz [1], we define the quasi-sure no-arbitrage condition NA($\mathcal{P}$) by

$$(H\circ S)_T \ge 0,\ \mathcal{P}\text{-q.s.} \ \Longrightarrow \ (H\circ S)_T = 0,\ \mathcal{P}\text{-q.s.} \quad \text{for all } H \in \mathcal{H}. \tag{5.6.43}$$

Further, for each $t = 0, \cdots, T-1$ and $\omega^t \in \Omega_t$, we define the no-arbitrage condition NA($\mathcal{P}_t(\omega^t)$) by

$$h\cdot\Delta S_{t+1}(\omega^t,\cdot) \ge 0,\ \mathcal{P}_t(\omega^t)\text{-q.s.} \implies h\cdot\Delta S_{t+1}(\omega^t,\cdot) = 0,\ \mathcal{P}_t(\omega^t)\text{-q.s.} \ \text{ for all } h\in\mathbb{R}^d. \tag{5.6.44}$$

Recall also that (by Lemma 4.6 of [1]) the set $N_t = \{\omega^t \in \Omega_t : \text{NA}(\mathcal{P}_t(\omega^t)) \text{ fails}\}$ is $\mathcal{P}$-polar if NA($\mathcal{P}$) holds.

Let us denote by $\mathcal{Q}_0$ the collection of measures $\mathbb{Q} \in \mathfrak{B}(\Omega)$ such that $\mathbb{Q} \ll\ll \mathcal{P}$ and S is an $(\mathbb{F},\mathbb{Q})$-martingale. Then given a universally measurable random variable $\varphi : \Omega \to \mathbb{R}_+$, we define

$$\mathcal{Q}_0^* \ := \big\{\mathbb{Q}\in\mathcal{Q}_0 \ : \mathbb{E}^{\mathbb{Q}}\big[g^-\big] + \mathcal{E}(\mathbb{Q},\mathcal{P}) < \infty\big\} \text{ and } \mathcal{Q}_\varphi^* := \{\mathbb{Q}\in\mathcal{Q}_0^* : \mathbb{E}^{\mathbb{Q}}[\varphi] < +\infty\}.$$

Theorem 5.6.1. *Let $g : \Omega \to (-\infty, +\infty]$ be upper semi-analytic and suppose that* NA($\mathcal{P}$) *holds. Then for any universally measurable $\varphi : \Omega \to \mathbb{R}_+$, one has*

$$V := \inf_{H\in\mathcal{H}} \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}\big[\exp\big(g + (H\circ S)_T\big)\big] = \sup_{\mathbb{Q}\in\mathcal{Q}_\varphi^*} \big\{\mathbb{E}^{\mathbb{Q}}[g] - \mathcal{E}(\mathbb{Q},\mathcal{P})\big\}.$$

Moreover, the infimum over $H \in \mathcal{H}$ is attained by some optimal trading strategy $\hat{H}$.

Remark 5.6.2. *In Bartl [2], the above result is proved under the condition that* NA($\mathcal{P}_t(\omega^t)$) *holds for all $t = 0, \cdots, T-1$ and all $\omega^t \in \Omega_t$. As explained in Remark 2.5 of [2], the main reason to use this ω-wise no-arbitrage condition (rather than the quasi-sure no-arbitrage condition* NA($\mathcal{P}$)*) is the measurability issue due to their dynamic programming procedure. Our alternative procedure allows to overcome this measurability difficulty.*

5.6.1 Some technical lemmas

In this section, we shall give some technical lemmas which will be used in both Section 5.6.2 and Section 5.6.3. Firstly, by using the same arguments as in Lemma 5.5.4, one obtains the next weak duality.

Lemma 5.6.3. *Under the same conditions as Theorem 5.6.1, one has*

$$\inf_{H\in\mathcal{H}}\sup_{\mathbb{P}\in\mathcal{P}}\log\mathbb{E}^{\mathbb{P}}\big[\exp\big(g+(H\circ S)_T\big)\big] \;\geqslant\; \sup_{\mathbb{Q}\in\mathcal{Q}_0^*}\big\{\mathbb{E}^{\mathbb{Q}}[g]-\mathcal{E}(\mathbb{Q},\mathcal{P})\big\}.$$

Next, for all $t\in\{0,\cdots,T-1\}$, we consider an upper semi-analytic function $g_{t+1}:\Omega_{t+1}\to\mathbb{R}\cup\{\infty\}$, and define

$$g_t(\omega^t) \;:=\; \sup_{\mathbb{Q}\in\mathcal{Q}_0^*(t,\omega^t)}\Big\{\mathbb{E}^{\mathbb{Q}}[g_{t+1}]-\mathcal{E}\big(\mathbb{Q},\mathcal{P}_t(\omega^t)\big)\Big\},\quad\text{for all }\omega^t\in\Omega_t,$$

where

$$\mathcal{Q}_0^*(t,\omega^t) \;:=\; \Big\{\delta_{\omega^t}\otimes\mathbb{Q}\in\mathfrak{B}(\Omega_{t+1}) \;:\; \mathbb{Q}\ll\mathcal{P}_t(\omega^t),\ \mathbb{E}^{\mathbb{Q}}\big[S_{t+1}(\omega^t,\cdot)-S_t(\omega^t)\big]=0,$$
$$\mathbb{E}^{\mathbb{Q}}\big[g_{t+1}^-(\omega^t,\cdot)+|S_{t+1}(\omega^t,\cdot)-S_t(\omega^t)|\big]+\mathcal{E}\big(\mathbb{Q},\mathcal{P}_t(\omega^t)\big)<\infty\Big\}.$$

Further, given a universally measurable random variable $Y_{t+1}:\Omega_{t+1}\to\mathbb{R}_+$, we introduce

$$\mathcal{Q}_{Y_{t+1}}^*(t,\omega^t) \;:=\; \big\{\mathbb{Q}\in\mathcal{Q}_0^*(t,\omega^t) \;:\; \mathbb{E}^{\mathbb{Q}}[Y_{t+1}(\omega^t,\cdot)]<\infty\big\}.$$

Moreover, for any universally measurable random variable $Y_t:\Omega_t\to\mathbb{R}_+$, we denote

$$\mathcal{Q}_{Y_t,t}^*:=\{\mathbb{Q}\in\mathcal{Q}_0|_{\Omega_t}:\mathbb{E}^{\mathbb{Q}}[g_t^-]+\mathcal{E}(\mathbb{Q},\mathcal{P})<+\infty,\ \mathbb{E}^{\mathbb{Q}}[Y_t]<+\infty\}.$$

Lemma 5.6.4. *For any universally measurable random variable $Y_{t+1}:\Omega_{t+1}\to\mathbb{R}_+$, one has*

$$g_t(\omega^t) \;=\; \sup_{\mathbb{Q}\in\mathcal{Q}_{Y_{t+1}}^*(t,\omega^t)}\Big\{\mathbb{E}^{\mathbb{Q}}[g_{t+1}]-\mathcal{E}\big(\mathbb{Q},\mathcal{P}_t(\omega^t)\big)\Big\},\quad\mathcal{P}\text{-q.s.}$$

In addition, if Y_{t+1} is assumed to be Borel measurable, the graph set

$$\big[\big[\mathcal{Q}_{Y_{t+1}}^*(t)\big]\big] \;:=\; \big\{(\omega,\mathbb{Q}):\omega\in\Omega_t,\mathbb{Q}\in\mathcal{Q}_{Y_{t+1}}^*(t,\omega)\big\}\ \textit{is analytic.}$$

Proof. The first result is consequent on the one-period duality result in Theorem 3.1 of Bartl [2] (see also our Remark 5.5.8), and the second result follows essentially the same arguments as in the proof of Lemma 5.5.9. □

Lemma 5.6.5. *Assume that* NA($\mathcal{P}$) *holds true. Then g_t is upper semi-analytic, and there exists a universally measurable map $h_{t+1}:\Omega_t\to\mathbb{R}^d$, together with a $\mathcal{P}$-polar set $N\subset\Omega_t$ such that, for all $\omega\in N^c$, one has*

$$g_t(\omega^t) \;=\; \sup_{\mathbb{P}\in\mathcal{P}_t(\omega^t)}\log\mathbb{E}^{\mathbb{P}}\Big[\exp\big(g_{t+1}(\omega^t,\cdot)+h_{t+1}(\omega^t)(S_{t+1}(\omega^t,\cdot)-S_t(\omega^t))\big)\Big]>-\infty.$$

Proof. The argument is similar to Lemma 5.5.10, so we shall provide here a sketch of the proof. As g_{t+1} is upper semi-analytic, $\big[\big[\mathcal{Q}_0^*(t)\big]\big]$ is analytic from Lemma 5.6.4 and $(\omega^t,\mathbb{Q})\in\Omega_t\times\mathfrak{B}(\Omega_1)\mapsto-\mathcal{E}(\mathbb{Q},\mathcal{P}_t(\omega^t))$ is upper semi-analytic by [2, Lemma 4.2] and [92, Proposition 7.47], it follows from Lemma 5.6.4 and a measurable selection argument(see e.g. [92, Proposition 7.26, 7.47, 7.48]) that $\omega^t\mapsto g_t$ is upper semi-analytic. By defining

$$V_t^*(\omega^t):=\inf_{h_{t+1}\in\mathbb{R}^d}\sup_{\mathbb{P}\in\mathcal{P}_t(\omega^t)}\log\mathbb{E}^{\mathbb{P}}\Big[\exp\big(g_{t+1}(\omega^t,\cdot)+h_{t+1}(S_{t+1}(\omega^t,\cdot)-S_t(\omega^t))\big)\Big],$$

and applying Theorem 3.1 of [2], we obtain that

$$g_t(\omega^t) \;=\; V_t^*(\omega^t) > -\infty, \quad \text{if NA}(\mathcal{P}_t(\omega^t)) \text{ holds true.}$$

As NA($\mathcal{P}$) holds, this is valid outside a $\mathcal{P}$-polar set N.

By defining $\phi_t(\omega^t, h_{t+1}) := \sup_{\mathbb{P}\in\mathcal{P}_t(\omega^t)} \log \mathbb{E}^{\mathbb{P}}\Big[\exp\big(g_{t+1}(\omega^t,\cdot) + h_{t+1}(S_{t+1}(\omega^t,\cdot) - S_t(\omega^t))\big)\Big]$, we can argue similarly as Lemma 5.5.10 to see that $(\omega^t, h_{t+1}) \mapsto \phi_t$ is in $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. Let us now consider the random set

$$\Phi(\omega^t) := \{h \in \mathbb{R}^d : \phi(\omega^t, h) = g_t(\omega^t)\}.$$

The previous arguments yield that $[[\Phi]]$ is in $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$. Thus by Lemma 4.11 of [1], Φ admits an $\mathcal{F}_t$-measurable selector h_{t+1} on the universally measurable set $\Phi(\omega^t) \neq \emptyset$. Moreover, Theorem 3.1 of [2] implies that $\Phi(\omega^t) \neq \emptyset$ holds true outside a $\mathcal{P}$-polar set N, thus h_{t+1} solves the infimum $\mathcal{P}$-q.s. □

5.6.2 Proof of Theorem 5.6.1 in a dominated case

We first provide the proof of Theorem 5.6.1 in a dominated case, where $\mathcal{P}$ is a singleton, i.e. $\mathcal{P} = \{\mathbb{P}\}$, for $\mathbb{P} = \mathbb{P}_0 \otimes \mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_{T-1}$, where $\mathbb{P}_t(\omega^t) \in \mathcal{P}_t(\omega^t)$ for all $\omega^t \in \Omega^t$ and all $t \leq T-1$. In particular, $\mathbb{P}_t : \Omega_t \to \mathfrak{B}(\Omega_{t+1})$ is a regular conditional probability distribution(r.c.p.d.) of $\mathbb{P}$ knowing $\mathcal{F}_t^0$. We can assume without loss of generality that $\mathcal{P}_t(\omega^t) = \{\mathbb{P}_t(\omega^t)\}$. Moreover, let $\mathcal{F}_t^{\mathbb{P}}$ denote the $\mathbb{P}$-completion of the σ-field $\mathcal{F}_t$, then any $\mathcal{F}_t^{\mathbb{P}}$-measurable random variable can be modified to a Borel measurable random variable in sense of $\mathbb{P}$-a.s.

The following lemma is an analogue of Lemma 5.5.11, and in this dominated context, the measurability issue is much easier to treat.

Lemma 5.6.6. *Assume the same conditions in Theorem 5.6.1 and that $\mathcal{P} = \{\mathbb{P}\}$. Then for all $t \leq T-1$ and all $\mathcal{F}_{t+1}^{\mathbb{P}}$-measurable random variable $Y_{t+1} : \Omega_{t+1} \to \mathbb{R}_+$, there is a Borel random variable $Y_t : \Omega_t \to \mathbb{R}_+$ such that*

$$\sup_{\mathbb{Q}\in\mathcal{Q}^*_{Y_t,t}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_t] - \mathcal{E}(\mathbb{Q}, \mathbb{P}|_{\Omega_t})\Big\} \;\leq\; \sup_{\mathbb{Q}\in\mathcal{Q}^*_{Y_{t+1},t+1}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_{t+1}] - \mathcal{E}(\mathbb{Q}, \mathbb{P}|_{\Omega_{t+1}})\Big\}. \tag{5.6.45}$$

Proof. Under the reference probability $\mathbb{P}$, for any $\mathcal{F}_{t+1}^{\mathbb{P}}$-measurable random variable Y_{t+1}, there exists a Borel measurable random variable Y_{t+1}^*, such that $Y_{t+1} = Y_{t+1}^*, \mathbb{P}$-a.s. and thus $\mathbb{E}^{\mathbb{Q}}[Y_{t+1}] = \mathbb{E}^{\mathbb{Q}}[Y_{t+1}^*]$, for all $\mathbb{Q} \in \mathcal{Q}^*_{Y_{t+1},t+1}$. So we can assume w.l.o.g. that Y_{t+1} is Borel measurable. By Lemma 5.6.4 together with a measurable selection argument (see e.g. Proposition 7.50 of [92]), for any $\varepsilon > 0$, there exists a universally measurable kernel $\mathbb{Q}_t^\varepsilon(\cdot) : \Omega_t \to \mathfrak{B}(\Omega_1)$ such that $\delta_\omega \otimes \mathbb{Q}_t^\varepsilon(\omega) \in \mathcal{Q}^*_{Y_{t+1}}(t,\omega)$ for all $\omega \in \Omega_t$, and

$$g_t(\omega) \;\leq\; \mathbb{E}^{\delta_\omega\otimes\mathbb{Q}_t^\varepsilon(\omega)}[g_{t+1}] - \mathcal{E}\big(\mathbb{Q}_t^\varepsilon(\omega), \mathbb{P}_t(\omega)\big) + \varepsilon.$$

The rest arguments are almost the same as in Step (ii) of the proof of Lemma 5.5.11 and we shall omit the details. □

Proof of Theorem 5.6.1 when $\mathcal{P} = \{\mathbb{P}\}$. We can argue by induction as in the proof of Proposition 5.5.2. Noticing NA($\{\mathbb{P}\}$) holds, the case $T = 1$ is proved by Theorem 3.1 of [2].

Suppose the case $T = t$ is verified with optimal strategy $\hat{H} := (\hat{H}_1, \cdots, \hat{H}_t)$:

$$\log \mathbb{E}^{\mathbb{P}}\big[\exp(g_t + (\hat{H} \circ S)_t)\big] \quad = \quad \sup_{\mathbb{Q} \in \mathcal{Q}^*_{Y_t,t}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_t] - \mathcal{E}\big(\mathbb{Q}, \mathbb{P}|_{\Omega_t}\big)\Big\}, \tag{5.6.46}$$

and we shall pass to the $T = t + 1$ case. Denoting $g_{t+1} := g$, defining $\hat{H}_{t+1}(\omega^t) := h_{t+1}$ as in Lemma 5.6.5, setting $Y_{t+1} := \varphi$ for any universally measurable random variable $\varphi : \Omega \to \mathbb{R}_+$, and letting Y_t be given as in Lemma 5.6.6, it follows that

$$\begin{aligned}
&\sup_{\mathbb{Q} \in \mathcal{Q}^*_\varphi} \Big\{\mathbb{E}^{\mathbb{Q}}[g_{t+1}] - \mathcal{E}(\mathbb{Q}, \mathbb{P}|_{\Omega_{t+1}})\Big\} \;\geq\; \sup_{\mathbb{Q} \in \mathcal{Q}^*_{Y_t,t}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_t] - \mathcal{E}\big(\mathbb{Q}, \mathbb{P}|_{\Omega_t}\big)\Big\} \\
= \;& \log \mathbb{E}^{\mathbb{P}}\big[\exp(g_t + (\hat{H} \circ S)_t)\big] \\
= \;& \log \mathbb{E}^{\mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_{t-1}}\Big[\exp\Big(\log \mathbb{E}^{\mathbb{P}_t}\Big[\exp\big(g_{t+1} + (\hat{H} \circ S)_{t+1}\big)\Big]\Big)\Big] \\
= \;& \log \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(g_{t+1} + (\hat{H} \circ S)_{t+1}\Big)\Big] \\
\geq \;& \inf_{H \in \mathcal{H}} \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(g_{t+1} + (H \circ S)_{t+1}\right)\right],
\end{aligned}$$

where the first inequality follows by Lemma 5.6.6 and the third line follows by Lemma 5.6.5. As the reverse inequality is the weak duality by Lemma 5.6.3, we have proved the case $T = t + 1$. In particular, $(\hat{H}_1, \cdots, \hat{H}_t, \hat{H}_{t+1})$ is the optimal trading strategy for the case $T = t + 1$. □

The following corollary serves as an important technical step in the proof of Lemma 5.5.11.

