The Kinetic Fokker-Planck Equation : Hypocoercivity and Hypoellipticity

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Contents

Chapter 1

Introduction

This thesis is mainly about the convergence to equilibrium for the kinetic equations. We first introduce the kinetic theory and Boltzmann's equation. Then we introduce a range of equations which will be discussed through out this work. After this we will give a review of some classical existing works on converge to equilibrium, mainly focusing on hypocoercivity. Lastly we briefly describe the contents of each of the chapters.

1.1 Kinetic Equations

Kinetic theory was developed in the 19th century, most notably by Boltzmann and Maxwell, in the modeling of dilute gases. Kinetic equations model the evolution of a gas in an intermediate scale between the microscopic description which is given by Newton's laws and a macroscopic fluid descriptions. If we have a system of N particles performing either deterministic or stochastic dynamics, we can write their joint distribution at time t

$$F_N(t, z_1, z_2..., z_n) \geq 0.$$

In this case z_i is either v_i, the velocity of the ith particle, or $(x_i; v_i)$, the position and velocity of the ith particle. We look at the situation where this equation models a large number of indistinguishable agents, for example gas particles. We study particles interacting in a gas by Newton's laws, then they will follow the equations

$$\begin{aligned} \dot{x_i} &= v_i \\ \dot{v_i} &= \sum_{i \neq j} F(i,j) + F, \end{aligned}$$

where $F(j,i)$ is the force acting on particle i due to particle j and F is an external force. For collisional gases the equation derived in this process, via the Boltzmann-Grad scaling, is Boltzmann's equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f),$$

where

$$Q(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|,(v-v_*)\cdot\sigma)(f(v')g(v'_*) - f(v)g(v_*))d\sigma dv_*$$

with

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

B is called the collision kernel. Here $f = f(t,x,v)$ represents the density in phase space of a single particle in the ensemble and $t \geq 0, x \in \mathbb{R}^d, v \in R^d$. We can observe here the general structure of a collisional kinetic equation

$$\partial_t f + v\cdot\nabla_x f = L(f),$$

where the $v\cdot\nabla_x f$ operator comes from the transport term in Newton's law. The $L(f)$ operator acts only on the velocity variable v and is the result of collisions between particles or between particles and a background statistical medium. We can derive macroscopic quantities from this density, by setting

$$\rho(x) = \int f(x,v)dv,$$

the local density,

$$u(x) = \frac{1}{\rho(x)}\int vf(x,v)dv,$$

the local speed and

$$T(x) = \frac{1}{\rho(x)}\int |u-u(x)|^2 f(x,v)dv,$$

the local temperature. The steady state solution of Boltzmann's equation was derived by Maxwell and it was shown to be the unique asymptotic equilibrium by Boltzmann. It is known as the Maxwellian

$$M(v) = \rho(2\pi T)^{-d/2}\exp(-\frac{1}{2T}|v|^2)$$

Here ρ, T are the spatial averages of the local quantities above. Since this is a steady state of the equation it becomes natural to ask whether the solution to Boltzmann's equation will eventually come close to the Maxwellian and if so how fast this will happen.

Throughout the rest of this thesis, we use the notation $M(v)$ to refer to the normalized Maxwellian with $u = 0, \rho = 1$ and $T = 1$. That is

$$M(v) = (2\pi)^{-d/2}\exp(-\frac{|v|^2}{2}). \tag{1.1.1}$$

We can also look at equations where instead of just transported, potentials are both transported and confined. This allows us to have equilibrium states where the x variable is in the whole of $\mathbb{R}^d$. The corresponding equation write

$$\partial_t f + v\cdot\nabla_x f - \nabla U(x)\cdot\nabla_v f = Q(f,f).$$

where $U(x)$ represents an external confinement force, in the following we introduce some equations that are studied in the thesis.

1.1.1 The kinetic Fokker-Planck equation

The kinetic Fokker-Planck equation is one of the simplest equations in kinetic theory. It is a kinetic version of the Fokker-Planck equation which was developed by Fokker [28], and Planck [59]. The Fokker-Planck equation is the PDE associated which is corresponding to the Orstein-Uhlenbeck process,

$$\partial_t f = \nabla_v \cdot (\nabla_v f + v f),$$

The kinetic Fokker-Planck equation wirtes

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + v f),$$

on the density $f = f(t, x, v), t \geq 0, v \in \mathbb{R}^d, x \in \mathbb{T}^d$ or $x \in \mathbb{R}^d$. The associated equilibrium state is

$$e^{-U(x)} M(v) = \exp(-(U(x) + \frac{|v|^2}{2})),$$

We look at the kinetic Fokker-Planck equation in Chapters 3 and 4.

1.1.2 The linear relaxation Boltzmann equation

The linear relaxation Boltzmann equation is the simplest example of a scattering equation from kinetic theory. It is also known as the linear BGK equation. We write it

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \Pi_M f - f,$$

on the density $f = f(t, x, v), t \geq 0, v \in \mathbb{R}^d, x \in \mathbb{T}^d$ or $x \in \mathbb{R}^d$, with

$$\Pi_M f = (\int_{\mathbb{R}^d} f(x, u) du) M(v),$$

Here again the phase space is either $\mathbb{R}^d \times \mathbb{R}^d$ or $\mathbb{T}^d \times \mathbb{R}^d$. The equilibrium state is the same as the kinetic Fokker-Planck equation.

$$e^{-U(x)} M(v) = \exp(-(U(x) + \frac{|v|^2}{2})),$$

There are variants of this equation that we do not study but exhibit similar behavior. First we can generalize the collision kernel as

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \int_{\mathbb{R}^d} f(x, u) k(v, u) du - f,$$

here $k(v, u)$ represents the rate of jumping from velocity u to velocity v.
Second we can look at the equation when the rate of collision depends on space, namely

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \sigma(x)(\Pi_M f - f).$$

We look at the linear relaxation Boltzmann equation in Chapters 5.

1.1.3 The linear Boltzmann equation

We also look at the linear Boltzmann equation. This is a scattering type equation which can be derived from microscopic dynamics with particles interacting with a heat bath. The equation is

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \mathcal{Q}(f, M),$$

where $\mathcal{Q}$ is the Boltzmann collision operator given before and M is defined in (1.1.1). This equation is much simpler than the full Boltzmann equation. It is linear and is the equation of the density for a Markov process provided B is sufficiently nice. The spatially homogeneous linear Boltzmann equation

$$\partial_t f = \mathcal{Q}(f, M),$$

has been well studied.
We look at the linear Boltzmann equation in Chapters 5.

1.2 Harris's Theorem

Harris's theorem [38, 41, 52] is a result from the theory of Markov processes. Reproving and rewriting Harris's theorem in a PDE context is a subject of an ongoing work from José Cañizo and Stéphane Mischler. Harris's theorem shows quantitative rates of convergence to equilibrium for processes satisfying two assumptions.

Now let us be more specific about Harris's Theorem. We give the theorems and assumption as in the setting of [41] where they make it clear how the rates depend on those in the assumptions. Markov operators can be defined by means of *transition probability functions*. We always assume that $(\Omega, \mathcal{S})$ is a measurable space. A function $S : \Omega \times \mathcal{S} \to \mathbb{R}$ is a transition probability function on a finite measure space if $S(x, \cdot)$ is a probability measure for every x and $x \mapsto S(x, A)$ is a measurable function for every $A \in \mathcal{S}$. We can then define $\mathcal{P}$, the associated stochastic operator on probability measures, by

$$\mathcal{P}\mu(\cdot) = \int \mu(\mathrm{d}x) S(x, \cdot).$$

Since we are looking at a process we have Markov transition kernel S_t for each $t > 0$, and then an associated Markov semigroup $\mathcal{P}_t$ where $\mathcal{P}_t$ is defined from S_t as above. In our situation $\mathcal{P}_t\mu$ is the weak solution to the PDE with initial data μ. If we define $\mathcal{M}(\Omega)$ as the space of finite measures on $(\Omega, \mathcal{S})$ then we have that $\mathcal{P}_t$ is a *linear* map

$$\mathcal{P}_t : \mathcal{M}(\Omega) \to \mathcal{M}(\Omega).$$

From the conditions on S_t we see that $\mathcal{P}_t$ will be *linear, mass preserving* and *positivity preserving*. We can define the *forward operator* $\mathcal{L}$, associated to S_t by

$$\frac{\mathrm{d}}{\mathrm{d}t} S_t \phi \Big|_{t=0} = \mathcal{L}\phi.$$

We begin by looking at Doeblin's Theorem. Harris's Theorem is a natural generator to Doeblin's Theorem. Harris's and Doeblin's theorems are usually stated for a fixed time t_*. In our theorems we work to choose an appropriate t_*.

Hypothesis 1.2.1 (Doeblin's Condition)**.** *We assume $(\mathcal{P}_t)_{t\geq 0}$ is a stochastic semigroup, coming from a Markov transition kernel, and that there exists $t_* > 0$, a probability distribution ν and $\alpha \in (0,1)$ such that for any z in the state space, we have*

$$\mathcal{P}_{t_*}\delta_z \geq \alpha\nu.$$

Using this we prove

Theorem 1.2.2 (Doeblin's Theorem)**.** *If $(\mathcal{P}_t)_{t\geq 0}$ is a stochastic semigroup satisfying Doeblin's condition (Hypothesis 1.2.1) then for any two probability measures μ_1 and μ_2 and any integer $n \geq 0$, we have*

$$\|\mathcal{P}_{t_*}^n\mu_1 - \mathcal{P}_{t_*}^n\mu_2\|_{\mathrm{TV}} \leq (1-\alpha)^n\|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{1.2.2}$$

As a consequence, the semigroup has a unique equilibrium probability measure μ_*, and for all μ

$$\|\mathcal{P}_t(\mu - \mu_*)\|_{\mathrm{TV}} \leq \frac{1}{1-\alpha}e^{-\lambda t}\|\mu - \mu_*\|_{\mathrm{TV}}, \qquad t \geq 0, \tag{1.2.3}$$

where

$$\lambda := \frac{\log(1-\alpha)}{t_*} > 0.$$

Proof. This proof is classical and can be found in various versions in [41] and many other places. We now sketch the proof.

Firstly we show that if $\mathcal{P}_t\delta_z \geq \alpha\nu$ for every z, then we also have $\mathcal{P}_t\mu \geq \alpha\nu$ for every μ. Here since $\mathcal{P}_t$ comes from a Markov transition kernel we have

$$\mathcal{P}_t\delta_z(\cdot) = \int S_t(z', \cdot)\delta_z(\mathrm{d}z') = S_t(z, \cdot).$$

Therefore our condition says that

$$S_t(z, \cdot) \geq \alpha\nu(\cdot)$$

for every z. Therefore,

$$\mathcal{P}_t\mu(\cdot) = \int S_t(z, \cdot)\mu(\mathrm{d}z) \geq \alpha\int \nu(\cdot)\mu(\mathrm{d}z) = \alpha\nu(\cdot).$$

By the triangle inequality we have

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{\mathrm{TV}} \leq \|\mathcal{P}_{t_*}\mu_1 - \alpha\nu\|_{\mathrm{TV}} + \|\mathcal{P}_{t_*}\mu_2 - \alpha\nu\|_{\mathrm{TV}}.$$

Now, since $\mathcal{P}_{t_*}\mu_1 \geq \alpha\nu$, we can write

$$\|\mathcal{P}_{t_*}\mu_1 - \alpha\nu\|_{\mathrm{TV}} = \int(\mathcal{P}_{t_*}\mu_1 - \alpha\nu) = \int \mu_1 - \alpha = 1 - \alpha,$$

due to mass conservation, and similarly for the term $\|\mathcal{P}_{t_*}\mu_2 - \alpha\nu\|_{\mathrm{TV}}$. This gives

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{\mathrm{TV}} \leq 2(1-\alpha) = (1-\alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}},$$

if μ_1, μ_2 have disjoint support. By homogeneity, this inequality is obviously also true for any nonnegative μ_1, μ_2 having disjoint support with $\int \mu_1 = \int \mu_2$. We obtain the inequality in general for any μ_1, μ_2 with the same integral by writing $\mu_1 - \mu_2 = (\mu_1 - \mu_2)_+ - (\mu_2 - \mu_1)_+$, which is a difference of nonnegative measures with the same integral. This proves

$$\|\mathcal{P}_{t_*}\mu_1 - \mathcal{P}_{t_*}\mu_2\|_{\mathrm{TV}} \leq (1-\alpha)\|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{1.2.4}$$

We then iterate this to obtain (1.2.2). The contractivity (1.2.4) shows that the operator $\mathcal{P}_{t_*}$ has a unique fixed point, which we call μ_*. In fact, μ_* is a stationary state of the whole semigroup since for all $s \geq 0$ we have

$$\mathcal{P}_{t_*}\mathcal{P}_s\mu_* = \mathcal{P}_s\mathcal{P}_{t_*}\mu_* = \mathcal{P}_s\mu_*,$$

which shows that $\mathcal{P}_s\mu_*$ (which is again a probability measure) is also a stationary state of $\mathcal{P}_{t_*}$; due to uniqueness,

$$\mathcal{P}_s\mu_* = \mu_*.$$

Hence the only stationary state of $\mathcal{P}_t$ must be μ_*, since any stationary state of $\mathcal{P}_t$ is in particular a stationary state of $\mathcal{P}_{t_*}$.

In order to show (1.2.3), for any probability measure μ and any $t \geq 0$ we write

$$k := \lfloor t/t_* \rfloor,$$

(where $\lfloor \cdot \rfloor$ denotes the integer part) so that

$$\frac{t}{t_*} - 1 < k \leq \frac{t}{t_*}.$$

Then,

$$\begin{aligned}\|\mathcal{P}_t(\mu - \mu_*)\|_{\mathrm{TV}} &= \|\mathcal{P}_{t-kt_*}\mathcal{P}_{kt_*}(\mu - \mu_*)\|_{\mathrm{TV}} \leq \|\mathcal{P}_{kt_*}(\mu - \mu_*)\|_{\mathrm{TV}} \\ &\leq (1-\alpha)^k\|\mu - \mu_*\|_{\mathrm{TV}} \leq \frac{1}{1-\alpha}\exp\left(t\log(1-\alpha)/t_*\right)\|\mu - \mu_*\|_{\mathrm{TV}}. \quad \square\end{aligned}$$

Harris's Theorem extends this to the setting where we cannot prove minorisation uniformly on the whole of the state space. The idea is to use the argument given above on the center of the state space then exploit the Lyapunov structure to show that any stochastic process will return to the center infinitely often.

We make two assumptions on the behaviour of $\mathcal{P}_{t_*}$, for some fixed t_*:

Hypothesis 1.2.3 (Lyapunov condition)**.** *There exists some function $V : \Omega \to [0, \infty)$ and constants $D \geq 0, \alpha \in (0,1)$ such that*

$$(\mathcal{P}_{t_*}V)(z) \leq \alpha V(z) + D.$$

Remark 1.2.4. We use the name Lyapunov condition as it is the standard name used for this condition in probability literature. However, we should stress this condition is not closely related to the Lyapunov method for proving convergence to equilibrium. We do not prove monotonicity on a functional.

Remark 1.2.5. In our situation where we have an equation on the law $f(t)$, this is equivalent to the statement

$$\int_S f(t,z)V(z)\mathrm{d}z \leq \alpha \int_S f(0,z)V(z)\mathrm{d}z + D. \tag{1.2.5}$$

We usually verify this by showing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S f(t,z)V(z)\mathrm{d}z \leq -\lambda \int_S f(t,z)V(z)\mathrm{d}z + K,$$

for some positive constants K and λ, which then implies (1.2.5) for $\alpha = e^{-\lambda t}$ and $D = \frac{K}{\lambda}(1-e^{-\lambda t}) \leq Kt$.

The idea behind verifying the Lyapunov structure in our case comes from [50] where they use similar Lyapunov structures for the kinetic Fokker-Planck equation. When we work on the torus the Lyapunov structure is only needed in the v variable and the result is purely about how moments in v are affected by the collision operator.

The next assumption is a minorisation condition as in Doeblin's Theorem

Hypothesis 1.2.6. *There exists a probability measure ν and a constant $\beta \in (0,1)$ such that*

$$\inf_{z\in\mathcal{C}} \mathcal{P}_{t_*}\delta_z \geq \beta\nu,$$

where

$$\mathcal{C} = \{z | V(z) \leq R\}$$

for some $R > 2D/(1-\alpha)$.

Remark 1.2.7. Production of quantitative lower bounds as a way to quantify the *positivity* of a solution has been proved and used in kinetic theory before. For example it is an assumption required for the works of Desvillettes and Villani [22, 23]. Such lower bounds have been proved for the non-linear Boltzmann equation in [7, 8].

This second assumption is more challenging to verify in our situations. Here we use a strategy based on our observation about how noise is transferred from the v to the x variable as described earlier. The actual calculations are based on the PDE governing the evolution and iteratively using Duhamel's formula.

We define a distance on probability measures for every $a > 0$:

$$\rho_a(\mu_1,\mu_2) = \int (1 + aV(x,v))|\mu_1 - \mu_2|(\mathrm{d}x\mathrm{d}v).$$

Theorem 1.2.8 (Harris's Theorem as in [50]). *If Hypotheses 1.2.3 and 1.2.6 hold then there exist $\bar{\alpha} \in (0,1)$ and $a > 0$ such that*

$$\rho_a(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq \bar{\alpha}\rho_a(\mu_1, \mu_2). \tag{1.2.6}$$

Explicitly if we choose $\beta_0 \in (0, \beta)$ and $\alpha_0 \in (\alpha + 2D/R, 1)$ then we can set $\gamma = \beta_0/K$ and $\bar{\alpha} = (1 - (\beta - \beta_0)) \vee (2 + R\gamma\alpha_0)/(2 + R\gamma)$.

Remark 1.2.9. We have

$$\min\{1, a\}\rho_1(\mu_1, \mu_2) \leq \rho_a(\mu_1, \mu_2) \leq \max\{1, a\}\rho_1(\mu_1, \mu_2).$$

The result follows if we can find an $\alpha_0 < 1$ such that

$$\rho_1(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq \alpha_0\rho_a(\mu_1, \mu_2).$$

Assuming that μ_1 and μ_2 have disjoint support and that $V(z) \geq R$. Then, by choosing any $\alpha_1 \in (\alpha, 1)$ and by Hypotheses 1.2.3 and 1.2.6 we obtain

$$\begin{aligned}\rho_1(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq 2 + a(\mathcal{P}V)(z) &\leq 2 + a\alpha(\mathcal{P}V)(z) + 2aD \\ &\leq 2 + a\alpha_1(\mathcal{P}V)(z) + a(2D - (\alpha_1 - \alpha)R).\end{aligned}$$

If we ensure that R is sufficiently large so that $(\alpha_1 - \alpha)R > 2D$, then there exists some $\beta_1 < 1$ (depending on a) such that we have

$$\rho_1(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq \beta_1\rho_a(\mu_1, \mu_2).$$

Now, we determine a choice for a. We consider the case $V(z) \leq R$. To treat this case, we split the measure μ_1 as

$$\mu_1 = \mu_1^{(1)} + \mu_1^{(2)} \text{ where } |\mu_1^{(1)}| \leq 1, \quad |\mu_1^{(2)}| \leq aV(z), \text{ for all } z \in \Omega.$$

Then we have

$$\begin{aligned}\rho_1(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq \rho_1(\mathcal{P}_{t_*}\mu_1^{(1)}, \mathcal{P}_{t_*}\mu_1^{(2)}) + \rho_1(\mathcal{P}_{t_*}\mu_2) &\leq 2(1 - \beta) + a\alpha V(z) + 2aD \\ &\leq 2 - 2\beta + a(\alpha R + 2D).\end{aligned}$$

Hence fixing for example $a = \beta/(\alpha R + 2D)$ we obtain

$$\rho_1(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq 2 - \beta \leq (1 - \beta/2)\rho_a(\mu_1, \mu_2),$$

since $\rho_a(\mu_1, \mu_2) \leq 2$. Setting $\alpha_0 = \max\{1 - \beta/2, \beta_1\}$ concludes the proof. □

We can also iterate Theorem 1.2.8 to get

$$\rho_a(\mathcal{P}_{nt_*}\mu_1, \mathcal{P}_{nt_*}\mu_2) \leq \bar{\alpha}^n\rho_a(\mu_1, \mu_2).$$

Therefore we have that

$$\rho_1(\mathcal{P}_{nt_*}\mu_1, \mathcal{P}_{nt_*}\mu_2) \leq \bar{\alpha}^n \frac{\max\{1, a\}}{\min\{1, a\}}\rho_1(\mu_1, \mu_2).$$

Remark 1.2.10. In this thesis we always consider functions V where $V(z) \to \infty$ as $|z| \to \infty$. In this case, we can replace $\mathcal{C}$ in Hypothesis 1.2.6 with some ball of radius R' which will contain $\mathcal{C}$.

There are versions of Harris's Theorem adapted to weaker Lyapunov conditions which give subgeometric convergence [21]. We use the following theorem which can be found in Section 4 of [39].

Theorem 1.2.11 (Subgeometric Harris's Theorem)**.** *Given the forwards operator, $\mathcal{L}$, of our Markov semigroup $\mathcal{P}$, suppose that there exists a continuous function V valued in $[1, \infty)$ with pre compact level sets such that*

$$\mathcal{L}V \leq K - \phi(V),$$

for some constant K and some strictly concave function $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(0) = 0$ and increasing to infinity. Assume that for every $C > 0$ we have the minorisation condition like Hypothesis 1.2.6. i.e. for some t_* a time and ν a probability distribution and $\alpha \in (0, 1)$, then for all z with $V(z) \leq C$:

$$\mathcal{P}_{t_*}\delta_z \geq \alpha\nu.$$

With these conditions we have that

- *There exists a unique invariant probability measure μ for the Markov process and it satisfies*
$$\int \phi(V(z))\mathrm{d}\mu \leq K.$$
- *Let H_ϕ be the function defined by*
$$H_\phi = \int_1^u \frac{\mathrm{d}s}{\phi(s)}.$$
Then there exists an constant C such that
$$\|\mathcal{P}_t\nu - \mu\|_{\mathrm{TV}} \leq \frac{C\nu(V)}{H_\phi^{-1}(t)} + \frac{C}{(\phi \circ H_\phi^{-1})(t)}$$
holds for every probability measure ν.

Remark 1.2.12. Since $\|\mathcal{P}_t\nu - \mu\|_{\mathrm{TV}} \leq \|\nu - \mu\|_{\mathrm{TV}}$ we can use the fact that the geometric mean of two numbers is greater than the minimum to see that

$$\|\mathcal{P}_t\nu - \mu\|_{\mathrm{TV}} \leq \sqrt{\|\nu - \mu\|_{\mathrm{TV}}}\sqrt{\frac{C\nu(V)}{(\phi \circ H_\phi^{-1})(t)}}.$$

We will apply this abstract theorem as well as Harris's Theorem to the PDEs we study to show convergence when they only satisfy a weaker confinement condition.

1.3 Hypocoercivity

The name hypocoercivity was first used by Villani in his mémoire *Hypocoercivity* [62]. He credits the name to Thierry Gallay to emphasize the link with hypoellipticity. Let us begin by giving a definition of hypocoercivity.

Definition 1.3.13. If L is a linear operator, and let $f(t)$ to be the solution to the equation

$$\partial_t f + Lf = 0, \quad f(0) = f_0,$$

at time t, if there exists constants C, λ, such that for all initial data f_0 we have

$$\|f(t)\| \leq Ce^{-\lambda t}\|f_0\|,$$

then we say the equation is hypocoercive in the norm $\|\cdot\|$.

Remark 1.3.14. The operator is called coercive if this inequality is true with $C = 1$.

This inequality on the semigroup is equivalent to the generator L having a spectral gap in the norm $\|\cdot\|$.

A key aspect of hypocoercivity is to try and prove constructive theorems which give explicitly C. In particular constructive estimates for C which are important because it allows us to know the time after which the convergence effects shown by the inequality will act. An inequality of the form shown in hypocoercivity does not give any convergence until so if C is unknown and potentially very large the result may not hold in the time frame for which the model is valid. This means it is not sufficient to know the spectral gap for a non symmetric operator. However, even the spectral gap is not computed when using many methods based on compactness.

Before hypocoercivity, there were many influential works showing convergence to equilibrium. For example [29, 30, 31, 32, 33, 45, 61]. Another very influential paper was [42] which studies convergence to equilibrium for the kinetic Fokker-Planck equation with a confining potential in L^2 norm. This paper shows what we would now call hypocoercivity as well as hypoellipticity for the kinetic Fokker-Planck equation with a confining potential. This is a first example of what we will call L^2-hypocoercivity which is theorems which show hypocoercivity directly in weighted L^2 distances. This paper was one of the influences for Villani's seminal mémoire Hypocoercivity, [62]. In this work Villani named and formalised the study of hypocoercivity. He proved a more general result for hypoelliptic type operator in both H^1 and entropy distance, as well as reformulating the work in [102] and discussing this in the context of his earlier work with Desvillettes [22, 23]. In this section we review in more detail the proofs of hypocoercivity in H^1, and in L^2.

1.3.1 H^1 Hypocoercivity

First we review equations in what Villani calls Hörmander sum of squares form. That is

$$\partial_t f + \sum_i A^* A f + Bf = 0,$$

where A^* is the conjugate of A in $L^2(\mu)$ for some probability measure μ and $B = -B^*$. A classical example of such an equation in this sum of squares form is the kinetic Fokker-Planck equation.

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + vf),$$

Showing convergence to equilibrium for the kinetic Fokker-Planck equation is the main goal of the methods based on Hörmander sum of squares form. The abstract form introduced in [62] allows us to see the importance of the commutator brackets.

Theorem 1.3.15. *Consider a linear operator $L = A^*A + B$ (B antisymmetric), let $\mathcal{K} = ker L$ and define $C := [A, B]$. Assume the existence of constants α, β such that*

(1) A and A^* commute with C; A commutes with A (i.e. each A_i commutes with each A_j).

(2) $[A, A^*]$ is α-bounded relatively to I and A, i.e

$$\forall h \in H^1/\mathcal{K}, \quad \|[A, A^*]h\| \leq \alpha(\|Ah\|^2 + \|h\|^2).$$

(3) $[B, C]$ is β-bounded relatively to A, A^2, C and AC, i.e

$$\forall h \in H^1/\mathcal{K}, \quad \|[B, C]h\| \leq \beta(\|Ah\| + \|A^2 h\| + \|Ch\| + \|ACh\|).$$

Then there is a scalar product $((\cdot, \cdot))$ on $H^1/\mathcal{K}$, defining a norm equivalent to the H^1 norm, such that

$$\forall h \in H^1/\mathcal{K}, \quad ((h, Lh)) \geq K(\|Ah\|^2 + \|Ch\|^2),$$

for some constant $K > 0$, only depending on α and β. If, in addition,

$$\forall h \in H^1/\mathcal{K}, \quad (\|Ah\|^2 + \|Ch\|^2) \geq \tau(h, h),$$

for some $\tau > 0$, then there is a constant $\lambda > 0$, only depending on α, β and τ, such that

$$\forall h \in H^1/\mathcal{K}, \quad ((h, Lh)) \geq \lambda((h, h)).$$

In particular, L is hypocoercive in $H^1/\mathcal{K}$:

$$\|e^{-tL}\|_{H^1/\mathcal{K} \to H^1/\mathcal{K}} \leq Ce^{-\lambda t},$$

where both λ and C can be estimated explicitly in terms of upper bounds on α and β, and a lower bound on k.

This theorem works well for kinetic Fokker-Planck equation on the whole space. Hypocoercivity is first shown for this equation in [55].

We recall the kinetic Fokker-Planck equation in the whole space

$$\partial_t f = -v \cdot \nabla_x f + \nabla_x U(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}(vf), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,$$

with the equilibrium is given by

$$f_\infty = \frac{1}{Z} e^{-(U(x)+\frac{1}{2}|v|^2)},$$

by a change of variable $h = f/f_\infty$, we have

$$\partial_t h = -v \cdot \nabla_x h + \nabla U(x) \cdot \nabla_v h + \Delta_v h - v \cdot \nabla_v h,$$

Denote

$$\mu(dxdv) = f_\infty dxdv$$

Here we can write the equation in the form

$$\mathcal{L} = A^* A + B.$$

In this case $A = \nabla_v$, $B = v \cdot \nabla_x - \nabla U(x) \cdot \nabla_v$ and define $\mathcal{H}^1 = H^1(\mu)$

$$\|h\|_{\mathcal{H}^1} = \int_{\mathbb{R}^n \times \mathbb{R}^n} (|\nabla_x h(x, v)|^2 + |\nabla_v h(x, v)|^2) \mu(dxdv),$$

by direct computation

$$[A, A^*] = I, \quad C := [A, B] = \nabla_x, \quad [A, C] = [A^*, C] = 0,$$

and

$$[B, C] = \nabla^2 U(x) \cdot \nabla_v.$$

We then have

Theorem 1.3.16. *Let L be the operator for the kinetic Fokker-Planck equation, U be a C^2 potential in $\mathbb{R}^n$, satisfying*

$$|\nabla^2 U| \leq c(1 + |\nabla U|),$$

for some constant $c > 0$ and there exist $k > 0$ such that

$$\int |\nabla_x h(x)|^2 e^{-U(x)} dx \geq k[\int h^2 e^{-U} - (\int h e^{-U})^2],$$

for all h good enough. Then there are constants $C \geq 0$ and $\lambda > 0$, explicitly computable, such that for all $h_0 \in H^1(\mu)$,

$$\|e^{-tL} h_0 - \int h_0 d\mu\|_{H^1(\mu)} \leq C e^{-\lambda t} \|h_0\|_{H^1(\mu)}.$$

1.3.2 L^2 Hypocoercivity

L^2 hypocoercivity was developed in [38] to show hypocoercivity for the linear relaxation Boltzmann equation. It was then generalized in [19] to give a strategy for showing hypocoercivity for a range of kinetic equations with one conservation law. We briefly describe the results of [19]. Here we write an abstract kinetic equation

$$\partial_t f + Tf = Lf,$$

The idea of this theorem is that hypocoercivity can be seen as the combination of two effects.

- Microscopic coercivity which is coercivity on the kinetic level. There exist $\lambda_m > 0$ such that

$$-\langle Lf, f\rangle_{L^2(\mu^{-1})} \geq \lambda_m \|(I-\Pi)f\|_{L^2(\mu^{-1})},$$

  i.e. the equation pushes the solution toward the set of local equilibria.

- Macroscopic coercivity which is coercivity on the level of the hydrodynamic limit equation. This is seen through coercivity of the operator T on the set of local equilibria, there exists $M > 0$ such that

$$\|T\Pi f\|_{L^2(\mu^{-1})} \geq \lambda_M \|\Pi f\|_{L^2(\mu^{-1})}$$

Theorem 1.3.17. *(Dolbeault-Mouhot-Schmeiser '15). Suppose that T, L satisfy the microscopic and macroscopic coercivity assumptions. Suppose further that $\Pi T \Pi = 0$ and various auxiliary operators are bounded. Then, there exists constants $C, \lambda > 0$ such that*

$$\|e^{t(L-T)} f\|_{L^2(\mu^{-1})} \leq Ce^{-\lambda t}\|f\|_{L^2(\mu^{-1})}.$$

Like in H^1 hypocoercivity the proof proceeds by showing an entropy-entropy production inequality for a functional which is equivalent to our desired distance. In this case the functional has a very different form. The proof of hypocoercivity in L^2 then begins by constructing the new norm to the space

$$H(f) = \frac{1}{2}\|f\|^2_{L^2(\mu^{-1})} + \epsilon\langle Af, f\rangle_{L^2(\mu^{-1})}$$

where

$$A = (1 + (T\Pi)^* T\Pi)^{-1}(T\Pi)^*.$$

In [38, 42] the new entropy constructed has a similar form. The main disadvantage of this approach is that it can currently only deal with equations with one conservation law. In general H_1 hypocoercivity methods do not work for equations with a confinement potential which are also not a diffusion. It can also be extended to work with equations where the equilibrium measure is not explicit and so no Poincaré inequality is known as in [4].

1.4 List of works

In this section we list some of the works we will present in the following chapters.

1.4.1 The Kinetic Fokker-Planck equation with weak confinement

This work is accepted by *Communication in Mathematical Sciences*. In this work, we consider the weak hypocoercivity issue for a solution f to the kinetic Fokker-Planck equation

$$\partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(vf), \tag{1.4.7}$$

on a function $f = f(t, x, v)$, with $t \geq 0$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$. The evolution equation (5.4.44) is complemented with an initial datum

$$f(0, \cdot) = f_0 \text{ on } \mathbb{R}^{2d},$$

and we make the assumption on the confinement potential V

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1),$$

where $\langle x \rangle^2 := 1 + |x|^2$. Let us make some elementary but fundamental observations. First, the equation is mass conservative, that is

$$\mathcal{M}(f(t, \cdot)) = \mathcal{M}(f_0), \quad \forall t \geq 0,$$

where we define the mass of f by

$$\mathcal{M}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv.$$

Next, we observe that the function

$$G = Z^{-1} e^{-W}, \quad W = \frac{|v|^2}{2} + V(x), \quad Z \in \mathbb{R}_+ \tag{1.4.8}$$

is a positive normalized steady state of the KFP model, precisely

$$\mathcal{L}G = 0, \quad G > 0, \quad \mathcal{M}(G) = 1,$$

by choosing the normalizing constant $Z > 0$ appropriately. Finally we observe that, contrary to the case $\gamma \geq 1$, a Poincaré inequality of the type

$$\exists c > 0, \quad \int_{\mathbb{R}^d} |\phi(x)|^2 \exp(-V(x)) dx \leq c \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 \exp(-V(x)) dx,$$

for any smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^d} \phi(x) \exp(-V(x)) dx = 0,$$

does not hold. Only a weaker version of this inequality remains true (see [43, 60]). In particular, there is no spectral gap for the associated operator $\mathcal{L}$, nor is there an exponential trend to the equilibrium for the associated semigroup.

For a given weight function m, we will denote $L^p(m) = \{f | fm \in L^p\}$ the associated Lebesgue space and $\|f\|_{L^p(m)} = \|fm\|_{L^p}$ the associated norm. The notation $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$.

The main result is writes as follows.

Theorem 1.4.18. *(1) For any initial datum $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1, \infty)$, $\epsilon > 0$ small, the associated solution $f(t, \cdot)$ to the kinetic Fokker-Planck equation (2.4.4) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b} \|f_0 - \mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})},$$

for any $b \in (0, \frac{\gamma}{2-\gamma})$ and some constant $C > 0$.
(2) For any initial datum $f_0 \in L^1(m)$, $m = H^k$, $H = |x|^2 + |v|^2 + 1$, $k \geq 1$, the associated solution $f(t, \cdot)$ to the kinetic Fokker-Planck equation (2.4.4) satisfies

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1} \lesssim (1+t)^{-\frac{k}{1-\frac{\gamma}{2}}} \|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)}.$$

The constants in the estimates only depend on $\gamma, d, \epsilon, p, k$.

Remark 1.4.19. Let us emphasize the lost of tail control in both estimate in Theorem 2.4.39, which is reminiscent of decay estimates in sub-geometric contexts.

Remark 1.4.20. In the results above the constants can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 1.4.21. Theorem 2.4.39 is also true when $V(x)$ behaves like $\langle x \rangle^\gamma$, that is for any $V(x)$ satisfying

$$C_1 \langle x \rangle^\gamma \leq V(x) \leq C_2 \langle x \rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3 |x| \langle x \rangle^{\gamma-1} \leq x \cdot \nabla_x V(x) \leq C_4 |x| \langle x \rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

with B_R is denote the ball centered at origin with radius R and $B_R^c = \mathbb{R}^d \setminus B_R$, and

$$|D_x^2 V(x)| \leq C_5 \langle x \rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

for some constants $C_i > 0$, $R > 0$.

We carry out the proof by a combination of the L^2 hypocoercivity method introduced in Section 1.3, an interpolation between weight spaces which is common in sub-geometric proofs and finally an hypoelliptic regularization. One advantage of the method in this

paper is that it can yield convergence on a wider range of initial conditions and L^p space, while previous proofs of convergence to equilibrium mainly using some strong L^1 norms (probability method) or L^2 norms (PDE methods). Also the method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative type argument such as the consequence of Krein-Rutman theorem. While our method also have some disadvantage, it requires the equation has an explicit steady state.

Perspectives

A natural next step would be try to extend the results to other spaces such as Wasserstein distance. And another next step might be to investigate weak hypocoercivity for other commonly studied kinetic equations. The work in Chapters 4 shows some results for some similar models with different methods. It would be interesting to see how this method will work for other kinetic models.

1.4.2 Kinetic Fokker-Planck type equation with general confinement

In this work, we consider the kinetic Fokker-Planck (KFP for short) equation with general force

$$\partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(\nabla_v W(v) f), \tag{1.4.9}$$

for a density function $f = f(t, x, v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$, with

$$V(x) = \frac{\langle x \rangle^\gamma}{\gamma}, \quad \gamma \geq 1, \quad W(v) = \frac{\langle v \rangle^\beta}{\beta}, \quad \beta \geq 2$$

where $\langle x \rangle^2 := 1 + |x|^2$, and the kinetic Fitzhugh-Nagumo equation

$$\partial_t f := \mathcal{L}f = \partial_x(A(x, v)f) + \partial_v(B(x, v)f) + \partial^2_{vv} f \tag{1.4.10}$$

with

$$A(x, v) = ax - bv, \quad B(x, v) = v(v - 1)(v - \lambda) + x$$

for some $a, b, \lambda > 0$. The evolution equations are complemented with an initial datum

$$f(0, x, v) = f_0(x, v) \quad \text{on } \mathbb{R}^{2d}.$$

It's easily seen that both equations are mass conservative, that is

$$\mathcal{M}(f(t, \cdot)) = \mathcal{M}(f_0),$$

And we can introduce the main result.

Theorem 1.4.22. *(1) When $2 \leq \beta, 1 \leq \gamma$, there exist a weight function $m > 0$ and a nonnegative normalized steady state $G \in L^1(m)$ such that for any initial datum $f_0 \in L^1(m)$, the associated solution $f(t,\cdot)$ of the kinetic Fokker-Planck equation (1.4.9) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \leq Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

for some constant $C, \lambda > 0$.
(2) The same conclusion holds for the kinetic Fitzhugh-Nagumo equation (1.4.10).

In the results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 1.4.23. Similarly as before, Theorem 1.4.22 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$ and $W(v)$ behaves like $\langle v\rangle^\beta$, that is for any $V(x)$ satisfying

$$C_1\langle x\rangle^\gamma \leq V(x) \leq C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \leq x\cdot\nabla_x V(x) \leq C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^n V(x)| \leq C_n\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d, \quad \forall n \geq 2,$$

for some constant $C_i > 0$, $R > 0$, and similar estimates holds for $W(v)$.

For the kinetic Fitzhugh-Nagumo equation (1.4.10), an exponential convergence with non-quantitative rate to the convergence has already been proved in [53], our method improves the result to a quantitative rate.

We carry out all of our proofs using variations of Harris's Theorem for Markov semigroup introduced in Section 1.2. One advantage of the Harris method is that it directly yields convergence for a wide range of initial conditions, while previous proofs of convergence to equilibrium mainly use some strongly weighted L^2 or H^1 norms (typically with a weight which is the inverse of a Gaussian). The Harris method also gives existence of stationary solutions under general conditions; in some cases these are explicit and easy to find, but in other cases such as the two models above they can be nontrivial. Also the Harris method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative type argument such as the consequence of Krein Rutman theorem.

Perspective

The most natural next step from this proof would be to try to extend it to subgeometric cases. This would be more challenging since the proof uses Nash's inequality, which does

not allow us to interpolate between weights, which is common in proof for subgeometric cases.

Another direction would be to extend this method to other kernels. For example, try to use such methods work for a fractional Laplacian operator.

Also a good direction would be to think about what would happen when at least one of γ and β is smaller.In a weak case there maybe no exponential convergence. When all γ and β are weak, the equation would probably behave like a heat equation.

1.4.3 Hypocoercivity via Harris's theorem for kinetic equations with jumps

This work is done in collaboration with José Cañizo, Josephine Evans and Havva Yoldaş. The work is accepted by *Kinetic and Related Models*.

Harris's theorem has been used to show convergence to equilibrium for kinetic equations. In [50] they show convergence for kinetic Fokker-Planck equations using Harris's theorem. They do not verify the minorisation condition with a quantitative method. Therefore the end result is not quantitative. In [5] they show convergence to equilibrium using Harris's theorem for various scattering equations that includes equations similar to the ones studied in these chapters. Again it gives rates which are not quantitative. In [16] the authors use Doeblin's theorem to prove quantitative rates of convergence for some non-linear kinetic equations on the torus with a non equilibrium steady state.

In this work we study 4 different kinetic equations with jumps. The linear relaxation Boltzmann equation and the linear Boltzmann equation both with a confining potential and on the torus. We show convergence to equilibrium in a weighted total variation distance with quantitative rates. The weighting is comparable to $U(x) + |x|^2 + |v|^2$ where $U(x)$ is the confining potential. A similar method using Harris's theorem to get quantitative rates for jump equation has been used in [14, 34] to show convergence to equilibrium for equations modelling biological processes. Now we briefly introduce the notations and the results.

