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## Stochastic control on networks

Isaac Ohavi

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# THÈSE DE DOCTORAT

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## Contrôle Stochastique sur les réseaux.

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# Chapter 1

## General Introduction

In the thesis we are interested in controlled diffusions on a network structure and on the associated partial differential equations. We address here three basic problems: the existence of a diffusion, the optimal control of this diffusion (dynamic programming principle) and the well-posedness of the associated Hamilton-Jacobi equation. For simplicity of notation, we will focus on a network consisting in a single junction, as the multi-junction setting can be treated with similar tools. In this framework the diffusion satisfies the following controlled reflected stochastic differential equation

$$dx(t) = \sigma_{i(t)}(t, x(t), \beta_{i(t)}(t))dW(t) + b_{i(t)}(t, x(t), \beta_{i(t)}(t))dt + dl(t), \quad 0 \leq t \leq T \quad (1.1)$$

where  $l(\cdot)$  is a non decreasing process starting from 0, satisfying  $\int_0^T \mathbf{1}_{\{x(s)>0\}} dl(s) = 0$ , and  $W$  is a standard one dimensional brownian motion.

The corresponding Ito's formula is given by

$$\begin{aligned} df_{i(t)}(t, x(t)) &= \partial_x f_{i(t)}(t, x(t)) \sigma_{i(t)}(t, x(t), \beta_{i(t)}(t)) dW(t) + \left( \partial_t f_{i(t)}(t, x(t)) + \right. \\ &\quad \left. \frac{1}{2} \partial_{x,x} f_{i(t)}(t, x(t)) \sigma_{i(t)}^2(t, x(t), \beta_{i(t)}(t)) + \partial_x f_{i(t)}(t, x(t)) b_{i(t)}(t, x(t), \beta_{i(t)}(t)) \right) dt \\ &\quad + \sum_{i=1}^I \partial_x f_i(t, 0) \alpha_i(t) dl(t), \quad 0 \leq t \leq T, \end{aligned} \quad (1.2)$$

for  $f$  regular enough.

We recall that a junction  $\mathcal{J}$  consists in a vertex and a finite number  $I \in \mathbb{N}^*$  of edges.

More precisely:

$$\mathcal{J} = \left\{ X = (x, i), \quad x \in \mathbb{R}_+ \text{ and } i \in \{1, \dots, I\} \right\},$$

where all the points  $(0, i)$ ,  $i = 1, \dots, I$ , are identified to the vertex denoted by 0. We can then write

$$\mathcal{J} = \bigcup_{i=1}^I J_i,$$

with  $J_i := \mathbb{R}_+ \times \{i\}$  and  $J_i \cap J_j = \{0\}$  for  $i \neq j$ .

They will be two types of control. The first ones are the  $\beta_i$  appearing on each edge  $J_i^*$  in (1.1), and are classical from a mathematical point of view for a problem of control. The second ones are the terms  $\alpha_i$  appearing in (1.2), in front of the term of reflection  $l$ , that we will call in the sequel the control at the junction point, and can be interpreted as the probabilities of moving to another edge as soon as the junction point is reached by the process.

The optimal control consists in minimizing the cost

$$\mathbb{E}^P \left[ \sum_{i=1}^I \int_t^T \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), \beta_i(u)) du + \int_t^T h_0(u, \alpha_1(u) \dots \alpha_I(u)) dl(u) + g(X_T) \right],$$

with cost  $h_i$  on each edge  $J_i^*$ , cost  $h_0$  at the junction point, and the terminal condition  $g$ .

The value function  $v$  associated with this problem satisfies (at least formally) the following backward Hamilton-Jacobi equation at the junction, with non linear Kirchoff condition at the vertex

$$\begin{cases} \partial_t u_i(t, x) + H_i(t, x, u_i(t, x), \partial_x u_i(t, x), \partial_{x,x} u_i(t, x)) = 0, & \text{if } (t, x) \in (0, T) \times (0, +\infty), \\ F(t, u(t, 0), \partial_x u(t, 0)) = 0, & \text{if } t \in (0, T), \\ u(T, \cdot) = g(\cdot), \end{cases} \quad (1.3)$$

where

$$H_i(t, x, u, p, S) = \inf_{k_i \in K_i} \left\{ \frac{1}{2} \sigma_i^2(t, x, k_i) S + b_i(t, x, k_i) p + h_i(t, x, k_i) \right\},$$

$$F(t, u, p) = \inf_{\alpha = (\alpha_1, \dots, \alpha_I) \in A_0} \left\{ \sum_{i=1}^I \alpha_i p + h_0(t, \alpha_1 \dots \alpha_I) \right\}$$

Let us recall that the concept of ramified spaces and the analysis of (linear) partial differential equation on these spaces were first introduced by Nikol'skii [34] and Lumer [31, 32]. They are naturally associated with stochastic processes as in (1.1) living on graphs. These processes were introduced in the seminal papers [15] and [16], where they appear as the singular perturbation of Hamiltonian systems. Since then there has been a large literature on the subject: see for instance the survey paper [27] on similar (multi-dimensional) stochastic systems and their interpretations.

On the PDE side, there has been several works on linear and quasilinear parabolic equations of the form (1.3), with more general Hamiltonians on the edges. For linear equations, von Below [38] shows that, under natural smoothness and compatibility conditions, linear boundary value problems on a network with a linear Kirchhoff are well-posed and enjoy a strong maximum principle. In [40] he also studies the classical global solvability for a class of semilinear parabolic equations on ramified networks, where a dynamical node condition is prescribed. Still in the linear setting, another approach, yielding similar existence results, was developed by Fijavz, Mugnolo and Sikolya in [12]: the idea is to use semi-group theory as well as variational methods to understand how the spectrum of the operator is related to the structure of the network.

Equations of the form (1.3) can also be analyzed in terms of viscosity solutions. The first results on viscosity solutions for Hamilton-Jacobi equations on networks have been obtained by Schieborn in [36] for the Eikonal equations and later discussed in many contributions on first order problems [8, 20, 28], elliptic equations [29] and second order problems with vanishing diffusion at the vertex [30]. Let us finally quote the very recent paper [2] which discusses the well-posedness of stationary Hamilton-Jacobi equations on a network and builds solutions to ergodic mean field game systems on this network. The same authors will in a forthcoming work treat the finite horizon MFGs, in the nonstationary case. Still in the MFGs theory on networks in the nonstationary case, we refer to the



following recent thesis [9], where the author studies MFG PDE system on a junction, with linear Kirchoff condition at the junction point, and build weak solutions of the system in Sobolev spaces, with Lipschitz Hamiltonians.

The main reason for studying equation (1.3) is the optimal control of a diffusion living on the junction. Control problems on stratified domains or networks have already been well-studied in the literature, most often for first order problems, and we refer for instance to [3], [4], [5], [17], [1], [33]... On the other hand, for stochastic control problems with reflection and controllability at the boundary, we refer to [11], where the author studied optimal reflection with some applications in financial markets.

Let us recall in the sequel, the main difficulties and motivations that we have faced out in this thesis:

First of all, to study a problem of control, with control at the junction point, which involves the behavior of the process  $l(\cdot)$  given in (1.2), we can not consider second order terms vanishing at the junction, since the quadratic variation of  $l(\cdot)$  is centrally related to the assumption of ellipticity. From a PDE point of view, we will study the quasi linear case, since in the literature, the viscosity context have been seldom considered, and a comparison Theorem for the problem (1.3), with non vanishing viscosity at the junction point is still an open problem. Moreover, in the quasi linear context, classical solutions and their uniqueness have been proved in [40], but with a dynamical Neumann condition at the junction point, which can not be used for our problem regarding to the generator giving in (1.2). The same author proves the existence and uniqueness of a solution, when the Neumann boundary condition is non dynamical, but with viscosity vanishing at the junction point. The proof uses classical fixed point arguments, and this method is not well adapted for our problem. This is why, we will consider an elliptic scheme, since the non degenerate elliptic problem is well known in literature.

The second main difficulty for studying a problem of control at the junction, is that we should be able to get the existence of a diffusion with measurable, and time dependent coefficients. This the main motivation of Chapter 3, where we propose another proof of the existence on a diffusion, since semi-groups techniques used in [15] are not adapted to this context. The main problem we face out, is that we are not able to have measurable coefficients  $\alpha_i$  at the junction point, and this the key point to formulate a verification

Theorem in the theory of stochastic control. However, in the last Chapter [4](#), we will prove the dynamic programming principle, adapting classical arguments of compactification methods, to our problem which is a good starting point. We get several technical results, that will be used in a future framework to improve this theory.

Our contributions to the topic are the following:

- *Well-posedness of the Hamilton-Jacobi equation [\(1.3\)](#), in the time dependent and uniformly parabolic setting.*

In Chapter [2](#), we prove the existence of classical quasi linear solution to equation [\(1.3\)](#). Our main assumptions are that the equation is uniformly parabolic with smooth coefficients and that the term  $F = F(u, p)$  at the junction is either decreasing with respect to  $u$  or increasing with respect to  $p$ . Note that, in contrast with most previous works, we assume that the diffusion is not degenerate (in particular at the junction) and of evolution type. We also prove a comparison principle, from which we derive the uniqueness of the solution.

The main idea of the proof is to use a time discretization, exploiting at each step the solvability of the associated elliptic problem:

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, \\ F(u(0), \partial_x u(0)) = 0. \end{cases} \quad (1.4)$$

- *Construction of a solution to the diffusion [\(1.1\)](#).*

Before discussing the optimal control problem [\(1.3\)](#), we show how to build a stochastic process of the form [\(1.1\)](#). Let us recall that in [\[16\]](#), this process is built by semi-group techniques.

The aim of Chapter [3](#) is to provide a different and more intuitive method for the construction of the diffusion [\(1.1\)](#). We explain this construction in the time-independent framework, where the coefficient of the diffusion at the junction  $\alpha = (\alpha_i)$  are fixed. Our idea is to build the process as the limit, as the small parameter  $\delta$  tends to 0, of processes which jump at a position  $\delta$  on a branch as soon as it touches the junction point 0. The branch  $i$  is chosen randomly (and independently of the process) with probability  $\alpha_i$ . We also describe the process  $l$  as the limit of the quadratic variation

of  $X$  over the times spent at the neighborhood of 0.

- *Dynamic programming principle for the optimal control problem (1.3).*

In Chapter 4, we study stochastic control problems at the junction, with control at the junction point. This kind of problem has not been studied before and our main result is a dynamic programming principle. For this we follow the classical strategy of proof introduced in [25]. We use a weak formulation in the problem, the main novelty (as well as the main difficulty) being to treat the reflexion at the junction point, which is responsible for the process  $l$  in (1.1). Our first main step is to prove the compactness of the class of admissible controls. Then we show stability properties of the set of controls by conditioning and concatenation at stopping times, from which we derive the dynamic programming principle.

Beside this introduction, the thesis is organized in 3 Chapters: in the first one we analyze the Hamilton-Jacobi equation with a Neumann boundary condition at the junction; in the second one we build a solution to the reflected process while the last one is dedicated to optimal control problems on the junction.

Different topics can be treated in futur frameworks, as the viscosity theory for equations of type (1.3), the verification theorem for the stochastic problem, and applications in MFGs.

## Chapter 2

# Quasi linear parabolic PDE in a junction with non linear Neumann vertex condition

This chapter is based on a paper written under the supervision of P. Cardaliaguet and submitted for publication: "I.Wahbi. Quasi linear parabolic PDE in a junction, with non linear Neumann boundary condition. ArXiv:1807.04032, 2018," [\[41\]](#).

### 2.1 Introduction

In this Chapter, we study non degenerate quasi linear parabolic partial differential equations on a junction, satisfying a non linear Neumann boundary condition at the junction point  $x = 0$ :

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x}^2 u_i(t, x) + H_i(x, u_i(t, x), \partial_x u_i(t, x)) = 0, \\ \text{for all } x > 0, \text{ and for all } i \in \{1 \dots I\}, \\ F(u(t, 0), \partial_x u(t, 0)) = 0. \end{cases} \quad (2.1)$$

The well-known Kirchhoff law corresponds to the case where  $F$  is linear in  $\partial_x u$  and independent of  $u$ .

Originally introduced by Nikol'skii [\[34\]](#) and Lumer [\[31, 32\]](#), the concept of ramified

spaces and the analysis of partial differential equation on these spaces have attracted a lot of attention in the last 30 years. As explained in [34], the main motivations are applications in physics, chemistry, and biology (for instance small transverse vibrations in a grid of strings, vibration of a grid of beams, drainage system, electrical equation with Kirchhoff law, wave equation, heat equation,...). Linear diffusions of the form (4.4), with a Kirchhoff law, are also naturally associated with stochastic processes living on graphs. These processes were introduced in the seminal papers [15] and [16]. Another motivation for studying (4.4) is the analysis of associated stochastic optimal control problems with a control at the junction. The result of this Chapter will allow us in a future work to characterize the value function of such problems.

There has been several works on linear and quasilinear parabolic equations of the form (4.4). For linear equations, von Below [38] shows that, under natural smoothness and compatibility conditions, linear boundary value problems on a network with a linear Kirchhoff condition at the vertex point, are well-posed. The proof consists mainly in showing that the initial boundary value problem on a junction is equivalent to a well-posed initial boundary value problem for a parabolic system, where the boundary conditions are such that the classical results on linear parabolic equations [26] can be applied. The same author investigates in [39] the strong maximum principle for semi linear parabolic operators with Kirchhoff condition, while in [40] he studies the classical global solvability for a class of semilinear parabolic equations on ramified networks, where a dynamical node condition is prescribed: Namely the Neumann condition at the junction point  $x = 0$  in (4.4), is replaced by the dynamic one

$$\partial_t u(t, 0) + F(t, u(t, 0), \partial_x u(t, 0)) = 0.$$

In this way the application of classical estimates for domains established in [26] becomes possible. The author then establish the classical solvability in the class  $\mathcal{C}^{1+\alpha, 2+\alpha}$ , with the aid of the Leray-Schauder-principle and the maximum principle of [39]. Let us note that this kind of proof fails for equation (4.4) because in this case one cannot expect an uniform bound for the term  $|\partial_t u(t, 0)|$  (the proof of Lemma 3.1 of [26] VI.3 fails). Still in the linear setting, another approach, yielding similar existence results, was developed by Fijavz, Mugnolo and Sikolya in [12]: the idea is to use semi-group theory as well as

variational methods to understand how the spectrum of the operator is related to the structure of the network.

Equations of the form (4.4) can also be analyzed in terms of viscosity solutions. The first results on viscosity solutions for Hamilton-Jacobi equations on networks have been obtained by Schieborn in [36] for the Eikonal equations and later discussed in many contributions on first order problems [8, 20, 28], elliptic equations [29] and second order problems with vanishing diffusion at the vertex [30].

In contrast second order Hamilton-Jacobi equations with a non vanishing viscosity at the boundary have seldom been studied in the literature and our aim is to show the well-posedness of classical solutions for (4.4) in suitable Höder spaces: see Theorem 2.2.2 for the existence and Theorem 2.2.4 for the comparison, and thus the uniqueness. Our main assumptions are that the equation is uniformly parabolic with smooth coefficients and that the term  $F = F(u, p)$  at the junction is either decreasing with respect to  $u$  or increasing with respect to  $p$ .

The main idea of the proof is to use a time discretization, exploiting at each step the solvability in  $C^{2+\alpha}$  of the elliptic problem

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x}^2 u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, \\ F(u(0), \partial_x u(0)) = 0. \end{cases} \quad (2.2)$$

The Chapter is organized as follows. In section 2.2, we introduce the notations and state our main results. In Section 2.3, we review the mains results of existence and uniqueness of the elliptic problem (2.2). Finally Section 2.4, is dedicated to the proof of our main results.

## 2.2 Main results

In this section we state our main result Theorem [2.2.2](#), on the solvability of the parabolic problem with Neumann boundary condition at the vertex, on a bounded junction

$$\left\{ \begin{array}{l} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x} u_i(t, x) + \\ H_i(x, u_i(t, x), \partial_x u_i(t, x)) = 0, \quad \text{if } (t, x) \in (0, T) \times (0, a_i), \\ F(u(t, 0), \partial_x u(t, 0)) = 0, \quad \text{if } t \in [0, T], \\ \forall i \in \{1 \dots I\}, \quad u_i(t, a_i) = \phi_i(t), \quad \text{if } t \in [0, T], \\ \forall i \in \{1 \dots I\}, \quad u_i(0, x) = g_i(x), \quad \text{if } x \in [0, a_i]. \end{array} \right. \quad (2.3)$$

There will be two typical assumptions for  $F = F(u, p)$ : either  $F$  is decreasing with respect to  $u$  or  $F$  is increasing with respect to  $p$  (Kirchhoff conditions).

### 2.2.1 Notations and preliminary results

Let us start by introducing the main notation used in this Chapter as well as an interpolation result.

Let  $I \in \mathbb{N}^*$  be the number of edges, and  $a = (a_1, \dots, a_I) \in (0, \infty)^I$  be the length of each edge.

The bounded junction is defined by

$$\mathcal{J}^a = \left\{ X = (x, i), \quad x \in [0, a_i] \text{ and } i \in \{1, \dots, I\} \right\},$$

where all the points  $(0, i)$ ,  $i = 1, \dots, I$ , are identified to the vertex denoted by 0. We can then write

$$\mathcal{J}^a = \bigcup_{i=1}^I J_i^{a_i},$$

with  $J_i^{a_i} := [0, a_i] \times \{i\}$ ,  $J_i^{a_i} \cap J_j^{a_j} = \{0\}$ . For  $T > 0$ , the time-space domain  $\mathcal{J}_T^a$  is defined by

$$\mathcal{J}_T^a = [0, T] \times \mathcal{J}^a.$$

The interior of  $\mathcal{J}_T^a$  set minus the junction point 0 is denoted by  $\overset{\circ}{\mathcal{J}}_T^a$ , and is defined by

$$\overset{\circ}{\mathcal{J}}_T^a = (0, T) \times \left( \bigcup_{i=1}^I \overset{\circ}{\mathcal{J}}_i^{a_i} \right).$$

For the functional spaces that will be used in the sequel, we use here the notations of Chapter 1.1 of [26]. For the convenience of the reader, we recall these notations in Appendix A.

In addition we introduce the parabolic Hölder space on the junction  $(\mathcal{C}^{\frac{1}{2}, l}(\mathcal{J}_T^a), \|\cdot\|_{\mathcal{C}^{\frac{1}{2}, l}(\mathcal{J}_T^a)})$  and the space  $\mathcal{C}_b^{\frac{1}{2}, l}(\overset{\circ}{\mathcal{J}}_T^a)$ , defined by (where  $l > 0$ , see Appendix A for more details)

$$\begin{aligned} \mathcal{C}^{\frac{1}{2}, l}(\mathcal{J}_T^a) &:= \left\{ f : \mathcal{J}_T^a \rightarrow \mathbb{R}, \quad (t, (x, i)) \mapsto f_i(t, x), \quad \forall (i, j) \in \{1 \dots I\}^2, \quad \forall t \in (0, T), \right. \\ &\quad \left. f_i(t, 0) = f_j(t, 0) = f(t, 0), \quad \forall i \in \{1 \dots I\}, \quad (t, x) \mapsto f_i(t, x) \in \mathcal{C}^{\frac{1}{2}, l}([0, T] \times [0, a_i]) \right\}, \\ \mathcal{C}_b^{\frac{1}{2}, l}(\overset{\circ}{\mathcal{J}}_T^a) &:= \left\{ f : \mathcal{J}_T^a \rightarrow \mathbb{R}, \quad (t, (x, i)) \mapsto f_i(t, x), \right. \\ &\quad \left. \forall i \in \{1 \dots I\}, \quad (t, x) \mapsto f_i(t, x) \in \mathcal{C}_b^{\frac{1}{2}, l}((0, T) \times (0, a_i)) \right\}, \end{aligned}$$

with

$$\|u\|_{\mathcal{C}^{\frac{1}{2}, l}(\mathcal{J}_T^a)} = \sum_{1 \leq i \leq I} \|u_i\|_{\mathcal{C}^{\frac{1}{2}, l}([0, T] \times [0, a_i])}.$$

We will use the same notations, when the domain does not depend on time, namely  $T = 0$ ,  $\Omega_T = \Omega$ , removing the dependence on the time variable.

We continue with the definition of a nondecreasing maps  $F : \mathbb{R}^I \rightarrow \mathbb{R}$ .

Let  $(x = (x_1, \dots, x_I), y = (y_1, \dots, y_I)) \in \mathbb{R}^{2I}$ , we say that

$$x \leq y, \text{ if } \forall i \in \{1 \dots I\}, \quad x_i \leq y_i,$$

and

$$x < y, \text{ if } x \leq y, \text{ and there exists } j \in \{1 \dots I\}, \quad x_j < y_j.$$



We say that  $F \in \mathcal{C}(\mathbb{R}^I, \mathbb{R})$  is nondecreasing if

$$\forall (x, y) \in \mathbb{R}^I, \text{ if } x \leq y, \text{ then } F(x) \leq F(y),$$

increasing if

$$\forall (x, y) \in \mathbb{R}^I, \text{ if } x < y, \text{ then } F(x) < F(y).$$

Next we recall an interpolation inequality, which will be useful in the sequel.

**Lemma 2.2.1.** *Suppose that  $u \in \mathcal{C}^{0,1}([0, T] \times [0, R])$  satisfies an Hölder condition in  $t$  in  $[0, T] \times [0, R]$ , with exponent  $\alpha \in (0, 1]$ , constant  $\nu_1$ , and has derivative  $\partial_x u$ , which for any  $t \in [0, T]$  are Hölder continuous in the variable  $x$ , with exponent  $\gamma \in (0, 1]$ , and constant  $\nu_2$ . Then the derivative  $\partial_x u$  satisfies in  $[0, T] \times [0, R]$ , an Hölder condition in  $t$ , with exponent  $\frac{\alpha\gamma}{1+\gamma}$ , and constant depending only on  $\nu_1, \nu_2, \gamma$ . More precisely*

$$\begin{aligned} & \forall (t, s) \in [0, T]^2, \quad |t - s| \leq 1, \quad \forall x \in [0, R], \\ |\partial_x u(t, x) - \partial_x u(s, x)| & \leq \left( 2\nu_2 \left( \frac{\nu_1}{\gamma\nu_2} \right)^{\frac{\gamma}{1+\gamma}} + 2\nu_1 \left( \frac{\gamma\nu_2}{\nu_1} \right)^{-\frac{1}{1+\gamma}} \right) |t - s|^{\frac{\alpha\gamma}{1+\gamma}}. \end{aligned}$$

This is a special case of Lemma II.3.1, in [26], (see also [34]). The main difference is that we are able to get global Hölder regularity in  $[0, T] \times [0, R]$  for  $\partial_x u$  in its first variable. Let us recall that this kind of result fails in higher dimensions.

*Proof.* Let  $(t, s) \in [0, T]^2$ , with  $|t - s| \leq 1$ , and  $x \in [0, R]$ . Suppose first that  $x \in [0, \frac{R}{2}]$ . Let  $y \in [0, R]$ , with  $y \neq x$ , we write

$$\begin{aligned} & \partial_x u(t, x) - \partial_x u(s, x) = \\ & \frac{1}{y - x} \int_x^y (\partial_x u(t, x) - \partial_x u(t, z)) + (\partial_x u(t, z) - \partial_x u(s, z)) + (\partial_x u(s, z) - \partial_x u(s, x)) \, dz. \end{aligned}$$

Using the Hölder condition in time satisfied by  $u$ , we have

$$\left| \frac{1}{y - x} \int_x^y (\partial_x u(t, z) - \partial_x u(s, z)) dz \right| \leq \frac{2\nu_1 |t - s|^\alpha}{|y - x|}.$$

On the other hand, using the Hölder regularity of  $\partial_x u$  in space satisfied, we have

$$\left| \frac{1}{y-x} \int_x^y (\partial_x u(t, x) - \partial_x u(t, z)) + (\partial_x u(s, z) - \partial_x u(s, x)) dz \right| \leq 2\nu_2 |y-x|^\gamma.$$

It follows

$$|\partial_x u(t, x) - \partial_x u(s, x)| \leq 2\nu_2 |y-x|^\gamma + \frac{2\nu_1 |t-s|^\alpha}{|y-x|}.$$

Assuming that  $|t-s| \leq \left( (\frac{3R}{2})^{1+\gamma} \frac{\gamma\nu_2}{\nu_1} \right)^{\frac{1}{\alpha}} \wedge 1$ , minimizing in  $y \in [0, R]$ , for  $y > x$ , the right side of the last equation, we get that the infimum is reached for

$$y^* = x + \left( \frac{\nu_1 |t-s|^\alpha}{\gamma\nu_2} \right)^{\frac{1}{1+\gamma}},$$

and then

$$|\partial_x u(t, x) - \partial_x u(s, x)| \leq C(\nu_1, \nu_2, \gamma) |t-s|^{\frac{\alpha\gamma}{1+\gamma}},$$

where the constant  $C(\nu_1, \nu_2, \gamma)$ , depends only on the data  $(\nu_1, \nu_2, \gamma)$ , and is given by

$$C(\nu_1, \nu_2, \gamma) = 2\nu_2 \left( \frac{\nu_1}{\gamma\nu_2} \right)^{\frac{\gamma}{1+\gamma}} + 2\nu_1 \left( \frac{\gamma\nu_2}{\nu_1} \right)^{-\frac{1}{1+\gamma}}.$$

For the cases  $y < x$ , and  $x \in [\frac{R}{2}, R]$ , we argue similarly, that completes the proof.  $\square$

## 2.2.2 Assumptions and main results

We state in this subsection the central Theorem of this Chapter, namely the solvability and uniqueness of [\(2.3\)](#) in the class  $\mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1+\frac{\alpha}{2}, 2+\alpha}(\overset{\circ}{\mathcal{J}}_T^a)$ . **In the rest of these Chapter we fix  $\alpha \in (0, 1)$ .**

Let us state the assumptions we will work on.

### Assumption ( $\mathcal{P}$ )

We introduce the following data

$$\begin{cases} F \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^I, \mathbb{R}) \\ g \in \mathcal{C}^1(\mathcal{J}^a) \cap \mathcal{C}_b^2(\overset{\circ}{\mathcal{J}}^a) \end{cases},$$

and for each  $i \in \{1 \dots I\}$

$$\begin{cases} \sigma_i \in \mathcal{C}^1([0, a_i] \times \mathbb{R}, \mathbb{R}) \\ H_i \in \mathcal{C}^1([0, a_i] \times \mathbb{R}^2, \mathbb{R}) \\ \phi_i \in \mathcal{C}^1([0, T], \mathbb{R}) \end{cases}.$$

We suppose furthermore that the data satisfy

(i) Assumption on  $F$

$$\begin{cases} a) F \text{ is decreasing with respect to its first variable,} \\ b) F \text{ is nondecreasing with respect to its second variable,} \\ c) \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b, B) = 0, \end{cases}$$

or satisfies the Kirchhoff condition

$$\begin{cases} a) F \text{ is nonincreasing with respect to its first variable,} \\ b) F \text{ is increasing with respect to its second variable,} \\ c) \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b, B) = 0. \end{cases}$$

We suppose moreover that there exists a parameter  $m \in \mathbb{R}$ ,  $m \geq 2$  such that we have

(ii) The (uniform) ellipticity condition on the  $(\sigma_i)_{i \in \{1 \dots I\}}$ : there exists  $\underline{\nu}, \bar{\nu}$ , strictly positive constants such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x, p) \in [0, a_i] \times \mathbb{R}, \\ \underline{\nu}(1 + |p|)^{m-2} \leq \sigma_i(x, p) \leq \bar{\nu}(1 + |p|)^{m-2}.$$

(iii) The growth of the  $(H_i)_{i \in \{1 \dots I\}}$  with respect to  $p$  exceed the growth of the  $\sigma_i$  with respect to  $p$  by no more than two, namely there exists  $\mu$  an increasing real continuous

function such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x, u, p) \in [0, a_i] \times \mathbb{R}^2, \quad |H_i(x, u, p)| \leq \mu(|u|)(1 + |p|)^m.$$

(iv) We impose the following restrictions on the growth with respect to  $p$  of the derivatives for the coefficients  $(\sigma_i, H_i)_{i \in \{1 \dots I\}}$ , which are for all  $i \in \{1 \dots I\}$ ,

$$\begin{aligned} a) \quad & |\partial_p \sigma_i|_{[0, a_i] \times \mathbb{R}^2} (1 + |p|)^2 + |\partial_p H_i|_{[0, a_i] \times \mathbb{R}^2} \leq \gamma(|u|)(1 + |p|)^{m-1}, \\ b) \quad & |\partial_x \sigma_i|_{[0, a_i] \times \mathbb{R}^2} (1 + |p|)^2 + |\partial_x H_i|_{[0, a_i] \times \mathbb{R}^2} \leq \left( \varepsilon(|u|) + P(|u|, |p|) \right) (1 + |p|)^{m+1}, \\ c) \quad & \forall (x, u, p) \in [0, a_i] \times \mathbb{R}^3, \quad -C_H \leq \partial_u H_i(x, u, p) \leq \left( \varepsilon(|u|) + P(|u|, |p|) \right) (1 + |p|)^m, \end{aligned}$$

where  $\gamma$  and  $\varepsilon$  are continuous non negative increasing functions.  $P$  is a continuous function, increasing with respect to its first variable, and tends to 0 for  $p \rightarrow +\infty$ , uniformly with respect to its first variable, from  $[0, u_1]$  with  $u_1 \in R$ , and  $C_H > 0$  is real strictly positive number. We assume that  $(\gamma, \varepsilon, P, C_H)$  are independent of  $i \in \{1 \dots I\}$ .

(v) A compatibility conditions for  $g$  and  $(\phi_i)_{\{1 \dots I\}}$

$$F(g(0), \partial_x g(0)) = 0 ; \quad \forall i \in \{1 \dots I\}, \quad g_i(a_i) = \phi_i(0).$$

**Theorem 2.2.2.** *Assume  $(\mathcal{P})$ . Then system [\(2.3\)](#) is uniquely solvable in the class  $\mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1+\frac{\alpha}{2}, 2+\alpha}(\overset{\circ}{\mathcal{J}}_T^a)$ . There exist constants  $(M_1, M_2, M_3)$ , depending only the data introduced in assumption  $(\mathcal{P})$ ,*

$$\begin{aligned} M_1 &= M_1 \left( \max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0, a_i)} | -\sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x)) | + \right. \right. \\ &\quad \left. \left. |\partial_t \phi_i|_{(0, T)} \right\}, \max_{i \in \{1 \dots I\}} |g_i|_{(0, a_i)}, C_H \right), \\ M_2 &= M_2 \left( \bar{\nu}, \underline{\nu}, \mu(M_1), \gamma(M_1), \varepsilon(M_1), \sup_{|p| \geq 0} P(M_1, |p|), |\partial_x g_i|_{(0, a_i)}, M_1 \right), \\ M_3 &= M_3 \left( M_1, \underline{\nu} (1 + |p|)^{m-2}, \mu(|u|)(1 + |p|)^m, |u| \leq M_1, |p| \leq M_2 \right), \end{aligned}$$

such that

$$\|u\|_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_1, \quad \|\partial_x u\|_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_2, \quad \|\partial_t u\|_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_1, \quad \|\partial_{x,x} u\|_{\mathcal{C}(\mathcal{J}_T^a)} \leq M_3.$$

Moreover, there exists a constant  $M(\alpha)$  depending on  $(\alpha, M_1, M_2, M_3)$  such that

$$\|u\|_{\mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}(\mathcal{J}_T^a)} \leq M(\alpha).$$

We continue this Section by giving the definitions of super and sub solution, and stating a comparison Theorem for our problem.

**Definition 2.2.3.** We say that  $u \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T^a)$ , is a super solution (resp. sub solution) of

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x} u_i(t, x) + \\ H_i(x, u_i(t, x), \partial_x u_i(t, x)) = 0, & \text{if } (t, x) \in (0, T) \times (0, a_i), \\ F(u(t, 0), \partial_x u(t, 0)) = 0, & \text{if } t \in (0, T), \end{cases} \quad (2.4)$$

if

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x} u_i(t, x) + \\ H_i(x, u_i(t, x), \partial_x u_i(t, x)) \geq 0, \text{ (resp. } \leq 0), \quad \forall (t, x) \in (0, T) \times (0, a_i), \\ F(u(t, 0), \partial_x u(t, 0)) \leq 0, \text{ (resp. } \geq 0), \quad \forall t \in (0, T) \end{cases}$$

**Theorem 2.2.4.** *Parabolic comparison.*

Assume  $(\mathcal{P})$ . Let  $u \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}}_T^a)$  (resp.  $v \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}}_T^a)$ ) a super solution (resp. a sub solution) of (2.4), satisfying for all  $i \in \{1 \dots I\}$ ,  $u_i(t, a_i) \geq v_i(t, a_i)$ , for all  $t \in [0, T]$ , and  $u_i(0, x) \geq v_i(0, x)$ , for all  $x \in [0, a_i]$ .

Then for each  $(t, (x, i)) \in \mathcal{J}_T^a$  :  $u_i(t, x) \geq v_i(t, x)$ .

*Proof.* We start by showing that for each  $0 \leq s < T$ , for all  $(t, (x, i)) \in \mathcal{J}_s^a$ ,  $u_i(t, x) \geq v_i(t, x)$ .

Let  $\lambda > 0$ . Suppose that  $\lambda > C_1 + C_2$ , where the expression of the constants  $(C_1, C_2)$  are given in the sequel (see (2.5), and (2.6)). We argue by contradiction assuming that

$$\sup_{(t, (x, i)) \in \mathcal{J}_s^a} \exp(-\lambda t + x) (v_i(t, x) - u_i(t, x)) > 0.$$

Using the boundary conditions satisfied by  $u$  and  $v$ , the supremum above is reached at a

point  $(t_0, (x_0, j_0)) \in (0, s] \times \mathcal{J}$ , with  $0 \leq x_0 < a_{j_0}$ .

Suppose first that  $x_0 > 0$ , the optimality conditions imply that

$$\begin{aligned} \exp(-\lambda t_0 + x_0) \left( -\lambda(v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0)) + \partial_t v_{j_0}(t_0, x_0) - \partial_t u_{j_0}(t_0, x_0) \right) &\geq 0, \\ \exp(-\lambda t_0 + x_0) \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) + \partial_x v_{j_0}(t_0, x_0) - \partial_x u_{j_0}(t_0, x_0) \right) &= 0, \\ \exp(-\lambda t_0 + x_0) \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) + 2 \left( \partial_x v_{j_0}(t_0, x_0) - \partial_x u_{j_0}(t_0, x_0) \right) \right. \\ &\quad \left. + \left( \partial_{x,x} v_{j_0}(t_0, x_0) - \partial_{x,x} u_{j_0}(t_0, x_0) \right) \right) = \\ \exp(-\lambda t_0 + x_0) \left( - \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) \right) + \partial_{x,x} v_{j_0}(t_0, x_0) - \partial_{x,x} u_{j_0}(t_0, x_0) \right) &\leq 0. \end{aligned}$$

Using assumptions  $(\mathcal{P})$  (iv) a), (iv) c) and the optimality conditions above we have

$$\begin{aligned} H_{j_0}(x_0, u_i(t_0, x_0), \partial_x u_{j_0}(t_0, x_0)) - H_{j_0}(x_0, v_{j_0}(t_0, x_0), \partial_x v_{j_0}(t_0, x_0)) &\leq \\ \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) \right) \left( C_H + \gamma(|\partial_x v_{j_0}(t_0, x_0)|) \right) \left( (1 + |\partial_x u_{j_0}(t_0, x_0)| \vee |\partial_x v_{j_0}(t_0, x_0)|) \right)^{m-1} & \\ \leq C_1 \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) \right), & \end{aligned}$$

where

$$C_1 := \max_{i \in \{1 \dots I\}} \left\{ \sup_{(t,x) \in [0,T] \times [0,a_i]} \left\{ \left( C_H + \gamma(|\partial_x v_i(t, x)|) \right) \left( 1 + |\partial_x u_i(t, x)| \vee |\partial_x v_i(t, x)| \right)^{m-1} \right\} \right\}. \quad (2.5)$$

On the other hand we have using assumption  $(\mathcal{P})$  (ii), (iv) a), (iv) c), and the optimality conditions

$$\begin{aligned} \sigma_{j_0}(x_0, \partial_x v_{j_0}(t_0, x_0)) \partial_{x,x} v_{j_0}(t_0, x_0) - \sigma_{j_0}(x_0, \partial_x u_{j_0}(t_0, x_0)) \partial_{x,x} u_{j_0}(t_0, x_0) &\leq \\ \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) \right) \left( \bar{\nu} (1 + |\partial_x v_{j_0}(t_0, x_0)|)^{m-2} + \left| \partial_{x,x} u_{j_0}(t_0, x_0) \right| \right. \\ &\quad \left. + \gamma(|\partial_x u_{j_0}(t_0, x_0)|) (1 + |\partial_x u_{j_0}(t_0, x_0)| \vee |\partial_x v_{j_0}(t_0, x_0)|)^{m-1} \right) \\ &\leq C_2 \left( v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0) \right), \end{aligned}$$

where

$$C_2 := \max_{i \in \{1 \dots I\}} \left\{ \sup_{(t,x) \in [0,T] \times [0,a_i]} \left\{ \bar{v}(1 + |\partial_x v_i(t,x)|)^{m-2} + \left| \partial_{x,x} u_i(t,x) \right| + \gamma(|\partial_x u_i(t,x)|)(1 + |\partial_x u_i(t,x)| + |\partial_x v_i(t,x)|)^{m-1} \right\} \right\}. \quad (2.6)$$

Using now the fact that  $v$  is a sub-solution while  $u$  is a super-solution, we get

$$\begin{aligned} 0 &\leq \\ &\partial_t u_{j_0}(t_0, x_0) - \sigma_{j_0}(x_0, \partial_x u_{j_0}(t_0, x_0)) \partial_{x,x} u_{j_0}(t_0, x_0) + H_{j_0}(x_0, u_i(t_0, x_0), \partial_x u_{j_0}(t_0, x_0)) \\ &- \partial_t v_{j_0}(t_0, x_0) + \sigma_{j_0}(x_0, \partial_x v_{j_0}(t_0, x_0)) \partial_{x,x} v_{j_0}(t_0, x_0) - H_{j_0}(x_0, v_{j_0}(t_0, x_0), \partial_x v_{j_0}(t_0, x_0)) \\ &\leq -(\lambda - (C_1 + C_2))(v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0)) < 0, \end{aligned}$$

which is a contradiction. Therefore the supremum is reached at  $(t_0, 0)$ , with  $t_0 \in (0, s]$ .

We apply a first order Taylor expansion in space, in the neighborhood of the junction point 0. Since for all  $(i, j) \in \{1 \dots I\}$ ,  $u_i(t_0, 0) = u_j(t_0, 0) = u(t_0, 0)$ , and  $v_i(t_0, 0) = v_j(t_0, 0) = v(t_0, 0)$ , we get from

$$\begin{aligned} \forall (i, j) \in \{1, \dots, I\}^2, \quad \forall h \in (0, \min_{i \in \{1 \dots I\}} a_i] \\ v_j(t_0, 0) - u_j(t_0, 0) \geq \exp(h) \left( v_i(t_0, h) - u_i(t_0, h) \right), \end{aligned}$$

that

$$\begin{aligned} \forall (i, j) \in \{1, \dots, I\}^2, \quad \forall h \in (0, \min_{i \in \{1 \dots I\}} a_i] \\ v_j(t_0, 0) - u_j(t_0, 0) \geq v_i(t_0, 0) - u_i(t_0, 0) + \\ h \left( v_i(t_0, 0) - u_i(t_0, 0) + \partial_x v_i(t_0, 0) - \partial_x u_i(t_0, 0) \right) + h \varepsilon_i(h), \end{aligned}$$

where

$$\forall i \in \{1, \dots, I\}, \quad \lim_{h \rightarrow 0} \varepsilon_i(h) = 0.$$

We get then

$$\forall i \in \{1, \dots, I\}, \quad \partial_x v_i(t_0, 0) \leq \partial_x u_i(t_0, 0) - \left( v_i(t_0, 0) - u_i(t_0, 0) \right) < \partial_x u_i(t_0, 0).$$

Using the growth assumptions on  $F$  (assumption  $(\mathcal{P})(i)$ ), and the fact that  $v$  is a sub-solution while  $u$  is a super-solution, we get

$$0 \leq F(t_0, v(t_0, 0), \partial_x v(t_0, 0)) < F(t_0, u(t_0, 0), \partial_x u(t_0, 0)) \leq 0,$$

and then a contradiction.

