



HAL
open science

Contrôle Stochastique Impulsionnel avec Incertitude en Finance et en Assurance

Nicolas Baradel

► **To cite this version:**

Nicolas Baradel. Contrôle Stochastique Impulsionnel avec Incertitude en Finance et en Assurance. Analyse classique [math.CA]. Université Paris sciences et lettres, 2018. Français. NNT : 2018PSLED078 . tel-03222702

HAL Id: tel-03222702

<https://theses.hal.science/tel-03222702>

Submitted on 10 May 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE DE DOCTORAT

de l'Université de recherche Paris Sciences et Lettres
PSL Research University

Préparée à Université Paris-Dauphine

Contrôle Stochastique Impulsionnel avec
Incertitude en Finance et en Assurance

École Doctorale de Dauphine — ED 543

Spécialité Sciences

Soutenue le 20.12.2018
par Nicolas BARADEL

Dirigée par Bruno BOUCHARD

COMPOSITION DU JURY :

M. Bruno BOUCHARD
Université Paris-Dauphine
Directeur de thèse

M. Romuald ELIE
Université Paris-Est
Rapporteur

M. Huyên PHAM
Université Paris-Diderot
Rapporteur

M. Charles-Albert LEHALLE
Capital Fund Management
Membre du jury

M. Stéphane LOISEL
Université Claude Bernard Lyon 1
Président du jury

Remerciements

Je tiens avant tout à remercier très chaleureusement Bruno Bouchard pour avoir accepté d'encadrer ma thèse, pour sa très grande disponibilité et ses nombreux conseils, mais surtout, parce que travailler avec lui a été, depuis le début, un réel plaisir.

Je remercie également très chaleureusement Caroline Hillairet, travailler avec elle pour la voie Actuariat de l'ENSAE a été à la fois très agréable et très enrichissant.

Pendant ces années à l'ENSAE, j'ai également eu le plaisir d'interagir avec Peter Tankov que je remercie pour les discussions très intéressantes, que ce soit sur le plan mathématique ou autre. J'en profite également pour remercier la direction des études de l'ENSAE, en particulier Arthur Cazaubiel, Morgane Cure, Christophe Gaillac, Anne Grillot, Jérémy L'Hour, Lucie Neirac, Claude Petit, Rosalinda Solotareff, Corentin Trevien, Jérôme Trinh, Lionel Wilner, pour le plaisir de travailler avec eux. Mes remerciements se dirigent naturellement vers le Laboratoire de Finance-Assurance du CREST, en particulier Alexis Duremigny, Jean-David Fermanian, Christian Francq, Sébastien Fries, Christian Gouriéroux, Alain Monfort, Fanda Traore, Jean-Michel Zakoian ; sans oublier les anciens : Olivier Guéant, Olivier Lopez, Xavier Milhaud. Enfin, je remercie également Pierre Biscourp, le directeur de l'Ecole, et tous les enseignants-chercheurs et doctorants du CREST pour toutes les discussions intéressantes, que ce soit en séminaire, dans les couloirs, devant la table de ping-pong ou au babyfoot.

Je remercie également Romuald Elie et Huyên Pham pour avoir accepté de rédiger les rapports de cette thèse et le temps accordé, ainsi que Charles-Albert Lehalle et Stéphane Loisel pour avoir accepté de participer à ce jury.

Un grand merci à William Jouot pour l'aventure RCode et RPGM et son incroyable efficacité.

Je remercie Roxana Dumitrescu sans qui la vie n'aurait pas la même saveur.

Enfin, mes meilleures pensées se dirigent vers ma famille.

Table des matières

1	Introduction	1
2	Optimal control under uncertainty and Bayesian parameters adjustments : Application to trading algorithms	13
2.1	Introduction	13
2.2	The impulse problem with parameters adjustment	15
2.2.1	The controlled system	15
2.2.2	Bayesian updates	17
2.2.3	Gain function	20
2.3	Value function characterization	21
2.4	Viscosity solution properties	22
2.4.1	Sub-solution property	23
2.4.2	Discrete time approximation and dynamic programming	26
2.4.3	Super-solution property as the time step vanishes	29
2.4.4	Conclusion of the proof of Theorem 2.3.1	31
2.5	A sufficient condition for the comparison	31
2.6	Numerical approximation	33
2.6.1	An example of numerical scheme	33
2.6.2	Construction of ε -optimal controls	36
2.7	Applications to optimal trading	36
2.7.1	Immediate impact of aggressive orders	36
2.7.2	Random execution times : application to strategies using limit-orders	45
3	Optimal Control under uncertainty : Application to the issue of CAT bonds	51
3.1	Introduction	51
3.2	The framework	52
3.2.1	General framework	52
3.2.2	CAT Bond framework	53
3.2.3	The controlled system	53
3.2.4	The CAT bonds process	55
3.2.5	Bayesian updates	57
3.2.6	Parametrization of the set \mathbf{M}^λ	63
3.2.7	Gain function	64

3.3	Value function characterization	65
3.4	Viscosity solution properties	68
3.4.1	Sub-solution property	68
3.4.2	Super-solution property	69
3.5	A sufficient condition for the comparison	71
3.6	Numerical Scheme	73
3.7	Example : CAT bonds in a <i>per event</i> framework for Hurricanes in Florida . .	75
3.7.1	Intensity of Hurricanes : the Gamma case	76
3.7.2	Intensity of Hurricanes : the Bernoulli case	77
3.7.3	Severity of the Hurricanes	78
3.7.4	The set of controls and the output process	78
3.7.5	Gain function and dimension reduction	80
3.7.6	The choice of the parameters	80
3.7.7	Results	82
4	Optimal inventory management and order book modeling	91
4.1	Introduction	91
4.2	General order book presentation and priors of the market participants	93
4.3	Market maker's optimal control problem	96
4.3.1	Market maker's strategy and state dynamics	96
4.3.2	The optimal control problem	99
4.3.3	The dynamic programming equation	101
4.3.4	Dimension reduction, symmetries and numerical resolution	102
4.3.5	Approximate optimal controls	104
4.3.6	Numerical experiments	105
4.4	High frequency trader's pair trading problem	110
4.4.1	The optimal control problem	110
4.4.2	The dynamic programming equation	112
4.4.3	Dimension reduction, symmetries and numerical resolution	113
4.4.4	Approximate optimal controls	114
4.4.5	Numerical experiments	115
4.5	Institutional broker strategies for portfolio liquidation	117
4.5.1	Volume strategy	117
4.5.2	VWAP	120
4.6	Simulation of the full market	125

Chapitre 1

Introduction

Cette thèse s'intéresse à des problèmes de gestion de portefeuille avec *incertitude* et à leurs applications en finance et en assurance. Le modèle le plus célèbre est celui proposé en 1951 par Markowitz [54]. Il est encore très utilisé de nos jours. Sa force réside dans sa simplicité : il n'a que deux périodes. Il offre une solution optimale explicite et le modèle ne dépend que du vecteur de moyennes et de la matrice de variance-covariance des rendements. Il suffit de calibrer le modèle et de choisir un niveau de risque, complètement caractérisé par la variance, et on obtient le portefeuille à acquérir en période initiale afin d'avoir le meilleur rendement moyen en période terminale sous la contrainte de risque choisie.

Toutefois, ce modèle très simple repose sur des hypothèses très fortes. L'une d'entre elles est la liquidité parfaite des actifs. Acquérir ou liquider une position importante ne peut pas se faire de façon immédiate sans payer un coût élevé, voir par exemple [35]. De plus, en fonction de l'aversion au risque ou des conditions de marché par exemple, l'agent peut avoir intérêt à rebalancer son portefeuille avant la date terminale. Ainsi, le cadre mathématique naturel pour une gestion de portefeuille avancée est celui des modèles mathématiques en temps continu.

Lorsque les actifs sont supposés être des semi-martingales d'Itô, l'agent observe $d \in \mathbb{N}^*$ actifs sur $[0, T]$, notés $(S_u)_{0 \leq u \leq T}$. Ils sont supposés être solution forte d'une équation différentielle stochastique brownienne. Si $(W_u)_{0 \leq u \leq T}$ est un mouvement brownien de dimension d , on suppose que :

$$S^{t,s} := s + \int_t^\cdot \mu_S(u, S_u^{t,s}) du + \int_t^\cdot \sigma_S(u, S_u^{t,s}) dW_u. \quad (1.0.1)$$

Une *stratégie* à un instant $0 \leq u \leq T$ consiste à posséder le vecteur d'actif $\alpha_u \in \mathbf{A}$ où \mathbf{A} est un sous-ensemble de \mathbb{R}^d . Dans ce cas, la valeur du portefeuille évolue en fonction de la variation des actifs possédés à chaque instant :

$$V^{t,v} := v + \int_t^\cdot \alpha_u dS_u^{t,s} = v + \int_t^\cdot \alpha_u \mu_S(u, S_u^{t,s}) du + \int_t^\cdot \alpha_u \sigma_S(u, S_u^{t,s}) dW_u,$$

où $(\alpha_u)_{0 \leq u \leq T}$ est un processus progressivement mesurable à valeurs dans \mathbf{A} . L'objectif est de trouver la stratégie $(\alpha_u)_{0 \leq u \leq T}$ qui maximise l'espérance de $V_T^{t,v}$ à travers une fonction

d'utilité.

Ce problème est un cas particulier du problème de contrôle stochastique suivant :

$$v(t, x) := \sup_{\alpha \in \mathcal{A}^{t,x}} \mathbb{E} \left[g(X_T^{t,x,\alpha}) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d \quad (1.0.2)$$

avec $g : \mathbb{R}^d \mapsto \mathbb{R}$ une fonction à croissance au plus polynomiale dont la régularité sera donnée par le contexte, et X est défini par l'équation différentielle stochastique :

$$X^{t,x,\alpha} := x + \int_t^\cdot \mu(s, X_s^{t,x,\alpha}, \alpha_s) ds + \int_t^\cdot \sigma(s, X_s^{t,x,\alpha}, \alpha_s) dW_s \quad (1.0.3)$$

où α est fixé dans $\mathcal{A}^{t,x}$ qui est l'ensemble des *contrôles admissibles*.

Dans un cas un peu plus général¹, l'agent peut avoir à payer un coût instantané lié au choix de son contrôle (par exemple, un coût de liquidité). Si $f : [0, T] \times \mathbb{R}^d \times \mathbf{A} \mapsto \mathbb{R}$ est une fonction continue qui décrit le gain (si f est positive) ou un coût (si f est négative), le problème de contrôle stochastique devient :

$$v(t, x) := \sup_{\alpha \in \mathcal{A}^{t,x}} \mathbb{E} \left[\int_t^T f(s, X_s^{t,x,\alpha}, \alpha_s) ds + g(X_T^{t,x,\alpha}) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (1.0.4)$$

Dans ce genre de problème, l'objectif premier est de trouver un contrôle optimal (ou presque optimal) $\alpha : [0, T] \times \Omega \mapsto \mathbf{A}$, il s'agit de l'approche standard. Ici, par définition de α et de $X^{t,x,\alpha}$, l'agent agit en temps continu et le processus X est continu. Cette approche ne contient pas, par exemple, les modèles avec coût de transaction non proportionnel qui font sauter la composante d'argent liquide de l'agent. Et dans ce cas, par construction, il ne pourra agir qu'un nombre fini de fois : il supporterait un coût infini, sinon. Le contrôle devient une suite $\phi := (\tau_i, \alpha_i)_{i \geq 1}$ où, pour tout $i \geq 1$, τ_i est un temps d'arrêt et α_i est \mathcal{F}_{τ_i} -mesurable et à valeurs dans un ensemble \mathbf{A} . Le processus X peut s'écrire par exemple :

$$\begin{aligned} dX_s^{t,x,\phi} &:= \mu(s, X_s^{t,x,\phi}) ds + \sigma(s, X_s^{t,x,\phi}) dW_s && \text{sur } \mathcal{N}_\phi^0 \\ X_{\tau_i}^{t,x,\phi} &:= F(\tau_i, X_{\tau_i^-}^{t,x,\phi}, \alpha_i) && i \geq 1 \end{aligned} \quad (1.0.5)$$

où $\mathcal{N}_\phi^0 := \bigcup_{i \geq 0} [\tau_i, \tau_{i+1}] \cap [0, T]$ avec $\tau_0 := 0$.

Dans ce cas, le problème de contrôle optimal est :

$$v(t, x) := \sup_{\phi \in \Phi^{t,x}} \mathbb{E} \left[g(X_T^{t,x,\phi}) \right], \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (1.0.6)$$

avec $\Phi^{t,x}$ l'ensemble des contrôles admissibles.

Le **Chapitre 2** repose sur deux articles écrits en collaboration avec Bruno Bouchard et Ngoc Minh Dang : *Optimal control under uncertainty and bayesian parameters adjustments*

1. Le coût instantané peut toutefois être ajouté directement dans le vecteur X .

(publié dans *SIAM Journal on Control and Optimization*) [10] et *Optimal trading with online parameter revisions* (publié dans *Market Microstructure and Liquidity*) [9].

Par rapport à (1.0.5), nous considérons qu'il y a de l'incertitude sur la réaction lorsqu'on joue le contrôle. Nous traduisons cette incertitude par l'ajout d'un paramètre inconnu de l'agent $v \in U$, où U est un espace polonais, et une suite i.i.d. (ε_i) dont chaque composante est à valeurs dans un espace polonais E et de loi \mathbb{P}_ε .

L'équation de $X^{t,x,\phi}$ en τ_i se réécrit :

$$X_{\tau_i}^{t,x,\phi} = F(\tau_i, X_{\tau_i-}^{t,x,\phi}, \alpha_i, v, \varepsilon_i), \quad i \geq 1,$$

De tels problèmes de contrôle optimal ont été étudiés en temps discret, voir par exemple [37] pour des références. L'objectif de ce chapitre est d'étudier ce problème de contrôle optimal en temps continu.

On s'autorise également à ce que notre action ne soit pas immédiate, voir par exemple [21], mais puisse intervenir après un délai aléatoire : par exemple, lorsqu'on place un ordre à cours limité et qu'on attend son éventuelle exécution. On définit les variables aléatoires :

$$\vartheta_i := \varpi(\tau_i, X_{\tau_i-}, \alpha_i, v, \varepsilon_i), \quad i \geq 1,$$

avec ϖ une fonction mesurable qui vérifie $\varpi(t, \cdot) \geq t$. L'équation différentielle stochastique (1.0.5) se réécrit :

$$\begin{aligned} dX_s^{t,x,\phi} &:= \mu_X(s, X_s^{t,x,\phi})ds + \sigma_X(s, X_s^{t,x,\phi})dW_s && \text{sur } \mathcal{N}_\phi \\ X_{\vartheta_i}^{t,x,\phi} &:= F(\tau_i, X_{\tau_i-}^{t,x,\phi}, \alpha_i, v, \varepsilon) && i \geq 1 \end{aligned} \tag{1.0.7}$$

où $\mathcal{N}_\phi := \bigcup_{i \geq 0} [\vartheta_i, \tau_{i+1}[\cap [0, T]$ avec $\vartheta_0 := 0$.

On introduit $(\mathcal{F}_s^{t,x,\phi})_{s \geq t}$ la filtration générée par $X^{t,x,\phi}$. C'est celle observée par l'agent et sur laquelle va s'appuyer la stratégie de l'agent. La filtration dépend de manière non triviale des conditions initiales ce qui se révélera être une difficulté afin d'établir un principe de programmation dynamique.

L'ajout du paramètre inconnu $v \in U$ n'est pas anodin. Il est commun à toutes les impulsions. À chaque fois que l'agent agit sur le système, il obtient de l'information sur v . La fonction valeur ne peut pas seulement dépendre de (t, x) , elle doit aussi dépendre de l'information sur v .

Étant donnée une loi à priori $m \in \mathbf{M}$ sur v où \mathbf{M} est un sous-ensemble des mesures de probabilité boréliennes sur U , on introduit le processus $M^{t,x,m,\phi}$ défini pour tout borélien $B \in \mathcal{B}(U)$ par :

$$M_s^{t,x,m,\phi}(B) := \mathbb{E} \left(\mathbf{1}_{\{v \in B\}} \mid \mathcal{F}_s^{t,x,m,\phi} \right). \tag{1.0.8}$$

La fonction valeur va naturellement dépendre de la loi de v à l'instant t . Cette définition amène à une révision Bayésienne de M en ϑ_i sachant M_{τ_i-} et en dehors, le processus est constant. C'est en agissant sur le système qu'on obtient de l'information nouvelle sur v et que M s'actualise, ce qui permet d'éviter toute diffusion infinitésimale sur M .

L'objectif est de caractériser sous forme de solution de viscosité la solution du problème :

$$v(t, x, m) := \sup_{\phi \in \Phi^{t,x}} \mathbb{E} \left[g(T[\phi], X_{T[\phi]}^{t,x,\phi}, M_{T[\phi]}^{t,x,m,\phi}) \right], \quad (t, x, m) \in [0, T] \times \mathbb{R}^d \times \mathbf{M}, \quad (1.0.9)$$

où $g : [T, +\infty[\times \mathbb{R}^d \times \mathbf{M} \mapsto \mathbb{R}$ est une fonction continue bornée. Puisque, une fois la dernière action effectuée en $\tau_i \leq T$, on récupère $(X^{t,x,\phi}, M^{t,x,m,\phi})$ en ϑ_i , il se peut que $\vartheta_i > T$. Dans ce cas $T[\phi] = \vartheta_i$, sinon $T[\phi] = T$. La fonction g peut ne pas dépendre de $M_{T[\phi]}^{t,x,m,\phi}$, dans ce cas la fonction valeur dépendra encore naturellement de m . L'introduction de l'espace des mesures dans la définition de la fonction valeur amène à des difficultés techniques résolues à travers les résultats de [17].

Pour introduire l'équation aux dérivées partielles (au sens de la viscosité) satisfaite par v , supposons-la suffisamment régulière. Supposons avoir à notre disposition un principe de programmation dynamique, en considérant un contrôle ϕ° tel que $\mathbb{P}\{\tau_1 > T\} = 1$, on devrait avoir $v(t, x, m) \geq \mathbb{E}[v(t+h, X_{t+h}^{t,x,\phi^\circ}, m)]$. En appliquant Itô, puis en divisant par h et en le faisant tendre vers 0, on devrait avoir $-\mathcal{L}v(t, x, m) \geq 0$ où \mathcal{L} est l'opérateur de Dynkin associé à X^{t,x,ϕ° ,

$$\mathcal{L}\varphi := \partial_t \varphi + \langle \mu, D\varphi \rangle + \frac{1}{2} \text{Tr}[\sigma \sigma^\top D^2 \varphi].$$

D'autre part, en considérant un contrôle ϕ^a tel que $\mathbb{P}(\tau_1 = t, \alpha_1 = a) = 1$ avec $a \in \mathbf{A}$ compact, on devrait avoir

$$v(t, x, m) \geq \mathcal{K}v(t, x, m),$$

où $\mathcal{K}\varphi := \sup_{a \in \mathbf{A}} \mathcal{K}^a \varphi$ avec $\mathcal{K}^a \varphi := \mathbb{E} \left[\varphi(\vartheta_1, X_{\vartheta_1}^{t,x,\phi^a}, M_{\vartheta_1}^{t,x,m,\phi^a}) \right]$.

De même, à la frontière en T , on devrait avoir $v(T, \cdot) \geq \mathcal{K}_T g$ et $v(T, \cdot) \geq \mathcal{K}v(T, \cdot)$, où $\mathcal{K}_T g(\cdot, m) := \int_U \int_E g(\cdot, m, u, e) d\mathbb{P}_\epsilon(e) dm(u)$.

Enfin, par optimalité, v devrait être solution (de viscosité) de l'équation quasi-variationnelle

$$\min \{-\mathcal{L}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \quad \text{sur } [0, T) \times \mathbb{R}^d \times \mathbf{M} \quad (1.0.10)$$

$$\min \{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}\varphi\} = 0 \quad \text{sur } \{T\} \times \mathbb{R}^d \times \mathbf{M}. \quad (1.0.11)$$

Nous supposons tout d'abord l'existence d'un principe de comparaison.

Hypothèse 1.0.1. *Soit u (resp. v) une sous- (resp. sur-) solution de viscosité semi-continue supérieurement (resp. inférieurement) et bornée de (1.0.10)-(1.0.11). On suppose de plus que $u \leq v$ sur $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Alors, $u \leq v$ sur $[0, T] \times \mathbb{R}^d \times \mathbf{M}$.*

Le résultat principal que nous établissons est le suivant.

Théorème 1.0.1. *Supposons que l'Hypothèse 1.0.1 est vérifiée. Alors, v est continue sur $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ et est l'unique solution de viscosité bornée de (1.0.10)-(1.0.11).*

La difficulté principale afin d’obtenir ce résultat est d’établir un principe de programmation dynamique. Nous n’avons pas d’hypothèse de régularité a priori, nous pourrions établir un principe de programmation dynamique faible. Nous ne pouvons cependant pas utiliser Bouchard et Touzi [22] (voir aussi Dumitrescu et al. [36]) car la filtration dépend des conditions initiales de manière non triviale.

L’inégalité majorante du principe de programmation dynamique (faisant intervenir l’enveloppe semi-continue supérieure) dans notre problème ne pose pas de difficulté particulière, car la démonstration repose sur un argument de conditionnement. Nous établissons une partie du principe de programmation dynamique faible pour des temps d’arrêt de la forme : $\chi = \theta \wedge \tau_1$ où θ est le temps de sortie de $(X^{t,x,\phi}, M^{t,x,m,\phi})$ d’une boule (Proposition 2.4.2). Il est construit pour en déduire la propriété de sous-solution de viscosité (Proposition 2.4.1).

L’autre sens du principe de programmation dynamique est plus délicat. Nous approximons notre problème de contrôle optimal en temps continu par une suite de versions discrètes $(v_n)_{n \geq 1}$ dans lesquelles l’agent joue à temps discret. Sur cette version discrète, afin d’établir le principe de programmation dynamique sans régularité a priori, l’argument principal est un argument de sélection mesurable. Nous montrons que pour tout $n \geq 1$, v_n satisfait un principe de programmation dynamique. Nous montrons ensuite que la limite inférieure relaxée de (v_n) est sur-solution de notre problème et nous en déduisons le Théorème 1.0.1 grâce à l’Hypothèse 1.0.1.

Nous donnons ensuite une condition suffisante à l’Hypothèse 1.0.1 (Proposition 2.5.1) qui nécessite l’existence d’une sur-solution stricte Ψ .

Nous proposons ensuite un schéma numérique convergent (Section 2.6) qui repose sur une adaptation des arguments de [14] à notre contexte.

Nous terminons ce chapitre avec deux exemples d’application. Le premier porte sur l’acquisition optimale d’une position en passant des ordres *au marché*. Cela se traduit par la possibilité, à chaque instant, de passer un ordre d’achat immédiat pour plusieurs actions. Le prix de marché est endogène : à chaque fois que nous achetons il y a une augmentation immédiate du prix de manière aléatoire. C’est cette augmentation qui dépend du paramètre inconnu v qu’on apprend au fil de l’eau à chaque fois que nous agissons.

Dans le second exemple, nous envoyons des ordres à *cours limité* et nous attendons qu’ils soient exécutés ou non, en nous fixant un temps d’attente maximum $\ell > 0$. Plus la limite de l’ordre est basse, plus le temps d’attente est long en moyenne. Le temps d’attente est supposé suivre une loi exponentielle, et le paramètre inconnu est le paramètre de cette loi exponentielle que nous apprenons au fil de l’eau.

Le **Chapitre 3** repose sur l’article *Optimal control under uncertainty : Application to the issue of CAT bonds (en prépublication)* [7].

Nous nous plaçons cette fois-ci dans un cadre actuariel. Le risque n’est plus associé à l’évolution d’un mouvement brownien mais à une mesure aléatoire de Poisson à activité finie : il y a un nombre fini de sauts sur $[0, T]$ et la mesure aléatoire de Poisson s’assimile à un processus de Poisson composé qui représente l’arrivée des sinistres. Pour un état de l’art sur les solutions de viscosité avec des mesures de aléatoires Poisson générales, voir [13].

Le cadre est le suivant : un assureur (ou un réassureur) possède un portefeuille qui est exposé à un risque de catastrophes naturelles. Il peut, afin de se couvrir, émettre des obligations catastrophes : des *CAT bonds*² (voir par exemple [32] ou [33] pour une introduction générale).

Nous introduisons à nouveau un contrôle impulsif de la même forme que précédemment : $\phi := (\tau_i, \alpha_i)_{i \geq 1}$ où τ_i correspond à la date d'émission du $i^{\text{ème}}$ CAT bond et α_i représente les caractéristiques de ce CAT bond (région et péril, tranche, etc). La maturité des CAT bonds est fixée à une durée $\ell > 0$. On introduit la suite $(\vartheta_i)_{i \geq 1}$ qui sont les dates de fin des CAT bonds émis en $(\tau_i)_{i \geq 1}$, par arrivée à l'échéance ou par défaut.

Lors de l'émission, le coupon n'est pas parfaitement connu à l'avance (le coupon sera déterminé au moment de l'émission sur le marché), et on s'autorise à avoir une incertitude sur la loi comme au Chapitre 2. Nous introduisons à nouveau un paramètre $v_0 \in U^v$ où U^v est un espace polonais. Mais cette fois-ci, il y a également incertitude sur la diffusion : on considère une intensité inhomogène qui dépend d'un paramètre inconnu $\lambda_0 \in U^\lambda$ et, en cas de sinistre, ce sinistre suit une loi qui dépend d'un paramètre $\gamma_0 \in U^\gamma$ où U^λ et U^γ sont des ensembles polonais. Le processus $X^{t,x,\phi}$ est solution de :

$$\begin{aligned}
X &:= x + \int_t^\cdot \mu(s, X_s) ds + \int_t^\cdot \int_{\mathbb{R}^d} \beta(s, X_{s-}, u) N(ds, du) \\
&+ \sum_{i \geq 1} \mathbf{1}_{\{t \leq \tau_i < \cdot\}} H(\tau_i, X_{\tau_i}, \alpha_i) \\
&+ \sum_{i \geq 1} \mathbf{1}_{\{t \vee \tau_i \leq \cdot\}} \int_{t \vee \tau_i}^{\cdot \wedge \vartheta_i} \bar{C}(s, r_i) ds \\
&+ \sum_{i \geq 1} \mathbf{1}_{\{t \leq \vartheta_i \leq \cdot\}} F(\vartheta_i, X_{\vartheta_i-}, X_{\tau_i}, r_i, \alpha_i, \vartheta_i - \tau_i, u_i) \mathbf{1}_{\{\vartheta_i - \tau_i \neq \ell\}},
\end{aligned} \tag{1.0.12}$$

où $r_i := \mathfrak{C}_0(\tau_i, X_{\tau_i-}, \alpha_i, v, \epsilon_i)$ et u_i est la taille du sinistre associé à la mesure de Poisson N à la date ϑ_i (en cas de sinistre).

On fixe un nombre $\kappa \in \mathbb{N}^*$ de CAT bonds que l'assureur peut avoir en cours d'exécution simultanément. On remarque que, si $\kappa = 1$, dès que l'assureur émet un CAT bond, il est obligé d'attendre la fin de celui-ci pour pouvoir en lancer un autre. Dans ce cas particulier, on pourrait reformuler (1.0.12) afin de retrouver une équation de la forme (1.0.5) et avoir un problème de contrôle impulsif identique au Chapitre 2, à l'exception du mouvement brownien remplacé par une mesure aléatoire de Poisson.

Dans ce chapitre, on s'autorise à avoir plusieurs CAT bonds en cours, ce qui correspond à plusieurs impulsions en cours.

Le processus (1.0.12) n'est pas markovien, même sans paramètre inconnu. En effet, il faut retenir les contrats signés afin d'appliquer le paiement des coupons et les *payoff* éventuels en cas de sinistre. Nous introduisons un processus (C, L) où C est le processus des caractéristiques des CAT bonds en cours et L les durées associées. Nous séparons les caractéristiques

2. Les obligations catastrophe, ou CAT bonds, sont des obligations négociables à taux variable. Le risque associé à CAT bond n'est pas lié à la défaillance d'une entité (État ou société) mais est lié à la survenance d'une catastrophe.

de la durée car elles joueront un rôle différent dans l'EDP. Le processus C est un processus constant par morceaux qui saute à l'émission ou à l'échéance d'un CAT bond. Le processus L évolue continument : c'est le temps écoulé depuis l'émission. Nous notons \mathbf{CL} l'ensemble dans lequel le processus (C, L) prend ses valeurs.

On introduit $(\mathcal{F}_s^{t,x,\phi})_{s \geq t}$ la filtration générée par $(X^{t,x,\phi}, N)$. Une fois cette définition introduite, comme dans le chapitre précédent, afin d'établir un principe de programmation dynamique et de caractériser la solution comme unique solution de viscosité d'une certaine équation, nous introduisons les processus M^λ, M^γ et M^v qui sont définis par :

$$\begin{aligned} M_s^{\lambda,t,m}(B) &:= \mathbb{E} \left(\mathbf{1}_{\{\lambda \in B\}} \mid \mathcal{F}_s^{t,x,\phi} \right), & M_s^{\gamma,t,m}(B) &:= \mathbb{E} \left(\mathbf{1}_{\{\gamma \in B\}} \mid \mathcal{F}_s^{t,x,\phi} \right), \\ M_s^{v,x,m,\phi}(B) &:= \mathbb{E} \left(\mathbf{1}_{\{v \in B\}} \mid \mathcal{F}_s^{t,x,\phi} \right). \end{aligned} \tag{1.0.13}$$

Comme M^γ et M^v n'évoluent que lors de la survenance de sauts (par la mesure de Poisson pour le premier, par le contrôle pour le second), la révision Bayésienne se définit comme précédemment. Pour M^λ , le processus évolue continument et saute lors de la survenance de sinistres. Afin d'avoir un générateur infinitésimal *classique*, nous supposons que M^λ prend ses valeurs dans un ensemble \mathbf{M}^λ homéomorphe à un sous-ensemble \mathbf{P} de \mathbb{R}^d , puis nous travaillons sur le processus défini sur \mathbb{R}^d et nous le notons P . On pose $M := (M^\gamma, M^v)$ à valeurs dans un ensemble de mesure de probabilité boréliennes \mathbf{M} .

L'objectif est de caractériser sous forme de solution de viscosité la solution du problème :

$$v(z, p, m) := \sup_{\phi \in \Phi^{t,x}} \mathbb{E} \left[g(X_T^{z,\phi}, C_T^{z,\phi}, L_T^{z,\phi}, P_T^{t,p}, M_T^{z,m,\phi}) \right], \quad (z, p, m) \in [0, T] \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M},$$

où $g : \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$ est une fonction continue bornée et $z := (t, x, c, l)$.

On introduit $\mathcal{L}_* \varphi := \mathbb{E} [\mathcal{L} \varphi]$ qui est l'espérance de l'opérateur de Dynkin associé à notre problème. L'espérance est prise à travers m^λ et m^γ .

En supposant que v est régulière et qu'on a à notre disposition un principe de programmation dynamique, si on pose $\pi(c, l)$ le nombre de CAT bonds en cours, on s'attend à avoir

$$v \geq -\mathcal{L}_* v \mathbf{1}_{\{\pi(c,l)=\kappa\}} + \min \{ -\mathcal{L}_* v, v - \mathcal{K}v \} \mathbf{1}_{\{\pi(c,l) \neq \kappa\}}, \tag{1.0.14}$$

où $\mathcal{K} \varphi := \sup_{a \in \mathbf{A}} \mathcal{K}^a \varphi$ avec $\mathcal{K}^a \varphi := \mathbb{E} \left[\varphi(\tau_1, X_{\tau_1+}^{z,\phi^a}, C_{\tau_1+}^{z,\phi^a}, L_{\tau_1+}^{z,\phi^a}, P_{\tau_1+}^{t,p}, M_{\tau_1+}^{z,m,\phi^a}) \right]$ avec ϕ^a qui vérifie $\mathbb{P}(\tau_1 = t, \alpha_1 = a) = 1$ (pour $a \in \mathbf{A}$ compact).

De même, à la frontière en T , on devrait avoir $v(T, \cdot) \geq g$ et $v(T, \cdot) \geq \mathcal{K}v(T, \cdot)$ si $\pi(c, l) \neq \kappa$.

Pour des contrats en cours d'exécution dans les indices $\mathbf{J} \subset \{1, \dots, \kappa\}$, on introduit l'opérateur $\mathfrak{L}^{\mathbf{J}'}$, avec $\mathbf{J}' \subset \mathbf{J}$, défini par :

$$\mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'} : [0, \ell]^{\mathbf{J}} \rightarrow [0, \ell]^{\mathbf{J}} \quad (1.0.15)$$

$$(l_j)_{1 \leq j \leq \kappa} \mapsto (\ell \mathbf{1}_{\{j \in \mathbf{J}'\}} + l_j \mathbf{1}_{\{j \notin \mathbf{J}'\}})_{1 \leq j \leq \kappa}. \quad (1.0.16)$$

avec $[0, \ell]^{\mathbf{J}} := \{l \in ([0, \ell] \cup \partial)^{\kappa} : l_j \neq \partial \Leftrightarrow j \in \mathbf{J}\}$, $l_j = \partial$ s'il n'y a pas de contrat dans cet indice. Cet opérateur représente les contrats arrivant à maturité sans déclenchement de *payoff*.

Lorsqu'un CAT bond arrive à maturité, il n'y a pas de *payoff*. On devrait avoir, si on note \mathfrak{C}_{-}^{ℓ} la fonction qui, à un ensemble de contrats (c, l) , supprime ceux arrivés à maturité ℓ ,

$$\lim_{l' \rightarrow \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)} v(\cdot, c, l', \cdot) = \max\{v(\cdot, \mathfrak{C}_{-}^{\ell}[c, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)], \cdot), \mathcal{K}v(\cdot, \mathfrak{C}_{-}^{\ell}[c, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)], \cdot)\}.$$

Enfin pour $\mathbf{J} \subset \{1, \dots, \kappa\}$ et $\mathbf{J} \neq \emptyset$, par optimalité, v devrait être solution (de viscosité) de l'équation quasi-variationnelle

$$\mathbf{1}_{\{\pi(c, l) = \kappa\}} [-\mathcal{L}_{\star} \varphi] + \mathbf{1}_{\{\pi(c, l) \neq \kappa\}} \min\{-\mathcal{L}_{\star} \varphi, \varphi - \mathcal{K} \varphi\} = 0 \quad \text{sur } \mathbf{D}_{\circ}. \quad (1.0.17)$$

$$\varphi = \mathbf{1}_{\{\pi(c, l) = \kappa\}} g + \mathbf{1}_{\{\pi(c, l) \neq \kappa\}} \max\{\mathcal{K}g, g\} \quad \text{sur } \mathbf{D}_T \quad (1.0.18)$$

$$\lim_{l' \rightarrow \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)} \varphi(\cdot, c, l', \cdot) = \max\{\varphi(\cdot, \mathfrak{C}_{-}^{\ell}[c, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)], \cdot), \mathcal{K} \varphi(\cdot, \mathfrak{C}_{-}^{\ell}[c, \mathfrak{L}_{\mathbf{J}}^{\mathbf{J}'}(l)], \cdot)\} \quad \text{sur } \mathbf{D}_{\circ} \cup \mathbf{D}_T, \quad (1.0.19)$$

où

$$\mathbf{D}_{\circ} := [0, T) \times \mathbb{R}^d \times \mathbf{CL}_{\mathbf{J}} \times \mathbf{P} \times \mathbf{M},$$

$$\mathbf{D}_T := \{T\} \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}.$$

Le résultat principal que nous établissons est le suivant.

Théorème 1.0.2. *Supposons qu'il existe un principe de comparaison. Alors, v est continue sur $[0, T] \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$ et est l'unique solution de viscosité bornée de (1.0.17)-(1.0.18)-(1.0.19).*

Outre les sauts, la première différence avec le Chapitre 2 est que le générateur infinitésimal dépend de λ_0 et de γ_0 qui sont aléatoires. C'est \mathcal{L}_{\star} qui apparaît dans l'équation, c'est-à-dire son espérance. On montre que le générateur est continu sous des hypothèses raisonnables (Lemme 6), propriété qui intervient dans l'établissement des propriétés de solution de viscosité. Une autre différence est la condition au bord associée à l'arrivée à échéance d'un CAT bond qui n'a pas fait défaut.

Afin d'établir le principe de programmation dynamique, nous avons la même difficulté de dépendance non triviale de la filtration aux conditions initiales. On s'appuie sur les résultats du Chapitre 2 afin de l'établir. On caractérise ensuite la fonction valeur comme unique solution de viscosité de (1.0.17)-(1.0.18)-(1.0.19) en adaptant les arguments du Chapitre 2.

À nouveau, nous donnons une condition suffisante à l'hypothèse d'existence d'un principe de comparaison (Proposition 3.5.1) où nous construisons une sur-solution stricte au moyen d'une fonction Ψ .

Nous établissons ensuite un schéma numérique convergent (Section 3.6) qui repose sur [14].

Enfin, on s'intéresse à une application d'assurance face au péril des ouragans en Floride. L'assureur a une part de marché de la valeur assurable en Floride et fait face au risque d'ouragans. Il peut souscrire des CAT bonds à différentes tranches afin de couvrir son risque. Toutefois, il ne connaît pas non plus parfaitement l'intensité d'arrivée des sinistres. Dans un premier cas, on ne considère pas de réchauffement climatique et l'incertitude est sur le niveau d'intensité, contrainte liée au fait qu'on se fixe une loi a priori de la famille Gamma. Dans le second cas, on suppose que l'ensemble des paramètres possibles pour l'intensité est discret et fini. Ce second cas, plus simple en terme de loi a priori, permet d'intégrer une inconnue sur le seul facteur d'accroissement de l'intensité en conséquence du réchauffement climatique.

Le **Chapitre 4** repose sur un article écrit en collaboration avec Bruno Bouchard, David Evangelista et Othmane Mounjid : *Optimal inventory management and order book modeling* (à paraître dans *ESAIM : Proceedings and Surveys*) [11].

Dans ce chapitre, on n'utilise plus une approche avec prix de marché unique mais on modélise le carnet d'ordre. L'objectif est de proposer une modélisation de carnet d'ordre endogène aux agents. Ce sont les actions d'agents économiques rationnels qui font bouger le carnet d'ordre et par conséquent le prix. Ces agents économiques résolvent des problèmes de programmation dynamique qu'ils appliquent ensuite sur le carnet d'ordre. Nous mettons ensuite ces agents ensemble sur le même carnet d'ordre, ce qui construit notre carnet d'ordre endogène.

En finance, le carnet d'ordre est la rencontre de la courbe d'offre et de demande sur un actif. Les offres d'achat et de vente sont discrètes : l'écart minimum entre deux prix est appelé le *tick*. Si on note le tick $\delta > 0$, l'ensemble des prix possibles est $\delta\mathbb{Z}$.

Un agent peut passer un ordre d'achat au prix p_b pour une quantité n_b . Cet ordre est enregistré et, sauf s'il venait à être annulé, il sera exécuté si, à un instant, personne n'a de meilleure offre d'achat et qu'une offre de vente se fait au prix p_b . La meilleure offre d'achat est appelée prix *bid*, que l'on note p_b . La meilleure offre de vente est appelée prix *ask*, que l'on note p_a . Par construction, $p_b < p_a$. En effet, s'ils étaient égaux, il y aurait échange au prix $p_b = p_a$ jusqu'à ce qu'ils soient distincts. La valeur $p_a - p_b \in \delta\mathbb{N}^*$ est appelée le *bid-ask spread*.

Un agent économique peut placer un ordre d'achat à cours limité dans le carnet au meilleur prix, p_b . Toutefois, si d'autres ordres sont placés avant lui au même prix, ils sont prioritaires et l'ensemble des ordres à ce prix forme une queue. Si une offre de vente se fait au prix p_b , c'est l'ordre d'arrivée dans la queue qui prime. Nous souhaitons calculer la stratégie optimale de placement d'ordre sur un tel carnet : il nous faut tenir compte de la quantité placée ainsi que de la position dans la queue.

Le modèle est proche de ceux utilisés dans [29], [46], [47] et [52]. Empiriquement, la plupart des ordres sont placés aux meilleurs limites, voir par exemple [19]. Ainsi, par parcimonie, nous

ne modélisons que la queue associée à la meilleure offre d'achat, p_b , et la queue associée à la meilleure offre de vente, p_a . Lorsqu'une queue est détruite, par exemple celle d'achat, nous appliquons une loi aléatoire de découverte de queue à un tick plus bas, au prix $p_b - \delta$. Nous modélisons jusqu'à deux ticks, on a $p_a - p_b \in \{\delta, 2\delta\}$. Enfin, des ordres d'achat et de vente arrivent selon une mesure aléatoire de Poisson qui dépend de la structure du carnet d'ordre à travers l'*imbalance*, voir par exemple [18].

Les agents peuvent placer des ordres à cours limité, des ordres au marché (aussi appelés ordres agressifs) et, lorsque $p_a - p_b = 2\delta$, placer des ordres à cours limité au prix $p_b + \delta$. Ils peuvent également annuler leurs ordres à cours limité non exécutés ou partiellement exécutés. Ils peuvent placer jusqu'à un ordre à cours limité à l'achat et un autre ordre à cours limité à la vente.

Face à ce carnet d'ordre, nous nous intéressons à plusieurs agents économiques. Le premier est le teneur de marché, ou *market maker*. Celui-ci va essayer de se faire acheter au prix p_b tout en vendant au prix p_a et ainsi, d'empocher le *bid-ask spread* $p_a - p_b$. La mesure aléatoire de Poisson dépend des paramètres du carnet d'ordre : en particulier de la taille des queues. Le teneur de marché pourra également être amené à jouer sur une anticipation de mouvement du prix. Intuitivement, il placera des ordres à cours limité de chaque côté du carnet d'ordre et pourra être amené à placer des ordres au marché si son inventaire devient trop déséquilibré.

Nous construisons un problème de contrôle optimal pour le teneur de marché où celui-ci gère son inventaire et vise à maximiser la valeur de ses opérations sur le carnet d'ordre sur une période $[0, T]$ à travers une fonction d'utilité. Nous proposons un schéma numérique convergent et complètement explicite afin de le résoudre.

Nous nous intéressons ensuite au comportement d'un trader haute fréquence, ou *High Frequency Trader*. Celui-ci ne cherche pas à empocher le *bid-ask spread* mais à profiter d'un écart de cours entre l'actif d'une part, et un future sur ce même actif d'autre part. Lorsqu'il réalise une transaction sur le marché de l'actif, il réalise l'opération inverse immédiatement sur le marché des futures. Le marché des futures est modélisé sans carnet d'ordre, plus précisément, on modélise uniquement l'écart de prix entre le future et l'action. Cet écart profite d'un effet *retour à la moyenne* vers 0 que le trader haute fréquence va chercher à exploiter.

Nous construisons un problème de contrôle optimal pour le trader haute fréquence où celui-ci gère son inventaire composé d'actifs et de futures en sens opposé et vise à maximiser la valeur de ses opérations mixtes sur les deux marchés, en plaçant des ordres sur le carnet d'ordre de l'actif. Si un ordre est exécuté, il réalise automatiquement l'opération inverse sur le marché des futures. Nous proposons un schéma numérique convergent et complètement explicite afin de le résoudre.

À ces deux types d'agent, nous ajoutons des investisseurs institutionnels qui suivent des stratégies classiques d'acquisition ou de liquidation d'actifs : la stratégie VWAP (*Volume Weighted Average Price*) et des stratégies qui reposent sur le volume de marché (robots).

Enfin, grâce à un unique programme (en C++ avec OpenMP) qui résout dans un premier temps les problèmes de contrôle optimal, nous faisons jouer tous ces agents ensemble sur un

marché. Seul l'écart de prix entre l'action et le future sur celle-ci est simulé. Les agents interagissent sur le carnet d'ordre avec leurs contrôles pré-calculés et nous observons l'évolution de celui-ci et les stratégies des différents agents.

Chapitre 2

Optimal control under uncertainty and Bayesian parameters adjustments : Application to trading algorithms

We propose a general framework for the optimal control/design of trading algorithms in situations where market conditions or impact parameters are uncertain. Given a prior on the distribution of the unknown parameters, we explain how it should evolve according to the classical Bayesian rule after each sequence of trades. Taking these progressive prior-adjustments into account, we characterize the optimal policy through a quasi-variational parabolic equation, which can be solved numerically. From the mathematical point of view, we indeed treat a quite general impulse control problem with unknown parameters, and the derivation of the dynamic programming equation seems to be new in this context. The main difficulty lies in the nature of the set of controls which depends in a non trivial way on the initial data through the filtration itself. Typical examples of application are discussed.

2.1 Introduction

When trading at a high frequency level, several market parameters become of major importance. It can be the nature of the market impact of aggressive orders, or the time to be executed when entering a book order queue, see e.g. [51] and the references therein. However, the knowledge of these execution conditions is in general not perfect. One can try to estimate them but they remain random and can change from one market/platform to another one, or depending on the current market conditions. Most importantly, they can only be estimated by actually acting on the market. We therefore face the typical problem of estimating a reaction parameters (impact/execution time) while actually controlling a system (trading) that depends on these parameters.

Such problems have been widely studied in the discrete time stochastic optimal control

literature, see e.g. [37] for references. One fixes a certain prior distribution on the unknown parameter, and re-evaluate it each time an action is taken, by applying the standard Bayesian rule to the observed reactions. The optimal strategy generically results from a compromise between acting on the system, to get more information, and being not too aggressive, because of the uncertainty on the real value of the parameters. If the support of the initial prior contains the true value of the parameters, one can expect (under natural identification conditions) that the sequence of updated priors actually converges to it in the long range.

It is a-priori much more difficult to handle in a continuous time framework with continuous time monitoring, as it leads to a filtering problem, leaving on an infinite dimensional space. However, optimal trading under market impact can very naturally be considered in the impulse form, as robots send orders in a discrete time manner. In a sense, we are back to a discrete time problem which dimension can be finite (depending on the nature of the uncertainty), although interventions on the system may occur at any time.

In this paper, we thus consider a general impulse control problem with an unknown parameter, under which an initial prior law is set. Given this prior, we aim at maximizing a certain gain functional. We show that the corresponding value function can be characterized as the unique viscosity solution (in a suitable class) of a quasi-variational parabolic equation, for which a convergent numerical scheme is constructed. To better fit with market practices, we allow for (possibly) not observing immediately the effect of an impulse. This applies for instance to trading robots that are launched for a certain time period and whose impact will be observed only at the end of this period, or to dark pools in which nothing is observed but the execution time.

The study of such non-classical impulse control problems seems to be new in the literature. From the mathematical point of view, the main difficulty consists in establishing a dynamic programming principle. The principal reason lies in the choice of the filtration. Because of the uncertainty on the parameter driving the dynamics, the only natural filtration to which the control policy should be adapted is the one generated by the controlled process himself. This implies in particular that the set of admissible controls depends heavily (and in a very non trivial way) on the initial state of the system at the starting time of the strategy. Hence, no a priori regularity nor good measurability properties can be expected to construct explicitly measurable almost optimal controls, see e.g. [22], or to apply a measurable selection theorem, see e.g. [17]. We therefore proceed differently. The (usually considered as) easy part of the dynamic programming can actually be proved, as it only requires a conditioning argument. It leads as usual to a sub-solution characterization. We surround the difficulty in proving the second (difficult) part by considering a discrete time version of our initial continuous time control problem. When the time step goes to 0, it provides a super-solution of the targeted dynamic programming equation. Using comparison and the natural ordering on the value functions associated to the continuous and the discrete time model, we show that the two coincide at the limit.

We consider two examples of applications. In the first one, aggressive orders are send in a model with immediate and resilient impact. The unknown are the parameters of the impact and liquidity costs functions. In the second one, we only consider limit orders. The unknown is the distribution of the time to be executed. In both situations, the problems

can be solved numerically without much difficulties and we provide numerical illustrations showing the dependence of the optimal strategies on the current priors.

The rest of the chapter is organized as follows. The model is described in Section 2.2. In Section 2.3, we provide the PDE characterization of the value function. Proofs are collected in Section 2.4 (viscosity solution properties) and in Section 2.5 (comparison). An example of numerical scheme is provided in Section 2.6. Section 2.7 is dedicated to two examples of application.

2.2 The impulse problem with parameters adjustment

All over this paper, $C([0, T], \mathbb{R}^d)$ is the space of continuous functions from $[0, T]$ into \mathbb{R}^d which start at 0 at the origin. Recall that it is a Polish space for the sup-norm topology. We denote by $W(\omega) = \omega$ the canonical process and let \mathbb{P} be the Wiener measure. We also consider a Polish space $(U, \mathcal{B}(U))$ that will support an unknown parameter v . We denote by \mathbf{M} a locally compact subset¹ of the set of Borel probability measures on U endowed with the topology of weak convergence. In particular, it is Polish. A prior on the unknown parameter v will be an element $m \in \mathbf{M}$. To allow for additional randomness in the measurement of the effects of actions on the system, we consider another Polish space E on which is defined a family $(\epsilon_i)_{i \geq 0}$ of i.i.d. random variables with common measure \mathbb{P}_ϵ on E . On the product space $\Omega := C([0, T], \mathbb{R}^d) \times U \times E^{\mathbb{N}}$, we consider the family of measures $\{\mathbb{P} \times m \times \mathbb{P}_\epsilon^{\otimes \mathbb{N}} : m \in \mathbf{M}\}$ and denote by \mathbb{P}_m an element of this family whenever $m \in \mathbf{M}$ is fixed. The operator \mathbb{E}_m is the expectation associated to \mathbb{P}_m . Note that W , v and $(\epsilon_i)_{i \geq 0}$ are independent under each \mathbb{P}_m . For $m \in \mathbf{M}$ given, we let $\mathbb{F}^m = (\mathcal{F}_t^m)_{t \geq 0}$ denote the \mathbb{P}_m -augmentation of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ defined by $\mathcal{F}_t = \sigma((W_s)_{s \leq t}, v, (\epsilon_i)_{i \geq 0})$ for $t \geq 0$. Hereafter, all the random variables are considered with respect to the probability space $(\Omega, \mathcal{F}_T^m)$ with $m \in \mathbf{M}$ given by the context, and where T is a fixed time horizon.

2.2.1 The controlled system

Let $\mathbf{A} \subset [0, T] \times \mathbb{R}^d$ be a (non-empty) compact set. Given $N \in \mathbb{N}$ and $m \in \mathbf{M}$, we denote by $\Phi_N^{\circ, m}$ the collection of sequences of random variables $\phi = (\tau_i, \alpha_i)_{i \geq 1}$ on $(\Omega, \mathcal{F}_T^m)$ with values in $\mathbb{R}_+ \times \mathbf{A}$ such that $(\tau_i)_{i \geq 1}$ is a non-decreasing sequence of \mathbb{F}^m -stopping times satisfying $\tau_j > T$ \mathbb{P}_m -a.s. for $j > N$. We set

$$\Phi^{\circ, m} := \bigcup_{N \geq 1} \Phi_N^{\circ, m}.$$

An element $\phi = (\tau_i, \alpha_i)_{1 \leq i \leq N} \in \Phi^{\circ, m}$ will be our impulse control and we write α_i in the form

$$\alpha_i = (\ell_i, \beta_i) \text{ with } \ell_i \in [0, T] \text{ and } \beta_i \in \mathbb{R}^d \mathbb{P}_m \text{- a.s.}$$

More precisely, the τ_i 's will be the times at which an impulse is made on the system (e.g. a trading robot is launched), β_i will model the nature of the order send at time τ_i (e.g. the

1. In many situations, the family of probability measures of interest will in fact be parameterized or be the set of measures on a compact metrizable space, see Remark 2.2.1 below.

parameters used for the trading robot), and ℓ_i will stand for the maximal time length during which no new intervention on the system can be made (e.g. the time prescribed to the robot to send orders on the market). Later on we shall impose more precise non-anticipativity conditions.

From now on, we shall always use the notation $(\tau_i^\phi, \alpha_i^\phi)_{i \geq 1}$ with $\alpha_i^\phi = (\ell_i^\phi, \beta_i^\phi)$ to refer to a control $\phi \in \Phi^{\circ, m}$.

We allow for not observing nor being able to act on the system before a random time ϑ_i^ϕ defined by

$$\vartheta_i^\phi := \varpi(\tau_i^\phi, X_{\tau_i^{\phi-}}^\phi, \alpha_i^\phi, v, \epsilon_i),$$

where X^ϕ is the controlled state process that will be described below, and

$$\varpi : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{A} \times \mathbf{U} \times \mathbf{E} \rightarrow [0, T] \text{ is measurable, such that } \varpi(t, \cdot) \geq t \text{ for all } t \geq 0 \quad (2.2.1)$$

In the case where the actions consist in launching a trading robot at τ_i^ϕ during a certain time ℓ_i^ϕ , we can naturally take $\vartheta_i^\phi = \tau_i^\phi + \ell_i^\phi$. If the action consists in placing a limit order during a maximal duration ℓ_i^ϕ , ϑ_i^ϕ is the time at which the limit order is executed if it is less than $\tau_i^\phi + \ell_i^\phi$, and $\tau_i^\phi + \ell_i^\phi$ otherwise.

We say that $\phi \in \Phi^{\circ, m}$ belongs to Φ^m if $\vartheta_i^\phi \leq \tau_{i+1}^\phi$ and $\tau_i^\phi < \tau_{i+1}^\phi$ \mathbb{P}_m -a.s. for all $i \geq 1$, and define

$$\mathcal{N}^\phi := \left[\bigcup_{i \geq 1} [\tau_i^\phi, \vartheta_i^\phi] \right]^c. \quad (2.2.2)$$

We are now in a position to describe our controlled state process. Given some initial data $z := (t, x) \in \mathbf{Z} := [0, T] \times \mathbb{R}^d$, and $\phi \in \Phi^m$, we let $X^{z, \phi}$ be the unique strong solution on $[t, 2T]$ of

$$\begin{aligned} X &= x + \left(\int_t^\cdot \mathbf{1}_{\mathcal{N}^\phi}(s) \mu(s, X_s) ds + \int_t^\cdot \mathbf{1}_{\mathcal{N}^\phi}(s) \sigma(s, X_s) dW_s \right) \\ &\quad + \sum_{i \geq 1} \mathbf{1}_{\{t \leq \vartheta_i^\phi \leq \cdot\}} [F(\tau_i^\phi, X_{\tau_i^{\phi-}}, \alpha_i^\phi, v, \epsilon_i) - X_{\tau_i^{\phi-}}]. \end{aligned} \quad (2.2.3)$$

In the above, the function

$$\begin{aligned} (\mu, \sigma, F) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{A} \times \mathbf{U} \times \mathbf{E} &\mapsto \mathbb{R}^d \times \mathbb{M}^d \times \mathbb{R}^d \text{ is measurable.} \\ \text{The map } (\mu, \sigma) &\text{ is continuous, and Lipschitz with linear growth} \\ &\text{in its second argument, uniformly in the first one,} \end{aligned} \quad (2.2.4)$$

with \mathbb{M}^d defined as the set of $d \times d$ matrices. This dynamics means the following. When no action is currently made on the system, i.e. on the intervals in \mathcal{N}^ϕ , the system evolves according to a stochastic differential equation driven by the Brownian motion W :

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s \quad \text{on } \mathcal{N}^\phi.$$

When an impulse is made at τ_i^ϕ , we freeze the dynamics up to the end of the action at time ϑ_i^ϕ . This amounts to saying that we do not observe the current evolution up to ϑ_i^ϕ . At the end of

the action, the state process takes a new value $X_{\vartheta_i^\phi} = F(\tau_i^\phi, X_{\tau_i^\phi-}, \alpha_i^\phi, v, \epsilon_i)$. The fact that F depends on the unknown parameter v and the additional noise ϵ_i models the fact the correct model is not known with certainty, and that the exact value of the unknown parameter v can (possibly) not be measured precisely just by observing $(\vartheta_i^\phi - \tau_i^\phi, X_{\vartheta_i^\phi} - X_{\tau_i^\phi-})$.

In order to simplify the notations, we shall now write :

$$Z^{z,\phi} := (\cdot, X^{z,\phi}) \quad \text{and} \quad Z^{z,\circ} := (\cdot, X^{z,\circ}) \quad (2.2.5)$$

in which $X^{z,\circ}$ denotes the solution of (2.2.3) for ϕ such that $\tau_1^\phi > T$ and satisfying $X_t^{z,\circ} = x$. This corresponds to the stochastic differential equation (2.2.3) in the absence of impulse. Note in particular that

$$Z_{\vartheta_1^\phi}^{z,\phi} = z'(Z_{\tau_1^\phi-}^{z,\circ}, \alpha_1^\phi, v, \epsilon_1) \quad \text{on} \quad \{\tau_1^\phi \geq t\}, \quad \text{with} \quad z' := (\varpi, F). \quad (2.2.6)$$

From now on, we denote by $\mathbb{F}^{z,m,\phi} = (\mathcal{F}_s^{z,m,\phi})_{t \leq s \leq 2T}$ the \mathbb{P}_m -augmentation of the filtration generated by $(X^{z,\phi}, \sum_{i \geq 1} \mathbf{1}_{[\vartheta_i^\phi, \infty)})$ on $[t, 2T]$. We say that $\phi \in \Phi^m$ belongs to $\Phi^{z,m}$ if $(\tau_i^\phi)_{i \geq 1}$ is a sequence of $\mathbb{F}^{z,m,\phi}$ -stopping times and α_i^ϕ is $\mathcal{F}_{\tau_i^\phi}^{z,m,\phi}$ -measurable, for each $i \geq 1$. Hereafter an admissible control will be an element of $\Phi^{z,m}$.

2.2.2 Bayesian updates

Obviously, the prior m will evolve with time, as the value of the unknown parameter is partially revealed through the observation of the impacts of the actions on the system : at time t , one has observed $\{z'(Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi, v, \epsilon_i) : i \geq 1, \vartheta_i^\phi \leq t\}$. It should therefore be considered as a state variable, in any case, as its dynamics will naturally appear in any dynamic programming principle related to the optimal control of $X^{z,\phi}$, see Proposition 2.4.2 below. Moreover, its evolution can be of interest in itself. One can for instance be interested by the precision of our (updated) prior at the end of the control period, as it can serve as a new prior for another control problem.

In this section, we describe how it is updated with time, according to the usual Bayesian procedure. Given $z = (t, x) \in \mathbf{Z}$, $u \in \mathbf{U}$ and $a \in \mathbf{A}$, we assume that the law under \mathbb{P}_ϵ of $z'[z, a, u, \epsilon_1]$, recall (2.2.6), is given by $q(\cdot|z, a, u)d\mathbf{Q}(\cdot|z, a)$, in which $q(\cdot|\cdot)$ is a Borel measurable map and $\mathbf{Q}(\cdot|z, a)$ is a dominating measure on \mathbf{Z} for each $(z, a) \in \mathbf{Z} \times \mathbf{A}$. For $z = (t, x) \in \mathbf{Z}$, $m \in \mathbf{M}$ and $\phi \in \Phi^{z,m}$, let $M^{z,m,\phi}$ be the process defined by

$$M_s^{z,m,\phi}[C] := \mathbb{P}_m[v \in C | \mathcal{F}_s^{z,m,\phi}], \quad C \in \mathcal{B}(\mathbf{U}), \quad s \geq t. \quad (2.2.7)$$

As no new information is revealed in between the end of an action and the start of the next one, the prior should remain constant on these time intervals :

$$M^{z,m,\phi} = M_{\vartheta_i^\phi}^{z,m,\phi} \quad \text{on} \quad [\vartheta_i^\phi, \tau_{i+1}^\phi), \quad i \geq 0, \quad (2.2.8)$$

with the conventions $\vartheta_0^\phi = 0$ and $M_0^{z,m,\phi} = m$. But, $M^{z,m,\phi}$ should jump from each τ_i^ϕ to each ϑ_i^ϕ , $i \geq 1$, according to the Bayes rule :

$$M_{\vartheta_i^\phi}^{z,m,\phi} = \mathfrak{M}(M_{\tau_i^\phi-}^{z,m,\phi}; Z_{\vartheta_i^\phi}^{z,\phi}, Z_{\tau_i^\phi-}^{z,\phi}, \alpha_i^\phi), \quad i \geq 1, \quad (2.2.9)$$

in which

$$\mathfrak{M}(m_o; z'_o, z_o, a_o)[C] := \frac{\int_C q(z'_o | z_o, a_o, u) dm_o(u)}{\int_U q(z'_o | z_o, a_o, u) dm_o(u)}, \quad (2.2.10)$$

for almost all $(z_o, z'_o, a_o, m_o) \in \mathbf{Z}^2 \times \mathbf{A} \times \mathbf{M}$ and $C \in \mathcal{B}(U)$.

Note that we did not specify $M^{z,m,\phi}$ on each $[\tau_i^\phi, \vartheta_i^\phi)$ since the controller must wait until ϑ_i^ϕ before being able to make another action. A partial information on v through ϑ_i^ϕ is known as a right-censored observation of ϑ_i^ϕ is revealed through the interval $[\tau_i^\phi, \vartheta_i^\phi)$.

