



# Transfer operators of random partially hyperbolic dynamical systems

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## ► To cite this version:

Luc Gossart. Transfer operators of random partially hyperbolic dynamical systems. Dynamical Systems [math.DS]. Université Grenoble Alpes [2020-..], 2020. English. NNT : 2020GRALM062 . tel-03223781

**HAL Id: tel-03223781**

**<https://theses.hal.science/tel-03223781>**

Submitted on 11 May 2021

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## THÈSE

Pour obtenir le grade de

**DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES**

Spécialité : Mathématiques

Arrêté ministériel : 25 mai 2016

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préparée au sein du **Laboratoire Institut Fourier**  
dans l'**École Doctorale Mathématiques, Sciences et**  
**technologies de l'information, Informatique**

**Opérateurs de transfert de systèmes  
dynamiques partiellement hyperboliques  
aléatoires**

**Transfer operators of random partially  
hyperbolic dynamical systems**

Thèse soutenue publiquement le **12 novembre 2020**,  
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## Remerciements

## Introduction en français

## Contexte

On étudie dans cette thèse les traces plates des itérés d'opérateurs de transferts associés à des applications partiellement hyperboliques. Etant donné une application dilatante ou un difféomorphisme Anosov  $T$  sur une variété riemannienne compacte  $M$ , l'étude de l'opérateur  $\mathcal{L}$ , tirant les fonctions en arrière par le biais de  $T$  :

$$(0.1) \quad \mathcal{L} : \begin{cases} \mathcal{C}^\infty(M) & \longrightarrow \mathcal{C}^\infty(M) \\ u & \longmapsto u \circ T \end{cases}$$

s'est révélée intéressante pour la connaissance des propriétés statistiques de l'application  $T$ . Ce cadre a été étudié en détail au cours des décennies passées (voir par exemple [Rue89], [Bow75], [PP90], [BKL02], et se référer au chapitre 1 pour une présentation plus détaillée et des références plus fournies).

L'opérateur  $\mathcal{L}$  sera appelé dans tout ce qui suit **opérateur de transfert**, bien que ce terme ait à l'origine désigné, et désigne encore le plus souvent dans la littérature, son adjoint.

Les propriétés statistiques de l'application  $T$  sont liées au spectre de  $\mathcal{L}$  : Si  $T$  est lisse, la résolvante de l'opérateur de transfert admet un prolongement méromorphe à  $\mathbb{C} \setminus \{0\}$  et les fonctions de corrélation admettent un développement asymptotique s'exprimant de manière explicite à l'aide des pôles et résidus de la résolvante (voir par exemple à cet égard [Rue89] et [BKL02]; le chapitre 1 contient une description plus détaillée de ces principes, notamment la section 1, et la section 5 présente des résultats précis concernant pour des applications  $\mathcal{C}^k$ ). Les pôles de la résolvante (c'est-à-dire de son prolongement méromorphe) sont appelées **résonances de Ruelle-Pollicott**, et leur collection **spectre de Ruelle-Pollicott**. L'existence d'un tel prolongement méromorphe est généralement obtenue par le biais de la construction d'espaces dans lesquels le spectre de l'opérateur de transfert est discret en dehors d'un voisinage arbitrairement petit de l'origine.

Dans le cas de flots ou d'applications partiellement hyperboliques, la présence d'une direction neutre complique les choses. La section 5 du chapitre 1 présente des résultats de la littérature à ce sujet. La façon la plus élémentaire de construire un système qui sort du cadre hyperbolique est peut-être de partir d'une application hyperbolique et d'en construire une extension en cercles : Si  $T$  est une application dilatante ou un difféomorphisme d'une variété riemannienne compacte  $M$ , et si l'on dispose d'une fonction continue  $\tau : M \rightarrow \mathbb{T} := \mathbb{R}/\mathbb{Z}$ , alors on peut considérer le système dynamique

$$(0.2) \quad F : \begin{cases} M \times \mathbb{T} & \longrightarrow M \times \mathbb{T} \\ (x, y) & \longmapsto (T(x), y + \tau(x)) \end{cases} .$$

Ce genre de systèmes a été étudié notamment par Dolgopyat [Dol02], qui en a montré le caractère génériquement mélangeant pour des mesures naturelles.

Avec cette définition, les modes de Fourier dans la direction  $y$  sont préservés : si  $u \in \mathcal{C}^\infty(M)$  et  $p \in \mathbb{Z}$ ,

$$(0.3) \quad (u \otimes e^{2i\pi p \cdot})(F(x, y)) = e^{2i\pi p \tau(x)} u(T(x)) e^{2i\pi p y} .$$

De ce fait, on se ramène à l'étude d'une famille d'opérateurs de transfert

$$(0.4) \quad \mathcal{L}_{\xi, \tau} : \begin{cases} \mathcal{C}^\infty(M) & \longrightarrow \mathcal{C}^\infty(M) \\ u & \longmapsto e^{i\xi\tau} u \circ T \end{cases}$$

associés à une application dilatante ou hyperbolique, et indexés par un paramètre fréquentiel  $\xi \in \mathbb{R}$ . Comme première étape dans l'exploration du spectre de Ruelle, on prend pour  $\tau$  une fonction aléatoire et on s'intéresse aux traces plates des itérés de l'opérateur de transfert : Si  $\mathcal{L}_{\xi, \tau}$  n'est en général pas un opérateur à trace, il admet une trace plate (voir chapitre 1 section 5.1), définie comme l'intégrale de son noyau de Schwartz le long de la diagonale. Pour les opérateurs à trace, cette définition coïncide avec la notion habituelle de trace. La trace plate d'un itéré de l'opérateur de transfert s'exprime comme une somme sur les orbites périodiques :

$$(0.5) \quad \mathrm{Tr}^b(\mathcal{L}_{\xi, \tau}^n) = \sum_{x, T^n(x)=x} \frac{e^{i\xi\tau_x^n}}{|\det(1 - dT_x^n)|},$$

où  $\tau_x^n$  est la somme de Birkhoff

$$(0.6) \quad \tau_x^n = \tau(x) + \dots + \tau(T^{n-1}(x)).$$

Notons  $l_x$  la période (minimale) d'un point périodique  $x$ , et

$$(0.7) \quad A_n := \left( \sum_{x, T^n(x)=x} \frac{l_x}{|\det(1 - dT_x^n)|^2} \right)^{-\frac{1}{2}} \gg 1.$$

Remarquons qu' $A_n$  est exponentiellement grand :

$$(0.8) \quad A_n = e^{-\frac{n}{2} \mathrm{Pr}(-2J_u) + o(n)}$$

où  $J_u = \log \det(dT|_{E^u})$  est le jacobien instable, et  $\mathrm{Pr}(-2J_u) < 0$  sa pression topologique (voir (3.B.24)).

Dans l'hypothèse où les phases comprenant les sommes de Birkhoff, qui interviennent dans l'expression de la trace plate (0.5), se comporteraient comme des variables aléatoires indépendantes, uniformes sur  $S^1$ , un rapide calcul montre que les traces plates  $\mathrm{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$ , renormalisées par le facteur  $A_n$  convergent vers des variables aléatoires gaussiennes complexes de variance 1.

Dans [FW17], Faure et Weich établissent une forme normale globale, dans un modèle ouvert très proche, appelé IFS (*iterated function scheme*) : ils expriment les itérés  $\mathcal{L}_{\xi, \tau}^n$  de l'opérateur de transfert comme une somme d'opérateurs de rang 1 indexés par les points périodiques, dans la limite où  $n$  et  $\xi$  tendent vers l'infini, avec  $n$  plus grand qu'un temps d'Ehrenfest  $c \log \xi$ . Ils font usage de cette forme normale pour obtenir une borne supérieure sur le rayon spectral de  $\mathcal{L}_{\xi, \tau}$  pour  $\xi$  large, ce qui fournit un trou spectral asymptotique. Ils rencontrent les mêmes phases, faisant intervenir de sommes de Birkhoff, que dans la formule des traces (0.5), et obtiendraient une borne optimale si les phases se comportaient comme des variables aléatoires uniformes indépendantes.

Cependant, il n'est pas très raisonnable de considérer des fonctions aléatoires  $\tau$  pour lesquelles les valeurs prises en différents points périodiques d'une même période sont indépendantes, car cela mènerait à une grande irrégularité (si par exemple  $\tau$  est un champ gaussien, cette hypothèse implique que  $\tau$  est partout localement non

borné). Il n'y a alors plus lieu de parler de traces plates. De plus, l'ensemble captif, qui apparaît de manière centrale dans le papier [FW17] est sensible aux perturbations  $\mathcal{C}^1$ , il est donc naturel de souhaiter prendre pour fonction toit  $\tau$  une perturbation, petite en norme  $\mathcal{C}^1$ , d'une fonction  $\tau_0$  donnée.

### Résultats

Les résultats contenus dans cette thèse sont les deux théorèmes suivants:

**Applications dilatantes du cercle, chapitre 2.** Ce théorème est démontré dans le chapitre 2, qui est une reproduction de [Gos20a]. Soit  $k \in \mathbb{N}$ . Soit  $E$  une application dilatante  $\mathcal{C}^k$  du cercle, de degré  $l$ , préservant l'orientation. Soit  $M = \max E'$ .

**THEOREM 0.1.** *Soit  $\tau_0 \in \mathcal{C}^k(\mathbb{T})$ . Soit  $(c_p)_{p \in \mathbb{N}}$  une famille variables aléatoires gaussiennes complexes vérifiant  $\mathbb{E}[|c_p|^2] = O(p^{-2-\nu})$  pour un certain  $\nu > 0$  (afin de s'assurer que  $\delta\tau$  est presque sûrement continu). Soit  $c_{-p} = \bar{c}_p$  et*

$$(0.9) \quad \delta\tau(x) = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p x}.$$

Si

$$(0.10) \quad \exists \epsilon > 0, \exists C > 0, \forall p \in \mathbb{Z}^*, \mathbb{E}[|c_p|^2] \geq \frac{C}{p^{2k+2+\epsilon}},$$

alors on a la convergence en loi

$$(0.11) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

pour  $n$  et  $\xi$  tendant tous deux vers l'infini, en suivant la relation

$$(0.12) \quad \exists 0 < c < 1, \forall n, \xi, \quad n \leq c \frac{\log \xi}{\log l + (k + \frac{1}{2} + \frac{\epsilon}{2}) \log M}.$$

On verra que si l'on inverse l'inégalité dans la condition (0.10), c'est-à-dire, s'il existe  $\alpha > 0$  et  $C > 0$  tels que

$$(0.13) \quad \forall p \in \mathbb{Z}^*, \mathbb{E}[|c_p|^2] \leq \frac{C}{p^{2k+2+\alpha}},$$

alors la fonction  $\delta\tau$  est presque sûrement de classe  $\mathcal{C}^k$ .

Prendre  $k$  plus grand permet donc d'énoncer le théorème pour des champs plus réguliers, la contrepartie étant qu'il est alors valable pour des temps plus courts.

**Transitive Anosov diffeomorphisms, Chapter 3.** Ce théorème est démontré dans le chapitre 3 qui est une reproduction de [Gos20b]. Soit  $M$  une variété riemannienne compacte, soient  $T : M \longrightarrow M$  un difféomorphisme Anosov transitif, et  $h_{\text{top}}$  son entropie topologique. Soit  $\Lambda := \max(\Lambda^+, \Lambda^-)$  où

$$(0.14) \quad \Lambda^{\pm} := \lim_{n \rightarrow \infty} \max_{x \in M} \|dT_x^{\pm n}\|^{\frac{1}{n}}.$$

Soit  $(\phi_j)_{j \in \mathbb{N}}$  la suite des fonctions propres du laplacien, ordonnée par valeur propre croissante.

**THEOREM 0.2.** *Fixons une fonction quelconque  $\tau_0 \in \mathcal{C}^0(M)$  et  $\varepsilon > 0$ . Soient  $\zeta_j$  des variables aléatoires i.i.d.  $\mathcal{N}(0, 1)$  et  $(c_j)_{j \in \mathbb{N}}$  une suite de nombres réels telle que*

$$(0.15) \quad \frac{1}{C} j^{-\alpha} \leq c_j \leq C j^{-\beta}$$

*pour certaines constantes  $C > 0$ ,  $\alpha > \beta > 1$ . La condition impliquant  $\beta$  permet de s'assurer que*

$$(0.16) \quad \delta\tau = \sum_{j \geq 0} c_j \zeta_j \phi_j$$

*est presque sûrement continu, et que l'on peut bien ainsi définir  $\mathcal{L}_{\xi, \tau}^n$  pour  $\tau := \tau_0 + \varepsilon \delta\tau$ . Alors, pour tout  $0 < c < 1$ , on a la convergence en loi suivante :*

$$(0.17) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

*lorsque  $n$  et  $\xi$  tendent vers l'infini sous la relation*

$$(0.18) \quad n \leq c \frac{\log \xi}{h_{\text{top}} + \frac{d}{2}(\alpha - \frac{1}{2}) \log \Lambda}.$$

**Remarques.** Ces deux résultats sont très proches, et le théorème 0.4 est essentiellement une généralisation (et légère amélioration, voir la remarque 1.8 du chapitre 3) du théorème 0.3, à ceci près qu'il requiert l'inversibilité. Sa démonstration peut toutefois être adaptée sans grande difficulté au cas d'une application dilatante.

Les traces plates sont reliées au spectre de Ruelle-Pollicott comme suit (voir corollaire 5.13 dans le chapitre 1 ou [Jéz17]) : soit  $\varepsilon > 0$  tel que l'opérateur de transfert  $\mathcal{L}_V$  avec potentiel  $V$  n'ait pas de résonance de module  $\varepsilon$ , alors

$$(0.19) \quad \text{Tr}^b \mathcal{L}_V^n = \sum_{\substack{\lambda \in \text{Res}(\mathcal{L}_V) \\ |\lambda| > \varepsilon}} \lambda^n + O(\varepsilon^n).$$

De plus, par le théorème 5.22 du chapitre 1, la collection des traces plates de tous les itérés  $\mathcal{L}_{\xi, \tau}^n$  caractérise le spectre de Ruelle-Pollicott. On ne dispose toutefois ici que d'informations concernant un nombre fini de temps  $n$ , et il n'est pas clair que de ces résultats sur les traces plates découle un énoncé concernant le spectre de Ruelle-Pollicott. Naud [Nau16] montre, dans le cadre d'applications Anosov analytiques du tore, avec pour fonction toit un polynôme trigonométrique, qu'avec probabilité positive le module de la trace plate est moralement d'ordre au moins  $A_n^{-1}$ , et parvient à en déduire une borne inférieure sur le module maximal des résonances, en tirant avantage du caractère à trace de l'opérateur de transfert pour des applications analytiques.

L'étude de ces phases présente des analogies avec la conjecture Bohigas-Giannoni-Schmidt [BGS84] issue du chaos quantique. Cette conjecture énonce que pour des systèmes quantiques, dont la dynamique classique associée est chaotique, le spectre de l'opérateur hamiltonien présente des propriétés statistiques typiques de matrices aléatoires : Il y a une correspondance en courbure négative constante entre le spectre de Ruelle-Pollicott pour le flot géodésique et le spectre du laplacien, qui est un opérateur hamiltonien ayant pour dynamique «classique» associée le flot



géodésique (voir [DFG15] et les références qui y figurent). Un projet en cours avec Faure et Rivera vise à approximer l'opérateur de transfert  $\mathcal{L}_{\xi,\tau}^n$  pour des temps  $n$  supérieurs à un temps d'Ehrenfest  $C \log \xi$  par une matrice aléatoire, et de chercher les conséquences que cela pourrait induire sur le spectre de Ruelle-Pollicott.

Les théorèmes sont formulés avec une perturbation  $\delta\tau$  indépendante de  $n$ . Ils pourraient au lieu de cela faire intervenir une suite de fonctions  $\tau_n$  en ne gardant que les termes  $j \leq n$  dans l'équation (2.27) du chapitre 3. Ainsi, les fonctions  $\tau_n$  sont lisses (analytiques dans le cas du tore), mais les opérateurs de transfert ne proviennent plus d'une même application partiellement hyperbolique.

**Idées de démonstration.** Les preuves suivent les mêmes lignes directrices : on écrit la fonction aléatoire  $\delta\tau$  comme une somme de fonctions aléatoires indépendantes  $\delta\tau_j$ ,  $j \in \mathbb{N}$ , de sorte que les images par  $\delta\tau_j$  de deux points périodiques de période  $j$  soient décorréélées lorsque  $j$  tend vers l'infini. Ceci impose la présence de hautes fréquences dans la décomposition en Fourier de  $\delta\tau_j$ . Cette décomposition étant établie, au temps  $n$ , on sépare la somme de Birkhoff  $\delta\tau_x^n$  en une somme  $(\delta\tau_n)_x^n + (\sum_{j \neq n} \delta\tau_j)_x^n$ . Le premier terme forme une famille de variables aléatoires quasiment indépendantes, lorsque  $x$  parcourt des points périodiques associés à différentes orbites, et on montre que le second terme ne joue pas de rôle si  $\xi$  est assez grand devant  $n$ . Voir la section 1.7 du chapitre 2 pour une heuristique plus détaillée.

## Chapitre premier

Le chapitre 1 ne contient pas de nouveaux résultats, et constitue une présentation non-exhaustive du domaine et de résultats liés au sujet des articles.

**Section 1.** La première section présente des idées générales et basiques concernant les systèmes dynamiques et les résonances de Ruelle-Pollicott.

**Section 2.** La section 2 présente l'analyse semi-classique. D'abord quelques heuristiques à propos de la transformée FBI, qui est utilisée dans la troisième section; puis une brève présentation des opérateurs pseudo-différentiels qui interviennent dans la construction des fonctions aléatoires dans [Gos20b] (chapitre 3).

**Section 3.** La section 3 montre, dans un cadre très simple et concret, une construction d'espaces fournissant de bonnes propriétés spectrales pour l'opérateur de transfert, ainsi qu'un spectre de Ruelle et des états propres explicites. Bien qu'étant très simpliste, ce modèle met en lumière des stratégies et outils proches de ceux employés dans des circonstances bien plus générales, comme par exemple dans [FT17a]. Le choix de ce modèle lui-même n'est pas nouveau, celui-ci était présenté dans [Fau19].

**Section 4.** La quatrième section présente les notions d'entropie topologique, de pression topologique et d'entropie métrique d'applications. Elle énonce les principes variationnels qui jouent un rôle important dans les chapitres suivants. D'autres propriétés de la pression utilisées dans les articles, comme l'existence et l'unicité des mesures d'équilibre pour les applications dilatantes et les difféomorphismes d'Anosov transitifs sont présentés dans la section suivante.

**Section 5.** La dernière section du premier chapitre est dédiée à un résumé de résultats connus sur le spectre de Ruelle-Pollicott dans le cadre hyperbolique, entre autres existence d'espaces dans lesquels les opérateurs sont quasi-compacts, résultats concernant les valeurs propres extérieures, traces plates.

## English introduction

### Context

The main object of study of this thesis are flat traces of transfer operators of expanding and hyperbolic maps. Given an expanding map or an Anosov diffeomorphism  $T$  of a compact Riemannian manifold, it has been proven interesting to study the action of the pull-back operator  $\mathcal{L}$  associated to  $T$

$$(0.20) \quad \mathcal{L} : \begin{cases} \mathcal{C}^\infty(M) & \longrightarrow \mathcal{C}^\infty(M) \\ u & \longmapsto u \circ T \end{cases}$$

This framework has been extensively studied across the previous decades by several authors (for example [Rue89], [Bow75], [PP90], [BKL02], see Chapter 1 for additional references).

We will refer to the operator  $\mathcal{L}$  as **transfer operator**, although this term usually designates its adjoint.

The statistical properties of the map  $T$  are linked to the spectral properties of the operator  $\mathcal{L}$ : If  $T$  is smooth, the resolvent of the transfer operator extends meromorphically to  $\mathbb{C} \setminus \{0\}$  and the correlations functions have an asymptotic expansion given explicitly in terms of poles and residues of this resolvent (see *e.g.* [Rue89] and [BKL02]; more details are given in Section 1 of Chapter 1 about the resolvent and in Section 5 of Chapter 1 about specific results in that regard). The poles of the meromorphic continuation of the resolvent are called **Ruelle-Pollicott spectrum**, or **Ruelle-Pollicott resonances**. Practically, the existence of this meromorphic continuation involves the construction of spaces in which the transfer operator has discrete spectrum outside an arbitrarily small neighbourhood of the origin.

For flows or partially hyperbolic systems, the presence of a neutral direction adds some difficulty. See Section 5 of Chapter 1 for some historical remarks about this setting. Probably the simplest way of extending the hyperbolic framework is to study a circle extension of a hyperbolic map: Let as before  $T$  be an expanding map or an Anosov diffeomorphism of a compact Riemannian manifold  $M$  and let  $\tau : M \rightarrow \mathbb{T} := \mathbb{R}/\mathbb{Z}$  be a continuous function. Then one can consider the dynamical system

$$(0.21) \quad F : \begin{cases} M \times \mathbb{T} & \longrightarrow M \times \mathbb{T} \\ (x, y) & \longmapsto (T(x), y + \tau(x)) \end{cases} .$$

This kind of models has been studied notably in [Dol02], where generic rapid mixing with respect to natural measures is established.

With this construction, the transfer operator preserves Fourier modes in the neutral  $y$  direction: for  $u \in \mathcal{C}^\infty(M)$  and  $p \in \mathbb{Z}$ ,

$$(0.22) \quad (u \otimes e^{2i\pi p \cdot})(F(x, y)) = e^{2i\pi p \tau(x)} u(T(x)) e^{2i\pi p y} .$$

So the study of the transfer operator amounts to the study of a family of transfer operators

$$(0.23) \quad \mathcal{L}_{\xi, \tau} : \begin{cases} \mathcal{C}^\infty(M) & \longrightarrow \mathcal{C}^\infty(M) \\ u & \longmapsto e^{i\xi \tau} u \circ T \end{cases}$$

associated to expanding or Anosov maps, indexed by a frequency parameter  $\xi \in \mathbb{R}$ . As a first step towards the exploration of the Ruelle-Pollicott spectrum, we consider

for  $\tau$  a random function, and investigate the flat traces of the iterates of the transfer operators:  $\mathcal{L}_{\xi,\tau}$  is not trace-class, yet it admits a flat trace (see Section 5.1 of Chapter 1), defined as the integral of its Schwartz kernel along the diagonal. This definition coincides with the usual trace for trace-class operators. It is given by a sum over periodic orbits

$$(0.24) \quad \mathrm{Tr}^b(\mathcal{L}_{\xi,\tau}^n) = \sum_{x, T^n(x)=x} \frac{e^{i\xi\tau_x^n}}{|\det(1 - dT_x^n)|},$$

where  $\tau_x^n$  is the Birkhoff sum

$$(0.25) \quad \tau_x^n = \tau(x) + \dots + \tau(T^{n-1}(x)).$$

Let  $l_x$  denote the prime period of a periodic point  $x$ , and let

$$(0.26) \quad A_n := \left( \sum_{x, T^n(x)=x} \frac{l_x}{|\det(1 - dT_x^n)|^2} \right)^{-\frac{1}{2}} \gg 1.$$

$A_n$  is exponentially large:

$$(0.27) \quad A_n = e^{-\frac{n}{2} \mathrm{Pr}(-2J_u) + o(n)}$$

where  $J_u = \log \det(dT|_{E^u})$  is the unstable Jacobian, and  $\mathrm{Pr}(-2J_u) < 0$  its topological pressure (see (3.B.24)). If the phases involving these Birkhoff sums in (0.24) were independent uniform random variables on the circle  $S^1$ , then a simple computation shows that the flat trace  $\mathrm{Tr}^b(\mathcal{L}_{\xi,\tau}^n)$ , rescaled by  $A_n$  converges towards complex Gaussian random variables of variance 1.

In [FW17], Faure and Weich establish a global normal form for the related open model of one dimensionnal IFS: They express the iterates  $\mathcal{L}_{\xi,\tau}^n$  of the transfer operator as a sum of rank one operators, indexed by periodic orbits, as  $n, \xi \rightarrow \infty$ , for  $n$  larger than an Ehrenfest time  $c \log \xi$ . They use it to deduce an upperbound on the spectral radius of the transfer operator  $\mathcal{L}_{\xi,\tau}$  for large  $\xi$ , which gives an asymptotic spectral gap. They face similar phases involving Birkhoff sums, and would obtain the conjectural optimal bound, if these phases would behave as independent uniform random variables.

It is however not quite reasonable to consider random functions  $\tau$ , for which the values at different periodic points are independent, since the repartition of periodic orbits then would impose for it to be extremely irregular (nowhere locally bounded if  $\tau$  is a Gaussian random field). Then the flat traces from (0.24) do not make sense. Moreover, the trapped set that appears centrally in [FW17] is sensitive to  $\mathcal{C}^1$  perturbations, so it is natural to want the function  $\tau$  of the form  $\tau = \tau_0 + \varepsilon \delta\tau$ , for a deterministic function  $\tau_0$ , a small constant  $\varepsilon$  and a random  $\mathcal{C}^1$  function  $\delta\tau$ .

## Results

The results of this thesis are the following theorems:

**Expanding maps of the circle, Chapter 2.** This theorem is proved in Chapter 2, which is a reproduction of [Gos20a]. Let  $k \in \mathbb{N}$ . Let  $E$  be an orientation preserving  $\mathcal{C}^k$  expanding map of the circle of degree  $l$ , and let  $M := \max E'$ .