We deduce then for all  $0 \leq s < T$ , for all  $(t, (x, i)) \in [0, s] \times \mathcal{J}^a$ ,

$$\exp(-\lambda t + x) \left( v_i(t, x) - u_i(t, x) \right) \leq 0.$$

Using the continuity of  $u$  and  $v$ , we deduce finally that for all  $(t, (x, i)) \in [0, T] \times \mathcal{J}^a$ ,

$$v_i(t, x) \leq u_i(t, x).$$

□

## 2.3 The elliptic problem

As explained in the general Introduction [2.1](#), the construction of a solution for our parabolic problem [\(2.3\)](#) relies on a time discretization and on the solvability of the associated elliptic problem. We review in this section the well-posedness of the elliptic problem

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, \\ F(u(0), \partial_x u(0)) = 0 \end{cases}, \quad (2.7)$$

which is formulated for regular maps  $(x, i) \mapsto u_i(x)$ , continuous at the junction point, namely for each  $i \neq j \in \{1 \dots I\}$ ,  $u_i(0) = u_j(0) = u(0)$ , that follows at each edge

$$-\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0,$$

and  $u_i$  satisfy the following non linear Neumann boundary condition at the vertex

$$F(u(0), \partial_x u(0)) = 0, \quad \text{where} \quad \partial_x u(0) = (\partial_x u_1(0), \dots, \partial_x u_I(0)).$$



We introduce the following data for  $i \in \{1 \dots I\}$

$$\left\{ \begin{array}{l} F \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^I, \mathbb{R}), \\ \sigma_i \in \mathcal{C}^1([0, a_i] \times \mathbb{R}, \mathbb{R}) \\ H_i \in \mathcal{C}^1([0, a_i] \times \mathbb{R}^2, \mathbb{R}) \\ \phi_i \in \mathbb{R} \end{array} \right. ,$$

satisfying the following assumptions

**Assumption ( $\mathcal{E}$ )**

(i) Assumption on  $F$

$$\left\{ \begin{array}{l} a) \ F \text{ is decreasing with respect to its first variable,} \\ b) \ F \text{ is nondecreasing with respect to its second variable,} \\ c) \ \exists(b, B) \in \mathbb{R} \times \mathbb{R}^I, \text{ such that : } F(b, B) = 0, \end{array} \right.$$

or  $F$  satisfies the Kirchhoff condition

$$\left\{ \begin{array}{l} a) \ F \text{ is nonincreasing with respect to its first variable,} \\ b) \ F \text{ is increasing with respect to its second variable,} \\ c) \ \exists(b, B) \in \mathbb{R} \times \mathbb{R}^I, \text{ such that : } F(b, B) = 0. \end{array} \right.$$

(ii) The ellipticity condition on the  $\sigma_i$

$$\exists c > 0, \quad \forall i \in \{1 \dots I\}, \quad \forall(x, p) \in [0, a_i] \times \mathbb{R}, \quad \sigma_i(x, p) \geq c.$$

(iii) For the Hamiltonians  $H_i$ , we suppose

$$\begin{aligned} \exists C_H > 0, \quad \forall i \in \{1 \dots I\}, \quad \forall(x, u, v, p) \in (0, a_i) \times \mathbb{R}^3, \\ \text{if } u \leq v, \quad C_H(u - v) \leq H_i(x, u, p) - H_i(x, v, p). \end{aligned}$$

For each  $i \in \{1 \dots I\}$ , we define the following differential operators  $(\delta_i, \bar{\delta}_i)_{i \in \{1 \dots I\}}$  acting on  $\mathcal{C}^1([0, a_i] \times \mathbb{R}^2, \mathbb{R})$ , for  $f = f(x, u, p)$  by

$$\delta_i := \partial_u + \frac{1}{p} \partial_x; \quad \bar{\delta}_i := p \partial_p.$$

(iv) We impose the following restrictions on the growth with respect to  $p$  for the coefficients  $(\sigma_i, H_i)_{i \in \{1 \dots I\}} = (\sigma_i(x, p), H_i(x, u, p))_{i \in \{1 \dots I\}}$ , which are for all  $i \in \{1 \dots I\}$

$$\begin{aligned} \delta_i \sigma_i &= o(\sigma_i), \\ \bar{\delta}_i \sigma_i &= \mathcal{O}(\sigma_i), \\ H_i &= \mathcal{O}(\sigma_i p^2), \\ \delta_i H_i &\leq o(\sigma_i p^2), \\ \bar{\delta}_i H_i &\leq \mathcal{O}(\sigma_i p^2), \end{aligned}$$

where the limits behind are understood as  $p \rightarrow +\infty$ , uniformly in  $x$ , for bounded  $u$ .

The main result of this section is the following Theorem, for the solvability and uniqueness of the elliptic problem at the junction, with non linear Neumann condition at the junction point.

**Theorem 2.3.1.** *Assume  $(\mathcal{E})$ . The following elliptic problem at the junction, with Neumann boundary condition at the vertex*

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, & \text{if } x \in (0, a_i), \\ F(u(0), \partial_x u(0)) = 0, \\ \forall i \in \{1 \dots I\}, \quad u_i(a_i) = \phi_i, \end{cases} \quad (2.8)$$

is uniquely solvable in the class  $\mathcal{C}^{2+\alpha}(\mathcal{J}^a)$ .

Theorem [2.3.1](#) is stated without proof in [\[29\]](#). For the convenience of the reader, we sketch its proof in the Appendix [B](#).

The uniqueness of the solution of [\(2.8\)](#), is a consequence of the elliptic comparison Theorem for smooth solutions, for the Neumann problem, stated in this Section, and whose proof uses the same arguments of the proof of the parabolic comparison Theorem [2.2.4](#). We complete this section by recalling the definition of super and sub solution for the

elliptic problem (2.8), and the corresponding elliptic comparison Theorem.

**Definition 2.3.2.** Let  $u \in \mathcal{C}^2(\mathcal{J}^a)$ . We say that  $u$  is a super solution (resp. sub solution) of

$$\begin{cases} -\sigma_i(x, \partial_x f_i(x)) \partial_{x,x} f_i(x) + H_i(x, f_i(x), \partial_x f_i(x)) = 0, & \text{if } x \in (0, a_i), \\ F(f(0), \partial_x f(0)) = 0, \end{cases} \quad (2.9)$$

if

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) \geq 0, & (\text{resp. } \leq 0), \quad \text{if } x \in (0, a_i), \\ F(u(0), \partial_x u(0)) \leq 0, & (\text{resp. } \geq 0). \end{cases}$$

**Theorem 2.3.3.** Elliptic comparison Theorem, see for instance Theorem 2.1 of [29].

Assume  $(\mathcal{E})$ . Let  $u \in \mathcal{C}^2(\mathcal{J}^a)$  (resp.  $v \in \mathcal{C}^2(\mathcal{J}^a)$ ) a super solution (resp. a sub solution) of (2.9), satisfying for all  $i \in \{1 \dots I\}$ ,  $u_i(a_i) \geq v_i(a_i)$ . Then for each  $(x, i) \in \mathcal{J}^a$  :  $u_i(x) \geq v_i(x)$ .

## 2.4 The parabolic problem

In this Section, we prove Theorem 2.2.2. The construction of the solution is based on the results obtained in Section 2.3 for the elliptic problem, and is done by considering a sequence  $u^n \in \mathcal{C}^2(\mathcal{J}^a)$ , solving on a time grid an elliptic scheme defined by induction. We will prove that the solution  $u^n$  converges to the required solution.

### 2.4.1 Estimates on the discretized scheme

Let  $n \in \mathbb{N}^*$ , we consider the following time grid,  $(t_k^n = \frac{kT}{n})_{0 \leq k \leq n}$  of  $[0, T]$ , and the following sequence  $(u_k)_{0 \leq k \leq n}$  of  $\mathcal{C}^{2+\alpha}(\mathcal{J}^a)$ , defined recursively by

$$\text{for } k = 0, \quad u_0 = g,$$

and for  $1 \leq k \leq n$ ,  $u_k$  is the unique solution of the following elliptic problem

$$\begin{cases} n(u_{i,k}(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x u_{i,k}(x)) \partial_{x,x} u_{i,k}(x) + \\ H_i(x, u_{i,k}(x), \partial_x u_{i,k}(x)) = 0, & \text{if } x \in (0, a_i), \\ F(u_k(0), \partial_x u_k(0)) = 0, \\ \forall i \in \{1 \dots I\}, \quad u_{i,k}(a_i) = \phi_i(t_k^n). \end{cases} \quad (2.10)$$

The solvability of the elliptic scheme (2.10) can be proved by induction, using the same arguments as for Theorem 2.3.1. The next step consists in obtaining uniform estimates of  $(u_k)_{0 \leq k \leq n}$ . We start first by getting uniform bounds for  $n|u_k - u_{k-1}|_{(0,a_i)}$  using the comparison Theorem 2.3.3.

**Lemma 2.4.1.** *Assume (P). There exists a constant  $C > 0$ , independent of  $n$ , depending only the data  $C = C\left(\max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0, a_i)} |-\sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))| + |\partial_t \phi_i|_{(0,T)} \right\}, C_H\right)$ , such that*

$$\sup_{n \geq 0} \max_{k \in \{1 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ n|u_{i,k} - u_{i,k-1}|_{(0,a_i)} \right\} \leq C,$$

and then

$$\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0,a_i)} \right\} \leq C + \max_{i \in \{1 \dots I\}} \left\{ |g_i|_{(0,a_i)} \right\}.$$

*Proof.* Let  $n > \lfloor C_H \rfloor$ , where  $C_H$  is defined in assumption (P) (iv) c). Let  $k \in \{1 \dots n\}$ , we define the following sequence

$$\begin{cases} M_0 = \max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0, a_i)} |-\sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x))| + |\partial_t \phi_i|_{(0,T)} \right\}, \\ M_{k,n} = \frac{n}{n - C_H} M_{k-1,n}, \quad k \in \{1 \dots n\}. \end{cases}$$

We claim that for each  $k \in \{1 \dots n\}$

$$\max_{i \in \{1 \dots I\}} \left\{ n|u_{i,k} - u_{i,k-1}|_{(0,a_i)} \right\} \leq M_{k,n}.$$

We give a proof by induction. For this, if  $k = 1$ , let us show that the map  $h$  defined on

the junction by

$$h := \begin{cases} \mathcal{J}^a \rightarrow \mathbb{R} \\ (x, i) \mapsto \frac{M_{1,n}}{n} + g_i(x), \end{cases}$$

is a super solution of (2.10), for  $k = 1$ . For this we will use the Elliptic Comparison Theorem 2.3.3.

Using the compatibility conditions satisfied by  $g$ , namely assumption (P) (v), and the assumptions of growth on  $F$ , assumption (P) (i), we get for the boundary conditions

$$\begin{aligned} F(h(0), \partial_x h(0)) &= F\left(\frac{M_{1,n}}{n} + g(0), \partial_x g(0)\right) \leq F(g(0), \partial_x g(0)) = 0, \\ h(a_i) &= \frac{M_{1,n}}{n} + g_i(a_i) \geq \frac{M_{0,n}}{n} + g_i(a_i) \geq \phi_i(t_1^n). \end{aligned}$$

For all  $i \in \{1 \dots I\}$ , and  $x \in (0, a_i)$ , we get using assumption (P) (iii)

$$\begin{aligned} n(h_i(x) - g_i(x)) - \sigma_i(x, \partial_x h_i(x)) \partial_x^2 h_i(x) + H_i(x, h_i(x), \partial_x h_i(x)) &= \\ M_{1,n} - \sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, \frac{M_{1,n}}{n} + g_i(x), \partial_x g_i(x)) &\geq \\ M_{1,n} - \sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x)) - \frac{M_{1,n} C_H}{n} &\geq 0. \end{aligned}$$

It follows from the comparison Theorem 2.3.3, that for all  $i \in \{1 \dots I\}$ , and  $x \in [0, a_i]$

$$u_{1,i}(x) \leq \frac{M_{1,n}}{n} + g_i(x).$$

Using the same arguments, we show that

$$h := \begin{cases} \mathcal{J}^a \rightarrow \mathbb{R} \\ (x, i) \mapsto -\frac{M_{1,n}}{n} + g_i(x), \end{cases}$$

is a sub solution of (2.10) for  $k = 1$ , and we then get

$$\max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0, a_i)} n |u_{1,i}(x) - g_i(x)| \right\} \leq M_{1,n}.$$

Let  $2 \leq k \leq n$ , suppose that the assumption of induction holds true. Let us show that

the following map

$$h := \begin{cases} \mathcal{J}^a \rightarrow \mathbb{R} \\ (x, i) \mapsto \frac{M_{k,n}}{n} + u_{i,k-1}(x), \end{cases}$$

is a super solution of (2.10). For the boundary conditions, using assumption (P) (i), we get

$$\begin{aligned} F(h(0), \partial_x h(0)) &= F\left(\frac{M_{k,n}}{n} + u_{k-1}(0), \partial_x u_{k-1}(0)\right) \leq F(u_{k-1}(0), \partial_x u_{k-1}(0)) \leq 0, \\ h(a_i) &= \frac{M_{k,n}}{n} + u_{i,k-1}(a_i) \geq \frac{M_{0,n}}{n} + u_{i,k-1}(a_i) \geq \phi_i(t_k^n). \end{aligned}$$

For all  $i \in \{1 \dots I\}$ , and  $x \in (0, a_i)$

$$\begin{aligned} n(h_i(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x h(x)) \partial_x^2 h(x) + H_i(x, h(x), \partial_x h(x)) &= \\ M_{k,n} - \sigma_i(x, \partial_x u_{i,k-1}(x)) \partial_x^2 u_{i,k-1}(x) + H_i\left(x, \frac{M_{k,n}}{n} + u_{i,k-1}(x), \partial_x u_{k-1}(x)\right) &\geq \\ M_{k,n} - \sigma_i(x, \partial_x u_{i,k-1}(x)) \partial_x^2 u_{i,k-1}(x) + H_i(x, u_{i,k-1}(x), \partial_x u_{k-1}(x)) - \frac{C_H M_{k,n}}{n}. \end{aligned}$$

Since we have for all  $x \in (0, a_i)$

$$-\sigma_i(x, \partial_x u_{i,k-1}(x)) \partial_x^2 u_{i,k-1}(x) + H_i(x, u_{i,k-1}(x), \partial_x u_{i,k-1}(x)) = -n(u_{i,k-1}(x) - u_{i,k-2}(x)),$$

using the induction assumption we get

$$\begin{aligned} n(h_i(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x h(x)) \partial_x^2 h(x) + H_i(x, \partial_x h(x), \partial_x h(x)) &\geq \\ M_{k,n} - n(u_{i,k-1}(x) - u_{i,k-2}(x)) - \frac{C_H M_{k,n}}{n} &\geq M_{k,n} \frac{n - C_H}{n} - M_{k-1,n} \geq 0. \end{aligned}$$

It follows from the comparison Theorem 2.3.3, that for all  $(x, i) \in \mathcal{J}^a$

$$u_{i,k}(x) \leq \frac{M_{k,n}}{n} + u_{i,k-1}(x).$$

Using the same arguments, we show that

$$h := \begin{cases} \mathcal{J}^a \rightarrow \mathbb{R} \\ (x, i) \mapsto -\frac{M_{k,n}}{n} + u_{i,k-1}(x), \end{cases}$$

is a sub solution of (2.10), and we get

$$\max_{i \in \{1 \dots I\}} \left\{ n |u_{i,k}(x) - u_{i,k-1}(x)|_{(0,a_i)} \right\} \leq M_{k,n}.$$

We obtain finally using that for all  $k \in \{1 \dots n\}$

$$\begin{cases} M_{k,n} & \leq M_{n,n}, \\ M_{k,n} & = \left( \frac{n}{n - C_H} \right)^k M_0, \end{cases}$$

and

$$M_{n,n} \xrightarrow{n \rightarrow +\infty} M := \exp(C_H) \max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0,a_i)} | -\sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x)) | + |\partial_t \phi_i|_{(0,T)} \right\},$$

that

$$\begin{aligned} \sup_{n \geq 0} \max_{k \in \{1 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ n |u_{i,k} - u_{i,k-1}|_{(0,a_i)} \right\} & \leq C, \\ \sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0,a_i)} \right\} & \leq C + \max_{i \in \{1 \dots I\}} \left\{ |g_i|_{(0,a_i)} \right\}. \end{aligned}$$

That completes the proof. □

The next step consists in obtaining uniform estimates for  $|\partial_x u_k|_{(0,a_i)}$ , in terms of  $n|u_k - u_{k-1}|_{(0,a_i)}$  and the quantities  $(\underline{\nu}, \bar{\nu}, \mu, \gamma, \varepsilon, P)$  introduced in assumption  $(\mathcal{P})$  (ii), (iii) and (iv). More precisely, we use similar arguments as for the proof of Theorem 14.1 of [19], using a classical argument of upper and lower barrier functions at the boundary. The assumption of growth  $(\mathcal{P})$  (ii) and (iii) are used in a key way to get an uniform bound on the gradient at the boundary. Finally to conclude, we appeal to a gradient maximum principle, using the growth assumption  $(\mathcal{P})$  (iv), adapting Theorem 15.2 of [19] to our elliptic scheme. The basic idea, which goes back as far as Bernstein's work, involves differentiation of each quasi linear equation in each edge of (2.10), with respect to  $x$ . Thereafter, the maximum principle is applied to the resulting equation in the function  $|\partial_x u_{i,k}|^2$ .

**Lemma 2.4.2.** *Assume  $(\mathcal{P})$ . There exists a constant  $C > 0$ , independent of  $n$ , depending*

only the data

$$\begin{aligned} & \left( \bar{\nu}, \underline{\nu}, \mu(|u|), \gamma(|u|), \varepsilon(|u|), \sup_{|p| \geq 0} P(|u|, |p|), |\partial_x g_i|_{(0, a_i)}, \right. \\ & |u| \leq \sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0, a_i)} \right\}, \\ & \left. \sup_{n \geq 0} \max_{k \in \{1 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ n |u_{i,k} - u_{i,k-1}|_{(0, a_i)} \right\} \right), \end{aligned}$$

such that

$$\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |\partial_x u_{i,k}|_{(0, a_i)} \right\} \leq C.$$

*Proof. Step 1 :* We claim that, for each  $k \in \{1 \dots n\}$ ,  $\max_{i \in \{1 \dots I\}} \left\{ |\partial_x u_{i,k}|_{\partial(0, a_i)} \right\}$  is bounded by the data, uniformly in  $n$ .

It follows from Lemma [2.4.1](#), that there exists  $M > 0$  such that

$$\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0, a_i)} + n |u_{i,k} - u_{i,k-1}|_{(0, a_i)} \right\} \leq M.$$

We fix  $i \in \{1 \dots I\}$ . We apply a barrier method consisting in building two functions  $w_{i,k}^+, w_{i,k}^-$  satisfying in a neighborhood of 0, for example  $[0, \kappa]$ , with  $\kappa \leq a_i$

$$\begin{aligned} Q_i(x, w_{i,k}^+(x), \partial_x w_{i,k}^+(x), \partial_x^2 w_{i,k}^+(x)) &\geq 0, \quad \forall x \in [0, \kappa], \quad w_{i,k}^+(0) = u_{i,k}(0), \quad w_{i,k}^+(\kappa) \geq M, \\ Q_i(x, w_{i,k}^-(x), \partial_x w_{i,k}^-(x), \partial_x^2 w_{i,k}^-(x)) &\leq 0, \quad \forall x \in [0, \kappa], \quad w_{i,k}^-(0) = u_{i,k}(0), \quad w_{i,k}^-(\kappa) \leq -M, \end{aligned}$$

where we recall that for each  $(x, u, p, S) \in [0, a_i] \times R^3$

$$Q_i(x, u, p, S) = n(u - u_{i,k-1}(x)) - \sigma_i(x, p)S + H_i(x, u, p).$$

For  $n > \lfloor C_H \rfloor$ , where  $C_H$  is defined in assumption  $\mathcal{P}$  (iv) c), it follows then from the comparison principle that

$$w_{i,k}^-(x) \leq u_{i,k}(x) \leq w_{i,k}^+(x), \quad \forall x \in [0, \kappa],$$

and then

$$\partial_x w_{i,k}^-(0) \leq \partial_x u_{i,k}(0) \leq \partial_x w_{i,k}^+(0).$$



We look for  $w_{i,k}^+$  defined on  $[0, \kappa]$  of the form

$$\begin{aligned} w_{i,0}^+ &= g_i(x) \\ w_{i,k}^+ : x &\mapsto u_{i,k}(0) + \frac{1}{\beta} \ln(1 + \theta x), \end{aligned}$$

where the constants  $(\beta, \theta, \kappa)$  will be chosen in the sequel independent of  $k$ .

Remark first that for all  $x \in [0, \kappa]$ ,  $\partial_x^2 w_{i,k}^+(x) = -\beta \partial_x w_{i,k}^+(x)^2$ , and  $w_{i,k}^+(0) = u_{i,k}(0)$ . Let us choose  $(\theta, \kappa)$ , such that

$$\forall k \in \{1 \dots n\}, \quad 0 < \kappa \leq \min_{i \in \{1 \dots I\}} a_i, \quad w_{i,k}^+(\kappa) \geq M, \quad \partial_x w_{i,k}^+(\kappa) \geq \beta. \quad (2.11)$$

We choose for instance

$$\begin{aligned} \theta &= \beta^2 \exp(2\beta M) + \frac{1}{\min_{i \in \{1 \dots I\}} a_i} \exp(2\beta M) \\ \kappa &= \frac{1}{\theta} \left( \exp(2\beta M) - 1 \right). \end{aligned} \quad (2.12)$$

The constant  $\beta$  will be chosen in order to get

$$\beta \geq \sup_{k \in \{1 \dots n\}} \sup_{x \in [0, \kappa]} \frac{\mu(w_{i,k}^+(x))(1 + \partial_x w_{i,k}^+(x))^m + M}{\underline{\nu}(1 + \partial_x w_{i,k}^+(x))^{m-2} \partial_x w_{i,k}^+(x)^2}, \quad (2.13)$$

where  $(\mu(\cdot), \underline{\nu}, m)$  are defined in assumption  $(\mathcal{P})$  (ii) and (iii). Since we have

$$\begin{aligned} \forall x \in [0, \kappa], \quad w_{i,k}^+(x) &\leq w_{i,k}^+(\kappa) = 2M, \\ \beta \leq \partial_x w_{i,k}^+(\kappa) &\leq \partial_x w_{i,k}^+(x) \leq \partial_x w_{i,k}^+(0). \end{aligned}$$

We can then choose  $\beta$  large enough to get (2.13), for instance

$$\beta \geq \frac{\mu(2M)}{\underline{\nu}} \left( 1 + \frac{1}{\beta^2} \right) + \frac{M}{\underline{\nu} \beta^2}.$$

It is easy to show by induction that  $w_{i,k}^+$  is lower barrier of  $u_{i,k}$  in the neighborhood  $[0, \kappa]$ .

More precisely, since  $w_{i,0}^+ = u_{i,0}$ , and for all  $k \in \{1 \dots n\}$

$$\begin{aligned} w_{i,k}^+(0) &= u_{i,k}(0), \quad w_{i,k}^+(\kappa) \geq u_{i,k}(\kappa), \\ w_{i,k}^+(x) &= w_{i,k-1}^+(x) + u_{i,k}(0) - u_{i,k-1}(0) \geq w_{i,k-1}^+(x) - \frac{M}{n}, \end{aligned}$$

we get using the assumption of induction, assumption  $(\mathcal{P})$  (ii) and (iii), and (2.13) that for all  $x \in (0, \kappa)$

$$\begin{aligned} n(w_{i,k}^+(x) - u_{i,k-1}(x)) - \sigma_i(x, \partial_x w_{i,k}^+(x)) \partial_{x,x} w_{i,k}^+(x) + H_i(x, w_{i,k}^+(x), \partial_x w_{i,k}^+(x)) &\geq \\ -M + \beta \sigma_i(x, \partial_x w_{i,k}^+(x)) \partial_x w_{i,k}^+(x)^2 + H_i(x, w_{i,k}^+(x), \partial_x w_{i,k}^+(x)) &\geq \\ -M + \beta \underline{\nu} (1 + \partial_x w_{i,k}^+(x))^{m-2} \partial_x w_{i,k}^+(x)^2 + \mu(w_{i,k}^+(x)) (1 + \partial_x w_{i,k}^+(x))^m &\geq 0. \end{aligned}$$

We obtain therefore

$$\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \partial_x u_{i,k}(0) \leq \frac{\theta}{\beta} \vee \partial_x g_i(0).$$

With the same arguments we can show that

$$\begin{aligned} w_{i,0}^- &= g_i(x) \\ w_{i,k}^- : x &\mapsto u_{i,k}(0) - \frac{1}{\beta} \ln(1 + \theta x), \end{aligned}$$

is a lower barrier in the neighborhood of 0. Using the same method, we can show that  $\partial_x u_{i,k}(a_i)$  is uniformly bounded by the same upper bounds, that completes the proof of

**Step 1.**

**Step 2 :** For the convenience of the reader, we do not detail all the computations of this Step, since they can be found in the proof of Theorem 15.2 of [19]. It follows from Lemma 2.4.1 that there exists  $M > 0$  such that

$$\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0, a_i)} \right\} \leq M.$$

We set furthermore

$$\forall (x, u, p) \in [0, a_i] \times \mathbb{R}^2, \quad H_{i,k}^n(x, u, p) = n(u - u_{i,k-1}(x)) + H_i(x, u, p).$$

Let  $u$  be a solution of the elliptic equation, for  $x \in (0, a_i)$

$$\sigma_i(x, \partial_x u(x)) \partial_{x,x} u(x) - H_{i,k}^n(x, u(x), \partial_x u(x)) = 0,$$

and assume that  $|u|_{(0, a_i)} \leq M$ . The main key of the proof will be in the use of the following

equalities

$$\delta_i H_{i,k}^n(x, u, p) = \delta_i H_i(x, u, p) + \frac{n(p - \partial_x u_{i,k-1}(x))}{p}, \quad \bar{\delta}_i H_{i,k}^n(x, u, p) = \bar{\delta}_i H_i(x, u, p), \quad (2.14)$$

where we recall that the operators  $\delta_i$  and  $\bar{\delta}_i$  are defined in assumption  $(\mathcal{E})$  (iii). We follow the proof of Theorem 15.2 in [19]. Let  $\bar{u}$  such that  $u = \psi(\bar{u})$ , where  $\psi \in \mathcal{C}^3[\bar{m}, \bar{M}]$ , is strictly increasing and  $\bar{m} = \psi(-M)$ ,  $\bar{M} = \psi(M)$ . In the sequel, we will set  $v = \partial_x u^2$  and  $\bar{v} = \partial_x \bar{u}^2$ . To simplify the notations, we will omit the variables  $(x, u(x), \partial_x u(x))$  in the functions  $\sigma_i$  and  $H_{i,k}^n$ , and the variable  $\bar{u}$  for  $\psi$ . We assume first that the solution  $u \in \mathcal{C}^3([-M, M])$ , and we follow exactly all the computations that lead to equation of (15.25) of [19] to get the following inequality

$$\sigma_i \partial_{x,x} \bar{v} + B_i \partial_x \bar{v} + G_{i,k}^n \geq 0, \quad (2.15)$$

where  $B_i$  and  $G_{i,k}^n$  have the same expression in (15.26) of [19] with  $(\sigma_i = \sigma_i^*, c_i = 0)$ . We choose  $(r = 0, s = 0)$ , since we will see in the sequel (2.17), that condition (15.32) of [19] holds under assumption assumption  $(\mathcal{P})$ . We have more precisely

$$\begin{aligned} B_i &= \psi' \partial_p \sigma_i \partial_{x,x} \bar{u} - \partial_p H_i + \omega \partial_p (\sigma_i p^2), \\ G_{i,k}^n &= \frac{\omega'}{\psi'} + \kappa_i \omega^2 + \beta_i \omega + \theta_{i,k}^n, \\ \omega &= \frac{\psi''}{\psi'^2} \in \mathcal{C}^1([\bar{m}, \bar{M}]), \\ \kappa_i &= \frac{1}{\sigma_i p^2} \left( \bar{\delta}_i (\sigma_i p^2) + \frac{p^2}{4\sigma_i} |(\bar{\delta}_i + 1)\sigma_i|^2 \right), \\ \beta_i &= \frac{1}{\sigma_i p^2} \left( \delta_i (\sigma_i p^2) - \bar{\delta}_i H_i + \frac{p^2}{2\sigma_i} ((\bar{\delta}_i + 1)\sigma_i)(\delta_i \sigma_i) \right), \\ \theta_{i,k}^n &= \frac{1}{\sigma_i p^2} \left( \frac{p^2}{4\sigma_i} |\delta_i \sigma_i|^2 - \delta_i H_{i,k}^n \right) = \theta_i - \frac{1}{\sigma_i p^2} \left( \frac{n(p - \partial_x u_{i,k-1})}{p} \right), \\ \theta_i &= \frac{1}{\sigma_i p^2} \left( \frac{p^2}{4\sigma_i} |\delta_i \sigma_i|^2 - \delta_i H_i \right). \end{aligned}$$

We set in the sequel

$$G_i = \frac{\partial_x \omega}{\partial_x \psi} + \kappa_i \omega^2 + \beta_i \omega + \theta_i, \quad \text{in order to get } G_{i,k}^n = G_i - \frac{1}{\sigma_i p^2} \left( \frac{n(p - \partial_x u_{i,k-1})}{p} \right). \quad (2.16)$$

More precisely, we see from (2.14) that all the coefficients  $(B_i, \kappa_i, \beta_i, \theta_i)$  can be chosen

independent of  $n$  and  $u_{i,k-1}$ . The main argument then to get a bound of  $\partial_x u$  is to apply a maximum principle for  $\bar{v}$  in (2.15), and this will be done as soon as we ensure

$$G_{i,k}^n \leq 0, \text{ for } |\partial_x u| \geq L_k^n.$$

Hence, we have

$$\sup_{x \in \Omega_L} \bar{v} = \sup_{x \in \partial \Omega_L} \bar{v}, \quad \Omega_L = \{x \in [0, a_i], |\partial_x u(x)| \geq L_k^n\}$$

On the other hand, using assumption (P) (ii) (iii) and (iv), it is easy to check that there exists a constants  $(a, b, c)$ , depending only on the data

$$\left( \bar{v}, \underline{\nu}, \mu(M), \gamma(M), \varepsilon(M), \sup_{|p| \geq 0} P(M, |p|) \right),$$

such that

$$\begin{aligned} \sup_{x \in [0, a_i], |u| \leq M} \limsup_{|p| \rightarrow +\infty} \kappa_i(x, u, p) &\leq a, \\ \sup_{x \in [0, a_i], |u| \leq M} \limsup_{|p| \rightarrow +\infty} \beta_i(x, u, p) &\leq b, \\ \sup_{x \in [0, a_i], |u| \leq M} \limsup_{|p| \rightarrow +\infty} \theta_i(x, u, p) &\leq c, \end{aligned}$$

where

$$\begin{aligned} a &= \frac{1}{\underline{\nu}}(\gamma(M) + \bar{v}) + \frac{1}{2} + \frac{\gamma(M)^2}{\underline{\nu}^2}, \\ b &= \frac{\varepsilon(M) + \sup_{|p| \geq 0} P(M, |p|) + \gamma(M)}{\underline{\nu}} + \frac{(\varepsilon(M) + \sup_{|p| \geq 0} P(M, |p|))(\bar{v} + \gamma(M))}{\underline{\nu}^2}, \\ c &= \frac{(\varepsilon(M) + \sup_{|p| \geq 0} P(M, |p|))^2}{4\underline{\nu}^2} + \frac{2(\varepsilon(M) + \sup_{|p| \geq 0} P(M, |p|))}{\underline{\nu}}. \end{aligned}$$

Therefore, fixing  $\varepsilon > 0$ , using (2.16), we can find  $L = L(a, b, c)$ , such that for all  $p \geq L(a, b, c)$

$$G_i \leq \frac{\partial_x \omega}{\partial_x \psi} + a\omega^2 + b\omega + c + \varepsilon.$$

As it has been done in the proof of Theorem 15.2 of [19], using Riccati's equation, we

choose  $\psi(\cdot) = \psi(a, b, c)(\cdot)$  such that we ensure

$$G_i \leq 0, \quad \text{if } |\partial_x u(x)| \geq L(a, b, c).$$

We see then from the expression of  $\theta_{i,k}^n$  that we get

$$G_{i,k}^n \leq 0, \quad \text{if } |\partial_x u(x)| \geq L(a, b, c) \vee |\partial_x u_{i,k-1}(x)|.$$

Therefore applying the maximum principle to  $\bar{v}$  in [\(2.15\)](#), and from the relation  $u = \psi(\bar{u})$ ,  $\bar{v} = \partial_x \bar{u}^2$  we get finally

$$|\partial_x u|_{(0,a_i)} \leq \max \left( \frac{\max \psi'(a, b, c)(\cdot)}{\min \psi'(a, b, c)(\cdot)} |\partial_x u|_{\partial(0,a_i)}, L(a, b, c), |\partial_x u_{i,k-1}|_{(0,a_i)} \right).$$

This upper bound still holds if  $u \in \mathcal{C}^2([0, a_i])$ , (cf. (15.30) and (15.31) of the proof of Theorem 15.2 in [\[19\]](#)). Finally applying the upper bound above to the solution  $u_k$ , we get by induction that

$$\begin{aligned} & \sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |\partial_x u_{i,k}|_{(0,a_i)} \right\} \\ & \leq \max \left( \frac{\max \psi'(a, b, c)(\cdot)}{\min \psi'(a, b, c)(\cdot)} |\partial_x u_{i,k}|_{\partial(0,a_i)}, L(a, b, c), |\partial_x g_i|_{(0,a_i)} \right). \end{aligned}$$

This completes the proof. □

The following Proposition follows from Lemmas [2.4.1](#) and [2.4.2](#), assumption  $(\mathcal{P})$  (ii) (iii), and from the relation

$$\begin{aligned} \forall x \in [0, a_i], \quad |\partial_{x,x} u_{i,k}(x)| & \leq \frac{|n(u_{i,k}(x) - u_{i,k-1}(x))| + |H_i(x, u_{i,k}(x), \partial_x u_{i,k}(x))|}{\sigma_i(x, \partial_x u_{i,k}(x))} \\ & \leq \frac{|n(u_{i,k}(x) - u_{i,k-1}(x))| + \mu(|u_{i,k}(x)|)(1 + |\partial_x u_{i,k}(x)|^m)}{\underline{\nu}(1 + |\partial_x u_{i,k}(x)|^{m-2})}. \end{aligned}$$

**Proposition 2.4.3.** *Assume  $(\mathcal{P})$ . There exist constants  $(M_1, M_2, M_3)$ , depending only*

the data introduced in assumption  $(\mathcal{P})$

$$\begin{aligned}
M_1 &= M_1 \left( \max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0, a_i)} | -\sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x)) | + \right. \right. \\
&\quad \left. \left. |\partial_t \phi_i|_{(0, T)} \right\}, \max_{i \in \{1 \dots I\}} |g_i|_{(0, a_i)}, C_H \right), \\
M_2 &= M_2 \left( \bar{\nu}, \underline{\nu}, \mu(M_1), \gamma(M_1), \varepsilon(M_1), \sup_{|p| \geq 0} P(M_1, |p|), |\partial_x g_i|_{(0, a_i)}, M_1 \right), \\
M_3 &= M_3 \left( M_1, \underline{\nu} (1 + |p|)^{m-2}, \mu(|u|) (1 + |p|)^m, |u| \leq M_1, |p| \leq M_2 \right),
\end{aligned}$$

such that

$$\begin{aligned}
\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0, a_i)} \right\} &\leq M_1, \\
\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |\partial_x u_{i,k}|_{(0, a_i)} \right\} &\leq M_2, \\
\sup_{n \geq 0} \max_{k \in \{1 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |n(u_{i,k} - u_{i,k-1})|_{(0, a_i)} \right\} &\leq M_1, \\
\sup_{n \geq 0} \max_{k \in \{0 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |\partial_{x,x} u_{i,k}|_{(0, a_i)} \right\} &\leq M_3.
\end{aligned}$$

Unfortunately, we are unable to give an upper bound of the modulus of continuity of  $\partial_{x,x} u_{i,k}$  in  $\mathcal{C}^\alpha([0, a])$  independent of  $n$ . However, we are able to formulate in the weak sense a limit solution. From the regularity of the coefficients, using some tools introduced in Section [2.2.1](#), Lemma [2.2.1](#), we get interior regularity, and a smooth limit solution.

#### 2.4.2 Proof of Theorem [2.2.2](#).

*Proof.* The uniqueness is a result of the comparison Theorem [2.2.4](#). To simplify the notations, we set for each  $i \in \{1 \dots I\}$ , and for each  $(x, q, u, p, S) \in [0, a_i] \times \mathbb{R}^4$

$$Q_i(x, u, q, p, S) = q - \sigma_i(x, p)S + H_i(x, u, p).$$

Let  $n \geq 0$ . Consider the subdivision  $(t_k^n = \frac{kT}{n})_{0 \leq k \leq n}$  of  $[0, T]$ , and  $(u_k)_{0 \leq k \leq n}$  the solution of [\(2.10\)](#).

From estimates of Proposition [2.4.3](#), there exists a constant  $M > 0$  independent of  $n$ ,

such that

$$\sup_{n \geq 0} \max_{k \in \{1 \dots n\}} \max_{i \in \{1 \dots I\}} \left\{ |u_{i,k}|_{(0,a_i)} + |n(u_{i,k} - u_{i,k-1})|_{(0,a_i)} + |\partial_x u_{i,k}|_{(0,a_i)} + |\partial_{x,x} u_{i,k}|_{(0,a_i)} \right\} \leq M. \quad (2.17)$$

We define the following sequence  $(v_n)_{n \geq 0}$  in  $\mathcal{C}^{0,2}(\mathcal{J}_T^a)$ , piecewise differentiable with respect to its first variable by

$$\begin{aligned} \forall i \in \{1 \dots I\}, \quad v_{i,0}(0, x) &= g_i(x) \quad \text{if } x \in [0, a_i], \\ v_{i,n}(t, x) &= u_{i,k}(x) + n(t - t_k^n)(u_{i,k+1}(x) - u_{i,k}(x)) \quad \text{if } (t, x) \in [t_k^n, t_{k+1}^n] \times [0, a_i]. \end{aligned}$$

We deduce then from (2.17), that there exists a constant  $M_1$  independent of  $n$ , depending only on the data of the system, such that for all  $i \in \{1 \dots I\}$

$$|v_{i,n}|_{[0,T] \times [0,a_i]}^\alpha + |\partial_x v_{i,n}|_{x,[0,T] \times [0,a_i]}^\alpha \leq M_1.$$

Using Lemma 2.2.1, we deduce that there exists a constant  $M_2(\alpha) > 0$ , independent of  $n$ , such that for all  $i \in \{1 \dots I\}$ , we have the following global Hölder condition

$$|\partial_x v_{i,n}|_{t,[0,T] \times [0,a_i]}^{\frac{\alpha}{2}} + |\partial_x v_{i,n}|_{x,[0,T] \times [0,a_i]}^\alpha \leq M_2(\alpha).$$

We deduce then from Ascoli's Theorem, that up to a sub sequence  $n$ ,  $(v_{i,n})_{n \geq 0}$  converge in  $\mathcal{C}^{0,1}([0, T] \times [0, a_i])$  to  $v_i$ , and then  $v_i \in \mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}([0, T] \times [0, a_i])$ .

Since  $v_n$  satisfies the following continuity condition at the junction point

$$\forall (i, j) \in \{1 \dots I\}^2, \quad \forall n \geq 0, \quad \forall t \in [0, T], \quad v_{i,n}(t, 0) = v_{j,n}(t, 0) = v_n(t, 0),$$

we deduce then  $v \in \mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}(\mathcal{J}_T^a)$ .

We now focus on the regularity of  $v$  in  $\mathring{\mathcal{J}}_T^a$ , and we will prove that  $v \in \mathcal{C}^{1+\frac{\alpha}{2}, 2+\alpha}(\mathring{\mathcal{J}}_T^a)$ , and satisfies on each edge

$$Q_i(x, v_i(t, x), \partial_t v_i(t, x), \partial_x v_i(t, x), \partial_{x,x} v_i(t, x)) = 0, \quad \text{if } (t, x) \in (0, T) \times (0, a_i).$$

Using once again (2.17), there exists a constant  $M_3$  independent of  $n$ , such that for each  $i \in \{1 \dots I\}$

$$\|\partial_t v_{i,n}\|_{L_2((0,T) \times (0,a_i))} \leq M_3, \quad \|\partial_{x,x} v_{i,n}\|_{L_2((0,T) \times (0,a_i))} \leq M_3.$$

Hence we get up to a sub sequence, that

$$\partial_t v_{i,n} \rightharpoonup \partial_t v_i, \quad \partial_{x,x} v_{i,n} \rightharpoonup \partial_{x,x} v_i,$$

weakly in  $L_2((0, T) \times (0, a_i))$ .