In order to ensure that $M^{z,m,\phi}$ remains in \mathbf{M} whenever $m \in \mathbf{M}$, we need the following standing assumption :

Assumption 2.2.1 (Standing Assumption).

$$\mathfrak{M}(\mathbf{M}; \cdot) \subset \mathbf{M}.$$

Remark 2.2.1. *The above assumption means that we have to define a locally compact space \mathbf{M} such the initial prior belongs to \mathbf{M} , and that is stable under the operator \mathfrak{M} . It is important for the use of viscosity solutions. This is clearly a limitation of our approach, from a theoretical point of view. An alternative would be to lift \mathbf{M} to the space of square integrable random variables, and then use the methodologies developed in the context of mean-field games (see e.g. [25, Section 6]). We prevent from doing this for sake of clarity. On the other hand, our assumptions are satisfied in many practical applications where \mathbf{M} is either a set of measures defined on a metrizable compact space, see e.g. [17, Proposition 7.22 p130], or a parameterized family (which needs to be the case eventually if a numerical resolution is performed). If it is a parameterized family, it suffices to find an homeomorphism f from an open set of \mathbb{R}^k , $k \geq 1$, to \mathbf{M} to ensure that \mathbf{M} is locally compact. On the other hand, the stability of \mathbf{M} with respect to \mathfrak{M} can be ensured by using conjugate families, as explained in e.g. [16, Chapter 5.2]. The simplest example being the convex hull of a family of Dirac masses. See Section 2.7 for examples of applications.*

We formalize the dynamics of $M^{z,m,\phi}$ in the next proposition.

Proposition 2.2.1. *For all $z = (t, x) \in \mathbf{Z}$, $m \in \mathbf{M}$ and $\phi \in \Phi^{z,m}$, the process $M^{z,m,\phi}$ is \mathbf{M} valued and follows the dynamics (2.2.8)-(2.2.9) on $[t, 2T]$.*

Proof. Let C be a Borel set of U and φ be a Borel bounded function on the Skorohod space D^{d+1} of càdlàg functions with values in \mathbb{R}^{d+1} . Set $\xi^\phi := \sum_{i \geq 1} \mathbf{1}_{\{\vartheta_i^\phi, \infty\}}$ and set $\delta X^i := X_{\cdot \vee \vartheta_i^\phi}^{z,\phi} - X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}$. One can find a Borel measurable map $\bar{\varphi}$ on D^{2d+1} such that

$$\varphi(X_{\cdot \wedge s}^{z,\phi}, \xi_{\cdot \wedge s}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} = \bar{\varphi}(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \delta X_{\cdot \wedge s}^i, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}.$$

Then, the independence of v with respect to $\sigma(W_{\cdot \vee \vartheta_i^\phi} - W_{\vartheta_i^\phi})$ given $\mathcal{F}_{\vartheta_i^\phi}^{z,m,\phi}$, and the fact that τ_{i+1}^ϕ is measurable with respect to the sigma-algebra generated by $\sigma(W_{\cdot \vee \vartheta_i^\phi} - W_{\vartheta_i^\phi})$ and $\mathcal{F}_{\vartheta_i^\phi}^{z,m,\phi}$ imply that, for $s \geq 0$,

$$\begin{aligned} \mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \varphi(X_{\cdot \wedge s}^{z,\phi}, \xi_{\cdot \wedge s}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right] &= \mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \bar{\varphi}(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \delta X_{\cdot \wedge s}^i, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right] \\ &= \mathbb{E}_m \left[M_{\vartheta_i^\phi}^{z,m,\phi}[C] \bar{\varphi}(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \delta X_{\cdot \wedge s}^i, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right] \\ &= \mathbb{E}_m \left[M_{\vartheta_i^\phi}^{z,m,\phi}[C] \varphi(X_{\cdot \wedge s}^{z,\phi}, \xi_{\cdot \wedge s}^\phi) \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} \right]. \end{aligned}$$

This shows that $M_s^{z,m,\phi}[C] \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}} = M_{\vartheta_i^\phi}^{z,m,\phi}[C] \mathbf{1}_{\{\vartheta_i^\phi \leq s < \tau_{i+1}^\phi\}}$ \mathbb{P}_m - a.s.

It remains to compute $M_{\vartheta_i^\phi}^{z,m,\phi}$. Note that (2.2.3) implies that $(X_{\tau_i^\phi}^{z,\phi}, \xi_{\tau_i^\phi}^\phi) = (X_{\vartheta_i^\phi}^{z,\phi}, \xi_{\vartheta_i^\phi}^\phi)$. Let φ be as above, and let $\bar{\varphi}$ be a Borel measurable map on $\mathbb{D}^{d+1} \times \mathbb{R}_+ \times \mathbb{R}^d$ such that

$$\varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) = \bar{\varphi}(X_{\cdot \wedge \tau_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi}^\phi, \vartheta_i^\phi, X_{\vartheta_i^\phi}^{z,\phi}) = \bar{\varphi}(X_{\cdot \wedge \tau_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi}^\phi, z'[\tau_i^\phi, X_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi, v, \epsilon_i]).$$

Then, since ϵ_i is independent of $\mathcal{F}_{\tau_i^\phi}^{z,m,\phi}$ and has the same law as ϵ_1 ,

$$\begin{aligned} &\mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \right] \\ &= \mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \bar{\varphi}(X_{\cdot \wedge \tau_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi}^\phi, z'[\tau_i^\phi, X_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi, v, \epsilon_i]) \right] \\ &= \mathbb{E}_m \left[\int \mathbf{1}_{\{v \in C\}} \bar{\varphi}(X_{\cdot \wedge \tau_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi}^\phi, z') q(z' | Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi, v) dQ(z' | Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi) \right] \\ &= \mathbb{E}_m \left[\int \bar{\varphi}(X_{\cdot \wedge \tau_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi}^\phi, z') \left(\int_C q(z' | Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi, u) dM_{\tau_i^\phi}^{z,m,\phi}(u) \right) dQ(z' | Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi) \right]. \end{aligned}$$

Let us now introduce the notation $\mathfrak{M}_i[C](z') := \mathfrak{M}(M_{\tau_i^\phi}^{z,m,\phi}; z', Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi)$. Then,

$$\begin{aligned} &\mathbb{E}_m \left[\mathbf{1}_{\{v \in C\}} \varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \right] \\ &= \mathbb{E}_m \left[\int \bar{\varphi}(X_{\cdot \wedge \tau_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \tau_i^\phi}^\phi, z') \mathfrak{M}_i[C](z') q(z' | Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi, v) dQ(z' | Z_{\tau_i^\phi}^{z,\phi}, \alpha_i^\phi) \right] \\ &= \mathbb{E}_m \left[\varphi(X_{\cdot \wedge \vartheta_i^\phi}^{z,\phi}, \xi_{\cdot \wedge \vartheta_i^\phi}^\phi) \mathfrak{M}_i[C](Z_{\vartheta_i^\phi}^{z,\phi}) \right]. \end{aligned}$$

This concludes the proof. \square

Remark 2.2.2. For later use, note that the above provides the joint conditional distribution of $(Z_{\vartheta_i^\phi}^{z,\phi}, M_{\vartheta_i^\phi}^{z,m,\phi})$ given $\mathcal{F}_{\tau_i^\phi}^{z,m,\phi}$. Namely, for Borel sets $B \in \mathcal{B}([t, T] \times \mathbb{R}^d)$ and $D \in \mathcal{B}(\mathbf{M})$, a simple application of Fubini's Lemma implies that

$$\mathbb{P}[(Z_{\vartheta_i^\phi}^{z,\phi}, M_{\vartheta_i^\phi}^{z,m,\phi}) \in B \times D | \mathcal{F}_{\tau_i^\phi}^{z,m,\phi}] = \mathbf{k}(B \times D | Z_{\tau_i^\phi}^{z,\phi}, M_{\tau_i^\phi}^{z,m,\phi}, \alpha_i^\phi) \quad (2.2.11)$$

in which

$$k(B \times D | z_o, m_o, a_o) := \int_U \int_B \mathbf{1}_D(\mathfrak{M}(m_o; z', z_o, a_o)) q(z' | z_o, a_o, u) dQ(z' | z, a) dm_o(u), \quad (2.2.12)$$

for $(z_o, m_o, a_o) \in \mathbf{Z} \times \mathbf{M} \times \mathbf{A}$.

2.2.3 Gain function

Given $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$, the aim of the controller is to maximize the expected value of the gain functional

$$\phi \in \Phi^{z,m} \mapsto G^{z,m}(\phi) := g(Z_{T[\phi]}^{z,\phi}, M_{T[\phi]}^{z,m,\phi}, v, \epsilon_0),$$

in which $T[\phi]$ is the end of the last action after T :

$$T[\phi] := \sup\{\vartheta_i^\phi : i \geq 1, \tau_i^\phi \leq T\} \vee T.$$

As suggested earlier, the gain may not only depend on the value of the original time-space state process $Z_{T[\phi]}^{z,\phi}$ but also on $M_{T[\phi]}^{z,m,\phi}$, to model the fact that we are also interested by the precision of the estimation made on v at the final time. One also allows for terminating the last action after T . However, since g can depend on $T[\phi]$ through $Z_{T[\phi]}^{z,\phi}$, one can penalize the actions that actually terminates strictly after T .

Hereafter, the function g is assumed to be measurable and bounded² on $\mathbf{Z} \times \mathbf{M} \times \mathbf{U} \times \mathbf{E}$.

Given $\phi \in \Phi^{z,m}$, the expected gain is

$$J(z, m; \phi) := \mathbb{E}_m [G^{z,m}(\phi)],$$

and

$$v(z, m) := \sup_{\phi \in \Phi^{z,m}} J(z, m; \phi) \mathbf{1}_{\{t \leq T\}} + \mathbf{1}_{\{t > T\}} \mathbb{E}_m [g(z, m, v, \epsilon_0)] \quad (2.2.13)$$

is the corresponding value function. Note that v depends on m through the set of admissible controls $\Phi^{z,m}$ and the expectation operator \mathbb{E}_m , even if g does not depend on $M_{T[\phi]}^{z,m,\phi}$.

Remark 2.2.3. *Note that a running gain term could be added without any difficulty. One usually reduces to a Mayer formulation by adding a component to the space process and by modifying the terminal reward accordingly. Here, if this running gain only covers the period $[0, T]$, it should be added explicitly because of the modified time horizon $T[\phi]$ at which the terminal gain is computed.*

2. Boundedness is just for sake of simplicity. Much more general frameworks could easily be considered.

2.3 Value function characterization

The aim of this section is to provide a characterization of the value function v . As usual, it should be related to a dynamic programming principle. In our setting, it corresponds to : Given $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$, then

$$v(z, m) = \sup_{\phi \in \Phi^{z, m}} \mathbb{E}_m[v(Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi})], \quad (2.3.1)$$

for all collection $(\theta^\phi, \phi \in \Phi^{z, m})$ of $\mathbb{F}^{z, m, \phi}$ -stopping times with values in $[t, 2T]$ such that $\theta^\phi \in \mathcal{N}^\phi \cap [t, T[\phi]]$ $\mathbb{P}_m - \text{a.s.}$, recall the definition of \mathcal{N}^ϕ in (2.2.2).

Let us comment this. First, one should restrict to stopping times such that $\theta^\phi \in \mathcal{N}^\phi$. The reason is that no new impulse can be made outside of \mathcal{N}^ϕ , each interval $[\tau_i^\phi, \vartheta_i^\phi]$ is a latency period. Second, the terminal gain is evaluated at $T[\phi]$, which in general is different from T . Hence, the fact that θ^ϕ is only bounded by $T[\phi]$.

A partial version of (2.3.1) will be proved in Proposition 2.4.2 below and will be used to provide a sub-solution property. As already mentioned in the introduction, we are not able to prove a full version (2.3.1). The reason is that the value function v depends on $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$ through the set of admissible controls $\Phi^{z, m}$, and more precisely through the choice of the filtration $\mathbb{F}^{z, m, \phi}$, which even depends on ϕ itself. This makes this dependence highly singular and we are neither in position to play with any a-priori smoothness, see e.g. [22], nor to apply a measurable selection theorem, see e.g. [17].

We continue our discussion, assuming that (2.3.1) holds and that v is sufficiently smooth. Then, it should in particular satisfy $v(z, m) \geq \mathbb{E}_m[v(Z_{t+h}^{z, \circ}, m)]$ whenever $z = (t, x) \in [0, T] \times \mathbb{R}^d$ and $0 < h \leq T - t$ ($Z^{z, \circ}$ is defined after (2.2.5)). This corresponds to the sub-optimality of the control consisting in making no impulse on $[t, t + h]$. Applying Itô's lemma, dividing by h and letting h go to 0, we obtain $-\mathcal{L}v(z, m) \geq 0$ in which \mathcal{L} is the Dynkin operator associated to $X^{z, \circ}$,

$$\mathcal{L}\varphi := \partial_t \varphi + \langle \mu, D\varphi \rangle + \frac{1}{2} \text{Tr}[\sigma \sigma^\top D^2 \varphi].$$

On the other hand, it follows from (2.3.1) and Remark 2.2.2 that

$$v(z, m) \geq \sup_{a \in \mathbf{A}} \mathbb{E}_m[v(z'[z, a, v, \epsilon_1], \mathfrak{M}(m; z'[z, a, v, \epsilon_1], z, a))] = \mathcal{K}v(z, m)$$

$$\text{where } \mathcal{K}\varphi := \sup_{a \in \mathbf{A}} \mathcal{K}^a \varphi \quad \text{with } \mathcal{K}^a \varphi := \int \varphi(z', m') d\mathbf{k}(z', m' | \cdot, a) \text{ for } a \in \mathbf{A}. \quad (2.3.2)$$

As for the time- T boundary condition, the same reasoning as above implies $v(T, \cdot) \geq \mathcal{K}_T g$ and $v(T, \cdot) \geq \mathcal{K}v(T, \cdot)$, in which

$$\mathcal{K}_T g(\cdot, m) = \int_{\mathbf{U}} \int_{\mathbf{E}} g(\cdot, m, u, e) d\mathbb{P}_\epsilon(e) dm(u). \quad (2.3.3)$$

By optimality, v should therefore solve the quasi-variational equations

$$\min \{-\mathcal{L}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \quad \text{on } [0, T] \times \mathbb{R}^d \times \mathbf{M} \quad (2.3.4)$$

$$\min \{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}\varphi\} = 0 \quad \text{on } \{T\} \times \mathbb{R}^d \times \mathbf{M}, \quad (2.3.5)$$

in the sense of the following definition (given for sake of clarity).

Definition 2.3.1. We say that a lower-semicontinuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity super-solution of (2.3.4)-(2.3.5) if for any $z_o = (t_o, x_o) \in \mathbf{Z}$, $m_o \in \mathbf{M}$, and $\varphi \in C^{1,2,0}([0, T] \times \mathbb{R}^d \times \mathbf{M})$ such that $\min_{\mathbf{Z} \times \mathbf{M}}(U - \varphi) = (U - \varphi)(z_o, m_o) = 0$ we have

$$\left[\min \{-\mathcal{L}\varphi, \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_o < T\}} + \min \{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_o = T\}} \right] (z_o, m_o) \geq 0.$$

We say that a upper-semicontinuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity sub-solution of (2.3.4)-(2.3.5) if for any $z_o = (t_o, x_o) \in \mathbf{Z}$, $m_o \in \mathbf{M}$ and $\varphi \in C^{1,2,0}([0, T] \times \mathbb{R}^d \times \mathbf{M})$ such that $\max_{\mathbf{Z} \times \mathbf{M}}(U - \varphi) = (U - \varphi)(z_o, m_o) = 0$ we have

$$\left[\min \{-\mathcal{L}\varphi, \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_o < T\}} + \min \{\varphi - \mathcal{K}_T g, \varphi - \mathcal{K}U\} \mathbf{1}_{\{t_o = T\}} \right] (z_o, m_o) \leq 0.$$

We say that a continuous function U on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$ is a viscosity solution of (2.3.4)-(2.3.5) if it is a super- and a sub-solution.

To ensure that the above operator is continuous, we assume from now on that, on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$,

$$\begin{aligned} \mathcal{K}_T g \text{ is continuous, and } \mathcal{K}\varphi \text{ is upper- (resp. lower-) semicontinuous,} \\ \text{for all upper- (resp. lower-) semicontinuous bounded function } \varphi. \end{aligned} \quad (2.3.6)$$

A sufficient condition for (2.3.6) to hold is that k defined in (2.2.12) is a continuous stochastic kernel, see [17, Proposition 7.31 and 7.32 page 148].

Finally, we assume that comparison holds for (2.3.4)-(2.3.5).

Assumption 2.3.1. Let U (resp. V) be a upper- (resp. lower-) semicontinuous bounded viscosity sub- (resp. super-) solution of (2.3.4)-(2.3.5). Assume further that $U \leq V$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Then, $U \leq V$ on $\mathbf{Z} \times \mathbf{M}$.

See Proposition 2.5.1 below for a sufficient condition. We are now in position to state the main result of this paper. The proof is provided in the next section.

Theorem 2.3.1. Let Assumption 2.3.1 (or the conditions of Proposition 2.5.1 below) hold. Then, v is continuous on $\mathbf{Z} \times \mathbf{M}$ and is the unique bounded viscosity solution of (2.3.4)-(2.3.5).

Remark 2.3.1. We do not discuss here the issue of existence of an optimal control, we refer to subsection 2.6.2 for the construction of approximately optimal controls. Note also that the construction of Section 2.4.2 below produces an almost optimal control as the arguments of Section 2.4.4 show that the sequence of value functions $(v_n)_{n \geq 1}$ actually converges to v .

2.4 Viscosity solution properties

This part is dedicated to the proof of the viscosity solution characterization of Theorem 2.3.1. We start with the sub-solution property, which is the more classical part. As for the super-solution property, we shall later on introduce a discrete time version of the model that will provide a natural lower bound. We will then show that the sequence of corresponding value functions converges to a super-solution of our quasi-variational equation as the time step goes to 0. By comparison, we will finally identify this (limit) lower bound to the original value function, thus showing that the later is also a super-solution.

2.4.1 Sub-solution property

We start with the sub-solution property and show that it is satisfied by the upper-semicontinuous envelope of v defined in (2.2.13) :

$$v^*(z, m) := \limsup_{(z', m') \rightarrow (z, m)} v(z', m') \quad , \quad (z, m) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}.$$

Proposition 2.4.1. v^* is a viscosity subsolution of (2.3.4)-(2.3.5).

The proof is rather standard. As usual, it is based on the partial dynamic programming principle contained in Proposition 2.4.2 below, that can be established by adapting standard lines of arguments, see e.g. [22]. For this part, the dependency of the filtration on the initial data is not problematic as it only requires a conditioning argument. Before to state it, let us make an observation.

Remark 2.4.1. Note that, given $z = (t, x) \in \mathbf{Z}$, the process $X^{z, \circ}$ defined in (2.2.5) is predictable with respect to the \mathbb{P} -augmentation of the raw filtration $\mathbb{F}^{t, W}$ generated by $(W_{\cdot \vee t} - W_t)$. By [34, Lemma 7, Appendix I], it is indistinguishable from a $\mathbb{F}^{t, W}$ -predictable process. Using this identification, $X_s^{z, \circ}(\omega) = X_s^{z, \circ}(\omega^{t, s})$ for $s \geq t$, with $\omega^{t, s} := \omega_{t \vee \cdot \wedge s} - \omega_t$. Similarly, τ_1^ϕ and α_1^ϕ can be identified to Borel measurable maps on $C([0, T]; \mathbb{R}^d)$ that depends only on $\omega^{t, \tau_1^\phi(\omega^{t, T})}$ so that $(Z_{\vartheta_1^\phi}^{z, \phi}, M_{\vartheta_1^\phi}^{z, m, \phi})$ can be seen as a Borel map on $C([0, T]; \mathbb{R}^d) \times \mathbf{U} \times \mathbf{E}$, while $(Z_{\tau_1^{\phi-}}^{z, \phi}, M_{\tau_1^{\phi-}}^{z, m, \phi})$ can be seen as a Borel map on $C([0, T]; \mathbb{R}^d)$ that only depends on $\omega^{t, \tau_1^\phi(\omega^{t, T})}$, recall (2.2.6), (2.2.8) and (2.2.9). Iterating this argument, we also obtain that $(Z_{T[\phi]}^{z, \phi}, M_{T[\phi]}^{z, m, \phi})$ is equal, up to \mathbb{P}_m -null sets, to a Borel map on $C([0, T]; \mathbb{R}^d) \times \mathbf{U} \times \mathbf{E}^N$, for some $N \geq 1$ that depends on ϕ .

We use the notations introduced in (2.2.5), (2.3.2) and (2.3.3) in the following.

Proposition 2.4.2. Fix $(z, m) \in \mathbf{Z} \times \mathbf{M}$, and let θ be the first exit time of $Z^{z, \circ}$ from a Borel set $B \subset \mathbf{Z}$ containing (z, m) . Then,

$$v(z, m) \leq \sup_{\phi \in \Phi_{\geq t}^{z, m}} \mathbb{E}_m[f(Z_\theta^{z, \circ}, m) \mathbf{1}_{\{\theta < \tau_1^\phi\}} + \mathcal{K}^{\alpha_1^\phi} f(Z_{\tau_1^{\phi-}}^{z, \circ}, m)] \mathbf{1}_{\{\theta \geq \tau_1^\phi\}} \quad (2.4.1)$$

in which $z := (t, x)$, $\Phi_{\geq t}^{z, m} := \{\phi \in \Phi^{z, m} : \tau_1^\phi \geq t\}$ and

$$f(z', m') := v^*(z', m') \mathbf{1}_{\{t' < T\}} + \mathcal{K}_T g(z', m') \mathbf{1}_{\{t' \geq T\}} \quad (2.4.2)$$

for $z' = (t', x') \in \mathbf{A}$ and $m' \in \mathbf{M}$.

Proof. Let $N \geq 1$ be such that $\tau_i^\phi > T$ for $i \geq N$. By right continuity of $(Z^{z, \phi}, M^{z, m, \phi})$ and upper-semicontinuity of f and $\mathcal{K}f$ on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$, see (2.3.6), it suffices to prove the result for the projections on the right of θ and τ_1^ϕ on a deterministic time grid. Then, it is enough to consider the case where $(\theta, \tau_1^\phi) \equiv (s, s') \in [t, T]^2$, by arguing as below and conditioning by the values taken by (θ, τ_1^ϕ) on the grid. In the following, we use regular conditional expectation

operators. We shall make use of Remark 2.4.1. In particular, we write $\phi(\omega, u, (e_i)_{i \leq N})$ to denote the Borel map $(\omega, u, (e_i)_{i \leq N}) \in C([0, T]; \mathbb{R}^d) \times \mathbf{U} \times \mathbf{E}^N \mapsto \{(\tau_i^\phi, \alpha_i^\phi)(\omega^{t,T}, u, (e_j)_{j \leq i-1}), i \leq N\}$ associated to ϕ . If $s < s'$, we have \mathbb{P}_m -a.s.

$$\begin{aligned} \mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_s^{z,m,\phi}](\omega, u, (e_i)_{i \geq 1}) &= \mathbb{E}_m[G^{Z_s^{z,\circ}(\omega^{t,s}),m}(\phi_{\omega^{t,s}})] \\ &= \mathbb{E}_m[\mathcal{K}_T g(X_T^{Z_s^{z,\circ}(\omega^{t,s}),\phi_{\omega^{t,s}}}, M_T^{Z_s^{z,\circ}(\omega^{t,s}),m,\phi_{\omega^{t,s}}})] \end{aligned}$$

in which \mathcal{K}_T is defined in (2.3.3) and

$$\phi_{\omega^{t,s}} : (\omega', u, (e_i)_{i \leq N}) \in C([s, T]; \mathbb{R}^d) \times \mathbf{U} \times \mathbf{E}^N \mapsto \phi(\omega^{t,s} + \omega'_{\cdot \vee s} - \omega'_s, u, (e_i)_{i \leq N})$$

is an element of $\Phi^{Z_s^{z,\circ}(\omega^{t,s}),m,\phi_{\omega^{t,s}}}$. It follows that $\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_s^{z,m,\phi}]\mathbf{1}_{s < s'} \leq f(Z_s^{z,\circ}, m)\mathbf{1}_{s < s'}$ \mathbb{P}_m -a.s. Similarly, if $s \geq s'$, we have \mathbb{P}_m -a.s.

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_{s'-}^{z,m,\phi}](\omega, u, (e_i)_{i \leq N}) = \mathbb{E}_m[G^{\xi(\omega^{t,s'}, v, \epsilon_1, \alpha_1^\phi(\omega^{t,s'}))}(\phi_{\omega^{t,s'}})]$$

with

$$\xi(\omega^{t,s'}, v, \epsilon_1, \alpha_1^\phi(\omega^{t,s'})) = (\cdot, \mathfrak{M}(m; \cdot, Z_{s'-}^{z,\circ}(\omega^{t,s'}), \alpha_1^\phi(\omega^{t,s'}))) \circ z'(Z_{s'-}^{z,\circ}(\omega^{t,s'}), \alpha_1^\phi(\omega^{t,s'}), v, \epsilon_1),$$

recall the notations in (2.2.6) and (2.2.10). Hence, \mathbb{P}_m -a.s.,

$$\mathbb{E}_m[G^{z,m}(\phi)|\mathcal{F}_{s'-}^{z,m,\phi}](\omega, u, (e_i)_{i \leq N}) \leq \mathbb{E}_m[f(\xi(\omega^{t,s'}, v, \epsilon_1, \alpha_1^\phi(\omega^{t,s'})))] = \mathcal{K}^{\alpha_1^\phi(\omega^{t,s'})} f(Z_{s'-}^{z,\circ}(\omega^{t,s'}), m),$$

in which $a \in \mathbf{A} \mapsto \mathcal{K}^a$ is defined in (2.3.2). \square

Proof of Proposition 2.4.1 As already mentioned, the proof is standard, we provide it for completeness. Let φ be a (bounded) $C^{1,2,0}$ function and fix $(z_o, m_o) \in \mathbf{Z} \times \mathbf{M}$ such that

$$0 = (v^* - \varphi)(z_o, m_o) = \max_{\mathbf{Z} \times \mathbf{M}} (v^* - \varphi). \quad (2.4.3)$$

We use the notation $z_o = (t_o, x_o) \in [0, T] \times \mathbb{R}^d$.

Step 1. We first assume that $t_o < T$. Let us suppose that $\min\{-\mathcal{L}\varphi, \varphi - \mathcal{K}v^*\}(z_o, m_o) > 0$, and work towards a contradiction to Proposition 2.4.2. Let $d_{\mathbf{M}}$ be a metric compatible with the weak topology and let $\|\cdot\|_{\mathbf{Z}}$ be the Euclidean norm on \mathbf{Z} . We define

$$\bar{\varphi}(z', m') := \varphi(z', m') + \|z' - z_o\|_{\mathbf{Z}}^4 + d_{\mathbf{M}}(m', m_o).$$

If the above holds, then $\min\{-\mathcal{L}\bar{\varphi}, \bar{\varphi} - \mathcal{K}v^*\}(z_o, m_o) > 0$. By our continuity assumption (2.3.6), we can find $\iota, \eta > 0$, such that

$$\min\{-\mathcal{L}\bar{\varphi}, \bar{\varphi} - \mathcal{K}v^*\} \geq \eta \quad \text{on } B_\iota, \quad (2.4.4)$$

in which

$$B_\iota := \{(z', m') \in \mathbf{Z} \times \mathbf{M} : \|z' - z_o\|_{\mathbf{Z}}^4 + d_{\mathbf{M}}(m', m_o) < \iota\} \subset [0, T] \times \mathbb{R}^d \times \mathbf{M}.$$

Note that, after possibly changing $\eta > 0$, we can assume that

$$(v^* - \bar{\varphi}) \leq -\eta \text{ on } (B_\iota)^c. \quad (2.4.5)$$

In the following, we let $(z, m) \in B_\iota$ be such that

$$|v(z, m) - \bar{\varphi}(z, m)| \leq \eta/2, \quad (2.4.6)$$

recall (2.4.3). As above, we write $z = (t, x) \in [0, T] \times \mathbb{R}^d$. Fix $\phi \in \Phi^{z, m}$. We write $(\tau_i, \alpha_i, \vartheta_i)_{i \geq 1}$, Z and M for $(\tau_i^\phi, \alpha_i^\phi, \vartheta_i^\phi)_{i \geq 1}$, $Z^{z, \phi}$ and $M^{z, m, \phi}$. Let θ be the first time when (Z, M) exits B_ι . Without loss of generality, one can assume that $\tau_1 \geq t$. Define $\chi := \theta \mathbf{1}_{\{\theta < \tau_1\}} + \mathbf{1}_{\{\theta \geq \tau_1\}} \vartheta_1$. In view of (2.4.4), (2.4.5) and (2.4.6),

$$\begin{aligned} \mathbb{E}_m[v^*(Z_\chi, M_\chi)] &= \mathbb{E}_m[v^*(Z_{\vartheta_1}, M_{\vartheta_1}) \mathbf{1}_{\{\chi \neq \theta\}} + v^*(Z_\theta, M_\theta) \mathbf{1}_{\{\chi = \theta\}}] \\ &\leq \mathbb{E}_m[\mathcal{K}v^*(Z_{\tau_1-}, M_{\tau_1-}) \mathbf{1}_{\{\chi \neq \theta\}} + v^*(Z_\theta, M_\theta) \mathbf{1}_{\{\chi = \theta\}}] \\ &\leq \mathbb{E}_m[\bar{\varphi}(Z_{\theta \wedge \tau_1-}, M_{\theta \wedge \tau_1-})] - \eta \\ &\leq \bar{\varphi}(z, m) - \eta \\ &\leq v(z, m) - \eta/2. \end{aligned}$$

Since $\chi < T$, this contradicts Proposition 2.4.2 by arbitrariness of ϕ .

Step 2. We now consider the case $t_o = T$. We assume that $\min\{\varphi - \mathcal{K}v^*, \varphi - \mathcal{K}Tg\}(z_o, m_o) > 0$, and work toward a contradiction. Let us define

$$\bar{\varphi}(t', x', m') := \bar{\varphi}(t', x', m') + C(T - t') + \|(t', x') - z_o\|_{\mathbf{Z}}^4 + d_{\mathbf{M}}(m', m_o)$$

and note that, for C large enough, $\min\{-\mathcal{L}\bar{\varphi}, \bar{\varphi} - \mathcal{K}v^*, \bar{\varphi} - \mathcal{K}Tg\}(z_o, m_o) > 0$. Then, as in Step 1, we can find $\iota, \eta > 0$, such that

$$\min\{-\mathcal{L}\bar{\varphi}, \bar{\varphi} - \mathcal{K}v^*, \bar{\varphi} - \mathcal{K}Tg\} \geq \eta \text{ on } B_\iota,$$

in which

$$B_\iota := \{(t', x', m') \in (T - \iota, T] \times \mathbf{M} : \|x' - x_o\|_{\mathbb{R}^d}^4 + d_{\mathbf{M}}(m', m_o) < \iota\}.$$

After possibly changing $\eta > 0$, one can assume that

$$(v^* - \bar{\varphi}) \leq -\eta \text{ on } (B_\iota)^c.$$

Let $(t, x, m) \in B_\iota$ be such that

$$|v(t, x, m) - \bar{\varphi}(t, x, m)| \leq \eta/2.$$

One can assume that $t < T$. Otherwise, this would mean that

$$v^*(z_o, m_o) = \limsup_{(T, x', m') \rightarrow (z_o, m_o)} v(T, x', m') = \limsup_{(T, x', m') \rightarrow (z_o, m_o)} \mathcal{K}_T(T, x', m') = \mathcal{K}_Tg(z_o, m_o),$$

recall (2.3.6), and there is nothing to prove. Given $\phi \in \Phi^{z, m}$, with $z := (t, x)$, let $(\tau_1, \vartheta_1, Z = (\cdot, X), M)$ be defined as in Step 1 with respect to ϕ and (z, m) , and consider $\chi := \theta \mathbf{1}_{\{\theta < \tau_1\}} + \mathbf{1}_{\{\theta \geq \tau_1\}} \vartheta_1$, where θ is the first exit time of (X, M) from $\{(x', m') \in \mathbb{R}^d \times \mathbf{M} : \|x' - x_o\|_{\mathbb{R}^d}^4 + d_{\mathbf{M}}(m', m_o) < \iota\}$. As in Step 1, the above implies that $\mathbb{E}_m[v^*(Z_\chi, M_\chi)] \leq v(z, m) - \eta/2$, which contradicts Proposition 2.4.2 by arbitrariness of ϕ . \square

2.4.2 Discrete time approximation and dynamic programming

In this part, we prepare for the proof of the super-solution property. As already mentioned above, we could not provide the opposite inequality in (2.4.1), with v^* replaced by the lower-semicontinuous envelope of v , because of the non-trivial dependence of $\mathbb{F}^{z,m,\phi}$ with respect to the initial data. Instead, we use the natural idea of approximating our continuous time control problem by a sequence of discrete time counterparts defined on a sequence of time grids. In discrete time, the dynamic programming principle can be proved along the lines of [17] for the corresponding value functions $(v_n)_{n \geq 1}$. Passing to the limit as the time mesh vanishes provides a super-solution v_\circ of (2.3.4)-(2.3.5). As v^* is a sub-solution of the same equation, Assumption 2.3.1 will imply that $v_\circ \geq v^*$, while the opposite will hold by construction. Then, we will conclude that v is actually a super-solution, and is even continuous. This approach is similar to the one used in [38] in the context of differential games.

We first construct the sequence of discrete time optimal control problems. For $n \geq 1$, let $\pi_n := \{t_j^n, j \leq 2^n\}$ with $t_j^n := jT/2^n$, and let $\Phi_n^{z,m}$ be the set of controls $\phi = (\tau_i^\phi, \alpha_i^\phi)_{i \geq 1}$ in $\Phi^{z,m}$ such that $(\tau_i^\phi)_{i \geq 1}$ takes values in $\pi_n \cup \{t\} \cup [T, \infty)$, if $z = (t, x)$. The corresponding value function is

$$v_n(z, m) = \sup_{\phi \in \Phi_n^{z,m}} J(z, m, \phi), \quad (z, m) \in \mathbf{Z} \times \mathbf{M}.$$

We extend v_n by setting

$$v_n := \mathcal{K}_T g, \quad \text{on } (T, \infty) \times \mathbf{M}, \quad (2.4.7)$$

Remark 2.4.2. *Note that $v_n \leq v \leq v^*$ by construction.*

We first prove that v_n satisfies a dynamic programming principle. This requires additional notations. We first define the next time on the grid at which a new action can be made, given that a is plaid :

$$s^{n,a}[t, x] := \min\{s \in \pi_n \cup [T, \infty) : s \geq \varpi(t, x, a, v, \epsilon_j) \text{ and } s > t\}.$$

Let ∂ denote a cemetery point that does not belong to \mathbf{A} . Given $a \in \mathbf{A} \cup \{\partial\}$, we make a slight abuse of notation by denoting by $(Z^{(t,x),a}, M^{(t,x),m,a})$ the process defined as $(Z^{(t,x),\phi}, M^{(t,x),m,\phi})$ for ϕ such that

$$(\tau_1^\phi, \alpha_1^\phi) = (t, a)\mathbf{1}_{\{a \neq \partial\}} + (T+1, a_\star)\mathbf{1}_{\{a = \partial\}}$$

in which $a_\star \in \mathbf{A}$ and $\tau_i^\phi > T+1$ for $i > 1$. Then, we set

$$\bar{J}(T, \cdot; a) := \mathcal{K}_T \mathcal{K}^a g, \quad \bar{v}_n(T, \cdot) := \sup_{a \in \mathbf{A} \cup \{\partial\}} \bar{J}(T, \cdot; a) \quad \text{on } \mathbb{R}^d \times \mathbf{M} \times (\mathbf{A} \cup \{\partial\}),$$

with the convention that \mathcal{K}^∂ is the identity, and define by backward induction on the intervals $[t_j^n, T)$, $j = n-1, \dots, 0$,

$$\bar{J}(z, m; a) := \mathbb{E}_m[\bar{v}_n(Z_{s^n, a[z]}^{z,a}, M_{s^n, a[z]}^{z,m,a})], \quad \bar{v}_n := \sup_{a \in \mathbf{A} \cup \{\partial\}} \bar{J}(\cdot; a),$$

together with the extension

$$\bar{v}_n := \mathcal{K}_T g \quad \text{on } (T, \infty) \times \mathbb{R}^d \times \mathbf{M}.$$

Lemma 2.4.1. Fix $\iota > 0$. Then, there exists a universally measurable map $(z, m) \in \mathbf{Z} \times \mathbf{M} \mapsto \hat{a}^{n,\iota}[z, m] \in \mathbf{A} \cup \{\partial\}$ such that $\bar{J}(\cdot; \hat{a}^{n,\iota}[\cdot]) \geq \bar{v}_n - \iota$ on $\mathbf{Z} \times \mathbf{M}$. Moreover, the map \bar{v}_n is upper semi-analytic.

Proof. Since \mathcal{K}_{Tg} is assumed to be upper semi-analytic (indeed continuous), it follows from [17, Proposition 7.48 page 180] that \bar{J} is upper semi-analytic on $[t_{n-1}^n, T] \times \mathbb{R}^d \times \mathbf{M} \times (\mathbf{A} \cup \{\partial\})$. Then, the required result holds on $[t_{n-2}^n, T] \times \mathbb{R}^d \times \mathbf{M}$ by [17, Proposition 7.50 page 184]. It is then extended to $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ by a backward induction. \square

Proposition 2.4.3. $\bar{v}_n = v_n$ on $\mathbf{Z} \times \mathbf{M}$. Moreover, given a random variable (ζ, μ) with values in $\mathbf{Z} \times \mathbf{M}$ and $\iota > 0$, there exists a measurable map $(z, m) \mapsto \phi^\iota[z, m]$ such that

$$J(\zeta, \mu; \phi^\iota[\zeta, \mu]) \geq v_n(\zeta, \mu) - \iota \quad \mathbb{P}_m - \text{a.s.}$$

Proof. The proof proceeds by induction. Our claim follows from definitions on $[t_n^n, T] \times \mathbb{R}^d \times \mathbf{M}$. Assume that it holds on $[t_{j+1}^n, T] \times \mathbb{R}^d \times \mathbf{M}$ for some $j \leq n-1$. For the following, we fix $z = (t, x) \in \mathbf{Z}$ with $t \in [t_j^n, t_{j+1}^n)$ and $m \in \mathbf{M}$.

Step 1 : In this step, we first construct a suitable candidate to be an almost-optimal control. Fix $\varepsilon_1, \dots, \varepsilon_n > 0$, $\varepsilon_0 := 0$, and set $\varepsilon(i) := (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_i)$. Let $(\hat{a}^{n,\iota})_{\iota>0}$ be as in Lemma 2.4.1, and consider its extension defined by $\hat{a}^{n,\iota} = a_\star$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$. Define $r_1^{\varepsilon(0)} := t$ and $\phi_1^{\varepsilon(1)} \in \Phi_n^{z,m}$ by

$$(\tau_i^{\phi_1^{\varepsilon(1)}}, \alpha_i^{\phi_1^{\varepsilon(1)}}) = (r_1^{\varepsilon(0)}, \tilde{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m]) \mathbf{1}_{\{i=1\}} + \mathbf{1}_{\{i>1\}}(T + i, a_\star), \quad i \geq 1.$$

where

$$\tilde{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m] := \hat{a}^{n,\varepsilon_1}[r_1^{\varepsilon(0)}, x, m].$$

We then set

$$r_2^{\varepsilon(1)} := \min \pi_n \cap [\vartheta_1^{\phi_1^{\varepsilon(1)}}, 2T] \cap (r_1^{\varepsilon(0)}, \infty).$$

By Lemma 2.4.1 and [17, Lemma 7.27 page 173] applied to the pull-back measure of $(Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z, m, \phi_1^{\varepsilon(1)}})$, we can find a Borel measurable map $(t', x', m') \in \mathbf{Z} \times \mathbf{M} \mapsto \tilde{a}_2^{n,\varepsilon_2}[t', x', m'] \in \mathbf{A} \cup \{\partial\}$ such that

$$\tilde{a}^{n,\varepsilon_2}[Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z, m, \phi_1^{\varepsilon(1)}}] = \hat{a}^{n,\varepsilon_2}[Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z, m, \phi_1^{\varepsilon(1)}}] \quad \mathbb{P}_m - \text{a.s.}$$

We define $\phi_2^{\varepsilon(2)}$ by

$$(\tau_i^{\phi_2^{\varepsilon(2)}}, \alpha_i^{\phi_2^{\varepsilon(2)}}) = (r_2^{\varepsilon(1)}, \tilde{a}^{n,\varepsilon_2}[Z_{r_2^{\varepsilon(1)}}^{z, \phi_1^{\varepsilon(1)}}, M_{r_2^{\varepsilon(1)}}^{z, m, \phi_1^{\varepsilon(1)}}]) \mathbf{1}_{\{i=2, r_2^{\varepsilon(1)} \leq T\}} + (\tau_i^{\phi_1^{\varepsilon(1)}}, \alpha_i^{\phi_1^{\varepsilon(1)}}) \mathbf{1}_{\{i \neq 2\} \cup \{r_2^{\varepsilon(1)} > T\}},$$

for $i \geq 1$. We then define recursively for $k \geq 2$

$$\begin{aligned} r_{k+1}^{\varepsilon(k)} &:= \inf \pi_n \cap [\vartheta_k^{\phi_k^{\varepsilon(k)}}, 2T] \cap (r_k^{\varepsilon(k-1)}, \infty) \\ (\tau_i^{\phi_{k+1}^{\varepsilon(k+1)}}, \alpha_i^{\phi_{k+1}^{\varepsilon(k+1)}}) &= (r_{k+1}^{\varepsilon(k)}, \tilde{a}^{n, \varepsilon_{k+1}} [Z_{r_{k+1}^{\varepsilon(k)}}^{z, \phi_k^{\varepsilon(k)}}, M_{r_{k+1}^{\varepsilon(k)}}^{z, m, \phi_k^{\varepsilon(k)}}]) \mathbf{1}_{\{i=k+1, r_{k+1}^{\varepsilon(k)} \leq T\}} \\ &\quad + (\tau_i^{\phi_k^{\varepsilon(k)}}, \alpha_i^{\phi_k^{\varepsilon(k)}}) \mathbf{1}_{\{i \neq k+1\} \cup \{r_{k+1}^{\varepsilon(k)} > T\}}, \end{aligned}$$

for $i \geq 1$, in which $(t', x', m') \in \mathbf{Z} \times \mathbf{M} \mapsto \tilde{a}_{k+1}^{n, \varepsilon_{k+1}}[t', x', m'] \in \mathbf{A} \cup \{\partial\}$ is a Borel measurable map such that

$$\tilde{a}^{n, \varepsilon_{k+1}} [Z_{r_{k+1}^{\varepsilon(k)}}^{z, \phi_k^{\varepsilon(k)}}, M_{r_{k+1}^{\varepsilon(k)}}^{z, m, \phi_k^{\varepsilon(k)}}] = \hat{a}^{n, \varepsilon_{k+1}} [Z_{r_{k+1}^{\varepsilon(k)}}^{z, \phi_k^{\varepsilon(k)}}, M_{r_{k+1}^{\varepsilon(k)}}^{z, m, \phi_k^{\varepsilon(k)}}] \quad \mathbb{P}_m - \text{a.s.}$$

We finally set

$$\phi^\varepsilon := (\tau_i^{\phi_i^{\varepsilon(i)}}, \alpha_i^{\phi_i^{\varepsilon(i)}})_{i \geq 1} \in \Phi_n^{z, m}.$$

Step 2 : We now prove that $\bar{v}_n(z, m) \geq v_n(z, m)$. By the above construction and Lemma 2.4.1,

$$\bar{v}_n(z, m) \geq \bar{J}(z, m; \alpha_1^{\phi_1^{\varepsilon(1)}}) \geq \bar{v}_n(z, m) - \varepsilon_1.$$

Since $v_n(t_k, \cdot) = \bar{v}_n(t_k, \cdot)$ for $k > j$ by our induction hypothesis, we obtain

$$\bar{v}_n(z, m) \geq \sup_{a \in \mathbf{A} \cup \{\partial\}} \mathbb{E}_m[v_n(Z_{r_2^{\varepsilon(1)}}^{z, a}, M_{r_2^{\varepsilon(1)}}^{z, m, a})] - \varepsilon_1 \geq v_n(z, m) - \varepsilon_1,$$

in which the last inequality follows from a simple conditioning argument as in the proof of Proposition 2.4.2. By arbitrariness of $\varepsilon_1 > 0$, this implies that $\bar{v}_n(z, m) \geq v_n(z, m)$.

Step 3 : It remains to prove that $\bar{v}_n(z, m) \leq v_n(z, m)$. Define

$$Y_i^{\varepsilon(i-1)} := (Z_{r_i^{\varepsilon(i-1)}}^{z, \phi_i^\varepsilon}, M_{r_i^{\varepsilon(i-1)}}^{z, m, \phi_i^\varepsilon}), \quad i \geq 1,$$

with $Y_0^{\varepsilon(-1)} := (z, m)$, and observe that $Y_i^{\varepsilon(i-1)}$ and $\mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z, m, \phi_i^\varepsilon}$ only depend on $\varepsilon(i-1)$. Then, for each $i \geq 0$,

$$\begin{aligned} \bar{v}_n(Y_i^{\varepsilon(i-1)}) &= \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\bar{v}_n(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}) | \mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z, m, \phi_i^\varepsilon}] \\ &= \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} \leq T\}} \bar{v}_n(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}) | \mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z, m, \phi_i^\varepsilon}] \\ &\quad + \lim_{\varepsilon_i \downarrow 0} \mathbb{E}_m[\mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} > T\}} g(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}), v, \varepsilon_0) | \mathcal{F}_{r_i^{\varepsilon(i-1)}}^{z, m, \phi_i^\varepsilon}] \quad \mathbb{P}_m - \text{a.s.} \end{aligned}$$

on $\{r_i^{\varepsilon(i-1)} \leq T\}$. Since g is bounded, so is \bar{v}_n . The above combined with the dominated convergence theorem then implies

$$\begin{aligned} \bar{v}_n(z, m) &= \lim_{\varepsilon_n \downarrow 0} \cdots \lim_{\varepsilon_n \downarrow 0} \mathbb{E}_m[\sum_{i=0}^n \mathbf{1}_{\{r_{i+1}^{\varepsilon(i)} > T \geq r_i^{\varepsilon(i-1)}\}} g(Z_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}, M_{r_{i+1}^{\varepsilon(i)}}^{Y_i^{\varepsilon(i-1)}, \phi_i^{\varepsilon(i)}}), v, \varepsilon_0)] \\ &= \lim_{\varepsilon_n \downarrow 0} \cdots \lim_{\varepsilon_n \downarrow 0} J(z, m; \phi^\varepsilon) \leq v_n(z, m), \end{aligned}$$

which concludes the proof that $\bar{v}_n = v_n$.

Step 4. The second assertion of the proposition is obtained by observing that, given a random variable (ζ, μ) with values in $\mathbf{Z} \times \mathbf{M}$, one can choose $\tilde{a}^{n, \varepsilon_1}$ Borel measurable such that $\tilde{a}^{n, \varepsilon_1}[\zeta, \mu] = \hat{a}^{n, \varepsilon_1}[\zeta, \mu]$ \mathbb{P}_m - a.s. \square

We are now in position to conclude that v_n satisfies a dynamic programming principle.

Corollary 2.4.1. *Fix $z = (t, x) \in \mathbf{Z}$ and $m \in \mathbf{M}$. Let $(\theta^\phi, \phi \in \Phi_n^{z, m})$ be such that each θ^ϕ is a $\mathbb{F}^{z, m, \phi}$ -stopping time with values in $[t, 2T] \cap (\pi_n \cup [T, \infty))$ such that $\theta^\phi \in \mathcal{N}^\phi \cap [t, T[\phi]]$ \mathbb{P}_m - a.s. for $\phi \in \Phi_n^{z, m}$. Then,*

$$v_n(z, m) = \sup_{\phi \in \Phi_n^{z, m}} \mathbb{E}_m[v_n(Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi})].$$

Proof. The inequality \leq can be obtained trivially by a conditioning argument. Fix $\phi \in \Phi_n^{z, m}$. By Proposition 2.4.3, we can find a Borel measurable map $(z', m') \mapsto \phi^t[z', m']$ such that

$$J(Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi}; \phi^t[Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi}]) \geq v_n(Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi}) - \iota.$$

Let us now simply write ϕ^t for $\phi^t[Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi}]$. Without loss of generality, one can assume that $\tau_1^\phi \geq t$ and that $\tau_1^{\phi^t} \geq \theta^\phi$. Let $I := \text{card}\{i \geq 1 : \tau_i^\phi < \theta^\phi\}$. Then, $J(z, m; \tilde{\phi}^t) \geq \mathbb{E}_m[v_n(Z_{\theta^\phi}^{z, \phi}, M_{\theta^\phi}^{z, m, \phi})] - \iota$ in which $(\tau_i^{\tilde{\phi}^t}, \alpha_i^{\tilde{\phi}^t}) = \mathbf{1}_{i \leq I}(\tau_i^\phi, \alpha_i^\phi) + \mathbf{1}_{i > I}(\tau_{i-I}^{\phi^t}, \alpha_{i-I}^{\phi^t})$, $i \geq 1$. Sending $\iota \rightarrow 0$ leads to the required result. \square

2.4.3 Super-solution property as the time step vanishes

We now consider the limit $n \rightarrow \infty$. Let us set, for $(z, m) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbf{M}$,

$$v_o(z, m) := \liminf_{(t', x', m', n) \rightarrow (z, m, \infty)} v_n(t', x', m').$$

Remark 2.4.3. *Note that (2.4.7) and (2.3.6) implies that $v_o = \mathcal{K}_T g$ on $(T, \infty) \times \mathbb{R}^d \times \mathbf{M}$.*

Proposition 2.4.4. *The function v_o is a viscosity super-solution of (2.3.4)-(2.3.5).*

Proof. Let $n_k \rightarrow \infty$ and $(z_k, m_k) \rightarrow (z_o, m_o)$ be such that $v_{n_k}(z_k, m_k) \rightarrow v_o(z_o, m_o)$.

Step 1. We first show that $v_o(z_o, m_o) \geq \mathcal{K}v_o(z_o, m_o)$. By Corollary 2.4.1 applied to v_{n_k} with a control ϕ^k defined by $(\tau_i^k, \alpha_i^k) = (t_k, a_k)\mathbf{1}_{\{i=1\}} + \sum_{j>1} (T+j, a_\star)\mathbf{1}_{\{i=j\}}$, $i \geq 1$, with $a_k \in \mathbf{A}$, we obtain

$$v_{n_k}(z_k, m_k) \geq \sup_{a_k \in \mathbf{A}} \int \mathbb{E}[v_{n_k}(Z_{s_+^{n_k}[z']}^{z', o}, m')] dk(z', m' | z_k, m_k, a_k) = \mathcal{K} \mathbb{E}[v_{n_k}(Z_{s_+^{n_k}[\cdot]}^{z', o}, \cdot)](z_k, m_k),$$

in which $s_+^{n_k}[t, x] := \min \pi_{n_k} \cap [t, \infty)$. Let φ_{k_o} be the lower-semicontinuous envelope of $\inf\{\mathbb{E}[v_{n_k}(Z_{s_+^{n_k}[\cdot]}^{z', o}, \cdot)], k \geq k_o\}$. Then, for $k \geq k_o$, $v_{n_k}(z_k, m_k) \geq \int \varphi_{k_o}(z', m') dk(z', m' | z_k, m_k, a_k)$, and, by (2.3.6), passing to the limit $k \rightarrow \infty$ leads to $v_o(z_o, m_o) \geq \int \varphi_{k_o}(z', m') dk(z', m' | z_o, m_o, a_o)$. We shall prove in step 3 that $\lim_{k_o \rightarrow \infty} \varphi_{k_o} \geq v_o$. These maps are bounded, since g is. Dominated convergence then implies that $v_o(z_o, m_o) \geq \int v_o(z', m') dk(z', m' | z_o, m_o, a_o)$.

Step 2. Let φ be a (bounded) $C^{1,2,0}([0, T] \times \mathbb{R}^d \times \mathbf{M})$ function and $(z_o, m_o) \in [0, T] \times \mathbb{R}^d \times \mathbf{M}$ be a minimal point of $v_o - \varphi$ on $\mathbf{Z} \times \mathbf{M}$. Without loss of generality, one can assume that $(v_o - \varphi)(z_o, m_o) = 0$. Let B and $(z_k, m_k, n_k)_{n \geq 1}$ be as in Lemma 2.4.2 below. We write $z_k = (t_k, x_k), z_o = (t_o, x_o) \in [0, T] \times \mathbb{R}^d$. On the other hand, by considering the control ϕ^k defined by $(\tau_i^k, \alpha_i^k) = (T + i, a_*)$, $i \geq 1$, we obtain from Corollary 2.4.1 that

$$v_{n_k}(z_k, m_k) \geq \mathbb{E}_m[v_{n_k}(Z_{t_k+h_k}^{z_k, \circ}, m)]$$

with $h_k \in T2^{-n_k}(\mathbb{N} \cup \{0\})$ such that $t_k + h_k < T$ if $t_o \neq T$ and $t_k + h_k = T$ otherwise. Let $C > 0$ be a common bound for $(v_n)_{n \geq 1}$ and φ . Then we can choose $(h_k)_{k \geq 1}$ such that

$$\delta_k := (\varphi(z_k, m_k) - v_{n_k}(z_k, m_k) - 2C \mathbb{P}[Z_{t_k+h_k}^{z_k, \circ} \notin B])/h_k \rightarrow 0.$$

This follows from standard estimates on the solution of sde's with Lipschitz coefficients. Then, if $t_o < T$,

$$0 \geq h_k^{-1} \mathbb{E}_m[\varphi(Z_{t_k+h_k}^{z_k, \circ}, m_k) - \varphi_{n_k}(z_k, m_k)] + \delta_k = \mathbb{E}_m[h_k^{-1} \int_{t_k}^{t_k+h_k} \mathcal{L}\varphi(Z_s^{z_k, \circ}, m_k) ds] + \delta_k,$$

sending $k \rightarrow \infty$ leads to $\mathcal{L}\varphi(z_o, m_o) \leq 0$. If $t_o = T$, $v_{n_k}(z_k, m_k) \geq \mathbb{E}_m[g(Z_T^{z_k, \circ}, m_k, v, \epsilon_0)] = \mathbb{E}_m[\mathcal{K}_T g(Z_T^{z_k, \circ}, m_k)]$ and passing to the limit leads to $\varphi(z_o, m_o) \geq \mathcal{K}_T g(z_o, m_o)$, recall (2.3.6). Finally, $\varphi(z_o, m_o) \geq \mathcal{K}\varphi(z_o, m_o)$ by Step 1. \square

Step 3 : It remains to prove the claim used in Step 1. Let us set

$$\bar{\varphi}_{k_o}(z', m') := \inf_{k \geq k_o} \left\{ \mathbb{E} \left[v_{n_k} \left(Z_{s_+^{n_k}[z']}^{z', \circ}, m' \right) \right] \right\},$$

so that φ_{k_o} is the lower-semicontinuous envelope of $\bar{\varphi}_{k_o}$. Note that $Z_{s_+^{n_k}[z']}^{z', \circ}$ converges a.s. to z as $(z', k) \rightarrow (z, \infty)$. Hence, for all $\varepsilon > 0$, there exist open neighborhoods $B_\varepsilon(z, m)$ and $B_{\frac{\varepsilon}{2}}(z, m)$ of (z, m) , as well as $k_\varepsilon \in \mathbb{N}$ such that $\mathbb{P}[(Z_{s_+^{n_k}[z']}^{z', \circ}, m') \notin B_\varepsilon(z, m)] \leq \varepsilon$ for $k \geq k_\varepsilon$ and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$. One can also choose k_ε and $B_{\frac{\varepsilon}{2}}(z, m)$ such that $\inf_{k \geq k_\varepsilon} v_{n_k}(z', m') \geq v_o(z, m') - \varepsilon$ for all $k \geq k_\varepsilon$ and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$. Let $C > 0$ be a bound for $(|v_n|)_{n \geq 1}$ and $|v_o|$, recall that g is bounded. Then, for k_o large enough and $(z', m') \in B_{\frac{\varepsilon}{2}}(z, m)$,

$$\bar{\varphi}_{k_o}(z', m') \geq v_o(z, m) - \varepsilon - 2C \sup_{k \geq k_o} \mathbb{P}[(Z_{s_+^{n_k}[z']}^{z', \circ}, m') \notin B_\varepsilon(z, m)] \geq v_o(z, m) - \varepsilon(1 + 2C).$$

Hence, since v_o is lower-semicontinuous,

$$\lim_{k_o \rightarrow \infty} \varphi_{k_o}(z, m) = \lim_{k_o \rightarrow \infty} \liminf_{(z', m') \rightarrow (z, m)} \bar{\varphi}_{k_o}(z', m') \geq v_o(z, m).$$

\square

We conclude this section with the technical lemma that was used in the above proof.

Lemma 2.4.2. *Let $(u_n)_{n \geq 1}$ be a sequence of lower semi-continuous maps on $\mathbf{Z} \times \mathbf{M}$ and define $u_o := \liminf_{(z', m', n) \rightarrow (\cdot, \infty)} u_n(z', m')$ on $\mathbf{Z} \times \mathbf{M}$. Assume that u_o is locally bounded. Let φ be a continuous map and assume that (z_o, m_o) is a strict minimal point of $u_o - \varphi$ on $\mathbf{Z} \times \mathbf{M}$. Then, one can find a bounded open set B of $[0, T] \times \mathbb{R}^d$ and a sequence $(z_k, m_k, n_k)_{n \geq 1} \subset B \times \mathbf{M} \times \mathbb{N}$ such that $n_k \rightarrow \infty$, (z_k, m_k) is a minimum point of $u_{n_k} - \varphi$ on $B \times \mathbf{M}$ and $(z_k, m_k, u_{n_k}(z_k, m_k)) \rightarrow (z_o, m_o, u_o(z_o, m_o))$.*

Proof. Since \mathbf{M} is assumed to be locally compact, it suffices to repeat the arguments in the proof of [12, p80, Proof of Lemma 6.1]. \square

2.4.4 Conclusion of the proof of Theorem 2.3.1

We already know from Proposition 2.4.1 and Proposition 2.4.4 that v^* and v_o are respectively a bounded viscosity sub- and super-solution of (2.3.4)-(2.3.5). By (2.2.13), Remark 2.4.3 and (2.3.6), we also have $v_o \geq v^*$ on $(0, T) \times \mathbb{R}^d \times \mathbf{M}$. In view of Assumption 2.3.1 and Remark 2.4.2, v is continuous on $\mathbf{Z} \times \mathbf{M}$ and is the unique bounded viscosity solution of (2.3.4)-(2.3.5). \square

Remark 2.4.4. *The above arguments actually show that $(v_n)_{n \geq 1}$ converges to v .*

2.5 A sufficient condition for the comparison

In this section, we provide a sufficient condition for Assumption 2.3.1 to hold.

Proposition 2.5.1. *Assumption 2.3.1 holds whenever there exists a continuous function Ψ on $[0, 2T] \times \mathbb{R}^d \times \mathbf{M}$ satisfying*

- (i) $\Psi(\cdot, m) \in C^{1,2}([0, T] \times \mathbb{R}^d)$, for all $m \in \mathbf{M}$.
- (ii) $\varrho \Psi \geq \mathcal{L}\Psi$ on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ for some constant $\varrho > 0$,
- (iii) $\Psi - \mathcal{K}\Psi \geq \delta$ on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$ for some $\delta > 0$,
- (iv) $\Psi \geq \mathcal{K}_T[\tilde{g}]$ on $[T, \infty) \times \mathbb{R}^d \times \mathbf{M}$ with $\tilde{g}(t, \cdot) := e^{\varrho t} g(t, \cdot)$ and ϱ is defined in (ii),
- (v) Ψ^- is bounded.

The idea of the proof is the same as in [20, Proposition 4.12]. Note that their condition H2 (v) is not required here because we only consider bounded sub and super-solutions and we take a different approach. To avoid it, we slightly reinforce the hypothesis H2 (iii) and asked for Ψ^- to be bounded.