We recall the *linear relaxation Boltzmann equation* is given by

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}^+ f - f, \tag{1.4.11}$$

where

$$\mathcal{L}^+ f = \left(\int f(t,x,u) \mathrm{d}u \right) M(v),$$

Φ is a $\mathcal{C}^2$ potential and $\mathbb{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$ as before, similarly the linear Boltzmann equation is given by

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = Q(f, \mathcal{M}), \tag{1.4.12}$$

where Q is the Boltzmann operator

$$Q(f,g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, \sigma) \left(f(v') g(v'_*) - f(v) g(v_*) \right) \mathrm{d}\sigma \, \mathrm{d}v_*,$$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

and B is the *collision kernel*. We assume that B is a hard kernel and can be written as a product

$$B(|v-v_*|,\sigma) = |v-v_*|^\gamma \; b\left(\sigma \cdot \frac{v-v_*}{|v-v_*|}\right), \tag{1.4.13}$$

for some $\gamma \geq 0$ and b integrable and uniformly positive on $[-1,1]$; that is, there exists $C_b > 0$ such that

$$b(z) \geq C_b \qquad \text{for all } z \in [-1,1]. \tag{1.4.14}$$

As before, alternatively we consider the same equation posed for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, without any potential Φ:

$$\partial_t f + v \cdot \nabla_x f = Q(f, \mathcal{M}).$$

The precise results are the following.

Theorem 1.4.24 (Exponential convergence results on the torus)**.** *Suppose that $t \mapsto f_t$ is the solution to* (1.4.9) *or* (1.4.12) *with initial data $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. In the case of equation* (1.4.12) *we also assume* (1.4.13) *with $\gamma \geq 0$ and* (1.4.14)*. Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that*

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (that is, $\mu(x,v) = \mathbb{M}(v)$). The norm $\|\cdot\|_*$ is just the total variation norm $\|\cdot\|_{\mathrm{TV}}$ for equation (1.4.11)*,*

$$\|f_0 - \mu\|_* = \|f_0 - \mu\|_{\mathrm{TV}} := \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |f_0 - \mu| \,\mathrm{d}x \,\mathrm{d}v \qquad \textit{for equation } (1.4.11),$$

and it is a weigthed total variation norm in the case of equation (1.4.12)*:*

$$\|f_0 - \mu\|_* = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} (1+|v|^2)|f_0 - \mu| \,\mathrm{d}x \,\mathrm{d}v \qquad \textit{for equation } (1.4.12).$$

Theorem 1.4.25 (Exponential convergence results with a confining potential)**.** *Suppose that $t \mapsto f_t$ is the solution to* (1.4.11) *or* (1.4.12) *with initial data $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and a potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ which is bounded below, and satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d,$$

for some positive constants γ_1, γ_2, A. Define $\langle x \rangle = \sqrt{1+|x|^2}$. In the case of equation (1.4.12) *we also assume* (1.4.13)*,* (1.4.14) *and*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\mathrm{d}\mu = \mathcal{M}(v)e^{-\Phi(x)}\mathrm{d}v\mathrm{d}x.$$

The norm $\|\cdot\|_*$ is a weighted total variation norm defined by

$$\|f_t - \mu\|_* := \int \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) |f_t - \mu| \,\mathrm{d}v \,\mathrm{d}x.$$

Theorem 1.4.26 (Subgeometric convergence results with weak confining potentials)**.** *Suppose that $t \mapsto f_t$ is the solution to* (1.4.11) *in the whole space with a confining potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$. Define $\langle x\rangle = \sqrt{1+|x|^2}$. Assume that for some β in $(0,1)$ the confining potential satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x\rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then we have that there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C\int f_0(x,v)\left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right)(1+t)^{-\beta/(1-\beta)}\right\}.$$

Similarly if $t \mapsto f_t$ is the solution to (1.4.12) *in the whole space, satisfies* (1.4.13)*,* (1.4.14) *and*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x\rangle^{\beta+1} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \leq \gamma_3 \langle x\rangle^{1+\beta},$$

for some positive constants $\gamma_1, \gamma_2, A, \beta, \gamma_3$. Then we have that there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C\int f_0(x,v)\left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|\right)(1+t)^{-\beta}\right\}.$$

In fact, our methods works for any kinetic equation with jump as

$$\partial_t f = -v \cdot \nabla_x f + \nabla U(x) \cdot \nabla_v f = \int_{\mathbb{R}^d} K(v,u) f(x,v) dv - f,$$

and true when $\mathbb{R}^d$ is replaced by some ball contains the origin. For example this method works for the "run-and-tumble" equation discussed in [51].

$$\partial_t f = \mathcal{L} f = -v \cdot \nabla_x f + \int_{\mathcal{V}} K' f' - K f dv',$$

with

$$K = 1 + \frac{1}{2}\mathrm{sign}(x \cdot v), \quad \mathcal{V} = B(0,R)$$

and $K = K(x,v,v') \geq 0$. Here we used the shorthands $f' = f(t,x,v')$ and $K' = K(x,v',v)$.

Perspectives

This method works very well for kinetic equations with jumps. It seems hopeful that it could be applied to many more models. It seems likely that these ideas could be applied to relaxation equations with spatially inhomogeneous jump rates. For example it is an open problem to show quantitative rates of convergence to equilibrium for the equations

$$\partial_t f = -v \cdot \nabla_x f + \nabla U(x) \cdot \nabla_v f + \sigma(x)(\Pi f - f),$$

where $\sigma(x)$ is a function which vanishes at more than isolated points. It is known in [2, 3] that this equation will converge exponentially fast to the equilibrium in L^1 provided $\sigma(x)$ satisfies the geometric control condition but there is no quantitative rate.

Another possible application would be to look at the linear Boltzmann equation with soft potentials. In this chapter we look at the hard potential case and with cut off. There is a version of Harris's theorem which will give sub geometric rates of convergence given a weaker Lyapunov condition. It seems likely that we could apply this to the linear Boltzmann equation with cut off and soft potentials to establish rates of convergence.

Also another direction would be to discuss if the results would work for other spaces than L^1 (Harris method) or L^2 (DMS methods).

Chapter 2

Introduction en Français

Cette thése concerne principalement la convergence vers l'équilibre pour les équations cinétiques. Nous introduisons d'abord la théorie cinétique et l'équation de Boltzmann. Ensuite, nous introduisons une gamme d'équations qui seront discutées tout au long de ce travail. Aprés cela, nous passerons en revue quelques travaux classiques existants sur la convergence vers l'équilibre, en mettant l'accent sur l'hypocoercivité. Enfin, nous décrivons briévement le contenu de chacun des chapitres.

2.1 Équation Cinétique

La théorie cinétique a été développée au 19éme siécle, notamment par Boltzmann et Maxwell, dans la modélisation des gaz dilués. Les équations cinétiques modélisent l'évolution d'un gaz á une échelle intermédiaire entre la description microscopique donnée par les lois de Newton et une description fluide macroscopique. Si nous avons un systéme de N particules effectuant une dynamique déterministe ou stochastique, nous pouvons écrire leur distribution conjointe au temps t

$$F_N(t, z_1, z_2..., z_n) \geq 0.$$

Dans ce cas, z_i est soit v_i, la vitesse de la $i$éme particule, soit $(x_i; v_i)$, la position et la vitesse de la $i$éme particule. Nous observons le cas oú cette équation modélise un grand nombre d'agents impossibles á distinguer, par exemple des particules de gaz. Nous étudions les particules qui interagissent dans un gaz selon les lois de Newton, puis elles suivront les équations suivantes:

$$\begin{aligned} \dot{x_i} &= v_i \\ \dot{v_i} &= \sum_{i \neq j} F(i,j) + F, \end{aligned}$$

oú $F(j,i)$ est la force agissant sur la particule i due á la particule j et F est une force externe. Pour les gaz de collision, l'équation obtenue dans ce processus, via l'échelle de Boltzmann-Grad, est l'équation de Boltzmann

$$\partial_t f + v \cdot \nabla_x f = Q(f,f),$$

oú

$$Q(f,g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v-v_*|, (v-v_*)\cdot\sigma)(f(v')g(v'_*) - f(v)g(v_*))d\sigma dv_*$$

avec

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

B est appelé le noyau de collision. Ici $f = f(t,x,v)$ représente la densité dans l'espace de phase d'une seule particule dans l'ensemble et $t \geq 0, x \in \mathbb{R}^d, v \in \mathbb{R}^d$. Nous pouvons observer ici la structure générale d'un équation cinétique de collision

$$\partial_t f + v \cdot \nabla_x f = L(f),$$

oú l'opérateur $v \cdot \nabla_x f$ provient du terme de transport défini dans la loi de Newton. L'opérateur $L(f)$ n'agit que sur la variable de vitesse v, en mettant

$$\rho(x) = \int f(x,v)dv,$$

la densité locale

$$u(x) = \frac{1}{\rho(x)} \int v f(x,v)dv,$$

la vitesse locale et

$$T(x) = \frac{1}{\rho(x)} \int |u-u(x)|^2 f(x,v)dv,$$

la température locale. La solution á l'équilibre de l'équation de Boltzmann a été dérivée par Maxwell

$$M(v) = \rho(2\pi T)^{-d/2} \exp(-\frac{1}{2T}|v|^2)$$

Dans la suite de cette thése, nous utilisons la notation $M(v)$ pour désigner la normalisée Maxwellien avec $u = 0, \rho = 1$ et $T = 1$. C'est

$$M(v) = (2\pi)^{-d/2} \exp(-\frac{|v|^2}{2}).$$

Nous pouvons également regarder les équations oú les potentiels sont confiné.

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = Q(f,f).$$

oú $U(x)$ représente une force de confinement externe. Dans ce qui suit, nous introduisons quelques équations qui sont étudiées dans la thése.

2.1.1 Équation de Fokker-Planck Cinétique

L'équation de Fokker-Planck cinétique est l'une des équations les plus simples de la théorie cinétique. C'est une version cinétique de l'équation de Fokker-Planck développée par Fokker [28] et Planck [59]

$$\partial_t f = \nabla_v \cdot (\nabla_v f + v f),$$

L'équation de Fokker-Planck cinétique est

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + v f),$$

sur la densité $f = f(t, x, v), t \geq 0, v \in \mathbb{R}^d, x \in \mathbb{T}^d$ ou $x \in \mathbb{R}^d$. L'état d'équilibre associé est

$$e^{-U(x)} M(v) = \exp(-(U(x) + \frac{|v|^2}{2})),$$

Nous examinons l'équation de Fokker-Planck cinétique en Chapitre 3 et 4.

2.1.2 Équation de relaxation linéaire

L'équation de relaxation linéaire est l'exemple le plus simple d'équation de diffusion de théorie cinétique. Elle est également connue sous le nom d'équation de BGK linéaire

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \Pi_M f - f,$$

sur la densité $f = f(t, x, v), t \geq 0, v \in \mathbb{R}^d, x \in \mathbb{T}^d$ ou $x \in \mathbb{R}^d$, avec

$$\Pi_M f = (\int_{\mathbb{R}^d} f(x, u) du) M(v),$$

l'espace de la phase est soit $\mathbb{R}^d \times \mathbb{R}^d$ ou $\mathbb{T}^d \times \mathbb{R}^d$. L'état d'équilibre est

$$e^{-U(x)} M(v) = \exp(-(U(x) + \frac{|v|^2}{2})),$$

Nous examinons l'équation de relaxation linéaire en Chapitre 5.

2.1.3 Équation de Boltzmann linéaire

L'équation de Boltzmann linéaire est

$$\partial_t f + v \cdot \nabla_x f - \nabla U(x) \cdot \nabla_v f = \mathcal{Q}(f, M),$$

oú $\mathcal{Q}$ est l'opérateur de collision de Boltzmann donné par

$$Q(f,g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, (v - v_*) \cdot \sigma)(f(v')g(v'_*) - f(v)g(v_*))d\sigma dv_*$$

et M est défini avant. Cette équation est beaucoup plus simple que l'équation de Boltzmann.

Nous examinons l'équation de Boltzmann linéaire en Chapitre 5.

2.2 Théoréme de Harris

Le théoréme de Harris [38, 41, 52] est un résultat de la théorie des processus de Markov.

Nous commençons par regarder le théoréme de Doeblin. Le théoréme de Harris est un générateur naturel du théoréme de Doeblin. Les théorémes de Harris et de Doeblin sont généralement énoncés pour un temps fixe t_*. Dans nos théorémes, nous travaillons pour choisir un t_* approprié.

Hypothesis 2.2.27 (Doeblin)**.** *Si $(\mathcal{P}_t)_{t\geq 0}$ est un semi-groupe stochastique, et il existe $t_* > 0$, une distribution de probabilité ν et $\alpha \in (0,1)$ tel que, pour tout z dans l'espace d'état, nous avons*

$$\mathcal{P}_{t_*}\delta_z \geq \alpha\nu.$$

Theorem 2.2.28 (Doeblin)**.** *Si $(\mathcal{P}_t)_{t\geq 0}$ est un semi-groupe stochastique satisfaisant l'hypothése avant, pour deux mesures de probabilité μ_1 et μ_2 et pour tout entier $n \geq 0$*

$$\|\mathcal{P}^n_{t_*}\mu_1 - \mathcal{P}^n_{t_*}\mu_2\|_{\mathrm{TV}} \leq (1-\alpha)^n \|\mu_1 - \mu_2\|_{\mathrm{TV}}. \tag{2.2.1}$$

En conséquence, le semigroupe a un équilibre unique μ_*, et pour tous les μ

$$\|\mathcal{P}_t(\mu - \mu_*)\|_{\mathrm{TV}} \leq \frac{1}{1-\alpha} e^{-\lambda t} \|\mu - \mu_*\|_{\mathrm{TV}}, \qquad t \geq 0, \tag{2.2.2}$$

oú

$$\lambda := \frac{\log(1-\alpha)}{t_*} > 0.$$

Le théoréme de Harris étend le théoréme de Doeblin.

Hypothesis 2.2.29 (Lyapunov)**.** *Il existe une fonction* $V : \Omega \to [0, \infty)$ *et constantes* $D \geq 0, \alpha \in (0, 1)$ *tel que*

$$(\mathcal{P}_{t_*} V)(z) \leq \alpha V(z) + D.$$

Remark 2.2.30. Dans notre situation oú nous avons une équation sur la loi $f(t)$, cela équivaut á la déclaration

$$\int_S f(t, z)V(z)\mathrm{d}z \leq \alpha \int_S f(0, z)V(z)\mathrm{d}z + D.$$

Nous vérifions cela par

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S f(t, z)V(z)\mathrm{d}z \leq -\lambda \int_S f(t, z)V(z)\mathrm{d}z + K,$$

pour certaines constantes K and λ, ce qui implique la condition pour $\alpha = e^{-\lambda t}$ et $D = \frac{K}{\lambda}(1 - e^{-\lambda t}) \leq Kt$.

Hypothesis 2.2.31. *(Harris) Il existe une mesure de probabilité* ν *et une constante* $\beta \in (0, 1)$ *tel que*

$$\inf_{z \in \mathcal{C}} \mathcal{P}_{t_*} \delta_z \geq \beta \nu,$$

oú

$$\mathcal{C} = \{z | V(z) \leq R\}$$

pour une $R > 2D/(1 - \alpha)$.

Dénoter

$$\rho_a(\mu_1, \mu_2) = \int (1 + aV(x, v))|\mu_1 - \mu_2|(\mathrm{d}x\mathrm{d}v).$$

Theorem 2.2.32 (Harris)**.** *Si les deux hypothéses sont vérifiées, il existe* $\bar{\alpha} \in (0, 1)$ *et* $a > 0$ *tels que*

$$\rho_a(\mathcal{P}_{t_*}\mu_1, \mathcal{P}_{t_*}\mu_2) \leq \bar{\alpha}\rho_a(\mu_1, \mu_2). \tag{2.2.3}$$

Il existe des versions du théoréme de Harris adaptées aux conditions de Lyapunov plus faibles.

Theorem 2.2.33. *Il existe une fonction V tel que*

$$\mathcal{L}V \leq K - \phi(V),$$

pour une constantes K and une fonction strictement concave $\phi : \mathbb{R}_+ \to \mathbb{R}$ avec $\phi(0) = 0$ and $\phi(\infty) = \infty$ et Il existe une mesure de probabilité ν et une constante $\beta \in (0, 1)$ tel que

$$\inf_{z \in \mathcal{C}} \mathcal{P}_{t_*} \delta_z \geq \beta \nu,$$

oú

$$\mathcal{C} = \{z | V(z) \leq R\}$$

. Nous avons

- *Il existe une mesure invariante μ et*

$$\int \phi(V(z)) \mathrm{d}\mu \leq K.$$

- *H_ϕ est*

$$H_\phi = \int_1^u \frac{\mathrm{d}s}{\phi(s)}.$$

. Ensuite, il existe une constante C telle que

$$\|\mathcal{P}_t \nu - \mu\|_{\mathrm{TV}} \leq \frac{C\nu(V)}{H_\phi^{-1}(t)} + \frac{C}{(\phi \circ H_\phi^{-1})(t)}$$

pour toute mesure de probabilité ν.

2.3 Hypocoercivité

Le nom hypocoercivité a été utilisé pour la premiére fois par C. Villani dans sa mémoire em Hypocoercivity [62]. Commençons par donner un définition de l'hypocoercivité.

Definition 2.3.34. Si L est un opérateur linéaire, et que $f(t)$ soit la solution á l'équation

$$\partial_t f + Lf = 0, \quad f(0) = f_0,$$

au temps t, s'il existe des constantes C, λ, telles que pour toutes les données initiales f_0 nous avons

$$\|f(t)\| \leq Ce^{-\lambda t} \|f_0\|,$$

on dit alors que l'équation est hypocoercive dans la norme $\|\cdot\|$.

Remark 2.3.35. L'opérateur est appelé coercitif si cette inégalité est vraie avec $C = 1$.

Cette inégalité sur le semi-groupe est équivalente au générateur L ayant un intervalle spectral dans la norme $\|\cdot\|$.

2.3.1 Hypocoercivité dans H^1

On considére les équations du type

$$\partial_t f + \sum_i A^* A f + Bf = 0,$$

oú A^* est le conjugué de A dans $L^2(\mu)$ pour une mesure de probabilité μ et $B = -B^*$. Un exemple classique est l'équation de Fokker-Planck cinétique

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + vf),$$

Theorem 2.3.36. *Considérons un opérateur linéaire $L = A^*A + B$ ($B^* = -B$), définir $\mathcal{K} = ker L$, $C := [A, B]$. Supposer l'existence de constantes α, β tel que*

(1) Pour A et A^*

$$[A, C] = [A^*, C] = [A, A^*] = 0,$$

(2) Pour $[A, A^*]$

$$\forall h \in H^1/\mathcal{K}, \quad \|[A, A^*]h\| \leq \alpha(\|Ah\|^2 + \|h\|^2).$$

(3) Pour $[B, C]$

$$\forall h \in H^1/\mathcal{K}, \quad \|[B, C]h\| \leq \beta(\|Ah\| + \|A^2 h\| + \|Ch\| + \|ACh\|).$$

Ensuite, il existe un produit scalaire$((\cdot, \cdot))$ sur $H^1/\mathcal{K}$, équivalent á H^1 norm, tel que

$$\forall h \in H^1/\mathcal{K}, \quad ((h, Lh)) \geq K(\|Ah\|^2 + \|Ch\|^2),$$

pour une constante $K > 0$, seulement dépendant de α et β. Si, en plus

$$\forall h \in H^1/\mathcal{K}, \quad (\|Ah\|^2 + \|Ch\|^2) \geq \tau(h, h),$$

pour une $\tau > 0$, il existe une constante $\lambda > 0$, seulement dépendant de α, β et τ, tel que

$$\forall h \in H^1/\mathcal{K}, \quad ((h, Lh)) \geq \lambda((h, h)).$$

En particulier, L est hypocoercif dans $H^1/\mathcal{K}$:

$$\|e^{-tL}\|_{H^1/\mathcal{K} \to H^1/\mathcal{K}} \leq Ce^{-\lambda t}.$$

Ce théoréme fonctionne bien pour l'équation de Fokker-Planck cinétique. Nous rappelons l'équation de Fokker-Planck cinétique

$$\partial_t f = -v \cdot \nabla_x f + \nabla_x U(x) \cdot \nabla_v f + \Delta_v f + \operatorname{div}(vf), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,$$

avec l'équilibre est

$$f_\infty = \frac{1}{Z} e^{-(U(x) + \frac{1}{2}|v|^2)},$$

dénoter $h = f/f_\infty$, nous avons

$$\partial_t h = -v \cdot \nabla_x h + \nabla U(x) \cdot \nabla_v h + \Delta_v h - v \cdot \nabla_v h,$$

dénoter

$$\mu(dxdv) = f_\infty dxdv$$

nous avons

$$\mathcal{L} = A^* A + B.$$

avec

$$A = \nabla_v, B = v \cdot \nabla_x - \nabla U(x) \cdot \nabla_v$$

et $\mathcal{H}^1 = H^1(\mu)$

$$\|h\|_{\mathcal{H}^1} = \int_{\mathbb{R}^n \times \mathbb{R}^n} (|\nabla_x h(x,v)|^2 + |\nabla_v h(x,v)|^2) \mu(dxdv),$$

nous avons

$$[A, A^*] = I, \quad C := [A, B] = \nabla_x, \quad [A, C] = [A^*, C] = 0,$$

et

$$[B, C] = \nabla^2 U(x) \cdot \nabla_v.$$

Nous avons

Theorem 2.3.37. *Soit L l'opérateur de l'équation de Fokker-Planck cinétique, U est un potentiel C^2 dans $\mathbb{R}^n$, et*

$$|\nabla^2 U| \leq c(1 + |\nabla U|),$$

pour une constante $c > 0$ et il existe $k > 0$ tel que

$$\int |\nabla_x h(x)|^2 e^{-U(x)} dx \geq k[\int h^2 e^{-U} - (\int h e^{-U})^2],$$

pour tout h. Ensuite, il existe des constantes $C \geq 0$ et $\lambda > 0$, telles que pour tout $h_0 \in H^1(\mu)$

$$\|e^{-tL} h_0 - \int h_0 d\mu\|_{H^1(\mu)} \leq C e^{-\lambda t} \|h_0\|_{H^1(\mu)}.$$

2.3.2 Hypocoercivité dans L^2

L'hypocoercivité en L^2 a été développée dans [38] pour montrer une hypocoercivité pour la équation de relaxation linéaire. Il a ensuite été généralisé dans [19] pour donner une stratégie de démonstration de l'hypocoercivité pour une gamme d'équations cinétiques avec une loi de conservation. Pour

$$\partial_t f + Tf = Lf,$$

Theorem 2.3.38. *(Dolbeault-Mouhot-Schmeiser '15). Si nous avons*

- *Il existe $\lambda_m > 0$ tel que*

$$-\langle Lf, f\rangle_{L^2(\mu^{-1})} \geq \lambda_m \|(I-\Pi)f\|_{L^2(\mu^{-1})},$$

- *Il existe $M > 0$ tel que*

$$\|T\Pi f\|_{L^2(\mu^{-1})} \geq \lambda_M \|\Pi f\|_{L^2(\mu^{-1})}$$

Supposons en outre que $\Pi T \Pi = 0$ et que les opérateurs sont borné. Ensuite, il existe des constantes $C, \lambda > 0$ tel que

$$\|e^{t(L-T)} f\|_{L^2(\mu^{-1})} \leq C e^{-\lambda t} \|f\|_{L^2(\mu^{-1})}.$$

La preuve commence par la construction d'une nouvelle norme

$$H(f) = \frac{1}{2}\|f\|^2_{L^2(\mu^{-1})} + \epsilon \langle Af, f\rangle_{L^2(\mu^{-1})}$$

avec

$$A = (1 + (T\Pi)^* T\Pi)^{-1} (T\Pi)^*.$$

2.4 Liste des résultats

Dans cette section, nous énumérons certains des résultats présentés dans les chapitres suivants.

2.4.1 Équation de Fokker-Planck cinétique avec confinement faible

Ce travail est accepté par *Communication in Mathematical Sciences*. Dans ce travail, nous considérons la équation de Fokker-Planck cinétique avec confinement faible

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(vf), \tag{2.4.4}$$

sur une fonction $f = f(t, x, v)$, avec $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$. et

$$f(0, \cdot) = f_0,$$

le potentiel de confinement V est

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1),$$

oú $\langle x \rangle^2 := 1 + |x|^2$. La masse est conservée

$$\mathcal{M}(f(t, \cdot)) = \mathcal{M}(f_0), \quad \forall t \geq 0,$$

oú la masse de f est

$$\mathcal{M}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv.$$

Et nous avons

$$G = Z^{-1} e^{-W}, \quad W = \frac{|v|^2}{2} + V(x), \quad Z \in \mathbb{R}_+ \tag{2.4.5}$$

est un état stable normalisé et positif

$$\mathcal{L}G = 0, \quad G > 0, \quad \mathcal{M}(G) = 1,$$

pour une constante $Z > 0$.

Pour une fonction m, on note $L^p(m) = \{f | fm \in L^p\}$ l'espace de Lebesgue associé et $\|f\|_{L^p(m)} = \|f\|_{L^p}$ la norme associée. La notation $A \lesssim B$ signifie $A \leq CB$ pour une constante $C > 0$.

Le résultat principal est

Theorem 2.4.39. *(1) Pour toute $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1, \infty)$, $\epsilon > 0$ small, la solution $f(t, \cdot)$ á l'équation de Fokker-Planck cinétique est*

$$\|f(t, \cdot) - \mathcal{M}(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b} \|f_0 - \mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})},$$

Pour toute $b \in (0, \frac{\gamma}{2-\gamma})$ pour une constante $C > 0$.
(2) Pour toute $f_0 \in L^1(m)$, $m = H^k$, $H = |x|^2 + |v|^2 + 1$, $k \geq 1$, la solution $f(t, \cdot)$ á l'équation de Fokker-Planck cinétique est

$$\|f(t, \cdot) - \mathcal{M}(f_0)G\|_{L^1} \lesssim (1 + t)^{-\frac{k}{1-\frac{\gamma}{2}}} \|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)}.$$

Remark 2.4.40. Dans les résultats ci-dessus, les constantes peuvent être explicitement estimées en termes de les paramétres apparaissant dans l'équation en suivant les calculs dans les preuves. Nous ne les donnons pas explicitement car nous ne nous attendons pas á ce qu'ils soient optimaux, mais ils sont néanmoins complétement constructifs.

Remark 2.4.41. Le théoréme est également vrai si $V(x)$ satisfait

$$C_1\langle x\rangle^\gamma \le V(x) \le C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \le x\cdot\nabla_x V(x) \le C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

avec B_R, la balle centrée á l'origine avec le rayon R et $B_R^c = \mathbb{R}^d \setminus B_R$, et

$$|D_x^2 V(x)| \le C_5\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

pour une constante $C_i, R > 0$.

2.4.2 Équation de Fokker-Planck cinétique avec force générale

Dans ce travail, nous considérons la équation de Fokker-Planck cinétique avec force générale

$$\partial_t f = \mathcal{L}f := -v\cdot\nabla_x f + \nabla_x V(x)\cdot\nabla_v f + \Delta_v f + \mathrm{div}_v(\nabla_v W(v) f),$$

sur une fonction $f = f(t,x,v)$, avec $t \ge 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$. et

$$f(0,\cdot) = f_0,$$

et avec

$$V(x) = \frac{\langle x\rangle^\gamma}{\gamma}, \quad \gamma \ge 1, \quad W(v) = \frac{\langle v\rangle^\beta}{\beta}, \quad \beta \ge 2,$$

oú $\langle x\rangle^2 := 1 + |x|^2$, et l'équation de Fitzhugh-Nagumo cinétique

$$\partial_t f := \mathcal{L}f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \partial_{vv}^2 f$$

avec

$$A(x,v) = ax - bv, \quad B(x,v) = v(v-1)(v-\lambda) + x$$

pour une constante $a, b, \lambda > 0$. La masse est conservée

$$\mathcal{M}(f(t,\cdot)) = \mathcal{M}(f_0), \quad \forall t \ge 0,$$

Le résultat principal est

Theorem 2.4.42. *(1) Oú $2 \le \beta, 1 \le \gamma$, il existe une fonction $m > 0$ et un état stable normalisé non négatif $G \in L^1(m)$. Pour toute $f_0 \in L^1(m)$, la solution $f(t,\cdot)$ á l'équation de Fokker-Planck cinétique est*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \le Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

pour une constante $C, \lambda > 0$.
(1) Il existe une fonction $m > 0$ *et un état stable normalisé non négatif* $G \in L^1(m)$. *Pour toute* $f_0 \in L^1(m)$, *la solution* $f(t,\cdot)$ *á l'équation de Fitzhugh-Nagumo cinétique est*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \leq Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)}.$$

Remark 2.4.43. Dans les résultats ci-dessus, les constantes peuvent être explicitement estimées en termes de les paramétres apparaissant dans l'équation en suivant les calculs dans les preuves. Nous ne les donnons pas explicitement car nous ne nous attendons pas á ce qu'ils soient optimaux, mais ils sont néanmoins complétement constructifs.

Remark 2.4.44. Le théoréme est également vrai si $V(x)$ satisfait

$$C_1\langle x\rangle^\gamma \leq V(x) \leq C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \leq x\cdot\nabla_x V(x) \leq C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

avec B_R, la balle centrée á l'origine avec le rayon R et $B_R^c = \mathbb{R}^d \setminus B_R$, et

$$|D_x^2 V(x)| \leq C_5\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

pour une constante $C_i, R > 0$, et pareil pour $W(v)$.

2.4.3 Hypocoercivité via le théoréme de Harris pour les équations cinétiques

Ce travail est en collaboration avec José Cañizo, Josephine Evans and Havva Yoldaş. Ce travail est accepté par *Kinetic and Related Models*.

L'équation de relaxation linéaire est

$$\partial_t f + v\cdot\nabla_x f - (\nabla_x\Phi\cdot\nabla_v f) = \mathcal{L}^+ f - f,$$

oú

$$\mathcal{L}^+ f = \left(\int f(t,x,u)\mathrm{d}u\right) M(v),$$

Φ est $\mathcal{C}^2$ and $\mathbb{M}(v) := (2\pi)^{-d/2}\exp(-|v|^2/2)$. L'équation de Boltzmann linéaire est

$$\partial_t f + v\cdot\nabla_x f - (\nabla_x\Phi\cdot\nabla_v f) = Q(f,\mathcal{M}),$$

oú Q est le opérateur Boltzmann

$$Q(f,g) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v-v_*|,\sigma)\left(f(v')g(v'_*) - f(v)g(v_*)\right)\,\mathrm{d}\sigma\,\mathrm{d}v_*,$$

avec

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

Nous supposons que B est

$$B(|v-v_*|,\sigma) = |v-v_*|^\gamma\; b\left(\sigma\cdot\frac{v-v_*}{|v-v_*|}\right),$$

pour une $\gamma \geq 0$ et il existe $C_b > 0$ tel que

$$b(z) \geq C_b \qquad \forall z \in [-1,1].$$

nous considérons aussi pour $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, sans potentiel Φ:

$$\partial_t f + v\cdot\nabla_x f = Q(f,\mathcal{M}).$$

Le résultat principal est

Theorem 2.4.45. *Si $t \mapsto f_t$ est la solution aux deux équations avec les données initiales $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. Ensuite, il existe des constantes $C > 0, \lambda > 0$ tel que*

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*,$$

oú μ est l'état d'équilibre dans $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ ($\mu(x,v) = \mathbb{M}(v)$). La norme $\|\cdot\|_*$ est la norme de variation totale $\|\cdot\|_{\mathrm{TV}}$ pour l'équation de relaxation linéaire

$$\|f_0 - \mu\|_* = \|f_0 - \mu\|_{\mathrm{TV}} := \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |f_0 - \mu|\,\mathrm{d}x\,\mathrm{d}v$$

et c'est une norme de variation totale pondérée pour l'équation de Boltzmann linéaire

$$\|f_0 - \mu\|_* = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} (1+|v|^2)|f_0 - \mu|\,\mathrm{d}x\,\mathrm{d}v$$

Theorem 2.4.46. *Si $t \mapsto f_t$ est la solution aux deux équations avec les données initiales $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ et*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d,$$

pour une constantes γ_1, γ_2, A. Et dans le cas de l'équation de Boltzmann linéaire,

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

pour une constantes γ_1, γ_2, A.Ensuite, il existe des constantes $C > 0, \lambda > 0$ tel que

$$\|f_t - \mu\|_* \leq C e^{-\lambda t} \|f_0 - \mu\|_*,$$

oú μ est l'état d'équilibre dans $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$

$$\mathrm{d}\mu = \mathcal{M}(v) e^{-\Phi(x)} \mathrm{d}v \mathrm{d}x.$$

La norme $\|\cdot\|_*$ est la norme de variation totale pondérée

$$\|f_t - \mu\|_* := \int \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) |f_t - \mu| \, \mathrm{d}v \, \mathrm{d}x.$$

Theorem 2.4.47. *Si $t \mapsto f_t$ est la solution aux l'équation de relaxation linéaire avec les données initiales $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ et*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

pour une constantes $\gamma_1, \gamma_2, A, \beta \in (0,1)$. Ensuite, il existe des constantes $C > 0$ tel que

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) (1+t)^{-\beta/(1-\beta)}\right\}.$$

Si $t \mapsto f_t$ est la solution aux l'équation de Boltzmann linéaire avec les données initiales $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$, $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ et

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\beta+1} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \leq \gamma_3 \langle x \rangle^{1+\beta},$$

pour une constantes $\gamma_1, \gamma_2, A, \beta, \gamma_3$. Ensuite, il existe des constantes $C > 0$ tel que

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|\right) (1+t)^{-\beta}\right\}.$$

En fait, notre résultat est vrai pour toute équation cinétique avec

$$\partial_t f = -v \cdot \nabla_x f + \nabla U(x) \cdot \nabla_v f = \int_{\mathbb{R}^d} K(v,u) f(x,v) dv - f,$$

et vrai si $\mathbb{R}^d$ est remplacé par une balle. Par exemple le équation de "run-and-tumble" dans [51].

$$\partial_t f = \mathcal{L} f = -v \cdot \nabla_x f + \int_{\mathcal{V}} K' f' - K f dv',$$

avec

$$K = 1 + \frac{1}{2}\,\mathrm{sign}(x \cdot v), \quad \mathcal{V} = B(0,R)$$

et $K = K(x,v,v') \geq 0$, avec $f' = f(t,x,v')$ et $K' = K(x,v',v)$.

Chapter 3

The Kinetic Fokker-Planck Equation with Weak Confinement Force

3.1 Introduction

In this chapter, we consider the weak hypocoercivity issue for a solution f to the kinetic Fokker-Planck (KFP for short) equation

$$\partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(vf), \tag{3.1.1}$$

on a function $f = f(t, x, v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$. The evolution equation (3.1.1) is complemented with an initial datum

$$f(0, \cdot) = f_0 \text{ on } \mathbb{R}^{2d}.$$

In all the paper, we make the assumption on the confinement potential V

$$V(x) = \langle x \rangle^\gamma, \quad \gamma \in (0, 1),$$

where $\langle x \rangle^2 := 1 + |x|^2$. we observe that the mass is conservative

$$\mathcal{M}(f(t, \cdot)) = \mathcal{M}(f_0), \quad \forall t \geq 0$$

where we recall the mass of f by

$$\mathcal{M}(f) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv.$$

and that the function

$$G = Z^{-1} e^{-W}, \quad W = \frac{|v|^2}{2} + V(x), \quad Z \in \mathbb{R}_+ \tag{3.1.2}$$

is a positive normalized steady state of 3.1.1.

The main result of this chapter writes as follows.

Theorem 3.1.1. *(1) For any initial datum $f_0 \in L^p(G^{-(\frac{p-1}{p}+\epsilon)})$, $p \in [1,\infty)$, $\epsilon > 0$ small, the associated solution $f(t,\cdot)$ to the kinetic Fokker-Planck equation (3.1.1) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-Ct^b}\|f_0 - \mathcal{M}(f_0)G\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)})},$$

for any $b \in (0, \frac{\gamma}{2-\gamma})$ and some constant $C > 0$.
(2) For any initial datum $f_0 \in L^1(m)$, $m = H^k$, $H = |x|^2+|v|^2+1$, $k \geq 1$, the associated solution $f(t,\cdot)$ to the kinetic Fokker-Planck equation (3.1.1) satisfies

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1} \lesssim (1+t)^{-\frac{k}{1-\frac{\gamma}{2}}}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)}.$$

The constants in the estimates only depend on $\gamma, d, \epsilon, p, k$.

Remark 3.1.2. Let us emphasize the lost of tail control in both estimate in Theorem 3.1.1, which is reminiscent of decay estimates in sub-geometric contexts.

Remark 3.1.3. In the results above the constants can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 3.1.4. Theorem 3.1.1 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$, that is for any $V(x)$ satisfying

$$C_1\langle x\rangle^\gamma \leq V(x) \leq C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \leq x\cdot\nabla_x V(x) \leq C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

with B_R is denote the ball centered at origin with radius R and $B_R^c = \mathbb{R}^d \setminus B_R$, and

$$|D_x^2 V(x)| \leq C_5\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d,$$

for some constants $C_i > 0$, $R > 0$.

Remark 3.1.5. There are many classical results about the strong confinement framework corresponding to $\gamma \geq 1$. In this case there is a spectral gap on the operator $\mathcal{L}$ and exponentially decay estimates on the associated semigroup $S_\mathcal{L}$, we refer the interested readers to [62, 19, 20, 42, 36, 38].

Remark 3.1.6. For the Fokker-Planck equation with weak confinement force, a sub-geometric convergence to equilibrium is established in [43, 60]

Remark 3.1.7. There are already some convergence results for the KFP equation with weak confinement case considered in the present paper proved by probability method. In [21] a polynomial rate of convergence to the equilibrium is established, in total variation distance with some weight norm, and in [9], a sub-geometric rate of convergence in total variation distance with some weight norm. Both papers use Harris theorem introduced in Section 1.2.

One advantage of the method in this paper is that it can yield convergence on a wider range of initial conditions and L^p space, while previous proofs of convergence to equilibrium mainly use some strong L^1 norms (probability method) or L^2 norms (PDE methods). Also the method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative type argument such as the consequence of Krein-Rutman theorem. While our method also have some disadvantage, it requires the equation has an explicit steady state.

Let us briefly explain the main ideas behind our method of proof.

We introduce four spaces $E_1 = L^2(G^{-1/2})$, $E_2 = L^2(G^{-1/2}e^{\epsilon_1 V(x)})$, $E_3 = L^2(G^{-(1+\epsilon_2)/2})$ and $E_0 = L^2(G^{-1/2}\langle x\rangle^{\gamma-1})$, with $\epsilon_1 > 0$ and $\epsilon_2 > 0$ small such that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset L^2$. Thus E_1 is an "interpolation" space between E_0 and E_2. We first use a hypocoercivity argument as in [19, 20] to prove that, for any $f_0 \in E_3$, the solution f to the KFP equation (3.1.1) satisfies

$$\frac{d}{dt}\|f(t)\|_{\tilde{E}_1} \leq -\lambda\|f(t)\|_{E_0},$$

for some constant $\lambda > 0$, where the norm of $\tilde{E}_1$ is equivalent to the norm of E_1. We use this and the Duhamel formula to prove

$$\|f(t)\|_{E_2} \lesssim \|f_0\|_{E_3}.$$

Combining the two inequalities and using a interpolation argument as in [43], we get

$$\|f(t)\|_{E_1} \lesssim e^{-at^b}\|f_0\|_{E_3}, \tag{3.1.3}$$

for some $a > 0, b \in (0, 1)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [56] and developed in [47, 35]. For any operator $\mathcal{L}$, denote $S_{\mathcal{L}}(t)$ the associated semigroup. We introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is an appropriately defined bounded operator so that $\mathcal{B}$ becomes a dissipative operator. Moreover we prove that $S_{\mathcal{B}}$ satisfies some regularization estimate

$$\|S_{\mathcal{B}}(t)\|_{L^p(m_1)\to L^2(m_2)} \lesssim t^{-\alpha}, \quad \forall t \in [0, \eta],$$

for any $p \in [1, 2)$, some weight function m_1, m_2 and some $\alpha, \eta > 0$, and using the iterated Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \sum_{l=1}^{n-1}(S_{\mathcal{B}}) * (\mathcal{A}S_{\mathcal{B}})^{(*l)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*n)}, \tag{3.1.4}$$

we deduce the L^p convergence on $S_{\mathcal{L}}$. Here and below $\mathcal{U} * \mathcal{V}$ denotes the convolution of two operators valued function $\mathcal{U}$, $\mathcal{V}$ defined by

$$(\mathcal{U} * \mathcal{V})(t) = \int_0^t \mathcal{U}(s)\mathcal{V}(t-s)ds,$$

and we set $\mathcal{U}^{(*0)} = I, \mathcal{U}^{(*1)} = \mathcal{U}$ and for any $k \geq 2$, $\mathcal{U}^{(*k)} = \mathcal{U}^{(*(k-1))} * \mathcal{U}$.

Let us end the introduction by describing the plan of this chapter. In Section 3.2, we will develop a hypocoercivity argument to prove a weighted L^2 estimate for the KFP model. In section 3.3, we introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and using the L^2 estimate deduced in Section 3.2 to prove a L^2 convergence. In Section 3.4, we present the proof of a hypoelliptic estimate on $S_{\mathcal{B}}$ from L^p to L^2. In Section 3.5, we prove some L^1 estimate on the semigroup $S_{\mathcal{B}}$. Finally in Section 3.6 we use the above hypoelliptic estimate to conclude the L^p convergence for the KFP equation.

3.2 L^2 framework: Dirichlet form and rate of convergence estimate

For later discussion, we introduce some notations for the whole chapter.