Corollary 5.6.7. *Assume the same conditions in Theorem 5.6.1 and let $\mathbb{P} \in \mathcal{P}$ be fixed. Then for any universally measurable random variables $g : \Omega \to \mathbb{R}$ and $\varphi : \Omega \to \mathbb{R}_+$, and any $\mathbb{Q}^* \in \mathcal{Q}^*_0$, one has*

$$\mathbb{E}^{\mathbb{Q}^*}[g] - \mathcal{E}\big(\mathbb{Q}^*, \mathbb{P}\big) \;\leqslant\; \sup_{\mathbb{Q} \in \mathcal{Q}^*_\varphi} \Big\{\mathbb{E}^{\mathbb{Q}}[g] - \mathcal{E}\big(\mathbb{Q}, \mathbb{P}\big)\Big\}. \tag{5.6.47}$$

Proof. Without loss of generality, we can assume that $\mathcal{E}(\mathbb{Q}^*, \mathbb{P}) < \infty$. In this case, one has $\mathbb{Q}^* \ll \mathbb{P}$ and the classical no-arbitrage condition NA($\{\mathbb{Q}^*\}$) holds. Let us denote

$$\mathcal{Q}^{**}_\varphi := \{\mathbb{Q} \in \mathcal{Q}^*_\varphi : \mathcal{E}(\mathbb{Q}, \mathbb{Q}^*) < +\infty\},$$

Using the weak duality of Lemma 5.6.3 and applying Theorem 5.6.1 (in the context of the fixed probability space $(\Omega, \mathcal{F}, \mathbb{Q}^*)$), we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^*}[g] - \mathcal{E}\big(\mathbb{Q}^*, \mathbb{P}\big) \quad = \quad \mathbb{E}^{\mathbb{Q}^*}\left[g - \log \frac{d\mathbb{Q}^*}{d\mathbb{P}}\right] - \mathcal{E}(\mathbb{Q}^*, \mathbb{Q}^*) \\
\leq \;& \inf_{H \in \mathcal{H}} \log \mathbb{E}^{\mathbb{Q}^*}\left[\exp\left(g - \log \frac{d\mathbb{Q}^*}{d\mathbb{P}} + (H \circ X)_T\right)\right] \\
= \;& \sup_{\mathbb{Q} \in \mathcal{Q}^{**}_\varphi} \left(\mathbb{E}^{\mathbb{Q}}\left[g - \log \frac{d\mathbb{Q}^*}{d\mathbb{P}}\right] - \mathcal{E}\left(\mathbb{Q}, \mathbb{Q}^*\right)\right) \\
\leqslant \;& \sup_{\mathbb{Q} \in \mathcal{Q}^*_\varphi} \left(\mathbb{E}^{\mathbb{Q}}\left[g - \log \frac{d\mathbb{Q}^*}{d\mathbb{P}}\right] - \mathcal{E}\left(\mathbb{Q}, \mathbb{Q}^*\right)\right) \quad = \quad \sup_{\mathbb{Q} \in \mathcal{Q}^*_\varphi} \left(\mathbb{E}^{\mathbb{Q}}[g] - \mathcal{E}\left(\mathbb{Q}, \mathbb{P}\right)\right).
\end{aligned}$$

□

5.6.3 Proof of Theorem 5.6.1

We now provide the proof of Theorem 5.6.1 in the general (possibly) non-dominated case.

Lemma 5.6.8. *Let $t+1 \leq T$, then for any universally measurable random variable $Y_{t+1} : \Omega_{t+1} \to \mathbb{R}_+$ and $\varepsilon > 0$, there is a universally measurable random variable $Y_t^\varepsilon : \Omega_t \to \mathbb{R}_+$ such that*

$$\sup_{\mathbb{Q}\in\mathcal{Q}^*_{Y_t^\varepsilon,t}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_t] - \mathcal{E}\big(\mathbb{Q},\mathcal{P}|_{\Omega_t}\big)\Big\} \leq \sup_{\mathbb{Q}\in\mathcal{Q}^*_{Y_{t+1},t+1}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_{t+1}] - \mathcal{E}\big(\mathbb{Q},\mathcal{P}|_{\Omega_{t+1}}\big)\Big\} + \varepsilon. \tag{5.6.48}$$

Proof. In view of Corollary 5.6.7, we can assume w.l.o.g. that $Y_{t+1} \equiv 0$. By Lemma 5.6.4 and a measurable selection argument (see e.g. Proposition 7.50 of [92]), for any $\varepsilon > 0$, there exists a universally measurable kernel $\mathbb{Q}_t^\varepsilon(\cdot) : \Omega_t \to \mathfrak{B}(\Omega_1)$ such that $\delta_\omega \otimes \mathbb{Q}_t^\varepsilon(\omega) \in \mathcal{Q}^*_{Y_{t+1}}(t,\omega)$ for all $\omega \in \Omega_t$, and

$$g_t(\omega) \;\leq\; \mathbb{E}^{\delta_\omega\otimes\mathbb{Q}_t^\varepsilon(\omega)}[g_{t+1}] - \mathcal{E}\big(\mathbb{Q}_t^\varepsilon(\omega),\mathcal{P}_t(\omega)\big) + \varepsilon.$$

The rest argument is similar to Step (ii) of the proof of Lemma 5.5.11, and we omit it here. □

Proof of Theorem 5.6.1. Notice that under NA($\mathcal{P}$), the results in case $T = 1$ follows from Theorem 3.1 of [2]. Suppose that when $T = t$ the duality holds with optimal strategy $(\hat{H}_1, \cdots, \hat{H}_t)$. Denoting $g_{t+1} := g$, defining $\hat{H}_{t+1}(\omega^t) := h_{t+1}$ as in Lemma 5.6.5, setting $Y_{t+1} := \varphi$ and letting Y_t be given in Lemma 5.6.8, we apply similar argument as in the dominated context in Section 5.6.2 to obtain

$$\begin{aligned}
\sup_{\mathbb{Q}\in\mathcal{Q}^*_\varphi} \Big\{\mathbb{E}^{\mathbb{Q}}[g_{t+1}] - \mathcal{E}(\mathbb{Q},\mathcal{P}|_{\Omega_{t+1}})\Big\} \;&\geq\; \sup_{\mathbb{Q}\in\mathcal{Q}^*_{Y_t,t}} \Big\{\mathbb{E}^{\mathbb{Q}}[g_t] - \mathcal{E}\big(\mathbb{Q},\mathcal{P}|_{\Omega_t}\big)\Big\} \\
&=\; \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}\big[\exp(g_t + (\hat{H}\circ S)_t)\big] \\
&\geq\; \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}\Big[\exp\Big(g_{t+1} + (\hat{H}\circ S)_{t+1}\Big)\Big] \\
&\geq\; \inf_{H\in\mathcal{H}} \sup_{\mathbb{P}\in\mathcal{P}} \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(g_{t+1} + (H\circ S)_{t+1}\right)\right].
\end{aligned}$$

This together with Lemma 5.6.3 implies the duality result. Moreover, $(\hat{H}_1, \cdots, \hat{H}_{t+1})$ is the optimal trading strategy for the case $T = t+1$. □

Chapter 6

A sparse grid approach to balance sheet risk measurement

6.1 Introduction

The main goal of this note is to present a robust and efficient method to numerically assess risks on the balance sheet distribution of, say, an insurance company, at a given horizon which is in practice chosen to be one year, consistently with the Solvency 2 regulation (prudential framework for assessing the required solvency capital for an European insurance company).

On a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, the balance sheet of the company is a random process summarised, at any time $t \geq 0$, by the value of the assets of the company $(A_t)_{t\geq 0}$ and the value of the liabilities $(L_t)_{t\geq 0}$. The quantity of interest is the Profit and Loss (PnL in the sequel) associated to the balance sheet, which is given by

$$P_t = L_t - A_t \ , \quad t \geq 0 \, .$$

By convention, and adopting the point of view of risk management, we measure the loss as a positive quantity.

On the Liability side, the insurance company has sold a structured financial product which depends on the evolution of a one-dimensional stock price (S_t) and the risk-free interest rate (r_t). Several insurance products could be of this type, in particular Unit-Linked (with or without financial guarantees) and Variable Annuity contracts. For those contracts, client's money is invested in equity and bond markets while the insurance company might also provide with financial guarantees similar to long-term put options. The long maturity of those contracts requires the introduction of a model for interest rate as they are very sensitive to Interest Rate curve movements. The value L_1 is just the price of this product taking into account the value of some risk factors $\mathcal{X}_1$ (stock price, interest rate curve etc.) at time $t = 1$ used to calibrate the pricing model.

On the Asset side, the insurance company manages some assets to hedge the risk associated to the product sale (the pricing actually includes a margin which is secured through hedging). The hedging assets are the stock and swaps of several maturities, in practice mostly concentrated on the long term (bond futures are also included sometimes). The hedging portfolio is typically rebalanced on a weekly basis and the hedging quantities are determined by a financial model, taken to be the same as the liability pricing model, whose inputs are the risk factors $\mathcal{X}_t$ at the time t when the hedge is computed.

We describe precisely in Section 6.2, the pricing and hedging model, the dynamics of the risk factor $\mathcal{X}$ and the value of the asset and liability side of the balance sheet. Let us stress that the risk factor model is given under the so-called *real-world* probability measure $\mathbb{P}$, which might be objectively calibrated using time series of financial markets or represents the management view. This *real-world* model may be (and most of the time is) completely different from the pricing and hedging model which might be simplified for runtime/trackability purposes, prudent (pricing and hedging include a margin) or being constrained by regulation.

Our goal is then to compute various risk indicator for the loss distribution of the balance sheet at one year namely the distribution of P_1 under the *real-world* probability measure $\mathbb{P}$, that we denote hereafter η.

Precisely, we measure the risk associated to η using a (law invariant) risk measure defined over the class of square integrable measures $\varrho : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$. First, we consider for ϱ the so called Value-at-Risk ($V@R$), which is defined by the left-side quantile:

$$V@R_p(\eta) = \inf \{q \in \mathbb{R} \mid \eta((-\infty, q]) \geq p\} \ . \tag{6.1.1}$$

We will also work with the class of *spectral risk measures*: a spectral risk measure is defined as

$$\varrho_h(\eta) = \int_0^1 V@R_p(\eta)h(p)\mathrm{d}p \ , \tag{6.1.2}$$

where h is a non-decreasing probability density on $[0,1]$. In the numerics, we will focus on the Average Value-at-Risk ($AV@R$) which is given by

$$AV@R_\alpha(\eta) = \frac{1}{1-\alpha}\int_\alpha^1 V@R_p(\eta)\mathrm{d}p \ , \tag{6.1.3}$$

and is a special case of a spectral risk measure.

For a law invariant risk measure ϱ, we denote by $\Re$ its "lift" on $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}) =: L^2$, namely $\Re[X] = \varrho([X])$ for any $X \in L^2$, where $[X]$ denotes the law of X. The lift $\Re_h$ from a spectral risk measure ϱ_h satisfies the following properties:

1. *Monotonicity*: $\Re_h[X] \leq \Re_h[Y]$, for $X \leq Y \in L^2$;
2. *Cash invariance*: $\Re_h[X+c] = \Re_h[X] + c$ for $X \in L^2$ and $c \in \mathbb{R}$;
3. *Positive homogeneity*: $\Re[tX] = t\Re[X]$, $t \geq 0$ and $X \in L^2$.
4. *Convexity*: $\Re[tX + (1-t)Y] \leq t\Re[X] + (1-t)\Re[Y]$, whenever $0 \leq t \leq 1$, for $X, Y \in L^2$;

Let us stress the fact that $V@R$ only satisfies 1-3. We refer to [118] and the references therein for more insights on risk measures and spectral risk measures.

In our setting, the loss distribution η of the balance sheet PnL is obtained through the following expression:

$$\eta = p_1 \sharp \nu \ ,$$

where $\sharp$ denotes the push-forward operator, $p_1 : \mathbb{R}^\theta \to \mathbb{R}$ is the *function* describing the PnL in terms of the risk factors, and ν stands for the distribution of the risk factors $\mathcal{X}$. In practice the estimation of $\varrho(\eta)$ requires to sample from η. In turn, this demands for a sample of the model parameter distribution ν and for a numerical approximation of p_1. In this note, we compare two main approaches to form the sample of η given one of ν.

The first one is known as the *nested simulation* approach: It is a two-step method. First, a set of "outer simulation", describing the random values of the risk factors, is drawn. Then, for each value of the risk factors, a sample of "inner simulation" is drawn to compute the various hedge and prices. In this approach, all computations are realised "online". The main advantage of this approach is its simplicity to implement in practice, described in the first paragraph of Subsection 6.3.1. However, it is well known that this approach is quite greedy, even if optimised as in [119]. We also want to stress the fact that when computing the η-sample, no information about p_1 is stored for future work: for example if ν is modified, due to time or a model change, a full recalculation would be required.

The other approach we chose to adopt and would like to promote is a *grid approach* where the approximation of p_1 is made "offline", by a Monte Carlo approach, and then stored. The numerical computation is then done through a (multi-linear) interpolation on a grid. The main drawback of this approach is that the size of the grid, in high dimension, can become untractable, especially if one uses regular grid. To partially circumconvent this difficulty, we introduce a *sparse grid* [120] which reduces drastically the number of point to be used (equivalently, values to be stored) with only small reduction of the accuracy of the method.

We prove that, for a spectral risk measure, the two approaches give an estimation of $\varrho(\eta)$ which converges to the true value, see Theorem 6.3.8.

Furthermore, we show in the numerical Section 6.4 that using the grid approach together with a sparse grid of low level allows to get a good approximation of the loss distributions η, and of some related risk measures, while reducing drastically the computational time and allowing to keep information about the balance sheet function p_1. Last, this permits to numerically quantify uncertainty. Indeed, since the computations on the grid are stored, the computation of the distribution of the PnL under other distributions for the parameters is almost instantaneous and can be compared with the results obtained with the initial one. An application to uncertainty estimation is given in the last numerical application.

The rest of the paper is organised as follows. In the second section, we first describe the mathematical models that are used to describe the evolution of the prices under the risk-neutral measure $\mathbb{Q}$. We then describe precisely how A and L are specified. In the third section, we describe the two numerical methods used to compute L_t and A_t at any given time $t \geq 0$. In particular, we show how to efficiently compute, at time t, the quantities to hold in the hedging portfolio, which are expressed in term of the derivatives of the claim's price. We also explain how to compute the price of the product and of the assets used to construct the hedging porfolio, leading to the computation of L_t and A_t. We show how to obtain an approximation of the distribution of P_1 under the physical measure $\mathbb{P}$, and we prove an upper bound for the mean square error of the overall procedure. Finally, in Section 4, we present our numerical results.

6.2 Financial Model

In this section, we give the precise specification of the asset and liability sides of the balance sheet. We also present the risk-neutral model and the real-world model that are used.