Proof. Step 1. As usual, we shall argue by contradiction. We assume that there exists $(z_0, m_0) \in \mathbf{Z} \times \mathbf{M}$ such that $(U - V)(z_0, m_0) > 0$, in which U and V are as in Assumption 2.3.1. Recall the definition of Ψ , ϱ and \tilde{g} in Proposition 2.5.1. We set $\tilde{u}(t, x, m) := e^{\varrho t} U(t, x, m)$ and $\tilde{v}(t, x, m) := e^{\varrho t} V(t, x, m)$ for all $(t, x, m) \in \mathbf{Z} \times \mathbf{M}$. Then, there exists $\lambda \in (0, 1)$ such that

$$(\tilde{u} - \tilde{v}^\lambda)(z_0, m_0) > 0, \tag{2.5.1}$$

in which $\tilde{v}^\lambda := (1 - \lambda)\tilde{v} + \lambda\Psi$. Note that \tilde{u} and \tilde{v} are sub and supersolution on $\mathbf{Z} \times \mathbf{M}$ of

$$\min \{\varrho\varphi - \mathcal{L}\varphi, \varphi - \mathcal{K}\varphi\} = 0 \quad (2.5.2)$$

associated to the boundary condition

$$\min \{\varphi - \mathcal{K}_T\tilde{g}, \varphi - \mathcal{K}\varphi\} = 0. \quad (2.5.3)$$

Step 2. Let $d_{\mathbf{M}}$ be a metric on \mathbf{M} compatible with the topology of weak convergence. For $(t, x, y, m) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$, we set

$$\Gamma_\varepsilon(t, x, y, m) := \tilde{u}(t, x, m) - \tilde{v}^\lambda(t, y, m) - \varepsilon \left(\|x\|^2 + \|y\|^2 + d_{\mathbf{M}}(m) \right) \quad (2.5.4)$$

with $\varepsilon > 0$ small enough such that $\Gamma_\varepsilon(t_0, x_0, x_0, m_0) > 0$. Note that the supremum of $(t, x, m) \mapsto \Gamma_\varepsilon(t, x, x, m)$ over $\mathbf{Z} \times \mathbf{X} \times \mathbf{M}$ is achieved by some $(t_\varepsilon, x_\varepsilon, x_\varepsilon, m_\varepsilon)$. This follows from the the upper semi-continuity of Γ_ε and the fact that $\tilde{u}, -\tilde{v}, -\Psi$ are bounded from above. Recall that \mathbf{M} is locally compact. For $(t, x, y, m) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$, we set

$$\Theta_\varepsilon^n(t, x, y, m) := \Gamma_\varepsilon(t, x, y, m) - n\|x - y\|^2.$$

Again, there is $(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}$ such that $\sup_{\mathbf{Z} \times \mathbf{X} \times \mathbf{M}} \Theta_\varepsilon^n = \Theta_\varepsilon^n(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon)$. It is standard to show that, after possibly considering a subsequence,

$$\begin{aligned} (t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) &\rightarrow (\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) \in \mathbf{Z} \times \mathbf{X} \times \mathbf{M}, \quad n\|x_n^\varepsilon - y_n^\varepsilon\|^2 \rightarrow 0, \\ \text{and } \Theta_\varepsilon^n(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) &\rightarrow \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) = \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, m_\varepsilon), \end{aligned} \quad (2.5.5)$$

see e.g. [31, Lemma 3.1].

Step 3. We first assume that, up to a subsequence, $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) \leq 0$, for $n \geq 1$. It follows from the supersolution property of \tilde{v} and Condition (iii) of Proposition 2.5.1 that

$$\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) - \lambda\delta.$$

Passing to the limsup and using (2.5.5) and (2.3.6), we obtain $(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) + \lambda\delta \leq \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$. In particular, by (2.5.4), $\Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) + \lambda\delta \leq \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$.

Now let us observe that

$$\sup_{\mathbf{Z} \times \mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) = \lim_{\varepsilon \rightarrow 0} \sup_{(t, x, m) \in \mathbf{Z} \times \mathbf{M}} \Gamma_\varepsilon(t, x, x, m) = \lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, m_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon), \quad (2.5.6)$$

in which the last identity follows from (2.5.5). Combined with the above inequality, this shows that $\sup_{\mathbf{Z} \times \mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) + \lambda\delta \leq \lim_{\varepsilon \rightarrow 0} \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$, which leads to a contradiction for ε small enough.

Step 4. We now show that there is a subsequence such that $t_n^\varepsilon < T$ for all $n \geq 1$. If not, one can assume that $t_n^\varepsilon = T$ and it follows from the boundary condition (2.5.3) and step 3 that

$\tilde{u}(T, x_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}_T \tilde{g}(T, x_n^\varepsilon, m_n^\varepsilon)$ for all $n \geq 1$. Since, by (2.5.3) and Condition (iv) of Proposition 2.5.1, $\tilde{v}^\lambda(T, y_n^\varepsilon, m_n^\varepsilon) \geq \mathcal{K}_T \tilde{g}(T, y_n^\varepsilon, m_n^\varepsilon)$, it follows that $\tilde{u}(T, x_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(T, y_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}_T \tilde{g}(T, x_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}_T \tilde{g}(T, y_n^\varepsilon, m_n^\varepsilon)$. Hence, $\Gamma_\varepsilon(T, x_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}_T \tilde{g}(T, x_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}_T \tilde{g}(T, y_n^\varepsilon, m_n^\varepsilon)$. Combining (2.3.6), (2.5.5) and (2.5.6) as above, we obtain $\sup(\tilde{u} - \tilde{v}^\lambda) \leq 0$, a contradiction.

Step 5. In view of step 3 and 4, we may assume that $t_n^\varepsilon < T$ and $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) > 0$ for all $n \geq 1$. Using Ishii's Lemma and following standard arguments, see Theorem 8.3 and the discussion after Theorem 3.2 in [31], we deduce from the sub- and supersolution viscosity property of \tilde{u} and \tilde{v}^λ , and the Lipschitz continuity assumptions on μ and σ , that

$$\varrho \left(\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, m_n^\varepsilon) \right) \leq C \left(n \|x_n^\varepsilon - y_n^\varepsilon\|^2 + \varepsilon \left(1 + \|x_n^\varepsilon\|^2 + \|y_n^\varepsilon\|^2 \right) \right),$$

for some $C > 0$, independent on n and ε . In view of (2.5.4) and (2.5.5), we get

$$\varrho \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon) \leq 2C\varepsilon \left(1 + \|\hat{x}_\varepsilon\|^2 \right). \quad (2.5.7)$$

We shall prove in next step that the right-hand side of (2.5.7) goes to 0 as $\varepsilon \rightarrow 0$, up to a subsequence. Combined with (2.5.6), this leads to a contradiction to (2.5.1).

Step 6. We conclude the proof by proving the claim used above. First note that we can always construct a sequence $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon)_{\varepsilon>0}$ such that

$$\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon) \rightarrow \sup_{\mathbf{Z} \times \mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) \quad \text{and} \quad \varepsilon (\|\tilde{x}_\varepsilon\|^2 + d_{\mathbf{M}}(\tilde{m}_\varepsilon)) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By (2.5.5), $\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{x}_\varepsilon, \tilde{m}_\varepsilon) \leq \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{m}_\varepsilon)$. Hence, $\sup_{\mathbf{Z} \times \mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) \leq \sup_{\mathbf{Z} \times \mathbf{M}} (\tilde{u} - \tilde{v}^\lambda) - 2 \liminf_{\varepsilon \rightarrow 0} \varepsilon \|\hat{x}_\varepsilon\|^2$. \square

2.6 Numerical approximation

2.6.1 An example of numerical scheme

When the comparison result of Assumption 2.3.1 holds, one can easily derive a convergent finite different scheme for (2.3.4)-(2.3.5).

We consider here a simple explicit scheme based on [23, 24]. We let h_0 be a time-discretization step so that T/h_0 is an integer, and set $\mathbf{T}^{h_0} := \{t_j^{h_0} := jh_0, j \leq T/h_0\}$. The space \mathbb{R}^d is discretized with a space step h_1 on a rectangle $[-c, c]^d$, containing $N_{h_1}^x$ points on each direction. The corresponding finite set is denoted by $\mathbf{X}_c^{h_1}$.

The first order derivatives $\partial_i \varphi$ and $(\partial \varphi / \partial x^i)_{i \leq d}$ are approximated by using the standard up-wind approximations :

$$\begin{aligned} \Delta_t^{h_0} \varphi(t, x, m) &:= h_0^{-1} (\varphi(t + h_0, x, m) - \varphi(t, x, m)) \\ \Delta_{h_1, i}^{h_0} \varphi(t, x, m) &:= \begin{cases} h_1^{-1} (\varphi(t + h_0, x + e_i h_1, m) - \varphi(t, x, m)) & \text{if } \mu^i(x) \geq 0 \\ h_1^{-1} (\varphi(t, x, m) - \varphi(t + h_0, x - e_i h_1, m)) & \text{if } \mu^i(x) < 0, \end{cases} \end{aligned}$$

in which e_i is i -th unit vector of \mathbb{R}^d .

As for the second order term, we use the fact that each point $x \in \mathbb{R}^d$ can be approximated as a weighted combination

$$x = \sum_{x' \in C_{h_1}(x)} x' \omega(x'|x)$$

of the points x' lying on the corners $C_{h_1}(x)$ of the cube formed by the partition of \mathbb{R}^d it belongs too. Then, given another small parameter $h_2 > 0$, we approximate $\text{Tr}[\sigma(x)\sigma(x)^\top D^2\varphi(t, x, m)]$ by $\mathbf{T}_{h_0, h_1}^{h_2}[\varphi](t, x, m)$ defined as

$$(h_2 d)^{-1} \sum_{i=1}^d [\varphi]_{h_1}(t + h_0, x + \sqrt{h_2} \sigma^i(x), m) + [\varphi]_h(t + h_0, x - \sqrt{h_2} \sigma^i(x), m) - 2h_2^{-1} \varphi(t, x, m)$$

in which σ^i is the i -th column of σ and

$$[\varphi]_h(t, x, m) := \sum_{x' \in C_{h_1}(x)} \omega(x'|x) \varphi([t]_h, x', m) \text{ with } [t]_h := \min[t, 2T] \cap (\mathbf{T}^{h_0} \cup [T, 2T]),$$

is a piecewise linear approximation of φ . In the case where only the first row σ^1 of σ is not identically equal to 0, one can use the usual simpler approximation

$$(h_1)^{-1} \|\sigma^1\|^2 \left(\varphi(t + h_0, x + \sqrt{h_1} e_1, m) + \varphi(t + h_0, x - \sqrt{h_1} e_1, m) \right) - 2(h_1)^{-1} \|\sigma^1\|^2 \varphi(t, x, m).$$

Similarly, we approximate $\mathcal{K}\varphi$ by

$$\mathcal{K}_{h_0, h_1} \varphi(t, x, m) := \sup_{a \in \mathbf{A}} \int [\varphi]_h(\max(t + h_0, t'), x', m') dk(t', x', m'|t, x, m, a).$$

Letting $h := (h_0, h_1, h_2)$, and setting

$$\mathcal{L}^h \varphi := \Delta_t^{h_0} \varphi + \sum_{i \leq d} \mu^i \Delta_{h_1, i}^{h_0} \varphi + \frac{1}{2} \mathbf{T}_{h_0, h_1}^{h_2}[\varphi], \quad (2.6.1)$$

our numerical scheme consists in solving

$$\min \left\{ -\mathcal{L}^h \varphi, \varphi - \mathcal{K}_{h_1} \varphi \right\} = 0 \text{ on } (\mathbf{T}^{h_0} \setminus \{T\}) \times (\mathbf{X}_c^{h_1} \setminus \partial \mathbf{X}_c^{h_1}) \times \mathbf{M}, \quad (2.6.2)$$

$$\min \{ \varphi - \mathcal{K}_T g, \varphi - \mathcal{K}_{h_1} \varphi \} = 0 \text{ on } \{T\} \times (\mathbf{X}_c^{h_1} \setminus \partial \mathbf{X}_c^{h_1}) \times \mathbf{M}, \quad (2.6.3)$$

$$\varphi - \mathcal{K}_T g := 0 \text{ on } ([0, T] \times \partial \mathbf{X}_c^{h_1} \times \mathbf{M}) \cup ((T, 2T] \times \mathbb{R}^d \times \mathbf{M}). \quad (2.6.4)$$

We specify here a precise boundary condition on $\partial \mathbf{X}_c^{h_1}$ but any other (bounded) boundary condition could be used. Finally, we extend v_h^c to the whole space by setting $v_h^c = [v_h^c]_h$ on $[0, T] \times \mathbb{R}^d \times \mathbf{M}$.

This scheme is always convergent as $(h_2, h_1/h_2, h_0/h_1) \rightarrow 0$ and $c \rightarrow \infty$.

Proposition 2.6.1. *Let v_h^c denote the solution of (2.6.2)-(2.6.3)-(2.6.4). If Assumptions 2.3.1 holds, then $v_h^c \rightarrow v$ as $(h_2, h_1/h_2, h_0/h_1) \rightarrow 0$ and then $c \rightarrow \infty$.*

Proof. Using Lemma 2.6.1, one easily checks that our scheme satisfies the conditions of [14, Theorem 2.1.]. In particular, $|v_h^c| \leq \sup |g| < \infty$. Then, the convergence holds by the same arguments as in [14, Theorem 2.1.], it suffices to replace their assertion (2.7) by Lemma 2.4.2 stated below. \square

Remark 2.6.1. *We did not discuss in the above the problem of the discrete approximation of \mathbf{M} . Applications will typically be based on a parameterized family $\mathbf{M} = \{m_\theta, \theta \in \Theta\}$, for a subset Θ of a finite dimensional space. We can then further approximate Θ by a sequence of finite sets to build up a numerical scheme. Similarly, the set of control values \mathbf{A} need to be approximated in practice. If the corresponding sequences of approximations are dense, then convergence of the numerical scheme will still hold.*

We conclude this section with the technical lemmas that were used in the above proof.

Lemma 2.6.1. *If $(u_n)_{n \geq 1}$ is a bounded sequence of functions on $\mathbf{Z} \times \mathbf{M}$ and $(z_n, m_n)_{n \geq 1}$ is a sequence in $\mathbf{Z} \times \mathbf{M}$ that converges to (z_\circ, m_\circ) , then*

$$\liminf_{\substack{n \rightarrow \infty \\ (h_0, h_1) \rightarrow (0, 0)}} \mathcal{K}_{h_0, h_1} u_n(z_n, m_n) \geq \mathcal{K}u_\circ(z_\circ, m_\circ) \quad , \text{ where } u_\circ := \liminf_{\substack{n \rightarrow \infty \\ (z', m') \rightarrow \cdot}} u_n(z', m'),$$

and

$$\limsup_{\substack{n \rightarrow \infty \\ (h_0, h_1) \rightarrow (0, 0)}} \mathcal{K}_{h_0, h_1} u_n(z_n, m_n) \leq \mathcal{K}u^\circ(z_\circ, m_\circ) \quad , \text{ where } u^\circ := \limsup_{\substack{n \rightarrow \infty \\ (z', m') \rightarrow \cdot}} u_n(z', m').$$

Proof. We first rewrite

$$\mathcal{K}_{h_0, h_1} u_n(z_n, m_n) = \sup_{a \in \mathbf{A}} \int u_{n, h}(z', m') dk(z', m' | z_n, m_n, a) \quad (2.6.5)$$

where $u_{n, h}(z', m') := [u_n]_{h_1}(\max(t_n + h_0, t'), x', m')$. Let $\bar{u}_{n_\circ, h_\circ}$ be the lower-semicontinuous envelope of $\inf_{n \geq n_\circ, h \leq h_\circ} u_{n, h}$. From (2.6.5), we get, for $n \geq n_\circ$ and $h \leq h_\circ$,

$$\mathcal{K}u_{n, h}(z_n, m_n) \geq \mathcal{K}\bar{u}_{n_\circ, h_\circ}(z_n, m_n),$$

and, by (2.3.6), passing to the limit inf as $(n, h) \rightarrow (+\infty, 0)$ leads to

$$\liminf_{(n, h) \rightarrow (+\infty, 0)} \mathcal{K}u_{n, h}(z_n, m_n) \geq \mathcal{K}\bar{u}_{n_\circ, h_\circ}(z_\circ, m_\circ).$$

Moreover, $\bar{u}_{n_\circ, h_\circ} \uparrow u_\circ$ point-wise. The required result is then obtained by monotone convergence.

2.6.2 Construction of ε -optimal controls

It remains to explain how to deduce the optimal policy. At each of point (t, x) of the time-space grid and for each prior m , one computes

$$(\hat{\ell}(t, x, m), \hat{b}(t, x, m)) \in \arg \max \left\{ \int v_h^c(z', m') dk(z', m' | (t, x), m, (\ell, b)), (\ell, b) \in \mathbf{A} \right\}.$$

If $v_h^c(t, x, m)$ is equal to the above maximum, then we play the control $(\hat{\ell}(t, x, m), \hat{b}(t, x, m))$, otherwise we wait for the next time step. This is the usual philosophy : we act on the system only if this increases the expected gain. As already argued, here the gain should not only be considered as an improvement of the current future reward, it can also be a gain in the precision of our prior which will then lead to better future rewards.

This produces a Markovian control which is optimal for the discrete time problem associated to our numerical scheme, and asymptotically optimal for the original control problem. We shall use this algorithm for the toy examples presented in the next section.

2.7 Applications to optimal trading

This section is devoted to the study of two examples of application. Each of them corresponds to an idealized model, the aim here is not to come up with a good model but rather to show the flexibility of our approach, and to illustrate numerically the behavior of our backward algorithm.

2.7.1 Immediate impact of aggressive orders

We consider first a model in which the impact of each single order sent to the market is taken into account. It means that α_i represents the number of shares bought exactly at time τ_i , so that $\ell_i = 0$, for each i . This corresponds to $\mathbf{A} = \{0\} \times \mathbf{B}$ in which $\mathbf{B} \subset \mathbb{R}_+$ is a compact set of values of admissible orders. Therefore, one can identify \mathbf{A} to \mathbf{B} in the following, and we will only write b for $a = (0, b) \in \mathbf{A}$ and β_i for $\alpha_i = (\ell_i, \beta_i)$.

Our model can be viewed as a scheduling model or as a model for illiquid market. The first component of X represents the stock price. We consider a simple linear impact : when a trade of size β_i occurs at τ_i , the stock price jumps by

$$X_{\tau_i}^1 = X_{\tau_i-}^1 + \beta_i(v + \epsilon_i)/2$$

in which $v \in \mathbb{R}$ is the unknown linear impact parameter, $(\epsilon_i)_{i \geq 1}$ is a sequence of independent noises following a centered Gaussian distribution with standard deviation σ_ϵ . The coefficient $1/2$ in the dynamics of X^1 stands for a 50% proportion of immediate resilience.

It evolves according to a Brownian diffusion between two trades and has a residual resilience effect :

$$dX_t^1 = \sigma dW_t^1 + dX_t^4 \text{ and } dX_t^4 = -\rho X_t^4 dt, \quad (2.7.1)$$

where $\sigma, \rho > 0$ and $X_0^1 \in \mathbb{R}$ are constants. The process X^4 represents the drift of X^1 due to the non immediate resilience and $X_0^4 = 0$. When a trade occurs, it jumps according to

$$X_{\vartheta_i}^4 = X_{\tau_i-}^4 + \beta_i(v + \epsilon_i)/2.$$

We call it spread hereafter. This is part of the deviation from the un-impacted dynamic. The third component, which describes the total cost, evolves as

$$X_{\vartheta_i}^2 = X_{\tau_i-}^2 + X_{\tau_i-}^1 \beta_i + (v + \epsilon_i) \frac{\beta_i^2}{2}.$$

Finally, the last component is used to keep track of the cumulative number of shares bought :

$$X_{\vartheta_i}^3 = X_{\tau_i-}^3 + \beta_i.$$

We are interest in the cost of buying N shares, and minimize the criteria

$$\mathbb{E}_m[e^{\eta L(X_T, v)} \wedge C]$$

where $\eta > 0$ is a risk aversion parameter, $C > 0$, and

$$L(X_T, v) := X_T^2 + X_T^1(N - X_T^3) + (v + \epsilon_0) \frac{(N - X_T^3)^2}{2}$$

represents the total cost after setting the total number of shares bought to N at T . If the prior law m on v is a Gaussian distribution, then $q(\cdot|t, x, b, u)$ is a Gaussian density with respect to

$$dQ(x'|t, x, b) = dx^{1'} d\delta_{x^2+bx^{1'}}(x^{2'}) d\delta_{x^3+b}(x^{3'}) d\delta_{x^4+(x^{1'}-x^1)}(x^{4'})$$

and the transition map

$$\mathfrak{M}(m; t', x', t, x, b)[C] = \frac{\int_C q(x'|t, x, b, u) dm(u)}{\int_{\mathbb{R}} q(x'|t, x, b, u) dm(u)},$$

maps Gaussian distributions into Gaussian distributions, which, in practice, enables us to restrict \mathbf{M} to the set of Gaussian distributions. More precisely, if $(m_v(\tau_i-), \sigma_v(\tau_i-))$ are the mean and the standard deviation of M_{τ_i-} , then the values corresponding to the posterior distribution M_{ϑ_i} are

$$\begin{aligned} \sigma_v(\vartheta_i) &= \mathbf{1}_{\{\sigma_v(\tau_i-) \neq 0\}} \left(\frac{1}{\sigma_v(\tau_i-)^2} + \frac{1}{\sigma_\epsilon^2} \right)^{-\frac{1}{2}}, \\ m_v(\vartheta_i) &= m_v(\tau_i-) \mathbf{1}_{\{\sigma_v(\tau_i-) = 0\}} + \left(\frac{\sigma_\epsilon^2 m_v(\tau_i-) + \sigma_v(\tau_i-)^2 (X_{\vartheta_i}^1 - X_{\tau_i-}^1)}{\sigma_\epsilon^2 + \sigma_v(\tau_i-)^2} \right) \mathbf{1}_{\{\sigma_v(\tau_i-) \neq 0\}}. \end{aligned}$$

Comparing to the general result of the previous section, we add a boundary condition $v(t, x^1, x^2, N, x^4) = 1$ and restrict the domain of X^3 to be $\{0, \dots, N\}$. Since this parameter x^3 is discrete this does not change the nature of our general results.

Note also that the map $\Psi(t, x, m) = N - x^3$ defined on $[0, T] \times \mathbb{R}^2 \times \{0, \dots, N\} \times \mathbb{R} \times \mathbf{M}$ actually satisfies the conditions of Proposition 2.5.1 to ensure that Assumption 2.3.1 holds.

We now discuss a numerical illustration. We consider 30 seconds of trading and $N = 25$ shares to buy. We take $\eta = 1$, $x_0 = 100$ and $\sigma = 0.4x_0$ which corresponds to a volatility of 40% in annual terms. The trading period is divided into intervals of 1 second-length. The size of an order β_i ranges in $\{1, 2, 3, 4, 5\}$. We take $\sigma_\varepsilon = 10^{-4}$ and ρ such that the spread X^4 is divided by 3 every second if no new order is sent. We start with a prior given by a Gaussian distribution with mean $m_v(0)$ and standard deviation $\sigma_v(0)$. Finally, we take $C = 10^{200}$ which makes this threshold parameter essentially inefficient while still ensuring that the terminal condition is bounded.

In Figure 2.1, we plot the optimal strategy for $\sigma_v(0) = 5.10^{-4}$ and $m_v(0) = 5.10^{-2}$ in terms of (X^2, X^3) . Clearly, the level of spread X^4 has a significant impact : when it is large, it is better to wait for it to decrease before sending a new order. This can also be observed in Figure 2.2 which provides a simulated path corresponding to an initial prior $(m_v(0) = 2.10^{-2}, \sigma_v(0) = 10^{-3})$: after 15 seconds the algorithm alternates between sending an order and doing nothing, i.e. waiting for the spread to be reduced at the next time step. On the top right graph, we can also observe that the low mean of the initial prior combined with a zero initial resilience leads to sending an order of size 3 at first, then the mean of the prior is quickly adjusted to a higher level and the algorithm slows down immediately.

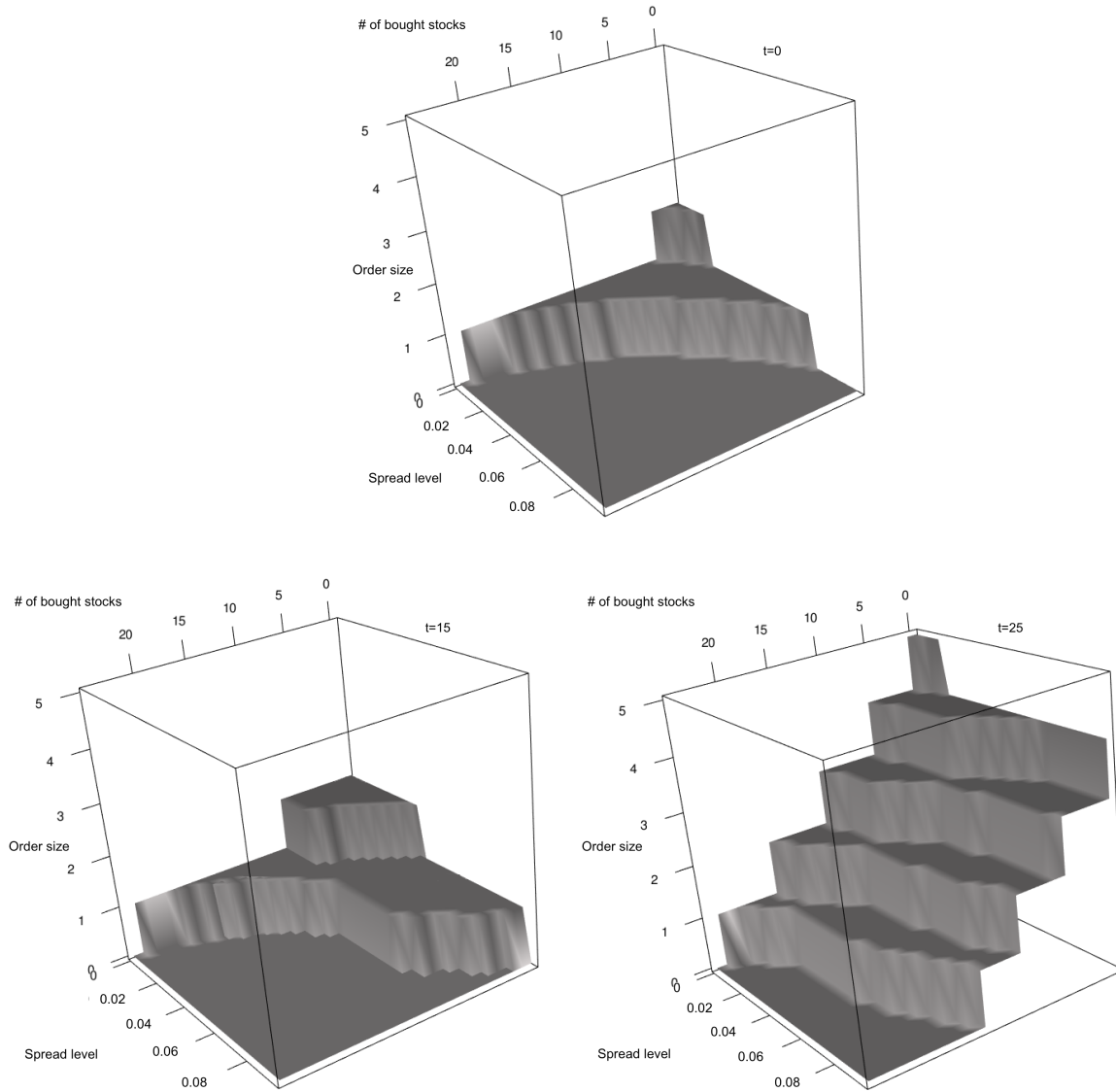


FIGURE 2.1 – Evolution of β in terms of (X^3, X^4) at time 0s (top), 15s (left) and 25s (right), for $(m_v, \sigma_v) = (5 \cdot 10^{-2}, 5 \cdot 10^{-4})$.

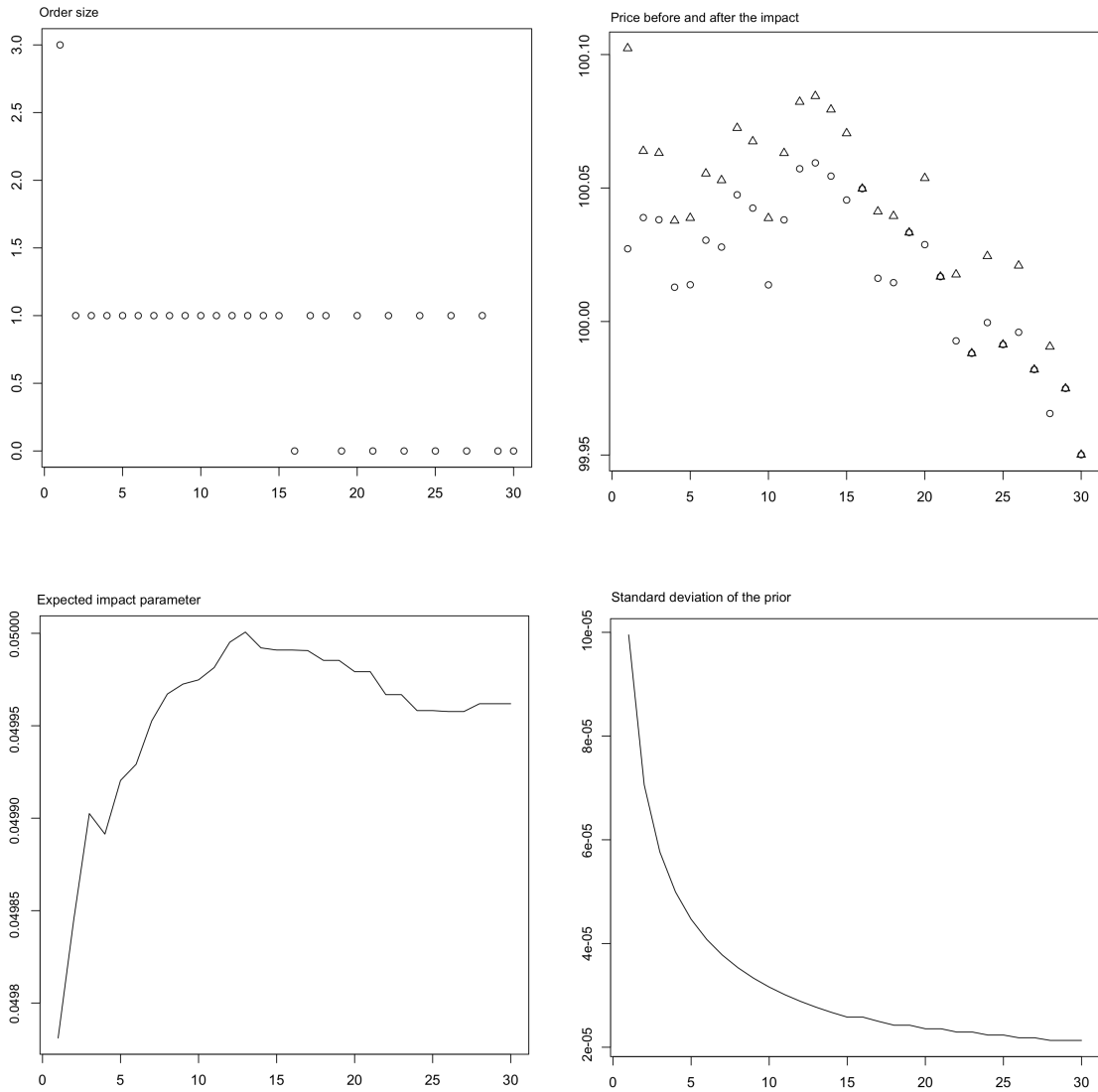


FIGURE 2.2 – Evolution of β (top left), price before (circles) and after (triangles) the impact (top right), m_v (bottom left), σ_v (bottom right), with time in second. The true value of v is 5.10^{-2} . x -axis : time in seconds.

Let us now consider the case $\rho = 0$, i.e. without dynamic resilience, with a trading period of 60 seconds and $N = 50$. In Figure 2.3, we provide the optimal policy (number of traded shares) in terms of the number X^3 of already traded shares and the prior's mean parameter m_v for different times. Not surprisingly the algorithm is more aggressive as the prior's mean decreases and the remaining number of shares to buy increases. It is rather stable in time (compare $t = 0$ s with $t = 30$ s) up to the end where it is forced to accelerate to avoid a large final impact cost. It is also much more aggressive compared to the case $\rho > 0$ presented above : we can no more make profit of the decrease of the resilience term X^4 , and there is no reason to wait.

In Figure 2.4, we provide a simulated path of $(X, \alpha, m_v, \sigma_v)$ that shows how the prior on the unknown coefficients v can adapt to changing market conditions. The red dashed lines and circles correspond to the same path of Brownian motion and the same realized noises $(\epsilon_i)_{i \geq 1}$ as the black solid lines and crosses, but the true parameter is changed from $5 \cdot 10^{-2}$ to $5 \cdot 10^{-4}$ after 5 seconds. It is more aggressive quite quickly after the shock as the prior adapts to the new small level of impact. Note that the total number of shares is bought slightly before 30s, so that the prior do not change anymore after this date.

In Figure 2.5, we plot the log of the value function minus the cost $5 \cdot 10^3$ of buying the total shares without impact (similar to the *implementation shortfall*), in terms of the different quantities of interest.

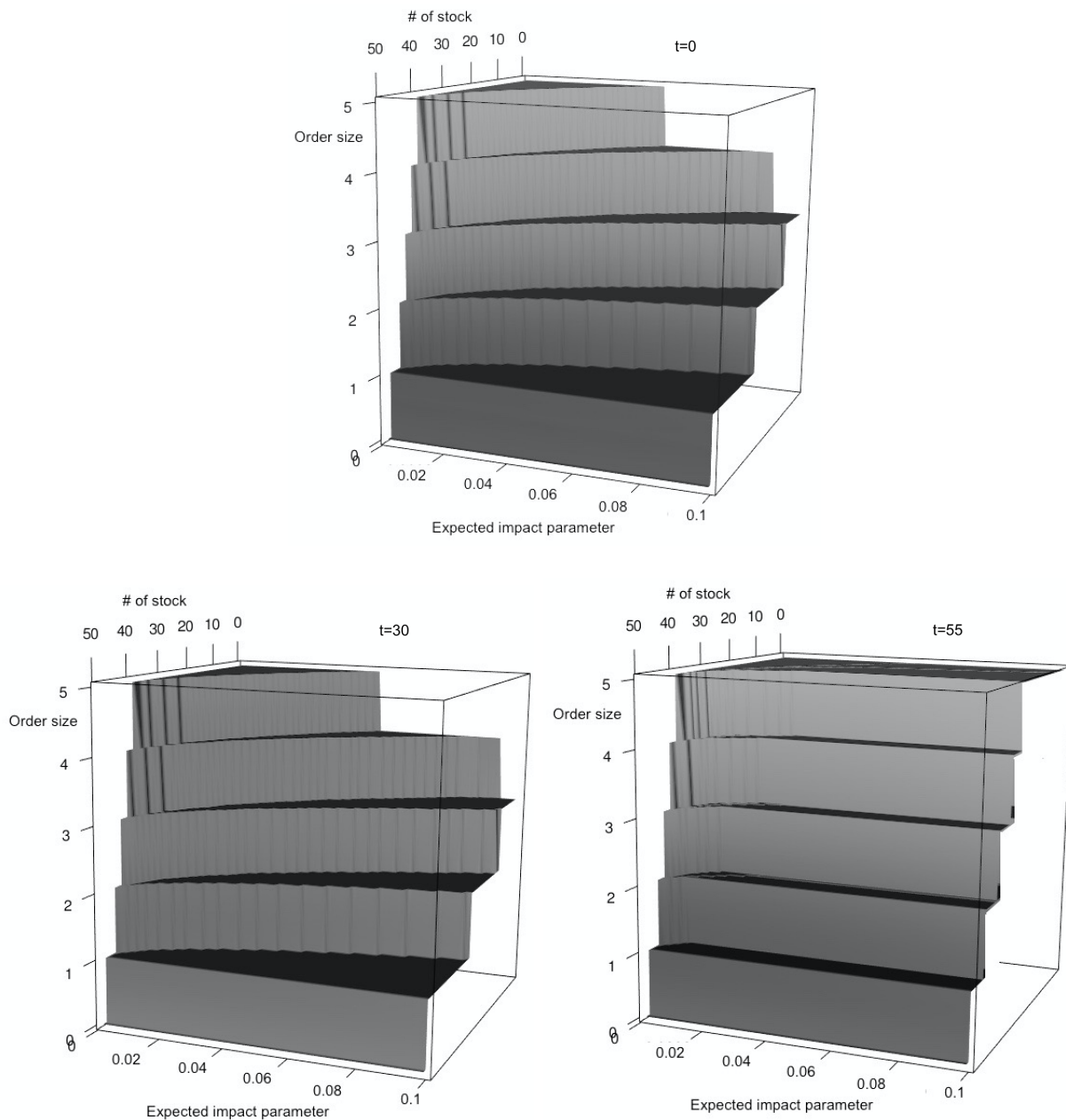


FIGURE 2.3 – Evolution of α in terms of (m_v, X^3) at time 0s (top), 30s (left) and 55s (right), for $\sigma_v = 5.10^{-4}$.

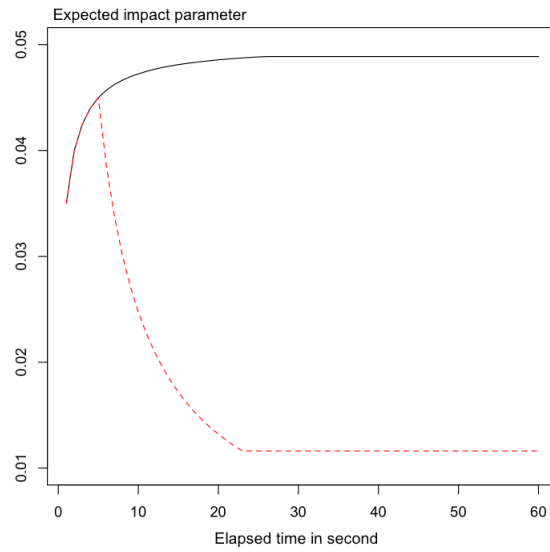
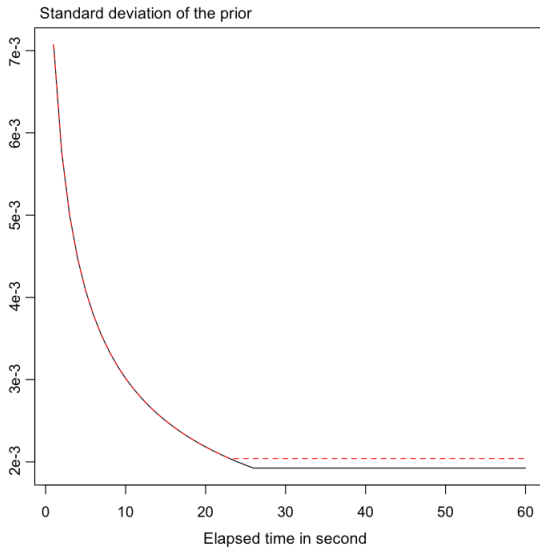
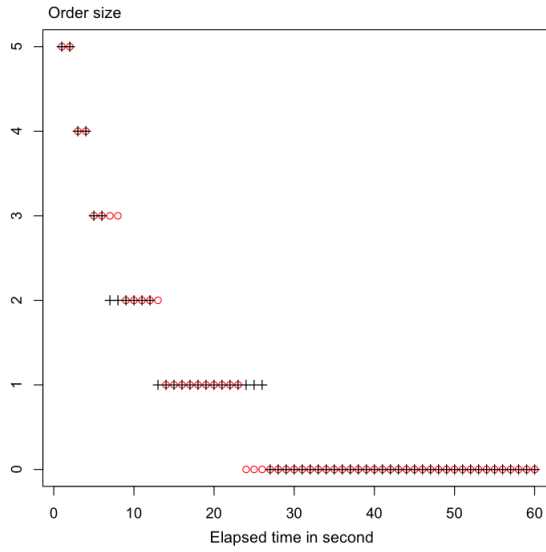


FIGURE 2.4 – Evolution of α (top), σ_v (left), m_v (right) with time. Black crosses and black solid lines : the true value of v is $5 \cdot 10^{-2}$. Red circles and red dashed lines : the true value of v is $5 \cdot 10^{-2}$ for the first 5 seconds, and then jumps to $5 \cdot 10^{-4}$.

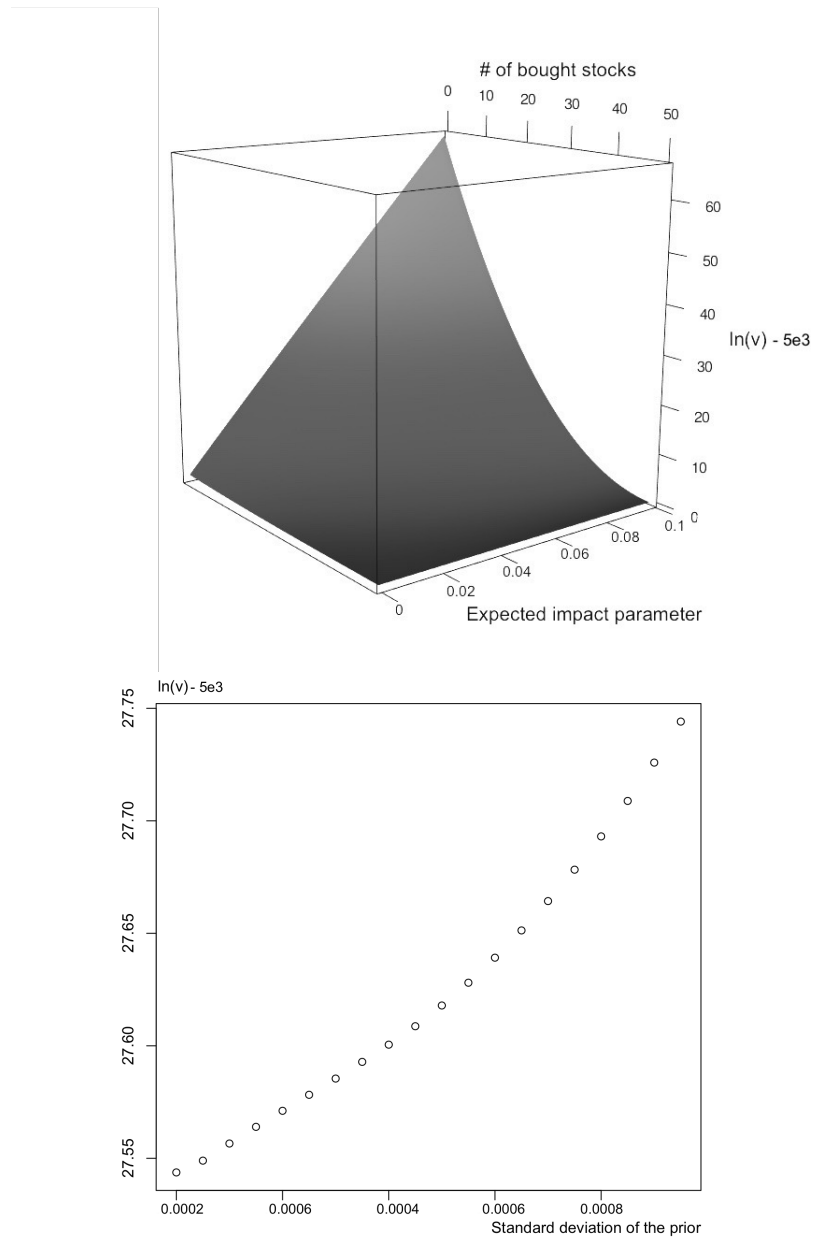


FIGURE 2.5 – Top : $\ln(v) - 5.10^3$ in terms of (x^3, m_v) for $\sigma_v = 5.10^{-4}$ and $t = 0$. Bottom : $\ln(v) - 5.10^3$ in terms of σ_v for $(x^3, m_v) = (0, 2.10^{-2})$ at $t = 0$.

2.7.2 Random execution times : application to strategies using limit-orders

In this section, we consider a limit-order trading model. X^1 now represents a mid-price (of reference) and, between two trades, has the dynamic

$$dX_t^1 = \sigma dW_t^1. \quad (2.7.2)$$

An order is of the form (ℓ, β) in which ℓ is the maximal time we are ready to wait before being executed, while β is the price at which the limit order is sent³. For simplicity, each order corresponds to buying one share.

We assume that the time θ it takes to be executed follows an exponential distribution of parameter $\rho(v, X_\tau^1 - \beta)$, given the information at time τ . One can send a new order only after $\vartheta := \tau + \ell \wedge \theta$.

Hence, given a flow of orders $\phi = (\tau_i, \ell_i, \beta_i)_{i \geq 1}$, the number X^3 of shares bought evolves according to

$$\begin{aligned} X^3 &= X_{\vartheta_i}^3 \quad \text{on } [\vartheta_i, \tau_{i+1}) \\ X_{\vartheta_i}^3 &= X_{\tau_i-}^3 + \mathbf{1}_{\{\theta_i \leq \ell_i\}}, \end{aligned}$$

in which $\vartheta_i := \tau_i + \ell_i \wedge \theta_i$. Each θ_i follows an exponential distribution of parameter $\rho(v, X_{\tau_i}^1 - \beta_i)$ given $\mathcal{F}_{\tau_i-}^{z, m, \phi}$. As in the previous model, X^3 is restricted to $\{0, \dots, N\}$. The total cost X^2 of buying the shares has the dynamics

$$\begin{aligned} X^2 &= X_{\vartheta_i}^2 \quad \text{on } [\vartheta_i, \tau_{i+1}) \\ X_{\vartheta_i}^2 &= X_{\tau_i-}^2 + \beta_i \mathbf{1}_{\{\theta_i \leq \ell_i\}}. \end{aligned}$$

We want to minimize

$$\mathbb{E} \left[e^{X_{T[\phi]}^2 + 1.02(N - X_{T[\phi]}^3) + \frac{5.10^2}{2}(N - X_{T[\phi]}^3)^2} \wedge C \right],$$

in which 1.02 is the best ask (kept constant) and 5.10^2 is an impact coefficient. This corresponds to the cost of liquidating instantaneously the remaining shares $(N - x^3)^+$ at T . This model is a version of [6], [42], [45], see also [43].

Direct computations show that the prior process M evolves according to

$$\begin{aligned} M &= M_{\vartheta_i} \quad \text{on } [\vartheta_i, \tau_{i+1}) \\ M_{\vartheta_i} &= \mathfrak{M}_1(M_{\tau_i-}; Z_{\vartheta_i}, Z_{\tau_i-}, \alpha_i) \mathbf{1}_{\{\theta_i \leq \ell_i\}} + \mathfrak{M}_2(M_{\tau_i-}; Z_{\vartheta_i}, Z_{\tau_i-}, \alpha_i) \mathbf{1}_{\{\theta_i > \ell_i\}} \end{aligned}$$

in which

$$\mathfrak{M}_1(m; t', x', t, x, l, b)[B] := \frac{\int_B \rho(u, x^1 - b) e^{-\rho(u, x^1 - b)t'} dm(u)}{\int_{\mathbb{R}^+} \rho(u, x^1 - b) e^{-\rho(u, x^1 - b)t'} dm(u)}$$

3. Dark pool strategies could be considered similarly, in this case, β would rather describe the choice of the trading platform

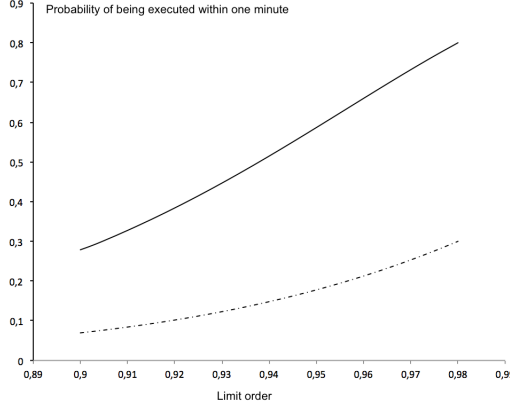


FIGURE 2.6 – Solid : $u = 0.8$. Dashed : $u = 0.3$

and

$$\mathfrak{M}_2(m; t', x', t, x, l, b)[B] := \frac{\int_B e^{-\rho(u, x^1 - b)l} dm(u)}{\int_{\mathbb{R}^+} e^{-\rho(u, x^1 - b)l} dm(u)}$$

for all Borel set B .

In the case where \mathbf{M} is the convex hull of a finite number of Dirac masses, then the weights associated to M can be computed explicitly.

Here again, the map $\Psi(t, x, m) = N - x^3$ satisfies the conditions of Proposition 2.5.1 to ensure that Assumption 2.3.1 holds.

We now consider a numerical illustration. We take $C = 10^{200}$. The time horizon is $T = 15$ minutes. To simplify, we fix the reference mid-price to be $X^1 \equiv 1$ (i.e. $\sigma = 0$) and restrict to $\ell = 1$, i.e. an order is sent each minute. We take $N = 10$. One can send limit buy orders in the range $B := \{0.90, 0.92, 0.94, 0.96, 0.98\}$.

As for the intensity of the execution time, we use an exponential form as in [42] : $\rho(u, x^1 - b) = \lambda(u)e^{-20(0.98 - b)}$ in which $\lambda(u) = -\ln(1 - u)$. This means that the probability to be executed at the price 0.98 within one minute is u . Orders are sent each minute, but we use a finer time grid in order to take into account that it can be executed before this maximal time-length. The original prior is supported by two Dirac masses at $u = 0.3$ and $u = 0.8$. The corresponding probabilities of being executed within one minute are plotted in Figure 2.6.

Our time step corresponds to 15 seconds, so that every 15 seconds the controller can launch a new order if the previous one has been executed before the maximal 1 minute time-length. In Figure 2.7, we plot the difference, in logarithms, between the value functions obtained in the latter case and for a time step of 1 minute (in which case a new order cannot be launched before one minute). Clearly, the possibility of launching new orders in advance is an advantage.

In Figure 2.8, we plot the optimal policy at time $t = 0$ and $t = 7.5$ minutes. As expected, the algorithm is more aggressive when the probability of having $v = 0.8$ is higher.

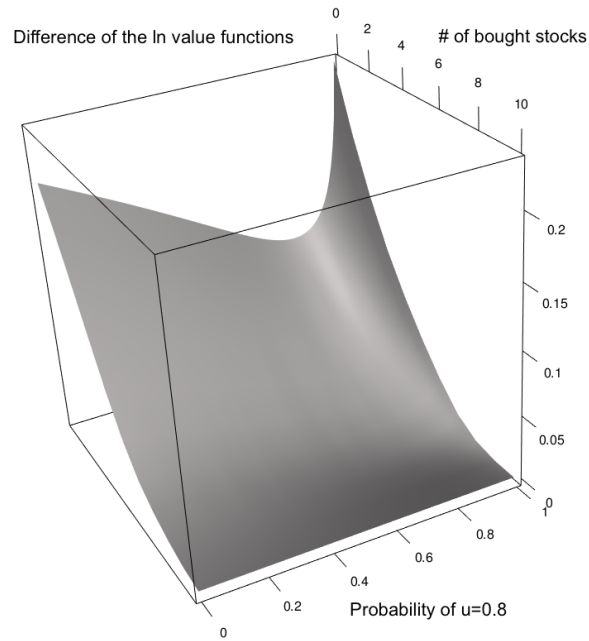


FIGURE 2.7 – Launching new orders in advance, if the previous one is executed, is an advantage.

In Figure 2.9, we plot a simulated path. The red and black lines and points correspond to the same realization of the random variables at hand, but for different values of the real value of v . Black corresponds to the most favorable case $v = 0.8$, while red corresponds to $v = 0.8$ for the first 7.5 minutes and $v = 0.3$ for the remaining time. The initial prior is $\mathbb{P}[v = 0.8] = 9\%$. Again, the algorithm adapts pretty well to this shock on the true parameter. We also see that it is more aggressive when the prior probability of being in the favorable case is high.

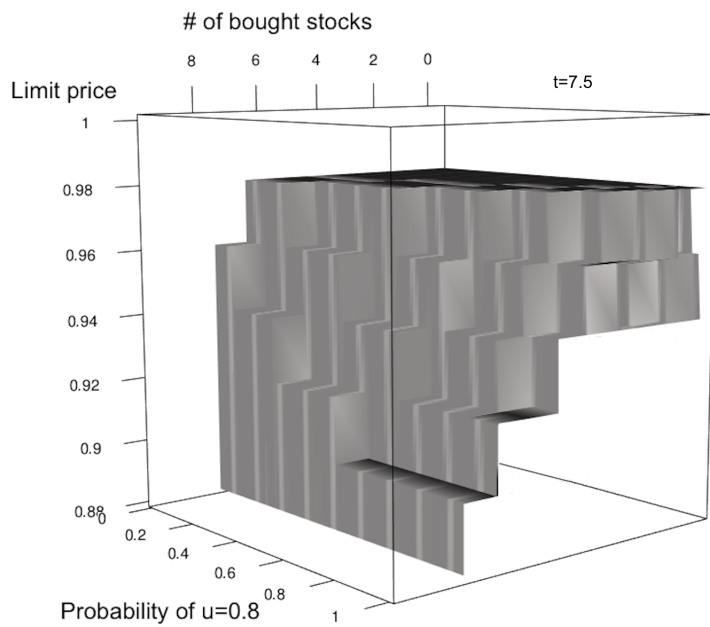
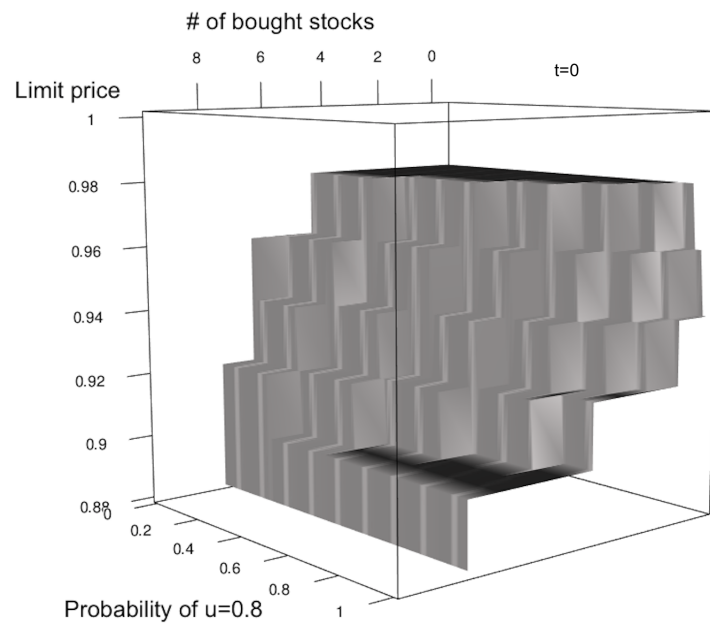


FIGURE 2.8 – Top : $t = 0$. Bottom : $t = 7.5$ minutes

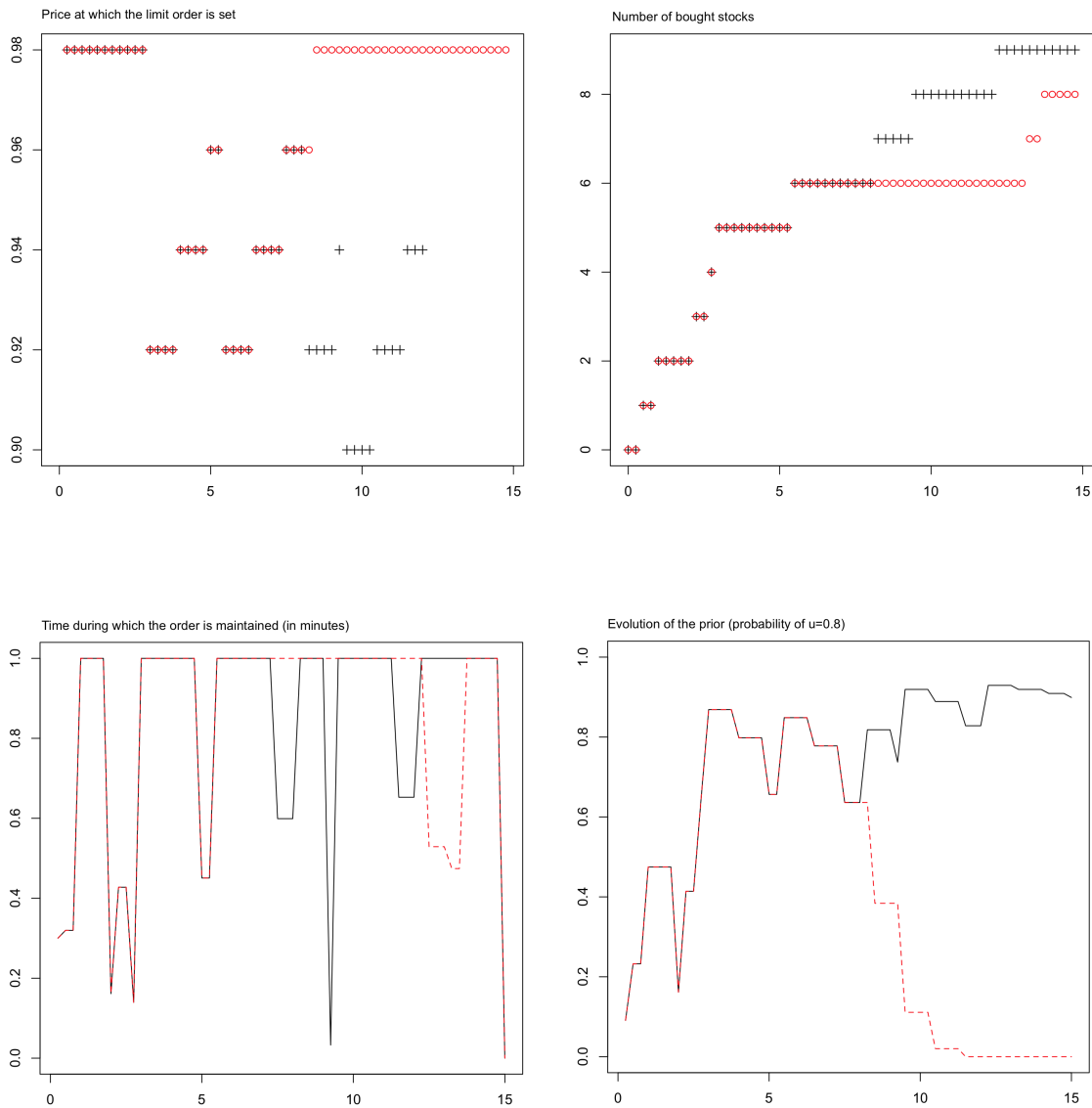


FIGURE 2.9 – Black crosses and solid lines : $v = 0.8$. Red circles and dashed lines : $v = 0.8$ before $t = 7.5$ minutes and $v = 0.3$ after ; x -axis= time in minutes.

Chapitre 3

Optimal Control under uncertainty : Application to the issue of CAT bonds

We propose a general framework for studying optimal issue of CAT bonds in the presence of uncertainty on the parameters. In particular, the intensity of arrival of natural disasters is inhomogeneous and may depend on unknown parameters. Given a prior on the distribution of the unknown parameters, we explain how it should evolve according to the classical Bayes rule. Taking these progressive prior-adjustments into account, we characterize the optimal policy through a quasi-variational parabolic equation, which can be solved numerically. We provide examples of application in the context of hurricanes in Florida.

3.1 Introduction

We consider an insurer or a reinsurer who holds a portfolio in non-life insurance exposed to one or several natural disasters. He can issue one or several CAT bonds¹ in order to reduce the risk taken, see e.g. [32] or [33] for a general introduction to CAT bonds.

The first CAT bonds were issued at the end of the 1990s and the market is globally increasing, with a total risk capital outstanding greater than USD 30 billion at the end of 2017, see [5] and [26]. CAT bonds give a strong alternative to the classical reinsurance market.

However, issuing a CAT bond leads to the choice of several parameters, as the layer e.g. and the date of issuance. The coupon is not a priori perfectly known as well as the claim distribution. Moreover, the global warming will lead to an increase of several natural disasters which is a source of uncertainty on the distribution of future claims. For example, in [55], the authors estimate that if the temperature rises of 2.5 degrees in the next decades, the frequency of Hurricanes in North Atlantic will rise by 30%.

The aim of this paper is to provide a rigorous continuous-time framework in which we can establish the optimal behavior policy in issuing CAT bonds, taking into account the

1. Catastrophe bonds, or CAT bonds, are tradable floating rate notes. The risk associated with a CAT bond is not linked to the default of one entity (state or corporate) but is related to the occurrence of a catastrophe.

uncertainty described above as the risk evolution.

The coupon of the CAT bond is generally not known in advance, even its distribution is not always clearly fixed. We therefore need to model it as a random variable whose distribution depends on unknown parameters. It is the same for the distribution of the natural disasters.

The particular case of acting on a system with partially unknown response distributions has been studied in [10] in a Brownian framework, see the references therein for the case of discrete settings. They fix a prior distribution on the unknown parameter and introduce a stochastic process on the space of measures which leads to a dynamic programming principle and a PDE characterization of the value function (in the viscosity solution sense).

In this paper, the natural disasters will be represented by a random Poisson measure² and two parameters are unknown : the distribution of the severity of the natural disasters and the intensity of their arrivals. As in [10], we allow the agent to issue new CAT bonds at any time, the actions are discrete but chosen in a continuous time framework.