**THEOREM 0.3.** *Let  $\tau_0 \in \mathcal{C}^k(\mathbb{T})$ . Let  $(c_p)_{p \in \mathbb{N}}$  be random complex Gaussian variables such that  $\mathbb{E}[|c_p|^2] = O(p^{-2-\nu})$  for some  $\nu > 0$  (so that  $\delta\tau$  is almost surely continuous). Let  $c_{-p} = \overline{c_p}$  and*

$$(0.28) \quad \delta\tau(x) = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p x}.$$

*If*

$$(0.29) \quad \exists \epsilon > 0, \exists C > 0, \forall p \in \mathbb{Z}^*, \mathbb{E}[|c_p|^2] \geq \frac{C}{p^{2k+2+\epsilon}},$$

*then one has the convergence in law of the flat traces*

$$(0.30) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

*as  $n$  and  $\xi$  go to infinity, under the constraint*

$$(0.31) \quad \exists 0 < c < 1, \forall n, \xi, \quad n \leq c \frac{\log \xi}{\log l + (k + \frac{1}{2} + \frac{\epsilon}{2}) \log M}.$$

We will see that reversing the inequality in (0.29), that is, if there exists  $\alpha > 0$  and  $C > 0$  such that

$$(0.32) \quad \forall p \in \mathbb{Z}^*, \mathbb{E}[|c_p|^2] \geq \frac{C}{p^{2k+2+\alpha}},$$

then  $\delta\tau$  is almost surely  $\mathcal{C}^k$ .

Having  $k$  larger allows consequently to state the theorem for more regular fields, with the counterpart that it is then valid for shorter times.

**Transitive Anosov diffeomorphisms, Chapter 3.** This theorem is proved in Chapter 3, which is a reproduction of [Gos20b]. Let  $M$  be a compact Riemannian manifold, let  $T : M \rightarrow M$  be a transitive Anosov diffeomorphism, and  $h_{\text{top}}$  be its topological entropy. Let  $\Lambda := \max(\Lambda^+, \Lambda^-)$  where

$$(0.33) \quad \Lambda^{\pm} := \lim_{n \rightarrow \infty} \max_{x \in M} \|dT_x^{\pm n}\|^{\frac{1}{n}}.$$

Let  $(\phi_j)_{j \in \mathbb{N}}$  be the sequence of eigenfunctions of the Laplace-Beltrami operator on  $M$ , indexed by increasing eigenvalues.

**THEOREM 0.4.** *Let us fix any  $\tau_0 \in \mathcal{C}^0(M)$  and  $\varepsilon > 0$ . Let  $\zeta_j$  be i.i.d.  $\mathcal{N}(0, 1)$  random variables and  $(c_j)_{j \in \mathbb{N}}$  be a real sequence such that*

$$(0.34) \quad \frac{1}{C} j^{-\alpha} \leq c_j \leq C j^{-\beta}$$

*for some constants  $C > 0$ ,  $\alpha > \beta > 1$ . The condition involving  $\beta$  ensures that*

$$(0.35) \quad \delta\tau = \sum_{j \geq 0} c_j \zeta_j \phi_j$$

*is almost surely continuous, and that we can define the flat trace of  $\mathcal{L}_{\xi, \tau}^n$  for  $\tau := \tau_0 + \varepsilon \delta\tau$ . Then, for any  $0 < c < 1$ , we have the following convergence in law*

$$(0.36) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

*as  $n$  and  $\xi$  go to infinity under the relation*

$$(0.37) \quad n \leq c \frac{\log \xi}{h_{\text{top}} + \frac{d}{2}(\alpha - \frac{1}{2}) \log \Lambda}.$$

**Comments.** These results are very similar, and Theorem 0.4 is essentially a generalisation (and slight improvement, see Remark 1.8 of Chapter 3) of Theorem 0.3, except for the fact that it does not cover non-invertibility. Its proof can however be adapted with very little modifications to the general expanding case.

The flat traces are related to the Ruelle-Pollicott spectrum as follows (see Corollary 5.13 in Chapter 1 or [Jéz17]): Let  $\varepsilon > 0$  be such that the transfer operator  $\mathcal{L}_V$  with potential  $V$  has no resonance of modulus  $\varepsilon$ , then

$$(0.38) \quad \mathrm{Tr}^b \mathcal{L}_V^n = \sum_{\substack{\lambda \in \mathrm{Res}(\mathcal{L}_V) \\ |\lambda| > \varepsilon}} \lambda^n + O(\varepsilon^n).$$

Moreover, the collection of flat traces of all iterates  $\mathcal{L}_{\xi, \tau}^n$  characterizes the Ruelle-Pollicott spectrum by Theorem 5.22 of Chapter 1. However, we only have information about the iterates for a finite number of  $n$ , and it is not clear whether these results give any information about the Ruelle resonances. Naud [Nau16], in the framework of analytic Anosov maps of the torus, with a random trigonometric polynomial as roof function  $\tau$ , shows that with positive probability the modulus of the flat trace is morally of order at least  $A_n^{-1}$ , and is able to deduce from this fact a lower bound on the outtermost resonance, using the trace-class property for analytic maps.

This study of phases shows some analogies with the Bohigas-Giannoni-Schmidt conjecture from quantum chaos [BGS84], that states that for quantum systems whose associated classical dynamic is chaotic, the spectrum of the Hamiltonian operator exhibits typical statistical properties of random matrices: There is a correspondence in constant negative curvature between Ruelle-Pollicott spectrum for the geodesic flow and spectrum of the Laplacian, which is the Hamiltonian corresponding to the “classical” geodesic flow (see [DFG15] and references therein). An ongoing project with Faure and Rivera aims to approximate the transfer operator  $\mathcal{L}_{\xi, \tau}^n$  for random  $\tau$  and times of order at least  $n \approx C \log \xi$  by a random matrix, and to look for implications that this might have on the Ruelle-Pollicott spectrum.

The theorems are formulated with a perturbation  $\delta\tau$  independent of  $n$ . Yet, it could be stated with a sequence  $\tau_n$  by keeping only terms up to  $j = n$  in (2.27) in Chapter 3. This way, the sequence  $\tau_n$  is smooth (analytic in the case of the torus), but the transfer operators no longer come from a given partially hyperbolic map.

**Ideas of the proof.** The proofs follow the same pattern: the random function  $\delta\tau$  is written as a sum of independent random functions  $\delta\tau_j$ ,  $j \in \mathbb{N}$ , so that the images by  $\delta\tau_j$  of two periodic points of period  $j$  are decorrelating as  $j \rightarrow \infty$  (they are actually independent at all time in the circle case). This imposes the presence of high frequencies in  $\delta\tau_j$ . Then, at time  $n$ , we split  $\delta\tau_x^n$  into a sum  $(\delta\tau_n)_x^n + (\sum_{j \neq n} \delta\tau_j)_x^n$ . The first term behaves as independent random variables, when  $x$  runs over periodic points associated to different orbits, and we show that the second term plays no role if  $\xi$  is large enough with respect to  $n$ . See Section 1.7 of Chapter 2 for a more detailed heuristic.

## Chapter one

Chapter one does not contain any new result, and is meant as a non-exhaustive presentation of some topics and references related to the subject of the articles.

**Section 1.** Its first section presents general basics ideas on dynamical systems and Ruelle-Pollicott spectrum.

**Section 2.** The second section briefly introduces semi-classical analysis. First some heuristics about the FBI transform, that is later used in the third section; and then some context about pseudo-differential operators, which are involved in the contruction of random functions in [Gos20b] (Chapter 3).

**Section 3.** Section 3 presents, in a very simple and concrete framework, an explicit construction of spaces giving discrete spectrum for the transfer operator of a flow, as well as the Ruelle-Pollicott eigenvalues and eigendistributions. This model, in spite of being perhaps oversimplified, shows some tools and strategies close to those employed in broader settings, like in [FT17a]. The choice of this model itself, is not new, and was presented before in [Fau19].

**Section 4.** The fourth section presents the notions of topological entropy, topological pressure and metric entropy for maps. It states the variational principles, that play a role in both papers. Other features of the pressure used in the article, like existence and uniqueness of equilibrium measures for expanding maps and Anosov diffeomorphisms are presented in the next section.

**Section 5.** The final section of this chapter is devoted to a summary of some known results about the Ruelle-Pollicott spectrum in the hyperbolic framework, namely existence of spaces in which the operators are quasi-compact, results about outter eigenvalues and equilibrium measures, flat traces.



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## CHAPTER 1

# Presentation of the field

This chapter is an introductory chapter and does not provide any new results. The first and third sections are intended to a reader new to the field and introduce general ideas with some details. The other sections are more concise and give some context and references to topics related to the last two chapters.

## 1. Generalities about dynamical systems and chaos

**1.1. Chaos, ergodicity, mixing.** Dynamical systems are deterministic transformations: here we will mainly be interested in  $C^k$  maps or flows generated by a  $C^k$  vector field, always on a compact (except for the introductory example where we consider  $\mathbb{R}$ ) Riemannian manifold. The case of maps  $T$  is referred to as discrete time while flows  $\phi^t$  form the so-called continuous time dynamical systems. In both cases the short term behaviour is known and the main object of interest is the properties of  $T^n$  (resp.  $\phi^t$ ) for large time  $n \in \mathbb{N}$  or  $\mathbb{Z}$  (resp. large  $t \in \mathbb{R}_+$  or  $\mathbb{R}$ ). There is no consensus about a precise definition of chaos, but this term is generally used for systems who exhibit a great instability, in the sense that close points see their trajectories split quickly and behave somehow independently from each other. There is consequently a seeming unpredictability and decorrelation between the evolution of a point and the initial condition. The simplest example of such a system might be the angle multiplying map on  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ :

$$(1.1) \quad E_l : \begin{cases} \mathbb{T} & \longrightarrow \mathbb{T} \\ x & \longmapsto lx \end{cases}$$

for some integer  $l \geq 2$ . A point  $x$  in  $\mathbb{T}$  can be seen as a number between 0 and 1, written for instance in decimal form  $x = 0, x_1 x_2 \dots$ . Then  $E_{10}$  acts as a shift on the sequence of decimals:

$$(1.2) \quad E_{10}(x) = 0, x_2 x_3 \dots$$

It is therefore clear that if we know the position of a point with a precision  $10^{-n}$ , then after  $n$  iterations, we have lost all information about the position of the point. The behaviour of the trajectory may then seem random after a short time, in spite of the fully deterministic nature of the system. A remarkable feature of many chaotic systems is that this instability of individual trajectories often gives rise to predictable collective properties, and probabilistic phenomena. The proportion of time spent by the trajectory of almost every point of  $\mathbb{T}$  under the action of  $E_{10}$  in any Borel subset converges for example to the Lebesgue measure of this subset [BS02, Proposition 4.4.2].

**DEFINITION 1.1.** Let  $T$  be a measure-preserving ( $T_*\mu = \mu$ ) map of a measured space  $(X, \mu)$ .  $T$  is said to be

(1) ergodic if for every  $L^1$  function  $f$  and  $\mu$  almost every  $x \in X$ ,

$$(1.3) \quad \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) \xrightarrow{n \rightarrow \infty} \int f \, d\mu.$$

(2) weakly mixing if for every  $L^2$  functions  $f, g$

$$(1.4) \quad \frac{1}{n} \sum_{k=0}^{n-1} \left| \int f \circ T^k \cdot g \, d\mu - \int g \, d\mu \int f \, d\mu \right| \longrightarrow 0.$$

(3) mixing if for every  $L^2$  functions  $f, g$

$$(1.5) \quad \int f \circ T^n \cdot g \, d\mu \longrightarrow \int f \, d\mu \int g \, d\mu.$$

It is not hard to see that

$$\text{mixing} \implies \text{weak mixing} \implies \text{ergodicity}.$$

REMARK 1.2. We have obvious analogs for the continuous time case.

Ergodicity expresses that almost all orbits (with respect to  $\mu$ ) have the same statistical behaviour in some sense: if we take for instance the indicator of a measurable set  $A$  for  $f$  in the definition, we obtain that for almost every orbit, the proportion of time spent in  $A$  is asymptotically  $\mu(A)$ . We say indifferently, according to the context, that  $T$  is ergodic with respect to  $\mu$ , or that  $\mu$  is ergodic with respect to  $T$ . Note that some ergodic measure are of limited interest: For instance, if  $x$  is a fixed point of  $T$ , a Dirac measure on  $x$  is ergodic, but does not necessarily reflect the statistical behaviour of the whole system. The ergodic basin of an ergodic measure is the set of points for which (1.3) is satisfied. Of prime interest will be the Radon measures on compact Riemannian manifolds whose ergodic basin is of positive Lebesgue measure. Such measures are called physical measures.

Mixing can be interpreted in probability terms of decorrelation as mentioned earlier: if  $\mu$  is for instance a probability measure, and  $f, g$  two  $L^2$  functions,  $f$  and  $g$  can be seen as random variables, then  $T$  acts on  $f$  by composition and mixing states that

$$(1.6) \quad \mathbb{E}[f \circ T^n \cdot g] \xrightarrow{n \rightarrow \infty} \mathbb{E}[f] \mathbb{E}[g].$$

Thus, any  $L^2$  random variable becomes asymptotically decorrelated with everything (including itself) under iterations of a mixing map  $T$ . There is therefore a loss of information induced by the chaotic dynamic.

PROPOSITION 1.3 ([KH97] Proposition 4.2.7). *The maps  $E_l$  defined in (1.1) are mixing with respect to the Lebesgue measure on  $\mathbb{T}$ .*

DEFINITION 1.4. For  $f, g \in L^2(\mu)$ , we call correlation function the function

$$(1.7) \quad c_{f,g} : \begin{cases} \mathbb{N} & \longrightarrow \mathbb{R} \\ n & \longmapsto \int f \circ T^n \cdot g \, d\mu - \int f \, d\mu \int g \, d\mu \end{cases}$$

REMARK 1.5. Sometimes it will be more convenient to use instead the definition

$$(1.8) \quad c_{f,g}(n) = \int f \circ T^n \cdot g \, d\mu.$$

It is in general interesting to consider the speed at which the decorrelation occurs for mixing dynamical systems. Often, the more regular the observables are, the faster is the decay.

Ergodicity is a form of law of large numbers for (not random) variables  $f \circ T^k$ . And a fast enough decay of correlations may imply a central limit theorem [Liv96].

We can mention the following result:

PROPOSITION 1.6. *Let  $f, g$  be two smooth functions on  $(\mathbb{T}, \text{Leb})$ . Then the correlations of  $f$  and  $g$  with respect to the expanding map  $E_l$  decay superexponentially fast:*

$$(1.9) \quad \forall \alpha > 0, \exists C_{\alpha, f, g} > 0, \forall n \in \mathbb{N}, c_{f,g}(n) \leq C_{\alpha, f, g} e^{-\alpha n}.$$

This is however a very particular case, due to the fact that, thanks to the linearity of the map  $E_l$ , the image of a Fourier mode is exactly a Fourier mode. For general expanding maps, the decay is exponential as we will see later.

PROOF. We will denote by  $c_p(h)$  the  $p$ -th Fourier coefficient of a function  $h$  on  $\mathbb{T}$ . It is a classical fact that if  $f$  is smooth, its Fourier coefficients observe fast decay:

$$(1.10) \quad \forall N \in \mathbb{N}, \exists C_N > 0, \forall p \in \mathbb{Z}, |c_p(f)| \leq C_N(1 + |p|)^{-N}.$$

Moreover, a quick computation shows that

$$(1.11) \quad c_q(f \circ E_l^n) = \begin{cases} c_p(f) & \text{if } q = l^n p \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$(1.12) \quad \begin{aligned} c_{f,g}(n) &= \sum_{|q| \geq 1} \overline{c_q(f \circ E_l^n)} c_q(g) \\ &= \sum_{|p| \geq 1} \overline{c_p(f)} c_{l^n p}(g) \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

superexponentially fast in virtue of (1.10).  $\square$

The strategy followed here, to observe the action of the dynamics in frequency, more precisely to see which trajectories escape to infinity (high frequencies) and to consider observables that decay along those trajectories has been widely used to obtain informations about the asymptotic behaviour of the correlation functions. A lot more technicality is of course involved in general.

**1.2. Transfer operator and Ruelle-Pollicott resonances.** We have talked in the previous subsection about the fact that chaotic dynamics tend to exhibit interesting statistical properties. The examples cited, ergodicity and mixing, both involved the action of the map  $T$  on functions, and more precisely the long-term behaviour of  $f \circ T^n$ , for different classes of functions  $f$ . It is therefore natural to investigate the spectrum of the operator  $f \mapsto f \circ T$  on different spaces.

DEFINITION 1.7. Let  $M$  be a compact Riemannian manifold and  $T : M \rightarrow M$  be a smooth map. We call transfer operator, or Koopman operator the operator

$$(1.13) \quad \mathcal{L} : \begin{cases} \mathcal{C}^\infty(M) & \rightarrow \mathcal{C}^\infty(M) \\ f & \mapsto f \circ T \end{cases}$$

It can be extended to distribution by duality. Its  $L^2$  adjoint  $\mathcal{L}^*$  is called Ruelle-Perron-Frobenius transfer operator and satisfies the formula

$$(1.14) \quad \mathcal{L}^* u(x) = \sum_{y \in T^{-1}(\{x\})} \frac{u(y)}{|\det dT_y|}.$$

Ruelle was the first to study extensively transfer operators and to introduce various other techniques from statistical physics, called thermodynamical formalism, to study chaotic dynamics [Rue04].

In order to obtain the asymptotic behaviour of correlations, one studies the resolvent of the transfer operator, for instance on  $L^2(\mu)$  for some appropriate invariant Radon measure  $\mu$  (In some cases, taking for  $\mu$  the Riemannian volume  $dx$

will be interesting, even when it is not invariant). Since  $\mu$  is invariant,  $\mathcal{L}$  has norm 1 on  $L^2(\mu)$ , and therefore its resolvent is analytic on  $\{z \in \mathbb{C}, |z| > 1\}$ :

$$(1.15) \quad \forall z \in \mathbb{C}, |z| > 1 \implies \sum_{n=0}^{\infty} z^{-n-1} \mathcal{L}^n = \frac{1}{z} \left(1 - \frac{1}{z} \mathcal{L}\right)^{-1} = (z - \mathcal{L})^{-1}.$$

In general it is not meromorphic inside the unit disk (see Proposition 1.18 below). It might however admit a meromorphic extension as a family of operators  $\mathcal{C}^\infty(M) \rightarrow \mathcal{D}'(M)$  to a domain larger than  $\{z \in \mathbb{C}, |z| > 1\}$ .

REMARK 1.8. Analyticity (resp. meromorphy) for operators  $\mathcal{C}^\infty(M) \rightarrow \mathcal{D}'(M)$  is understood as analyticity (resp. meromorphy) in  $\mathcal{D}'(M \times M)$  of the kernel  $K(z)$  defined against tensor product of smooth functions as

$$(1.16) \quad \langle K(z), f \otimes g \rangle = \langle (z - \mathcal{L})^{-1} f, g \rangle_{L^2(\mu)}.$$

Analyticity (resp. meromorphy) in  $\mathcal{D}'$  is defined as usual, with all the limits taken in the  $\mathcal{D}'$  topology. The same properties as in the complex valued case remain true (complex derivability, local Taylor series expansions, Cauchy formula...). In particular, meromorphic functions admit at each pole an operator valued Laurent series expansion, thereby defining the multiplicity of the pole.

Moreover,  $z \mapsto (z - \mathcal{L})^{-1}$  is analytic (resp. meromorphic) on a domain if, and only if, for any smooth functions  $f, g \in \mathcal{C}^\infty(M)$ , the complex-valued application  $z \mapsto \langle (z - \mathcal{L})^{-1} f, g \rangle$  is analytic (resp. meromorphic) on this domain ([Con80] Theorem 8.4).

Now, remark that for every  $z$  of modulus greater than 1 and every smooth functions  $f, g$  on  $M$ , the calculus made in (1.15) gives a relation between the resolvent and the generating function on the correlations:

$$(1.17) \quad \begin{aligned} \langle (z - \mathcal{L})^{-1} f, g \rangle_{L^2(\mu)} &= \sum_{n \in \mathbb{N}} \int \mathcal{L}^n f \cdot g \, d\mu \, z^{-n-1} \\ &= \sum_{n \in \mathbb{N}} c_{f,g}(n) z^{-n-1}. \end{aligned}$$

As a consequence, an hypothetic meromorphic continuation of the resolvent gives an asymptotic expansion of the correlations.

COROLLARY 1.9. *Let  $\mu$  be an invariant Radon measure. Let  $\varepsilon \geq 0$ . If the resolvent admits a meromorphic continuation to  $\{z \in \mathbb{C}, |z| > \varepsilon\}$ , with a discrete set of poles  $\lambda \in J$ , depending on the measure  $\mu$ , of respective multiplicity  $m_\lambda$ , then the correlation functions admit for every smooth functions  $f, g$  an asymptotic expansion of the form*

$$(1.18) \quad \begin{aligned} c_{f,g}(n) &:= \int f \circ T^n \, g \, d\mu \\ &= \sum_{\lambda \in J} P_{f,g,\lambda}(n) \lambda^n + o((\varepsilon')^n) \end{aligned}$$

for some polynomials  $P_{f,g,\lambda}$  of degree at most  $m_\lambda - 1$ .

The collection of all possible poles of a meromorphic continuation of the resolvent is called **Ruelle-Pollicott spectrum**, its elements are called **resonances**. We will write it  $\text{Res}(\mathcal{L})$ .

REMARK 1.10. For flows the situation is more complicated: The Ruelle-Pollicott resonances are defined as the poles of the biggest meromorphic continuation of the resolvent of the vector field  $X$  seen as differential operator. However, this is not enough to recover informations about the resolvent of the transfer operator or the correlation functions. Precisely, for any time  $t_0$ , just as in Corollary 1.9, if the resolvent of  $\mathcal{L}^{t_0}$  admits a meromorphic continuation to  $\{z \in \mathbb{C}, |z| > e^{t_0\alpha}\}, \alpha \in \mathbb{R}$  with poles  $e^{t_0\lambda}, \lambda \in J$  of respective multiplicities  $m_\lambda$ , then the correlation functions

$$(1.19) \quad c_{f,g}(t) = \int f \circ \mathcal{L}^t \cdot g \, d\mu$$

admit an expansion

$$(1.20) \quad c_{f,g}(t) = \sum_{\lambda} P_{f,g}(t) e^{\lambda t} + O(e^{\alpha' t})$$

for some polynomial functions  $P_{f,g}$  of degree at most  $m_\lambda - 1$ , and any  $\alpha' > \alpha$ .

This meromorphic continuation of the resolvent of the transfer operator implies that the resolvent of the vector field  $X$  admits a meromorphic continuation to  $\{\operatorname{Re}(z) > \alpha\}$  with Ruelle-Pollicott resonances precisely  $\lambda \in J$ . This is a consequence of Theorem II.1.10 p.42 in [EN99] applied to the semigroup  $\mathcal{L}^t - \sum_{e^{t\lambda} \in \operatorname{Res}(\mathcal{L}^t)} e^{t\lambda} \Pi_\lambda$  whose generator is  $X - \sum \lambda \Pi_\lambda$ . However, the converse is not true in general.

REMARK 1.11. Practically, the resonances are constructed as eigenvalues of the transfer operator in certain spaces. See Section 5.

PROOF OF COROLLARY 1.9. Let us assume that the resolvent admits a meromorphic continuation to  $\{z \in \mathbb{C}, |z| > \varepsilon\}$ , for some  $\varepsilon \geq 0$ . In other terms, that there exist an analytic family of operators  $F(z) : \mathcal{C}^\infty(M) \rightarrow \mathcal{D}'(M)$ , a discrete set of resonances  $\operatorname{Res}(\mathcal{L})$  of modulus less than or equal to one, and for each  $\lambda \in \operatorname{Res}(\mathcal{L})$  an integer  $m_\lambda \in \mathbb{N}$  and operators  $A_{\lambda,j} : \mathcal{C}^\infty \rightarrow \mathcal{D}'(M)$  for  $j = 1, \dots, m_\lambda$  such that

$$(1.21) \quad (z - \mathcal{L})^{-1} = F(z) + \sum_{\lambda \in J} \sum_{j=1}^{m_\lambda} \frac{A_{\lambda,j}}{(z - \lambda)^j}.$$

We can note that this decomposition necessarily takes the following form:

REMARK 1.12. Note that each residue  $A_{\lambda,1}$  is given by the Cauchy formula

$$(1.22) \quad A_{\lambda,1} = \Pi_\lambda := \frac{1}{2i\pi} \int (z - \mathcal{L})^{-1} dz$$

where the integral is taken over a small path around  $\lambda$ .  $\Pi_\lambda$  is a projector commuting with  $\mathcal{L}$ . Moreover, writing

$$(1.23) \quad (z - \lambda + \lambda - \mathcal{L})(z - \mathcal{L})^{-1} = id$$

gives by identification for each  $j$

$$(1.24) \quad A_{\lambda,j} = (\mathcal{L} - \lambda)^{j-1} \Pi_\lambda$$

and

$$(1.25) \quad (\mathcal{L} - \lambda)^{m_\lambda} \Pi_\lambda = 0.$$



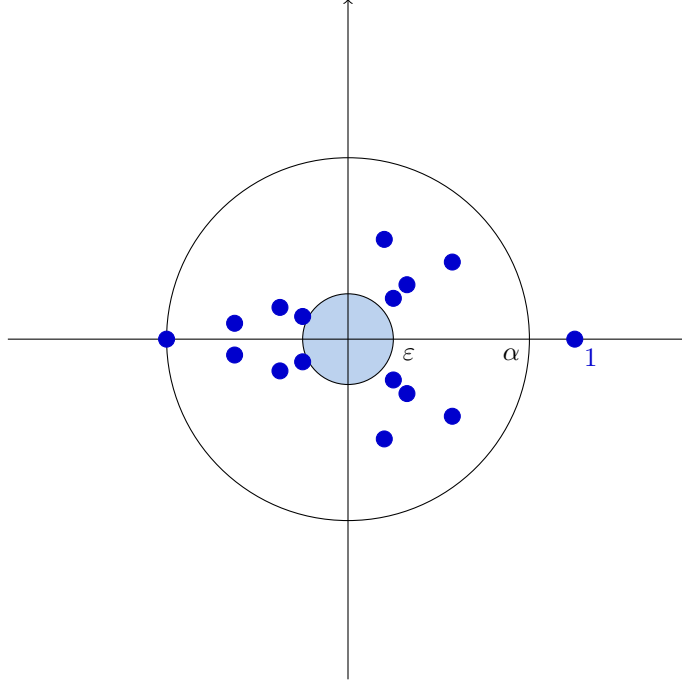


FIGURE 1. Meromorphic extension of the resolvent of a map outside a circle of radius  $\varepsilon$ . If 1 is a simple pole, the spectral gap depicted implies exponential mixing at rate  $\alpha^n$  for smooth observables. For smooth expanding and Anosov maps this is the case, with  $\varepsilon = 0$ . For  $C^k$  maps, this is also the case but for  $\varepsilon$  decaying algebraically with  $k$  (see Section 5.1). The symmetry with respect to the origin comes from the fact that  $\overline{z - \mathcal{L}} = \bar{z} - \mathcal{L}$ .