6.2.1 Description of the sold product

Let us assume that a company sells a contingent claim at time $t = 0$ which is a (discretely) path-dependent option with a payoff function G paid at the maturity $T > 0$, depending upon the evolution of a one-dimensional risky asset's price S. We focus here on:
A put lookback option, that is a discretely path-dependent option whose strike at maturity T is given by the maximum of the asset's price S over the times $t \in \{\tau_0 = 0, \tau_1, \cdots, \tau_\kappa = T\}$ where $\kappa \geq 1$:

$$G(S_{\tau_0}, \ldots, S_{\tau_\kappa}) = \left(\max_{0 \leq \ell \leq \kappa} S_{\tau_\ell} \right) - S_T. \tag{6.2.4}$$

Remark 6.2.1. *The proxy provided above is close to financial guarantees offered in Variable Annuity contracts. Those contracts are structured insurance products composed of a fund investment on top of which both insurance and financial protection are added. In our case, the contract is a* Guaranteed Minimum Accumulation Benefit including a ratchet mechanism. *At time $t = 0$, the customer invests his/her money in the underlying fund and will receive at a given maturity the maximum between the terminal fund value and its* terminal benefit base *in case she is still alive. The terminal benefit base is equal to the maximum of the underlying fund values observed at each anniversary date of the contract (ratchet mechanism). We do not consider the modeling of death/survival in this proceedings, neither the possibility that client can surrender at any time during the life of the contract.*

6.2.2 Market model under the risk-neutral measure

We assume that all pricing and hedging is done with market risk-neutral measure $\mathbb{Q}$.

The derivative with a payoff function G as above is depending upon a one-dimensional stock's price $S = (S_t)_{t\in[0,T]}$. We assume here that the dynamics of the asset under $\mathbb{Q}$ are of the Black & Scholes type as described in Section 6.2.2 with a stochastic interest rate $r = (r_t)_{t\in[0,T]}$ which follows a Hull & White model.

As the payoff G is a proxy of Variable Annuity guarantee which is a long term Savings product (in practice maturity ranges from 10 to 30 years depending on product type), the modeling (and hedging) of interest rate is essential as the product and therefore the overall balance sheet of the company is very sensitive to this risk.

The short rate model

Let $\Theta \in \mathbb{R}^d$ ($d := 3$ in the sequel) be a set of parameters representing some market observations. The short rate evolution is governed by the Hull & White dynamics

$$r_s^{t,\Theta} = r_t^{t,\Theta} + \int_t^s a\left(\mu_u^{t,\Theta} - r_u^{t,\Theta}\right) \mathrm{d}u + b\left(B_s - B_t\right), \ s \in [t,T], \tag{6.2.5}$$

where B is a $\mathbb{Q}$-Brownian motion, a and b are real constants and $\mu^{t,\Theta} : [t,T] \to \mathbb{R}$ is a function. We refer to [121] for a more complete analysis of the Hull & White short rate model.

The parameter $\mu^{t,\Theta}$ is calibrated using the market observations Θ, so that the model reproduces the interest rate curve observed on the market. It is given by

$$\mu_s^{t,\Theta} = f^{\Theta}(t,s) + \frac{1}{a}\frac{\partial f^{\Theta}(t,s)}{\partial s} + \frac{b^2}{2a^2}\left(1 - e^{-2a(s-t)}\right), s \in [t,T]. \tag{6.2.6}$$

We refer to the Appendix for a derivation of (6.2.6).

As a consequence, the Θ parameter must be chosen in order to represent adequately the forward rate curve observed on the market.

We suppose here that the forward rate curve $f^{\Theta}(t,\cdot)$ is directly observed and is a linear combination of three elementary functions $h^{t,1}, h^{t,2}, h^{t,3}$ from $[t,T]$ to $\mathbb{R}$, given by

$$h^{t,1}(s) := h^1(s-t), \ h^{t,2}(s) := h^2(s-t), \text{ and } h^{t,3}(s) := h^3(s-t), \ s \in [t,T],$$

where, for $u \in [0,T]$:

$$h^1(u) = \begin{cases} 1 & \text{if } u \le \frac{t_1+t_2}{2}, \\ 2\frac{\frac{t_2+t_3}{2}-u}{t_3-t_1} & \text{if } u \in [\frac{t_1+t_2}{2}, \frac{t_2+t_3}{2}], \\ 0 & \text{otherwise,} \end{cases} \qquad h^3(u) = \begin{cases} 0 & \text{if } u \le \frac{t_2+t_3}{2} \\ 2\frac{u-\frac{t_2+t_3}{2}}{t_4-t_2} & \text{if } u \in [\frac{t_2+t_3}{2}, \frac{t_3+t_4}{2}], \\ 1 & \text{otherwise,} \end{cases}$$

$$\text{and } h^2(u) = 1 - h^1(u) - h^3(u).$$

where $0 \le t_1 < t_2 < t_3 < t_4 \le T$ are four fixed real numbers.

The function $h^{t,1}$ (resp. $h^{t,2}, h^{t,3}$) model the short (resp. middle, long) term structure of the interest rates curve.

In a nutshell, the short rate model is determined by:

(i) the time of observation $t \in [0,T]$,

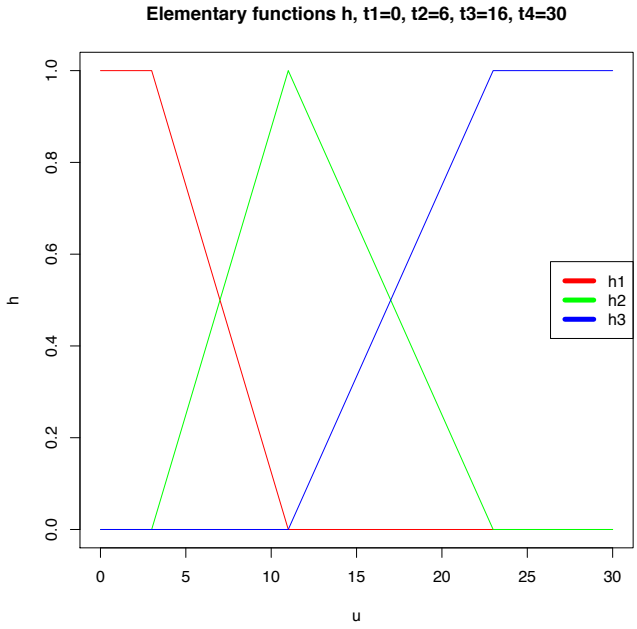

Figure 6.1: Building blocks for the forward interest rate curve.

(ii) the three-dimensional parameter $\Theta := \{\theta_1, \theta_2, \theta_3\} \in \mathbb{R}^3$, where $\theta_1, \theta_2, \theta_3$ are such that

$$f^{\Theta}(t, \cdot) = \theta_1 h^{t,1} + \theta_2 h^{t,2} + \theta_3 h^{t,3}, \tag{6.2.7}$$

$f^{\Theta}(t, \cdot)$ being the observed forward rates curve.

As a result, given an observation (t, Θ) as above, the short rate process $r^{t,\Theta}$ has the dynamics (6.2.5), where $\mu^{t,\Theta}$ is computed using (6.2.6).

Remark 6.2.2. *In practice, the parameters a, b appearing in* (6.2.5) *can be calibrated so that the model reproduces the prices (observed on the market) of some contracts such as swaps or swaptions. We could more generally allow the parameters a, b of the Hull & White model to depend upon the market observations Θ. The parameter Θ should live in a higher-dimensional space to take into account the observed swap(tion)s prices. In practice, regular recalibration of parameters is largely performed by practitioners, in particular when they performed dynamic hedging.*

There are several reasons explaining the choice of this model. First of all, it is quite simple to calibrate using the data. In fact, the function μ is directly given as a function of the forward rate curve. We should note again that the choice of keeping a, b fixed through time simplifies the calibration. Secondly, we will see later in Proposition 6.3.2 that this short rate model, associated to the stock model described below, leads to an exact simulation under the risk-neutral measure. Lastly, closed and easily tractable formulas can be obtained for the prices of the zero-coupon bonds and swaps which are the products used to construct the hedging portfolio and then to compute the value of the company's assets A.

These prices are as follows.

Proposition 6.2.3. *Let $(t,\Theta)\in[0,1]\times\mathbb{R}^3$ be a market observation, and consider the process $\left(r_s^{t,\Theta}\right)_{s\in[t,T]}$ given by (6.2.5), where the parameter $\mu^{t,\Theta}$ is defined with (6.2.6) and (6.2.7).*

1. *The price at time t of a zero-coupon maturing at time $u\in[t,T]$ is given by:*

$$P^{t,\Theta,u}=\exp\left(-\int_t^u f^{\Theta}(t,s)\mathrm{d}s\right), \tag{6.2.8}$$

and its derivatives with respect to $\Theta:=(\theta_1,\theta_2,\theta_3)$ are given by:

$$\frac{\partial P}{\partial\theta_i}^{t,\Theta,u}=-P^{t,\Theta,u}\int_t^u h^{t,i}(s)\mathrm{d}s. \tag{6.2.9}$$

2. *Let $(0,\Theta_0)$ be the observation made at time 0. Consider a swap contract issued in $s=0$, with maturity $M>0$, rate $R>0$, with coupons versed at every time $i\in\{1,\dots,M\}$. Then, the price of this contract at time t is given by:*

$$SW^{t,\Theta,M,R}=\frac{P^{t,\Theta,1}}{P^{0,\Theta_0,1}}-P^{t,\Theta,M}-R\sum_{i=1}^{M}P^{t,\Theta,i}, \tag{6.2.10}$$

and its derivatives with respect to Θ are given by:

$$\frac{\partial SW}{\partial\theta_j}^{t,\Theta,M,R}=-\frac{P^{t,\Theta,1}}{P^{0,\Theta_0,1}}\int_t^1 h^{t,j}(s)\mathrm{d}s+P^{t,\Theta,M}\int_t^M h^{t,j}(r)\mathrm{d}r+R\sum_{i=1}^{M}\left(P^{t,\Theta,i}\int_t^{t+i}h^{t,j}(s)\mathrm{d}s\right). \tag{6.2.11}$$

The stock model

Given the observations Θ of the interest rate factors and the risky asset's price $x\in(0,\infty)$, the evolution of the price under the neutral-risk measure $\mathbb{Q}$ is given by

$$S_s^{t,x,\Theta}=x+\int_t^s r_u^{t,\Theta}S_u^{t,x,\Theta}\mathrm{d}u+\int_t^s\sigma S_u^{t,x,\Theta}\mathrm{d}\tilde{W}_u,\ s\in[t,T], \tag{6.2.12}$$

where $\sigma>0$, $\tilde{W}$ is another $\mathbb{Q}$-Brownian motion, whose quadratic covariation with B is given by

$$\langle B,\tilde{W}\rangle_t:=\rho\,t,t\in[0,T],$$

where $\rho\in[-1,1]$. Equivalently, $S^{t,x,\Theta}$ can be written as:

$$S_s^{t,x,\Theta}=x+\int_t^s r_u^{t,\Theta}S_u^{t,x,\Theta}\mathrm{d}u+\int_t^s\rho\sigma S_u^{t,x,\Theta}\mathrm{d}B_u+\int_t^s\sqrt{1-\rho^2}\sigma S_u^{t,x,\Theta}\mathrm{d}W_u,\ s\in[t,T] \tag{6.2.13}$$

where W is a $\mathbb{Q}$-Brownian motion, independent of B.

Remark 6.2.4. *In practice, the parameter σ appearing in (6.2.13) can be calibrated so that the model reproduces the prices of some derivatives over the risky asset. This can be taken into account by increasing the dimension of the space where Θ lives, and by adding this calibration procedure.*

Remark 6.2.5. *Naturally, the general sparse grid approach can be applied to different models and functional representations. We made the choice of using a Black & Scholes model for the stock*

value and a Hull & White model with the given functional representation in terms of h^1, h^2, h^3 for the short rate, since they are convenient to obtain explicit pricing and sensitivities formulae as we show in the following.

6.2.3 Modeling the Balance Sheet

The key point for us is to approximate the distribution of the balance sheet of an insurance company at time $t = 1$ (here a year) given the market observations at $t = 0$. As mentionned in the introduction, the PnL is a process P which can be decomposed as

$$P_t = L_t - A_t,\ t \in [0, 1], \tag{6.2.14}$$

where L is the value of the liabilities of the company and A is the value of the assets.
We assume that at time $t = 0$, the balance sheet is clear, meaning that the company has no asset nor liability, that is $L_0 = A_0 = 0$.

We describe precisely in the two subsequent sections how these quantities are defined. Importantly, we denote by $\bar{\mathcal{X}}_t := (\bar{S}_t, \bar{\Theta}_t)$, $0 \le t \le 1$, the stochastic process representing the random evolution of the market parameter under the *real-world* measure $\mathbb{P}$. Namely, $\bar{S}$ is the stock price and $\bar{\Theta}$ the interest rate curve parameters as described above in Section 6.2.2. It is important to have in mind that the model chosen for the stock price S (under $\mathbb{Q}$) and $\bar{S}$ (under $\mathbb{P}$) will be completely different as they do not serve the same purpose (pricing-hedging on one hand, risk management of the Balance Sheet or regulatory assessment of required capital on the other hand).

Liability side

For any market observation $\bar{\mathcal{X}}_t := (\bar{S}_t, \bar{\Theta}_t)$, the value $L_t =: \ell(t, \bar{S}_t, \bar{\Theta}_t)$ of the liabilities has to be computed, especially at time $t = 1$ in our application. The company's liabilities are reduced to one derivative product sold at $t = 0$. In our setting, the contingent claim's price is simply given by:

$$\ell(t, x, \Theta) = \mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta}), \tag{6.2.15}$$

where $(r^{t,\Theta}, S^{t,x,\Theta})_{t \le s \le T}$ are the risk neutral dynamics of the short rate and stock price, see Section 6.2.2 and Section 6.2.2, calibrated from the observed market parameter (x, Θ) at time t.

We recall that the payoff G depends on $S^{t,x,\Theta}$ only through the values $S_\tau^{t,x,\Theta}$, $\tau \in \Gamma_G := \{\tau_0, \dots, \tau_\kappa\}$ ($\kappa \ge 0$), see (6.2.4).

As explained in more detail below, the computation of P_1 first requires to approximate $\ell(t, x, \Theta)$ for (t, x, Θ) on a (possibly stochastic) discrete grid of $[0, 1] \times (0, \infty) \times \mathbb{R}^3$. This approximation L at any point $(t, x, \Theta) \in [0, 1] \times (0, \infty) \times \mathbb{R}^3$ basically follows from the simulation of the processes $r^{t,\Theta}$ and $S^{t,x,\Theta}$ under the risk-neutral measure $\mathbb{Q}$ and a Monte Carlo procedure. We will see in Section 6.3 that the simulation can be done in an exact manner in our model.

Remark 6.2.6. *A classical approach to compute ℓ would be to use a dynamic programming principle. Step by step, it requires*

1. *To numerically obtain $\ell(1, x, \Theta)$ for all (x, Θ) on the grid with* (6.2.15)*,*
2. *Then to iteratively compute $\ell(t_{k+1}, x, \Theta)$ for all x, Θ on the grid using*

$$\ell(t_k, x, \Theta) = \mathbb{E}^{\mathbb{Q}} e^{-\int_{t_k}^{t_{k+1}} r_s^{t_k,\Theta} \mathrm{d}s} \mathbb{E}^{\mathbb{Q}} e^{-\int_{t_{k+1}}^{T} r_s^{t_k,\Theta} \mathrm{d}s} G(S^{t_k,x,\Theta}) | \mathcal{F}_{t_{k+1}}. \tag{6.2.16}$$

However, the inner conditional expectation is not *of the form* $\ell(t_{k+1}, x, \Theta)$ *for* $x > 0$ *and* $\Theta \in \mathbb{R}^3$. *In fact, the time of the market observation* Θ *is still* t_k, *while the discount factor goes only from* t_{k+1} *to* T, *in contrast with* (6.2.15). *Therefore, using the dynamic programming equation* (6.2.16) *would require to introduce some additional and artificial parameters, namely the time of calibration and the value of* $r^{t_k,\Theta}$ *at time* t_{k+1}. *This would made the overall procedure heavier that is why we compute* ℓ *using simply* (6.2.15).

Asset side

The company wants to replicate the product with payoff G. (in practice, the pricing embeds a margin and the objective of replicating the payoff G is to secure it). The classical theory of mathematical finance ensures that it is equivalent, in theory, to possess a portfolio which negates the variations of the price of the product with respect to the evolution of the underlying parameters.
In our context, the insurer wants to be protected against the variations with respect to the stock price S_t and the interest rate curve, which is modeled through the parameter Θ.
The dynamic hedging portfolio is constructed and rebalanced in discrete time, on the time grid $\Gamma := \{t_0 = 0 < t_1 < \cdots < t_n = 1\}$ (in practice, the porfolio will be rebalanced up to the maturity of the product, but in our setting, we are only interested in the portfolio's value up to $t = 1$). At each time $t \in \Gamma$, the insurer computes the derivatives of the price with respect to S_t and Θ_t, and then buys some financial assets (the stock and swaps) in order to construct a portfolio whose derivatives match those computed.
To model this framework, we decompose the hedging portfolio's value A in two parts:

$$A_t = A_t^{\Delta} + A_t^{\rho}. \tag{6.2.17}$$

The process A^{Δ} is the value of the portfolio obtained to cancel the variations of the price with respect to S, while A^{ρ} is defined to deal with the variations with respect to Θ.