To the best of our knowledge, the study of such a general problem with an application to the CAT bonds seems to be new in the literature, even in the case where all parameters are known. From a mathematical point of view, the main difficulty comes from the fact that the conditional distribution on the unknown parameters evolves continuously and jumps at the occurrence times of a catastrophic event. In [10], it was only evolving when an action was taken on the system. For tractability, we assume that the associated process remains in a finite-dimensional space which can be linked smoothly to a subset of \mathbb{R}^d for some $d \geq 1$.

Although the model presented below has been designed for the particular case of CAT bonds, it is quite general from a mathematical view-point and can be applied to all cases where the agent faces a random Poisson measure and can issue contracts from which he pays a premium and receives a specific payoff depending on some event.

3.2 The framework

3.2.1 General framework

All over this paper, $D([0, T], \mathbb{R}^d)$ is the Skorohod space of càdlàg³ functions from $[0, T]$ into \mathbb{R}^d , \mathbb{P} is a probability measure on this space, and $T > 0$ is a fixed time horizon.

We consider three Polish spaces : $(U^\lambda, \mathcal{B}(U^\lambda))$, $(U^\gamma, \mathcal{B}(U^\gamma))$ and $(U^\nu, \mathcal{B}(U^\nu))$ that will support three unknown parameters, respectively λ_0 , γ_0 and ν_0 . Here $\mathcal{B}(\cdot)$ denotes the Borel σ -algebra. We set $U := (U^\lambda, U^\gamma, U^\nu)$.

Let $N(dt, du)$ be a random Poisson measure with compensator $\nu(dt, du)$ such that ν is finite on $(\mathbb{R}^{d^*}, \mathcal{B}(\mathbb{R}^{d^*}))$ where $\mathbb{R}^{d^*} := \mathbb{R}^d \setminus \{0_{\mathbb{R}^d}\}$. The intensity of the random Poisson measure is supposed to be inhomogeneous of intensity $s \mapsto \Lambda(s, \lambda_0)$ where λ_0 is a random variable valued in U^λ . The jump distribution is assumed to be $\Upsilon(\gamma_0, \cdot)$ where γ_0 is a random variable valued in U^γ . We denote by \mathbf{M}^λ a subset of the set of Borel probability measures on U^λ and

2. The activity of the random Poisson measure will be finite, by construction

3. continue à droite, limite à gauche (Right continuous with left limits)

by $\mathbf{M}^\gamma \otimes \mathbf{M}^\nu =: \mathbf{M}$ the product of two locally compact subsets of the set of Borel probability measures, respectively on U^γ and U^ν , endowed with the weak topology.

We also allow an additional randomness when acting on the system and consider another Polish space $(E, \mathcal{B}(E))$ on which is defined a family $(\epsilon_i)_{i \geq 1}$ of i.i.d. random variables with common probability measure \mathbb{P}_ϵ on $\mathcal{B}(E)$.

On the product space $\Omega := D([0, T], \mathbb{R}^d) \times U \times E^{\mathbb{N}^*}$, we consider the family of measures $\{\mathbb{P} \times \bar{m} \times \mathbb{P}_\epsilon^{\otimes \mathbb{N}^*}, \bar{m} \in \bar{\mathbf{M}}\}$ where $\bar{\mathbf{M}} := \mathbf{M}^\lambda \otimes \mathbf{M}$. We denote by $\mathbb{P}_{\bar{m}}$ an element of this family whenever $\bar{m} \in \bar{\mathbf{M}}$ is fixed. The operator $\mathbb{E}_{\bar{m}}$ is the expectation associated to $\mathbb{P}_{\bar{m}}$. Note that $N(dt, du)$ and $(\epsilon_i)_{i \geq 1}$ are independent under each $\mathbb{P}_{\bar{m}}$. For $\bar{m} \in \bar{\mathbf{M}}$ given, we let $\mathbb{F}^{\bar{m}} := (\mathcal{F}_t^{\bar{m}})_{t \geq 0}$ denote the $\mathbb{P}_{\bar{m}}$ -augmentation of the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ defined by $\mathcal{F}_t := \sigma(N([0, s] \times \cdot)_{s \leq t}, \lambda_0, \gamma_0, \nu_0, (\epsilon_i)_{i \geq 1})$. Hereafter, all random variables are considered with respect to the probability space $(\Omega, \mathcal{F}_T^{\bar{m}}, \mathbb{P}_{\bar{m}})$ with $\bar{m} \in \bar{\mathbf{M}}$ given by the context.

3.2.2 CAT Bond framework

In this framework, $d \in \mathbb{N}^*$ is the number of perils. The insurer has some exposure related to these perils and may issue CAT bonds to reduce the risk taken. The random Poisson measure represents the arrival of claims. The intensity of arrival is $s \mapsto \Lambda(s, \lambda_0)$ in which λ_0 , valued in U^λ , may be unknown to the insurer. The dependence in time may represent the seasonality or a structural change, for example caused by the global warming.

The measure $m^\lambda \in \mathbf{M}^\lambda$ is the initial knowledge of the insurer on λ_0 and will evolve through the observations of N , whose jumps model the arrival of natural disasters. The severity distribution of the claims may also be unknown, it depends on the unknown parameter γ_0 , valued in U^γ . An initial prior is given as an element $m^\gamma \in \mathbf{M}^\gamma$. Acting on the system consists in issuing a CAT bond, which means transferring a part of the risk to the market. The equilibrium premium that the insurer will pay is random (since it comes from the law of supply and demand and is not known when the decision to issue is taken), and the distribution may not be perfectly known. We assume that it depends on the unknown parameter ν_0 , valued in U^ν . Its prior distribution is represented by some $m^\nu \in \mathbf{M}^\nu$.

We fix a maximum of $n \in \mathbb{N}$ possible CAT bonds in term of risk covered. The possible risk coverages are denoted by $(A_j)_{1 \leq j \leq n}$ with $A_j \subset \mathcal{B}(\mathbb{R}^{d^*})$ in which $\mathcal{B}(\mathbb{R}^{d^*})$ denotes all Borel sets of \mathbb{R}^{d^*} . In practice, it will represent the layer of one peril for one region, and then, if for $j = 1$, it is the first dimension (risk) of N which is covered, A_j will have the form $[a, +\infty[\times \mathbb{R} \times \dots \times \mathbb{R}$ with $a > 0$. If a claim $u \in \mathbb{R}^{d^*}$ satisfies $u \in A_1$, it will give a payoff of the form $(u_1 - a)$ bounded by some $b > 0$ associated to this layer (the layer is $[a, a + b]$).

3.2.3 The controlled system

Let $\mathbf{A} \subset \mathbb{R}^{d+1}$ be a non-empty compact set. Let $\ell \in \mathbb{R}_+^*$ be the time-length of each action on the controlled system. Given $\bar{m} \in \bar{\mathbf{M}}$, we denote by $\Phi^{\circ, \bar{m}}$ the collection of random variables $\phi = (\tau_i^\phi, \alpha_i^\phi)_{i \geq 1}$ on $(\Omega, \mathcal{F}_T^{\bar{m}})$ with values in $\mathbb{R}_+ \times \mathbf{A}$ such that $(\tau_i^\phi)_{i \geq 1}$ is a non-decreasing sequence of $\mathbb{F}^{\bar{m}}$ -stopping times and each α_i is $\mathcal{F}_{\tau_i}^{\bar{m}}$ -measurable for $i \geq 1$. We shall

write $\alpha_i^\phi := (k_i^\phi, n_i^\phi) \in \mathbf{A}$ where k_i^ϕ and n_i^ϕ are \mathbb{R}^d and \mathbb{R} -valued. To each k_i^ϕ , we associate a non-empty closed set $A_{k_i^\phi} \subset \mathbb{R}^{d*}$ through a one-to-one map.

The τ_i^ϕ 's will be the times at which a CAT bond is issued. The fixed value ℓ is the time-length (or maturity) of all CAT bonds. In $\alpha_i^\phi := (k_i^\phi, n_i^\phi) \in \mathbf{A}$, n_i^ϕ is related to the notional and $A_{k_i^\phi}$ is the layer chosen for one peril and one region : it is the characteristics of the CAT bonds associated to the risk covered. If a natural disaster occurs and its severity is in the layer $A_{k_i^\phi}$, i.e. the random Poisson measure has a jump in $A_{k_i^\phi}$, then the CAT bonds ends and the reinsurer gains a payoff proportional to the notional n_i^ϕ .

We denote by ϑ_i^ϕ the end of the i -th CAT Bond defined by :

$$\vartheta_i^\phi := \inf\{t > \tau_i^\phi, N(\{t\} \times A_{k_i^\phi}) = 1\} \wedge (\tau_i^\phi + \ell). \quad (3.2.1)$$

Remark 3.2.1. According to the definition of $(\vartheta_i^\phi)_{i \geq 1}$, it can happen that $\vartheta_{i_1}^\phi = \vartheta_{i_2}^\phi$ for $i_1 \neq i_2$. Moreover,

$$\tau_i^\phi < \vartheta_i^\phi \leq \tau_i^\phi + \ell.$$

We are now in position to describe the controlled state process. Given some initial data $(t, x) \in [0, T] \times \mathbb{R}^d$, and $\phi \in \Phi^{\circ, \bar{m}}$, we let $X^{t, x, \phi}$ be a strong solution on $[t, T]$ of

$$\begin{aligned} X &:= x + \int_t^\cdot \mu(s, X_s) ds + \int_t^\cdot \int_{\mathbb{R}^d} \beta(s, X_{s-}, u) N(ds, du) \\ &\quad + \sum_{i \geq 1} \mathbf{1}_{\{t \leq \tau_i^\phi < \cdot\}} H(\tau_i^\phi, X_{\tau_i^\phi}, \alpha_i^\phi) \\ &\quad + \sum_{i \geq 1} \mathbf{1}_{\{t \vee \tau_i^\phi \leq \cdot\}} \int_{t \vee \tau_i^\phi}^{\cdot \wedge \vartheta_i^\phi} \bar{C}(s, r_i^\phi) ds \\ &\quad + \sum_{i \geq 1} \mathbf{1}_{\{t \leq \vartheta_i^\phi \leq \cdot\}} F(\vartheta_i^\phi, X_{\vartheta_i^\phi-}, X_{\tau_i^\phi}, r_i^\phi, \alpha_i^\phi, \vartheta_i^\phi - \tau_i^\phi, u_i) \mathbf{1}_{\{\vartheta_i^\phi - \tau_i^\phi \neq \ell\}}, \end{aligned} \quad (3.2.2)$$

in which $r_i^\phi := \mathfrak{C}_0(\tau_i^\phi, X_{\tau_i^\phi-}, \alpha_i^\phi, \nu, \epsilon_i)$ with $\mathfrak{C}_0 : [0, T] \times \mathbb{R}^d \times \mathbf{A} \times U^\nu \times E \rightarrow \mathbb{R}$ a measurable function and u_i is the jump size of the random Poisson measure N at ϑ_i .

To guarantee existence and uniqueness of the above, we make the following standard assumptions.

Assumption 3.2.1. $\mu : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\beta : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{M}^d$ and $\bar{C} : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$, are assumed to be measurable, continuous, and Lipschitz with linear growth in their second argument, uniformly in the other ones.

The maps $H : [0, T] \times \mathbb{R}^d \times \mathbf{A}$, and $F : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbf{A} \times [0, \ell] \times \mathbb{R}^d$ are assumed to be measurable. Moreover, H (resp. F) has linear growth in its second (resp. third) component.

This dynamics means the following. Without any CAT bond, the process X follows a pure jump process with a drift on the first line of (3.2.2). The second line refers to a jump of the whole process when a CAT bond is issued, for example, a fixed cost. The third line represents the instantaneous cash flows generated by the closed and current active CAT bonds. The last line represents the final cash flow if the policy ends before the maturity.

The first component of the process X will be the cash. The second may record the aversion of the market for the CAT bonds : when a natural disaster occurs, it jumps and then decrease again over time. The function μ can be the drift associated to some interest rate or the decrease of the risk aversion of the market when no natural disaster occurs. The function β can represent the exposure in cash of the reinsurer for each peril, and also the sensitivity of the CAT bond market (for the second component of X) when a natural disaster occurs. The function H represents an initial cost to issue the CAT bond. The function \bar{C} is the continuous premium paid by the reinsurer for the CAT bond and r_i^ϕ is the level of the coupon (a random variable which is determined by an unknown parameter v and a noise ϵ_i). The function F is the payout, $\vartheta_i^\phi - \tau_i^\phi \neq \ell$ means that the CAT Bond ends with an event.

We denote by $\mathbb{F}^{t,x,\bar{m},\phi} := \left(\mathcal{F}_s^{t,x,\bar{m},\phi} \right)_{s \geq 0}$ the $\mathbb{P}_{\bar{m}}$ -augmentation of the filtration generated by $\left(X^{t,x,\phi}, \sum_{i \geq 1} r_i^\phi \mathbf{1}_{[\tau_i^\phi, +\infty[}, N([t, s] \times \cdot)_{s \geq t} \right)$.

For $\kappa \in \mathbb{N}^*$, we say that $\phi \in \Phi^{\circ, \bar{m}}$ belongs to $\Phi_\kappa^{t,x,\bar{m}}$ if the condition

$$\sum_{i \geq 1} \mathbf{1}_{\{\tau_i^\phi < t \leq \vartheta_i^\phi\}} \leq \kappa \quad \forall t \leq T \quad (3.2.3)$$

holds. The set $\Phi_\kappa^{t,x,\bar{m}}$ is the set of admissible controls. The constraint (3.2.3) refers to the fact that the controller cannot have more that κ simultaneous running CAT bonds at each time.

Note that $X^{t,x,\phi}$ has a jump of size $H(\tau_i^\phi, X_{\tau_i^\phi}^{t,x,\phi}, \alpha_i^\phi)$ at each τ_i^ϕ and is left-continuous at this point, whereas it is right-continuous at each ϑ_i . This allows to observe a jump from the left from the random Poisson measure and then issue immediately a new CAT bond, leading to an immediate jump of X from the right. The process $X^{t,x,\phi}$ defined above is l\`a d\`a g.

3.2.4 The CAT bonds process

We need to keep track of how many CAT bonds are running, and which parameters are associated to. Corresponding to the definition of the process X in (3.2.2), the effect of a CAT bond will be measured by the value of $(X_{\tau_i^\phi}^{t,x,\phi}, r_i^\phi, \alpha_i^\phi)$ determined at τ_i , for $(t, x) \in [0, T] \times \mathbb{R}^d$, $\phi \in \Phi_\kappa^{t,x,\phi}$. Moreover, a CAT bond will end from a jump or after the time-length ℓ . We need to define a process which will keep track of this information. We introduce the sets $\mathbf{C} := \left((\mathbb{R}^d \times \mathbb{R} \times \mathbf{A}) \cup \partial \right)^\kappa$, $\mathbf{L} := ([0, \ell \cup \partial)^\kappa$, in which

- An element of the set $\mathbb{R}^d \times \mathbb{R} \times \mathbf{A}$ represents the initial parameters of the CAT bond ;
- An element of the set $[0, \ell[$ represents the time-length elapsed of a running CAT bond ;
- The point ∂ represents the absence of CAT bond, it is a cemetery point.

The set of CAT bonds is

$$\mathbf{CL} := \{(c, l) \in \mathbf{C} \times \mathbf{L} \mid c_j = \partial \iff l_j = \partial, \forall 1 \leq j \leq \kappa\}$$

and we denote by $\overline{\mathbf{CL}}$ its closure. We set $\mathbf{K} := \{0, \dots, \kappa\}$ and we define by $\mathcal{P}(\mathbf{K})$ the set of subsets of \mathbf{K} . We can now define the sets $\mathbf{CL}_{\mathbf{J}}$ with $\mathbf{J} \in \mathcal{P}(\mathbf{K})$:

$$\mathbf{CL}_{\mathbf{J}} := \{(c, l) \in \mathbf{CL} \mid j \in \mathbf{J} \iff c_j \neq \partial, \forall 1 \leq j \leq \kappa\}$$

which represent the sets of CAT Bonds in which there is CAT Bonds running exactly in the indexes of \mathbf{J} .

Moreover, for $(c, l) \in \mathbf{CL} \setminus \mathbf{CL}_{\mathbf{K}}$, we introduce :

$$\Pi^0(c, l) := \min\{1 \leq j \leq \kappa : c^j = \partial\},$$

which is the first index with no CAT bond.

For $z := (t, x, c, l) \in \mathbf{Z} := [0, T] \times \mathbb{R}^d \times \mathbf{CL}$ and a control $\phi \in \Phi_{\kappa}^{t, x, \bar{m}}$, we now define the process $((C, L)_s^{z, \phi, j})_{\substack{1 \leq j \leq \kappa \\ t \leq s \leq T}}$ valued in \mathbf{CL} and denoted hereafter (C, L) for ease of notation. The process (C, L) will jump at the τ_i 's (new CAT bond) and at the ϑ_i 's (end of one or several CAT bonds). C will be a pure jump process whereas the indexes of L will evolve continuously over time, recall that it represents the elapsed time-length of the CAT bonds.

We now define the functions associated to the jumps of (C, L) . The first one, denoted by \mathfrak{C}_+ , represents the arrival of one new CAT bond with parameters $(x, r, a) \in \mathbb{R}^d \times \mathbb{R} \times \mathbf{A}$ and is defined by

$$\begin{aligned} \mathfrak{C}_+ : (\mathbf{CL} \setminus \mathbf{CL}_{\mathbf{K}}) \times \mathbb{R}^d \times \mathbb{R} \times \mathbf{A} &\rightarrow \mathbf{CL} \\ (c, l; x, r, a) &\mapsto \mathfrak{C}_+(c, l; x, r, a) \end{aligned}$$

where, if we write (c_+, l_+) for $\mathfrak{C}_+(c, l; x, r, a)$,

$$\begin{aligned} (c_+, l_+)_{\Pi^0(c, l)} &:= ((x, r, a), 0), \\ (c_+, l_+)_j &= (c, l)_j \quad j \neq \Pi^0(c, l). \end{aligned} \tag{3.2.4}$$

The second function, denoted by \mathfrak{C}_- , represents the end of the CAT bonds by an event associated to the random Poisson measure, of severity $u \in \mathbb{R}^{d^*}$, and is defined by

$$\begin{aligned} \mathfrak{C}_- : \mathbf{CL} \times \mathbb{R}^{d^*} &\rightarrow \mathbf{CL} \\ (c, l; u) &\mapsto \mathfrak{C}_-(c, l; u). \end{aligned}$$

Nonetheless, several CAT bonds may end with a single event. We define the set of indexes in $c \in \mathbf{C}$ which end after the natural disaster $u \in \mathbb{R}^{d^*}$, by

$$\mathcal{J}(c; u) := \{j \in \{1, \dots, \kappa\} \mid c_j \neq \partial, u \in A_{k_j}\}. \tag{3.2.5}$$

Using this set, $\mathfrak{C}_-(c, l; u)$ is defined simply through its j -component

$$\mathfrak{C}_-(c, l; u)_j := \begin{cases} \partial \times \partial & \text{if } j \in \mathcal{J}(c; u) \\ (c, l)_j & \text{if } j \notin \mathcal{J}(c; u) \end{cases}, \quad 1 \leq j \leq \kappa. \tag{3.2.6}$$

It remains to consider the case where a CAT Bond ends because $l_j = \ell$ for some $1 \leq j \leq \kappa$. We define :

$$\begin{aligned} \mathfrak{C}_-^\ell : (\overline{\mathbf{CL}} \setminus \overline{\mathbf{CL}}_\emptyset) &\rightarrow \mathbf{CL} \\ (c, l) &\mapsto \mathfrak{C}_-^\ell(c, l), \end{aligned}$$

where, for all $1 \leq j \leq \kappa$,

$$\mathfrak{C}_-^\ell(c, l)_j = (\partial \times \partial) \mathbf{1}_{\{l_j = \ell\}} + (c, l)_j \mathbf{1}_{\{l_j \neq \ell\}}.$$

We are now in position to define the processes $C^{z,\phi}$ and $L^{z,\phi}$ for $\phi \in \Phi_\kappa^{t,x,\bar{m}}$. The process evolves at τ_i^ϕ and ϑ_i^ϕ , for $i \geq 1$, according to :

$$\begin{aligned} (C, L)_{\tau_i^\phi+}^{z,\phi} &:= \mathfrak{C}_+((C, L)_{\tau_i^\phi}^{z,\phi}; X_{\tau_i^\phi}^{z,\phi}, r_i^\phi, \alpha_i^\phi); \\ (C, L)_{\vartheta_i^\phi}^{z,\phi} &:= \mathbf{1}_{\{\vartheta_i^\phi < \tau_i^\phi + \ell\}} \mathfrak{C}_-((C, L)_{\vartheta_i^\phi-}^{z,\phi}, u_i) + \mathbf{1}_{\{\vartheta_i^\phi = \tau_i^\phi + \ell\}} \mathfrak{C}_-^\ell((C, L)_{\vartheta_i^\phi-}^{z,\phi}). \end{aligned} \quad (3.2.7)$$

Elsewhere, $C^{z,\phi}$ is constant. For $1 \leq j \leq \kappa$, $L^{z,\phi,j}$ evolves according to :

$$dL_t^{z,\phi,j} = \mathbf{1}_{\{L_t^{z,\phi,j} \neq \partial\}} dt.$$

This closes the definition of the process (C, L) . Note that we separated both the initial parameters with the elapsed time-length since the second one will play a different role in the PDE characterization in consequence of its continuous part.

Remark 3.2.2. *If $c \mapsto \Pi(c) := \#\{j \in \mathbf{K} : c^j \neq \partial\}$, the process $C^{z,\phi}$ (and also, by construction, $L^{z,\phi}$) satisfies :*

$$\begin{aligned} \Pi(C_s^{z,\phi}) &\leq \kappa, \quad \forall s \in [t, T], \quad \mathbb{P}_{\bar{m}} - a.s. \\ \Pi(C_{\tau_i^\phi}^{z,\phi}) &\leq \kappa - 1, \quad \forall i \geq 1, \quad \mathbb{P}_{\bar{m}} - a.s.. \end{aligned}$$

We also give a metric on $\overline{\mathbf{CL}}$.

Definition 3.2.1. *We associate to $\overline{\mathbf{CL}}$ the metric \mathfrak{d} defined by*

$$\begin{aligned} \mathfrak{d}[(c, l), (c', l')] &:= \sum_{j \in \mathbf{J} \cap \mathbf{J}'} [\|c_j - c'_j\|^2 + (l_j - l'_j)^2] + \sum_{j \in \mathbf{J} \setminus \mathbf{J}'} (\|c_j\|^2 + l_j^2) \\ &+ \sum_{j \in \mathbf{J}' \setminus \mathbf{J}} (\|c'_j\|^2 + (l'_j)^2) + \text{Card}(\mathbf{J} \Delta \mathbf{J}'), \end{aligned}$$

where \mathbf{J} and \mathbf{J}' are respectively the set of running CAT bonds of parameters (c, l) and (c', l') .

Remark 3.2.3. *For $z := (t, x, c, l) \in \mathbf{Z}$, we shall write $X^{z,\phi}$ for the process X starting with the CAT bonds (c, l) and $\mathbb{F}^{z,\phi}$ the same filtration as $\mathbb{F}^{t,x,\bar{m},\phi}$ but also starting with the CAT bonds (c, l) . Note that (C, L) is adapted $\mathbb{F}^{z,\bar{m},\phi}$ -adapted. Moreover, we define $\Phi_\kappa^{z,\bar{m}}$ as $\Phi_\kappa^{t,x,\bar{m}}$ but, again, starting with CAT bonds (c, l) .*

3.2.5 Bayesian updates

Obviously, the prior $\bar{m} \in \overline{\mathbf{M}}$ will evolve over time. Recall that $\overline{\mathbf{M}} := \mathbf{M}^\lambda \otimes \mathbf{M}$ and denote by $\bar{m} := (m^\lambda, m^\gamma, m^\nu)$ the corresponding element. The observation of X over time will lead to a continuous update of m^λ , whereas m^γ will be updated by observing the size of a jump from N and the measure m^ν will be updated by acting on the system at times τ_i^ϕ . This leads to the definition of the process $M := (M^\lambda, M^\gamma, M^\nu)$ valued in $\overline{\mathbf{M}}$. We first focus on m^λ .

Evolution of the intensity

We start with the assumption associated to the unknown and inhomogeneous intensity of the random Poisson measure.

Assumption 3.2.2. For all $m^\lambda \in \mathbf{M}^\lambda$,

- i) $\int_s^t \Lambda(u, \lambda_0) du < +\infty$ $m^\lambda - a.s.$, for all $0 \leq s \leq t$.
- ii) $t \mapsto \Lambda(t, \lambda_0)$ is a càdlàg process $m^\lambda - a.s.$
- iii) For almost every $s \geq 0$ such that $\Lambda(s, \lambda_0) > 0$ $m^\lambda - a.s.$, there exists $h_0 > 0$ and $K > 0$ such that $\int_s^{s+h} \Lambda(u, \lambda_0) du \leq Kh\Lambda(s, \lambda_0)$ for all $h \leq h_0$.
- iv) $\int_0^{+\infty} \Lambda(u, \lambda_0) du = +\infty$ $m^\lambda - a.s.$

Between two jumps of the random Poisson measure, the probability measure associated to λ_0 will evolve continuously. When a jump occurs, it jumps as well. We first deal with what happens between two jumps.

Remark 3.2.4. Remark that, since a càdlàg function has at most a countable set of points of discontinuity, under ii) of Assumption 3.2.2 we have

$$\int_s^t \Lambda(u, \lambda_0) e^{-\int_\alpha^u \Lambda(v, \lambda_0) dv} du = e^{-\int_\alpha^s \Lambda(v, \lambda_0) dv} - e^{-\int_\alpha^t \Lambda(v, \lambda_0) dv} \quad m^\lambda\text{-a.e.} \quad (3.2.8)$$

for almost all $0 \leq \alpha \leq s \leq t$.

Given $B \in \mathcal{B}(U^\lambda)$, we set $M_s^{t, m^\lambda}(B) := \mathbb{E}_{\bar{m}}(\mathbf{1}_{\{\lambda_0 \in B\}} | \mathcal{F}_s^{z, \bar{m}, \phi})$ for $z = (t, x, c, l)$ and $\phi \in \Phi_{\kappa}^{z, \bar{m}}$. We shall see below that M_s^{t, m^λ} does not depend on x and ϕ . From now on, we denote by $(\zeta_i)_{i \geq 1}$ the jump times associated to the random Poisson measure.

Lemma 1. For all $z = (t, x, c, l) \in \mathbf{Z}$ and $s > t$,

$$M_s^{t, m^\lambda}(B) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} = \mathfrak{M}_\lambda(B; \zeta_i, s) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}}$$

where

$$\mathfrak{M}_\lambda(B; \zeta_i, s) := \frac{\int_B e^{-\int_{\zeta_i}^s \Lambda(u, \lambda) du} M_{\zeta_i}^{t, m^\lambda}(d\lambda)}{\int_{\mathbb{R}_+} e^{-\int_{\zeta_i}^s \Lambda(u, \lambda) du} M_{\zeta_i}^{t, m^\lambda}(d\lambda)} \mathbf{1}_{\{\zeta_i \leq s\}}.$$

Proof. Let φ be a Borel bounded function on $D([0, T], \mathbb{R}^{d+1})$. Set $\xi^\phi := \sum_{i \geq 1} r_i^\phi \mathbf{1}_{[r_i^\phi, +\infty[}$, $\delta X^i := X_{\cdot, \vee \zeta_i}^{z, \phi} - X_{\zeta_i}^{z, \phi}$, and $\delta \xi^i := \xi_{\cdot, \vee \zeta_i} - \xi_{\zeta_i}$. Note that $\delta \xi_{\cdot, \wedge s}^i \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}}$ is $\sigma(\mathcal{F}_{\zeta_i}^{z, m, \phi} \cup \sigma(v, (\epsilon_j)_{1 \leq j \leq K}))$ -measurable. We can find a Borel measurable map $\bar{\varphi}$ such that

$$\varphi(X_{\cdot, \wedge s}^{z, \phi}, \xi_{\cdot, \wedge s}^\phi) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} = \bar{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^\phi, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}}.$$

In view of Remark 3.2.4, it then follows :

$$\begin{aligned}
& \mathbb{E}_{\overline{m}} \left(\mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} \varphi(X_{\cdot, \wedge s}^{z, \phi}, \xi_{\cdot, \wedge s}^{\phi}) \right) \\
&= \mathbb{E}_{\overline{m}} \left(\mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} \overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \right) \\
&= \mathbb{E}_{\overline{m}} \left(\int_{\mathbb{R}_+} \mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{\{\zeta_i \leq s < u\}} \overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \Lambda(u, \lambda_0) e^{-\int_{\zeta_i}^u \Lambda(v, \lambda_0) dv} du \right) \\
&= \mathbb{E}_{\overline{m}} \left(\mathbf{1}_{\{\lambda_0 \in B\}} \overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \int_{\mathbb{R}_+} \mathbf{1}_{\{\zeta_i \leq s < u\}} \Lambda(u, \lambda_0) e^{-\int_{\zeta_i}^u \Lambda(v, \lambda_0) dv} du \right) \\
&= \mathbb{E}_{\overline{m}} \left(\mathbf{1}_{\{\lambda_0 \in B\}} \overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \mathbf{1}_{\{\zeta_i \leq s\}} e^{-\int_{\zeta_i}^s \Lambda(v, \lambda_0) dv} \right) \\
&= \mathbb{E}_{\overline{m}} \left(\overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \mathbf{1}_{\{\zeta_i \leq s\}} \int_B e^{-\int_{\zeta_i}^s \Lambda(v, \lambda) dv} M_{\zeta_i}^{t, m^\lambda}(d\lambda) \right) \\
&= \mathbb{E}_{\overline{m}} \left(\overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \mathbf{1}_{\{\zeta_i \leq s\}} \mathfrak{M}_\lambda(B; \zeta_i, s) \int_{\mathbb{R}_+} e^{-\int_{\zeta_i}^s \Lambda(v, \lambda) dv} M_{\zeta_i}^{t, m^\lambda}(d\lambda) \right) \\
&= \mathbb{E}_{\overline{m}} \left(\overline{\varphi}(X_{\cdot, \wedge \zeta_i}^{z, \phi}, \xi_{\cdot, \wedge \zeta_i}^{\phi}, \delta X_{\cdot, \wedge s}^i, \delta \xi_{\cdot, \wedge s}^i) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} \mathfrak{M}_\lambda(B; \zeta_i, s) \right) \\
&= \mathbb{E}_{\overline{m}} \left(\varphi(X_{\cdot, \wedge s}^{z, \phi}, \xi_{\cdot, \wedge s}^{\phi}) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} \mathfrak{M}_\lambda(B; \zeta_i, s) \right)
\end{aligned}$$

This shows that $M_s^{t, m^\lambda}(B) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} = \mathfrak{M}_\lambda(B; \zeta_i, s) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}}$ $\mathbb{P}_{\overline{m}}$ -a.s.

Lemma 2. For all $m^\lambda \in \mathbf{M}^\lambda$ and almost all $s \geq t$, we have

i)

$$\int_{U^\lambda} \Lambda(s, \lambda) M_s^{t, m^\lambda}(d\lambda) < +\infty \quad \mathbb{P}_{\overline{m}} - a.s.$$

ii)

$$\int_{U^\lambda} \Lambda(\zeta_i, \lambda) M_{\zeta_i}^{t, m^\lambda}(d\lambda) < +\infty \quad \mathbb{P}_{\overline{m}} - a.s., \quad i \geq 1.$$

iii)

$$\int_{U^\lambda} \Lambda(s, \lambda) M_{s-}^{t, m^\lambda}(d\lambda) < +\infty \quad \mathbb{P}_{\overline{m}} - a.s.$$

Proof. Step 1. For almost all $\lambda \in U^\lambda$, we fix $\mathcal{N}_\lambda \subset [0, T]$ the set of discontinuity of $t \mapsto \Lambda(t, \lambda)$ which is, at most, countable. We introduce :

$$\mathcal{N}^c := \{\forall i \geq 1, \zeta_i \notin \mathcal{N}_{\lambda_0}\}.$$

We shall show that $\mathbb{P}(\mathcal{N}^c) = 1$ by showing that $\mathbb{P}(\zeta_i \in \mathcal{N}_{\lambda_0}) = 0$ for all $i \geq 1$. Fix $i \geq 1$ and remark that, given $\lambda \in U^\lambda$, the distribution of $\zeta_i \mid \{\lambda_0 = \lambda\}$ is absolutely continuous with respect to the Lebesgue measure. Denote by $f_{i|\lambda}$ a corresponding density function. Then,

$$\mathbb{P}_{\overline{m}}(\zeta_i \in \mathcal{N}_{\lambda_0}) = \int_{U^\lambda} \left[\int_{\mathbb{R}_+} \mathbf{1}_{\mathcal{N}_\lambda}(z) f_{i|\lambda}(z) dz \right] dm^\lambda(\lambda) = \int_{U^\lambda} 0 dm^\lambda(\lambda) = 0.$$

Step 2. We show *i*). We set :

$$K_i(s) := \left(\int_{U^\lambda} e^{-\int_{\zeta_i}^s \Lambda(u, \lambda) du} M_{\zeta_i}^{t, m^\lambda}(d\lambda) \right)^{-1} \leq K_i(\zeta_{i+1}) \quad \text{on } \{\zeta_i \leq s < \zeta_{i+1}\}.$$

We have, by *i*) of Assumption 3.2.2,

$$K_i(\zeta_{i+1}) < +\infty.$$

Moreover, by Fubini's Lemma and Remark 3.2.4,

$$\begin{aligned} \int_{\zeta_i}^{\zeta_{i+1}} \int_{U^\lambda} \Lambda(s, \lambda) e^{-\int_{\zeta_i}^s \Lambda(u, \lambda) du} M_{\zeta_i}^{t, m^\lambda}(d\lambda) ds &= \int_{U^\lambda} \int_{\zeta_i}^{\zeta_{i+1}} \Lambda(s, \lambda) e^{-\int_{\zeta_i}^s \Lambda(u, \lambda) du} ds M_{\zeta_i}^{t, m^\lambda}(d\lambda) \\ &= \int_{U^\lambda} [1 - e^{-\int_{\zeta_i}^{\zeta_{i+1}} \Lambda(u, \lambda) du}] M_{\zeta_i}^{t, m^\lambda}(d\lambda) < +\infty, \end{aligned}$$

on \mathcal{N}^c . On the other hand, using Lemma 1,

$$\int_{\zeta_i}^{\zeta_{i+1}} \int_{U^\lambda} \Lambda(s, \lambda) M_s^{t, m^\lambda}(d\lambda) ds \leq K_i(\zeta_{i+1}) \int_{\zeta_i}^{\zeta_{i+1}} \int_{U^\lambda} \Lambda(s, \lambda) e^{-\int_{\zeta_i}^s \Lambda(u, \lambda) du} M_{\zeta_i}^{t, m^\lambda}(d\lambda) ds < +\infty$$

on \mathcal{N}^c . This shows that, for almost all $s \geq t$,

$$\mathbf{1}_{\{\zeta_i < s < \zeta_{i+1}\}} \int_{U^\lambda} \Lambda(s, \lambda) M_s^{t, m^\lambda}(d\lambda) < +\infty \quad \text{on } \mathcal{N}^c.$$

This leads to the result since $\zeta_i \rightarrow +\infty$ when $i \rightarrow +\infty$ for almost all ω .

Step 3. We show *ii*). Since M^{t, m^λ} evolves continuously on all $]\zeta_i, \zeta_{i+1}[$, we also have,

$$\int_{U^\lambda} \Lambda(\zeta_i^-, \lambda) M_{\zeta_i^-}^{t, m^\lambda}(d\lambda) < +\infty \quad \mathbb{P}_{\bar{m}} - a.s.$$

Moreover, on \mathcal{N}^c , ζ_i cannot be on a discontinuity of Λ by construction, $i \geq 1$. Then, we have, on \mathcal{N}^c ,

$$\int_{U^\lambda} \Lambda(\zeta_i, \lambda) M_{\zeta_i^-}^{t, m^\lambda}(d\lambda) < +\infty.$$

Step 4. We show *iii*). We introduce :

$$A := \{s \in [t, T] : m^\lambda [\Lambda(s, \lambda_0) = 0] < 1\}.$$

Recall that, by construction, $M_s^{t, m^\lambda} \ll m^\lambda$ for all $s \geq t$. If $s \in A$, $\int_{U^\lambda} \Lambda(s, \lambda) M_s^{t, m^\lambda}(d\lambda) = 0 < +\infty$. If $s \notin A$, the distribution of ζ_i is equivalent to the Lebesgue measure and then, by *ii*), we get the result.

We now look at the intensity at the observation of a jump ζ_i .

Lemma 3. For all $z = (t, x, c, l) \in \mathbf{Z}$ and $B \in \mathcal{B}(U^\lambda)$,

$$M_{\zeta_i}^{t, m^\lambda}(B) = \frac{\int_B \Lambda(\zeta_i, \lambda) M_{\zeta_i^-}^{t, m^\lambda}(d\lambda)}{\int_{U^\lambda} \Lambda(\zeta_i, \lambda) M_{\zeta_i^-}^{t, m^\lambda}(d\lambda)}, \quad i \geq 1.$$

Proof. We use the same notations as in the proof of Lemma 1.

1. For ease of notation, we set $B_i(\zeta) := \{\zeta_{i-1} < s, \zeta_i \in [s, s+h], s+h < \zeta_{i+1}\}$. For $s > 0$, we show that, with $\zeta_0 := 0$,

$$M_{s+h}^{t,m^\lambda}(B)\mathbf{1}_{B_i(\zeta)} = \mathfrak{M}'_\lambda(B; M_{s-}^{t,m^\lambda}, s, h)\mathbf{1}_{B_i(\zeta)}, \quad (3.2.9)$$

where

$$\mathfrak{M}'_\lambda(B; M_{s-}^{t,m^\lambda}, s, h) := \frac{\int_B \left[\int_s^{s+h} \Lambda(u, \lambda) du \right] e^{-\int_s^{s+h} \Lambda(u, \lambda) du} M_{s-}^{t,m^\lambda}(d\lambda)}{\int_{U^\lambda} \left[\int_s^{s+h} \Lambda(u, \lambda) du \right] e^{-\int_s^{s+h} \Lambda(u, \lambda) du} M_{s-}^{t,m^\lambda}(d\lambda)}.$$

Let φ be a Borel bounded function of $D([0, T], \mathbb{R}^{d+1})$, we can find a Borel measurable map $\bar{\varphi}$ such that

$$\varphi(X_{\cdot \wedge s+h}^{z, \phi}, \xi_{\cdot \wedge s+h}^\phi)\mathbf{1}_{B_i(\zeta)} = \bar{\varphi}(X_{\cdot \wedge s}^{z, \phi}, \xi_{\cdot \wedge s}^\phi, \delta X_{\cdot \wedge s+h}^i, \delta \xi_{\cdot \wedge s+h}^i)\mathbf{1}_{B_i(\zeta)}.$$

We shall write $\bar{\varphi}(X, \xi)$ for $\bar{\varphi}(X_{\cdot \wedge s}^{z, \phi}, \xi_{\cdot \wedge s}^\phi, \delta X_{\cdot \wedge s+h}^i, \delta \xi_{\cdot \wedge s+h}^i)$. It then follows :

$$\begin{aligned} & \mathbb{E}_{\bar{m}} \left(\mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{B_i(\zeta)} \varphi(X_{\cdot \wedge s+h}^{z, \phi}, \xi_{\cdot \wedge s+h}^\phi) \right) \\ &= \mathbb{E}_{\bar{m}} \left(\mathbf{1}_{\{\lambda_0 \in B\}} \mathbf{1}_{B_i(\zeta)} \bar{\varphi}(X, \xi) \right) \\ &= \mathbb{E}_{\bar{m}} \left(\int_{U^\lambda} \mathbf{1}_{\{\lambda \in B\}} \mathbf{1}_{\{\zeta_{i-1} < s\}} \bar{\varphi}(X, \xi) \left[\int_s^{s+h} \Lambda(u, \lambda) du \right] e^{-\int_s^{s+h} \Lambda(u, \lambda) du} M_{s-}^{t,m^\lambda}(d\lambda) \right) \\ &= \mathbb{E}_{\bar{m}} \left(\bar{\varphi}(X, \xi) \mathbf{1}_{\{\zeta_{i-1} < s\}} \int_B \left[\int_s^{s+h} \Lambda(u, \lambda) du \right] e^{-\int_s^{s+h} \Lambda(u, \lambda) du} M_{s-}^{t,m^\lambda}(d\lambda) \right) \\ &= \mathbb{E}_{\bar{m}} \left(\bar{\varphi}(X, \xi) \mathbf{1}_{\{\zeta_{i-1} < s\}} \mathfrak{M}'_\lambda(B; M_{s-}^{t,m^\lambda}, s, h) \int_{U^\lambda} \left[\int_s^{s+h} \Lambda(u, \lambda) du \right] e^{-\int_s^{s+h} \Lambda(u, \lambda) du} M_{s-}^{t,m^\lambda}(d\lambda) \right) \\ &= \mathbb{E}_{\bar{m}} \left(\bar{\varphi}(X, \xi) \mathbf{1}_{B_i(\zeta)} \mathfrak{M}'_\lambda(B; M_{s-}^{t,m^\lambda}, s, h) \right) \\ &= \mathbb{E}_{\bar{m}} \left(\varphi(X_{\cdot \wedge s+h}^{z, \phi}, \xi_{\cdot \wedge s+h}^\phi) \mathbf{1}_{B_i(\zeta)} \mathfrak{M}'_\lambda(B; M_{s-}^{t,m^\lambda}, s, h) \right) \end{aligned}$$

This shows that (3.2.9) hold $\mathbb{P}_{\bar{m}}$ -a.s.

2. For $i = 1$, on $\{\zeta_1 \geq s\}$, by Lemma 2, $\Lambda(s, \lambda_0) \in L^1(M_{s-}^{t,m^\lambda})$ for almost all s . Using *iii*) of Assumption 3.2.2, by the dominated convergence theorem, we deduce that

$$M_s^{t,m^\lambda}(B)\mathbf{1}_{\{\zeta_0 < s, \zeta_1 = s\}} = \frac{\int_B \Lambda(s, \lambda) M_{s-}^{t,m^\lambda}(d\lambda)}{\int_{U^\lambda} \Lambda(s, \lambda) M_{s-}^{t,m^\lambda}(d\lambda)},$$

i.e., since the law of ζ_1 is absolutely continuous with respect to the Lebesgue measure,

$$M_{\zeta_1}^{t,m^\lambda}(B) = \frac{\int_B \Lambda(\zeta_1, \lambda) M_{\zeta_1-}^{t,m^\lambda}(d\lambda)}{\int_{U^\lambda} \Lambda(\zeta_1, \lambda) M_{\zeta_1-}^{t,m^\lambda}(d\lambda)} \quad \mathbb{P}_{\bar{m}} \text{ - a.s.}$$

Since almost surely, $\zeta_{i+1} > \zeta_i, i \geq 1$, and since the law of each ζ_i is absolutely continuous with respect to the Lebesgue measure, we deduce the result by a straightforward induction.

We provide a sufficient condition for Assumption 3.2.2 to hold.

Lemma 4. *Assume that Λ can be written as follows :*

$$\Lambda(s, \lambda) = \mathbf{1}_A(s) \sum_{i=1}^n f_i(s) g_i(\lambda),$$

for all $(s, \lambda) \in [0, T] \times U^\lambda$ where :

- A is a Borel set of $[0, T]$ such that $s \mapsto \mathbf{1}_A(s)$ is càdlàg,
 - $(g_i)_{1 \leq i \leq n} : U^\lambda \mapsto \mathbb{R}_+$ are measurable and positive,
 - $(f_i)_{1 \leq i \leq n} : [0, T] \mapsto \mathbb{R}_+$ are càdlàg, positive and locally bounded by below.
- Then Assumption 3.2.2 holds.

Proof. Let $\epsilon > 0$. Since for each $1 \leq i \leq p$, f_i is right continuous and locally bounded by below, there exists $h_0^i > 0$ and $c_i > 0$ such that, for all $0 \leq h \leq h_0^i$, $c_i \leq f_i(s+h) \leq f_i(s) + \epsilon$. Let $h_0 := \min_{1 \leq i \leq p} h_0^i$ and $c := \min_{1 \leq i \leq p} c^i$. Then, for $0 \leq h \leq h_0$

$$\begin{aligned} \int_s^{s+h} \Lambda(u, \lambda) du &= \sum_{i=1}^p g_i(\lambda) \int_s^{s+h} \mathbf{1}_A(u) f_i(u) du \leq \sum_{i=1}^p g_i(\lambda) \int_s^{s+h} (f_i(s) + \epsilon) du \\ &\leq h \Lambda(s, \lambda) \left(1 + \frac{\epsilon \sum_{i=1}^p g_i(\lambda)}{\sum_{i=1}^p f_i(s) g_i(\lambda)} \right) \leq h \Lambda(s, \lambda) (1 + c^{-1} \epsilon). \end{aligned}$$

Evolution of the parameters γ_0 and v_0

We use the notations of Section 3.2.5. We define $M_s^{t, m^\gamma}(B) := \mathbb{E}_{\bar{m}}(\mathbf{1}_{\{\gamma \in B\}} | \mathcal{F}_s^{z, \bar{m}, \phi})$ and $M_s^{z, m^v, \phi}(B) := \mathbb{E}_{\bar{m}}(\mathbf{1}_{\{v \in B\}} | \mathcal{F}_s^{z, \bar{m}, \phi})$.

Between two jumps of the random Poisson measure, no information about the size distribution of the jumps is revealed, and therefore, about γ_0 . Whereas no information is revealed about v between two jumps from our control. In this case, both processes should remain constant. At the i -th Poisson jump of size u_i , the process M^{t, m^γ} should evolve according to the classical Bayes rule. The process $M^{z, m^v, \phi}$ should evolve at the time the j -th CAT bonds with the coupon c_j is issued according to, again, the Bayes rule.

Lemma 5. *Fix $s \geq 0$. Assume that, for almost all $\gamma \in U^\gamma$, the claim size distribution is dominated by some common measure μ_\circ . We have*

$$\begin{aligned} M_s^{t, m^\gamma}(B) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} &= M_{\zeta_i}^{t, m^\gamma}(B) \mathbf{1}_{\{\zeta_i \leq s < \zeta_{i+1}\}} \\ M_{\zeta_i}^{t, m^\gamma}(B) &= \mathfrak{M}_\gamma(M_{\zeta_i^-}^{t, m^\gamma}(B); U_i) \end{aligned}$$

in which

$$\mathfrak{M}_\gamma(m_\circ^\gamma; u_\circ) = \frac{\int_B q_\gamma(u_\circ | \gamma) dm_\circ^\gamma(\gamma)}{\int_U q_\gamma(u_\circ | \gamma) dm_\circ^\gamma(\gamma)}.$$

for almost all $(m_\circ^\gamma, u_\circ) \in \mathbf{M}^\gamma \times \mathbb{R}^{d^*}$, in which $q_\gamma(u_\circ | \gamma)$ is the conditional density, with respect to m_\circ^γ , of observing a jump of size u_\circ knowing $\{\gamma_0 = \gamma\}$.

Moreover,

$$\begin{aligned} M_s^{t,m^v,\phi}(B) \mathbf{1}_{\{\tau_j \leq s < \tau_{j+1}\}} &= M_{\tau_j}^{t,m^v,\phi}(B) \mathbf{1}_{\{\tau_j \leq s < \tau_{j+1}\}} \\ M_{\tau_j}^{t,m^v,\phi}(B) &= \mathfrak{M}_v(M_{\tau_j^-}^{t,m^v,\phi}(B); r_j, \tau_j, X_{\tau_j^-}^{z,\phi}, \alpha_j) \end{aligned}$$

in which

$$\mathfrak{M}_v(m_\circ^v; r_\circ, t_\circ, x_\circ, a_\circ) = \frac{\int_C q_v(r_\circ \mid t_\circ, x_\circ, a_\circ, v) dm_\circ^v(v)}{\int_U q_v(r_\circ \mid t_\circ, x_\circ, a_\circ, v) dm_\circ^v(v)}.$$

for almost all $(m_\circ^v, r_\circ, t_\circ, x_\circ, a_\circ) \in \mathbf{M}^v \times \mathbb{R} \times [0, T] \times \mathbb{R}^d \times \mathbf{A}$, in which $q_v(r_\circ \mid t_\circ, x_\circ, a_\circ, v)$ is the conditional density, with respect to m_\circ^v , of observing a jump of size r_\circ knowing $\{\tau_j = t_\circ, X_{\tau_j^-}^{z,\phi} = x_\circ, \alpha_i = a_\circ, v_0 = v\}$.

Proof. Use the same arguments as in the proof of Proposition 2.2.1 in Chapter 2.

3.2.6 Parametrization of the set \mathbf{M}^λ

Here, we have three measures on which will depend the value function. The one associated to the distribution of the jumps of the Poisson measure and the one from the unknown parameter evolve by a finite number of jumps on each bounded interval : the first one according to the jumps of the random Poisson process and the second one according to the impulses from the control. Those will not lead to deal with derivatives on the space of measures and a specific Itô formula nor generator of the diffusion. However, the measure associated to the parameter of the intensity evolves continuously. To deal with this, we will assume that the associated space of measures can be linked smoothly to a subset of \mathbb{R}^k for some $k \geq 1$.

Assumption 3.2.3. *We assume that there exists an open or compact set $\mathbf{P} \subset \mathbb{R}^k$, for some $k \in \mathbb{N}^*$, and a function*

$$\begin{aligned} f : \mathbf{P} &\rightarrow \mathbf{M}_\lambda \\ \theta &\mapsto f(\theta), \end{aligned}$$

which is a homeomorphism between \mathbf{P} and \mathbf{M}_λ .

Remark 3.2.5. *The process $P^{t,p}$ defined by :*

$$p = f^{-1}(m^\lambda), \quad P_s^{t,p} := f^{-1}(M_s^{t,m^\lambda}), \quad s \geq t,$$

remains, by construction, in \mathbf{P} . Moreover, Lemma 1 and 3 provide that M^{t,m^λ} only depends on the stopping times of the jumps of the random Poisson measure on $[0, t]$, thus, M^{t,m^λ} is $\mathcal{F}^N := s \mapsto \sigma(N(u, \cdot), t \leq u \leq s)$ -adapted. Then, from Assumption 3.2.3, $P^{t,p}$ is also \mathcal{F}^N -adapted. Moreover, $P^{t,p}$ does not depend on the size of the jumps.

According to Remark 3.2.5, we formulate the following assumption.

Assumption 3.2.4. *Let $P^{t,p}$ be the process defined in Remark 3.2.5.*

There exists Lipschitz maps $h_1 : [0, T] \times \mathbf{P} \rightarrow \mathbb{R}^k$ and $h_2 : [0, T] \times \mathbf{P} \rightarrow \mathbb{R}^k$ with linear growth such that

$$\begin{aligned} P^{t,p} &= p + \int_t^\cdot h_1(s, P_s^{t,p}) ds + \int_t^\cdot \int_{\mathbb{R}^{d^*}} h_2(s, P_{s-}^{t,p}) N(ds, du) \\ &= p + \int_t^\cdot h_1(s, P_s^{t,p}) ds + \int_t^\cdot h_2(s, P_{s-}^{t,p}) dN_s, \end{aligned}$$

where we use the notation : $dN_s := N(ds, \mathbb{R}^{d^*})$.

We provide two examples in which the Assumptions 3.2.3 and 3.2.4 are fulfilled.

Example 1. Assume that there exists a càdlàg function $h : [0, T] \mapsto \mathbb{R}^+$ such that $\Lambda(t, \lambda) = \lambda h(t)$ for all $t \geq 0, \lambda \in U^\lambda$. Set $m^\lambda = M_t^{t, m^\lambda} := \mathcal{G}(\alpha_t, \beta_t)$, where \mathcal{G} denotes the Gamma distribution. Then, if we define

$$(\alpha, \beta) := \left(\alpha_t + N - N_t, \beta_t + \int_t^\cdot h(u) du \right),$$

it follows that

$$M^{t, m^\lambda} = \mathcal{G}(\alpha, \beta),$$

and $P^{t,p} = (\alpha, \beta)$ satisfies Assumption 3.2.4.

Example 2. Assume that $U^\lambda := \{\lambda_1, \dots, \lambda_n\} \in (\mathbb{R}_+^*)^n$. Define, for $p = (p_i)_{1 \leq i \leq n}$ with $p_i > 0, 1 \leq i \leq n$, the distribution $\mathcal{D}(p)$ by :

$$\mathcal{D}(p) := \frac{\sum_{i=1}^n p_i \delta_{\lambda_i}}{\sum_{i=1}^n p_i}.$$

Set, for $s \geq t$,

$$P_s^{t,p,i} := p^i \left[\prod_{j=N_t+1}^{N_s} \Lambda(\zeta_j, \lambda_i) \right] e^{-\int_s^t \Lambda(u, \lambda_i) du}, \quad 1 \leq i \leq n.$$

Then $M^{t, m^\lambda} = \mathcal{D}(P^{t,p})$ and the process above satisfies the stochastic differential equation :

$$P^{t,p,i} = p^i - \int_t^\cdot P_s^{t,p,i} \Lambda(s, \lambda_i) ds + \int_t^\cdot P_{s-}^{t,p,i} [\Lambda(s, \lambda_i) - 1] dN_s, \quad 1 \leq i \leq n.$$

3.2.7 Gain function

Given $z = (t, x, c, l) \in \mathbf{Z}$ and $(p, m) \in \mathbf{P} \times \mathbf{M}$, the aim of the controller is to maximize the expected value of the gain functional

$$\phi \in \Phi^{z, \bar{m}} \mapsto G^{z,p,m}(\phi) := g(X_T^{z,\phi}, C_T^{z,\phi}, L_T^{z,\phi}, P_T^{t,p}, M_T^{z,m,\phi}),$$

in which g is a continuous and bounded function on $\mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$. Recall that $C_T^{z,\phi}$ is the random variable which represents all CAT bonds which are still active at the end and that $L_T^{z,\phi}$ is the elapsed time. If there is an initial cost when a CAT bond is issued⁴, recall the function H , one should not issue any CAT bond too close to the end, this allows to compensate it.

Given $\phi \in \Phi_\kappa^{z,\bar{m}}$, the expected gain is

$$J(z, p, m; \phi) := \mathbb{E}_{\bar{m}}[G^{z,p,m}(\phi)],$$

and

$$v(z, p, m) := \sup_{\phi \in \Phi_\kappa^{z,\bar{m}}} J(z, p, m; \phi)$$

is the corresponding value function. Note that v is bounded.

3.3 Value function characterization

In order to introduce the PDE, we first need the definition of a new function. Recall the set $\mathcal{J}(c; u)$ defined in (3.2.5). Then,

$$\mathfrak{F}(z; u) := \sum_{j \in \mathcal{J}(c; u)} F(t, x, c^j, l^j; u), \quad z := (t, x, c, l),$$

represents the total *payoff* for the ends of the CAT bonds according to the jump u . Recall that $\Pi : \mathbf{C} \rightarrow \mathbf{K}$ gives the number of running policies where $\mathbf{K} := \{0, \dots, \kappa\}$.

For ease of notation, we define $\mathbf{D} := [0, T] \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$, and for $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, $\mathbf{D}_{\mathbf{J}} := [0, T] \times \mathbb{R}^d \times \mathbf{CL}_{\mathbf{J}} \times \mathbf{P} \times \mathbf{M}$. To $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we denote by $\mathbf{1}_{\mathbf{J}} = (\mathbf{1}_{\mathbf{J}}(j))_{1 \leq j \leq \kappa}$ the vector in \mathbb{R}^κ in which $\mathbf{1}_{\mathbf{J}}(j) = 1$ if $j \in \mathbf{J}$, 0 else.

For $(z, p, m) \in \mathbf{D}$ and $u \in \mathbb{R}^{d^*}$, we introduce the operator \mathcal{I} defined, for all $(z, p, m) \in \mathbf{D}$, by :

$$\mathcal{I}[\varphi, u](z, p, m) := \varphi(t, x + \beta(t, x, u) + \mathfrak{F}(z; u), \mathfrak{C}_-(c, l; u), p + h_2(t, p), \mathfrak{M}_\gamma(m^\gamma; u), m^v).$$

Thus, the Dynkin operator associated to our problem with policies running in indexes \mathbf{J} is :

$$\begin{aligned} \mathcal{L}^{\mathbf{J}}\varphi &:= \partial_t \varphi + \langle \mu + \sum_{j=1}^{\kappa} \mathbf{1}_{\mathbf{J}}(j) \bar{C}(t, c^j), D\varphi \rangle + \langle \mathbf{1}_{\mathbf{J}}, D_l \varphi \rangle + \langle h_1, D_p \varphi \rangle + \\ &\int_{\mathbb{R}^d} [\mathcal{I}[\varphi, u] - \varphi] \Lambda(t, \lambda_0) \Upsilon(\gamma_0, du), \end{aligned}$$

in which recall that Υ denotes the size distribution of the jumps of the random Poisson measure N . Moreover, we introduce :

$$\mathcal{L}_*^{\mathbf{J}}\varphi := \mathbb{E}_{\bar{m}}[\mathcal{L}^{\mathbf{J}}\varphi],$$

4. To issue a CAT bond has a cost.

and

$$\begin{aligned}\mathbf{D}_\circ &:= [0, T) \times \mathbb{R}^d \times \mathbf{CL}_\mathbf{J} \times \mathbf{P} \times \mathbf{M}, \\ \mathbf{D}_T &:= \{T\} \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}.\end{aligned}$$

Then, we expect that v is a viscosity solution of, for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ and non-empty $\mathbf{J}' \subset \mathbf{J}$,

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} \left[-\mathcal{L}_*^{\mathbf{K}} \varphi \right] + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \min \{ -\mathcal{L}_*^{\mathbf{J}} \varphi, \varphi - \mathcal{K} \varphi \} = 0 \quad \text{on } \mathbf{D}_\circ \quad (3.3.1)$$

$$\varphi = \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} g + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \max \{ \mathcal{K} g, g \} \quad \text{on } \mathbf{D}_T \quad (3.3.2)$$

$$\lim_{l' \rightarrow \mathfrak{L}_\mathbf{J}^{\mathbf{J}'}(l)} \varphi(\cdot, c, l', \cdot) = \max \{ \varphi(\cdot, \mathfrak{C}_-^\ell[c, \mathfrak{L}_\mathbf{J}^{\mathbf{J}'}(l)], \cdot), \mathcal{K} \varphi(\cdot, \mathfrak{C}_-^\ell[c, \mathfrak{L}_\mathbf{J}^{\mathbf{J}'}(l)], \cdot) \} \quad \text{on } \mathbf{D} \setminus \mathbf{D}_\emptyset \quad (3.3.3)$$

in which, for $(z, p, m) \in \mathbf{D}_\mathbf{J}$ and $\phi^a \in \Phi^{z, \bar{m}}$ a control such that $\{\tau_1^{\phi^a} = t, \alpha_1^{\phi^a} = a\}$ holds with probability one,

$$\mathcal{K} \varphi := \sup_{a \in \mathbf{A}} \mathcal{K}^a \varphi, \quad \mathcal{K}^a \varphi(z, p, m) := \mathbb{E}_{\bar{m}}[\varphi(Z_{t+}^{z, \phi^a}, p, M_{t+}^{z, m, \phi^a})];$$

and, for $\mathbf{J}' \subset \mathbf{J}$,

$$\mathfrak{L}_\mathbf{J}^{\mathbf{J}'} : [0, \ell]^\mathbf{J} \rightarrow [0, \ell]^\mathbf{J} \quad (3.3.4)$$

$$(l_j)_{1 \leq j \leq \kappa} \mapsto (\ell \mathbf{1}_{\{j \in \mathbf{J}'\}} + l_j \mathbf{1}_{\{j \notin \mathbf{J}'\}})_{1 \leq j \leq \kappa}, \quad (3.3.5)$$

where $[0, \ell]^\mathbf{J} := \{l \in ([0, \ell] \cup \partial)^\kappa : l_j \neq \partial \Leftrightarrow j \in \mathbf{J}\}$.