By (1.17), this gives

$$(1.26) \quad \sum_{n=0}^{\infty} c_{f,g}(n) z^{-n-1} = F(z) + \sum_{\lambda \in J} \sum_{j=1}^{m_\lambda} \frac{\langle (\mathcal{L} - \lambda)^{j-1} \Pi_\lambda f, g \rangle}{(z - \lambda)^j}.$$

Since  $F$  is analytic on  $\{z, |z| > \varepsilon\}$ , we can write it as a series

$$(1.27) \quad F(z) = \sum_{n=0}^{\infty} r_n z^{-n}$$

with

$$(1.28) \quad r_n = O(\varepsilon'^n)$$

for every  $\varepsilon' > \varepsilon$ .

Then, we have explicit Taylor expansions of the fractions  $\frac{1}{(z-\lambda)^j}$ : the coefficient in front of  $z^{-n-1}$  in this decomposition is for  $n \geq j+1$  a polynomial  $p_j$  of degree  $j-1$  multiplied by  $\lambda^{n-3}$ . This gives the expected expansion of correlations.

□

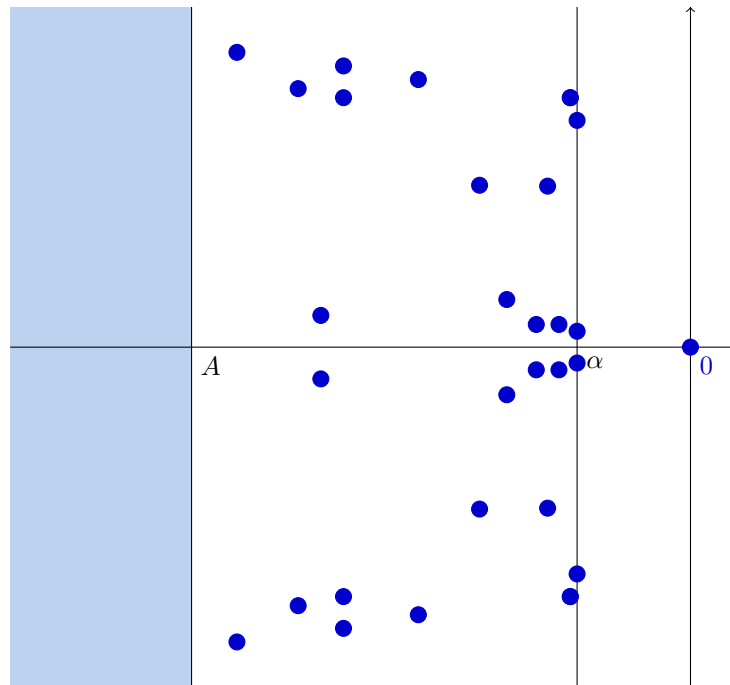


FIGURE 2. Meromorphic continuation of the resolvent of a vector field to  $\{\operatorname{Re}(z) > A\}$ . The spectral gap pictured here is not sufficient to obtain an exponential decay of correlations. An additional ingredient, such as a bound on the norm of the resolvent on a vertical line between  $\alpha$  and 0 would be required. Similarly as before, since  $\overline{z - X} = \bar{z} - X$ , the resonances are symmetric with respect to the real axis.

REMARK 1.13. We can also see that, conversely, Proposition 1.6 tells us that the resolvent of the transfer operator associated to  $E_t$  admits a meromorphic continuation to  $\mathbb{C} \setminus \{0\}$  and that the only Ruelle resonance is 1, of multiplicity one, with associated projector  $|\mathbb{1}\rangle\langle\mathbb{1}|$ .

The existence of a meromorphic continuation of the resolvent has far reaching consequences. Note that the constant function  $\mathbb{1}$  on  $M$  is always an eigenfunction of  $\mathcal{L}$  associated to the eigenvalue 1.

PROPOSITION 1.14. *Let  $T : M \rightarrow M$  be a smooth map of a compact Riemannian manifold  $M$  and  $\mu$  be an invariant Radon measure. If the resolvent of the transfer operator associated with  $T$  admits a meromorphic extension to  $\{|z| > 1 - \varepsilon\}$  for some  $\varepsilon > 0$ , then we have the following properties:*

- (1)  *$T$  is ergodic with respect to  $\mu$  if, and only if the Ruelle-Pollicott resonance 1 has multiplicity 1.*
- (2)  *$T$  is weakly mixing with respect to  $\mu$  if, and only if there is no Ruelle-Pollicott resonance on the unit circle, except for 1, of multiplicity 1. This in turn means that  $T$  is mixing at exponential rate, given by the second largest modulus of the resonances.*

REMARK 1.15. (Recall that, in almost all cases, the Ruelle-Pollicott resonances will be defined as spectrum of the transfer operator in some Banach spaces)

SKETCH OF PROOF. First note that, since the correlations remain bounded, the polynomials  $P_{f,g,\lambda}$  of Corollary 1.9 need to be constant for  $|\lambda| = 1$ . By taking the Cesaro average, this implies that for any  $f, g \in \mathcal{C}^\infty(M)$ ,

$$(1.29) \quad \frac{1}{n} \sum_{k=0}^{n-1} \int f \circ T^k g \, d\mu \xrightarrow{n \rightarrow \infty} \int \Pi_1 f \cdot g \, d\mu$$

$\Pi_\lambda$  being the projector defined in (1.22) associated to the resonance  $\lambda$ . Then,  $\mu$  is ergodic if and only if for any smooth functions  $f, g$ ,

$$(1.30) \quad \frac{1}{n} \sum_{k=0}^{n-1} \int f \circ T^k g \, d\mu \xrightarrow{n \rightarrow \infty} \int f \, d\mu \int g \, d\mu,$$

this proves the first claim. Similarly, if we fix  $f, g \in \mathcal{C}^\infty(M)$ , and if we write as in the proof of Corollary 1.9

$$(1.31) \quad \Pi_\lambda = \int (z - \mathcal{L})^{-1} \, dz$$

the projector associated to a resonance  $\lambda$ , we have

$$(1.32) \quad \int f \circ T^k g \, d\mu = \sum_{\substack{\lambda \in \text{Res}(\mathcal{L}) \\ |\lambda|=1}} \lambda^k \langle \Pi_\lambda f, g \rangle + o(1)$$

If  $T$  is weakly mixing, then it is ergodic and 1 is a simple resonance and  $\Pi_1 f = \int f \, d\mu \, 1$ . Thus,

$$(1.33) \quad \int f \circ T^k g \, d\mu - \int f \, d\mu \int g \, d\mu = \sum_{\substack{\lambda \in \text{Res}(\mathcal{L}) \\ |\lambda|=1, \lambda \neq 1}} \lambda^k \langle \Pi_\lambda f, g \rangle + o(1).$$

Now, if we write  $\lambda_0, \dots, \lambda_{J-1}$  the resonances of modulus 1 different from one,

$$(1.34) \quad \begin{aligned} \frac{1}{n} \sum_{k=0}^{n-1} \left| \int f \circ T^k g \, d\mu - \int f \, d\mu \int g \, d\mu \right| &= \frac{1}{n} \sum_{p=0}^{\lfloor \frac{n}{J} \rfloor} \sum_{i=0}^{J-1} \left| \sum_{j=0}^{J-1} \lambda_j^{pJ+i} \int \Pi_{\lambda_j} f g \, d\mu \right| + o(1) \\ &= \frac{1}{n} \sum_{p=0}^{\lfloor \frac{n}{J} \rfloor} \left\| A \begin{pmatrix} \lambda_1^{pJ} \int \Pi_{\lambda_1} f g \, d\mu \\ \vdots \\ \lambda_J^{pJ} \int \Pi_{\lambda_J} f g \, d\mu \end{pmatrix} \right\|_1 + o(1) \end{aligned}$$

where  $A$  is the Vandermonde matrix

$$(1.35) \quad A := (\lambda_j^i)_{0 \leq i, j \leq J-1},$$

and  $\|\cdot\|_1$  is the sum of the modulus of the coefficients.

All the  $\lambda_j$  are different, so  $A$  is invertible and there exists  $C > 0$  such that for all  $X \in \mathbb{R}^J$ ,  $\|AX\|_1 \geq C\|X\|_1$ . Consequently,

$$\begin{aligned}
(1.36) \quad \frac{1}{n} \sum_{k=0}^{n-1} \left| \int f \circ T^k g \, d\mu - \int f \, d\mu \int g \, d\mu \right| &\geq \tilde{C} \left\| \begin{pmatrix} \lambda_1^{p_J} \int \Pi_{\lambda_1} f \, g \, d\mu \\ \vdots \\ \lambda_J^{p_J} \int \Pi_{\lambda_J} f \, g \, d\mu \end{pmatrix} \right\|_1 \\
&= \tilde{C} \left\| \begin{pmatrix} \int \Pi_{\lambda_1} f \, g \, d\mu \\ \vdots \\ \int \Pi_{\lambda_J} f \, g \, d\mu \end{pmatrix} \right\|_1
\end{aligned}$$

for some  $\tilde{C} > 0$ .

If  $T$  is weakly mixing, it follows that

$$(1.37) \quad \forall \lambda \in S^1 \setminus \{1\}, \forall f, g \in \mathcal{C}^\infty(M), \int \Pi_\lambda f \, g \, d\mu = 0.$$

Reciprocally, it is clear that if  $T$  has no resonance of modulus 1 but 1 with  $\Pi_1 f = \int f \, d\mu \, \mathbb{1}$ , then  $T$  mixes exponentially.  $\square$

**REMARK 1.16.** For flows once again, the meromorphic extension of the resolvent of the vector field to a set of the form  $\{\operatorname{Re}(z) > -\varepsilon\}$  for some  $\varepsilon > 0$  is not enough to obtain equivalence between weak mixing and exponential mixing. For instance, although Anosov vector fields admit such a meromorphic continuation (see Section 5.4) it is not known whether every mixing Anosov flow is exponentially mixing. Tsujii and Zhang [TZ20] have recently showed that this is the case in dimension 3.

### 1.3. Discrete spectrum.

**DEFINITION 1.17.** Let  $\mathcal{L}$  be a linear operator on a Hilbert space  $\mathcal{H}$ . The discrete spectrum is the set of isolated eigenvalues of finite multiplicity of  $\mathcal{L}$ . Its complement in the spectrum is called the essential spectrum. The essential spectral radius  $r_{\text{ess}}(T)$  is the smallest non-negative number such that every element of the spectrum of modulus greater than  $r_{\text{ess}}(T)$  belongs to the discrete spectrum.

**1.3.1. Case of maps.** The general approach to get a meromorphic continuation of the resolvent of the transfer operator associated to a chaotic map is to construct spaces of distributions containing smooth functions, in which the transfer operator is quasicompact, that is, has an essential spectral radius smaller than its spectral radius. Then the resolvent extends meromorphically outside the essential spectral radius, its poles are precisely the eigenvalues of the operator in this region, and the projectors  $\Pi_\lambda$  are the spectral projectors of  $\mathcal{L}$  associated to  $\lambda$ .

In order to do this, it has been proven useful to construct norms that decay along the trajectories escaping towards infinity in frequency, since correlation functions involve testing against regular functions, that “do not see” quick oscillations. We have seen for instance in Proposition 1.6 that in the simple case of linear expanding maps on the circle, non-constant Fourier modes are sent to Fourier modes of higher frequency, and their trajectory thus escapes towards high frequencies.

Consequently, Sobolev spaces of negative order  $-m$ , for which the norm of a Fourier mode  $\phi_n = e^{2i\pi n \cdot}$  is  $\langle n \rangle^{-m}$ , (where  $\langle x \rangle = \sqrt{1+x^2}$  denotes the “Japanese bracket”), furnish for expanding maps an essential spectral radius that goes to zero as  $m \rightarrow \infty$ , while the essential spectral radius in  $L^2$  is 1 and does not reflect the superexponential decay of correlations for smooth observables:

PROPOSITION 1.18. *The spectrum of the transfer operator  $\mathcal{L}$  associated to the map  $E_l$  in  $L^2(\mathbb{T})$  is the whole unit disk. Its essential spectral radius in the Sobolev space  $\mathcal{H}^{-m}$  is smaller than  $\left(\frac{2}{1+l^2}\right)^{\frac{m}{2}}$ . Moreover, in  $\mathcal{H}^{-m}$ , the whole disk  $\{|z| \leq \frac{1}{l^m}\}$  is contained in the essential spectrum.*

PROOF. Let us write for  $p \in \mathbb{Z}$   $\phi_p$  the corresponding Fourier mode:

$$(1.38) \quad \phi_p(x) = e^{2i\pi px}$$

On an orbit of the form  $\{\phi_{l^n p}\}_{n \in \mathbb{N}}$ , the operator  $\mathcal{L}$  acts as a one-sided shift operator, it is therefore not surprising to find the same spectrum. We can indeed show that every  $z$  of modulus smaller than one is an eigenvalue of the  $L^2$  adjoint  $\mathcal{L}^*$ . The formula

$$(1.39) \quad \langle \mathcal{L}\phi_p, \phi_{p'} \rangle_{L^2(\mathbb{T})} = \delta_{p'=lp}$$

implies that for every  $k \geq 1$ ,  $\mathcal{L}^* \phi_{l^k} = \phi_{l^{k-1}}$  and  $\mathcal{L}^* \phi_1 = 0$ . Thus, formally

$$(1.40) \quad \mathcal{L}^* \sum_{k=0}^{\infty} z^k \phi_{l^k} = z \sum_{k=0}^{\infty} z^k \phi_{l^k}.$$

This sum converges in  $L^2$  as soon as  $|z| < 1$ , which gives the first part of the result.

Moreover, it converges in  $\mathcal{H}^m = (\mathcal{H}^{-m})^*$  when  $|z| < \frac{1}{l^m}$ , and we can see that on  $\mathbf{1}^\perp = \overline{\text{Vect}(\phi_p, p \neq 0)}$  equipped with the  $\mathcal{H}^{-m}$  norm, we have

$$(1.41) \quad \|\mathcal{L}\| = \frac{\|\mathcal{L}\phi_1\|}{\|\phi_1\|} = \left(\frac{2}{1+l^2}\right)^{\frac{m}{2}}.$$

Consequently, the essential spectrum  $\sigma_{\text{ess}, \mathcal{H}^{-m}}(\mathcal{L}) (= \sigma_{\text{ess}, \mathcal{H}^m}(\mathcal{L}^*))$  satisfies

$$(1.42) \quad \left\{|z| \leq \frac{1}{l^m}\right\} \subset \sigma_{\text{ess}, \mathcal{H}^{-m}}(\mathcal{L}) \subset \left\{|z| \leq \left(\frac{2}{1+l^2}\right)^{\frac{m}{2}}\right\}.$$

□

This approach translates well to the case of expanding maps, that is, maps who expand the metric:

DEFINITION 1.19. An expanding map on a compact Riemannian manifold  $M$  is a map  $T$  such that:

$$(1.43) \quad \exists C > 1, \forall v \in TM, \|T \cdot v\| \geq C\|v\|.$$

For these maps, the Sobolev spaces gives similar result (see Section 5.1 for references and more precise statements). The analysis of the frequency dynamics is however more subtle and requires tools from semiclassical analysis (see Section 2), such as pseudodifferential operators, or invariant cone families.

DEFINITION 1.20. An Anosov diffeomorphism of a compact Riemannian manifold  $(M, g)$  is a diffeomorphism  $T : M \rightarrow M$  such that there exists an invariant splitting of the tangent bundle:

$$(1.44) \quad T_x M = E_u \oplus E_s$$

$$(1.45) \quad \forall x \in M, dT_x \cdot E^i(x) = E^i(f(x)), i \in \{u, s\}$$

satisfying the following property: there exist constants  $0 < \lambda < 1$  and  $C > 0$ ,

$$(1.46) \quad \forall v \in E^u, \forall n \in \mathbb{N}, \|dT^{-n} \cdot v\| \leq C\lambda^n \|v\|$$

and

$$(1.47) \quad \forall v \in E^s, \forall n \in \mathbb{N}, \|dT^n \cdot v\| \leq C\lambda^n \|v\|.$$

The simplest examples of Anosov diffeomorphisms are hyperbolic toral automorphisms: A matrix  $A \in SL_n(\mathbb{Z})$  induces a map  $\mathbb{T}^n \rightarrow \mathbb{T}^n$ , invertible since  $A^{-1}$  has integer coefficients by the comatrix formula. If  $A$  has no eigenvalues of modulus 1, it defines an Anosov diffeomorphism on  $\mathbb{T}^n$ ; the unstable bundle  $E_u$  is  $\mathbb{T}^n \times F_{>1} \subset \mathbb{T}^n \times \mathbb{R}^n \cong T\mathbb{T}^n$ , where  $F_{>1}$  denotes the sum of the eigenspaces associated to eigenvalues of moduli greater than 1,  $E_s$  is  $\mathbb{T}^n \times F_{<1}$  with obvious notations.

In the case of Anosov maps, there is also a meromorphic continuation of the resolvent (see Section 5.3 for further references and details). The classical Sobolev space no longer provide quasicompactness of the transfer operator. Studying what happens **microlocally**, that is taking into account both position and frequency, and applying a weight alongside trajectories that escape towards infinity, leads to "anisotropic" Sobolev spaces, of negative order in the unstable direction and positive order in the stable direction.

### 1.3.2. Case of flows.

DEFINITION 1.21. A vector field  $X$  on a compact Riemannian manifold  $M$  is called Anosov if the tangent bundle admits a splitting  $TM = E^u \oplus E^s \oplus \mathbb{R} \cdot X$  such that there exist constants  $C > 0, 0 < \lambda < 1$  such that

$$(1.48) \quad \begin{aligned} &\bullet \forall v \in E^s, \forall t \in \mathbb{R}_+, \|d\phi^t \cdot v\| \leq C\lambda^t \|v\| \\ &\bullet \forall v \in E^u, \forall t \in \mathbb{R}_+, \|d\phi^{-t} \cdot v\| \leq C\lambda^t \|v\|. \end{aligned}$$

Anosov flows add a substantial difficulty, due to the fact that the flow direction is neither contracting nor expanding:

$$(1.49) \quad \forall x \in M, \forall t \in \mathbb{R}, d\phi_x^t \cdot X(x) = X(\phi^t(x)).$$

The tools developed to study these vector fields usually involve Fourier expansion in the neutral direction of the flow. Indeed, in local flow coordinates  $(x, z)$  such that

$$(1.50) \quad \phi^t(x, z) = (x, z + t),$$

we can observe that in a Fourier decomposition with respect to the variable  $z$ , if a function  $u$  writes locally  $u(x, z) = v(x)e^{i\omega z}$ , then we have

$$(1.51) \quad u(\phi^t(x, z)) = (v(x)e^{i\omega t}) e^{i\omega z} :$$

The frequency along the flow is preserved. Applying a suitable escape function, that decreases along trajectories of the flow lifted to cotangent space (taking simultaneously account of position at frequency, for instance with the FBI transform, see Section 2) gives discrete spectrum of the vector field (see Section 5.4 for a brief history and references).

In the following subsection we present a very simplified toy model of Anosov flow for which everything is explicit and easily computable.

The results of this thesis involve a toy model exhibiting in a simpler way a neutral direction. Namely, we are interested in skew products of expanding or Anosov diffeomorphisms: if  $T : M \rightarrow M$  is Anosov or expanding, we consider

$$(1.52) \quad F : \begin{cases} M \times \mathbb{T} & \longrightarrow & M \times \mathbb{T} \\ (x, y) & \longmapsto & (T(x), y + \tau(y)) \end{cases}$$

where  $\tau \in \mathcal{C}^k(M)$ . The  $y$  direction is a neutral direction, and for a given frequency  $\xi$  in this direction, the restriction of the transfer operator is  $\mathcal{L}_{\xi, \tau} u = e^{i\xi\tau} u \circ T$ . We then try to get informations about its spectral properties in the semiclassical limit of large  $\xi$ , when  $\tau$  exhibits some randomness.

## 2. Semiclassical and microlocal analysis

A fundamental concept of semiclassical analysis [dV05] [Zwo12] is the link between classical and quantum dynamics. Namely, if  $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$  is a function, seen as the Hamiltonian, that is the energy of a Hamiltonian system, then a question is to know how to describe the underlying quantum mechanics. In quantum mechanics, the Hamiltonian is a self-adjoint operator acting on a Hilbert space.

Procedures to build a self-adjoint operator from a function on phase space are known as quantizations. For instance the natural quantization of the kinetic energy  $a(x, \xi) = \|\xi\|^2$  would be the operator  $-h^2\Delta$  (see for instance [Mar02] Chapter 1 for a brief discussion on that topic). The former describes the classical motion of a point evolving freely, while the latter governs the quantum evolution of a free particle.

Semiclassical analysis studies the relationships that link the quantum Hamiltonian and the classical Hamiltonian. For example, the mathematical study of Bohr's correspondence principle [Rob87] states that taking  $h \rightarrow 0$  in Schrödinger's equation gives the classical behaviour.

**2.1. FBI transform.** While Fourier transform gives a useful representation of functions in terms of frequencies, it hides thereby informations about position. For instance, the Fourier transform of the indicator of  $[0, 1] \subset \mathbb{R}$  has a Fourier transform that decays slowly (essentially as  $\xi^{-1}$ ). This shows for instance immediately that the derivative of the function is not in  $L^2$ . However, at first sight it is not easy to see in this frequential data that the function is smooth everywhere but at 0, 1.

Microlocal analysis aims to take account of both position and frequency. There are several formalisms for this. Here we briefly present a simple model of FBI-transform [Mar02, Chapter 3] [Sjö95] on  $\mathbb{R}^d$  as described in [FT15].

The idea is to decompose distributions on  $\mathbb{R}^d$  as superposition of wave packets, that are localized both in space and frequency. We can for instance take Gaussian wave packets:

**DEFINITION 2.1.** Let  $(x, \xi) \in T^*\mathbb{R}^d$ . Let  $h > 0$ , called semiclassical parameter, whose purpose is to be small. Define

$$(2.1) \quad \phi_{(x, \xi)}^h : \begin{cases} \mathbb{R}^d & \longrightarrow \mathbb{R} \\ y & \longmapsto a_h e^{-\frac{|x-y|^2}{2h} + \frac{i}{h} \langle \xi, y \rangle} \end{cases} ,$$

where  $a_h$  is a constant chosen so that  $\|\phi_{(x, \xi)}^h\|_{L^2(\mathbb{R}^d)} = 1$ .

Those wave packets are isotropic Gaussian functions of variance  $h$  or equivalently standard deviation  $h^{1/2}$ . Henceforth their Fourier transform is a Gaussian of large variance  $h^{-1}$ . There is consequently an impossibility for those wave packets to be arbitrarily localized in space and frequency. This is in fact a general phenomenon, called uncertainty principle [Zwo12, Theorem 3.9 p.39]. Rescaling the cotangent space by a homothety  $(x, \xi) \mapsto (x, h\xi)$  gives a picture in which the wave packets “occupy” a volume of order  $h^d$ .

We can then decompose functions on  $\mathbb{R}^d$  with respect to wave packets:

**DEFINITION 2.2 (FBI transform).** For  $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ , let us define  $\mathcal{B}^h u \in \mathcal{C}^\infty(T^*\mathbb{R}^d)$  by

$$(2.2) \quad \mathcal{B}^h u(x, \xi) := \langle \phi_{(x, \xi)}^h, u \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \overline{\phi_{(x, \xi)}^h(y)} u(y) \, dy.$$



PROPOSITION 2.3 ([Mar02] Proposition 3.1.1).  $B^h$  extends to a (non-surjective) linear isometry  $B : L^2(\mathbb{R}^d) \longrightarrow L^2(T^*\mathbb{R}^d, \frac{dx d\xi}{(2\pi h)^d})$ :  $(\mathcal{B}^h)^* \mathcal{B}^h = \text{Id}_{L^2(\mathbb{R}^d)}$ , where its adjoint  $(\mathcal{B}^h)^*$  is given by

$$(2.3) \quad \forall v \in L^2(T^*\mathbb{R}^d), (\mathcal{B}^h)^* v(y) = \int_{T^*\mathbb{R}^d} \phi_{(x,\xi)}(y) v(x, \xi) \frac{dx d\xi}{(2\pi h)^d}.$$

Moreover, for every  $u \in \mathcal{S}'(\mathbb{R}^d)$ ,  $\mathcal{B}^h u \in \mathcal{C}^\infty(T^*\mathbb{R}^d)$ .