Remark 6.2.7. *Obviously, since in practice the hedging is done in discrete time and some underlying parameter are not considered, the payoff* G *is not exactly replicated, nor super-replicated. Therefore the PnL of the company is not null, nor always positive, and the goal of this proceedings is precisely to propose a new numerical method to estimate the distribution of this quantity at time* $t = 1$.

To construct the hedging portfolio, the insurer can buy the underlying stock, together with three swap contracts, defined by some rates $R_1, R_2, R_3 > 0$ and maturity dates $T_1, T_2, T_3 \in [1, T]$. In practice, interest rate hedging is performed so that the portfolio is insensitive to the variations of the main maturities of the interest rate curve. For long-term products, this means building an hedging portfolio containing several different maturities from 1 year to 30 year. Here, only 3 maturities representing short, medium and long-term part of the curve are considered for simplicity. The formula for their price $\mathrm{SW}^{t,\Theta,T_i,R_i}, i = 1, 2, 3$ is given in Proposition 6.2.3 above. We now describe how to compute the quantities of assets and swaps to buy at a time $t \in \Gamma$, to rebalance the hedging portfolio. Denote by Δ (resp. $\rho_i, i = 1, 2, 3$) the quantities of stock (resp. swap with rate R_i and maturity date T_i, $i = 1, 2, 3$). Then the value of the portfolio of the company is given by:

$$\Pi_t = \Delta S_t + \sum_{i=1}^{3} \rho_i \mathrm{d}SW^{t,\Theta_t,T_i,R_i} - \ell(t, S_t, \Theta_t). \tag{6.2.18}$$

By Ito's formula, assuming a semimartingale decomposition for the process Θ under $\mathbb{P}$, we get:

$$\begin{aligned}
\mathrm{d}\Pi_t &= (\Delta - \Delta(t, S_t, \Theta_t))\,\mathrm{d}S_t \\
&+ \left(\rho_1 \frac{\partial \mathrm{SW}^{t,\Theta,T_1,R_1}}{\partial \theta_1} + \rho_2 \frac{\partial \mathrm{SW}^{t,\Theta,T_2,R_2}}{\partial \theta_1} + \rho_3 \frac{\partial \mathrm{SW}^{t,\Theta,T_3,R_3}}{\partial \theta_1} - \frac{\partial \ell}{\partial \theta_1}(t,x,\Theta)\right)\mathrm{d}\theta_1 \\
&+ \left(\rho_1 \frac{\partial \mathrm{SW}^{t,\Theta,T_1,R_1}}{\partial \theta_2} + \rho_2 \frac{\partial \mathrm{SW}^{t,\Theta,T_2,R_2}}{\partial \theta_2} + \rho_3 \frac{\partial \mathrm{SW}^{t,\Theta,T_3,R_3}}{\partial \theta_2} - \frac{\partial \ell}{\partial \theta_2}(t,x,\Theta)\right)\mathrm{d}\theta_2 \\
&+ \left(\rho_1 \frac{\partial \mathrm{SW}^{t,\Theta,T_1,R_1}}{\partial \theta_3} + \rho_2 \frac{\partial \mathrm{SW}^{t,\Theta,T_2,R_2}}{\partial \theta_3} + \rho_3 \frac{\partial \mathrm{SW}^{t,\Theta,T_3,R_3}}{\partial \theta_3} - \frac{\partial \ell}{\partial \theta_3}(t,x,\Theta)\right)\mathrm{d}\theta_3 \\
&+ \mathrm{d}t \text{ terms.}
\end{aligned}$$

To cancel the risks induced by the variations of the stock price and the interest rate curve, it is needed that the four first terms in the previous equation cancel.
Those considerations lead to the following construction for the hedging portfolio $A = A^{\Delta} + A^{\rho}$:

Δ-hedging: The value of A^{Δ} at time 1 is

$$A_1^{\Delta} = \sum_{i=0}^{n-1} \Delta(t_i, \bar{S}_{t_i}, \bar{\Theta}_{t_i}) \left(\bar{S}_{t_{i+1}} - \bar{S}_{t_i}\right) \quad \text{where} \quad \Delta(t,x,\Theta) := \frac{\partial L}{\partial x}(t,x,\Theta). \tag{6.2.19}$$

Note that the function Δ has to be computed, at each time $t_i, i = 0, \ldots, n$, and for any market situation (x, Θ) at this time. A method leading to a numerical estimation of Δ is proposed in Section 6.3.

ρ-hedging: The value of A_1^{ρ} is

$$A_1^{\rho} = \sum_{i=0}^{n-1} \sum_{j=1}^{3} \rho^j(t_i, \bar{S}_{t_i}, \bar{\Theta}_{t_i}) \left(\mathrm{SW}^{t_{i+1}, \bar{\Theta}_{t_{i+1}}, T_j, R_j} - \mathrm{SW}^{t_i, \bar{\Theta}_{t_i}, T_j, R_j}\right). \tag{6.2.20}$$

At time $t \in [0,T]$, for a market at (x,Θ), the quantities $\rho^i(t,x,\Theta)$, $i = 1,2,3$ of each swap contract required for the hedging are given by the solution of the linear system

$$\begin{cases}
\rho_1 \frac{\partial \mathrm{SW}^{t,\Theta,T_1,R_1}}{\partial \theta_1} + \rho_2 \frac{\partial \mathrm{SW}^{t,\Theta,T_2,R_2}}{\partial \theta_1} + \rho_3 \frac{\partial \mathrm{SW}^{t,\Theta,T_3,R_3}}{\partial \theta_1} &= \frac{\partial \ell}{\partial \theta_1}(t,x,\Theta) \\
\rho_1 \frac{\partial \mathrm{SW}^{t,\Theta,T_1,R_1}}{\partial \theta_2} + \rho_2 \frac{\partial \mathrm{SW}^{t,\Theta,T_2,R_2}}{\partial \theta_2} + \rho_3 \frac{\partial \mathrm{SW}^{t,\Theta,T_3,R_3}}{\partial \theta_2} &= \frac{\partial \ell}{\partial \theta_2}(t,x,\Theta) \\
\rho_1 \frac{\partial \mathrm{SW}^{t,\Theta,T_1,R_1}}{\partial \theta_3} + \rho_2 \frac{\partial \mathrm{SW}^{t,\Theta,T_2,R_2}}{\partial \theta_3} + \rho_3 \frac{\partial \mathrm{SW}^{t,\Theta,T_3,R_3}}{\partial \theta_3} &= \frac{\partial \ell}{\partial \theta_3}(t,x,\Theta)
\end{cases} \tag{6.2.21}$$

One key quantity to compute for us in this setting is thus the vector of sensitivities $(\frac{\partial \ell}{\partial \theta_i}(t,x,\Theta))$, $i = 1,2,3$.

Remark 6.2.8. *(i) We choose to always use the same swap contracts issued at $t = 0$ as hedging instruments. We could have decided to enter for free in swaps (at the swap rate) at each rebalancing time. However, this strategy requires to keep the memory of the swap rate in order to compute the swap price at the next rebalancing date.*
(ii) The T_i, R_i should be chosen so that they represent some liquid contracts.

The global PnL function

From the previous two sections, we conclude that the PnL of the balance sheet at time 1 can be expressed as,

$$P_1 = p_1\left((t, \bar{S}_t, \bar{\Theta}_t)_{t\in\Gamma}\right)$$

where $(\bar{S}, \bar{\Theta})$ are the market parameters (risk factors) and the PnL function $p_1 : \mathbb{R}^\gamma \to \mathbb{R}$, with $\gamma = 4 \times (n+1)$, is given by

$$\ell(t_n, x_n, \Theta_n) - \sum_{i=0}^{n-1} \Delta(t_i, x_i, \Theta_i)(x_{i+1} - x_i) - \sum_{i=0}^{n-1}\sum_{j=1}^{3} \rho^j(t_i, x_i, \Theta_i)\left(\mathrm{SW}^{t_{i+1},\Theta_{i+1},T_j,R_j} - \mathrm{SW}^{t_i,\Theta_i,T_j,R_j}\right) . \tag{6.2.22}$$

In the next section, we describe what model we will consider for the market parameter $(\bar{S}, \bar{\Theta})$.

6.2.4 Market parameters under the *real-world* measure

We describe here the model that will be used for the simulation of the market parameters in the numerical part. Let us insist that this *real-world* measure $\mathbb{P}$ might represent the view of the management (possibly risk management) on the evolution of the market parameter on the period $[0, 1]$. As already mentioned, it can be completely different from the model used for the *risk-neutral* pricing.

We assume that we know, or at least are able to estimate, the first two moments of the distribution of $(X_1 := \log(S_1), \Theta_1) = (X_1, (\theta_1)_1, (\theta_2)_1, (\theta_3)_1)$ under $\mathbb{P}$. More precisely, we assume that under $\mathbb{P}$, this random vector has mean and covariance matrix given by

$$\mu = (\mu_X, \mu_1, \mu_2, \mu_3)\,, \tag{6.2.23}$$
$$V = (V_{ij})_{i,j=0,1,2,3}. \tag{6.2.24}$$

To model the random process $(X_t, (\theta_1)_t, (\theta_2)_t, (\theta_3)_t)_{t\in[0,1]}$ under $\mathbb{P}$, we assume that its dynamics are given by

$$X_t = X_0 + b_0 t + c_{00}W_t^0 + c_{01}W_t^1 + c_{02}W_t^2 + c_{03}W_t^3, \tag{6.2.25}$$
$$(\theta_1)_t = (\theta_1)_0 + b_1 t + c_{11}W_t^1 + c_{12}W_t^2 + c_{13}W_t^3 \tag{6.2.26}$$
$$(\theta_2)_t = (\theta_2)_0 + b_2 t + c_{22}W_t^2 + c_{23}W_t^3 \tag{6.2.27}$$
$$(\theta_3)_t = (\theta_3)_0 + b_3 t + c_{33}W_t^3, \tag{6.2.28}$$

where $W^i, i = 0, 1, 2, 3$ are independent $\mathbb{P}$-Brownian motions.

Proposition 6.2.9. *There exists at most one set of coefficients $b_i, c_{ij}, i, j = 0, 1, 2, 3$, such that the random vector $(X_1, (\theta_1)_1, (\theta_2)_2, (\theta_3)_2)$ has mean μ and covariance matrix V.*

Proof. We refer to the appendix for a proof, cf. Proposition 6.5.1. □

Remark 6.2.10. *It is well-known that it is difficult to estimate accurately the drift parameter in a Black-Scholes model. This makes our computation subject to model risk. We leave it to a future research work to find a robust way to approximate the law of P_1 under $\mathbb{P}$. Nevertheless, let us point out that the grid approach allows us to compute with minimal re-computation risk measures for different approximations of the law of P_1. This is one of the advantages of this method with respect to using "nested simulations", as illustrated in Section 6.4.*

6.3 Numerical methods

In this section, we describe the two numerical methods that we use to compute the risk indicator on the balance sheet PnL. The first one is known as *nested simulation* approach and is already used in the industry, see the seminal paper [119]. The second one is a *sparse grid* approach and is designed to be more efficient that the *nested simulation* approach in the case of moderate dimensions. In the next section, we present numerical simulations that confirms this fact for the model with moderate dimension that we consider here.

6.3.1 Estimating the risk measure

Given a risk measure ϱ and the loss distribution η of the balance sheet at one year, we estimate the quantity of interest $\varrho(\eta)$ by simulating a sample of N i.i.d random variables $(\Psi_j)_{1\leq j\leq N}$ with law η and then computing simply $\varrho(\eta^N)$ using the formulae (6.1.1), (6.1.2) and (6.1.3) with η^N instead of η. Here, η^N stands for the empirical measure associated to the Ψ_j i.e.

$$\eta^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\Psi_j} \ ,$$

where δ_x is the Dirac mass at the point x.

Recall, that in our work, the loss distribution η is obtained through the following expression:

$$\eta = p_1 \sharp \nu$$

p_1 being described in (6.2.22) and ν stands for the distribution of the market parameters. Namely, ν is the law of the random variable

$$\bar{\mathcal{X}} := (\bar{S}_t, \bar{\Theta}_t)_{t\in\Gamma} \tag{6.3.29}$$

under the *real world* probability measure $\mathbb{P}$.

In order to estimate $\varrho(\eta)$ for a chosen risk measure, we need to be able to sample from η which implies two steps in our setting. First, we need to be able to sample $\bar{\mathcal{X}}$ and then we use an approximation p_1^υ of p_1:

- $p_1^{\mathcal{N}}$ if one chooses the *nested simulation* approach;
- $p_1^{\mathcal{S}}$ if one chooses the *sparse grid* approach.

Eventually, the estimator of $\varrho(\mu)$ is given by

$$R^\upsilon := \varrho(p_1^\upsilon \sharp \nu^N) \ , \quad \text{for } \upsilon \in \{\mathcal{N}, \mathcal{S}\} \, . \tag{6.3.30}$$

To summarise, the two numerical methods have the following steps.

Nested simulation approach

1. Outer step: Simulate the model parameters $(\mathcal{X}_j)_{j=1,\dots,N}$.
2. Inner step: Simulate $\Psi_j = p^{\mathcal{N}}(\mathcal{X}_j)$ using MC simulations. This requires to compute the option prices with Monte Carlo estimates, the interest rate derivative prices, and the various sensitivities of the price, see subsection 6.3.2. All these computations are done using closed-form formulae that are derived below.

3. Estimate the risk measure.

All computations are made "online".

Sparse grid approach

1. Fix a (sparse) grid $\mathcal{V}$ and compute the approximation $p^{\mathcal{S}}$ at each required value on the grid by an MC simulation. This involves exactly the same computations as 2. above.

2. Simulate the N model parameter samples $(\mathcal{X}_j)$ and evaluate $\Psi_j = p^{\mathcal{S}}(\mathcal{X}_j)$.

3. Estimate the risk measure.

Computations at step 1. are done "offline". The next two steps are done "online".

We now describe precisely how to compute $p^{\mathcal{N}}$ and $p^{\mathcal{S}}$.

6.3.2 Nested Simulation approach

In the *nested simulation* approach, once the market parameters $\bar{\mathcal{X}}$ have been simulated, the function $p_1^{\mathcal{N}}$ has itself to be computed. Recalling (6.2.22), this requires to evaluate the function $\ell^{\mathcal{N}}, \Delta^{\mathcal{N}}, \rho^{\mathcal{N}}$ (approximations of ℓ, Δ, ρ) at the value of the market parameters. This computation is made using again a Monte Carlo approach.

In order to compute the risk-neutral expectations in the above formulae, one has to sample the short rate process and compute its integral over $[t, T]$, and also to simulate the stock price S at least at the times $\tau_\ell \in \Gamma_G \cap [t, T]$. Under the model described in section 6.2.2, the simulation is exact and is described in the two following statements whose proofs are postponed to the appendix.

Let $(t, x, \Theta) \in [0, T] \times (0, \infty) \times \mathbb{R}^3$ be a market observation, and consider the processes $\left(r_s^{t,\Theta}\right)_{s\in[t,T]}$ and $\left(S_s^{t,x,\Theta}\right)_{s\in[t,T]}$ defined by (6.2.5), (6.2.13). We first start with an easy and well known observation.

Lemma 6.3.1. *We have the following decomposition for the short rate process:*

$$r^{t,\Theta} = \xi^t + \alpha^{t,\Theta}, \tag{6.3.31}$$

where $(\alpha_s^{t,\Theta})_{s\in[t,T]}$ *is the deterministic function of time*

$$\alpha_s^{t,\Theta} = \frac{1}{a} f^{\Theta}(t, s) + \frac{b^2}{2a^3}\left[1 - e^{-a(s-t)}\right]^2, s \in [t, T], \tag{6.3.32}$$

and $(\xi_s^t)_{s\in[t,T]}$ *is the solution of the SDE*

$$d\xi_s^t = -a\xi_s^t \mathrm{d}s + b\mathrm{d}B_s, s \in [t, T] \tag{6.3.33}$$

$$\xi_t^t = 0. \tag{6.3.34}$$

The following proposition provides a recursive way to produce sample paths of the triplet $(\xi^t, A^t, X^{t,x,\Theta})$ on the discrete grid $\{t\} \cup (\Gamma_G \cap [t, 1])$. We are thus in a position to simulate the evolution of $(r^{t,\Theta}, S^{t,x,\Theta})$ and the discount factor $\beta_T^{t,\theta} := e^{-\int_t^T r_s^{t,\Theta}\mathrm{d}s}$ under the risk-neutral measure $\mathbb{Q}$.