The continuity of the coefficients  $(\sigma_i, H_i)_{i \in \{1 \dots I\}}$ , Lebesgue's Theorem, the linearity of  $Q_i$  in the variable  $\partial_t$  and  $\partial_{x,x}$ , allows us to get for each  $i \in \{1 \dots I\}$ , up to a subsequence  $n_p$

$$\begin{aligned} & \int_0^T \int_0^{a_i} \left( Q_i(x, v_{i,n_p}(t, x), \partial_t v_{i,n_p}(t, x), \partial_x v_{i,n_p}(t, x), \partial_{x,x} v_{i,n_p}(t, x)) \right) \psi(t, x) dx dt \\ & \xrightarrow{p \rightarrow +\infty} \int_0^T \int_0^{a_i} \left( Q_i(x, v_i(t, x), \partial_t v_i(t, x), \partial_x v_i(t, x), \partial_{x,x} v_i(t, x)) \right) \psi(t, x) dx dt, \\ & \quad \forall \psi \in \mathcal{C}_c^\infty((0, T) \times (0, a_i)). \end{aligned}$$

We now prove that for any  $\psi \in \mathcal{C}_c^\infty((0, T) \times (0, a_i))$

$$\int_0^T \int_0^{a_i} \left( Q_i(x, v_{i,n_p}(t, x), \partial_t v_{i,n_p}(t, x), \partial_x v_{i,n_p}(t, x), \partial_{x,x} v_{i,n_p}(t, x)) \right) \psi(t, x) dx dt \xrightarrow{p \rightarrow +\infty} 0.$$

Using that  $(u_k)_{0 \leq k \leq n}$  is the solution of (2.10), we get for any  $\psi \in \mathcal{C}_c^\infty((0, T) \times (0, a_i))$

$$\begin{aligned} & \int_0^T \int_0^{a_i} \left( Q_i(x, v_{i,n}(t, x), \partial_t v_{i,n}(t, x), \partial_x v_{i,n}(t, x), \partial_{x,x} v_{i,n}(t, x)) \right) \psi(t, x) dx dt = \\ & \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \int_0^{a_i} \left( \sigma_i(x, \partial_x u_{i,k+1}(x)) \partial_{x,x} u_{i,k+1}(x) - \sigma_i(x, \partial_x v_{i,n}(t, x)) \partial_{x,x} v_{i,n}(t, x) \right. \\ & \quad \left. + H_i(x, v_{i,n}(t, x), \partial_x v_{i,n}(t, x)) - H_i(x, u_{i,k+1}(x), \partial_x u_{i,k+1}(x)) \right) \psi(t, x) dx dt. \quad (2.18) \end{aligned}$$

Using assumption (P) more precisely the Lipschitz continuity of the Hamiltonians  $H_i$ , the Hölder equicontinuity in time of  $(v_{i,n}, \partial_x v_{i,n})$ , there exists a constant  $M_4(\alpha)$  independent of  $n$ , such that for each  $i \in \{1 \dots I\}$ , for each  $(t, x) \in [t_k^n, t_{k+1}^n] \times [0, a_i]$

$$|H_i(x, u_{i,k+1}(x), \partial_x u_{i,k+1}(x)) - H_i(x, v_{i,n}(t, x), \partial_x v_{i,n}(t, x))| \leq M_4(\alpha)(t - t_k^n)^{\frac{\alpha}{2}},$$



and therefore for any  $\psi \in \mathcal{C}_c^\infty((0, T) \times (0, a_i))$

$$\left| \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \int_0^{a_i} \left( H_i(x, u_{i,k+1}(x), \partial_x u_{i,k+1}(x)) - H_i(x, v_{i,n}(t, x), \partial_x v_{i,n}(t, x)) \right) \psi(t, x) dx dt \right| \leq a_i M_4(\alpha) |\psi|_{(0,T) \times (0,a_i)} n^{-\frac{\alpha}{2}} \xrightarrow{n \rightarrow +\infty} 0.$$

For the last term in (2.18), we write for each  $i \in \{1 \dots I\}$ , for each  $(t, x) \in (t_k^n, t_{k+1}^n) \times (0, a_i)$

$$\begin{aligned} \sigma_i(x, \partial_x u_{i,k+1}(x)) \partial_{x,x} u_{i,k+1}(x) - \sigma_i(x, \partial_x v_{i,n}(t, x)) \partial_{x,x} v_{i,n}(t, x) &= \\ \left( \sigma_i(x, \partial_x u_{i,k+1}(x)) - \sigma_i(x, \partial_x v_{i,n}(t, x)) \right) \partial_{x,x} u_{i,k}(x) + & \quad (2.19) \end{aligned}$$

$$\left( \sigma_i(x, \partial_x u_{i,k+1}(x)) - n(t - t_k^n) \sigma_i(x, \partial_x v_{i,n}(t, x)) \right) \left( \partial_{x,x} u_{i,k+1}(x) - \partial_{x,x} u_{i,k}(x) \right). \quad (2.20)$$

Using again the Hölder equicontinuity in time of  $(v_{i,n}, \partial_x v_{i,n})$  as well as the uniform bound on  $|\partial_{x,x} u_{i,k}|_{[0,a_i]}$  (2.17), we can show that for (2.19), for any  $\psi \in \mathcal{C}_c^\infty((0, T) \times (0, a_i))$ ,

$$\left| \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \int_0^{a_i} \left( \sigma_i(x, \partial_x u_{i,k+1}(x)) - \sigma_i(x, \partial_x v_{i,n}(t, x)) \right) \partial_{x,x} u_{i,k}(x) \psi(t, x) dx dt \right| \xrightarrow{n \rightarrow +\infty} 0.$$

Finally, from assumptions  $(\mathcal{P})$ , for all  $i \in \{1 \dots I\}$ ,  $\sigma_i$  is differentiable with respect to all its variable, integrating by part we get for (2.20)

$$\begin{aligned} & \left| \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \int_0^{a_i} \left( \sigma_i(x, \partial_x u_{i,k+1}(x)) - n(t - t_k^n) \sigma_i(x, \partial_x v_{i,n}(t, x)) \right) \right. \\ & \quad \left. \left( \partial_{x,x} u_{i,k+1}(x) - \partial_{x,x} u_{i,k}(x) \right) \psi(t, x) dx dt \right| = \\ & \left| \sum_{k=0}^{n-1} \int_{t_k^n}^{t_{k+1}^n} \int_0^{a_i} \left( \partial_x \left( \sigma_i(x, \partial_x u_{i,k+1}(t, x)) \psi(t, x) \right) - n(t - t_k^n) \partial_x \left( \sigma_i(x, \partial_x v_{i,n}(t, x)) \psi(t, x) \right) \right) \right. \\ & \quad \left. \left( \partial_x u_{i,k+1}(x) - \partial_x u_{i,k}(x) \right) dx dt \right| \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

We conclude that for any  $\psi \in \mathcal{C}_c^\infty((0, T) \times (0, a_i))$

$$\int_0^T \int_0^{a_i} \left( Q_i(x, v_i(t, x), \partial_t v_i(t, x), \partial_x v_i(t, x), \partial_{x,x} v_i(t, x)) \right) \psi(t, x) dx dt = 0.$$

It is then possible to consider the last equation as a linear one, with coefficients  $\tilde{\sigma}_i(t, x) = \sigma_i(x, \partial_x v_i(t, x))$ ,  $\tilde{H}_i(t, x) = H_i(x, v_i(t, x), \partial_x v_i(t, x))$  belonging to the class  $\mathcal{C}^{\frac{\alpha}{2}, \alpha}((0, T) \times$

$(0, a_i)$ ), and using Theorem III.12.2 of [26], we get finally that for all  $i \in \{1 \dots I\}$ ,  $v_i \in \mathcal{C}^{1+\frac{\alpha}{2}, 2+\alpha}((0, T) \times (0, a_i))$ , which means that  $v \in \mathcal{C}^{1+\frac{\alpha}{2}, 2+\alpha}(\mathring{\mathcal{J}}_T^a)$ .

We deduce that  $v_i$  satisfies on each edge

$$Q_i(x, v_i(t, x), \partial_t v_i(t, x), \partial_x v_i(t, x), \partial_{x,x} v_i(t, x)) = 0, \quad \text{if } (t, x) \in (0, T) \times (0, a_i).$$

From the estimates (2.17), we know that  $\partial_t v_{i,n}$  and  $\partial_{x,x} v_{i,n}$  are uniformly bounded by  $n$ . We deduce finally that  $v \in \mathcal{C}_b^{1+\frac{\alpha}{2}, 2+\alpha}(\mathring{\mathcal{J}}_T^a)$ .

We conclude by proving that  $v$  satisfies the non linear Neumann boundary condition at the vertex. For this, let  $t \in (0, T)$ ; we have up to a sub sequence  $n_p$

$$F(v_{n_p}(t, 0), \partial_x v_{n_p}(t, 0)) \xrightarrow{p \rightarrow +\infty} F(v(t, 0), \partial_x v(t, 0)).$$

On the other hand, using that  $F(u_k(0), \partial_0 u_k(x)) = 0$ , we know from the continuity of  $F$  (assumption  $(\mathcal{P})$ ), the Hölder equicontinuity in time of  $t \mapsto v_n(t, 0)$ , and  $t \mapsto \partial_x v(t, 0)$ , that there exists a constant  $M_5(\alpha)$  independent of  $n$ , such that if  $t \in [t_k^n, t_{k+1}^n]$

$$\begin{aligned} |F(v_n(t, 0), \partial_x v_n(t, 0))| &= |F(v_n(t, 0), \partial_x v_n(t, 0)) - F(u_k(0), \partial_x u_k(0))| \leq \\ &\sup \left\{ |F(u, x) - F(v, y)|, \quad |u - v| + \|x - y\|_{\mathbb{R}^I} \leq M_5(\alpha) n^{-\frac{\alpha}{2}} \right\} \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Therefore, we conclude once more from the continuity of  $F$  (assumption  $(\mathcal{P})$ ), the compatibility condition (assumption  $(\mathcal{P})$  (v)), that for each  $t \in [0, T)$

$$F(v(t, 0), \partial_x v(t, 0)) = 0.$$

On the other hand, it is easy to get

$$\forall i \in \{1 \dots I\}, \quad \forall x \in [0, a_i], \quad v_i(0, x) = g_i(x), \quad \forall t \in [0, T], \quad v_i(t, a_i) = \phi_i(t).$$

Finally, the expression of the upper bounds of the solution given in Theorem 2.2.2, are a consequence of Proposition 2.4.3, and Lemma 2.2.1, that completes the proof.  $\square$

### 2.4.3 On the existence for unbounded junction

We give in this subsection a result on the existence and the uniqueness of the solution for the parabolic problem (2.3), in a unbounded junction  $\mathcal{J}$  defined for  $I \in \mathbb{N}^*$  edges by

$$\mathcal{J} = \left\{ X = (x, i), x \in \mathbb{R}_+ \text{ and } i \in \{1, \dots, I\} \right\}.$$

In the sequel,  $\mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T)$  is the class of function with regularity  $\mathcal{C}^{0,1}([0, T] \times [0, +\infty)) \cap \mathcal{C}^{1,2}((0, T) \times (0, +\infty))$  on each edge, and  $L^\infty(\mathcal{J}_T)$  is the set of measurable real bounded maps defined on  $\mathcal{J}_T$ .

We introduce the following data

$$\begin{cases} F \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R}^I, \mathbb{R}) \\ g \in \mathcal{C}_b^1(\mathcal{J}) \cap \mathcal{C}_b^2(\overset{\circ}{\mathcal{J}}) \end{cases},$$

and for each  $i \in \{1 \dots I\}$

$$\begin{cases} \sigma_i \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}) \\ H_i \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R}) \\ \phi_i \in \mathcal{C}^1([0, T], \mathbb{R}) \end{cases}.$$

We suppose furthermore that the data satisfy the following assumption

#### Assumption ( $\mathcal{P}_\infty$ )

(i) Assumption on  $F$

$$\begin{cases} a) F \text{ is decreasing with respect to its first variable,} \\ b) F \text{ is nondecreasing with respect to its second variable,} \\ c) \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, F(b, B) = 0, \end{cases}$$

or the Kirchhoff condition

$$\left\{ \begin{array}{l} a) \ F \text{ is nonincreasing with respect to its first variable,} \\ b) \ F \text{ is increasing with respect to its second variable,} \\ c) \ \exists(b, B) \in \mathbb{R} \times \mathbb{R}^I, \quad F(b, B) = 0. \end{array} \right.$$

We suppose moreover that there exist a parameter  $m \in \mathbb{R}$ ,  $m \geq 2$  such that we have

(ii) The (uniform) ellipticity condition on the  $(\sigma_i)_{i \in \{1 \dots I\}}$  : there exists  $\underline{\nu}, \bar{\nu}$ , strictly positive constants such that

$$\begin{aligned} \forall i \in \{1 \dots I\}, \quad \forall (x, p) \in \mathbb{R}_+ \times \mathbb{R}, \\ \underline{\nu}(1 + |p|)^{m-2} \leq \sigma_i(x, p) \leq \bar{\nu}(1 + |p|)^{m-2}. \end{aligned}$$

(iii) The growth of the  $(H_i)_{i \in \{1 \dots I\}}$  with respect to  $p$  exceed the growth of the  $\sigma_i$  with respect to  $p$  by no more than two, namely there exists  $\mu$  an increasing real continuous function such that

$$\forall i \in \{1 \dots I\}, \quad \forall (x, u, p) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad |H_i(x, u, p)| \leq \mu(|u|)(1 + |p|)^m.$$

(iv) We impose the following restrictions on the growth with respect to  $p$  of the derivatives for the coefficients  $(\sigma_i, H_i)_{i \in \{1 \dots I\}}$ , which are for all  $i \in \{1 \dots I\}$ ,

$$\begin{aligned} a) \quad & |\partial_p \sigma_i|_{\mathbb{R}_+ \times \mathbb{R}^2} (1 + |p|)^2 + |\partial_p H_i|_{\mathbb{R}_+ \times \mathbb{R}^2} \leq \gamma(|u|)(1 + |p|)^{m-1}, \\ b) \quad & |\partial_x \sigma_i|_{\mathbb{R}_+ \times \mathbb{R}^2} (1 + |p|)^2 + |\partial_x H_i|_{\mathbb{R}_+ \times \mathbb{R}^2} \leq \left( \varepsilon(|u|) + P(|u|, |p|) \right) (1 + |p|)^{m+1}, \\ c) \quad & \forall (x, u, p) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad -C_H \leq \partial_u H_i(x, u, p) \leq \left( \varepsilon(|u|) + P(|u|, |p|) \right) (1 + |p|)^m, \end{aligned}$$

where  $\gamma$  and  $\varepsilon$  are continuous non negative increasing functions.  $P$  is a continuous function, increasing with respect to its first variable, and tends to 0 for  $p \rightarrow +\infty$ , uniformly with respect to its first variable, from  $[0, u_1]$  with  $u_1 \in R$ , and  $C_H > 0$  is real strictly positive number. We assume that  $(\gamma, \varepsilon, P, C_H)$  are independent of  $i \in \{1 \dots I\}$ .

(v) A compatibility conditions for  $g$

$$F(g(0), \partial_x g(0)) = 0.$$

We state here a comparison Theorem for the problem (2.3), in a unbounded junction.

**Theorem 2.4.4.** *Assume  $(\mathcal{P}_\infty)$ . Let  $u \in \mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T) \cap L^\infty(\mathcal{J}_T)$  (resp.  $v \in \mathcal{C}^{0,1}(\mathcal{J}_T) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T) \cap L^\infty(\mathcal{J}_T)$ ) be a super solution (resp. a sub solution) of (2.4) (where  $a_i = +\infty$ ), satisfying for all  $i \in \{1 \dots I\}$  for all  $x \in [0, +\infty)$ ,  $u_i(0, x) \geq v_i(0, x)$ . Then for each  $(t, (x, i)) \in \mathcal{J}_T$  :  $u_i(t, x) \geq v_i(t, x)$ .*

*Proof.* Let  $s \in [0, T)$ ,  $K = (K \dots K) > (1, \dots 1)$  in  $\mathbb{R}^I$ , and  $\lambda = \lambda(K) > 0$ , that will be chosen in the sequel. We argue as in the proof of Theorem 2.2.4, assuming

$$\sup_{(t, (x, i)) \in \mathcal{J}_s^K} \exp\left(-\lambda t - \frac{(x-1)^2}{2}\right) (v_i(t, x) - u_i(t, x)) > 0.$$

Using the boundary conditions satisfied by  $u$  and  $v$ , the above supremum is reached at a point  $(t_0, (x_0, j_0)) \in (0, s] \times \mathcal{J}$ , with  $0 \leq x_0 \leq K$ .

If  $x_0 \in [0, K)$ , the optimality conditions are given for  $x_0 \neq 0$  by

$$\begin{aligned} -\lambda(v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0)) + \partial_t v_{j_0}(t_0, x_0) - \partial_t u_{j_0}(t_0, x_0) &\geq 0, \\ -(x_0 - 1)(v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0)) + \partial_x v_{j_0}(t_0, x_0) - \partial_x u_{j_0}(t_0, x_0) &= 0, \\ (v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0)) - 2(x_0 - 1)^2(v_{j_0}(t_0, x_0) - u_{j_0}(t_0, x_0)) \\ + (\partial_{x,x} v_{j_0}(t_0, x_0) - \partial_{x,x} u_{j_0}(t_0, x_0)) &\leq 0, \end{aligned}$$

and if  $x_0 = 0$ ,

$$\forall i \in \{1, \dots, I\}, \quad \partial_x v_i(t_0, 0) \leq \partial_x u_i(t_0, 0) - (v_i(t_0, 0) - u_i(t_0, 0)) < \partial_x u_i(t_0, 0).$$

If  $x_0 = 0$ , we obtain a contradiction exactly as in the proof of Theorem 2.2.4. On the other hand if  $x_0 \in (0, K)$ , using assumptions  $(\mathcal{P})$  (iv) a), (iv) c) and the optimality conditions, we can choose  $\lambda(K)$  of the form  $\lambda(K) = C(1 + K^2)$ , (see (2.5) and (2.6)), where  $C > 0$  is a constant independent of  $K$ , to get again a contradiction. We deduce that, if

$$\sup_{(t, (x, i)) \in \mathcal{J}_s^K} \exp\left(-\lambda(K)t - \frac{(x-1)^2}{2}\right) (v_i(t, x) - u_i(t, x)) > 0,$$

then for all  $(t, (x, i)) \in [0, T] \times \mathcal{J}^K$

$$\exp(-\lambda(K)t - \frac{(x-1)^2}{2}) \left( v_i(t, x) - u_i(t, x) \right) \leq \exp(-\lambda(K)t - \frac{(K-1)^2}{2}) \left( v_i(t, K) - u_i(t, K) \right).$$

Hence for all  $(t, (x, i)) \in [0, T] \times \mathcal{J}^K$

$$\exp(-\frac{(x-1)^2}{2}) \left( v_i(t, x) - u_i(t, x) \right) \leq \exp(-\frac{(K-1)^2}{2}) \left( v_i(t, K) - u_i(t, K) \right).$$

On the other hand, if

$$\sup_{(t, (x, i)) \in \mathcal{J}_s^K} \exp(-\lambda(K)t - \frac{(x-1)^2}{2}) \left( v_i(t, x) - u_i(t, x) \right) \leq 0,$$

then for all  $(t, (x, i)) \in [0, T] \times \mathcal{J}^K$

$$\exp(-\lambda(K)t - \frac{(x-1)^2}{2}) \left( v_i(t, x) - u_i(t, x) \right) \leq 0.$$

So

$$\exp(-\frac{(x-1)^2}{2}) \left( v_i(t, x) - u_i(t, x) \right) \leq 0.$$

Finally we have, for all  $(t, (x, i)) \in [0, T] \times \mathcal{J}^K$

$$\max \left( 0, \exp(-\frac{(x-1)^2}{2}) \left( v_i(t, x) - u_i(t, x) \right) \right) \leq \exp(-\frac{(K-1)^2}{2}) \left( \|u\|_{L^\infty(\mathcal{J}_T)} + \|v\|_{L^\infty(\mathcal{J}_T)} \right).$$

Sending  $K \rightarrow \infty$  and using the boundedness of  $u$  and  $v$ , we deduce the inequality  $v \leq u$  in  $[0, T] \times \mathcal{J}$ .  $\square$

**Theorem 2.4.5.** *Assume  $(\mathcal{P}_\infty)$ . The following parabolic problem with Neumann boundary condition at the vertex*

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x} u_i(t, x) + \\ H_i(x, u_i(t, x), \partial_x u_i(t, x)) = 0, & \text{if } (t, x) \in (0, T) \times (0, +\infty), \\ F(u(t, 0), \partial_x u(t, 0)) = 0, & \text{if } t \in [0, T], \\ \forall i \in \{1 \dots I\}, \quad u_i(0, x) = g_i(x), & \text{if } x \in [0, +\infty), \end{cases} \quad (2.21)$$

is uniquely solvable in the class  $\mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}(\mathcal{J}_T) \cap \mathcal{C}^{1+\frac{\alpha}{2}, 2+\alpha}(\overset{\circ}{\mathcal{J}}_T)$ . There exist constants  $(M_1, M_2, M_3)$ , depending only the data introduced in assumption  $(\mathcal{P}_\infty)$

$$\begin{aligned} M_1 &= M_1 \left( \max_{i \in \{1 \dots I\}} \left\{ \sup_{x \in (0, +\infty)} | -\sigma_i(x, \partial_x g_i(x)) \partial_x^2 g_i(x) + H_i(x, g_i(x), \partial_x g_i(x)) | \right\}, \right. \\ &\quad \left. \max_{i \in \{1 \dots I\}} |g_i|_{(0, +\infty)}, C_H \right), \\ M_2 &= M_2 \left( \bar{\nu}, \underline{\nu}, \mu(M_1), \gamma(M_1), \varepsilon(M_1), \sup_{|p| \geq 0} P(M_1, |p|), |\partial_x g_i|_{(0, +\infty)}, M_1 \right), \\ M_3 &= M_3 \left( M_1, \underline{\nu}(1 + |p|)^{m-2}, \mu(|u|)(1 + |p|)^m, |u| \leq M_1, |p| \leq M_2 \right), \end{aligned}$$

such that

$$\|u\|_{\mathcal{C}(\mathcal{J}_T)} \leq M_1, \quad \|\partial_x u\|_{\mathcal{C}(\mathcal{J}_T)} \leq M_2, \quad \|\partial_t u\|_{\mathcal{C}(\mathcal{J}_T)} \leq M_1, \quad \|\partial_{x,x} u\|_{\mathcal{C}(\mathcal{J}_T)} \leq M_3.$$

Moreover, there exists a constant  $M(\alpha)$  depending on  $(\alpha, M_1, M_2, M_3)$  such that for any  $a \in (0, +\infty)^I$

$$\|u\|_{\mathcal{C}^{\frac{\alpha}{2}, 1+\alpha}(\mathcal{J}_T^a)} \leq M(\alpha).$$

*Proof.* Assume  $(\mathcal{P}_\infty)$  and let  $a = (a, \dots, a) \in (0, +\infty)^I$ . Applying Theorem [2.2.2](#), we can define  $u^a \in \mathcal{C}^{0,1}(\mathcal{J}_T^a) \cap \mathcal{C}^{1,2}(\overset{\circ}{\mathcal{J}}_T^a)$  as the unique solution of

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x} u_i(t, x) + \\ H_i(x, u_i(t, x), \partial_x u_i(t, x)) = 0, & \text{if } (t, x) \in (0, T) \times (0, a), \\ F(u(t, 0), \partial_x u(t, 0)) = 0, & \text{if } t \in [0, T], \\ \forall i \in \{1 \dots I\}, u_i(t, a) = g_i(a), & \text{if } t \in [0, T], \\ \forall i \in \{1 \dots I\}, u_i(0, x) = g_i(x), & \text{if } x \in [0, a]. \end{cases} \quad (2.22)$$

Using assumption  $(\mathcal{P}_\infty)$  and Theorem [2.2.2](#), we get that there exists a constant  $C > 0$  independent of  $a$  such that

$$\sup_{a \geq 0} \|u^a\|_{\mathcal{C}^{1,2}(\mathcal{J}_T^a)} \leq C.$$

We are going to send  $a$  to  $+\infty$  in [\(2.22\)](#).

Following the same argument as for the proof of Theorem 2.2.2, we get that, up to a subsequence,  $u^a$  converges locally uniformly to some map  $u$  which solves (2.21). On the other hand, uniqueness of  $u$  is a direct consequence of the comparison Theorem 2.4.4, since  $u \in L^\infty(\mathcal{J}_T)$ . Finally the expression of the upper bounds of the derivatives of  $u$  given in Theorem 2.4.5, are a consequence of Theorem 2.2.2 and assumption  $(\mathcal{P}_\infty)$ .  $\square$



# Appendices

# Appendix A

## Functionnal spaces

Let  $l \in \mathbb{R}_+^*$ ,  $T > 0$ , and  $\Omega$  a smooth open and bounded subset of  $\mathbb{R}^n$ , ( $n > 0$ ). We set  $\Omega_T = (0, T) \times \Omega$ , and we introduce the following spaces :

-if  $l \in 2\mathbb{N}^*$ , the Banach space  $\left(\mathcal{C}^{\frac{l}{2}, l}(\overline{\Omega_T}), \|\cdot\|_{\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)}\right)$ , whose elements are continuous functions  $(t, x) \rightarrow u(t, x)$  in  $\overline{\Omega_T}$ , together with all its derivatives of the form  $\partial_t^r \partial_x^s u$ , with  $2r + s < l$ . The norm  $\|\cdot\|_{\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)}$  is defined for all  $u \in \mathcal{C}^{\frac{l}{2}, l}(\overline{\Omega_T})$  by

$$\|u\|_{\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)} = \sum_{2r+s=j} \sup_{(t,x) \in \Omega_T} |\partial_t^r \partial_x^s u(t, x)|.$$

-if  $l \notin \mathbb{N}^*$ , the Banach space  $\left(\mathcal{C}^{\frac{l}{2}, l}(\overline{\Omega_T}), \|\cdot\|_{\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)}\right)$ , whose elements are continuous functions  $(t, x) \rightarrow u(t, x)$  in  $\overline{\Omega_T}$ , together with all its derivatives of the form  $\partial_t^r \partial_x^s u$ , with  $2r + s < l$ , and satisfying an Hölder condition with exponent  $\frac{l-2r-s}{2}$  in their first variable, and with exponent  $(l - [l])$  in their second variable, over all the connected components of  $\Omega_T$  whose radius is smaller than 1.

The norm  $\|\cdot\|_{\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)}$  is defined for all  $u \in \mathcal{C}^{\frac{l}{2}, l}(\overline{\Omega_T})$  by

$$\|u\|_{\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)} = |u|_{\Omega_T}^l + \sum_{j=0}^{[l]} |u|_{\Omega_T}^j,$$

with

$$\begin{aligned}
\forall j \in \{0, \dots, l\}, \quad |u|_{\Omega_T}^j &= \sum_{2r+s=j} \sup_{(t,x) \in \Omega_T} |\partial_t^r \partial_x^s u(t, x)|, \\
|u|_{\Omega_T}^l &= |u|_{x, \Omega_T}^l + |u|_{t, \Omega_T}^{\frac{l}{2}}, \\
|u|_{x, \Omega_T}^l &= \sum_{2r+s=l} |\partial_t^r \partial_x^s u(t, x)|_{x, \Omega_T}^{l-|l|}, \\
|u|_{t, \Omega_T}^l &= \sum_{0 < l-2r-s < 2} |\partial_t^r \partial_x^s u(t, x)|_{t, \Omega_T}^{\frac{l-2r-s}{2}}, \\
|u|_{x, \Omega_T}^\alpha &= \sup_{t \in (0, T)} \sup_{x, y \in \Omega, x \neq y, |x-y| \leq 1} \frac{|u(t, x) - u(t, y)|}{|x - y|^\alpha}, \quad 0 < \alpha < 1, \\
|u|_{t, \Omega_T}^\alpha &= \sup_{x \in \Omega} \sup_{t, s \in (0, T), t \neq s, |t-s| \leq 1} \frac{|u(t, x) - u(s, x)|}{|t - s|^\alpha}, \quad 0 < \alpha < 1.
\end{aligned}$$

$\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)$ , the set whose elements  $f$  verify, for any open set  $O_T$  separated from the boundary of  $\Omega_T$  by a strictly positive distance, namely

$$\inf_{y \in \partial \Omega_T, x \in \overline{O_T}} \|x - y\|_{\mathbb{R}^n} > 0,$$

then  $f \in \mathcal{C}^{\frac{l}{2}, l}(\overline{O_T})$ .

$\mathcal{C}_b^{\frac{l}{2}, l}(\Omega_T)$ , the subset of  $\mathcal{C}^{\frac{l}{2}, l}(\Omega_T)$ , whose derivatives of the form  $\partial_t^r \partial_x^s u$ , (with  $2r + s < l$ ) are bounded, namely  $\sup_{(t,x) \in \Omega_T} |\partial_t^r \partial_x^s u(t, x)| < +\infty$ .

We use the same notations, when the domain does not depend on time, namely  $T = 0$ ,  $\Omega_T = \Omega$ , removing the dependence on the time variable.

For  $R > 0$ , we denote by  $L_2((0, T) \times (0, R))$  the usual space of square integrable maps, and  $\mathcal{C}_c^\infty((0, T) \times (0, R))$  the set of infinite continuous differentiable functions on  $(0, T) \times (0, R)$ , vanishing at the lateral surface of  $(0, T) \times (0, R)$ , namely  $[0, T] \times \partial(0, R)$ .

## Appendix B

### The Elliptic problem

**Proposition B.0.1.** *Let  $\theta \in \mathbb{R}$  and assume assumption  $(\mathcal{E})$ . Suppose that  $u \in \mathcal{C}^2([0, a_i])$  is the solution of*

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, & \text{if } x \in (0, a_i) \\ \forall i \in \{1 \dots I\}, u_i(0) = u(0) = \theta, \\ \forall i \in \{1 \dots I\}, u_i(a_i) = \phi_i. \end{cases} \quad (\text{B.1})$$

then the following map

$$\Psi := \begin{cases} \mathbb{R} \mapsto \mathcal{C}^2([0, a_i]) \\ \theta \rightarrow u^\theta \end{cases}$$

is continuous.

*Proof.* Let  $\theta_n$  a sequence converging to  $\theta$ . Using the Schauder estimates Theorem 6.6 of [19], we get that there exists a constant  $M > 0$  independent of  $n$ , depending only the data, such that for all  $\alpha \in (0, 1)$

$$\|u^{\theta_n}\|_{\mathcal{C}^{2+\alpha}([0, a_i])} \leq M.$$

From Ascoli's Theorem,  $u^{\theta_n}$  converges up to a subsequence to  $v$  in  $\mathcal{C}^2([0, a_i])$  solution of (B.1). By uniqueness of the solution of (B.1),  $u^{\theta_n}$  converges necessary to the solution  $u^\theta$  of (B.1) in  $\mathcal{C}^2([0, a_i])$ , that completes the proof.  $\square$

**Proof of Theorem 2.3.1.**

*Proof.* The uniqueness of (2.8) results from the elliptic comparison Theorem 2.3.3.

We turn to the solvability, and for this let  $\theta \in \mathbb{R}$ . We consider the elliptic Dirichlet problem at the junction

$$\begin{cases} -\sigma_i(x, \partial_x u_i(x)) \partial_{x,x} u_i(x) + H_i(x, u_i(x), \partial_x u_i(x)) = 0, & \text{if } x \in (0, a_i) \\ \forall i \in \{1 \dots I\}, u_i(0) = u(0) = \theta, \\ u_i(a_i) = \phi_i. \end{cases} \quad (\text{B.2})$$

For all  $i \in \{1 \dots I\}$ , each elliptic problem is uniquely solvable on each edge in  $\mathcal{C}^{2+\alpha}([0, a_i])$ , then (3.49) is uniquely solvable in the class  $\mathcal{C}^{2+\alpha}(\mathcal{J}^a)$ , and we denote by  $u^\theta$  its solution.

We turn to the Neumann boundary condition at the vertex. Let us recall assumption (E)(i)

$$\begin{cases} F \text{ is strictly decreasing in its first variable, increasing in its second variable,} \\ \exists (b, B) \in \mathbb{R} \times \mathbb{R}^I, \text{ such that : } F(b, B) = 0. \end{cases}$$

Fix now

$$\begin{aligned} K_i &= \sup_{(x,u) \in (0,a_i) \times (-a_i B_i, a_i B_i)} |H_i(x, u, B_i)|, \\ \theta &\geq |b| + \max_{i \in \{1 \dots I\}} \left\{ |\phi_i| + |a_i B_i| + \frac{K_i}{C_H} \right\}, \end{aligned}$$

and let us show that  $f : x \mapsto \theta + B_i x$ , is a super solution on each edge  $J_i^{a_i}$  of (3.49).

We have the boundary conditions

$$f(0) = \theta, \quad f(a_i) = \theta + a_i B_i \geq |\phi_i| + |a_i B_i| + a_i B_i \geq \phi_i,$$

and using assumption (E) (iii), we have for all  $x \in (0, a_i)$

$$\begin{aligned} Q_i(x, \theta + B_i x, B_i, 0) &= H_i(x, \theta + B_i x, B_i) \geq H_i(x, B_i x, B_i) + C_H \theta \geq \\ &H_i(x, B_i x, B_i) + K_i \geq 0. \end{aligned}$$

We then get that for each  $i \in \{1 \dots I\}$ ,  $x \in [0, a_i]$ ,  $u_i^\theta(x) \leq \theta + B_i x$ , and a Taylor expansion

in the neighborhood of the junction point gives that for each  $i \in \{1 \dots I\}$ ,  $\partial_x u_i^\theta(0) \leq B_i$ . Since  $u^\theta(0) = \theta \geq b$ , we then get from assumption  $(\mathcal{E})$  (i)

$$F(u^\theta(0), \partial_x u^\theta(0)) \leq F(b, B) \leq 0.$$

Similarly, fixing

$$\theta \leq -|b| - \min_{i \in \{1 \dots I\}} \left\{ -|\phi_i| - |a_i B_i| - \frac{K_i}{C_H} \right\},$$

the map  $f : x \mapsto \theta + xB_i$  is a sub solution on each vertex  $J_i^{a_i}$  of (3.49), then for each  $i \in \{1 \dots I\}$ ,  $\partial_x u_i^\theta(0) \geq B_i$ , which means

$$F(u^\theta(0), \partial_x u^\theta(0)) \geq 0.$$

From Proposition B.0.1, we know that the real maps  $\theta \rightarrow u^\theta(0)$  and  $\theta \rightarrow \partial_x u^\theta(0)$  are continuous. Using the continuity of  $F$  (assumption  $(\mathcal{E})$ ), we get that  $\theta \rightarrow F(u^\theta(0), \partial_x u^\theta(0))$  is continuous, and therefore there exists  $\theta^* \in \mathbb{R}$  such that

$$F(u^{\theta^*}(0), \partial_x u^{\theta^*}(0)) = 0.$$

We remark that  $\theta^*$  is bounded by the data, namely  $\theta^*$  belongs to the following interval

$$\left[ \begin{array}{l} -|b| - \max_{i \in \{1 \dots I\}} \left\{ |\phi_i| + |a_i B_i| + \frac{\sup_{(x,u) \in (0, a_i)} |H_i(x, B_i x, B_i)|}{C_H} \right\}, \\ |b| + \max_{i \in \{1 \dots I\}} \left\{ |\phi_i| + |a_i B_i| + \frac{\sup_{(x,u) \in (0, a_i)} |H_i(x, B_i x, B_i)|}{C_H} \right\} \end{array} \right],$$

This completes the proof. Finally, since the solution  $u^{\theta^*}$  of (2.8) is unique, we get the uniqueness of  $\theta^*$ .  $\square$

## Chapter 3

# Diffusion on a junction

### 3.1 Introduction

Originally introduced by Freidlin and Sheu in [15] and Freidlin and Wetzell in [16], stochastic diffusions in graphs have attracted a lot of attention in the last 20 years. In [16] the authors studied an elliptic linear operator on a junction, with Kirchoff condition, and the process generated by its semi group. More precisely, given a junction  $\mathcal{J} = \bigcup_{i=1}^I J_i$ , with  $J_i = \{0, \{i\} \times (0, +\infty)\}$ ,  $i = 1, \dots, I$ , and  $(\sigma_i, b_i)$  regular functions from  $\mathbb{R}_+$  to  $\mathbb{R}$ , the authors have defined the elliptic operator  $L$  on the set of twice continuous differentiable function at the junction  $\mathcal{C}^2(\mathcal{J})$ , by

$$L : \begin{cases} \mathcal{C}^2(\mathcal{J}) \rightarrow \mathcal{C}(\mathcal{J}) \\ f = f_i(x) \mapsto \left( x \rightarrow \frac{1}{2} \sigma_i^2(x) \partial_{x,x} f_i(x) + b_i(x) \partial_x f_i(x), \quad (x, i) \in J_i \right), \end{cases}$$

with domain

$$D(L) := \left\{ f \in \mathcal{C}^2(\mathcal{J}), \sum_{i=1}^I \alpha_i \partial_x f(0, i) = 0 \right\},$$

where  $\alpha_1, \dots, \alpha_I$  are non-negative constants such that  $\alpha_1 + \dots + \alpha_I = 1$ . The authors of [16] have proved that there exists a continuous Markov process  $X = (x, i)$  defined on  $\mathcal{J}$  generated by  $L$ . Then [15] showed that there exists a one dimensional Wiener process  $W$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , adapted to the natural filtration of  $X = (x, i)$ , such that the process  $x$  satisfies on each time interval  $[0, T]$  the stochastic differential

equation:

$$dx(t) = \sigma_{i(t)}(x(t))dW(t) + b_{i(t)}(x(t))dt + dl(t), \quad 0 \leq t \leq T, \quad (3.1)$$

where  $l$  is a nondecreasing process starting from zero and satisfying:

$$\mathbb{P}\left(\left(\int_0^t \mathbf{1}_{\{x(s)>0\}} dl(s)\right)_{0 \leq t \leq T} = 0\right) = 1.$$

Moreover, [15] gives the following Itô's formula:

$$df(X(t)) = L\left(f_{i(t)}(x(t))\right)dt + \partial_x f_{i(t)}(x(t))\sigma_{i(t)}(x(t))dW(t) + \sum_{i=1}^I \alpha_i \partial_x f_i(0)dl(t), \quad (3.2)$$

for any sufficiently smooth test function  $f$ .

The aim of this Chapter is to provide a different method for the construction of the diffusion on the junction  $\mathcal{J}$ . Our main idea is to build the process as the limit of the càdlàg<sup>1</sup> process  $X^\delta$  (with parameter  $\delta > 0$ ) which jumps at position  $\delta$  on the edge  $J_i$  with probability  $\alpha_i$  each time it reaches the junction 0. We prove that the process  $X^\delta$  converges weakly, as  $\delta$  tends to 0, to a continuous diffusion process  $X$  valued in  $\mathcal{J}$  and with dynamics similar to (4.1). The additional randomness due to the process  $i$ , prevent us from completely describing the behavior of the component  $x$  with its dynamics (4.1). We therefore use the previous convergence to establish an Itô's formula for the process  $X$ , which completely characterizes it. We then focus on the process  $l$  involved in (4.1). This process can be seen as the local time of  $x$  at the vertex. We therefore provide an estimate of this process as the limit of the quadratic variation of  $x$  over the times spent at the neighborhood of 0.

The Chapter is organized as follows. In Section 3.2, we introduce the mathematical material that will be used throughout this work. Section 3.3 deals with the construction of the non-homogeneous diffusion on the junction. Finally, Section 3.4 is dedicated to Itô's formula, and local time estimates on the vertex.

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<sup>1</sup>Acronym for right-continuous and left-limited



## 3.2 Introductory material

### 3.2.1 Notations and preliminary results

Let  $I \in \mathbb{N}^*$ , be the number of edges. We recall that a junction  $\mathcal{J}$  consists in a vertex and a finite number  $I$  of edges. More precisely:

$$\mathcal{J} = \left\{ X = (x, i), \quad x \in \mathbb{R}_+ \text{ and } i \in \{1, \dots, I\} \right\},$$

where all the points  $(0, i)$ ,  $i = 1, \dots, I$ , are identified to the vertex denoted by 0. We can then write

$$\mathcal{J} = \bigcup_{i=1}^I J_i,$$

with  $J_i := \mathbb{R}_+ \times \{i\}$  and  $J_i \cap J_j = \{0\}$  for  $i \neq j$ .

Let  $T > 0$ , we set

$$\mathcal{J}_T = [0, T] \times \mathcal{J}.$$

We use here the notations of Section [2.2.1](#), and we recall for the convenience of the reader the definition of following spaces

$$\begin{aligned} \mathcal{C}_b^{0,1}(\mathcal{J}_T) &:= \left\{ f : \mathcal{J}_T \mapsto \mathbb{R}, \quad (t, (x, i)) \rightarrow f_i(t, x), \quad \forall i \in \{1 \dots I\}, \right. \\ &\quad (t, x) \rightarrow f_i(t, x) \in \mathcal{C}_b^{0,1}([0, T] \times [0, +\infty)), \text{ and } \forall (i, j) \in \{1 \dots I\}^2, \\ &\quad \left. \forall t \in [0, T], \quad f_i(t, 0) = f_j(t, 0) = f(t, 0) \right\}, \end{aligned}$$

$$\mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}}_T) := \left\{ f : \mathcal{J}_T \mapsto \mathbb{R}, \quad \forall i \in \{1 \dots I\}, \quad (t, x) \rightarrow f_i(t, x) \in \mathcal{C}_b^{1,2}((0, T) \times (0, +\infty)) \right\}.$$

We introduce the following Polish spaces, that will be useful in the sequel for the construction of the process in Section [3.3](#):

(i)  $(\mathcal{J}, d^{\mathcal{J}})$ , where  $d^{\mathcal{J}}$  is defined by

$$\forall \left( (x, i), (y, j) \right) \in \mathcal{J}^2, \quad d^{\mathcal{J}} \left( (x, i), (y, j) \right) = \begin{cases} |x - y| & \text{if } i = j, \\ x + y & \text{if } i \neq j. \end{cases}$$

(ii)  $(\mathcal{U}, d^{\mathcal{U}})$ , the set of sequences valued in  $\{1, \dots, I\}$ , where  $d^{\mathcal{U}}$  is defined by

$$\forall (u, z) \in \mathcal{U}^2, \quad d^{\mathcal{U}}(u, z) = \sum_{n=0}^{+\infty} \frac{|u_n - z_n|}{2^n}.$$

(iii)  $(\mathbb{N}, d^{\mathbb{N}})$ , where  $d^{\mathbb{N}}$  is defined by

$$\forall (u, p) \in \mathbb{N}, \quad d^{\mathbb{N}}(u, p) = |u - p|.$$

As explained in the Introduction [3.1](#), the limit process is valued in the space of continuous functions from  $[0, T]$  to  $(\mathcal{J}, d^{\mathcal{J}})$ , which we denote by  $\mathcal{C}^{\mathcal{J}}([0, T])$ . This space is endowed with the following uniform norm  $d_{(0, T)}^{\mathcal{J}}$ , defined by

$$\forall \left( (x(\cdot), i(\cdot)), (y(\cdot), j(\cdot)) \right) \in \mathcal{C}^{\mathcal{J}}([0, T])^2, \quad d_{(0, T)}^{\mathcal{J}}(X, Y) = \sup_{0 \leq s \leq T} d^{\mathcal{J}}\left( (x(s), i(s)), (y(s), j(s)) \right),$$

and then is Polish.

We recall that for  $(E, d^E)$  a Polish space  $E$ , endowed with its metric  $d^E$ , the Polish space  $(\mathcal{D}^E([0, T]), d_{sko}^E)$ , consists of càdlàg maps defined on  $[0, T]$ , valued in  $E$  endowed with the Skohokhod's metric  $d_{sko}^E$  defined by

$$\begin{aligned} & \forall \left( z(\cdot), u(\cdot) \right) \in \mathcal{D}^E([0, T]), \\ d_{sko}^E(z, u) &= \inf_{\lambda \in \Lambda} \left\{ \sup_{t \neq s} \left| \ln \left( \frac{\lambda(t) - \lambda(s)}{t - s} \right) \right| \vee \sup_{0 \leq t \leq T} d^E \left( z(\lambda(t)), u(t) \right) \right\}, \end{aligned}$$

where  $\Lambda$  denotes the set of continuous increasing homeomorphism from  $[0, T]$  to itself, and  $|\cdot|_{(0, T)}$  the standard uniform norm of  $\mathcal{C}([0, T])$  (see, for instance, [\[7\]](#)).