Remark 3.3.1. *Note that the above corresponds to the definition of a system of PDEs linked by the common boundary conditions.*

We now define what is a viscosity solution of (3.3.1)-(3.3.2)-(3.3.3). For $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we define :

$$\mathcal{C}_\mathbf{J}^1 := \left\{ \varphi : \mathbf{D}_\mathbf{J} \mapsto \mathbb{R}, \quad \varphi \in C^{1,1,(0,1),1,0}(\mathbf{D}_\mathbf{J}) \right\}.$$

Definition 3.3.1. *We say that a upper-semicontinuous function u on \mathbf{D} is a viscosity sub-solution of (3.3.1)-(3.3.2)-(3.3.3) if, for any $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, $(z_\circ, p_\circ, m_\circ) \in \mathbf{D}_\mathbf{J}$, and $\varphi \in \mathcal{C}_\mathbf{J}^1$ such that $\max_{\mathbf{D}_\mathbf{J}}(u - \varphi) = (u - \varphi)(z_\circ, p_\circ, m_\circ) = 0$ we have, if $t_\circ < T$,*

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} \left[-\mathcal{L}_*^{\mathbf{K}} \varphi \right] + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \min \{ -\mathcal{L}_*^{\mathbf{J}} \varphi, \varphi - \mathcal{K} u \} (z_\circ, p_\circ, m_\circ) \leq 0,$$

if $\mathbf{J} \neq \emptyset$, for any non-empty $\mathbf{J}' \in \mathcal{P}(\mathbf{J})$, with $d_\circ = (t_\circ, x_\circ, c_\circ, \mathfrak{L}_\mathbf{J}^{\mathbf{J}'}(l_\circ), p_\circ, m_\circ)$ and $d'_\circ = (t_\circ, x_\circ, \mathfrak{C}_-^\ell[c_\circ, \mathfrak{L}_\mathbf{J}^{\mathbf{J}'}(l_\circ)], p_\circ, m_\circ)$,

$$\limsup_{(z,p,m) \rightarrow d_\circ} u(z, p, m) \leq \max \{ u(d'_\circ), \mathcal{K} u(d'_\circ) \},$$

and, if $t_\circ = T$,

$$u(z_\circ, p_\circ, m_\circ) \leq \left\{ \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} g + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \max(\mathcal{K} g, g) \right\} (x_\circ, c_\circ, l_\circ, p_\circ, m_\circ).$$

We say that a lower-semicontinuous function v on \mathbf{D} is a viscosity super-solution of (3.3.1)-(3.3.2)-(3.3.3) if, for any $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, $(z_o, p_o, m_o) \in \mathbf{D}_{\mathbf{J}}$, and $\varphi \in \mathcal{C}_{\mathbf{J}}^1$ such that $\min_{\mathbf{D}_{\mathbf{J}}}(v - \varphi) = (v - \varphi)(z_o, p_o, m_o) = 0$ we have, if $t_o < T$,

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} \left[-\mathcal{L}_{\star}^{\mathbf{K}} \varphi \right] + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \min \{ -\mathcal{L}_{\star}^{\mathbf{J}} \varphi, \varphi - \mathcal{K}v \} (z_o, p_o, m_o) \geq 0,$$

if $\mathbf{J} \neq \emptyset$, for any non-empty $\mathbf{J}' \in \mathcal{P}(\mathbf{J})$, with $d_o = (t_o, x_o, c_o, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}}(l_o), p_o, m_o)$ and $d'_o = (t_o, x_o, \mathfrak{C}_{-}^{\ell}[c_o, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}}(l_o)], p_o, m_o)$,

$$\liminf_{(z,p,m) \rightarrow d_o} v(z, p, m) \geq \max \{ v(d'_o), \mathcal{K}v(d'_o) \}$$

and, if $t_o = T$,

$$v(z_o, p_o, m_o) \geq \left\{ \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} g + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \max(\mathcal{K}g, g) \right\} (x_o, c_o, l_o, p_o, m_o).$$

We say that a function u is a viscosity solution of (3.3.1)-(3.3.2)-(3.3.3) if its upper-semicontinuous envelope u^* is a viscosity sub-solution and its lower-semicontinuous envelope u_* is a viscosity super-solution of (3.3.1)-(3.3.2)-(3.3.3).

To ensure that the above operator is continuous, we first assume that :

Assumption 3.3.1. $\mathcal{K}\varphi$ is upper- (resp. lower-) semicontinuous, for all upper- (resp. lower-) semicontinuous bounded function φ .

A sufficient condition for Assumption 3.3.1 to hold is provided in Chapter 2, see the discussion after equation (2.3.6).

In order to ensure that $\mathcal{L}_{\star}^{\mathbf{J}}$ is continuous for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we make the following assumption.

Assumption 3.3.2. We assume that

- The functions F and \mathfrak{M}_{γ} are continuous;
- The stochastic kernel $\gamma \mapsto \Upsilon(\gamma, du)$ is continuous;
- There map $(t, \lambda) \mapsto \Lambda(t, \lambda)$ is continuous.

Lemma 6. Assume that Assumption 3.3.2 holds. Then, for all $(c, m) \in \mathbf{C} \times \mathbf{M}$, with $\mathbf{J} := \{j \in \mathbf{K} : c_j \neq \partial\}$, and for all bounded function $\varphi \in \mathcal{C}_{\mathbf{J}}^1$, the operator $\mathcal{L}_{\star}^{\mathbf{J}}\varphi$ is continuous.

Proof. Let $(c, m) \in \mathbf{C} \times \mathbf{M}$ and \mathbf{J} defined as above. Recall that

$$\begin{aligned} \mathcal{L}_{\star}^{\mathbf{J}}\varphi &= \partial_t \varphi + \langle \mu + \sum_{j=1}^{\kappa} \mathbf{1}_{\mathbf{J}}(j) \bar{C}(t, c^j), D\varphi \rangle + \langle \mathbf{1}_{\mathbf{J}}, D_l \varphi \rangle + \langle h_1, D_p \varphi \rangle \\ &\quad + \mathbb{E}_{\bar{m}} \left[\int_{\mathbb{R}^d} [\mathcal{I}[\varphi, u] - \varphi] \Lambda(t, \lambda_0) \Upsilon(\gamma_0, du) \right]. \end{aligned}$$

For the first line above, since all involved functions are continuous, the operator is continuous. For the second line, since φ is bounded, one easily checks that the expected value with respect to (λ, γ) is well defined and one can apply Fubini's theorem. This is rewritten :

$$\bar{\Lambda}(t, p) \int_{U^\gamma} \left[\int_{\mathbb{R}^d} [\mathcal{I}[\varphi, u] - \varphi] \Upsilon(\gamma, du) \right] dm^\gamma(\gamma)$$

with $\bar{\Lambda}(t, p) := \int_{U^\lambda} \Lambda(t, \lambda) dm^\lambda(\lambda)$ which is continuous, see [17, Proposition 7.30 p145].

Now, remark that the function integrated through $\Upsilon(\gamma, du)$ with $\gamma \in U^\gamma$ fixed is continuous by definition. Since the stochastic kernel $\gamma \mapsto \Upsilon(\gamma, du)$ is assumed to be continuous, we get again from [Proposition 7.30 p145] that the function integrated through m^γ is continuous and bounded. And then, the operator is continuous.

We now assume that we have a comparison principle. A sufficient condition is provided in Proposition 3.5.1 below.

Assumption 3.3.3. *Let U (resp. V) be a upper- (resp. lower-) semicontinuous bounded viscosity sub- (resp. super-) solution of (3.3.1)-(3.3.2)-(3.3.3). Assume further that $U \leq V$ on \mathbf{D}_T . Then, $U \leq V$ on \mathbf{D} .*

Theorem 1. *The function v is the unique viscosity solution of (3.3.1)-(3.3.2)-(3.3.3).*

3.4 Viscosity solution properties

This part is dedicated to the proof of the viscosity solution characterization of Theorem 1. We start with the sub-solution property and continue with the super-solution property. The main difficulty relies on the fact that the filtration depends on the initial data. The results can be obtained along the lines of Chapter 2.

3.4.1 Sub-solution property

Proposition 3.4.1. *The function v is a viscosity sub-solution of (3.3.1)-(3.3.2)-(3.3.3).*

The proof of this proposition, as usual, relies on a dynamic programming principle. For this part, the dependency of the filtration on the initial data is not problematic as it only requires a conditioning argument. We have the following result :

Proposition 3.4.2. *Fix $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ and $(z, p, m) \in \mathbf{D}_{\mathbf{J}}$, and let θ be the first exit time of $(Z^{z, \phi^0}, P^{t, p})$ from a Borel set $B \subset \mathbf{D}_{\mathbf{J}}$ containing (z, p, m) where $\phi^0 \in \Phi^{z, m}$ is a control such that $\tau_1^{\phi^0} > t$. Then,*

$$v(z, p, m) \leq \sup_{\phi \in \Phi_{\geq t}^{z, m}} \mathbb{E}_m \left[v^*(Z_\theta^{z, \phi}, P_\theta^{t, p}, m) \mathbf{1}_{\{\theta < \tau_1^\phi\}} + \mathcal{K}^{\alpha_1^\phi} v^*(Z_{\tau_1^\phi -}^{z, \phi}, P_{\tau_1^\phi -}^{t, p}, m) \mathbf{1}_{\{\theta \geq \tau_1^\phi\}} \right] \quad (3.4.1)$$

in which $z := (t, x, c, l)$, $\Phi_{\geq t}^{z, \bar{m}} := \{\phi \in \Phi_{\kappa}^{z, \bar{m}} : \tau_1^\phi \geq t\}$.

Proof. It suffices to follow the arguments of Proposition 2.4.2 in Chapter 2.

We now prove Proposition 3.4.1.

Proof.

Since, for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, the operator $\mathcal{L}_*^{\mathbf{J}}$ is continuous, the proof of (3.3.1) and (3.3.2) can be obtained by using the same arguments as in Proposition 2.4.1 in Chapter 2.

To prove (3.3.3), one can use the same arguments used in order to prove (3.3.2).

3.4.2 Super-solution property

Because of the non-trivial dependence of the filtration $\mathbb{F}^{z,m,\phi}$ with respect to the initial data, in order to prove the super-solution property associated to Theorem 1, we shall use a discrete version of our impulse control problem, as in Chapter 2. We shall show that the limit problem is a super-solution of (3.3.1)-(3.3.2)-(3.3.3). Proposition 3.4.1 and the comparison assumption will show that the limit problem is v.

We shall use a dynamic programming principle in some discrete form defined below.

Proposition 3.4.3. *Fix $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ and $(z, p, m) \in \mathbf{D}_{\mathbf{J}}$. Let $\Phi_n^{z,\bar{m}}$ be the subset of elements of $\Phi_n^{z,\bar{m}}$ such that the stopping times $\tau_i^\phi, i \geq 1$ are valued in $\{t\} \cup \pi_n \cap [t, T]$ with $\pi_n := \{kT/2^n; 0 \leq k \leq 2^n\}$. The corresponding value function is :*

$$v_n(z, p, m) := \sup_{\phi \in \Phi_n^{z,\bar{m}}} J(z, p, m; \phi), \quad (z, p, m) \in \mathbf{D}.$$

Let $(\theta^\phi, \phi \in \Phi_n^{z,m})$ be such that θ^ϕ is a $\mathbb{F}^{z,\bar{m},\phi}$ -stopping time valued in $\{t\} \cup \pi_n \cap [t, T]$. Then,

$$v_n(z, p, m) = \sup_{\phi \in \Phi_n^{z,\bar{m}}} \mathbb{E}_{\bar{m}} \left[v_n(Z_{\theta^\phi}^{z,\phi}, P_{\theta^\phi}^{t,p}, M_{\theta^\phi}^{z,m,\phi}) \right].$$

Proof. It suffices to follow the arguments of Lemma 2.4.1, Proposition 2.4.3 and Corollary 2.4.1 in Chapter 2.

We now consider the limit $n \rightarrow +\infty$. Let us set, for $(z, p, m) \in \mathbf{D}$,

$$v_\circ(z, p, m) := \liminf_{(z', p', m', n) \rightarrow (z, p, m, +\infty)} v_n(z', p', m').$$

Proposition 3.4.4. *The function v_\circ is a viscosity super-solution of (3.3.1)-(3.3.2)-(3.3.3).*

Proof.

The equations (3.3.1) and (3.3.2) can be obtained by using Proposition 3.4.3 and following the arguments in the proof of Proposition 2.4.4 in Chapter 2.

We now prove the boundary condition (3.3.3).

Step 1. Fix $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ and $(z, p, m) \in \mathbf{D}_{\mathbf{J}}$.

Let $n_k \rightarrow +\infty$ and $(z_k, p_k, m_k) \rightarrow (z, p, m)$ such that $v_{n_k}(z_k, p_k, m_k) \rightarrow v_\circ(z, p, m)$. Let $k_\circ \geq 1$ and define the lower semi-continuous function φ_{k_\circ} as in the proof of Proposition 2.4.4 in Chapter 2. Then, from Proposition 3.4.3, with $\phi^0 \in \Phi^{t,x,m}$ a control such that $\tau_1^{\phi^0} > T$, we get for $k \geq k_\circ$

$$v_{n_k}(z_k, p_k, m_k) \geq \mathbb{E}_{\bar{m}} \left[\varphi_{k_\circ}(Z_{\theta^{\phi^0}}^{z_k, \phi^0}, P_{\theta^{\phi^0}}^{t, p_k}, M_{\theta^{\phi^0}}^{z_k, m_k, \phi^0}) \right].$$

Then, $k \rightarrow +\infty$ leads to

$$v_\circ(z, p, m) \geq \mathbb{E}_{\bar{m}} \left[\varphi_{k_\circ}(Z_{\theta^{\phi^0}}^{z, \phi^0}, P_{\theta^{\phi^0}}^{t, p}, M_{\theta^{\phi^0}}^{z, m, \phi^0}) \right]$$

and, again from the proof of Proposition 2.4.4 in Chapter 2, we get that $\lim_{k_o \rightarrow +\infty} \varphi_{k_o} \geq v_o$. By Fatou's lemma we have

$$v_o(z, p, m) \geq \mathbb{E}_{\bar{m}} \left[v_o(Z_{\theta\phi^0}^{z, \phi^0}, P_{\theta\phi^0}^{t, p}, M_{\theta\phi^0}^{z, m, \phi^0}) \right].$$

Step 2. Now fix $\mathbf{J}' \subset \mathbf{J}$ and $(z_o, p_o, m_o) \in \mathbf{D}_{\mathbf{J}}$. Let $k \rightarrow +\infty$ and $(z_k, p_k, m_k) \rightarrow (t_o, x_o, c_o, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o), p_o, m_o)$ such that

$$v_o(z_k, p_k, m_k) \rightarrow \liminf_{(z, p, m) \rightarrow (t_o, x_o, c_o, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o), p_o, m_o)} v_o(z, p, m).$$

We introduce $h_k := \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o) - l_k$. Then, for k_o large enough, we can find $\varepsilon > 0$ such that $\sup_{k \geq k_o} \max_{j \in \mathbf{J}'} h_k^j < \varepsilon < \inf_{k \geq k_o} \max_{j \in \mathbf{J} \setminus \mathbf{J}'} (\ell - l_k^j)$. Then, for $k \geq k_o$,

$$v_o(t_k, x_k, c_k, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o) - h_k, p_k, m_k) \geq \mathbb{E} \left[v_o(Z_{t+\varepsilon}^{z_k, \phi^0}, P_{t+\varepsilon}^{t_k, p_k}, M_{t+\varepsilon}^{z_k, m_k, \phi^0}) \right].$$

Now, we send $k \rightarrow +\infty$, since the functions in the diffusion are Lipschitz, using Fatou's lemma leads to

$$\lim_{k \rightarrow +\infty} v_o(t_k, x_k, c_k, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o) - h_k, p_k, m_k) \geq \mathbb{E} \left[v_o(Z_{t+\varepsilon}^{z, \phi^0}, P_{t+\varepsilon}^{t, p}, M_{t+\varepsilon}^{z, m, \phi^0}) \right].$$

Since, under the control ϕ^0 , the processes X , P and M are driven here by the random Poisson measure with finite activity, they satisfy the stochastic continuity property. Moreover, since the probability of observing a jump decreases to 0 when $\varepsilon \rightarrow 0$, one easily shows that,

$$\lim_{k \rightarrow +\infty} v_o(t_k, x_k, c_k, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o) - h_k, p_k, m_k) \geq v_o(t_o, x_o, \mathfrak{C}_{-}^{\ell}[c_o, \mathfrak{L}_{\mathbf{J}'}^{\mathbf{J}'}(l_o)], p_o, m_o),$$

by using the fact that v_o is bounded and the definition of the process C and L after the end of one or several CAT bonds.

Step 3. In order to show the second inequality, repeat Step 1. and Step 2. using, instead of ϕ^0 , a control $\phi^a \in \Phi_{\kappa}^{z, \bar{m}}$ such that $\{\tau_1^{\phi^a} = t, \alpha_1^{\phi^a} = a, \tau_2^{\phi^a} > T\}$ holds with probability one.

We now prove Theorem 1.

Proof. [Proof of Theorem 1.] We already know that v^* and v_o are respectively a bounded sub- and super-solution of (3.3.1)-(3.3.2)-(3.3.3). Then, under Assumption 3.3.3, $v^* \leq v_o$. Moreover, by construction, $v_o \leq v \leq v^*$. Then, v is continuous and the unique solution of (3.3.1)-(3.3.2)-(3.3.3).

Remark 3.4.1. *If we denote by $\mathcal{S}_{\mathbf{K}}$ the set of permutation of $\{1 \leq k \leq \kappa\}$, then, by symmetry,*

$$v(z, p, m) = v(t, x, (c, l) \circ \Sigma, p, m)$$

for each $\Sigma \in \mathcal{S}_{\mathbf{K}}$, $(z, p, m) \in \mathbf{D}$. From a numerical point of view, this allows to only compute the value function on $\kappa + 1$ different dimensions for the CAT bonds space \mathbf{CL} on which we can order them, instead of 2^{κ} different dimensions with no order.

3.5 A sufficient condition for the comparison

In this section, we provide a sufficient condition for Assumption 3.3.3 to hold.

Proposition 3.5.1. *Assumption 3.3.3 holds whenever there exists a function Ψ on $[0, T) \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P} \times \mathbf{M}$ such that, for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$,*

- (i) $(t, x, l, p) \mapsto \Psi(t, x, c, l, p, m) \in C^{1,1,1,1}([0, T) \times \mathbb{R}^d \times [0, \ell) \times \mathbf{P})$ for all $(c, m) \in \mathbf{C} \times \mathbf{M}$,
- (ii) $\varrho\Psi \geq \mathcal{L}_*^{\mathbf{J}}\Psi$ on $\mathbf{D}_{\mathbf{J}}$ for some $\varrho > 0$,
- (iii) $\Psi - \mathcal{K}\Psi \geq \delta$ on $\mathbf{D}_{\mathbf{J}}$ for some $\delta > 0$,
- (iv) $\Psi \geq \max(\mathcal{K}\tilde{g}, \tilde{g})$ on $\mathbb{R}^d \times \mathbf{CL}_{\mathbf{J}} \times \mathbf{P} \times \mathbf{M}$ with $\tilde{g}(t, \cdot) := e^{gt}g(t, \cdot)$ and ϱ is defined in (ii),
- (v) $\liminf_{\nu \rightarrow \mathfrak{L}_{\mathbf{J}}^{\nu}(l)} \Psi(\cdot, c, l', \cdot) - \Psi(\cdot, \mathfrak{C}_{-}^{\ell}(c, \mathfrak{L}_{\mathbf{J}}^{\nu}(l)), \cdot) \geq 0$ for all $\mathbf{J}' \subset \mathbf{J}$,
- (vi) $\Psi^- \leq \bar{\Psi}(x) = o(\|x\|^2)$ as $\|x\|^2 \rightarrow +\infty$ for some $\bar{\Psi} : \mathbb{R}^d \rightarrow \mathbb{R}$.

Proof.

Step 1. As usual, we shall argue by contradiction. We assume that there exists some $\mathbf{J}_0 \in \mathcal{P}(\mathbf{K})$ and some $(z_0, p_0, m_0) \in \mathbf{D}_{\mathbf{J}_0}$ such that $(U - V)(z_0, p_0, m_0) > 0$, in which U is a sub-solution of (3.3.1)-(3.3.2)-(3.3.3) and V is a super-solution of (3.3.1)-(3.3.2)-(3.3.3). Recall the definition of Ψ , ϱ and \tilde{g} in Proposition 3.5.1. We set $\tilde{u}(t, \cdot) = e^{gt}U(t, \cdot)$ and $\tilde{v}(t, \cdot) = e^{gt}V(t, \cdot)$ for all $(t, \cdot) \in \mathbf{D}_{\mathbf{J}}$ for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$. Then, there exists $\lambda \in (0, 1)$ such that

$$(\tilde{u} - \tilde{v}^{\lambda})(z_0, p_0, m_0) > 0, \quad (3.5.1)$$

in which $\tilde{v}^{\lambda} := (1 - \lambda)\tilde{v} + \lambda\Psi$. Note that \tilde{u} and \tilde{v} are sub and super-solution on $\mathbf{D}_{\mathbf{J}}$ of

$$\min \{ \varrho\varphi - \mathcal{L}_*^{\mathbf{J}}\varphi, \varphi - \mathcal{K}\varphi \} = 0$$

for each $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, with the boundary conditions

$$\mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}}(\varphi(T, \cdot) - \tilde{g}) + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \min \{ \varphi(T, \cdot) - \tilde{g}, \varphi(T, \cdot) - \mathcal{K}\tilde{g} \} = 0, \quad (3.5.2)$$

and

$$\lim_{\nu \rightarrow \mathfrak{L}_{\mathbf{J}}^{\nu}(l)} \varphi(\cdot, c, l', \cdot) = \varphi(\cdot, \mathfrak{C}_{-}^{\ell}[c, \mathfrak{L}_{\mathbf{J}}^{\nu}(l)], \cdot) \quad \forall \mathbf{J}' \subset \mathbf{J}, (c, l) \in \mathbf{CL}_{\mathbf{J}} \quad (3.5.3)$$

Step 2. Let $d_{\mathbf{M}}$ be a metric on \mathbf{M} compatible with the weak topology. For $(t, x, y, c, l, p, q, m) \in \mathbf{D}' := [0, T) \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{CL} \times \mathbf{P}^2 \times \mathbf{M}$, we set :

$$\begin{aligned} \Gamma_{\varepsilon}(t, x, y, c, l, p, q, m) := & \tilde{u}(t, x, c, l, p, m) - \tilde{v}^{\lambda}(t, y, c, l, q, m) \\ & - \varepsilon \left(\|x\|^2 + \|y\|^2 + \mathfrak{d}(c, l) + \|p\|^2 + \|q\|^2 + d_{\mathbf{M}}(m) \right) \end{aligned} \quad (3.5.4)$$

with $\varepsilon > 0$ small enough such that $\Gamma_{\varepsilon}(t_0, x_0, x_0, c_0, l_0, p_0, p_0, m_0) > 0$. Although $[0, \ell)$ is not closed, note that the supremum is achieved for some $\mathbf{J}_{\varepsilon} \in \mathcal{P}(\mathbf{K})$ by some $(t_{\varepsilon}, x_{\varepsilon}, y_{\varepsilon}, c_{\varepsilon}, l_{\varepsilon}, p_{\varepsilon}, q_{\varepsilon}, m_{\varepsilon}) \in \mathbf{D}_{\mathbf{J}_{\varepsilon}}$. This follows from the upper-semicontinuity of Γ_{ε} , the fact that \tilde{u} , $-\tilde{v}$ and $-\Psi$ are bounded from above, and by the fact that

$$\limsup_{\nu \rightarrow \mathfrak{L}_{\mathbf{J}}^{\nu}(l)} (\tilde{u} - \tilde{v}^{\lambda})(\cdot, c, l', \cdot) \leq (\tilde{u} - \tilde{v}^{\lambda})(\cdot, \mathfrak{C}_{-}^{\ell}(c, \mathfrak{L}_{\mathbf{J}}^{\nu}(l)), \cdot).$$

For $(t, x, y, c, l, p, q, m) \in \mathbf{D}'$, we set

$$\Theta_\varepsilon^n(t, x, y, c, l, p, q, m) = \Gamma_\varepsilon(t, x, y, c, l, p, q, m) - n \left(\|x - y\|^2 + \|p - q\|^2 \right).$$

Again, there is $(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \in \mathbf{D}'$ such that

$$\sup_{\mathbf{D}'} \Theta_\varepsilon^n = \Theta_\varepsilon^n(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon).$$

It is standard to show that, after possibly considering a subsequence,

$$(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \rightarrow (\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{q}_\varepsilon, \hat{m}_\varepsilon) \in \mathbf{D}',$$

$$n \left(\|x_n^\varepsilon - y_n^\varepsilon\|^2 + \|p_n^\varepsilon - q_n^\varepsilon\|^2 \right) \rightarrow 0, \text{ and}$$

$$\Theta_\varepsilon^n(t_n^\varepsilon, x_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \rightarrow \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{y}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{q}_\varepsilon, \hat{m}_\varepsilon) = \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, y_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, q_\varepsilon, m_\varepsilon), \quad (3.5.5)$$

see e.g. [31, Lemma 3.1]. Moreover, up to a subsequence, there exists $n_0 \in \mathbb{N}$, such that, for all $n \geq n_0$, $(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) \in D_{\mathbf{J}_\varepsilon}$ and $(t_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \in D_{\mathbf{J}_\varepsilon}$.

Step 3. We first assume that, up to a subsequence, $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) \leq 0$, for $n \geq 1$. Then, it follows from the supersolution property of \tilde{u} and Condition (iii) that

$$\begin{aligned} & \tilde{u}(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \leq \\ & \mathcal{K}\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) - \lambda\delta. \end{aligned}$$

Passing to the lim sup and using (3.5.5) and (3.3.1), we obtain

$$(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon) + \lambda\delta \leq \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon)$$

Now let us observe that

$$\begin{aligned} \sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) &= \lim_{\varepsilon \rightarrow 0} \sup_{(t, x, c, l, p, m) \in \mathbf{D}} \Gamma_\varepsilon(t, x, x, c, l, p, p, m) \\ &= \lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, p_\varepsilon, m_\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0} \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon), \end{aligned} \quad (3.5.6)$$

in which the last identity follows from (3.5.5). Combined with the above inequality, this shows that $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) + \lambda\delta \leq \limsup_{\varepsilon \rightarrow 0} \mathcal{K}(\tilde{u} - \tilde{v}^\lambda)(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon)$, which leads to a contradiction for ε small enough.

Step 4. We now show that there is a subsequence such that $t_n^\varepsilon < T$ for all $n \geq 1$. If not, one can assume that $t_n^\varepsilon = T$. If, up to a subsequence, one can have $\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) \leq \tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon)$, then it follows from (3.5.2) and Condition (iv) that,

$$\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \leq \tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon).$$

Hence,

$$\Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, x_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, p_\varepsilon, m_\varepsilon) \leq \tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon),$$

and (3.5.5) with (3.5.6) leads to $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \leq 0$, a contradiction. If, up to a subsequence, $\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}\tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon)$, by (3.5.2) and Condition (iv),

$$\tilde{u}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \leq \mathcal{K}\tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon).$$

Hence,

$$\Gamma_\varepsilon(t_\varepsilon, x_\varepsilon, c_\varepsilon, l_\varepsilon, p_\varepsilon, m_\varepsilon) \leq \mathcal{K}\tilde{g}(T, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \mathcal{K}\tilde{g}(T, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon),$$

and combining Assumption 3.3.1 with (3.5.5) and (3.5.6) leads to $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \leq 0$, the same contradiction.

Step 5. In view of step 2, 3, 4, one can assume that $t_n^\varepsilon < T$, $(\tilde{u} - \mathcal{K}\tilde{u})(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) > 0$ and $(c_n^\varepsilon, l_n^\varepsilon) \in \mathbf{CL}_{\mathbf{J}^\varepsilon}$ for all $n \geq 1$. Using Ishii's Lemma and following standard arguments, see Theorem 8.3 and the discussion after Theorem 3.2 in [31], we deduce from the sub- and supersolution viscosity solutions property of \tilde{u} and \tilde{v}^λ , and the Lipschitz continuity assumptions on μ, σ and β , that

$$\begin{aligned} & \varrho \left(\tilde{u}(t_n^\varepsilon, x_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, p_n^\varepsilon, m_n^\varepsilon) - \tilde{v}^\lambda(t_n^\varepsilon, y_n^\varepsilon, c_n^\varepsilon, l_n^\varepsilon, q_n^\varepsilon, m_n^\varepsilon) \right) \leq \\ & C \left(n(\|x_n^\varepsilon - y_n^\varepsilon\|^2 + \|p_n^\varepsilon - q_n^\varepsilon\|^2) + \varepsilon(1 + \|x_n^\varepsilon\|^2 + \|y_n^\varepsilon\|^2 + \|p_n^\varepsilon\|^2 + \|q_n^\varepsilon\|^2) \right), \end{aligned}$$

for some $C > 0$, independent of n and ε . In view of (3.5.4) and (3.5.5), we get

$$\varrho \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon) \leq 2C\varepsilon \left(1 + \|\hat{x}_\varepsilon\|^2 + \|\hat{p}_\varepsilon\|^2 \right). \quad (3.5.7)$$

We shall prove in next step that the right-hand side of (3.5.7) goes to 0 as $\varepsilon \rightarrow 0$, up to a subsequence. Combined with (3.5.6), this leads to a contradiction of (3.5.1).

Step 6. We conclude the proof by proving the claim used above. First note that we can always construct a sequence $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{c}_\varepsilon, \tilde{l}_\varepsilon, \tilde{p}_\varepsilon, \tilde{m}_\varepsilon)_{\varepsilon>0}$ such that

$$\begin{aligned} & \Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{c}_\varepsilon, \tilde{l}_\varepsilon, \tilde{p}_\varepsilon, \tilde{m}_\varepsilon) \rightarrow \sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \quad \text{and} \\ & \varepsilon \left(\|\tilde{x}_\varepsilon\|^2 + \mathfrak{d}(\tilde{c}_\varepsilon, \tilde{l}_\varepsilon) + \|\tilde{p}_\varepsilon\|^2 + d_{\mathbf{M}}(\tilde{m}_\varepsilon) \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

By (3.5.5), $\Gamma_\varepsilon(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{c}_\varepsilon, \tilde{l}_\varepsilon, \tilde{p}_\varepsilon, \tilde{m}_\varepsilon) \leq \Gamma_\varepsilon(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{c}_\varepsilon, \hat{l}_\varepsilon, \hat{p}_\varepsilon, \hat{m}_\varepsilon)$. Hence, $\sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) \leq \sup_{\mathbf{D}}(\tilde{u} - \tilde{v}^\lambda) - 2 \liminf_{\varepsilon \rightarrow 0} \varepsilon (\|\hat{x}_\varepsilon\|^2 + \|\hat{p}_\varepsilon\|^2)$.

3.6 Numerical Scheme

We let h_o be a time-discretization step such that both T/h_o and ℓ/h_o are an integer. In order to ensure the existence of such a h_o , we shall assume that $(T/\ell) \in \mathbb{Q}_+^*$ which does not appear as a restriction from a practical point of view. We set $\mathbf{T}^{h_o} := \{t_i^{h_o} := ih_o, i \leq T/h_o\}$ and, for $\mathbf{J} \in \mathcal{P}(\mathbf{K})$, we set $\mathbf{L}_{\mathbf{J}}^{h_o} = \prod_{j=1}^{\kappa} (\partial \mathbf{1}_{\mathbf{J}^c}(j) + \mathbf{L}^{h_o} \mathbf{1}_{\mathbf{J}}(j))$ in which $\mathbf{L}^{h_o} := \{l_i^{h_o} := ih_o, i < \ell/h_o\}$.

The space \mathbb{R}^d is discretized with a space step h_\star on a rectangle $[-c, c]^d$ containing $N_{h_\star}^x$ points on each direction. The corresponding set is denoted by $\mathbf{X}_c^{h_\star}$. Recall that \mathbf{P} is a subset of \mathbb{R}^d . We again discretise \mathbb{R}^d with the same step space h_\star on a rectangle $[-c, c]^d$ containing $N_{h_\star}^p$ points. The corresponding set is denoted by $\mathbf{P}_c^{o, h_\star}$, thus, the discretization of \mathbf{P} is $\mathbf{P}_c^{h_\star} := \mathbf{P}_c^{o, h_\star} \cap \mathbf{P}$.

We set $h = (h_o, h_\star)$. The first order derivatives $(\partial_t \varphi)$, $(\partial_{x_i} \varphi)_{i \leq d}$, $(\partial_{l_i} \varphi)_{i \leq \kappa}$ and $(\partial_{p_i} \varphi)_{i \leq d}$ are approximated by using the standard up-wind approximations :

$$\begin{aligned} \Delta_i^{h_o, t} \varphi(z, p, m) &:= h_o^{-1} (\varphi(t + h_o, \cdot) - \varphi) \\ \Delta_i^{h_\star, x} \varphi(z, p, m) &:= \begin{cases} h_\star^{-1} (\varphi(\cdot, x + e_i h_\star, \cdot) - \varphi) & \text{if } \mu_i + \sum_{j=1}^\kappa \bar{C} \geq 0 \\ h_\star^{-1} (\varphi - \varphi(\cdot, x - e_i h_\star, \cdot)) & \text{else} \end{cases} \\ \Delta_i^{h_\star, l} \varphi(z, p, m) &:= \begin{cases} h_\star^{-1} (\varphi(\cdot, l + e_i h_\star, \cdot) - \varphi) & \text{if } i \in \mathbf{J} \\ 0 & \text{else} \end{cases} \\ \Delta_i^{h_\star, p} \varphi(z, p, m) &:= \begin{cases} h_\star^{-1} (\varphi(\cdot, p + e_i h_\star, \cdot) - \varphi) & \text{if } h_1 \geq 0 \\ h_\star^{-1} (\varphi - \varphi(\cdot, p - e_i h_\star, \cdot)) & \text{else} \end{cases} \end{aligned}$$

in which e_i is i -th unit vector of \mathbb{R}^d .

We shall assume that \mathbf{A} is finite. We introduce :

$$\mathbf{C}_\mathbf{J}^{h_\star} := \prod_{j=1}^\kappa (\partial \mathbf{1}_{\mathbf{J}^c}(j) + (\mathbf{X}_c^{h_\star} \times \mathbf{R}_c^{h_\star} \times \mathbf{A}) \mathbf{1}_\mathbf{J}(j)),$$

in which $\mathbf{R}_c^{h_\star} := \{ih_\star : -c/h_\star \leq i \leq c/h_\star\}$.

Then, the discrete counter-part of the set of policies running in indexes \mathbf{J} is defined by

$$\mathbf{CL}_\mathbf{J}^h := \mathbf{C}_\mathbf{J}^{h_\star} \times \mathbf{L}_\mathbf{J}^{h_o}.$$

We introduce :

$$\bar{\Lambda}[h_o](t, p) = h_o^{-1} \int_t^{t+h_o} \int_{U^\lambda} \Lambda(s, \lambda) dm^\lambda(\lambda) ds,$$

in which m^λ is completely determined by p , recall Assumption 3.2.3.

Note that, for $u \in U^\gamma$, we may have $x + \beta(\cdot, u) + \mathfrak{F}(\cdot; u) \notin \mathbf{X}_{c_x}^{h_\star}$. One needs to approximate φ with the *closest* points in $\mathbf{X}_{c_x}^{h_\star}$. We have the same issue with $\mathbf{P}_{c_p}^{h_\star}$. We define $[\varphi]_{h_\star}$ as an approximation of φ by

$$[\varphi]_{h_\star} = \sum_{(x', p') \in C_{h_\star}(x) \times C_{h_\star}(p)} \omega(x', p' | x, p) \varphi(\cdot, x', \cdot, p', \cdot).$$

in which $C_{h_\star}(x)$ (resp. $C_{h_\star}(p)$) denotes the corners of the cube of \mathbb{R}^d (resp. \mathbb{R}^d) in which x (resp. p) belongs too and $\omega(\cdot | x, p)$ is a weight function.

Moreover, in order to integrate the boundary condition when $l_j \rightarrow \ell$ for some $j \in \mathbf{J}$, we define $\bar{\mathbf{L}}^{h_o} = \mathbf{L}^{h_o} \cup \ell$ and $\mathbf{L}_\mathbf{J}^{h_o} = \prod_{j=1}^\kappa (\partial \mathbf{1}_{\mathbf{J}^c}(j) + \mathbf{L}^{h_o} \mathbf{1}_\mathbf{J}(j))$. We introduce

$$[\varphi]^\ell(\cdot, c, l, \cdot) = \varphi(\cdot, \mathfrak{C}_-^\ell(c, l), \cdot), \quad (c, l) \in \mathbf{C}_\mathbf{J}^{h_\star} \times \bar{\mathbf{L}}_\mathbf{J}^{h_o}.$$

And finally,

$$[\varphi]_{h_\star}^\ell = [[\varphi]^\ell]_{h_\star}.$$

The discrete counterpart of $\mathcal{L}_\star^{\mathbf{J}}$ for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$ is

$$\begin{aligned} \mathcal{L}_h^{\mathbf{J}}\varphi := & \Delta_i^{h_\circ, t}[\varphi]^\ell + \sum_{1 \leq i \leq d} \mu^i \Delta_i^{h_\star, x}[\varphi]^\ell + \sum_{i \in \mathbf{J}} \Delta_i^{h_\star, \ell}[\varphi]^\ell + \sum_{1 \leq i \leq d} h_1 \Delta_i^{h_\star, p}[\varphi]^\ell \\ & + \bar{\Lambda}[h_\circ] \int_{U^\gamma} \int_{\mathbb{R}^d} [\mathcal{I} [[\varphi]_{h_\star}^\ell, u] (t + h_\circ, \cdot) - \varphi] \Upsilon(\gamma, du) dm^\gamma(\gamma). \end{aligned} \quad (3.6.1)$$

For the sequel, we set $\phi^\circ \in \Phi_{\kappa}^{z, \bar{m}}$ a control such that $\tau_1^{\phi^\circ} > T$ a.s. and $\phi^a \in \Phi_{\kappa}^{z, \bar{m}}$ a control such that $\tau_1^{\phi^a} = t$ a.s. and $\tau_2^{\phi^a} > T$ a.s. for $a \in \mathbf{A}$. Thus, the discrete counterpart of \mathcal{K} is

$$\mathcal{K}^h \varphi := \sup_{a \in \mathbf{A}} \mathbb{E}_{\bar{m}} \left[[\varphi]_{h_\star}^\ell (Z_{t+h_\circ}^{z, \phi^a}, P_{t+h_\circ}^{t, p}, M_{t+h_\circ}^{z, m, \phi^a}) \right]. \quad (3.6.2)$$

We set $\mathring{\mathbf{X}}_{c_x}^{h_\star} := (\mathbf{X}_{c_x}^{h_\star} \setminus \partial \mathbf{X}_{c_x}^{h_\star})$, and $\mathring{\mathbf{P}}_{c_p}^{h_\star} := (\mathbf{P}_{c_p}^{h_\star} \setminus \partial \mathbf{P}_{c_p}^{\circ, h_\star})$.

Our numerical scheme consists in solving, for all $\mathbf{J} \in \mathcal{P}(\mathbf{K})$:

$$0 = \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} \left[-\mathcal{L}_h^{\mathbf{J}}\varphi \right] + \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \min \left\{ -\mathcal{L}_h^{\mathbf{J}}\varphi, \varphi - \mathcal{K}^h \varphi \right\} \quad \text{on } (\mathbf{T}^{h_\circ} \setminus T) \times \mathring{\mathbf{X}}_{c_x}^{h_\star} \times \mathbf{CL}_{\mathbf{J}}^h \times \mathring{\mathbf{P}}_{c_p}^{h_\star} \times \mathbf{M} \quad (3.6.3)$$

$$\varphi = g \mathbf{1}_{\{\mathbf{J}=\mathbf{K}\}} + (g \vee \mathcal{K}[g]_{h_\star}) \mathbf{1}_{\{\mathbf{J} \neq \mathbf{K}\}} \quad \text{on } \{T\} \times \mathring{\mathbf{X}}_{c_x}^{h_\star} \times \mathbf{CL}_{\mathbf{J}}^h \times \mathring{\mathbf{P}}_{c_p}^{h_\star} \times \mathbf{M} \quad (3.6.4)$$

$$\varphi = g \quad \text{on } \mathbf{T}^{h_\circ} \times \partial \mathbf{X}_c^{h_\star} \times \mathbf{CL}_{\mathbf{J}}^h \times \mathring{\mathbf{P}}_c^{h_\star} \times \mathbf{M} \quad (3.6.5)$$

Proposition 3.6.1. *Let v_h^c denote the solution of (3.6.3)-(3.6.4)-(3.6.5). Then $v_h^c \rightarrow v$ when $(h_\star, h_\circ/h_\star) \rightarrow 0$ and $c \rightarrow +\infty$.*

Proof. We check that the conditions of [14, Theorem 2.1.] are satisfied as in [9].

3.7 Example : CAT bonds in a *per event* framework for Hurricanes in Florida

Here we focus on a simple example where the controller is an insurance or a reinsurance company which can issue CAT bonds in order to cover its risk in natural disasters.

We will consider CAT bonds of *per event* type. The time-unit will be the year and we fix $\ell = 3$ which corresponds to the average maturity of CAT bonds in years.

We will consider the case of hurricanes occurring on the US Atlantic coast. More specifically, on Florida. The motivation comes from the fact that this region is well exposed, about one hurricane every two years in average, see [53]; and has an important and increasing insured value about 4000 billion in 2015, see [60].

Thus, we build an example in which an insurer has a strong exposition in Florida against the hurricanes, and can launch CAT bonds to cover it.

We consider a 1-dimension random Poisson measure N , which represents the intensity of arrival and the severity of Hurricanes. We first on the case with a Gamma distribution as a prior.

3.7.1 Intensity of Hurricanes : the Gamma case

We define the intensity Λ as the function :

$$\Lambda(t, \lambda) = \lambda h(t), \quad (t, \lambda) \in [0, T] \times \mathbb{R}_+^*,$$

in which $h : t \mapsto h(t)$ is a positive continuous function which represents the seasonality of the arrival of hurricanes and some growth according to the global warming. The parameter $\lambda \in U^\lambda := \mathbb{R}_+^*$, which is unknown, represents a level of intensity.

We set $m_0^\lambda = \mathcal{G}(\alpha_0, \beta_0)$ with $(\alpha_0, \beta_0) \in (\mathbb{R}_+^*)^2$ as an initial prior on λ .

Thus, by Example 1, we deduce that the process M^{t, m^λ} , starting from $m^\lambda := \Gamma(\alpha_t, \beta_t)$ at $t \in [0, T]$, remains in the family of Gamma distributions and, for all $s \geq t$,

$$M_s = \mathcal{G}\left(\alpha_t + N_s - N_t, \beta_t + \int_t^s h(u)du\right).$$

Moreover, we can define two processes P^α and P^β :

$$\begin{aligned} P^\alpha &= P_t^\alpha + \int_t^\cdot dN_s, \\ P^\beta &= P_t^\beta + \int_t^\cdot h(s)ds. \end{aligned}$$

and, by construction, $M = \mathcal{G}(P^\alpha, P^\beta)$.

For the function h , we need to add seasonality. We will add growth's intensity in the Bernoulli case. For the seasonality, especially on big Hurricanes, we refer to [56] in which the authors give a curve based on a kernel density estimation. One close parametric density function over one year can be found in the form :

$$h_0 : [0, 1] \rightarrow \mathbb{R}_+ \tag{3.7.1}$$

$$t \mapsto \begin{cases} f_{\hat{\alpha}, \hat{\beta}}\left(\frac{t-d_0}{d_1-d_0}\right) & \text{if } t \in (d_0, d_1) \\ 0 & \text{else} \end{cases} \tag{3.7.2}$$

in which $f_{\hat{\alpha}, \hat{\beta}}$ is the density function of the Beta distribution of parameters $(\hat{\alpha}, \hat{\beta}) \in (\mathbb{R}_+^*)^2$. The Figure 3.1 shows a representation of h_0 close to the one obtained in [56].

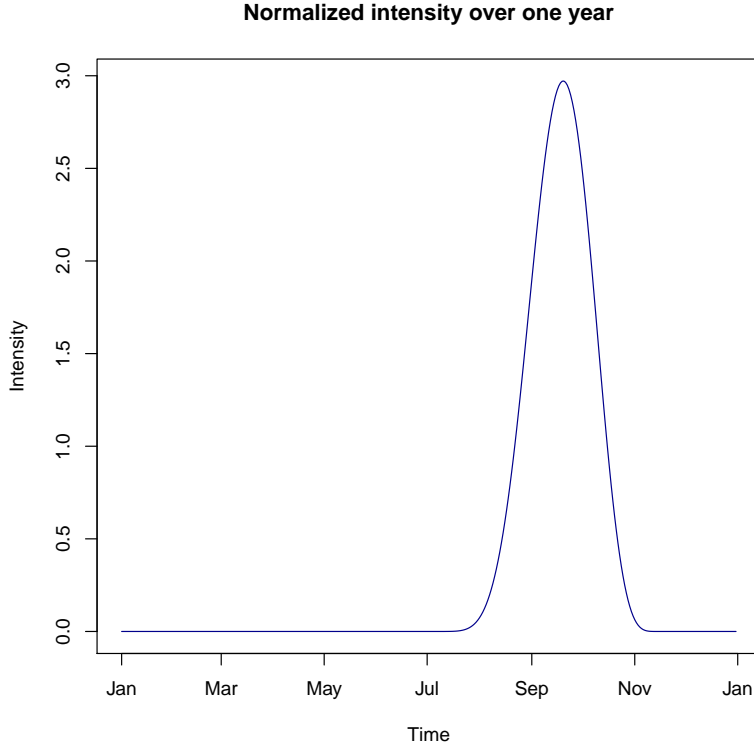


FIGURE 3.1 – Representation of h over one year with $d_0 = 1^{\text{st}}$ July, $d_1 = 15^{\text{th}}$ November, $\hat{\alpha} = 8$ and $\hat{\beta} = 6$.

3.7.2 Intensity of Hurricanes : the Bernoulli case

Although the Gamma prior gives parameters that belongs in \mathbb{R}_+ , in order to remains in the Gamma distribution over time, it requires the form $(t, \lambda) \mapsto \lambda h(t)$ and then the intensity of the whole period is proportional in λ . We introduce a Bernoulli case with three alternatives in which one can place any function depending on time.

With $E : \mathbb{R}_+ \mapsto \mathbb{N}$ the integer part function, we define the intensity as :

$$\Lambda(t, \lambda) = \frac{1}{2}h(t) \left(1 + \frac{E(t)}{T}\lambda \right), \quad (t, \lambda) \in [0, T] \times \{\lambda_1, \lambda_2, \lambda_3\}, \quad (3.7.3)$$

in which the parameter $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\} \subset \mathbb{R}_+$ represents 3 scenarios of the evolution of the intensity, as a consequence of the global warming.

Following Example 2, we can define 3 processes, starting from $p := (p^1, p^2, p^3) \in \mathbb{R}_+^3$ at time $t \in [0, T]$:

$$P^i := p^i - \int_t^\cdot P_s \Lambda(s, \lambda_i) ds + \int_t^\cdot P_{s-} [\Lambda(s, \lambda_i) - 1] dN_s \quad (3.7.4)$$

3.7.3 Severity of the Hurricanes

As in [53], we use a Generalized Pareto Distribution for the simulation of the severity of the claim, over the exposure of 4000 billion. Their threshold (minimum claim size) is $\mu = 0.25$ billion for an exposure of 2000 billion. Here, we shall use : $\mu = 0.5$, $\sigma = 5$ and $\xi = 0.5$. To fix ideas, the median is 4.5 billion, the quantile at 90% is 22 billion and the quantile at 99.5% is 132 billion. We also bound the distribution by the total exposure of 4000 billion.

Now we define the possible CAT bonds to issue. We will work with *per event* CAT bonds.

We introduce the so-called Occurrence Exceedance Probability (OEP) curve. To this aim, we introduce the random variable :

$$\iota_t := \max_{t \leq s \leq t+1} \int_{\mathbb{R}^*} uN(du, \{s\}),$$

which is the greatest Hurricane in $[t, t + 1]$ for $t \in [0, T - 1]$. The OEP curve is simply :

$$OEP_t^t := \inf \left\{ x \in \mathbb{R} : \mathbb{P}(\iota_t \leq x) \geq 1 - \frac{1}{t} \right\}, \quad t \geq 0,$$

in which t is called the *Return period*. By construction, OEP_t^t is the quantile of order $1 - 1/t$ of ι_t .

The Figure 3.2 shows the corresponding OEP curve with the prior $(p^\alpha, p^\beta) := (25, 50)$.

We now define the set of controls and the output process.

3.7.4 The set of controls and the output process

Recall that a control ϕ has the form $(\tau_i^\phi, k_i^\phi, n_i^\phi)$. Here n_i is the percentage of the Insured Value in the portfolio of the Insurer and is the notional of the CAT bond. It is fixed to one. We introduce $\{K_1, K_2, K_3, K_4\} := \{10, 50, 200, 1000\}$. We introduce what will be the capacity of the CAT bonds : $\iota_{K_j}^t = OEP_{K_{j+1}}^t - OEP_{K_j}^t$ for $1 \leq j \leq 3$ and $t \in [0, T - 1]$.

The value k_i can be chosen in $\{K_1, K_2, K_3\}$ and the associated sets A_{k_i} are defined by :

$$A_{k_i}^t = [OEP_{k_i}^t, +\infty[, \quad i \geq 1.$$

If a Hurricane leads to a cost in $A_{k_i}^t$, then the default of the CAT bond is activated. It remains to define the payout for the insurer in the default case. It corresponds to cover the layer $[OEP_{k_i}^t, OEP_{k_i}^t + \iota_{k_i}^t]$ at a ratio of n_i . We define the payout of the $j - th$ CAT bonds as :

$$F_j(t, x, c, l, n, k, u) := n_j \left[\left(u - OEP_{k_j}^{t-l_j} \right)^+ \wedge \iota_{k_j}^t \right], \quad j \in \{1, 2\}.$$

Note that, in our example, the risk cannot be covered above the return period of 1000.

We consider the process $X := (X^1, X^2)$ valued in \mathbb{R}^2 . The first component represents the cash of the Insurer/Reinsurer and the second component represents the risk premium, in term of percentage of the pure premium, of the market about the CAT bonds.

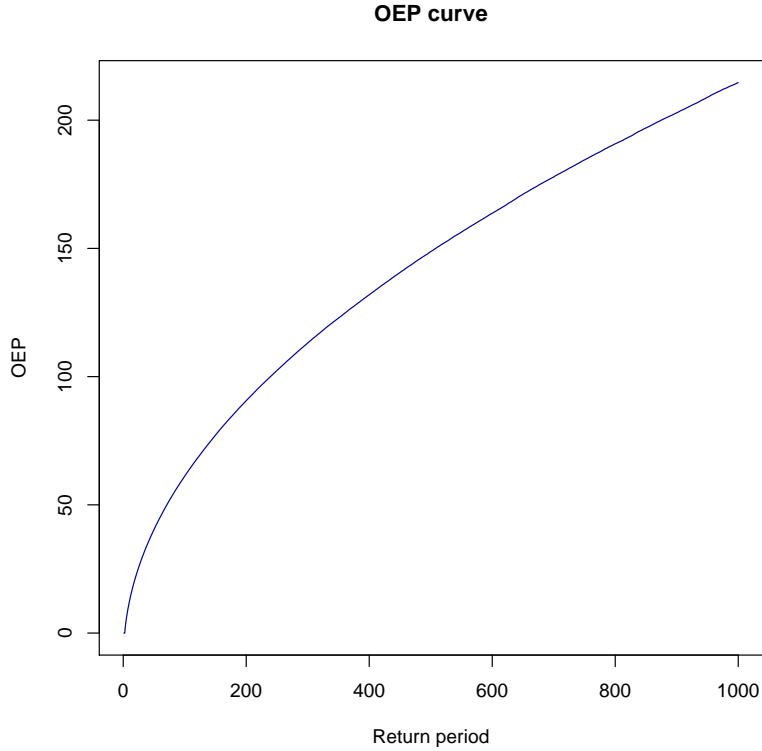


FIGURE 3.2 – Representation of an OEP curve, with the parameter (μ, σ, ξ) defined in the text and with the prior $(p^\alpha, p^\beta) := (25, 50)$.

We shall denote by $\rho > 0$ the speed mean return of the price of CAT bonds, by $\rho^* : \mathbb{R} \mapsto \mathbb{R}$ the increase function of the price after a claim and by $H_0 > 0$ the initial cost of issuing a CAT bond. We set, with $x := (x^1, x^2)$:

$$\begin{aligned} \mu(t, x) &= \begin{pmatrix} \mu + rx^1 \\ -\rho x^2 \end{pmatrix}, \\ \beta(t, x, u) &= \begin{pmatrix} u \\ \rho^*(u) \end{pmatrix}, \\ H(t, x, a) &= \begin{pmatrix} -H_0 \\ 0 \end{pmatrix}, \\ \bar{C}(t, c) &= \begin{pmatrix} c \\ 0 \end{pmatrix}, \\ F &= \begin{pmatrix} \sum_{j=1}^{\kappa} F_j \\ 0 \end{pmatrix}. \end{aligned}$$

The parameter μ represents the premium rate, the insurer is profitable if $\mu > \mathbb{E}_{\bar{m}} [\Lambda(t, \lambda)] \int_{\mathbb{R}^*} u \Upsilon(du)$, and $r > 0$ is the constant interest rate.

3.7.5 Gain function and dimension reduction

The controller wants to maximize, for some $\gamma > 0$, the criteria

$$g(x, c, l, p, m) := -\exp \left[-\gamma \left(x^1 + \frac{H_0}{\ell} \sum_{k=1}^{\kappa} \mathbf{1}_{\{l_k \neq \partial\}} (\ell - l_k) \right) \right] \vee \hat{C}.$$

The right part inside the exponential function compensates the initial cost for remaining CAT bonds, in order to avoid particular behavior of issuing nothing close to the end. We take $\hat{C} := -10^{300}$ which ensures that g is bounded and big enough such that it will not play an essential role.

Note that in the Gamma prior case, we have $P^\beta = P_t^\beta + \int_t^{\cdot} h_s ds$ which is a function of time. Then, one can avoid it in the numerical scheme since it is a function of time fully characterized by the initial prior.

In the Bernoulli case, one can see that, if we set for the prior

$$p' := \delta p,$$

for some $\delta > 0$, then, for all $s \geq t$, we have $P'_s = \delta P_s$ and then $\mathcal{D}(P'_s) = \mathcal{D}(P_s)$. One can normalized P such that the sum is 1 and avoid the last component.

3.7.6 The choice of the parameters

We choose here the form and the functions and the parameters for our toy examples. We first describe the Gamma case (for the prior) and then describes the Bernoulli case.

Just after the occurrence of Katrina, the price of the reinsurance was about two or three times greater with a persistence of about two years and can be also seen on the CAT bond market, see Figure 9 in [33]. Thus, we set

$$\rho := 2.$$

Moreover, the estimated return-period of such event is about 20-year return period, see [48]. Since the increase was about two of three times greater, we set

$$\rho^*(u) := \frac{0.05}{1 - F_{\mu, \sigma, \xi}(u)},$$

in which $F_{\mu, \sigma, \xi}$ denotes the cumulative distribution function of the Pareto distribution of parameters (μ, σ, ξ) . Then, here, for a return period of 40 years (recall that we have in average one claim each 2-year period), it gives an increase of 100% of the price.

The insurer has a market share of $e_0 \in]0, 1]$ that we fix at 10%. We shall assume that, the insurer is profitable until $\lambda = 0.65$. Then, the premium rate is

$$\mu := 0.65 e_0 \int_{\mathbb{R}^*} u \Upsilon(du) = 0.65 \times e_0 \times \left(\mu_0 + \frac{\sigma_0}{1 - \xi} \right) = 0.6825.$$

If $k_i = K_j$ with $j \in \{1, 2, 3\}$,

$$r_i = \mathfrak{C}_0(\tau_i, X_{\tau_i}, \alpha_i, \varepsilon_i) = n_i \left[e_0 \left(\frac{1}{K_{j+1}} + \frac{1}{2} \left(\frac{1}{K_j} - \frac{1}{K_{j+1}} \right) \right) \right] \mathfrak{I}_{K_j} (1 + x^2 + \varepsilon_i). \quad (3.7.5)$$

Thus, the CAT bond price is decomposed by :

- The part $\frac{1}{K_{j+1}}$ which is the probability that a claim is above the layer within one year and then the payout is the layer
- The part $\frac{1}{K_j} - \frac{1}{K_{j+1}}$ which is the probability that the greatest claim is in the layer, and we multiply it by one half like if it was uniformly distributed in the layer, which is greater than the true value.
- The factor x^2 is the risk aversion of the market, and ε_i is some random value about the price the coupon.

Finally, the cost of issuing a CAT bond is fixed at : $H_0 := 0.0025$, the interest rate is fixed at $r := 1\%$ and the market share at $e_0 := 10\%$.

Remark 3.7.1. *In these examples, we deal with per event CAT bonds. One also can deal with aggregated losses within the period. In this case, one needs to remember the current accumulation of claims and to introduce another dimension in the output process X .*

Remark 3.7.2. *In practice, in general, a partial default below 70%-80% of the capacity does not end the CAT bonds : the coupon is reduced by the proportional loss and another loss may lead to the complete default, using the same limits. Here, for simplification, the CAT bond ends whenever the layer is attained.*

Remark 3.7.3. *Note that the function $\Psi(x, c, l, p) := \frac{\mu}{r} + x^1 + \delta$ satisfies the conditions of Proposition 3.5.1, for $\delta > 0$ great enough.*

Note that, in this example, we did not add any global warming effect, it will be added in the Bernoulli case. Actually here, we could only add a deterministic global warming effect since the Bayes stability requires an intensity of the form $\Lambda(t, \lambda) = \lambda h(t)$.

With a convex hull of Dirac masses

In this case, the intensity grows over time, recall (3.7.3). We fix $\lambda_1 = 0.2$, $\lambda_2 = 0.3$, $\lambda_3 = 0.4$ and $P_0^1 = P_0^2 = P_0^3 = \frac{1}{3}$, recall (3.7.4).

To be consistent, we say that the premium rate also rises over time following the rise of intensity, but by 35%, and then is :

$$\mu(t, x) = \left(\mu \left(1 + 0.35 \frac{t}{T} \right) + r x^1 \right) - \rho x^2, \quad (t, x) \in [0, T] \times \mathbb{R}^2.$$

We assume that the market is updating the OEP with :

$$OEP^t := OEP^0 \left(1 + 0.35 \frac{t}{T} \right), \quad t \in [0, T].$$

3.7.7 Results

Recall that, for each CAT bond that the insurer can issue, we need to add its characteristics and then the complexity increases hugely in κ , depending on possible policies. Thus, in our simulation, we use $\kappa = 2$ and thus, the controller can choose at most 2 layers among the three available (recall them in term of return periods : [10, 50], [50, 200] and [200, 1000] which correspond to [1.23, 4], [4, 9], and [9, 21.5] in billion dollars).

With the Gamma prior

In Figure 3.3, we provide a simulated path of the optimal strategy in which the Pareto distribution is discretized in 2500 points (the highest possible value is 49 billion dollars). The top left graphic describes the control played by the insurer. The top part represents the issue of CAT bonds, the level is the lower bound of the layer. The bottom part represents the running CAT bonds with respect to the layer. The double dash says that two CAT bonds at the same layer are running. The top right graphic describes the arrival of natural disasters. The bottom part gives the size of the claim of the insurer while the top part gives the payoff of the CAT bond(s). The middle left graphic describes the evolution of the cash of the insurer. The middle right graphic gives the evolution of X^2 , the price penalty of the CAT bonds which appears in (3.7.5). The bottom left graphic gives the evolution of the mean of the estimated distribution of λ_0 , defined by $\frac{P^\alpha}{P^\beta}$, and the bottom right graphic gives the evolution of the standard deviation, defined by $\frac{\sqrt{P^\alpha}}{P^\beta}$.

At the beginning, the insurer does not issue any CAT bond. Since we start in January, there is no risk to experiment a claim and thus the insurer delay the issue. Just when the season starts, he first chooses to issue two CAT bonds on the layer [200, 1000]. Recall that it is the highest layer which corresponds to [9, 21.5] in billion dollars. It is possible to have a claim highly above the layer and having a double cover on this big layer gives, indirectly, a cover against huge claims above the layer (recall that the maximum claim size is 49 billion dollars). He renews each CAT bond at the maturity until he meets a claim with a return period above 1000 during the 5th year. He gets the associated payoff. Despite the huge increase of the price of CAT bonds, by almost 400%, he immediately issues a new one on the layer [200, 1000], but only one. He waits the next season, with a better expected price, to issue the other one. After, he follows this strategy to the end, except very close to the end where he optimizes the cost of CAT bonds.

In Figure 3.4, we represent the approximated density (by kernel estimation) of the total cash of the insurer at the end of the 30 years. On the left, it is the case with $\lambda_0 = 0.6$ (as it is also the case in Figure 3.3) and on the right with $\lambda_0 = 0.5$, i.e. what believes the insurer at the beginning. The solid curve is the case when the insurer plays the optimal control and the dashed curve is when he never issues any CAT bond. We also add the quantiles at 99.5% in term of losses, see the legend. In the case with $\lambda_0 = 0.6$ (left), from which the paths in Figure 3.3 come from, we can see that the standard deviation is reduced. And the quantile at 99.5% is strongly reduced. One can observe that the case $\lambda_0 = 0.6$ strongly reduces the expected net return in average.