Proposition 6.3.2. *Fix $t \leq s \leq u \leq T$. Conditionally upon $\mathcal{F}_s$, the triplet*

$$\left(\xi_u^t, A_u^{t,s} := \int_s^u \xi_r^t \mathrm{d}r, X_u^{t,x,\Theta} := \log\left(S_u^{t,x,\Theta}\right)\right)$$

is a Gaussian vector of dimension 3, with mean vector and covariance matrix respectively given by

$$\begin{aligned}
\mathbb{E}_s \xi_u^t &= \xi_s^t e^{-a(u-s)}, \\
\mathbb{E}_s A_u^{t,s} &= \frac{\xi_s^t}{a}(1 - e^{-a(u-s)}), \\
\mathbb{E}_s X_u^{t,x,\Theta} &= X_s^{t,x,\Theta} + \int_s^u \alpha_r \mathrm{d}r - \frac{\sigma^2}{2}(u-s) + \frac{\xi_s^t}{a}(1 - e^{-a(u-s)}),
\end{aligned} \tag{6.3.35}$$

and

$$\begin{aligned}
\mathrm{Var}_s(\xi_u^t) &= \frac{b^2}{2a}(1 - e^{-2a(u-s)}), \\
\mathrm{Cov}_s(\xi_u^t, A_u^{t,s}) &= \frac{b^2}{a^2}(1 - e^{-a(u-s)}) - \frac{b^2}{2a^2}(1 - e^{-2a(u-s)}), \\
\mathrm{Var}_s(A_u^{t,s}) &= \frac{b^2}{a^2}(u-s) - \frac{2b^2}{a^3}(1 - e^{-a(u-s)}) + \frac{b^2}{2a^3}(1 - e^{-2a(u-s)}), \\
\mathrm{Cov}_s(\xi_u^t, X_u^{t,x,\Theta}) &= \frac{b}{a}(\frac{b}{a} + \rho\sigma)(1 - e^{-a(u-s)}) - \frac{b^2}{2a^2}(1 - e^{-2a(u-s)}), \\
\mathrm{Cov}_s(A_u^{t,s}, X_u^{t,x,\Theta}) &= -\frac{1}{a}\mathrm{Cov}(\xi_u^t, X_u^{t,x,\Theta}) + \frac{b}{a}(\frac{b}{a} + \rho\sigma)(u-s) - \frac{b^2}{a^3}(1 - e^{-a(u-s)}), \\
\mathrm{Var}_s(X^{t,x,\Theta}) &= (\rho\sigma + \frac{b}{a})^2(u-s) + (1-\rho^2)\sigma^2(u-s) - 2\frac{b}{a^2}(\rho\sigma + \frac{b}{a})(1 - e^{-a(u-s)}) + \frac{b^2}{2a^3}(1 - e^{-2a(u-s)}).
\end{aligned} \tag{6.3.36}$$

Approximation of the Liability side

With the above results, the approximation at any time t of the liability function ℓ, denoted $\ell^{\mathcal{N}}$, is straightforward. It is given by

$$\ell^{\mathcal{N}}(t,x,\Theta) = \frac{1}{M}\sum_{k=1}^{M} \beta_T^{t,\Theta,k} G(S^{t,x,\Theta,k}), \tag{6.3.37}$$

with $((\beta_\tau^{t,\Theta,k}, S_\tau^{t,x,\Theta,k})_{\tau\in\Gamma_G})_{1\leq k\leq M}$ are i.i.d realisations of $(\beta_\tau^{t,\Theta}, S_\tau^{t,x,\Theta})_{\tau\in\Gamma_G}$, recall (6.2.4).

Approximation of the Asset side

The approximation of the asset side is slightly more involveld as it requires the computation of sensitivities with respect to the model parameters: $\frac{\partial \ell}{\partial x}$ and $\frac{\partial \ell}{\partial \theta_i}$, $i = 1, 2, 3$, see (6.2.19), (6.2.20) and (6.2.21). We choose to compute the sensitivities using a "weight" approach namely we express them as expectation of the discounted payoff times a well chosen random weight. Note that other techniques are available to compute these sensitivities e.g. *Automatic differentiation*. In our context, we have compared the two methods and observed that the weight approach is more efficient, see Section 6.5.4 in the Appendix.

We now describe how to obtain the sensitivities in a convenient form for Monte Carlo simulation. Recall that we consider a liability function ℓ of the form:

$$\ell(t,x,\Theta) = \mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta}), \tag{6.3.38}$$

where G depends upon $S^{t,x,\Theta}$ through its values on a finite set $\Gamma_G = \{\tau_0 = 0, \dots, \tau_\kappa = T\} \subset [0,T]$. In the following, we assume for simplicity that $\tau_1 > t$. Otherwise, there are deterministic terms that are to be added, but the method remains the same.

The Delta: We want to compute:

$$\frac{\partial \ell}{\partial x}(t,x,\Theta) = \frac{\partial}{\partial x} \mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta}) \tag{6.3.39}$$

and we have the following result.

Proposition 6.3.3. *For all $(t,x,\Theta) \in [0,1] \times \mathbb{R} \times \mathbb{R}^3$, the following holds:*

$$\frac{\partial \ell}{\partial x}(t,x,\Theta) = \mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta}) H^{x,\Theta}\left(e^{-\int_t^{\tau_1} \xi_s^t \mathrm{d}s}, S_{\tau_1}^{t,x,\Theta}\right), \tag{6.3.40}$$

with

$$H^{x,\Theta}(a,y) = \frac{\Sigma_{1,2}^{-1} a + \Sigma_{2,2}^{-1} \times \left(\log(y/x) - \int_t^{\tau_1} \alpha_r^{t,\Theta} \mathrm{d}r + \frac{\sigma^2}{2}(\tau_1 - t)\right)}{x}, \tag{6.3.41}$$

where Σ is the covariance matrix of $(A_{\tau_1}^t, X_{\tau_1}^{t,x,\Theta})$, see (6.3.46).

Proof. We write the expectation as an integral, remembering that we know the law of the couple $(\xi_u^t, A_u^t, X_u^{t,x,\Theta})$ conditionally upon $\mathcal{F}_s$:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta} \mathrm{d}s} G(S^{t,x,\Theta}) \qquad (6.3.42)\\
&= e^{-\int_t^T \alpha_s^{t,\Theta} \mathrm{d}s} \mathbb{E}^{\mathbb{Q}} e^{-\int_t^{\tau_1} \xi_s^t \mathrm{d}s} \prod_{\ell=1}^{\kappa-1} e^{-\int_{\tau_\ell}^{\tau_{\ell+1}} \xi_s^t \mathrm{d}s} G(S^{t,x,\Theta})\\
&= e^{-\int_t^T \alpha_s^{t,\Theta} \mathrm{d}s} \int_{\mathbb{R}^{2\kappa}} e^{-a_1} \prod_{\ell=1}^{\kappa-1} e^{-a_{\ell+1}} G(e^{x_1}, \dots, e^{x_\kappa}) \mathrm{d}\mathbb{Q}_{(A_u^t, X_u^{t,x,\Theta})_{u\in\Gamma_G}}(a_1, \cdots, a_\kappa, x_1, \cdots, x_\kappa)\\
&= e^{-\int_t^T \alpha_s^{t,\Theta} \mathrm{d}s} \int_{\mathbb{R}^{2\kappa}} e^{-a_1} \prod_{\ell=1}^{\kappa-1} e^{-a_{\ell+1}} G(e^{x_1}, \dots, e^{x_\kappa}) p^\Theta(t,0,\log(x),1,a_1,x_1) \times \dots\\
&\qquad\qquad \times p^\Theta(t_{\kappa-1},0,x_{\kappa-1},t_\kappa,a_\kappa,x_\kappa) \mathrm{d}a_1 \cdots \mathrm{d}a_\kappa \mathrm{d}x_1 \cdots \mathrm{d}x_\kappa,
\end{aligned}$$

where $p^\Theta(s,a,x,u,.,.)$ is the density of the couple $(A_u^{t,a}, X_u^{t,x,\Theta})$, conditionally upon $\mathcal{F}_s$. We have previously seen that it is a Gaussian vector with explicit mean vector and covariance matrix. Thus, using Fubini's theorem, we get, since there is no dependence on x except in the first density:

$$\frac{\partial \ell(t,x,\Theta)}{\partial x} = e^{-\int_t^T \alpha_s^{t,\Theta} \mathrm{d}s} \int_{\mathbb{R}^{2\kappa}} e^{-a_1} \prod_{\ell=1}^{\kappa-1} e^{-a_{\ell+1}} G(e^{x_1},\dots,e^{x_\kappa}) \frac{\partial p^\Theta(t,0,\log(x),1,a_1,x_1)}{\partial x} \times \dots \\ \times p^\Theta(t_{\kappa-1},0,x_{\kappa-1},t_\kappa,a_\kappa,x_\kappa)\mathrm{d}a_1\cdots\mathrm{d}a_\kappa \mathrm{d}x_1\cdots\mathrm{d}x_\kappa. \tag{6.3.43}$$

Consequently, the sensibility of the discounted price with respect to the initial stock price is computed only by calculating the derivative of the density with respect to x.

We have

$$p^\Theta(t,0,\log(x),s,a,y) = \frac{1}{\det(2\pi\Sigma)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\left(((a,y)-\mu)\,\Sigma^{-1}\,((a,y)-\mu)^\top\right)\right), \tag{6.3.44}$$

where, thanks to (6.3.35)-(6.3.36),

$$\mu = \left(\mathbb{E}_t A^t_{\tau_1}, \mathbb{E}_t X^{t,x,\Theta}_{\tau_1}\right) \tag{6.3.45}$$

$$\Sigma = \begin{pmatrix} \mathrm{Var}(A^t_{\tau_1})_t & \mathrm{Cov}(A^t_{\tau_1}, X^{t,x,\Theta}_{\tau_1})_t \\ \mathrm{Cov}(A^t_{\tau_1}, X^{t,x,\Theta}_{\tau_1})_t & \mathrm{Var}(X^{t,x,\Theta}_{\tau_1})_t \end{pmatrix}. \tag{6.3.46}$$

Still by (6.3.35)-(6.3.36), we see that only $\mathbb{E}_t X^{t,x,\Theta}_{\tau_1}$ depends upon x. Thus we get:

$$\frac{\partial f(t,0,\log(x),s,a,y)}{\partial x} = \frac{\Sigma^{-1}_{1,2}a + \Sigma^{-1}_{2,2} \times \left(\log(y/x) - \int_t^{\tau_1} \alpha_r^{t,\Theta}\mathrm{d}r + \frac{\sigma^2}{2}(\tau_1 - t)\right)}{x} f(t,0,\log(x),s,a,y). \tag{6.3.47}$$

Reinjecting this equality into (6.3.43) and rewriting the result in term of expectations, we finally get the result. □

The function $\Delta(t_i,\cdot)$ is computed using the Monte Carlo estimator given in (6.3.40) as in (6.3.37) where we simulate in addition the weight H.

Sensitivities with respect to the interest rates curve. We consider now the derivatives with respect to the interest rates curve. For $i = 1,2,3$, we want to compute:

$$\frac{\partial \ell}{\partial \theta_i}(t,x,\Theta) = \frac{\partial}{\partial \theta_i} \mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta}\mathrm{d}s} G(S^{t,x,\Theta}),\ i = 1,2,3.$$

Proposition 6.3.4. *For all $(t,x,\Theta) \in [0,1]\times(0,\infty)\times\mathbb{R}^3$ and all $i = 1,2,3$, we have the following identity (where we have set $\tau_0 = t$):*

$$\frac{\partial \ell}{\partial \theta_i}(t,x,\Theta) = \mathbb{E}^{\mathbb{Q}} e^{-\int_t^T r_s^{t,\Theta}\mathrm{d}s} G(S^{t,x,\Theta}) H_i^{x,\Theta}\left((\xi^t_{\tau_l}, e^{-\int_{\tau_{l-1}}^{\tau_l} \xi^t_u \mathrm{d}u}, S^{t,x,\Theta}_{\tau_l})_{l=1,\dots,\kappa}\right), \tag{6.3.48}$$

with

$$H_i^{t,x,\Theta}\left((r_\ell,a_\ell,s_\ell)_{\ell=1,\dots,\kappa}\right) = -\int_t^{\tau_\kappa} h_s^{t,i}\mathrm{d}s + \sum_{\ell=1}^{\kappa}\left(\int_{\tau_{\ell-1}}^{\tau_\ell} h_s^{t,i}\mathrm{d}s\right)\Big((\Sigma^{\tau_{\ell-1},\tau_\ell})^{-1}_{1,3}(r_\ell-\mu_1^{\tau_{\ell-1},\tau_\ell}) + (\Sigma^{\tau_{\ell-1},\tau_\ell})^{-1}_{2,3}(a_\ell-\mu_2^{\tau_{\ell-1},\tau_\ell}) + (\Sigma^{\tau_{\ell-1},\tau_\ell})^{-1}_{3,3}(\log(s_\ell)-\mu_3^{\tau_{\ell-1},\tau_\ell})\Big), \tag{6.3.49}$$

where $\mu^{s,u}$ and $\Sigma^{s,u}$ are the mean and the covariance matrix of the Gaussian vector $(\xi_u^t, A_u^t, X^{t,x,\Theta})$ conditionally upon $\mathcal{F}_s$.

Proof. Performing a similar analysis as the one above,

$$\frac{\partial \ell}{\partial\theta_i}(t,x,\Theta) = \frac{\partial e^{-\int_t^T \alpha_s^{t,\Theta}\mathrm{d}s}}{\partial\theta_i}\mathbb{E}^{\mathbb{Q}}e^{-\int_t^T \xi_s^t\mathrm{d}s}G(S^{t,x,\Theta}) + e^{-\int_t^T \alpha_s^{t,\Theta}\mathrm{d}s}\int_{\mathbb{R}^{2\kappa}} e^{-a_1}\prod_{\ell=1}^{\kappa-1} e^{-a_{\ell+1}}G(e^{x_1},\dots,e^{x_\kappa}) \frac{\partial\left(p^\Theta(t,0,\log(x),1,a_1,x_1)\dots p^\Theta(t_{\kappa-1},a_{\kappa-1},x_{\kappa-1},t_\kappa,a_\kappa,x_\kappa)\right)}{\partial\theta_i}\mathrm{d}a_1\cdots\mathrm{d}a_\kappa\mathrm{d}x_1\cdots\mathrm{d}x_\kappa. \tag{6.3.50}$$

Here, the computations are more involved since every density depends upon the $a_i, i=1,2,3$, but the idea is the same as before.

The only difference is that, when we use (6.3.35)-(6.3.36) to differentiate, we see that the short term itself appears in the formulae. This is not a problem as we can rewrite the previous integral as an integral over $\mathbb{R}^{3\kappa}$, with the short rate process taken at times $\tau_l, l=1,\dots,\kappa$, as new variables to integrate against. □

The quantity $\frac{\partial\ell}{\partial\theta_i}$ $(i=1,2,3)$ is computed using the Monte Carlo estimator of the formula (6.3.48). Then solving the system (6.2.21) allows to obtain the coefficients ρ_1,ρ_2,ρ_3 .

This method to calculate derivatives allows us to compute the function $\ell(t,x,\Theta)$ and its four derivatives with only one Monte Carlo simulation. Furthermore, given a risk-neutral scenario, each quantity involved in formulae (6.3.41) and (6.3.49) can be exactly computed by integrating the elementary functions $h^{t,i}$ and by inverting real symmetric matrices of size 3×3. Thus, the weight functions $H^{t,x,\Theta}, H_i^{t,x,\Theta}$ are easily and accurately computed.

6.3.3 Sparse Grid Approach

The *nested simulation* approach requires the approximation of the function ℓ and its derivative $\frac{\partial\ell}{\partial x}, \frac{\partial\ell}{\partial\theta_i}, i=1,2,3$ for each path $(\bar{S}_t^j,\bar{\Theta}_t^j)_{t\in\Gamma}$ of the market parameters. These values are computed on the fly which is quite time consuming.

We suggest here an alternative method which will pre-compute the quantities of interest (ℓ and its derivatives) and store them. The requested value for a given market parameter will then be obtained by an interpolation procedure.