**Remark 3.2.1.** We recall that  $d_{sko}^E$  is equivalent to the following metric defined on  $\mathcal{D}^E([0, T])$  by

$$\begin{aligned} & \forall \left( z(\cdot), u(\cdot) \right) \in \mathcal{D}^E([0, T]), \\ d_{sko}^{E, o}(z, u) &= \inf_{\lambda \in \Lambda} \left\{ |\lambda - Id|_{(0, T)} \vee \sup_{0 \leq t \leq T} d^E \left( z(\lambda(t)), u(t) \right) \right\}, \end{aligned}$$

see for instance Theorem 12.1 in [\[7\]](#).

We get therefore, that  $\left( (\mathcal{D}^{\mathcal{J}}([0, T]), d_{sko}^{\mathcal{J}}), (\mathcal{D}^{\mathcal{U}}([0, T]), d_{sko}^{\mathcal{U}}), (\mathcal{D}^{\mathbb{N}}([0, T]), d_{sko}^{\mathbb{N}}) \right)$  are Polish spaces.

As it has been said in the general introduction (contributions ??), for  $\delta > 0$ , the approximate càdlàg process  $X^\delta$  jumps at position  $\delta$  on the edge  $J_i$  with probability  $\alpha_i$  each time it reaches the junction. Therefore it is valued in the space  $\mathcal{D}_\delta^{\mathcal{J}}([0, T])$  defined by

$$\mathcal{D}_\delta^{\mathcal{J}}([0, T]) := \left\{ \begin{array}{l} (x(\cdot), i(\cdot)) \in \mathcal{D}^{\mathcal{J}}([0, T]), \text{ if } d^{\mathcal{J}}((x(s), i(s)), (x(s-), i(s-))) \neq 0, \\ \text{then } x(s-) = 0 \text{ and } x(s) = \delta \end{array} \right\},$$

namely all the càdlàg maps whose jumps are of size  $\delta$  and can only occur at the junction point. This set inherits of the structure of  $\mathcal{D}^{\mathcal{J}}([0, T])$ .

**Proposition 3.2.2.**  $(\mathcal{D}_\delta^{\mathcal{J}}([0, T]), d_{sko}^{\mathcal{J}})$  is closed in  $(\mathcal{D}^{\mathcal{J}}([0, T]), d_{sko}^{\mathcal{J}})$ , and then a Polish space.

*Proof.* Let  $X_n(\cdot) = (x_n(\cdot), i_n(\cdot))$  be a sequence in  $\mathcal{D}_\delta^{\mathcal{J}}([0, T])$  which converges to  $X(\cdot) = (x(\cdot), i(\cdot)) \in \mathcal{D}^{\mathcal{J}}([0, T])$ , and let us show that  $X(\cdot) \in \mathcal{D}_\delta^{\mathcal{J}}([0, T])$ . There exists a sequence of continuous increasing homeomorphism  $\lambda_n : [0, T] \rightarrow [0, T]$ , such that :

$$\sup_{t \neq s} \left| \ln \left( \frac{\lambda_n(t) - \lambda_n(s)}{t - s} \right) \right| \vee d_{(0, T)}^{\mathcal{J}}(X_n \circ \lambda_n, X) \xrightarrow{n \rightarrow +\infty} 0.$$

Using Remark [3.2.1](#), we get that

$$|\lambda_n - Id|_{(0, T)} \vee d_{(0, T)}^{\mathcal{J}}(X_n \circ \lambda_n, X) \xrightarrow{n \rightarrow +\infty} 0. \quad (3.3)$$

Let  $t \in [0, T]$ , we have

$$x(t) - x(t-) = x(t) - x_n(\lambda_n(t)) + x_n(\lambda_n(t)) - x_n(\lambda_n(t)-) + x_n(\lambda_n(t)-) - x(t-). \quad (3.4)$$

From [\(3.3\)](#), we get

$$|x(t) - x_n(\lambda_n(t))| + |x_n(\lambda_n(t)-) - x(t-)| \xrightarrow{n \rightarrow +\infty} 0.$$

Knowing that the sequence  $X_n \circ \lambda_n(\cdot)$  is valued too in  $\mathcal{D}_\delta^{\mathcal{J}}([0, T])$ , we have therefore using

[\(3.4\)](#)

$$\begin{aligned} x(t) - x(t-) &= x(t) - x_n(\lambda_n(t)) + x_n(\lambda_n(t)-) - x(t-), \\ \text{or, } x(t) - x(t-) &= x(t) - x_n(\lambda_n(t)) + \delta + x_n(\lambda_n(t)-) - x(t-), \end{aligned}$$

and sending  $n$  to  $+\infty$  we obtain finally

$$x(t) - x(t-) \in \{0, \delta\} .$$

We now prove that a discontinuity of  $x(\cdot)$  only can occurs for  $x(\cdot-) = 0$ . Fix  $t \in [0, T]$  such that  $x(t) - x(t-) = \delta$ . Let  $\varepsilon \leq \frac{\delta}{4}$ , there exists  $n_0 \in \mathbb{N}$ , such that for all  $n \geq n_0$ ,  $|x_n \circ \lambda_n - x|_{(0, T)} \leq \varepsilon$ . We therefore get using (3.4)

$$x_n(\lambda_n(t)) - x_n(\lambda_n(t)-) \geq \delta - 2\varepsilon,$$

and hence

$$x_n(\lambda_n(t)) - x_n(\lambda_n(t)-) \geq \frac{\delta}{2},$$

and then necessarily, for all  $n \geq n_0$ ,  $x_n(\lambda_n(t)) = \delta$  and  $x_n(\lambda_n(t)-) = 0$ . Hence  $x(t) = \delta$ ,  $x(t-) = 0$ , that completes the proof.  $\square$

### 3.3 Construction of the process on the junction

In this section, we fix a terminal time  $T > 0$ , and we build a diffusion process  $(X(t) = (x(t), i(t)))_{0 \leq t \leq T}$  valued in  $\mathcal{C}^{\mathcal{J}}([0, T])$ . The continuous process  $(X(t) = (x(t), i(t)))_{0 \leq t \leq T}$  is approximated by a classical càdlàg process  $X^\delta$  with parameter  $\delta > 0$  which jumps at position  $\delta$  on the edge  $J_i$  with probability  $\alpha_i$  each time it reaches the junction.

To this end we consider a family of processes  $(X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T}$ , depending on a parameter  $\delta > 0$ , such that  $(X^\delta(t))_{0 \leq t \leq T}$  is valued in  $\mathcal{D}_\delta^{\mathcal{J}}([0, T])$ . We prove the tightness of this sequence and get a continuous limit satisfying a stochastic differential equation on  $\mathcal{J}$ .

Let us introduce the following data

$$\left\{ \begin{array}{l} (\Omega, \mathcal{A}, \mathbb{P}) \text{ a probability space,} \\ W \text{ a standard one dimensional Brownian motion defined on } (\Omega, \mathcal{A}, \mathbb{P}), \\ Z = (Z_n)_{n \geq 0} \text{ a random variable valued in } \mathcal{U}, \text{ defined on } (\Omega, \mathcal{A}, \mathbb{P}), \\ (\alpha_i) \in (0, 1)^I, \\ x_0 \in L^2(\Omega), \text{ such that } x_0 > 0, \mathbb{P} \text{ a.s.} \end{array} \right.$$

and for each  $i \in \{1 \dots I\}$

$$\left\{ \begin{array}{l} \sigma_i \in W^{1,\infty}([0, +\infty), \mathbb{R}) \\ b_i \in W^{1,\infty}([0, +\infty), \mathbb{R}) \end{array} \right. .$$

We suppose furthermore that the data satisfy the following assumptions:

**Assumption ( $\mathcal{H}$ )**

$$(i) \quad \exists c > 0, \quad \forall i \in \{1 \dots I\}, \quad \forall x \in [0, +\infty), \quad \sigma_i(x) \geq c,$$

$$(ii) \quad \forall i \in \{1 \dots I\}, \quad (\sigma_i, b_i) \in L^\infty((0, +\infty), \mathbb{R}),$$

$$(iii) \quad \sum_{i=1}^I \alpha_i = 1,$$

$$(iv) \quad x_0 \text{ is independent of } W \text{ and } Z,$$

$$(v) \quad Z = (Z_n)_{n \geq 0} \text{ are independent and identically distributed (i.i.d.), with } \mathbb{P}(Z_n = i) = \alpha_i.$$

### 3.3.1 Diffusion with jumps at the vertex

We consider the sequence of diffusion processes  $(x_n^\delta)_{n \geq 0}$  and stopping times  $(\tau_n^\delta)_{n \geq 0}$  defined recursively by

$$\left\{ \begin{array}{l} x_0^\delta(t) = x_0 + \int_0^t b_{Z_0}(x_0^\delta(s)) ds + \int_0^t \sigma_{Z_0}(x_0^\delta(s)) dW(s), \quad 0 \leq t \leq T, \\ \tau_0^\delta = 0, \end{array} \right. \quad \mathbb{P} \text{ a.s.}$$

and, for  $n \geq 0$ ,

$$\begin{cases} x_{n+1}^\delta(t) = \delta + \int_0^t \mathbf{1}_{[\tau_{n+1}, +\infty)}(s) b_{Z_{n+1}}(x_{n+1}^\delta(s)) ds + \\ \int_0^t \mathbf{1}_{[\tau_{n+1}, +\infty)}(s) \sigma_{Z_{n+1}}(x_{n+1}^\delta(s)) dW(s), & 0 \leq t \leq T, & \mathbb{P} \text{ a.s.} \\ \tau_{n+1}^\delta = \inf \left\{ t > \tau_n^\delta : x_n^\delta(t) = 0 \right\}, \end{cases} \quad (3.5)$$

We now define the process  $X^\delta(t) = \left( x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T}$  valued in  $\mathcal{D}_\delta^\mathcal{J}([0, T])$  by

$$\begin{cases} x^\delta(t) = \sum_{n \geq 0} x_n^\delta(t) \mathbf{1}_{[\tau_n^\delta, \tau_{n+1}^\delta)}(t), \\ i^\delta(t) = \sum_{n \geq 0} Z_n \mathbf{1}_{[\tau_n^\delta, \tau_{n+1}^\delta)}(t), \end{cases} \quad \mathbb{P} \text{ a.s.}$$

We notice that the process  $\left( x^\delta(t) \right)_{0 \leq t \leq T}$  is valued in  $(0, +\infty)$  and satisfies the following equation, for all  $t \in [0, T]$

$$\begin{aligned} x^\delta(t) &= x_0 + \int_0^{\tau_1^\delta \wedge t} b_{i^\delta(s)}(x^\delta(s)) ds + \int_0^{\tau_1^\delta \wedge t} \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) \\ &+ \sum_{n \geq 1} \mathbf{1}_{\{\tau_n^\delta \leq t < \tau_{n+1}^\delta\}} \left( \delta + \int_{\tau_n^\delta}^t b_{i^\delta(s)}(x^\delta(s)) ds + \int_{\tau_n^\delta}^t \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) \right), \quad \mathbb{P} \text{ a.s.} \end{aligned} \quad (3.6)$$

From the ellipticity assumption  $(\mathcal{H})$  (i), we underline that each stopping time  $(\tau_n^\delta)_{n \geq 0}$  is almost surely finite. We denote by  $(\mathcal{F}_t^\delta)_{0 \leq t \leq T}$  the right continuous and complete filtration generated by  $\left( X^\delta(t) = x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T}$ . The dynamics can be rewritten as

$$\begin{cases} X^\delta(0) = (x_0, Z_0) \\ dx^\delta(t) = b_{i^\delta(t)}(x^\delta(t)) dt + \sigma_{i^\delta(t)}(x^\delta(t)) dW(t) + \delta d \left( \sum_{0 \leq s \leq t} \mathbf{1}_{\{x^\delta(s-) = 0\}} \right) \\ i^\delta(t) = Z \sum_{0 \leq s \leq t} \mathbf{1}_{\{x^\delta(s-) = 0\}} \end{cases}, \quad \mathbb{P} \text{ a.s.} \quad (3.7)$$

In the next subsection we prove that the sequence  $\left( X^\delta(t) = x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T}$  is C-tight, namely up to a subsequence  $\left( X^\delta(t) = x^\delta(t), i^\delta(t) \right)_{0 \leq t \leq T}$  converges weakly to a random variable  $\left( X(t) = x(t), i(t) \right)_{0 \leq t \leq T}$ , with continuous paths. In other words the

limit  $\left( X(t) = x(t), i(t) \right)_{0 \leq t \leq T}$  is valued in  $\mathcal{C}^{\mathcal{J}}([0, T])$ .

### 3.3.2 Weak convergence

For  $Y \in \mathcal{D}^{\mathcal{J}}([0, T])$ , we denote by  $\omega_T(Y, \cdot)$  the modulus of continuity of  $Y$  on  $[0, T]$  defined by

$$\omega_T(Y, \theta) = \sup \left\{ d^{\mathcal{J}}(Y(s), Y(u)), (u, s) \in [t, t + \theta], 0 \leq t \leq t + \theta \leq T \right\}, \quad (3.8)$$

for all  $\theta \in [0, T]$ .

**Proposition 3.3.1.** *There exists a constant  $C$  independent of  $\delta$ , depending only on*

$$\left( T, \max_{i \in \{1 \dots I\}} \{ |\sigma_i|_{(0, +\infty)}, |b_i|_{(0, +\infty)} \} \right),$$

such that

$$\mathbb{E}^{\mathbb{P}} \left[ |x^\delta|_{(0, T)}^2 \right] \leq C \left( 1 + \mathbb{E}^{\mathbb{P}} \left[ x_0^2 \right] + \delta^2 \right), \quad (3.9)$$

$$\forall \theta \in (0, T], \quad \mathbb{E}^{\mathbb{P}} \left[ \omega_T(X^\delta, \theta)^2 \right] \leq C \left( \delta^2 + \theta \ln \left( \frac{2T}{\theta} \right) \right). \quad (3.10)$$

*Proof.* Let  $t \in [0, T]$ . We use the stopping times  $(\tau_n^\delta)$  defined in (3.5) and we introduce the following random times

$$\bar{\tau}_t^\delta := \inf \left\{ \tau_n^\delta, \tau_n^\delta \geq t \right\} \quad \text{and} \quad \underline{\tau}_t^\delta := \sup \left\{ \tau_n^\delta; \tau_n^\delta \leq t \right\}.$$

Using (3.5) and (3.7), we have

$$\begin{aligned} |x^\delta(t)|^2 &\leq 3 \left( |x^\delta(\underline{\tau}_t^\delta)|^2 + \left| \int_{\underline{\tau}_t^\delta}^t b_{i^\delta(s)}(x^\delta(s)) ds \right|^2 + \left| \int_{\underline{\tau}_t^\delta}^t \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) \right|^2 \right) \leq \\ &3 \left( |x_0|^2 + \delta^2 + T^2 \max_{i \in \{1 \dots I\}} \{ |b_i|_{(0, +\infty)}^2 \} + 2 \sup_{t \in [0, T]} \left| \int_0^t \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) \right|^2 \right), \quad \mathbb{P} \text{ a.s.} \end{aligned}$$

Using assumption  $(\mathcal{H})$  and Doob-Meyer inequality, we get that there exists a constant  $C$  independent of  $\delta$ , depending only on  $\left( T, \max_{i \in \{1 \dots I\}} \{ |\sigma_i|_{(0, +\infty)}, |b_i|_{(0, +\infty)} \} \right)$  such that

$$\mathbb{E}^{\mathbb{P}} \left[ |x^\delta|_{(0, T)}^2 \right] \leq C \left( 1 + \mathbb{E}^{\mathbb{P}} \left[ x_0^2 \right] + \delta^2 \right).$$

Let  $\theta \in (0, T]$ , and  $(t, s) \in [0, T]^2$ , such that  $s \leq t$  and  $|t - s| \leq \theta$ . We define the process  $(V^\delta(t))_{0 \leq t \leq T}$  by

$$V^\delta(t) = \int_0^t b_{i^\delta(s)}(x^\delta(s)) ds + \int_0^t \sigma_{i^\delta(s)}(x^\delta(s)) dW(s), \quad \mathbb{P} \text{ a.s.}$$

We have:

$$\begin{aligned} & \text{if } \sum_{s \leq u \leq t} \mathbf{1}_{\{x^\delta(s-) = 0\}} = 0, \text{ then } d^{\mathcal{J}}(X^\delta(s), X^\delta(t)) \leq \omega_T(V^\delta, \theta), \\ & \text{if } \sum_{s \leq u \leq t} \mathbf{1}_{\{x^\delta(s-) = 0\}} \geq 1, \text{ then } d^{\mathcal{J}}(X^\delta(s), X^\delta(t)) \leq \\ & d^{\mathcal{J}}(X^\delta(s), X^\delta(\bar{\tau}_s^\delta -)) + d^{\mathcal{J}}(X^\delta(\underline{\tau}_t^\delta -), X^\delta(\underline{\tau}_t^\delta +)) + d^{\mathcal{J}}(X^\delta(\underline{\tau}_t^\delta +), X^\delta(t)) \leq \delta + 2\omega_T(V^\delta, \theta), \\ & \mathbb{P} \text{ a.s.} \end{aligned}$$

Using Theorem 1 of [13], and assumption  $(\mathcal{H})$ , we get that there exists a constant  $C$  independent of  $\delta$ , depending only on  $(T, \max_{i \in \{1 \dots I\}} \{|\sigma_i|_{(0, +\infty)}, |b_i|_{(0, +\infty)}\})$ , such that

$$\mathbb{E}^{\mathbb{P}} \left[ \omega_T(V^\delta, \theta)^2 \right] \leq C\theta \ln \left( \frac{2T}{\theta} \right).$$

Therefore

$$\mathbb{E}^{\mathbb{P}} \left[ \omega_T(X^\delta, \theta)^2 \right] \leq C \left( \delta^2 + \theta \ln \left( \frac{2T}{\theta} \right) \right).$$

This completes the proof. □

Now we have the necessary tools to prove the C-tightness of the sequence  $(X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T}$ .

**Theorem 3.3.2.** *The sequence  $(X^\delta(t) = (x^\delta(t), i^\delta(t)))_{0 \leq t \leq T}$  is C-tight.*

*Proof.* Let  $\delta > 0$ ,  $\theta > 0$ ,  $\eta > 0$  and  $h > 0$ . Using Proposition 3.3.1 and Markov inequality, there exists a constant  $C$ , independent of  $\delta$  and depending only on  $T$  and on



$\max_{i \in \{1 \dots I\}} \{|\sigma_i|_{(0,+\infty)}, |b_i|_{(0,+\infty)}\}$ , such that

$$\begin{aligned} \mathbb{P}\left( |x^\delta|_{(0,T)} > h \right) &\leq C \frac{\sqrt{1 + \mathbb{E}^\mathbb{P} \left[ x_0^2 \right] + \delta^2}}{h}, \\ \mathbb{P}\left( \omega_T(X^\delta, \theta) \geq \eta \right) &\leq C \frac{\sqrt{\delta^2 + \theta \ln \left( \frac{2T}{\theta} \right)}}{\eta}. \end{aligned}$$

Therefore we get

$$\begin{aligned} \lim_{h \rightarrow +\infty} \limsup_{\delta \rightarrow 0} \mathbb{P}\left( |x^\delta|_{(0,T)} > h \right) &= 0 ; \\ \lim_{\eta \rightarrow 0+} \limsup_{\theta \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathbb{P}\left( \omega_T(X^\delta, \theta) \geq \eta \right) &= 0. \end{aligned}$$

Theorem 3.21 and Proposition 3.26 of [23] then imply the tightness of  $\left( X^\delta(t) = (x^\delta(t), i^\delta(t)) \right)_{0 \leq t \leq T}$  as  $\delta$  tends to 0.  $\square$

From Prokhorov's Theorem, there is a subsequence, denoted in the same way, such that

$$\mathbb{P}_{X^\delta} \xrightarrow[\delta \rightarrow 0]{law} \mu,$$

where  $\mu$  is a probability measure defined on  $\mathcal{C}^\mathcal{J}([0, T])$ .

### 3.3.3 Identification of the limit law

The aim of this subsection is to identify the law  $\mu$ .

To this end, we transform the weak convergence of  $X^\delta$  into a strong one using Skorokhod's representation Theorem. Using Tikhonov's Theorem, we know that  $\mathcal{U}$  is compact, therefore it is easy to get that the random variable  $(x_0, X^\delta, W, Z)$  is precompact for the weak topology. There exists a probability space  $(E, \mathcal{E}, \mu)$  and random variables  $\left( y_0^\delta, Y^\delta = (y^\delta, j^\delta), B^\delta, A^\delta \right)$  and  $\left( y_0, Y = (y, j), B, A \right)$ , taking their values in the following Polish spaces

$$\left( [0, +\infty) \times \mathcal{D}_\delta^\mathcal{J}([0, T]) \times \mathcal{C}([0, T]) \times \mathcal{U}, |\cdot| + d_{sko}^\mathcal{J} + |\cdot|_{(0,T)} + d^\mathcal{U} \right),$$

and respectively

$$\left( [0, +\infty) \times \mathcal{C}^{\mathcal{J}}([0, T]) \times \mathcal{C}([0, T]) \times \mathcal{U}, \quad |\cdot| + d_{(0,T)}^{\mathcal{J}} + |\cdot|_{(0,T)} + d^{\mathcal{U}} \right),$$

such that

$$\begin{aligned} \forall \delta > 0, \quad (y_0^\delta, Y^\delta, B^\delta, A^\delta) &\stackrel{law}{=} (x_0, X^\delta, W, Z) \\ (y_0, Y, B, A) &\stackrel{law}{=} (x_0, X, W, Z) \end{aligned}$$

and

$$|y_0^\delta - y_0| + d_{sko}^{\mathcal{J}}(Y^\delta, Y) + |B^\delta - B|_{(0,T)} + d^U(A^\delta, A) \xrightarrow[\delta \rightarrow 0]{\mu\text{-a.s.}} 0. \quad (3.11)$$

To make easier the reading, in the rest of these Chapter, we denote by  $\left(N^\delta(t)\right)_{0 \leq t \leq T}$  the following process valued in  $\mathbb{N}$ , whose paths are defined by

$$N^\delta(t) = \sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s-) = 0\}}, \quad \mu \text{ a.s.} \quad (3.12)$$

**Proposition 3.3.3.** *For any  $\delta > 0$ , the process  $\left(y^\delta(t), j^\delta(t)\right)_{0 \leq t \leq T}$  satisfies*

$$\begin{aligned} \left( \begin{aligned} y^\delta(t) &= y_0^\delta + \int_0^t b_{j^\delta(s)}(y^\delta(s)) ds + \int_0^t \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) + \delta N^\delta(t) \\ j^\delta(t) &= A_{N^\delta(t)}^\delta \end{aligned} \right)_{0 \leq t \leq T}, \quad \mu \text{ a.s.} \end{aligned} \quad (3.13)$$

*Proof.* Let  $i \in \{1 \dots I\}$ , and  $\gamma > 0$ . We define  $(b_i^\gamma, \sigma_i^\gamma)$  by

$$\begin{aligned} \sigma_i^\gamma(x) &= \sigma_i(x) \mathbf{1}_{x > \gamma} + \frac{\sigma_i(\gamma)}{\gamma} x \mathbf{1}_{0 \leq x \leq \gamma}, \quad \text{if } x \in [0, +\infty), \\ b_i^\gamma(x) &= b_i(x) \mathbf{1}_{x > \gamma} + \frac{b_i(\gamma)}{\gamma} x \mathbf{1}_{0 \leq x \leq \gamma}, \quad \text{if } x \in [0, +\infty). \end{aligned}$$

Therefore for all  $i \in \{1 \dots I\}$ ,  $(b_i^\gamma, \sigma_i^\gamma) \in W^{1,\infty}((0, +\infty))$  and satisfies the following continuity condition at the junction point

$$\forall i, j \in \{1 \dots I\}, \quad i \neq j, \quad \sigma_i^\gamma(0) = \sigma_j^\gamma(0), \quad b_i^\gamma(0) = b_j^\gamma(0).$$

**Step 1.** For  $n \geq 0$ , we start by proving that the following maps

$$\begin{aligned} \rho &:= \begin{cases} \mathcal{D}_\delta^{\mathcal{J}}([0, T]) \rightarrow \mathcal{D}^{\mathbb{N}}([0, T]) \\ (x(\cdot), i(\cdot)) \mapsto \left( \sum_{0 \leq s \leq t} \mathbf{1}_{\{x(s) - x(s-) = \delta\}} \right)_{0 \leq t \leq T} \end{cases}, \\ \kappa &:= \begin{cases} \mathcal{U} \times \mathcal{D}_\delta^{\mathcal{J}}([0, T]) \rightarrow \mathcal{D}^{\mathcal{U}}([0, T]) \\ (Z, (x(\cdot), i(\cdot))) \mapsto \left( Z \sum_{0 \leq s \leq t} \mathbf{1}_{\{x(s) - x(s-) = \delta\}} \right)_{0 \leq t \leq T} \end{cases}, \\ \psi_{n, \gamma} &:= \begin{cases} \mathcal{C}([0, T]) \times \mathcal{D}_\delta^{\mathcal{J}}([0, T]) \rightarrow \mathcal{D}^{\mathbb{R}}([0, T]) \\ (w(\cdot), (x(\cdot), i(\cdot))) \mapsto \left( \sum_{j=0}^{2^n-1} \sigma_{i(\frac{jt}{2^n})}^\gamma(x(\frac{jt}{2^n})) (w_{\frac{(j+1)t}{2^n}} - w_{\frac{jt}{2^n}}) \right)_{0 \leq t \leq T} \end{cases}, \\ \phi_{n, \gamma} &:= \begin{cases} \mathcal{D}_\delta^{\mathcal{J}}([0, T]) \rightarrow \mathcal{D}^{\mathbb{R}}([0, T]) \\ (x(\cdot), i(\cdot)) \mapsto \left( x(0) + \frac{1}{n} \sum_{j=0}^{n-1} b_{i(\frac{jt}{2^n})}^\gamma(x(\frac{jt}{2^n})) \right)_{0 \leq t \leq T} \end{cases}, \end{aligned}$$

are continuous.

To this end let  $(Z^k, w^k(\cdot), (x^k(\cdot), i^k(\cdot)))$  converge to  $(Z, w(\cdot), (x(\cdot), i(\cdot)))$  in  $\mathcal{U} \times \mathcal{C}[0, T] \times \mathcal{D}_\delta^{\mathcal{J}}([0, T])$ .

Using Remark [3.2.1](#), we get that for all  $\varepsilon > 0$ , there exists  $k_\varepsilon \in \mathbb{N}$ , and a sequence  $\lambda_k$  valued in  $\Lambda$  (the set of continuous increasing homeomorphism from  $[0, T]$  to itself), such that for all  $k \geq k_\varepsilon$

$$\begin{aligned} & \sum_{n \geq 0} \frac{|Z_n^k - Z_n|}{2^n} + |w_k - w|_{(0, T)} + \\ & |\lambda_k - Id|_{(0, T)} \vee d_{(0, T)}^{\mathcal{J}} \left( (x^k \circ \lambda_k(\cdot), i^k \circ \lambda_k(\cdot)), (x(\cdot), i(\cdot)) \right) \leq \varepsilon. \end{aligned} \quad (3.14)$$

**Substep 1.1**  $\rho$  and  $\kappa$  are continuous:

Let  $\varepsilon < \frac{\delta}{4} \wedge \frac{1}{2}$ . We have

$$\frac{\delta}{2} \geq x_k(\lambda_k(s)) - x_k(\lambda_k(s-)) - (x(s) - x(s-)) \geq -\frac{\delta}{2}. \quad (3.15)$$

for each  $s \in [0, T]$ .

Since, from Proposition [3.2.2](#),  $\mathcal{D}_\delta^{\mathcal{J}}([0, T])$  is closed,  $(x(\cdot), i(\cdot)) \in \mathcal{D}_\delta^{\mathcal{J}}([0, T])$ . Using [\(3.15\)](#)

and the assumption  $\varepsilon < \delta/4$ , we then have

$$\left\{ s \in [0, t], \quad x(s) - x(s-) = \delta \right\} = \left\{ s \in [0, t], \quad x_k(\lambda_k(s)) - x_k(\lambda_k(s)-) = \delta \right\}$$

which implies that

$$\sum_{0 \leq s \leq \lambda_k(t)} \mathbf{1}_{\{x_k(s) - x_k(s-) = \delta\}} = \sum_{0 \leq s \leq t} \mathbf{1}_{\{x(s) - x(s-) = \delta\}}, \quad (3.16)$$

and

$$|\lambda_k - Id|_{(0,T)} \vee \sup_{0 \leq t \leq T} d^{\mathbb{N}} \left( \sum_{0 \leq s \leq \lambda_k(t)} \mathbf{1}_{\{x_k(s) - x_k(s-) = \delta\}}, \sum_{0 \leq s \leq t} \mathbf{1}_{\{x(s) - x(s-) = \delta\}} \right) \leq \varepsilon,$$

namely  $\rho$  is continuous.

On the other hand we get using (3.14) and (3.16), that for all  $t \in [0, T]$

$$\begin{aligned} & d^{\mathcal{U}} \left( \kappa \left( Z^k, (x^k(\cdot), i^k(\cdot)) \right) \left( \lambda_k(t) \right), \kappa \left( Z, (x(\cdot), i(\cdot)) \right) \left( t \right) \right) = \\ & \sum_{n \geq 0} \frac{\left| Z_n^k \mathbf{1}_{\left\{ \sum_{0 \leq s \leq \lambda_k(t)} \mathbf{1}_{\{x_k(s) - x_k(s-) = \delta\}} = n \right\}} - Z_n \mathbf{1}_{\left\{ \sum_{0 \leq s \leq t} \mathbf{1}_{\{x(s) - x(s-) = \delta\}} = n \right\}} \right|}{2^n} \leq \varepsilon. \end{aligned}$$

Therefore

$$|\lambda_k - Id|_{(0,T)} \vee \sup_{0 \leq t \leq T} d^{\mathcal{U}} \left( \kappa \left( Z^k, (x^k(\cdot), i^k(\cdot)) \right) \left( \lambda_k(t) \right), \kappa \left( Z, (x(\cdot), i(\cdot)) \right) \left( t \right) \right) \leq \varepsilon,$$

and  $\kappa$  is continuous.

**Substep 1.2** Let us show now that  $\psi_{n,\gamma}$  and  $\phi_{n,\gamma}$  are continuous.

Let  $j \in \{0, \dots, 2^n - 1\}$ , and  $t \in [0, T]$ , we write

$$\begin{aligned} \left| \sigma_{i(\frac{j\lambda_k(t)}{2^n})}^{\gamma} \left( x_k \left( \frac{j\lambda_k(t)}{2^n} \right) \right) - \sigma_{i(\frac{jt}{2^n})}^{\gamma} \left( x \left( \frac{jt}{2^n} \right) \right) \right| & \leq \left| \sigma_{i(\frac{j\lambda_k(t)}{2^n})}^{\gamma} \left( x_k \left( \frac{j\lambda_k(t)}{2^n} \right) \right) - \sigma_{i(\frac{j\lambda_k(t)}{2^n})}^{\gamma} (0) \right| + \\ & \left| \sigma_{i(\frac{j\lambda_k(t)}{2^n})}^{\gamma} (0) - \sigma_{i(\frac{jt}{2^n})}^{\gamma} (0) \right| + \left| \sigma_{i(\frac{jt}{2^n})}^{\gamma} (0) - \sigma_{i(\frac{jt}{2^n})}^{\gamma} \left( x \left( \frac{jt}{2^n} \right) \right) \right|. \end{aligned}$$

Since

$$\forall i, j \in \{1 \dots I\}, \quad i \neq j, \quad \sigma_i^\gamma(0) = \sigma_j^\gamma(0),$$

we get from assumption  $(\mathcal{H})$  and [\(3.14\)](#)

$$|\sigma_{i(\frac{j\lambda_k(t)}{2^n})}^\gamma(x_k(\frac{j\lambda_k(t)}{2^n})) - \sigma_{i(\frac{jt}{2^n})}^\gamma(x(\frac{jt}{2^n}))| \leq \max_{i \in \{1 \dots I\}} \|\partial_x \sigma_i^\gamma\|_{L^\infty((0, +\infty))} \varepsilon.$$

Therefore, simple computations allows us to write that there exists a constant  $C > 0$ , independent of  $k$ , depending only on the data

$$(n, \max_{i \in \{1 \dots I\}} \|\sigma_i^\gamma\|_{L^\infty((0, +\infty))}, \max_{i \in \{1 \dots I\}} \|\partial_x \sigma_i^\gamma\|_{L^\infty((0, +\infty))}, |w|_{(0, T)}),$$

such that

$$|\lambda_k - Id|_{(0, T)} \vee \sup_{0 \leq t \leq T} \left( \psi_{n, \gamma}(w^k, (x^k, i^k)) \circ \lambda_k(t), \psi_{n, \gamma}(w(t), (x(t), i(t))) \right) \leq C\varepsilon.$$

We conclude then that  $\psi_{n, \gamma}$  is continuous.

With the same arguments used for the continuity of  $\psi_{n, \gamma}$ , we can show then that  $\phi_{n, \gamma}$  is continuous.

**Step 2.**

We have now from the results of **Step 1** the mains tools to prove Proposition [3.3.3](#)

Let  $(g, h) \in \mathcal{C}_b(\mathcal{D}([0, T]), \mathbb{R}) \times \mathcal{C}_b(\mathcal{D}^\mu([0, T]), \mathbb{R})$ . Since  $\rho, \kappa, \psi_{n, \gamma}, \phi_{n, \gamma}$  are continuous, we have

$$\begin{aligned} & \mathbb{E}^\mathbb{P} \left[ g \left( x^\delta(\cdot) - \left( \phi_{n, \gamma}(X^\delta)(\cdot) + \psi_{n, \gamma}(W, X^\delta)(\cdot) + \delta\rho(X^\delta)(\cdot) \right) \right) \right] \\ = & \mathbb{E}^\mu \left[ g \left( y^\delta(\cdot) - \left( \phi_{n, \gamma}(Y^\delta)(\cdot) + \psi_{n, \gamma}(B^\delta, Y^\delta)(\cdot) + \delta\rho(Y^\delta)(\cdot) \right) \right) \right], \end{aligned} \quad (3.17)$$

and

$$\mathbb{E}^\mathbb{P} \left[ h \left( i^\delta(\cdot) - \kappa(Z, X^\delta)(\cdot) \right) \right] = \mathbb{E}^\mu \left[ h \left( j^\delta(\cdot) - \kappa(A^\delta, Y^\delta)(\cdot) \right) \right]. \quad (3.18)$$

From assumption  $(\mathcal{H})$  and Burkholder-Davis-Gundy inequality we have, up to a subse-

quence  $n_k$

$$\left| \psi_{n_k, \gamma}(W, X^\delta)(\cdot) - \int_0^\cdot \sigma_{i^\delta(s)}^\gamma(x^\delta(s)) dW(s) \right|_{(0, T)} \xrightarrow[n_k \rightarrow +\infty]{\mathbb{P}\text{-a.s.}} 0,$$

and

$$\left| \psi_{n_k, \gamma}(B^\delta, Y^\delta)(\cdot) - \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) \right|_{(0, T)} \xrightarrow[n_k \rightarrow +\infty]{\mu\text{-a.s.}} 0.$$

Still using Burkholder-Davis-Gundy inequality and  $(\mathcal{H})$ , there exists a constant  $C$  independent of  $\gamma$ , depending only on  $(\max_{i \in \{1 \dots I\}} \|\sigma_i\|_{L^\infty((0, \infty))})$ , such that

$$\mathbb{E}^\mathbb{P} \left[ \left| \int_0^\cdot \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) - \int_0^\cdot \sigma_{i^\delta(s)}^\gamma(x^\delta(s)) dW(s) \right|_{(0, T)} \right] \leq C \mathbb{E}^\mathbb{P} \left[ \int_0^T \mathbf{1}_{\{x^\delta(s) \leq \gamma\}} ds \right],$$

and

$$\mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) \right|_{(0, T)} \right] \leq C \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y^\delta(s) \leq \gamma\}} ds \right].$$

Using that  $y^\delta(\cdot)$  and  $x^\delta(\cdot)$  have the same law, we have for each  $s \in [0, T]$

$$\mu(y^\delta(s) = 0) = \mathbb{P}(x^\delta(s) = 0) = 0.$$

We get therefore from the dominated convergence's Theorem

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \mathbb{E}^\mathbb{P} \left[ \left| \int_0^\cdot \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) - \int_0^\cdot \sigma_{i^\delta(s)}^\gamma(x^\delta(s)) dW(s) \right|_{(0, T)} \right] &= \\ \lim_{\gamma \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) \right|_{(0, T)} \right] &= 0. \end{aligned}$$

Therefore up to a subsequence  $\gamma_p$

$$\left| \int_0^\cdot \sigma_{i^\delta(s)}(x^\delta(s)) dW(s) - \int_0^\cdot \sigma_{i^\delta(s)}^{\gamma_p}(x^\delta(s)) dW(s) \right|_{(0, T)} \xrightarrow[\gamma_p \rightarrow 0]{\mathbb{P}\text{-a.s.}} 0,$$

and

$$\left| \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j^\delta(s)}^{\gamma_p}(y^\delta(s)) dB^\delta(s) \right|_{(0, T)} \xrightarrow[\gamma_p \rightarrow 0]{\mu\text{-a.s.}} 0.$$

Using the same arguments, we can show that up to another subsequence  $(\gamma_{p'}, n_{k'})$

$$\begin{aligned} \lim_{\gamma_{p'} \rightarrow 0} \limsup_{n_{k'} \rightarrow +\infty} \left| \int_0^\cdot b_{i^\delta}(s)(x^\delta(s))dW(s) - \phi_{n_{k'}, \gamma_{p'}}(X^\delta)(\cdot) \right|_{(0, T)} &\xrightarrow{\mathbb{P}\text{-a.s.}} 0, \\ \lim_{\gamma_{p'} \rightarrow 0} \limsup_{n_{k'} \rightarrow +\infty} \left| \int_0^\cdot b_{j^\delta}(s)(y^\delta(s))dB^\delta(s) - \phi_{n_{k'}, \gamma_{p'}}(Y^\delta)(\cdot) \right|_{(0, T)} &\xrightarrow{\mu\text{-a.s.}} 0. \end{aligned}$$

On the other hand using that

$$\sum_{0 \leq s \leq \cdot} \mathbf{1}_{\{x(s) - x(s-) = \delta\}} = \sum_{0 \leq s \leq \cdot} \mathbf{1}_{\{x(s-) = 0\}}, \quad \mathbb{P} \text{ a.s.},$$

$$\forall s \in [0, T], \quad \mu\left(y^\delta(s) - y^\delta(s-) = \delta, y^\delta(s-) > 0\right) = \mathbb{P}\left(x^\delta(s) - x^\delta(s-) = \delta, x^\delta(s-) > 0\right) = 0,$$

and applying Lebesgue's Theorem, we get from [\(3.17\)](#), that for any  $g \in \mathcal{C}_b(\mathcal{D}([0, T]), \mathbb{R})$

$$\begin{aligned} &\lim_{\gamma_{p'} \rightarrow 0} \limsup_{n_{k'} \rightarrow +\infty} \mathbb{E}^\mathbb{P} \left[ g\left(x^\delta(\cdot) - \left(\phi_{n_{k'}, \gamma_{p'}}(X^\delta)(\cdot) + \psi_{n_{k'}, \gamma_{p'}}(W, X^\delta)(\cdot) + \delta\rho(X^\delta)(\cdot)\right)\right) \right] \\ = &\mathbb{E}^\mathbb{P} \left[ g\left(x^\delta(\cdot) - x_0 - \int_0^\cdot b_{i^\delta}(s)(x^\delta(s))ds - \int_0^\cdot \sigma_{i^\delta}(s)(x^\delta(s))dW(s) + \delta\left(\sum_{0 \leq s \leq \cdot} \mathbf{1}_{\{x^\delta(s-) = 0\}}\right)\right) \right], \end{aligned}$$

and

$$\begin{aligned} &\lim_{\gamma_{p'} \rightarrow 0} \limsup_{n_{k'} \rightarrow +\infty} \mathbb{E}^\mu \left[ g\left(y^\delta(\cdot) - \left(\phi_{n_{k'}, \gamma_{p'}}(Y^\delta)(\cdot) + \psi_{n_{k'}, \gamma_{p'}}(B^\delta, Y^\delta)(\cdot) + \delta\rho(Y^\delta)(\cdot)\right)\right) \right] \\ = &\mathbb{E}^\mu \left[ g\left(y^\delta(\cdot) - y_0^\delta - \int_0^\cdot b_{j^\delta}(s)(y^\delta(s))ds - \int_0^\cdot \sigma_{j^\delta}(s)(y^\delta(s))dB^\delta(s) + \delta\left(\sum_{0 \leq s \leq \cdot} \mathbf{1}_{\{y^\delta(s-) = 0\}}\right)\right) \right]. \end{aligned}$$

Therefore for any  $g \in \mathcal{C}_b(\mathcal{D}([0, T]), \mathbb{R})$

$$\begin{aligned} &\mathbb{E}^\mathbb{P} \left[ g\left(x^\delta(\cdot) - x_0 - \int_0^\cdot b_{i^\delta}(s)(x^\delta(s))ds - \int_0^\cdot \sigma_{i^\delta}(s)(x^\delta(s))dW(s) + \delta\left(\sum_{0 \leq s \leq \cdot} \mathbf{1}_{\{x^\delta(s-) = 0\}}\right)\right) \right] \\ = &\mathbb{E}^\mu \left[ g\left(y^\delta(\cdot) - y_0^\delta - \int_0^\cdot b_{j^\delta}(s)(y^\delta(s))ds - \int_0^\cdot \sigma_{j^\delta}(s)(y^\delta(s))dB^\delta(s) + \delta\left(\sum_{0 \leq s \leq \cdot} \mathbf{1}_{\{y^\delta(s-) = 0\}}\right)\right) \right]. \end{aligned}$$

Considering a sequence  $g_n$  of  $\mathcal{C}_b(\mathcal{D}([0, T]), \mathbb{R})$ , uniformly bounded, converging to  $x \rightarrow$

$\mathbf{1}_{\{x=0\}}$  in the pointwise sense, we get that

$$\mu\left(\left(y^\delta(t) = y_0^\delta - \int_0^t b_{j^\delta(s)}(y^\delta(s))ds - \int_0^t \sigma_{j^\delta(s)}(y^\delta(s))dB^\delta(s) + \delta\left(\sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s-)=0\}}\right)\right)_{0 \leq t \leq T}\right) = 1.$$

Using the same arguments and the continuity of  $\kappa$  we also infer that

$$\mu\left(\left(j^\delta(t) = A^\delta \sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s-)=0\}}\right)_{0 \leq t \leq T}\right) = 1,$$

that completes the proof of Proposition [3.3.3](#).  $\square$

**Remark 3.3.4.** As a consequence of [\(3.13\)](#) and Proposition [3.3.1](#), it is easy to get that there exists a constant  $C$  independent of  $\delta$ , depending only on

$$\left(T, \max_{i \in \{1 \dots I\}} \{|\sigma_i|_{(0,+\infty)}, |b_i|_{(0,+\infty)}\}, \mathbb{E}^\mu[y_0^2]\right),$$

such that

$$\begin{aligned} \mathbb{E}^\mu \left[ |y^\delta|_{(0,T)}^2 \right] &\leq C \left( 1 + \delta^2 \right), \\ \mathbb{E}^\mu \left[ |\delta N^\delta(\cdot)|_{(0,T)}^2 \right] &\leq C \left( 1 + \delta^2 \right). \end{aligned}$$

For simplicity, we denote by  $(y_0^\delta, Y^\delta = (y^\delta, j^\delta), B^\delta, A^\delta)$  the subsequence converging almost surely to  $(y_0, Y = (y, j), B, A)$ . Using Proposition [3.3.3](#), we know that the process  $Y^\delta = (y^\delta, j^\delta)$  satisfies

$$\left\{ \begin{aligned} y^\delta(\cdot) &= y_0^\delta + \int_0^{\gamma_1^\delta \wedge \cdot} b_{j^\delta(s)}(y^\delta(s))ds + \int_0^{\gamma_1^\delta \wedge \cdot} \sigma_{j^\delta(s)}(y^\delta(s))dB^\delta(s) \\ &+ \sum_{n \geq 1} \mathbf{1}_{\{\gamma_n^\delta \leq \cdot < \gamma_{n+1}^\delta\}} \left( \delta + \int_{\gamma_n^\delta}^{\gamma_{n+1}^\delta \wedge \cdot} b_{j^\delta(s)}(y^\delta(s))ds + \int_{\gamma_n^\delta}^{\gamma_{n+1}^\delta \wedge \cdot} \sigma_{j^\delta(s)}(y^\delta(s))dB^\delta(s) \right) \quad , \quad \mu \text{ a.s.} \end{aligned} \right. \quad (3.19)$$

$$j^\delta(\cdot) = \sum_{n \geq 0} A_n^\delta \mathbf{1}_{[\gamma_n^\delta, \gamma_{n+1}^\delta)}(\cdot)$$



where we have defined recursively the following stopping times

$$\gamma_0^\delta = 0, \quad \gamma_{n+1}^\delta = \inf \left\{ T \geq t > \gamma_n^\delta : y^\delta(t) = 0 \right\}, \quad \mu \text{ a.s.} \quad (3.20)$$

We denote by  $(\mathcal{G}_t^\delta)_{0 \leq t \leq T}$  the right continuous and complete filtration generated by  $\left( Y^\delta(t) = (y^\delta(t), j^\delta(t)) \right)_{0 \leq t \leq T}$ . We also denote by  $(\mathcal{G}_t)_{0 \leq t \leq T}$  the right continuous and complete filtration generated by the process  $\left( Y(t) = (y(t), j(t)) \right)_{0 \leq t \leq T}$ . We now state some properties of the process  $\left( Y(t) = (y(t), j(t)) \right)_{0 \leq t \leq T}$ .