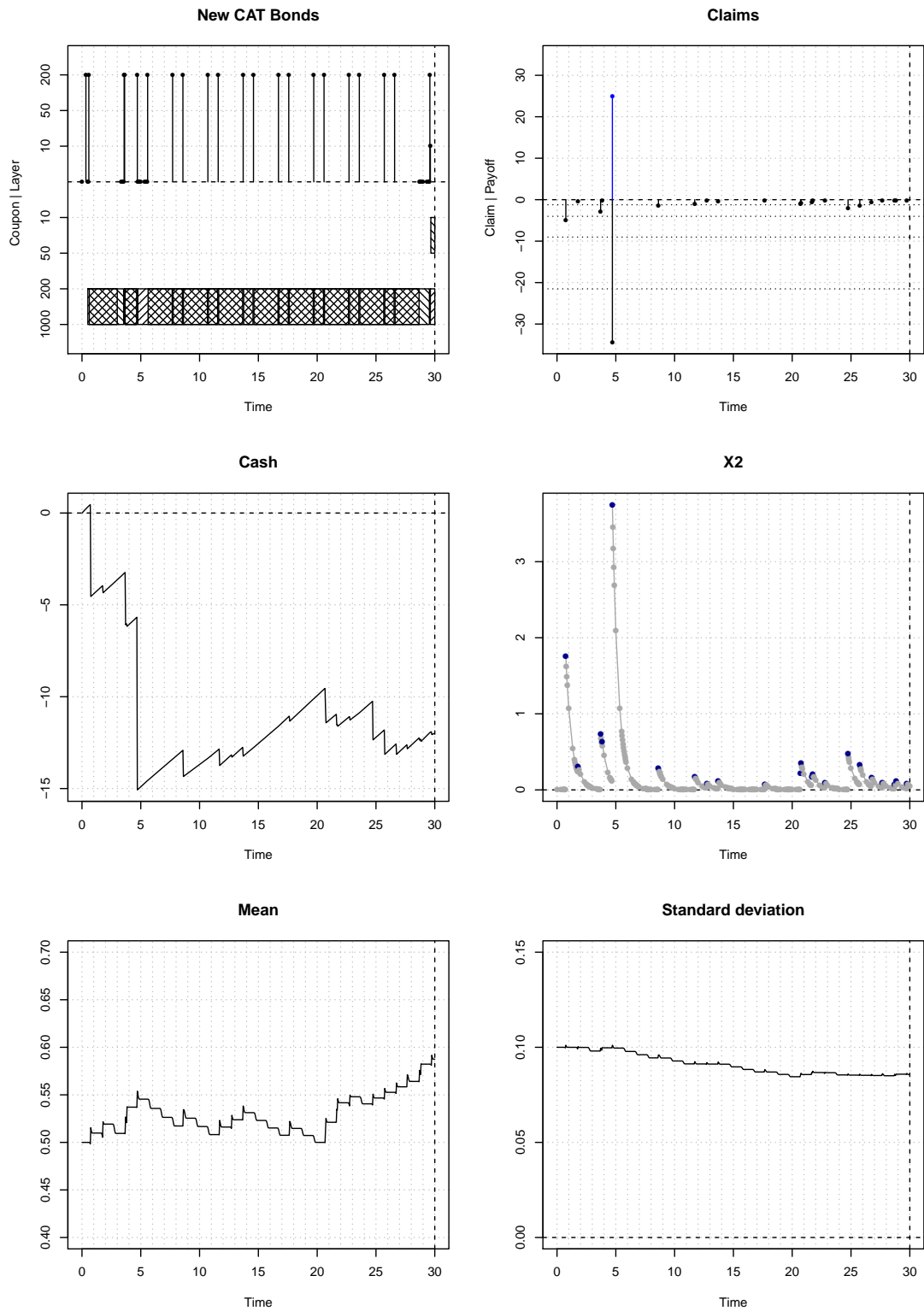


FIGURE 3.3 – Simulated path of the optimal strategy of the insurer.

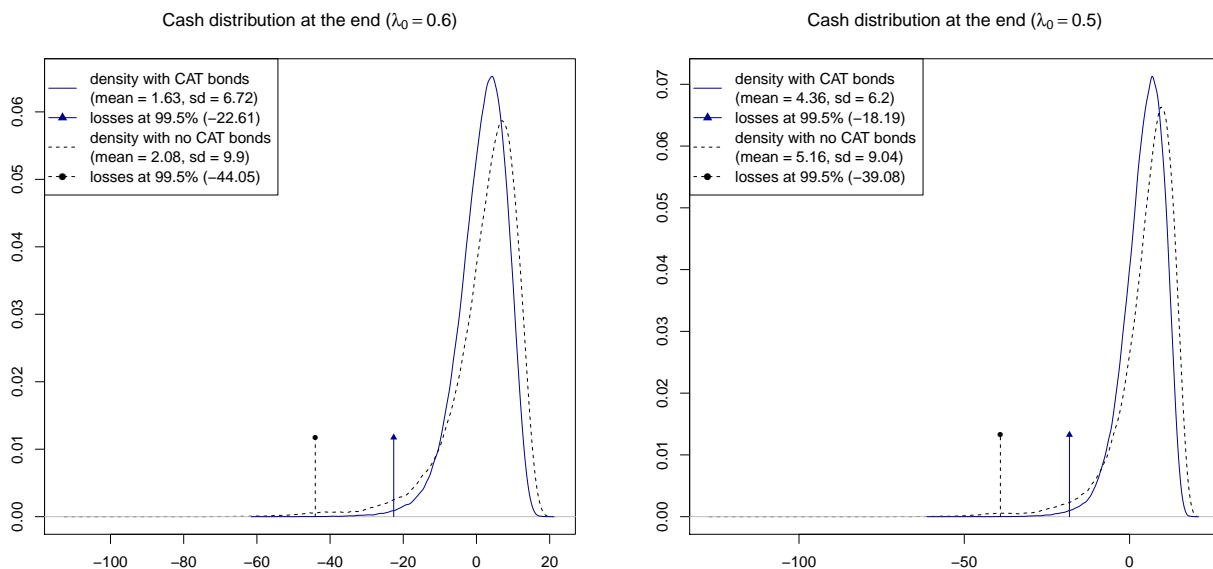


FIGURE 3.4 – Cash distribution (with 200 000 simulations) for $\lambda_0 = 0.6$ (left) and $\lambda_0 = 0.5$ (right) with the optimal control (solid dark blue) and without any CAT bond (dashed black).

We now look at the case with a discretization of 500 of the Pareto distribution. In particular, the maximum claim size is 21.4 billion which does not exceed the maximum layer [9.0, 21.5]. In general, the risk is lower. In Figure 3.5, we show a simulated path. This time, the insurer chooses to get two CAT bonds at the layer [50, 200]. Actually, with this discretization, the layer [200, 1000] appears to be less competitive since the discretization of 500 leads to a lower expected payoff. In the first years, the expected intensity is revised higher and the relative price of the layer [10, 50] decreases (this layer requires the highest coupon since it is frequently hit). At the 4th year, he changes his strategy and gets one CAT bond on the layer [10, 50] and the other one on the layer [50, 200]. A catastrophe above the return period of 200 occurs at the 20th year and both CAT bonds end. He prefers to wait the next season because of the consecutive price increase. Note that, in the previous cases (with Pareto distribution discretized in 2500 points), he was never without any CAT bond, even after an increase of 400%. Then, he continues his strategy to get a CAT bond on the layer [10, 50] and the other one on the layer [50, 200], until the end.

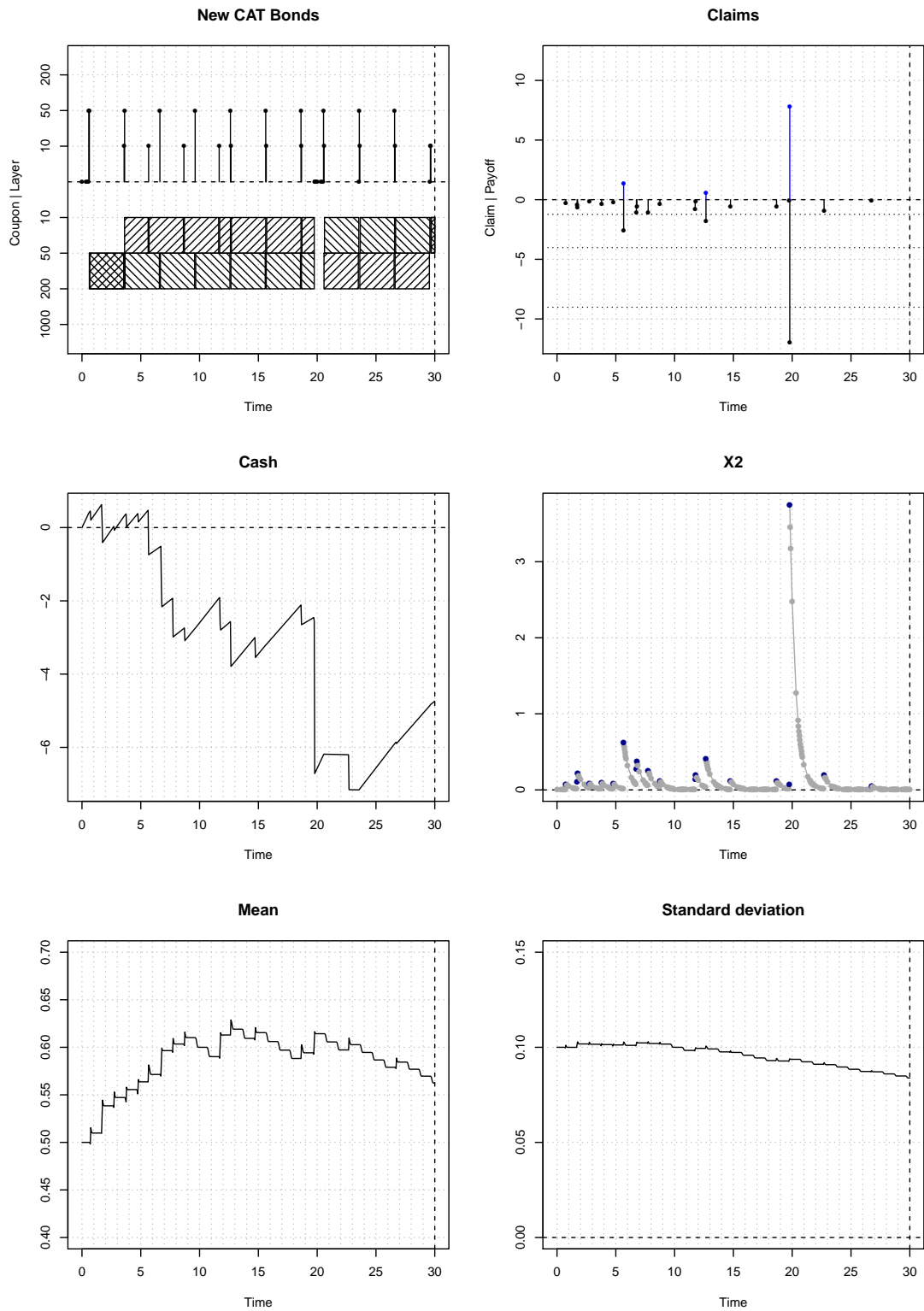


FIGURE 3.5 – Simulated path of the optimal strategy of the insurer.

With the Bernoulli prior

In Figure 3.6, we provide a simulated path of the optimal strategy in which the Pareto distribution is discretized in 2500 points (recall that the highest possible value is 49 billion dollars). As in the Gamma prior case, the insurer chooses to get two CAT bonds at the higher layer. When he experiences a huge claim during the second year, he still gets twice the layer but prefers to wait before to take a new CAT bond, according to the huge rise of the price. He waits the next year and restarts the same strategy until the 12th year. Then, he issues CAT bonds on the layer [50, 200] and [200, 1000] until close the end.

The estimated probabilities on λ_0 evolve slowly at the beginning since λ_0 has an impact which rises over time.

In Figure 3.7, we represent the approximated density (by kernel estimation) of the total cash of the insurer at the end of the 30 years. On the left, it is the case with $\lambda_0 = 0.4$ (as it is also the case in Figure 3.3) and on the right with $\lambda_0 = 0.3$. The legend is the same as in Figure 3.4 and we get close distributions.

We now look at the case with a discretization of 500 of the Pareto distribution and show a simulated path in Figure 3.8. As in the Gamma prior case, at the beginning, the insurer chooses to get two CAT bonds at the layer [50, 200]. He follows this strategy until he meets a huge claim in the 16th year. He waits the next season and restarts the same strategy. At the 24th year, he chooses to issue CAT bonds on two different layers, at [50, 200] and [10, 50]. As in Figure 3.5, this results in a change on the belief on the intensity.

Finally, in Figure 3.9, we display the distribution of the probabilities on λ_0 . This highlights the fact that it is very difficult to estimate it with observations through time.

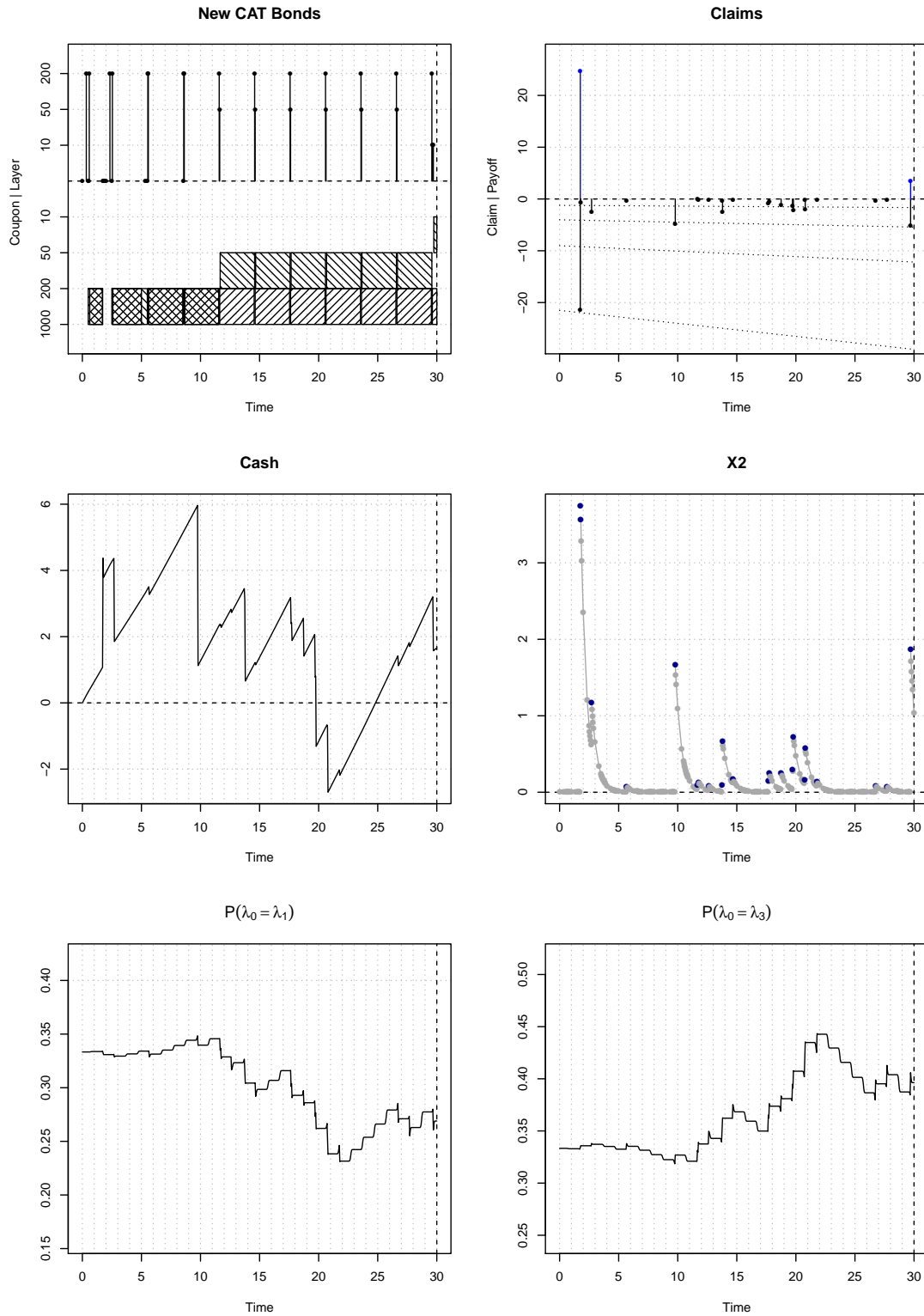


FIGURE 3.6 – Simulated path of the optimal strategy of the insurer.

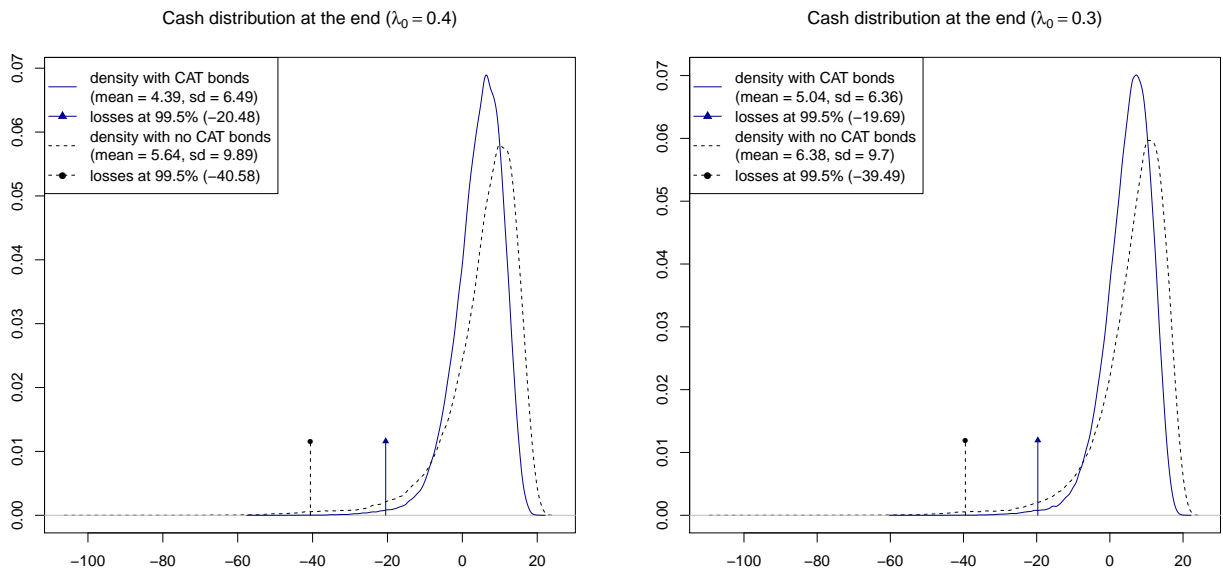


FIGURE 3.7 – Cash distribution (with 200 000 simulations) for the increase parameter $\lambda_0 = 0.4$ (left) and $\lambda_0 = 0.3$ (right) with the optimal control (solid dark blue) and without any CAT bonds (dashed black).

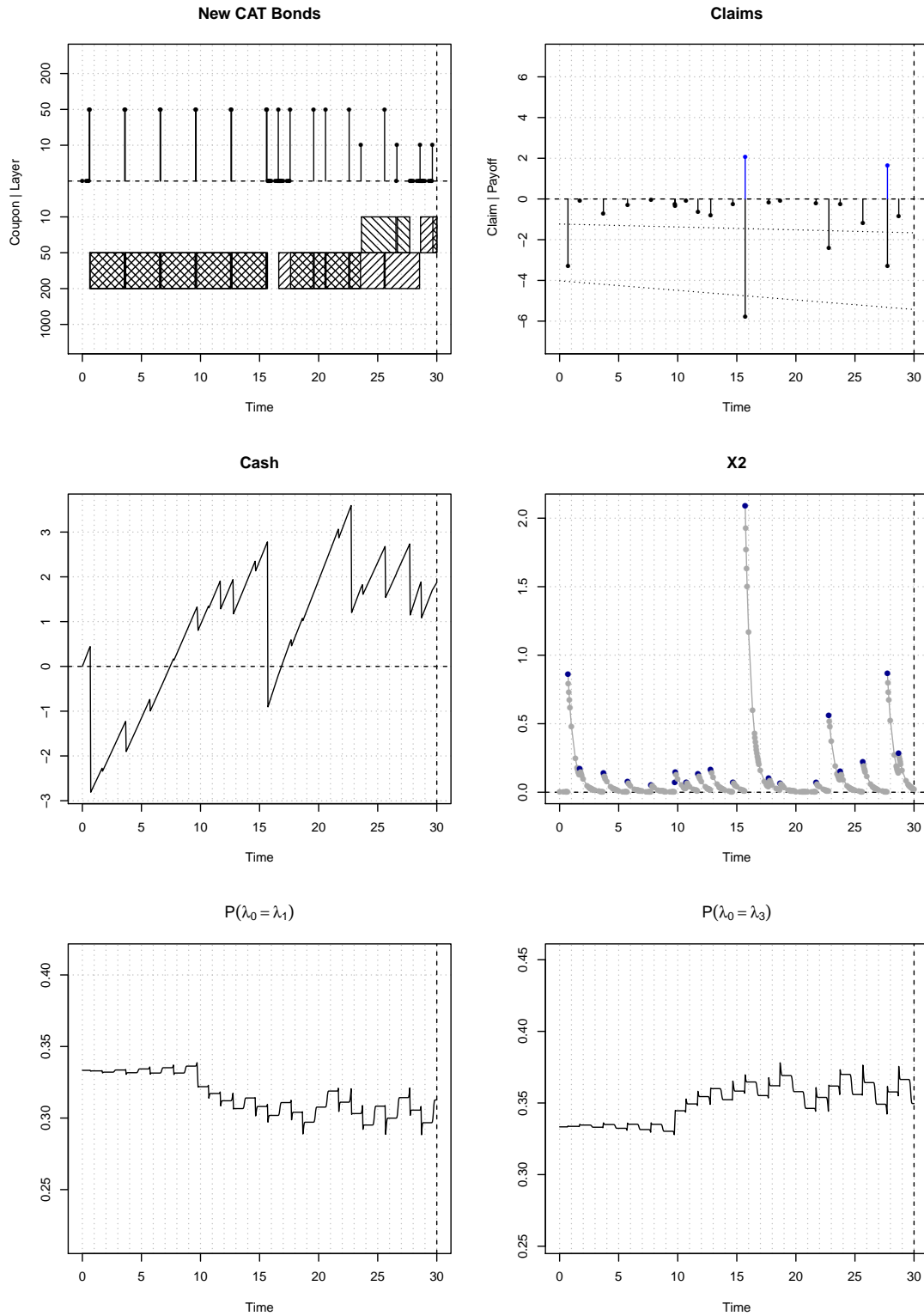


FIGURE 3.8 – Simulated path of the optimal strategy of the insurer.

Prior distribution at the end ($\lambda_0 = 0.4$)

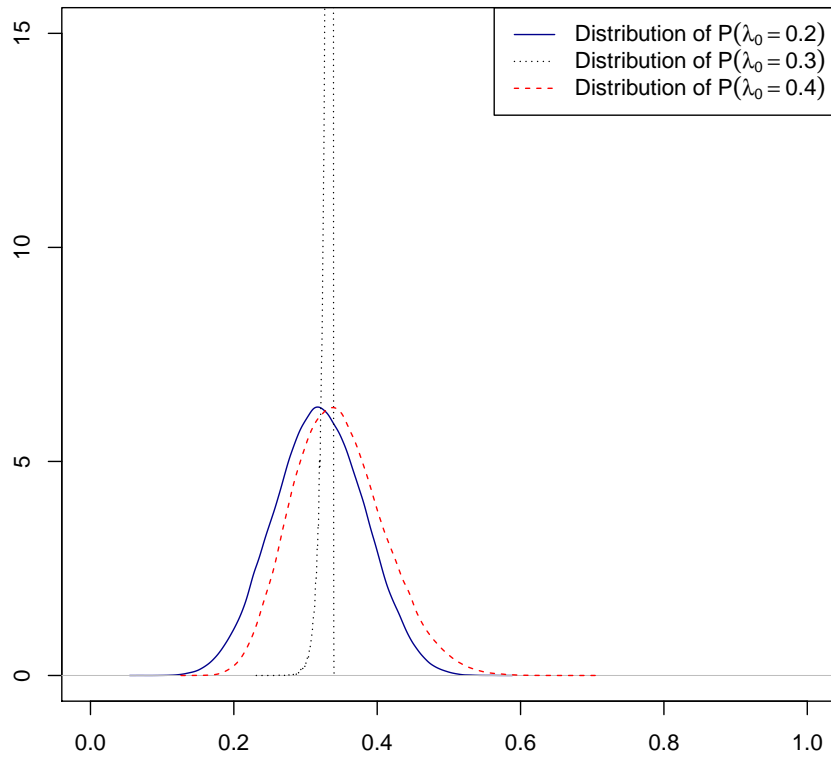


FIGURE 3.9 – Distribution of the probabilities on λ_0 at the end (with 200 000 simulations).

Chapitre 4

Optimal inventory management and order book modeling

We model the behavior of three agent classes acting dynamically in a limit order book of a financial asset. Namely, we consider market makers (MM), high-frequency trading (HFT) firms, and institutional brokers (IB). Given a prior dynamic of the order book, similar to the one considered in the Queue-Reactive models [29, 46, 47], the MM and the HFT define their trading strategy by optimizing the expected utility of terminal wealth, while the IB has a prescheduled task to sell or buy many shares of the considered asset. We derive the variational partial differential equations that characterize the value functions of the MM and HFT and explain how almost optimal control can be deduced from them. We then provide a first illustration of the interactions that can take place between these different market participants by simulating the dynamic of an order book in which each of them plays his own (optimal) strategy.

4.1 Introduction

The comprehension of the order book dynamic has become a fundamental issue for all market participants and for regulators that try to increase the market transparency and efficiency. A deep understanding of the order book dynamic and agents behaviors enables : market makers to ensure liquidity provision at cheaper prices, high-frequency traders to reduce arbitrage opportunities, investors to reduce their transaction costs, policy makers to design relevant rules, to strengthen market transparency and to reduce market manipulation. Moreover, modeling the order book provides insights on the behavior of the price at larger time scales since the price formation process starts at the order book level, see e.g. [28] for Brownian diffusion asymptotic of rescaled price processes. Recently, the widespread market electronification has facilitated the access of high quality data describing market participants decisions and interactions at the finest time scale, on which statistics can be based. The availability of the order book data certainly allows a better understanding of the market activity. On the other hand, the recent market fragmentation and the increase of trading

frequency rise the complexity of agents actions and interactions.

The main objective of the present paper is to propose a flexible order book dynamics model close to the one of [29, 46, 47, 52], construct a first building block towards a realistic order book modeling, and try to better understand the various regimes related to the presence of different market participants. Instead of considering a pure statistical dynamics as in e.g. [29], we construct an endogenous dynamics, see e.g. [27, 52], based on the optimal behavior of agents that are assumed to be rational. For numerical tractability, we simplify the market in three classes of (most significant) participants : the market makers (MM), the high-frequency trading (HFT) firms, and the institutional brokers (IB). Each of them decides of his policy in an optimal way, given prior statistics, and then interact with the others given the endogenous realizations of the market.

More precisely, we postulate that they assume an order book model similar to the one suggested by the Queue-Reactive model [46], see also [1, 29, 44], in which we restrict to the best bid and ask queues¹ : limit and aggressive orders arrive with certain intensities, when a queue is depleted it is regenerated according to a certain law and possibly after a price move, the spread can take two different values². Importantly, we also take the order book's imbalance into account in the modeling of the different transition probabilities, see e.g. Besson et al. [18].

The market participants can either put limit or aggressive orders. The aim of a market maker is to gain the spread. He should therefore essentially put limit orders, aggressive orders being used when his inventory is too disequibrated. In our model, he can only acts on the given order book. The high-frequency trader is assumed to play on the correlation between the order book dynamics, viewed as the stock price, and the price of another asset, called futures hereafter. Indeed, he believes that the difference between the stock and the futures prices is mean-reverting. Whenever he buys/sells one unit of the stock, he sells/buys back one unit of the futures. We do not handle the order book associated to the futures but simply model the price of the futures as the mid-price of the stock to which a mean-reverting process is added. Still, we introduce a (possibly equal to 0) transaction cost proportional to the size of the transaction. As the market-makers, he seeks for a zero inventory at the end of the trading period. Finally, institutional brokers are simply assumed to play VWAP (Volume Weighted Average Price)- or Volume-based strategies (robots). Again, they essentially use limit orders and become aggressive when they are too late in their schedule.

We focus on the derivation of the optimal strategies of the market makers and the high-frequency traders, and on how they can be computed numerically by solving the associated variational partial differential equations. Note that it is important to consider their strategies within a dynamical model as current actions impact the order book and therefore may modify its futures dynamics. We will actually see that, in certain situations, participants can place aggressive orders or limit orders in the spread just to try to manipulate the order book's dynamics in a favorable way. Note that the dimension of our control space is higher than

1. This limitation is for numerical tractability. It is already enough for most markets.

2. One could consider a larger set of possible spreads in theory, but refrains to doing this for numerical tractability.

in [1, 44, 27, 52] since more complex decisions are required to tackle the market making problem.

However, the ultimate goal of this work is to provide a market simulator. In the last section, we already present simulations of the market behavior given the pre-computed optimal strategies of the different actors. More precisely, we will only simulate the evolution of the mean-reverting process (driving the difference between the stock and the futures price) together with the reconstruction of the queues when prices move, and let the participants play their optimal strategies given the evolution of the order book due to their different actions. This should allow us to study how these different market participants may interact among each other if each of them is playing his optimal policy. In particular, one should observe different market regimes depending on the proportion of the different participants in the total population, on their risk aversion, etc. First simulations are provided in this paper, a more throughout study will be conducted in the companion paper [8].

Our approach therefore lies in between two current streams of literature. The first one is based on “general equilibrium models”, including economic models, where the market activity is generated by interactions between rational agents who take optimal decisions that interact through the market netting process, see e.g. [39, 57, 58]. The second stream of literature considers purely statistical models where the order book is seen as a random process, see e.g. [2, 3, 15, 29, 30, 41, 46, 47, 49, 50, 59]. The statistical models focus on reproducing many salient features of a real market rather than agents behaviors and interactions. In our approach, we take into account that the agent’s behavior are essentially based on statistical approaches, but that they eventually interact with each other.

We end this introduction with an outline of this chapter. In Section 4.2, we present the general order book dynamics. The marker maker control problem is studied in details in Section 4.3. There, we present the equations satisfied by their optimal strategy and propose a numerical solution for this problem, together with numerical illustrations. In Section 4.4, we formulate the high-frequency trader control problem and perform a similar analysis. The institutional broker strategy is described in Section 4.5, where we restrict to VWAP and Volume liquidation problems. Finally, in Section 4.6, we simulate a realistic market using the three agent’s optimal trading strategies.

4.2 General order book presentation and priors of the market participants

As mentioned above, we focus on a single order book and only model the best bid and ask prices, in a similar way to [29]. In this section, we describe the general market mechanisms as well as the priors on which the optimal strategies of the agents are based. We fix a terminal time horizon T and consider a probability space (Ω, \mathbb{P}) . Here, $\Omega := \Omega_1 \times \Omega_2$ and $\mathbb{P} := \mathbb{P}_1 \otimes \mathbb{P}_2$, where Ω_1 is the space of \mathbb{R}^{11} -valued càdlàg paths on $[0, T]$ endowed with a probability measure \mathbb{P}_1 with full support on Ω_1 , and Ω_2 is the one dimensional Wiener space endowed with the Wiener measure \mathbb{P}_2 .

We denote respectively by $(P_t^b)_{t \geq 0}$ and $(P_t^a)_{t \geq 0}$ the best bid offer and the best ask offer processes on the market. They are valued in $\mathfrak{d}\mathbb{Z}$ where $\mathfrak{d} > 0$ is the tick size. We denote by $(Q_t^b)_{t \geq 0}$ and $(Q_t^a)_{t \geq 0}$ the sizes of the corresponding queues valued in \mathbb{N}^* . To simplify the notation, we introduce $P := (P^b, P^a)$, $Q := (Q^b, Q^a)$ and define the spread process as $\delta P := P^a - P^b$. Moreover, we assume³ that $\delta P_t \in \{\mathfrak{d}, 2\mathfrak{d}\}$ for all $t \geq 0$.

We denote by $(\tau_i)_{i \geq 1}$ the times at which orders are sent to the market. We assume that this sequence is increasing and that $\#\{i \in \mathbb{N} : \tau_i < T\} < \infty$ a.s. The market participants can send different types of orders at each time τ_i :

- Aggressive orders of size $\alpha_i^b \in \mathbb{N} \cap [0, Q^b]$ at the bid or of size $\alpha_i^a \in \mathbb{N} \cap [0, Q^a]$ at the ask : the size of the corresponding queue, Q^b or Q^a , decreases by the size of the aggressive order, α_i^b or α_i^a .
- Limit orders of size $L_i^b \in \mathbb{N}$ at the bid or of size $L_i^a \in \mathbb{N}$ at the ask : the size of the corresponding queue, Q^b or Q^a , increases by the size of the limit order, L_i^b or L_i^a .
- When $\delta P = 2\mathfrak{d}$: Limit orders of size $L_i^{b, \frac{1}{2}} \in \mathbb{N}$ at the bid or of size $L_i^{a, \frac{1}{2}} \in \mathbb{N}$ at the ask : the order is placed inside the spread, at the price $P^b + \mathfrak{d} = P^a - \mathfrak{d}$, this generates a new queue at the bid or at the ask, of size $L_i^{b, \frac{1}{2}}$ or $L_i^{a, \frac{1}{2}}$, and a price move.
- Cancellations : Cancellations of $M_i^b \in \mathbb{N} \cap [0, Q^b]$ orders at the bid or of $M_i^a \in \mathbb{N} \cap [0, Q^a]$ orders at the ask. The difference between cancellations and aggressive orders is that aggressive orders consume the bottom of the limit while we see cancellations as only consuming the top of the limit first.

We assume that the sequence $(\alpha_i^b, \alpha_i^a, L_i^b, L_i^a, L_i^{b, \frac{1}{2}}, L_i^{a, \frac{1}{2}}, M_i^b, M_i^a)_{i \geq 1}$ is made of random variables leaving, with probability one, on the state space \mathbf{C}_o defined as the collection of elements $(a^b, a^a, \ell^b, \ell^a, \ell^{b, \frac{1}{2}}, \ell^{a, \frac{1}{2}}, m^b, m^a) \in \mathbb{N}^8$ such that

$$\begin{cases} a^b a^a = 0 \\ \ell^b = \ell^a = 0 \text{ if } \max\{a^b, a^a\} \geq 1 \\ \ell^{b, \frac{1}{2}} = 0 \text{ if } \max\{a^b, a^a, \ell^b\} \geq 1 \\ \ell^{a, \frac{1}{2}} = 0 \text{ if } \max\{a^b, a^a, \ell^a\} \geq 1 \\ m^b = 0 \text{ if } \max\{a^b, a^a, \ell^b, \ell^{b, \frac{1}{2}}\} \geq 1 \\ m^a = 0 \text{ if } \max\{a^b, a^a, \ell^a, \ell^{a, \frac{1}{2}}\} \geq 1 \end{cases} . \quad (4.2.1)$$

We interpret the above expression as follows. First, aggressive orders can not be sent simultaneously at the bid and at the ask. Next, limit orders (at the current bid/ask prices) can not be placed at the same time that aggressive orders are sent. Finally, one can not place limit orders within the spread if limit orders at the current bid/ask prices are placed. Because we only consider the first limits, these conditions are natural whenever one presumes that orders of different market participants do not arrive exactly at the same time.

Depending on the arrival of orders, queues can be depleted. In this case, new queues can be re-generated, at the same prices or at different prices, and possibly with a change of the

3. The extension to more possible spread values is straightforward. We stick to this setting for notational and computational simplicity. Note that this limit is also justified by empirical evidences for many stocks, see [29].

spread value. To model this, we introduce a sequence of random variables $(\epsilon_i, \epsilon_i^b, \epsilon_i^a)_{i \geq 1}$ with values in $\{0, 1\} \times (\mathbb{N}^*)^2$. The sequence $(\epsilon_i)_{i \geq 1}$ will describe possible jumps of the bid/ask prices, while the sequence $(\epsilon_i^b, \epsilon_i^a)_{i \geq 1}$ will describe the new sizes of the queues when they are re-generated, after one of them is depleted. More precisely, we postulate the dynamics

$$\begin{aligned}
P_{\tau_i}^b &= P_{\tau_{i-1}}^b + \mathfrak{d} \left[\mathbf{1}_{\{\epsilon_i=1\}} \left(-\mathbf{1}_{\{Q_{\tau_{i-1}}^b=\hat{\alpha}_i^b\}} + \mathbf{1}_{\{\delta P_{\tau_{i-1}}=2\mathfrak{d}\}} \mathbf{1}_{\{Q_{\tau_{i-1}}^a=\hat{\alpha}_i^a\}} \right) + \mathbf{1}_{\{L_i^{b,\frac{1}{2}}>0\}} \right] \\
P_{\tau_i}^a &= P_{\tau_{i-1}}^a + \mathfrak{d} \left[\mathbf{1}_{\{\epsilon_i=1\}} \left(\mathbf{1}_{\{Q_{\tau_{i-1}}^a=\hat{\alpha}_i^a\}} - \mathbf{1}_{\{\delta P_{\tau_{i-1}}=2\mathfrak{d}\}} \mathbf{1}_{\{Q_{\tau_{i-1}}^b=\hat{\alpha}_i^b\}} \right) - \mathbf{1}_{\{L_i^{a,\frac{1}{2}}>0\}} \right] \\
Q_{\tau_i}^b &= Q_{\tau_{i-1}}^b + L_i^b + (L_i^{b,\frac{1}{2}} - Q_{\tau_{i-1}}^b) \mathbf{1}_{\{L_i^{b,\frac{1}{2}}>0\}} - \hat{\alpha}_i^b \mathbf{1}_{\{\Delta P_{\tau_i}^b=0\}} + (\epsilon_i^b - Q_{\tau_{i-1}}^b) \mathbf{1}_{\{\Delta P_{\tau_i}^b \neq 0\} \cup \{\hat{\alpha}_i^b=Q_{\tau_{i-1}}^b\}} \\
Q_{\tau_i}^a &= Q_{\tau_{i-1}}^a + L_i^a + (L_i^{a,\frac{1}{2}} - Q_{\tau_{i-1}}^a) \mathbf{1}_{\{L_i^{a,\frac{1}{2}}>0\}} - \hat{\alpha}_i^a \mathbf{1}_{\{\Delta P_{\tau_i}^a=0\}} + (\epsilon_i^a - Q_{\tau_{i-1}}^a) \mathbf{1}_{\{\Delta P_{\tau_i}^a \neq 0\} \cup \{\hat{\alpha}_i^a=Q_{\tau_{i-1}}^a\}}
\end{aligned} \tag{4.2.2}$$

for $i \geq 1$, where

$$\hat{\alpha}_i^{b/a} := \alpha_i^{b/a} + M_i^{b/a},$$

with

$$(P_0^b, P_0^a, Q_0^b, Q_0^a) \in D_{P,Q} := \{(p^b, p^a, q) \in (\mathfrak{d}\mathbb{Z})^2 \times (\mathbb{N}^*)^2 : p^a - p^b \in \{\mathfrak{d}, 2\mathfrak{d}\}\},$$

and the convention $\tau_0 = 0-$. We refer to (4.2.1) to see that this dynamics is consistent. In particular, prices can move only if one of the queues is depleted because of the arrival of aggressive orders or if a new limit order is inserted within the spread. These two situations can not occur simultaneously. The ask price can move by \mathfrak{d} when the ask queue is depleted. If the spread was already $2\mathfrak{d}$, then the bid price moves up as well. The other way around if the bid queue is depleted. In the following, we extend the dynamics of (P, Q) by considering it as a step constant right-continuous process on $[0, T]$.

We now denote by E the \mathbb{N}^{12} -valued step constant right-continuous process defined by

$$\Delta E_{\tau_i} := E_{\tau_i} - E_{\tau_{i-1}} := (\alpha_i^b, \alpha_i^a, L_i^b, L_i^a, L_i^{b,\frac{1}{2}}, L_i^{a,\frac{1}{2}}, M_i^b, M_i^a, \epsilon_i, \epsilon_i^b, \epsilon_i^a, 1), \quad i \geq 1,$$

with $E_{\tau_0} := E_0 := 0$. Later on, we shall only write

$$(P_{\tau_i}, Q_{\tau_i}) = \mathcal{T}_{P,Q}(P_{\tau_{i-1}}, Q_{\tau_{i-1}}, \Delta E_{\tau_i}) \tag{4.2.3}$$

in which the map $\mathcal{T}_{P,Q}$ is defined explicitly by (4.2.2).

The process E models the flow of all the orders on the market. From the viewpoint of a market participant, it corresponds to its own orders and to the other participants' orders that we denote by \tilde{E} . The process \tilde{E} has jump times $(\tilde{\tau}_i)_{i \geq 1} \subset (\tau_i)_{i \geq 1}$ of sizes

$$\Delta \tilde{E}_{\tilde{\tau}_i} = \sum_{j \geq 1} \mathbf{1}_{\{\tilde{\tau}_i=\tau_j\}} \Delta E_{\tau_j}, \quad i \geq 1. \tag{4.2.4}$$

It induces a counting measure $\tilde{\nu}(dt, de)$. For a market participant, a prior on this measure is given by the compensator $\tilde{\mu}$ of $\tilde{\nu}$ under \mathbb{P} . We assume that it is state dependent. More

precisely, we consider a Borel kernel $(p, q) \in D_{P,Q} \mapsto \tilde{\mu}(\cdot|p, q) \in \mathcal{M}([0, T] \times \mathbb{N}^8)$, where $\mathcal{M}([0, T] \times \mathbb{N}^8)$ denotes the collection of non-negative measures on $[0, T] \times \mathbb{N}^8$. In particular, it can depend on the order book's imbalance, as observed in e.g. [18]. To be consistent with the constraints imposed above, it satisfies :

$$\tilde{\mu}(\cdot|p, q) \text{ is supported by } \mathbf{C}_\circ, \text{ for all } (p, q) \in D_{P,Q}. \quad (4.2.5)$$

It should also be such that aggressive orders and cancellations are never bigger than the corresponding queue size, which will be made more explicit in our numerical example sections, see Section 4.3.4.

Next, we shall denote by E^ϕ the flows corresponding to the trading strategy of either a market maker, a high-frequency trader, or an institutional broker. Thus, she will assume facing a global flow $E = \tilde{E} + E^\phi$.

Moreover, for simplicity, we shall assume that $\tilde{\mu}$ is of the form

$$d\tilde{\mu}(c, \varepsilon, \varepsilon^b, \varepsilon^a, dt|p, q) = d\lambda(\varepsilon, \varepsilon^b, \varepsilon^a|p, q, c)d\beta(c|p, q)dt \quad (4.2.6)$$

in which λ and β are bounded Borel non-negative kernels and (without loss of generality)

$$\int d\lambda(\varepsilon, \varepsilon^b, \varepsilon^a|p, q, c) = 1, \text{ for all } (p, q, c) \in D_{P,Q} \times \mathbf{C}_\circ. \quad (4.2.7)$$

Later on, when an order $(\alpha_i^b, \alpha_i^a, L_i^b, L_i^a, L_i^{b, \frac{1}{2}}, L_i^{a, \frac{1}{2}}, M_i^b, M_i^a)$ is send by a MM, an HFT or an IB, we shall also assume that the conditional law of $(\epsilon_i, \epsilon_i^b, \epsilon_i^a)$ is given by λ .

Remark 4.2.1. *Let $\gamma(p, q) := \int d\beta(c|p, q)$. Then, γ is uniformly bounded by the above assumption. Let τ be a stopping time and fix $h > 0$. Since λ integrates to one, it follows that $\mathbb{P}[\#\{t \in [\tau, \tau + h] : \Delta\tilde{E}_t \neq 0\} = 1 | \mathcal{F}_\tau] = h\gamma(P_\tau, Q_\tau) + o(h)$ and $\mathbb{P}[\#\{t \in [\tau, \tau + h] : \Delta\tilde{E}_t \neq 0\} > 1 | \mathcal{F}_\tau] = o(h)$. Moreover, the process counting the number of jumps of \tilde{E} is dominated by a Poisson process with intensity $\bar{\gamma} := \sup \gamma < \infty$. Hence, $\mathbb{P}[\#\{t \leq T : \Delta\tilde{E}_t \neq 0\} \geq k] \leq \bar{\gamma}T/k$, for $k \geq 1$, by Markov's inequality. Similarly, if g is a non-decreasing Borel map, then $\mathbb{E}[g(\#\{t \in [0, T] : \Delta\tilde{E}_t \neq 0\})] \leq \sum_{k \geq 1} g(k) \frac{(\bar{\gamma}T)^k}{k!} e^{-\bar{\gamma}T}$.*

4.3 Market maker's optimal control problem

In this section, we describe the optimal control problem of the market maker, the key tools to characterize the solution and how to numerically approximate the optimal control.

4.3.1 Market maker's strategy and state dynamics

The market maker typically places limit orders in order to make profit of the spread but can turn aggressive when his inventory is too important. At the end the trading period $[0, T]$, the later should be zero. In the following, we denote by G his gain process and by I his inventory. We also need to keep track of the sizes of his orders already placed at the bid

queue, N^b , and at the ask queue, N^a . For simplicity, we impose that new orders can not be placed at the bid (respectively at the ask) if he already has previously taken a position at the bid (respectively at the ask). Then, his position at the bid (resp. at the ask) is completely described by N^b (resp. N^a) and the number of units B^b before him in the bid-queue (resp. B^a before him in the ask-queue). Later on, we only write $N = (N^b, N^a)$ and $B = (B^b, B^a)$.

To define the market maker's control, we assume that he faces the exogenous process \tilde{E} described in Section 4.2.

For him, a control is a sequence of random variables $\phi = (\tau_i^\phi, \mathbf{c}_i^\phi)_{i \geq 1}$ where $(\tau_i^\phi)_{i \geq 1}$ is an increasing sequence of times and each $\mathbf{c}_i^\phi = (\alpha_i^{b,\phi}, \alpha_i^{a,\phi}, L_i^{b,\phi}, L_i^{a,\phi}, L_i^{b,\frac{1}{2},\phi}, L_i^{a,\frac{1}{2},\phi}, M_i^{b,\phi}, M_i^{a,\phi})$ is \mathbf{C}_o -valued, see below for more implicit restrictions. The times $(\tau_i^\phi)_{i \geq 1}$ are the times at which he sends orders : the action done at τ_i^ϕ is \mathbf{c}_i^ϕ , whose components have the same meanings as in Section 4.2.

Given his own orders and the other participants' orders, the sequence of times at which orders are sent to the market is $(\tau_i)_{i \geq 1}$ where $\tau_0 := 0-$ and $\tau_{i+1} = \min\{\tilde{\tau}_j > \tau_i, j \geq 1\} \wedge \min\{\tau_j^\phi > \tau_i, j \geq 1\}$, see (4.2.4) and above.

We denote by E^ϕ the càdlàg process that jumps only at the times τ_i^ϕ 's with jump size

$$\Delta E_{\tau_i^\phi}^\phi = (\mathbf{c}_i^\phi, \sum_{j \geq 1} (\epsilon_j, \epsilon_j^b, \epsilon_j^a) \mathbf{1}_{\{\tau_i^\phi = \tau_j\}}), \quad i \geq 1,$$

so that⁴

$$E = E^\phi + \tilde{E}$$

from his point of view. As usual, we impose that E^ϕ is predictable for the (completed) filtration $\mathbb{F}^\phi = (\mathcal{F}_t^\phi)_{t \geq 0}$ generated⁵ by E . We will also keep in mind the number of actions

$$J := \sum_{i \geq 1} \mathbf{1}_{\{\tau_i^\phi \leq \cdot\}}$$

from time 0 on, as it may induce a cost.

We now impose a minimum and maximum inventory size, denoted by $(-I^*, I^*) \in (-\mathbb{N}) \times \mathbb{N}$, and that

$$\#\{\tau_i^\phi \leq T, i \geq 1\} \leq k_\phi \wedge J_o \text{ a.s., for some } k_\phi \in \mathbb{N},$$

for some $J_o \in \mathbb{N} \cup \{\infty\}$. The constraint on the inventory is classic. The constraint on the number of operations can be justified by operational constraints. In the case $J_o = \infty$, it just means that each control should be of essentially bounded activity, but the bound is not uniform on the set of controls and can be as large as needed.

To be admissible, a control ϕ should therefore be such that each \mathbf{c}_i^ϕ is $\mathbf{C}(Z_{\tau_i^\phi -})$ -valued, where

$$Z := (P, Q, X) \text{ with } X := (G, I, N, B, J),$$

4. We keep in mind that E also depends of ϕ but dot not make this explicit for ease of notations.

5. Note that this creates a dependence of the filtration on the control itself, which is similar to Chapter 2.

and, for $z = (p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j)$, $\mathbf{C}(z)$ is the collection of elements $c := (a^b, a^a, \ell^b, \ell^a, \ell^{b, \frac{1}{2}}, \ell^{a, \frac{1}{2}}, m^b, m^a) \in \mathbf{C}_\circ \setminus \{0\}$ such that :

$$\begin{aligned} a^b &\leq \min \{i + I^* - b^a ; q^b\} , a^a \leq \min \{I^* - i - b^b ; q^a\} , \\ \ell^b &\leq (I^* - i) \mathbf{1}_{\{n^b=0\}} , \ell^a \leq (i + I^*) \mathbf{1}_{\{n^a=0\}} , \\ \ell^{b, \frac{1}{2}} &\leq (I^* - i) \mathbf{1}_{\{p^a - p^b = 2\delta\}} \mathbf{1}_{\{n^b=0\}} , \ell^{a, \frac{1}{2}} \leq (i + I^*) \mathbf{1}_{\{p^a - p^b = 2\delta\}} \mathbf{1}_{\{n^a=0\}} \\ m^b &\leq n^b , m^a \leq n^a \\ c &= 0 \text{ if } j = J_\circ . \end{aligned}$$

Note that the constraints on the first three lines correspond to the fact that we do not want to take a position that could lead to an inventory out of the limits $-I^*$ and I^* if it was suddenly executed. The indicator functions correspond to additional constraints on the controls, imposed for numerical tractability : no new limit order can be send on a side if one has not been executed or has not canceled the position on the same side before, no limit order can be send in the spread if it is not equal to two ticks. In this case, we write $\phi \in \mathcal{C}(0, Z_{0-})$.

The dynamics of X is given by

$$X_{\tau_i} = \mathcal{T}_X(P_{\tau_{i-1}}, Q_{\tau_{i-1}}, X_{\tau_{i-1}}, \Delta E_{\tau_i}^\phi) \mathbf{1}_{\{\Delta E_{\tau_i}^\phi \neq 0\}} + \tilde{\mathcal{T}}_X(P_{\tau_{i-1}}, Q_{\tau_{i-1}}, X_{\tau_{i-1}}, \Delta \tilde{E}_{\tau_i}) \mathbf{1}_{\{\Delta E_{\tau_i}^\phi = 0\}} , \quad (4.3.1)$$

in which $\mathcal{T}_X, \tilde{\mathcal{T}}_X : \mathbb{R}^{22} \mapsto \mathfrak{d}\mathbb{Z} \times \mathbb{N} \times \mathbb{N}^4 \times \mathbb{N}$. More precisely, consider the map

$$\text{exe}(a, n, b) := \min\{(a - b)^+, n\}, \quad a, b, n \in \mathbb{N}.$$

It represents the number of stocks set at a limit that are executed when an aggressive order of size a arrives, that the position in the queue is b , and the size of the posted block at this position is n .

Then, having in mind the constraints encoded in $\mathbf{C}(\cdot)$ above, see also (4.2.1), we can write

$$\mathcal{T}_X = (\mathcal{T}_G, \mathcal{T}_I, \mathcal{T}_{N^b}, \mathcal{T}_{N^a}, \mathcal{T}_{B^b}, \mathcal{T}_{B^a}, \mathcal{T}_J) , \quad \tilde{\mathcal{T}}_X = (\tilde{\mathcal{T}}_G, \tilde{\mathcal{T}}_I, \tilde{\mathcal{T}}_{N^b}, \tilde{\mathcal{T}}_{N^a}, \tilde{\mathcal{T}}_{B^b}, \tilde{\mathcal{T}}_{B^a}, \tilde{\mathcal{T}}_J)$$

where

$$\begin{aligned} \mathcal{T}_G(p, q, x, \delta) &= g + (a^b - \text{exe}(a^b, n^b, b^b))p^b - (a^a - \text{exe}(a^a, n^a, b^a))p^a \\ \mathcal{T}_I(p, q, x, \delta) &= i - (a^b - \text{exe}(a^b, n^b, b^b)) + (a^a - \text{exe}(a^a, n^a, b^a)) \\ \mathcal{T}_{N^{b/a}}(p, q, x, \delta) &= n^{b/a} + [\ell^{b/a} - n^{b/a}]^+ + \ell^{b/a, \frac{1}{2}} - m^{b/a} - \text{exe}(a^{b/a}, n^{b/a}, b^{b/a}) \\ \mathcal{T}_{B^{b/a}}(p, q, x, \delta) &= b^{b/a} + (q^{b/a} - b^{b/a}) \mathbf{1}_{\{\ell^{b/a} \neq 0\}} - b^{b/a} \mathbf{1}_{\{m^{b/a} = n^{b/a}\}} - (b^{b/a} \wedge a^{b/a}) \mathbf{1}_{\{a^{b/a} \neq 0\}} - b^{b/a} \mathbf{1}_{\{\ell^{b/a, \frac{1}{2}} \neq 0\}} \\ \mathcal{T}_J(p, q, x, \delta) &= j + 1 \end{aligned}$$

and

$$\begin{aligned}
\tilde{\mathcal{T}}_G(p, q, x, \delta) &= g - \text{exe}(a^b, n^b, b^b)p^b + \text{exe}(a^a, n^a, b^a)p^a \\
\tilde{\mathcal{T}}_I(p, q, x, \delta) &= i + \text{exe}(a^b, n^b, b^b) - \text{exe}(a^a, n^a, b^a) \\
\tilde{\mathcal{T}}_{N^{b/a}}(p, q, x, \delta) &= n^{b/a} - \text{exe}(a^{b/a}, n^{b/a}, b^{b/a}) \\
\tilde{\mathcal{T}}_{B^{b/a}}(p, q, x, \delta) &= [b^{b/a} - a^{b/a}]^+ \mathbf{1}_{\{m^{b/a}=0\}} + (b^{b/a} - [m^{b/a} - (q^{b/a} - b^{b/a} - n^{b/a})]^+)^+ \mathbf{1}_{\{m^{b/a} \neq 0\}} \\
\tilde{\mathcal{T}}_J(p, q, x, \delta) &= 0,
\end{aligned}$$

for $x = (g, i, n^b, n^a, b^b, b^a, j)$, $\delta = (a^b, a^a, \ell^b, \ell^a, \ell^{b, \frac{1}{2}}, \ell^{a, \frac{1}{2}}, m^b, m^a, \varepsilon, \varepsilon^b, \varepsilon^a)$, $p = (p^b, p^a)$ and $q = (q^b, q^a)$.

Remark 4.3.1. *It follows from (4.2.6) and the constraint that E^ϕ is predictable that the probability that E^ϕ and \tilde{E} jump at the same time on $[0, T]$ is zero. This justifies the formulation (4.3.1).*

For later use, note that it follows from (4.2.2) and (4.3.1) that

$$Z_{\tau_i} = \mathcal{T}(Z_{\tau_{i-1}}, \Delta E_{\tau_i}^\phi) \mathbf{1}_{\{\Delta E_{\tau_i}^\phi \neq 0\}} + \tilde{\mathcal{T}}(Z_{\tau_{i-1}}, \Delta \tilde{E}_{\tau_i}) \mathbf{1}_{\{\Delta E_{\tau_i}^\phi = 0\}}, \quad (4.3.2)$$

in which

$$\mathcal{T} = (\mathcal{T}_{P,Q}, \mathcal{T}_X) \quad \text{and} \quad \tilde{\mathcal{T}} = (\mathcal{T}_{P,Q}, \tilde{\mathcal{T}}_X).$$

4.3.2 The optimal control problem

The aim of the market maker is to maximize her expected utility

$$\mathbb{E}[U(Z_T)]$$

in which

$$U(z) := -\exp\left(-\eta\{g + i^+ p^b - i^- p^a - \kappa([i^+ - q^b]^+ + [i^- - q^a]^+) - \varrho j\}\right) \quad (4.3.3)$$

for $z = (p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j)$. In the above, $\eta > 0$ is the absolute risk aversion parameter, and $\kappa > 0$ is a penalty term taking into account that liquidating the current inventory may lead to a worse price than the one corresponding to the best bid or ask : the quantity $i^+ p^b - i^- p^a$ corresponds to the liquidation value of the inventory if the bid and ask queues are big enough to absorb it, the expression starting from κ takes into account the number of shares that will not be liquidated at the best limit. The coefficient $\varrho \geq 0$ penalizes the number of actions taken by the market maker.

To define the corresponding value function, we now extend the definition of our state processes by writing

$$Z^{t,z,\phi} = (P^{t,z,\phi}, Q^{t,z,\phi}, X^{t,z,\phi})$$

for the process satisfying (4.2.2)-(4.3.1) for the control ϕ and with initial condition $Z_{t-}^{t,z,\phi} = z \in D_Z$ where D_Z is the collection of elements $(p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j) \in D_{P,Q} \times \mathfrak{d}\mathbb{Z} \times \{-I^*, \dots, I^*\} \times \mathbb{N}^4 \times \{0, \dots, J_o\}$ such that

$$\begin{aligned} n^b + i &\leq I^*, \quad i - n^a \geq -I^* \\ b^b + n^b &\leq q^b, \quad b^a + n^a \leq q^a. \end{aligned}$$

The corresponding set of admissible controls is $\mathcal{C}(t, z)$, and the filtration associated to $\phi \in \mathcal{C}(t, z)$ is $\mathbb{F}^{t,z,\phi}$. We then set

$$v(t, z) := \sup_{\phi \in \mathcal{C}(t,z)} J(t, z; \phi) \quad \text{for } (t, z) \in [0, T] \times D_Z,$$

where

$$J(t, z; \phi) := \mathbb{E}[U(Z_T^{t,z,\phi})].$$

Remark 4.3.2. For later use, observe that

$$v(t, z) = e^{-\eta g} v(t, p^b, p^a, q^b, q^a, 0, i, n^b, n^a, b^b, b^a, j)$$

for all $t \leq T$ and $z = (p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j) \in D_Z$. Moreover, if $J_o = \infty$, we also have

$$v(t, z) = e^{-\eta(g-\varrho j)} \bar{v}(t, z) := e^{-\eta(g-\varrho j)} v(t, p^b, p^a, q^b, q^a, 0, i, n^b, n^a, b^b, b^a, 0).$$

Remark 4.3.3. Note that v is bounded from above by 0 by definition. On the other hand, for all $z = (p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j) \in D_Z$,

$$\begin{aligned} v(t, z) &\geq \min_{i \in [-I^*, I^*]} \mathbb{E}[U(P_T^{t,z,0}, 0, 0, g, i, 0, 0, 0, 0, j)] \\ &= e^{-\eta(g-\varrho j)} \min_{i \in [-I^*, I^*]} \mathbb{E}[U(P_T^{t,z,0}, 0, 0, 0, i, 0, 0, 0, 0, 0)], \end{aligned}$$

where $P^{t,z,0}$ corresponds to the dynamics in the case that the MM does not act on the order book up to T . Moreover, it follows from (4.3.3) that

$$\mathbb{E}[U(P_T^{t,z,0}, 0, 0, 0, i, 0, 0, 0, 0, 0)] \geq -e^{\eta I^* |p^b|} \mathbb{E}[e^{\eta I^* (|P_T^{t,z,0,b} - p^b| + 2\mathfrak{d} + \kappa)}]$$

where

$$\sup_{p^b \in \mathfrak{d}\mathbb{Z}} \mathbb{E}[e^{\eta I^* (|P_T^{t,z,0,b} - p^b| + 2\mathfrak{d} + \kappa)}] < \infty,$$

by Remark 4.2.1 and the fact that the price can jump only by \mathfrak{d} when a market event occurs. Thus, v belongs to the class $\mathbf{L}_\infty^{\text{exp}}$ of functions φ such that φ/L is bounded, in which

$$L(p^b, g, j) := e^{-\eta(g-I^*|p^b|-\varrho j)}$$

for $(p^b, g, j) \in \mathfrak{d}\mathbb{Z} \times \mathfrak{d}\mathbb{Z} \times \{0, \dots, J_o\}$.

4.3.3 The dynamic programming equation

The derivation of the dynamic programming equation is standard, and is based on the dynamic programming principle. We state below the weak version of Bouchard and Touzi [22], we let v_* and v^* denote the lower- and upper-semicontinuous envelopes of v .