A first simple approach is to consider an equidistant grid of the domain $A := \prod_{l=1}^d [m_p, M_p]$ which is a truncation of the support of $\mathcal{X}$ ($\mathbb{R}^d$, $d=4$ in our setting). Then one can use a multi-linear interpolation to reconstruct the function in the whole space. If one sets 2^p points in one dimension, the total number of points will be 2^{dp} for one function at one given time and overall $(4n+1)2^{dp}$ to store. This will become rapidly too big, especially if one allows the

number of market parameters to grow. This is a typical example of the "curse of dimensionality" encountered in numerical analysis when dealing with problem of high dimension.

Instead of considering a regular grid, we shall rely on the use of *sparse grid*, which allows to lower the number of points required to store the numerical approximation of the function. We now present rapidly the main concepts linked to sparse grids, see [120] for a comprehensive survey. In our numerical examples, see Section 6.4, the sparse grid will be computed using the StOpt C++ library [122].

For each multi-index $\mathbf{k} \leq \mathbf{l}$, we define a grid mesh $h_{\mathbf{k}} = 2^{-\mathbf{k}}$ and grid points

$$\check{y}_{\mathbf{k},\mathbf{i}} = (m_1 + i_1(M_1 - m_1)h_{k_1}, \ldots, m_d + i_d(M_d - m_d)h_{k_d}),\ \mathbf{0} \leqslant \mathbf{i} \leqslant 2^{\mathbf{k}}.$$

Using the *hat* function,

$$y \in \mathbb{R} \mapsto \phi(y) := \begin{cases} 1 - |y| & \text{if } y \in [-1, 1] \\ 0 & \text{otherwise} \end{cases} \tag{6.3.51}$$

and we can associate to the previous grid a set of nodal basis function:

$$y \in \mathbb{R}^d \mapsto \phi_{\mathbf{k},\mathbf{i}}(y; A) = \prod_{l=1}^{d} \phi(\frac{y_l - \check{y}^l_{\mathbf{k},\mathbf{i}}}{2^{-i_l}})\ .$$

We consider the sparse grid nodal space of order p defined by

$$\mathcal{V}_\kappa := \text{span}\{\phi_{\mathbf{l},\mathbf{j}}; (\mathbf{l},\mathbf{j}) \in \mathcal{I}_p(A)\},$$

where

$$\begin{aligned} \mathcal{I}_\kappa := \big\{(\mathbf{l},\mathbf{j}) :\quad & 0 \leqslant \sum_{i=1}^{d} l_i \leqslant p; \quad \mathbf{0} \leqslant \mathbf{j} \leqslant \mathbf{2^l}; \\ & (l_i > 0 \text{ and } j_i \text{ is odd}) \text{ or } (l_i = 0),\ \text{for } i = 1, \ldots, d\big\}. \end{aligned} \tag{6.3.52}$$

For a function $\psi : A \to \mathbb{R}$ with support in A, we define its associated $\mathcal{V}_\kappa$-interpolator by

$$\pi^A_{\mathcal{V}_\kappa}[\psi](y) := \sum_{(\mathbf{l},\mathbf{j}) \in \mathcal{I}_\kappa(A)} \theta_{\mathbf{l},\mathbf{j}}(\psi; A)\phi_{\mathbf{l},\mathbf{j}}(y; A) \tag{6.3.53}$$

where the operator $\theta_{\mathbf{l},\mathbf{j}}$ can be defined recursively in terms of r, the dimension of $\mathbf{l}$, by:

$$\theta_{\mathbf{l},\mathbf{j}}(\psi; A) = \begin{cases} \psi(\check{y}_{\mathbf{l},\mathbf{j}}); & r = 0 \\ \theta_{\mathbf{l}-,\mathbf{j}-}(\psi(\cdot, \check{y}^r_{l_r,j_r}); A-); & l_r = 0 \\ \theta_{\mathbf{l}-,\mathbf{j}-}(\psi(\cdot, \check{y}^r_{l_r,j_r}); A-) - \frac{1}{2}\theta_{\mathbf{l}-,\mathbf{j}-}(\psi(\cdot, \check{y}^r_{l_r,j_r-1}); A-) & \\ \qquad -\frac{1}{2}\theta_{\mathbf{l}-,\mathbf{j}-}(\psi(\cdot, \check{y}^r_{l_r,j_r+1}); A-); & l_r > 0 \end{cases} \tag{6.3.54}$$

where, for a hypercube $A = \prod_{l=1}^{d}[m_l, M_l]$, $A- := \prod_{l=1}^{d-1}[m_l, M_l]$ and for a multi-index $\mathbf{k}$ with dimension $r \geq 1$, $\mathbf{k}- = (k_1, \ldots, k_{r-1})$.

Let us now introduce the approximation that we use to compute the loss distribution, namely

$$\ell^{\mathcal{S}}(\cdot) := \pi_{\mathcal{V}_\kappa}[\ell^{\mathcal{N}}](\cdot), \left(\frac{\partial \ell}{\partial x}(t,\cdot)\right)^{\mathcal{S}} := \pi_{\mathcal{V}_\kappa}[\left(\frac{\partial \ell}{\partial x}(t,\cdot)\right)^{\mathcal{N}}](\cdot) \tag{6.3.55}$$
$$\text{and } \left(\frac{\partial \ell}{\partial \theta_i}(t,\cdot)\right)^{\mathcal{S}} := \pi_{\mathcal{V}_\kappa}[\left(\frac{\partial \ell}{\partial \theta_i}(t,\cdot)\right)^{\mathcal{N}}](\cdot)\ i \in \{1,2,3\}\ ,\ t \in \Gamma\ .$$

These functions are built by computing the coefficients appearing in (6.3.53), which are then stored in memory. For $\ell^{\mathcal{S}}$ say, this amounts to compute the function $\ell^{\mathcal{N}}$ on the sparse grid $\mathcal{V}_p$ and this is done by Monte Carlo simulation as in the previous approach, recall the definition of $\ell^{\mathcal{N}}$ in (6.3.37).

Complexity The main limitation of this method is the memory usage and the time to pre-compute the functions. This is proportional to the number of point in the grid. This number can be estimated to be of $O(2^{\kappa-d+1}\frac{(\kappa-d+1)^{d-1}}{(d-1)!})$, see Proposition 4.1 in [120]. Let us insist on the fact that this is done "offline" compared to the *nested simulation* approach. For the "online" computations, the main effort is put in the evaluation of the function which is slightly more evolved than a linear interpolation and is of $O(\kappa)$ where κ is the maximum level that is chosen.

Remark 6.3.5. *The computations at each point being independant, this Sparse Grid approach can be easily parallelised, hence improving further the gain of time observed in Subsection 6.4.1.*

6.3.4 Convergence study

The goal is to obtain a reasonable approximation of the risk associated to the loss distribution of the balance sheet in an efficient way. In this section, we explain why the methods introduced above are indeed good approximations of the risk indicators. We also study theoretically the numerical complexity of both methods in terms of memory and time consumption.

Error analysis

For the risk estimation, we will investigate a root Mean Square Error (rMSE) of the following form

$$\epsilon^{\upsilon} := \mathbb{E}|\varrho(p_1\sharp\eta) - \varrho(p_1^{\upsilon}\sharp\eta^N)|^{2\,\frac{1}{2}}\ ,\ \text{for } \upsilon \in \{\mathcal{N},\mathcal{S}\}.$$

The expectation operator $\mathbb{E}\cdot$ acts under $\mathbb{P}^{\otimes N}\otimes\mathbb{Q}^{\otimes M}$, namely it averages both on the simulation of the market parameters under the *real-world* measure used for calibration and the risk neutral evolution of the market model under the pricing measure.

The first observation is that under reasonable assumptions on the risk measure used in the risk indicator, the error performed in the numerical simulation can be separated in two main contribution: the error due to the sampling of the loss distribution coming from the sampling of the market parameters and the error made when approximating the different pricing and hedging functions.

Lemma 6.3.6. *Assume that ϱ has a* Monotonic *and* Cash Invariant *lift $\Re$, then*

$$\epsilon^{\upsilon} \le \mathbb{E}|\varrho(p_1\sharp\eta) - \varrho(p_1\sharp\eta^N)|^{2\,\frac{1}{2}} + \mathbb{E}\sup_{1\le j\le N}|p_1(\mathcal{X}^j) - p_1^{\upsilon}(\mathcal{X}^j)|^{2\,\frac{1}{2}}\ .$$

Proof. We denote by $\widehat{\mathcal{X}}^N(\omega)$ the random variable with distribution $\eta^N(\omega)$. Note that

$$p_1(\widehat{\mathcal{X}}^N(\omega)) \leq p_1^{\upsilon}(\widehat{\mathcal{X}}^N(\omega)) + \sup_{1\leq j\leq N} |p_1(\mathcal{X}^j(\omega)) - p_1^{\upsilon}(\mathcal{X}^j(\omega))| \tag{6.3.56}$$

which leads to

$$\Re(p_1(\widehat{\mathcal{X}}^N(\omega))) \leq \Re(p_1^{\upsilon}(\widehat{\mathcal{X}}^N(\omega))) + \sup_{1\leq j\leq N} |p_1(\mathcal{X}^j(\omega)) - p_1^{\upsilon}(\mathcal{X}^j(\omega))| \,.$$

By symmetry, we easily get

$$|\varrho(p_1\sharp\eta) - \varrho(p_1^{\upsilon}\sharp\eta^N(\omega))| \leq |\varrho(p_1\sharp\eta) - \varrho(p_1\sharp\eta^N(\omega))| + \sup_{1\leq j\leq N} |p_1(\mathcal{X}^j(\omega)) - p_1^{\upsilon}(\mathcal{X}^j(\omega))| \tag{6.3.57}$$

and then the proof is concluded using Minkowski inequality. □

The error due to the approximation of the function p_1 is well understood when the function is smooth enough. Note that the asset side of the function is quite involved and we will not attempt to obtain the condition for smoothness of the overall function p_1. We will now simply review the error done on the liability part $\ell(1,\cdot)$ assuming that the mapping G is bounded and

$$(x,\Theta) \to \beta_T^{1,\Theta} G(S^{1,x,\Theta}) \in \mathcal{C}_b^2. \tag{6.3.58}$$

Even though, this cannot be almost surely true in the model presented above, we will assume that $\beta_T^{1,\Theta}$ is bounded in the discussion below. A more precise analysis should take care of these extreme events arising with small probability. Another possibility would be to force the interest rate to be non-negative, by truncation or by considering a CIR type of model for (6.2.5).

Lemma 6.3.7. *Assume that* (6.3.58) *holds true. Recall the definition of* $\ell^{\mathcal{N}}$ *in* (6.3.37)*, then*

$$\mathbb{E} \max_{1\leq j\leq N} |\ell_1(\mathcal{X}^j) - \ell_1^{\mathcal{N}}(\mathcal{X}^j)|^{2\,\frac{1}{2}} \leq C\sqrt{\frac{\log(N)}{M}} \,. \tag{6.3.59}$$

Proof. We denote by c the bound on the mapping $(x,\Theta) \to \beta_T^{1,\Theta} G(S^{1,x,\Theta})$ (recall the discussion after equation (6.3.58)) and thus the bound on ℓ_1. For the reader's convenience, we introduce

$$\Sigma_M^j := \sum_{k=1}^{M} \ell_1(\mathcal{X}^j) - \beta_T^{t,\mathcal{X}^j,k} G(S_T^{t,\mathcal{X}^j,k}),$$

and observe that $\mathbb{E}\Sigma_M^j = 0$ and recall that the (Σ_M^j) are i.i.d. We have, using Hoeffding Inequality,

$$\mathbb{E}\mathbf{1}_{\{|\Sigma_M^j|^2 > z\}} \leq 2\exp\left(-\frac{z}{cM}\right) \,.$$

Using the independence property, we obtain

$$\mathbb{E}\mathbf{1}_{\{\max_j |\Sigma_M^j|^2 \leq z\}} \geq \left(1 - 2\exp\left(-\frac{z}{cM}\right)\right)^N \,. \tag{6.3.60}$$

Now set $\xi := cM\log(N)$ and compute

$$\begin{aligned}\mathbb{E}\max_{1\le j\le N}|\Sigma_M^j|^2 &= \int_0^\infty \mathbb{E}\mathbf{1}_{\{\max_j|\Sigma_M^j|^2>z\}}\mathrm{d}z\\ &\le \xi + \int_\xi^\infty \mathbb{E}\mathbf{1}_{\{\max_j|\Sigma_M^j|^2>z\}}\mathrm{d}z\\ &\le \xi + \int_\xi^\infty \left\{1-\left(1-2\exp\left(-\frac{z}{cM}\right)\right)^N\right\}\mathrm{d}z.\end{aligned}$$

Now, we observe that for $N\ge 2$, $2\exp\left(-\frac{z}{cM}\right)\le 1$ for $z\ge\xi$. Using the fact that $1-(1-u)^N\le Nu$, for $u\in[0,1]$, we get

$$\mathbb{E}\max_{1\le j\le N}|\Sigma_M^j|^2 \le \xi + 2N\int_\xi^\infty \exp\left(-\frac{z}{cM}\right)\mathrm{d}z$$

which leads to

$$\mathbb{E}\max_{1\le j\le N}|\Sigma_M^j|^2 \le \xi + 2cM,$$

and concludes the proof.

□

We conclude this section by giving the overall estimation error induced by the numerical procedure above. We will admit that the upper bound for the error given for $\ell(1,\cdot)$ in Lemma 6.3.7 holds true for the PnL function $p(1,\cdot)$ with a scaling by n coming from the number of rebalancing date.

Theorem 6.3.8. *Assume that ϱ_h is a spectral risk measure. Then, the following holds, for some $\alpha>0$,*

1. *for the* Nested Simulation *approach*

$$\epsilon^{\mathcal{N}} \le C\left(\frac{1}{N^\alpha}+n\sqrt{\frac{\log(N)}{M}}\right)\ ; \tag{6.3.61}$$

2. *for the* Sparse Grid *approach with maximum level κ*

$$\epsilon^{\mathcal{S}} \le C\left(\frac{1}{N^\alpha}+n\left\{\sqrt{\frac{\log(N)}{M}}+2^{-2\kappa}(\kappa-d+1)^{(d-1)}\right\}\right)\ . \tag{6.3.62}$$

Proof. 1. We first show that

$$\mathbb{E}\max_{1\le j\le N}|\ell_1(\mathcal{X}^j)-\ell_1^{\mathcal{S}}(\mathcal{X}^j)|^{2\,\frac{1}{2}} \le C\left(\sqrt{\frac{\log(N)}{M}}+2^{-2\kappa}\kappa^{(d-1)}\right)\ . \tag{6.3.63}$$

Indeed, we have that

$$\begin{aligned}\ell_1^{\mathcal{S}} &= \pi_{\mathcal{V}_\kappa}[\ell_1^{\mathcal{N}}]\\ &= \ell_1^{\mathcal{N}} + \pi_{\mathcal{V}_\kappa}[\ell_1^{\mathcal{N}}] - \ell_1^{\mathcal{N}}\ .\end{aligned}$$

And we observe that

$$\pi_{\mathcal{V}_\kappa}[\ell_1^{\mathcal{N}}] - \ell_1^{\mathcal{N}} = \frac{1}{M}\sum_{j=1}^{M} \pi_{\mathcal{V}_\kappa}[e^{-\int_1^T r_s^{1,\cdot,k}\mathrm{d}s} G(S^{1,\cdot,k})] - e^{-\int_1^T r_s^{1,\cdot,k}\mathrm{d}s} G(S^{1,\cdot,k})$$

Let us denote by $(x,\Theta) \mapsto \phi^k(x,\Theta) = e^{-\int_1^T r_s^{1,\Theta,k}\mathrm{d}s} G(S^{1,x,\Theta,k})$ which is a random function as it depends on the random realisation of the (r,S) process. Under (6.3.58), ϕ^k is smooth enough to apply the results in Proposition 4.1 in [120] and we obtain

$$|\pi_{\mathcal{V}_\kappa}[\ell_1^{\mathcal{N}}] - \ell_1^{\mathcal{N}}|_\infty \leq \frac{1}{M}\sum_{j=1}^{M} |\pi_{\mathcal{V}_\kappa}[\phi^k] - \phi^k|_\infty \leq C 2^{-2\kappa}\kappa^{d-1} \ . \tag{6.3.64}$$

We then observe that

$$\mathbb{E}\max_{1\leq j\leq N} |\ell_1(\mathcal{X}^j) - \ell_1^{\mathcal{S}}(\mathcal{X}^j)|^{2\frac{1}{2}} \leq C\left(\mathbb{E}\max_{1\leq j\leq N} |\ell_1(\mathcal{X}^j) - \ell_1^{\mathcal{N}}(\mathcal{X}^j)|^{2\frac{1}{2}} + |\pi_{\mathcal{V}_\kappa}[\ell_1^{\mathcal{N}}] - \ell_1^{\mathcal{N}}|_\infty\right) \ .$$

The proof of (6.3.63) is concluded by combining the above inequality with (6.3.64) and Lemma 6.3.7.