We split the KFP operator as

$$\mathcal{L} = \mathcal{T} + \mathcal{S},$$

where $\mathcal{T}$ stands for the transport part

$$\mathcal{T}f = -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f,$$

and $\mathcal{S}$ stands for the collision part

$$\mathcal{S}f = \Delta_v f + div_v(vf).$$

We will denote the cut-off function χ such that $\chi(x,v) \in [0,1]$, $\chi(x,v) \in C^\infty$, $\chi(x,v) = 1$ when $|x|^2 + |v|^2 \leq 1$, $\chi(x,v) = 0$ when $|x|^2 + |v|^2 \geq 2$, and then denote $\chi_R = \chi(x/R, v/R)$.

We may also define another splitting of the KFP operator $\mathcal{L}$ by

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} = K\chi_R(x,v). \tag{3.2.1}$$

with $K, R > 0$ to be chosen later.

For $f = f(x)$, we use $\int f$ in replace $\int_{\mathbb{R}^d} f dx$, and for $f = f(x,v)$, we use$\int f$ in place of $\int_{\mathbb{R}^d \times \mathbb{R}^d} f dx dv$ for short, for $f = f(x,v)$, $\int f dx$ means $\int_{\mathbb{R}^d} f dx$, $\int f dv$ means $\int_{\mathbb{R}^d} f dv$. $B_{|x| \leq \rho}$ is used to denote the ball such that $\{x \in \mathbb{R}^d | |x| \leq \rho\}$, similarly B_ρ means the ball such that $\{x, v \in \mathbb{R}^d | |x|^2 + v^2 \leq \rho\}$.

For $V(x) = \langle x \rangle^\gamma, 0 < \gamma < 1$, we also denote $\langle \nabla V \rangle$ for $\langle x \rangle^{\gamma - 1}$, and $\langle \nabla V \rangle^{-1}$ for $\langle x \rangle^{1-\gamma}$.

With these notations we introduce the Dirichlet form adapted to our problem. We define the 0 order and first order moments

$$\rho_f = \rho[f] = \int f dv, \quad j_f = j[f] = \int v f dv,$$

then we define a projection operator π by

$$\pi f = M\rho_f, \quad M = Ce^{-|v|^2/2}, \quad \int M dv = 1,$$

and the complement of π by

$$\pi^\perp = I - \pi, \quad f^\perp = \pi^\perp f.$$

We define an elliptic operator Δ_V and its dual Δ_V^* by

$$\Delta_V u := \mathrm{div}_x(\nabla_x u + \nabla_x V u), \quad \Delta_V^* u := \Delta_x u - \nabla_x V \cdot \nabla_x u,$$

let $u = (\Delta_V^*)^{-1}\xi$ be the solution to the above elliptic equation

$$\Delta_V^* u = \xi \quad \text{on } \mathbb{R}^d,$$

satisfies

$$\int u e^{-V} \langle \nabla V \rangle^{-2} = 0.$$

We will prove the existence and uniqueness to this elliptic equation in Lemma 3.2.6 below, we then define $\mathcal{H} = L^2(G^{-1/2})$, $\mathcal{H}_1 = L^2(G^{-1/2}\langle \nabla V \rangle)$ and

$$\mathcal{H}_0 = \{h \in \mathcal{H}, \int h dx dv = 0\}$$

where we recall that G has been introduced in (3.1.2). Using these notations, define a scalar product by

$$\begin{aligned}((f,g)) &:= (f,g)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j_f, (\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon((\rho_f e^V \langle \nabla V \rangle^2), \Delta_V^{-1}\nabla_x j_g)_{L^2} \\ &= (f,g)_{\mathcal{H}} + \epsilon(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon((\nabla_x(\Delta_V^*)^{-1}(\rho_f e^V \langle \nabla V \rangle^2), j_g)_{L^2},\end{aligned}$$

for some $\epsilon > 0$ to be specified later.
Finally we define the Dirichlet form

$$\begin{aligned}D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f)_{\mathcal{H}} + \epsilon(\Delta_V^{-1}\nabla_x j[-\mathcal{L}f], (\rho_f e^V \langle \nabla V \rangle^2))_{L^2} \\ &\quad + \epsilon((\rho[-\mathcal{L}f] e^V \langle \nabla V \rangle^2), \Delta_V^{-1}\nabla_x j_f)_{L^2}.\end{aligned}$$

With these notations we can come to our first theorem.

Theorem 3.2.1. *There exists $\epsilon > 0$ small enough, such that on $\mathcal{H}_0$ the norm $((f,f))^{\frac{1}{2}}$ defined above is equivalent to the norm of $\mathcal{H}$, moreover there exist $\lambda > 0$, such that*

$$D[f] \geq \lambda \|f\|_{\mathcal{H}_1}^2, \quad \forall f \in \mathcal{H}_0.$$

As a consequence, for any $f_0 \in \mathcal{H}_0$, the associated solution $f(t,\cdot)$ of the kinetic Fokker-Planck equation (3.1.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}((f,f)) \leq -C\int f^2 G^{-1}\langle x\rangle^{2(\gamma-1)}, \tag{3.2.2}$$

for some constant $C>0$. In particular for any $f_0 \in \mathcal{H}_0$, we have

$$\|f(t,\cdot)\|_{L^2(G^{-\frac{1}{2}})} \leq C\|f_0\|_{L^2(G^{-\frac{1}{2}})}, \tag{3.2.3}$$

for some constant $C>0$.

Remark 3.2.2. In $\mathcal{H}_0$ we have

$$\int \rho_f e^V \langle\nabla V\rangle^2 e^{-V}\langle\nabla V\rangle^{-2}dx = \int \rho_f dx = \int f dxdv = 0,$$

so the term $(\Delta_V^*)^{-1}(\rho_g e^V\langle\nabla V\rangle^2)$ is well defined in $\mathcal{H}_0$.

Remark 3.2.3. Our statement is a generalization of [19, 20].

Before proving the theorem, we need some lemmas.

We say that W satisfies a local Poincaré inequality on a bounded open set Ω if there exists some constant $\kappa_\Omega > 0$ such that:

$$\int_\Omega h^2 W \leq \kappa_\Omega \int_\Omega |\nabla h|^2 W + \frac{1}{W(\Omega)}\left(\int_\Omega hW\right)^2,$$

for any nice function $h:\mathbb{R}^d \to \mathbb{R}$ and where we denote $W(\Omega) := \langle W 1_\Omega\rangle$.

Lemma 3.2.4. *Under the assumption $W, W^{-1} \in L^\infty_{loc}(\mathbb{R}^d)$, the function W satisfies the local Poincaré inequality for any ball $\Omega \in \mathbb{R}^d$.*

For the proof of Lemma 3.2.4 we refer to [57] Lemma 2.3.

Lemma 3.2.5. *(weak Poincaré inequality) There exists a constant $\lambda > 0$ such that*

$$\|u\|_{L^2(\langle\nabla V\rangle e^{-V/2})} \leq \lambda \|\nabla u\|_{L^2(e^{-V/2})}$$

for any $u \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int u e^{-V}\langle\nabla V\rangle^{-2} = 0$$

Proof. We prove for any $h \in \mathcal{D}(\mathbb{R}^d)$ such that

$$\int h e^{-V}\langle\nabla V\rangle^{-2} = 0, \tag{3.2.4}$$

we have

$$\int |\nabla h|^2 e^{-V} \geq \frac{1}{\lambda}\int h^2 e^{-V}\langle x\rangle^{2(\gamma-1)},$$

for some $\lambda > 0$. Taking $g = he^{-\frac{1}{2}V}$, we have $\nabla g = \nabla h e^{-\frac{1}{2}V} - \frac{1}{2}\nabla V h e^{-\frac{1}{2}V}$, so that

$$\begin{aligned} 0 \leq \int |\nabla g|^2 &= \int |\nabla h|^2 e^{-V} + \int h^2 \frac{1}{4}|\nabla V|^2 e^{-V} - \int \frac{1}{2}\nabla(h^2)\cdot \nabla V e^{-V} \\ &= \int |\nabla h|^2 e^{-V} + \int h^2 \left(\frac{1}{2}\Delta V - \frac{1}{4}|\nabla V|^2\right) e^{-V}. \end{aligned}$$

Since

$$\frac{1}{4}|\nabla V|^2 - \Delta V \geq \frac{1}{8}\langle \nabla V\rangle^2 - K\mathbb{1}_{B_{R_0}}\langle \nabla V\rangle^{-2}$$

for some $K, R_0 > 0$. We deduce for some $K, R_0 > 0$

$$\int |\nabla h|^2 e^{-V} \geq \int \frac{1}{8} h^2 \langle \nabla V\rangle^2 e^{-V} - K\int_{B_{R_0}} h^2 e^{-V}\langle \nabla V\rangle^{-2}.$$

Defining

$$\epsilon_R := \int_{B_R^c} e^{-V}\langle \nabla V\rangle^{-6}, \quad Z_R := \int_{B_R} e^{-V}\langle \nabla V\rangle^{-2},$$

and using (3.2.4), we get

$$\begin{aligned} \left(\int_{B_R} h e^{-V}\langle \nabla V\rangle^{-2}\right)^2 &= \left(\int_{B_R^c} h e^{-V}\langle \nabla V\rangle^{-2}\right)^2 \\ &\leq \int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2 \int_{B_R^c} e^{-V}\langle \nabla V\rangle^{-6} \\ &\leq \epsilon_R \int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2. \end{aligned}$$

Using the local Poincaré inequality in Lemma 3.2.4, we deduce

$$\begin{aligned} \int_{B_R} h^2 e^{-V}\langle \nabla V\rangle^{-2} &\leq C_R \int_{B_R} |\nabla h|^2 e^{-V}\langle \nabla V\rangle^{-2} + \frac{1}{Z_R}\left(\int_{B_R} h e^{-V}\langle \nabla V\rangle^{-2}\right)^2 \\ &\leq C_R' \int_{B_R} |\nabla h|^2 e^{-V} + \frac{\epsilon_R}{Z_R}\int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2. \end{aligned}$$

Putting all the inequalities together and taking $R > R_0$, we finally get

$$\begin{aligned} \int h^2 e^{-V}\langle \nabla V\rangle^2 &\leq 8\int |\nabla h|^2 e^{-V} + 8K\int_{B_{R_0}} h^2 e^{-V}\langle \nabla V\rangle^{-2} \\ &\leq 8(1 + KC_R')\int |\nabla h|^2 e^{-V} + \frac{8K\epsilon_R}{Z_R}\int_{B_R^c} h^2 e^{-V}\langle \nabla V\rangle^2, \end{aligned}$$

and we conclude by taking R large such that: $\frac{8K\epsilon_R}{Z_R} \leq \frac{1}{2}$. □

Lemma 3.2.6. *(Elliptic Estimate) For any $\xi_1 \in L^2(\langle \nabla V\rangle^{-1}e^{-V/2})$ and $\xi_2 \in L^2(e^{-V/2})$, there exists a unique solution u to the elliptic equation*

$$-\Delta_V^* u = \xi_1 + \nabla\cdot\xi_2, \quad \int u e^{-V}\langle \nabla V\rangle^{-2} = 0, \tag{3.2.5}$$

which satisfies

$$\|u\|_{L^2(\langle \nabla V\rangle e^{-V/2})} + \|\nabla u\|_{L^2(e^{-V/2})} \lesssim \|\xi_1\|_{L^2(\langle \nabla V\rangle^{-1}e^{-V/2})} + \|\xi_2\|_{L^2(e^{-V/2})}. \tag{3.2.6}$$

In addition for any $\xi \in L^2(\langle\nabla V\rangle^{-1}e^{-V/2})$, the solution u to the elliptic problem

$$-\Delta_V^* u = \xi, \quad \int ue^{-V}\langle\nabla V\rangle^{-2} = 0,$$

satisfies

$$\|u\|_{L^2(\langle\nabla V\rangle^2 e^{-V/2})} + \|\nabla u\|_{L^2(\langle\nabla V\rangle e^{-V/2})} + \|D^2 u\|_{L^2(e^{-V/2})} \lesssim \|\xi\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})}. \quad (3.2.7)$$

Proof. Multiply (3.2.5) by ue^{-V} and observes that

$$e^V \mathrm{div}_x[e^{-V}\nabla_x u] = \Delta_x u - \nabla_x V\cdot\nabla_x u = \Delta_V^* u, \quad (3.2.8)$$

we have after integration

$$-\int e^V \mathrm{div}_x[e^{-V}\nabla_x u]ue^{-V} = \int(\xi_1 + \nabla\cdot\xi_2)ue^{-V}.$$

Performing one integration by parts, we deduce

$$\int e^{-V}|\nabla_x u|^2 = \int(\xi_1 u - \xi_2\cdot\nabla u + \xi_2\cdot\nabla V u)e^{-V},$$

using Lemma 3.2.5 and Lax-Milgram theorem we obtain (3.2.6), the existence and thus the uniqueness follows. In inequality (3.2.7), the first two terms are easily bounded by (3.2.6) and $\langle\nabla V\rangle \le 1$, we then only need to prove the bound for the third term. By integration by parts, we have

$$\begin{aligned}
\int |D^2 u|^2 e^{-V} &= \sum_{i,j=1}^d \int (\partial_{ij}^2 u)^2 e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_i u(\partial_{ij}^2 u\partial_j V - \partial_{ijj}^3 u)e^{-V} \\
&= \sum_{i,j=1}^d \int \partial_{jj}^2 u\partial_i(\partial_i u e^{-V}) - \frac{1}{2}\int(\partial_i u)^2\partial_j(\partial_j V e^{-V}) \\
&= \int(\Delta u)(-\Delta_V^* u)e^{-V} + \frac{1}{2}\int|\nabla u|^2(|\nabla V|^2 - \Delta V)e^{-V} \\
&\lesssim \|D^2 u\|_{L^2(e^{-V/2})}\|\xi\|_{L^2(e^{-V/2})} + \|\langle\nabla V\rangle\nabla u\|_{L^2(e^{-V/2})},
\end{aligned}$$

where in the third equality we have used

$$\begin{aligned}
\int \partial_{ij}^2 u\partial_i u\partial_j V e^{-V} &= -\int \partial_i u\partial_j(\partial_i u\partial_j V e^{-V}) \\
&= -\int \partial_{ij}^2 u\partial_i u\partial_j V e^{-V} - \int(\partial_i u)^2\partial_j(\partial_j V e^{-V}),
\end{aligned}$$

which implies

$$\int \partial_{ij}^2 u\partial_i u\partial_j V e^{-V} = -\frac{1}{2}\int(\partial_i u)^2\partial_j(\partial_j V e^{-V}),$$

and in the fourth equality we have used (3.2.8). That concludes the proof. □

Now we turn to the proof of Theorem 3.2.1.

Proof. (**Proof of Theorem 3.2.1.**) First we prove the equivalence of the norms associated to $((\ , \ ))$ and $(\ , \ )_{\mathcal{H}}$. By Cauchy-Schwarz inequality and Lemma 3.2.6, we have

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \leq \|j_f\|_{L^2(e^{V/2})} \|\rho_g e^V \langle \nabla V\rangle^2\|_{L^2(\langle \nabla V\rangle^{-1} e^{-V/2})},$$

and obviously

$$\|\rho_g e^V \langle \nabla V\rangle^2\|_{L^2(\langle \nabla V\rangle^{-1} e^{-V/2})} = \|\rho_g\|_{L^2(\langle \nabla V\rangle e^{V/2})} \leq \|\rho_g\|_{L^2(e^{V/2})} \lesssim \|g\|_{\mathcal{H}}.$$

Using the elementary observations

$$|j_f| \lesssim \|f\|_{L^2_v(e^{|v|^2/4})}, \quad |\rho_f| \lesssim \|f\|_{L^2_v(e^{|v|^2/4})},$$

we deduce

$$(j_f, \nabla_x(\Delta_V^*)^{-1}(\rho_g e^V \langle \nabla V\rangle^2))_{L^2} \lesssim \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}},$$

The third term in the definition of $((\ , \ ))$ can be estimated in the same way and that ends the proof of equivalence of norms. □

Now we prove the main estimate of the theorem. We split the Dirichlet term $D[f]$ into 3 parts

$$D[f] = T_1 + \epsilon T_2 + \epsilon T_3,$$

with

$$\begin{aligned} T_1 &:= (-\mathcal{L}f, f)_{\mathcal{H}} \\ T_2 &:= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ T_3 &:= ((\Delta_V)^{-1} \nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2}\langle \nabla V\rangle)} \ , \end{aligned}$$

and compute them separately.
For the T_1 term, using the classical Poincaré inequality, we have

$$\begin{aligned} T_1 &:= (-\mathcal{T}f - \mathcal{S}f, f)_{\mathcal{H}} = (-\mathcal{S}f, f)_{\mathcal{H}} \\ &= -\int [\Delta_v f + div_v(vf)] f M^{-1} e^V = \int |\nabla_v (f/M)|^2 M e^V \\ &\geq k_p \int |f/M - \rho_f|^2 M e^V = k_p \|f - \rho_f M\|^2_{\mathcal{H}} = k_p \|f^{\perp}\|^2_{\mathcal{H}}, \end{aligned}$$

for some $k_p > 0$. We split the T_2 term as

$$\begin{aligned} T_2 &:= (\Delta_V^{-1} \nabla_x j[-\mathcal{L}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &= (\Delta_V^{-1} \nabla_x j[-\mathcal{T}\pi f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &\quad + (\Delta_V^{-1} \nabla_x j[-\mathcal{T}f^{\perp}], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &\quad + (\Delta_V^{-1} \nabla_x j[-\mathcal{S}f], \rho_f)_{L^2(e^{V/2}\langle \nabla V\rangle)} \\ &:= T_{2,1} + T_{2,2} + T_{2,3}. \end{aligned}$$

First observe

$$\mathcal{T}\pi f = -v\cdot\nabla_x\rho_f M - \nabla_x V\cdot v\rho_f M = -e^{-V}Mv\cdot\nabla_x(\rho_f/e^{-V}),$$

so that we have

$$j[-\mathcal{T}\pi f] = \sum_{k=1}^{d}\langle vv_kM\rangle e^{-V}\partial_{x_k}(\rho_f/e^{-V}) = e^{-V}\nabla_x(\rho_f/e^{-V}).$$

Next by (3.2.8), we have

$$\begin{aligned} T_{2,1} &= (j[-\mathcal{T}\pi f], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= (\rho_f, [e^V div_x(e^{-V}\nabla)][(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)])_{L^2} \\ &= \|\rho_f e^{V/2}\langle\nabla V\rangle\|_{L^2}^2 = \|\pi f\|_{\mathcal{H}_1}^2. \end{aligned}$$

Using the notation $\eta_1 = \langle v\otimes vf^\perp\rangle$ and $\eta_{2,\alpha\beta} = \langle v_\alpha\partial_{v_\beta}f^\perp\rangle$, and observing that

$$|\eta_1| \lesssim \|f^\perp\|_{L^2_v(e^{|v|^2/4})}, |\eta_2| \lesssim \|f^\perp\|_{L^2_v(e^{|v|^2/4})},$$

we compute

$$\begin{aligned} T_{2,2} &= (j[-\mathcal{T}f^\perp], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= (D\eta_1 + \eta_2\nabla V, \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &= (\eta_1, D^2(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} + (\eta_2, \nabla V\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &\le \|\eta_1\|_{L^2(e^{V/2})}\|D^2(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})} \\ &\quad +\|\eta_2\|_{L^2(e^{V/2})}\|\nabla V\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})}. \end{aligned}$$

By Lemma 3.2.6, we estimate

$$\begin{aligned} T_{2,2} &\lesssim \|\eta_1\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} \\ &\quad +\|\eta_2\|_{L^2(e^{V/2})}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(e^{-V/2}\langle\nabla V\rangle^{-1})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

Using

$$\begin{aligned} j[-\mathcal{S}f] = j[-\mathcal{S}f^\perp] &= -\int v[\Delta_v f^\perp + div_v(vf^\perp)]dv \\ &= d\int f^\perp v dv \lesssim \|f^\perp\|_{L^2_v(e^{|v|^2/4})}, \end{aligned}$$

and Lemma 3.2.6, we have

$$\begin{aligned} T_{2,3} &= (j[-Sf], \nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2))_{L^2} \\ &\le \|j[-Sf]\|_{L^2(e^{V/2})}\|\nabla(\Delta_V^*)^{-1}(\rho_f e^V\langle\nabla V\rangle^2)\|_{L^2(e^{-V/2})} \\ &\lesssim \|f^\perp\|_{\mathcal{H}}\|\rho_f e^V\langle\nabla V\rangle^2\|_{L^2(\langle\nabla V\rangle^{-1}e^{-V/2})} \\ &= \|f^\perp\|_{\mathcal{H}}\|\rho_f\|_{L^2(\langle\nabla V\rangle e^{V/2})} \\ &= \|f^\perp\|_{\mathcal{H}}\|\pi f\|_{\mathcal{H}_1}. \end{aligned}$$

Finally we come to the T_3 term. Using

$$\rho[-Sf] = \int \nabla_v \cdot (\nabla_v f + vf) dv = 0,$$

and

$$\begin{aligned} \rho[-Tf] &= \rho[v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f] \\ &= \int v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f dv \\ &= \nabla_x j[f], \end{aligned}$$

since $\nabla(\langle \nabla V \rangle^2) \lesssim \langle \nabla V \rangle^2$ and $\langle \nabla V \rangle^2 \lesssim \langle \nabla V \rangle$, we get

$$\begin{aligned} T_3 &= ((\Delta_V)^{-1} \nabla_x j_f, \rho[-\mathcal{L}f])_{L^2(e^{V/2}\langle \nabla V \rangle)} \\ &= ((\Delta_V)^{-1} \nabla_x j[f^\perp], \rho[-\mathcal{T}f])_{L^2(e^{V/2}\langle \nabla V \rangle)} \\ &= (j[f^\perp], \nabla(\Delta_V^*)^{-1}(\nabla_x j[f] e^V \langle \nabla V \rangle^2))_{L^2} \\ &\leq \|j[f^\perp]\|_{L^2(e^{V/2})} \|\nabla(\Delta_V^*)^{-1}[\nabla_x(j_f e^V \langle \nabla V \rangle^2) \\ &\quad -\nabla V j_f e^V \langle \nabla V \rangle^2 - \nabla(\langle \nabla V \rangle^2) j_f e^V]\|_{L^2(e^{-V/2})}, \end{aligned}$$

using again Lemma 3.2.6, we have

$$\begin{aligned} T_3 &\lesssim \|j[f^\perp]\|_{L^2(e^{V/2})} (\|j_f e^V \langle \nabla V \rangle^2\|_{L^2(e^{-V/2}\langle \nabla V \rangle^{-1})} \\ &\quad + \|j_f e^V \nabla(\langle \nabla V \rangle^2)\|_{L^2(\langle \nabla V \rangle^{-1} e^{-V/2})}) \\ &\lesssim \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1}. \end{aligned}$$

Putting all the terms together and choosing $\epsilon > 0$ small enough, we can deduce

$$\begin{aligned} D[f] &\geq k_p \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|f\|_{\mathcal{H}_1} - \epsilon 2K \|f^\perp\|_{\mathcal{H}} \|\pi f\|_{\mathcal{H}_1} \\ &\geq k_p \|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2 - (2\epsilon + 4\epsilon^{1/2}) K \|f^\perp\|_{\mathcal{H}}^2 - \epsilon^{3/2} 4K \|\pi f\|_{\mathcal{H}_1}^2 \\ &\geq \frac{k_p}{2} (\|f^\perp\|_{\mathcal{H}}^2 + \epsilon \|\pi f\|_{\mathcal{H}_1}^2) \geq \lambda \|f\|_{\mathcal{H}_1}, \end{aligned}$$

for some $\lambda > 0$. □

3.3 L^2 sub-exponential decay for the kinetic Fokker-Planck equation based on a splitting trick

In this section we establish a first decay estimate on $S_{\mathcal{L}}$ which is a particular case in the result of Theorem 3.1.1.

Theorem 3.3.1. *Using the notation and results in Theorem 3.2.1, we have*

$$\|S_{\mathcal{L}}(t) f_0\|_{L^2(G^{-\frac{1}{2}})} \lesssim e^{-Ct^{\gamma/(2-\gamma)}} \|f_0\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})},$$

for any $f_0 \in L^2(G^{-(\frac{1}{2}+\epsilon)}) \cap \mathcal{H}_0$, $\epsilon > 0$ *small enough.*

Remark 3.3.2. It's worth emphasizing that we deduce immediately part (1) of Theorem 3.1.1 in the case $p = 2$ by considering the initial datum $f_0 - \mathcal{M}(f_0)G$ for any $f_0 \in L^2(G^{-\frac{1}{2}+\epsilon})$.

Recall the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ introduced in (3.2.1), we first prove some decay estimate on the semigroup $S_{\mathcal{B}}$.

Lemma 3.3.3. *Let us fix $p \in [1, \infty)$.*

(1) For any given smooth weight function m, we have

$$\int |f|^{p-1} \operatorname{sign} f(\mathcal{L}f) G^{-(p-1)} m \leq \frac{1}{p} \int |f|^p G^{-(p-1)} \tilde{m}, \tag{3.3.1}$$

with

$$\tilde{m} = \Delta_v m - \nabla_v m \cdot v - \nabla_x V(x) \cdot \nabla_v m + v \cdot \nabla_x m.$$

(2) Taking $m = e^{\epsilon H^\delta}$, $\epsilon > 0$ if $0 < \delta < \frac{\gamma}{2}$, ϵ small enough if $\delta = \frac{\gamma}{2}$, $H = 3v^2 + 2x \cdot v + x^2 + 1$, we have

$$\int |f|^{p-1} \operatorname{sign} f(\mathcal{B}f) G^{-(p-1)} e^{\epsilon H^\delta} \leq -C \int |f|^p G^{-(p-1)} e^{\epsilon H^\delta} H^{\frac{\delta}{2}+\gamma-1}, \tag{3.3.2}$$

for some K and R large.

(3) With the same notation as above, there holds

$$\|S_{\mathcal{B}}(t)\|_{L^p(e^{2\epsilon H^\delta} G^{-\frac{p-1}{p}}) \to L^p(e^{\epsilon H^\delta} G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \tag{3.3.3}$$

for some $a > 0$. In particular, this implies

$$\|S_{\mathcal{B}}(t)\|_{L^p(G^{-(\frac{p-1}{p}+\epsilon)}) \to L^p(G^{-\frac{p-1}{p}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Proof. Step 1. Recall (3.1.2), we write

$$\begin{aligned} \int |f|^{p-1} \operatorname{sign} f(\mathcal{L}f) G^{-(p-1)} m &= \int |f|^{p-1} \operatorname{sign} f(\mathcal{T}f) G^{-(p-1)} m \\ &+ \int |f|^{p-1} \operatorname{sign} f(\mathcal{S}f) G^{-(p-1)} m. \end{aligned}$$

We first compute the contribution of the term with operator $\mathcal{T}$

$$\begin{aligned} \int |f|^{p-1} \operatorname{sign} f(\mathcal{T}f) G^{-(p-1)} m &= \frac{1}{p} \int \mathcal{T}(|f|^p) G^{-(p-1)} m \\ &= -\frac{1}{p} \int |f|^p \mathcal{T}(G^{-(p-1)} m) \\ &= \frac{1}{p} \int |f|^p G^{-(p-1)} (v \cdot \nabla_x m - \nabla V(x) \cdot \nabla_v m). \end{aligned}$$

For the term with operator $\mathcal{S}$, we use one integration by parts, and we get

$$\begin{aligned} & \int |f|^{p-1} \operatorname{sign} f(\mathcal{S}f) G^{-(p-1)} m \\ = & \int |f|^{p-1} \operatorname{sign} f(\Delta_v f + \operatorname{div}_v(vf)) G^{-(p-1)} m \\ = & -\int \nabla_v (\operatorname{sign} f(|f|G^{-1})^{p-1} m) \cdot \nabla_v (fG^{-1}) G \\ = & -\int (p-1) |\nabla_v (fG^{-1})|^2 (|f|G^{-1})^{p-2} G m - \frac{1}{p} \nabla_v ((|f|G^{-1})^p) \cdot (\nabla_v m) G. \end{aligned}$$

Performing another integration by parts on the latter term, we have

$$\begin{aligned}
&\int f^{p-1}\,\mathrm{sign}\, f(\mathcal{S}f)G^{-(p-1)}m \\
=\ &\int -(p-1)|\nabla_v(fG^{-1})|^2(|f|G^{-1})^{p-2}Gm+\frac{1}{p}\nabla_v\cdot(G\nabla_v m)(|f|G^{-1})^p \\
=\ &\int -(p-1)|\nabla_v(fG^{-1})|^2(|f|G^{-1})^{p-2}Gm+\frac{1}{p}(\Delta_v m-v\cdot\nabla_v m)|f|^pG^{-(p-1)}.
\end{aligned}$$

Inequality (3.3.1) follows by putting together the two identities.
Step 2. We particular use $m=e^{\epsilon H^\delta}$ and we easily compute

$$\frac{\nabla_v m}{m}=\delta\epsilon\frac{\nabla_v H}{H^{1-\delta}},\quad \frac{\nabla_x m}{m}=\delta\epsilon\frac{\nabla_x H}{H^{1-\delta}},$$

and

$$\frac{\Delta_v m}{m}\le\delta\epsilon\frac{\Delta_v H}{H^{1-\delta}}+(\delta\epsilon)^2\frac{|\nabla_v H|^2}{H^{2(1-\delta)}}.$$

We deduce that $\phi=\frac{\tilde{m}}{m}$ satisfies

$$\frac{\phi H^{1-\delta}}{\epsilon\delta}\le\Delta_v H+\epsilon\delta\frac{|\nabla_v H|^2}{H^{1-\delta}}-v\cdot\nabla_v H+v\cdot\nabla_x H-\nabla_x V(x)\cdot\nabla_v H.$$

From the very definition of H, we have

$$\nabla_v H=6v+2x,\quad \nabla_x H=2v+2x,\quad \Delta_v H=6.$$

Choosing $\epsilon>0$ arbitrary if $0<2\delta<\gamma$, ϵ small enough if $2\delta=\gamma$,we deduce

$$\begin{aligned}
&\Delta_v H+2\epsilon\delta\frac{|\nabla_v H|^2}{H^{1-\delta}}+v\cdot\nabla_x H-v\cdot\nabla_v H-\nabla_x V(x)\cdot\nabla_v H \\
=\ &6+\epsilon\delta\frac{|6v+2x|^2}{H^{1-\delta}}+2|v|^2+2x\cdot v-6|v|^2-2x\cdot v-6v\cdot\nabla_x V(x)-2x\cdot\nabla_x V(x) \\
\le\ &(2|v|^2+C_1|v|+C_2|v|^{2\delta}-6|v|^2)+(C_3\epsilon\delta|x|^{2\delta}-2x\cdot\nabla_x V(x))+C \\
\le\ &-C_4|v|^2-C_5x\cdot\nabla_x V(x)+C_6 \\
\le\ &-C_7H^{\frac{\gamma}{2}}+K\chi_R,
\end{aligned}$$

for some constants $C_i,K,R>0$. As a consequence, we have proved

$$\phi-K\chi_R\le\frac{-C}{H^{1-\delta-\frac{\gamma}{2}}}\le 0,$$

which is nothing but (3.3.2).
Step 3. In the following, we use the "interpolation" argument from [43], denote $f_t=S_{\mathcal{B}}(t)f_0$ the solution to the evolution equation $\partial_t f=\mathcal{B}f, f(0)=f_0$. On the one hand, by (3.3.2) we have

$$\frac{d}{dt}\int|f_t|^pG^{-(p-1)}e^{2\epsilon H^\delta}=\int|f_t|^{p-1}\,\mathrm{sign}\, f_t(\mathcal{B}f_t)G^{-(p-1)}e^{2\epsilon H^\delta}\le 0,$$

which implies

$$\int|f_t|^pG^{-(p-1)}e^{2\epsilon H^\delta}\le\int|f_0|^pG^{-(p-1)}e^{2\epsilon H^\delta}:=Y_1,\quad\forall t\ge 0$$

On the other hand, defining

$$Y(t) := \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta},$$

using again (3.3.2), we have

$$\begin{aligned}
\frac{d}{dt}Y &= p\int |f_t|^{p-1} \operatorname{sign} f_t (\mathcal{B} f_t) G^{-(p-1)} e^{\epsilon H^\delta} \\
&\leq -a \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} H^{\delta+\frac{\gamma}{2}-1} \\
&\leq -a \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x\rangle^{2\delta+\gamma-2} \\
&\leq -a \int_{B_{|x|\leq\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} \langle x\rangle^{2\delta+\gamma-2},
\end{aligned}$$

for any $\rho > 0$ and for some $a > 0$. As $2\delta + \gamma < 2$, $0 \leq |x| \leq \rho$ implies $\langle x\rangle^{2\delta+\gamma-2} \geq \langle \rho\rangle^{2\delta+\gamma-2}$, we deduce

$$\begin{aligned}
\frac{d}{dt}Y &\leq -a\langle\rho\rangle^{2\delta+\gamma-2} \int_{B_{|x|\leq\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} \\
&\leq -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} \int_{B_{|x|\geq\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta},
\end{aligned}$$

Using that $e^{\epsilon\langle x\rangle^{2\delta}} \geq e^{\epsilon\langle\rho\rangle^{2\delta}}$ on $|x| \geq \rho$, we get

$$\begin{aligned}
\frac{d}{dt}Y &\leq -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} e^{-\epsilon\langle\rho\rangle^{2\delta}} \int_{B_{|x|\geq\rho}} |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon\langle x\rangle^{2\delta}} \\
&\leq -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} e^{-\epsilon\langle\rho\rangle^{2\delta}} \int |f_t|^p G^{-(p-1)} e^{\epsilon H^\delta} e^{\epsilon\langle x\rangle^{2\delta}} \\
&\leq -a\langle\rho\rangle^{2\delta+\gamma-2} Y + a\langle\rho\rangle^{2\delta+\gamma-2} e^{-\epsilon\langle\rho\rangle^{2\delta}} C Y_1.
\end{aligned}$$

Thanks to Grönwall's Lemma

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) \leq -\alpha X(t) + b \Rightarrow X(t) \leq e^{-\alpha t}X(0) + \frac{b}{\alpha}(1 - e^{-\alpha t}) \leq e^{-\alpha t}X(0) + \frac{b}{\alpha},$$

we obtain

$$\begin{aligned}
Y(t) &\leq e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t} Y(0) + C e^{-\epsilon\langle\rho\rangle^{2\delta}} Y_1 \\
&\lesssim (e^{-a\langle\rho\rangle^{2\delta+\gamma-2}t} + e^{-\epsilon\langle\rho\rangle^{2\delta}}) Y_1,
\end{aligned}$$

Choosing finally ρ such that $a\langle\rho\rangle^{2\delta+\gamma-2}t = \epsilon\langle\rho\rangle^{2\delta}$, that is $\langle\rho\rangle^{2-\gamma} = Ct$, we deduce

$$Y(t) \leq C_1 e^{-C_2 t^{\frac{2\delta}{2-\gamma}}} Y_1,$$

for some $C_1, C_2 > 0$, and we deduce the proof of (3.3.3). □

Now we come to prove Theorem 3.3.1.

Proof. (**Proof of Theorem 3.3.1.**) We recall that from (3.2.3), we have

$$\|S_{\mathcal{L}}(t)\|_{L^2(G^{-\frac{1}{2}})\to L^2(G^{-\frac{1}{2}})} \lesssim 1, \quad \forall t \geq 0$$

From the very definition of $\mathcal{A}$ we have

$$\|\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^2(e^{2\epsilon H^\delta}G^{-\frac{1}{2}})} \lesssim 1.$$

From Lemma 3.3.3 case $p = 2$, we have

$$\|S_{\mathcal{B}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-\frac{1}{2}})\to L^2(e^{\epsilon H^\delta}G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{2\delta}{2-\gamma}}}, \quad \forall t \geq 0.$$

Gathering the three estimates and using Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

we deduce

$$\|S_{\mathcal{L}}(t)\|_{L^2(e^{2\epsilon H^\delta}G^{-\frac{1}{2}})\to L^2(e^{\epsilon H^\delta}G^{-\frac{1}{2}})} \lesssim 1, \quad \forall t \geq 0.$$

In the following, we denote $f_t = S_{\mathcal{L}}(t)f_0$ the solution to the evolution equation $\partial_t f = \mathcal{L}f, f(0,\cdot) = f_0$. Taking $2\delta = \gamma$, ϵ small enough, we have in particular

$$\int |f_t|^2 G^{-1} e^{\epsilon H^{\frac{\gamma}{2}}} \leq C \int |f_0|^2 G^{-1} e^{2\epsilon H^{\frac{\gamma}{2}}} =: Y_3.$$

We define

$$Y_2(t) := ((f, f)),$$

with $((,))$ is defined in Theorem 3.2.1. Thanks to the result in (3.2.2), we have

$$\begin{aligned} \frac{d}{dt}Y_2 &\leq -a \int |f_t|^2 G^{-1} \langle x\rangle^{2(\gamma-1)} \\ &\leq -a \int_{B_{|x|\leq\rho}} |f_t|^2 G^{-1} \langle x\rangle^{2(\gamma-1)}, \end{aligned}$$

for any $\rho \geq 0$, using the same argument as Lemma 3.3.3, we deduce

$$\begin{aligned} Y_2(t) &\leq Ce^{-a\langle\rho\rangle^{2(\gamma-1)}t}Y_2(0) + Ce^{-\epsilon_2\langle\rho\rangle^\gamma}Y_3 \\ &\lesssim (e^{-a\langle\rho\rangle^{2(\gamma-1)}t} + e^{-\epsilon_2\langle\rho\rangle^\gamma})Y_3. \end{aligned}$$

Choosing ρ such that $a\langle\rho\rangle^{2(\gamma-1)}t = \epsilon_2\langle\rho\rangle^\gamma$, that is $\langle\rho\rangle^{2-\gamma} = Ct$, we conclude

$$Y_2(t) \leq C_1 e^{-C_2 t^{\gamma/(2-\gamma)}} Y_3,$$

for some constants $C_i > 0$. As $H^{\frac{\gamma}{2}} \lesssim C(\frac{v^2}{2} + V(x))$, we have

$$e^{\epsilon H^{\frac{\gamma}{2}}} \leq G^{-C\epsilon},$$

Taking ϵ small, the proof of Theorem 3.3.1 is done. □

3.4 Regularization property of $S_{\mathcal{B}}$

In this section we will denote $\mathcal{L}^* = \mathcal{L}^*_{G^{-1/2}} = \mathcal{S} - \mathcal{T}$ be the dual operator of $\mathcal{L}$ on $L^2(G^{-\frac{1}{2}})$. In other words, $\mathcal{L}^*$ is defined by the identity

$$\int (\mathcal{L}f) g G^{-1} = \int (\mathcal{L}^* g) f G^{-1}.$$

for any smooth function f, g. We also denote $\mathcal{B}^* = \mathcal{L}^* - K\chi_R$. The aim of this section is to establish the following regularization property. The proof closely follows the proof of similar results in [38, 47, 62]

Theorem 3.4.1. *For any $0 \le \delta < 1$, denote $m_1 = G^{-\frac{1}{2}(1+\delta)}$, there exist $\eta > 0$ such that*

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^2(m_1)} \lesssim \frac{1}{t^{\frac{3d+2}{4}}} \|f\|_{L^1(m_1)}, \quad \forall t \in [0, \eta].$$

Similarly, for any $0 \le \delta < 1$, there exist $\eta > 0$ such that

$$\|\mathcal{S}_{\mathcal{B}^*}(t) f\|_{L^2(m_1)} \lesssim \frac{1}{t^{\frac{3d+2}{4}}} \|f\|_{L^1(m_1)}, \quad \forall t \in [0, \eta].$$

As a consequence, there exist $\eta > 0$ such that

$$\|\mathcal{S}_{\mathcal{B}}(t) f\|_{L^\infty(G^{-\frac{1}{2}})} \lesssim \frac{1}{t^{\frac{3d+2}{4}}} \|f\|_{L^2(G^{-\frac{1}{2}})}, \quad \forall t \in [0, \eta].$$

We start with some elementary lemmas.

Lemma 3.4.2. *For any $0 \le \delta < 1$, we have*

$$\begin{aligned} \int (f(\mathcal{L}g) + g(\mathcal{L}f)) G^{-(1+\delta)} &= -2 \int \nabla_v (f G^{-1}) \cdot \nabla_v (g G^{-1}) G^{1-\delta} \\ &+ \int (\delta d - \delta(1-\delta)|v|^2) f g G^{-(1+\delta)} \end{aligned} \tag{3.4.1}$$

in particular, this implies

$$\begin{aligned} \int f(\mathcal{L}f) G^{-(1+\delta)} &= -\int |\nabla_v (f G^{-1})|^2 G^{1-\delta} + \frac{\delta d}{2} \int |f|^2 G^{-(1+\delta)} \\ &- \frac{\delta(1-\delta)}{2} \int |v|^2 |f|^2 G^{-(1+\delta)}, \end{aligned} \tag{3.4.2}$$

similarly, for any $0 \le \delta < 1$, we have

$$\begin{aligned} \int f(\mathcal{L}f) G^{-(1+\delta)} &= -\int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2} \int |v|^2 |f|^2 G^{-(1+\delta)} \\ &+ \frac{(2+\delta)d}{2} \int |f|^2 G^{-(1+\delta)}. \end{aligned} \tag{3.4.3}$$

All the equalities remain true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$.