Proposition 4.3.1. *Fix $(t, z) \in [0, T] \times D_Z$ and a family $\{\theta^\phi, \phi \in \mathcal{C}(t, z)\}$ such that each θ^ϕ is a $[t, T]$ -valued $\mathbb{F}^{t, z, \phi}$ -stopping time and $\|Z_{\theta^\phi}^{t, z, \phi}\|_{\mathbf{L}^\infty} < \infty$. Then,*

$$\sup_{\phi \in \mathcal{C}(t, z)} \mathbb{E} \left[v_*(\theta^\phi, Z_{\theta^\phi}^{t, z, \phi}) \right] \leq v(t, z) \leq \sup_{\phi \in \mathcal{C}(t, z)} \mathbb{E} \left[v^*(\theta^\phi, Z_{\theta^\phi}^{t, z, \phi}) \right].$$

Proof. The right-hand side inequality follows from a conditioning argument, see [22]. The left-hand side is more delicate because the set of admissible controls depends on the initial data. However, it can be easily proved along the lines of Section 2.4.2 in Chapter 2 when $\{\theta^\phi, \phi \in \mathcal{C}(t, z)\}$ is $[t, T] \cap (\mathbb{N} \cup \{t, T\})$ -valued. Then, the general case is obtained by approximating $[t, T]$ -valued stopping times from the right (recall that Z is right-continuous). \square

One can then derive the corresponding dynamic programming equation. For $z = (p, q, x) \in D_Z$, $c \in \mathbf{C}(z)$, $t \leq T$, and a continuous and bounded function φ , we set

$$\mathcal{I}\varphi(t, z) := \int (\mathcal{K}^c \varphi(t, z) - \varphi(t, z)) d\beta(c|p, q) \text{ and } \mathcal{K}\varphi(t, z) := \sup_{c \in \mathbf{C}(z)} \mathcal{K}^c \varphi(t, z)$$

where we use the convention that $\mathcal{K}^0 = \sup\{\emptyset\} = -\infty$, and

$$\mathcal{K}^c \varphi(t, z) := \int \varphi(t, \mathcal{T}(z, c, \varepsilon, \varepsilon^b, \varepsilon^a)) d\lambda(\varepsilon, \varepsilon^b, \varepsilon^a|p, q, c),$$

recall (4.2.6), (4.2.7) and (4.3.2).

The partial differential equation characterization of v is then at least formally given by

$$\begin{aligned} \min \{-\partial_t \varphi - \mathcal{I}\varphi, \varphi - \mathcal{K}\varphi\} &= 0 \text{ on } [0, T] \times D_Z \\ \min \{\varphi - U, \varphi - \mathcal{K}\varphi\} &= 0 \text{ on } \{T\} \times D_Z. \end{aligned} \tag{4.3.4}$$

In order to ensure that the above is correct, we need two additional conditions.

Assumption 4.3.1. *For all upper-semicontinuous (resp. lower-semicontinuous) $\varphi \in \mathbf{L}_\infty^{\text{exp}}$, the map $(t, z) \in [0, T] \times D_Z \mapsto (\mathcal{I}, \mathcal{K})\varphi(t, z)$ is upper-semicontinuous (resp. lower-semicontinuous) and belongs to $\mathbf{L}_\infty^{\text{exp}}$.*

Assumption 4.3.2. *There exists a Borel function ψ that is continuously differentiable in time and such that*

- (i) $0 \geq \partial_t \psi + \mathcal{I}\psi$ on $[0, T] \times D_Z$,
- (ii) $\psi - \mathcal{K}\psi \geq \iota$ on $[0, T] \times D_Z$ for some $\iota > 0$,
- (iii) $\psi \geq U$ on $\{T\} \times D_Z$,

(iv) $\liminf_{n \rightarrow \infty} (\psi/L)(t_n, z_n) = \infty$ if $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$, for all $(t_n, z_n)_{n \geq 1} \subset [0, T] \times D_Z$.

Then, one can actually prove that v is the unique solution of (4.3.4) in the class $\mathbf{L}_\infty^{\text{exp}}$ defined in Remark 4.3.3.

Theorem 4.3.1. *Let Assumption 4.3.1 hold. Then, v_* (resp. v^*) is a viscosity supersolution (resp. subsolution) of (4.3.4). If moreover Assumption 4.3.2 holds, then v is continuous on $[0, T] \times D_Z$ and is the unique viscosity solution of (4.3.4), in the class of (discontinuous) solutions in $\mathbf{L}_\infty^{\text{exp}}$.*

Proof. In view of Proposition 4.3.1, the derivation of the viscosity super- and subsolution properties is very standard under Assumption 4.3.1, see e.g. Section 2.4 or [22]. As for uniqueness, let us assume that v and w are respectively a super- and a subsolution. Let ψ be as in Assumption 4.3.2. Then, $(v - w - \psi)(t_n, z_n)$ converges to $-\infty$ if $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$, for any sequence $(t_n, z_n)_{n \geq 1} \subset [0, T] \times D_Z$, and showing that $v \geq w$ on $[0, T] \times D_Z$ can be done by, e.g., following the line of arguments of Proposition 2.5.1. Finally, $v^*, v_* \in \mathbf{L}_\infty^{\text{exp}}$ by Remark 4.3.3. \square

Remark 4.3.4. *If $J_\circ < \infty$ and the supports of $\lambda(\cdot|p, q, c)$ and $\gamma(\cdot|p, q)$ are bounded, uniformly in $(p, q, c) \in (\partial\mathbb{Z})^2 \times \mathbb{N}^2 \times \mathbf{C}_\circ$, then it is not difficult to see that the function defined by*

$$\psi(t, z) := e^{2\eta(1+I^*)|z|} e^{-r(j+t)}, \text{ for } z = (p, q, g, i, n, b, j) \in D_Z \text{ and } t \leq T,$$

satisfies the requirements of Assumption 4.3.2, for r large enough. Verifying Assumption 4.3.2 in the case $J_\circ = \infty$ seems much more difficult. On the other hand, the sequence of value functions associated to a sequence $(J_\circ^n)_{n \geq 1}$ increases to the value function associated to $J_\circ = \infty$ as $J_\circ^n \rightarrow \infty$. which provides a natural way to construct a convergent numerical scheme for the computation of v and the optimal control policy, see Sections 4.3.4 and 4.3.5 below. Standard arguments based on this approximation would also imply that $v_ = v$ and that v is the smallest supersolution of (4.3.4), in the class of (discontinuous) solutions in $\mathbf{L}_\infty^{\text{exp}}$.*

4.3.4 Dimension reduction, symmetries and numerical resolution

Before to provide a converging numerical scheme for (4.3.4), let us first recall that the variables g (and j if $J_\circ = \infty$) can be omitted, see Remark 4.3.2. If moreover, the transition kernels depend on prices only through the spread (which is a natural assumption at least on a rather short time horizon), then one more dimension can be eliminated.

Assumption 4.3.3. *The kernel $(p, q, c) \in D_{P,Q} \times \mathbf{C} \mapsto (\lambda(\cdot|p, q, c), \beta(\cdot|p, q))$ depend on $p = (p^b, p^a)$ only through the value of the mid-spread $\delta p := (p^a - p^b)/2$.*

Indeed, if Assumption 4.3.3 holds, then one easily checks that

$$\begin{aligned} e^{-\eta p^\circ} \bar{v}(t, -\delta p, \delta p, q^b, q^a, 0, i, \cdot) &= \bar{v}(t, p^\circ - \delta p, p^\circ + \delta p, q^b, q^a, 0, i, \cdot) \\ &= \bar{v}(t, p^b, p^a, q^b, q^a, 0, i, \cdot) \end{aligned}$$

with $p^\circ := (p^a + p^b)/2$, so that

$$e^{ni(p^b + \delta p)} \bar{v}(t, p^b, p^b + 2\delta p, q^b, q^a, 0, i, \cdot)$$

does not depend on p^b but only on δp .

The resolution of the equation can also be simplified by using potential symmetries, in the sense of the following assumption.

Assumption 4.3.4. *For all $(p, q) \in D_{P,Q}$, $c := (a^b, a^a, \ell^b, \ell^a, \ell^{b, \frac{1}{2}}, \ell^{a, \frac{1}{2}}, m^b, m^a) \in \mathbf{C}$ and all Borel sets $O \subset \{0, 1\}$, $O^b, O^a \subset \mathbb{N}$,*

$$\int_{O \times O^b \times O^a} d\lambda(\varepsilon, \varepsilon^b, \varepsilon^a | p, q, c) = \int_{O \times O^a \times O^b} d\lambda(\varepsilon, \varepsilon^a, \varepsilon^b | \bar{p}, \bar{q}, \bar{c})$$

where $\bar{p} = (-p^a, -p^b)$, $\bar{q} = (q^a, q^b)$ and $\bar{c} = (a^a, a^b, \ell^a, \ell^b, \ell^{a, \frac{1}{2}}, \ell^{b, \frac{1}{2}}, m^a, m^b)$. Moreover, for all Borel sets $O = O_a^b \times O_a^a \times O_\ell^b \times O_\ell^a \times O_{\ell^{\frac{1}{2}}}^b \times O_{\ell^{\frac{1}{2}}}^a \times O_m^b \times O_m^a \subset \mathbb{N}^8$,

$$\int_O d\beta(c | p, q) = \int_{\bar{O}} d\beta(c | \bar{p}, \bar{q})$$

where $\bar{O} := O_a^a \times O_a^b \times O_\ell^a \times O_\ell^b \times O_{\ell^{\frac{1}{2}}}^a \times O_{\ell^{\frac{1}{2}}}^b \times O_m^a \times O_m^b$.

The above assumption implies that the transition probabilities of the order book are symmetric at the bid and at the ask, whenever the configurations are. Then, v admits a symmetry which can be exploited to reduce the complexity of the numerical resolution of (4.3.4). Namely, under Assumption 4.3.4, we have

$$\bar{v}(t, p^b, p^a, q^b, q^a, 0, i, n^b, n^a, b^b, b^a, j) = \bar{v}(t, -p^a, -p^b, q^a, q^b, 0, -i, n^a, n^b, b^a, b^b, j).$$

Let us now turn to the definition of a numerical scheme for (4.3.4). We now make the additional assumption that the supports of λ and β are bounded (not that they are already discrete, by nature).

Assumption 4.3.5. *$J_o < \infty$ and there exists finite Borel sets $O_1 \subset \mathbb{N}^8$ and $O_2 \subset \mathbb{N}^3$ such that $\beta(\cdot | p, q)$ is supported by O_1 and $\lambda(\cdot | p, q, c)$ is supported by O_2 , for all $(p, q, c) \in D_{P,Q} \times \mathbf{C}$.*

Then, the operators \mathcal{I} and \mathcal{K} are explicit. Hence, the only required discretization is in time. For a time step $T/n > 0$, we define a time grid $\pi_n := \{t_i^n, i \leq n\}$ where $t_i^n = iT/n$ for $i \leq n$. We next consider the sequence of space domains $D_Z^k := D_Z \cap [-k, k]^{11}$ for $k \geq 1$, and we let v_n^k be the solution of

$$\begin{aligned} v_n^k(t_i^n, \cdot) &= \max \left\{ v_n^k(t_{i+1}^n, \cdot) + \frac{T}{n} \mathcal{I} v_n^k(t_{i+1}^n, \cdot), \max_{c \in \mathbf{C}(\cdot)} \mathcal{K}^{c,n} v_n^k(t_{i+1}^n, \cdot) \right\} = 0 \text{ on } D_Z^k, \quad i \leq n-1 \\ v_n^k - U &= 0 \text{ on } (\{T\} \times D_Z^k) \cup (\pi_n \times (D_Z \setminus D_Z^k)), \end{aligned} \tag{4.3.5}$$

where

$$\mathcal{K}^{c,n} = \mathcal{K}^c + \frac{T}{n} \mathcal{I} \circ \mathcal{K}^c.$$

This fully explicit scheme is convergent.

Proposition 4.3.2. *Let Assumption 4.3.5 hold, then the sequence $(v_n^k)_{k,n \geq 1}$ converges pointwise to v on $[0, T) \times D_Z$ as $k, n \rightarrow \infty$.*

Proof. First note that $(v_n^k/L)_{k,n \geq 1}$ is uniformly bounded, where L is defined in Remark 4.3.3. This follows from Assumption 4.3.5 and a simple induction argument, compare with Remark 4.3.3. Then, standard stability results, see e.g. [14] and Section 2.6.1, imply that the relaxed upper-limit v^∞ and lower-limit of v_∞ of $(v_n^k)_{k,n \geq 1}$ are respectively sub- and supersolution of (4.3.4) and belong to the class $\mathbf{L}_\infty^{\text{exp}}$. The comparison result mentioned in the proof of Theorem 4.3.1, see Remark 4.3.4, thus implies that $v_\infty \geq v \geq v^\infty$ while $v_\infty \leq v^\infty$ by definition. \square

4.3.5 Approximate optimal controls

In the following, we estimate the optimal control in a classical way. For each $i < n$, we choose a measurable map $\hat{c}_n^k(t_i^n, \cdot)$ such that

$$\begin{aligned} \hat{c}_n^k(t_i^n, \cdot) &\in \arg \max \{ \mathcal{K}^{c,n} v_n^k(t_{i+1}^n, \cdot), c \in \mathbf{C}(\cdot) \} \text{ on } D_Z^k \\ \hat{c}_n^k(t_i^n, \cdot) &= 0 \text{ on } D_Z \setminus D_Z^k, \end{aligned}$$

and define the sequence of stopping times

$$\hat{\tau}_{j+1}^{n,k} := \min \{ t_i^n : i \geq 0, t_i^n > \hat{\tau}_j^{n,k}, (v_n^k - \mathcal{K}^{\hat{c}_n^k} v_n^k(t_{i+1}^n, \cdot))(t_i^n, \hat{Z}_{t_i^n}^{n,k}) = 0 \}, j \geq 0,$$

with $\hat{\tau}_0^{n,k} := 0-$, and in which $\hat{Z}^{n,k} = (\hat{P}^{n,k}, \hat{Q}^{n,k}, \hat{X}^{n,k})$ is defined as in (4.3.2) for the initial condition Z_{0-} at 0 and the control associated to $\hat{\phi}_n^k := (\hat{\tau}_i^{n,k}, \hat{c}_n^k(\hat{\tau}_i^{n,k}, \hat{Z}_{\hat{\tau}_i^{n,k}}^{n,k}))_{i \geq 1}$ in a Markovian way. This provides a sequence of controls that is asymptotically optimal.

Proposition 4.3.3. *Let the conditions of Proposition 4.3.2 hold. Then,*

$$\lim_{n,k \rightarrow \infty} \mathbb{E}[U(\hat{Z}_T^{n,k})] = v(0, Z_{0-}).$$

Proof. Let $\gamma(p, q) := \int d\beta(c|p, q)$ and recall that γ is uniformly bounded by assumption, as well as the sequence $(v_n^k)_{k,n \geq 1}$. Then, it follows from Remark 4.2.1 that

$$\begin{aligned} v_n^k(t_{i+1}^n, \hat{Z}_{t_i^n}^{n,k}) + \frac{T}{n} \mathcal{I} v_n^k(t_{i+1}^n, \hat{Z}_{t_i^n}^{n,k}) &= v_n^k(t_{i+1}^n, \hat{Z}_{t_i^n}^{n,k}) (1 - \frac{T}{n} \gamma(\hat{P}_{t_i^n}^{n,k}, \hat{Q}_{t_i^n}^{n,k})) \\ &\quad + \frac{T}{n} \int \mathcal{K}^c v_n^k(t_{i+1}^n, \hat{Z}_{t_i^n}^{n,k}) d\beta(c | \hat{P}_{t_i^n}^{n,k}, \hat{Q}_{t_i^n}^{n,k}) \\ &= \mathbb{E}[v_n^k(t_{i+1}^n, \tilde{\mathcal{T}}(\hat{Z}_{t_i^n}^{n,k}, \Delta \tilde{E}_{t_{i+1}^n}^k)) | \mathcal{F}_{t_i^n}^{\hat{\phi}_n^k}] + o(n^{-1}) \end{aligned}$$

and, similarly,

$$\mathcal{K}^{\hat{c}_n^k(\hat{\tau}_i^{n,k}, \hat{Z}_{\hat{\tau}_i^{n,k}}^{n,k})} v_n^k(\hat{\tau}_i^{n,k} + \frac{T}{n}, \hat{Z}_{\hat{\tau}_i^{n,k}}^{n,k}) = \mathbb{E}[v_n^k(\hat{\tau}_i^{n,k} + \frac{T}{n}, \tilde{\mathcal{T}}(\hat{Z}_{\hat{\tau}_i^{n,k} + \frac{T}{n}}^{n,k}, \Delta \tilde{E}_{\hat{\tau}_i^{n,k} + \frac{T}{n}}^k)) | \mathcal{F}_{\hat{\tau}_i^{n,k}}^{\hat{\phi}_n^k}] + o(n^{-1})$$

with the convention that $\tilde{\mathcal{T}}(\cdot, 0)$ is the identity. Let θ_k^n be the first time when $\hat{Z}^{n,k}$ exists the domain $D_{\tilde{Z}}^k$. In view of (4.3.5), Assumption 4.3.5 (in particular that $J_0 < \infty$) and Remark 4.2.1 (see the arguments at the end of Remark 4.3.3), an induction implies that

$$v_n^k(0, Z_{0-}) = \mathbb{E}[U(\hat{Z}_{\theta_k^n}^{n,k})] + o_n(1) = \mathbb{E}[U(\hat{Z}_T^{n,k})] + o_n(1) + o_k(1)$$

in which $o_n(1)$ and $o_k(1)$ go to 0 as $n \rightarrow \infty$ and $k \rightarrow \infty$. It remains to appeal to Proposition 4.3.2. \square

4.3.6 Numerical experiments

We now turn to a numerical experiment. We compute an approximation of the optimal control as described in Section 4.3.5, using the simplifications detailed in Section 4.3.4.

Let us first describe a realistic prior distribution for the evolution of \tilde{E} . The coefficients we use are inspired from the behavior of the stock Société Générale (CLE FP)⁶ and from [18]. More precise and realistic statistics will be provided in our companion paper [8].

As for the prior on the dynamics of the market. We simply consider that both market and limit orders arrive according to a Poisson process. Both limit and aggressive orders arrive with an intensity of 0.6 per second. When a limit order arrives, it is assigned to the bid or the ask with probability 1/2. When a market order arrives, we assign it to the bid or the ask according to the statistics described for big caps in [18, Chart 8 p.10]. Namely, if

$$\text{Imb}_{\tilde{\tau}_i} := \frac{Q_{\tilde{\tau}_i}^b - Q_{\tilde{\tau}_i}^a}{Q_{\tilde{\tau}_i}^b + Q_{\tilde{\tau}_i}^a}$$

is the order book imbalance at the time $\tilde{\tau}_i$ at which the market order arrives, then it arrives at the ask with probability $0.5 + 0.35 * \text{Imb}_{\tilde{\tau}_i}$.

To describe the size of the orders and of the inventory, we take 1/2 of the ATS (mediAn⁷ Trade Size) as the unit.

The size of the market orders is also assumed to be dependent on the order book imbalance. We use [18, Chart 16 p.10] to estimate that the size of the trade arriving at the ask (if it arrives at the ask), represents a percentage of the queue (that we round to an integer number, by above). Namely, we set $\hat{f}_{\tilde{\tau}_i}^a := 0.7 + 0.3 * \text{Imb}_{\tilde{\tau}_i}$, and assign a (conditional) probability of 60% that the order is of size $\text{round}[\hat{f}_{\tilde{\tau}_i}^a * Q_{\tilde{\tau}_i}^a]$ and the same probability that the executed volume deviates from the latter by one unit (with equal probability to be by one more and one less unit - if quantities are negative or bigger than the size of the queue, we obviously set them to 0 or to the size of the queue).

We use a simpler modeling approach for the limit orders. With 55% probability a limit order (if it arrives) is of size of 2 (recall that the unit is 1/2 of an ATS). It is of size 3 with probability 10% and of size 1 with probability 35%. Again, it is based on [18].

6. We thank Chevreux-Kepler for providing us these data.

7. Following [18, Page 23].

When a queue is depleted, the probability of a price move is set to $\mathbb{P}[\epsilon_i = 1 | \mathcal{F}_{\bar{\tau}_i}] = 75\%$. If the bid is depleted but the bid price does not go done, the size of the new bid queue is set to 2 units with probability 60%, 1 unit with probability 25% and 3 units with probability 15%, otherwise it is set to 10 units with probability 60%, 5 units with probability 25% and 12 units with probability 15%. The same applies to the ask price if this is the ask queue that is depleted.

If both ask and bid prices move, recall (4.2.3), we take the same distribution for both queues and consider them as being (conditionally) independent. The distribution corresponds to the one of a price move, as described above.

When the spread is equal to two ticks, the next limit order arrives in the spread with probability 90%. This models the fact that a spread of two ticks is not common, see e.g. [29]. To be consistent with the probability of arrival of market orders at the bid or at the ask, we assume that a new bid limit is created in the spread with probability $0.5 + 0.35 * \text{Imb}_{\bar{\tau}_i}$ (otherwise, this is a new ask limit). The size of this new limit is 2 with 55%, of size 3 with probability 10% and of size 1 with probability 35%. This corresponds to the behavior of the stock Société Générale (CLE FP). We do not change the dynamic of aggressive orders with the size of the spread.

Let us now describe the other parameters of the Market Maker's optimal control problem. We set the bid price to 10 at time $0-$, the spread is one tick, and the tick equal to 0.01, recall from Section 4.3.4 that only the spread size matters (because we have here the required symmetry) and observe that the latter can be rescaled together with the level of risk aversion, which is here fixed to $\eta = 1$. We take $\kappa = 0.02$ and $\varrho = 10^{-20}$. The time horizon is $T = 59$ second, and the time step is 1 second. We keep a small time horizon for a better visibility of the evolution of the order book. We do not consider cancellations from the rest of the market for simplicity. Moreover, in order to reduce the computation time, we add the additional constraint that the MM can not cancel only part of position in a queue, he can only cancel the whole position. We also fix a maximal queue size of 12 and fix the maximal absolute value of the inventory to $I^* = 7$. The size of the limit and market orders sent by the MM are constrained to be less than 3. This corresponds to adding an additional constraint in the definition of $\mathcal{C}(\cdot)$ which does not change the above analysis.

In Figure 4.2, we provide a simulated path of the optimal strategy, starting from a symmetric configuration of the order book, with queue lengths equal to 6. In this simulation, the MM always play before the other (random) players⁸. The top left graphic describes the control played by the MM. Triangles pointing outward (with respect to the zero line) correspond to limit orders, the number of triangles giving their size. Arrows with triangles pointing inward are cancellations, again the number of triangles gives the size. Aggressive orders are symbolized by lines with squares, while limit orders within the spread correspond to the lines with dots. The top right graphic gives the state of the order book just after the MM has played, and before the nature (i.e. the other players) plays. The size of the lines gives the total length of each queues, while the dots symbolize the position of the MM in the queue. The middle left graphic describes the state of the order book after the nature plays. The bottom

8. This is just a convention, since the transition probabilities do not depend on time.

left graphic gives the value of the portfolio of the MM if he had to liquidate (by sending aggressive orders) his stock holding at the best bid/ask price. The final value gives the “true” liquidation value of the book. In the case that the final inventory can not be liquidated at the best bid/ask price, we liquidate the remaining part at the best bid/ask price minus/plus one tick. The middle right graphic is the inventory just before he plays and the bottom right one is the bid and ask prices just after he plays. Figure 4.1 provides the distribution of the gain made by the MM, it uses 10^5 simulated paths.

His strategy can seem difficult to interpret at first sight. But, we have to remember that he believes that the imbalance plays an important role in the book order dynamics, and that he not only should take care of it but that he can actually use it : some limit orders are send not to be executed but to influence the evolution of the price in a favorable direction. Also note that he should avoid the price to go down/up if his inventory is positive/negative. Having this in mind, it is not surprising that he can be sometimes at the limit of a price manipulation strategy. Finally, we have to keep in mind that his strategy is constrained. He sometimes cancels positions to be free to react more quickly to a more favorable market configuration later on.

Not surprisingly, the MM first send limit orders of equal sizes on both sides of the order book. Nothing happens until time $t = 9$. At this time is inventory is 2 and his position at the ask is still far from the beginning of the queue. In order to avoid increasing again his inventory, he cancels his remaining position of one unit at the bid. He puts a new position at the bid at time 10 after the queue has slightly increased, because of an exogenous limit order. Unfortunately for him, this new position is immediately executed, and his inventory jumps to 5. After the queue is regenerated, he immediately puts a limit order of size 2 at the bid. His reasoning is the following : 1. this position has little chance to be executed immediately, he does not take a risk of again increasing his inventory ; 2. by doing this he increases the imbalance and therefore the probability of being executed at the ask so has to gain the spread and reduce his inventory. This strategy is successful since immediately 2 units of his positions at the ask are executed. The imbalance is still good even if he cancels his last unit at the bid, so as to be free to play the control he wants once his last unit at the ask will be executed. In fact, it does not work and he decides to refresh his global position at time 16 by canceling his final unit at the ask and putting limit orders again at time 17 in a symmetric way. This is a limit spread order at the bid. This makes sense since he has to avoid the stock price to go down, because he has a positive inventory. The fact that just after he alternates between putting and canceling limit orders at the ask is probably a numerical artifact : on the one hand, he wants to have a limit position at the ask because his inventory is large, on the other hand he does not want to increase the imbalance to avoid having a too big probability of being executed at the bid and of seeing the price go down. Similarly, he wants to keep a position at the bid to avoid a downward price move, while he really does not want to be executed on this side. The aggressive order at the bid at time 19 is just a partial cancellation, that is completed at time 20. He just does not want to be executed. By doing so, he unfortunately causes a downward jump of the bid, which is not good for him. Just after he sends a limit spread order at the bid to fight against this downward pressure, and then cancels it once the bid is back at a distance one tick of the ask. This strategy is successful.

The rest of his orders can be interpreted in a similar manner. Just note that he starts to be aggressive at the bid side at time 49 because his inventory is too big and the maturity starts to be quite close, in particular the first market order of size 3 compensates the execution of a position of size 3 at the bid just before (so that the graphic of the inventory does not move, although the inventory temporary jumps to the upper limit 7).

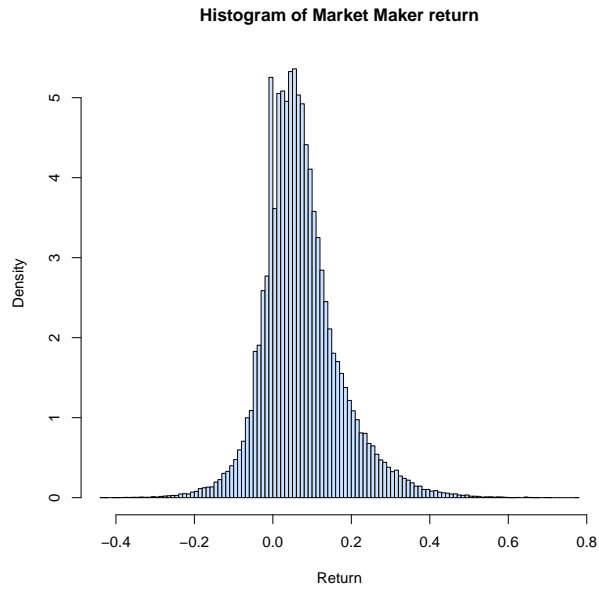


FIGURE 4.1 – Density estimation of the gain made by the Market Maker.

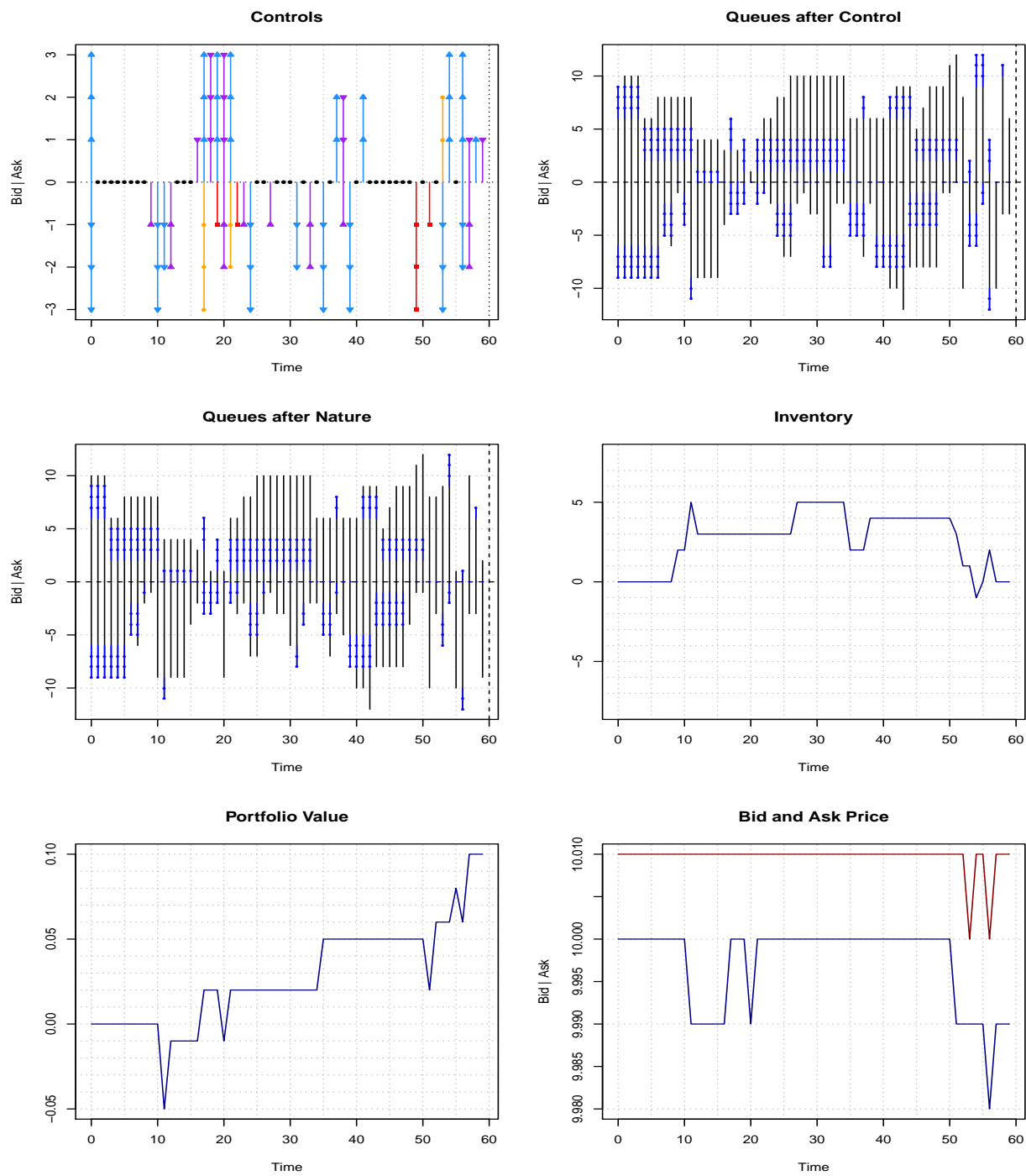


FIGURE 4.2 – Simulated path of the optimal strategy of the Market Maker.

4.4 High frequency trader's pair trading problem

We now turn to the HFT strategy. We consider a pair trading strategy where the trader invests in the difference of two highly correlated assets. Here, we choose the futures price of the stock to be the second asset. The order book defined in Section 4.2 represents the dynamics of the stock. Whenever the trader buys/sells n units of stocks (being by an aggressive order or by the execution of a limit order), she sells/buys immediately n units of the futures. Her inventory should be fully liquidated at T .

4.4.1 The optimal control problem

We assume that the reference price F of the futures is given by

$$F = \frac{P^b + P^a}{2} + S \quad (4.4.1)$$

where S is a mean-reverting process

$$S = S_0 + \int_0^\cdot \rho(\hat{s} - S_t)dt + \int_0^\cdot \sigma(S_t)dW_t. \quad (4.4.2)$$

Here, ρ is the strength of mean reversion, \hat{s} is the average of mean reversion and $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz bounded function representing the volatility of the process. In the following, we shall assume that

$$\text{the support of } \sigma \text{ is bounded.} \quad (4.4.3)$$

Remark 4.4.1. *The above implies in particular that S lies in a certain compact set D_S as soon as S_0 does. This could clearly be relaxed to the price of a finer analysis.*

The strategy of the HFT is described by the same quantities as the one of the MM in Section 4.3. The only difference lies in the fact that she constantly holds a number equal to $-I$ units of the futures F . We assume that buying/selling the futures leads to the payment of a proportional cost $\kappa \geq 0$. Then, the dynamics of the corresponding state process X is given by

$$X_{\tau_i} = \mathcal{T}_X(S_{\tau_i}, P_{\tau_{i-1}}, Q_{\tau_{i-1}}, X_{\tau_{i-1}}, \Delta E_{\tau_i}^\phi) \mathbf{1}_{\{\Delta E_{\tau_i}^\phi \neq 0\}} + \tilde{\mathcal{T}}_X(S_{\tau_i}, P_{\tau_{i-1}}, Q_{\tau_{i-1}}, X_{\tau_{i-1}}, \Delta \tilde{E}_{\tau_i}) \mathbf{1}_{\{\Delta E_{\tau_i}^\phi = 0\}}, \quad (4.4.4)$$

with \mathcal{T}_X and $\tilde{\mathcal{T}}_X$ now defined with respect to

$$\mathcal{T}_G(s, p, q, x, \delta) = g + (a^b - \text{exe}(a^b, n^b, b^b))\Delta_-^b - (a^a - \text{exe}(a^a, n^a, b^a))\Delta_+^a$$

$$\mathcal{T}_I(s, p, q, x, \delta) = i - (a^b - \text{exe}(a^b, n^b, b^b)) + (a^a - \text{exe}(a^a, n^a, b^a))$$

$$\mathcal{T}_{N^{b/a}}(s, p, q, x, \delta) = n^{b/a} + [\ell^{b/a} - n^{b/a}]^+ + \ell^{b/a, \frac{1}{2}} - m^{b/a} - \text{exe}(a^{b/a}, n^{b/a}, b^{b/a})$$

$$\mathcal{T}_{B^{b/a}}(p, q, x, \delta) = b^{b/a} + (q^{b/a} - b^{b/a}) \mathbf{1}_{\{\ell^{b/a} \neq 0\}} - b^{b/a} \mathbf{1}_{\{m^{b/a} = n^{b/a}\}} - (b^{b/a} \wedge a^{b/a}) \mathbf{1}_{\{a^{b/a} \neq 0\}} - b^{b/a} \mathbf{1}_{\{\ell^{b/a} \neq 0\}}$$

$$\mathcal{T}_J(s, p, q, x, \delta) = j + 1$$

and

$$\begin{aligned}
\tilde{\mathcal{T}}_G(s, p, q, x, \delta) &= g - \text{exe}(a^b, n^b, b^b) \Delta_+^b + \text{exe}(a^a, n^a, b^a) \Delta_-^a \\
\tilde{\mathcal{T}}_I(s, p, q, x, \delta) &= i + \text{exe}(a^b, n^b, b^b) - \text{exe}(a^a, n^a, b^a) \\
\tilde{\mathcal{T}}_{N^{b/a}}(s, p, q, x, \delta) &= n^{b/a} - \text{exe}(a^{b/a}, n^{b/a}, b^{b/a}) \\
\tilde{\mathcal{T}}_{B^{b/a}}(p, q, x, \delta) &= [b^{b/a} - a^{b/a}]^+ \mathbf{1}_{\{m^{b/a}=0\}} + (b^{b/a} - [m^{b/a} - (q^{b/a} - b^{b/a} - n^{b/a})]^+)^+ \mathbf{1}_{\{m^{b/a} \neq 0\}} \\
\tilde{\mathcal{T}}_J(s, p, q, x, \delta) &= 0,
\end{aligned}$$

in which

$$\Delta_{\pm}^b := p^b - \frac{p^b + p^a}{2} - s \pm \kappa, \quad \Delta_{\pm}^a := p^a - \frac{p^b + p^a}{2} - s \pm \kappa.$$

In the above, we used the notations $x = (g, i, n^b, n^a, b^b, b^a, j)$, $\delta = (a^b, a^a, \ell^b, \ell^a, \ell^{b, \frac{1}{2}}, \ell^{a, \frac{1}{2}}, m^b, m^a, \varepsilon, \varepsilon^b, \varepsilon^a)$, $p = (p^b, p^a)$ and $q = (q^b, q^a)$.

The set of admissible controls $\mathcal{C}(0, S_0, Z_{0-})$ is defined as in Section 4.3 but with respect to the (completed) filtration $\mathbb{F}^\phi = (\mathcal{F}_t^\phi)_{t \geq 0}$ generated by (S, E) .

We also assume that she has an exponential type utility function, with risk aversion parameter $\eta > 0$. Then, she wants to maximize over $\phi \in \mathcal{C}(0, S_0, Z_{0-})$ the expected utility

$$\mathbb{E}[U(S_T, Z_T^\phi)]$$

where

$$U(s, z) := -\exp\left(-\eta\{g + i^+ \Delta_-^b - i^- \Delta_+^a - \kappa([i^+ - q^b]^+ + [i^- - q^a]^+) - \varrho j\}\right) \quad (4.4.5)$$

for $z = (p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j)$, and where Δ_{\pm}^b and Δ_{\pm}^a are defined as above.

As in Section 4.3, we next extend the definition of our state processes by writing

$$(S^{t,s}, Z^{t,s,z,\phi}) = (S^{t,s}, P^{t,s,z,\phi}, Q^{t,s,z,\phi}, X^{t,s,z,\phi})$$

for the process satisfying (4.2.2)-(4.3.1)-(4.4.2) for the control ϕ and the initial condition $(S_t^{t,s}, Z_t^{t,s,z,\phi}) = (s, z) \in D_S \times D_Z$. The corresponding set of admissible controls is $\mathcal{C}(t, s, z)$, and the filtration associated to $\phi \in \mathcal{C}(t, s, z)$ is $\mathbb{F}^{t,s,z,\phi} = (\mathcal{F}_s^{t,s,z,\phi})_{s \in [t, T]}$. We finally define the value function

$$v(t, s, z) := \sup_{\phi \in \mathcal{C}(t,s,z)} J(t, s, z; \phi) \quad \text{for } (t, s, z) \in [0, T] \times D_S \times D_Z,$$

where

$$J(t, s, z; \phi) := \mathbb{E}[U(S_T^{t,s}, Z_T^{t,s,z,\phi})].$$

We close this section with Remarks that are the counterparts of Remarks 4.3.2 and 4.3.3.

Remark 4.4.2. For later use, observe that

$$v(t, s, z) = e^{-\eta g} v(t, s, p^b, p^a, q^b, q^a, 0, i, n^b, n^a, b^b, b^a, j)$$

for all $t \leq T$, $s \in D_S$ and $z = (p^b, p^a, q^b, q^a, g, i, n^b, n^a, b^b, b^a, j) \in D_Z$. Moreover, if $J_o = \infty$, we also have

$$v(t, s, z) = e^{-\eta(g-\varrho j)} \bar{v}(t, z) := e^{-\eta(g-\varrho j)} v(t, s, p^b, p^a, q^b, q^a, 0, i, n^b, n^a, b^b, b^a, 0).$$

Remark 4.4.3. Note that v is bounded from above by 0 by definition. It also follows from (4.4.3) that $S^{t,s}$ takes values in the compact set D_S so that $v \in \mathbf{L}_\infty^{\text{exp}}$ by the same reasoning as in Remark 4.3.3.

4.4.2 The dynamic programming equation

As in Section 4.3.3, we first provide a dynamic programming principle. Again, we let v_* and v^* denote the lower- and upper-semicontinuous envelopes of v .

Proposition 4.4.1. Fix $(t, s, z) \in [0, T] \times D_S \times D_Z$ and a family $\{\theta^\phi, \phi \in \mathcal{C}(t, s, z)\}$ such that each θ^ϕ is a $[t, T]$ -valued $\mathbb{F}^{t,s,z,\phi}$ -stopping time and $\|(S_{\theta^\phi}^{t,s}, Z_{\theta^\phi}^{t,s,z,\phi})\|_\infty < \infty$. Then,

$$\sup_{\phi \in \mathcal{C}(t,x,q)} \mathbb{E} \left[v_*(\theta^\phi, S_{\theta^\phi}^{t,s}, Z_{\theta^\phi}^{t,s,z,\phi}) \right] \leq v(t, s, z) \leq \sup_{\phi \in \mathcal{C}(t,x,q)} \mathbb{E} \left[v^*(\theta^\phi, S_{\theta^\phi}^{t,s}, Z_{\theta^\phi}^{t,s,z,\phi}) \right].$$

Proof. Let $\mathcal{C}_k(t, x, z)$ be the set of controls ϕ satisfying the additional constraint $\#\{\tau_i^\phi, i \geq 1\} \leq k$ a.s., and let v_k be the corresponding value function, for $k \geq 1$. Then, it is not difficult to see that v_k is continuous, and the arguments of [22] imply that

$$\begin{aligned} \sup_{\phi \in \mathcal{C}_k(t,x,z)} \mathbb{E} \left[v_k(\theta^\phi, S_{\theta^\phi}^{t,s}, Z_{\theta^\phi}^{t,s,z,\phi}) \right] &\leq v_k(t, s, z) \\ &\leq \sup_{\phi \in \mathcal{C}_k(t,x,z)} \mathbb{E} \left[v_k(\theta^\phi, S_{\theta^\phi}^{t,s}, Z_{\theta^\phi}^{t,s,z,\phi}) \right] \\ &\leq \sup_{\phi \in \mathcal{C}(t,x,z)} \mathbb{E} \left[v(\theta^\phi, S_{\theta^\phi}^{t,s}, Z_{\theta^\phi}^{t,s,z,\phi}) \right]. \end{aligned}$$

Since by definition $v = \lim_{k \rightarrow \infty} \uparrow v_k$ and $\mathcal{C}(t, x, z) = \cup_{k \geq 1} \mathcal{C}_k(t, x, z)$, sending $k \rightarrow \infty$ in the above leads to the required result, recall Remark 4.4.3. \square

The partial differential equation associated to v is then at least formally given by

$$\begin{aligned} \min \{-\mathcal{L}\varphi - \mathcal{I}\varphi, \varphi - \mathcal{K}\varphi\} &= 0 \text{ on } [0, T) \times D_S \times D_Z \\ \min \{\varphi - U, \varphi - \mathcal{K}\varphi\} &= 0 \text{ on } \{T\} \times D_S \times D_Z, \end{aligned} \tag{4.4.6}$$

in which \mathcal{L} is the Dynkin operator associated to (4.4.2) :

$$\mathcal{L}\varphi = \partial_t \varphi + \rho(\hat{s} - s) \partial_s \varphi + \frac{1}{2} \sigma^2 \partial_{ss}^2 \varphi.$$

To fully characterize the value function, we need the additional assumption, similar to Assumption 4.3.2.

Assumption 4.4.1. *There exists a Borel function $\psi \in C^{1,2}([0, T] \times D_S \times D_Z)$ such that*

- (i) $0 \geq \mathcal{L}\psi + \mathcal{I}\psi$ on $[0, T] \times D_S \times D_Z$,
- (ii) $\psi - \mathcal{K}\psi \geq \iota$ on $[0, T] \times D_S \times D_Z$ for some $\iota > 0$,
- (iii) $\psi \geq U$ on $\{T\} \times D_S \times D_Z$,
- (iv) $\liminf_{n \rightarrow \infty} (\psi/L)(t_n, s_n, z_n) = \infty$ if $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$, for all $(t_n, s_n, z_n)_{n \geq 1} \subset [0, T] \times D_S \times D_Z$.

Theorem 4.4.1. *Let Assumption 4.3.1 hold. Then, v_* (resp. v^*) is a viscosity supersolution (resp. subsolution) of (4.4.6). If moreover Assumption 4.4.1 holds, then v is continuous on $[0, T] \times D_Z$ and is the unique viscosity solution of (4.4.6), in the class $\mathbf{L}_\infty^{\text{exp}}$.*

Proof. In view of Proposition 4.4.1, the derivation of the viscosity super- and subsolution properties is very standard under Assumption 4.3.1, see e.g. [22]. As for uniqueness, this follows from a comparison principle that can be proved in the class $\mathbf{L}_\infty^{\text{exp}}$ by arguing as in the proof of Theorem 4.3.1. \square

As for the MM problem, Assumption 4.4.1 is easily checked when $J_o < \infty$.

Remark 4.4.4. *If $J_o < \infty$ and the supports of $\lambda(\cdot|p, q, c)$ and $\gamma(\cdot|p, q)$ are bounded, uniformly in $(p, q, c) \in (\mathfrak{D}\mathbb{Z})^2 \times \mathbb{N}^2 \times \mathbf{C}_o$, then the function ψ defined in Remark 4.4.4 also satisfies the requirements of Assumption 4.4.1, for r large enough.*

4.4.3 Dimension reduction, symmetries and numerical resolution

As in Section 4.3.4, one can use specificities of the value function to reduce the complexity of the resolution of (4.4.6). First, the variable g (and j if $J_o = \infty$) can be omitted, see Remark 4.4.2. Moreover, if Assumption 4.3.3 holds, then one easily checks that

$$\bar{v}(t, s, p^b, p^b + 2\delta p, q^b, q^a, 0, i, n^b, n^a, b^b, b^a, 0)$$

does not depend on p^b . The difference with the relation obtained in Section 4.3.4 is due to (4.4.1) and the fact that the HFT always hold a symmetric position in the stock and the futures (he is protected against evolutions of the mid-price). The other symmetry relations described in Section 4.3.4 do not hold because of the dependence on the process S .

Let us now turn to the definition of a numerical scheme for (4.4.6). Recall that, under Assumption 4.3.5, the operators \mathcal{I} and \mathcal{K} are explicit. Hence, the only required discretization is in time and in the variable s . We shall consider separately the diffusion part and the obstacle part of the pde. More precisely, we fix a time and a space grid $\pi_t^n := \{t_i^n, i \leq n_t\}$ and $\pi_s^n := \{s_i^n, i \leq n_s\}$ where $t_i^n = iT/n_t$ for $i \leq n_t$ and $s_i^n = \underline{s} + i(\bar{s} - \underline{s})/n_s$, $i \leq n_s$. Here, \underline{s} and \bar{s} are such that $D_S = [\underline{s}, \bar{s}]$, recall Remark 4.4.1, and $n := (n_t, n_s) \in \mathbb{N}^2$. We next define the sequence of space domains $D_Z^k := D_Z \cap [-k, k]^{11}$ for $k \geq 1$, and we let v_n^k be defined by

$$\begin{aligned} v_n^k &= \max \left\{ \check{v}_n^k, \mathcal{K}\check{v}_n^k \right\} = 0 \text{ on } \pi_t^n \times \pi_s^n \times D_Z^k, \\ v_n^k - U &= 0 \text{ on } (\{T\} \times \pi_s^n \times D_Z^k) \cup (\pi_t^n \times \pi_s^n \times (D_Z \setminus D_Z^k)). \end{aligned} \tag{4.4.7}$$

Here, for $i \leq n_t - 1$ and $i' \leq n_s$,

$$\check{v}_n^k(t_i^n, s_{i'}^n, \cdot) = \mathbb{E}[v_n^k(t_{i+1}^n, p_n(S_{t_{i+1}^n}^{t_i^n, s_{i'}^n}), \cdot)] + \frac{T}{n} \mathcal{I}v_n^k(t_{i+1}^n, s_{i'}^n, \cdot), \text{ on } D_Z^k, \quad (4.4.8)$$

where p_n is the left-hand side projection operator on π_s^n .

Note that the above numerical scheme is not fully explicit as it requires to compute conditional expectations. This can however be easily performed either by a finite difference scheme or by Monte-Carlo techniques in a very classical way. Note in particular that the randomness in these conditional expectations only comes from a one dimensional factor.

Proposition 4.4.2. *Let Assumptions 4.4.1 and 4.3.5 hold, then the sequence $(v_n^k)_{n \geq 1}$ converges pointwise to v as $n_t, n_s, k \rightarrow \infty$.*

Proof. Note that Assumption 4.4.1 ensures that comparison holds for (4.4.6) in the class $\mathbf{L}_\infty^{\text{exp}}$, see e.g., Proposition 2.5.1. Thus, as for Proposition 4.3.2, the result is an easy consequence of the stability result of [14]. \square

4.4.4 Approximate optimal controls

The optimal control can be numerically estimated as in Section 4.3.4. We first extend (v_n^k, \check{v}_n^k) to $\pi_t^n \times D_S \times D_Z$ by setting $(v_n^k, \check{v}_n^k)(\cdot, s, \cdot) := (v_n^k, \check{v}_n^k)(\cdot, p_n(s), \cdot)$. Then, we choose a measurable map $\hat{c}_n^k(t_i^n, \cdot)$ such that

$$\begin{aligned} \hat{c}_n^k(t_i^n, \cdot) &\in \arg \max \{ \mathcal{K}^c \check{v}_n^k(t_i^h, s, \cdot), c \in \mathbf{C}(\cdot) \}, \text{ on } D_S \times D_Z^k \\ \hat{c}_n^k(t_i^n, \cdot) &= 0 \text{ on } D_S \times (D_Z \setminus D_Z^k), \end{aligned}$$

and define the sequence of stopping times

$$\hat{\tau}_{j+1}^{n,k} := \min \{ t_i^n : i \geq 0, t_i^n > \hat{\tau}_j^{n,k}, (v_n^k - \mathcal{K}^{\hat{c}_n^k} \check{v}_n^k(t_i^n, \cdot))(t_i^n, S_{t_i^n}, \hat{Z}_{t_i^n}^{n,k}) = 0 \}, j \geq 0,$$

with $\hat{\tau}_0^{n,k} := 0-$, and in which $\hat{Z}^{n,k} = (\hat{P}^{n,k}, \hat{Q}^{n,k}, \hat{X}^{n,k})$ is defined as in (4.3.2)-(4.4.2) for the initial condition Z_{0-} and the control associated to $\hat{\phi}_n^k := (\hat{\tau}_i^{n,k}, \hat{c}_n^k(\hat{\tau}_i^{n,k}, S_{\hat{\tau}_i^{n,k}}, \hat{Z}_{\hat{\tau}_i^{n,k}}^{n,k}))_{i \geq 1}$ in a Markovian way. Again, this provides a sequence of controls that is asymptotically optimal.

Proposition 4.4.3. *Let the conditions of Proposition 4.4.2 hold. Then,*

$$\lim_{k \rightarrow \infty} \lim_{n_t, n_s \rightarrow \infty} \mathbb{E}[U(S_T, \hat{Z}_T^{n,k})] = v(0, S_0, Z_{0-}),$$

in which the limit is taken along sequences n such that $n_t^2 n_s^{-1} \rightarrow 0$.

Proof. Let $\gamma(p, q) := \int d\beta(c|p, q)$ and recall that γ is uniformly bounded by assumption. The family $\{(v_n^k, \check{v}_n^k)/L\}_{k, n \geq 1}$ is uniformly bounded, compare with Remark 4.4.3. Also note that $(s, z) \in D_S \times D_Z^k \mapsto U$ is C_k -Lipschitz, for some $C_k > 0$ that only depends on k . By induction (recall that the component s is projected on π_s^n),

$$|\check{v}_n^k(\cdot, s, \cdot) - \check{v}_n^k(\cdot, s', \cdot)| \leq C'_k [|s - s'| + n_t O(n_s^{-1})], \text{ for all } s, s' \in D_S, \text{ on } D_Z^k,$$

for some $C'_k > 0$ that does not depend on $n \in \mathbb{N}^2$. Since S is 1/2-Hölder in time in \mathbf{L}_2 , together with Assumption 4.3.5, this implies that

$$\begin{aligned} \mathcal{K}^c \mathcal{I} \check{V}_n^k(t_{i+1}^n, s_{i'}^n, z) &= \mathcal{K}^c \mathbb{E}[\mathcal{I} \check{V}_n^k(t_{i+1}^n, S_{t_{i+1}^n}^{t_i^n, s_{i'}^n}, z)] + O_k(n_t^{-\frac{1}{2}}) + n_t O_k(n_s^{-1}), \\ &= \mathbb{E}[\check{V}_n^k(t_{i+1}^n, S_{t_{i+1}^n}^{t_i^n, s_{i'}^n}, Z_{t_{i+1}^n}^{t_i^n, z, c})] + O_k(n_t^{-\frac{1}{2}}) + n_t O_k(n_s^{-1}) \text{ for } z \in D_Z^k, c \in \mathbf{C}(z), \end{aligned}$$

in which the exponent c in $Z_{t_{i+1}^n}^{t_i^n, z, c}$ means that the impulse c is given at t_i^n , and we use Remark 4.2.1 again. In the above, $O_k(\xi)$ is a function, that may depend on k , but such that $\xi \in \mathbb{R} \setminus \{0\} \mapsto |O_k(\xi)|/\xi$ is bounded in a neighborhood of 0. By the arguments already used in the proof of Proposition 4.3.3, it follows that

$$\begin{aligned} v_n^k(t_i^n, s_{i'}^n, z) &= \mathbb{E}[v_n^k(t_{i+1}^n, S_{t_{i+1}^n}^{t_i^n, s_{i'}^n}, Z_{t_{i+1}^n}^{t_i^n, z, 0})] \vee \max_{c \in \mathbf{C}(z)} \mathbb{E}[v_n^k(t_{i+1}^n, S_{t_{i+1}^n}^{t_i^n, s_{i'}^n}, Z_{t_{i+1}^n}^{t_i^n, z, c})] \\ &\quad + o_k(n_t^{-1}) + n_t O_k(n_s^{-1}), \end{aligned}$$

and therefore

$$\begin{aligned} v_n^k(0, S_0, Z_{0-}) &= \mathbb{E}[U(S_{\theta_k^n}, \hat{Z}_{\theta_k^n}^{n, k})] + o_n^k(1) + O_k(n_t^2 n_s^{-1}) \\ &= \mathbb{E}[U(S_T, \hat{Z}_T^{n, k})] + o_n^k(1) + O_k(n_t^2 n_s^{-1}) + o_k(1) \end{aligned}$$

by induction, in which $o_n^k(1)$ goes to 0 as $n \rightarrow \infty$, $o_k(1)$ goes to 0 as $k \rightarrow \infty$, and θ_k^n is as in the proof of Proposition 4.3.3. It remains to appeal to Proposition 4.4.2. \square

4.4.5 Numerical experiments

We use the same model as the one described in Section 4.3.6. As for the new parameters, we take $\hat{s} = 0$, $\rho = 50$ and $\sigma = 0.2$, in particular the mean reversion parameter is taken to be large.

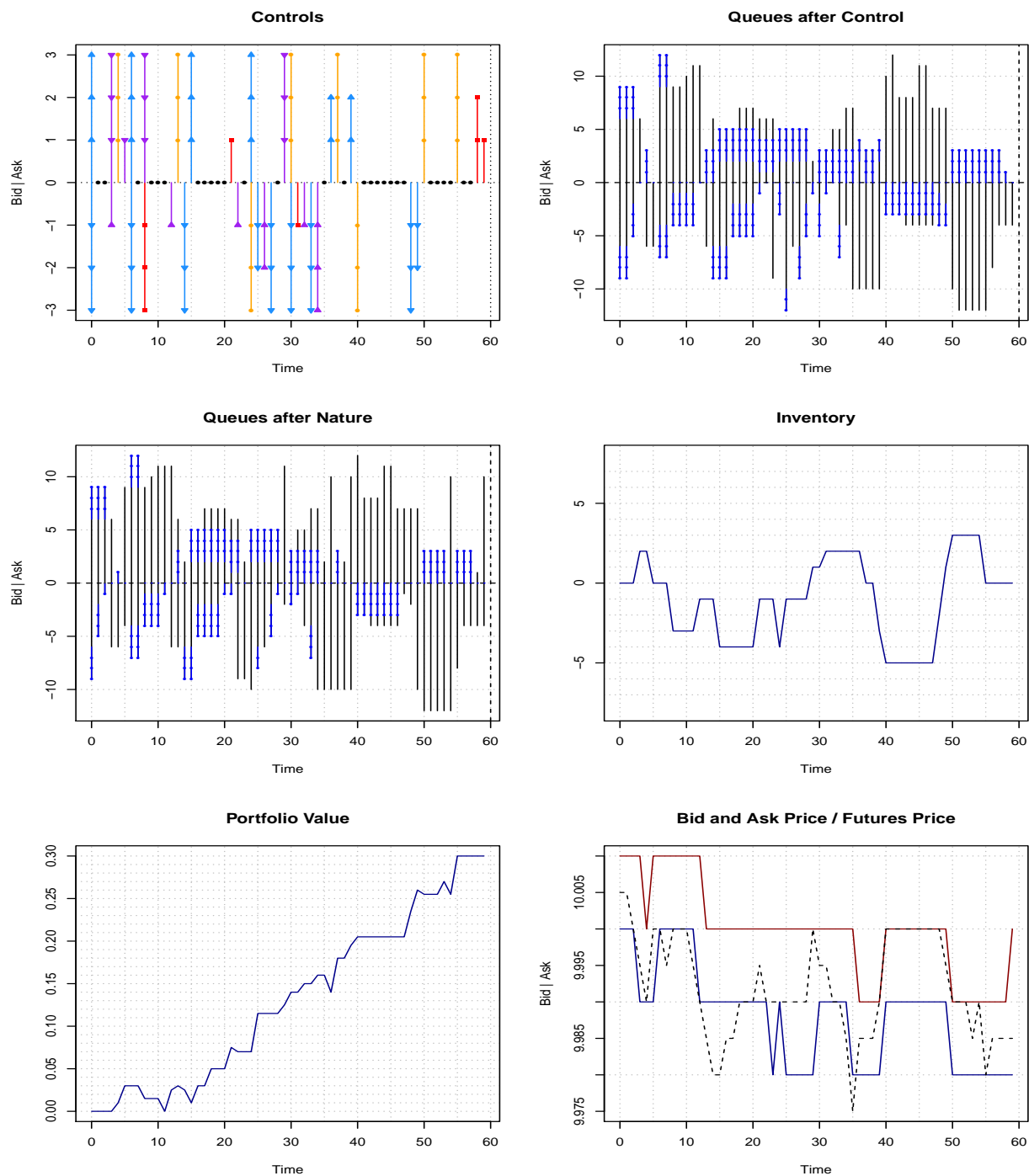


FIGURE 4.3 – Simulated path of the optimal strategy of the High Frequency Trader.

We approximate the behavior of the spread process S by a trinomial tree based on the transition probabilities associated to the diffusion (4.4.2), so that the expectation in (4.4.8) can be computed explicitly. More precisely, we consider a centered 6 points grid with mesh

equal to half a tick (i.e. 0.005).

The graphics in Figure 4.3 have the same interpretation as in Section 4.3.6 except that we now also provide the evolution of the Futures process F , this is the dashed line in the bottom right graphic.

Remember that the HFT does not gain from the evolution of the mid-price, as her position in stocks is always covered by a symmetric position in the futures. She only gains from the evolution of the spread process S or from the bid-ask spread of the stock if S does not move. Not surprisingly, her behavior is quite different from the one of the MM described in Section 4.3.4.

As the MM would do, she first positions herself on the limits in a symmetric ways, because the spread with the future is 0. She then tries selling at time 4 by sending a limit sell order in the spread to buy the futures whose price (compared to the mid-price of the stock) is gone low. She is starting to play on the stock-futures spread. She follows this strategy until time 14. She plays in a more symmetric way after this until time 35, with a slight tendency to resume her inventory. At time 35, she decides to clearly sell stocks again and buy the futures whose price is again very low. From time 40 on, she inverts her position on the book to try buying the stock and thus sell the futures whose price has grown up. She finally inverts her position again at time 50 when the futures price goes back to the mid price, to liquidate her position on the pair.

Compared to the positions of the MM in Figure 4.2, her positions in one direction are much more clear, and are clearly driven by the stock-futures spread. The HFT also does not seem to try to control the stock's mid-price as the MM did, again this is because it does not matter for her.

In Figure 4.4, we provide an estimation of the density of the gain of the HFT based on 10^5 simulated paths.

4.5 Institutional broker strategies for portfolio liquidation

We now turn to the Institutional Broker problem. We consider in the following the two mostly used strategies for buying/selling a block of stocks. We focus on the buying side, selling being performed in a symmetrical way.