This way, every Schwartz distribution on  $\mathbb{R}^d$  can be viewed as a function on  $T^*\mathbb{R}^d$ , where the value at each point represents the coefficient in front of the corresponding wave packet:

$$(2.4) \quad u(y) = \int_{T^*\mathbb{R}^d} \mathcal{B}^h u(x, \xi) \phi_{(x,\xi)}^h(y) dx d\xi.$$

If  $T$  is for instance a hyperbolic map  $\mathbb{R}^d \longrightarrow \mathbb{R}^d$ , then the transfer operator

$$(2.5) \quad \mathcal{L} : \begin{cases} \mathcal{C}^\infty(\mathbb{R}^d) & \longrightarrow \mathcal{C}^\infty(\mathbb{R}^d) \\ u & \longmapsto u \circ T, \end{cases}$$

has the following action on wave packets:

$$(2.6) \quad \mathcal{L} \phi_{(x,\xi)}^h(y) = a_h \exp \left( -\frac{|x - T(y)|^2}{2h} + \frac{i}{h} \langle \xi, T(y) \rangle \right).$$

So the image of  $\phi_{(x,\xi)}^h$  is localized in position near  $T^{-1}(x)$ . And around this point

$$(2.7) \quad \begin{aligned} \langle \xi, T(y) \rangle &= \langle \xi, x + dT_{T^{-1}(x)}(y - T^{-1}(x)) + O(\|y - T^{-1}(x)\|^2) \rangle \\ &= \langle \xi, x - dT_{T^{-1}(x)} T^{-1}(x) \rangle + \langle dT_{T^{-1}(x)}^* \xi, y \rangle + O(\|y - T^{-1}(x)\|^2). \end{aligned}$$

The first term is independent of  $y$ , it just induces a constant phase. So, the image of  $\phi_{(x,\xi)}^h$  is roughly microlocalized at the lift of  $T$  to the cotangent space  $T(x, \xi) := (T^{-1}(x), dT_{T^{-1}(x)}^* \xi)$ . The study of this extension of  $T$  can be used to adapt the ideas of Proposition 1.18 to much more complicated settings [FT15] [FT13]: If one considers a map or a flow on a compact Riemannian manifold, a general idea is to define an escape function [HS86] on  $T^*M$  that decays along trajectories of  $\tilde{\mathcal{L}}$ . Then, if such a function is constructed, it induces a norm

$$(2.8) \quad \|u\|_W^2 = \int_{T^*\mathbb{R}^d} |\mathcal{B}^h u(x, \xi)|^2 |W(x, \xi)|^2 dx d\xi$$

that may lead to a space in which the transfer operator is quasicompact. The resonances outside the essential spectrum are then described by what happens microlocally near the trapped set, set of points whose orbits under the action of  $\tilde{\mathcal{L}}$  remain in a bounded region. See Section 3 for a presentation of those ideas on a toy model.  $C^\infty(M)$  with respect to frequency  $\omega$  along the flow, and on each subspace take  $h = \frac{1}{\omega}$ .

**2.2. Pseudodifferential operators.** In order to study high frequencies, a convenient tool is pseudodifferential operators. They are a generalization of differential operators, stable under a certain functional calculus.

Pseudodifferential operators [Zwo12] [Mar02] were introduced in the 60's to study partial differential equations. They are a generalization of differential operators: A differential operator on a manifold, as for instance the Laplace-Beltrami

operator  $\Delta$ , is a finite sum of terms of the form  $V_1 \cdots V_k$ , where  $V_1, \dots, V_k$  are vector fields.

On  $\mathbb{R}^d$ , let us write  $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$  and  $D = (D_1, \dots, D_d)$ . A differential operator is then an operator of the form  $P(D)$  for some polynomial  $P \in \mathcal{C}^0(\mathbb{R}^d)[X_1, \dots, X_d]$ .

Using the relation with the Fourier transform  $\xi_j \mathcal{F} = \mathcal{F} D_j$ , a differential operator can thus be written under the form

$$(2.9) \quad \begin{aligned} P(x, D)u(x) &:= \mathcal{F}^{-1} (P(x, \cdot) \mathcal{F} u(\cdot)) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} P(x, \xi) e^{i\langle \xi, x-y \rangle} u(y) \, dy d\xi, \end{aligned}$$

for a function  $P : \mathbb{R}^{2d} \rightarrow \mathbb{R}$  polynomial in the second variable.

Differential operators have some limitations. For instance, they are not invariant under functional calculus; The square-root of the Laplacian is not a differential operator. It is however a pseudodifferential operator:

DEFINITION 2.4 (Symbol classes, pseudodifferential operators [Zwo12, p.72]).

Let  $m \in \mathbb{Z}$ . Let us define the symbol class

$$(2.10) \quad S^m := \{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}), \forall \alpha, \beta \in \mathbb{N}^d, \exists C_{\alpha, \beta}, \forall (x, \xi) \in \mathbb{R}^{2d}, |D_x^\alpha D_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\beta|}\}$$

with  $D_x^\alpha = D_1^{\alpha_1} \cdots D_d^{\alpha_d}$ ,  $D_\xi^\beta = D_{d+1}^{\beta_1} \cdots D_{2d}^{\beta_d}$ . (Recall that  $\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}$ ).

A symbol  $a \in S^m$  induces an operator  $a(x, D)$ , called pseudodifferential operator, by the formula

$$(2.11) \quad a(x, hD)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} a(x, \xi) e^{\frac{i}{h} \langle \xi, x-y \rangle} u(y) \, dy d\xi.$$

REMARK 2.5. There are various different classes of symbols, according to the context. We only present here an example, called Kohn-Nirenberg symbols.

REMARK 2.6. The symbols may be allowed to depend on  $h$ , the constant  $C_{\alpha, \beta}$  in (2.10) is then asked to be uniform with respect to  $h$  near 0.

REMARK 2.7. This definition of pseudodifferential operators has the downside that  $a(x, hD) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  might not be self-adjoint, even if  $a$  is real-valued. A slightly different definition, called Weyl quantization overcomes this flaw:

$$(2.12) \quad a^W(x, hD)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h} \langle \xi, x-y \rangle} u(y) \, dy d\xi.$$

If  $a \in S^0 \cap \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R})$ , then  $a^W(x, hD)$  extends to a bounded self-adjoint operator  $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  ([Zwo12, Theorem 4.1 and 4.23]).

Those symbol classes exhibit an invariance under change of coordinates ([Zwo12, Theorem 9.10]) that allow to define pseudo-differential operators on manifolds, independently of the choice of charts:

DEFINITION 2.8 ([Zwo12, p.347]). Let  $M$  be a compact Riemannian manifold. A pseudo-differential operator of order  $m$  on  $M$  is a linear application

$$(2.13) \quad A : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$$

such that

- for any chart  $\gamma : U \subset M \longrightarrow V \subset \mathbb{R}^d$ , there exists a symbol  $a \in S^m$ , such that, for any  $u \in \mathcal{C}^\infty(M)$ , and any smooth compactly supported functions  $\phi, \psi$  on  $U$ ,

$$(2.14) \quad \phi A(\psi u) = \phi \gamma^* a^W(x, hD)(\gamma^{-1})^*(\psi u)$$

- If  $\phi, \psi \in \mathcal{C}^\infty(M)$  are functions with disjoint supports, then, for all  $k \in \mathbb{N}$  and all  $N \in \mathbb{N}$ ,

$$(2.15) \quad \|\phi A \psi\|_{\mathcal{C}^k \rightarrow \mathcal{C}^k} = O(h^N).$$

The set of pseudo-differential operators of order  $m$  is written  $\Psi^m(M)$ , and  $\Psi^{-\infty}(M)$  denotes  $\bigcap_{m \in \mathbb{R}} \Psi^m(M)$ .

Symbols on the manifold can be defined independently of the atlas as follows: We write  $S^m(T^*M)$  the set of smooth functions  $a$  on  $T^*M$  such that, in any chart  $\gamma : U \subset M \rightarrow V \subset \mathbb{R}^d$ , and for any smooth function  $\phi$  with support in  $U$ , we have

$$(2.16) \quad a_{\phi, \gamma} : \begin{cases} T^*V \approx V \times \mathbb{R}^d & \longrightarrow \mathbb{C} \\ (x, \xi) & \longmapsto \phi \circ \gamma^{-1}(x) a(\gamma^{-1}(x), {}^t d\gamma_{\gamma^{-1}(x)} \cdot \xi) \end{cases} \in S^m(\mathbb{R}^{2d}).$$

**THEOREM 2.9** ([Zwo12, Theorem 14.1]). *If  $(\gamma_j : U_j \rightarrow V_j)_{j=1}^J$  is an atlas of  $M$ , and  $(\psi_j)_{j=1}^J$  a quadratic partition of unity subordinated to  $(\gamma_j)$ , that is, if*

$$(2.17) \quad \text{Supp } \psi_j \subset U_j \text{ and } \sum_j |\psi_j|^2 = 1,$$

*then to every symbol  $a \in S^m(T^*M)$  can be associated a pseudo-differential operator*

$$(2.18) \quad \text{Op}(a) := \sum_{j=1}^J \gamma_j^* a_{\psi_j, \gamma_j}^W(x, hD)(\gamma_j^{-1})^* \psi_j \in \Psi^m(M).$$

*Reciprocally, there is a linear mapping*

$$(2.19) \quad \sigma : \Psi^m(M) \longrightarrow S^m(T^*M) \bmod hS^{m-1}(T^*M)$$

*such that for any  $a \in S^m(T^*M)$*

$$(2.20) \quad \sigma(\text{Op}(a)) = a \bmod hS^{m-1}(T^*M).$$

*modulo  $hS^{m-1}$ , called **principal symbol** of  $A$ , such that  $A = \text{Op}(a)$ . Moreover, for any  $A, B \in \Psi^m(M)$ ,  $\sigma(AB) = \sigma(A)\sigma(B)$ .*

**EXAMPLE 2.10.** For instance, if  $\Delta$  is the Laplace-Beltrami operator and  $V \in \mathcal{C}^\infty(M)$ , the Schrödinger operator  $h^2\Delta + V$  has principal symbol  $a(x, \xi) = \|\xi\|^2 + V(x) \bmod hS^1(M)$ .

We recall the following classical theorem about Schrödinger operators on compact Riemannian manifolds:

**THEOREM 2.11.** *Let  $M$  be a compact Riemannian manifold and  $V \in \mathcal{C}^\infty(M, \mathbb{R})$ . The Schrödinger operator  $h^2\Delta + V$  with domain  $H^2(M)$  is non-negative, self-adjoint, and its spectrum is a sequence of eigenvalues  $\lambda_j^h$  of finite multiplicities going to infinity. The eigenfunctions  $u_j^h$  are moreover smooth on  $M$ .*

A functional calculus can then be applied by setting for  $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ ,

$$(2.21) \quad f(h^2\Delta + V) = \sum_j f(\lambda_j^h) |u_j^h\rangle \langle u_j^h|.$$

THEOREM 2.12 ([Str72]). *If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function satisfying*

$$(2.22) \quad \forall \alpha \in \mathbb{N}, \exists C_\alpha, \forall x \in \mathbb{R}, m^{(\alpha)}(x) \leq C_\alpha \langle x \rangle^{b-2\alpha}$$

*then*

$$(2.23) \quad f(h^2\Delta + V) \in \Psi^b(M).$$

Moreover, if  $f \in \mathcal{S}(\mathbb{R})$ , then  $f(h^2\Delta + V)$  is trace-class ([Zwo12, Theorem 14.10]), and its trace is given by

$$(2.24) \quad \text{Tr}(f(h^2\Delta + V)) = \int_{T^*M} f(|\xi|^2 + V(x)) \, dx d\xi + O(h).$$

Lastly, we mention Weyl's law:

THEOREM 2.13 ([Zwo12, Theorem 14.11]). *Let  $0 = \lambda_0 \leq \lambda_1 \leq \dots$  be the eigenvalues of the Laplace-Beltrami operator  $\Delta$  on a compact Riemannian manifold  $M$  of dimension  $d$ . There exists a constant  $C$  such that*

$$(2.25) \quad \#\{j, \lambda_j \leq r\} \sim Cr^{\frac{d}{2}}.$$

### 3. Toy model of hyperbolic flow

Let us describe briefly how FBI transform allows to obtain discrete spectrum for the very simplified model of Anosov flow on  $\mathbb{R}^2 \times S^1$  given by the vector field

$$(3.1) \quad X(x, y, z) = -ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

with  $a, b > 0$ .

This example is described in [Fau19] and gives an idea of some of the techniques used for example in [FT17a]. We reproduce here the main ideas.

**PROPOSITION 3.1.** *The set of Ruelle-Pollicott resonances of  $X$  is  $-a\mathbb{N} - b\mathbb{N}^* + 2i\pi\mathbb{Z}$ . And the projector on a resonance  $\lambda$  is*

$$(3.2) \quad \Pi_\lambda := \sum_{\substack{j \in \mathbb{N}, k \in \mathbb{N}, l \in \mathbb{Z} \\ -aj - b(k+1) + 2i\pi l = \lambda}} \left| \frac{x^j}{j!} \otimes \delta^{(k)} \otimes e^{2i\pi l z} \right\rangle \left\langle \delta^{(j)} \otimes \frac{y^k}{k!} \otimes e^{2i\pi l z} \right|,$$

$\delta^{(j)}$  being the  $j$ -th derivative of the Dirac distribution.

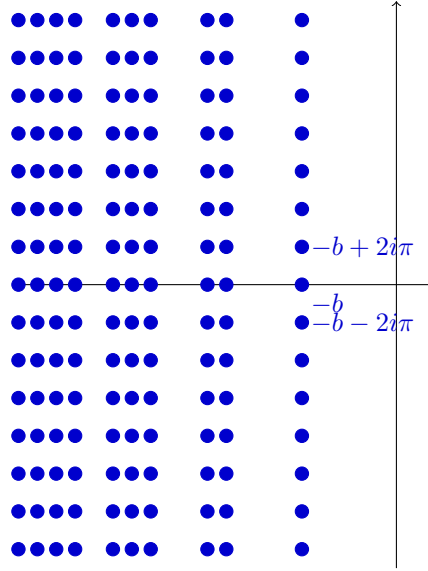


FIGURE 3. Ruelle spectrum of a linear hyperbolic Anosov flow on  $\mathbb{R}^2 \times \mathbb{T}$ . Here we drew the resonances for  $a = 4, b = 5$ .

**3.1. Contracting vector field.** Let us first consider the vector field on  $\mathbb{R}$  given by

$$(3.3) \quad X(x) = -ax \frac{d}{dx}.$$

It generates a flow  $\phi : (t, x) \mapsto \phi^t(x) = e^{-at}x$  and a semi-group of operators

$$(3.4) \quad \mathcal{L}^t := e^{tX} : u \mapsto u \circ \phi^t$$

acting for instance on  $\mathcal{C}_c^\infty(\mathbb{R})$ .

Formally, if we are not concerned with the space on which it acts as a differential operator, its “spectrum” can be considered to contain  $\mathbb{R}_-$ , since for every  $\lambda \in \mathbb{R}_+$ , at least as distributions,

$$(3.5) \quad X \cdot x^{\lambda/a} = -\lambda x^{\lambda/a}.$$

However, if we restrict to smoother functions,  $X$  seems to have a discrete set of eigenvalues of multiplicity one, as the following Taylor expansion suggests:

$$(3.6) \quad \begin{aligned} \forall u, v \in \mathcal{C}_c^\infty(\mathbb{R}), \forall k \in \mathbb{N}, \int_{\mathbb{R}} u \circ \phi^t(x) v(x) dx &= \int_{\mathbb{R}} u(xe^{-at}) v(x) dx \\ &= u(0) \int_{\mathbb{R}} v(x) dx + e^{-at} u'(0) \int_{\mathbb{R}} x v(x) dx + \cdots + e^{-akt} u^{(k)}(0) \int_{\mathbb{R}} \frac{x^k}{k!} v(x) dx \\ &\quad + e^{-at(k+1)} \int_{\mathbb{R}} \frac{x^k}{k!} v(x) dx \int_0^1 (1-s)^k u^{(k+1)}(se^{-t}x) dx. \end{aligned}$$

This gives the impression that the Ruelle-Pollicott resonances of  $X$  should be  $-a\mathbb{N}$ , the eigendistributions associated to  $-aj$ ,  $\mathbb{R} \frac{x^j}{j!}$  and the associated projector

$$(3.7) \quad \Pi_j = \left| \frac{x^j}{j!} \right\rangle \left\langle \delta^{(j)} \right|.$$

To obtain this, we can construct a space included in  $\mathcal{C}^k$  functions (in order to make sense of  $\delta^{(k)}u$ ), made of functions with fast enough algebraic decay (so that we can evaluate  $\int x^k v(x) dx$ ). A possible construction of such a space can be made with help of the FBI transform introduced in Section 2. This method may be a bit overcomplicated for this simple example, but is somewhat general.

We will write  $\mathcal{B} := \mathcal{B}^1$  the FBI transform from Definition 2.2 with constant parameter  $h = 1$ .

DEFINITION 3.2. Let

$$(3.8) \quad \tilde{\mathcal{L}}^t := \mathcal{B} \mathcal{L}^t \mathcal{B}^*$$

be an extension of  $\mathcal{L}^t$  to  $L^2(T^*\mathbb{R})$ . We have the following commutative diagram

$$(3.9) \quad \begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{L}^t} & L^2(\mathbb{R}) \\ \mathcal{B} \downarrow & & \downarrow \mathcal{B} \\ L^2(T^*\mathbb{R}) & \xrightarrow{\tilde{\mathcal{L}}^t} & L^2(T^*\mathbb{R}) \end{array}$$

$\tilde{\mathcal{L}}^t$  describes the action of  $\mathcal{L}^t$  on wave packets (2.1):

$$(3.10) \quad \tilde{\mathcal{L}}^t \delta_{(x,\xi)} = \frac{1}{2\pi} \mathcal{B} \mathcal{L}^t \phi_{(x,\xi)}^1.$$

The operator  $\tilde{\mathcal{L}}^t$  can be thought as transporting a Dirac in  $(x, \xi)$  towards a Dirac in  $\tilde{\phi}^t(x, \xi)$ , where

$$(3.11) \quad \tilde{\phi}^t(x, \xi) := (\phi^t(x), (d\phi_x^t)^{-1*} \xi) = (e^{-at}x, e^{at}\xi).$$

is the lift of  $\phi^t$  to the cotangent. However,  $\tilde{\mathcal{L}}^t$  has a smooth kernel, its maps consequently a Dirac  $\delta_{(x,\xi)}$  to a smooth function, localized near  $\tilde{\phi}^t(x,\xi)$  at a scale  $e^{at}$ . More precisely, a quick computation shows that

LEMMA 3.3 ([Fau19, Lemma 6.11]). *The kernel  $K_t$  of  $\tilde{\mathcal{L}}^t$ , given by*

$$(3.12) \quad K_t(x, \xi, y, \eta) = \langle \delta_{(x,\xi)}, \tilde{\mathcal{L}}^t \delta_{(y,\eta)} \rangle$$

*satisfies the estimate*

$$(3.13) \quad \forall N \in \mathbb{N}, \exists C_N, \forall t, |\langle \delta_{(x,\xi)}, \tilde{\mathcal{L}}^t \delta_{(y,\eta)} \rangle| \leq C_N \left\langle \frac{d((x,\xi), \tilde{\phi}^t(y,\eta))}{e^{at}} \right\rangle^{-N}.$$

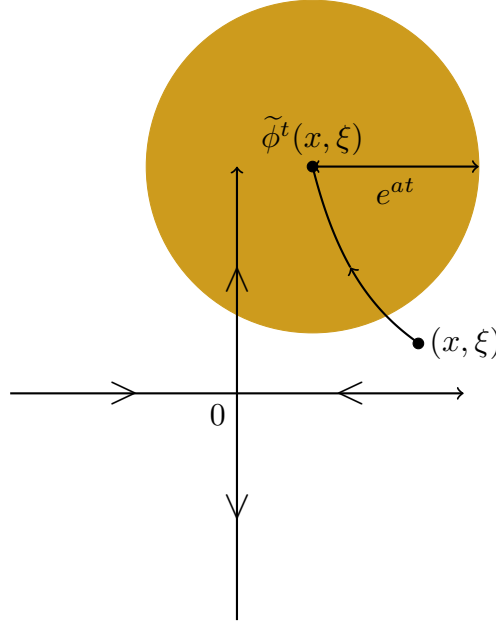


FIGURE 4. The image of a Dirac  $\delta_{(x,\xi)}$  by the operator  $\tilde{\mathcal{L}}$  is a smooth function microlocalized around the image  $\tilde{\phi}^t(x,\xi)$  of the corresponding point by the lifted flow. Most of its mass is contained at distance  $O(e^{at})$  of  $\tilde{\phi}^t(x,\xi)$ .

Under the flow  $\tilde{\phi}^t$ , every point, with the exception of the fixed point 0, escapes towards infinity (as  $t \rightarrow \infty$  or  $t \rightarrow -\infty$ ). As mentioned before, Ruelle resonances characterize low frequencies phenomena: they are linked to correlations for smooth observables, that cut out high frequencies and get rid of information going to small scales.

In order to reveal those resonances, it is then useful to introduce a weight function  $W$  that decays along trajectories outside a compact set, and to split the operator into a sum of two operators, one acting on low frequencies, that turns out to be compact, and the other one acting on high frequencies, that will have small norm in the weighted space  $L^2(T^*M, W^2 \, dx d\xi)$ .

DEFINITION 3.4. Let

$$(3.14) \quad W_r(x, \xi) = \frac{\langle e^{-at}x \rangle^r}{\langle e^{-at}\xi \rangle^r}.$$

REMARK 3.5. The factor  $e^{-at}$  is linked to the scaling factor  $e^{-at}$  in (3.13).

PROPOSITION 3.6. *Let  $H_{W_r}$  be the completion of  $\mathcal{C}_c^\infty(\mathbb{R})$  with respect to the norm*

$$(3.15) \quad \|u\|_{H_{W_r}} := \|\mathcal{B}u\|_{L^2(T^*\mathbb{R}, W_r^2 \, dx d\xi)}.$$

*Then,  $\mathcal{L}^t : H_{W_r} \rightarrow H_{W_r}$  is a bounded operator, and there exists  $C > 0$  such that it has essential spectral radius  $r_{\text{ess}}(\mathcal{L}^t) \leq Ce^{-(r-1)at}$ . Moreover, in this space, the spectrum  $\sigma(\mathcal{L}^t)$  satisfies*

$$(3.16) \quad \sigma(\mathcal{L}^t) \cap \{|z| > Ce^{-(r-1)at}\} = e^{-a\mathbb{N}} \cap \{|z| > Ce^{-(r-1)at}\}.$$

PROOF. The function  $W_r$  satisfies obviously

$$(3.17) \quad \exists C > 0, |x| \geq e^{at} \text{ or } |\xi| \geq 1 \implies \frac{W_r(\tilde{\phi}^t(x, \xi))}{W_r(x, \xi)} \leq Ce^{-art}.$$

Let us write  $\chi_t$  the indicator function of the corresponding rectangle

$$(3.18) \quad \chi_t(x, \xi) := \mathbb{1}_{|x| \leq e^{at}, |\xi| \leq 1}.$$

To study the spectral properties of  $\mathcal{L}^t$  in  $\mathcal{H}_{W_r}$ , we study  $\tilde{\mathcal{L}}^t : L^2(T^*\mathbb{R}, W_r^2 dx d\xi) \rightarrow L^2(T^*\mathbb{R}, W_r^2 dx d\xi)$  or equivalently  $W_r \tilde{\mathcal{L}}^t W_r^{-1} : L^2(T^*\mathbb{R}) \rightarrow L^2(T^*\mathbb{R})$ .

Let us split this operator as mentioned earlier:

$$(3.19) \quad W_r \tilde{\mathcal{L}}^t W_r^{-1} = \underbrace{W_r \tilde{\mathcal{L}}^t W_r^{-1} \chi_t}_{\tilde{\mathcal{L}}_{t, \text{in}}} + \underbrace{W_r \tilde{\mathcal{L}}^t W_r^{-1} (1 - \chi_t)}_{\tilde{\mathcal{L}}_{t, \text{out}}}.$$

Note that neither  $\tilde{\mathcal{L}}_{t, \text{in}}$  nor  $\tilde{\mathcal{L}}_{t, \text{out}}$  are semi-groups.

LEMMA 3.7.  $\tilde{\mathcal{L}}_{t, \text{in}}$  is a compact operator.