Proof. Recall $\mathcal{T}(G^{-(1+\delta)}) = 0$, we have

$$\int f(\mathcal{T}g)G^{-(1+\delta)} = \int \mathcal{T}(fG^{-(1+\delta)})g = -\int (\mathcal{T}f)gG^{-(1+\delta)},$$

which implies

$$\int f(\mathcal{T}g)G^{-(1+\delta)} + \int (\mathcal{T}f)gG^{-(1+\delta)} = 0.$$

for the term with operator $\mathcal{S}$ we have

$$\begin{aligned}\int f(\mathcal{S}g)G^{-(1+\delta)} &= -\int \nabla_v(fG^{-(1+\delta)})\cdot(\nabla_v g + vg)\\ &= -\int (\nabla_v f + (1+\delta)vf)\cdot(\nabla_v g + vg)G^{-(1+\delta)}\\ &= -\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta}\\ &\quad -\int (\delta|v|^2 fg + \delta f v\cdot\nabla_v g)G^{-(1+\delta)},\end{aligned}$$

using integration by parts

$$\begin{aligned}\int \delta f v\cdot\nabla_v g G^{-(1+\delta)} &= -\int \delta g\nabla_v\cdot(vfG^{-(1+\delta)})\\ &= -\int \delta g v\cdot\nabla_v f G^{-(1+\delta)}\\ &\quad -\int (\delta d + \delta(1+\delta)|v|^2)fgG^{-(1+\delta)},\end{aligned}$$

so we deduce

$$\begin{aligned}&\int (f(\mathcal{S}g) + g(\mathcal{S}f))G^{-(1+\delta)}\\ =\ & -2\int \nabla_v(fG^{-1})\cdot\nabla_v(gG^{-1})G^{1-\delta} + \int (\delta d - \delta(1-\delta)|v|^2)fgG^{-(1+\delta)},\end{aligned}$$

so (3.4.1) and (3.4.2) are thus proved by combining the two terms above. Finally, we compute

$$\begin{aligned}&\int f\mathcal{S}fG^{-(1+\delta)}\\ =\ & -\int (\nabla_v f + (1+\delta)vf)\cdot(\nabla_v f + vf)G^{-(1+\delta)}\\ =\ & -\int |\nabla_v f|^2 G^{-(1+\delta)} - \int (1+\delta)|v|^2|f|^2G^{-(1+\delta)} - \int (2+\delta)fv\cdot\nabla_v f G^{-(1+\delta)}\\ =\ & -\int |\nabla_v f|^2 G^{-(1+\delta)} - \int (1+\delta)|v|^2|f|^2G^{-(1+\delta)} + \frac{2+\delta}{2}\int \nabla_v\cdot(vG^{-(1+\delta)})|f|^2\\ =\ & -\int |\nabla_v f|^2 G^{-(1+\delta)} + \frac{\delta(1+\delta)}{2}\int |v|^2|f|^2G^{-(1+\delta)} + \frac{(2+\delta)d}{2}\int |f|^2G^{-(1+\delta)},\end{aligned}$$

so (3.4.3) follows by putting together the above equality with

$$\int f\mathcal{T}fG^{-(1+\delta)} = 0.$$

Since the term associated with $\mathcal{T}$ is 0, by $\mathcal{L} = \mathcal{S} + \mathcal{T}, \mathcal{L}^* = \mathcal{S} - \mathcal{T}$, we know the same equalities will remain true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$. □

Lemma 3.4.3. *When $f_t = S_{\mathcal{B}}(t)f_0$, denote $m_1 = G^{-\frac{1}{2}(1+\delta)}$, define an energy functional*

$$\begin{aligned}\mathcal{F}(t,f_t) &:= A\|f_t\|^2_{L^2(m_1)} + at\|\nabla_v f_t\|^2_{L^2(m_1)} \\ &+ 2ct^2(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} + bt^3\|\nabla_x f_t\|^2_{L^2(m_1)}, \end{aligned} \tag{3.4.4}$$

when $f_t = S_{\mathcal{B}^*}(t)f_0$, define another energy functional

$$\begin{aligned}\mathcal{F}^*(t,f_t) &:= A\|f_t\|^2_{L^2(m_1)} + at\|\nabla_v f_t\|^2_{L^2(m_1)} \\ &- 2ct^2(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} + bt^3\|\nabla_x f_t\|^2_{L^2(m_1)}, \end{aligned} \tag{3.4.5}$$

with $a, b, c > 0, c \le \sqrt{ab}$ and A large enough. Then for both cases, there exists $\eta > 0$ such that

$$\frac{d}{dt}F(t,f_t) \le -L\left(\|\nabla_v f_t\|^2_{L^2(m_1)} + t^2\|\nabla_x f_t\|^2_{L^2(m_1)}\right) + \|f_t\|^2_{L^2(m_1)},$$

for all $t \in [0,\eta]$, for some $L > 0, C > 0$ and $F = \mathcal{F}$ or $\mathcal{F}^*$.

Proof. We only prove the case $F = \mathcal{F}$, the proof for $F = \mathcal{F}^*$ is the same. We split the computation into several parts and then put them together. First using (3.4.2) and (3.4.3) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|f_t\|^2_{L^2(m_1)} \\ =\ & (f_t, (\mathcal{L} - K\chi_R)f_t)_{L^2(m_1)} \\ =\ & \frac{1-\delta}{2}(f_t, \mathcal{L}f_t)_{L^2(m_1)} + \frac{1+\delta}{2}(f_t, \mathcal{L}f_t)_{L^2(m_1)} - (f_t, K\chi_R f_t)_{L^2(m_1)} \\ \le\ & \frac{1-\delta}{2}\|\nabla_v f_t\|^2_{L^2(m_1)} - \frac{1+\delta}{2}\|f_t G^{-1}\|^2_{L^2(m_1 G^{1/2})} + C\|f_t\|^2_{L^2(m_1)} \\ \le\ & \frac{1-\delta}{2}\|\nabla_v f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}\end{aligned}$$

By

$$\partial_{x_i}\mathcal{L}f = \mathcal{L}\partial_{x_i}f + \sum_{j=1}^{d}\partial^2_{x_i x_j}V\partial_{v_j}f, \tag{3.4.6}$$

and (3.4.2) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_i}f_t\|^2_{L^2(m_1)} \\ =\ & (\partial_{x_i}f_t, \partial_{x_i}(\mathcal{L} - K\chi_R)f_t)_{L^2(m_1)} \\ \le\ & \|\nabla_v(\partial_{x_i}f_t G^{-1})\|^2_{L^2(m_1 G)} + \frac{\delta d}{2}\|\partial_{x_i}f_t\|^2_{L^2(m_1)} - \frac{\delta(1-\delta)}{2}\|\partial_{x_i}f_t\|^2_{L^2(m_1|v|)} \\ &+ \left(\partial_{x_i}f_t, \sum_{j=1}^{d}\partial^2_{x_i x_j}V\partial_{v_j}f_t\right)_{L^2(m_1)} - (\partial_{x_i}f_t, K\partial_{x_i}\chi_R f_t)_{L^2(m_1)}\end{aligned}$$

Using Cauchy-Schwarz inequality and summing up by i, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f_t\|^2_{L^2(m_1)} &\le \sum_{i=1}^{d}\|\nabla_v(\partial_{x_i}f_t G^{-1})\|^2_{L^2(m_1 G)} - \frac{\delta(1-\delta)}{2}\|\nabla_x f_t\|^2_{L^2(m_1|v|)} \\ &+ C\|\nabla_v f_t\|^2_{L^2(m_1)} + C\|\nabla_x f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}\end{aligned}$$

for some $C > 0$. Similarly using

$$\partial_{v_i}\mathcal{L}f = \mathcal{L}\partial_{v_i}f - \partial_{x_i}f + \partial_{v_i}f, \tag{3.4.7}$$

and (3.4.2), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{v_i}f_t\|^2_{L^2(m_1)}\\
=\;& (\partial_{v_i}f_t, \partial_{v_i}(\mathcal{L} - K\chi_R)f_t)_{L^2(m_1)}\\
\leq\;& \|\nabla_v(\partial_{v_i}f_tG^{-1})\|^2_{L^2(m_1G)} + \frac{\delta d}{2}\|\partial_{v_i}f_t\|^2_{L^2(m_1)} - \frac{\delta(1-\delta)}{2}\|\partial_{v_i}f_t\|^2_{L^2(m_1|v|)}\\
-\;& (\partial_{x_i}f_t, \partial_{v_i}f_t)_{L^2(m_1)} + \|\partial_{v_i}f_t\|^2_{L^2(m_1)} - (\partial_{v_i}f_t, K\partial_{v_i}\chi_Rf_t)_{L^2(m_1)}
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing up by i we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_vf_t\|^2_{L^2(m_1)} \;\leq\;& \sum_{i=1}^d \|\nabla_v(\partial_{v_i}f_tG^{-1})\|^2_{L^2(m_1G)} - \frac{\delta(1-\delta)}{2}\|\nabla_vf_t\|^2_{L^2(m_1|v|)}\\
+\;& C\|\nabla_vf_t\|^2_{L^2(m_1)} + C(|\nabla_xf_t|, |\nabla_vf_t|)_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}
\end{aligned}$$

For the crossing term, we split it also into two parts

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}2(\partial_{x_i}f_t, \partial_{v_i}f_t)_{L^2(m_1)} &= (\partial_{x_i}f_t, \partial_{v_i}\mathcal{L}f_t)_{L^2(m_1)} + (\partial_{v_i}f_t, \partial_{x_i}\mathcal{L}f_t)_{L^2(m_1)}\\
&- (\partial_{x_i}f_t, \partial_{v_i}(K\chi_Rf_t))_{L^2(m_1)} - (\partial_{v_i}f_t, \partial_{x_i}(K\chi_Rf_t))_{L^2(m_1)}\\
&:= W_1 + W_2.
\end{aligned}$$

Using (3.4.6) and (3.4.7) we have

$$\begin{aligned}
W_1 &= (\partial_{x_i}f_t, \mathcal{L}(\partial_{v_i}f_t))_{L^2(m_1)} + (\partial_{v_i}f_t, \mathcal{L}(\partial_{x_i}f_t))_{L^2(m_1)}\\
&+ (\partial_{v_i}f_t, \sum_{j=1}^d \partial^2_{x_ix_j}V\partial_{v_j}f_t)_{L^2(m_1)} - \|\partial_{x_i}f_t\|^2_{L^2(m_1)} + (\partial_{x_i}f_t, \partial_{v_i}f_t)_{L^2(m_1)}
\end{aligned}$$

By (3.4.1), we deduce

$$\begin{aligned}
W_1 \;\leq\;& (\nabla_v(\partial_{x_i}f_tG^{-1}), \nabla_v(\partial_{v_i}f_tG^{-1}))_{L^2(m_1G)} + \delta d(\partial_{v_i}f_t, \partial_{x_i}f_t)_{L^2(m_1)}\\
-\;& \delta(1-\delta)(\partial_{v_i}f_t, \partial_{x_i}f_t)_{L^2(m_1|v|)} + (\partial_{v_i}f_t, \sum_{j=1}^d \partial^2_{x_ix_j}V\partial_{v_j}f_t)_{L^2(m_1)}\\
-\;& \|\partial_{x_i}f_t\|^2_{L^2(m_1)} + (\partial_{x_i}f_t, \partial_{v_i}f_t)_{L^2(m_1)}
\end{aligned}$$

For the W_2 term we have

$$\begin{aligned}
W_2 &= -2(\partial_{x_i}f_t, K\chi_R\partial_{v_i}f_t)_{L^2(m_1)} - (\partial_{x_i}f_t, K\partial_{v_i}\chi_Rf_t)_{L^2(m_1)} - (\partial_{v_i}f_t, K\partial_{x_i}\chi_Rf_t)_{L^2(m_1)}\\
&\leq C(|\partial_{x_i}f_t|, |f_t|)_{L^2(m_1)} + C(|f_t|, |\partial_{v_i}f_t|)_{L^2(m_1)} + C(|\partial_{x_i}f_t|, |\partial_{v_i}f_t|)_{L^2(m_1)}
\end{aligned}$$

Combining the two parts, using Cauchy-Schwarz inequality, and summing up by i we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}2(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} \\
\leq\ & 2\sum_{i=1}^{d}(\nabla_v(\partial_{x_i} f_t G^{-1}), \nabla_v(\partial_{v_i} f_t G^{-1}))_{L^2(m_1 G)} - \delta(1-\delta)(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1|v|)} \\
-\ & \frac{1}{2}\|\nabla_x f_t\|^2_{L^2(m_1)} + C\|\nabla_v f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}
\end{aligned}$$

For the very definition of $\mathcal{F}$ in (3.4.4), we easily compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t, f_t) \ =\ & A\frac{\mathrm{d}}{\mathrm{d}t}\|f_t\|^2_{L^2(m_1)} + at\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f_t\|^2_{L^2(m_1)} + 2ct^2\frac{\mathrm{d}}{\mathrm{d}t}(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} \\
& + bt^3\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f_t\|^2_{L^2(m_1)} + a\|\nabla_v f_t\|^2_{L^2(m_1)} + 4ct(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1)} \\
& + 3bt^2\|\nabla_x f_t\|^2_{L^2(m_1)}.
\end{aligned}$$

Gathering all the inequalities above together, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t, f_t) \\
\leq\ & (2a - \frac{A(1-\delta)}{2} + Cat + 2Ct^2c + Cbt^3)\|\nabla_v f_t\|^2_{L^2(m_1)} \\
+\ & (3bt^2 - \frac{c}{2}t^2 + Cbt^3)\|\nabla_x f_t\|^2_{L^2(m_1)} + (4ct + Cat)(|\nabla_v f_t|, |\nabla_x f_t|)_{L^2(m_1)} \\
-\ & \sum_{i=1}^{d}(at\|\nabla_v(\partial_{v_i} f_t G^{-1})\|^2_{L^2(m_1 G)} + bt^3\|\nabla_v(\partial_{x_i} f_t G^{-1})\|^2_{L^2(m_1 G)} \\
+\ & 2ct^2(\nabla_v(\partial_{x_i} f_t G^{-1}), \nabla_v(\partial_{v_i} f_t G^{-1}))_{L^2(m_1 G)}) - \frac{\delta(1-\delta)}{2}(at\|\nabla_v f_t\|^2_{L^2(m_1|v|)} \\
+\ & bt^3\|\nabla_x f_t\|^2_{L^2(m_1|v|)} + 2ct^2(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1|v|)}) + C\|f_t\|^2_{L^2(m_1)},
\end{aligned}$$

for some $C > 0$. We observe that

$$|2ct^2(\nabla_v f_t, \nabla_x f_t)_{L^2(m_1|v|)}| \leq at\|\nabla_v f_t\|^2_{L^2(m_1|v|)} + bt^3\|\nabla_x f_t\|^2_{L^2(m_1|v|)},$$

and

$$\begin{aligned}
&|2ct^2(\nabla_v(\partial_{x_i} f_t G^{-1}), \nabla_v(\partial_{v_i} f_t G^{-1}))_{L^2(m_1 G)}| \\
\leq\ & at\|\nabla_v(\partial_{v_i} f_t G^{-1})\|^2_{L^2(m_1 G)} + bt^3\|\nabla_v(\partial_{x_i} f_t G^{-1})\|^2_{L^2(m_1 G)}.
\end{aligned}$$

by our choice on a, b, c. So by taking A large, $12b \leq c$, and $0 < \eta$ small ($t \in [0, \eta]$), as a consequence

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t, f_t) \leq -L(\|\nabla_v f_t\|^2_{L^2(m_1)} + t^2\|\nabla_x f_t\|^2_{L^2(m_1)}) + C\|f_t\|^2_{L^2(m_1)},$$

for some $L, C > 0$, and that ends the proof. □

Remark 3.4.4. For the case $F = \mathcal{F}^*$, the only difference in the proof is to change (3.4.6) and (3.4.7) into

$$\partial_{x_i}\mathcal{L}^* f = \mathcal{L}^*\partial_{x_i} f - \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f,$$

and

$$\partial_{v_i}\mathcal{L}^* f = \mathcal{L}^*\partial_{v_i} f + \partial_{x_i} f + \partial_{v_i} f.$$

The following proof of this section is true for both cases.

Lemma 3.4.5. *Denote $m_1 = G^{-\frac{1}{2}(1+\delta)}$, then for any $0 < \delta < 1$ we have*

$$\int |\nabla_{x,v}(f m_1)|^2 \leq \int |\nabla_{x,v} f|^2 m_1^2 + C \int f^2 m_1^2,$$

for some constant $C > 0$.

Proof. We have

$$\begin{aligned}
\int |\nabla(f m_1)|^2 &= \int |\nabla f m_1 + \nabla m_1 f|^2 \\
&= \int |\nabla f|^2 m_1^2 + \int |\nabla m_1|^2 f^2 + \int 2 f m_1 \nabla f \cdot \nabla m_1 \\
&= \int |\nabla f|^2 m_1^2 + \int (|\nabla m_1|^2 - \frac{1}{2}\Delta(m_1^2)) f^2, \\
&= \int |\nabla f|^2 m_1^2 - \int \frac{\Delta m_1}{m_1} f^2 m_1^2,
\end{aligned}$$

since

$$\frac{\Delta m_1}{m_1} = \frac{(1+\delta)^2}{4}(|v|^2 + |\nabla_x V(x)|^2) + \frac{1+\delta}{2}(\Delta_x V(x) + d) \geq -C,$$

for some $C > 0$, we are done. □

Lemma 3.4.6. *Nash's inequality: for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, there exists a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \leq C_d \|f\|_{L^1}^{\frac{2}{d}} \|\nabla_v f\|_{L^2},$$

For the proof of Nash's inequality, we refer to [44], Section 8.13 for instance. □

Lemma 3.4.7. *Denote $m_1 = G^{-\frac{1}{2}(1+\delta)}$, then for any $0 < \delta < 1$ we have*

$$\frac{d}{dt}\|f\|_{L^1(m_1)} \leq d\|f\|_{L^1(m_1)} \tag{3.4.8}$$

which implies

$$\|f_t\|_{L^1(m_1)} \leq e^{dt}\|f_0\|_{L^1(m_1)}$$

In particular we have

$$\|f_t\|_{L^1(m_1)} \leq C\|f_0\|_{L^1(m_1)}, \quad \forall t \in [0, \eta], \tag{3.4.9}$$

for some constant $C > 0$.

Proof. By Lemma 3.5.2 in the next section, letting $p = 1$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int |f|m_1 &= \int |f|(\Delta_v m_1 - v\cdot\nabla_v m_1 \\ &\quad + v\cdot\nabla_x m_1 - \nabla_x V(x)\cdot\nabla_v m_1 - K\chi_R m_1 \\ &\leq \int |f|(\frac{1+\delta}{2}d - \frac{(1+\delta)(1-\delta)}{4}v^2)m_1 \\ &\leq d\int |f|m_1.\end{aligned}$$

so (3.4.8) is proved. As $\mathcal{T}m_1 = 0$, the result is still true when $F = \mathcal{F}^*$. □

Now we come to the proof of Theorem 3.4.1.

Proof. (**Proof of Theorem 3.4.1.**) We define

$$\mathcal{G}(t, f_t) = B\|f_t\|^2_{L^1(m_1)} + t^Z\mathcal{F}(t, f_t),$$

with $B, Z > 0$ to be fixed and $\mathcal{F}$ is defined in Lemma 3.4.2. We choose $t \in [0, \eta]$, η small such that

$$(a + b + c)Z\eta^{Z+1} \leq \frac{1}{2}L\eta^Z,$$

where a, b, c, L are also defined Lemma 3.4.2, by (3.4.8) and Lemma 3.4.2 we have

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t, f_t) &\leq dB\|f_t\|^2_{L^1(m_1)} + Zt^{Z-1}\mathcal{F}(t, f_t) \\ &\quad -Lt^Z(\|\nabla_v f_t\|^2_{L^2(m_1)} + t^2\|\nabla_x f_t\|^2_{L^2(m_1)}) + Ct^Z\|f_t\|^2_{L^2(m_1)} \\ &\leq dB\|f_t\|^2_{L^1(m_1)} + Ct^{Z-1}\|f_t\|^2_{L^2(m_1)} \\ &\quad -\frac{L}{2}t^Z(\|\nabla_v f_t\|^2_{L^2(m_1)} + t^2\|\nabla_x f_t\|^2_{L^2(m_1)}).\end{aligned}$$

Nash's inequality and Lemma 3.4.5 implies

$$\begin{aligned}\|f_t m_1\|_{L^2} &\leq C\|f_t m_1\|_{L^1}^{\frac{2}{d+2}}\|\nabla_{x,v}(f_t m_1)\|_{L^2}^{\frac{d}{d+2}} \\ &\leq C\|f_t m_1\|_{L^1}^{\frac{2}{d+2}}(\|\nabla_{x,v} f_t m_1\|_{L^2} + C\|f_t m_1\|_{L^2})^{\frac{d}{d+2}}\end{aligned}$$

Using Young's inequality, we have

$$\|f_t\|^2_{L^2(m_1)} \leq C_\epsilon t^{-\frac{3}{2}d}\|f\|^2_{L^1(m_1)} + \epsilon t^3(\|\nabla_{x,v} f_t\|^2_{L^2(m_1)} + C\|f_t\|^2_{L^2(m_1)}).$$

Taking ϵ small such that $C\epsilon\eta^3 \leq \frac{1}{2}$, we deduce

$$\|f_t\|^2_{L^2(m_1)} \leq 2C_\epsilon t^{-\frac{3}{2}d}\|f\|^2_{L^1(m_1)} + 2\epsilon t^3\|\nabla_{x,v} f_t\|^2_{L^2(m_1)}.$$

Taking ϵ small we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t, f_t) \leq dB\|f_t\|^2_{L^1(m_1)} + C_1 t^{Z-1-\frac{3}{2}d}\|f_t\|^2_{L^1(m_1)},$$

for some $C_1 > 0$. Choosing $Z = 1 + \frac{3}{2}d$, and using (3.4.9), we deduce

$$\forall t \in [0, \eta], \quad \mathcal{G}(t, f_t) \leq \mathcal{G}(0, f_0) + C_2\|f_0\|^2_{L^1(m_1)} \leq C_3\|f_0\|^2_{L^1(m_1)},$$

which ends the proof. □

3.5 $S_{\mathcal{B}}$ decay in larger spaces

The aim of this section is to prove the following decay estimate for the semigroup $S_{\mathcal{B}}$ which will be useful in the last section where we will prove Theorem 3.1.1 in full generally.

Theorem 3.5.1. *Let $H = 1 + x^2 + 2v \cdot x + 3v^2$, for any $\theta \in (0,1)$ and for any $l > 0$, we have*

$$\|S_{\mathcal{B}}(t)\|_{L^1(H^l)\to L^1(H^{l\theta})} \lesssim (1+t)^{-a},$$

where

$$a = \frac{l(1-\theta)}{1-\frac{\gamma}{2}}.$$

We start with an elementary identity.

Lemma 3.5.2. *For the kinetic Fokker Planck operator $\mathcal{L}$, let m be a weight function, for any $p \in [1,\infty]$ we have*

$$\int |f|^{p-1} \operatorname{sign} f(\mathcal{L}f)m^p = -(p-1)\int |\nabla_v(mf)|^2(m|f|)^{p-2} + \int |f|^p m^p \phi,$$

with

$$\phi = \frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

In particular when $p = 1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m}.$$

Proof. We split the integral as

$$\int \operatorname{sign} f(\mathcal{L}f)|f|^{p-1}m^p = \int \operatorname{sign} f|f|^{p-1}(\mathcal{S}f)m^p + \int \operatorname{sign} f|f|^{p-1}(\mathcal{T}f)m^p.$$

First compute the contribution of the term with operator $\mathcal{T}$

$$\int \operatorname{sign} f|f|^{p-1}(\mathcal{T}f)m^p = \frac{1}{p}\int \mathcal{T}(|f|^p)m^p = -\int |f|^p m^p \frac{\mathcal{T}m}{m}.$$

Concerning the term with operator $\mathcal{S}$, we split it also into two parts

$$\int (\mathcal{S}f)\operatorname{sign} f|f|^{p-1}m^p = \int \operatorname{sign} f|f|^{p-1}m^p(\Delta_v f + \operatorname{div}_v(vf)) := C_1 + C_2.$$

We first compute the C_2 term, we have

$$\begin{aligned} C_2 &= \int \operatorname{sign} f|f|^{p-1}m^p(df + v\cdot\nabla_v f) \\ &= \int d|f|^p m^p - \frac{1}{p}\int |f|^p \operatorname{div}_v(vm^p) \\ &= \int |f|^p\left[(1-\frac{1}{p})d - v\cdot\frac{\nabla_v m}{m}\right]m^p. \end{aligned}$$

Then we turn to the C_1 term, we have

$$\begin{aligned} C_1 &= \int \operatorname{sign} f|f|^{p-1}m^p\Delta_v f = -\int \nabla_v(\operatorname{sign} f|f|^{p-1}m^p)\cdot\nabla_v f \\ &= \int -(p-1)|\nabla_v f|^2|f|^{p-2}m^p - \frac{1}{p}\int \nabla_v|f|^p\cdot\nabla_v(m^p). \end{aligned}$$

Using $\nabla_v(mf) = m\nabla_v f + f\nabla_v m$, we deduce

$$\begin{aligned} C_1 &= -(p-1)\int|\nabla_v(mf)|^2|f|^{p-2}m^{p-2} + (p-1)\int|\nabla_v m|^2|f|^p m^{p-2} \\ &\quad + \frac{2(p-1)}{p^2}\int\nabla_v(|f|^p)\cdot\nabla_v(m^p) - \frac{1}{p}\int\nabla_v(|f|^p)\cdot\nabla_v(m^p) \\ &= -(p-1)\int|\nabla_v(mf)|^2|f|^{p-2}m^p + (p-1)\int|\nabla_v m|^2|f|^p m^{p-2} \\ &\quad - \frac{p-2}{p^2}\int|f|^p\Delta_v m^p. \end{aligned}$$

Using that $\Delta_v m^p = p\Delta_v m\ m^{p-1} + p(p-1)|\nabla_v m|^2 m^{p-2}$, we obtain

$$\begin{aligned} C_1 &= -(p-1)\int|\nabla_v(mf)|^2|f|^{p-2}m^{p-2} \\ &\quad + \int|f|^p m^p\left[(\frac{2}{p}-1)\frac{\Delta_v m}{m} + 2(1-\frac{1}{p})\frac{|\nabla_v m|^2}{m^2}\right]. \end{aligned}$$

We conclude by combining the above equalities. □

Proof. (**Proof of Theorem 3.5.1.**) From Lemma 3.5.2, we have

$$\begin{aligned} &\int \operatorname{sign} f(\mathcal{B}f)|f|^{p-1}m^p \qquad (3.5.1)\\ =\ &\int \operatorname{sign} f((\mathcal{L}-K\chi_R)f)|f|^{p-1}m^p \\ =\ &-(p-1)\int|\nabla_v(mf)|^2(m|f|)^{p-2} + \int|f|^p m^p\phi, \end{aligned}$$

with

$$\phi = \left[\frac{2}{p'}\frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + \frac{d}{p'} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - K\chi_R\right].$$

When $p=1$, we have

$$\phi = \frac{\Delta_v m}{m} - v\cdot\frac{\nabla_v m}{m} - \frac{\mathcal{T}m}{m} - K\chi_R.$$

Let $m = H^k$. We have

$$\frac{\nabla_v m}{m} = k\frac{\nabla_v H}{H}, \quad \frac{\nabla_x m}{m} = k\frac{\nabla_x H}{H},$$

and

$$\frac{\Delta_v m}{m} = \frac{k\Delta_v H}{H} + \frac{k(k-1)|\nabla_v H|^2}{H^2}.$$

Summing up, we have for ϕ

$$\frac{\phi H}{k} = \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} - v\cdot\nabla_v H + v\cdot\nabla_x H - \nabla_x V(x)\cdot\nabla_v H - K\chi_R,$$

From the very definition of H, we have

$$\nabla_v H = 6v + 2x, \quad \nabla_x H = 2v + 2x, \quad \Delta_v H = 6.$$

We then compute

$$\begin{aligned}
& \Delta_v H + (k-1)\frac{|\nabla_v H|^2}{H} + v\cdot\nabla_x H - v\cdot\nabla_v H - \nabla_x V(x)\cdot\nabla_v H \\
= \;& 6 + (k-1)\frac{|6v+2x|^2}{H} + 2|v|^2 + 2x\cdot v - 6|v|^2 \\
& -2x\cdot v - 6v\cdot\nabla_x V(x) - 2x\cdot\nabla_x V(x) \\
\leq \;& (2|v|^2 + Cv - 6|v|^2) - 2x\cdot\nabla_x V(x) + C \\
\leq \;& -C_1|v|^2 - C_2 x\cdot\nabla_x V(x) + C_3 \\
\leq \;& -C_4 H^{\frac{\gamma}{2}} + K_1\chi_{R_1},
\end{aligned}$$

for some $C_i > 0$. Taking K and R large enough, we have $\phi \leq -CH^{\frac{\gamma}{2}-1}$, using this inequality in equation (3.5.1), we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}Y_4(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int |f_{\mathcal{B}}(t)|H^k &= \int \mathrm{sign}(f_{\mathcal{B}}(t))(\mathcal{B}f_{\mathcal{B}}(t))H^k \\
&\leq -C\int |f_B(t)|H^{k-1+\frac{\gamma}{2}},
\end{aligned} \tag{3.5.2}$$

for any $k > 1$. In particular for any $l \geq 1$, we can find K and R large enough such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |f_{\mathcal{B}}(t)|H^l \leq 0,$$

which readily implies

$$\int |f_{\mathcal{B}}(t)|H^l \leq \int |f_0|H^l := Y_5.$$

Take $k \leq l$, denoting

$$\alpha = \frac{l-k}{l-k+1-\frac{\gamma}{2}} \in (0,1),$$

the Hölder's inequality

$$\int |f_B(t)|H^k \leq \left(\int |f_B(t)|H^{k-1+\frac{\gamma}{2}}\right)^{\alpha}\left(\int |f_B(t)|H^l\right)^{1-\alpha},$$

implies

$$\left(\int |f_B(t)|H^k\right)^{\frac{1}{\alpha}}\left(\int |f_B(t)|H^l\right)^{\frac{\alpha-1}{\alpha}} \leq \int |f_B(t)|H^{k-1+\frac{\gamma}{2}},$$

From this inequality and (3.5.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_4(t) \leq -C(Y_4(t))^{\frac{1}{\alpha}}Y_5^{\frac{\alpha-1}{\alpha}}.$$

Using $Y_4(0) \leq Y_5$, after an integration, we deduce

$$Y_4(t) \leq C_\alpha \frac{1}{(1+t)^{\frac{\alpha}{1-\alpha}}} Y_5,$$

which is nothing but the polynomial decay on $S_{\mathcal{B}}$

$$\|S_{\mathcal{B}}(t)\|_{L^p(H^l)\to L^p(H^k)} \lesssim (1+t)^{-a},$$

with

$$a = \frac{l-k}{1-\frac{\gamma}{2}}, \quad \forall 0 < k < l, \quad 1 \leq l.$$

We conclude Theorem 3.5.1 by writing $k = l\theta$, $0 < \theta < 1$. □

3.6 L^p convergence for the KFP model

Before going to the proof of our main theorem, we need two last deduced results.

Lemma 3.6.1. *For any $\epsilon > 0$ small enough, we have*

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

and

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0,$$

for some $a > 0$. Similarly for any $0 < b < \frac{\gamma}{2-\gamma}$ and for any $\epsilon > 0$ small enough, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} e^{-at^b}, \quad \forall t \geq 0,$$

for $\alpha = \frac{3d+2}{4}$ and some $a > 0$.

Proof. The first two inequalities are obtained obviously by Lemma 3.3.3 and the property of $\mathcal{A} = K\chi_R$. For the third inequality we split it into two parts, $t \in [0, \eta]$ and $t > \eta$, where η is defined in Theorem 3.4.1. When $t \in [0, \eta]$, we have $e^{-at^{\frac{\gamma}{2-\gamma}}} \geq e^{-a\eta^{\frac{\gamma}{2-\gamma}}}$, by Theorem 3.4.1, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{-\alpha} \lesssim t^{-\alpha} e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \in [0, \eta],$$

for some $a > 0$. When $t \geq \eta$, by Theorem 3.4.1, we have

$$\|S_{\mathcal{B}}(\eta)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim \eta^\alpha \lesssim 1,$$

and by Lemma 3.3.3

$$\|S_{\mathcal{B}}(t-\eta)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})} \lesssim e^{-a(t-\eta)^{\frac{\gamma}{2-\gamma}}} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

gathering the two inequalities, we have

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}} \lesssim t^{-\alpha} e^{-at^b}, \quad \forall t > \eta,$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, the proof is ended by combining the two cases above. □

Lemma 3.6.2. *Similarly as Lemma 3.6.1. For any $p \in (2, \infty)$, we have*

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^2(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0.$$

and

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^p(G^{-\frac{1}{2}})\to L^p(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \quad \forall t \geq 0.$$

for some $a > 0$. And for any $0 < b < \frac{\gamma}{2-\gamma}$ we have

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^2(G^{-\frac{1}{2}})\to L^p(G^{-\frac{1}{2}})} \lesssim t^{-\beta}e^{-at^b}, \quad \forall t \geq 0.$$

for some $\beta > 0$ and some $a > 0$.

The proof of Lemma 3.6.2 is similar to the proof of Lemma 3.6.1 and thus skipped. □

Lemma 3.6.3. *let X, Y be two Banach spaces, $S(t)$ a semigroup such that for all $t \geq 0$ and some $0 < a, 0 < b < 1$ we have*

$$\|S(t)\|_{X\to X} \leq C_X e^{-at^b}, \quad \|S(t)\|_{Y\to Y} \leq C_Y e^{-at^b},$$

and for some $0 < \alpha$, we have

$$\|S(t)\|_{X\to Y} \leq C_{X,Y} t^{-\alpha} e^{-at^b}.$$

Then we can have that for all integer $n > 0$

$$\|S^{(*n)}(t)\|_{X\to X} \leq C_{X,n} t^{n-1} e^{-at^b},$$

similarly

$$\|S^{(*n)}(t)\|_{Y\to Y} \leq C_{Y,n} t^{n-1} e^{-at^b},$$

and

$$\|S^{(*n)}(t)\|_{X\to Y} \leq C_{X,Y,n} t^{n-\alpha-1} e^{-at^b}.$$

In particular for $\alpha + 1 < n$, and for any $b^* < b$

$$\|S^{(*n)}(t)\|_{X\to Y} \leq C_{X,Y,n} e^{-at^{b^*}}.$$

The proof of Lemma 3.6.3 is the same as Lemma 2.5 in [53], plus the fact $t^b \leq s^b + (t-s)^b$ for any $0 \leq s \leq t, 0 < b < 1$. □

Then we come to the final proof.

Proof. (**Proof of Theorem 3.1.1.**) We only prove the case when $m = G^{\frac{p-1}{p}(1+\epsilon)}, \quad p \in [1, 2]$, for the proof of the other cases, one need only replace the use of Lemma 3.6.1 in the following proof by Lemma 3.6.2 and Theorem 3.4.1. We will prove $p = 1$ first, this time we need to prove

$$\|S_{\mathcal{L}}(I - \Pi)(t)\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^b},$$

for any $0 < b < \frac{\gamma}{2-\gamma}$, where I is the identity operator and Π is a projection operator defined by

$$\Pi(f) = \mathcal{M}(f)G.$$

First, Iterating the Duhamel's formula we split it into 3 terms

$$\begin{aligned} S_{\mathcal{L}}(I-\Pi) &= (I-\Pi)\{S_{\mathcal{B}} + \sum_{l=1}^{n-1}(S_{\mathcal{B}}\mathcal{A})^{(*l)} * (S_{\mathcal{B}})\} \\ &\quad + \{(I-\Pi)S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}}(t))^{(*n)}, \end{aligned}$$

and we will estimate them separately. By Lemma 3.3.3, we have

$$\|S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon})\to L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}, \tag{3.6.1}$$

the first term is thus estimated. For the second term, still using Lemma 3.3.3, we get

$$\|S_{\mathcal{B}}(t)\mathcal{A}\|_{L^1\to L^1} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}},$$

by Lemma 3.6.3, we have

$$\|(S_{\mathcal{B}}(t)\mathcal{A})^{(*l)}\|_{L^1\to L^1} \lesssim t^{l-1}e^{-at^{\frac{\gamma}{2-\gamma}}},$$

together with Lemma 3.6.1 the second term is estimated. For the last term by Lemma 3.3.3

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^1(G^{-\epsilon})\to L^1(G^{-(\frac{1}{2}+\epsilon)})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

By Lemma 3.6.1 and 3.6.3, for any $0 < b < \frac{\gamma}{2-\gamma}$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*(n-1))}(t)\|_{L^1(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-(\frac{1}{2}+\epsilon)})} \lesssim t^{n-\alpha-2}e^{-at^b},$$

finally by Theorem 3.3.1, we have

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{L^2(G^{-(\frac{1}{2}+\epsilon)})\to L^2(G^{-\frac{1}{2}})} \lesssim e^{-at^{\frac{\gamma}{2-\gamma}}}.$$

Taking $n > \alpha + 2$ the third term is estimated, thus the proof of case $p = 1$ is concluded by gathering the inequalities above. As the case $p = 2$ is already proved in Theorem 3.3.1, the case $p \in (1, 2)$ follows by interpolation. □

Chapter 4

The Kinetic Fokker-Planck Equation with General Force

4.1 Introduction

In this paper, we consider the kinetic Fokker-Planck (KFP for short) equation with general force

$$\partial_t f = \mathcal{L} f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(\nabla_v W(v) f), \tag{4.1.1}$$

for a density function $f = f(t, x, v)$, with $t \geq 0,\ x \in \mathbb{R}^d,\ v \in \mathbb{R}^d$, with

$$V(x) = \frac{\langle x \rangle^\gamma}{\gamma}, \quad \gamma \geq 1, \quad W(v) = \frac{\langle v \rangle^\beta}{\beta}, \quad \beta \geq 2$$

where $\langle x \rangle^2 := 1 + |x|^2$, and the kinetic Fitzhugh-Nagumo equation

$$\partial_t f := \mathcal{L} f = \partial_x(A(x, v) f) + \partial_v(B(x, v) f) + \partial_{vv}^2 f \tag{4.1.2}$$

with

$$A(x, v) = ax - bv, \quad B(x, v) = v(v - 1)(v - \lambda) + x$$

for some $a, b, \lambda > 0$. The evolution equations are complemented with an initial datum

$$f(0, x, v) = f_0(x, v) \quad \text{on } \mathbb{R}^{2d}.$$

It's easily seen that both equations are mass conservative, that is

$$\mathcal{M}(f(t, \cdot)) = \mathcal{M}(f_0),$$

With these notations, we can introduce the main result of this paper.

Theorem 4.1.1. *(1) When $2 \le \beta, 1 \le \gamma$, there exist a weight function $m > 0$ and a non-negative normalized steady state $G \in L^1(m)$ such that for any initial datum $f_0 \in L^1(m)$, the associated solution $f(t,\cdot)$ of the kinetic Fokker-Planck equation (4.1.1) satisfies*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \le Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)},$$

for some constant $C, \lambda > 0$.
(2) The same conclusion holds for the kinetic Fitzhugh-Nagumo equation (4.1.2).

In the results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

Remark 4.1.2. Theorem 4.1.1 is also true when $V(x)$ behaves like $\langle x\rangle^\gamma$ and $W(v)$ behaves like $\langle v\rangle^\beta$, that is for any $V(x)$ satisfying

$$C_1\langle x\rangle^\gamma \le V(x) \le C_2\langle x\rangle^\gamma, \quad \forall x \in \mathbb{R}^d,$$

$$C_3|x|\langle x\rangle^{\gamma-1} \le x\cdot\nabla_x V(x) \le C_4|x|\langle x\rangle^{\gamma-1}, \quad \forall x \in B_R^c,$$

and

$$|D_x^n V(x)| \le C_n\langle x\rangle^{\gamma-2}, \quad \forall x \in \mathbb{R}^d, \quad \forall n \ge 2,$$

for some constant $C_i > 0$, $R > 0$, and similar estimates holds for $W(v)$.

In fact, Theorem 4.1.1 is a special case of the following theorem.

Theorem 4.1.3. *Consider the following equation*

$$\partial_t f := \mathcal{L}f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \Delta_v f \tag{4.1.3}$$

with

$$A(x,v) = -v + \Phi(x)$$

where $\Phi(x)$ is Lipschitz

$$|\Phi(x) - \Phi(y)| \le M|x-y|,$$

for some $M > 0$, define

$$\begin{aligned}\phi_2(m) &= v\cdot\frac{\nabla_x m}{m} - \Phi(x)\cdot\frac{\nabla_x m}{m} + \frac{1}{2}div_x\Phi(x) + \frac{|\nabla_v m|^2}{m^2}\\ &+ \frac{\Delta_v m}{m} - B(x,v)\cdot\frac{\nabla_v m}{m} + \frac{1}{2}div_v B(x,v)\end{aligned}$$

then if we can find a weight function m and a function $H \ge 1$ such that

$$\mathcal{L}^* m \le -\alpha m + b,$$

for some $\alpha, b > 0$

$$-C_1 Hm \le \phi_2(m) \le -C_2 Hm + C_3,$$

for some $C_1, C_2, C_3 > 0$*, and for any integer* $n \ge 2$ *fixed, for any* $\epsilon > 0$ *small, we can find a constant* $C_{\epsilon,n}$ *such that*

$$\sum_{k=2}^{n} |D_x^k \Phi(x)| + \sum_{k=2}^{n} |D_{x,v}^k B(x,v)| \le C_{n,\epsilon} + \epsilon H$$

and

$$\frac{\Delta_{x,v} m}{m} \ge -C_4$$

for some $C_4 > 0$*, then we have there exist a steady state* G *such that*

$$\|f(t,\cdot) - \mathcal{M}(f_0)G\|_{L^1(m)} \le Ce^{-\lambda t}\|f_0 - \mathcal{M}(f_0)G\|_{L^1(m)}.$$

for some $C, \lambda > 0$*.*

Remark 4.1.4. In fact $\phi_2(m)$ satisfies

$$\int (f(\mathcal{L}g) + g(\mathcal{L}f))m^2 \;=\; -2\int \nabla_v f \cdot \nabla_v g m^2 + 2\int fg\phi_2(m)m^2.$$

the computation can be found in Section 4.8.