**Proposition 3.3.5.** *The process  $\left( Y^\delta(t) = (y^\delta(t), j^\delta(t)) \right)_{0 \leq t \leq T}$  converges uniformly to  $\left( Y(t) = (y(t), j(t)) \right)_{0 \leq t \leq T}$ , namely:*

$$d_{(0,T)}^{\mathcal{J}}(Y^\delta, Y) \xrightarrow[\delta \rightarrow 0]{\mu\text{-a.s.}} 0, \quad (3.21)$$

and therefore

$$|y^\delta - y|_{(0,T)} \xrightarrow[\delta \rightarrow 0]{\mu\text{-a.s.}} 0. \quad (3.22)$$

*Proof.* From (3.11) and Remark 3.2.1, there exists a (random) change of time  $\lambda_\delta$  such that

$$|\lambda_\delta - Id|_{(0,T)} \vee d_{(0,T)}^{\mathcal{J}}(Y_\delta \circ \lambda_\delta, Y) \xrightarrow[\delta \rightarrow 0]{\mu\text{-a.s.}} 0.$$

We then have

$$\begin{aligned} d_{(0,T)}^{\mathcal{J}}(Y^\delta, Y) &\leq d_{(0,T)}^{\mathcal{J}}(Y^\delta \circ \lambda_\delta \circ \lambda_\delta^{-1}, Y \circ \lambda_\delta^{-1}) + d_{(0,T)}^{\mathcal{J}}(Y \circ \lambda_\delta^{-1}, Y) \\ &\leq d_{(0,T)}^{\mathcal{J}}(Y^\delta \circ \lambda_\delta, Y) + d_{(0,T)}^{\mathcal{J}}(Y \circ \lambda_\delta^{-1}, Y), \quad \mu \text{ a.s.} \end{aligned}$$

From Theorem 3.3.2,  $Y$  is continuous and hence uniformly continuous on  $[0, T]$ . Therefore we get

$$d_{(0,T)}^{\mathcal{J}}(Y^\delta, Y) \xrightarrow[\delta \rightarrow 0]{\mu\text{-a.s.}} 0,$$

and consequently

$$|y^\delta - y|_{(0,T)} \xrightarrow[\delta \rightarrow 0]{\mu\text{-a.s.}} 0.$$

□

**Lemma 3.3.6.** *Fix  $M > 0$ . There exists a constant  $C > 0$ , independent of  $\delta$ , depending only on the data*

$$\left( T, M, \max_{i \in \{1 \dots I\}} |b_i|_{(0,+\infty)}, \max_{i \in \{1 \dots I\}} |\sigma_i|_{(0,+\infty)}, y(0) \right),$$

introduced in assumption  $(\mathcal{H})$ , such that

$$\mathbb{E}^\mu \left[ \exp(My^\delta(T)) \right] \leq C \exp(M\delta). \quad (3.23)$$

*Proof.* We define the following map  $\phi$  by

$$\phi := \begin{cases} [0, +\infty) \rightarrow \mathbb{R} \\ x \mapsto \exp(Mx) - Mx - 1 \end{cases}.$$

Let  $k \geq 0$ , we introduce the following stopping times

$$\begin{aligned} \theta_k &:= \inf \{ s \in [t, T], \quad x(s) \geq k \}, \\ \underline{\gamma}_T^\delta &:= \sup \left\{ \gamma_n^\delta; \gamma_n^\delta \leq T \right\}, \end{aligned}$$

where the sequence  $(\gamma_n^\delta)_{n \geq 0}$  is defined in (3.20). Hence, applying Ito's formula in the following interval  $(\underline{\gamma}_T^\delta, T)$ , using (3.19) and Remark 3.3.4, we get

$$\begin{aligned} \mathbb{E}^\mu \left[ \exp(My^\delta(T \wedge \theta_k)) \right] &= \left( \exp(M\delta) - M\delta \right) \mathbf{1}_{\{\underline{\gamma}_T^\delta > 0\}} + \left( \exp(My^\delta(0)) - My^\delta(0) \right) \mathbf{1}_{\{\underline{\gamma}_T^\delta = 0\}} \\ &+ \mathbb{E}^\mu \left[ My^\delta(T \wedge \theta_k) \right] + \mathbb{E}^\mu \left[ \int_{\underline{\gamma}_T^\delta}^{T \wedge \theta_k} \left( \frac{1}{2} \sigma_i^2(x(u)) \partial_{x,x} \phi(x(u)) + \right. \right. \\ &\left. \left. b_i(x(u)) \partial_x \phi(x(u)) \right) du \right] \leq C \exp(M\delta) \left( 1 + \mathbb{E}^\mu \left[ \int_{\underline{\gamma}_T^\delta}^{T \wedge \theta_k} \exp(My^\delta(u)) du \right] \right), \end{aligned}$$

where  $C$  is a constant depending only on

$$\left( T, M, \max_{i \in \{1 \dots I\}} |b_i|_{(0,+\infty)}, \max_{i \in \{1 \dots I\}} |\sigma_i|_{(0,+\infty)}, y(0) \right),$$

since from: (3.11)  $|y^\delta(0) - y(0)| \rightarrow 0$ .

Hence sending  $k \rightarrow +\infty$ , we get using monotone convergence's Theorem, and Fubini's Theorem

$$\begin{aligned} \mathbb{E}^\mu \left[ \exp(My^\delta(T)) \right] &\leq C \exp(M\delta) \left( 1 + \mathbb{E}^\mu \left[ \int_{\frac{\delta}{2}}^T \exp(My^\delta(u)) du \right] \right) \\ &\leq C \exp(M\delta) \left( 1 + \mathbb{E}^\mu \left[ \int_0^T \exp(My^\delta(u)) du \right] \right) \leq C \exp(M\delta) \left( 1 + \int_0^T \mathbb{E}^\mu \left[ \exp(My^\delta(u)) \right] du \right). \end{aligned}$$

We conclude finally using Gronwall's Lemma to the following measurable map

$$\rho := \begin{cases} [0, T] \rightarrow \mathbb{R} \\ s \mapsto \mathbb{E}^\mu \left[ \exp(My^\delta(s)) \right] \end{cases}.$$

□

We now study the behavior of the jump part of  $y^\delta(\cdot)$ . To this end, we estimate the time spent by  $y(\cdot)$  at the junction point, and we state the following proposition that will be useful in the sequel.

**Proposition 3.3.7.** *We have*

$$\lim_{\varepsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y^\delta(s) \leq \varepsilon\}} ds \right] = 0, \quad (3.24)$$

and therefore

$$\mu \left( \int_0^T \mathbf{1}_{\{y(s)=0\}} ds = 0 \right) = 1.$$

*Proof.* Let  $\varepsilon > \delta$ , and  $\beta^\varepsilon \in \mathcal{C}([0, +\infty), \mathbb{R}_+)$  satisfying

$$\forall x \geq 2\varepsilon, \quad \beta_\varepsilon(x) = 0, \quad \forall x \geq 0, \quad \mathbf{1}_{\{x \leq \varepsilon\}} \leq \beta_\varepsilon(x) \leq 1. \quad (3.25)$$

For all  $i \in \{1 \dots I\}$ , we define  $u_i^\varepsilon \in \mathcal{C}^2([0, +\infty))$  as the unique solution of the following

ordinary second order differential equation

$$\begin{cases} \frac{\sigma_i(x)^2}{2} \partial_{x,x} u_i^\varepsilon(x) + b_i(x) \partial_x u_i^\varepsilon(x) = \beta^\varepsilon(x), & \text{if } x \in (0, +\infty), \\ \partial_x u_i^\varepsilon(0) = 0, \\ u_i^\varepsilon(0) = 0. \end{cases} \quad (3.26)$$

For each  $i \in \{1 \dots I\}$ , the solution is given by

$$u_i^\varepsilon(x) = \int_0^x \exp\left(-\int_0^z \frac{2b_i(u)}{\sigma_i^2(u)} du\right) \int_0^z \frac{2\beta_i^\varepsilon(u)}{\sigma_i^2(u)} \exp\left(\int_0^u \frac{2b_i(r)}{\sigma_i^2(r)} dr\right) dudz.$$

By the assumption on  $\beta_\varepsilon$ , and using assumption  $(\mathcal{H})$ , we get that there exists a constant  $M > 0$  independent of  $\varepsilon$  and  $i$ , depending only on the data  $(c, \max_{i \in \{1 \dots I\}} |b_i|_{L^\infty((0, +\infty))})$  introduced in assumption  $(\mathcal{H})$ , such that

$$\forall x \geq 0, \quad 0 \leq \partial_x u^\varepsilon(x) \leq 2\varepsilon \exp(Mx), \quad 0 \leq u^\varepsilon(x) \leq \frac{2\varepsilon}{M} (\exp(Mx) - 1) \quad (3.27)$$

We define the function  $u^\varepsilon$  on the junction by

$$u^\varepsilon := \begin{cases} \mathcal{J} \mapsto \mathbb{R} \\ (x, i) \rightarrow u_i^\varepsilon(x) \end{cases}.$$

We recall the definition of the stopping times [\(3.20\)](#)

$$\gamma_0^\delta = 0, \quad \gamma_{n+1}^\delta = \inf \left\{ T \geq t > \gamma_n^\delta : y^\delta(t) = 0 \right\}, \quad \mu \text{ a.s.}$$

We write then

$$\begin{aligned} & u_{j^\delta(T)}^\varepsilon(y^\delta(T)) - u_{j^\delta(0)}^\varepsilon(y^\delta(0)) = \\ & \left( \sum_{n \geq 0} u_{j^\delta(\gamma_{n+1}^\delta -)}^\varepsilon(y^\delta(\gamma_{n+1}^\delta -)) - u_{j^\delta(\gamma_n^\delta +)}^\varepsilon(y^\delta(\gamma_{n+1}^\delta +)) \right) \end{aligned} \quad (3.28)$$

$$+ \left( \sum_{n \geq 0} u_{j^\delta(\gamma_n^\delta +)}^\varepsilon(y^\delta(\gamma_n^\delta +)) - u_{j^\delta(\gamma_n^\delta -)}^\varepsilon(y^\delta(\gamma_n^\delta -)) \right) \quad \mu \text{ a.s.} \quad (3.29)$$

For [\(3.28\)](#), we apply Itô's formula and use [\(4.12\)](#) on each interval  $(\gamma_n^\delta, \gamma_{n+1}^\delta -)$  to get

(after an argument of localization with stopping times)

$$\begin{aligned}
& \mathbb{E}^\mu \left[ \sum_{n \geq 0} u_{j^\delta(\gamma_{n+1}^\delta-)}^\varepsilon \left( y^\delta(\gamma_{n+1}^\delta-) \right) - u_{j^\delta(\gamma_n^\delta+)}^\varepsilon \left( y^\delta(\gamma_n^\delta+) \right) \right] \\
&= \mathbb{E}^\mu \left[ \int_0^T \beta_{j^\delta(s)}^\varepsilon(y^\delta(s)) ds + \int_0^T \sigma_{j^\delta(s)}(y^\delta(s)) \partial_x u_{j^\delta(s)}^\varepsilon(y^\delta(s)) dB^\delta(s) \right] \\
&= \mathbb{E}^\mu \left[ \int_0^T \beta_{j^\delta(s)}^\varepsilon(y^\delta(s)) ds \right].
\end{aligned}$$

We now study the term (3.29). Using the continuity of  $u^\varepsilon$  at the junction point, namely

$$\forall (i, j) \in \{1 \dots I\}^2, \quad u_i^\varepsilon(0) = u_j^\varepsilon(0) = 0,$$

and (3.27), we get

$$\begin{aligned}
\mathbb{E}^\mu \left[ |(3.29)| \right] &\leq \max_{i \in \{1 \dots I\}} \left\{ |u_i^\varepsilon(\delta) - u_i^\varepsilon(0)| \right\} \mathbb{E}^\mu \left[ N^\delta(T) \right] \leq \\
&2\varepsilon \exp(M\delta) \delta \mathbb{E}^\mu \left[ N^\delta(T) \right].
\end{aligned}$$

Using Remark 3.3.4, it is easy to get that there exists a constant  $K > 0$  such that

$$\mathbb{E}^\mu \left[ \delta N^\delta(T) \right] \leq K(1 + \delta^2)^{1/2},$$

hence

$$\begin{aligned}
\mathbb{E}^\mu \left[ \int_0^T \beta_{j^\delta(s)}^\varepsilon(y^\delta(s)) ds \right] &\leq \mathbb{E}^\mu \left[ u_{j^\delta(T)}^\varepsilon \left( y^\delta(T) \right) - u_{j^\delta(0)}^\varepsilon \left( y^\delta(0) \right) \right] + 2\varepsilon \exp(M\delta) \delta \mathbb{E}^\mu \left[ N^\delta(T) \right] \\
&\leq \frac{2\varepsilon}{M} \mathbb{E}^\mu \left[ (\exp(My^\delta(T)) - 1) \right] + 2\varepsilon \exp(M\delta) K(1 + \delta^2)^{1/2}.
\end{aligned}$$

We get then using Lemma 3.3.6

$$\lim_{\varepsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \int_0^T \beta_{j^\delta(s)}^\varepsilon(y^\delta(s)) ds \right] = 0.$$

Finally, using that  $x \rightarrow \mathbf{1}_{\{0 \leq x < \varepsilon\}}$  is lower semicontinuous,  $x \rightarrow \mathbf{1}_{\{0 \leq x < \varepsilon\}} \leq \beta^\varepsilon(x)$ , Proposition 3.3.5, and Fatou Lemma we get

$$\mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y(s)=0\}} ds \right] \leq \lim_{\varepsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y^\delta(s) \leq \varepsilon\}} ds \right] = 0.$$

Therefore

$$\mu\left(\int_0^T \mathbf{1}_{\{y(s)=0\}} ds = 0\right) = 1,$$

that completes the proof.  $\square$

**Proposition 3.3.8.** *The following map*

$$\psi := \begin{cases} \mathcal{D}^{\mathbb{R}^+}([0, T]) \times \mathcal{D}_c^{\mathbb{R}^+}([0, T]) \rightarrow \mathbb{R}_+, \\ (y, l) \mapsto \int_0^T \mathbf{1}_{\{y(s)>0\}} dl(s) \end{cases},$$

(where  $\mathcal{D}_c^{\mathbb{R}^+}([0, T])$  is the set of càdlàg nondecreasing maps valued in  $\mathbb{R}_+$ ), is lower semi-continuous at any  $(y, l) \in \mathcal{C}([0, T], \mathbb{R}_+) \times (\mathcal{C}([0, T], \mathbb{R}_+) \cap \mathcal{D}_c^{\mathbb{R}^+}([0, T]))$ .

*Proof.* Let  $(y_k(\cdot))$  be a sequence in  $\mathcal{D}^{\mathbb{R}^+}([0, T])$  converging to  $y(\cdot) \in \mathcal{C}([0, T], \mathbb{R}_+)$ , and  $(l_k(\cdot))$  in  $\mathcal{D}_c^{\mathbb{R}^+}([0, T])$  converging to  $l(\cdot) \in (\mathcal{C}([0, T], \mathbb{R}_+) \cap \mathcal{D}_c^{\mathbb{R}^+}([0, T]))$ .

Using Proposition 3.3.5, we get that  $(y_k(\cdot))$  converges uniformly to  $y(\cdot)$ , and with the same arguments  $(l_k(\cdot))$  converges uniformly to  $l(\cdot)$ . Let  $p \geq 0$  and  $\phi_p \in \mathcal{C}([0, +\infty))$  a continuous sequence uniformly bounded, converging from below to  $x \rightarrow \mathbf{1}_{\{x>0\}}$  in the pointwise sense, as  $p \rightarrow +\infty$ . We write then

$$\begin{aligned} & \left| \int_0^T \phi_p(y^k(u)) dl_k(u) - \int_0^T \phi_p(y(u)) dl(u) \right| \leq \int_0^T \left| \phi_p(y_k(u)) - \phi_p(y(u)) \right| dl_k(u) \\ & + \left| \int_0^T \phi_p(y(u)) dl_k(u) - \int_0^T \phi_p(y(u)) dl(u) \right| \leq \left| \phi_p(y_k(\cdot)) - \phi_p(y(\cdot)) \right|_{(0, T)} l_k(T) \\ & \quad + \left| \int_0^T \phi_p(y(u)) dl_k(u) - \int_0^T \phi_p(y(u)) dl(u) \right|. \end{aligned}$$

Hence we get using that  $l_k(T)$  is uniformly bounded in  $k$ , and that  $l_k$  converges weakly to  $l$

$$\forall p \geq 0, \quad \lim_{k \rightarrow +\infty} \int_0^T \phi_p(y^k(u)) dl_k(u) = \int_0^T \phi_p(y(u)) dl(u).$$

Finally writing

$$\int_0^T \mathbf{1}_{\{y_k(u)>0\}} dl_k(u) \geq \int_0^T \phi_p(y_k(u)) dl_k(u),$$

we get

$$\forall p \geq 0, \quad \liminf_{k \rightarrow +\infty} \int_0^T \mathbf{1}_{\{y_k(u) > 0\}} dl_k(u) \geq \int_0^T \phi_p(y(u)) dl(u),$$

and hence using Lebesgue's Theorem

$$\liminf_{k \rightarrow +\infty} \int_0^T \mathbf{1}_{\{y_k(u) > 0\}} dl_k(u) \geq \limsup_{p \rightarrow +\infty} \int_0^T \phi_p(y(u)) dl(u) = \int_0^T \mathbf{1}_{\{y(u) > 0\}} dl(u).$$

That completes the proof.  $\square$

We are now ready to identify the SDE satisfied by the limit process  $(y(t))_{0 \leq t \leq T}$ .

**Theorem 3.3.9.** *There exists a nondecreasing process  $(l(t))_{0 \leq t \leq T}$  such that  $(y(t))_{0 \leq t \leq T}$  satisfies the stochastic differential equation*

$$\begin{cases} y(0) = y_0 \\ dy(t) = b_{j(t)}(y(t))dt + \sigma_{j(t)}(j(t))dB(t) + dl(t), \quad 0 \leq t \leq T \end{cases} \quad \mu \text{ a.s.}, \quad (3.30)$$

with

$$\begin{cases} l(0) = 0 \\ \int_0^t \mathbf{1}_{\{y(s) > 0\}} dl(s) = 0, \quad 0 \leq t \leq T \end{cases} \quad \mu \text{ a.s.}$$

Moreover the process  $(N^\delta(t))_{0 \leq t \leq T}$  converges uniformly to  $(l(t))_{0 \leq t \leq T}$ , namely

$$\left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0,T)} \xrightarrow{\delta \rightarrow 0} 0, \quad \mu \text{ a.s.} \quad (3.31)$$

*Proof. Step 1 :* We start to prove that up to a subsequence again denoted  $\delta$

$$\begin{aligned} \lim_{\delta \rightarrow 0} \left| y^\delta(0) + \int_0^\cdot b_{j^\delta(s)}(y^\delta(s))ds + \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s))dB^\delta(s) \right. \\ \left. - \left( y_0 + \int_0^\cdot b_{j(s)}(y(s))ds + \int_0^\cdot \sigma_{i(s)}(s, y(s))dB(s) \right) \right|_{(0,T)} = 0. \end{aligned}$$

For all  $\gamma > 0$ , we introduce the map  $(x, i) \rightarrow \sigma_i^\gamma(x)$  defined by

$$\sigma_i^\gamma(x) = \sigma_i(x)\mathbf{1}_{x > \gamma} + \frac{\sigma_i(\gamma)}{\gamma}x\mathbf{1}_{0 \leq x \leq \gamma}$$

for  $(x, i) \in \mathcal{J}$ .

For  $n \geq 0$ , we define the functions  $\psi_{n,\gamma}$  from  $\mathcal{C}[0, T] \times \mathcal{D}^{\mathcal{J}}([0, T])$  to  $\mathcal{D}([0, T])$  by

$$\psi_{n,\gamma}\left(w(\cdot), (x(\cdot), i(\cdot))\right) = \left( \sum_{j=0}^{2^n-1} \sigma_{i(\frac{jT}{2^n})}^{\gamma}\left(\frac{jT}{2^n}, x\left(\frac{jT}{2^n}\right)\right) (w_{\frac{(j+1)T}{2^n} \wedge t} - w_{\frac{jT}{2^n} \wedge t}) \right)_{0 \leq t \leq T},$$

for all  $(w(\cdot), (x(\cdot), i(\cdot))) \in \mathcal{C}[0, T] \times \mathcal{D}^{\mathcal{J}}([0, T])$ . We write then

$$\left| \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j(s)}(y(s)) dB(s) \right|_{(0,T)} \leq \quad (3.32)$$

$$\left| \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j^\delta(s)}^{\gamma}(y^\delta(s)) dB^\delta(s) \right|_{(0,T)} + \quad (3.33)$$

$$\left| \int_0^\cdot \sigma_{j^\delta(s)}^{\gamma}(y^\delta(s)) dB^\delta(s) - \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) \right|_{(0,T)} + \quad (3.34)$$

$$\left| \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) - \psi_{n,\gamma}(B, (y, i))(\cdot) \right|_{(0,T)} + \quad (3.35)$$

$$\left| \psi_{n,\gamma}(B, (y, i))(\cdot) - \int_0^\cdot \sigma_{j(s)}^{\gamma}(y(s)) dB(s) \right|_{(0,T)} + \quad (3.36)$$

$$\left| \int_0^\cdot \sigma_{j(s)}^{\gamma}(y(s)) dB(s) - \int_0^\cdot \sigma_{j(s)}(y(s)) dB(s) \right|_{(0,T)} . \quad (3.37)$$

Now we prove that (3.32) tends to 0 as  $\delta$  goes to 0 in  $L^2(E)$ . Using Burkholder-Davis-Gundy inequality and assumption  $(\mathcal{H})$ , there exists a constant  $C > 0$ , independent of  $\delta, n, \gamma$ , such that for (3.33) and (3.37)

$$\begin{aligned} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j^\delta(s)}^{\gamma}(y^\delta(s)) dB^\delta(s) \right|_{(0,T)}^2 \right] &\leq \\ & C \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y^\delta(s) \leq \gamma\}} ds \right] , \\ \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j(s)}^{\gamma}(y(s)) dB(s) - \int_0^\cdot \sigma_{j(s)}(y(s)) dB(s) \right|_{(0,T)}^2 \right] &\leq \\ & C \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y(s) \leq \gamma\}} ds \right] . \end{aligned}$$

Using then Proposition 3.3.5 and Proposition 3.3.7, we get

$$\lim_{\gamma \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ (3.33) \right] \leq \lim_{\gamma \rightarrow 0} \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y(s) \leq \gamma\}} ds \right] = \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{y(s)=0\}} ds \right] = 0,$$

and with the same arguments  $\lim_{\gamma \rightarrow 0} \mathbb{E}^\mu \left[ (3.37) \right] = 0$ . For (3.34), we get that there exists



a constant  $C(\gamma)$  independent of  $\delta, n$ , depending only on  $\gamma$  such that

$$\mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) - \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) \right|_{(0,T)}^2 \right] \leq \\ C(\gamma) \mathbb{E}^\mu \left[ \sup \left\{ \left| \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) - \sigma_{j^\delta(t)}^\gamma(y^\delta(t)) \right|^2, \quad |t-s| \leq \frac{1}{2^n}, \quad (t,s) \in [0,T] \right\} \right].$$

Using Proposition [3.3.5](#) and the continuity of  $\sigma_i^\gamma$ , we deduce then by Lebesgue's Theorem

$$\limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) - \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) \right|_{(0,T)}^2 \right] \leq \\ C(\gamma) \mathbb{E}^\mu \left[ \sup \left\{ \left| \sigma_{j(s)}^\gamma(y(s)) - \sigma_{j(t)}^\gamma(y(t)) \right|^2, \quad |t-s| \leq \frac{1}{2^n}, \quad (t,s) \in [0,T] \right\} \right].$$

Finally using the continuity of the paths of the process  $Y(\cdot)$  and once again the continuity of  $\sigma^\gamma$  we get

$$\lim_{n \rightarrow +\infty} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) - \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) \right|_{(0,T)}^2 \right] = 0.$$

Similarly we have  $\lim_{n \rightarrow +\infty} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \text{[\(3.36\)](#) \right] = 0$ . Finally for [\(3.35\)](#) we get that there exists a constant  $C(\gamma)$  independent of  $(n, \delta)$  such that

$$\left| \psi_{n,\gamma}(B^\delta, (y^\delta, i^\delta))(\cdot) - \psi_{n,\gamma}(B, (y, i))(\cdot) \right|_{(0,T)} \leq \\ 2^{n+1} C(\gamma) \left( |B^\delta - B|_{(0,T)} + |\sigma_{j^\delta(\cdot)}^\gamma(y^\delta(\cdot)) - \sigma_{j(\cdot)}^\gamma(y(\cdot))|_{(0,T)} \right).$$

Using once more Proposition [3.3.5](#), assumption  $(\mathcal{H})$ , and the continuity of  $\sigma^\gamma$  we get by Lebesgue's Theorem that  $\lim_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \text{[\(3.35\)](#) \right] = 0$ . Therefore we have

$$\lim_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j^\delta(s)}^\gamma(y^\delta(s)) dB^\delta(s) - \int_0^\cdot \sigma_{j(s)}^\gamma(y(s)) dB(s) \right|_{(0,T)}^2 \right] \leq \\ \lim_{\gamma \rightarrow 0} \limsup_{n \rightarrow +\infty} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left( \text{[\(3.32\)](#)} + \text{[\(3.33\)](#)} + \text{[\(3.34\)](#)} + \text{[\(3.35\)](#)} + \text{[\(3.36\)](#)} + \text{[\(3.37\)](#)} \right) \right] = 0.$$

With the same arguments, we can show that

$$\lim_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot b_{j^\delta(s)}(y^\delta(s)) ds - \int_0^\cdot b_{j(s)}(y(s)) ds \right|_{(0,T)} \right] = 0.$$

Choosing again a subsequence  $\delta$ , to get an almost convergence sense, we have  $\mu$  almost

surely

$$\lim_{\delta \rightarrow 0} \left| \begin{aligned} & y^\delta(0) + \int_0^\cdot b_{j^\delta(s)}(y^\delta(s))ds + \int_0^\cdot \sigma_{j^\delta(s)}(y^\delta(s))dB^\delta(s) \\ & - \left( y(0) + \int_0^\cdot b_{j(s)}(y(s))ds + \int_0^\cdot \sigma_{j(s)}(y(s))dB(s) \right) \Big|_{(0,T)} = 0. \end{aligned} \right.$$

We recall that from Proposition [3.3.3](#), we have

$$\mu \left( \left( y^\delta(t) = y_0^\delta + \int_0^t b_{j^\delta(s)}(y^\delta(s))ds + \int_0^t \sigma_{j^\delta(s)}(y^\delta(s))dB^\delta(s) + \delta \left( \sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s^-)=0\}} \right) \right)_{0 \leq t \leq T} \right) = 1.$$

The continuity of both processes  $(y(t))_{0 \leq t \leq T}$  and  $(\int_0^t b_{j(s)}(y(s))ds + \int_0^t \sigma_{j(s)}(y(s))dB(s))_{0 \leq t \leq T}$ , implies that  $(\delta(\sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s^-)=0\}}))_{0 \leq t \leq T}$  tends almost surely uniformly to a continuous process  $(l(t))_{0 \leq t \leq T}$ . Since  $(\delta(\sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s^-)=0\}}))_{0 \leq t \leq T}$  has nondecreasing trajectories, it implies that  $(l(t))_{0 \leq t \leq T}$  is a nondecreasing process.

**Step 2 :** We now prove that

$$\mu \left( \begin{cases} l(0) = 0 \\ \left( \int_0^t \mathbf{1}_{\{y(s)>0\}} dl(s) \right)_{0 \leq t \leq T} = 0 \end{cases} \right) = 1,$$

namely  $(l(t))_{0 \leq t \leq T}$  can increase only when the process  $(Y(t) = (y(t), j(t)))_{0 \leq t \leq T}$  reaches the junction point 0. The fact that  $l(0) = 0$ ,  $\mu$  almost surely is obvious. Since the paths of  $y^\delta(\cdot)$  and  $\delta N^\delta(\cdot)$  converge uniformly to the continuous processes  $y(\cdot)$  and  $l(\cdot)$ , we have using Proposition [3.3.8](#) that

$$\liminf_{\delta \rightarrow 0} \int_0^T \mathbf{1}_{\{y^\delta(s)>0\}} d\delta N^\delta(s) \geq \int_0^T \mathbf{1}_{\{y(s)>0\}} dl(s), \quad \mu \text{ a.s.}$$

Hence, using the lower semi continuity of the real map  $x \mapsto \mathbf{1}_{\{x>0\}}$ , we obtain

$$\liminf_{\delta \rightarrow 0} \mathbf{1} \left\{ \int_0^T \mathbf{1}_{\{y^\delta(s)>0\}} d\delta N^\delta(s) > 0 \right\} \geq \mathbf{1} \left\{ \int_0^T \mathbf{1}_{\{y(s)>0\}} dl(s) > 0 \right\}, \quad \mu \text{ a.s.}$$

Therefore, using Fatou's Lemma we have

$$\begin{aligned} \liminf_{\delta \rightarrow 0} \mu \left( \int_0^T \mathbf{1}_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right) &\geq \mu \left( \liminf_{\delta \rightarrow 0} \int_0^T \mathbf{1}_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right) \\ &\geq \mu \left( \int_0^T \mathbf{1}_{\{y(s) > 0\}} dl(s) > 0 \right), \end{aligned}$$

and using that  $\mu \left( \int_0^T \mathbf{1}_{\{y^\delta(s) > 0\}} d\delta N^\delta(s) > 0 \right) = 0$ , we get finally

$$\mu \left( \left( \int_0^t \mathbf{1}_{\{y(s) > 0\}} dl(s) \right)_{0 \leq t \leq T} = 0 \right) = 1.$$

We conclude then that  $\left( y(t) \right)_{0 \leq t \leq T}$  satisfies the differential stochastic equation [\(3.30\)](#), with the required conditions on  $\left( l(t) \right)_{0 \leq t \leq T}$ .  $\square$

### 3.4 Itô's formula and local time estimate at the junction

#### 3.4.1 Itô's formula.

The stochastic differential equation [\(3.30\)](#) satisfied by the process  $\left( y(t) \right)_{0 \leq t \leq T}$ , does not completely characterize the process, since the randomness due to  $\left( j(t) \right)_{0 \leq t \leq T}$  is hidden in the process  $\left( l(t) \right)_{0 \leq t \leq T}$ . In order to better characterize the behavior of the process  $\left( Y(t) = (y(t), j(t)) \right)_{0 \leq t \leq T}$ , we test its dynamic against regular maps  $f$  defined on the junction. This is the aim of the following Itô's formula.

**Theorem 3.4.1.** *Let  $f \in \mathcal{C}_b^{0,1}(\mathcal{J}_T) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}}_T)$ . We have*

$$\begin{aligned} \left( f(t, Y(t)) - f(0, Y(0)) \right) &= \int_0^t L(f)(s, Y(s)) ds \\ + \int_0^t \partial_y f_{j(s)}(s, y(s)) \sigma_{j(s)}(y(s)) dB(s) &+ \sum_{i=1}^I \alpha_i \int_0^t \partial_y f_i(s, 0) dl(s) \Big)_{0 \leq t \leq T}, \quad \mu \text{ a.s.} \end{aligned} \quad (3.38)$$

The Dynkin operator  $L(f)$  is defined by

$$L(f)(t, (y, j)) = \partial_t f_j(t, y) + \partial_y f_j(t, y) b_j(y) + \frac{1}{2} \partial_{y,y}^2 f_j(t, y) \sigma_j^2(y),$$

for any  $(t, (y, j)) \in \mathcal{J}_T$ .

*Proof.* Suppose first that  $f$  does not depend on time:

$$\forall (t, (y, j)) \in \mathcal{J}_T, \quad f_j(t, y) = f_j(y).$$

We first show that

$$\left( f(Y(t)) - f(Y(0)) = \int_0^t L(f)(y(s), j(s)) ds + \int_0^t \partial_y f_{j(s)}(y(s)) \sigma_{j(s)}(y(s)) dB(s) + \sum_{i=1}^I \alpha_i \partial_y f_i(0) l(t) \right)_{0 \leq t \leq T} \quad \mu \text{ a.s.}$$

We recall the definition of the stopping times [\(3.20\)](#)

$$\gamma_0^\delta = 0, \quad \gamma_{n+1}^\delta = \inf \left\{ t > \gamma_n^\delta : y^\delta(t) = 0 \right\}, \quad \mu \text{ a.s.}$$

Let  $t \in [0, T]$ , we have then the following decomposition

$$f_{j^\delta(t)}(y^\delta(t)) - f_{j^\delta(0)}(y^\delta(0)) = \sum_{n \geq 0} f_{j^\delta(t \wedge \gamma_{n+1}^\delta -)}(y^\delta(t \wedge \gamma_{n+1}^\delta -)) - f_{j^\delta(t \wedge \gamma_n^\delta +)}(y^\delta(t \wedge \gamma_n^\delta +)) \quad (3.39)$$

$$+ \sum_{n \geq 0} f_{j^\delta(t \wedge \gamma_n^\delta +)}(y^\delta(t \wedge \gamma_n^\delta +)) - f_{j^\delta(t \wedge \gamma_n^\delta -)}(y^\delta(t \wedge \gamma_n^\delta -)), \quad \mu \text{ a.s.} \quad (3.40)$$

For [\(3.39\)](#), we apply Itô's formula on each interval  $(t \wedge \gamma_n^\delta +, t \wedge \gamma_{n+1}^\delta -)$ , to get

$$\begin{aligned} & \sum_{n \geq 0} f_{j^\delta(t \wedge \gamma_{n+1}^\delta -)}(y^\delta(t \wedge \gamma_{n+1}^\delta -)) - f_{j^\delta(t \wedge \gamma_n^\delta +)}(y^\delta(t \wedge \gamma_n^\delta +)) \\ &= \int_0^t L(f)(y^\delta(s), j^\delta(s)) ds + \int_0^t \sigma_{j^\delta(s)}(y^\delta(s)) \partial_y f_{j^\delta(s)}(y^\delta(s)) dB^\delta(s), \quad \mu \text{ a.s.} \end{aligned}$$

Using the same arguments as in the proof of Theorem [3.3.9](#), it is easy to show that up to a subsequence  $\delta$

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left| \int_0^t L(f)(y^\delta(s), j^\delta(s)) ds + \int_0^t \sigma_{j^\delta(s)}(y^\delta(s)) \partial_x f_{j^\delta(s)}(y^\delta(s)) dB^\delta(s) \right. \\ & \left. - \left( \int_0^t L(f)(y(s), j(s)) ds + \int_0^t \sigma_{j(s)}(y(s)) \partial_y f_{j(s)}(y(s)) dB(s) \right) \right|_{(0, T)} = 0, \quad \mu \text{ a.s.} \end{aligned}$$

Using the continuity of  $f$  at the junction point, namely  $\forall (i, j) \in \{1 \dots I\}^2, \quad f_i(0) =$

$f_j(0) = f(0)$ , we get for (3.40) (where we recall that  $N^\delta(t) = \sum_{0 \leq s \leq t} \mathbf{1}_{\{y^\delta(s-, \omega)=0\}}$ ,  $\mu$  a.s.)

$$\begin{aligned} & \sum_{n \geq 0} f_{j^\delta(t \wedge \gamma_n^\delta+)} \left( y^\delta(t \wedge \gamma_n^\delta+) \right) - f_{j^\delta(t \wedge \gamma_n^\delta-)} \left( y^\delta(t \wedge \gamma_n^\delta-) \right), \\ &= \left( \sum_{n=0}^{N^\delta(t)} f_{j^\delta(\gamma_n^\delta)}(\delta) - f_{j^\delta(\gamma_n^\delta)}(0) \right) \mathbf{1}_{\{N^\delta(t) > 0\}} \\ &= \left( \sum_{n=0}^{N^\delta(t)} \delta \partial_y f_{j^\delta(\gamma_n^\delta)}(0) + \delta \varepsilon_{j(\gamma_n^\delta)}(\delta) \right) \mathbf{1}_{\{N^\delta(t) > 0\}}, \quad \mu \text{ a.s.} \end{aligned}$$

where for each  $i \in \{1 \dots I\}$ ,  $\lim_{\delta \rightarrow 0} \varepsilon_i(\delta) = 0$ .

From Remark 3.3.4, we know that there exists a constant  $C > 0$  independent of  $\delta$ , such that

$$\mathbb{E}^\mu \left[ \delta N^\delta(\cdot) \right] \leq C(1 + \delta^2)^{1/2}, \quad (3.41)$$

hence

$$\mathbb{E}^\mu \left[ \left| \sum_{n=0}^{N^\delta(\cdot)} \delta \varepsilon_{j(\gamma_n^\delta)}(\delta) \right|_{(0,T)} \right] \leq C(1 + \delta^2)^{1/2} \max_{i \in \{1 \dots I\}} |\varepsilon_i(\delta)|,$$

which means that, up to a subsequence  $\delta$ , we have

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \left| \sum_{n \geq 0} f_{j^\delta(\cdot \wedge \gamma_n^\delta+)} \left( y^\delta(\cdot \wedge \gamma_n^\delta+) \right) - f_{j^\delta(\cdot \wedge \gamma_n^\delta-)} \left( y^\delta(\cdot \wedge \gamma_n^\delta-) \right) \right. \\ & \quad \left. - \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \delta \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} \right) \mathbf{1}_{\{N^\delta(\cdot) > 0\}} \right|_{(0,T)} = 0, \quad \mu \text{ a.s.} \end{aligned}$$

We recall that from assumption  $(\mathcal{H})$ ,  $(\sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}})_{n \geq 0}$  are i.i.d. Using the law of large numbers, there exists  $\mathcal{A} \in \mathcal{E}$  with  $\mu(\mathcal{A}) = 1$ , such that for all  $\omega \in \mathcal{A}$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^n \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_k^\delta)=i\}} = \sum_{i=1}^I \alpha_i \partial_x f_i(0),$$

since for all  $\delta$ , for all  $i \in \{1 \dots I\}$ ,  $\mathbb{E}^\mu[\mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}}] = \alpha_i$ .