2. We now prove (6.3.61). Applying Lemma 6.3.6, we obtain

$$\epsilon^{\mathcal{N}} \leq \mathbb{E}|\varrho(p_1\sharp\eta) - \varrho(p_1\sharp\eta^N)|^{2\frac{1}{2}} + \mathbb{E}\max_{1\leq j\leq N} |\ell_1(\mathcal{X}^j) - \ell_1^{\mathcal{N}}(\mathcal{X}^j)|^{2\frac{1}{2}} \ . \tag{6.3.65}$$

The second term in the right-hand side of the above inequality is controlled by using Lemma 6.3.7. We now study the first term in the right-hand side, which is the error introduced by the sampling of the loss distribution. Applying Corollary 11 in [118] to the spectral risk measure, we first get

$$\mathbb{E}|\varrho(p_1\sharp\eta) - \varrho(p_1\sharp\eta^N)|^{2\frac{1}{2}} \leq C\mathbb{E}\mathcal{W}_2(\eta,\eta^N)^{2\frac{1}{2}}.$$

We then use Theorem 1 in [123] to bound the Wasserstein distance, which concludes the proof for this step.

3. To prove (6.3.62), we follow similar arguments as in step 2. but using (6.3.63) instead of invoking Lemma 6.3.7. □

Remark 6.3.9. *We can compare the bound obtained for the nested simulation with the ones in [119]. Using a different approach, the authors prove a very nice bound on the overall error given by*

$$C\left(\frac{1}{\sqrt{N}} + \frac{1}{M}\right) \ ,$$

for the V@R (which is not a spectral risk measure) and AV@R. Note that the term $\frac{1}{M}$ is obtained by cancellation of the first order term through an error expansion. It would be interesting to understand under which assumptions their bound can be retrieved in our setting of general spectral risk measure. This topic is left for further research.

We conclude this Section by a short account on the numerical complexity of the two methods.

The *Nested Simulation* approach is a pure "online" method which is very simple to implement

but has a huge drawback in term of running time. Each time an estimation is requested the numerical complexity is overall of nNM, where recall n is the number of rebalancing date, M the number of sample for the risk neutral simulation and N the number of sample for the real-world simulation. The memory requirements comes only from the estimation of the loss distribution and are of order N.

As already mentioned, the *Sparse Grid* approach is both an "online" and "offline" method. In terms of memory requirement, it is thus greedier than the *nested simulation* approach. On top of the memory needed to store the sample distribution (of order N), memory is also needed to store the sparse grid approximation $p^{\mathcal{S}}$, the requirement are of order $O(n2^{\kappa-d+1}\frac{(\kappa-d+1)^{d-1}}{(d-1)!})$. In term of running time, the gain is important as the complexity of evaluating $p^{\mathcal{S}}$ is of $O(\kappa)$ only, where κ is the maximum level used.

6.4 Numerics

In the numerical applications below, we will compare the loss distribution obtained via our two numerical procedures by computing the Wasserstein distance between the two empirical distributions. Since the loss distribution is one-dimensional, we use the following formula [124]: for two probability distribution on $\mathbb{R}$, η and $\tilde{\eta}$,

$$W_2(\eta,\tilde{\eta}) = (\int_0^1 |F_\eta^{-1}(u) - F_{\tilde{\eta}}^{-1}(u)|^2 \mathrm{d}u)^{\frac{1}{2}} \ . \tag{6.4.66}$$

In the setting of empirical distributions, the above distance is easily computed. Suppose $\eta = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $\tilde{\eta} = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$.
We straightforwardly compute

$$W_2(\eta,\tilde{\eta})^2 = \sum_{i=1}^N \int_{\frac{i-1}{N}}^{\frac{i}{N}} |F_\eta^{-1}(u) - F_{\tilde{\eta}}^{-1}(u)|^2 \mathrm{d}u = \frac{1}{N}\sum_{i=1}^N |x_{(i)} - y_{(i)}|^2, \tag{6.4.67}$$

where the subscript (i) refers to the i-th order statistic of the distribution, since $x_{(i)}$ (resp. $y_{(i)}$) is simply the $\frac{i}{N}$-th quantile of η (resp. $\tilde{\eta}$).

Besides the Wasserstein distance between the two empirical distributions, we will also compare the estimated V@R and AV@R, which are computed in a similar way. Indeed, for $\alpha \in (0,1]$, we have:

$$V@R_\alpha(\eta) = F_\eta^{-1}(\alpha) = x_{(i_\alpha)}, \tag{6.4.68}$$

where $\frac{i_\alpha - 1}{N} < \alpha \leq \frac{i_\alpha}{N}$, $i_\alpha \in \{1,\dots,N\}$.
For a given $\alpha \in (0,1]$, we observe that

$$AV@R_\alpha(\eta) = \frac{1}{1-\alpha}\int_\alpha^1 V@R_p(\eta)\mathrm{d}p = \frac{1}{1-\alpha}\left(\int_\alpha^{\frac{i_\alpha}{N}} V@R_p(\eta)\mathrm{d}p + \sum_{i=i_\alpha}^{N-1}\int_{\frac{i}{N}}^{\frac{i+1}{N}} V@R_p(\eta)\mathrm{d}p\right)$$

which leads to

$$AV@R_{\bar{\alpha}}(\eta) = \frac{1}{1-\alpha}\left(\{\frac{i_\alpha}{N} - \alpha\}x_{(i_\alpha)} + \frac{1}{N}\sum_{i=i_\alpha}^{N-1} x_{(i+1)}\right). \tag{6.4.69}$$

Using the formulae (6.4.67), (6.4.68) and (6.4.69), we will now present numerical results showing the efficiency and usefulness of the *sparse grid* approach. We first start with a comparison with the classically used *nested simulation* approach.

6.4.1 Sparse grid approach versus nested simulations approach

We computed the empirical distribution of the PnL at horizon 1 year using the *nested simulations* approach, recall Section 6.3.2, and the *sparse grid* approach, recall Section 6.3.3.

For both methods, we used a sample of size $N = 11000$ describing the *real-world* evolution of S and Θ, recall Section 6.2.4, and a sample of size $M = 10000$ for the *risk-neutral* Monte Carlo simulations. Following Proposition 6.2.9, we calibrated a Gaussian model such that $(X_1, (\theta_1)_1, (\theta_2)_1, (\theta_3)_1)$ has mean and covariance matrix given by:

$$\mu = (4.1 \times 10^{-5}, 0.01, 0.03, 0.01), \tag{6.4.70}$$

$$V = \begin{pmatrix} 0.004 & 3.2 \times 10^{-5} & 6.76 \times 10^{-6} & 0.000008 \\ 3.2 \times 10^{-5} & 3.1 \times 10^{-5} & 1.82 \times 10^{-5} & 1.5 \times 10^{-5} \\ 6.76 \times 10^{-6} & 1.82 \times 10^{-5} & 7.5 \times 10^{-5} & 8.1 \times 10^{-6} \\ 0.000008 & 1.5 \times 10^{-5} & 8.1 \times 10^{-6} & 2.7 \times 10^{-5} \end{pmatrix} \tag{6.4.71}$$

The risk-neutral simulations were computed by a Monte Carlo procedure, using each time a sample of size $M = 10000$, computed with the exact formulae in the Hull & White and Black & Scholes setting we used, recall Proposition 6.3.2. The volatility parameter used in the Black & Scholes model is set to $\sigma = 0.3$ while the parameters defining the Hull & White model are set to $a = 0.05$ and $b = 0.01$. Last, the covariation parameter between the two Brownian motions is set to $\rho = 0$.

In this setting, the nested simulations method was tested with the Put Lookback option described in 6.2.1, with maturity $T = 30$ years. Figure 6.2 shows the outcome PnL's distribution.

We next looked at the grid method. Figure 6.3 shows the outcome PnL's distributions for sparse grids of level $1, 2, 3$, which respectively have cardinal 81, 297, 945. Figure 6.4 compares the distribution obtained with nested simulations with the distribution obtained with the sparse grid of level 3. Table 6.1 shows computational times comparison, and Table 6.2 shows V@R and AV@R comparison for the empirical distributions obtained in each case.

We observe a significant gain in time obtained by the use of the sparse grid method instead of the nested simulations one. As already observed in Remark 6.3.5, this gain in time can be further improved by parallelisation of the computations on the Sparse Grid. In addition, we observe that, once the computations on the grid are done, then the PnL distribution is almost straightforwardly obtained. This is a key feature of the method since the computations on the grid are to be kept. Indeed, if one needs to change the distribution of (S, Θ) under $\mathbb{P}$, say because the view of the risk management on the evolution of the market parameters has changed, then they can be re-used easily. In the next section, we give an application in this direction.

Level of the sparse grid	$l = 1$	$l = 2$	$l = 3$	Nested simulations
Computations on the grid	7 min 30 sec	26 min 30 sec	1h 08 min	
Computation of the PnL distribution	2 sec	4 sec	8 sec	9h 50min

Table 6.1: Computational times

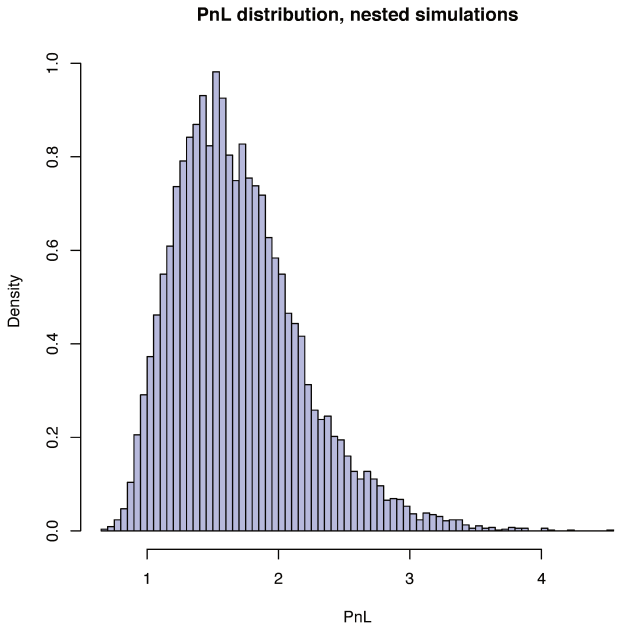

Figure 6.2: PnL distribution, nested simulations

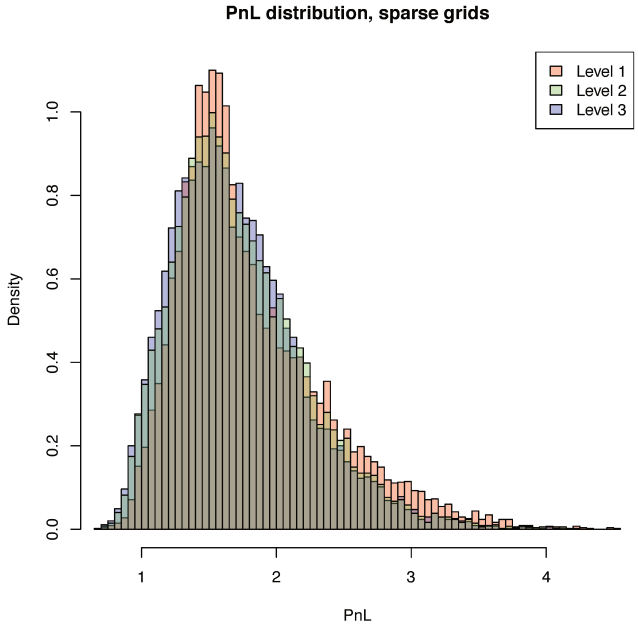

Figure 6.3: PnL distribution, sparse grids

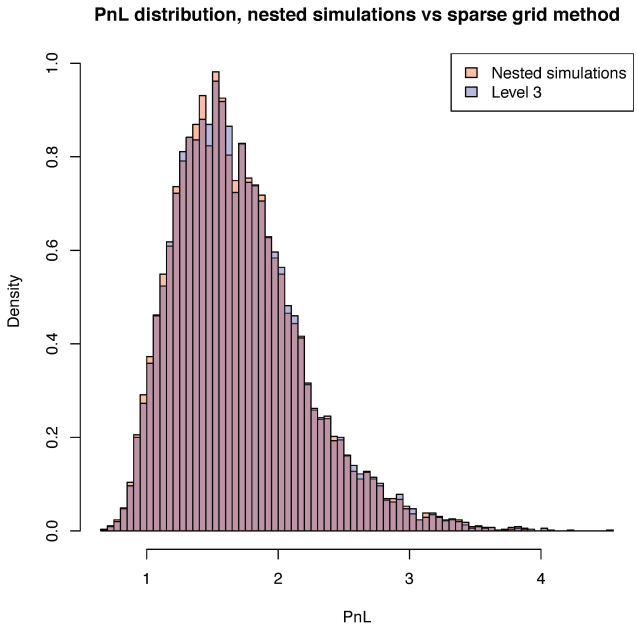

Figure 6.4: PnL distribution, nested simulations versus sparse grid method

Level of the sparse grid	$l = 1$	$l = 2$	$l = 3$	Nested simulations
Wasserstein distance	0.13	0.030	0.0070	
V@R/AV@R $\alpha = 0.005$	3.65 / 3.88	3.42 / 3.69	3.37 / 3.63	3.35 / 3.62
V@R/AV@R $\alpha = 0.01$	3.47 / 3.71	3.21 / 3.50	3.18 / 3.45	3.16 / 3.43
V@R/AV@R $\alpha = 0.05$	2.89 / 3.23	2.64 / 2.97	2.61 / 2.95	2.60 / 2.94
V@R/AV@R $\alpha = 0.1$	2.57 / 2.97	2.39 / 2.73	2.34 / 2.71	2.34 / 2.70

Table 6.2: Comparison of the empirical distributions

6.4.2 Model risk

As mentioned above, an interesting feature of the *sparse grid* approach developed in this paper is the ability to change the distribution of the processes $X = log(S)$ and Θ under the *real-world* probability $\mathbb{P}$. In this Section, we use the model described in Section 6.2.4 to simulate a first sample. Then, we consider some uncertainty over the estimated moments μ, V of (X_1, Θ_1) used to calibrate the gaussian model: we only assume that the true moments lie in centered intervals around the estimations. In practice, we consider intervals of the form $[m \times 0.95, m \times 1.05]$, where m is the estimated moment under consideration.

To better understand the risk associated with this uncertainty under $\mathbb{P}$, we simulate two "extreme" new samples of (X, Θ), where every moment taken into account to calibrate the model are multiplied by 0.95 (resp. 1.05), and, thanks to the grid computations done before with the initial model, we are in position to compute almost instantaneously the empirical PnL distributions associated with these two new samples.

Table 6.3 shows the Wasserstein distance between the initial distributions and the two obtained for the shifted parameters, and the V@R and AV@R obtained at different quantile levels.

We observe that with these small change the distribution are quite close to each other. The main discrepancy are obtained for the diminished moments.