Remark 4.1.5. For the kinetic Fokker-Planck equation with general force 4.1.1, we can take

$$m = e^{H_1}, \quad H_1 = |v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle, \quad H = \langle v \rangle^\beta + \langle x \rangle^{\gamma - 1},$$

for some $\epsilon > 0$ small, the computation can be found in Section 4.7 below. For the kinetic Fitzhugh-Nagumo equation (4.1.2), we can take

$$m = e^{\lambda(x^2+v^2)}, \quad H = |v|^4 + |x|^2,$$

for some constant $\lambda > 0$, the computation can be found in Section 4.6

For the kinetic Fitzhugh-Nagumo equation (4.1.2), an exponential convergence with non-quantitative rate to the convergence has already been proved in [53], our method improves the result to a quantitative rate.

If $\beta = 2$, the equation (4.1.1) will turn to the classical KFP equation

$$\partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(vf),$$

This time we observe that

$$G = Z^{-1}e^{-W}, \quad W = \frac{v^2}{2} + V(x), \quad Z \in \mathbb{R}_+,$$

is an explicit steady state. There are many classical results on the case $\gamma \ge 1$, where there is an exponentially decay. We refer the interested readers to [62, 19, 20, 42, 36, 38, 9, 47], and for the weak confinement case $\gamma \in (0,1)$, there are also some polynomial

or sub-geometric convergence results proved in [9, 10, 21]. We also emphasize that our results for kinetic Fokker-Planck equation with general potentials are to our knowledge new.

We carry out all of our proofs using variations of Harris's Theorem for Markov semigroup. One advantage of the Harris method is that it directly yields convergence for a wide range of initial conditions, while previous proofs of convergence to equilibrium mainly use some strongly weighted L^2 or H^1 norms (typically with a weight which is the inverse of a Gaussian). The Harris method also gives existence of stationary solutions under quite general conditions; in some cases these are explicit and easy to find, but in other cases such as the two models in our paper they can be nontrivial. Also the Harris method provides a quantitative rate of convergence to the steady state, which is better than non-quantitative type argument such as the consequence of Krein Rutman theorem.

Let us end the introduction by describing the plan of the chapter. In Section 4.2, we introduce a PDE translate of the Harris Theorem which is a little stronger than 1.2. In Section 4.3 we present the proof of a regularization estimate on $S_{\mathcal{L}}$. In Section 4.4 we prove the Harris-Doeblin condition for the general kinetic Fokker- Planck equation. In Section 4.5, we prove a lemma for the spreading of positivity. In Section 4.6 , 4.7 and 4.8 we make the computation for the kinetic Fokker Planck equation and the Fitzhugh-Nagumo equation.

4.2 Harris Theorem and existence of steady state

In this section we introduce a PDE translation version of the Harris-Doeblin theorem by José Cañizo and S. Mischler, then give a proof of the existence of steady state under Lyapunov condition.

Theorem 4.2.1. *(Harris- Doeblin Theorem) We consider a semigroup S_t in a Banach space X with generator $\mathcal{L}$ and we assume that*
(H1)(Lyapunov condition) There exists some weight function $m : \mathbb{R}^d \to [1, \infty)$ satisfying $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \leq -\alpha m + b,$$

(H2)(Harris condition) For any $R > 0$ there exist a constant $T = T(R) > 0$ and a positive, nonzero measure $\mu = \mu(R)$ such that

$$S_T f \geq \mu \int_{B_R} f, \quad \forall f \in X, \quad f \geq 0.$$

where B_R denotes the ball centered at origin with radius R. There exist some constants $C \geq 1$ and $a < 0$ such that

$$\|S_t f\|_{L^1(m)} \leq C e^{at} \|f\|_{L^1(m)}, \quad \forall t \geq 0, \quad \forall f \in X, \quad \mathcal{M}(f) = 0.$$

Remark 4.2.2. In fact this version of Harris-Doeblin Theorem is a little stronger than Theorem 1.2.8 because this version do not require a minimum of T for all R, in this version it may happen that

$$T(R) \to 0, \text{ as } R \to \infty,$$

while in Theorem 1.2.8 they require a minimum t_*.

Before proving the theorem, we first prove a useful lemma.

Lemma 4.2.3. *(Doeblin's variant). Under assumption (H2), if $f \in L^1(m)$, with $m(x) \to \infty$ as $|x| \to \infty$, satisfies*

$$\|f\|_{L^1} \geq \frac{4}{m(R)}\|f\|_{L^1(m)}, \quad \textit{and} \quad \langle f \rangle = 0, \tag{4.2.1}$$

we then have

$$\|S_T f\|_{L^1} \leq (1 - \frac{\langle \mu \rangle}{2})\|f\|_{L^1}$$

Proof. From the hypothesis (4.2.1), we have

$$\begin{aligned} \int_{B_R} f_\pm &= \int f_\pm - \int_{B_R^c} f_\pm \\ &\geq \frac{1}{2}\int |f| - \frac{1}{m(R)}\int |f| m \geq \frac{1}{4}\int |f| \end{aligned}$$

Together with (H2), we get

$$S_T f_\pm \geq \frac{\mu}{4}\int |f| := \eta$$

We deduce

$$|S_T f| \leq |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T f_+ - \eta + S_T f_- - \eta = S_T|f| - 2\eta,$$

and next

$$\int |S_T f| \leq \int S_T|f| - 2\eta = \int |f| - \frac{\mu}{2}\int |f|$$

which is nothing but the announced estimate.

□

Then we come to the proof of Theorem 4.2.1

Proof. Proof of Theorem 4.2.1. We split the proof in several steps.
Step 1. We fix $f_0 \in L^1(m), \langle f \rangle = 0$, and we denote $f_t := S_t f_0$. From (H1), we have

$$\frac{d}{dt}\|f\|_{L^1(m)} \leq -\alpha\|f_t\|_{L^1(m)} + b\|f_t\|_{L^1}, \quad \forall t \geq 0,$$

from what we deduce

$$\|S_T f\|_{L^1(m)} \leq e^{-\alpha t}\|f_0\|_{L^1(m)} + (1 - e^{-\alpha t})\frac{b}{\alpha}\|f_0\|_{L^1}, \quad \forall t \geq 0,$$

In other words, we have proved

$$\|S_T f\|_{L^1(m)} \leq \gamma \|f_0\|_{L^1(m)} + K\|f_0\|_{L^1}, \quad \forall t \geq 0, \tag{4.2.2}$$

with $\gamma \in (0,1)$ and $K > 0$. We fix $R > 0$ large enough such that $\frac{8b}{\alpha} \leq m(R)$, then take $T = T(R)$ and $\mu = \mu(R)$, define

$$\gamma := e^{-\alpha T}, \quad K := (1 - e^{-\alpha T})\frac{b}{\alpha}$$

Then we have $K/A \leq (1-\gamma)/2$ with $A := m(R)/4$. We also recall that

$$\|S_T f\|_{L^1} \leq \|f_0\|_{L^1}, \quad \forall t \geq 0, \tag{4.2.3}$$

We define

$$\|f\|_\beta = \|f_0\|_{L^1} + \beta\|f_0\|_{L^1(m)}$$

and we observe that the following altenative holds

$$\|f_0\|_{L^1(m)} \leq A\|f_0\|_{L^1} \tag{4.2.4}$$

or

$$\|f_0\|_{L^1(m)} > A\|f_0\|_{L^1} \tag{4.2.5}$$

Step 2. We observe that under condition (4.2.4), there holds

$$\|S_T f_0\|_{L^1} \leq \gamma_1 \|f_0\|_{L^1}, \quad \gamma_1 \in (0,1) \tag{4.2.6}$$

and more precisely $\gamma_1 := 1 - \langle \mu \rangle/2$, which is nothing but the conclusion of Lemma 2.3.
Step 3. We claim that under condition (4.2.4), there holds

$$\|S_T f_0\|_\beta \leq \gamma_2 \|f_0\|_\beta, \quad \gamma_2 := max(\frac{\gamma_1 + 1}{2}, \gamma) \tag{4.2.7}$$

for $\beta > 0$ small enough. Indeed, using (4.2.2) and (4.2.7), we compute

$$\begin{aligned} \|S_T f_0\|_\beta &= \|S_T f_0\|_{L^1} + \beta\|S_T f_0\|_{L^1(m)} \\ &\leq (\gamma_1 + K\beta)\|f_0\|_{L^1} + \gamma\beta\|f_0\|_{L^1(m)} \end{aligned}$$

and we take $\beta > 0$ such that $\gamma_1 + K\beta \leq \gamma_2$.
Step 4. We claim that under condition (4.2.5), there holds

$$\|S_T f_0\|_{L^1(m)} \leq \gamma_3 \|f_0\|_{L^1(m)}, \quad \gamma_3 := \frac{\gamma + 1}{2}. \tag{4.2.8}$$

Indeed we compute

$$\|S_T f_0\|_{L^1(m)} \leq \gamma\|f_0\|_{L^1(m)} + \frac{K}{A}\|f_0\|_{L^1(m)} = \gamma_3\|f_0\|_{L^1(m)}.$$

Step 5. We claim that under condition (4.2.5), there holds

$$\|S_T f_0\|_\beta \leq \gamma_4 \|f_0\|_\beta, \quad \gamma_4 := \frac{\gamma_3 + 1/\beta}{1 + 1/\beta}. \tag{4.2.9}$$

Indeed, using (4.2.3) and (4.2.8), we compute

$$\begin{aligned}\|S_T f_0\|_\beta &= \|S_T f_0\|_{L^1} + \beta\|S_T f_0\|_{L^1(m)} \\ &\leq \|f_0\|_{L^1} + \gamma_3\beta\|f_0\|_{L^1(m)} \\ &\leq (1-\epsilon)\|f_0\|_{L^1} + (\epsilon + \gamma_3\beta)\|f_0\|_{L^1(m)}\end{aligned}$$

and we choose $\epsilon \in (0,1)$ such that $1 - \epsilon = \epsilon/\beta + \gamma_3$.
Step 6. By gathering (4.2.7) and (4.2.9), we see that we have

$$\|S_T f_0\|_\beta \leq \gamma_5\|f_0\|_\beta, \quad \gamma_5 := max(\gamma_2, \gamma_4) \in (0,1),$$

for some well chosen $\beta > 0$. By iteration, we get

$$\|S_{nT} f_0\|_\beta \leq \gamma_5^n\|f_0\|_\beta,$$

and we then conclude in a standard way.

□

The Lyapunov condition also provides a sufficient condition for the existence of an invariant measure (for the dual semigroup).

Theorem 4.2.4. *Any mass conserving positive Markov semigroup (S_t) which fulfills the above Lyapunov condition has at least one invariant borelian measure $G \in M^1(m)$, where M^1 is the space of measures.*

Proof. Step 1. We prove that (S_t) is a bounded semigroup. For $f_0 \in M^1(m)$, we define $f_t := S_{\mathcal{L}}(t)f_0$, and we easily compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |f_t|m \leq \int |f_t|\mathcal{L}^* m \leq \int |f_t|(-am + b).$$

Using the mass conservation and positivity, integrating the above differential inequality, we get

$$\begin{aligned}\int |f_t|m &\leq e^{-at}\int |f_0|m + \frac{b}{a}(1 - e^{-at})\int |f_0| \\ &\leq \max(1, \frac{b}{a})\int |f_0|m, \quad \forall t \geq 0,\end{aligned}$$

so that (S_t) is bounded in $M^1(m)$.
Step 2. We prove the existence of a steady state, more precisely, we start proving that there exists a positive and normalized steady state $G \in M^1(m)$. For the equivalent norm $|||\cdot|||$ defined on $M^1(m)$ by

$$|||f||| := \sup_{t>0}\|S_{\mathcal{L}}(t)f\|_{M^1(m)},$$

and since we have

$$|||S_{\mathcal{L}}(t)f||| \leq |||f|||, \quad \forall t \geq 0,$$

t hat is the semigroup $S_{\mathcal{L}}$ is a contraction semigroup on $(M^1(m), |||\cdot|||)$. There exists $R > 0$ large enough such that the intersection of the closed hyperplane

$$\{f \in M^1(m); \langle f \rangle = 1\}$$

and the closed ball of radius R in $(M^1(m), |||\cdot|||)$ is a convex, non-empty subset. Then consider the closed, weakly * compact convex set

$$\mathbb{K} := \{f \in M^1(m); |||f||| \leq R, f \geq 0, \langle f \rangle = 1\},$$

Since $S_{\mathcal{L}}(t)$ is a linear, weakly * continuous, contraction in $(M^1(m), |||\cdot|||)$ and

$$\langle S_{\mathcal{L}}(t) f \rangle = \langle f \rangle, \quad \forall t \geq 0,$$

we see that $\mathbb{K}$ is stable under the action of the semigroup. Therefore we apply the Markov-Kakutani fixed point theorem and we conclude that there exists $G \in \mathbb{K}$ such that $S_{\mathcal{L}}(t)G = G$. Therefore we have in particular $G \in D(\mathcal{L})$ and $\mathcal{L}G = 0$. □

4.3 Regularization property of $S_{\mathcal{L}}$

The aim of this section is to establish the following regularization property, which is similar to Section 3.4. The proof closely follows the proof of similar results in [38, 47, 62]

Theorem 4.3.1. *Consider the weight function m as defined in Theorem 4.1.3, there exist $\eta, C > 0$ such that*

$$\|S_{\mathcal{L}}(t) f\|_{L^2(m)} \leq \frac{C}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta].$$

for some weight function m. In addition, for any integer $k > 0$ there exist we some $\alpha(k), C(k) > 0$ such that

$$\|S_{\mathcal{L}}(t) f\|_{H^k(m)} \leq \frac{C}{t^{\alpha}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta].$$

as a consequence we have

$$\|S_{\mathcal{L}}(t) f\|_{C^{2,\delta}(m)} \leq \frac{C}{t^{\zeta}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta],$$

for some $\delta \in (0, 1), \zeta > 0$

We start with some elementary lemmas.

Lemma 4.3.2. *For $f_t = S_{\mathcal{L}}(t)f_0$, define an energy functional*

$$\begin{aligned}\mathcal{F}(t,f_t) &:= A\|f_t\|^2_{L^2(m)} + at^2\|\nabla_v f_t\|^2_{L^2(m)} \\ &+ 2ct^4(\nabla_v f_t, \nabla_x f_t)^2_{L^2(m)} + bt^6\|\nabla_x f_t\|^2_{L^2(m)},\end{aligned} \tag{4.3.1}$$

with $a, b, c > 0, c \leq \sqrt{ab}$ and A large enough. Then there exist $\eta > 0$ such that

$$\frac{d}{dt}\mathcal{F}(t,f_t) \leq -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

for all $t \in [0,\eta]$, for some $L > 0, C > 0$, as a consequence, we have

$$\|S_{\mathcal{L}}f_0\|_{H^1(m)} \leq Ct^{-6}\|f_0\|_{L^2(m)},$$

for all $t \in [0,\eta]$, iterating k times we get

$$\|S_{\mathcal{L}}f_0\|_{H^k(m)} \leq Ct^{-6k}\|f_0\|_{L^2(m)}.$$

Proof. We only prove the case $k = 1$, for $k = 2$, one need only replace f by $\partial_{x_i} f$ and $\partial_{v_i} f$, similarly for $k > 2$. First by Theorem 4.1.3 and Remark 4.1.4 we have

$$(f, \mathcal{L}g)_{L^2(m)} + (g, \mathcal{L}f)_{L^2(m)} = -2(\nabla_v f, \nabla_v g)_{L^2(m)} + (f, g\phi_2(m))_{L^2(m)},$$

for any $f, g \in L^2(m)$. As a consequence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2(m)} = (f, \mathcal{L}f)_{L^2(m)} \leq -\|\nabla_v f\|^2_{L^2(m)} - C_1\|f\|^2_{L^2(mH^{1/2})} + C_2\|f\|^2_{L^2(m)}.$$

By

$$\partial_{x_i}\mathcal{L}f = \mathcal{L}\partial_{x_i} f + \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f, \tag{4.3.2}$$

and since

$$|\nabla^2_x V(x)| \leq CH_1,$$

for some $C > 0$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_i} f\|_{L^2(m)} \\ =\ & (\partial_{x_i} f, \mathcal{L}\partial_{x_i} f)_{L^2(m)} + (\partial_{x_i} f, \sum_{j=1}^{d} \partial^2_{x_i x_j} V \partial_{v_j} f)_{L^2(m)} \\ \leq\ & -\|\nabla_v(\partial_{x_i} f)\|^2_{L^2(m)} - C_1\|\partial_{x_i} f\|^2_{L^2(mH^{1/2})} + C_2\|\partial_{x_i} f\|^2_{L^2(m)} + C(|\nabla_x f|, |\nabla_v f|)_{L^2(mH^{1/2})}.\end{aligned}$$

Using Cauchy-Schwarz inequality and summing over $i = 1, 2, 3, ..., n$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2_{L^2(m)} \\ \leq\ & -\sum_{i=1}^{n} \|\nabla_v(\partial_{x_i} f)\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_x f\|^2_{L^2(mH^{1/2})} + C_2\|\nabla_x f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})}.\end{aligned}$$

for some $C > 0$. Similarly using

$$\partial_{v_i}\mathcal{L}f = \mathcal{L}\partial_{v_i}f - \partial_{x_i}f + \sum_{j=1}^{d}\partial^2_{v_iv_j}W\partial_{v_j}f, \tag{4.3.3}$$

and since

$$|\nabla_v^2 W(v)| \le \frac{C_1}{2d}H + C,$$

we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{v_i}f\|^2_{L^2(m)}\\
=\;& (\partial_{v_i}f, \mathcal{L}\partial_{v_i}f)_{L^2(m)} - (\partial_{x_i}f, \partial_{v_i}f)_{L^2(m)} + (\partial_{v_i}f, \sum_{j=1}^{d}\partial^2_{v_iv_j}W\partial_{v_j}f)_{L^2(m)}\\
\le\;& -\|\nabla_v(\partial_{v_i}f)\|^2_{L^2(m)} - C_1\|\partial_{v_i}f\|^2_{L^2(mH^{1/2})} + C_2\|\partial_{v_i}f\|^2_{L^2(m)} + \frac{C_1}{2d}\||\nabla_v f\|^2_{L^2(mH^{1/2})}\\
&+C\|\nabla_v f\|^2_{L^2(m)} - (\partial_{x_i}f, \partial_{v_i}f)_{L^2(m)}.
\end{aligned}$$

Using Cauchy-Schwarz inequality and summing over $i = 1, 2, ..., n$ we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f\|^2_{L^2(m)}\\
\le\;& -\sum_{i=1}^{n}\|\nabla_v(\partial_{v_i}f)\|^2_{L^2(m)} - \frac{C_1}{2}\|\nabla_v f\|^2_{L^2(mH^{1/2})}\\
&+C\||\nabla_v f\|^2_{L^2(m)} - (\nabla_v f, \nabla_x f)_{L^2(m)}.
\end{aligned}$$

For the crossing term, we split it also into two parts. Using (4.3.2) and (4.3.3), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}2(\partial_{v_i}f, \partial_{x_i}f)_{L^2(m)}\\
=\;& (\partial_{x_i}f, \mathcal{L}\partial_{v_i}f)_{L^2(m)} - (\partial_{x_i}f, \partial_{x_i}f)_{L^2(m)} + (\partial_{x_i}f, \sum_{j=1}^{d}\partial^2_{v_iv_j}W\partial_{v_j}f)_{L^2(m)}\\
&+(\partial_{v_i}f, \mathcal{L}\partial_{x_i}f)_{L^2(m)} + (\partial_{v_i}f, \sum_{j=1}^{d}\partial^2_{x_ix_j}V\partial_{v_j}f)_{L^2(m)}\\
\le\;& -2(\nabla_v(\partial_{x_i}f), \nabla(\partial_{v_i}f))_{L^2(m)} - \|\partial_{x_i}f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})}\\
+\;& C(|\nabla_v f|, |\nabla_x f|)_{L^2(mH^{1/2})},
\end{aligned}$$

Combining the two parts, using Cauchy-Schwarz inequality, and summing over i we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}2(\nabla_v f, \nabla_x f)_{L^2(m)}\\
\le\;& -2\sum_{i=1}^{n}(\nabla_v(\partial_{x_i}f), \nabla(\partial_{v_i}f))_{L^2(m)} - \|\nabla_x f\|^2_{L^2(m)} + C\|\nabla_v f\|^2_{L^2(mH^{1/2})}\\
+\;& C(|\nabla_v f|, |\nabla_x f|)_{L^2(mH^{1/2})},
\end{aligned}$$

For the very definition of $\mathcal{F}$ in (4.3.1), we easily compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t, f_t) \;=\;& A\frac{\mathrm{d}}{\mathrm{d}t}\|f_t\|^2_{L^2(m)} + at^2\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f_t\|^2_{L^2(m)} + 2ct^4\frac{\mathrm{d}}{\mathrm{d}t}(\nabla_v f_t, \nabla_x f_t)^2_{L^2(m)}\\
&+bt^6\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f_t\|^2_{L^2(m)} + 2at\|\nabla_v f_t\|^2_{L^2(m)} + 8ct^3(\nabla_v f_t, \nabla_x f_t)^2_{L^2(m)}\\
&+6bt^5\|\nabla_x f_t\|^2_{L^2(m)}.
\end{aligned}$$

Gathering all the inequalities above together, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t,f_t) &\leq (2at - A + Cat^2)\|\nabla_v f_t\|^2_{L^2(m)} + (6bt^5 - \frac{c}{2}t^4 + Cbt^6)\|\nabla_x f_t\|^2_{L^2(m)}\\ &+ (8ct^3 - Cat^2)(\nabla_v f_t, \nabla_x f_t)_{L^2(m_2)} + CA\|f_t\|^2_{L^2(m)},\\ &- \sum_{i=1}^{d}(at^2\|\nabla_v(\partial_{v_i} f_t)\|^2_{L^2(m)} + bt^6\|\nabla_v(\partial_{x_i} f_t)\|^2_{L^2(m)}\\ &+ 2ct^4(\nabla_v(\partial_{x_i} f_t), \nabla_v(\partial_{v_i} f_t))_{L^2(m)}) - \frac{C_1}{2}ct^6\|\nabla_x f\|^2_{L^2(mH^{1/2})}\\ &+ (-\frac{C_1}{2}at^2 + 2Cbt^6 + Cct^4)\|\nabla_v f\|^2_{L^2(mH^{1/2})} + 2bt^4C(|\nabla_v f|, |\nabla_x f|)_{L^2(mH^{1/2})},\end{aligned}$$

for some $C > 0$. We observe that

$$\begin{aligned}&|2ct^4(\nabla_v(\partial_{x_i} f_t), \nabla_v(\partial_{v_i} f_t))_{L^2(m)})|\\ \leq\ & at^2\|\nabla_v(\partial_{v_i} f_t)\|^2_{L^2(m)} + bt^6\|\nabla_v(\partial_{v_i} f_t)\|^2_{L^2(m)},\end{aligned}$$

by our choice on a, b, c. So by taking A large and $0 < \eta$ small ($t \in [0, \eta]$), we conclude to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t,f_t) \leq -L(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}) + C\|f_t\|^2_{L^2(m)},$$

for some $L, C > 0$, and that ends the proof. □

Lemma 4.3.3. *We have*

$$\int |\nabla_{x,v}(fm)|^2 \leq \int |\nabla_{x,v} f|^2 m^2 + C\int f^2 m^2,$$

Proof. We have

$$\begin{aligned}\int |\nabla(fm)|^2 &= \int |\nabla f m + \nabla m f|^2\\ &= \int |\nabla f|^2 m^2 + \int |\nabla m|^2 f^2 + \int 2f\nabla f m \nabla_x m\\ &= \int |\nabla f|^2 m^2 + \int (|\nabla m|^2 - \frac{1}{2}\Delta(m^2))f^2,\\ &= \int |\nabla f|^2 m^2 - \int \frac{\Delta m}{m} f^2 m^2,\end{aligned}$$

since

$$\frac{\Delta m}{m} \geq -C,$$

for some $C > 0$, we are done. □

Lemma 4.3.4. *Nash's inequality: for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$,there exist a constant C_d such that:*

$$\|f\|_{L^2}^{1+\frac{2}{d}} \leq C_d\|f\|_{L^1}^{\frac{2}{d}}\|\nabla_v f\|_{L^2},$$

For the proof of Nash's inequality, we refer to [44], Section 8.13 for instance. □

Lemma 4.3.5. *There exist $\lambda > 0$ such that*

$$\frac{d}{dt}\|f\|_{L^1(m)} \leq \lambda \|f\|_{L^1(m)} \tag{4.3.4}$$

which implies

$$\|f_t\|_{L^1(m)} \leq Ce^{\lambda t}\|f_0\|_{L^1(m)}$$

In particular we have

$$\|f_t\|_{L^1(m)} \leq C\|f_0\|_{L^1(m)}, \quad \forall t \in [0, \eta], \tag{4.3.5}$$

for some constant $C > 0$.

Proof. It' s an immediate consequence of the Lyapunov condition (H1). □

Now we come to the proof of Theorem 4.3.1.

Proof. (**Proof of Theorem 4.3.1.**) We define

$$\mathcal{G}(t, f_t) = B\|f_t\|^2_{L^1(m)} + t^Z \mathcal{F}(t, f_t),$$

with $B, Z > 0$ to be fixed and $\mathcal{F}$ defined in Lemma 4.3.2. We choose $t \in [0, \eta]$, η small enough such that $(a + b + c)Z\eta^{Z+1} \leq \frac{1}{2}L\eta^Z$ (a, b, c, L are also defined Lemma 4.3.2). By (4.3.4) and Lemma 4.3.2, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}(t, f_t) &\leq \lambda B\|f_t\|^2_{L^1(m)} + Zt^{Z-1}\mathcal{F}(t, f_t) \\
&\quad -Lt^Z(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}) + Ct^Z\|f_t\|^2_{L^2(m)} \\
&\leq \lambda B\|f_t\|^2_{L^1(m)} + Ct^{Z-1}\|f_t\|^2_{L^2(m)} \\
&\quad -\frac{L}{2}t^Z(\|\nabla_v f_t\|^2_{L^2(m)} + t^4\|\nabla_x f_t\|^2_{L^2(m)}),
\end{aligned}$$

where λ is defined in Lemma 4.3.5. Nash's inequality and Lemma 4.3.2 imply

$$\|fm\|_{L^2} \leq C\|fm\|_{L^1}^{\frac{2}{d+2}}\|\nabla_{x,v}(fm)\|_{L^2}^{\frac{d}{d+2}} \leq C\|fm\|_{L^1}^{\frac{2}{d+2}}(\|\nabla_{x,v}fm\|_{L^2} + C\|fm\|_{L^2})^{\frac{d}{d+2}}.$$

Using Young's inequality, we have

$$\|f_t\|^2_{L^2(m)} \leq C_\epsilon t^{-5d}\|f\|^2_{L^1(m)} + \epsilon t^5(\|\nabla_{x,v}f_t\|^2_{L^2(m)} + C\|f_t\|^2_{L^2(m)}).$$

Taking ϵ small such that $C\epsilon\eta^5 \leq \frac{1}{2}$, we deduce

$$\|f_t\|^2_{L^2(m)} \leq 2C_\epsilon t^{-5d}\|f\|^2_{L^1(m)} + 2\epsilon t^5\|\nabla_{x,v}f_t\|^2_{L^2(m)}.$$

Taking ϵ small we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t, f_t) \leq dB\|f_t\|^2_{L^1(m)} + C_1 t^{Z-1-5d}\|f_t\|^2_{L^1(m)},$$

for some $C_1 > 0$. Choosing $Z = 1 + 5d$, and using (4.3.5), we deduce

$$\forall t \in [0, \eta], \quad \mathcal{G}(t, f_t) \leq \mathcal{G}(0, f_0) + C_2 \|f_0\|^2_{L^1(m)} \leq C_3 \|f_0\|^2_{L^1(m)},$$

which proves

$$\|S_{\mathcal{L}}(t) f\|_{L^2(m)} \leq \frac{C}{t^{\frac{5d+1}{2}}} \|f\|_{L^1(m)}, \quad \forall t \in [0, \eta].$$

together with Lemma 4.3.2 ends the proof. □

4.4 Proof of Harris condition

In this section we prove the Harris condition (H2) for equation (4.1.3). Before the proof of the theorem, we first prove a useful lemma.

Lemma 4.4.1. *For any $R > 0$, there exist a λ, ρ such that for any $t > 0$, there exists $(x_0, v_0) \in B_\rho$ such that*

$$f(t, x_0, v_0) \geq \lambda \int_{B_R} f_0.$$

Proof. From conservation of mass, we classically show that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f(t, x, v) dx dv = 0,$$

so we have

$$\|S_{\mathcal{L}}(t)\|_{L^1 \to L^1} \leq 1, \quad \forall t \geq 0, \tag{4.4.1}$$

Define the splitting of the KFP operator $\mathcal{L}$ by

$$\mathcal{B} = \mathcal{L} - \mathcal{A}, \quad \mathcal{A} = M \chi_R(x, v).$$

with $M, R > 0$ large, where χ is the cut-off function such that $\chi(x, v) \in [0, 1]$, $\chi(x, v) \in C^\infty$, $\chi(x, v) = 1$ when $x^2 + v^2 \leq 1$, $\chi(x, v) = 0$ when $x^2 + v^2 \geq 2$, and $\chi_R = \chi(x/R, v/R)$. From the Lyapunov function condition (H1) and taking M, R large, we have

$$\|S_{\mathcal{B}}(t)\|_{L^1(m) \to L^1(m)} \leq C e^{-\lambda t}, \quad \forall t \geq 0. \tag{4.4.2}$$

By Duhamel's formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{L}},$$

we directly deduce from (4.4.1) and 4.4.2 that

$$\|S_{\mathcal{L}}(t)\|_{L^1(m) \to L^1(m)} \leq A, \quad \forall t \geq 0,$$

for some $A > 0$. We fix $R > 0$ and take $g_0 = f_0 1_{B_R} \in L^1(\mathbb{R}^d)$ with that supp $g_0 \subset B_R$, ,denote $g_t = S_{\mathcal{L}} g_0$, $f_t = S_{\mathcal{L}} f_0$, then we have

$$\int_{\mathbb{R}^d} g_t = \int_{\mathbb{R}^d} g_0 = \int_{B_R} g_0 = \int_{B_R} f_0.$$

Moreover, since there exists $A > 0$ such that

$$\int_{\mathbb{R}^d} g_t m \leq A \int_{\mathbb{R}^d} g_0 m \leq Am(R) \int_{B_R} g_0.$$

For any $\rho > 0$, we write

$$\begin{aligned} \int_{B_\rho} g_t &= \int_{\mathbb{R}^d} g_t - \int_{B_\rho^c} g_t \\ &\geq \int_{\mathbb{R}^d} g_0 - \frac{1}{\rho} \int_{\mathbb{R}^d} g_t m \\ &\geq \int_{\mathbb{R}^d} g_0 - \frac{Am(R)}{\rho} \int_{B_R} g_0 \geq \frac{1}{2} \int_{B_R} g_0, \end{aligned}$$

by taking $\rho = 2Am(R)$. As a consequence, for any $t > 0$, there exist a $(x_0, v_0) \in B_\rho$ which may depend on g_0 such that

$$g(t, x_0, v_0) \geq \frac{1}{|B_\rho|} \int_{B_\rho} g_t \geq \frac{1}{2|B_{2Am(R)}|} \int_{B_R} g_0 := \lambda \int_{B_R} g_0.$$

By the maximum principle we have

$$f(t, x_0, v_0) \geq g(t, x_0, v_0) \geq \lambda \int_{B_R} g_0 = \lambda \int_{B_R} f_0.$$

□

Theorem 4.4.2. *The equation (4.1.3) satisfies the Harris condition.*

Proof. By Theorem 4.3.1 we now take for $t > \frac{\eta}{2}$, we have

$$\Delta_v f, \nabla_x f, \nabla_v f \in C^{0,\alpha},$$

and by equation

$$\partial_t f := \mathcal{L} f = \partial_x (A(x, v) f) + \partial_v (B(x, v) f) + \Delta_v f,$$

we deduce that for any $R_1 > 0$ we have,

$$|\partial_t f| + |\partial_x f| + |\partial_v f| \leq C \quad , \quad \text{on } [\frac{\eta}{2}, \eta] \times B_{R_1},$$

for some constant $C > 0$. By continuity for every $R > 0$, there exist $t_1, t_2, r_0, \rho, \lambda > 0$ which do not depend on f and $(x_0, v_0) \in B_\rho$ which may depend on f, such that for all $t \in (t_1, t_2)$, we have

$$f(t, x, v) \geq \frac{\lambda}{2} 1_{B_{r_0}(x_0, v_0)} \int_{B_R} f_0,$$

where $B_{r_0}(x_0, v_0)$ denotes the ball centered at (x_0, v_0) with radius r_0, to make x_0, v_0 f independent we use

Theorem 4.4.3. *Let $f(t,x,v)$ be a classical nonnegative solution of*

$$\partial_t f - \Delta_v f = -(v+\Phi(x))\cdot\nabla_x f + A(t,x,v)\cdot\nabla_v f + B(t,x,v)f,$$

in $[0,T)\times\Omega$, where $\Phi(x)$ is Lipschitz

$$|\Phi(x)-\Phi(y)| \le M|x-y|, \quad \forall x,y\in\Omega,$$

Ω is an open subset of $\mathbb{R}^2$, and $A,B : [0,T)\times\mathbb{R}^d$ and bounded continuous functions. Let $(x_0,v_0)\in\Omega$, let

$$V = \max\{v_0+\Phi(x_0)|(v_0,x_0)\in\Omega\}, \quad \bar{A} = \|A\|_{L^\infty(\Omega)}, \quad \bar{B} = \|B\|_{L^\infty(\Omega)},$$

then for any $r,\tau>0$ there are constants $\lambda,K>0$, only depending on $\bar{A},V,\bar{B}$and r^2/τ, , such that the following holds: If

$$B_{\lambda r}(x_0,v_0)\in\Omega, \quad \tau < min(1/2, r^3/4V, 1/M)$$

and

$$f\ge\delta>0, \quad in \ \ [0,\tau)\times B_r(x_0,v_0),$$

then

$$f\ge K\delta, \quad in\ [\tau/2,\tau)\times B_{2r}(x_0,v_0).$$

in fact it is possible to take λ such that

$$2068\times 16\times 8\max(\frac{r^2}{\tau},\frac{r^6}{\tau^3}) = \lambda\min(\frac{r^2}{\tau},\frac{r^6}{\tau^3}).$$

Proof. See Section 4.5. □

Coming back to the proof of Theorem 4.4.2. Define

$$T = \min(t_2-t_1, 1/2, r_0^3/4R, 1/M),$$

iterate n times we have for any $t\in(t_2-\frac{T}{2^n},t_2)$

$$f(t,x,v)\ge\frac{\lambda}{2}\prod_{i=1}^{n}K_i 1_{B_{2^n r_0}(x_0,v_0)}\int_{B_R} f_0,$$

take n large such that $2^n r_0 > 2\rho$, since $(x_0,v_0)\in B_\rho$ implies that $B_\rho\subset B_{2\rho}(x_0,v_0)$, we have

$$f(t,x,v)\ge\frac{\lambda}{2}\prod_{i=1}^{n}K_i 1_{B_\rho}\int_{B_R} f_0,$$

for any $t\in(t_2-\frac{T}{2^n},t_2)$, which is just Harris condition. □

4.5 Proof of spreading of positivity

In this chapter we prove the spreading of positivity

Theorem 4.5.1. *Let $f(t,x,v)$ be a classical nonnegative solution of*

$$\partial_t f - \Delta_v f = -(v+\Phi(x))\cdot\nabla_x f + A(t,x,v)\cdot\nabla_v f + B(t,x,v)f,$$

in $[0,T)\times\Omega$, where $\Phi(x)$ is Lipschitz

$$|\Phi(x)-\Phi(y)| \le M|x-y|, \quad \forall x,y\in\Omega,$$

Ω is an open subset of $\mathbb{R}^2$, and $A,B : [0,T)\times\mathbb{R}^d$ and bounded continuous functions. Let $(x_0,v_0)\in\Omega$, let

$$V = \max\{v_0+\Phi(x_0)|(v_0,x_0)\in\Omega\}, \quad \bar{A} = \|A\|_{L^\infty(\Omega)}, \quad \bar{B} = \|B\|_{L^\infty(\Omega)},$$

then for any $r,\tau>0$ there are constants $\lambda, K>0$, only depending on $\bar{A}, V, \bar{B}$ and r^2/τ, , such that the following holds: If

$$B_{\lambda r}(x_0,v_0)\in\Omega, \quad \tau < min(1/2, r^3/4V, 1/M)$$

and

$$f\ge\delta>0, \quad in \ \ [0,\tau)\times B_r(x_0,v_0),$$

then

$$f\ge K\delta, \quad in\ [\tau/2,\tau)\times B_{2r}(x_0,v_0).$$

in fact it is possible to take λ such that

$$2068\times 16\times 8\max(\frac{r^2}{\tau},\frac{r^6}{\tau^3}) = \lambda\min(\frac{r^2}{\tau},\frac{r^6}{\tau^3}).$$

Proof. This proof is similar to the proof in [62] Appendix A. 22. Let $g(t,x,v) = e^{\bar{B}t}f(t,x,v)$, then $g\ge f$ and $\mathcal{L}g\ge 0$ in $(0,T)\times\Omega$, where

$$\mathcal{L} = \frac{\partial}{\partial t} + (v+\Phi(x))\cdot\nabla_x - \Delta_v - A(t,x,v)\cdot\nabla_v,$$

Let us construct a particular subsolution for $\mathcal{L}$. In the sequel, B_r will stand for $B_r(x_0,v_0)$. For $t\in(0,\tau]$ and $(x,v)\in\Omega\setminus B_r$ let

$$Q(t,x,v) = a\frac{|v-v_0|^2}{2t} - b\frac{\langle v-v_0, x-X_t(x_0,v_0)\rangle}{t^2} + c\frac{|x-X_t(x_0,v_0)|^2}{2t^3},$$

where $X_t(x_0,v_0) = x_0 + t(v_0+\Phi(x_0))$ (abbreviated X_t in the sequel) is the position at time t of the geodesic flow starting from (x_0,v_0), and $a,b,c>0$ will be chosen later on. Let further

$$\phi(t,x,v) = \delta e^{-\mu Q(t,x,v)} - \epsilon,$$

where $\mu, \epsilon > 0$ will be chosen later on. Let us assume $b^2 < ac$, so that Q is a positive definite quadratic form in the two variables $v - v_0$ and $x - X_t$. Then

$$\mathcal{L}\phi = \mu\delta e^{-\mu Q}\mathcal{A}(Q),$$

where

$$\mathcal{A}(Q) = \partial_t Q + (v + \Phi(x)) \cdot \nabla_x Q - \Delta_v Q + \mu|\nabla_v Q|^2 - A(t,x,v) \cdot \nabla_v Q.$$

By computation,

$$\begin{aligned}
\mathcal{A}(Q) &= -a\frac{|v-v_0|^2}{2t^2} + 2b\frac{\langle v - v_0, x - X_t\rangle}{t^3} - 3c\frac{|x - X_t(x_0,v_0)|^2}{2t^4} \\
&+ b\frac{\langle v - v_0, v_0 + \Phi(x_0)\rangle}{t^2} - c\frac{\langle x - X_t, v_0 + \Phi(x_0)\rangle}{t^3} \\
&- b\frac{\langle v - v_0, v + \Phi(x)\rangle}{t^2} + c\frac{\langle x - X_t, v + \Phi(x)\rangle}{t^3} - a\frac{n}{t} \\
&+ \mu|a\frac{v - v_0}{t} - b\frac{x - X_t}{t^2}|^2 - a\frac{\langle A, v - v_0\rangle}{t} + b\frac{\langle A, x - X_t\rangle}{t^2} \\
&= \mathcal{B}\left(\frac{v - v_0}{t}, \frac{x - X_t}{t^2}\right) - a\frac{\langle A, v - v_0\rangle}{t} + b\frac{\langle A, x - X_t\rangle}{t^2} - a\frac{d}{t} \\
&- b\frac{\langle v - v_0, \Phi(x) - \Phi(x_0)\rangle}{t^2} + c\frac{\langle x - X_t, \Phi(x) - \Phi(x_0)\rangle}{2t^3},
\end{aligned}$$

where $\mathcal{B}$ is a quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ with matrix $p \otimes I_n$,

$$P = \begin{pmatrix} \mu a^2 - \frac{a}{2} + b & -\mu ab + b + \frac{c}{2} \\ -\mu ab + b + \frac{c}{2} & \mu b^2 - \frac{3c}{2} \end{pmatrix}$$

If a, b, c are given, then as $\mu \to \infty$

$$\begin{cases} \mathrm{tr}p = \mu(a^2 + b^2) + O(1), \\ \det P = \mu(\frac{3ab^2}{2} + abc - b^3 - \frac{3a^2c}{2}) + O(1). \end{cases}$$

Both quantities are positive if $b \geq a$ and $ac \geq b^2$, for example we can take $b = 2a, c = 8a$, then as $\mu \to \infty$ the eigenvalues of P are of order μb^2 and $ac/b > b$. So for any fixed C we may choose a, b, c and μ so that

$$\mathcal{B}(\frac{v - v_0}{t}, \frac{x - X_t}{t^2}) \geq Cb(\frac{|v - v_0|^2}{t^2} + \frac{|x - X_t|^2}{t^4}).$$

where C is arbitrarily large. If $t \in (0, \min\{\frac{1}{M}, 1\})$, we have

$$\epsilon(Q) \geq -8b\frac{|x - X_t|^2}{t^4} - 3b\frac{|v - v_0|^2}{t^2} - 3b\bar{A}^2 - 3bV^2 - \frac{\beta d}{2t},$$

gathering the two terms, we have

$$\mathcal{A}(Q) \geq const.\frac{b}{t}[C(\frac{|v - v_0|^2}{t} + \frac{|x - X_t|^2}{t^3}) - 1].$$

with C arbitrarily large. Recall that $(x,v) \notin B_r$, so
- either $|v-v_0| \geq r$, and then $\mathcal{A}(Q) \geq const.(b/t)[Cr^2/\tau - 1]$, which is positive if $C > \tau/r^2$;
- or $|x-x_0| \geq r^3$, and then, for any $\tau \leq r^3/(4V)$, then we have

$$\frac{|x-X_t|^2}{t^2} \geq \frac{|x-x_0|^2}{2t^2} - 2|v_0+\Phi(x_0)|^2 \geq \frac{r^6}{2\tau^2} - 2V^2 \geq \frac{r^6}{4\tau^2},$$

so $\mathcal{A}(Q) \geq const.(b/t)[Cr^6/4\tau^3 - 1]$, which is positive as soon as $C > 4(\tau/r^2)^3$.