On the other hand we have

$$\mathcal{A} = \bigcap_{\eta \in \mathbb{Q}_+^*} \bigcup_{K \in \mathbb{N}^*} \mathcal{A}_{\eta, K},$$

where

$$\mathcal{A}_{\eta, K} = \left\{ \omega \in \mathcal{A}, \forall n \geq K, \left| \frac{1}{n} \sum_{k=0}^n \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} - \sum_{i=1}^I \alpha_i \partial_y f_i(0) \right| \leq \eta \right\}.$$

Given  $\eta \in \mathbb{Q}_+^*$  and  $K \in \mathbb{N}^*$ , we have

$$\left| \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \delta \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} \right) \mathbf{1}_{\{N^\delta(\cdot) > 0\}} - l(\cdot) \left( \sum_{i=1}^I \alpha_i \partial_y f_i(0) \right) \right|_{(0, T)} \leq \quad (3.42)$$

$$\left| \frac{l(\cdot) - \delta N^\delta(\cdot)}{N^\delta(\cdot)} \left( \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} \right) \mathbf{1}_{\{N^\delta(\cdot) > 0\}} \right|_{(0, T)} + \quad (3.43)$$

$$\begin{aligned} & \left| l(\cdot) \right|_{(0, T)} \left| \left( \frac{1}{N^\delta(\cdot)} \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} - \left( \sum_{i=1}^I \alpha_i \partial_y f_i(0) \right) \right) \right| \times \quad (3.44) \\ & \left( \mathbf{1}_{\{0 < N^\delta(\cdot) \leq K\}} + \mathbf{1}_{\{N^\delta(\cdot) > K\}} \left( \mathbf{1}_{\{\mathcal{A}_{\eta, K}\}} + \mathbf{1}_{\{\overline{\mathcal{A}_{\eta, K}}\}} \right) \right) \Big|_{(0, T)}, \quad \mu \text{ a.s.} \end{aligned}$$

For (3.43), we have the following upper bound

$$(3.43) \leq \max_{i \in \{1 \dots I\}} \left\{ \left| \partial_x f_i(0) \right| \right\} \left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0, T)}.$$

Recall that from (3.31) in Theorem 3.3.9, we know that

$$\left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0, T)} \xrightarrow{\delta \rightarrow 0} 0, \quad \mu \text{ a.s.},$$

hence using Lebesgue's Theorem

$$\lim_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ (3.43) \right] = 0.$$

We now consider the term (3.44), we have

$$\begin{aligned} \forall t \in [0, T], \quad 0 \leq l(t) \mathbf{1}_{\{0 < N^\delta(t) \leq K\}} &= (l(t) - \delta N^\delta(t) + \delta N^\delta(t)) \mathbf{1}_{\{0 < N^\delta(t) \leq K\}} \\ &\leq \left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0, T)} + \delta K, \quad \mu \text{ a.s.}, \end{aligned}$$

and then using Lebesgue's Theorem and once again (3.31), we have  
 (where  $C = C(\max_{i \in \{1, \dots, I\}} \left\{ \left| \partial_x f_i(0) \right| \right\})$  is a constant independent of  $\delta$ )

$$\begin{aligned} \lim_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left| l(\cdot) \left( \frac{1}{N^\delta(\cdot)} \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} - \left( \sum_{i=1}^I \alpha_i \partial_y f_i(0) \right) \right) \mathbf{1}_{\{0 < N^\delta(\cdot) \leq K\}} \right|_{(0,T)} \right] \\ \leq C \lim_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ \left| \delta N^\delta(\cdot) - l(\cdot) \right|_{(0,T)} + \delta K \right] = 0. \end{aligned}$$

Finally we have the following upper bounds

$$\begin{aligned} \mathbb{E}^\mu \left[ \left| l(\cdot) \left( \frac{1}{N^\delta(\cdot)} \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} - \left( \sum_{i=1}^I \alpha_i \partial_y f_i(0) \right) \right) \mathbf{1}_{\{N^\delta(\cdot) > K\}} \mathbf{1}_{\{\mathcal{A}_{\eta,K}\}} \right|_{(0,T)} \right] \\ \leq \eta \mathbb{E}^\mu \left[ \left| l(\cdot) \right|_{(0,T)} \right], \\ \mathbb{E}^\mu \left[ \left| l(\cdot) \left( \frac{1}{N^\delta(\cdot)} \sum_{n=0}^{N^\delta(\cdot)} \sum_{i=1}^I \partial_y f_i(0) \mathbf{1}_{\{j^\delta(\gamma_n^\delta)=i\}} - \left( \sum_{i=1}^I \alpha_i \partial_y f_i(0) \right) \right) \mathbf{1}_{\{N^\delta(\cdot) > K\}} \mathbf{1}_{\{\overline{\mathcal{A}_{\eta,K}}\}} \right|_{(0,T)} \right] \\ \leq C \mathbb{E}^\mu \left[ \left| l(\cdot) \right|_{(0,T)} \mathbf{1}_{\{\overline{\mathcal{A}_{\eta,K}}\}} \right]. \end{aligned}$$

Therefore

$$\lim_{\eta \rightarrow 0} \limsup_{K \rightarrow +\infty} \limsup_{\delta \rightarrow 0} \mathbb{E}^\mu \left[ (3.42) \right] = 0.$$

We finally get, that up to a subsequence  $\delta$

$$\begin{aligned} \lim_{\delta \rightarrow 0} \left| f_{j^\delta(\cdot)}(y^\delta(\cdot)) - f_{j^\delta(0)}(y^\delta(0)) - \left( \int_0^\cdot L(f)(s, (y(s), j(s))) ds + \right. \right. \\ \left. \left. \int_0^\cdot \partial_y f_{j(s)}(y(s)) \sigma_{j(s)}(y(s)) dB(s) + \sum_{i=1}^I \alpha_i \partial_y f_i(0) l(\cdot) \right) \right|_{(0,T)} = 0, \quad \mu \text{ a.s.} \end{aligned}$$

This completes the proof of **Step 1**.

**Step 2.** We generalize the result obtained in **Step 1**, for  $f \in \mathcal{C}_b^{0,1}(\mathcal{J}_T) \cap \mathcal{C}_b^{1,2}(\overset{\circ}{\mathcal{J}}_T)$ . To this end, fix  $t \in [0, T]$  and set  $(t_k^n := kt/n)_{0 \leq k \leq n}$ . We have

$$f_{j(t)}(t, y(t)) - f_{j(0)}(0, y(0)) = \sum_{k=1}^n f_{j(t_k^n)}(t_k^n, y(t_k^n)) - f_{j(t_k^n)}(t_{k-1}^n, y(t_k^n)) \quad (3.45)$$

$$+ \sum_{k=1}^n f_{j(t_k^n)}(t_{k-1}^n, y(t_k^n)) - f_{j(t_{k-1}^n)}(t_{k-1}^n, y(t_{k-1}^n)). \quad (3.46)$$

Using the continuity of  $f$  at the junction point, the interior regularity of the  $(\partial_t f_i)_{i \in \{1 \dots I\}}$ , the continuity of the paths of  $y(\cdot)$  and that  $\int_0^T \mathbf{1}_{\{y(s)=0\}} ds = 0$   $\mu$  a.s, we get

$$\left| \sum_{k=1}^n f_{j(t_k^n)}(t_k^n, y(t_k^n)) - f_{j(t_{k-1}^n)}(t_{k-1}^n, y(t_k^n)) - \int_0^{\cdot} \partial_t f_{j(s)}(s, y(s)) ds \right|_{(0,T)} \xrightarrow{n \rightarrow +\infty} 0, \quad \mu \text{ a.s.}$$

Using the result obtained in **Step 1**, we get for [\(3.46\)](#)

$$\begin{aligned} & \sum_{k=1}^n f_{j(t_{k-1}^n)}(t_{k-1}^n, y(t_k^n)) - f_{j(t_{k-1}^n)}(t_{k-1}^n, y(t_{k-1}^n)) = \\ & \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \left( \partial_y f_{j(s)}(t_{k-1}^n, y(s)) b_{j(s)}(y(s)) + \frac{1}{2} \partial_{y,y}^2 f_{j(s)}(t_{k-1}^n, y(s)) \sigma_j^2(y(s)) \right) ds + \\ & \int_{t_{k-1}^n}^{t_k^n} \partial_y f_{j(s)}(t_{k-1}^n, y(s)) \sigma_{j(s)}(y(s)) dB(s) + (l(t_k^n) - l(t_{k-1}^n)) \left( \sum_{i=1}^I \alpha_i \partial_y f_i(t_{k-1}^n, 0) \right). \end{aligned}$$

Finally, using the regularity of  $f$ , and the fact that  $\int_0^T \mathbf{1}_{\{y(s)\}} ds = 0$ ,  $\mu$  a.s (Proposition [3.3.7](#)), it is easy to check that up to a subsequence  $n_p$

$$\begin{aligned} & \left| \sum_{k=1}^{n_p} f_{j(t_{k-1}^{n_p})}(t_{k-1}^{n_p}, y(t_k^{n_p})) - f_{j(t_{k-1}^{n_p})}(t_{k-1}^{n_p}, y(t_{k-1}^{n_p})) - \right. \\ & \left( \int_0^{\cdot} \left( \partial_y f_{j(s)}(s, y(s)) b_{j(s)}(y(s)) + \frac{1}{2} \partial_{y,y}^2 f_{j(s)}(s, y(s)) \sigma_j^2(y(s)) \right) ds \right. \\ & \left. \left. + \int_0^{\cdot} \partial_y f_{j(s)}(s, y(s)) \sigma_{j(s)}(y(s)) dB(s) + \sum_{i=1}^I \alpha_i \int_0^{\cdot} \partial_y f_i(s, 0) dl(s) \right) \right|_{(0,T)} \xrightarrow{p \rightarrow +\infty} 0, \quad \mu \text{ a.s.} \end{aligned}$$

that completes the proof.  $\square$

### 3.4.2 Local time estimate at the junction

We complete this Chapter by giving a local time estimate of the process  $Y(t) = \left( (y(t), j(t)) \right)_{0 \leq t \leq T}$  at the junction point. This estimate is reminiscent of the local time for the reflected Brownian motion.

**Theorem 3.4.2.** *We have:*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \left( \frac{1}{2\varepsilon} \sum_{j=1}^I \int_0^{\cdot} \sigma_j^2(0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds \right) - l(\cdot) \right|_{(0,T)}^2 \right] = 0,$$



and, more generally, for any subset  $\mathcal{I} \subset \{1, \dots, I\}$ :

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \left( \frac{1}{2\varepsilon(\sum_{k \in \mathcal{I}} \alpha_k)} \sum_{j \in \mathcal{I}} \int_0^\cdot \sigma_j^2(0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds \right) - l(\cdot) \right|_{(0,T)}^2 \right] = 0.$$

*Proof.* Let  $\varepsilon > 0$ . We define the following sequence  $\phi^\varepsilon$  on  $\mathcal{J}$  by

$$\forall (y, j) \in \mathcal{J}, \quad \phi^\varepsilon((y, j)) = \phi_\varepsilon(y) = \begin{cases} \frac{y^2}{2\varepsilon} & \text{if } 0 \leq y \leq \varepsilon, \\ y - \frac{\varepsilon}{2} & \text{if } y \geq \varepsilon \end{cases}, \quad \text{and } y \in J_j.$$

**Step 1 :** We start by showing that

$$\left( \phi^\varepsilon(Y(\cdot)) - \phi^\varepsilon(Y(0)) = \int_0^\cdot \left( \partial_y \phi^\varepsilon(y(s)) b_{j(s)}(y(s)) + \frac{1}{2} \partial_{y,y} \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s))^2 \right) ds + \int_0^\cdot \partial_y \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s)) dB(s) \right)_{0 \leq t \leq T}, \quad \mu \text{ a.s.} \quad (3.47)$$

Let  $\eta > 0$ , and  $\phi^{\varepsilon, \eta} \in \mathcal{C}^2(\mathcal{J})$ , the sequence of smooth functions satisfying the following conditions ( $\mathcal{T}$ ):

$$\begin{cases} \phi^{\varepsilon, \eta} = \phi^\varepsilon \text{ on } [0, \varepsilon/2] \cup [2\varepsilon, +\infty), \\ \partial_{y,y} \phi^{\varepsilon, \eta} - \partial_{y,y} \phi^\varepsilon \xrightarrow[\eta \rightarrow 0]{L^1((0, +\infty))} 0, \\ |\partial_y \phi^{\varepsilon, \eta} - \partial_y \phi^\varepsilon|_{[0, +\infty)} \xrightarrow[\eta \rightarrow 0]{} 0, \\ |\phi^{\varepsilon, \eta} - \phi^\varepsilon|_{[0, +\infty)} \xrightarrow[\eta \rightarrow 0]{} 0 \end{cases}.$$

Let  $a > \varepsilon > 0$ . We introduce the following stopping time

$$\gamma_a := \inf\{t \in [0, T], \quad y(t) = a\}, \quad \mu \text{ a.s.}$$

Applying Itô's formula of Theorem 3.4.1 to  $\phi^{\varepsilon, \eta} \in \mathcal{C}^2(\mathcal{J})$ , we have

$$\begin{aligned} \left( \phi^{\varepsilon, \eta}(Y(\cdot \wedge \gamma_a)) - \phi^{\varepsilon, \eta}(Y(0)) = \int_0^{\cdot \wedge \gamma_a} \left( \partial_y \phi^{\varepsilon, \eta}(y(s)) b_{j(s)}(y(s)) + \frac{1}{2} \partial_{y,y} \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s))^2 \right) ds \right. \\ \left. + \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s)) dB(s) \right)_{0 \leq t \leq T}, \quad \mu \text{ a.s.} \end{aligned} \quad (3.48)$$

We are going to send  $\eta \rightarrow 0$  in [\(3.48\)](#). We start by showing that, up to a sub sequence  $\eta$ ,

$$\lim_{\eta \rightarrow 0} \left| \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s)) dB(s) - \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s)) dB(s) \right|_{(0, T)} = 0, \quad \mu \text{ a.s.}$$

Using assumption  $(\mathcal{H})$  and Burkholder-Davis-Gundy inequality, we get that there exists a constant  $C > 0$ , independent of  $a$ ,  $\varepsilon$ , and  $\eta$ , such that

$$\begin{aligned} \mathbb{E}^\mu \left[ \left| \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s)) dB(s) - \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s)) dB(s) \right|_{(0, T)}^2 \right] &\leq \\ C \mathbb{E}^\mu \left[ \int_0^{T \wedge \gamma_a} \left( \partial_y \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s)) - \partial_y \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s)) \right)^2 ds \right] &\leq \\ C \left( \left| \partial_y \phi^{\varepsilon, \eta}(\cdot) - \partial_y \phi^\varepsilon(\cdot) \right|_{(0, a)}^2 \right). \end{aligned}$$

Hence using conditions  $(\mathcal{T})$ , we get that up to a sub sequence  $\eta$

$$\lim_{\eta \rightarrow 0} \left| \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s)) dB(s) - \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s)) dB(s) \right|_{(0, T)} = 0, \quad \mu \text{ a.s.}$$

With the same arguments, we get that up to a sub sequence ( denoted in the same way by )  $\eta$

$$\lim_{\eta \rightarrow 0} \left| \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^{\varepsilon, \eta}(y(s)) b_{j(s)}(y(s)) ds - \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^\varepsilon(y(s)) b_{j(s)}(y(s)) ds \right|_{(0, T)} = 0, \quad \mu \text{ a.s.}$$

Finally, as the stochastic differential equation [\(3.30\)](#) has continuous and bounded coefficients, the process  $(y(t))_{0 \leq t \leq T}$  has a continuous density on  $(0, +\infty)$  (see Proposition 1.1.2 and Theorem 1.8.3 of [\[22\]](#)). Let us denote by  $C$  its bound on  $[\varepsilon/2, a]$ . We have then using conditions assumption  $(\mathcal{H})$

$$\begin{aligned} \mathbb{E}^\mu \left[ \left| \int_0^{\cdot \wedge \gamma_a} \frac{1}{2} \partial_{y,y} \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s))^2 ds - \int_0^{\cdot \wedge \gamma_a} \frac{1}{2} \partial_{y,y} \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s))^2 ds \right|_{(0, T)} \right] & \\ \leq \mathbb{E}^\mu \left[ \int_0^T \left| \frac{1}{2} \sigma_{j(s)}(y(s))^2 (\partial_{y,y} \phi^\varepsilon(y(s)) - \partial_{y,y} \phi^{\varepsilon, \eta}(y(s))) \mathbf{1}_{\{\varepsilon/2 \leq y(s) \leq a\}} \right| ds \right] & \\ \leq \max_{i \in \{1 \dots I\}} \left\| \frac{1}{2} \sigma_i^2 \right\|_{L^\infty((0, +\infty))} TC \|\partial_{y,y} \phi^{\varepsilon, \eta} - \partial_{y,y} \phi^\varepsilon\|_{L^1((\varepsilon/2, a))}. \end{aligned}$$

Then, by condition  $(\mathcal{T})$ , we have

$$\lim_{\eta \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^{\cdot \wedge \gamma_a} \frac{1}{2} \partial_{y,y} \phi^{\varepsilon, \eta}(y(s)) \sigma_{j(s)}(y(s))^2 ds - \int_0^{\cdot \wedge \gamma_a} \frac{1}{2} \partial_{y,y} \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s))^2 ds \right|_{(0, T)} \right] = 0.$$

This proves that, for any  $a > \varepsilon$ ,

$$\left( \phi^\varepsilon(Y(\cdot \wedge \gamma_a)) - \phi^\varepsilon(Y(0)) = \int_0^{\cdot \wedge \gamma_a} \left( \partial_y \phi^\varepsilon(y(s)) b_{j(s)}(y(s)) + \frac{1}{2} \partial_{y,y} \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s))^2 \right) ds + \int_0^{\cdot \wedge \gamma_a} \partial_y \phi^\varepsilon(y(s)) \sigma_{j(s)}(y(s)) dB(s) \right)_{0 \leq t \leq T}, \quad \mu \text{ a.s.}$$

Finally, since the process  $y(\cdot)$  has continuous paths, we have

$$\lim_{a \rightarrow +\infty} \gamma_a = +\infty, \quad \mu \text{ a.s.}$$

Hence sending  $a$  to  $+\infty$ , we obtained (3.47), and that completes the proof of **Step 1**.

**Step 2** : we prove the main result of this Proposition.

Using the result obtained in **Step 1** and Theorem 3.3.9, we get using the expression of the derivatives of  $\phi^\varepsilon$ :

$$\mathbb{E}^\mu \left[ \left| \frac{1}{2\varepsilon} \int_0^\cdot \sigma_{j(s)}^2(s, y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds - l(\cdot) \right|_{(0,T)}^2 \right] \leq 5 \left( \mathbb{E}^\mu \left[ \left| \phi^\varepsilon(Y(\cdot)) - \phi^\varepsilon(Y(0)) - (y(\cdot) - y(0)) \right|_{(0,T)}^2 \right] \right) \quad (3.49)$$

$$+ \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j(s)}(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} dB(s) \right|_{(0,T)}^2 \right] \quad (3.50)$$

$$+ \mathbb{E}^\mu \left[ \left| \int_0^\cdot b_{j(s)}(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds \right|_{(0,T)}^2 \right] \quad (3.51)$$

$$+ \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j(s)}(y(s)) \mathbf{1}_{\{y(s) \geq \varepsilon\}} dB(s) - \int_0^\cdot \sigma_{j(s)}(y(s)) dB(s) \right|_{(0,T)}^2 \right] \quad (3.52)$$

$$+ \mathbb{E}^\mu \left[ \left| \int_0^\cdot b_{j(s)}(y(s)) \mathbf{1}_{\{y(s) \geq \varepsilon\}} ds - \int_0^\cdot b_{j(s)}(y(s)) ds \right|_{(0,T)}^2 \right] \right). \quad (3.53)$$

For (3.49), recalling that, for all  $Y = (y, j) \in \mathcal{J}$ ,  $|\phi^\varepsilon(Y) - y| \leq 2\varepsilon$ , we get by Lebesgue's Theorem

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \phi^\varepsilon(Y(\cdot)) - \phi^\varepsilon(Y(0)) - (y(\cdot) - y(0)) \right|_{(0,T)}^2 \right] = 0.$$

On the other hand for (3.50), using assumption  $(\mathcal{H})$  and Burkholder-Davis-Gundy inequality, there exists a constant  $C > 0$  independent of  $\varepsilon$ , such that

$$\mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j(s)}(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} dB(s) \right|_{(0,T)}^2 \right] \leq C \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds \right],$$

and therefore using Proposition (3.3.7), we get

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j(s)}(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} dB(s) \right|_{(0,T)}^2 \right] = 0.$$

Similarly, we have for (3.51)

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot b_{j(s)}(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds \right|^2 \right] = 0.$$

Finally, for (3.52) and (3.53), it easy to check using assumption ( $\mathcal{H}$ ) that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot \sigma_{j(s)}(y(s)) \mathbf{1}_{\{y(s) \geq \varepsilon\}} dB(s) - \int_0^\cdot \sigma_{j(s)}(y(s)) dB(s) \right|_{(0,T)}^2 \right] &= 0, \\ \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \int_0^\cdot b_{j(s)}(y(s)) \mathbf{1}_{\{y(s) \geq \varepsilon\}} ds - \int_0^\cdot b_{j(s)}(y(s)) ds \right|_{(0,T)}^2 \right] &= 0. \end{aligned}$$

We get finally

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \frac{1}{2\varepsilon} \int_0^\cdot \sigma_{j(s)}^2(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds - l(\cdot) \right|_{(0,T)}^2 \right] = 0.$$

To conclude, we remark that all  $t \in [0, T]$

$$\begin{aligned} \frac{1}{2\varepsilon} \int_0^t \sigma_{j(s)}^2(y(s)) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds &= \sum_{j=1}^I \frac{1}{2\varepsilon} \int_0^t \sigma_j^2(0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds + \\ &\sum_{j=1}^I \frac{1}{2\varepsilon} \int_0^t \left( \sigma_j^2(y(s)) - \sigma_j^2(0) \right) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds, \quad \mu \text{ a.s.} \end{aligned}$$

Using the uniform Lipschitz continuity of all the  $(\sigma_i)_{i \in \{1 \dots I\}}$  at each edge  $J_i$  (assumption ( $\mathcal{H}$ )), we get that there exists a constant  $C > 0$ , independent of  $\varepsilon$  such that for all  $j \in \{1 \dots I\}$ , for all  $s \in [0, T]$

$$\left| \sigma_j^2(y(s)) - \sigma_j^2(0) \right| \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} \leq C\varepsilon \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}}, \quad \mu \text{ a.s.},$$

and then using Proposition (3.3.7)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \sum_{j=1}^I \frac{1}{2\varepsilon} \int_0^\cdot \left( \sigma_j^2(y(s)) - \sigma_j^2(0) \right) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds \right|_{(0,T)} \right] &\leq \\ C \lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \int_0^T \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon\}} ds \right] &= 0. \end{aligned}$$

We get then the required result, namely

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \left( \frac{1}{2\varepsilon} \sum_{j=1}^I \int_0^\cdot \sigma_j^2(0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds \right) - l(\cdot) \right|_{(0,T)}^2 \right] = 0.$$

We notice that the second approximation, which is for any  $\mathcal{I} \subset \{1, \dots, I\}$

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^\mu \left[ \left| \left( \frac{1}{2\varepsilon(\sum_{k \in \mathcal{I}} \alpha_k)} \sum_{j \in \mathcal{I}} \int_0^\cdot \sigma_j^2(0) \mathbf{1}_{\{0 \leq y(s) \leq \varepsilon, j(s)=j\}} ds \right) - l(\cdot) \right|_{(0,T)}^2 \right] = 0,$$

can be proved with the same arguments above considering the same map  $\phi^\varepsilon$ , but vanishing on each edge whose indexes belong to  $\{1, \dots, I\} \setminus \mathcal{I}$ . □

## Chapter 4

# Stochastic optimal control at the junction

### 4.1 Introduction

Originally introduced by Freidlin and Sheu in [15] and Freidlin and Wetzell in [16], stochastic diffusions in graphs have attracted a lot of attention in the last 20 years. More precisely, given a junction  $\mathcal{J} = \bigcup_{i=1}^I J_i$ ,  $(\sigma_i, b_i)$  regular functions from  $\mathbb{R}_+$  to  $\mathbb{R}$ , and  $\alpha_1 \dots \alpha_I$  non-negative constants such that  $\alpha_1 + \dots + \alpha_I = 1$ , the authors in [16] have proved that there exists a continuous Markov process  $X = (x, i)$  defined on  $\mathcal{J}$ .

Thereafter in [15], it is shown that there exists a one dimensional Wiener process  $W$  defined on a probability space  $(Y(\cdot), \mathcal{F}, \mathbb{P})$ , adapted to the natural filtration of  $X = (x, i)$ , such that the process  $x$  satisfies the following stochastic differential equation for a finite time horizon  $T > 0$ ,

$$dx(t) = \sigma_{i(t)}(x(t))dW(t) + b_{i(t)}(x(t))dt + dl(t), \quad 0 \leq t \leq T, \quad (4.1)$$

where  $l$  is a nondecreasing process starting from zero satisfying

$$\mathbb{P}\left(\left(\int_0^t \mathbf{1}_{\{x(s)>0\}} dl(s)\right)_{0 \leq t \leq T} = 0\right) = 1.$$

Moreover, [15] gives the following Itô's formula

$$\begin{aligned}
df_{i(t)}(x(t)) &= \left( b_{i(t)}(x(t))\partial_x f_{i(t)}(x(t)) + \frac{1}{2}\sigma_{i(t)}^2(x(t))\partial_{x,x} f_{i(t)}(x(t)) \right) dt + \\
&\quad \partial_x f_{i(t)}(x(t))\sigma_{i(t)}(x(t))dW_t + \sum_{i=1}^I \alpha_i \partial_x f_i(0) dl(t), \tag{4.2}
\end{aligned}$$

for  $f$  regular enough.

The process  $l$  can be interpreted as the local time of the process  $X$  at the vertex, whose quadratic approximation is given by

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}} \left[ \left| \left( \frac{1}{2\varepsilon} \sum_{j=1}^I \int_0^\cdot \sigma_j^2(0) \mathbf{1}_{\{0 \leq x(s) \leq \varepsilon, j(s)=j\}} ds \right) - l(\cdot) \right|_{(0,T)}^2 \right] = 0. \tag{4.3}$$

In this Chapter, we study a stochastic control problem with control at the junction point. We use a weak martingale formulation, and the method of compactification of the controls, as it has been introduced in [25]. Such a method is a classical one in the deterministic case and even in the stochastic case in [6], however it is not often used any more. For our problem, the method differs from what it has already done in the literature, since we will add a relaxation at the junction point, due to the process  $l$  introduced in equation (4.1), which takes into account its behavior. This new method of relaxation is introduced in Section 4.2, where a criterion a compactness is given in Theorem 4.2.1. Thereafter, we establish the compactness of the admissible controls in Theorem 4.3.6. As in [25], both stability properties of the set of rules by conditioning and concatenation at stopping times, are the main tools to formulate the dynamic programming principle, which is proved in Theorem 4.4.5. On the other hand, the value function of this problem of control, will allows us to make the link with the theory of non linear parabolic partial differential equations at a junction. Due to the process  $l$  and the quadratic approximation (4.3), we will get that the parabolic equation that characterized the value function, has non degenerate viscosity at the junction point  $x = 0$ , and satisfy a non linear Neumann and non dynamical boundary condition at  $x = 0$ ;

$$F(u(t, 0), \partial_x u(t, 0)) = \inf_{\alpha_i \in [0,1]^I, \sum_i \alpha_i = 1} \left\{ \sum_i \alpha_i \partial_x u_i(t, 0) \right\} = 0.$$

Until now, the only result of existence and uniqueness of these type of equation has been given in [41], where the author has shown well-posedness of classical solutions for the following problem

$$\begin{cases} \partial_t u_i(t, x) - \sigma_i(x, \partial_x u_i(t, x)) \partial_{x,x}^2 u_i(t, x) + H_i(x, u_i(t, x), \partial_x u_i(t, x)) = 0, \\ \text{for all } x > 0, \text{ and for all } i \in \{1 \dots I\}, \\ F(u(t, 0), \partial_x u(t, 0)) = 0, \end{cases} \quad (4.4)$$

in suitable Hölder spaces: see Theorem 2.2 for the existence and Theorem 2.4 for the comparison in [41], and thus the uniqueness. The main assumptions are that the equation is uniformly parabolic with smooth coefficients and that the term  $F = F(u, p)$  is increasing with respect to  $p$ , which is a natural assumption regarding to the set where the controls  $(\alpha_1 \dots \alpha_I)$  are valued. This result will allow us to formulate a Feynman-Kac's representation formula in a futur framework.

Let us mention that the control theory on stratified domains or networks have already been well-studied in the literature, for first order problems, and we refer for instance to [3], [4], [5], [17], [1], [33]... On the other hand, for stochastic control problems with reflection and controllability at the boundary, we refer to [11], where the author studied optimal reflection with some applications in financial markets.

The Chapter is organized as follows. We introduce the set of generalized actions that will be used for the compactification method, and the formulation of our martingale problem in Section 4.2. Thereafter, we prove the compactness of the admissible rules in Section 4.3. The dynamic programming principle is established in Section 4.4.

## 4.2 The set of generalized action, and the martingale problem

### 4.2.1 The set of generalized actions

In this sub section we define the set of generalized actions at the junction point, and we give a criterion of compactness Theorem 4.2.1. Let us introduce

$$L[0, T] := \left\{ l : [0, T] \rightarrow \mathbb{R}, \text{ continuous nondecreasing such that } l(0) = 0 \right\},$$



which is the space where the process  $l(\cdot)$  introduced in (4.1) takes its value. We define furthermore the following compact set  $A_0$  of  $\mathbb{R}^I$  by

$$A_0 := \left\{ (\alpha_i) \in [0, 1]^I, \sum_{i=1}^I \alpha_i = 1 \right\},$$

which is the set where the controls  $\alpha_i$  at the junction point appearing in the Ito's formula Theorem (3.4.1) are valued.

In the sequel, we use the notations introduced in Appendix C, and for the convenience of the reader we recall that

$$L_{mc}^\infty([0, T] \times A_0) := \left\{ f \in L^\infty([0, T] \times A_0), (t, \alpha_1 \dots \alpha_I) \mapsto f(t, (\alpha_1 \dots \alpha_I)) \in \mathcal{C}(A_0), \forall t \in [0, T] \right\},$$

and  $\mathcal{M}_{mc}([0, T] \times A_0)$  is the set consisting of non negative finite measures on  $([0, T] \times A_0, \mathbb{B}([0, T]) \otimes \mathbb{B}(A_0))$ , endowed with the finest topology making continuous the following family of linear forms  $(\theta_f)_{f \in L_{mc}^\infty([0, T] \times A_0)}$ ,

$$\theta_f : \begin{cases} \mathcal{M}_{mc}([0, T] \times A_0) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{[0, T] \times A_0} f d\nu \end{cases}.$$

The set of generalized actions at the junction point, denoted  $V([0, T] \times A_0)$  is defined by

$$V([0, T] \times A_0) := \left\{ \nu \in \mathcal{M}_{mc}([0, T] \times A_0), \exists l_\nu \in L[0, T], \nu^{[0, T]}(dt) = l_\nu(dt) \right\},$$

where  $\nu^{[0, T]}(dt) = \int_{A_0} \nu(dt, d\alpha_1 \dots d\alpha_I)$ .

As a consequence of the disintegration Theorem of a measure, (see for instance [24]), we will use the following notation for  $\nu \in V([0, T] \times A_0)$ :

$$\nu(dt, d\alpha_1 \dots d\alpha_I) = l_\nu(dt) \nu_t(d\alpha_1 \dots d\alpha_I),$$

where  $\nu_t$  is a measurable kernel of mass 1 on  $(A_0, \mathbb{B}(A_0))$ . As explained in the general Introduction 4.1, we establish here a criterion of compactness for  $V([0, T] \times A_0)$ , that will be useful in the proof of the compactness of the admissible rules in Section 4.3.

**Theorem 4.2.1.** *Let  $\mathcal{V}$  be a subset of  $V([0, T] \times A_0)$ . Assume that there exists a constant*

$C > 0$ , and a modulus of continuity  $w \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ , with  $w(0) = 0$ , such that

$$\begin{aligned} \forall \nu \in \mathcal{V}, \quad l_\nu(T) &\leq C, \\ \forall \nu \in \mathcal{V}, \quad \forall (t, s) \in [0, T], \quad |l_\nu(t) - l_\nu(s)| &\leq w(|t - s|), \end{aligned}$$

then  $\mathcal{V}$  is compact for the weak topology  $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$ .

*Proof.* Since the  $\sigma$  Borel algebra  $\mathbb{B}([0, T])$  of  $[0, T]$  is countably generated, we get from Proposition [C.0.3](#) that  $\mathcal{M}_{mc}([0, T] \times A_0)$  is metrizable, therefore  $\mathcal{V}$  is metrizable and the compactness can be proved sequentially.

Let  $\nu_n$  be a sequence of  $\mathcal{V}$ , we know that there exists a sequence  $l_{\nu_n}$  of  $L[0, T]$ , such that

$$\nu_n^{[0, T]}(dt) = \int_{A_0} \nu_n(dt, d\alpha_1 \dots d\alpha_I) = l_{\nu_n}(dt).$$

Using the assumptions satisfied by the sequence  $l_{\nu_n}$ , applying Ascoli's Theorem, we get that  $l_{\nu_n}$  converges uniformly up to a sub sequence to  $l \in \mathcal{C}[0, T]$ , and since  $L[0, T]$  is closed in  $\mathcal{C}[0, T]$  for the uniform convergence, we deduce that  $l \in L[0, T]$ .

Let us now show that  $\mathcal{V}$  is relatively compact for  $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$ , and for this we are going to apply Theorem [C.0.4](#).

We now show that  $\nu_n^{[0, T]}$  and (resp.  $\nu_n^{A_0} = \int_{[0, T]} \nu_n(dt, d\alpha_1 \dots d\alpha_I)$ ) are relatively compact in  $\mathcal{M}_m([0, T])$  (resp.  $\mathcal{M}_c(A_0)$ ), for the weak topologies  $*\sigma\left(L^{\infty, 1}([0, T])', L^{\infty, 1}([0, T])\right)$ , (resp.  $*\sigma\left(\mathcal{C}(A_0)', \mathcal{C}(A_0)\right)$ ), where we recall that

$$L^{\infty, 1}([0, T]) := \left\{ f \in L^\infty([0, T]), \exists B \in \mathbb{B}([0, T]), f(t) = \mathbf{1}_B(t) \right\},$$

and  $\mathcal{M}_m([0, T])$ , (resp.  $\mathcal{M}_c(A_0)$ ), are the set of finite positive finite measures on  $[0, T]$  (resp.  $A_0$ ), endowed with the finest topology making continuous the following family of linear forms  $(\theta_f)_{f \in L^{\infty, 1}([0, T])}$ , defined by

$$\theta_f : \begin{cases} \mathcal{M}_m([0, T]) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{[0, T]} f d\nu \end{cases}.$$

(resp.  $(\theta_f)_{f \in \mathcal{C}(A_0)}$ )

$$\theta_f : \left\{ \begin{array}{l} \mathcal{M}_c(A_0) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{A_0} f d\nu \end{array} \right\}.$$

Since  $l_n$  converges uniformly to  $l$  up to a sub sequence  $n_k$ , it is easy to get that for any  $f \in L^{\infty,1}([0, T])$

$$\int_{[0, T]} f(t) l_{\nu_{n_k}}(dt) \xrightarrow[n_k]{k \rightarrow +\infty} \int_{[0, T]} f(t) l(dt),$$

namely  $\nu_{n_k}^{[0, T]}(dt) \xrightarrow{*} l(dt)$  for  $*\sigma(L^{\infty,1}([0, T])', L^{\infty,1}([0, T]))$ .

On the other hand, we have

$$\|\nu_n^{A_0}\|_{\mathcal{C}(A_0)'} = \sup_{f \in \mathcal{C}(A_0), \|f\| \leq 1} \left| \int_{[0, T] \times A_0} f(t) \nu_n(dt, d\alpha_1 \dots d\alpha_I) \right| \leq l_n(T) \leq C,$$

and then we deduce that  $\nu_n^{A_0}$  is relatively compact for the weak topology  $*\sigma(\mathcal{C}(A_0)', \mathcal{C}(A_0))$ .

We deduce finally using Theorem [C.0.4](#), that  $\nu_n$  is relatively compact, and then converges up to a sub sequence (denoted in the same way by  $n_k$ ) to  $\phi \in L_{mc}^{\infty,1}([0, T] \times A_0)'$ , for  $*\sigma(L_{mc}^{\infty,1}([0, T] \times A_0)', L_{mc}^{\infty,1}([0, T] \times A_0))$ , where

$$L_{mc}^{\infty,1}([0, T] \times A_0) := \left\{ f \in L^\infty([0, T] \times A_0), \exists B \in \mathbb{B}([0, T]), g \in \mathcal{C}(A_0), f = \mathbf{1}_B(t)g(\alpha_1 \dots \alpha_I) \right\}.$$

We now turn to prove that  $\phi$  can be represented by an element of  $\nu \in \mathcal{M}_{mc}([0, T] \times A_0)$ , namely

$$\begin{aligned} & \exists \nu \in \mathcal{M}_{mc}([0, T] \times A_0), \quad \forall f \in L_{mc}^{\infty,1}([0, T] \times A_0), \\ \phi(f) &= \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu(dt, d\alpha_1 \dots d\alpha_I). \end{aligned}$$

For this, we use a Riesz representation Theorem, and more precisely we are going to prove that  $\phi$  satisfies (i) and (ii) of Theorem [C.0.5](#).

Let  $B \in \mathbb{B}([0, T])$ , we have

$$(t, \alpha_1 \dots \alpha_I) \mapsto \mathbf{1}_B \otimes 1(t, \alpha_1 \dots \alpha_I) := \begin{cases} 1, & \text{if } t \in B, \\ 0, & \text{if } t \notin B, \end{cases}$$

belongs to  $L^{\infty,1}([0, T] \times A_0)$ , and

$$\begin{aligned} \nu_{n_k}(\mathbf{1}_B \otimes 1) &\xrightarrow{k \rightarrow +\infty} \phi(\mathbf{1}_B \otimes 1), \\ \nu_{n_k}(\mathbf{1}_B \otimes 1) &= l_{n_k}(B) \xrightarrow{k \rightarrow +\infty} l(B). \end{aligned}$$

By uniqueness of the weak limit, we get that  $\phi(\mathbf{1}_B \otimes 1) = l(B)$ , and since  $l \in L[0, T]$ ,  $l$  defines a Borel measure on  $([0, T], \mathbb{B}([0, T]))$ , which means that (i) of Theorem [C.0.5](#) holds true.

On the other hand, since  $A_0$  is compact, we deduce easily that (ii) of Theorem [C.0.5](#) holds true.

We deduce then that there exists  $\nu \in \mathcal{M}_{mc}([0, T] \times A_0)$ , such that

$$\forall f \in L_{mc}^{\infty,1}([0, T] \times A_0), \quad \phi(f) = \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu(dt, d\alpha_1 \dots d\alpha_I).$$

Since  $\phi$  is a continuous linear form on  $Span(L_{mc}^{\infty,1}([0, T] \times A_0))$ , which is dense in  $L_{mc}^{\infty}([0, T] \times A_0)$  for the uniform convergence (see Lemma [C.0.6](#)), we deduce that

$$\forall f \in L_{mc}^{\infty}([0, T] \times A_0), \quad \phi(f) = \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu(dt, d\alpha_1 \dots d\alpha_I).$$

Finally, to complete the proof, it is enough to show that the projection  $\nu^{[0, T]}(dt)$  is equal to  $l(dt)$ . For this we use that, for any  $B \in \mathbb{B}([0, T])$

$$\begin{aligned} \int_{[0, T]} \mathbf{1}_B(t) l_{\nu_{n_k}}(dt) &\xrightarrow[n_k]{k \rightarrow +\infty} \int_{[0, T]} \mathbf{1}_B(t) l(dt), \\ \int_{[0, T] \times A_0} \mathbf{1}_B(t) \nu_{n_k}(dt, d\alpha_1 \dots d\alpha_I) &\xrightarrow[n_k]{k \rightarrow +\infty} \int_{[0, T] \times A_0} \mathbf{1}_B(t) \nu(dt, d\alpha_1 \dots d\alpha_I). \end{aligned}$$

Using the uniqueness of the weak limit, we get

$$\forall B \in \mathbb{B}([0, T]), \quad \int_{[0, T]} \mathbf{1}_B(t) l(dt) = \int_{[0, T] \times A_0} \mathbf{1}_B(t) \nu(dt, d\alpha_1 \dots d\alpha_I)$$

and then

$$l(dt) = \int_{A_0} \nu(dt, d\alpha_1 \dots d\alpha_I),$$

and that completes the proof.  $\square$

**Theorem 4.2.2.**  $V([0, T] \times A_0)$  endowed with the weak topology  $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$  is Polish.

*Proof.* Recall that  $\mathcal{M}_{mc}([0, T] \times A_0)$  endowed with the weak topology  $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$  is separable since

$$\mathcal{M}_{mc}([0, T] \times A_0) \subset \bigcup_{n \geq 0} \left\{ \phi \in L_{mc}^\infty([0, T] \times A_0)', \|\phi\| \leq n \right\},$$

and from Banach-Alaoglu-Bourbaki's Theorem

$$\forall n \geq 0, \left\{ \phi \in L_{mc}^\infty([0, T] \times A_0)', \|\phi\| \leq n \right\}$$

is compact for  $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$ .

As a subset of  $\mathcal{M}_{mc}([0, T] \times A_0)$ , we deduce that  $V([0, T] \times A_0)$  is separable for the weak topology  $*\sigma\left(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0)\right)$ .