PROOF. This can be seen from (3.13): the kernel of  $\tilde{\mathcal{L}}_{t, \text{in}}$  is compactly supported in one variable, and decays fast in the other. It is consequently  $L^2$ , thus  $\tilde{\mathcal{L}}_{t, \text{in}}$  is Hilbert-Schmidt. It is in fact trace-class [Fau19, Lemma 6.18].  $\square$

LEMMA 3.8.

$$(3.20) \quad \exists C > 0, \forall t \geq 0, \left\| \tilde{\mathcal{L}}_{t, \text{out}} \right\| \leq Ce^{-(r-1)at}.$$

PROOF. By the Schur test,

$$(3.21) \quad \begin{aligned} \left\| \tilde{\mathcal{L}}_{t, \text{out}} \right\|_{L^2}^2 &\leq \sup_{(x, \xi)} \int \left| \langle \delta_{(x, \xi)}, W_r \tilde{\mathcal{L}}_{t, \text{out}} W_r^{-1} \chi_t \delta_{(y, \eta)} \rangle \right| dy d\eta \\ &\quad \times \sup_{(y, \eta)} \int \left| \langle \delta_{(x, \xi)}, W_r \tilde{\mathcal{L}}_{t, \text{out}} W_r^{-1} \chi_t \delta_{(y, \eta)} \rangle \right| dx d\xi. \end{aligned}$$

Now,



(3.22)

$$\begin{aligned}
\langle \delta_{(x,\xi)}, W_r \tilde{\mathcal{L}}_{t,out} W_r^{-1} \chi_t \delta_{(y,\eta)} \rangle &= \frac{W(x,\xi)}{W(y,\eta)} \chi_t(y,\eta) K_t(x,\xi,y,\eta) \\
&= \frac{W(x,\xi)}{W(\tilde{\phi}^t(y,\eta))} \frac{W(\tilde{\phi}^t(y,\eta))}{W(y,\eta)} \chi_t(y,\eta) K_t(x,\xi,y,\eta).
\end{aligned}$$

LEMMA 3.9.

$$(3.23) \quad \frac{W(x,\xi)}{W(\tilde{\phi}^t(y,\eta))} \leq C \left\langle \frac{d\left((x,\xi), \tilde{\phi}^t(y,\eta)\right)}{e^{at}} \right\rangle^{2r}.$$

PROOF. For every  $u, v \in \mathbb{R}$ ,

$$(3.24) \quad \frac{\langle u \rangle}{\langle v \rangle} \leq 2\langle u - v \rangle.$$

Indeed,

$$(3.25) \quad u^2 \leq 2((u - v)^2 + v^2)$$

so

$$(3.26) \quad \begin{aligned} 1 + u^2 &\leq 2(1 + (u - v)^2 + v^2) \leq 2(1 + (u - v)^2 + v^2 + (u - v)^2 v^2) \\ &= 2(1 + (u - v)^2)(1 + v^2). \end{aligned}$$

Thus,

$$(3.27) \quad \begin{aligned} \frac{W(x,\xi)}{W(\tilde{\phi}^t(y,\eta))} &= \left( \frac{\langle e^{-at}x \rangle}{\langle e^{-2at}y \rangle} \frac{\langle \eta \rangle}{\langle e^{-at}\xi \rangle} \right)^r \\ &\leq 4^r \langle e^{-at}(x - e^{-at}y) \rangle^r \langle e^{-at}(\xi - e^{at}\eta) \rangle^r \\ &\leq 4^{r-\frac{1}{2}} \left( \langle e^{-at}(x - e^{-at}y) \rangle^{2r} + \langle e^{-at}(\xi - e^{at}\eta) \rangle^{2r} \right). \end{aligned}$$

Now,

$$(3.28) \quad \begin{aligned} &\langle e^{-at}(x - e^{-at}y) \rangle^{2r} + \langle e^{-at}(\xi - e^{at}\eta) \rangle^{2r} \\ &= \left(1 + e^{-2at}(x - e^{-at}y)^2\right)^r + \left(1 + e^{-2at}(\xi - e^{at}\eta)^2\right)^r \\ &\leq 2 \left(1 + e^{-2at}(x - e^{-at}y)^2 + e^{-2at}(\xi - e^{at}\eta)^2\right)^r \\ &= 2 \left\langle \frac{d\left((x,\xi), \tilde{\phi}^t(y,\eta)\right)}{e^{at}} \right\rangle^{2r}. \end{aligned}$$

□

End of proof of Lemma 3.8: Injecting (3.13) with  $N > 2r + 2$ , (3.17) and the previous Lemma 3.9 in (3.22), we get after a change of variables

$$(3.29) \quad \left\| \tilde{\mathcal{L}}_{t,out} \right\| \leq C e^{-(r-1)at}.$$

□

End of proof of Proposition 3.6:

$$(3.30) \quad \mathcal{L}^t = \mathcal{B}^* \tilde{\mathcal{L}}^t \mathcal{B}$$

with

$$(3.31) \quad \|\mathcal{B}\| = \|\mathcal{B}^*\| = 1$$

so  $\mathcal{L}^t$  writes as the sum of a compact operator and an operator of norm at most  $e^{(r-1)at}$ . Adding a compact operator does not change the essential spectrum, so  $r_{\text{ess}}(\mathcal{L}^t) = r_{\text{ess}}(\mathcal{L}_{t,\text{out}}) \leq Ce^{-(r-1)at}$ . But  $\mathcal{L}^t$  is a semi-group, so for any  $s, t > 0$ ,  $r_{\text{ess}}(\mathcal{L}^t) = r_{\text{ess}}(\mathcal{L}^s)^{t/s} \leq C^{t/s} e^{-(r-1)at}$ . Letting  $s \rightarrow \infty$ , we obtain the wanted bound.

The spectral data outside the disk of radius  $e^{-a(r-1)t}$  can now be deduced from the Taylor expansion (3.6): If we write  $\Pi_j = \left| \frac{x^j}{j!} \right\rangle \langle \delta^{(j)}|$ ,  $\Pi_j : \mathcal{H}_{W_r} \rightarrow \mathcal{H}_{W_r}$  is a bounded operator and it is clear that it is the spectral projector associated to the eigenvalue  $e^{-ajt}$ . (3.6) tells us moreover that for every  $u, v \in \mathcal{C}_c^\infty(\mathbb{R})$ ,

$$(3.32) \quad \left\langle v, \left( \mathcal{L}^t - \sum_{j < r-1} e^{-ajt} \Pi_j \right) u \right\rangle \leq \frac{e^{-a[r-2]t}}{[r-3]!} \|x^{[r-3]} v\|_{L^1(\mathbb{R})} \|u\|_{\mathcal{C}^{[r-2]}(\mathbb{R})}.$$

As a consequence, the restriction of the spectrum of  $\mathcal{L}^t$  to  $\{|z| > e^{-a(r-1)t}\}$  is  $\{e^{-ajt}, j < r-1\}$ .  $\square$

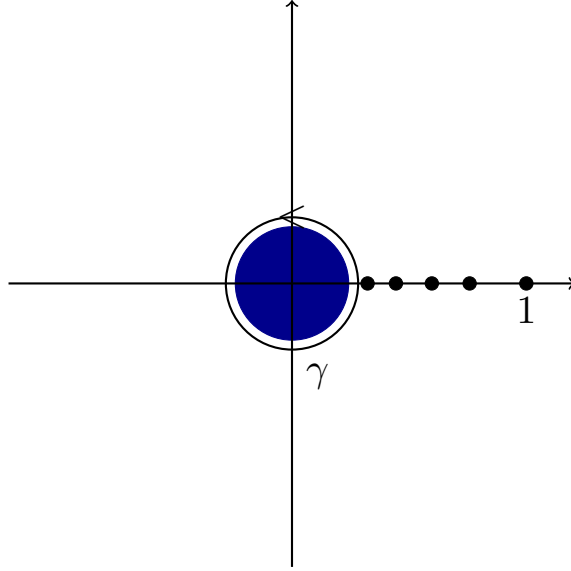


FIGURE 5.  $\sum_{j < r-1} e^{-ajt} \Pi_j = \left( 1 - \int_{\gamma} (z - \mathcal{L}^t)^{-1} dz \right) \mathcal{L}^t$ .

3.1.1. *Case of a non-linear contracting vector field.* The previous computation can be extended to the case of a non-linear contracting field, since this setting also provides an explicit expansion. The spectrum still forms a regular lattice: Let  $X = f \frac{d}{dx}$ , where  $f \in C^\infty(\mathbb{R})$ , satisfies  $f(0) = 0$  and

$$\exists C > 0, \forall x \in \mathbb{R}_+, f(x) \leq -Cx \text{ and } \forall x \in \mathbb{R}_-, f(x) \geq -Cx \geq 0.$$

The Ruelle-Pollicott resonances are the set  $-f'(0)\mathbb{N}$  (they are governed by a neighbourhood of the fixed point as we mentioned) and the spectral projector on  $-jf'(0)$  is explicitly given by

$$(3.33) \quad \Pi_j = c_j \left| x^j \exp \left( j \int_0^x \left( \frac{f'(0)}{f(z)} - 1 \right) dz \right) \right\rangle \langle \delta^{(j)} |,$$

where  $c_j$  is an explicit constant depending on the derivatives of  $f$  at 0. The expansion of the flow is indeed the following:

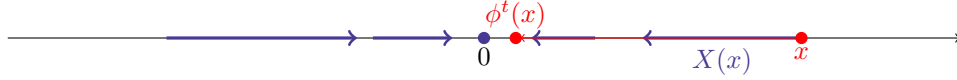


FIGURE 6. Vector field  $f \frac{d}{dx}$  and flow  $\phi^t(x) \approx C_x e^{f'(0)t}$ .

Let  $x > 0$  and let  $y$  be the solution of  $y' = X(y)$  satisfying  $y(0) = x$ .  $y' = f(y)$  and  $y(t) > 0$  for all  $t$ , so

$$(3.34) \quad \frac{y'}{f(y)} = 1$$

and if we write

$$(3.35) \quad F_x(y) = \int_x^y \frac{1}{f(z)} dz$$

the primitive of  $\frac{1}{f}$  vanishing at  $x$ , we have  $F_x(y(t)) = t$  for all  $t \geq 0$ , and thus

$$(3.36) \quad y(t) = F_x^{-1}(t).$$

In order to obtain a formula similar to (3.6), we want the asymptotic behaviour of  $F_x^{-1}(t)$  for large  $t$ . We can write around 0

$$(3.37) \quad f(z) = az + O(z^2),$$

so that

$$(3.38) \quad \frac{1}{f(z)} = \frac{1}{az} (1 + O(z)) = \frac{1}{az} + g(z)$$

for some smooth function  $g$ . Thus we have the expansion

$$(3.39) \quad F_x(y) = \frac{1}{a} \log y - \frac{1}{a} \log x - \int_y^x g(z) dz.$$

Since clearly  $y(t) \xrightarrow[t \rightarrow \infty]{} 0$ , taking  $y = F_x^{-1}(t)$  gives

$$(3.40) \quad t = \frac{1}{a} \log F_x^{-1}(t) - \frac{1}{a} \log x - \int_0^x g(z) dz + o(1).$$

Thus,

$$(3.41) \quad y(t) = F_x^{-1}(t) = x \exp \left( a \int_0^x g(z) dz \right) e^{at} (1 + o(1)).$$

Let us write

$$(3.42) \quad X(t) := x \exp \left( a \int_0^x g(z) dz \right) e^{at}.$$

In order to get a finer expansion than (3.41), we can inject it in (3.39) and perform a bootstrap:

$$(3.43) \quad t = \frac{1}{a} \log F_x^{-1}(t) - \frac{1}{a} \log x - \int_0^x g(z) dz + g(0)X(t) + o(X(t)).$$

Thus,

$$(3.44) \quad y(t) = X(t) e^{-aX(t)g(0)+o(X(t))} = X(t) - ag(0)X(t)^2 + o(X(t)^2).$$

Expanding the term  $\int_0^y g(z) dz$  as  $yg(0) + \dots + \frac{y^k}{k!} g^{(k-1)}(0)$ , and injecting inductively in (3.39), we see that  $y(t)$  can be written

$$(3.45) \quad y(t) = X(t) + \alpha_0 X(t)^2 + \dots + \alpha_{k-1} X(t)^{k+1} + o(X(t)^{k+1})$$

for some  $\alpha_0, \dots, \alpha_{k-1} \in \mathbb{R}$ .

**3.2. Expanding vector field, potential.** Let us remark that for any vector field  $X = f \frac{d}{dx}$ ,  $f \in \mathcal{C}^\infty(\mathbb{R})$  and any smooth compactly supported functions  $u, v \in \mathcal{C}_c^\infty(\mathbb{R})$ , we have

$$(3.46) \quad \begin{aligned} \int_{\mathbb{R}} Xu(x)v(x)dx &= \int_{\mathbb{R}} u'(x)f(x)v(x)dx \\ &= - \int_{\mathbb{R}} u(x)(Xv(x) + f'(x)v(x))dx \end{aligned}$$

Therefore, the  $L^2$  adjoint of  $X$  is

$$(3.47) \quad X^* = -X - f',$$

where  $f'$  is seen as a multiplication operator. Therefore, the study of an expanding vector field amounts to the one of a contracting vector field with a potential.

**COROLLARY 3.10.** *The vector field defined for  $(x, y) \in \mathbb{R}^2$  by*

$$X(x, y) = -ax\partial_x + by\partial_y$$

*has a set of Ruelle-Pollicott resonances*

$$(3.48) \quad \sigma := -b\mathbb{N}^* - a\mathbb{N}$$

*with projectors*

$$(3.49) \quad \Pi_\lambda = \sum_{\substack{j \in \mathbb{N}, k \in \mathbb{N}^* \\ \lambda = -aj - bk}} \left| \frac{x^j}{j!} \otimes \delta^{(k-1)} \right\rangle \left\langle \delta^{(j)} \otimes \frac{x^{k-1}}{(k-1)!} \right|$$

PROOF. Indeed, in this case the dynamics can be decoupled:  $e^{tX}$  is the tensor product  $e^{-atx\partial_x} \otimes e^{bty\partial_y}$ . The spectrum of  $e^{-atx\partial_x}$  on the space  $\mathcal{H}_{W_r}$  is given outside the disk of radius  $e^{-a(r-1)t}$  by  $\{e^{-ajt}, 0 \leq j < r-1\}$ . Similarly, by equation (3.47), (since here the potential is just the constant  $-b$ , and thus  $(e^{bty\partial_y})^* = e^{-bt}e^{-bty\partial_y}$ ),  $e^{bty\partial_y}$  has in  $\mathcal{H}_{W_r}^*$  the spectrum  $\{e^{-b(j+1)t}, 0 \leq j < r-2\}$  outside the disk of radius  $e^{-b(r-1)t}$ .

By [BP66], it follows that the spectrum of  $e^{-atx\partial_x} \otimes e^{bty\partial_y}$  in  $\mathcal{H}_{W_r} \otimes \mathcal{H}_{W_r}^*$  is the product of each of those spectra. The spectral projectors are moreover the  $\Pi_\lambda$  as we can see from the integral Taylor expansion of

$$(3.50) \quad \langle v, e^{tX} u \rangle = e^{-bt} \int_{\mathbb{R}^2} v(x, e^{-bt}y) u(e^{-at}x, y) \, dx dy.$$

As before, [EN99, Theorem II.1.10] implies that the spectrum of  $X$  in this space is discrete on  $\{\operatorname{Re}(z) > -\min(a, b)(r-1)\}$  and constituted in this region of the  $-aj - bk$ ,  $j \in \mathbb{N}, k \in \mathbb{N}^*$ .  $\square$

REMARK 3.11. For general bounded smooth potentials  $V \in \mathcal{C}^\infty(\mathbb{R})$ , the differential operator  $-a\partial_x + V$  generates a strongly continuous semi-group  $\mathcal{L}^t = e^{tX}$  defined for  $t \geq 0$ ,  $u \in \mathcal{C}^\infty(\mathbb{R})$  and  $x \in \mathbb{R}$  by

$$(3.51) \quad \mathcal{L}^t u(x) := e^{tX} u(x) = \exp \left( \int_0^t V(e^{-as}x) ds \right) u(e^{-at}x).$$

The spectrum of  $X + V$  in  $\mathcal{H}_{W_r}$  is similarly discrete on  $\{\operatorname{Re}(z) > -a(r-1) + \sup V\}$ , since the results of the beginning of the section remain valid, but the upper bound of Lemma 3.8 has to be multiplied by  $e^{t \sup V}$ . The spectral data on this right half-plane is once more explicit:

$$(3.52) \quad \sigma(X+V) \cap \{\operatorname{Re}(z) \geq -a(r-1) + \sup V\} = (V(0) - a\mathbb{N}) \cap \{\operatorname{Re}(z) \geq -a(r-1) + \sup V\}.$$

and the spectral projector onto the eigenspace of  $V(0) - j$  is

$$(3.53) \quad \Pi_j = \exp \left( \int_0^\infty (V(e^{-s}x) - V(0)) ds \right) \left| \frac{x^j}{j!} \right\rangle \left\langle \delta^{(j)} - \frac{1}{a} \left( V'(0)\delta^{(j-1)} - \dots - \frac{V^{(j-1)}(0)}{j-1} \delta' - \frac{V^{(j)}(0)}{j} \delta \right) \right| :$$

Let  $u$  and  $v$  be smooth compactly supported functions. The Taylor expansion for  $u$  is the same as before, and in addition, we have

$$(3.54) \quad \begin{aligned} \int_0^t V(e^{-as}x) ds &= tV(0) + \int_0^\infty (V(e^{-as}x) - V(0)) ds - \frac{x}{a} e^{-at} V'(0) - \frac{x^2}{2 \cdot 2a} e^{-2at} V''(0) \\ &\quad - \dots - \frac{x^k}{ka \cdot k!} e^{-akt} V^{(k)}(0) + o(x^k e^{-akt}). \end{aligned}$$

Therefore,

$$(3.55) \quad \mathcal{L}^t u(x) = \exp \left( tV(0) + \int_0^\infty (V(e^{-as}x) - V(0))ds \right) \\ \left( 1 - \frac{x}{a} e^{-at} V'(0) - \frac{x^2}{2 \cdot 2a} e^{-2at} V''(0) - \dots - \frac{x^k}{ka \cdot k!} e^{-akt} V^{(k)}(0) + o(x^k e^{-akt}) \right) \\ \left( u(0) + x e^{-at} u'(0) + \dots + \frac{x^k}{k!} e^{-akt} u^{(k)}(0) + o(x^k e^{-akt}) \right)$$

which gives the expression of  $\Pi_j$  when we develop and truncate.

In a similar fashion as previously, we can also obtain the spectral data in the non-linear case.

**3.3. Adding a neutral direction.** If we now consider the vector field  $X = -ax\partial_x + by\partial_y + \partial_z$  on  $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ , we obtain similarly that its spectrum in  $\mathcal{H}_{W_r} \otimes \mathcal{H}_{W_r}^* \otimes L^2(\mathbb{T})$  in the half-plane  $\{\operatorname{Re}(z) > -\min(a, b)(r-1)\}$  is given by

$$(3.56) \quad -b\mathbb{N}^* - a\mathbb{N} + 2i\pi\mathbb{Z}.$$

and the spectral projector on an eigenvalue  $\lambda$  is the sum over all possible way of writing  $\lambda$  as  $\lambda = -aj - (1+b)l + 2i\pi l$ ,  $j, k \in \mathbb{N}, l \in \mathbb{Z}$  of the mutually orthogonal rank one projectors

$$(3.57) \quad \Pi_{j,k,l} := \left| \frac{x^j}{j!} \otimes \delta^{(k)} \otimes e^{2i\pi lz} \right\rangle \left\langle \delta^{(j)} \otimes \frac{x^k}{k!} \otimes e^{2i\pi lz} \right|.$$

This concludes the proof of Proposition 3.1.

#### 4. Entropy and topological pressure

**4.1. Topological entropy.** Topological entropy [KH97, Section 3.1] of a dynamical system offers a measurement of the instability of trajectories. Let  $T : (X, d) \rightarrow (X, d)$  be a continuous map of a compact metric space. We can define for a time  $n \in \mathbb{N}$  the distance  $d_n$  on  $X \times X$  as

$$(4.1) \quad d_n(x, y) = \max_{0 \leq j \leq n-1} d(T^j(x), T^j(y)) \geq d(x, y).$$

It induces balls

$$(4.2) \quad B_n(x, \varepsilon) := \{y \in X, d_n(x, y) \leq \varepsilon\} = \bigcap_{0 \leq j \leq n-1} T^{-j} B(T^j(x), \varepsilon),$$

where  $B$  denotes balls for the metric  $d$ .

For example, for  $T = E_l$ :

$$(4.3) \quad T : \begin{cases} \mathbb{T} & \longrightarrow \mathbb{T} \\ x & \longmapsto lx \end{cases},$$

$$(4.4) \quad B_n(x, \varepsilon) = B(x, l^{-n}\varepsilon)$$

for  $\varepsilon < \frac{1}{2l}$ .

A way to measure this exponential shrinking is to introduce the notion of  $(n, \varepsilon)$ -spanning set: A finite subset  $F$  of  $X$  is  $(n, \varepsilon)$ -spanning if

$$(4.5) \quad X = \bigcup_{x \in F} B_n(x, \varepsilon).$$

In other terms if every point in  $X$  admits a representative in  $F$  that stays at distance at most  $\varepsilon$  of the orbit of  $x$  up to time  $n$ .

**DEFINITION 4.1.** The topological entropy is then defined as the exponential growth of the minimal cardinality  $\mathcal{N}(\varepsilon, n)$  of a  $(n, \varepsilon)$ -spanning set:

$$(4.6) \quad h_{\text{top}}(T) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{N}(\varepsilon, n).$$

This limit is well defined since  $\limsup_n \frac{1}{n} \log \mathcal{N}(\varepsilon, n)$  is non-decreasing with respect to  $\varepsilon$ .

(4.4) implies for instance that the minimal cardinality of a  $(n, \varepsilon)$ -spanning set is of order  $\varepsilon^{-1}l^n$ . Thus,  $h_{\text{top}}(E_l) = \log l$ .

**PROPOSITION 4.2** ([KH97, Proposition 3.1.2]). *The topological entropy depends only on the topology: it is the same for two equivalent metrics.*

As a consequence, if two maps  $T$  and  $S$  on compact metrizable topological spaces are conjugated by a homeomorphism  $h$

$$(4.7) \quad \begin{array}{ccc} X & \xrightarrow{T} & X \\ h \downarrow & & \downarrow h \\ Y & \xrightarrow{S} & Y \end{array},$$

then, choosing a metric  $d$  on  $X$  gives a metric  $h_*d$  on  $Y$  so that  $h$  is an isometry. Thus, it is clear that  $h_{\text{top}}(T) = h_{\text{top}}(S)$ .

Every expanding map  $E$  of the circle is conjugated to the map  $E_l$  for  $l = \deg E$  [KH97, Theorem 2.4.6], so the topological entropy of an expanding map  $E$  of the circle is always  $\log \deg E$ .

**4.2. Topological pressure.** Topological pressure is defined in a similar fashion as topological entropy, but involves more generally weighted sum over  $(n, \varepsilon)$ -spanning sets, instead of their cardinalities.

**DEFINITION 4.3.** Let  $T : X \rightarrow X$  be a continuous map of a compact metric space. Let  $\phi$  be a continuous function  $X \rightarrow \mathbb{C}$ , and

$$(4.8) \quad \phi_x^n := \sum_{j=0}^{n-1} \phi(T^j(x))$$

be the Birkhoff sum of  $\phi$  along the orbit of a point  $x \in X$ . Let moreover

$$(4.9) \quad \mathcal{S}(n, \varepsilon, \phi) := \inf \left\{ \sum_{x \in \mathcal{S}} e^{\phi_x^n}, \mathcal{S} \text{ } (n, \varepsilon)\text{-spanning subset of } X \right\}.$$

The topological pressure of  $\phi$  is then defined as

$$(4.10) \quad \text{Pr}(\phi) := \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathcal{S}(n, \varepsilon, \phi).$$

It is clear from the definitions that  $h_{\text{top}} = \text{Pr}(0)$ .

**4.3. Measure theoretic entropy.** Let  $(X, \mathcal{B}, \mu)$  be a probability space. The **entropy** of a finite partition  $A = \{A_1, \dots, A_n\}$  of  $X$  is defined as

$$(4.11) \quad H_\mu(A) = - \sum_{j=1}^n \mu(A_j) \log \mu(A_j).$$

It can be understood in some sense as the average information brought by the knowledge that a uniform random point belongs to  $A_j$ : Let  $x$  be a uniformly distributed (that is, with law  $\mu$ ) random variable on  $X$  and  $A(x)$  the element of the partition containing  $x$ . If, to simplify, every  $A_j$  has a measure of the form  $\mu(A_j) = 2^{-k_j}$  for some  $k_j \in \mathbb{N}$ , then the optimal way on average to determine to which  $A_j$   $x$  belongs by asking yes/no questions is to proceed by dichotomy (see Figure 2.2).

The knowledge that  $x \in A_j$  then contains  $k_j = -\log_2 \mu(A_j)$  bits of information. The average information brought by the partition would then be

$$(4.12) \quad \mathbb{E}[-\log \mu(A(x))] = - \sum_j \mu(A_j) \log_2 \mu(A_j).$$

**REMARK 4.4.** Among partitions of cardinality  $n$ , the minimal entropy is zero, attained if and only if every subset but one has measure zero. And the maximal entropy is  $\log n$ , attained if, and only if every subset has the same measure  $\frac{1}{n}$  [KH97, Proposition 4.3.3].



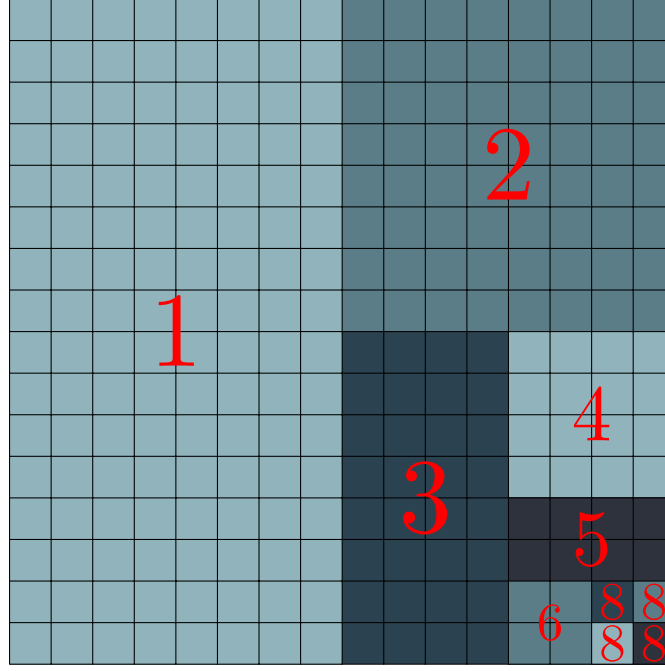


FIGURE 7. To determine one of the subsets by asking yes/no questions, one would ask if it is on the left half. If no, if it is on the top half etc. A subset  $A_j$  of measure  $2^{-k_j}$  then requires to ask at most  $k_j = -\log_2(\mu(A_j))$  questions.

DEFINITION 4.5. Given two partitions  $A = \{A_1, \dots, A_n\}$   $B = \{B_1, \dots, B_m\}$  of  $X$ , one can define their **joint partition**

$$(4.13) \quad A \vee B := \{A_j \cap B_k, 0 \leq j \leq n, 0 \leq k \leq m\}.$$

Let  $T$  be a  $\mu$ -preserving transformation. The iterations of  $T$  induce a refined partition

$$(4.14) \quad \bigvee_{j=0}^{n-1} T^{-j} A$$

of  $A$ .

The entropy of  $T$  with respect to a partition then measures the growth of information of these successive refined partitions:

DEFINITION 4.6. The sequence of entropies  $(H_\mu(\bigvee_{j=0}^{n-1} T^{-j} A))_{n \in \mathbb{N}}$  is subadditive [KH97, Proposition 4.3.6]. We can then define the entropy

$$(4.15) \quad h_\mu(T, A) := \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left( \bigvee_{j=0}^{n-1} T^{-j} A \right).$$

Obviously, if for example  $A$  is the trivial partition with only one element, this entropy is zero and does not convey any information about the map  $T$ .