4.5.1 Volume strategy

We first consider a simple volume strategy. The aim of the broker is to buy I_0 stocks. By convention, we set $I_0 = -I_o$. She fixes a participation rate $f \in (0, 1)$. It corresponds to the percentage of the total volume of trades that should correspond to trades done by the broker between the initial time 0 and the time at which the I_0 stocks are bought. Therefore, she should buy $f/(1 - f)$ of the total of the trades of the other market participant. To do so, she considers subintervals $[t_i, t_{i+1}]$ of \mathbb{R}_+ . At the beginning of each of this subintervals she estimates the conditional probability $p_{i,i+1}$ that a trade arrives at the bid, given that

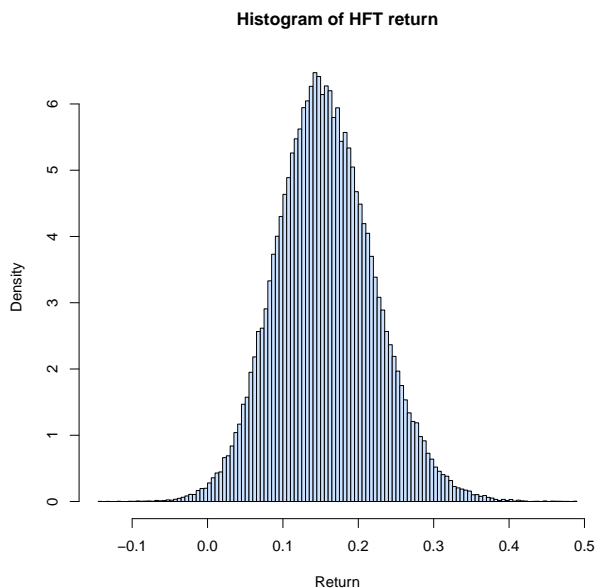


FIGURE 4.4 – Density estimation of the gain made by the High Frequency Trader.

a trade arrives and given the order book at t_i . Then, she puts a limit bid order of size $(f/(1-f))Q_{t_i}^b/p_{i,i+1}$ at t_i . When all these units are executed, she adds new limit orders to keep her position to a level at least equal to $(f/(1-f))Q_{t_i}^b/p_{i,i+1}$, or do nothing if this level is still achieved, and so on. In the meantime, she compares her realized volume $\Delta I := I - I_0$ to the volume $\tilde{v}(0, \cdot) + \Delta I$ realized on the market from time 0 on. Given a threshold $\delta_I > 0$, she stops adding limit orders and cancels her orders already at the bid if $\Delta I(1-f) > f\tilde{v}(0, \cdot) + \delta_I$ and wait until $\Delta I(1-f) \leq f\tilde{v}(0, \cdot) + \delta_I$. If $\Delta I(1-f) < f\tilde{v}(0, \cdot) - \delta_I$, then she sends an aggressive order so as to turn to a position $\Delta I(1-f) \geq f\tilde{v}(0, \cdot) - \delta_I$ as soon as possible. She stops trading when $I \geq 0$.

For our numerical experiment, we take the same model as in Section 4.3.6, we consider a participation rate $f = 0.2$ and simply take $p_{i,i+1} = 1/2$. The time intervals $[t_i, t_{i+1}]$ have a length of 60 seconds and the time step is $1/2$ second. We take $\delta_I = 4$ (i.e. 2 ATS). The number of units to buy is $I_0 = 250$ units. A simulated path of the strategy is provided in Figure 4.5. In the top left graphic, the dashed lines correspond to the target volume $\pm\delta_I$, while the solid curve is the volume traded by the IB. The top right graphic gives the control of the IB : lines with inward pointing arrows are limit orders, lines with squares are market orders. The bottom left graphic provides the evolution of the average price at which stocks have been bought by the IB from time 0 on. The bottom right graphic gives the evolution of the bid and ask prices. One can see that this very simple strategy is quite efficient in the sense that only a limited number of market orders are send. On the other hand, the systematic position of the IB at the bid limit creates an important imbalance that contributes to push up the price.

Figure 4.6 provides an histogram of the relative error (in %) of the VWAP obtained by

this strategy with respect to the VWAP realized at the level of the whole market⁹. It is based on 10^4 simulated paths. One can see that this average price is typically slightly higher than the VWAP of the market.

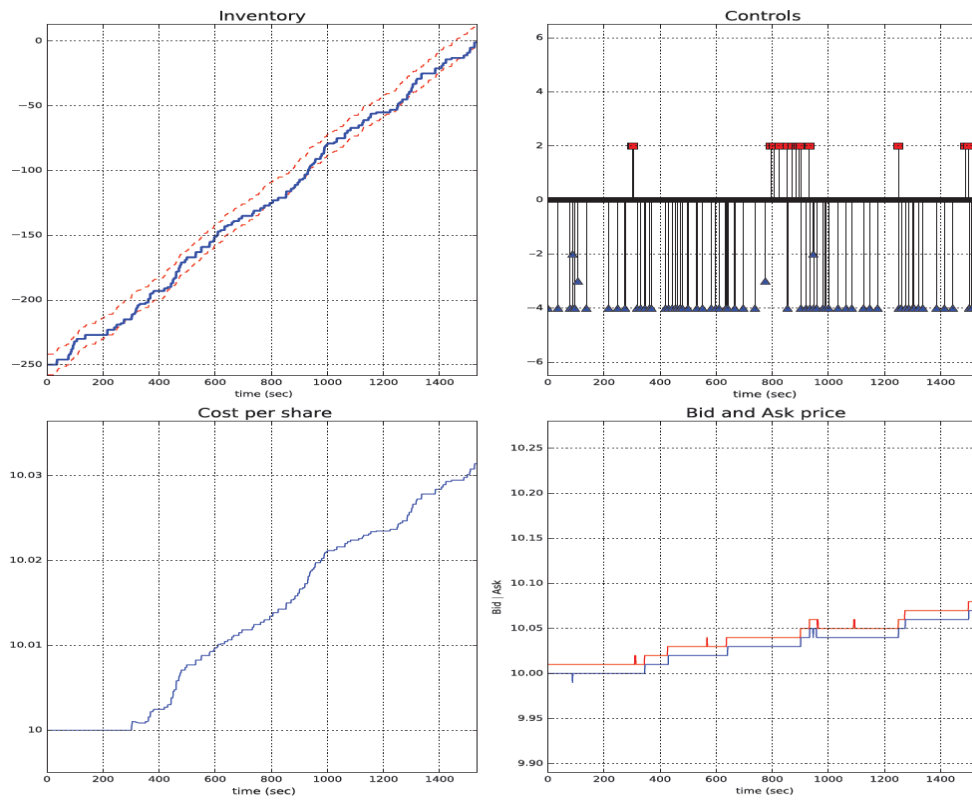


FIGURE 4.5 – Simulated path of the volume strategy.

9. Namely, $(\text{VWAP}_{\text{Vol}} - \text{VWAP}_{\text{Market}}) / \text{VWAP}_{\text{Market}}$ in which VWAP_{Vol} is the VWAP obtained by the IB by playing his volume strategy.

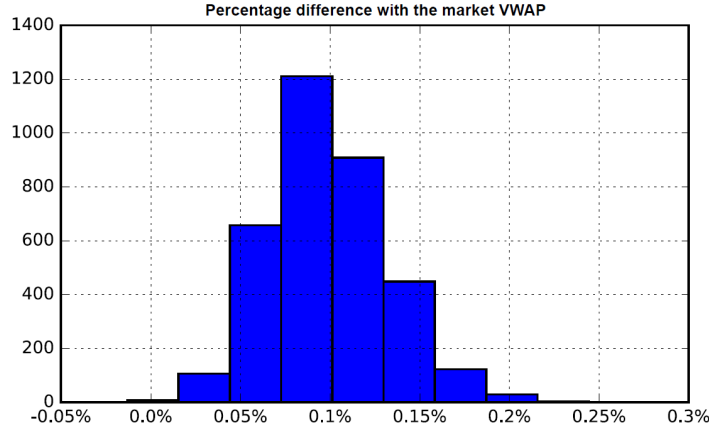


FIGURE 4.6 – Histogram of the VWAP percentage error with respect to the VWAP of the market, for the Volume strategy.

4.5.2 VWAP

We now present a VWAP (Volume Weighted Average Price) based trading strategy frequently practiced by institutional brokers. Suppose that, at initial time 0, an institutional broker decides to buy a quantity $I_0 \in \mathbb{N}$ of a tradable asset using a VWAP based strategy, i.e. by trying to obtain an average execution cost not more than

$$\text{VWAP}_T = \frac{w_T}{v(0, T)} \quad \text{where } w := \int_0^\cdot v_t P_t dt \text{ and } v(\cdot, T) := \int_\cdot^T v_s ds, \quad (4.5.1)$$

in which v is a deterministic non-negative continuous process such that $\int_0^T v_t dt > 0$, which represents the trading volume of the market, and P models the stock price.¹⁰

Abstract continuous time resolution

Following the seminal work of Almgren and Chriss [4], see also [40, Section 4.4], we first consider the idealized world in which trading is done continuously at a bounded intensity, the trading speed ϑ , taken as a process in the class \mathcal{C} of non-negative processes adapted to the (completed) filtration generated by W . She assumes that the dynamics of the asset reference price P has a permanent linear price impact generated by the agent's trading activity, see [4]. More precisely, she assumes that the stock price evolves according to

$$dP_t = \beta \vartheta_t dt + \sigma dW_t, \quad (4.5.2)$$

with $\beta, \sigma > 0$, while her inventory I follows the dynamics

$$dI_t = \vartheta_t dt$$

10. We take it deterministic for simplicity. In practice, it generally corresponds to a market volume curve estimated by the broker.

with the initial condition $I_0 = -I_0$, meaning that she is short at 0 of the I_0 stocks she has to buy on $[0, T]$.

She also assumes that her wealth G is affected by a temporary linear market impact $\kappa\vartheta$, for some $\kappa > 0$, i.e.

$$dG_t = -\vartheta_t [P_t + \kappa\vartheta_t] dt. \quad (4.5.3)$$

Her goal is to maximize over $\vartheta \in \mathcal{C}$ the expected utility

$$\mathbb{E} \left[-\exp[-\eta \{G_T + I_T(P_T - \tilde{\kappa}I_T) - I_0 \frac{w_T}{v(0, T)}\}] \right], \quad (4.5.4)$$

for some $\tilde{\kappa} > 0$ which represents a penalty in case the inventory does not match 0 at T .

This type of problems has been widely studied, see the book [40] for references. In the present form, it can be solved explicitly by using a verification argument based on the explicit solution of the Hamilton-Jacobi-Bellman equation corresponding to the value function $(t, p, i, g, w) \in [0, T] \times \mathbb{R}^4 \mapsto v(t, p, i, g, w)$ associated to the initial condition $(P_t, I_t, G_t, w_t) = (t, p, i, g, w)$:

$$0 = \sup_{u \geq 0} \left(\partial_t \varphi + \frac{1}{2} \sigma^2 \partial_{pp}^2 \varphi + p v_t \partial_w \varphi + u (\beta \partial_p \varphi + \partial_i \varphi - p \partial_g \varphi) - u^2 \kappa \partial_g \varphi \right) \quad \text{on } [0, T) \times \mathbb{R}^4$$

with terminal condition

$$\varphi(T, p, i, g, w) = -e^{-\eta \{g + i(p - \tilde{\kappa}i) - \bar{m}ow\}} \quad \text{for } (p, i, g, w) \in \mathbb{R}^4,$$

where

$$\bar{m} := I_0 / v(0, T).$$

To simplify the above, we first write v is the form

$$v(t, p, i, g, w) = e^{-\eta(g - \bar{m}w - p v(t, T) \bar{m})} \bar{v}(t, p, i) \quad \text{with } \bar{v}(t, p, i) := v(t, p, i, 0, 0),$$

so that \bar{v} formally solves

$$0 = \partial_t \varphi + \frac{1}{2} \sigma^2 (\partial_{pp}^2 \varphi + 2\eta v(t, T) \bar{m} \partial_p \varphi + \eta^2 v(t, T)^2 \bar{m}^2 \varphi) - \frac{(\beta \partial_p \varphi + \beta \eta v(t, T) \bar{m} \varphi + \partial_i \varphi + \eta p \varphi)^2}{4\kappa \eta \varphi}$$

on $[0, T) \times \mathbb{R}^4$, with terminal condition

$$\varphi(T, p, i, g, w) = -e^{-\eta \{i(p - \tilde{\kappa}i)\}} \quad \text{for } (p, i, g, w) \in \mathbb{R}^4.$$

One possible solution is given by

$$\bar{v}(t, p, i) = -e^{-\eta \{i(p - \tilde{\kappa}i)\}} e^{h_0(t) + h_1(t)i + h_2(t)i^2}$$

in which h_0, h_1 and h_2 solve¹¹ on $[0, T)$

$$\begin{aligned}\partial_t h_0 &= -\frac{1}{2}\sigma^2(\eta v(t, T)\bar{m})^2 + \frac{(h_1 + \beta\eta v(t, T)\bar{m})^2}{4\kappa\eta} \\ \partial_t h_1 &= \sigma^2\eta^2 v(t, T)\bar{m} + (h_1 + \beta\eta v(t, T)\bar{m})\frac{-\beta\eta + 2\eta\tilde{\kappa} + 2h_2}{2\kappa\eta} \\ \partial_t h_2 &= -\frac{\eta^2\sigma^2}{2} + \frac{[-\beta\eta + 2\eta\tilde{\kappa} + 2h_2]^2}{4\kappa\eta}\end{aligned}$$

with $h_0(T) = h_1(T) = h_2(T) = 0$. The solution of the last equation is of the form

$$h_2(t) = \frac{1}{c_1 + c_2 e^{c_3(T-t)}} - c_4$$

in which the constants c_1, \dots, c_4 can be computed explicitly by using the above differential equation and the terminal condition $h_2(T) = 0$. Namely, set

$$a_0 := -\frac{\eta^2\sigma^2}{2} + \frac{\eta(2\tilde{\kappa} - \beta)^2}{4\kappa}, \quad a_1 := \frac{2\tilde{\kappa} - \beta}{\kappa}, \quad a_2 = \frac{1}{\kappa\eta}$$

and let y_\circ be a root¹² of

$$(4a_0a_2 - a_1^2)y^2 + (a_1^2 - 4a_0a_2)y + a_0a_2 = 0,$$

then

$$c_3 = \frac{a_1}{1 - 2y_\circ}, \quad c_4 = \frac{a_0}{(1 - y_\circ)c_3}, \quad c_1 = \frac{y_\circ}{c_4}, \quad c_2 = \frac{1}{c_4} - c_1.$$

Existence of y_\circ is guaranteed since

$$a_1^2 - 4a_0a_2 = \frac{\eta^2\sigma^2}{2} > 0.$$

Then, h_0 and h_1 are fully characterized.

An easy verification argument shows that this is actually the correct function and that the optimal control policy is given by

$$\hat{v} = \hat{v}(\cdot, I)$$

where

$$\begin{aligned}\hat{v}(t, i) &:= \frac{\beta\partial_p v(t, p, i, g, w) + \partial_i v(t, p, i, g, w) - p\partial_g v(t, p, i, g, w)}{2\kappa\partial_g v(t, p, i, g, w)} \\ &= \frac{\beta[i - v(t, T)\bar{m}] - 2\tilde{\kappa}i - [h_1(t) + 2h_2(t)]i/\eta}{2\kappa}.\end{aligned}$$

11. Just insert the above in the PDE of \bar{v} and match the orders in the i variable.

12. The function h_2 does not depend on the choice of the root.

Strategy in practice and simulations

In practice, this optimal strategy can not be implemented within an order book. We therefore consider a “discrete” version. In this version, we assume that the IB tries to keep her inventory I equal to the optimal inventory $\hat{I} := \int_0^T \hat{v}_t dt - I_0$. To do this, she fixes a time grid $\{t_i, i \leq n\}$ with $t_0 = 0$ and $t_n = T$. At time t_i , she estimates that she has to execute a volume of $V_{i,i+1} := \int_{t_i}^{t_{i+1}} \hat{v}(t, I_{t_i}) dt$ on $[t_i, t_{i+1}]$ while the volume of the market will be $\tilde{V}_{i,i+1} = \int_{t_i}^{t_{i+1}} v_t dt$. Then, she follows a volume strategy with threshold $\delta_I = +\infty$, see Section 4.5.1 above, on $[t_i, t_{i+1}]$ with a target participation rate of $f = V_{i,i+1}/(\tilde{V}_{i,i+1} + V_{i,i+1})$. When $\Delta I := I - I_0 > \hat{I} + \bar{\delta}_I$, for some $\bar{\delta}_I > 0$, orders are canceled and she waits until $\Delta I \leq \hat{I} + \bar{\delta}_I$. In the case where $\Delta I < \hat{I} - \bar{\delta}_I$, she puts market orders to reduce to the situation $\Delta I \geq \hat{I} - \bar{\delta}_I$ as soon as possible.

For our numerical experiment, we take the same configuration as in Section 4.5.1, with $\bar{\delta}_I = 4$, i.e. 2 ATS. The optimization of the VWAP strategy is done with a time horizon of 30 minutes and a flat volume curve (so that the control does in fact not depend on it). The number of units to buy is $I_0 = 250$ units. The additional parameters¹³ are set to $\eta = 1$, $\sigma = 0.2$, $\beta = 0,0004$, $\kappa = 0.003$ and $\tilde{\kappa} = \kappa * 60$. The latter corresponds to the cost incurred when buying the remaining shares I_T in 1 additional minute after T , at a flat intensity in the theoretical continuous time model of Section 4.5.2. The volume intensity v corresponds to 0.6 ATS per second, i.e. 1.2 units per second.

The interpretation of the different graphics in Figure 4.7 is the same as in Section 4.5.1, except that the dashed lines in the top left graphic correspond now to the optimal VWAP trading curve ± 2 ATS. Again, we see that only a limited number of market orders were needed to be send, but that the imbalance created by the robots drives the price up.

Figure 4.8 provides an histogram of the relative error (in %) of the VWAP obtained by this strategy with respect to the VWAP of the whole market¹⁴. It is based on 10^4 simulated paths. One can see that he actually typically performs better than the market. Not surprisingly, this strategy performs better than the volume strategy in terms of VWAP.

13. The coefficients β and κ are estimated for our book dynamic. Given priors with simulate a bunch of paths and estimated them by a moment matching approach based on (4.5.2)-(4.5.3). We then use the updated values to simulate a new bunch of paths and we re-estimate them. And so on, until convergence.

14. Namely, $(\text{VWAP}_{\text{MM}} - \text{VWAP}_{\text{Market}})/\text{VWAP}_{\text{Market}}$, in which VWAP_{MM} is the VWAP obtained by the IB by playing his optimal VWAP strategy

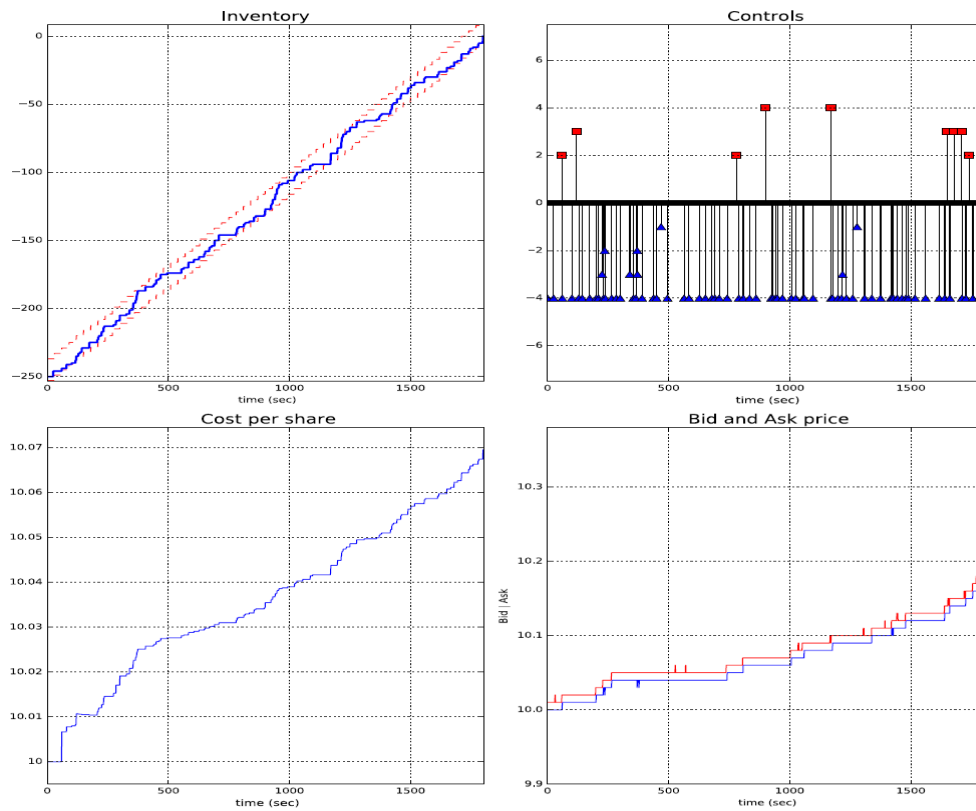


FIGURE 4.7 – Simulated path of the VWAP strategy.

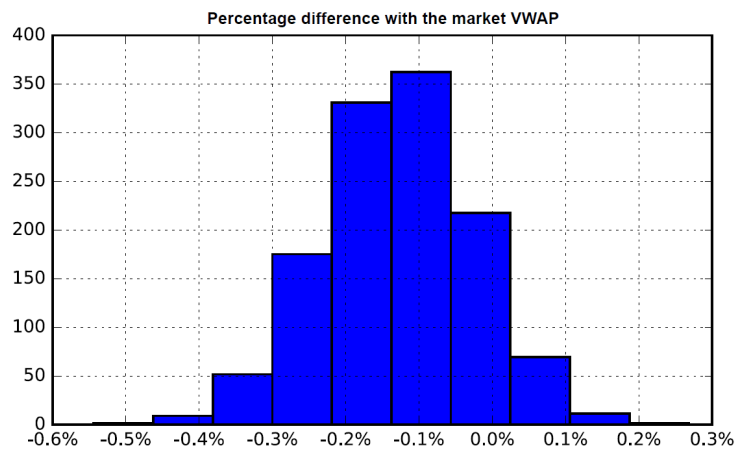


FIGURE 4.8 – Histogram of the percentage error with respect to the VWAP of the market, for the VWAP strategy.

4.6 Simulation of the full market

We now provide an illustration of possible interactions of several market participants. Again, a more realistic and general study will be performed in our companion paper [8].

The prior and parameters of the participants are the same as the one that have been used in the above numerical experiments. More precisely, we consider one MM, one HFT and four IB. Among the IB, two are playing volume strategies (a buy and a sell) and two are playing VWAP strategies (a buy and a sell). At each time step, each participant decides of his optimal control given the current state of the order book. They send their orders at the same time. The control of the HFT is executed first, then this is the turn of the MM, and finally the controls of the robots are executed (we choose the order among robots randomly, according to a uniform distribution). Since the MM and the robots decide before seeing the action of the HFT, their controls may not be applied or only partially, depending of the state of the order book after the previous participants have played. The exogenous randomness only comes from the simulation of the stock-futures spread process S and from the new queues created when one queue is depleted. If the bid queue is depleted, then the bid price moves down. If the bid-ask spread is equal to two ticks, then the ask price moves down as well. The other way round if the ask queue is depleted. If the bid (resp. the ask) price move down (resp. up), we consider that this is a discovered limit and its size is chosen as in Section 4.3.4 : 10 units with probability 60%, 5 units with probability 25% and 12 units with probability 15%. If the bid (resp. the ask) price move up (resp. down), which can happen if the bid-ask spread is already of two ticks, we consider that this is a created limit and its size is chosen again as in Section 4.3.4 : 2 units with probability 60%, 1 unit with probability 25% and 3 units with probability 15%. There is no other randomness, the rest of the dynamics is due to the HFT, the MM and the four robots.

Note that only the VWAP robots are forced to trade, when they reach the (upper or lower) limit of their prescribed inventory path. In principle, the MM plays aggressive orders only when he needs to adjust quickly his inventory. For a typical path of the stock-futures spread, the HFT also has no incentive to send aggressive orders, except to adjust his inventory. Without trades, the Volume robots do not act as well (because the market volume does not move). Therefore, if the MM and the HFT have a zero initial inventory, we expect to have to wait for the VWAP robots to initiate first aggressive orders, and starts the whole dynamics.

In our illustration, we start with an initial state in which the HFT and the MM have a zero initial inventory. Each VWAP robots has to buy/sell 75 stocks within 5 minutes of trading. This corresponds to an average of 0.5 trades per second, which is consistent with the priors of the MM and the HFT.

The Figures 4.9, 4.10, 4.11, 4.12, 4.13 and 4.14 have the same interpretation as in Section 4.3.4 except that now the position of the agent in the queue is in blue in the top right and middle left graphics (the black part corresponding to the other participants). The top right graphic is the state of the book just after the control of the agent is executed, the middle left is the state of the order book after the controls of all the participants have been executed. Note surprisingly, the trades are essentially due to the VWAP robots, to which the Volume robots

need to adjust. The aggressive orders send by the HFT and the MM are either cancellations or corrections of their inventory near the terminal time, except around time 35 at which the HFT clearly wants to take a position on the stock-futures spread because it is very low. Note that the first price jump is due to an aggressive sell order of the HFT, slightly before time 275, which pushes the bid price down. He just after sends a limit sell order in the spread, and the ask price moves down as well. Since he is executed first and all players decide at the same time, he does not give to the others the opportunity to take this position in the spread. The last price move, just before T , is due to the HFT and the MM. This time the HFT and the MM send aggressive buy orders that deplete the queue (we do not see the price move on the graphics of the HFT, because they provide the prices after the execution of the order of the agent). As in Section 4.4.5, his position follows the stock-futures spread. In particular, between time 50 and 100, one can observe that he waits before sending a new sell limit order because the spread is gone up and he wants to buy stocks to shortsell the futures. When the spread starts going down, he takes a stronger position at the ask than at the bid, to start selling back the stock. As for the MM, he finishes with an inventory equal to -1 , that will eventually be liquidated at 10.010, which explains the downward jump of his portfolio value at T . When his inventory is very low, he balances between putting limit buy orders and letting the price go down by not supporting the bid limit. It is not successful until the price eventually moves down around time 275.

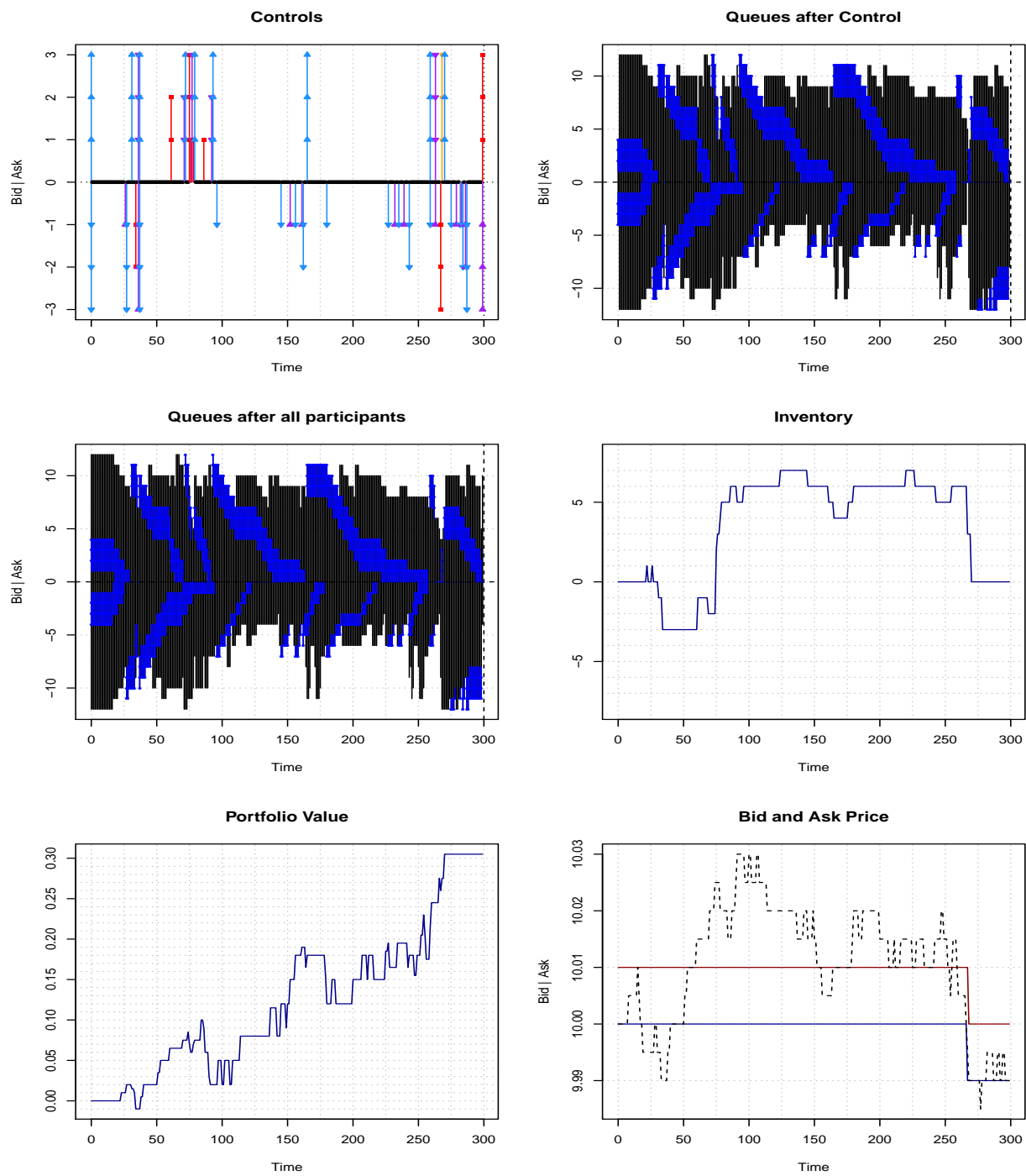


FIGURE 4.9 – Optimal strategy of the High Frequency Trader when agents play together.

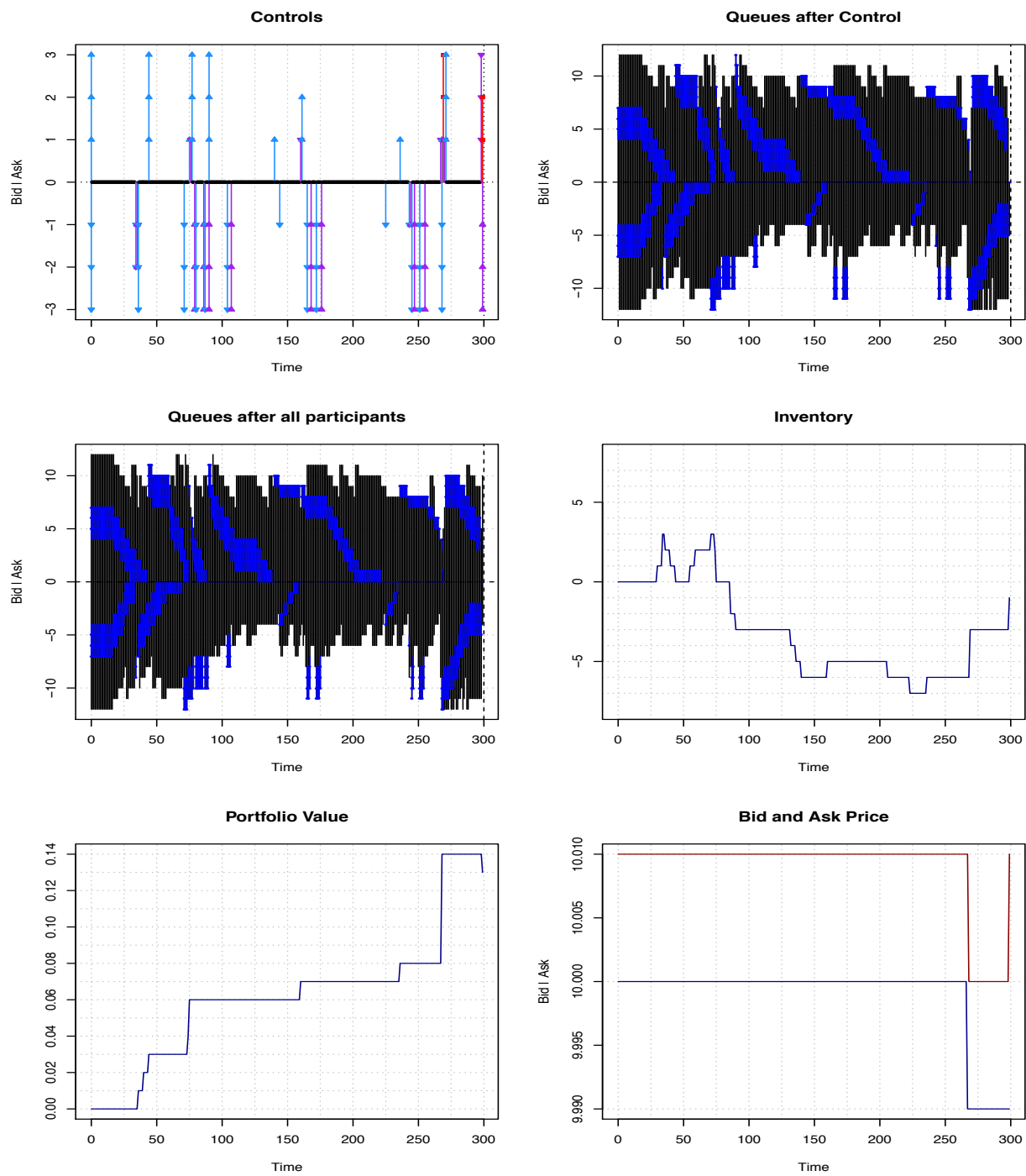


FIGURE 4.10 – Optimal strategy of the Market Maker when agents play together.

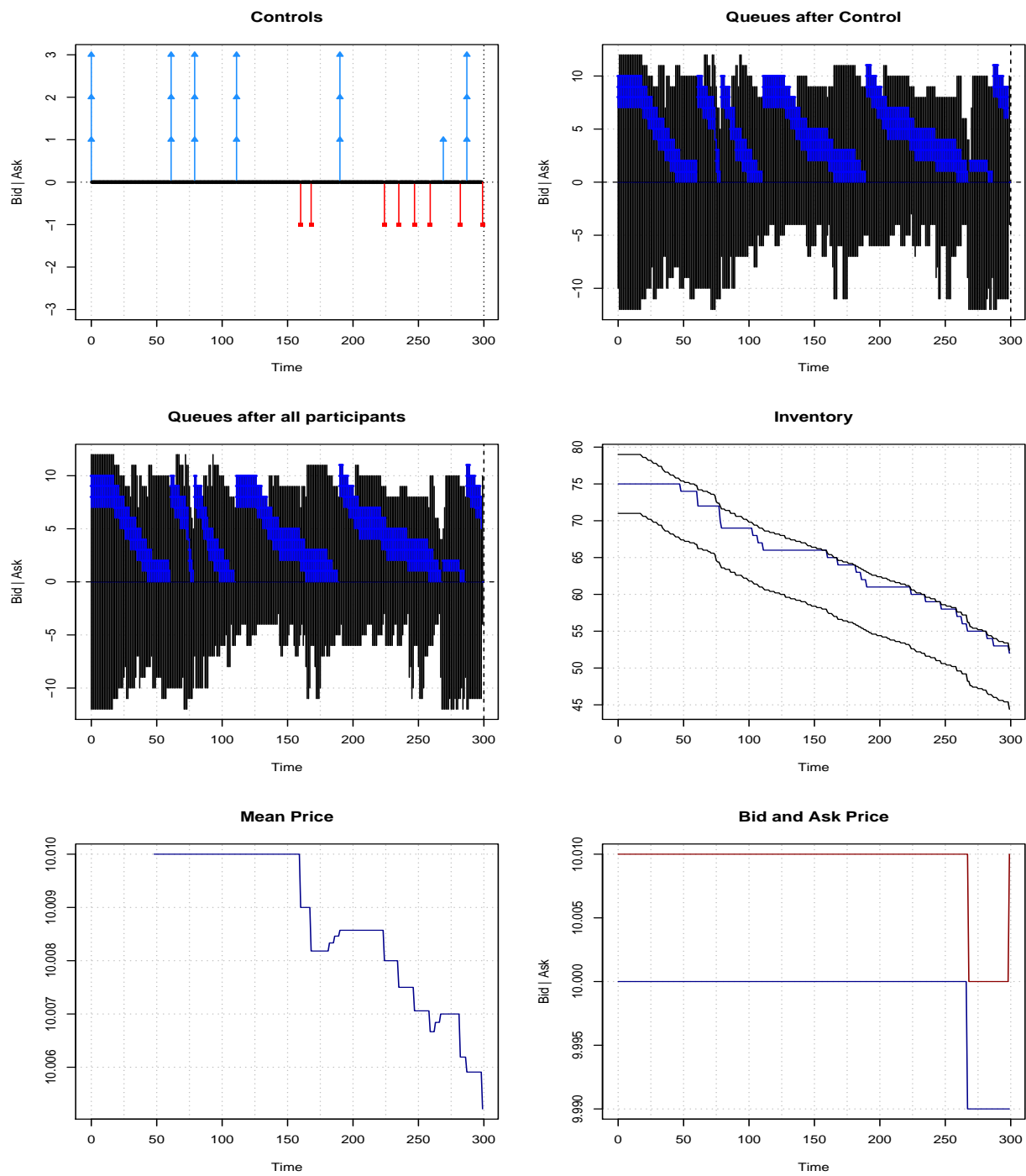


FIGURE 4.11 – Optimal strategy of the Volume robot (seller) when agents play together.

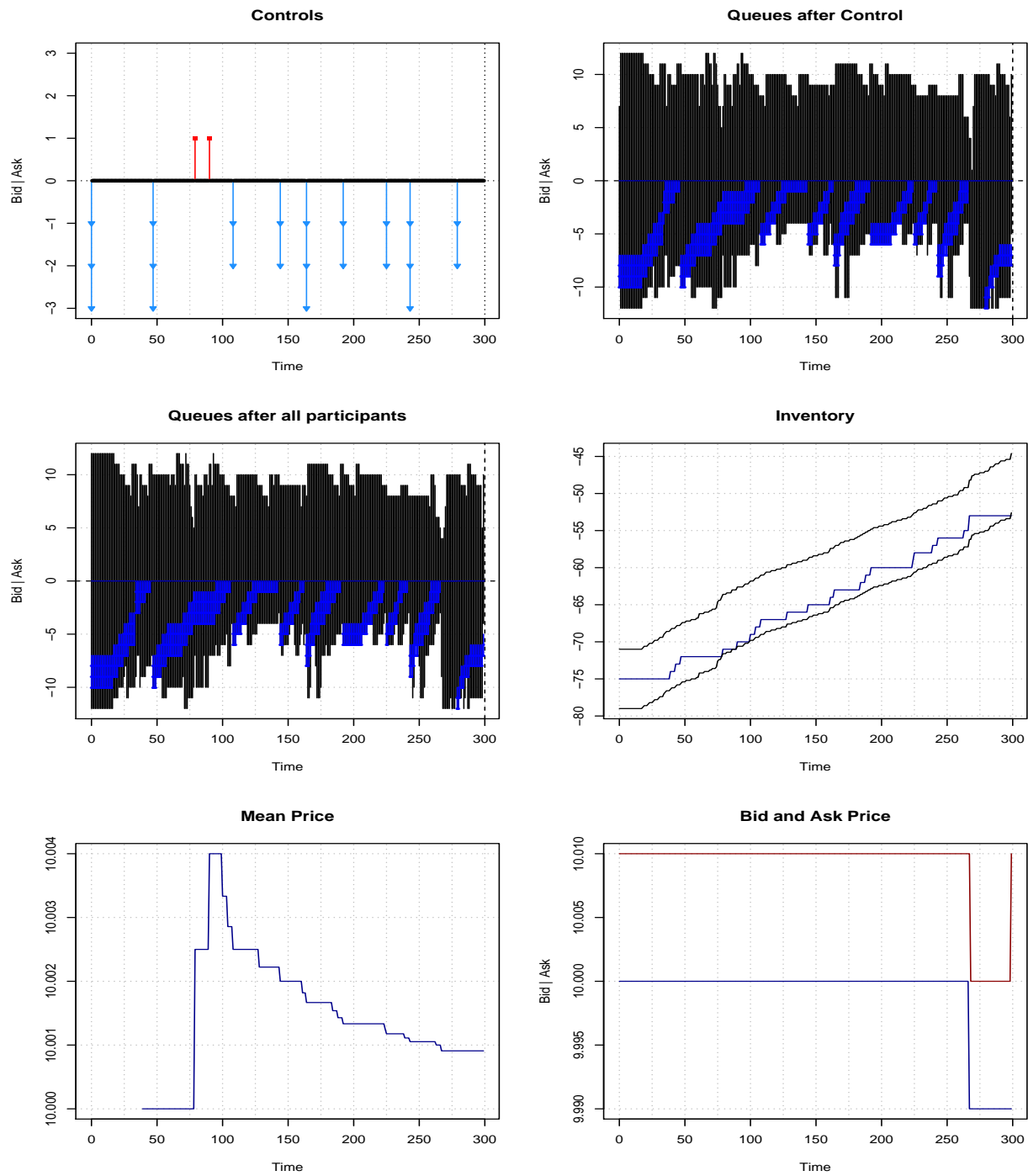


FIGURE 4.12 – Optimal strategy of the Volume robot (buyer) when agents play together.

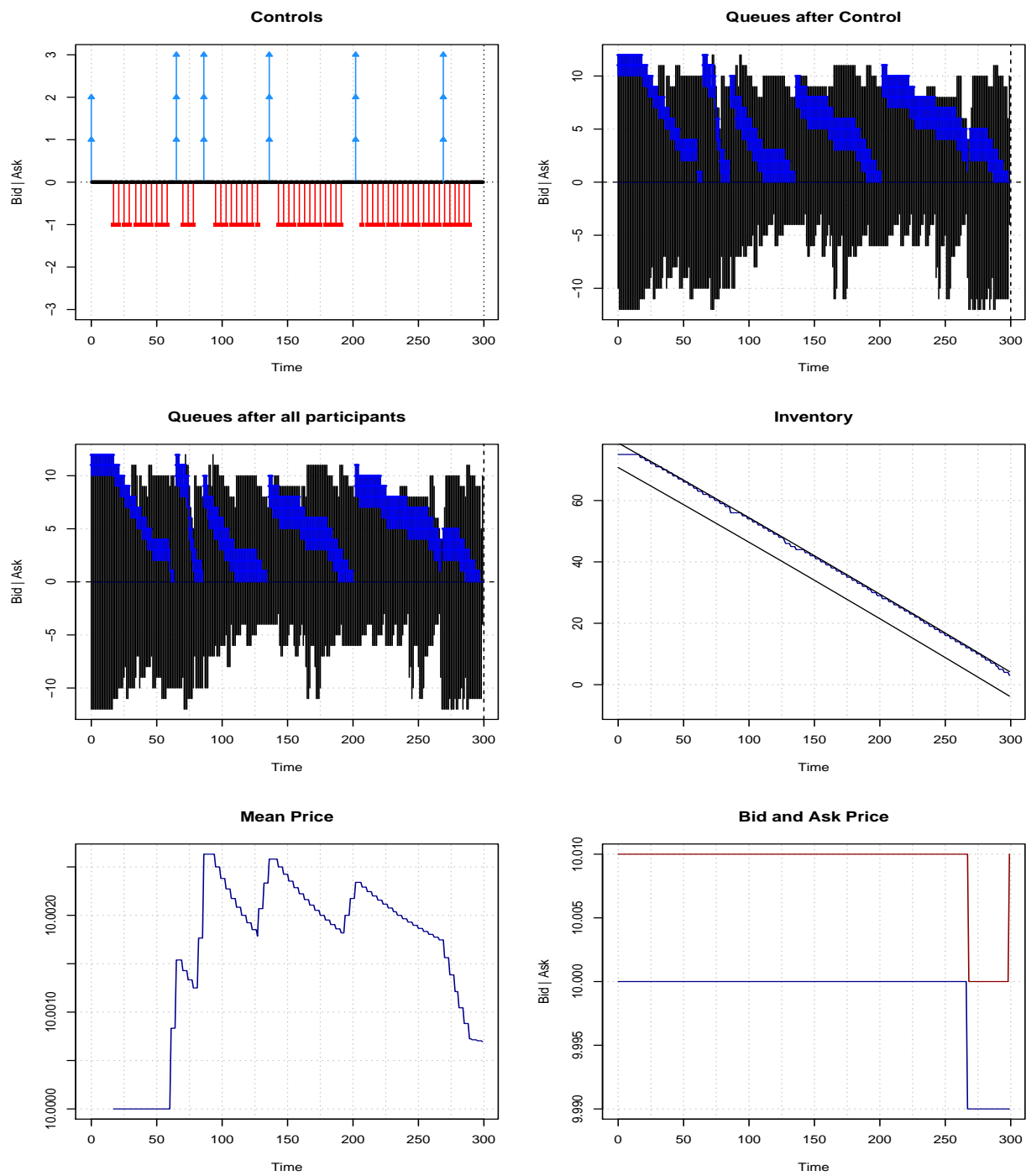


FIGURE 4.13 – Optimal strategy of the VWAP robot (seller) when agents play together.

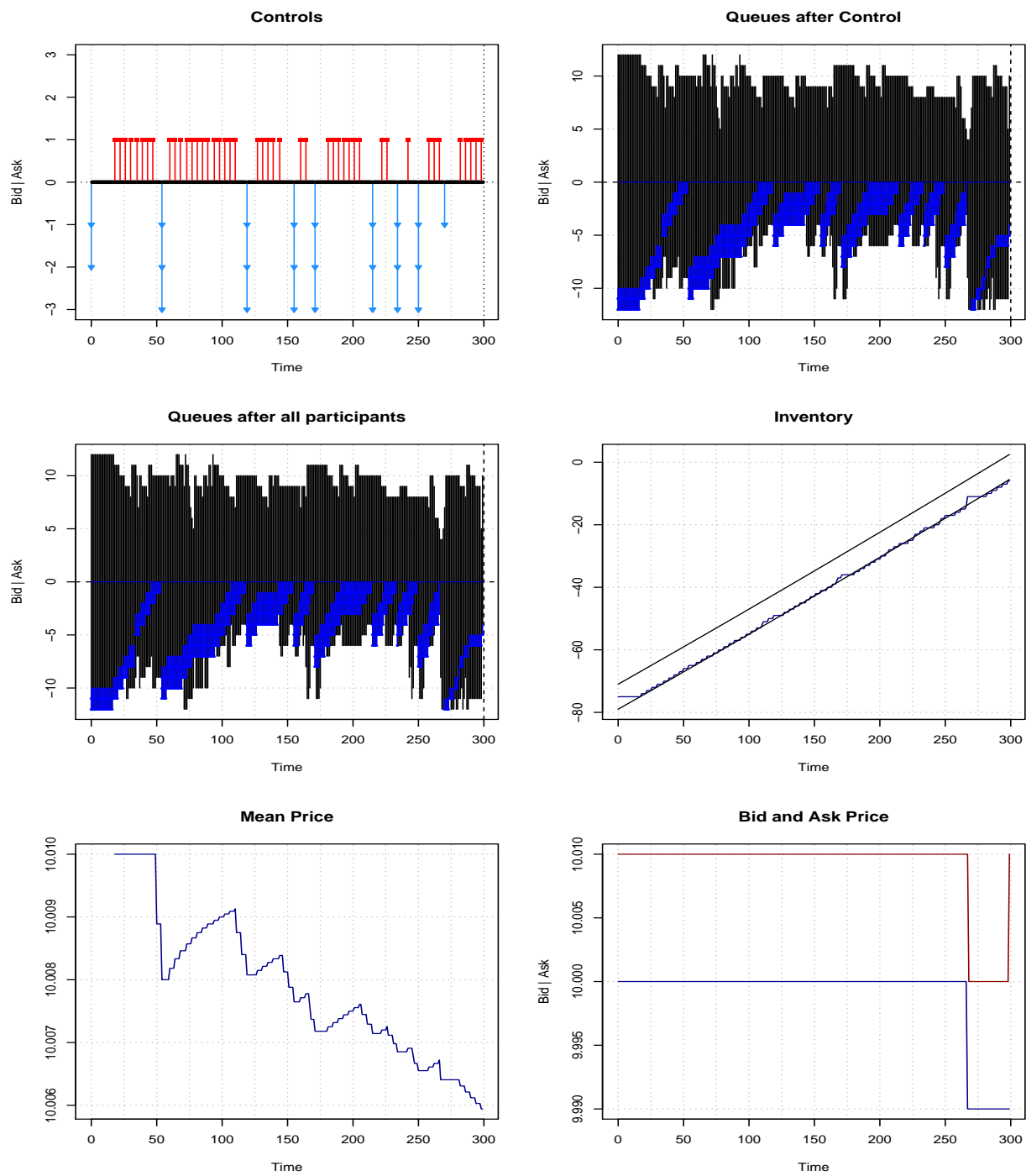


FIGURE 4.14 – Optimal strategy of the VWAP robot (buyer) when agents play together.

Bibliographie

- [1] Frédéric Abergel, Côme Huré, and Huyên Pham. Algorithmic trading in a microstructural limit order book model. *arXiv preprint arXiv:1705.01446*, 2017.
- [2] Frederic Abergel and Aymen Jedidi. A Mathematical Approach to Order Book Modeling. *Social Science Research Network Working Paper Series*, January 2011.
- [3] Frederic Abergel and Aymen Jedidi. Long time behaviour of hawkes process-based limit order book. *Social Science Research Network Working Paper Series*, 2015.
- [4] Robert Almgren and Neil Chriss. Value under liquidation. *Risk*, 12(12):61–63, 1999.
- [5] ARTEMIS. Q4 2017 catastrophe bond & ils market report. Technical report, 2018.
- [6] Marco Avellaneda and Sasha Stoikov. High-frequency trading in a limit order book. *Quantitative Finance*, 8(3):217–224, 2008.
- [7] Nicolas Baradel. Optimal control under uncertainty : Application to the issue of cat bonds. *arXiv preprint arXiv:1809.09545*, 2018.
- [8] Nicolas Baradel, Paul Besson, Bruno Bouchard, and Othmane Mounjid. Strategic interactions in a limit order book : statistical and numerical analysis. *In preparation*.
- [9] Nicolas Baradel, Bruno Bouchard, and Ngoc Minh Dang. Optimal trading with online parameter revisions. *Market Microstructure and Liquidity*, page 1750003, 2016.
- [10] Nicolas Baradel, Bruno Bouchard, and Ngoc Minh Dang. Optimal control under uncertainty and bayesian parameters adjustments. *SIAM Journal on Control and Optimization*, 56(2):1038–1057, 2018.
- [11] Nicolas Baradel, Bruno Bouchard, David Evangelista, and Othmane Mounjid. Optimal inventory management and order book modeling. *arXiv preprint arXiv:1802.08135*, 2018.
- [12] Guy Barles. An introduction to the theory of viscosity solutions for first-order hamilton–jacobi equations and applications. In *Hamilton-Jacobi equations : approximations, numerical analysis and applications*, pages 49–109. Springer, 2013.
- [13] Guy Barles and Cyril Imbert. Second-order elliptic integro-differential equations : viscosity solutions’ theory revisited. In *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis*, volume 25, pages 567–585. Elsevier Masson, 2008.
- [14] Guy Barles and Panagiotis E Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic analysis*, 4(3):271–283, 1991.

- [15] Bayer, Christian, Horst, Ulrich, and Qiu, Jinniao . A functional limit theorem for limit order books with state dependent price dynamics. 2015.
- [16] José M Bernardo and Adrian FM Smith. Bayesian theory, 2001.
- [17] Dimitir P Bertsekas and Steven Shreve. *Stochastic optimal control : the discrete-time case*. 2004.
- [18] Paul Besson, Stéphanie Pelin, and Matthieu Lasnier. To cross or not to cross the spread : that is the question ! Technical report, Kepler Cheuvreux, Paris, June 2016.
- [19] Bruno Biais, Pierre Hillion, and Chester Spatt. An empirical analysis of the limit order book and the order flow in the paris bourse. *the Journal of Finance*, 50(5):1655–1689, 1995.
- [20] Bruno Bouchard. A stochastic target formulation for optimal switching problems in finite horizon. *Stochastics : An International Journal of Probability and Stochastic Processes*, 81(2):171–197, 2009.
- [21] Bruno Bouchard, Ngoc-Minh Dang, and Charles-Albert Lehalle. Optimal control of trading algorithms : a general impulse control approach. *SIAM J. Financial Mathematics*, 2(1):404–438, 2011.
- [22] Bruno Bouchard and Nizar Touzi. Weak dynamic programming principle for viscosity solutions. *SIAM Journal on Control and Optimization*, 49(3):948–962, 2011.
- [23] Fabio Camilli and Maurizio Falcone. An approximation scheme for the optimal control of diffusion processes. *ESAIM : Mathematical Modelling and Numerical Analysis*, 29(1):97–122, 1995.
- [24] Fabio Camilli and Espen R Jakobsen. A finite element like scheme for integro-partial differential hamilton–jacobi–bellman equations. *SIAM Journal on Numerical Analysis*, 47(4):2407–2431, 2009.
- [25] Pierre Cardaliaguet. Notes on mean field games (from P.-L. Lions’ lectures at Collège de France). Technical report, 2010.
- [26] Guy Carpenter. Catastrophe bond update : Fourth quarter and full year 2015. Technical report, 2016.
- [27] Lehalle Charles-Albert and Mounjid Othmane. Limit Order Strategic Placement with Adverse Selection Risk and the Role of Latency. March 2016.
- [28] Rama Cont and Adrien De Larrard. Order book dynamics in liquid markets : limit theorems and diffusion approximations. 2012.
- [29] Rama Cont and Adrien De Larrard. Price dynamics in a markovian limit order market. *SIAM Journal on Financial Mathematics*, 4(1):1–25, 2013.
- [30] Rama Cont, Sasha Stoikov, and Rishi Talreja. A Stochastic Model for Order Book Dynamics. *Social Science Research Network Working Paper Series*, September 2008.
- [31] Michael G Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, 27(1):1–67, 1992.

- [32] J David Cummins. Cat bonds and other risk-linked securities : state of the market and recent developments. *Risk Management and Insurance Review*, 11(1):23–47, 2008.
- [33] J David Cummins. Cat bonds and other risk-linked securities : product design and evolution of the market. *The Geneva Reports*, 39, 2012.
- [34] Claude Dellacherie and Paul-André Meyer. Probabilities and potential. b, volume 72 of north-holland mathematics studies, 1982.
- [35] Jonathan Donier, Julius Bonart, Iacopo Mastromatteo, and J-P Bouchaud. A fully consistent, minimal model for non-linear market impact. *Quantitative finance*, 15(7):1109–1121, 2015.
- [36] Roxana Dumitrescu, Marie-Claire Quenez, and Agnès Sulem. A weak dynamic programming principle for combined optimal stopping/stochastic control with \mathcal{E}^f -expectations. *SIAM Journal on Control and Optimization*, 54(4):2090–2115, 2016.
- [37] David Easley and Nicholas M Kiefer. Controlling a stochastic process with unknown parameters. *Econometrica : Journal of the Econometric Society*, pages 1045–1064, 1988.
- [38] Wendell H Fleming and Panagiotis E Souganidis. On the existence of value functions of two-player, zero-sum stochastic differential games. *Indiana University Mathematics Journal*, 38(2):293–314, 1989.
- [39] Thierry Foucault. Order flow composition and trading costs in a dynamic limit order market1. *Journal of Financial Markets*, 2(2):99–134, 1999.
- [40] Olivier Guéant. *The Financial Mathematics of Market Liquidity : From optimal execution to market making*, volume 33. CRC Press, 2016.
- [41] Olivier Guéant. Optimal market making. *Applied Mathematical Finance*, 24(2):112–154, 2017.
- [42] Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Optimal portfolio liquidation with limit orders. *SIAM Journal on Financial Mathematics*, 3(1):740–764, 2012.
- [43] Olivier Guéant, Charles-Albert Lehalle, and Joaquin Fernandez-Tapia. Dealing with the inventory risk : a solution to the market making problem. *Mathematics and financial economics*, 7(4):477–507, 2013.
- [44] Fabien Guilbaud and Huyen Pham. Optimal high-frequency trading with limit and market orders. *Quantitative Finance*, 13(1):79–94, 2013.
- [45] Thomas Ho and Hans R Stoll. Optimal dealer pricing under transactions and return uncertainty. *Journal of Financial economics*, 9(1):47–73, 1981.
- [46] Weibing Huang, Charles-Albert Lehalle, and Mathieu Rosenbaum. Simulating and analyzing order book data : The queue-reactive model. *Journal of the American Statistical Association*, 10(509), December 2015.
- [47] Weibing Huang and Mathieu Rosenbaum. Ergodicity and diffusivity of Markovian order book models : a general framework, May 2015.

- [48] Robert William Kates, Craig E Colten, Shirley Laska, and Stephen P Leatherman. Reconstruction of new orleans after hurricane katrina : a research perspective. *Proceedings of the national Academy of Sciences*, 103(40):14653–14660, 2006.
- [49] Aimé Lachapelle, Jean-Michel Lasry, Charles-Albert Lehalle, and Pierre-Louis Lions. Efficiency of the Price Formation Process in Presence of High Frequency Participants : a Mean Field Game analysis. *Mathematics and Financial Economics*, 10(3):223–262, June 2016.
- [50] Peter Lakner, Josh Reed, and Sasha Stoikov. High Frequency Asymptotics for the Limit Order Book. Technical report, New York University Stern School of Business, July 2013.
- [51] Charles-Albert Lehalle and Sophie Laruelle. *Market microstructure in practice*. World Scientific, 2013.
- [52] Charles-Albert Lehalle, Othmane Mounjid, and Mathieu Rosenbaum. Optimal liquidity-based trading tactics. *arXiv preprint arXiv:1803.05690*, 2018.
- [53] Jill Malmstadt, Kelsey Scheitlin, and James Elsner. Florida hurricanes and damage costs. *southeastern geographer*, 49(2):108–131, 2009.
- [54] Harry Markowitz. Portfolio selection. *The journal of finance*, 7(1):77–91, 1952.
- [55] Kazuyoshi Oouchi, Jun Yoshimura, Hiromasa Yoshimura, Ryo Mizuta, Shoji Kusunoki, and Akira Noda. Tropical cyclone climatology in a global-warming climate as simulated in a 20 km-mesh global atmospheric model : Frequency and wind intensity analyses. *Journal of the Meteorological Society of Japan. Ser. II*, 84(2):259–276, 2006.
- [56] Francis Parisi and Robert Lund. Seasonality and return periods of landfalling atlantic basin hurricanes. *Australian & New Zealand Journal of Statistics*, 42(3):271–282, 2000.
- [57] Christine A. Parlour. Price Dynamics in Limit Order Markets. *The Review of Financial Studies*, 11(4):789–816, 1998.
- [58] Ioanid Roşu. A dynamic model of the limit order book. *Review of Financial studies*, 22:4601–4641, 2009.
- [59] Eric Smith, Doyne J. Farmer, László Gillemot, and Supriya Krishnamurthy. Statistical Theory of the Continuous Double Auction. *Quantitative Finance*, 3(6):481–514, 2003.
- [60] AIR Worldwide. The coastline at risk : 2016 update to the estimated insured value of u.s. coastal properties. Technical report, 2016.

Résumé

Cette thèse se compose de trois chapitres qui portent sur des problématiques de contrôle impulsif. Dans le premier chapitre, nous introduisons un cadre général de contrôle impulsif avec incertitude. Sachant une loi a priori sur des paramètres inconnus, nous expliquons comment celle-ci doit évoluer et l'intégrons au problème de contrôle optimal. Nous caractérisons la solution à travers une équation parabolique quasi-variationnelle qui se résout numériquement puis donnons des exemples d'application à la finance. Dans le deuxième chapitre, nous introduisons un problème de contrôle impulsif avec incertitude dans un cadre actuariel. Un (ré)assureur fait face à des catastrophes naturelles et peut émettre des CAT bonds afin de réduire le risque pris. Nous caractérisons à nouveau le problème de contrôle optimal à travers une équation parabolique quasi-variationnelle qui se résout numériquement et donnons des exemples d'application. Dans le dernier chapitre, nous proposons une modélisation du prix à travers un carnet d'ordre complètement endogène. Nous résolvons des problèmes de contrôle optimal impulsif (placement d'ordre) d'agents économiques rationnels que nous rassemblons sur un même marché.

Mots Clés

Contrôle optimal, trading optimal, incidence sur le marché, incertitude, filtrage Bayésien, obligation catastrophe

Abstract

This PhD thesis is composed of three chapters, which deal with applications of impulse control in Finance and Insurance. In the first chapter, we introduce a general framework of impulse control with uncertainty. Knowing a priori on unknown parameters, we explain how it should evolve and we integrate it in the formulation of the optimal control problem. We characterize the solution via a parabolic quasi-variational partial differential equation, which can be solved numerically. We give examples of application in finance. In the second chapter, we define an impulse control problem with uncertainty arising in actuarial sciences. A (re-)insurer faces natural disasters and may issue CAT bonds in order to reduce the risk taken. The problem is solved using a PDE approach. A numerical scheme and different examples of application are provided. In the last chapter, we propose a price model defined through a completely endogenous order book. We solve optimal impulse control problems (order placement) of rational economic agents that we gather on the same market.

Keywords

Optimal control, optimal trading, market impact, uncertainty, Bayesian filtering, CAT bond