To conclude, let  $\nu_n(dt, d\alpha_1 \dots d\alpha_I) := l_n(dt)\nu_{t,n}(dz)$  a Cauchy sequence of  $V([0, T] \times A_0)$ , we have then

$$\begin{aligned} \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall p \geq 0, \forall f \in L_{mc}^\infty([0, T] \times A_0), \\ \left| \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu_{n+p}(dt, d\alpha_1 \dots d\alpha_I) - \int_{[0, T] \times A_0} f(t, \alpha_1 \dots \alpha_I) \nu_n(dt, d\alpha_1 \dots d\alpha_I) \right| \leq \varepsilon. \end{aligned}$$

Let  $s \in [0, T]$ , choosing  $f(t, \alpha_1 \dots \alpha_I) = \mathbf{1}_{[0, s]}(t)$ , we get that  $l_n$  is a Cauchy sequence of  $L([0, T])$ , and then converges uniformly to  $l \in L([0, T])$ . Therefore using the converse of

Ascoli's Theorem, we get that the sequence  $l_n$  satisfies

$$\exists C > 0, \quad \forall n \geq 0, \quad l_n(T) \leq C,$$

$$\exists w \in \mathcal{C}([0, T]), \quad w(0) = 0, \quad \forall n \geq 0, \quad \forall (t, s) \in [0, T], \quad |l_n(t) - l_n(s)| \leq w(|t - s|).$$

We conclude then using Theorem [4.2.1](#), that  $\nu_n$  converges to  $\nu \in V([0, T] \times A_0)$  for the weak topology  $*\sigma(L_{mc}^\infty([0, T] \times A_0)', L_{mc}^\infty([0, T] \times A_0))$ , and that completes the proof.  $\square$

#### 4.2.2 Weak martingale formulation of the problem of control

In this sub section we define the martingale problem. We use a classical relaxation on each edge. Let then  $(K_i)_{1 \leq i \leq I}$ ,  $I$  compact sets of  $\mathbb{R}$ , we denote by  $U([0, T] \times K_i)$  the set of generalized actions on each edge  $J_i$

$$U([0, T] \times K_i) := \left\{ \nu \in \mathcal{M}_{mc}([0, T] \times K_i), \quad \nu^{[0, T]}(dt) = \int_{K_i} \nu(dt, dk) = dt \right\}.$$

As it has been done in the proof of Proposition [4.2.1](#), we can show that for each  $i \in \{1 \dots I\}$ ,  $U([0, T] \times K_i)$  are compact for the weak topology  $*\sigma(L_{mc}^\infty([0, T] \times K_i)', L_{mc}^\infty([0, T] \times K_i))$ . To formulate the martingale problem, we introduce in the sequel the following data, for each  $i \in \{1 \dots I\}$ ,

$$\left\{ \begin{array}{l} \sigma_i \in L^\infty([0, T] \times [0, +\infty) \times K_i) \\ b_i \in L^\infty([0, T] \times [0, +\infty) \times K_i) \\ h_i \in L^\infty([0, T] \times [0, +\infty) \times K_i) \\ h_0 \in L^\infty([0, T] \times A_0) \\ g \in \mathcal{C}_b^1(\mathcal{J}, \mathbb{R}) \cap \mathcal{C}_b^2(\overset{\circ}{\mathcal{J}}, \mathbb{R}) \end{array} \right. ,$$

satisfying the following assumptions

**Assumption** ( $\mathcal{H}$ )

- (i)  $\exists c > 0, \forall i \in \{1 \dots I\}, \forall (t, x, k_i) \in [0, T] \times [0, +\infty) \times K_i, \sigma_i(t, x, k_i) \geq c,$   
(ii)  $\forall i \in \{1 \dots I\}, x \mapsto (\sigma_i, b_i, h_i) \in C^1([0, +\infty)),$  and  
 $(\partial_x \sigma_i, \partial_x b_i, \partial_x h_i) \in L^\infty([0, T] \times [0, +\infty) \times K_i)).$

We recall that  $\mathcal{C}^{\mathcal{J}}([0, T])$  is the set of continuous maps defined in  $[0, T]$ , valued in the junction  $(J, d^{\mathcal{J}})$ , where  $d^{\mathcal{J}}$  is defined by

$$\forall ((x, i), (y, j)) \in \mathcal{J}^2, \quad d^{\mathcal{J}}((x, i), (y, j)) = \begin{cases} |x - y| & \text{if } i = j, \\ x + y & \text{if } i \neq j. \end{cases}$$

In the sequel,  $\mathcal{C}_b^{1,2}(\mathcal{J}_T)$  is the class of continuous functions defined on  $[0, T] \times \mathcal{J}$ , having a regularity of class  $\mathcal{C}^{1,2}([0, T] \times [0, +\infty))$  on each edge, and bounded together with all its derivatives.

The canonical space where we will define our process, is the following Polish space

$$\Phi = \mathcal{C}^{\mathcal{J}}[0, T] \times \left( \prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0),$$

endowed with its Borel  $\sigma$  algebra  $\mathbb{B}(\Phi)$ .

The canonical process is then defined on the measurable space  $(\Phi, \mathbb{B}(\Phi))$  by

$$X : \begin{cases} [0, T] \times \Phi \rightarrow \mathcal{J} \times \left( \prod_{i=1}^I \mathcal{M}_{mc}([0, T] \times K_i) \right) \times \mathcal{M}_{mc}([0, T] \times A_0) \\ \left( s, Y(\cdot) \right) \mapsto \left( X(s, Y(\cdot)) = \left( \left( y(s), j(s) \right), \nu_1(s) \dots \nu_I(s), \nu_0(s) \right) \right), \end{cases}$$

where for each  $i \in \{0, \dots, I\}$ ,  $\nu_i(s)(dt, d\alpha_1 \dots d\alpha_I) = \mathbf{1}_{[0, s]}(t) \nu_i(dt, d\alpha_1 \dots d\alpha_I)$ .

It is easy to check that the process  $\left( X(s) \right)_{0 \leq s \leq T}$  has continuous paths. We denote in the sequel by  $(\Psi_t)_{0 \leq t \leq T}$  the right continuous filtration generated by this process.

Let  $(t, (x, i)) \in [0, T] \times \mathcal{J}$ , we define the set of admissible rules  $\mathcal{A}(t, (x, i))$ , as the set of all the probability measures  $P_t^{(x, i)}$  defined on the filtered probability space  $(\Phi, (\Psi_t)_{0 \leq t \leq T})$  satisfying

**Conditions  $(\mathcal{S}_0)$**

-(i) For each  $u \leq t$ ,  $X(u) = \left( (x, i), \nu_1(t) \dots \nu_I(t), \nu_0(t) \right)$ ,  $P_t^{(x,i)}$  a.s.

-(ii) For each  $s \geq t$ ,

$$\int_t^s \int_{A_0} \mathbf{1}_{\{x(u)>0\}} \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) = \int_t^s \mathbf{1}_{\{x(u)>0\}} l_{\nu_0(s)}(du) = 0, \quad P_t^{(x,i)} \text{ a.s.}$$

-(iii) For any  $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$ , the following process  $(M_s^f)_{0 \leq s \leq T}$  defined on the filtered probability space  $(\Phi, \mathbb{B}(\Phi), (\Psi_t)_{0 \leq t \leq T}, P_t^{(x,i)})$  by

$$\begin{aligned} \forall s \in [t, T], \quad M_s^f - M_t^f &= f(s, X(s)) - f(t, X(t)) - \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_t f_i(u, x(u)) \right. \\ &\quad \left. + \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} f_i(u, x(u)) + b_i(u, x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) \\ &\quad - \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \end{aligned}$$

is a  $(\Psi_s)_{t \leq s \leq T}$  continuous martingale under the probability measure  $P_t^{(x,i)}$ , after time  $t$ .

**Remark 4.2.3.** We assume that  $\mathcal{A}(t, (x, i))$  is non empty, and we will prove it in a future work. More precisely, we will show that there exists  $P \in \mathcal{A}(t, (x, i))$  with a constant control at the junction point: namely for  $(a_1 \dots a_I) \in A_0$

$$\forall s \in [0, T], \quad \forall \nu_0 \in V([0, T] \times A_0), \quad \nu_0(s)(dt, d\alpha_1 \dots d\alpha_I) = \mathbf{1}_{[0, s]}(t) \delta_{(a_1 \dots a_I)}(\alpha_1 \dots \alpha_I),$$

then

$$\begin{aligned} \int_{A_0} \delta_{(a_1 \dots a_I)}(\alpha_1 \dots \alpha_I) &= 1, \quad \text{and} \quad \int_t^s \int_{A_0} \langle a, \partial_x f(u, 0) \rangle \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) = \\ &= \sum_{i=1}^I \int_t^s \int_{A_0} a_i \partial_x f_i(u, 0) dl_{\nu_0(s)}(u), \end{aligned}$$

with  $a = (a_1 \dots a_I)$ ,  $\partial_x f(u, 0) = (\partial_x f_1(u, 0) \dots \partial_x f_I(u, 0))$ , and  $\langle \cdot, \cdot \rangle$  denotes the classical scalar product in  $\mathbb{R}^I$ .

We can then define the following reward function  $\Lambda$  of our problem, with cost  $h_0$  at



the junction point and  $h_i$  on each edge by

$$\Lambda : \begin{cases} \mathcal{A}(t, (x, i)) \rightarrow \mathbb{R} \\ P_t^{(x, i)} \mapsto \mathbb{E}^{P_t^{(x, i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) \right. \\ \left. + \int_t^T \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right]. \end{cases} \quad (4.5)$$

The corresponding value function  $v$  is defined by

$$v : \begin{cases} [0, T] \times \mathcal{J} \rightarrow \mathbb{R} \\ (t, (x, i)) \mapsto \inf_{P_t^{(x, i)} \in \mathcal{A}(t, (x, i))} \Lambda(P_t^{(x, i)}) \end{cases}. \quad (4.6)$$

### 4.3 Compactness of the admissible rule

In this section, we will prove the compactness of the set of admissible rules  $\mathcal{A}(t, (x, i))$ , for the weak topology.

**Proposition 4.3.1.** *Define the following maps*

$$\rho : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0) \rightarrow \mathbb{R} \\ ((x(\cdot), i(\cdot)), \nu_0) \mapsto \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1 \dots d\alpha_I) \end{cases},$$

and for all  $i \in \{0 \dots I\}$

$$\rho_0 : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0) \rightarrow \mathbb{R} \\ ((x(\cdot), i(\cdot)), \nu_0) \mapsto \int_t^T \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(du, d\alpha_1 \dots d\alpha_I) \end{cases},$$

$$\rho_i : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times K_i) \rightarrow \mathbb{R} \\ ((x(\cdot), i(\cdot)), \nu_i) \mapsto \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(du, dk_i) \end{cases}.$$

$\rho, (\rho_i)_{i \in \{1 \dots I\}}$  are lower semi continuous and  $\rho_0$  is continuous.

*Proof.* We start by showing that  $\rho$  is lower semi continuous, and for this let  $((x^n(\cdot), i^n(\cdot)),$

$\nu_0^n(dt, d\alpha_1 \dots d\alpha_I)$  in  $\mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0)$  converging to  $((x(\cdot), i(\cdot)), \nu_0(dt, d\alpha_1 \dots d\alpha_I))$ . Let  $p \geq 0$  and  $\phi_p \in \mathcal{C}([0, +\infty))$  a sequence converging from below to  $x \rightarrow \mathbf{1}_{\{x>0\}}$  in the pointwise sense, as  $p \rightarrow +\infty$ . Since  $\nu_n(dt, d\alpha_1 \dots d\alpha_I) \xrightarrow{*} \nu(dt, d\alpha_1 \dots d\alpha_I)$ , we can find  $\theta \in \mathcal{M}_{mc}([0, T] \times A_0)$ , such that

$$\forall f \in L_{mc}^{\infty}([0, T] \times A_0), \quad \forall n \geq 0,$$

$$\int_{[0, T] \times A_0} |f(u, \alpha_1 \dots \alpha_I)| \nu_n(dx, d\alpha_1 \dots d\alpha_I) \leq \int_{[0, T] \times A_0} |f(u, \alpha_1 \dots \alpha_I)| \theta(dx, d\alpha_1 \dots d\alpha_I).$$

We write then

$$\begin{aligned} & \left| \int_t^T \int_{A_0} \phi_p(x^n(u)) \nu_0^n(du, d\alpha_1 \dots d\alpha_I) - \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1 \dots d\alpha_I) \right| \leq \\ & \quad \int_t^T \int_{A_0} \left| \phi_p(x^n(u)) - \phi_p(x(u)) \right| \nu_0^n(du, d\alpha_1 \dots d\alpha_I) \\ + & \left| \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0^n(du, d\alpha_1 \dots d\alpha_I) - \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1 \dots d\alpha_I) \right| \leq \\ & \quad \int_t^T \int_{A_0} \left| \phi_p(x^n(u)) - \phi_p(x(u)) \right| \theta(du, d\alpha_1 \dots d\alpha_I) \\ + & \left| \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0^n(du, d\alpha_1 \dots d\alpha_I) - \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1 \dots d\alpha_I) \right|. \end{aligned}$$

Therefore we get that

$$\forall p \geq 0, \quad \lim_{n \rightarrow +\infty} \int_t^T \int_{A_0} \phi_p(x^n(u)) \nu_0^n(du, d\alpha_1 \dots d\alpha_I) = \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1 \dots d\alpha_I).$$

Finally writing

$$\int_t^T \int_{A_0} \mathbf{1}_{\{x^n(u)>0\}} \nu_0^n(du, d\alpha_1 \dots d\alpha_I) \geq \int_t^T \int_{A_0} \phi_p(x^n(u)) \nu_0^n(du, d\alpha_1 \dots d\alpha_I),$$

we get

$$\forall p \geq 0, \quad \liminf_{n \rightarrow +\infty} \int_t^T \int_{A_0} \mathbf{1}_{\{x^n(u)>0\}} \nu_0^n(du, d\alpha_1 \dots d\alpha_I) \geq \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1 \dots d\alpha_I),$$

and hence

$$\begin{aligned} \liminf_{n \rightarrow +\infty} \int_t^T \int_{A_0} \mathbf{1}_{\{x^n(u) > 0\}} \nu_0^n(du, d\alpha_1 \dots d\alpha_I) &\geq \limsup_{p \rightarrow +\infty} \int_t^T \int_{A_0} \phi_p(x(u)) \nu_0(du, d\alpha_1 \dots d\alpha_I) \\ &= \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1 \dots d\alpha_I). \end{aligned}$$

We conclude then that  $\rho$  is lower semi continuous. We use the same arguments to show that the  $(\rho_i)_{i \in \{1 \dots I\}}$  are lower semi continuous and  $\rho_0$  is continuous.  $\square$

In the next Proposition, we characterize the paths of the process  $x(\cdot)$ , by showing that its martingale part can be represented by a Brownian martingale.

**Proposition 4.3.2.** *Let  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ , and  $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$ , we have*

$$\begin{aligned} \forall s \in [t, T], \quad d \langle f(\cdot, X(\cdot)) \rangle_s = \\ \left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_x f_i(s, x(s)) \sigma_i(s, x(s), k_i) \right)^2 \nu_{i,s}(s)(dk_i) \right) ds, \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

Moreover there exists a standard one dimensionnal Brownian motion  $W(\cdot)$ ,  $(\Psi_s)_{t \leq s \leq T}$  measurable, such that

$$\begin{aligned} \forall s \in [t, T], \quad x(s) = x + \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(u, x(u), i(u)) \in J_i^*\}} b_i(u, x(u), k_i) \nu_i(s)(du, dk_i) \\ + \int_t^s \left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \sigma_i(u, x(u), k_i)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}} dW(u) \\ + l_{\nu_0(s)}(s), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned} \quad (4.7)$$

*Proof.* Let  $g = g(x) \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R})$ , we have using the classical Itô's formula

$$\begin{aligned} \forall s \in [t, T], \quad g \circ f(s, X(s)) = g \circ f(s, x) + \int_t^s \partial_x g \circ f(u, X(u)) df(u, X(u)) \\ + \frac{1}{2} \int_t^s \partial_{x,x} g \circ f(u, X(u)) d \langle f(\cdot, X(\cdot)) \rangle_u, \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

On the other hand we have

$$\begin{aligned}
\int_t^s \partial_x g \circ f(u, X(u)) df(u, X(u)) &= \sum_{i=1}^I \int_{K_i} \int_t^s \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_x g \circ f(u, X(u)) \left( \partial_t f_i(u, x(u)) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} f_i(u, x(u)) + b_i(u, x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s) (du, dk_i) \right. \\
&\quad \left. + \sum_{i=1}^I \int_{A_0} \int_t^s \alpha_i \partial_x g \circ f(u, X(u)) \partial_x f_i(u, 0) \nu_0(s) (du, d\alpha_1 \dots d\alpha_I) + \right. \\
&\quad \left. \int_t^s \partial_x g \circ f(u, X(u)) dM^f(u), \quad P_t^{(x,i)} \text{ a.s.} \right)
\end{aligned}$$

Using condition  $(\mathcal{S}_0)$  (ii), namely:  $\int_t^s \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(s) (du, d\alpha_1 \dots d\alpha_I) = 0$ ,  $P_t^{(x,i)}$  a.s, we get

$$\begin{aligned}
&\sum_{i=1}^I \int_t^s \int_{A_0} \partial_x g \circ f(u, X(u)) \alpha_i \partial_x f_i(u, 0) \nu_0(s) (du, d\alpha_1 \dots d\alpha_I) = \\
&\sum_{i=1}^I \int_t^s \int_{A_0} \partial_x g \circ f(u, 0) \alpha_i \partial_x f_i(u, 0) \nu_0(s) (du, d\alpha_1 \dots d\alpha_I), \quad P_t^{(x,i)} \text{ a.s.}
\end{aligned}$$

On the other hand, using that  $g \circ f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$ , we know that

$$\begin{aligned}
&\left( g \circ f(s, X(s)) - g \circ f(t, X(t)) - \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_t (g \circ f_i)(u, x(u)) \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} (g \circ f_i)(u, x(u)) + b_i(u, x(u), k_i) \partial_x (g \circ f_i)(u, x(u)) \right) \nu_i(s) (du, dk_i) \right. \\
&\quad \left. - \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x (g \circ f_i)(u, 0) \nu_0(s) (du, d\alpha_1 \dots d\alpha_I) \right)_{t \leq s \leq T},
\end{aligned}$$

is a  $(\Psi_s)_{t \leq s \leq T}$  continuous martingale under the probability measure  $P_t^{(x,i)}$ . Simple computations allows to get that, at each vertex, for all  $x \in J_i^*$  and for all  $s \in [t, T]$

$$\begin{aligned}
\partial_t (g \circ f)_i(s, x) + b_i(s, x, k_i) \partial_x (g \circ f)_i(s, x) + \frac{1}{2} \sigma_i^2(s, x, k_i) \partial_{x,x} (g \circ f)_i(s, x) = \\
\partial_t f_i(s, x) \partial_x g \circ f_i(s, x) + b_i(s, x, k_i) \partial_x f_i(s, x) \partial_x g \circ f_i(s, x) + \\
\frac{1}{2} \sigma_i^2(s, x, k_i) \left( \partial_{x,x} f_i(s, x) \partial_x g \circ f_i(s, x) + \partial_x f_i(s, x)^2 \partial_{x,x} g \circ f_i(s, x) \right).
\end{aligned}$$

Identifying the martingale and finite variation terms, we get that

$$\forall s \geq t, \quad d \langle f(\cdot, X(\cdot)) \rangle_s = \left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_x f_i(s, x(s)) \sigma_i(s, x(s), k_i) \right)^2 \nu_{i,s}(s)(dk_i) \right) ds, \quad P_t^{(x,i)} \text{ a.s.}$$

Considering the special case when  $f(x) = x$ , if  $x \in J_i^*$ , after an argument of localization with stopping times, and (using the ellipticity assumption (i)  $(\mathcal{H})$ ), if we set

$$\forall s \geq t, \quad W(s) = \int_t^s \frac{1}{\left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \sigma_i(x(u), k_i)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}}} df(u, X(u)), \quad P_t^{(x,i)} \text{ a.s.}$$

we get that

$$d \langle W(\cdot) \rangle_s = ds, \quad P_t^{(x,i)} \text{ a.s.}$$

Therefore using Paul Levy's Theorem characterization of the Brownian motion,  $W(\cdot)$  is a standard one dimensional Brownian motion,  $(\Psi_s)_{t \leq s \leq T}$  measurable and that completes the proof.  $\square$

Next, we get upper bounds of the modulus of continuity of both processes  $x(\cdot)$  and  $l(\cdot)$ , which are useful for the proof of the compactness of the admissible rules  $\mathcal{A}(t, (x, i))$  for the weak topology.

**Proposition 4.3.3.** *Let  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ , and  $s \in [t, T]$ . There exists a constant  $C$ , depending only on the data  $(T, (\|b_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)})_{1 \leq i \leq I}$ , such that*

$$\begin{aligned} \mathbb{E}^{P_t^{(x,i)}} \left[ \left| x(\cdot)^2 \right|_{(t,s)} \right] &\leq C(1 + x^2), \\ \mathbb{E}^{P_t^{(x,i)}} \left[ \left| l_{\nu_0(\cdot)}(\cdot)^2 \right|_{(t,s)} \right] &\leq C(1 + x^2), \\ \mathbb{E}^{P_t^{(x,i)}} \left[ \omega(X(\cdot), \theta)^2 \right] &\leq C\theta \ln\left(\frac{2T}{\theta}\right), \\ \mathbb{E}^{P_t^{(x,i)}} \left[ \omega(l_{\nu_0(\cdot)}(\cdot), \theta)^2 \right] &\leq C\theta \ln\left(\frac{2T}{\theta}\right), \end{aligned}$$

where we have defined the following modulus of continuity

$$\begin{aligned}\omega(X, \theta) &= \sup \left\{ d^{\mathcal{J}}(X(s), X(u)), (u, s) \in [t, T], |u - s| \leq \theta, \theta \in [0, T] \right\}, \\ \omega(l, \theta) &= \sup \left\{ |l(u) - l(s)|, (u, s) \in [t, T], |u - s| \leq \theta, \theta \in [0, T] \right\}.\end{aligned}$$

*Proof.* Let  $s \geq t$ . We define the following map  $f \in \mathcal{C}^{1,2}(\mathcal{J}^T)$ , by  $f(x, i) = x^2$ , if  $x \in J_i^*$ . After an argument of localization with stopping times, and using condition  $(\mathcal{S}_0)$  (iii), we get

$$\begin{aligned}\frac{1}{2} \left| x(s)^2 - x^2 \right| &\leq \\ \left| \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \left( b_i(u, x(u), k_i)x(u) + \sigma_i(u, x(u), k_i) \right) \nu_i(s)(du, dk_i) \right| + |M_s^f| &\leq \\ \left| \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \left( b_i(u, x(u), k_i)x(u) + \sigma_i(u, x(u), k_i) \right) \nu_i(\cdot)(du, dk_i) \right|_{(t,s)} + |M^f|_{(t,s)}.\end{aligned}$$

From Burkholder-Davis-Gundy inequality, and Proposition [4.3.2](#) we have

$$\begin{aligned}\mathbb{E}^{P_t^{(x,i)}} \left[ |M^f|_{(t,s)} \right] &= \\ \mathbb{E}^{P_t^{(x,i)}} \left[ \left| \int_t^s \left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \left( 2x(u)\sigma_i(u, x(u), k_i) \right)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}} dW(u) \right|_{(t,s)} \right] & \\ \leq 4\mathbb{E}^{P_t^{(x,i)}} \left[ \int_t^s \left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \left( 2x(u)\sigma_i(u, x(u), k_i) \right)^2 \nu_{i,u}(s)(dk_i) \right) du \right] & \\ \leq 16 \max_{i \in \{1, \dots, I\}} \|\sigma_i\|_{L^\infty([0, T] \times [0, +\infty) \times K_i)}^2 \mathbb{E}^{P_t^{(x,i)}} \left[ \int_t^s \left| x(\cdot)^2 \right|_{(t,u)} du \right].\end{aligned}$$

On the other hand it is easy to see that there exists a constant  $C$ , depending only on the data

$$(T, (\|b_i\|_{L^\infty([0, T] \times [0, +\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0, T] \times [0, +\infty) \times K_i)})_{1 \leq i \leq I}),$$

such that

$$\begin{aligned}\left| \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \left( b_i(u, x(u), k_i)x(u) + \sigma_i(u, x(u), k_i) \right) \nu_i(\cdot)(du, dk_i) \right|_{(t,s)} & \\ \leq C \left( 1 + \int_t^s \left| x(\cdot)^2 \right|_{(t,u)} du \right).\end{aligned}$$

Therefore there exists a constant  $C$ , depending only on the data

$$(T, (\|b_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)})_{1 \leq i \leq I}),$$

such that

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \left| x(\cdot)^2 \Big|_{(t,s)} - x^2 \right| \right] \leq C \left( 1 + \int_t^s \mathbb{E}^{P_t^{(x,i)}} \left[ \left| x(\cdot) \right|_{(t,u)}^2 du \right] \right).$$

Applying Gronwall's Lemma to the following measurable function

$$\rho := \begin{cases} [t, T] \rightarrow \mathbb{R} \\ s \mapsto \mathbb{E}^{P_t^{(x,i)}} \left[ \left| x(\cdot)^2 \right|_{(t,s)} \right] \end{cases},$$

we get that there exists a constant  $C$ , depending only on the data

$$(T, (\|b_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)})_{1 \leq i \leq I}),$$

such that

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \left| x(\cdot)^2 \right|_{(t,s)} \right] \leq C(1 + x^2).$$

On the other hand, using [\(4.7\)](#), it is easy to see that there exists a constant  $C$ , depending only on the data

$$(T, (\|b_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0,T] \times [0,+\infty) \times K_i)})_{1 \leq i \leq I}),$$

such that

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \left| l_{\nu_0(\cdot)}(\cdot)^2 \Big|_{(t,s)} \right| \right] \leq C(1 + x^2).$$

We turn now to prove the required upper bounds for the modulus of continuity of the process  $\left( x(s) \right)_{t \leq s \leq T}$ , and  $\left( l_{\nu_0(s)}(s) \right)_{t \leq s \leq T}$ . For this end, let  $\varepsilon > 0$ , we introduce the

following sequence of stopping times

$$\begin{aligned} \theta_0^\varepsilon = t \ ; \ \tau_0^\varepsilon = \inf \left\{ t < u \leq T; \ x(u) = 0 \right\} \ ; \ \theta_1^\varepsilon = \inf \left\{ \tau_0^\varepsilon < u \leq T; \ x(u) = \varepsilon \right\} \\ \dots \\ \tau_n^\varepsilon = \inf \left\{ \theta_n^\varepsilon < u \leq T; \ x(u) = 0 \right\} \ ; \ \theta_{n+1}^\varepsilon = \inf \left\{ \tau_n^\varepsilon < u \leq T; \ x(u) = \varepsilon \right\}, \end{aligned}$$

and for each  $u \in [t, T]$

$$\bar{\theta}_u := \inf \left\{ \theta_n; \ \theta_n^\varepsilon \geq u \right\}, \text{ and } \underline{\theta}_u := \sup \left\{ \theta_n; \ \theta_n^\varepsilon \leq u \right\}.$$

Let  $(u, s) \in [t, T]^2$  such that  $s \leq u$ , and  $u - s \leq \theta$ ,  $\theta \in (0, T]$ , we have (assuming that the process  $X(\cdot)$  has reached the junction point between time  $[s, u]$ , else inequality (4.8) is still available)

$$d^{\mathcal{J}}(X(u), X(s)) \leq d^{\mathcal{J}}(X(u), X(\underline{\theta}_u)) + d^{\mathcal{J}}(X(\underline{\theta}_u), X(\bar{\theta}_s)) + d^{\mathcal{J}}(X(\bar{\theta}_s), X(s)), \quad P_t^{(x,i)} \text{ a.s.}$$

We get therefore for any  $\varepsilon > 0$

$$\omega(X, \theta) \leq 2\omega(\tilde{M}, \theta) + 2\varepsilon, \quad P_t^{(x,i)} \text{ a.s.} \quad (4.8)$$

where we have defined the process  $\left(\tilde{M}(s)\right)_{t \leq s \leq T}$  by

$$\begin{aligned} \forall s \in [t, T], \quad \tilde{M}(s) = \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} b_i(u, x(u), k_i) \nu_i(s)(du, dk_i) \\ + \int_t^s \left( \sum_{i=1}^I \int_{K_i} \mathbf{1}_{\left\{ (x(u), i(u)) \in J_i^* \right\}} \sigma_i(u, x(u), k_i)^2 \nu_{i,u}(s)(dk_i) \right)^{\frac{1}{2}} dW(u), \quad P_t^{(x,i)} \text{ a.s.} \end{aligned}$$

The process  $\left(\tilde{M}(s)\right)_{t \leq s \leq T}$  satisfies assumptions of Theorem 3.1 of [13], therefore we know that there exists a constant  $C$ , depending only on the data

$$(T, (\|b_i\|_{L^\infty([0, T] \times [0, +\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0, T] \times [0, +\infty) \times K_i)})_{1 \leq i \leq I}),$$



such that

$$\forall \varepsilon > 0, \quad \mathbb{E}^{P_t^{(x,i)}} \left[ \omega(X(\cdot), \theta)^2 \right] \leq C\theta \ln\left(\frac{2T}{\theta}\right) + 2\varepsilon,$$

and then

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \omega(X(\cdot), \theta) \right] \leq C\theta \ln\left(\frac{2T}{\theta}\right).$$

We conclude finally using that

$$l_{\nu_0(u)}(u) - l_{\nu_0(s)}(s) = x(u) - x(s) - (\tilde{M}_u - \tilde{M}_s), \quad P_t^{(x,i)} \text{ a.s.}$$

□

**Lemma 4.3.4.** *Let  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ , and  $M > 0$ . There exists a constant  $C > 0$ , depending only on the data*

$$\left( T, M, \max_{i \in \{1 \dots I\}} \|b_i\|_{L^\infty((0,T) \times (0,+\infty) \times K_i)}, \max_{i \in \{1 \dots I\}} \|\sigma_i\|_{L^\infty((0,T) \times (0,+\infty) \times K_i)}, x \right),$$

introduced in assumption  $(\mathcal{H})$ , such that

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \exp(Mx(T)) \right] \leq C. \quad (4.9)$$

*Proof.* We define the following map  $\phi$  by

$$\phi := \begin{cases} [0, +\infty) \rightarrow \mathbb{R} \\ x \mapsto \exp(Mx) - Mx - 1 \end{cases}.$$

Let  $k \geq 0$ , we introduce the following stopping time

$$\theta_k := \inf\{s \in [t, T], \quad x(s) \geq k\}.$$

Hence, using conditions  $(\mathcal{S}_0)$  (iii) with  $\phi$  and Proposition [4.3.3](#), we get

$$\begin{aligned} \mathbb{E}^{P_t^{(x,i)}} \left[ \exp(Mx(T \wedge \theta_k)) \right] &= \exp(Mx) - Mx + \mathbb{E}^{P_t^{(x,i)}} \left[ Mx(T \wedge \theta_k) \right] + \\ &\mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^{T \wedge \theta_k} \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} \phi(x(u)) + \right. \right. \\ &\left. \left. b_i(u, x(u), k_i) \partial_x \phi(x(u)) \right) \nu_i(s)(du, dk_i) \right] \leq C \left( 1 + \mathbb{E}^{P_t^{(x,i)}} \left[ \int_t^{T \wedge \theta_k} \exp(Mx(u)) du \right] \right), \end{aligned}$$

where  $C$  is a constant depending only on

$$\left( T, M, \max_{i \in \{1 \dots I\}} \|b_i\|_{L^\infty((0,T) \times (0, +\infty) \times K_i)}, \max_{i \in \{1 \dots I\}} \|\sigma_i\|_{L^\infty((0,T) \times (0, +\infty) \times K_i)}, x \right).$$

Hence sending  $k \rightarrow +\infty$ , we get using monotone convergence's Theorem and Fubini's Theorem

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \exp(Mx(T)) \right] \leq C \left( 1 + \int_t^T \mathbb{E}^{P_t^{(x,i)}} \left[ \exp(Mx(u)) \right] du \right).$$

We conclude finally using Gronwall's Lemma to the following measurable map

$$\rho := \begin{cases} [t, T] \rightarrow \mathbb{R} \\ s \mapsto \mathbb{E}^{P_t^{(x,i)}} \left[ \exp(Mx(s)) \right] \end{cases}.$$

□

**Proposition 4.3.5.** *Let  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ . There exists a constant  $C > 0$ , depending only on the data  $\left( T, \max_{i \in \{1 \dots I\}} \|b_i\|_{L^\infty((0,T) \times (0, +\infty) \times K_i)}, \max_{i \in \{1 \dots I\}} \|\sigma_i\|_{L^\infty((0,T) \times (0, +\infty) \times K_i)}, C, x \right)$ , introduced in assumption  $(\mathcal{H})$ , such that*

$$\forall \varepsilon > 0, \quad \mathbb{E}^{P_t^{(x,i)}} \left[ \int_t^T \mathbf{1}_{\{x(s) \leq \varepsilon\}} ds \right] \leq C\varepsilon. \quad (4.10)$$

*Proof.* Let  $\varepsilon > 0$ , and  $\beta^\varepsilon \in \mathcal{C}([0, +\infty), \mathbb{R}_+)$  satisfying

$$\forall x \geq 2\varepsilon, \quad \beta_\varepsilon(x) = 0, \quad \forall x \geq 0, \quad \mathbf{1}_{\{x \leq \varepsilon\}} \leq \beta_\varepsilon(x) \leq 1. \quad (4.11)$$

We define  $u^\varepsilon \in \mathcal{C}^2([0, +\infty))$  as the unique solution of the following ordinary second order

differential equation

$$\begin{cases} \partial_{x,x}u^\varepsilon(x) - M\partial_x u^\varepsilon(x) = 2\beta^\varepsilon(x)/c^2, & \text{if } x \in (0, +\infty), \\ \partial_x u^\varepsilon(0) = 0, \\ u^\varepsilon(0) = 0. \end{cases} \quad (4.12)$$

where  $c$  is the constant of ellipticity defined in assumption  $(\mathcal{H})(i)$ , and  $M$  is given by

$$M = \max_{i \in \{1 \dots I\}} \frac{\|b_i\|_{L^\infty((0,T) \times (0,+\infty) \times K_i)}}{\frac{1}{2}c^2}.$$

For each  $i \in \{1 \dots I\}$ , the solution is given by

$$u^\varepsilon(x) = \int_0^x \exp(Mz) \int_0^z \frac{2\beta_i^\varepsilon(u)}{c^2} \exp(-Mu) du dz.$$

By the assumption on  $\beta_\varepsilon$ , and assumption  $(\mathcal{H})$ , we get

$$\forall x \geq 0, \quad 0 \leq \partial_x u^\varepsilon(x) \leq 4\varepsilon/c^2 \exp(Mx), \quad 0 \leq u^\varepsilon(x) \leq \frac{4\varepsilon}{Mc^2} (\exp(Mx) - 1). \quad (4.13)$$

Hence applying condition  $(\mathcal{S}_0)$  (iii) (with  $f = u^\varepsilon$ , after an argument of localization with

stopping times), we get using (4.11), (4.12) and (4.13)

$$\begin{aligned}
& \mathbb{E}^{P_t^{(x,i)}} \left[ u^\varepsilon(x(T)) - u^\varepsilon(x) \right] = \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} \left( \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} u^\varepsilon(x(u)) + \right. \right. \\
& \quad \left. \left. b_i(u, x(u), k_i) \partial_x u^\varepsilon(x(u)) \right) \nu_i(T)(du, dk_i) \right] = \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} \frac{1}{2} \sigma_i^2(u, x(u), k_i) \left( \partial_{x,x} u^\varepsilon(x(u)) + \right. \right. \\
& \quad \left. \left. \frac{b_i(u, x(u), k_i)}{\frac{1}{2} \sigma_i^2(u, x(u), k_i)} \partial_x u^\varepsilon(x(u)) \right) \nu_i(T)(du, dk_i) \right] \geq \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} \frac{1}{2} \sigma_i^2(u, x(u), k_i) \left( \partial_{x,x} u^\varepsilon(x(u)) - M \partial_x u^\varepsilon(x(u)) \right) \nu_i(T)(du, dk_i) \right] \\
& \geq \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} \frac{1}{2} c^2 \left( 2\beta^\varepsilon(x(u))/c^2 \right) \nu_i(T)(du, dk_i) \right] \\
& \geq \mathbb{E}^{P_t^{(x,i)}} \left[ \int_t^T \beta^\varepsilon(x(u)) du \right] \geq \mathbb{E}^{P_t^{(x,i)}} \left[ \int_0^T \mathbf{1}_{\{x(u) \leq \varepsilon\}} du \right].
\end{aligned}$$

Hence we get using (4.13)

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \int_0^T \mathbf{1}_{\{x(s) \leq \varepsilon\}} ds \right] \leq \frac{4\varepsilon}{Mc^2} \mathbb{E}^{P_t^{(x,i)}} \left[ \exp(Mx(T)) - 1 \right].$$

We conclude using Lemma 4.3.4. □

We are able now to prove the main result of this section, namely the compactness of  $\mathcal{A}(t, (x, i))$ .

**Theorem 4.3.6.** *The set of probability measures  $\mathcal{A}(t, (x, i))$ , endowed with the weak topology is non empty, convex and compact. Moreover, the value function  $v(\cdot, \cdot)$  attains its minimum. Finally the set of optimal rules is non empty convex and compact.*

*Proof.* We recall that the fact that  $\mathcal{A}(t, (x, i))$  is non empty is a consequence of Remark 4.2.3. Let us show first that  $\mathcal{A}(t, (x, i))$  is precompact for the weak topology.

It is enough to show that all the following projections

$$\left\{ P_t^{(x,i)}|_{\mathcal{C}\mathcal{J}[0,T]}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\},$$

$$\left( \left\{ P_t^{(x,i)}|_{V([0,T] \times K_i)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\} \right)_{i \in \{1, \dots, I\}},$$

$$\left\{ P_t^{(x,i)}|_{V([0,T] \times A_0)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\},$$

are precompact. The precompactness of  $\left\{ P_t^{(x,i)}|_{\mathcal{C}\mathcal{J}[0,T]}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}$  is a consequence of the upper bounds obtained in Proposition [4.3.3](#), and Ascoli's Theorem.

We focus on the precompactness of  $\left\{ P_t^{(x,i)}|_{V([0,T] \times A_0)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}$ . Let  $\varepsilon > 0$ . It follows from Proposition [4.3.3](#), that there exists a constant  $C > 0$ , depending only on the data

$$(T, (\|b_i\|_{L^\infty([0,T] \times [0, +\infty) \times K_i)}, \|\sigma_i\|_{L^\infty([0,T] \times [0, +\infty) \times K_i)})_{1 \leq i \leq I}),$$

such that

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \left| l_{\nu_0(\cdot)}(\cdot)^2 \right|_{(t,T)} \right] \leq C(1 + x^2),$$

$$\forall \theta \in (0, T], \quad \mathbb{E}^{P_t^{(x,i)}} \left[ \omega(l_{\nu_0(\cdot)}(\cdot), \theta)^2 \right] \leq C\theta \ln\left(\frac{2T}{\theta}\right).$$

Let  $r_\varepsilon \in \mathbb{R}_+^*$  such that

$$\forall \theta \in (0, T], \quad \frac{C(1 + x^2)}{r_\varepsilon^2} + \frac{C\theta \ln\left(\frac{2T}{\theta}\right)}{r_\varepsilon^2} \leq \varepsilon,$$

and let us set

$$K_\varepsilon := \left\{ \nu_0 \in V([0, T] \times A_0), \quad l_{\nu_0}(T) \leq \frac{C(1 + x^2)}{r_\varepsilon}, \quad \forall \theta \in (0, T], \quad w(\theta, l_{\nu_0}) \leq \frac{C\theta \ln\left(\frac{2T}{\theta}\right)}{r_\varepsilon} \right\}.$$

Using Proposition [4.2.1](#), we know that  $K_\varepsilon$  is compact for the weak topology  $*\sigma\left(L_{mc}^\infty([0, T] \times$

$A_0)'$ ,  $L_{mc}^\infty([0, T] \times A_0)$ ). Moreover, using Tchebychev's inequality, we get that

$$\begin{aligned} & P_t^{(x,i)}|_{V([0,T] \times A_0)} \left( \left( \nu_0(s) \right)_{t \leq s \leq T} \notin K_\varepsilon \right) \leq \\ & \frac{\mathbb{E}^{P_t^{(x,i)}} \left[ \left| l_{\nu_0(\cdot)}(\cdot)^2 \right|_{(t,T)} \right]}{r_\varepsilon^2} + \frac{\mathbb{E}^{P_t^{(x,i)}} \left[ Y(\cdot)(l_{\nu_0(\cdot)}(\cdot), \theta)^2 \right]}{r_\varepsilon^2} \leq \varepsilon, \end{aligned}$$

and that proves the precompactness of  $\left\{ P_t^{(x,i)}|_{\mathcal{C}^{\mathcal{J}}[0,T]}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}$ . Finally, knowing that all  $(V([0, T] \times K_i)_{1 \leq i \leq I})$  are compact, we can show that  $\left( \left\{ P_t^{(x,i)}|_{V([0,T] \times K_i)}, P_t^{(x,i)} \in \mathcal{A}(t, (x, i)) \right\}_{i \in \{1, \dots, I\}} \right)$  are precompact.

We turn now to prove that  $\mathcal{A}(t, (x, i))$  is closed, and for this let  $P_{t,n}^{(x,i)}$  converging weakly to  $P_t^{(x,i)}$ . We are going to show that  $P_t^{(x,i)}$  satisfies condition  $(\mathcal{S}_0)$ .

Let  $f_p \in \mathcal{C}_b(\Phi, R)$ , uniformly bounded in  $p$ , converging to  $\mathbf{1}_{(X(u)_{0 \leq u \leq t} = ((x,i), \nu_1(t) \dots \nu_I(t), \nu_0(t))}$  in the pointwise sense, and from above. We have

$$\begin{aligned} \forall p \geq 0, \quad \mathbb{E}^{P_t^{(x,i)}} \left[ f_p(X(\cdot)) \right] &= \lim_{n \rightarrow +\infty} \mathbb{E}^{P_{t,n}^{(x,i)}} \left[ f_p(X(\cdot)) \right] \geq \\ & \lim_{n \rightarrow +\infty} \mathbb{E}^{P_{t,n}^{(x,i)}} \left[ \mathbf{1}_{(X(u)_{0 \leq u \leq t} = ((x,i), \nu_1(t) \dots \nu_I(t), \nu_0(t))} \right] = 1. \end{aligned}$$

Therefore we get

$$\lim_{p \rightarrow +\infty} \mathbb{E}^{P_t^{(x,i)}} \left[ f_p(X(\cdot)) \right] = 1,$$

and using Lebesgue's Theorem we have

$$\mathbb{E}^{P_t^{(x,i)}} \left[ \mathbf{1}_{(X(u)_{0 \leq u \leq t} = ((x,i), \nu_1(t) \dots \nu_I(t), \nu_0(t))} \right] = 1,$$

which means that (i) of conditions  $(\mathcal{S}_0)$  holds true.