DEFINITION 4.7. The entropy of  $T$  with respect to  $\mu$  (or equivalently of  $\mu$  with respect to  $T$ ) is

$$(4.16) \quad h_\mu(T) := \sup\{h_\mu(T, A), \text{ } A \text{ finite partition of } X\}.$$

Topological and metric entropy are linked by the variational principle:

THEOREM 4.8 ([KH97, Theorem 4.5.3]). *Let  $T : X \longrightarrow X$  be a continuous map on a compact metric space  $X$ .*

$$(4.17) \quad h_{\text{top}}(T) = \sup_{\mu} h_\mu(T)$$

*where the supremum is taken over all  $T$ -invariant Borel probability measures  $\mu$ .*

Similarly,

THEOREM 4.9. [Bow75, Theorem 2.17] *Let  $T : X \longrightarrow X$  be a continuous map on metric space and  $\phi \in \mathcal{C}(X)$  be a continuous function on  $X$ .*

$$(4.18) \quad \text{Pr}(\phi) = \sup_{\mu} \left( h_\mu(T) + \int_X \phi \, d\mu \right),$$

*the supremum being taken as before over the invariant Borel probability measures.*

## 5. Discrete spectrum

**5.1. Expanding maps.** Expanding maps  $E : M \longrightarrow M$  of a compact Riemannian manifold  $(M, g)$  are maps which expand the metric, *i.e.* such that:

$$\exists c > 1, \forall v \in TM, \|dE \cdot v\|_g \geq c \|v\|_g.$$

The simplest examples of such maps are the angle multiplying maps of the circle

$$E_l : \begin{cases} \mathbb{R}/\mathbb{Z} & \longrightarrow & \mathbb{R}/\mathbb{Z} \\ x & \longmapsto & lx \end{cases}$$

presented in 1.1.

The expanding maps have an exponentially big number of periodic orbits:

LEMMA 5.1 ([Rue92], Lemma 4.1). *Let  $h_{\text{top}} > 0$  be the topological entropy of an expanding map  $E$  on a compact Riemannian manifold  $M$ , and let  $\text{Per}(n)$  be the set of periodic orbits of period  $n$ . There exists  $C > 0$  such that*

$$(5.1) \quad \forall n \in \mathbb{N}^*, \frac{1}{C} e^{nh_{\text{top}}} \leq \#\text{Per}(n) \leq C e^{nh_{\text{top}}}$$

A first important fact about these maps is that they admit a unique invariant measure that is absolutely continuous with respect to Lebesgue's measure.

LEMMA 5.2 (Mixing of expanding maps). *Let  $E$  be a  $C^k$  expanding map of a compact Riemannian manifold  $M$ . Then  $E$  admits a unique invariant measure  $\mu_{\text{SRB}}$  absolutely continuous with respect to the Riemannian volume  $dx$  [Bal18, Theorem 2.2]. The density of  $\mu_{\text{SRB}}$  is positive, and  $\mu_{\text{SRB}}$  is mixing.*

*Moreover, this measure can be expressed in terms of periodic points [Rue92]: If we write  $J(y) := \log |\det dE_y|$  and*

$$(5.2) \quad J_x^n := J(x) + J(E(x)) + \cdots + J(E^{n-1}(x))$$

*its Birkhoff sum, then*

$$(5.3) \quad \mu_{\text{SRB}} = \text{weak} - * \lim_{n \rightarrow \infty} \frac{1}{\#\text{Per}(n)} \sum_{x \in \text{Per}(n)} e^{-J_x^n} \delta_x.$$

REMARK 5.3. In fact, if  $E$  is of class  $C^k$ , then the density of the invariant absolutely continuous measure is  $C^{k-1}$  by results of Sacksteder [Sac74] and Krzyzewski [Krz77].

REMARK 5.4. As a consequence of the positivity of the density, we can, given two functions  $f$  and  $g$  consider the correlations

$$(5.4) \quad c_{f,g, \frac{dx}{d\mu_{\text{SRB}}}}(n) = \int_M f \circ E^n \cdot g \, dx - \int f \, d\mu_{\text{SRB}} \int g \, dx.$$

They also converge to zero.

Let us introduce transfer operators associated to expanding maps:

DEFINITION 5.5 (Transfer operator). Let  $E$  be a  $C^k$  expanding map of a compact Riemannian manifold  $M$ , and  $V \in C^k(M, \mathbb{C})$  that will be referred to as potential. we define

$$(5.5) \quad \mathcal{L}_V : \begin{cases} C^k(M, \mathbb{C}) & \longrightarrow & C^k(M, \mathbb{C}) \\ u & \longmapsto & e^V \cdot u \circ E \end{cases}.$$

For all  $u \in \mathcal{C}^k(M)$ , the  $L^2$  adjoint (with respect to the Riemannian volume), called Ruelle-Perron-Frobenius operator is given by

$$(5.6) \quad \mathcal{L}_V^* u(x) = \sum_{E(y)=x} e^{(V-J)(y)} u(y).$$

Both  $\mathcal{L}_V$  and  $\mathcal{L}_V^*$  fix  $\mathcal{C}^k(M)$ ; they can be extended to  $(\mathcal{C}^k(M))'$  by duality.

**5.2. Ruelle-Pollicott spectrum of expanding maps.** The following theorem is due to Ruelle [Rue86], for the Banach spaces  $\mathcal{C}^\alpha$ . This version is detailed in [Bal18].

**THEOREM 5.6** ([Rue86],[Bal18, Thm 2.15 and Lemma 2.16]). *Let  $k \geq 1$ . Let  $E$  be a  $\mathcal{C}^k$  expanding map of a Riemannian manifold  $M$  and let  $V$  be a complex  $\mathcal{C}^k$  potential on  $M$ . Then for every  $0 \leq s < k$ ,  $\mathcal{L}_V : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$  is bounded and its essential spectral radius  $r_{\text{ess}}$  satisfies*

$$(5.7) \quad r_{\text{ess}} \leq e^{\text{Pr}(\text{Re}(V) - \frac{1}{2}J)} m^s$$

where  $0 < m < 1$  is the inverse of the expansion rate:

$$(5.8) \quad m^{-1} := \inf_{v \in TM} \frac{\|dE \cdot v\|_g}{\|v\|_g}.$$

**REMARK 5.7.** Notice that this implies that the resolvent admits a meromorphic continuation outside the disk of radius  $r_{\text{ess}}$ . Its poles, called Ruelle-Pollicott resonances, are then the eigenvalues of the transfer operator and the associated residue is the spectral projector on the corresponding eigenspace.

For smooth maps, the resolvent can therefore be meromorphically extended to  $\mathbb{C}^*$ .

For  $V = 0$ , this gives a precise asymptotic expansion of correlations: Let  $f \in \mathcal{H}^{-s}(M)$ ,  $g \in \mathcal{H}^s(M)$ , if we denote by  $\Pi_\lambda$  the finite rank spectral projector on a resonance  $\lambda$ , for every  $\varepsilon > r_{\text{ess}}$  such that there is no resonance of modulus  $\varepsilon$ ,

$$(5.9) \quad \begin{aligned} c_{f,g}(n) &= \int \mathcal{L}^n f \cdot g \, d\mu_{\text{SRB}} \\ &= \sum_{\substack{\lambda \in \text{Res}(\mathcal{L}) \\ |\lambda| > \varepsilon}} \int \mathcal{L}^n \Pi_\lambda f \cdot g \, d\mu_{\text{SRB}} + O(\varepsilon^n \|f\|_{\mathcal{H}^{-m}} \|g\|_{L^1}). \end{aligned}$$

**REMARK 5.8.** On each  $\text{Ran}(\Pi_\lambda)$ ,  $\mathcal{L}$  acts as a Jordan block: there exists a nilpotent finite rank operator  $Q_\lambda$  commuting with  $\Pi_\lambda$ , of index  $d_\lambda \leq \dim \text{Ran}(\Pi_\lambda)$  such that

$$\mathcal{L}\Pi_\lambda = (\lambda \text{Id} + Q_\lambda)\Pi_\lambda,$$

thus

$$\mathcal{L}^n \Pi_\lambda = \lambda^n \sum_{j=0}^{d_\lambda-1} \binom{n}{j} Q_\lambda^j \Pi_\lambda.$$

**REMARK 5.9.** In virtue of Remark 5.4, the expansion of (5.9) remains valid if we replace  $\mu_{\text{SRB}}$  with the Riemannian volume  $dx$ .

The eigenvalues of largest moduli and their associated eigendistributions give the information about the equilibrium state and the decay of correlations and have been extensively studied. The following result, due to Ruelle, extends the mixing properties we have seen to the case of a non-zero potential:

**THEOREM 5.10** (Ruelle-Perron-Frobenius theorem, [Bal18, Proposition 2.5]). *Let  $E$  be a  $\mathcal{C}^{1+\theta}$  expanding map on a compact Riemannian manifold  $M$  for some  $\theta > 0$ . Let  $V \in \mathcal{C}^\infty(M, \mathbb{R})$ . Then*

- (1)  $\mathcal{L}_V$  has a positive simple leading resonance  $e^{\text{Pr}(V-J)}$ : Every other resonance has strictly smaller modulus.
- (2) The spectral projector associated to  $\lambda_0$  is  $\langle h_V, \cdot \rangle \mu_V$ , where
  - $\mu_V$  is the probability measure satisfying

$$\mathcal{L}_V \mu_V = e^{\text{Pr}(V-J)} \mu_V,$$

- $h_V$  is the positive function satisfying

$$\mathcal{L}_V^* h_V = e^{\text{Pr}(V-J)} h_V$$

and

$$\int h_V d\mu_V = 1.$$

- (3)  $h_V \mu_V$  is an invariant mixing probability measure for  $E$  called Gibbs measure, or equilibrium measure. It is the unique probability measure  $\mu$  such that

$$(5.10) \quad \text{Pr}(V-J) = h_\mu(E) + \int (V-J) d\mu$$

- REMARK 5.11.** (1) If  $V \in \mathcal{C}^\infty(M, \mathbb{C})$ , the theorem does not apply. But the spectral radius is not bigger than the one of  $\mathcal{L}_{\text{Re}(V)}$ : every Ruelle resonance has modulus smaller than  $e^{\text{Pr}(\text{Re}V-J)}$ .
- (2) If  $V=0$ , as we have seen,

$$\lambda_0 = e^{\text{Pr}(V-J)} = e^{\text{Pr}(-J)} = 1$$

and  $\mu_0$  is the Riemannian volume  $dx$  ( $\mathbb{1} \circ E = \mathbb{1}$ ).  $h_0$  is the density of the SRB measure:

$$\mu_{\text{SRB}} = h_0 dx.$$

- (3) The equilibrium measure is given, like the SRB measure, by the vague limit of sum over periodic orbits, weighted, this time, by the potential:

$$(5.11) \quad h_V \mu_V = \text{weak-}^* \lim_{n \rightarrow \infty} \frac{1}{Z_V} \sum_{x \in \text{Per}(n)} e^{(V-J)_x^n} \delta_x$$

where

$$(5.12) \quad Z_V = \sum_{x \in \text{Per}(n)} e^{(V-J)_x^n} \longrightarrow \text{Pr}(V-J)$$

is a normalisation constant.

Let us mention that in the case of analytic maps, the transfer operator is nuclear in a certain Banach space of hyperfunctions [Rue76]. The modulus of the eigenvalues satisfies exponential decay.

Having lower bounds on the modulus of eigenvalues is generally difficult. Naud and Bandtlow [BN19] showed that for a dense set of analytic expanding maps of the circle, for every  $\varepsilon > 0$ , there exists a subsequence of resonances such that the modulus of the  $n$ -th eigenvalue decays slower than  $C_\varepsilon e^{-n^{1+\varepsilon}}$ . Moreover, Bandtlow,

Just and Slipantschuk [SBJ13] constructed for any  $\lambda$  of modulus less than one an analytic expanding maps of the circle whose spectrum is precisely

$$\{\lambda^n, n \in \mathbb{N}\} \cup \{\bar{\lambda}^n, n \in \mathbb{N}\}.$$

For a matrix  $A \in \mathcal{M}_N(\mathbb{C})$  with eigenvalues  $\lambda_1, \dots, \lambda_N$ , we can write for  $z$  smaller than the inverse of the spectral radius of  $A$

$$\begin{aligned} \det(1 - zA) &= \prod_{k=1}^N (1 - z\lambda_k) \\ &= \exp \left( \sum_{k=1}^N \log(1 - z\lambda_k) \right) \\ (5.13) \quad &= \exp \left( - \sum_{k=1}^N \sum_{n \geq 0} \frac{z^n}{n} \lambda_k^n \right) \\ &= \exp \left( - \sum_{n \geq 1} \operatorname{Tr}(A^n) \frac{z^n}{n} \right). \end{aligned}$$

This determinant is a polynomial function vanishing on the inverse of the non-zero eigenvalues of  $A$ .

For a trace-class operator  $A$  on a Hilbert space, we can as well [Sim10] write the spectral determinant as

$$(5.14) \quad \det(1 - zA) = \exp \left( - \sum_{n \geq 1} \operatorname{Tr}(A^n) \frac{z^n}{n} \right)$$

and moreover, if  $A^n$  is an operator  $\mathcal{C}^\infty \rightarrow \mathcal{D}'$  with continuous kernel  $K_n$ , then

$$(5.15) \quad \operatorname{Tr}(A^n) = \int K_n(x, x) \, dx.$$

In our case, the transfer operator is in general not trace-class, and its kernel is not continuous:

$$(5.16) \quad \mathcal{L}_V u(x) = e^{V(x)} u(E(x)) = \int_M e^{V(x)} \delta_{E(x)}(y) u(y) \, dy.$$

However, mollifying and performing a change of variables in local charts in (5.16) allows to make sense of the integral of the kernel  $K_n$  of  $\mathcal{L}_V^n$ ,  $n \geq 1$  along the diagonal (see [Bal18] Section 3.2.2). This gives rise to the flat trace

$$(5.17) \quad \operatorname{Tr}^b(\mathcal{L}_V^n) := \int K_n(x, x) \, dx$$

which admits a simple expression in terms of periodic points:

$$(5.18) \quad \operatorname{Tr}^b(\mathcal{L}_V^n) = \sum_{x \in \operatorname{Per}(n)} \frac{e^{V_x^n}}{|\det(dE^n - 1)|}.$$

These flat traces grow at most exponentially with  $n$ , so the formula

$$(5.19) \quad \det(1 - z\mathcal{L}_V) := \exp \left( - \sum_{n \geq 0} \frac{z^n}{n} \operatorname{Tr}^b(\mathcal{L}_V^n) \right)$$

defines a holomorphic non-vanishing function in a neighbourhood of 0 called dynamical determinant.

Ruelle [Rue90] then proved the following theorem about dynamical determinants (this version with Sobolev spaces is taken from [Bal18], Theorem 3.3)

**THEOREM 5.12.** *Let  $E$  be a  $C^r$  expanding map of a compact Riemannian manifold  $M$ , and let  $V : M \rightarrow \mathbb{C}$  be a potential such that  $V - J$  is  $C^\alpha$  for  $0 < \alpha \leq r$ . Let as in (5.8)*

$$(5.20) \quad m^{-1} := \inf_{v \in TM} \frac{\|dE \cdot v\|_g}{\|v\|_g}.$$

Then  $\det(1 - z\mathcal{L}_V)$  admits a holomorphic extension to

$$(5.21) \quad \{z \in \mathbb{C}, |z| \leq m^{-\alpha} e^{-\Pr(V-J)}\}.$$

Moreover, in this disk, the zeros of  $\det(1 - z\mathcal{L}_V)$  are precisely the inverse of the Ruelle-Pollicott resonances (defined in Corollary 1.9), with the same multiplicities.

This gives an expansion of the flat traces in terms of the resonances:

**COROLLARY 5.13.** *Under the hypothesis of the previous theorem,*

$$(5.22) \quad \mathrm{Tr}^b(\mathcal{L}_V^n) = \sum_{\substack{\lambda \in \mathrm{Res}(\mathcal{L}_V) \\ |\lambda| \geq m^\alpha e^{\Pr(V-J)}}} \lambda^n + O(\varepsilon^n)$$

for every  $\varepsilon > m^\alpha e^{\Pr(V-J)}$ . Note that the constant involved in the  $O$  term depends on the potential  $V$ , and the choice of  $\varepsilon$  in a way that is not controlled here.

**PROOF.** Indeed, writing  $m_\lambda$  the multiplicity of a resonance  $\lambda$ , the previous theorem implies that

$$(5.23) \quad \frac{\exp\left(-\sum_{n \geq 0} \frac{z^n}{n} \mathrm{Tr}^b(\mathcal{L}_V^n)\right)}{\prod_{\substack{\lambda \in \mathrm{Res}(\mathcal{L}_V) \\ |\lambda| \geq m^{-\alpha} \exp(-\Pr(V-J))}} (1 - z\lambda)^{m_\lambda}}$$

has a non-vanishing holomorphic extension  $f$  to the disk of radius  $m^{-\alpha} e^{-\Pr(V-J)}$  centered at the origin. Thus,  $\frac{f'}{f}$  admits a primitive  $\log f$  on that disk.

$$(5.24) \quad \log f(z) = - \sum_{n \geq 1} \frac{z^n}{n} \left( \mathrm{Tr}^b(\mathcal{L}_V^n) - \sum_{\substack{\lambda \in \mathrm{Res}(\mathcal{L}_V) \\ |\lambda| \geq m^{-\alpha} \exp(-\Pr(V-J))}} m_\lambda \lambda^n \right)$$

on a neighbourhood of zero and this sum has convergence radius at least  $m^{-\alpha} e^{-\Pr(V-J)}$ , which gives the result.  $\square$

### 5.3. Anosov diffeomorphisms.

**DEFINITION 5.14.** An Anosov diffeomorphism of a compact Riemannian manifold  $(M, g)$  is a diffeomorphism  $T : M \rightarrow M$  such that there exists an invariant splitting of the tangent bundle:

$$(5.25) \quad T_x M = E_u \oplus E_s$$

$$(5.26) \quad \forall x \in M, dT_x \cdot E^i(x) = E^i(f(x)), \quad i \in \{u, s\}$$

satisfying the following property: there exist constants  $0 < m < 1$  and  $C > 0$ ,

$$(5.27) \quad \forall v \in E^u, \forall n \in \mathbb{N}, \|dT^{-n} \cdot v\| \leq Cm^n \|v\|$$

and

$$(5.28) \quad \forall v \in E^s, \forall n \in \mathbb{N}, \|dT^n \cdot v\| \leq Cm^n \|v\|.$$

It is not known if every Anosov diffeomorphism of a compact manifold is topologically mixing. However it is the case if the diffeomorphism is topologically transitive, *i.e.* if it has a dense orbit [KH97, Corollary 18.3.5].

**THEOREM 5.15** ([KH97, Theorem 20.1.6]). *If  $T$  is a topologically transitive Anosov diffeomorphism, then the growth of its periodic points is exponential and given in terms of its topological entropy by*

$$(5.29) \quad \exists 0 < \alpha < h_{\text{top}}, \quad \#\text{Per}(n) = e^{nh_{\text{top}}} + O(e^{\alpha n})$$

In addition, like expanding maps, they have interesting invariant measures:

**THEOREM 5.16** ([Bow75] Theorem 4.1, [KH97] Theorem 20.3.7). *Let  $T$  be an Anosov diffeomorphism of a compact Riemannian manifold  $M$ . For each Hölder function  $\phi$ ,  $T$  has a unique invariant measure  $m_\phi$  such that*

$$(5.30) \quad \text{Pr}(\phi) = h_{m_\phi}(T) + \int \phi \, dm_\phi.$$

$m_\phi$  is ergodic. If  $T$  is moreover transitive, then  $m_\phi$  mixing, and is given by

$$(5.31) \quad m_\phi = \text{weak} - * \lim_{n \rightarrow \infty} \frac{1}{Z_n} \sum_{x \in \text{Per}(n)} e^{\phi_x^n} \delta_x$$

where  $Z_n$  is the normalisation constant.

Those measures, called equilibrium measures, are, like for expanding maps before, linked to the transfer operator (see next subsection). They need not be absolutely continuous with respect to the Riemannian volume:

**THEOREM 5.17** ([Bow75, Theorem 4.14]).  *$T$  admits an invariant measure absolutely continuous with respect to the Riemannian volume if, and only if the Birkhoff sums  $J_x^n$  of the Jacobian  $J := \det dT$  satisfy*

$$(5.32) \quad J_x^n = 0$$

for every point  $x \in \text{Per}(n)$ .

This condition is not stable by perturbation:

**COROLLARY 5.18** ([Bow75, Corollary 4.15]). *The subset of  $\mathcal{C}^2$  Anosov diffeomorphisms that admit no invariant measure absolutely continuous with respect to the volume is open and dense.*



5.3.1. *Discrete spectrum for Anosov diffeomorphisms.* For an Anosov diffeomorphism  $T$  of a compact Riemannian manifold  $M$ , and a complex potential  $V : M \rightarrow \mathbb{C}$ , Let

$$\mathcal{L}_V : u \mapsto e^V u \circ T$$

be the transfer operator with potential  $V$ .

Transfer operators for Anosov diffeomorphisms were first studied by Ruelle and Bowen, Parry, Pollicott, with symbolic dynamics (see [Bow75] and [PP90]). The symbolic dynamics approach does not allow to take advantage of the smoothness of the maps. Banach spaces of distributions that provide quasi-compactness of the transfer operator for hyperbolic maps were first introduced by Blank-Keller-Liverani [BKL02], then Gouëzel-Liverani [GL06], Baladi-Tsujii [BT07], Faure-Roy-Sjöstrand [FRS08] in the smooth case. See [Bal18] p.153-155 for a more detailed historical review. These spaces are made of distributions exhibiting some smoothness in the unstable direction, and less regularity in the stable direction.

**THEOREM 5.19** ([Bal18, Theorem 5.1]). *Let  $r > 1$ . Let  $T$  be a  $C^r$  Anosov diffeomorphism on a connected compact Riemannian manifold  $M$ , and  $V \in C^{r-1}(M, \mathbb{C})$  be a potential. For any  $s < \frac{r-1}{2}$ , there exists a Banach space  $B^s$  such that*

- $\mathcal{L}_V : B^s \rightarrow B^s$  is a bounded operator,
- For any  $\alpha > s$ ,

$$(5.33) \quad C^\alpha(M) \subset B^s \subset C^\alpha(M)',$$

- The spectral radius  $r(\mathcal{L}_V)$  of  $\mathcal{L}_V$  in  $B^s$  is

$$(5.34) \quad r(\mathcal{L}_V) \leq e^{\text{Pr}(\text{Re}(V) - J_u)},$$

- The essential spectral radius  $r_{\text{ess}}(\mathcal{L}_V)$  of  $\mathcal{L}_V$  in  $B^s$  satisfies

$$(5.35) \quad r_{\text{ess}}(\mathcal{L}_V) \leq e^{\text{Pr}(\text{Re}(V) - J_u)} m^s,$$

where  $m$  is the smallest constant such that (5.27) and (5.28) are satisfied, and  $J_u = \log |\det dT|_{E^u}| : M \rightarrow \mathbb{R}$ .

As in the case of linear expanding maps of the circle, the strategy of Proposition 1.6 shows that linear toral hyperbolic diffeomorphisms have a single Ruelle-Pollicott resonance 1. Adam showed [Ada17], based on an idea of Naud [Nau15] that an open dense subset of perturbations of such maps has non trivial Ruelle resonances.

Similarly to expanding maps, the peripheral spectrum of transitive Anosov diffeomorphisms is as follows:

**THEOREM 5.20** ([Bal18, Theorem 7.5]). *Under the hypothesis of the previous theorem, if the potential  $V$  is **real**, then  $e^{\text{Pr}(V - J_u)}$  is a simple eigenvalue of  $\mathcal{L}_V$  on  $B^s$ . The associated spectral projector is of the form*

$$(5.36) \quad \Pi = |\mu_V\rangle \langle \nu_V|$$

where  $\mu_V \in B^s$  and  $\nu_V \in (B^s)^*$  are Radon measures. Moreover, there are no other eigenvalues of modulus  $e^{\text{Pr}(V - J_u)}$ . The application

$$(5.37) \quad \begin{cases} C^{r-1}(M) & \longrightarrow & \mathbb{R} \\ \phi & \longmapsto & \langle \mu_V, \phi \nu_V \rangle \end{cases}$$

is a well-defined linear form on  $C^{r-1}(M)$  that extends to the equilibrium measure  $m_{V-J_u}$  defined in Theorem 5.16.

As in the case of expanding maps, for smooth maps, the Ruelle resonances need not be summable. In fact [Jéz17], they can decay arbitrarily slowly to zero. But we can make sense of the flat trace:

LEMMA 5.21 (Flat trace). *The operators  $\mathcal{L}_V^n, n \geq 1$  are kernel integral operators with distribution kernel*

$$(5.38) \quad K^n(x, y) = e^{V_x^n} \delta(y - T^n(x))$$

*They admit flat traces:*

$$(5.39) \quad \mathrm{Tr}^b \mathcal{L}_V^n := \int_M K^n(x, x) dx = \sum_{x, T^n(x)=x} \frac{e^{V_x^n}}{|\det(1 - d(T^n)_x)|}.$$

And there is a holomorphic continuation of the dynamical determinant (see [Bal18] Theorem 6.2 for more precise bounds):

THEOREM 5.22. *The dynamical determinant*

$$(5.40) \quad \det(1 - z\mathcal{L}) = \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Tr}^b(\mathcal{L}_V^n) \right)$$

*admits a holomorphic continuation to the disk centered in 0 of radius  $m^{-s} e^{-\mathrm{Pr}(\mathrm{Re}(V) - J_u)}$ . It vanishes precisely at the inverses of the Ruelle-Pollicott resonances, and the multiplicities of the zeros coincide with the multiplicities of the eigenvalues.*

Just like Corollary 5.13, we have

COROLLARY 5.23 (Link with the Ruelle-Pollicott resonances). *Let  $\varepsilon > 0$  be such that  $\mathcal{L}_V$  has no resonance of modulus  $\varepsilon$ , then*

$$(5.41) \quad \mathrm{Tr}^b \mathcal{L}_V^n = \sum_{\substack{\lambda \in \mathrm{Res}(\mathcal{L}_V) \\ |\lambda| > \varepsilon}} \lambda^n + O(\varepsilon^n).$$

#### 5.4. Flows.

DEFINITION 5.24. A vector field  $X$  on a compact Riemannian manifold  $M$ , or equivalently the flow that it generates is called Anosov if the tangent bundle admits a splitting  $TM = E^u \oplus E^s \oplus \mathbb{R} \cdot X$  such that there exist constants  $C > 0$ ,  $0 < \lambda < 1$  such that

$$(5.42) \quad \begin{aligned} &\bullet \forall v \in E^s, \forall t \in \mathbb{R}_+, \|d\phi^t \cdot v\| \leq C\lambda^t \|v\| \\ &\bullet \forall v \in E^u, \forall t \in \mathbb{R}_+, \|d\phi^{-t} \cdot v\| \leq C\lambda^t \|v\|. \end{aligned}$$

Anosov [A<sup>+</sup>67] proved that the geodesic flow on a compact Riemannian manifold with negative curvature is Anosov. The geodesic flow can in general be seen as a Hamiltonian flow on the unit cotangent space. It preserves moreover the canonical contact form. Anosov flows satisfying this property are called **contact** Anosov flows.