To summarize: under our assumptions there is a way to choose the constants a, b, c, μ, depending only on $d, \bar{A}, r^2/\tau$, satisfying $c > b > a > 1$ and $ac > b^2$, so that

$$\mathcal{L}\phi \leq 0, \quad \text{in } [0,\tau) \times (B_{\lambda r} \setminus B_r),$$

as soon as $\tau \leq min(1, r^3/(4V), \frac{1}{M})$. We now wish to enforce $\phi \leq g$ for $t = 0$ and for $(x,v) \in \partial(B_{\lambda r} \setminus B_r)$; then the classical maximum principle will imply $g \geq \phi$, in $[0,\tau) \times (B_{\lambda r} \setminus B_r)$.

The boundary condition at $t = 0$ is obvious since ϕ vanishes identically there (more rigorously, ϕ can be extended by continuity by 0 at $t = 0$). The condition is also true on ∂B_r since $\phi \leq \delta$ and $g \geq \delta$. It remains to impose it on $\partial B_{\lambda r}$. For that we estimate Q from below: as soon as ac/b^2 is large enough, it's easily easy to seen that for any $(x,v) \in \partial B_{\lambda r}$

$$Q(t,x,v) \geq \frac{a}{4}(\frac{|v-v_0|^2}{t} + \frac{|x-X_t|^2}{t^3}) \geq \frac{a}{4}\min(\frac{\lambda^2 r^2}{\tau}, \frac{\lambda^6 r^6}{4\tau^3}) \geq \frac{\alpha\lambda^2}{16}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}),$$

Thus if we choose

$$\epsilon = \delta \exp(-\frac{\mu\alpha\lambda^2}{16}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})),$$

we make sure that $\phi = \delta e^{-\mu Q} - \epsilon \leq 0$ on $\partial B_{\lambda r}$, a fortiori $\phi \leq g$ on this set, and then we can apply the maximum principle.

So now we have $\phi \leq g$, and this will yield a lower bound for g in $[\tau/2,\tau) \times (B_{2r} \setminus B_r)$: indeed, if $t \geq \tau/2$ and $(x,v) \in B_{2r} \setminus B_r$ then

$$Q(t,x,v) \leq 2c(\frac{|v-v_0|^2}{t} + \frac{|x-X_t|^2}{t^3}) \leq 2c(8\frac{r^2}{\tau} + \frac{1026r^6}{\tau^3}) \leq 2068c\max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})$$

For λ large enough we find $K_0 > 0$ such that

$$\phi(t,x,v) \geq \delta[\exp(-2068\mu c\max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3})) - \exp(-\frac{-\mu a\lambda^2}{16}\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}))] \geq K_0\delta,$$

because $c = 8a$, to find such λ it suffices that

$$2068 \times 16 \times 8\max(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}) \leq \lambda\min(\frac{r^2}{\tau}, \frac{r^6}{\tau^3}),$$

by consequence λ depends only on r^2/τ.
Finally we find $K, \lambda > 0$ depending on $\bar{A}, \bar{C}$ and r^2/τ such that

$$f \geq K_0\delta e^{-\tau\bar{C}} \quad \text{on} \quad [\tau/2,\tau) \times (B_{2r} \setminus B_r),$$

which conclude the proof. □

4.6 Lyapunov function for the kinetic Fitzhugh-Nagumo equation

Theorem 4.6.1. *Denote $\mathcal{L}$ the operator of the kinetic Fitzhugh-Nagumo equation (4.1.3), then there exist a weight function m and a function H satisfies Theorem 4.1.3.*

Proof. We recall that the kinetic Fitzhugh-Nagumo equation

$$\partial_t f := \mathcal{L}f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \partial^2_{vv} f,$$

with

$$A(x,v) = ax - bv, \quad B(x,v) = v(v-1)(v-c) + x,$$

for some $a, b, c > 0$. This time we have

$$\begin{aligned}\phi_2(m) &= v\cdot\frac{\nabla_x m}{m} - x\cdot\frac{\nabla_x m}{m} + \frac{d}{2} + \frac{|\nabla_v m|^2}{m^2} \\ &\quad + \frac{\Delta_v m}{m} - (v(v-1)(v-c)+x)\cdot\frac{\nabla_v m}{m} + \frac{1}{2}(3v^2 + 2(1+c)v + c),\end{aligned}$$

We can take $m = e^{\frac{r}{2}(|x|^2+|v|^2)}$, with $r > 0$ to be fixed later, then we have

$$\frac{\nabla_x m}{m} = rx, \quad \frac{\nabla_v m}{m} = rv, \quad \frac{\Delta_v m}{m} = r + r^2|v|^2,$$

and this time we have

$$\begin{aligned}\phi_2(m) &= rx\cdot v - r|x|^2 + \frac{2}{d} + r^2|x|^2 + r + r^2|v|^2 \\ &\quad - r|v|^2(v-1)(v-c) - rx\cdot v + \frac{1}{2}(3v^2 + 2(1+c)v + c),\end{aligned}$$

if we take $r < 1$ and

$$H = |v|^4 + |x|^2$$

then we have

$$-C_1 Hm \le \phi_2(m) \le -C_2 Hm + C_3,$$

for some $C_1, C_2, C_3 > 0$, and it's easily seen that for any integer $n \ge 2$ fixed, for any $\epsilon > 0$ small, we can find a constant $C_{\epsilon,n}$ such that

$$\sum_{k=2}^{n} |D^k_x(x)| + \sum_{k=2}^{n} |D^k_{x,v}(v(v-1)(v-c)+x))| \le C_{n,\epsilon} + \epsilon H,$$

since

$$\frac{\Delta_{x,v} m}{m} = r + r^2|v|^2 \ge 0,$$

all the conditions of Theorem 4.1.3 is satisfied, the proof is finished. □

4.7 Lyapunov function for the KFP equation

In this section we will give Lyapunov condition for the kinetic Fokker-Planck equation.

Theorem 4.7.1. *Denote $\mathcal{L}$ the operator of the kinetic Fokker-Planck equation (4.1.1), then there exist a weight function m satisfies Theorem 4.1.1.*

Proof. First we recall the kinetic Fokker-Planck equation with general force

$$\partial_t f = \mathcal{L}f := -v \cdot \nabla_x f + \nabla_x V(x) \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(\nabla_v W(v) f), \tag{4.7.1}$$

with

$$V(x) = \frac{\langle x \rangle^\gamma}{\gamma}, \quad \gamma \geq 1, \quad W(v) = \frac{\langle v \rangle^\beta}{\beta}, \quad \beta \geq 2$$

where $\langle x \rangle^2 := 1 + |x|^2$, and

$$\mathcal{L}^* f = v \cdot \nabla_v f - \nabla_x V(x) \cdot \nabla_v f + \Delta_v f - \nabla_v W(v) \cdot \nabla_v f,$$

we compute

$$\mathcal{L}^*(v^2 + V(x)) = d - v \cdot \nabla_v W(v),$$

and

$$\mathcal{L}^*(v \cdot \nabla_x \langle x \rangle) = v \nabla_x^2 \langle x \rangle v - \nabla_x V(x) \cdot \nabla_x \langle x \rangle + \nabla_v W(v) \cdot \nabla_x \langle x \rangle,$$

since

$$\nabla_x^2 \langle x \rangle \leq CI,$$

where I is the $d \times d$ identity matrix, combine the two terms together we have

$$\mathcal{L}^*(|v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle) \leq C - C(\langle v \rangle^\beta + \langle x \rangle^{\gamma - 1}),$$

with $\epsilon > 0$ small, denote

$$H = |v|^2 + V(x) + \epsilon v \cdot \nabla_x \langle x \rangle,$$

and since

$$\frac{\mathcal{L}^* e^{\lambda H}}{e^{\lambda H}} = \lambda(v \cdot \nabla_x H - \nabla_x V(x) \cdot \nabla_v H + \Delta_v H + \lambda |\nabla_v H|^2 - \nabla_v W(v) \cdot \nabla_v H),$$

take $\lambda > 0$ small, we have

$$\mathcal{L}^*(e^{\lambda H}) \leq -C_1 H_1 e^{\lambda H} + C_2,$$

for some constant $C_1, C_2 > 0$, with $H_1 = \langle v \rangle^\beta + \langle x \rangle^{\gamma - 1}$, then the Lyapunov condition follows. For the second inequality, by Lemma 4.8.1 we have

$$\begin{aligned} \phi_2(e^{\lambda H}) &= \lambda(v \cdot \nabla_x H + \nabla_x V(x) \cdot \nabla_v H + \frac{1}{2} \Delta_v W(v) \\ &+ \Delta_v H + (\lambda^2 + \lambda) |\nabla_v H|^2 - \nabla_v W(v) \cdot \nabla_v H), \end{aligned}$$

and we still have

$$-C_3H_1e^{\lambda H}\phi_2(e^{\lambda H}) \leq -C_1H_1e^{\lambda H} + C_2,$$

for some constant $C_1, C_2, C_3 > 0$, thus the theorem is proved.

□

4.8 Computation for $\phi_2(m)$

In this section we only make the computation for Remark 4.1.4.

Lemma 4.8.1. *Define*

$$\partial_t f := \mathcal{L}f = \partial_x(A(x,v)f) + \partial_v(B(x,v)f) + \Delta_v f, \tag{4.8.1}$$

with

$$A(x,v) = -v + \Phi(x),$$

Then for any weight function m we have

$$\int (f(\mathcal{L}g) + g(\mathcal{L}f))m^2 = -2\int \nabla_v f \cdot \nabla_v g m^2 + 2\int fg\phi_2(m)m^2, \tag{4.8.2}$$

with

$$\begin{aligned}\phi_2(m) &= v\cdot\frac{\nabla_x m}{m} - \Phi(x)\cdot\frac{\nabla_x m}{m} + \frac{1}{2}div\ \Phi(x) + \frac{|\nabla_v m|^2}{m^2}\\ &+ \frac{\Delta_v m}{m} - B(x,v)\cdot\frac{\nabla_v m}{m} + \frac{1}{2}div_v B(x,v),\end{aligned}$$

where we use $\int f$ in place of $\int_{\mathbb{R}^d\times\mathbb{R}^d} f dxdv$ for short.

Proof. Define

$$\mathcal{T}f = -v\cdot\nabla_x f,$$

we have

$$\int f(\mathcal{T}g)m^2 + \int(\mathcal{T}f)gm^2 = \int \mathcal{T}(fg)m^2 = -\int fg\mathcal{T}(m^2) = -2\int fgm^2\frac{\mathcal{T}m}{m},$$

for the term with operator Δ we have

$$\begin{aligned}\int(f\Delta_v g + g\Delta_v f)m^2 &= -\int \nabla_v(fm^2)\cdot\nabla_v g + \nabla_v(gm^2)\cdot\nabla_v f\\ &= -2\int \nabla_v f\cdot\nabla_v g m^2 + \int fg\Delta_v(m^2)\\ &= -2\int \nabla_v f\cdot\nabla_v g m^2 + 2\int fg|\nabla_v m|^2 + \Delta_v m m,\end{aligned}$$

using integration by parts

$$\begin{aligned}
& \int f\mathrm{div}_v(B(x,v)g)m^2 + g\mathrm{div}_v(B(x,v)f)m^2 \\
= & \int fB(x,v)\cdot\nabla_v g m^2 + gB(x,v)\cdot\nabla_v f m^2 + 2\mathrm{div}_v B(x,v)fgm^2 \\
= & \int -fg\nabla_v\cdot(B(x,v)m^2) + 2\mathrm{div}_v B(x,v)fgm^2 \\
= & \int -2fgB(x,v)\cdot\frac{\nabla_v m}{m}m^2 + \mathrm{div}_v B(x,v)fgm^2,
\end{aligned}$$

similarly

$$\begin{aligned}
& \int f\mathrm{div}_x(\Phi(x)g)m^2 + g\mathrm{div}_x(\Phi(x)f)m^2 \\
= & \int -2fg\Phi(x)\cdot\frac{\nabla_v m}{m}m^2 + \mathrm{div}_x\Phi(x)fgm^2,
\end{aligned}$$

so (4.8.2) are proved by combining the terms above. □

Chapter 5

Harris for jump process

5.1 Introduction

The goal of this chapter is to give quantitative rates of convergence to equilibrium for some linear kinetic equations, using a method based on Harris's Theorem from the theory of Markov processes [39, 52, 41] that we believe is very well adapted to hypocoercive, nonlocal equations. We consider equations of the type

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f,$$

where $f = f(t, x, v)$, with time $t \geq 0$, space $x \in \mathbb{T}^d$ (the d-dimensional unit torus), and velocity $v \in \mathbb{R}^d$. The operator $\mathcal{L}$ acts only on the v variable, and it must typically be the generator of a stochastic semigroup for our method to work. We give explicit results for $\mathcal{L}$ equal to the linear relaxation Boltzmann operator (sometimes known as linear BGK operator), and for $\mathcal{L}$ equal to the linear Boltzmann operator (see below for a full description). We also consider the equations posed on the whole space $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ with a confining potential Φ:

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L} f.$$

We are able to give exponential convergence results on the d-dimensional torus, or with confining potentials growing at least quadratically at ∞, always in total variation or weighted total variation norms (alternatively, L^1 or weighted L^1 norms). For subquadratic potentials we give algebraic convergence rates, again in the same kind of weighted L^1 norms. Some results were already available for these equations [15, 55, 19, 38]. We will give a more detailed account of them after we describe them more precisely. Previous proofs of convergence to equilibrium used strongly weighted L^2 norms (typically with a weight which is the inverse of a Gaussian), so one advantage of our method is that it directly yields convergence for a much wider range of initial conditions. The result works, in particular, for initial conditions with slow decaying tails,

and for measure initial conditions with very bad local regularity. The method gives also existence of stationary solutions under quite general conditions; in some cases these are explicit and easy to find, but in other cases they can be nontrivial. We also note that our results for subquadratic potentials are to our knowledge new. Apart from these new results, our aim is to present a new application of a probabilistic method, using mostly PDE arguments, and which is probably useful for a wide range of models.

The study of the speed of relaxation to equilibrium for kinetic equations is a well known problem, both for linear and nonlinear models. The central obstacle is that dissipation happens only on the v variable via the effect of the operator $\mathcal{L}$, while only transport takes place in x. The transport then "mixes" the dissipation into the x variable, and one has to find a way to estimate this effect. The theory of hypocoercivity was developed in [62, 38, 42] precisely to overcome these problems for linear operators. In a landmark result, [23] proved that the full nonlinear Boltzmann equation converges to equilibrium at least at an algebraic rate. Exponential convergence results for the (linear) Fokker-Planck equation were given in [22], and a theory for a range of linear kinetic equations has been given in [19]. All of these results give convergence in exponentially weighted L^2 norms or H^1 norms; convergence to equilibrium in weighted L^1 norms can then be proved for several kinetic models by using the techniques in [35].

Let us describe our equations more precisely. The *linear relaxation Boltzmann equation* is given by

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = \mathcal{L}^+ f - f, \tag{5.1.1}$$

where

$$\mathcal{L}^+ f = \left(\int f(t,x,u) \mathrm{d}u \right) \mathcal{M}(v),$$

and $\mathbb{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2)$. The function $f = f(t,x,v)$ depends on time $t \geq 0$, space $x \in \mathbb{R}^d$, and velocity $v \in \mathbb{R}^d$, and the potential $\Phi\, \mathbb{R}^d \to \mathbb{R}$ is a $\mathcal{C}^2$ function of x. Alternatively, we consider this equation on the torus; that is, for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, assuming periodic boundary conditions. In that case we omit Φ (which corresponds to $\Phi = 0$ in the above equation):

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - f. \tag{5.1.2}$$

This simple equation is well studied in kinetic theory and can be thought of as a toy model with similar properties to either the non-linear BGK equation or linear Boltzmann equation. It is also one of the simplest examples of a hypocoercive equation. Convergence to equilibrium in H^1 for this equation has been shown in [15], at a rate faster than any function of t. It was then shown to converge exponentially fast in both H^1 and L^2 using hypocoercivity techniques in [38, 55, 19].

The linear Boltzmann equation is of a similar type:

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi \cdot \nabla_v f) = Q(f, \mathcal{M}), \tag{5.1.3}$$

where Φ is a $\mathcal{C}^2$ potential and $\mathbb{M}(v) := (2\pi)^{-d/2}\exp(-|v|^2/2)$ as before, and Q is the Boltzmann operator

$$Q(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|,\sigma)\left(f(v')g(v'_*) - f(v)g(v_*)\right)\,\mathrm{d}\sigma\,\mathrm{d}v_*,$$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

and B is the *collision kernel*. We always assume that B is a hard kernel and can be written as a product

$$B(|v-v_*|,\sigma) = |v-v_*|^\gamma\; b\left(\sigma\cdot\frac{v-v_*}{|v-v_*|}\right), \tag{5.1.4}$$

for some $\gamma \geq 0$ and b integrable and uniformly positive on $[-1,1]$; that is, there exists $C_b > 0$ such that

$$b(z) \geq C_b \qquad \text{for all } z \in [-1,1]. \tag{5.1.5}$$

As before, alternatively we consider the same equation posed for $x \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, without any potential Φ:

$$\partial_t f + v\cdot\nabla_x f = Q(f,\mathcal{M}). \tag{5.1.6}$$

This equation models gas particles interacting with a background medium which is already in equilibrium. Moreover, it has been used in describing many other systems like radiative transfer, neutron transportation, cometary flow and dust particles. The spatially homogeneous case has been studied in [46, 1, 11]. The kinetic equations (5.1.3) or (5.1.6) fit into the general framework in [55, 19], so convergence to equilibrium in weighted L^2 norms may be proved by using the techniques described there.

We denote by $\mathcal{P}(\Omega)$ the set of probability measures on a set $\Omega \subseteq \mathbb{R}^k$ (that is, the probability measures defined on the Borel σ-algebra of Ω). We state our main results on the torus, and then on $\mathbb{R}^d$ with a confining potential:

Theorem 5.1.2 (Exponential convergence results on the torus)**.** *Suppose that $t \mapsto f_t$ is the solution to* (5.1.2) *or* (5.1.6) *with initial data $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. In the case of equation* (5.1.6) *we also assume* (5.1.4) *with $\gamma \geq 0$ and* (5.1.5)*. Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that*

$$\|f_t - \mu\|_* \leq Ce^{-\lambda t}\|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ (that is, $\mu(x,v) = M(v)$). The norm $\|\cdot\|_*$ is just the total variation norm $\|\cdot\|_{\mathrm{TV}}$ for equation (5.1.2)*,*

$$\|f_0 - \mu\|_* = \|f_0 - \mu\|_{\mathrm{TV}} := \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} |f_0 - \mu|\,\mathrm{d}x\,\mathrm{d}v \qquad \textit{for equation } (5.1.2),$$

and it is a weighted total variation norm in the case of equation (5.1.6)*:*

$$\|f_0 - \mu\|_* = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} (1+|v|^2)|f_0 - \mu|\,\mathrm{d}x\,\mathrm{d}v \qquad \textit{for equation } (5.1.6).$$

Theorem 5.1.3 (Exponential convergence results with a confining potential on the whole space)**.** *Suppose that $t \mapsto f_t$ is the solution to* (5.1.1) *or* (5.1.3) *with initial data $f_0 \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and a potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ which is bounded below, and satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A, \qquad x \in \mathbb{R}^d,$$

for some positive constants γ_1, γ_2, A. Define $\langle x \rangle = \sqrt{1 + |x|^2}$. In the case of equation (5.1.6) *we also assume* (5.1.4), (5.1.5) *and*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then there exist constants $C > 0, \lambda > 0$ (independent of f_0) such that

$$\|f_t - \mu\|_* \leq C e^{-\lambda t} \|f_0 - \mu\|_*,$$

where μ is the only equilibrium state of the corresponding equation in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$,

$$\mathrm{d}\mu = \mathcal{M}(v) e^{-\Phi(x)} \mathrm{d}v \mathrm{d}x.$$

The norm $\|\cdot\|_*$ is a weighted total variation norm defined by

$$\|f_t - \mu\|_* := \int \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) |f_t - \mu| \, \mathrm{d}v \, \mathrm{d}x.$$

In all results above the constants C and λ can be explicitly estimated in terms of the parameters appearing in the equation by following the calculations in the proofs. We do not give them explicitly since we do not expect them to be optimal, but they are nevertheless completely constructive.

We also look at Harris type theorems with weaker controls on moments to give analogues of all our theorems when the confining potential is weaker and give algebraic rates of convergence with rates depending on the assumption we make on the confining potential. Subgeometric convergence for kinetic Fokker-Planck equations with weak confinement has been shown in [21, 9, 10].

Theorem 5.1.4 (Subgeometric convergence results with weak confining potentials)**.** *Suppose that $t \mapsto f_t$ is the solution to* (5.1.1) *in the whole space with a confining potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$. Define $\langle x \rangle = \sqrt{1 + |x|^2}$. Assume that for some β in $(0, 1)$ the confining potential satisfies*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

for some positive constants γ_1, γ_2, A. Then we have that there exists a constant $C > 0$ such that

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{ \|f_0 - \mu\|_{\mathrm{TV}}, \; C \int f_0(x,v) \left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|^2\right) (1+t)^{-\beta/(1-\beta)} \right\}.$$

Similarly if $t \mapsto f_t$ is the solution to (5.1.3) *in the whole space, satisfies* (5.1.4), (5.1.5) *and*

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\beta+1} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \leq \gamma_3 \langle x \rangle^{1+\beta},$$

for some positive constants $\gamma_1, \gamma_2, A, \beta, \gamma_3$. *Then we have that there exists a constant* $C > 0$ *such that*

$$\|f_t - \mu\|_{\mathrm{TV}} \leq \min\left\{\|f_0 - \mu\|_{\mathrm{TV}},\ C\int f_0(x,v)\left(1 + \frac{1}{2}|v|^2 + \Phi(x) + |x|\right)(1+t)^{-\beta}\right\}.$$

This method is also applicable to some integro-PDEs describing several biological and physical phenomena. In [29], Doeblin's argument is used to show exponential relaxation to equilibrium for the conservative renewal equation which is a common model in population dynamics, often referred as the McKendrick-von Foerster equation. In [14], the authors show existence of a spectral gap property in the linear (no-connectivity) setting for elapsed-time structured neuron networks by using Doeblin's Theorem. Relaxation to the stationary state for the original nonlinear equation is then proved by a perturbation argument where the non-linearity is weak. Moreover, in [24] the authors consider a nonlinear model which is derived from mean-field description of an excitatory network made up of leaky integrate-and-fire neurons. In the case of weak connectivity, the authors demonstrate the uniqueness of a stationary state and its global exponential stability by using Doeblin's type of contraction argument for the linear case. Also in [9], the authors extend similar ideas to obtain quantitative estimates in total variation distance for positive semigroups, that can be non-conservative and non-homogeneous. They provide a speed of convergence for periodic semigroups and new bounds in the homogeneous setting.

Using Harris's Theorem gives an alternative and very different strategy for proving quantitative exponential decay to equilibrium. It allows us to look at hypocoercive effects on the level of stochastic processes and to look at specific trajectories which might allow one to produce quantitative theorems based on more trajectorial intuition. Another difference is that the confining behaviour is shown here by exploiting good behaviour of moments rather than a Poincaré inequality, this means looking at point wise bounds rather than integral controls on the operator. These are often equivalent for time reversible processes [9, 17] and have advantages and disadvantages. However, the condition on the moments used here might be much easier to verify in the case where the equilibrium state cannot be made explicit. This is the motivation behind [5, 16]. These works also allow us to look at a large class of initial data. We only need f_0 to be a probability measure where $\|f_0 - \mu\|$ is finite. Harris's Theorem has a restriction which is that we can only consider Markov processes. Many kinetic equations are linear Markov processes but this excludes the study of linearized non-linear equations which are not necessarily mass preserving.

The plan of the chapter is as follows. We deal the convergence of linear relaxation Boltzmann equation in Section 5.2, and we deal the convergence of linear Boltzmann equation in Section 5.3, finally in Section 5.4, we introduce an abstract theorem which could cover more jump process cases.

5.2 The linear relaxation Boltzmann equation

This is the simplest operator on the torus, so we do not in fact need to use Harris's Theorem. We can instead use Doeblin's Theorem where we have a uniform minorisation condition.

5.2.1 On the flat torus

We consider

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f, \tag{5.2.7}$$

posed for $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, where $\mathbb{T}^d$ is the d-dimensional torus of side 1 and

$$\mathcal{L} f(x, v) := \mathcal{L}^+ f(x, v) - f(x, v) := \left(\int_{\mathbb{R}^d} f(x, u) \, \mathrm{d}u \right) \mathcal{M}(v) - f(x, v), \tag{5.2.8}$$

which is a well defined operator from $L^1(\mathbb{T}^d \times \mathbb{R}^d)$ to $L^1(\mathbb{T}^d \times \mathbb{R}^d)$, and can also be defined as an operator from $\mathbb{M}(\mathbb{T}^d \times \mathbb{R}^d)$ to $\mathbb{M}(\mathbb{T}^d \times \mathbb{R}^d)$ with the same expression (where $\int_{\mathbb{R}^d} f(x, u) \, \mathrm{d}u$ now denotes the marginal of the measure f with respect to u). We define $(T_t)_{t \geq 0}$ as the transport semigroup associated to the operator $-v \cdot \nabla_x f$ in the space of measures with the bounded Lipschitz topology; that is, $t \mapsto T_t f_0$ solves the equation $\partial_t f + v \cdot \nabla_x f = 0$ with initial condition f_0. In this case one can write T_t explicitly as

$$T_t f_0(x, v) = f_0(x - tv, v). \tag{5.2.9}$$

Using Duhamel's formula repeatedly one can obtain that, if f is a solution of (5.2.7) with initial data f_0, then

$$e^t f_t \geq \int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}r \, \mathrm{d}s. \tag{5.2.10}$$

We will now check two properties, which we list as lemmas. The first one says that the operator $\mathcal{L}$ always allows jumps to any small velocity. We always use the notation $\mathbb{1}_A$ to denote the characteristic function of a set A (if A is a set), or the function which is 1 where the condition A is met, and 0 otherwise (if A is a condition).

Lemma 5.2.5. *For all $\delta_L > 0$ there exists $\alpha_L > 0$ such that for all nonnegative functions $g \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$ we have*

$$\mathcal{L}^+ g(x, v) \geq \alpha_L \left(\int_{\mathbb{R}^d} g(x, u) \, \mathrm{d}u \right) \mathbb{1}_{\{|v| \leq \delta_L\}} \tag{5.2.11}$$

for almost all $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$.

Proof. Given any δ_L it is enough to choose $\alpha_L := \mathbb{M}(v)$ for any v with $|v| = \delta_L$. □

The second one is regarding to the behaviour of the transport part alone. It says that if we start at any point inside a ball of radius R, and we are allowed to start with any small velocity, then we can reach any point in the ball of radius R with a predetermined bound on the final velocity:

Lemma 5.2.6. *Given any time $t_0 > 0$ and radius $R > 0$ there exist $\delta_L, R' > 0$ such that for all $t \geq t_0$ it holds that*

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\leq\delta_L\}}\Big)\,\mathrm{d}v \geq \frac{1}{t^d}\mathbb{1}_{\{|x|\leq R\}} \qquad \text{for all } x_0 \text{ with } |x_0| < R. \tag{5.2.12}$$

In particular, if we take $R > \sqrt{d}$, there exist $\delta_L, R' > 0$ such that

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\leq\delta_L\}}\Big)\,\mathrm{d}v \geq \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d. \tag{5.2.13}$$

Proof. Take $t, R > 0$. We have

$$T_t\left(\delta_{x_0}(x)\mathbb{1}_{B(\delta_L)}(v)\right) = \delta_{x_0}(x - vt)\mathbb{1}_{B(\delta_L)}(v),$$

where $B(\delta)$ denotes the open ball $\{x \in \mathbb{R}^d \mid |x| < \delta\}$, and in general we will use the notation $B(z, \delta)$ to denote the open ball of radius δ centered at $z \in \mathbb{R}^d$. Integrating this and changing variables gives that

$$\int_{B(R')} T_t\left(\delta_{x_0}(x)\mathbb{1}_{B(\delta_L)}(v)\right)\,\mathrm{d}v = \frac{1}{t^d}\int_{B(x,tR')} \delta_{x_0}(y)\mathbb{1}_{B(\delta_L)}\left(\frac{x-y}{t}\right)\mathrm{d}y.$$

Since $|x - y| \leq |x| + |y|$ we have that

$$\mathbb{1}_{B(\delta_L)}\left(\frac{x-y}{t}\right) \geq \mathbb{1}_{B(\delta_L/2)}\left(\frac{x}{t}\right)\,\mathbb{1}_{B(\delta_L/2)}\left(\frac{y}{t}\right).$$

Therefore if we take $\delta_L > 2R/t$ we have

$$\mathbb{1}_{B(\delta_L)}\left(\frac{x-y}{t}\right) \geq \mathbb{1}_{B(R)}(x)\mathbb{1}_{B(R)}(y).$$

On the other hand, if we take $|x| < R$ and $R' > 2R/t$ then

$$B(x, tR') \supseteq B(x, 2R) \supseteq B(R).$$

Hence if $\delta_L > 2R/t$ and $R' > 2R/t$,

$$\int_{B(R')} T_t\left(\delta_{x_0}(x)\mathbb{1}_{B(\delta_L)}(v)\right)\,\mathrm{d}v \geq \frac{1}{t^d}\mathbb{1}_{B(R)}(x),$$

which proves the result. □

Lemma 5.2.7 (Doeblin condition for the linear relaxation Boltzmann equation on the torus)**.** *For any $t_* > 0$ there exist constants $\alpha, \delta_L > 0$ (depending on t_*) such that any solution f to equation* (5.2.7) *with initial condition $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ satisfies*

$$f(t_*, x, v) \geq \alpha\mathbb{1}_{\{|v|\leq\delta_L\}}, \tag{5.2.14}$$

where the inequality is understood in the sense of measures.

Proof. It is enough to prove it for $f_0 := \delta_{(x_0,v_0)}$, where $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ is an arbitrary point. From Lemma 5.2.6 (with $R > \sqrt{d}$ and $t_0 := t_*/3$) we will use that there exists $\delta_L > 0$ such that

$$\int_{\mathbb{R}^d} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\le\delta_L\}}\Big)\,\mathrm{d}v \ge \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d,\ t > t_0.$$

Also, Lemma 5.2.5 gives an $\alpha_L > 0$ such that

$$\mathcal{L}^+ g \ge \alpha_L \left(\int_{\mathbb{R}^d} g(x,u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\le\delta_L\}}.$$

Take any $r > 0$. Since $T_r f_0 = \delta_{(x_0 - v_0 r, v_0)}$, using this shows

$$\mathcal{L}^+ T_r f_0 \ge \alpha_L\, \delta_{x_0 - v_0 r}(x)\, \mathbb{1}_{\{|v|\le\delta_L\}}.$$

Hence, whenever $s - r > t_0$ we have

$$\begin{aligned}
\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 &\ge \alpha_L \left(\int_{\mathbb{R}^d} T_{s-r}\mathcal{L}^+ T_r f_0\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\le\delta_L\}} \\
&\ge \alpha_L^2 \left(\int_{\mathbb{R}^d} T_{s-r}\big(\delta_{x_0 - v_0 r}(x)\,\mathbb{1}_{\{|u|\le\delta_L\}}\big)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\le\delta_L\}} \\
&\ge \frac{1}{(s-r)^d}\alpha_L^2\ \mathbb{1}_{\{|v|\le\delta_L\}}.
\end{aligned}$$

Finally, for the movement along the flow T_{t-s}, notice that

$$T_t\Big(\mathbb{1}_{\mathbb{T}^d}(x)\mathbb{1}_{|v|<\delta_L}(v)\Big) = \mathbb{1}_{\mathbb{T}^d}(x)\mathbb{1}_{\{|v|<\delta_L\}}(v) \qquad \text{for all } t \ge 0.$$

This means that for all $t > s > r > 0$ such that $s - r > t_0$ we have

$$T_{t-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0 \ge \frac{1}{(s-r)^d}\alpha_L^2\ \mathbb{1}_{\{|v|\le\delta_L\}}.$$

For any t_* we have then, recalling that $t_0 = t_*/3$,

$$\begin{aligned}
\int_0^{t_*}\int_0^s T_{t_*-s}\mathcal{L}^+ T_{s-r}\mathcal{L}^+ T_r f_0\,\mathrm{d}r\,\mathrm{d}s &\ge \alpha_L^2\ \mathbb{1}_{\{|v|\le\delta_L\}} \int_{2t_0}^{t_*}\int_0^{t_0} \frac{1}{(s-r)^d}\,\mathrm{d}r\,\mathrm{d}s \\
&\ge \frac{t_0^2}{t_*^d}\alpha_L^2 \mathbb{1}_{\{|v|\le\delta_L\}} = \frac{1}{9}t_*^{2-d}\alpha_L^2 \mathbb{1}_{\{|v|\le\delta_L\}}.
\end{aligned}$$

Finally, from Duhamel's formula (5.2.10) we obtain

$$f(t_*, x, v) \ge \frac{1}{9}e^{-t_*}t_*^{2-d}\alpha_L^2 \mathbb{1}_{\{|v|\le\delta_L\}},$$

which gives the result. □

Proof of Theorem 5.1.2 in the case of the linear relaxation Boltzmann equation. Lemma 5.2.7 allows us to apply directly Doeblin's Theorem 1.2.2 to obtain fast exponential convergence to equilibrium in the total variation distance. This rate is also explicitly calculable. Therefore, the proof follows. □

5.2.2 On the whole space with a confining potential

Now we consider the equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f = \mathcal{L} f, \tag{5.2.15}$$

where $\mathcal{L}$ is defined as in the previous section and $x, v \in \mathbb{R}^d$. We want to use a slightly different strategy to show the minorisation condition based on the fact that we instantaneously produce large velocities. We first need a result on the trajectories of particles under the action of the potential Φ. Always assuming that Φ is a $\mathcal{C}^2$ function, we consider the characteristic ordinary differential equations associated to the transport part of (5.4.42):

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\nabla \Phi(x), \end{aligned} \tag{5.2.16}$$

and we denote by $(X_t(x_0, v_0), V_t(x_0, v_0))$ the solution at time t to (5.4.43) with initial data $x(0) = x_0$, $v(0) = v_0$. Performing time integration twice, it clearly satisfies

$$X_t(x_0, v_0) = x_0 + v_0 t + \int_0^t \int_0^s \nabla \Phi(X_u(x_0, v_0)) \, \mathrm{d}u \, \mathrm{d}s \tag{5.2.17}$$

for any $x_0, v_0 \in \mathbb{R}^d$ and any t for which it is defined. Intuitively the idea is that for small times we can approximate (X_t, V_t) by $(X_t^{(0)}, V_t^{(0)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= 0, \end{aligned} \tag{5.2.18}$$

whose explicit solution is $(X_t^{(0)}, V_t^{(0)}) = (x_0 + v_0 t, v_0)$. If we want to hit a point x_1 in time t then if we travel with the trajectory $X^{(0)}$ we just need to choose $v_0 = (x_1 - x_0)/t$. Now we choose an interpolation between $(X^{(0)}, V^{(0)})$ and (X, V). We denote it by $(X^{(\epsilon)}, V^{(\epsilon)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\epsilon^2 \nabla \Phi(x), \end{aligned} \tag{5.2.19}$$

still with initial data (x_0, v_0). We calculate that

$$X_t^{(\epsilon)}(x_0, v_0) = X_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right), \qquad V_t^{(\epsilon)}(x_0, v_0) = \epsilon V_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right).$$

Now we can see from the ODE representation (and we will make this more precise later) that (X, V) is a C^1 map of (t, ϵ, x, v). Therefore if we fix t and x_0 we can define a C^1 map

$$F \, [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d,$$

by

$$F(\epsilon, v) = X_t^{(\epsilon)}(x_0, v).$$

Then for $\epsilon = 0$ we can find v_* such that $F(0, v_*) = x_1$ as given above. Furthermore $\nabla F(0, v_*) \neq 0$ so by the implicit function theorem for all ϵ less than some ϵ_* we have a C^1 function $v(\epsilon)$ such that $F(\epsilon, v(\epsilon)) = x_1$. This means that

$$X_{\epsilon t}\left(x_0, \frac{v(\epsilon)}{\epsilon}\right) = x_1.$$

So if we take $s < \epsilon_* t$ then we can choose v such that $X_s(x_0, v) = x_1$. We now need to get quantitative estimates on ϵ_*, and we do this by tracking the constants in the proof of the contraction mapping theorem.

In order to make these ideas quantitative and to check that the solution is in fact C^1 we need to get bounds on (X_t, V_t) and $\nabla\Phi(X_t)$ for t is some fixed intervals. For the potentials of interest we will have that the solutions to these ODEs will exist for infinite time. We prove bounds on the solutions and $\nabla\Phi(X_t)$ for any potential:

Lemma 5.2.8. *Assume that the potential Φ is $\mathcal{C}^2$ in $\mathbb{R}^d$. Take $\lambda > 1$, $R > 0$ and $x_0, v_0 \in \mathbb{R}^d$ with $|x_0| \leq R$. The solution $t \mapsto X_t(x_0, v_0)$ to (5.4.43) is defined (at least) for $|t| \leq T$, with*

$$T := \min\left\{\frac{(\lambda-1)R}{2|v_0|}, \frac{\sqrt{(\lambda-1)R}}{\sqrt{2C_{\lambda R}}}\right\}, \qquad C_{\lambda R} := \max_{|x|\leq \lambda R} |\nabla\phi(x)|.$$

(It is understood that any term in the above minimum is $+\infty$ if the denominator is 0.) Also, it holds that

$$|X_t(x_0, v_0)| \leq \lambda R \qquad \text{for } |t| \leq T.$$

Proof. By standard ODE theory, the solution is defined in some maximal (open) time interval I containing 0; if this maximal interval has any finite endpoint t_*, then $X_t(x_0, v_0)$ has to blow up as t approaches t_*. Hence if the statement is not satisfied, there must exist $t \in I$ with $|t| \leq T$ such that $|X_t(x_0, v_0)| \geq \lambda R$. By continuity, one may take $t_0 \in I$ to be the "smallest" time when this happens: that is, $|t_0| \leq T$ and

$$\begin{aligned} X_{t_0}(x_0, v_0) &= \lambda R, \\ |X_{t_0}(x_0, v_0)| &\leq \lambda R \qquad \text{for } |t| \leq |t_0|. \end{aligned}$$

By (5.4.44) and using that $|t_0| \leq T$ we have

$$\begin{aligned} \lambda R = |X_{t_0}(x_0, v_0)| &\leq |x_0| + |v_0 t_0| + \frac{t_0^2}{2} \max\{|\nabla\phi(X_t(x_0, v_0))| \colon t \leq t_0\} \\ &\leq R + \frac{(\lambda-1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2 = \frac{(\lambda+1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2, \end{aligned}$$

which implies that

$$(\lambda - 1)R \leq C_{\lambda R} t_0^2.$$

If $C_{\lambda R} = 0$ this is false; if $C_{\lambda R} > 0$, then this contradicts with that $|t_0| \leq T$. □

We now follow the intuition given at the beginning of this section. However we collapse the variables ϵ and t together and consequently look at $X_t\left(x, \frac{v}{t}\right)$ which is intuitively less clear but algebraically simpler.