Recall that from Proposition [4.3.1](#), the following map

$$\rho : \begin{cases} \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0) \rightarrow \mathbb{R} \\ \left( (x(\cdot), i(\cdot)), \nu_0 \right) \mapsto \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1 \dots d\alpha_I) \end{cases}$$

is lower semi continuous. Consequently, the following set  $O$  defined by

$$O := \left\{ \left( (x(\cdot), i(\cdot)), \nu_0 \right) \in \mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0), \int_t^T \int_{A_0} \mathbf{1}_{\{x(u) > 0\}} \nu_0(du, d\alpha_1 \dots d\alpha_I) > 0 \right\},$$

is open in  $\mathcal{C}^{\mathcal{J}}[0, T] \times V([0, T] \times A_0)$ . We get then

$$P_t^{(x,i)}(O) \leq \liminf_{n \rightarrow +\infty} P_{t,n}^{(x,i)}(O) = 0,$$

which means that (ii) of condition  $(\mathcal{S}_0)$  holds true.

Now let us show that (iii) of condition  $(\mathcal{S}_0)$  holds true. For this let  $q \in \mathcal{C}_b(\Phi, \mathbb{R})$ ,  $\Psi_s$  measurable, and  $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$ , we have using Lemma [4.3.5](#)

$$\mathbb{E}^{P_{t,n}^{(x,i)}} \left[ \int_t^T \mathbf{1}_{\{x(s)=0\}} ds \right] = \mathbb{E}^{P_t^{(x,i)}} \left[ \int_t^T \mathbf{1}_{\{x(s)=0\}} ds \right] = 0.$$

Hence

$$0 = \mathbb{E}^{P_{t,n}^{(x,i)}} \left[ q(M_t^f - M_s^f) \right] \xrightarrow{n \rightarrow +\infty} \mathbb{E}^{P_t^{(x,i)}} \left[ q(M_t^f - M_s^f) \right],$$

which means that the process

$$\begin{aligned} \left( M_s^f - M_t^f = f(s, X(s)) - f(t, X(t)) - \sum_{i=1}^I \int_t^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_t f_i(u, x(u)) \right. \right. \\ \left. \left. + \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} f_i(u, x(u)) + b_i(u, x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) \right. \\ \left. - \sum_{i=1}^I \int_t^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \right)_{t \leq s \leq T}, \end{aligned}$$

is a  $(\Psi_s)_{t \leq s \leq T}$  continuous martingale under the probability measure  $P_t^{(x,i)}$ , after time  $t$ , and that finally proves that  $\mathcal{A}(t, (x, i))$  is closed for the weak topology.

We end the proof by showing that the value function  $v(\cdot, \cdot)$  attains its minimum, and the set of optimal rules is convex and compact. Using Proposition [4.3.1](#), it is easy to check

that the reward function  $\Lambda$

$$\Lambda : \begin{cases} \mathcal{A}(t, (x, i)) \rightarrow \mathbb{R} \\ P_t^{(x, i)} \mapsto \mathbb{E}^{P_t^{(x, i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(u, x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) \right. \\ \left. + \sum_{i=1}^I \int_t^T \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right], \end{cases}$$

is lower semi continuous for the weak topology. Therefore the value function  $v(\cdot, \cdot)$  attains its minimum on the compact set  $\mathcal{A}(t, (x, i))$ . Finally, the fact that the set of optimal rules is convex and compact, is a consequence of the compactness of  $\mathcal{A}(t, (x, i))$ , the lower semi continuity of  $\Lambda$ , and the linearity of  $P_t^{(x, i)} \mapsto \Lambda(P_t^{(x, i)})$ .  $\square$

**Proposition 4.3.7.** *The following map*

$$\begin{cases} [0, T] \times \mathcal{J} \rightarrow \mathcal{P}(\Phi, (\Phi)) \\ (t, (x, i)) \mapsto \mathcal{A}(t, (x, i)) \end{cases} \quad (4.14)$$

(where  $\mathcal{P}(\Phi, (\Phi))$  is the set of probability measures defined on  $\Phi$ ), is upper semi continuous.

*Proof.* We endow  $\mathcal{P}(\Phi, (\Phi))$  with the Hausdorff metric defined over all its compact sets. Since we have shown that  $\mathcal{A}(t, (x, i))$  is compact for the weak topology, we follow then the same arguments of the proof of Proposition 5.10 in [25].  $\square$

Therefore as a consequence of the Proposition 4.3.7, Proposition 4.3.1 and Theorem 5.11 in [25], the value function defined in (4.6) by

$$v := \begin{cases} [0, T] \times \mathcal{J} \rightarrow \mathbb{R} \\ (t, (x, i)) \mapsto v_i(t, x) \end{cases},$$

is lower semi continuous.



## 4.4 Dynamic Programming Principle

The following section is dedicated to the proof of the dynamic programming principle.

Both stability of the set  $\mathcal{A}(t, (x, i))$  by conditioning and concatenation are proved.

We recall first a lemma of measurable selection, that will be useful in the sequel, (see for instance Corollary 5.4 in [25]).

**Proposition 4.4.1.** *Let  $\mathcal{G}, \mathcal{H}$  be two separable metric spaces. Let  $w$  a lower semi continuous real function on  $\mathcal{G} \times \mathcal{H}$  and  $h \mapsto K_h$  a measurable map from  $\mathcal{H}$  into  $\text{comp}(\mathcal{G})$ , (the set of compact sets of  $\mathcal{G}$ , endowed with the Hausdorff metric). Then*

*-the map  $v(h) := \inf \{ w(g, h), g \in K_h \}$  is a Borel function and  $h \mapsto M_h := \{ g, v(h) = w(h, g), g \in K_h \}$  is a measurable map of  $\mathcal{H}$  into  $\text{comp}(\mathcal{G})$ .*

*-for each probability measure  $P$  on  $\mathcal{H}$*

$$\begin{aligned} \int v(h) dP(h) &= \int \inf \{ w(g, h), g \in K_h \} dP(h) \\ &= \inf \left\{ \int w(\beta(h), h) dP(h), \beta : \mathcal{H} \rightarrow \mathcal{G}, \text{ measurable}, \beta(h) \in K_h \right\}. \end{aligned}$$

**Proposition 4.4.2.** *Let  $\tau$  a  $(\Psi_t)_{0 \leq t \leq T}$  a stopping time, then  $\Psi_\tau = \sigma(X(s \wedge \tau), s \leq T)$  is countably generated.*

*Proof.* Recall that

$$\Psi_\tau = \left\{ B \in \Psi_T, B \cap \{\tau \leq t\} \in \Psi_t, \forall t \in [0, T] \right\},$$

and the space where is defined our canonical process  $X(\cdot)$

$$\Phi = \mathcal{C}^{\mathcal{J}}[0, T] \times \left( \prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0),$$

is Polish.

We can use then the same arguments of the proof of Lemma 1.3.3 in [37], to get the result.

□

We start first by showing the stability of the set  $\mathcal{A}(t, (x, i))$  by conditioning.

**Proposition 4.4.3.**  $\mathcal{A}(t, (x, i))$  is stable under conditioning. More precisely, let  $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$ , and  $\tau$  a  $(\Psi_s)_{s \geq t}$  stopping time, then there exists a r.c.p.d (regular conditional probability distribution) of  $P_t^{(x, i)}$  conditionally to  $\Psi_\tau$ , denoted by  $P_\tau^{(x, i_\tau)}$ , such that  $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$ ,  $P_t^{(x, i)}$  a.s.

*Proof.* Let  $\tau : \Phi \rightarrow [t, T]$ ,  $Z(\cdot) \mapsto \tau(Z(\cdot))$  be a  $(\Psi_s)_{t \leq s \leq T}$  stopping time, and let  $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$ . Since the canonical space  $\Phi$  is Polish, we know from Theorem 1.3.4 of [37], that there exist a r.c.p.d of  $P_t^{(x, i)}$  respectively to the sub algebra  $\Psi_\tau$ , that we denote  $\bar{P}_\tau^{(x_\tau, i_\tau)}$ .

We define the following map

$$\mathcal{K} := \left\{ \begin{array}{l} \Phi \rightarrow \mathbb{B}(\Phi), \\ Z(\cdot) \mapsto \mathcal{K}(Z(\cdot)) := \left\{ Y(\cdot) \in \Phi, X(s, Y(\cdot)) = X(\tau(Z(\cdot)), Y(\cdot)), \forall s \in [t, \tau(Z(\cdot))] \right\} \end{array} \right\}.$$

On the other hand we set

$$\forall B \in \mathbb{B}(\Phi), P_\tau^{(x_\tau, i_\tau)}(B) = \int_\Phi \mathbf{1}_{\{Y(\cdot) \in B \cap \mathcal{K}(Z(\cdot))\}} d\bar{P}_\tau^{(x_\tau, i_\tau)}(Y(\cdot)), P_t^{(x, i)} \text{ a.s.}$$

We are going to prove that  $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$ ,  $P_t^{(x, i)}$  a.s.

From the definition of  $\mathcal{K}(\cdot)$ , it is easy to get that (i) of condition  $(\mathcal{S}_0)$  holds true. On the other hand writing

$$\begin{aligned} & \mathbb{E}^{P_t^{(x, i)}} \left[ \mathbf{1}_{\left\{ \int_{\tau(Z(\cdot))}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(s)}(du) = 0 \right\}} \right] = \\ & \mathbb{E}^{P_t^{(x, i)}} \left[ \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ \mathbf{1}_{\left\{ \int_{\tau(Z(\cdot))}^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0(T)}(du) = 0 \right\}} \right] \right] = 0, \end{aligned}$$

we get

$$P_\tau^{(x_\tau, i_\tau)} \left( \left\{ Y(\cdot) \in \Phi, \int_\tau^T \mathbf{1}_{\{x(u, Y(\cdot)) > 0\}} l_{\nu_0(T, Y(\cdot))}(du) = 0 \right\} \right) = 0, P_t^{(x, i)} \text{ a.s.},$$

and (ii) of condition  $(\mathcal{S}_0)$  holds true. Finally, let  $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$ . Using Theorem 1.2.10 of

[37], we get

$$\left( M_s^f := f(s, X(s)) - f(\tau, X(\tau)) - \sum_{i=1}^I \int_{\tau}^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} \left( \partial_t f_i(u, x(u)) + \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} f_i(u, x(u)) + b_i(u, x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) - \sum_{i=1}^I \int_{\tau}^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \right)_{\tau \leq s \leq T},$$

is a  $(\Psi_s)_{\tau \leq s \leq T}$  continuous martingale under the probability measure  $P_{\tau}^{(x_{\tau}, i_{\tau})}$ , after the stopping time  $\tau$ ,  $P_t^{(x, i)}$  a.s, but on a negligible set depending on  $f$ , that we denote by  $\mathcal{N}(f)$ .

Assume then first that  $f \in \mathcal{C}_0^{1,2}(\mathcal{J}_T)$ : the class of continuous functions defined on  $[0, T] \times \mathcal{J}$ , having a regularity of class  $\mathcal{C}^{1,2}([0, T] \times [0, +\infty))$  on each edge, and vanishing at each edge at  $+\infty$ . We get then that  $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$  is separable with the following norm  $\|\cdot\|_{\mathcal{C}_0^{1,2}(\mathcal{J}_T)}$ , defined by

$$\forall f \in \mathcal{C}_0^{1,2}(\mathcal{J}_T), \quad \|f\|_{\mathcal{C}_0^{1,2}(\mathcal{J}_T)} = \sum_{1 \leq i \leq I} \|f_i\|_{\mathcal{C}^{1,2}([0, T] \times [0, +\infty))},$$

$$\text{with : } \|f_i\|_{\mathcal{C}^{1,2}(\mathcal{J}_T)} = \sup_{(t,x) \in [0, T] \times [0, +\infty)} |f_i(t, x)| + \sup_{(t,x) \in [0, T] \times [0, +\infty)} |\partial_t f_i(t, x)| + \sup_{(t,x) \in [0, T] \times [0, +\infty)} |\partial_x f_i(t, x)| + \sup_{(t,x) \in [0, T] \times [0, +\infty)} |\partial_{x,x} f_i(t, x)|.$$

Hence, let  $f_n$  a sequence of  $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$ , dense in  $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$ , we set

$$\mathcal{N} = \bigcup_{n \geq 0} \mathcal{N}(f_n).$$

Thereafter, using that following functional

$$\kappa := \begin{cases} \mathcal{C}_0^{1,2}(\mathcal{J}_T) \rightarrow \mathbb{R} \\ f \mapsto \left( f(s, X(s)) - \sum_{i=1}^I \int_{\tau}^s \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i\}} \left( \partial_t f_i(u, x(u)) + \frac{1}{2} \sigma_i^2(u, x(u), k_i) \partial_{x,x} f_i(u, x(u)) + b_i(u, x(u), k_i) \partial_x f_i(u, x(u)) \right) \nu_i(s)(du, dk_i) - \sum_{i=1}^I \int_{\tau}^s \int_{A_0} \alpha_i \partial_x f_i(u, 0) \nu_0(s)(du, d\alpha_1 \dots d\alpha_I) \right)_{\tau \leq s \leq T}, \end{cases}$$

is continuous for any

$$\left( \left( x(\cdot), i(\cdot) \right), \nu_1 \dots \nu_I, \nu_0 \right) \in \mathcal{C}^{\mathcal{J}}[0, T] \times \left( \prod_{i=1}^I U([0, T] \times K_i) \right) \times V([0, T] \times A_0),$$

it is easy to check using Lebesgue's Theorem that  $(M_s^f)_{\tau \leq s \leq T}$  is a  $(\Psi_s)_{\tau \leq s \leq T}$  continuous martingale under the probability measure  $P_\tau^{(x_\tau, i_\tau)}$ , after the stopping time  $\tau$ ,  $P_t^{(x, i)}$  a.s, on the negligible set  $\mathcal{N}$ , using once again that from Lemma [4.3.5](#)

$$\mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ \int_\tau^T \mathbf{1}_{\{x(s)=0\}} ds \right] = 0, \quad P_t^{(x, i)} \text{ a.s.}$$

To conclude, let  $n \geq 0$ ,  $f \in \mathcal{C}_b^{1,2}(\mathcal{J}_T)$ , and  $f_n \in \mathcal{C}_0^{1,2}(\mathcal{J}_T)$  a sequence converging in the pointwise sens to  $f$ , and equal to  $f$  on each edge  $J_i \cap [0, n]$ .

Let then  $\theta$  a  $\Psi_\tau$  stopping time, using Proposition [4.3.3](#), Tchebychev's inequality and assumption  $(\mathcal{H})$ , it is easy to get that there exists a constant  $C > 0$  independent of  $n$  such that

$$\left| \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ M_s^{f_n} \mathbf{1}_{\{x(s) \geq n\}} \right] \right| \leq \frac{C}{n^2}. \quad (4.15)$$

We write then

$$\begin{aligned} \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ M_s^{f_n} \middle| \Psi_\theta \right] &= \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ M_s^{f_n} \mathbf{1}_{\{x(s) \leq n\}} \middle| \Psi_\theta \right] + \\ \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ M_s^{f_n} \mathbf{1}_{\{x(s) \geq n\}} \middle| \Psi_\theta \right] &= M_\theta^{f_n}, \quad P_t^{(x, i)} \text{ a.s,} \end{aligned}$$

and we conclude using Lebesgue's Theorem and [\(4.15\)](#). □

The second step is to prove the stability by concatenation.

**Proposition 4.4.4.**  $\mathcal{A}(t, (x, i))$  is stable under concatenation. Namely, let  $\tau$  a  $(\Psi_s)_{t \leq s \leq T}$  stopping time,  $Q_t^{(x, i)} \in \mathcal{A}(t, (x, i))$  and  $Q_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$ ,  $Q_t^{(x, i)}$  a.s. Suppose that  $Q_\tau^{(x_\tau, i_\tau)}$ , is a transition probability kernel from  $(\Phi, \Psi_\tau)$  to  $(\Phi, \Psi_T)$ . We can then compute a concatenated probability with  $Q_t^{(x, i)}$  and  $Q_\tau^{(x_\tau, i_\tau)}$ , denoted by  $P_t^{(x, i)}$ , such that

$$P_t^{(x, i)} = Q_t^{(x, i)} \oplus_\tau Q_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(t, (x, i)).$$

*Proof.* Let  $\tau : \Phi \mapsto [t, T]$ ,  $Z(\cdot) \rightarrow \tau(Z(\cdot))$  be a  $(\Psi_s)_{t \leq s \leq T}$  stopping time, and  $Q_t^{(x, i)}$  and

$Q_\tau^{(x_\tau, i_\tau)}$  satisfying the required assumptions of Proposition 4.4.4.

We define the following map

$$\mathcal{K} := \left\{ \begin{array}{l} \Phi \rightarrow \mathbb{B}(\Phi), \\ Z(\cdot) \mapsto \mathcal{N}(Z(\cdot)) := \left\{ Y(\cdot) \in \Phi, \quad X(\tau(Z(\cdot)), Y(\cdot)) = X(\tau(Z(\cdot)), Z(\cdot)) \right\} \end{array} \right\}.$$

Consider now  $\left( Y(\cdot) = \left( (y(\cdot), j(\cdot)), \nu_1 \dots \nu_I, \nu_0 \right), \tilde{Y}(\cdot) = \left( (\tilde{y}(\cdot), \tilde{j}(\cdot)), \tilde{\nu}_1 \dots \tilde{\nu}_I, \tilde{\nu}_0 \right) \right) \in \mathcal{K}(Z(\cdot))^2$ . We define the following concatenated  $(Y \oplus_\tau \tilde{Y})(\cdot)$  variable of  $\Phi$ , whose projection on  $\mathcal{C}^{\mathcal{J}}([0, T])$  is given by

$$\begin{aligned} (Y \oplus_\tau \tilde{Y})(\cdot)|_{\mathcal{C}^{\mathcal{J}}([0, T])} &:= (y(\cdot), j(\cdot)) \oplus_\tau (\tilde{y}(\cdot), \tilde{j}(\cdot)) \\ &:= \begin{cases} (y(s), j(s)), & \text{if } s \leq \tau(Z(\cdot)) \\ (\tilde{y}(s), \tilde{j}(s)), & \text{if } \tau(Z(\cdot)) \leq s \leq T \end{cases}. \end{aligned}$$

On the other, for each  $i \in \{1 \dots I\}$ , the projections of  $(Y \oplus_\tau \tilde{Y})(\cdot)$  on each  $V([0, T] \times K_i)$  are given by

$$(Y \oplus_\tau \tilde{Y})(\cdot)|_{V([0, T] \times K_i)} := \nu_i \oplus_\tau \tilde{\nu}_i = \mathbf{1}_{\{\cdot \leq \tau(Z(\cdot))\}}(\cdot) \nu_i(ds, dk_i) + \mathbf{1}_{\{\cdot \geq \tau(Z(\cdot))\}}(\cdot) \tilde{\nu}_i(ds, dk_i).$$

Finally, the projection of  $(Y \oplus_\tau \tilde{Y})(\cdot)$  on  $V([0, T] \times A_0)$  is given by

$$\begin{aligned} (Y \oplus_\tau \tilde{Y})(\cdot)|_{V([0, T] \times A_0)} &:= \nu_0 \oplus_\tau \tilde{\nu}_0 = \mathbf{1}_{\{\cdot \leq \tau(Z(\cdot))\}} l_{\nu_0(\cdot)}(ds) \nu_{0,\cdot}(d\alpha_1 \dots d\alpha_I) + \\ &\quad \mathbf{1}_{\{\cdot \geq \tau(Z(\cdot))\}} \bar{l}_{\tilde{\nu}_0(\cdot)}(ds) \tilde{\nu}_{0,\cdot}(d\alpha_1 \dots d\alpha_I). \end{aligned}$$

where we have defined for each  $s \geq \tau(Z(\cdot))$

$$\bar{l}_{\tilde{\nu}_0(s)}(s) = \tilde{l}_{\tilde{\nu}_0(s)}(s) \mathbf{1}_{\{s \geq \tau(Z(\cdot))\}} + \left( l_{\nu_0(s)}(\tau(Z(\cdot))) - \tilde{l}_{\tilde{\nu}_0(s)}(s)(\tau(Z(\cdot))) \right) \mathbf{1}_{\{s \leq \tau(Z(\cdot))\}}.$$

We define then the following concatenated Borel probability measure  $P_t^{(x, i)} = Q_t^{(x, i)} \oplus_\tau$

$Q_\tau^{(x_\tau, i_\tau)}$  on  $(\Phi, \mathbb{B}(\Phi))$  by

$$\forall B \in \mathbb{B}(\Phi), \quad P_t^{(x, i)}(B) = Q_t^{(x, i)} \oplus_\tau Q_\tau^{(x_\tau, i_\tau)}(B) = \int_{Z(\cdot) \in \Phi} \left( \int_{Y(\cdot) \in \Phi} \left( \int_{\tilde{Y}(\cdot) \in \Phi} \mathbf{1}_B((Y \oplus_\tau \tilde{Y})(\cdot)) dQ_{\tau(Z(\cdot))}^{(x_\tau(Z(\cdot)), i_\tau(Z(\cdot)))} \right) \mathbf{1}_{\{\mathcal{K}(Z(\cdot))\}} dQ_t^{(x, i)} \right) dQ_t^{(x, i)}$$

Let us show now that  $P_t^{(x, i)} \in \mathcal{A}(t, (x, i))$ , and that conditions  $(\mathcal{S}_0)$  holds true.

We have

$$\begin{aligned} & P_t^{(x, i)} \left( \left\{ \left( Y \oplus_{\tau(Z(\cdot))} \tilde{Y} \right)(\cdot) \in \Phi \cap \mathcal{K}(Z(\cdot)), \quad \forall s \leq t, \right. \right. \\ & \left. \left. X \left( s, \left( Y \oplus_{\tau(Z(\cdot))} \tilde{Y} \right)(\cdot) \right) = X \left( t, \left( Y \oplus_{\tau(Z(\cdot))} \tilde{Y} \right)(\cdot) \right) \right\} \right) \\ &= Q_t^{(x, i)} \left( \left\{ Y(\cdot) \in \Phi, \quad \forall s \leq t, \quad X \left( s, Y(\cdot) \right) = X \left( t, Y(\cdot) \right) \right\} \right) = 1, \end{aligned}$$

which means that (i) of  $(\mathcal{S}_0)$  holds true.

On the other hand, we get

$$\begin{aligned} & \mathbb{E}^{P_t^{(x, i)}} \left[ \int_t^T \mathbf{1}_{\{x(u, Y \oplus_{\tau(Z(\cdot))} \tilde{Y})(\cdot)) > 0\}} l_{\nu_0 \oplus_{\tau(Z(\cdot))} \tilde{\nu}_0}(u)(T)(du) \right] \\ &= \mathbb{E}^{Q_t^{(x, i)}} \left[ \int_t^{\tau(Z(\cdot))} \mathbf{1}_{\{x(u, Y(\cdot)) > 0\}} l_{\nu_0}(u)(\tau(Z(\cdot)))(du) \right] + \\ & \mathbb{E}^{Q_t^{(x, i)}} \left[ \mathbb{E}^{Q_t^{(x, i)}} \left[ \mathbb{E}^{Q_{\tau(Z(\cdot))}^{(x_\tau(Z(\cdot)), i_\tau(Z(\cdot)))}} \left[ \int_{\tau(Z(\cdot))}^T \mathbf{1}_{\{x(u, \tilde{Y}(\cdot)) > 0\}} l_{\tilde{\nu}_0}(u)(T)(du) \right] \right] \right] = 0. \end{aligned}$$

Therefore

$$\int_t^T \mathbf{1}_{\{x(u) > 0\}} l_{\nu_0 \oplus_{\tau} \tilde{\nu}_0}(u)(T)(du) = 0, \quad P_t^{(x, i)} \text{ a.s.}, \quad (4.16)$$

which means that (ii) of conditions  $(\mathcal{S}_0)$  is true. We finish with the martingale conditions (iii) of  $(\mathcal{S}_0)$ . For this, we use once again as in the proof of Proposition [4.4.3](#), the reverse of Theorem 1.2.10 of [\[37\]](#), Lemma [4.3.5](#), and the argument of separability of  $\mathcal{C}_0^{1,2}(\mathcal{J}_T)$ . We can conclude that (iii) conditions of  $(\mathcal{S}_0)$  holds true and that completes the proof.  $\square$

Now we have the necessary tools in order prove the main result of this Section, namely the dynamic programming principle.

**Theorem 4.4.5.** *Dynamic Programming equation : let  $\tau$  be a  $(\Psi)_{s \geq t}$  stopping time, we*

have

$$\begin{aligned}
v_i(t, x) = & \\
\inf_{P_t^{(x,i)} \in \mathcal{A}(t, (x,i))} \left\{ \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(\tau)(du, dk_i) \right. \right. \\
& \left. \left. + \sum_{i=1}^I \int_t^\tau \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] \right\}. \quad (4.17)
\end{aligned}$$

*Proof.* Let  $\tau$  be a  $(\Psi_s)_{s \geq t}$  stopping time, and  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ , we have

$$\begin{aligned}
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \right. \\
& \quad \left. \sum_{i=1}^I \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] = \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \right. \right. \\
& \quad \left. \left. \sum_{i=1}^I \int_t^T \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \mid \Psi_\tau \right] \right].
\end{aligned}$$

Using Proposition [4.4.3](#), namely the stability by conditioning, we know that there exists a r.c.p.d  $P_\tau^{(x_\tau, i_\tau)}$  of  $P_t^{(x,i)}$ , conditionally to  $\Psi_\tau$ , such that  $P_\tau^{(x_\tau, i_\tau)} \in \mathcal{A}(\tau, (x_\tau, i_\tau))$ ,  $P_t^{(x,i)}$  a.s, which means therefore

$$\begin{aligned}
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \right. \\
& \quad \left. \sum_{i=1}^I \int_t^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \geq \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(\tau)(du, dk_i) + \right. \\
& \quad \left. \sum_{i=1}^I \int_t^\tau \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right].
\end{aligned}$$

Taking the infimum over all the  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ , we get then the following first

inequality

$$v_i(t, x) \geq \inf_{P_t^{(x,i)} \in \mathcal{A}(t, (x, i))} \left\{ \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(\tau)(du, dk_i) + \sum_{i=1}^I \int_t^\tau \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] \right\}.$$

Let  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ . We focus now on the reverse inequality. For this we use Proposition 4.4.1, with  $\mathcal{G} = \mathcal{P}(\Phi, \mathbb{B}(\Phi))$ ,  $\mathcal{H} = (\Phi, \mathbb{B}(\Phi))$ ,  $Z(\cdot) \rightarrow K_{Z(\cdot)} = \mathcal{A}(\tau(Z(\cdot)), (x_{\tau(Z(\cdot))}, i_{\tau(Z(\cdot))}))$  and

$$w : \begin{cases} \mathcal{P}(\Phi, \mathbb{B}(\Phi)) \times (\Phi, \mathbb{B}(\Phi)) \rightarrow \mathbb{R} \\ (P, Y(\cdot)) \mapsto \mathbb{E}^P \left[ \sum_{i=1}^I \int_t^T \int_{K_i} \mathbf{1}_{\{(y(u), j(u)) \in J_i^*\}} h_i(u, y(u), k_i) \nu_i(T)(du, dk_i) + \sum_{i=1}^I \int_t^T \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \end{cases},$$

From Proposition 4.3.1, we know that  $w$  is lower semi continuous. On the other hand, we know from Theorem 4.3.6, that for each  $Z(\cdot) \in \Phi$ ,  $K_{Z(\cdot)} = \mathcal{A}(\tau(Z(\cdot)), (x_{\tau(Z(\cdot))}, i_{\tau(Z(\cdot))}))$  is compact for the weak topology. We get then

$$\mathbb{E}^{P_t^{(x,i)}} \left[ v_{i_\tau}(\tau, x_\tau) \right] = \inf \left\{ \mathbb{E}^{P_t^{(x,i)}} \left[ \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)}} \left[ \sum_{i=1}^I \int_\tau^T \int_{K_i} \mathbf{1}_{\{(x(u), i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \sum_{i=1}^I \int_\tau^T \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right], Z(\cdot) \mapsto P_{\tau(Z(\cdot))}^{x(\tau(Z(\cdot))), i(\tau(Z(\cdot)))} \text{ measurable, } P_{\tau(Z(\cdot))}^{x(\tau(Z(\cdot))), i(\tau(Z(\cdot)))} \in \mathcal{A}(\tau(Z(\cdot)), (x(\tau(Z(\cdot))), i(\tau(Z(\cdot)))) \right\}.$$

Using Theorem 4.3.6, we know that last infimum is reached for a certain  $P_\tau^{(x_\tau, i_\tau)^*}$ ,  $P_t^{(x,i)}$



a.s, which means

$$\begin{aligned}
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(\tau)(du, dk_i) \right. \\
& + \left. \sum_{i=1}^I \int_t^\tau \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) + v_{i_\tau}(\tau, x_\tau) \right] = \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \sum_{i=1}^I \int_t^\tau \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(\tau)(du, dk_i) \right. \\
& \quad \left. + \sum_{i=1}^I \int_t^\tau \int_{A_0} h_0(u, \alpha_1 \dots \alpha_I) \nu_0(\tau)(du, d\alpha_1 \dots d\alpha_I) \right] + \\
& \mathbb{E}^{P_t^{(x,i)}} \left[ \mathbb{E}^{P_\tau^{(x_\tau, i_\tau)^*}} \left[ \sum_{i=1}^I \int_\tau^T \int_{K_i} \mathbf{1}_{\{(x(u),i(u)) \in J_i^*\}} h_i(u, x(u), k_i) \nu_i(T)(du, dk_i) + \right. \right. \\
& \quad \left. \left. \sum_{i=1}^I \int_\tau^T \int_{A_0} h_0(\alpha_1 \dots \alpha_I) \nu_0(T)(du, d\alpha_1 \dots d\alpha_I) + g(X_T) \right] \right] \geq v_i(t, x),
\end{aligned}$$

where we have used to conclude the stability by concatenation Proposition [4.4.4](#), namely

:

$P_t^{(x,i)} \oplus_\tau P_\tau^{(x_\tau, i_\tau)^*} \in \mathcal{A}(t, (x, i))$ . Taking the infimum over all the  $P_t^{(x,i)} \in \mathcal{A}(t, (x, i))$ , we conclude for the reverse inequality, and that completes the proof.  $\square$

# Appendices

# Appendix C

## Some analysis tools

We recall here some definitions and functional analysis tools. Let

$$\left\{ \begin{array}{l} (X, T) \text{ be a topological space and } \Sigma \text{ a } \sigma \text{ algebra on } X, \\ (E, \mathcal{E}) \text{ be a measurable space,} \\ (F, d) \text{ be a Polish space, endowed with its metric } d, \text{ and } \mathbb{B}(F) \text{ its Borel algebra.} \end{array} \right.$$

**Definition C.0.1.**  $(E, \mathcal{E})$  is said to be countably generated, if there exists a countable base generating  $\mathcal{E}$ . Namely there exists a sequence  $O_n$  of  $\mathcal{E}$ , such that  $\mathcal{E} = \sigma(O_n, n \in \mathbb{N})$ .

Since  $F$  is Polish, the measurable space  $(F, \mathbb{B}(F))$  is countably generated, (see for instance Proposition 3.1 in [35]).

**Definition C.0.2.** Let  $P$  be a measure on  $(X, \Sigma)$ . We say that  $P$  is regular if for any measurable subset  $B \in \Sigma$

$$P(B) = \sup \left\{ P(K), K \text{ closed}, K \in \Sigma, K \subset B \right\} = \inf \left\{ P(O), O \text{ open}, O \in \Sigma, B \subset O \right\}.$$

We recall that any Borel probability measure, or in other terms any probability measure on a metric space endowed with its  $\sigma$ -Borel algebra, is regular. (see for instance Proposition 2.3 in [35]).

We denote by :

$-L^\infty(E)$  the set consisting of all measurable real bounded maps on  $(E, \mathcal{E})$ .

$-\mathcal{C}(F)$ , (resp.  $\mathcal{C}_u(F)$ ), are the set of continuous (resp. uniformly continuous) bounded functions on  $F$ .

- $L^\infty(E \times F)$  is the set of measurable bounded real functions defined on  $(E \times F, \mathcal{E} \otimes \mathbb{B}(F))$ .

- $\mathcal{M}(E)$  the set consisting of non negative finite measures on  $(E, \mathcal{E})$ .

- $\mathcal{M}(F)$  the set consisting of non negative finite measures on  $(F, \mathbb{B}(F))$ .

- $\mathcal{M}(E \times F)$  the set consisting of non negative finite measures on  $(E \times F, \mathcal{E} \otimes \mathbb{B}(F))$ .

We set furthermore

$$\begin{aligned} L_{mc}^\infty(E \times F) &:= \left\{ f \in \mathcal{L}^\infty(E \times F), \quad x \mapsto f(s, x) \in \mathcal{C}(F), \quad \forall s \in E \right\}, \\ L_{mc}^{\infty,1}(E \times F) &:= \left\{ f \in L_{mc}^\infty(E \times F), \quad \exists A \in \mathcal{E}, \quad g \in \mathcal{C}_u(F), \quad f(x, z) = \mathbf{1}_A(x)g(z). \right\}, \\ L_{mc}^{\infty,2}(E \times F) &:= \left\{ f \in L_{mc}^\infty(E \times F), \quad \exists (A_n) \text{ a partition of } \mathcal{E}, \right. \\ &\quad \left. \text{and a sequence } (g_n) \text{ of } \in \mathcal{C}_u(F), \quad f(x, z) = \sum_n \mathbf{1}_{A_n}(x)g_n(z). \right\}. \end{aligned}$$

On the other hand  $\mathcal{M}(E)$  (resp.  $\mathcal{M}(F)$ ,  $\mathcal{M}(E \times F)$ ) are denoted by  $\mathcal{M}_m(E)$ , (resp.  $\mathcal{M}_c(F)$ ,  $\mathcal{M}_{mc}(E \times F)$ ) when they are endowed with the finest topology making continuous the following family of linear forms  $(\theta_f)_{f \in L^\infty(E)}$ , defined by

$$\theta_f : \begin{cases} \mathcal{M}(E) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_E f d\nu \end{cases} .$$

(resp.  $(\theta_f)_{f \in \mathcal{C}(F)}$ )

$$\theta_f : \begin{cases} \mathcal{M}(F) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_E f d\nu \end{cases} ,$$

$(\theta_f)_{f \in L_{mc}^\infty(E \times F)}$ ,

$$\theta_f : \begin{cases} \mathcal{M}(E \times F) \rightarrow \mathbb{R} \\ \nu \mapsto \nu(f) = \int_{E \times F} f d\nu \end{cases} .)$$

We identify  $\mathcal{M}_{mc}(E \times F)$  (resp.  $\mathcal{M}_m(E)$ ,  $\mathcal{M}_c(F)$ ), as subsets of the dual spaces  $L_{mc}^\infty(E \times F)'$  (resp.  $L_m^\infty(E)'$ ,  $\mathcal{C}(F)'$ ), endowed with the weak topologies  $*\sigma(L_{mc}^\infty(E \times F)', L_{mc}^\infty(E \times F))$  (resp.  $*\sigma(L_m^\infty(E)', L_m^\infty(E))$ ,  $*\sigma(\mathcal{C}(F)', \mathcal{C}(F))$ ).

We recall that a sequence  $\nu_n$  of  $L_{mc}^\infty(E \times F)'$  (resp.  $L_m^\infty(E)'$ ,  $\mathcal{C}(F)'$ ), converges to  $\nu \in$

$L_{mc}^\infty(E \times F)'$ , (resp.  $L_m^\infty(E)'$ ,  $\mathcal{C}(F)'$ ) for the weak topology  $*$ , and we denote  $\nu_n \xrightarrow{*} \nu$ , if and only if

$$\begin{aligned} & \forall f \in L_{mc}^\infty(E \times F), \quad \nu_n(f) \xrightarrow{n \rightarrow +\infty} \nu(f), \\ & \left( \text{resp. } \forall f \in L_m^\infty(E), \quad \nu_n(f) \xrightarrow{n \rightarrow +\infty} \nu(f), \quad \forall f \in \mathcal{C}(F), \quad \nu_n(f) \xrightarrow{n \rightarrow +\infty} \nu(f) \right). \end{aligned}$$

For any  $\nu \in \mathcal{M}(E \times F)$ , we denote by  $\nu^E$  (resp.  $\nu^F$ ), the marginal of  $\nu$  on  $E$  (resp. on  $F$ ), defined by

$$\nu^E(dx) = \int_{z \in F} \nu(dz), \quad \nu^F(dz) = \int_{x \in E} \nu(dx).$$

**Proposition C.0.3.** *Suppose that  $E$  is countably generated, then  $\mathcal{M}_{mc}(E \times F)$  is metrizable. (See for instance Proposition 2.10 in [23].)*

**Theorem C.0.4.** *Let  $\mathcal{N}$  be a subset of  $\mathcal{M}_{mc}(E \times F)$ . Then  $\mathcal{N}$  is relatively compact if and only if*

- (i)  $\left\{ \nu^F, \nu \in \mathcal{N} \right\}$  is relatively compact in  $\mathcal{M}_m(E)$ ,
- (ii)  $\left\{ \nu^E, \nu \in \mathcal{N} \right\}$  is relatively compact in  $\mathcal{M}_c(F)$ .

(See for instance Proposition 2.10 in [23].)

**Theorem C.0.5.** *Let  $\phi$  be a positive linear form defined on the vectorial space generated by  $L_{mc}^{\infty,1}(E \times F)$  satisfying*

- (i) 
$$\begin{cases} \mathcal{E} \rightarrow \mathbb{R} \\ A \mapsto \phi(\mathbf{1}_A \otimes 1) \end{cases}$$

is a measure on  $(E, \mathcal{E})$ , where we define for each  $(x, z) \in E \times F$ ,  $\mathbf{1}_A \otimes 1(x, z) = 1$ , if  $x \in A$  and  $\mathbf{1}_A \otimes 1(x, z) = 0$ , if  $x \notin A$ .

(ii) for each  $\varepsilon > 0$ , there exist a compact set  $K_\varepsilon$  of  $F$  such that  $\phi(1) - \phi(1 \otimes f) \leq \varepsilon$ , for any  $f \in \mathcal{C}_u(F)$ , satisfying  $\mathbf{1}_{K_\varepsilon} \leq f \leq 1$ , where we define for each  $(x, z) \in E \times F$ ,  $1(x, z) = 1$ , and  $1 \otimes f(x, z) = f(z)$ .

Then there exists  $\nu \in \mathcal{M}_{mc}(E \times F)$  such that

$$\forall f \in L_{mc}^{\infty,1}(E \times F), \quad \phi(f) = \int_{E \times F} f d\nu.$$

(See for instance Theorem 2.6 in [23]).

**Lemma C.0.6.** *Let  $K$  be a compact set of  $F$ , and  $f \in L_{mc}^\infty(E \times F)$ . Then there exist a sequence  $f_n$  of  $L_{mc}^{\infty,2}(E \times F)$  converging to  $f$  uniformly on  $E \times K$ . (See for instance Lemma 2.5 in [23]).*

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## Résumé

Cette thèse se décompose en trois grandes parties, qui traitent des EDP quasi linéaires paraboliques sur une jonction, des diffusions stochastiques sur une jonction, et du contrôle optimal également sur une jonction, avec contrôle au point de jonction. Nous commençons au premier Chapitre par introduire une nouvelle classe d'EDP non dégénérée et quasi linéaire, satisfaisant une condition de Neumann (ou de Kirchoff) non linéaire et non dynamique au point de jonction. Nous prouvons l'existence d'une solution classique, ainsi que son unicité. L'une des motivations portant sur l'étude de ce type d'EDP, est de faire le lien avec la théorie du contrôle optimale sur les jonctions, et de caractériser la fonction valeur de ce type de problème à l'aide des équations d'Hamilton Jacobi Bellman. Ainsi, au Chapitre suivant, nous formulons une preuve donnant l'existence d'une diffusion sur une Junction. Ce processus admet un temps local, dont l'existence et la variation quadratique dépendent essentiellement de l'hypothèse d'ellipticité des termes du second ordre au point de jonction. Nous formulerons une formule d'Itô pour ce processus. Ainsi, grâce aux résultats de ces deux Chapitres, nous formulerons dans le dernier Chapitre un problème de contrôle stochastique sur les jonctions, avec contrôle au point de jonction. L'espace des contrôles est celui des mesures de Probabilités résolvant un problème martingale. Nous prouvons la compacité de l'espace des contrôles admissibles, ainsi que le principe de la programmation dynamique.

## Mots Clés

Equations aux dérivées partiels paraboliques non linéaires, conditions aux bords de Neumann, Junction, équations d'Hamilton Jacobi Bellman, diffusion stochastique, temps local, contrôle stochastique, problème martingale, principe de la programmation dynamique.

## Abstract

This thesis consists of three parts which deal with quasi linear parabolic PDE on a junction, stochastic diffusion on a junction and stochastic control on a junction with control at the junction point. We begin in the first Chapter by introducing and studying a new class of non degenerate quasi linear parabolic PDE on a junction, satisfying a Neumann (or Kirchoff) non linear and non dynamical condition at the junction point. We prove the existence and the uniqueness of a classical solution. The main motivation of studying this new mathematical object is the analysis of stochastic control problems with control at the junction point, and the characterization of the value function of the problem in terms of Hamilton Jacobi Bellman equations. For this end, in the second Chapter we give a proof of the existence of a diffusion on a junction.

The process is characterized by its local time at the junction point, whose quadratic approximation is centrally related to the ellipticity assumption of the second order terms around the junction point. We then provide an Itô's formula for this process. Thanks to the previous results, in the last Chapter we study a problem of stochastic control on a junction, with control at the junction point. The set of controls is the set of the probability measures (admissible rules) satisfying a martingale problem. We prove the compactness of the admissible rules and the dynamic programming principle.

## Keywords

Non linear parabolic partial differential equations, Neumann boundary condition, Junction, Hamilton Jacobi Bellman equations, stochastic diffusion, local time, stochastic control, martingale problem, dynamic programming principle