THEOREM 5.25 ([KH97] Theorem 18.3.6). *Contact Anosov flows on connected compact manifolds are topologically mixing.*

Anosov flows in general need not be topologically mixing: for instance if  $T : M \rightarrow M$  is any Anosov map, one can define a suspension of  $T$  by a constant roof function by setting on the manifold  $M \times S^1$   $X(x, z) := \frac{\partial}{\partial z}$  the vertical vector field. Then  $X$  is an Anosov vector field which preserves the horizontal leaves. There is

however no chance for  $X$  to be topologically mixing since the  $z$ -projection of the image of a set by the flow has constant diameter.

Outside of this scope, they are however mixing: Anosov flows have a unique measure of maximal entropy [Bow73] [Mar70] and

**THEOREM 5.26** ([A<sup>+</sup>67],[AS67]). *An Anosov flow is either the suspension of an Anosov map by a constant roof function or mixing with respect to the measure of maximal entropy.*

Ruelle and Bowen made the following conjecture:

**CONJECTURE 5.27.** *Every topologically mixing Anosov flow is exponentially mixing with respect to the measure of maximal entropy.*

This conjecture is still open. Tsujii and Zhang [TZ20] recently made a breakthrough in this direction (see below).

Results on the speed of mixing of Anosov flows were first obtained for manifolds of constant negative curvature, where exponential mixing was shown in dimension 2 [CEG84], [Moo87], [Rat87] and 3 [Pol92].

Then, Chernov [Che98] was able to show subexponential decay of correlations for surfaces with variable negative curvature.

Shortly after, Dolgopyat was able to introduce transfer operator methods to show exponential decay of correlations when the strong stable and unstable manifolds are  $C^1$ . This covers the case of surfaces of variable negative curvature. However, in higher dimension and for more general flows, the foliations are in general Hölder continuous, but not differentiable. These restrictions on the regularity are due to the use of symbolic dynamics.

Liverani [Liv04] was then able to remove this hypothesis and showed exponential decay of correlations with respect to the contact volume, for all  $C^4$  contact Anosov flows, in every dimension, using anisotropic Banach spaces adapted to the vector field, following the work done in [BKL02].

Then, Tsujii [Tsu10] obtained an explicit spectral gap for the transfer operator acting on an anisotropic Hilbert space of distributions.

Butterley and Liverani [BW16] constructed Banach space suitable for general Anosov flows.

**COROLLARY 5.28.** *The resolvent of a smooth Anosov vector field consequently admits a meromorphic continuation to the whole complex plane, thereby defining a set of Ruelle resonances  $\text{Res}(\mathcal{L})$ .*

Faure and Sjöstrand [FS11] linked this construction to the theory of semiclassical resonances [HS86] and obtained a similar version of these spaces, Hilbert this time, as well as an upper bound on the number of resonances of large imaginary part of the Anosov vector field. Their technique studies the dynamics induced on the cotangent space  $T^*M$  by a  $C^\infty$  Anosov flow on a manifold  $M$ . They built spaces similar to those of Butterley and Liverani, by introducing an escape function decaying along trajectories outside a compact region. The frequency in the direction of the flow is preserved, and acts as a semiclassical parameter. Later, Faure and Tsujii [FT17b] [FT13] showed using semiclassical analysis that the spectrum of the generator of a contact Anosov flow was structured in bands.

Tsujii [Tsu18] recently showed that an open dense subset of the volume preserving  $C^r$  Anosov flows on a dimension 3 Riemannian manifolds was constituted

of exponentially mixing flows. And even more recently, with Zhang [TZ20], that for transitive Anosov flows in dimension 3, topological mixing is equivalent to exponential mixing.

Outside the scope of compact manifolds, Ruelle spectrum for flows has for instance been studied by Weich and Bonthonneau [BW17] in the framework of manifolds with cusps, and Guillarmou and Dyatlov [DG16] have shown meromorphic continuation of the resolvent for Axiom A flows.

5.4.1. *Trace formula for flows.* For Anosov Flows, there exist trace formulae as in the case of maps. The following expression, with a remainder term, is called “local trace formula”. Jezequel [Jéz19] has shown a global (without remainder) trace formula for a class of flows containing the Gevrey flows.

**THEOREM 5.29 ([JZ17]).** *Let  $X$  be a smooth Anosov vector field on a compact Riemannian manifold  $M$  of dimension  $d$ . Let  $\mathcal{G}$  be the set of its periodic orbits. For  $\gamma \in \mathcal{G}$ , let  $T_\gamma^\#$  be its prime period, and  $T_\gamma$  its period. For any  $A > 0$ , there exists a tempered distribution  $F_A \in \mathcal{S}'(\mathbb{R})$ , supported in  $\mathbb{R}_+$  such that the following distributions coincide:*

$$(5.43) \quad \sum_{\substack{\mu \in \text{Res}(X) \\ \text{Re}(\mu) > -A}} e^{\mu t} + F_A(t) = \sum_{\gamma \in \mathcal{G}} T_\gamma^\# \frac{\delta(t - T_\gamma)}{|\det(I - \mathcal{P}_\gamma)|}.$$

Moreover, the Fourier transform  $\hat{F}_A$  admits a holomorphic continuation to  $\text{Im} \lambda < A$  and for every  $\varepsilon > 0$ ,

$$(5.44) \quad |\hat{F}_A(\lambda)| = O(\langle \lambda \rangle^{2d+1})$$

on  $\{\text{Im}(\lambda) < A - \varepsilon\}$ , where the constant in  $O$  depends only on  $A$  and  $\varepsilon$ .

For ultradifferentiable flows, Jézéquel [Jéz19] showed that (5.43) actually holds without the term  $F_A$ .

**5.5. Compact group extensions of maps.** In this thesis we study  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ -extensions of expanding maps and transitive Anosov diffeomorphisms. Namely,

**DEFINITION 5.30.** Let  $T$  be an expanding map or a transitive Anosov diffeomorphism of a compact Riemannian manifold. Let  $\tau \in \mathcal{C}^k(M, \mathbb{T})$ . We call  $\mathbb{T}$ -extension of  $T$  by  $\tau$  the map

$$(5.45) \quad F : \begin{cases} M \times \mathbb{T} & \longrightarrow & M \times \mathbb{T} \\ (x, y) & \longmapsto & (T(x), y + \tau(x)) \end{cases}.$$

This model exhibits a neutral direction in the  $\mathbb{T}$  fiber, since the map  $F$  maps fibers to fibers and acts as a translation under the identification of fibers. The frequency in the neutral direction is preserved by the transfer operator: Let

$$(5.46) \quad \mathcal{L} : \begin{cases} \mathcal{C}^k(M \times \mathbb{T}) & \longrightarrow & \mathcal{C}^k(M \times \mathbb{T}) \\ u & \longmapsto & u \circ F \end{cases}.$$

Then, for every  $u \in \mathcal{C}^k(M)$  and every  $p \in \mathbb{Z}$ ,

$$(5.47) \quad \mathcal{L}(u \otimes e^{2i\pi p \cdot})(x, y) = e^{2i\pi \tau(x)} u \circ T(x) e^{2i\pi p y}.$$

Thus,  $\mathcal{C}^k(M \times \mathbb{T})$  splits as a sum of invariants subspaces

$$(5.48) \quad \mathcal{C}^k(M \times \mathbb{T}) = \bigoplus_p (\mathcal{C}^k(M) \otimes e^{2i\pi p \cdot})$$

on each of which the transfer operator can be reduced to a transfer operator on  $\mathcal{C}^k(M)$  with complex potential

$$(5.49) \quad \mathcal{L}_{p,\tau} : \begin{cases} M & \longrightarrow M \\ u & \longmapsto e^{2i\pi p \cdot} u \circ T \end{cases}$$

The properties of these operators regarding discrete spectrum and flat trace have been recalled in Subsections 5.1 and 5.3.

REMARK 5.31. In Chapter 2 and 3, we actually consider  $\mathbb{R}$  extensions instead of  $\mathbb{T}$  extensions, but this plays no other role than having a semiclassical parameter  $p$  in  $\mathbb{R}$  rather than  $\mathbb{Z}$ .

More general compact group extensions have first been studied by Brin [Bri75] in the volume preserving case, and then by Field, Parry and Pollicott [PP97] [FP99] in the general case. Writing  $\mu_{SRB}$  the measure of maximal entropy of  $T$ ,  $\mu_{SRB} \otimes \text{Leb}$  is an invariant measure of  $F$ . Dolgopyat [Dol02] showed generic exponential decay of correlations in the expanding maps setting and generic rapid mixing, *i.e.* superpolynomial decay for extensions of a hyperbolic map. De Simoi, Liverani, Poquet and Volk [DSL17] and de Simoi and Liverani [DSL16] [DSL18] studied fast-slow dynamical systems, namely functions of the form (5.45), where  $T$  and  $\tau$  depend on both  $x$  and  $y$ ,  $T$  is a family of expanding maps of the circle and  $\tau$  is multiplied by a small amplitude  $\epsilon$ .

## CHAPTER 2

# Flat traces for the skew product of an expanding map

**Acknowledgements**

The author thanks Alejandro Rivera for his many pieces of advice regarding probabilities, and Jens Wittsten, Masato Tsujii, Gabriel Rivière and Sébastien Gouëzel for interesting discussions about this work, as well as the anonymous referees for their careful reading and constructive suggestions.

## 1. Introduction

This paper focuses on the distribution of the flat traces of iterates of the transfer operator of a simple example of partially expanding map. It is motivated by the Bohigas-Gianonni-Schmidt **[BGS84]** conjecture in quantum chaos (see below). In chaotic dynamics, the transfer operator is an object of first importance linked to the asymptotics of the correlations. The collection of poles of its resolvent, called Ruelle-Pollicott spectrum, can be defined as the spectrum of the transfer operator in appropriate Banach spaces (see **[Rue76]** for analytic expanding maps, **[Kit99]**, **[BKL02]**, **[BT07]**, **[BT08]**, **[GL06]**, **[FRS08]** for the construction of the spaces for Anosov diffeomorphisms.)

The study of the Ruelle spectrum for Anosov flows is more difficult because of the flow direction that is neither contracting nor expanding. Dolgopyat has shown in particular in **[Dol98]** the exponential decay of correlations for the geodesic flow on negatively curved surfaces, and Liverani **[Liv04]** generalized this result to all  $\mathcal{C}^4$  contact Anosov flows. His method involved the construction of anisotropic Banach spaces in which the generating vector field has a spectral gap, and no longer relies on symbolic dynamics that prevented from using advantage of the smoothness of the flow. Tsujii **[Tsu10]** constructed appropriate Hilbert spaces for the transfer operator of  $\mathcal{C}^r$  contact Anosov flows,  $r \geq 3$  and gave explicit upper bounds for the essential spectral radii in terms of  $r$  and the expansion constants of the flow. Buterley and Liverani **[BL07]** and later Faure and Sjöstrand **[FS11]** constructed good spaces for Anosov flows, without the contact hypothesis. Weich and Bonthonneau defined in **[BW17]** Ruelle spectrum for geodesic flow on negatively curved manifolds with a finite number of cusps. Dyatlov and Guillarmou **[DG16]** handled the case of open hyperbolic systems. A simple example of Anosov flow is the suspension of an Anosov diffeomorphism, or the suspension semi-flow of an expanding map. Pollicott showed exponential decay of correlations in this setting under a weak condition in **[Pol85]** and Tsujii constructed suitable spaces for the transfer operator and gave an upper bound on its essential spectral radius in **[Tsu08]**.

In this article we study a closely related discrete time model, the skew product of an expanding map of the circle. It is a particular case of compact group extension **[Dol02]**, which are partially hyperbolic maps, with compact leaves in the neutral direction that are isometric to each other. Dolgopyat showed in **[Dol02]** that the correlation decrease generically rapidly for compact group extensions, and exponentially in the particular case of expanding maps. In our setting of skew-product of an expanding map of the circle, Faure **[Fau11]** has shown using semi-classical methods an upper bound on the essential spectral radius of the transfer operator under a condition shown to be generic by Nakano Tsujii and Wittsten **[NTW16]**. De Simoi, Liverani, Poquet and Volk **[DSLPV17]** and de Simoi and Liverani **[DSL16]** **[DSL18]** studied fast-slow dynamical systems, that generalize  $\mathbb{T}$ -extensions of circle expanding maps. The roof function, depending on two variables is multiplied by a small amplitude, and the authors obtained results about the statistical properties, for long time and small  $\varepsilon$ . Arnoldi, Faure, and Weich **[AFW17]** and Faure and Weich **[FW17]** studied the case of some open partially expanding maps, iteration function schemes, for which they found an explicit bound on the essential spectral radius of the transfer operator in a suitable space, and obtained a Weyl law



(upper bound on the number of Ruelle resonances outside the essential spectral radius). Naud [Nau16] studied a model close to the one presented in this paper, in the analytic setting, in which the transfer operator is trace-class, and used the trace formula, in the deterministic and random case to obtain a lower bound on the spectral radius of the transfer operator. In the more general framework of random dynamical systems in which the transfer operator changes randomly at each iteration, for the skew product of an expanding map of the circle, Nakano and Wittsten [NW15] showed exponential decay of correlations.

Semiclassical analysis describes the link between quantum dynamics and the associated classical dynamics in a symplectic manifold. The transfer operator happens to be a Fourier integral operator and the semi-classical approach has thus shown to be useful. The famous Bohigas-Giannoni-Schmidt [BGS84] conjecture of quantum chaos states that for quantum systems whose associated classical dynamic is chaotic, the spectrum of the Hamiltonian shows the same statistics as that of a random matrix (GUE, GOE or GSE according to the symmetries of the system)(see also [Gut13] and [GVZJ91]). We are interested analogously in investigating the possible links between the Ruelle-Pollicott spectrum and the spectrum of random matrices/operators. At first we try to get informations about the spectrum using a trace formula. More useful results could follow from the use of a global normal form as obtained by Faure-Weich in [FW17].

**1.1. Expanding map.** Let us consider a smooth orientation preserving expanding map  $E : \mathbb{T} \rightarrow \mathbb{T}$  on the circle  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ , that is, satisfying  $E' > 1$ , of degree  $l$ , and let us call

$$m := \inf E' > 1$$

and

$$M := \sup E'.$$

**1.2. Transfer operator.** Let us fix a function  $\tau \in C^k(\mathbb{T})$  for some  $k \geq 0$ . We are interested in the partially expanding dynamical system on  $\mathbb{T} \times \mathbb{R}$  defined by

$$(1.1) \quad F(x, y) = (E(x), y + \tau(x))$$

We introduce the transfer operator

$$\mathcal{L}_\tau : \begin{cases} C^k(\mathbb{T} \times \mathbb{R}) & \longrightarrow C^k(\mathbb{T} \times \mathbb{R}) \\ u & \longmapsto u \circ F \end{cases}.$$

**1.3. Reduction of the transfer operator.** Due to the particular form of the map  $F$ , the Fourier modes in  $y$  are invariant under  $\mathcal{L}_\tau$ : if for some  $\xi \in \mathbb{R}$  and some  $v \in C^k(\mathbb{T})$ ,

$$u(x, y) = v(x)e^{i\xi y},$$

then

$$\mathcal{L}_\tau u(x, y) = e^{i\xi\tau(x)}v(E(x))e^{i\xi y}.$$

Given  $\xi \geq 0$  and a function  $\tau$ , let us consequently consider the transfer operator  $\mathcal{L}_{\xi, \tau}$  defined on functions  $v \in C^k(\mathbb{T})$  by

$$\forall x \in \mathbb{T}, \mathcal{L}_{\xi, \tau} v(x) := e^{i\xi\tau(x)}v(E(x)),$$

**1.4. Spectrum and flat trace.** In appropriate spaces, the transfer operator has a discrete spectrum outside a small disk, the eigenvalues are called Ruelle resonances. It is in general not trace-class, but one can define its flat trace (see Appendix 2.C for a more precise discussion about Ruelle resonances, flat trace and their relationship).

LEMMA 1.1 (Trace formula, [AB67], [G<sup>+</sup>77]). *For any  $\mathcal{C}^0$  function  $\tau$  on  $\mathbb{T}$ , the flat trace of  $\mathcal{L}_{\xi, \tau}^n$  is well defined and*

$$(1.2) \quad \mathrm{Tr}^b \mathcal{L}_{\xi, \tau}^n = \sum_{x, E^n(x)=x} \frac{e^{i\xi \tau_x^n}}{(E^n)'(x) - 1},$$

where  $\tau_x^n$  denotes the Birkhoff sum: For a function  $\phi \in C(\mathbb{T})$  and a point  $x \in \mathbb{T}$  we define

$$(1.3) \quad \phi_x^n := \sum_{k=0}^{n-1} \phi(E^k(x)).$$

Note that for continuous  $\tau$ , there is no discrete spectrum. The trace formula makes still sense however.

**1.5. Gaussian random fields.** We define our random functions on the circle by means of their Fourier coefficients. We are only interested in  $\mathcal{C}^0$  functions. We will denote by  $\mathcal{N}(0, \sigma^2)$  (respectively  $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$ ) the real (respectively complex) centered Gaussian law of variance  $\sigma^2$ , with respective densities

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}x^2} \text{ and } \frac{1}{\sigma\pi} e^{-\frac{1}{\sigma^2}|z|^2}.$$

With these conventions, a random variable of law  $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$  has independent real and imaginary parts of law  $\mathcal{N}(0, \frac{\sigma^2}{2})$ , and the variance of its modulus is consequently  $\sigma^2$ .

DEFINITION 1.2. We will call centered stationary Gaussian random fields on  $\mathbb{T}$  the real random distributions  $\tau$  whose Fourier coefficients  $(c_p(\tau))_{p \geq 1}$  are independent complex centered Gaussian random variables, with variances growing at most polynomially, such that  $c_0(\tau)$  is a real centered Gaussian variable independent of the  $c_p(\tau)$ ,  $p \geq 1$ . The negative coefficients are necessarily given by

$$c_{-p}(\tau) = \overline{c_p(\tau)}.$$

The Gaussian fields are in general defined as distributions if their Fourier coefficients have variances with polynomial growth and the decay of the variances of the coefficients gives sufficient conditions for the regularity of the field.

LEMMA 1.3. *If  $\mathbb{E}[|c_p(\tau)|^2]$  has a polynomial growth,  $\tau = \sum_p c_p(\tau) e^{2i\pi p \cdot}$  defines almost surely a distribution: almost surely*

$$\forall \phi = \sum c_p(\phi) e^{2i\pi p \cdot} \in \mathcal{C}^\infty(\mathbb{T}), \quad \langle \tau, \phi \rangle := \sum_p \overline{c_p(\tau)} c_p(\phi) < \infty.$$

Let  $k \in \mathbb{N}$ . If for some  $\eta > 0$

$$(1.4) \quad \mathbb{E}[|c_p(\tau)|^2] = O\left(\frac{1}{p^{2k+2+\eta}}\right).$$

Then  $\tau$  is almost surely  $\mathcal{C}^k$ .

PROOF. See appendix 2.B.  $\square$

In what follows we will always assume that (1.4) is satisfied, at least for  $k = 0$ , so that our random fields are random variables on  $\mathcal{C}^0(\mathbb{T})$ . This will ensure the existence of flat traces.

Let us recall some basic facts about Gaussian random vectors [Gut09]. The characteristic function (essentially the Fourier transform of the law)

$$(1.5) \quad \phi : \xi \mapsto \mathbb{E} [e^{i\xi \cdot X}]$$

of a Gaussian random vector  $X : \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$  satisfies

$$(1.6) \quad \phi(\xi) = e^{-\frac{1}{2}\langle \xi, K\xi \rangle},$$

where the matrix  $K = (\mathbb{E} [\overline{X_i} X_j])_{(i,j)}$  is called covariance matrix of  $X$ . The characteristic function characterizes the probability law. A major consequence is that two components of a random Gaussian vector are independent if, and only if their covariance is 0.

If  $\tau$  is a centered Gaussian random field, then it is not hard to see that for any finite set  $\{x_1, \dots, x_n\} \subset \mathbb{T}$ ,  $X := \begin{pmatrix} \tau(x_1) \\ \vdots \\ \tau(x_n) \end{pmatrix}$  is a centered Gaussian random vector. A stationary centered Gaussian random field is thus characterized by its covariance function:

DEFINITION 1.4. Let  $\tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot}$  be a stationary centered Gaussian random field, satisfying

$$\mathbb{E}[|c_p|^2] = O\left(\frac{1}{p^{2+\eta}}\right)$$

for some  $\eta > 0$ , so that  $\tau$  is almost surely  $\mathcal{C}^0$  according to Lemma 1.3. Let us define its covariance function  $K$  by

$$(1.7) \quad K(x) := \sum_p \mathbb{E}[|c_p|^2] e^{2i\pi p x}.$$

For any pair of points  $x, y \in \mathbb{T}$ , we have

$$(1.8) \quad \mathbb{E}[\tau(x)\tau(y)] = K(x - y).$$

PROOF OF THE LAST STATEMENT. Remark from Appendix 2.B that the condition  $\mathbb{E}[|c_p|^2] = O\left(\frac{1}{p^{2+\eta}}\right)$  implies that  $\tau$  is almost surely equal to its Fourier series. Thus,

$$\begin{aligned} \mathbb{E}[\tau(x)\tau(y)] &= \sum_{p, q \in \mathbb{Z}} \mathbb{E}[c_p(\tau)c_q(\tau)] e^{2i\pi(px+qy)} \\ &= \sum_{p \in \mathbb{Z}} \left( \mathbb{E}[|c_p|^2] e^{2i\pi p(x-y)} + \mathbb{E}[c_p^2] e^{2i\pi p(x+y)} \right) \end{aligned}$$

from the independence relationships of the Fourier coefficients. Now, since the real and imaginary parts of  $c_p$  are independent and have the same variance, as we recalled at the beginning of Section 1.5,

$$\mathbb{E}[c_p^2] = \mathbb{E}[(\operatorname{Re}(c_p))^2] - \mathbb{E}[(\operatorname{Im}(c_p))^2] + 2i\mathbb{E}[(\operatorname{Re}(c_p))(\operatorname{Im}(c_p))] = 0.$$

□

In particular, in what follows, we will make extensive use of the fact that for a centered Gaussian random field and points  $x, y \in \mathbb{T}$ ,  $\tau(x)$  and  $\tau(y)$  are joint Gaussian random variable, both of variance  $\sum \mathbb{E}[|c_p|^2]$  and that they are independent if, and only if,  $\mathbb{E}[\tau(x)\tau(y)] = 0$ .

**1.6. Result.** If  $x$  is a periodic point, let us write its prime period

$$l_x := \min\{k \geq 1, E^k(x) = x\}.$$

Let us define for every  $n \in \mathbb{N}$ :

$$(1.9) \quad A_n := \left( \sum_{E^n(x)=x} \frac{l_x}{((E^n)'(x) - 1)^2} \right)^{-\frac{1}{2}}$$

THEOREM 1.5. Let  $k \in \mathbb{N}$ . Let  $\tau_0 \in \mathcal{C}^k(\mathbb{T})$ . Let

$$\delta\tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot}$$

be a centered Gaussian random field, such that  $\mathbb{E}[|c_p|^2] = O(p^{-2-\nu})$  for some  $\nu > 0$ . This way,  $\delta\tau$  is a.s.  $\mathcal{C}^0$ . If

$$(1.10) \quad \exists \epsilon > 0, \exists C > 0, \forall p \in \mathbb{Z}^*, \mathbb{E}[|c_p|^2] \geq \frac{C}{p^{2k+2+\epsilon}},$$

then one has the convergence in law of the flat traces

$$(1.11) \quad A_n \operatorname{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

as  $n$  and  $\xi$  go to infinity, under the constraint

$$(1.12) \quad \exists 0 < c < 1, \forall n, \xi, \quad n \leq c \frac{\log \xi}{\log l + (k + \frac{1}{2} + \frac{\epsilon}{2}) \log M},$$

(recall that  $l$  is the degree of the map  $E$ .) Note that condition (1.10) can allow  $\tau$  to be  $\mathcal{C}^k$  by Lemma 1.3.

REMARK 1.6. The statement implies that the convergence still holds if we multiply  $\delta\tau$  by an arbitrarily small number  $\eta > 0$ . For instance for  $\tau_0 = 0$ ,

$$A_n \operatorname{Tr}^b(\mathcal{L}_{\xi, 0}^n) \longrightarrow \infty$$

at exponential speed, uniformly in  $\xi$ , but if  $\delta\tau$  is an irregular enough Gaussian field in the sense of (1.10), then for any  $\eta > 0$  and  $c < 1$  holds

$$A_n \operatorname{Tr}^b(\mathcal{L}_{\xi, \eta \cdot \delta\tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

under condition (1.12).

REMARK 1.7. Condition (1.12) means that time  $n$  is smaller than a constant times the Ehrenfest time  $\log \xi$ , and this constant decreases with the regularity  $k$  of the field  $\delta\tau$ .

**1.7. Sketch of proof.** The proof is based around the following arguments:

- (1) Note first that the convergence (1.11) is satisfied if all the phases appearing in (1.2) are independent and uniformly distributed.