Lemma 5.2.9. *Assume that $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$, and take $x_0, x_1 \in \mathbb{R}^d$. Let $R := \max\{|x_0|, |x_1|\}$. There exists $0 < T_1 = T_1(R)$ such that for any $t \leq T_1$ we can find a $|v_0| \leq 4R$ such that*

$$X_t\left(x_0, \frac{v_0}{t}\right) = x_1.$$

In fact, it is enough to take $T_1 > 0$ such that

$$CT_1^2 e^{CT_1^2} \leq \frac{1}{4}, \qquad T_1 \leq \frac{\sqrt{R}}{\sqrt{2C_{2R}}}, \qquad T_1 \leq \frac{2\sqrt{R}}{\sqrt{C_{9R}}}, \qquad \text{where } C := \sup_{|x|\leq 9R} |D^2\Phi(x)|,$$

where $C_{\lambda R}$ is defined in Lemma 5.4.28 and $D^2\Phi$ denotes the Hessian matrix of Φ.

Proof. We define

$$f(t, v) = X_t\left(x_0, \frac{v}{t}\right) - x_1, \qquad t \neq 0,\ v \in \mathbb{R}^d,$$
$$f(0, v) := x_0 + v - x_1, \qquad v \in \mathbb{R}^d.$$

Notice that due to Lemma 5.4.28 with $\lambda = 9$, this is well-defined whenever

$$|t| \leq \frac{2\sqrt{R}}{\sqrt{C_{9R}}} =: T_2, \ \ |v| \leq 4R.$$

Our goal is to find a neighbourhood of $t = 0$ on which there exists $v = v(t)$ with $f(t, v(t)) = 0$, for which we will use the implicit function theorem.

Now, notice that we have

$$f(0, x_1 - x_0) = 0$$

and

$$\frac{\partial f}{\partial v_i}(0, x_1 - x_0) = 1, \quad i = 1, \ldots, d.$$

We can apply the implicit function theorem to find a neighbourhood I of $t = 0$ and a function $v = v(t)$ such that $f(t, v(t)) = 0$ for $t \in I$. However, since we need to estimate the size of I and of $v(t)$, we carry out a constructive proof.

Take $v_0, v_1 \in \mathbb{R}^d$ with $|v_0|, |v_1| \leq 4R$, and denote $\tilde{v}_0 := v_0/t$, $\tilde{v}_1 := v_1/t$. By (5.4.44), for all $0 < t \leq T_2$ we have

$$X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) = (\tilde{v}_1 - \tilde{v}_0)t + \int_0^t \int_0^s \nabla\phi(X_u(x_0, \tilde{v}_1)) - \nabla\phi(X_u(x_0, \tilde{v}_0))\, \mathrm{d}u\, \mathrm{d}s. \tag{5.2.20}$$

Take any $T_1 \leq T_2$, to be fixed later. Then Lemma 5.4.28 implies, for all $0 \leq t \leq T_1$,

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0)| \leq |\tilde{v}_1 - \tilde{v}_0|t + CT_1 \int_0^t |X_u(x_0, \tilde{v}_1) - X_u(x_0, \tilde{v}_0)|\, \mathrm{d}u.$$

by Gronwall's Lemma we have

$$|X_t(x_0,\tilde{v}_1) - X_t(x_0,\tilde{v}_0)| \le |\tilde{v}_1 - \tilde{v}_0| t e^{CT_1 t} \qquad \text{for } 0 < t \le T_1.$$

Using this again in (5.4.49) we have

$$\begin{aligned} |X_t(x_0,\tilde{v}_1) - X_t(x_0,\tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| &\le |\tilde{v}_1 - \tilde{v}_0| CT_1 \int_0^t u e^{CT_1 u}\,\mathrm{d}u \\ &\le |\tilde{v}_1 - \tilde{v}_0| t\, CT_1^2 e^{CT_1^2}. \end{aligned}$$

Taking T_1 such that

$$CT_1^2 e^{CT_1^2} \le \frac{1}{4} \tag{5.2.21}$$

we have

$$|X_t(x_0,\tilde{v}_1) - X_t(x_0,\tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| \le \frac{1}{4}|\tilde{v}_1 - \tilde{v}_0| t$$

which is the same as

$$\left| X_t\left(x_0, \frac{v_1}{t}\right) - X_t\left(x_0, \frac{v_0}{t}\right) - (v_1 - v_0)\right| \le \frac{1}{4}|v_1 - v_0|, \tag{5.2.22}$$

for any $0 < t \le T_1$ and any v_0, v_1 with $|v_0|, |v_1| \le 4R$. Now, for any $0 \le t \le T_1$ and $|v| \le 4R$ we define

$$A_t(v) = v - f(t,v).$$

A fixed point of $A_t(v)$ satisfies $f(t,v) = 0$, and by (5.4.51) $A_t(v)$ is contractive:

$$|A_t(v_1) - A_t(v_0)| \le \frac{1}{4}|v_1 - v_0| \qquad \text{for } 0 \le t \le T_1,\ |v| \le 4R.$$

(Equation (5.4.51) proves this for $0 < t \le T_1$, and for $t = 0$ it is obvious.) In order to use the Banach fixed-point theorem we still need to show that the image of A_t is inside the set with $|v| \le 4R$. Using (5.4.51) for $v_1 = 0$, $v_0 = v$ we also see that

$$\left| X_t(x_0,0) - X_t\left(x_0, \frac{v}{t}\right) + v\right| \le \frac{1}{4}|v|,$$

which gives

$$|A_t(v) + x_1 - X_t(x_0,0)| \le \frac{1}{4}|v|,$$

so

$$|A_t(v)| \le \frac{1}{4}|v| + |x_1| + |X_t(x_0,0)| \le 2R + |X_t(x_0,0)|. \tag{5.2.23}$$

If we take

$$T_1 \le \frac{\sqrt{R}}{\sqrt{2C_{2R}}} \tag{5.2.24}$$

then Lemma 5.4.28 (used for $\lambda = 2$) shows that

$$|X_t(x_0,0)| \le 2R \qquad \text{for } 0 \le t \le T_1,$$

and from (5.4.52) we have

$$|A_t(v)| \le 4R \quad \text{for} \quad 0 < t \le T_1.$$

Hence, as long as T_1 satisfies (5.4.50) and (5.4.53), A_t has a fixed point $|v|$ for any $0 < t \le T_1$, and this fixed point satisfies $|v| \le 4R$. □

Lemma 5.2.10. *Assume the potential $\Phi \in \mathcal{C}^2(\mathbb{R}^d)$ is bounded below, and let T_s denote the transport semigroup associated to the operator $f \mapsto -v \cdot \nabla_x f + \nabla_x \Phi(x) \cdot \nabla_v f$. Given any $R > 0$ there exists a time $T_1 > 0$ such that for any $0 < s < T_1$ one can find constants $\alpha, R', R_2 > 0$ (depending on s and R) such that*

$$\int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \ge \alpha \mathbb{1}_{\{|x| \le R\}}, \tag{5.2.25}$$

for any x_0 with $|x_0| \le R$. The constants α, R', R_2 are uniformly bounded in bounded intervals of time; that is, for any closed interval $J \subseteq (0, T_1)$ one can find α, R', R_2 for which the inequality holds for all $s \in J$.

Proof. Since the statement is invariant if Φ changes by an additive constant, we may assume that $\Phi \ge 0$ for simplicity. Using Lemma 5.4.30 we find T_1 such that for any $s < T_1$ and every $x_1 \in B(R)$ there exists $v \in B(4R)$ (depending on x_0, x_1 and s) such that

$$X_s\left(x_0, \frac{v}{s}\right) = x_1.$$

Since $v/s \in B(4R/s)$, call $R_2 := 4R/s$. We see that for every $x_1 \in B(0, R)$ there is at least one $u \in \mathbb{R}^d$ such that

$$(x_1, u) \in T_s\left(\{x_0\} \times \{|v| \le R_2\}\right).$$

In other words,

$$X_s(x_0, \{|v| \le R_2\}) \supseteq B(0, R). \tag{5.2.26}$$

This essentially contains our result, and we just need to carry out a technical argument to complete it and estimate the constants α and R'. For any compactly supported, continuous and positive $\varphi\, \mathbb{R}^d \to \mathbb{R}$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi(x) \int_{B(R')} & T_s(\delta_{x_0} \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{|V_s(x,v)| < R'\}} \, \varphi(X_s(x,v)) \delta_{x_0}(x) \mathbb{1}_{\{|v| \le R_2\}}) \, \mathrm{d}v \, \mathrm{d}x \\ &= \int_{|v| \le R_2} \mathbb{1}_{\{|V_s(x_0,v)| < R'\}} \, \varphi(X_s(x_0, v) \, \mathrm{d}v, \end{aligned} \tag{5.2.27}$$

since the characteristics map $(x, v) \mapsto (X_s(x, v), V_s(x, v))$ is measure-preserving. If we write the energy as $H(x, v) = |v|^2/2 + \Phi(x)$ and call

$$E_0 := \sup\{H(x, v) \,:\, |x| < R, |v| < R_2\}.$$

Then for all $s \ge 0$

$$E(X_s(x_0, v), V_s(x_0, v)) \le E_0,$$

and in particular

$$|V_s(x_0, v)| \le \sqrt{2E_0}.$$

If we take $R' > \sqrt{2E_0}$ then the term $\mathbb{1}_{\{|V_s(x_0,v)|<R'\}}$ is always 1 in (5.2.27) and we get

$$\int_{\mathbb{R}^d} \varphi(x) \int_{B(R')} T_s(\delta_{x_0} \mathbb{1}_{\{|v|\le R_2\}})\,\mathrm{d}v\,\mathrm{d}x = \int_{|v|\le R_2} \varphi(X_s(x_0,v))\,\mathrm{d}v.$$

Now, take an $M > 0$ such that $|\operatorname{Jac}_v X_s(x,v)| \le M$ for all (x,v) with $|x| \le R$ and $|v| \le R_2$. (Notice this M depends only on Φ, R and R_2.) Then

$$\begin{aligned}\int_{|v|\le R_2} \varphi(X_s(x_0,v))\,\mathrm{d}v &\ge \frac{1}{M}\int_{|v|\le R_2} \varphi(X_s(x_0,v))|\operatorname{Jac}_v X_s(x_0,v)|\,\mathrm{d}v \\ &= \frac{1}{M}\int_{X_s(x_0,\{|v|\le R_2\})} \varphi(x)\,\mathrm{d}x \ge \frac{1}{M}\int_{B(0,4R)} \varphi(x)\,\mathrm{d}x,\end{aligned}$$

where we have used (5.2.26) in the last step. In sum we find that

$$\int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} T_s(\delta_{x_0} \mathbb{1}_{\{|v|\le R_2\}})\,\mathrm{d}v\,\mathrm{d}x \ge \frac{1}{M}\int_{B(0,R)} \varphi(x)\,\mathrm{d}x$$

for all compactly supported, continuous and positive functions φ. This directly implies the result. □

Lemma 5.2.11 (Doeblin condition for linear relaxation Boltzmann equation with a confining potential)**.** *Let the potential $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function with compact level sets. Given $t > 0$ and $K > 0$ there exist constants $\alpha, \delta_X, \delta_V > 0$ such that any solution f to equation* (5.4.42) *with initial condition $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ supported on $B(0,K) \times B(0,K)$ satisfies*

$$f(t,x,v) \ge \alpha \mathbb{1}_{\{|x|<\delta_X\}}\ \mathbb{1}_{\{|v|<\delta_V\}}$$

for almost all $x, v \in \mathbb{R}^d$.

Proof. Fix any $t, K > 0$. Set $H_{\max}(K) = \max\{H(x,v) = |v|^2/2 + \Phi(x) : x \in B(0,K), v \in B(0,K)\}$ and then define $R := \max\{|x| : \Phi(x) \le H_{\max}(K)\}$. Since our conditions on Φ imply that its level sets are compact we know that R is finite. We use Lemma 5.4.31 to find constants $\alpha, R_2 > 0$ and an interval $[a,b] \subseteq (0,t)$ such that

$$\int_{\mathbb{R}^d} T_s(\delta_{x_0} \mathbb{1}_{\{|v|\le R_2\}})\,\mathrm{d}v \ge \alpha \mathbb{1}_{\{|x|\le R\}},$$

for any x_0 with $|x_0| \le R$ and any $s \in [a,b]$. From Lemma 5.2.5 we will use that there exists a constant $\alpha_L > 0$ such that

$$\mathcal{L}^+ g(x,v) \ge \alpha_L \left(\int_{\mathbb{R}^d} g(x,u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\le R_2\}} \tag{5.2.28}$$

for all nonnegative measures g. We first notice that we can do the same estimate as in formula (5.2.10), where now $(T_t)_{t\ge 0}$ represents the semigroup generated by the operator $-v \cdot \nabla_x f + \nabla_x \Phi(x) \cdot \nabla_v f$:

$$e^t f_t \ge \int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0\,\mathrm{d}r\,\mathrm{d}s. \tag{5.2.29}$$

Take $x_0, v_0 \in B(0,K)$, and call $f_0 := \delta_{(x_0,v_0)}$. For all r we have by the definition of R that

$$|X_r(x_0,v_0)| \leq R \qquad \text{for all } 0 \leq r. \tag{5.2.30}$$

For any $r > 0$, since $T_r f_0 = \delta_{(X_r(x_0,v_0),V_r(x_0,v_0))}$, using (5.4.37) gives

$$\mathcal{L}^+ T_r f_0 \geq \alpha_L \delta_{X_r(x_0,v_0)}(x) \mathbb{1}_{\{|v|\leq R_2\}}.$$

Then, using (5.2.30) and our two lemmas, whenever $s - r \in [a,b]$ we have

$$\begin{aligned}
\mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 &\geq \alpha_L \left(\int_{\mathbb{R}^d} T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}u \right) \mathbb{1}_{\{|v|\leq R_2\}} \\
&\geq \alpha_L^2 \left(\int_{\mathbb{R}^d} T_{s-r} \Big(\delta_{X_r(x_0,v_0)}(x) \mathbb{1}_{\{|u|\leq R_2\}} \Big) \, \mathrm{d}u \right) \mathbb{1}_{\{|v|\leq R_2\}} \\
&\geq \alpha_L^2 \alpha \, \mathbb{1}_{\{|x|\leq R\}} \mathbb{1}_{\{|v|\leq R_2\}}.
\end{aligned}$$

We now need to allow for a final bit of movement along the flow T_{t-s}. By the continuity of the flow, there exist $\epsilon > 0$ sufficiently small so that for all $0 \leq \tau \leq \epsilon$ we have

$$T_\tau \Big(\mathbb{1}_{B(R)}(x) \mathbb{1}_{B(R_2)}(v) \Big) \geq \mathbb{1}_{B(R/2)}(x) \mathbb{1}_{B(R_2/2)}(v).$$

Then for all t, s, r such that $t - s \leq \epsilon$ and $s - r \in (a,b)$ we have

$$T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \geq \alpha_L^2 \alpha \, \mathbb{1}_{\{|x|\leq R/2\}} \, \mathbb{1}_{\{|v|\leq R_2/2\}}.$$

We have then

$$\begin{aligned}
\int_0^t \int_0^s T_{t-s} \mathcal{L}^+ T_{s-r} \mathcal{L}^+ T_r f_0 \, \mathrm{d}r \, \mathrm{d}s &\geq \alpha_L^2 \alpha \int_{t-\epsilon}^t \int_{s-b}^{s-a} \mathbb{1}_{\{|x|\leq R/2\}} \mathbb{1}_{\{|v|\leq R_2/2\}} \, \mathrm{d}r \, \mathrm{d}s \\
&= \alpha_L^2 \alpha \epsilon (b-a) \mathbb{1}_{\{|x|\leq R/2\}} \mathbb{1}_{\{|v|\leq R_2/2\}}.
\end{aligned}$$

Finally, from Duhamel's formula (5.2.29) we obtain

$$f(t,x,v) \geq e^{-t} \alpha_L^2 \alpha \epsilon (b-a) \, \mathbb{1}_{\{|x|\leq R/2\}} \mathbb{1}_{\{|v|\leq R_2/2\}},$$

which gives the result. □

Lemma 5.2.12 (Lyapunov condition)**.** *Suppose that $\Phi(x)$ is a $\mathcal{C}^2$ function satisfying*

$$x \cdot \nabla \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A$$

for positive constants A, γ_1 γ_2. Then we have that

$$V(x,v) = 1 + \Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4} x \cdot v + \frac{1}{8}|x|^2$$

is a function for which the semigroup satisfies Hypothesis 1.2.3.

Remark 5.2.13. Φ is superquadratic at infinity (which is implied by earlier assumptions) then V is equivalent to $1+H(x,v)$ where the energy is defined as $H(x,v) = |v|^2/2 + \Phi(x)$. So the total variation distance weighted by V is equivalent to the total variation distance weighted by $1 + H(x,v)$.

Proof. We look at the forwards operator

$$Sf = v \cdot \nabla_x f - \nabla_x \Phi(x) \cdot \nabla_v f + \mathcal{L}^+ f - f.$$

We want a function $V(x, v)$ s.t

$$SV \leq -\lambda V + K$$

for some constants $\lambda > 0, K \geq 0$. We need to make the assumption that

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 |x|^2 + \gamma_2 \Phi(x) - A.$$

for some positive constant A, γ_1, γ_2. We then try the function

$$V(x, v) = \Phi(x) + \frac{1}{2}|v|^2 + ax \cdot v + b|x|^2.$$

We want this to be positive so we impose $a^2 < 2b$. We calculate that

$$\begin{aligned}
SV =& \frac{1}{2} - \frac{1}{2}|v|^2 - ax \cdot v + a|v|^2 - ax \cdot \nabla_x \Phi(x) + 2bx \cdot v \\
\leq& C' - \left(\frac{1}{2} - a\right)|v|^2 + (2b - a)x \cdot v - a\gamma_1 |x|^2 - a\gamma_2 \Phi(x) \\
(a = 1/4, b = 1/8) \qquad =& C' - \frac{1}{4}|v|^2 - \frac{\gamma_1}{4}|x|^2 - \frac{\gamma_2}{4}\Phi(x) \leq C' - \frac{\gamma_1}{4}(|x|^2 + |v|^2) - \frac{\gamma_2}{4}\Phi(x) \\
\leq& C' - \frac{\min(\gamma_1, 1)}{4}\left(\frac{1}{2}|v|^2 + \frac{1}{4}x \cdot v + \frac{1}{8}|x|^2\right) - \frac{\gamma_2}{4}\Phi(x)
\end{aligned}$$

So $V(x, v)$ works with

$$\lambda = \frac{\min(\gamma_1, \gamma_2, 1)}{4}.$$

□

Proof of Theorem 5.1.3 in the case of the linear relaxation Boltzmann equation. The proof follows by applying Harris's Theorem since Lemmas 5.2.11 and 5.2.12 show that the equation satisfies the hypotheses of the theorem. □

5.2.3 Subgeometric convergence

When we do not have the superquadratic behaviour of the confining potential at infinity we can still use a Harris type theorem to show convergence to equilibrium. This time we must pay the price of having subgeometric rates of convergence. We use the subgeometric Harris's Theorem given in Section 1.2. Now instead of our earlier assumption on the confining potential Φ, we instead make a weaker assumption that Φ is a C^2 function satisfying

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{2\beta} + \gamma_2 \Phi(x) - A,$$

for some positive constant A, γ_1, γ_2, where

$$\langle x \rangle = \sqrt{1 + |x|^2},$$

and $\beta \in (0, 1)$.

Proof of Theorem 5.1.4 in the case of the linear relaxation Boltzmann equation. We have already proved the minorisation condition. We can also replicate the calculations for the Lyapunov function to get that in this new situation, take the V in Lemma 5.2.12, we have for $a = 1/4, b = 1/8$ that

$$SV \leq C' - \frac{1}{4}|v|^2 - \frac{\gamma_1}{4}\langle x\rangle^{2\beta} - \frac{\gamma_2}{4}\Phi(x).$$

For $x, y \geq 1$

$$(x+y)^\beta \leq x^\beta + y^\beta.$$

So we have

$$\begin{aligned} SV \leq & C' - \frac{\min(\gamma_1, 1)}{4}\left(\langle v\rangle^2 + \langle x\rangle^{2\beta}\right) - \frac{\gamma_2}{4}\Phi(x) \\ \leq & C'' - \frac{\min(\gamma_1, 1)}{4}\left(1 + |x|^2 + |v|^2\right)^\beta - \frac{\gamma_2}{4}\Phi(x)^\beta \\ \leq & C'' - \lambda\left(1 + \frac{1}{2}|v|^2 + \frac{1}{4}x\cdot v + \frac{1}{8}|x|^2\right)^\beta - \lambda\Phi(x)^\beta \\ \leq & C'' - \lambda\left(\Phi(x) + \frac{1}{2}|v|^2 + \frac{1}{4}x\cdot v + \frac{1}{8}|x|^2\right)^\beta, \end{aligned}$$

for some constant $\lambda, C'' > 0$ that can be explicitly computed, so we have that

$$SV \leq -\lambda V^\beta + C''.$$

This means we can take $\phi(s) = 1 + s^\beta$. Therefore, for u large

$$H_\phi(u) = \int_1^u \frac{1}{1+t^\beta}\mathrm{d}t \sim 1 + u^{1-\beta},$$

and for t large

$$H_\phi^{-1}(t) \sim 1 + t^{1/(1-\beta)}$$

and

$$\phi \circ H_\phi^{-1}(t) \sim (1+t)^{\beta/(1-\beta)}.$$

□

5.3 The linear Boltzmann Equation

We now look at the linear Boltzmann equation. This has been studied in the spatially homogeneous case in [1, 11]. Here the interest is partly that this is a more complex and physically relevant operator. Also, it presents less globally uniform behaviour in v which means that we have to use a Lyapunov function even on the torus. Apart from this, the strategy is very similar to that from the linear relaxation Boltzmann equation. The full Boltzmann equation has been studied as a Markov process in [27], the linear

case is similar and more simple. It is well known that this equation preserves positivity and mass, which follows from standard techniques both in the spatially homogeneous case and the case with transport. The Lyapunov condition on the torus and the bound below on the jump operator have to be verified in this situation.

We consider for $x \in \mathbb{T}^d$

$$\partial_t f + v \cdot \nabla_x f = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma, |v - v_*|\right) \left(f(v')\mathcal{M}(v'_*) - f(v)\mathcal{M}(v_*)\right) \mathrm{d}\sigma \mathrm{d}v_*. \tag{5.3.31}$$

We assume that B splits as

$$B\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma, |v - v_*|\right) = b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) |v - v_*|^\gamma.$$

We make a cutoff assumption that b is integrable in σ. In fact, we make a much stronger assumption that b is bounded below by a constant. We also work in the hard spheres/Maxwell molecules regime that is to suppose $\gamma \geq 0$. We have

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L}^+ f - \sigma(v) f,$$

where $\sigma(v) \geq 0$ and $\sigma(v)$ behaves like $|v|^\gamma$ for large v; that is,

$$0 \leq \sigma(v) \leq (1 + |v|^2)^{\gamma/2}, \qquad v \in \mathbb{R}^d. \tag{5.3.32}$$

See [11] Lemma 2.1 for example.

We also look at the situation where the spatial variable is in $\mathbb{R}^d$ and we have a confining potential. With hard sphere, the operator $\mathcal{L}^+$ acting on $x \cdot v$ produces error terms which are difficult to deal with. We show that when we have hard spheres with $\gamma > 0$ we can still show exponential convergence when $\Phi(x)$ is growing at least as fast as $|x|^{\gamma+2}$. In the subgeometric case we suppose $\Phi(x)$ grows at least as fast as $|x|^{\epsilon+1}, \epsilon > 0$. The equation is

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \Phi(x) \cdot \nabla_v f) = Q(f, \mathcal{M}). \tag{5.3.33}$$

We begin by proving lemmas which are useful for proving the Doeblin condition in both situations. We want to reduce to a similar situation to the linear relaxation Boltzmann equation.

Lemma 5.3.14. *Let f be a solution to* (5.3.31) *or* (5.3.33)*, and define $H(x, v) := |v|^2/2$ on the torus for* (5.3.31) *or $H(x, v) := \Phi(x) + |v|^2/2$ in the whole space for* (5.3.33)*, where Φ is a $\mathcal{C}^2$ potential bounded below. Take $E_0 > 0$ and assume that f has initial condition $f_0 = \delta_{(x_0, v_0)}$ with*

$$H(x, v) \leq E_0.$$

Then there exists a constant $C_1 > 0$ such that

$$f(t, x, v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s} \widetilde{\mathcal{L}}^+ T_{s-r} \widetilde{\mathcal{L}}^+ T_r (\mathbb{1}_E f_0(x, v)) \, \mathrm{d}r \, \mathrm{d}s,$$

where

$$\widetilde{\mathcal{L}}^+ g := \mathbb{1}_E \mathcal{L}^+ g, \qquad E := \{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d \, : \, H(x, v) \leq E_0\}.$$

Proof. Call $(X_t(x,v), V_t(x,v))$ the solution to the backward characteristic equations obtained from the transport part of either (5.3.31) or (5.3.33). Let us call

$$\Sigma(s,t,x,v) = e^{\int_s^t \sigma(V_r(x,v))\mathrm{d}r}.$$

Looking at Duhamel's formula again we get

$$f(t,x,v) = \Sigma(0,t,x,v)T_t f_0 + \int_0^t \Sigma(0,t-s,x,v)(T_{t-s}\mathcal{L}^+ f_s)(x,v)\,\mathrm{d}s$$

If a function $g = g(x,v)$ has support on the set

$$E := \{(x,v) \,:\, H(x,v) \le E_0\},$$

then the same is true of $T_t g$ (since the transport part preserves energy). On the set E we have, using (5.3.32),

$$\int_s^t \sigma(V_r(x,v))\mathrm{d}r \le (t-s)C\left(1+2E_0\right)^{\gamma/2} =: (t-s)C_1, \qquad (x,v) \in E.$$

Hence

$$\begin{aligned} f(t,x,v) &\ge \Sigma(0,t,x,v)T_t(\mathbb{1}_E f_0) + \int_0^t \Sigma(0,t-s,x,v)(T_{t-s}(\mathbb{1}_E\mathcal{L}^+ f_s))(x,v)\,\mathrm{d}s \\ &\ge e^{-tC_1}T_t(\mathbb{1}_E f_0) + \int_0^t e^{-(t-s)C_1}(T_{t-s}(\mathbb{1}_E\mathcal{L}^+ f_s))(x,v)\,\mathrm{d}s \\ &= e^{-tC_1}T_t f_0 + \int_0^t e^{-(t-s)C_1}(T_{t-s}(\widetilde{\mathcal{L}}^+ f_s))(x,v)\,\mathrm{d}s, \end{aligned}$$

where we define

$$\widetilde{\mathcal{L}}^+ g := \mathbb{1}_E\mathcal{L}^+ g.$$

Iterating this formula we obtain the result. □

We have that

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b\left(\frac{v-v_*}{|v-v_*|}\cdot\sigma\right)|v-v*|^\gamma f(v')\mathcal{M}(v'_*)\mathrm{d}\sigma\mathrm{d}v_*.$$

Using the Carleman representation we rewrite this as

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d}\frac{f(v')}{|v-v'|^{d-1}}\int_{E_{v,v'}} B(|u|,\xi)\mathcal{M}(v'_*)\mathrm{d}v'_*.$$

We want to bound this in the manner of Lemma 5.2.5 from the first part. We look at hard spheres and no angular dependence, which means

$$B(|u|,\xi) = C|u|^\gamma \xi^{d-2}$$

with $\gamma \ge 0$. We also have that

$$\xi = \frac{|v-v'|}{|2v-v'-v'_*|}, \quad |u| = |2v-v'-v'_*|.$$

So we have that

$$\mathcal{L}^+ f = \int_{\mathbb{R}^d}\frac{f(v')}{|v-v'|}\int_{E_{v,v'}} |2v-v'-v'_*|^{\gamma-d-2}\mathcal{M}(v'_*)\mathrm{d}v'_*.$$

We want to prove a local version of Lemma 5.2.5: look at this localised so we want

Lemma 5.3.15. *Consider the positive part $\mathcal{L}^+$ of the linear Boltzmann operator for hard spheres, assuming* (5.1.4) *with $\gamma \geq 0$, and* (5.1.5)*. For all $R_L, r_L > 0$, there exists $\alpha > 0$ such that*

$$\mathcal{L}^+ g(v) \geq \alpha \int_{B(R_L)} g(u)\,\mathrm{d}u \qquad \text{for all } v \in \mathbb{R}^d \text{ with } |v| \leq r_L.$$

Proof. First we note that on $E_{(v,v')}$ we have

$$|2v - v' - v'_*|^{-d-2} \geq C_d \exp\left(-\frac{1}{2}|v - v'_*|^2 - \frac{1}{2}|v - v'|^2\right).$$

Then since $\gamma \geq 0$ we have

$$|2v - v' - v'_*|^{\gamma} = \left(|v - v'|^2 + |v - v'_*|^2\right)^{\gamma/2} \geq |v - v'_*|^{\gamma}.$$

So this means that

$$\begin{aligned}
\int_{E_{(v,v')}} |2v - v' - v'_*|^{\gamma-d-2} \mathcal{M}(v'_*)\mathrm{d}v'_* & \\
&\geq C e^{-|v-v'|^2/2} \int_{E_{(v,v')}} |v - v'_*|^{\gamma} \exp\left(-\frac{1}{2}|v - v'_*|^2 - \frac{1}{2}|v'_*|^2\right) \mathrm{d}v'_* \\
&\geq C e^{-|v-v'|^2/2-|v|^2/2} \int_{E_{(v,v')}} |v - v'_*|^{\gamma} e^{-|v-v'_*|^2} \mathrm{d}v'_* \\
&= C' e^{-|v-v'|^2/2-|v|^2/2}.
\end{aligned}$$

So we have that

$$\begin{aligned}
\mathcal{L}^+ f(v) &\geq C \int_{\mathbb{R}^d} f(v')|v - v'|^{-1} e^{-|v-v'|^2/2-|v|^2/2} \mathrm{d}v' \\
&\geq C \int_{\mathbb{R}^d} f(v') e^{-2|v'|^2-3|v|^2} \\
&\geq C e^{-2R_L^2} e^{-3|v|^2} \int_{B(0,R_L)} f(v') \mathrm{d}v',
\end{aligned}$$

which is a similar bound to the one we found in Lemma 5.2.5. This gives the result by choosing $\alpha := C \exp(-2R_L^2 - 3|r_L|^2)$. □

5.3.1 On the torus

Now we work specifically on the torus. For the minorisation we can argue almost exactly as for the linear relaxation Boltzmann equation.

Lemma 5.3.16 (Doeblin condition)**.** *Assume* (5.1.4) *with $\gamma \geq 0$, and* (5.1.5)*. Given $t_* > 0$ and $R > 0$ there exist constants $0 < \alpha < 1$, $\delta_L > 0$ such that any solution $f = f(t, x, v)$ to the linear Boltzmann equation* (5.3.31) *on the torus with initial condition $f_0 = \delta_{(x_0,v_0)}$ with $|v_0| \leq R$ satisfies*

$$f(t_*, x, v) \geq \alpha \mathbb{1}_{|v| \leq \delta_L},$$

for almost all $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$.

Proof. Take $f_0 := \delta_{(x_0,v_0)}$, where $(x_0, v_0) \in \mathbb{T}^d \times \mathbb{R}^d$ is an arbitrary point with $|v_0| \leq R$. From Lemma 5.2.6 (with $R > \sqrt{d}$ and $t_0 := t_*/3$) we will use that there exist $\delta_L, R' > 0$ such that

$$\int_{B(R')} T_t\Big(\delta_{x_0}(x)\mathbb{1}_{\{|v|\leq\delta_L\}}\Big)\,\mathrm{d}v \geq \frac{1}{t^d} \qquad \text{for all } x_0 \in \mathbb{T}^d,\ t > t_0. \tag{5.3.34}$$

Also, Lemma 5.3.15 gives an $\alpha > 0$ such that

$$\mathcal{L}^+ g \geq \alpha \left(\int_{B(R_L)} g(x,u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\leq\delta_L\}}, \tag{5.3.35}$$

where $R_L := \max\{R', R\}$. Finally, from Lemma 5.3.14 we can find $C_1 > 0$ (depending on R) such that

$$f(t,x,v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s}\widetilde{\mathcal{L}}^+ T_{s-r}\widetilde{\mathcal{L}}^+ T_r(\mathbb{1}_E \delta_{(x_0,v_0)})\,\mathrm{d}r\,\mathrm{d}s,$$

where E is the set of points with energy less than E_0, with

$$E_0 := \max\{R^2/2, \delta_L^2/2\},$$

and we recall that $\widetilde{\mathcal{L}}^+ f := \mathbb{1}_E \mathcal{L}^+ f$. Due to our choice of E_0, we see that equation (5.3.34) also holds with $\widetilde{\mathcal{L}}^+$ in the place of $\mathcal{L}^+$. One can then carry out the same proof as in Lemma 5.2.7, using estimates (5.3.34) and (5.3.35) instead of the corresponding ones there. □

Since our Doeblin condition holds only on sets which are bounded in $|v|$, we do need a Lyapunov functional in this case (as opposed to the linear relaxation Boltzmann equation, where Lemma 5.2.7 gives a lower bound for all starting conditions (x, v)). Testing with $V = v^2$ involves proving a result similar to the moment control result from [**?**]. Instead of the σ representation we use the n-representation for the collisions:

$$v' = v - n(u \cdot n), \qquad v'_* = v_* + n(u \cdot n).$$

By our earlier assumption, the collision kernel can be written as

$$\tilde{B}(|v - v_*|, |\xi|) = |v - v_*|^\gamma \tilde{b}(|\xi|),$$

where

$$\xi := \frac{u \cdot n}{|u|}, \qquad u := v - v_*.$$

Here the $\tilde{B}, \tilde{b}$ are different from those in the σ representation because of the change of variables. We also have by assumption that $\tilde{b}$ is normalised, that is,

$$\int_{\mathbb{S}^d} \tilde{b}(|w \cdot n|)\,\mathrm{d}n = 1$$

for all unit vectors $w \in \S^{d-1}$.

Lemma 5.3.17. *Let $\mathcal{L}$ be the linear Boltzmann operator. There are constants $C, K > 0$ such that*

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \,\mathrm{d}v \leq -C \int_{\mathbb{R}^d} |v|^2 f \,\mathrm{d}v + K \int_{\mathbb{R}^d} f$$

for all non-negative measures f.

Proof. Using the weak formulation of the operator,

$$\int_{\mathbb{R}^d} L(f)|v|^2 \,\mathrm{d}v = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} f(v)\mathbb{M}(v_*)|v - v_*|^\gamma \tilde{b}(|\xi|)(|v'|^2 - |v|^2) \,\mathrm{d}n \,\mathrm{d}v \,\mathrm{d}v_*.$$

Now we notice that

$$\begin{aligned} |v'|^2 - |v|^2 = |v_*|^2 - |v'_*|^2 &= -(u \cdot n)^2 - 2(v_* \cdot n)(u \cdot n) \\ &= -|u|^2\xi^2 - 2(v_* \cdot n)(v \cdot n) + 2(v_* \cdot n)^2 \\ &= -|v|^2\xi^2 - |v_*|^2\xi^2 + 2v \cdot v_*\xi^2 - 2(v_* \cdot n)(v \cdot n) + 2(v_* \cdot n)^2. \end{aligned}$$

Note that the first term is negative and quadratic in v, and the rest of the terms are of lower order in v. Hence, calling

$$\gamma_b := \int_{\mathbb{S}^{d-1}} \xi^2 b(|\xi|) d\xi$$

we have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \,\mathrm{d}v = &- \gamma_b \int_{\mathbb{R}^d} |v|^2 f(v) \int_{\mathbb{R}^d} \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \\ &- \gamma_b \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} |v_*|^2 \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \\ &+ 2\gamma_b \int_{\mathbb{R}^d} vf(v) \int_{\mathbb{R}^d} v_* \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \\ &- 2 \int_{\S^{d-1}} \int_{\mathbb{R}^d} (v \cdot n) f(v) \int_{\mathbb{R}^d} (v_* \cdot n)\mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \,\mathrm{d}n \\ &+ \int_{\S^{d-1}} \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} (v_* \cdot n)^2 \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \,\mathrm{d}n \\ \leq &- \gamma_b \int_{\mathbb{R}^d} |v|^2 f(v) \int_{\mathbb{R}^d} \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \\ &+ (2 + \gamma_b) \int_{\mathbb{R}^d} |v| f(v) \int_{\mathbb{R}^d} |v_*| \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v \\ &+ \int_{\mathbb{R}^d} f(v) \int_{\mathbb{R}^d} |v_*|^2 \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \,\mathrm{d}v. \end{aligned}$$

We can now use the following bound, which holds for all $k \geq 0$ and some constants $0 < A_k \leq C_k$ depending on k:

$$A_k(1 + |v|^\gamma) \leq \int_{\mathbb{R}^d} |v_*|^k \mathbb{M}(v_*)|v - v_*|^\gamma \,\mathrm{d}v_* \leq C_k(1 + |v|^\gamma), \qquad v \in \mathbb{R}^d.$$

We get

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \,\mathrm{d}v \leq & -A_0\gamma_b \int_{\mathbb{R}^d} |v|^2(1+|v|^\gamma) f(v)\,\mathrm{d}v + C_1(2+\gamma_b)\int_{\mathbb{R}^d} |v|(1+|v|^\gamma) f(v)\,\mathrm{d}v \\ & + C_2 \int_{\mathbb{R}^d} f(v)(1+|v|^\gamma)\,\mathrm{d}v \\ \leq & \int_{\mathbb{R}^d} f(v)(C_2 + C_1(1+\gamma_b/2)/\epsilon)\,(1+|v|^\gamma)\,\mathrm{d}v \\ & - (A_0\gamma_b - \epsilon C_1(1+\gamma_b/2)) \int_{\mathbb{R}^d} |v|^2(1+|v|^\gamma) f(v)\mathrm{d}v \\ \leq & \int_{\mathbb{R}^d} f(v)\left(C_2 + C_1(1+\gamma_b/2)/\epsilon + (\epsilon C_1(1+\gamma_b/2) - A_0\gamma_b)|v|^2\right)(1+|v|^\gamma) f(v)\mathrm{d}v \\ & - (A_0\gamma_b - \epsilon C_1(1+\gamma_b/2)) \int_{\mathbb{R}^d} |v|^2 f(v)\mathrm{d}v \\ \leq & \alpha_1 \int_{\mathbb{R}^d} f(v)\mathrm{d}v - \alpha_2 \int_{\mathbb{R}^d} |v|^2 f(v)\mathrm{d}v.\end{aligned}$$

Here we choose ϵ sufficiently small to make the constant in front of the second moment negative. This also means that

$$(C_2 + C_1(1+\gamma_b/2)/\epsilon + (\epsilon C_1(1+\gamma_b/2) - A_0\gamma_b)|v|^2(1+|v|^\gamma)$$

is bounded above. These things together give the final line. □

Proof of Theorem 5.1.2 in the case of the linear Boltzmann equation. We have the Doeblin condition from Lemma 5.3.16 and the Lyapunov structure from Lemma 5.3.17. Harris's Theorem gives the result. □

5.3.2 On the whole space with a confining potential

We now work on the whole space with a confining potential. As we stated earlier, we cannot verify the Lyapunov condition in the hard spheres case. However, the proof for the Doeblin's condition is the same in the hard sphere or Maxwell molecule case. We need to combine the Lemmas 5.4.31, 5.3.14 and 5.3.15.