Model	Initial model	Diminished moments	Augmented moments
Wasserstein distance	0	0.030	0.0070
V@R/AV@R $\alpha = 0.005$	3.37 / 3.63	3.43 / 3.81	3.33 / 3.67
V@R/AV@R $\alpha = 0.01$	3.18 / 3.45	3.17 / 3.54	3.16 / 3.45
V@R/AV@R $\alpha = 0.05$	2.61 / 2.95	2.63 / 2.97	2.60 / 2.93
V@R/AV@R $\alpha = 0.1$	2.34 / 2.71	2.37 / 2.73	2.32 / 2.69

Table 6.3: Comparison of the empirical distributions

6.5 Appendix

6.5.1 Proof of (6.2.6)

In this subsection, we shall give the proof of Proposition 2.1 for completeness. We remind that in the Hull-White model, the dynamics of the short rate is given by the following:

$$dr_s^{t,\Theta} = a(\mu_s^{t,\Theta} - r_s^{t,\Theta})\mathrm{d}s + b\mathrm{d}B_s \tag{6.5.72}$$

with $a, b \in \mathbb{R}$. We will prove that the mean-reverting θ_s can be calibrated by forward interest rate curve $f^\Theta(t,s)$ by:

$$\mu_s^{t,\Theta} = f^\Theta(t,s) + \frac{1}{a}\frac{\partial f^\Theta(t,s)}{\partial s} + \frac{b^2}{2a^2}(1 - e^{-2a(s-t)}) \tag{6.5.73}$$

The method is to express the price of the zero-coupon bond $P(t,s)$ in the following way:

$$\mathbb{E}[\exp(-\int_t^s r_u^{t,\Theta}\mathrm{d}u)] = P(t,s) = \exp(-\int_t^s f^\Theta(t,u)\mathrm{d}u) \tag{6.5.74}$$

Then by comparing both sides, we can determine $f^\Theta(t,s)$. First, it is easy to find out that the solution to (6.5.72) is

$$r_s^{t,\Theta} = r_t^{t,\Theta}e^{-a(s-t)} + a\int_t^s \mu_u^{t,\Theta}e^{-a(s-u)}\mathrm{d}u + b\int_t^s e^{-a(s-u)}\mathrm{d}B_u$$

Then by straightforward calculation we have that

$$\int_t^s r_u^{t,\Theta}\mathrm{d}u = \frac{r_t^{t,\Theta}}{a}(1 - e^{-a(s-t)}) + \int_t^s \mu_u^{t,\Theta}(1 - e^{-a(s-u)})\mathrm{d}u + \frac{b}{a}\int_t^s (1 - e^{-a(s-u)})\mathrm{d}B_u$$

So $\int_t^s r_u^{t,\Theta}\mathrm{d}u$ follows a normal distribution with mean

$$\mathbb{E}[\int_t^s r_u^{t,\Theta}\mathrm{d}u] = \frac{r_t^{t,\Theta}}{a}(1 - e^{-a(s-t)}) + \int_t^s \mu_u^{t,\Theta}(1 - e^{-a(s-u)})\mathrm{d}u$$

and variance

$$\mathbb{V}[\int_t^s r_u^{t,\Theta}\mathrm{d}u] = \frac{b^2}{a^2}\int_t^s (1-e^{-a(s-u)})^2\mathrm{d}u$$

Now comparing the both sides of (6.5.74), we have that

$$\int_t^s f^\Theta(t,u)\mathrm{d}u = \mathbb{E}[\int_t^s r_u^{t,\Theta}\mathrm{d}u] - \frac{1}{2}\mathbb{V}[\int_t^s r_u^{t,\Theta}\mathrm{d}u]$$

Thus

$$\begin{aligned} f^\Theta(t,s) &= \frac{\partial}{\partial s}\mathbb{E}[\int_t^s r_u^{t,\Theta}\mathrm{d}u] - \frac{1}{2}\frac{\partial}{\partial s}\mathbb{V}[\int_t^s r_u^{t,\Theta}\mathrm{d}u] \\ &= r_t^{t,\Theta}e^{-a(s-t)} + a\int_t^s \mu_u^{t,\Theta}e^{-a(s-u)}\mathrm{d}u - \frac{b^2}{2a^2}2a\int_t^s (e^{-a(s-u)} - e^{-2a(s-u)})\mathrm{d}u \\ &= r_t^{t,\Theta}e^{-a(s-t)} + a\int_t^s \mu_u^{t,\Theta}e^{-a(s-u)}du - \frac{b^2}{2a^2}(1-e^{-a(s-t)})^2 \end{aligned} \tag{6.5.75}$$

By straightforward differentiation, we have

$$\frac{\partial}{\partial s}f^\Theta(t,s) = -ar_t^{t,\Theta}e^{-a(s-t)} - a^2\int_t^s \mu_u^{t,\Theta}e^{-a(s-u)}\mathrm{d}u + a\mu_s^{t,\Theta} - \frac{b^2}{a}(e^{-a(s-t)} - e^{-2a(s-t)}) \tag{6.5.76}$$

Now by (6.5.75) and (6.5.76), we can easily verify that (6.5.73) is valid.

6.5.2 Proof of Proposition 6.2.9

We provide a recursive proof of Proposition 6.2.9, which allows to compute effectively the coefficients defining the processes.

Suppose more generally that a vector $\mu \in \mathbb{R}^n$ and a covariance matrix $V \in \mathbb{R}^{n\times n}$ is given. We look for n processes $X^i (i = 0, \ldots, n)$ defined by:

$$X_t^i = X_0^i + b_i t + \sum_{j=1}^n c_{ij} W_t^j, \tag{6.5.77}$$

where $W_t^j (j = 1, \ldots, n)$ are n independant Brownian motions, and $b \in \mathbb{R}^n, C = (c_{ij}) \in \mathbb{R}^{n\times n}$.

Proposition 6.5.1. *There is at most one $(b, C) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}$ such that:*

- $c_{ij} = 0$ *whenever* $i > j$,
- $\mathbb{E}X_1^i = \mu_i (i = 1, \ldots, n)$,
- $Cov(X_1^i, X_1^j) = V_{ij} (i, j = 1, \ldots, n)$.

Proof. We have $\mathbb{E}X_1^i = X_0^i + b_i$, so $b_i := \mu_i - X_0^i$ ensures $\mathbb{E}X_1^i = \mu_i$ for all i.

We next determine the matrix C thanks to a recursive algorithm:

Ascending step: Let $i, l \in \{1, \ldots, n\}$, and assume $c_{ik}, k > l$ and $c_{lk}, k \geq l$ are determined. Then we can determine c_{il}.

Indeed, if $i > l$, we set $c_{il} = 0$. If $i < l$, we have:

$$V_{il} = Cov(X_1^i, X_1^l) \tag{6.5.78}$$
$$= \sum_{j=1}^{n} c_{ij} c_{lj} \tag{6.5.79}$$
$$= \sum_{j=l}^{n} c_{ij} c_{lj} \tag{6.5.80}$$
$$= c_{il} c_{ll} + \sum_{j>l} c_{ij} c_{lj}. \tag{6.5.81}$$

Thus we set:

$$c_{il} = \frac{1}{c_{ll}} \left(V_{il} - \sum_{j>l} c_{ij} c_{lj} \right). \tag{6.5.82}$$

Back step: Let $l \in \{1, \dots, n\}$ and assume c_{lj} is determined, for $k > l$. Then we can determine c_{ll}.

Indeed:

$$V_{ll} = \mathbb{V}(X_1^l) = \sum_{j=1}^{n} c_{lj}^2 = \sum_{j=l}^{n} c_{lj}^2 = c_{ll}^2 + \sum_{j>l} c_{lj}^2. \tag{6.5.83}$$

Thus we set:

$$c_{ll} = \sqrt{V_{ll} - \sum_{j>l} c_{lj}^2}. \tag{6.5.84}$$

□

6.5.3 Proofs of Lemma 6.3.1 and Proposition 6.3.2

We prove here the Lemma 6.3.1 and the Proposition 6.3.2, which give a recursive procedure to simulate exactly under $\mathbb{Q}$.

Proof of Lemma 6.3.1. Let $(t, \Theta) \in [0, T] \times \mathbb{R}^3$ and consider the process $r^{t,\Theta} = (r_s^{t,\Theta})_{s \in [t,T]}$ defined by (6.2.5)-(6.2.6).

Let $s \in [t, T]$. An application of Itô's formula gives:

$$e^{as} r_s^{t,\Theta} = e^{at} r_t^{t,\Theta} + a \int_t^s e^{au} \mu_u^{t,\Theta} \mathrm{d}u + b \int_t^s e^{au} \mathrm{d}B_u, \tag{6.5.85}$$

and an easy computation using equality (6.2.6) shows that:

$$a \int_t^s e^{-a(s-u)} \mu_u^{t,\Theta} \mathrm{d}u = \alpha_s^{t,\Theta} - \alpha_t^{t,\Theta} e^{-a(s-t)}, \tag{6.5.86}$$

where $\alpha^{t,\Theta}$ is the defined by (6.3.32).

In addition, if ξ^t is defined by (6.3.33), applying Itô's formula again gives:

$$\xi_s^t = b\int_t^s e^{-a(s-u)}\mathrm{d}B_u. \tag{6.5.87}$$

Thus:

$$r_s^{t,\Theta} = e^{-a(s-t)}r_t^{t,\Theta} + \alpha_s^{t,\Theta} - e^{-a(s-t)}\alpha_t^{t,\Theta} + \xi_s^t, \tag{6.5.88}$$

which ends the proof as $r_t^{t,\Theta} = \alpha_t^{t,\Theta}$, by (6.5.75). □

We now turn to the proof of Proposition 6.3.2.

Proof of Proposition 6.3.2. Let $t \le s \le u \le T$. Itô's formula implies that the triplet $(\xi_r^t, A_r^{t,s}, X_r^{t,x,\Theta})_{r\in[s,u]}$ is the solution of the following linear stochastic differential equation:

$$d\begin{pmatrix}\xi_r\\A_r\\X_r\end{pmatrix} = \left[\begin{pmatrix}-a & 0 & 0\\1 & 0 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}\xi_r\\A_r\\X_r\end{pmatrix} + \begin{pmatrix}0\\0\\\alpha_r^{t,\Theta} - \frac{\sigma^2}{2}\end{pmatrix}\right]\mathrm{d}t + \begin{pmatrix}b & 0\\0 & 0\\\sigma\rho & \sigma\sqrt{1-\rho^2}\end{pmatrix}\begin{pmatrix}\mathrm{d}B_r\\\mathrm{d}W_r\end{pmatrix}, r\in[s,u], \tag{6.5.89}$$

with the initial conditions $\xi_s = \xi_s^t, A_s = 0, X_s = X_s^{t,x,\Theta}$. This linear equation has a closed form solution, and we find:

$$\xi_u^t = e^{-a(u-s)}\xi_s^t + b\int_s^u e^{-a(u-r)}\mathrm{d}B_r, \tag{6.5.90}$$

$$A_u^{t,s} = \frac{\xi_s^t}{a}(1-e^{-a(u-s)}) + \frac{b}{a}\int_s^u (1-e^{-a(u-r)})\mathrm{d}B_r, \tag{6.5.91}$$

$$X_u^{t,x,\Theta} = X_s^{t,x,\Theta} + \int_s^u \alpha_r^{t,\Theta}\mathrm{d}r - \frac{\sigma^2}{2}(u-s) + \frac{\xi_s^t}{a}(1-e^{-a(u-s)}) + \int_s^u \sqrt{1-\rho^2}\sigma\mathrm{d}W_r \tag{6.5.92}$$

$$+ \frac{b}{a}\int_s^u (1-e^{-a(u-r)})\mathrm{d}B_r + \int_s^u \rho\sigma\mathrm{d}B_r. \tag{6.5.93}$$

Conditionaly upon $\mathcal{F}_s$, the vector $(\xi_u^t, A_u^{t,s}, X_u^{t,x,\Theta})$ is Gaussian and the expectations and covariations given in the Proposition are easily computed thanks to the above formulae. □

6.5.4 Comparison with Automatic Differentiation.

We use the stan math C++ library [125] which allows to easily implement a (Reverse Mode) Automatic Differentiation procedure in order to deduce the derivatives directly from the Monte-Carlo computation of the function L. We compare the results obtained with the weights method developed here with the results obtained by Automatic Differentiation. We also provide a comparison about the computational times.

Precisely, we computed the derivatives of ℓ with respect to the 4 variables $(x, \theta_1, \theta_2, \theta_3)$ at 256 points $(x^i, \theta_1^i, \theta_2^i, \theta_3^i)_{i=1,\dots,256}$. In the case of the automatic differentiation, we only take 1000 risk-neutral simulations to compute ℓ, while for the approach involving the computation of weights, we took 10000 simulations to compute ℓ and its four derivatives.

Table 6.4 sums up the time taken for the computations. Clearly, the gain in time resulting by using the weights algorithm is really significant. Additionally, Figure 6.5 shows the accuracy in the computation using the weights derivatives in comparison with the Automatic Differentiation.

Algorithm - Option	Put Lookback
Automatic Differentiation	179 sec
Weights	97 sec

Table 6.4: Computational times

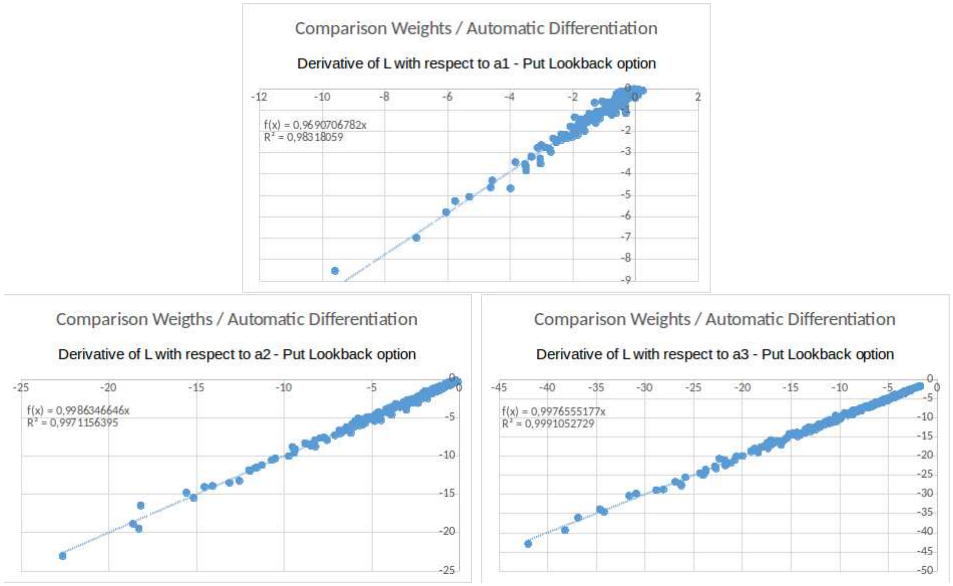

Figure 6.5: Results from the grid method versus results from Automatic Differentiation

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Résumé

Dans cette thèse, on considère trois sujets. Les deux premiers sujets sont liées avec la domaine de robuste finance et le dernier est une méthode numérique appliqué sur la gestion du risque des entreprises d'assurance.
Dans la première partie, on considère le problème de la sur-réplication des options américaines au temps discret. On considère une famille non-dominée des mesures de probabilité et les stratégies de trading sont dynamiques pour les sous-jacents et statiques pour les options. Pour obtenir la dualité de valorisation-couverture, on a deux méthodes. La première méthode est de réformuler les options américaines comme options européens dans un espace élargi. La deuxième méthode est de considérer un marché fictif dans lequel les stratégies pour tous les actifs sont dynamiques. Ensuite on applique le résultat général à deux exemples importants dans le context robust.
Dans la deuxième partie, on considère le problème de sur-réplication and maximisation d'utilité au temps discret avec coût de transaction sous l'incertitude du modèle. L'idée principale est de convertir le problème original à un problème sans friction dans un espace élargi en utilisant un argument de randomisation et le théorème de minimax. Pour le problème de sur-réplication, on obtient la dualité comme dans le cas classique. Pour le problème de maximisation d'utilité, en utilisant un argument de la programmation dynamique, on peut preuver à la fois l'existence de la stratégie optimale et le théorème de la dualité convexe.
Dans le troisième partie, on présente une méthode numérique basésur l'approximation du sparse grid pour calculer la distribution de la perte du bilan d'un entreprise d'assurance. On compare la nouvelle méthode numérique avec l'approche classique de la simulation et étudie la vitesse de la convergence des deux méthodes pour estimer l'indicateur du risque.

Abstract

This PhD dissertation presents three research topics. The first two topics are related to the domain of robust finance and the last is related to a numerical method applied in risk management of insurance companies.
In the first part, we focus on the problem of super-replication duality for American options in discrete time financial models. We consider the robust framework with a family of non-dominated probability measures and the trading strategies are dynamic on the stocks and static on the options. We use two different ways to obtain the pricing-hedging duality. The first insight is that we can reformulate American options as European options on an enlarged space. The second insight is that by considering a fictitious extensions of the market on which all the assets are traded dynamically. We then show that the general results apply in two important examples of the robust framework.
In the second part, we consider the problem of super-replication and utility maximization with proportional transaction cost in discrete time financial market with model uncertainty. Our key technique is to convert the original problem to a frictionless problem on an enlarged space by using a randomization technique together with the minimax theorem. For the super-replication problem, we obtain the duality results well-known in the classical dominated context. For the utility maximization problem, we are able to prove the existence of the optimal strategy and the convex duality theorem in our context with transaction costs.
In the third part, we present a numerical method based on a sparse grid approximation to compute the loss distribution of the balance sheet of an insurance company. We compare the new numerical method with the traditional nested simulation approach and review the convergence of both methods to estimate the risk indicators under consideration.

Mots Clés

incertitude du modèle, sur-réplication, maximisation d'utilité, coût de transaction, randomisation, option américaine

Keywords

model uncertainty, super-replication, utility maximization, transaction cost, randomization, american option