Lemma 5.3.18. *Let the potential $\Phi\,\mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$ function with compact level sets. Given $t > 0$ and $K > 0$ there exist constants $\alpha, \delta_X, \delta_V > 0$ such that for any (x_0, v_0) with $|x_0|, |v_0| < K$ the solution f to* (5.3.33) *with initial data $\delta_{(x_0,v_0)}$ satisfies*

$$f_t \geq \alpha \mathbb{1}_{\{|x|\leq\delta_X\}} \mathbb{1}_{\{|v|\leq\delta_V\}}.$$

Proof. We fix $R > 0$ as in Lemma 5.2.11. We use Lemma 5.4.31 to find constants $\alpha, R_2, R' > 0$ and an interval $[a, b] \subseteq (0, t)$ such that

$$\int_{B(R')} T_s(\delta_{x_0}\mathbb{1}_{\{|v|\leq R_2\}})\,\mathrm{d}v \geq \alpha \mathbb{1}_{\{|x|\leq R\}},$$

for any x_0 with $|x_0| \leq R$ and any $s \in [a, b]$. From Lemma 5.3.15 we will use that there exists a constant $\alpha_L > 0$ such that

$$\mathcal{L}^+ g(x, v) \geq \alpha_L \left(\int_{R_L} g(x, u)\,\mathrm{d}u\right) \mathbb{1}_{\{|v|\leq R_2\}} \tag{5.3.36}$$

for all nonnegative measures g, where $R_L := \max\{R, R'\}$. From Lemma 5.3.14 we can find $C_1 > 0$ (depending on R) such that

$$f(t,x,v) \geq e^{-tC_1} \int_0^t \int_0^s T_{t-s}\widetilde{\mathcal{L}}^+ T_{s-r}\widetilde{\mathcal{L}}^+ T_r(\mathbb{1}_E \delta_{(x_0,v_0)})\,\mathrm{d}r\,\mathrm{d}s,$$

where E is the set of points with energy less than E_0, with

$$E_0 := \max\{H(x,v) \,:\, |x| \leq R, |v| \leq \max\{R_L, R_2\}\},$$

and we recall that $\widetilde{\mathcal{L}}^+ f := \mathbb{1}_E \mathcal{L}^+ f$. These three estimates allow us to carry out a proof which is completely analogous to that of Lemma 5.2.11; notice that the only difference is the appearance of R' here, and the need to use $\widetilde{\mathcal{L}}^+$ (which still satisfies a bound of the same type). □

Now we need to find a Lyapunov functional. As before we will look at V of the form

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2.$$

We need $\Phi(x)$ to be stronger if we have hard spheres we want

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\gamma+2} + \gamma_2 \Phi(x) - A.$$

Lemma 5.3.19. *The function*

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \alpha x \cdot v + \beta |x|^2$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f(t,x,v)V(x,v)\mathrm{d}x\mathrm{d}v \leq -\lambda \int f(t,x,v)V(x,v)\mathrm{d}x\mathrm{d}v + K\int f(t,x,v)\mathrm{d}x\mathrm{d}v.$$

Proof. Lets look at how the collision operator acts on the different terms

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \mathrm{d}v = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v-v_*|^\gamma \left(|v'|^2 - |v|^2\right) \mathrm{d}n\mathrm{d}v\mathrm{d}v_*.$$

Repeating the calculation for the hard sphere case we notice that we in fact have that

$$\int_{\mathbb{R}^d} \mathcal{L}(f)|v|^2 \mathrm{d}v \leq -\alpha_1 \int_{\mathbb{R}^d} \langle v \rangle^{\gamma+2} f(v)\mathrm{d}v + \alpha_2 \int_{\mathbb{R}^d} f(v)\mathrm{d}v.$$

Similarly we have

$$\int_{\mathbb{R}^d} \mathcal{L}(f) x \cdot v \mathrm{d}v = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v-v_*|^\gamma \left(v' \cdot x - v \cdot x\right) \mathrm{d}n\mathrm{d}v\mathrm{d}v_*.$$

We can see that

$$v' \cdot x - v \cdot x = (v \cdot n)(x \cdot n) - (v_* \cdot n)(x \cdot n).$$

Integrating this gives that

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{L}(f) x \cdot v \mathrm{d}v &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} f(v)\mathcal{M}(v_*)\tilde{b}(|\xi|)|v-v_*|^\gamma (v \cdot n)(x \cdot n)\mathrm{d}v_*\mathrm{d}v\mathrm{d}n \\ &\leq \int_{\mathbb{R}^d} f(v)\langle v \rangle^{\gamma+1}|x|\mathrm{d}v.\end{aligned}$$

Therefore if we set

$$V(x,v) = H(x,v) + \alpha x \cdot v + \beta |x|^2,$$

where $H(x,v) = \Phi(x) + |v|^2/2$ is the energy, as above then using this we have that

$$\begin{aligned}
\int_{\mathbb{R}^d} (\mathcal{L} - T)(f) V(x,v) \mathrm{d}x\mathrm{d}v &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(-\alpha_1 \langle v \rangle^{\gamma+2} + \alpha_2 + \alpha \langle v \rangle^{\gamma+1} |x| + \alpha |v|^2 \right) \mathrm{d}x\mathrm{d}v \\
&\quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(-\alpha x \cdot \nabla_x \Phi(x) + 2\beta x \cdot v \right) \mathrm{d}v\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left((\alpha - \alpha_1) \langle v \rangle^{\gamma+2} + (\alpha + 2\beta) |x| \langle v \rangle^{\gamma+1} \right) \mathrm{d}v\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(-\alpha\gamma_1 \langle x \rangle^{\gamma+2} - \alpha\gamma_2 \Phi(x) + \alpha_2 + \alpha A \right) \mathrm{d}v\mathrm{d}x \\
\text{(setting} \quad \beta = \alpha, \alpha \leq \alpha_1/2) \qquad &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(\left(-\frac{\alpha_1}{2} + (3\alpha\epsilon)^{\frac{\gamma+2}{\gamma+1}} \frac{\gamma+1}{\gamma+2} \right) \langle v \rangle^{\gamma+2} \right) \mathrm{d}v\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(\left(\frac{(\alpha/\epsilon)^{\gamma+2}}{\gamma+2} - \alpha\gamma_2 \right) \langle x \rangle^{\gamma+2} \right) \mathrm{d}x\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(-\alpha\gamma_2 \Phi(x) + 1 - \frac{\gamma_1}{2} + \alpha A \right) \mathrm{d}x\mathrm{d}v.
\end{aligned}$$

Now we can set ϵ small enough so that the $\langle v \rangle^{\gamma+2}$ term is negative and then for this ϵ choose α small enough (since $\gamma + 2 \geq 1$ so that the $\langle x \rangle^{\gamma+2}$ term is negative. Then, since $\langle z \rangle^{\gamma+2}$ grows faster than $|z|^2$ at infinity this gives

$$\int_{\mathbb{R}^d} (\mathcal{L} - T)(f) V(x,v) \mathrm{d}x\mathrm{d}v \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,v) \left(-\lambda_1 (|x|^2 + |v|^2) - \lambda_2 \Phi(x) + K \right) \mathrm{d}x\, \mathrm{d}v.$$

Then using equivalence between the quadratic forms

$$|x|^2 + |v|^2 \quad \text{and} \quad \frac{1}{2}|v|^2 + \alpha x \cdot v + \alpha |x|^2,$$

when $\alpha < 1/2$ we have the result in the Lemma. □

Proof of Theorem 5.1.3 in the case of the linear Boltzmann equation. We have the minorisation condition in Lemma 5.3.18 and the Lyapunov condition from Lemma 5.3.19. Therefore we can apply Harris's Theorem. □

5.3.3 Subgeometric convergence

As with the linear relaxation Boltzmann equation, the minorisation results in Lemma 5.3.18 holds for Φ which are not sufficiently confining to prove the Lyapunov structure. However in this situation we can still prove subgeometric rates of convergence. Here in order to find a Lyapunov functional we need to be more precise about how $\mathcal{L}$ acts on the $x \cdot v$ moment.

We need $\Phi(x)$ to be stronger if we have hard spheres we want

$$x \cdot \nabla_x \Phi(x) \geq \gamma_1 \langle x \rangle^{\beta+1} + \gamma_2 \Phi(x) - A, \quad \Phi(x) \leq \gamma_3 \langle x \rangle^{1+\beta}$$

for some $\beta > 0$, then we have

Lemma 5.3.20. *The function*

$$V(x,v) = \Phi(x) + \frac{1}{2}|v|^2 + \frac{\alpha x \cdot v}{\langle x\rangle} + \beta\langle x\rangle$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int f(t,x,v)V(x,v)\mathrm{d}x\mathrm{d}v \le -\lambda\int f(t,x,v)V(x,v)^{\frac{\beta}{1+\beta}}\mathrm{d}x\mathrm{d}v + K\int f(t,x,v)\mathrm{d}x\mathrm{d}v.$$

Proof. Using the results in Lemma 5.3.19, we have

$$\begin{aligned}
\int_{\mathbb{R}^d}(\mathcal{L}-T)(f)V(x,v)\mathrm{d}x\mathrm{d}v \le & \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\left(-\alpha_1\langle v\rangle^{\gamma+2} + \alpha_2 + \alpha\langle v\rangle^{\gamma+1} + \alpha|v|^2\right)\mathrm{d}x\mathrm{d}v \\
& \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\left(-\frac{\alpha x\cdot\nabla_x\Phi(x)}{\langle x\rangle} + \frac{2\beta x\cdot v}{\langle x\rangle}\right)\mathrm{d}v\mathrm{d}x \\
\le & \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\left((\alpha-\alpha_1-2\beta)\langle v\rangle^{\gamma+2}\right)\mathrm{d}v\mathrm{d}x \\
& + \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\left(-\alpha\gamma_1\langle x\rangle^{\beta} - \alpha\gamma_2\frac{\Phi(x)}{\langle x\rangle} + \alpha_2 + \alpha A\right)\mathrm{d}v\mathrm{d}x \\
\le & \int_{\mathbb{R}^d\times\mathbb{R}^d} \lambda_1 f(x,v)\left(-|v|^2 - \langle x\rangle^{\beta} - \Phi(x)^{\beta/1+\beta} + C\right)\mathrm{d}x\mathrm{d}v \\
\le & \lambda\int f(t,x,v)V(x,v)^{\frac{\beta}{1+\beta}}\mathrm{d}x\mathrm{d}v + K\int f(t,x,v)\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

□

Proof of Theorem 5.1.4 in the linear Boltzmann case. We have the minorisation condition in Lemma 5.3.18 and the Lyapunov condition from Lemma 5.3.20. Therefore we can apply Harris's Theorem. □

5.4 Abstract Theorem

In this section we introduce a abstract theorem on the Harris condition, which covers the proofs above and may apply to more general models.

Theorem 5.4.21. *Suppose $f = f(t,x,v)$, with $t\in\mathbb{R}$, $x,v\in\mathbb{R}^d$, satisfies the following equation*

$$\partial_t f + v\cdot\nabla_x f - V(x)\cdot\nabla_v f = L^+ f + c(t,x,v)f, \quad f(0,x,v) = f_0(x,v),$$

with $V(x)$ is $\mathcal{C}^1$, L^+ is a linear positive operator, $c(t,x,v)$ is a function that is uniformly locally bounded, which means that, for any $R_1,R_2>0$, there exist a constant C_{R_1,R_2} such that for all $|x|\le R_1, |v|\le R_2, t\in\mathbb{R}$ we have

$$|c(t,x,v)| \le C_{R_1,R_2}.$$

Then we have, if L^+ satisfies Hypothesis 5.4.23 below, the solution to the original equation satisfies Harris-Doeblin condition, precisely for any f_0 smooth, for any $R>0$, there exist R_1,R_2,α_R,t which do not depend on f_0, such that

$$f(t,x,v) \geq \alpha_R \mathbb{1}[|x| \leq R_1]\mathbb{1}[|v| \leq R_2] \int_{B(0,R)\times B(0,R)} f_0(x,v)dxdv$$

and the constants R_1, R_2, α_R, t can be explicit computed.

Remark 5.4.22. $V(x)$ don't need to be a confinement potential.

We will now check several properties (which we list as hypotheses). The first one says that the operator L always allows jumps to any small velocity:

Hypothesis 5.4.23. *For all nonnegative functions g and for all $\delta, \beta > 0$, there exist $\alpha_{\beta,\delta} > 0$ such that*

$$L^+ g \geq \alpha_{\beta,\delta} \left(\int_{|u|\leq\beta} g(x,u)\,\mathrm{d}u \right) \mathbb{1}[|v| \leq \delta]. \tag{5.4.37}$$

In the following, we will denote the solution of the full transport part

$$\partial_t f + a(t,x,v)\cdot\nabla f = c(t,x,v)f, \quad f(0,x,v) = f_0(x,v),$$

by $T_t(f_0)$, and we denote the solution to

$$\partial_t f + a(t,x,v)\cdot\nabla f = 0, \quad f(0,x,v) = f_0(x,v),$$

by $S_t(f_0)$, we also denote $(S_t f_0)(x,v) = (X_t(x,v), V_t(x,v))$,it's easily seen that we have

$$T_t f_0(x,v) = S_t(x,v) e^{-\int_0^t c(r,X_r(x,v),V_r(x,v))dr}$$

The second one says that the transport part can move a particle to a neighborhood of 0, given that one starts out with the correct velocity:

Hypothesis 5.4.24. *For all $R > 0$ there exist $\beta_R, \gamma_R, \delta_R, T_R, b_R > 0, \epsilon_R > 0$ (possibly depending on R, independent of t) such that for all nonnegative functions $h = h(x)$ we have*

$$\int_{|v|\leq b_R} T_t\Big(h(x)\mathbb{1}[|v| \leq \beta_R]\Big)\,\mathrm{d}v \geq \gamma_R \left(\int_{|y|\leq R} h(y)\,\mathrm{d}y \right) \mathbb{1}[|x| \leq \delta_R] \ , \forall t \in (T_R - \epsilon_R, T_R) \tag{5.4.38}$$

We still need some continuity for the transport part.

Hypothesis 5.4.25. *(Continuity in short time) For all $R_1, R_2 > 0$ fixed, there exist $s, M > 0$ (possibly depending on R_1, R_2) such that for all $|x_0| \leq \frac{R_1}{2}$, $|v_0| \leq \frac{R_2}{2}$, $0 < t < s$ we have*

$$|X_t(x_0,v_0)| \leq R_1, \quad |V_t(x_0,v_0)| \leq R_2, \tag{5.4.39}$$

In particular we have

$$M\mathbb{1}[|x| \leq R_1]\mathbb{1}[|v| \leq R_2] \geq T_t \mathbb{1}[|x| \leq \frac{R_1}{2}]\mathbb{1}[|v| \leq \frac{R_2}{2}] \tag{5.4.40}$$

Hypothesis 5.4.26. *(Inverse Continuity in short time) For all $R_1, R_2 > 0$ there exist $s, M > 0$ (possibly depending on R_1, R_2) such that for all $0 < t < s$ we have*

$$T_t \mathbb{1}[|x| \leq R_1]\mathbb{1}[|v| \leq R_2] \geq M\mathbb{1}[|x| \leq \frac{R_1}{2}]\mathbb{1}[|v| \leq \frac{R_2}{2}] \tag{5.4.41}$$

5.4.1 Properties of the transport part

Theorem 5.4.27. *We denote the solution to the transport equation*

$$\partial_t f + v \cdot \nabla_x f - W(x) \cdot \nabla_v f = 0, \quad f(0,x,v) = f_0(x,v),$$

by $f_0(X_t, V_t) = T_t(f_0)$, if $W(x)$ is $\mathcal{C}^1$, then the transport equation satisfies Hypothesis 5.4.24, 5.4.25, 5.4.26.

□

Now we have the equation

$$\partial_t f + v \cdot \nabla_x f - W(x) \cdot \nabla_v f = Lf, \tag{5.4.42}$$

here L is defined as before and $x, v \in \mathbb{R}^d$. We want to use a slightly different strategy to show the minorisation condition based on the fact that we instantaneously produce large velocities. We first need a result on the trajectories of particles under the action of the potential W. Always assuming that W is a $\mathcal{C}^1$ function, we consider the characteristic ordinary differential equations associated to the transport part of (5.4.42):

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -W(x), \end{aligned} \tag{5.4.43}$$

and we denote by $(X_t(x_0,v_0), V_t(x_0,v_0))$ the solution at time t to (5.4.43) with initial data $x(0) = x_0$, $v(0) = v_0$. Performing time integration twice, it clearly satisfies

$$V_t(x_0,v_0) = v_0 - \int_0^t W(X_s(x_0,v_0))\ \mathrm{d}s \tag{5.4.44}$$

and

$$X_t(x_0,v_0) = x_0 + v_0 t - \int_0^t \int_0^s W(X_u(x_0,v_0))\,\mathrm{d}u\,\mathrm{d}s \tag{5.4.45}$$

for any $x_0, v_0 \in \mathbb{R}^d$ and any t for which it is defined. Intuitively the idea is that for small times we can approximate (X_t, V_t) by $(X_t^{(0)}, V_t^{(0)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -0. \end{aligned} \tag{5.4.46}$$

We can solve this explicitly $X_t^{(0)} = x_0 + v_0 t$. If we want to hit a point x_1 in time t then if we travel with the trajectory $X^{(0)}$ we just need to choose $v_0 = (x_1 - x_0)/t$. Now we choose an interpolation between $(X^{(0)}, V^{(0)})$ and (X, V). This is $(X^{(\epsilon)}, V^{(\epsilon)})$ which is a solution to the ordinary differential equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\epsilon^2 W(x), \end{aligned} \tag{5.4.47}$$

still with initial data (x_0, v_0). We calculate that

$$X_t^{(\epsilon)}(x_0, v_0) = X_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right), \qquad V_t^{(\epsilon)}(x_0, v_0) = \epsilon V_{\epsilon t}\left(x_0, \frac{v_0}{\epsilon}\right).$$

Now we can see from the ODE representation (and we will make this more precise later) that (X, V) is a C^1 map of (t, ϵ, x, v). Therefore if we fix t and x_0 we can define a C^1 map

$$F \quad : \quad [0,1] \times \mathbb{R}^d \to \mathbb{R}^d,$$

by

$$F(\epsilon, v) = X_t^{(\epsilon)}(x_0, v).$$

Then for $\epsilon = 0$ we can find v_* such that $F(0, v_*) = x_1$ as given above. Furthermore $\nabla F(0, v_*) \neq 0$ so by the implicit function theorem for all ϵ less than some ϵ_* we have a C^1 function $v(\epsilon)$ such that $F(\epsilon, v(\epsilon)) = x_1$. This means that

$$X_{\epsilon t}\left(x_0, \frac{v(\epsilon)}{\epsilon}\right) = x_1$$

. So if we take $s < \epsilon_* t$ then we can choose v such that $X_s(x_0, v) = x_1$. We now need to get quantitative estimates on ϵ_* we do this by traking the constants in the proof of the contraction mapping theorem.

In order to make these ideas quantitative and to check that the solution is in fact C^1 we need to get bounds on (X_t, V_t) and $W(X_t)$ for t is some fixed intervals. For the potentials of interest we will have that the solutions to these ODEs will exist for infinite time. We prove bounds on the solutions and $W(X_t)$ for any potential.

Lemma 5.4.28. *Assume that the potential W is $\mathcal{C}^1$ on $\mathbb{R}^d$. Take $\lambda > 1$, $R > 0$ and $x_0, v_0 \in \mathbb{R}^d$ with $|x_0| \leq R$. The solution $t \mapsto X_t(x_0, v_0)$ to (5.4.43) is defined (at least) for $|t| \leq T$, with*

$$T := \min\left\{\frac{(\lambda-1)R}{2|v_0|}, \frac{\sqrt{(\lambda-1)R}}{\sqrt{2C_{\lambda R}}}\right\}, \qquad C_{\lambda R} := \max_{|x| \leq \lambda R} |W(x)|.$$

(It is understood that any term in the above minimum is $+\infty$ if the denominator is 0.) Also, it holds that

$$|X_t(x_0, v_0)| \leq \lambda R \qquad \text{for } |t| \leq T.$$

from this we can deduce

$$|V_t(x_0, v_0)| \leq |v_0| + C_{\lambda R} t \qquad \text{for } |t| \leq T.$$

As a consequence, we have proved that T_t satisfies Hypothesis 5.4.25.

Proof. By standard ODE theory, the solution is defined in some maximal (open) time interval I containing 0; if this maximal interval has any finite endpoint t_*, then $X_t(x_0, v_0)$ has to blow up as t approaches t_*. Hence if the statement is not satisfied, there must

exist $t \in I$ with $|t| \leq T$ such that $|X_t(x_0, v_0)| \geq \lambda R$. By continuity, one may take $t_0 \in I$ to be the "smallest" time when this happens: that is, $|t_0| \leq T$ and

$$\begin{aligned} X_{t_0}(x_0, v_0) &= \lambda R, \\ |X_{t_0}(x_0, v_0)| &\leq \lambda R \qquad \text{for } |t| \leq |t_0|. \end{aligned}$$

By (5.4.44) and using that $|t_0| \leq T$ we have

$$\begin{aligned} \lambda R = |X_{t_0}(x_0, v_0)| &\leq |x_0| + |v_0 t_0| + \frac{t_0^2}{2} \max\{|W(X_t(x_0, v_0))| \colon t \leq t_0\} \\ &\leq R + \frac{(\lambda - 1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2 = \frac{(\lambda + 1)R}{2} + \frac{C_{\lambda R}}{2} t_0^2, \end{aligned}$$

which implies that

$$(\lambda - 1)R \leq C_{\lambda R} t_0^2.$$

If $C_{\lambda R} = 0$ this is false; if $C_{\lambda R} > 0$, then this contradicts that $|t_0| \leq T$. □

Theorem 5.4.29. *T_t has an inverse transform, precisely that there exist a transform G_t such that*

$$(G_t T_t)(x_0, v_0) = (x_0, v_0), \quad \forall t \geq 0,$$

moreover, if we denote $G_t(x_0, v_0) = (Y_t(x_0, v_0), Z_t(x_0, v_0))$, then Y_t, Z_t satisfies

$$\begin{aligned} \dot{y} &= -z \\ \dot{z} &= W(y), \end{aligned}$$

with initial condition $\{x_0, y_0\}$, as a consequence, T_t satisfies Hypothesis 5.4.26.

Proof. The first statement is obvious by taking $w = -t$, apply G_t on both side of equation (5.4.41), we turned Hypothesis 5.4.26 to, for all $R_1, R_2 > 0$ there exist $s > 0$ (possibly depending on R_1, R_2) such that for all $0 < t < s$ we have

$$\mathbb{1}[|x| \leq R_1]\mathbb{1}[|v| \leq R_2] \geq G_t \mathbb{1}[|x| \leq \frac{R_1}{2}]\mathbb{1}[|v| \leq \frac{R_2}{2}] \tag{5.4.48}$$

which is the continuity in short time (Hypothesis 5.4.25) for G_t, since G_t satisfies

$$\begin{aligned} \dot{y} &= -z \\ \dot{z} &= W(y), \end{aligned}$$

we know that G_t is only a change of sign of T_t, so Lemma 5.4.28 still holds for G_t. □

We now follow the intuition given at the beginning of this section. However we collapse the variables ϵ and t together and consequently look at $X_t(x, v/t)$ which is intuitively less clear but algebraically simpler.

Lemma 5.4.30. *Assume that $W \in \mathcal{C}^1(\mathbb{R}^d)$, and take $x_0, x_1 \in \mathbb{R}^d$. Let $R := \max\{|x_0|, |x_1|\}$. There exists $0 < T_1 = T_1(R)$ such that for any $0 < t \le T_1$ we can find a $|v_0| \le 4R$ such that*

$$X_t(x_0, \frac{v_0}{t}) = x_1.$$

In fact, it is enough to take $T_1 > 0$ such that

$$CT_1^2 e^{CT_1^2} \le \frac{1}{4}, \qquad T_1 \le \frac{\sqrt{R}}{\sqrt{2C_{2R}}}, \qquad \text{where} \quad C := \sup_{|x| \le 9R} |DW(x)|.$$

In addition we have for any $|x_0| \le R, |v_0| \le 4R, |v_1| \le 4R, 0 < t < T$, denote $\tilde{v_0} = v_0/t, \tilde{v_1} = v_1/t$, we have

$$|X_t(x_0, \tilde{v}_1) - X_t(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)t| \le \frac{1}{4}|\tilde{v}_1 - \tilde{v}_0|t,$$

which means that

$$\frac{3}{4}t \le |\frac{dX_t(x_0, v_0)}{dv_0}| \le \frac{5}{4}t, \quad 0 < t < T,$$

Proof. Due to Lemma 5.4.28 with $\lambda = 9$, we have

$$X_s(x_0, \frac{v}{t}), \qquad |v| \le 4R, \qquad |x_0| \le R,$$

is well-defined whenever

$$s \le \frac{2\sqrt{R}}{\sqrt{C_{9R}}} =: T_2, \qquad 0 < s \le t.$$

We define

$$f(t, v) = X_t(x_0, \frac{v}{t}) - x_1, \qquad t \ne 0,\ v \in \mathbb{R}^d,$$
$$f(0, v) := x_0 + v - x_1, \qquad v \in \mathbb{R}^d.$$

This is well-defined whenever

$$|t| \le T_2, \qquad |x_0| \le R, \qquad |v| \le 4R.$$

Our goal is to find a neighbourhood of $t = 0$ on which there exists $v = v(t)$ with $f(t, v(t)) = 0$, for which we will use the implicit function theorem.

Now, notice that we have

$$f(0, x_1 - x_0) = 0$$

and

$$\frac{\partial f}{\partial v_i}(0, x_1 - x_0) = 1, \qquad i = 1, \dots, d.$$

We can apply the implicit function theorem to find a neighbourhood I of $t = 0$ and a function $v = v(t)$ such that $f(t, v(t)) = 0$ for $t \in I$. However, since we need to estimate the size of I and of $v(t)$, we carry out a constructive proof.

Take $v_0, v_1 \in \mathbb{R}^d$ with $|v_0|, |v_1| \le 4R$, and denote $\tilde{v}_0 := v_0/t$, $\tilde{v}_1 := v_1/t$. By (5.4.51), for any $0 < t \le T_2$ fixed, for all $0 < \tau \le t$, we have

$$X_\tau(x_0, \tilde{v}_1) - X_\tau(x_0, \tilde{v}_0) = (\tilde{v}_1 - \tilde{v}_0)\tau + \int_0^\tau \int_0^s W(X_u(x_0, \tilde{v}_1)) - W(X_u(x_0, \tilde{v}_0))\,\mathrm{d}u\,\mathrm{d}s. \tag{5.4.49}$$

Take any $T_1 \le T_2$, to be fixed later. Then Lemma 5.4.28 implies, for all $0 < \tau \le t \le T_1$,

$$|X_\tau(x_0, \tilde{v}_1) - X_\tau(x_0, \tilde{v}_0)| \le |\tilde{v}_1 - \tilde{v}_0|\tau + CT_1 \int_0^\tau |X_u(x_0, \tilde{v}_1) - X_u(x_0, \tilde{v}_0)|\,\mathrm{d}u.$$

by Gronwall's Lemma we have

$$|X_\tau(x_0, \tilde{v}_1) - X_\tau(x_0, \tilde{v}_0)| \le |\tilde{v}_1 - \tilde{v}_0|\tau e^{CT_1\tau} \qquad \text{for } 0 < \tau \le t \le T_1.$$

Using this again in (5.4.49) we have

$$\begin{aligned} |X_\tau(x_0, \tilde{v}_1) - X_\tau(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)\tau| &\le |\tilde{v}_1 - \tilde{v}_0| CT_1 \int_0^\tau u e^{CT_1 u}\,\mathrm{d}u \\ &\le |\tilde{v}_1 - \tilde{v}_0|\tau CT_1^2 e^{CT_1^2}. \end{aligned}$$

Taking T_1 such that

$$CT_1^2 e^{CT_1^2} \le \frac{1}{4} \tag{5.4.50}$$

we have

$$|X_\tau(x_0, \tilde{v}_1) - X_\tau(x_0, \tilde{v}_0) - (\tilde{v}_1 - \tilde{v}_0)\tau| \le \frac{1}{4}|\tilde{v}_1 - \tilde{v}_0|\tau \qquad \text{for } 0 < \tau \le t \le T_1.$$

taking $\tau = t$ we have for all $0 \le t \le T_1$

$$|X_t(x_0, \frac{v_1}{t}) - X_t(x_0, \frac{v_0}{t}) - (v_1 - v_0)| \le \frac{1}{4}|v_1 - v_0|, \tag{5.4.51}$$

for any $0 < t \le T_1$ and any v_0, v_1 with $|v_0|, |v_1| \le 4R$. Now, for any $0 \le t \le T_1$ and $|v| \le 4R$ we define

$$A_t(v) = v - f(t, v).$$

A fixed point of $A_t(v)$ satisfies $f(t, v) = 0$, and by (5.4.51) $A_t(v)$ is contractive:

$$|A_t(v_1) - A_t(v_0)| \le \frac{1}{4}|v_1 - v_0| \qquad \text{for } 0 \le t \le T_1,\ |v| \le 4R.$$

(Equation (5.4.51) proves this for $0 < t \le T_1$, and for $t = 0$ it is obvious.) In order to use the Banach fixed-point theorem we still need to show that the image of A_t is inside the set with $|v| \le 4R$. Using (5.4.51) for $v_1 = 0$, $v_0 = v$ we also see that

$$|X_t(x_0, 0) - X_t(x_0, \frac{v}{t}) + v| \le \frac{1}{4}|v|,$$

which gives

$$|A_t(v) + x_1 - X_t(x_0, 0)| \le \frac{1}{4}|v|,$$

so

$$|A_t(v)| \le \frac{1}{4}|v| + |x_1| + |X_t(x_0,0)| \le 2R + |X_t(x_0,0)|. \tag{5.4.52}$$

If we take

$$T_1 \le \frac{\sqrt{R}}{\sqrt{2C_{2R}}} \tag{5.4.53}$$

then Lemma 5.4.28 (used for $\lambda = 2$) shows that

$$|X_t(x_0,0)| \le 2R \qquad \text{for } 0 \le t \le T_1,$$

and from (5.4.52) we have

$$|A_t(v)| \le 4R \quad \text{for} \quad 0 < t \le T_1.$$

Hence, as long as T_1 satisfies (5.4.50) and (5.4.53), A_t has a fixed point $|v|$ for any $0 < t \le T_1$, and this fixed point satisfies $|v| \le 4R$. □

Lemma 5.4.31. *T_t satisfies Hypothesis 5.4.24: For all $R > 0$ there exist $\beta_R, \gamma_R, \delta_R, T_R, b_R > 0, \epsilon_R > 0$ (possibly depending on R, independent of t) such that for all nonnegative functions $h = h(x)$ we have*

$$\int_{|v|\le b_R} T_t\Big(h(x)\mathbb{1}[|v| \le \beta_R]\Big)\,\mathrm{d}v \ge \gamma_R \left(\int_{|y|\le R} h(y)\,\mathrm{d}y\right) \mathbb{1}[|x| \le \delta_R] \ , \forall t \in (T_R - \epsilon_R, T_R),$$

Proof. Take $T_1(R)$ form Lemma 5.4.30, take

$$T_R = T_1, \quad \epsilon_R = \frac{T_R}{2}, \quad b_R = \frac{8R}{T_R}, \quad \gamma_R = \frac{4}{5T_R}, \quad \beta_R = b_R + T_R C_{9R}, \quad \delta_R = R,$$

first we have

$$\int_{|v|\le b_R} T_t\Big(h(x)\mathbb{1}[|v| \le \beta_R]\Big)\,\mathrm{d}v = \int_{|v|\le b_R} h(X_t(x,v))\mathbb{1}[|V_t(x,v)| \le \beta_R]\,\mathrm{d}v,$$

By Lemma 5.4.28 and the definition of T_1 in Lemma 5.4.30, for all $|x| \le R, |v| \le b_R, |t| \le T_R$ we have

$$|X_t(x,v)| \le 9R, \quad |V_t(x,v)| \le \beta_R,$$

so by

$$T_t f_0(x,v) = S_t(x,v) e^{-\int_0^t c(r, X_r(x,v), V_r(x,v))dr},$$

we have

$$T_t\Big(h(x)\mathbb{1}[|v| \le \beta_R]\Big) \ge e^{-C_{9R,\beta_R}T_R} h(X_t(x,v))\mathbb{1}[|V_t(x,v)| \le \beta_R],$$

which is

$$\int_{|v|\le b_R} T_t\Big(h(x)\mathbb{1}[|v| \le \beta_R]\Big)\,\mathrm{d}v \ge e^{-C_{9R,\beta_R}T_R} \int_{|v|\le b_R} h(X_t(x,v))\mathbb{1}[|V_t(x,v)| \le \beta_R]\,\mathrm{d}v,$$

and since $|x| \leq R, |v| \leq b_R, |t| \leq T_R$ implies $|V_t(x,v)| \leq \beta_R$, which means for any $0 \leq t \leq T_R$ we have

$$\mathbb{1}[|x| \leq R]\mathbb{1}[|v| \leq b_R] \leq \mathbb{1}[|V_t(x,v)| \leq \beta_R], \tag{5.4.54}$$

so

$$\int_{|v|\leq b_R} h(X_t(x,v))\mathbb{1}[|V_t(x,v)| \leq \beta_R]\,\mathrm{d}v \geq \int_{|v|\leq b_R} h(X_t(x,v))\,\mathrm{d}v\mathbb{1}[|x| \leq R],$$

Finally we make the change of variable $w = X_t(x,v)$, then using Lemma 5.4.30 we have, for every $x_1 \in B(0,R)$ there exists $v(x_0, x_1, s) \in B(0,4R)$ such that

$$X_s\left(x_0, \frac{v}{s}\right) = x_1.$$

Then since $s \geq T_R/2$ we have $v/s \in B(0, b_R)$, the range of X_t will be larger than $B(0,R)$, also by Lemma 5.4.30, we have

$$\frac{3}{8}T_R \leq \frac{3}{4}t \leq |\frac{dX_t(x_0,v_0)}{dv_0}| \leq \frac{5}{4}t \leq \frac{5}{4}T_R,$$

which means

$$|\frac{dv_0}{dX_t(x_0,v_0)}| \geq \frac{4}{5T_R},$$

so

$$\int_{|v|\leq b_R} h(X_t(x,v))\,\mathrm{d}v\mathbb{1}[|x| \leq R] \geq \frac{4}{5T_R}\int_{B(0,R)} h(x)dx\mathbb{1}[|x| \leq R],$$

which ends the proof of our theorem. □

Remark 5.4.32. When X_t, V_t satisfies

$$\dot{x} = v$$
$$\dot{v} = -W(x) + Mv,$$

for some constant M, then the integral form will turns to

$$V_t(x_0,v_0) = e^{Mt}v_0 - \int_0^t e^{M(t-s)}W(X_s(x_0,v_0))\ \mathrm{d}s,$$

and

$$X_t(x_0,v_0) = x_0 + \frac{e^{Mt}-1}{M}v_0 - \int_0^t\int_0^s e^{M(s-u)}W(X_u(x_0,v_0))\,\mathrm{d}u\,\mathrm{d}s,$$

The above proofs in the section will still be hold for this, as a conclusion, Theorem 5.4.21 will sitll holds for

$$\partial_t f - v\cdot\nabla_x f + \Phi(x)\cdot\nabla_v f = L^+ f + c(t,x,v)f + Mv\cdot f, \quad f(0,x,v) = f_0(x,v),$$

for some constant M, where the other notations are defined in 5.4.21. Similarly the result will holds for

$$\partial_t f - v\cdot\nabla_x f + \Phi(x,v)\cdot\nabla_v f = L^+ f + c(t,x,v)f, \quad f(0,x,v) = f_0(x,v),$$

where the term $\Phi(x,v)$ is uniformly Lipschitz in v.

5.4.2 Proof of Theorem 5.4.21

By the linearity of the operators, we only need to prove the theorem for all $f_0(x_0, v_0) = \delta_{x_0,v_0}$, precisely for any $R > 0$, for any $|x_0|, |v_0| \le R$, there exist R_1, R_2, α_R, t which do not depend on f_0, such that

$$f(t, x, v) \ge \alpha_R \mathbb{1}[|x| \le R_1]\mathbb{1}[|v| \le R_2]$$

First by Hypothesis 5.4.25 we have that there exist T_1 such that for any $0 < r < T_1$ we have

$$|X_r(x_0, v_0)| \le 2R, \quad |V_r(x_0, v_0)| \le 2R$$

since

$$S_r f_0 = \delta_{(X_r,V_r)}.$$

we have

$$T_r f_0 \ge e^{-C_{2R,2R}T_1}\delta_{(X_r,V_r)}.$$

Using Hypothesis 5.4.23,

$$L^+ S_r f_0 \ge \alpha_{2R,\beta_{2R}}\, e^{-C_{2R,2R}T_1}\delta_{X_r(x_0,v_0)}(x)\, \mathbb{1}[|v| \le \beta_{2R}] =: e^{-C_{2R,2R}T_1}\alpha_{2R,\beta_{2R}}\, h(x)\mathbb{1}[|v| \le \beta_{2R}],$$

where β_{2R} is defined in Hypothesis 5.4.24, using Hypotheses 5.4.23 and 5.4.24, whenever $s - r \in (T_{2R} - \epsilon_{2R}, T_{2R})$ we have

$$\begin{aligned}
L^+ T_{s-r} L^+ T_r f_0 &\ge e^{-C_{2R,2R}T_1}\alpha_{b_{2R},2R}\left(\int_{|u|\le b_{2R}} T_{s-r}L^+ T_r f_0\,\mathrm{d}u\right)\mathbb{1}[|v| \le 2R] \\
&\ge e^{-C_{2R,2R}T_1}\alpha_{2R,\beta_{2R}}\alpha_{b_{2R},2R}\left(\int_{|u|\le b_{2R}} T_{s-r}(h(x)\,\mathbb{1}[|u| \le \beta_{2R}])\,\mathrm{d}u\right)\mathbb{1}[|v| \le 2R] \\
&\ge e^{-C_{2R,2R}T_1}\alpha_{2R,\beta_{2R}}\alpha_{b_{2R},2R}\gamma_{2R}\,\mathbb{1}[|x| \le \delta_{2R}]\,\mathbb{1}[|v| \le 2R].
\end{aligned}$$

Then by Hypothesis 5.4.26 we have there exist $T_2 > 0$ such that for any $0 < t - s \le T_2$ we have

$$T_{t-s}L^+ T_{s-r} L^+ T_r f_0 \ge e^{-C_{2R,2R}T_1} M\alpha_{2R,\beta_{2R}}\alpha_{b_{2R},R}\gamma_{2R}\,\mathbb{1}[|x| \le \frac{\delta_{2R}}{2}]\,\mathbb{1}[|v| \le R].$$

Then we only need to using the Duhamel's formula. By the Duhamel's formula we have

$$f(t, x, v) \ge \int_0^t \int_0^s T_{t-s}L^+ T_{s-r} L^+ T_r f_0(x, v)\mathrm{d}r\mathrm{d}s.$$

fix first $T = T_1 + T_2 + T_{2R}$, we have for any fixed $R > 0$, for any $t < T, |x_0| < R, |v_0| < R$, let $f_0(x_0, v_0) = \delta_{x_0,v_0}$, we have

$$\begin{aligned}
f(t, x, v) \ge & e^{-C_{2R,2R}T_1} M\alpha_{2R,\beta_{2R}}\alpha_{b_{2R},R}\gamma_{2R}\mathbb{1}[|x| \le \frac{\delta_{2R}}{2}]\mathbb{1}[|v| \le R] \\
& \int_0^t \int_0^s \mathbb{1}[t - s \in [0, T_2]]\mathbb{1}[s - r \in [T_{2R} - \epsilon_{2R}, T_{2R}]]\mathbb{1}[r \in [0, T_1]]drds
\end{aligned}$$

the only thing left is to find a $t < T_1 + T_2 + T_{2R}$ such that

$$c \leq \int_0^t \int_0^s \mathbb{1}[t-s \in [0,T_2]]\mathbb{1}[s-r \in [T_{2R}-\epsilon_{2R}, T_{2R}]]\mathbb{1}[r \in [0,T_1]]drds$$

for some $c > 0$, let $\epsilon = min\{T_1, T_2, \epsilon_{2R}\}$, $t = T_{2R}$, we have

$$\begin{aligned}
& \int_0^t \int_0^s \mathbb{1}[t-s \in [0,T_2]]\mathbb{1}[s-r \in [T_{2R}-\epsilon_{2R}, T_{2R}]]\mathbb{1}[r \in [0,T_1]]drds \\
\geq \quad & \int_0^t \int_0^s \mathbb{1}[t-s \in [0,\epsilon]\mathbb{1}[r \in [T_{2R}-\epsilon, T_{2R}]]\mathbb{1}[s-r \in [0,\epsilon]]drds \\
\geq \quad & \int_{T_{2R}-\epsilon}^{T_{2R}} \int_{T_{2R}-\epsilon}^{s} \mathbb{1}[T_{2R}-s \in [0,\epsilon]\mathbb{1}[r \in [T_{2R}-\epsilon, T_{2R}]]\mathbb{1}[s-r \in [0,\epsilon]]drds \\
= \quad & \int_{T_{2R}-\epsilon}^{T_{2R}} \int_{T_{2R}-\epsilon}^{s} drds = \frac{\epsilon^2}{2}.
\end{aligned}$$

Then the proof for the whole theorem is ended. □

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RÉSUMÉ

Cette thèse porte principalement sur l'hypocoercivité et le comportement á long terme d'équations cinétiques. Nous considérons d'abord l'équation cinétique de Fokker-Planck avec la force de confinement faible et une classe de force générale. Nous prouvons l'existence et l'unicité d'un équilibre normalisé positif (dans le cas d'une force générale) et établissons un certain taux exponentiel ou sous-géométrique de convergence vers l'équilibre (et le taux peut être explicitement calculé). Ensuite, nous étudions la convergence vers l'équilibre de la relaxation Boltzmann linéaire (également appelé BGK linéaire) et le équations de Boltzmann linéaire soit sur le tore ou sur tout l'espace avec un confinement potentiel. Nous présentons des résultats de convergence explicites au normes de variation total ou de variation totale pondérée. Les taux de convergence sont exponentiels lorsque les équations sont posées sur le tore ou avec un potentiel de confinement grandir au moins quadratiquement à l'infini. De plus, nous donnons taux de convergence algébrique lorsque les potentiels sousquadratiqué pris en considération. Nous utilisons le théoréme de Harris.

MOTS CLÉS

Hypoellipticité, Hypocoercivité, Fokker-Planck cinétique, Boltzmann linéaire, Convergence vers l'équilibre

ABSTRACT

This thesis mainly study the hypocoercivity and long time behaviour of kinetic equations. We first consider the kinetic Fokker-Planck equation with weak confinement force and a class of general force. We prove the existence and uniqueness of a positive normalized equilibrium (in the case of a general force) and establish some exponential rate or sub-geometric rate of convergence to the equilibrium (and the rate can be explicitly computed). Then we study convergence to equilibrium of the linear relaxation Boltzmann (also known as linear BGK) and the linear Boltzmann equations either on the torus or on the whole space with a confining potential. We present explicit convergence results in total variation or weighted total variation norms. The convergence rates are exponential when the equations are posed on the torus, or with a confining potential growing at least quadratically at infinity. Moreover, we give algebraic convergence rates when subquadratic potentials considered. We use a method known as Harris's Theorem.

KEYWORDS

Hypoellipticity, Hypocoercivity, Kinetic Fokker-Planck, linear Boltzmann, Convergence to the equilibrium