REMARK 1.8. For sake of simplicity, in this sketch of proof, we will act as if the phases of (1.2) were pairwise independent. This cannot be the case: Birkhoff sums  $\phi_x^n$  are the same for all points of a given orbit. We will have to take this into account later, but this is a small technical detail. For instance this simplification would remove the factor  $l_x$  in the definition (1.9) of  $A_n$ , that comes from this multiplicity.

The convergence can be deduced from the standard proof of the central limit theorem showing pointwise convergence of the characteristic function. However, here, since the periodic points are dense in  $\mathbb{T}$ , requiring independence of the values  $(\delta\tau(x))_{E^n(x)=x}$  would lead to very bad regularity of the field (it is not hard to see that it would be almost surely nowhere locally bounded).

- (2) We fix a Gaussian field  $\delta\tau = \sum c_p e^{2i\pi p \cdot}$  fulfilling the hypothesis of Theorem 1.5 and start by constructing an auxiliary field with the same law and show that it satisfies the convergence (1.11). This is sufficient since the convergence in law only involves the law of the random field.
- (3) For each  $j \geq 1$ , we construct a smooth random field  $\delta\tau_j$ , such that for any pair of periodic points  $x \neq y$  of period  $j$ ,  $\delta\tau_j(x)$  and  $\delta\tau_j(y)$  are independent. Since by (1.2)  $\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$  only involves points of period  $n$ , the phases appearing at time  $n$ , for the function  $\delta\tau_n$ , in  $\mathcal{L}_{\xi, \delta\tau_n}^n$  are consequently all independent random variables on  $S^1$ . If moreover  $\xi$  is large enough, the variables  $\xi(\delta\tau_n)_x^n$  are Gaussian with large variances, so  $\xi(\delta\tau_n)_x^n \bmod 2\pi$  (and therefore the phases  $e^{i\xi(\delta\tau_n)_x^n}$ ) are close to be uniform. Thus, the convergence (1.11) should hold for  $\text{Tr}^b(\mathcal{L}_{\xi, \delta\tau_n}^n)$  under a certain relation between  $n$  and  $\xi$  that will be explained in number (8).
- (4) An important point is that if the phases  $(e^{i\xi(\delta\tau_n)_x^n})_{\{x \in \mathbb{T}, E^n(x)=x\}}$  are independent and close to be uniform, then adding to  $\delta\tau_n$  an independent field will not change this fact, as the following lemma suggests:

LEMMA 1.9. *Let  $X, X'$  be real independent random variables such that  $e^{iX}, e^{iX'}$  are uniform on  $S^1$ . Let  $Y, Y'$  be real random variables such that  $X$  and  $X'$  are independent of both  $Y$  and  $Y'$ . Then  $e^{i(X+Y)}$  and  $e^{i(X'+Y')}$  are still independent uniform random variables on  $S^1$ .*

Note that no independence between  $Y$  and  $Y'$  is needed. See appendix 2.D for the proof.

- (5) Using this analogy, if the fields  $\delta\tau_j$  are chosen independent, it should follow that the convergence (1.11) holds for  $\text{Tr}^b(\mathcal{L}_{\xi, \sum_{j \geq 1} \delta\tau_j}^n)$  for large  $\xi$ .
- (6) The fields  $\delta\tau_j$  are almost surely smooth. However, because the distance between periodic points decreases not faster than  $M^{-j}$  according to Lemma 2.A.1, if we want to be sure that  $\sum_j \delta\tau_j$  is  $\mathcal{C}^k$ , and  $\mathbb{E}[\delta\tau_j(x)\delta\tau_j(y)] = 0$  for all  $x \neq y$  of period  $j$ , let us see that we need to impose an exponential decay of the standard deviation (independent of the point  $x$ ):

$$(1.13) \quad \sqrt{\mathbb{E}[\delta\tau_j(x)^2]} \approx M^{-j(k+\frac{1}{2}+\varepsilon)}$$

for some  $\varepsilon > 0$ . This can be deduced heuristically from the fact (see Definition 1.4 below) that

$$(1.14) \quad \mathbb{E}[\delta\tau_j(x)\delta\tau_j(y)] = \sum_p \mathbb{E}[|c_p(\delta\tau_j)|^2] e^{ip(x-y)} =: K_j(x-y)$$

and the uncertainty principle: a localisation of  $K_j$  at a scale  $M^{-j}$  implies non negligible coefficients  $\mathbb{E}[|c_p(\delta\tau_j)|^2]$  for  $p$  of order  $M^j$ . Let us for instance assume that the Fourier coefficients  $\mathbb{E}[|c_p(\delta\tau_j)|^2]$  of  $K_j$  write

$$(1.15) \quad \mathbb{E}[|c_p(\delta\tau_j)|^2] = \alpha_j^2 f\left(\frac{p}{M^j}\right)^2$$

for some amplitudes  $\alpha_j$  to determine and some positive Schwartz function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then, since

$$\delta\tau_j = \sum_p \sqrt{\mathbb{E}[|c_p(\delta\tau_j)|^2]} \zeta_{j,p} e^{2i\pi p \cdot}$$

for i.i.d.  $\mathcal{N}(0,1)$  random variables  $\zeta_{j,p}$ , roughly,

$$\begin{aligned} \sup |\delta\tau_j^{(k)}| &\approx \alpha_j \sum_p |p|^k f\left(\frac{p}{M^j}\right) \\ &= \alpha_j M^{j(k+1)} \frac{1}{M^j} \sum_p \frac{|p|^k}{M^{jk}} f\left(\frac{p}{M^j}\right) \\ &\sim C \alpha_j M^{j(k+1)}. \end{aligned}$$

(The second line involved a Riemann sum.) Consequently, with those approximations, choosing  $\alpha_j = M^{-j(k+1+\varepsilon)}$  gives a  $\mathcal{C}^k$  function  $\sum_{j \geq 1} \delta\tau_j$ . Then,

$$\begin{aligned} \mathbb{E}[|\delta\tau_j(x)|^2] &\stackrel{(1.14)}{=} \sum_p \mathbb{E}[|c_p(\delta\tau_j)|^2] \\ &= \sum_p \alpha_j^2 f\left(\frac{p}{M^j}\right)^2 \\ &= \alpha_j^2 M^j \frac{1}{M^j} \sum_p f\left(\frac{p}{M^j}\right)^2 \\ &\sim C \alpha_j^2 M^j = M^{-j(2k+1+2\varepsilon)} \end{aligned}$$

as announced.

- (7) This condition, together with (1.10) can easily be shown to imply that the Fourier coefficients  $\tilde{c}_p$  of  $\sum_{j \geq 1} \delta\tau_j$  satisfy

$$\mathbb{E}[|\tilde{c}_p|^2] \leq C \mathbb{E}[|c_p|^2].$$

This allows us to define a field  $\delta\tau_0$ , that we chose independent from the other  $\delta\tau_j$ , by

$$\mathbb{E}[|c_p(\delta\tau_0)|^2] = C \mathbb{E}[|c_p|^2] - \mathbb{E}[|\tilde{c}_p|^2],$$

so that  $\frac{1}{C} \sum_{j \geq 0} \delta\tau_j$  has the same law as  $\delta\tau$  and still satisfies the convergence (1.11) for  $\xi$  large enough from (4) of this sketch.

- (8) To get an idea of the origin of the relation (1.12) between  $n$  and  $\xi$ , let us assume that we want all the arguments  $\xi(\delta\tau_n)_x$  in  $\text{Tr}^b(\mathcal{L}_{\xi, \delta\tau_n}^n)$  to go uniformly to infinity in order to get approximate uniformity of the phases and thus convergence towards a Gaussian law. Note that for any  $x$ ,

$$(1.16) \quad \mathbb{P} \left[ \frac{|(\delta\tau_n)_x|^n}{\sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]}} \leq \epsilon \right] \underset{\epsilon \rightarrow 0}{=} O(\epsilon).$$

Let  $(C_n)$  be a sequence going to infinity.

(1.16) implies

$$\begin{aligned} \mathbb{P} \left[ \bigcap_{E^n(x)=x} \{ \xi(\delta\tau_n)_x^n > C_n \} \right] &= 1 - \mathbb{P} [\exists x, E^n(x) = x, \xi(\delta\tau_n)_x^n \leq C_n] \\ &\geq 1 - \sum_{E^n(x)=x} \mathbb{P} [\xi(\delta\tau_n)_x^n \leq C_n] \\ &\underset{\text{Lemma 2.A.1}}{=} 1 - (l^n - 1) \mathbb{P} [\xi(\delta\tau_n)_x^n \leq C_n] \\ &\underset{(1.16)}{\geq} 1 - Cl^n \frac{C_n}{\xi \sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]}} \end{aligned}$$

if  $x$  denotes any point and  $\xi \gg \frac{C_n}{\sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]}}$ . By independence

$$\begin{aligned} \sqrt{\mathbb{E}[|(\delta\tau_n)_x|^2]} &= \left( \sum_{k=0}^{n-1} \mathbb{E}[|\delta\tau_n(E^k(x))|^2] \right)^{\frac{1}{2}} \\ &\underset{(1.13)}{\approx} \sqrt{n} M^{n(k+\frac{1}{2}+\varepsilon)} \end{aligned}$$

for some  $\varepsilon > 0$ . Thus

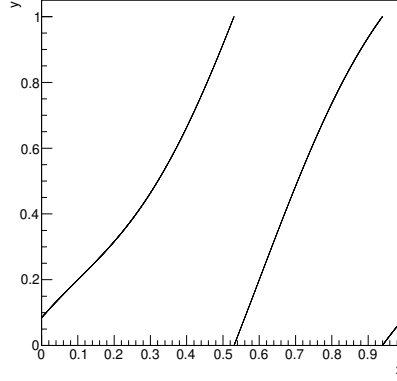
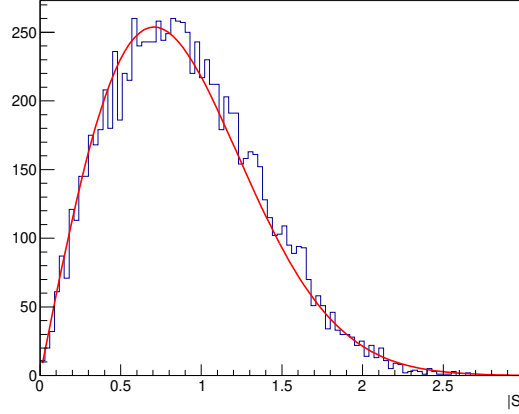
$$\begin{aligned} \mathbb{P} [\xi(\delta\tau_n)_x^n \rightarrow \infty \text{ uniformly w.r.t. } x \text{ s.t. } E^n(x) = x] &\rightarrow 1 \\ \text{for } \xi \gg l^n M^{n(k+\frac{1}{2}+\varepsilon)}, &\text{ which gives (1.12).} \end{aligned}$$

## 2. Numerical experiments

We consider an example with the non linear expanding map

$$(2.1) \quad E(x) = 2x + 0.9/(2\pi) \sin(2\pi(x + 0.4))$$

plotted on Figure 1. In Figure 2, we have the histogram of the modulus  $S = \left| A_n \text{Tr}^b \left( \mathcal{L}_{\xi, \tau_0 + \delta\tau}^n \right) \right|$  obtained after a sample of  $10^4$  random functions  $\delta\tau$ . We compare the histogram with the function  $CS \exp(-S^2)$  in red, i.e. the radial distribution of a Gaussian function, obtained from the prediction of Theorem 1.5. We took  $n = 11$ ,  $\xi = 2.10^6$ ,  $\tau_0 = \cos(2\pi x)$ . We also observe a good agreement for the (uniform) distribution of the arguments that is not represented here.

FIGURE 1. Graph of the expanding map  $E(x)$  in Eq.(2.1)FIGURE 2. In blue, the histogram of  $S = \left| A_n \text{Tr}^b \left( \mathcal{L}_{\xi, \tau_0 + \delta\tau}^n \right) \right|$  for  $n = 11$ ,  $\xi = 2.10^6$ ,  $\tau_0 = \cos(2\pi x)$  and the sample  $10^4$  random functions  $\delta\tau$ . The histogram is well fitted by  $C S \exp(-S^2)$  in red, as predicted by Theorem 1.5

### 3. Proof of Theorem 1.5

**3.1. Definition of a Gaussian field satisfying Theorem 1.5.** Let us fix a random centered Gaussian field  $\delta\tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot}$  satisfying the hypothesis of Theorem 1.5. Let us define the Gaussian fields mentioned in step (3) of the sketch of proof. Let  $K_{\text{init}} \in \mathcal{C}_c^\infty(\mathbb{R})$  be a smooth function supported in  $[-\frac{1}{3}, \frac{1}{3}]$ , with non negative Fourier transform, satisfying <sup>1</sup>

$$(3.1) \quad K_{\text{init}}(0) = 1.$$

<sup>1</sup>To construct such a function, take a non zero even function  $g \in \mathcal{C}_c^\infty(\mathbb{R})$ .  $g$  has a real Fourier transform. Then  $g * g \in \mathcal{C}_c^\infty(\mathbb{R})$  and its Fourier transform is  $\hat{g}^2 \geq 0$ . Moreover  $g * g(0) = \int \hat{g}^2 > 0$ .



Let  $k \geq 0$  be the integer involved in Theorem 1.5 giving the regularity of the field. Let  $\epsilon > 0$  be the constant appearing in Theorem 1.5 and define for any integer  $j \geq 1$

$$(3.2) \quad K_j(x) = \frac{1}{M^{j(2k+1+\epsilon)}} K_{\text{init}}(M^j x).$$

The Fourier transform of  $K_j$  is given by

$$(3.3) \quad \widehat{K_j}(\xi) = \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\text{init}}} \left( \frac{\xi}{M^j} \right) \geq 0$$

The functions  $K_j$ , for all  $j \geq 1$ , are supported in  $[-\frac{1}{3}, \frac{1}{3}]$  and can then be seen as functions on the circle  $\mathbb{T}$  by trivially periodizing them. Let  $c_{p,j}$ , for  $p \geq 0, j \geq 1$  be independent centered Gaussian random variables of respective variances  $\hat{K}_j(2\pi p)$ , and let us write

$$\delta\tau_j = \sum_p c_{p,j} e^{2i\pi p},$$

where  $c_{-p,j} := \overline{c_{p,j}}, p \geq 1$ . Note that, since  $K_j$  is smooth for all  $j$ , the variances  $\hat{K}_j(2\pi p)$  of  $c_{p,j}$  decay rapidly with  $p$  (for fixed  $j$ ), and therefore, each  $\delta\tau_j$  is almost surely smooth by Lemma 2.B.1.

LEMMA 3.1.  $\sum_{j \geq 1} \delta\tau_j$  is a centered Gaussian random field  $\sum \tilde{c}_p e^{2i\pi p}$  and

$$\mathbb{E} [|\tilde{c}_p|^2] = O(\mathbb{E} [|c_p|^2]).$$

PROOF. We have seen in Eq.(3.3) that

$$\widehat{K_j}(\xi) = \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\text{init}}} \left( \frac{\xi}{M^j} \right).$$

Since  $K_{\text{init}}$  is smooth, there exists a constant  $C > 0$  such that

$$\forall \xi \in \mathbb{R}, \widehat{K_{\text{init}}}(\xi) \leq \frac{C}{\langle \xi \rangle^{2k+2+\frac{\epsilon}{2}}},$$

with the usual notation  $\langle \xi \rangle = \sqrt{1 + \xi^2} \geq |\xi|$ . Thus,

$$\begin{aligned} \mathbb{E} [|c_{p,j}|^2] &= \frac{1}{M^{j(2k+2+\epsilon)}} \widehat{K_{\text{init}}} \left( \frac{2\pi p}{M^j} \right) \\ &\leq \frac{C}{M^{j\frac{\epsilon}{2}}} \frac{1}{|2\pi p|^{2k+2+\frac{\epsilon}{2}}}. \end{aligned}$$

Consequently, since by independence

$$\begin{aligned} \mathbb{E} [|\tilde{c}_p|^2] &= \mathbb{E} \left[ \left| \sum_{j \geq 1} c_{p,j} \right|^2 \right] = \sum_{j \geq 1} \mathbb{E} [|c_{p,j}|^2], \\ \mathbb{E} [|\tilde{c}_p|^2] &= O \left( \frac{1}{|2\pi p|^{2k+2+\frac{\epsilon}{2}}} \right) \stackrel{(1.10)}{=} O(\mathbb{E} [|c_p|^2]). \end{aligned}$$

□

Thus, fixing a constant  $C$  such that

$$C \mathbb{E} [|c_p|^2] \geq \mathbb{E} [|\tilde{c}_p|^2],$$

we can define a random Gaussian field  $\delta\tau_0 = \sum_{p \in \mathbb{Z}} c_{p,0} e^{2i\pi p \cdot}$  with coefficients  $c_{p,0}$  independent from the  $c_{p,j}$  such that

$$\mathbb{E}[|c_{p,0}|^2] = C \mathbb{E}[|c_p|^2] - \mathbb{E}[|\tilde{c}_p|^2].$$

This way  $\frac{1}{C} \sum_{j \geq 0} \delta\tau_j$  and  $\delta\tau$  have the same law. By this we mean that their Fourier coefficients have the same laws. By our hypothesis, the convergence of the Fourier series are almost surely normal, thus for any finite subset  $\{x_k\}_k$  of  $\mathbb{T}$ ,  $(\frac{1}{C} \sum \delta\tau_j(x_k))_k$  and  $(\delta\tau(x_k))_k$  have the same law. Therefore, the laws of  $\text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \delta\tau}^n)$  and  $\text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \frac{1}{C} \sum \delta\tau_j}^n)$  are the same, and the convergence of Theorem 1.5 is equivalent to

$$(3.4) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau_0 + \sum \delta\tau_j}^n) \xrightarrow{\mathcal{L}} \mathcal{N}_{\mathbb{C}}(0, 1)$$

under condition (1.10). (The constant  $\frac{1}{C}$  can be 'absorbed' in  $\xi$  up to the replacement of  $\tau_0$  by  $C\tau_0$  that has no consequence.) In the rest of the paper we will show (3.4) and will write

$$(3.5) \quad \tau := \tau_0 + \sum_{j \geq 0} \delta\tau_j.$$

**3.2. New expression for  $\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$ .** We will write the set of periodic orbits of (non primitive) period  $n$  as

$$(3.6) \quad \text{Per}(n) := \{ \{x, E(x), \dots, E^{n-1}(x)\}, E^n(x) = x, x \in \mathbb{T} \},$$

and the set of periodic orbits of primitive period  $n$  as

$$(3.7) \quad \mathcal{P}_n := \{ \{x, E(x), \dots, E^{n-1}(x)\}, n = \min\{k \in \mathbb{N}^*, E^k(x) = x\}, x \in \mathbb{T} \}.$$

This way,  $\text{Per}(n)$  is the disjoint union

$$(3.8) \quad \text{Per}(n) = \coprod_{m|n} \mathcal{P}_m.$$

Let us rewrite the sum  $\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$ , where  $\tau$  is given by (3.5). We know from (3.D.7) that

$$\begin{aligned} \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) &= \sum_{E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{(E^n)'(x) - 1} \\ &= \sum_{E^n(x)=x} \frac{e^{i\xi\tau_x^n}}{e^{J_x^n} - 1}, \end{aligned}$$

where  $J(x) = \log(E'(x)) > 0$  and  $J_x^n$  is the Birkhoff sum as defined in (1.3). If  $f_O^n$  stands for the Birkhoff sum  $f_x^n$  for any  $x \in O$ , let us write

$$(3.9) \quad \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) = \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi\tau_O^n}}{e^{J_O^n} - 1}.$$

For  $O \in \text{Per}(n)$ , we can write

$$\tau_O^n = (\delta\tau_n)_O^n + \sum_{j \neq n} (\delta\tau_j)_O^n + (\tau_0)_O^n.$$

Since the covariance function  $K_n$  is supported in  $[-\frac{1}{3M^n}, \frac{1}{3M^n}]$ , we deduce from Lemma 2.A.1 and (1.8) that the values taken by  $\delta\tau_n$  at different periodic points of

period dividing  $n$ , which have law  $\mathcal{N}(0, K_n(0))$  are independent random variables (recall from the discussion of Section 1.5 that the values taken by  $\delta\tau_n$  at two points  $x$  and  $y$  are Gaussian random variables of variance  $K_n(0) = \sum \mathbb{E}[|c_p(\delta\tau_n)|^2]$ , of covariance  $\mathbb{E}[\tau(x)\tau(y)] = K_n(x-y)$ , and that they are independent if, and only if  $\mathbb{E}[\tau(x)\tau(y)] = K_n(x-y) = 0$ .)

Thus, for  $n \in \mathbb{N}$ ,  $m|n$  and  $O \in \mathcal{P}_m$ ,  $(\delta\tau_n)_O^m$  is a centered Gaussian random variable of variance  $mK_n(0)$ , and  $(\delta\tau_n)_O^n = \frac{n}{m}(\delta\tau_n)_O^m$  has variance  $(\frac{n}{m})^2 mK_n(0) = \frac{n^2}{m} K_n(0)$ .

DEFINITION 3.2. We say that two families of real random variables  $(X_O^n)_{n \geq 1, O \in \text{Per}(n)}$  and  $(Y_O^n)_{n \geq 1, O \in \text{Per}(n)}$  satisfy condition (C) if

- (1) for every  $m|n$ , and  $O \in \mathcal{P}_m$ ,  $X_O^n$  has law  $\mathcal{N}(0, \frac{n^2}{m} K_n(0))$ ,
- (2) for every  $O' \neq O \in \text{Per}(n)$  and every  $O'' \in \text{Per}(n)$ ,  $X_O^n$  is independent of  $X_{O'}^n$  and  $Y_{O''}^n$ .

Writing  $X_O^n = (\delta\tau_n)_O^n$  and  $Y_O^n = \sum_{j \neq n} (\delta\tau_j)_O^n + (\tau_0)_O^n$ , we have obtained

LEMMA 3.3. *There exist families of random variables  $(X_O^n), (Y_O^n)$  satisfying condition (C) of Definition (3.2) such that for every  $n \geq 1$  and  $O \in \text{Per}(n)$*

$$(3.10) \quad \tau_O^n = X_O^n + Y_O^n$$

In order to adapt the proof of lemma 1.9, we want to show that for large  $\xi$ , the random variables  $e^{i\xi(X_O^n + Y_O^n)}$ ,  $O \in \text{Per}(n)$  are close to be independent and uniform on  $S^1$ .

REMARK 3.4. We have

$$(3.11) \quad \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \stackrel{(3.9), (3.10)}{=} \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi(X_O^n + Y_O^n)}}{e^{J_O^n} - 1}.$$

Our aim is to approximate the characteristic function of  $A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$  which is the expectation of

$$(3.12) \quad \exp \left( i A_n \left( \mu \text{Re}(\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)) + \nu \text{Im}(\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)) \right) \right) = \prod_{m|n} \prod_{O \in \mathcal{P}_m} \exp \left[ i \frac{m A_n}{e^{J_O^n} - 1} (\mu \cos(\xi(X_O^n + Y_O^n)) + \nu \sin(\xi(X_O^n + Y_O^n))) \right]$$

for fixed,  $\mu, \nu \in \mathbb{R}$ . The right hand side of (3.12) can be written as

$$\prod_{m|n} \left( \prod_{O \in \mathcal{P}_m} f_O \left( e^{i\xi(X_O^n + Y_O^n)} \right) \right)^m$$

for some continuous functions  $f_O : S^1 \rightarrow \mathbb{C}$  (depending on  $\mu, \nu$ ):

$$f_O(z) = \exp \left[ i \frac{A_n}{e^{J_O^n} - 1} (\mu \text{Re}(z) + \nu \text{Im}(z)) \right]$$

In the next Lemma we first consider indicator functions on  $S^1$  for  $f_O$ .

LEMMA 3.5. *Let  $(X_O^n)_{n \geq 1, O \in \text{Per}(n)}$  and  $(Y_O^n)_{n \geq 1, O \in \text{Per}(n)}$  be two families of real random variables satisfying satisfying condition (C) of Definition (3.2). Assume that  $n$  and*

$\xi$  satisfy (1.12). Then there is a constant  $C > 0$  such that for every  $n \in \mathbb{N}$  and every real numbers  $(\alpha_O)_{O \in \text{Per}(n)}, (\beta_O)_{O \in \text{Per}(n)}$  such that

$$\forall O \in \text{Per}(n), 0 < \beta_O - \alpha_O < 2\pi,$$

for every complex numbers  $(\lambda_O)_{O \in \text{Per}(n)}$ , if  $A_O := e^{i[\alpha_O, \beta_O[} \subset S^1 \subset \mathbb{C}$  and  $\mathbb{1}_{A_O} : S^1 \rightarrow \mathbb{C}$  is the characteristic function of  $A_O$ , we have

$$(3.13) \quad \left| \frac{\mathbb{E} \left[ \prod_{m|n} \left( \prod_{O \in \mathcal{P}_m} \lambda_O \mathbb{1}_{A_O} (e^{i\xi(X_O^n + Y_O^n)}) \right)^m \right]}{\prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_O^m \left( \frac{\beta_O - \alpha_O}{2\pi} \right)} - 1 \right| \leq \xi^{c-1} n^{-\frac{1}{2}} \left( \xrightarrow{(1.12)} 0 \right).$$

REMARK 3.6. In this expression, we compare the law of the family of random variables  $(e^{i\xi(X_O^n + Y_O^n)})_{O \in \text{Per}(n)}$  to the uniform law on the torus of dimension  $\#\text{Per}(n)$ . The proof of this lemma is given in the next subsection.

### 3.3. A normal law of large variance on the circle is close to uniform.

We will need the following lemma, which evaluates how much the law  $\mathcal{N}(0, 1) \bmod \frac{1}{t}$  differs from the uniform law on the circle  $\mathbb{R}/(\frac{1}{t}\mathbb{Z})$  for large values of  $t$ .

LEMMA 3.7. *There exists a constant  $C > 0$  such that for every real numbers  $\alpha, \beta$ , such that  $0 < \beta - \alpha < 2\pi$  and every real number  $t \geq 1$ ,*

$$\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbb{1}_{\frac{\alpha + 2k\pi}{t} \leq x \leq \frac{\beta + 2k\pi}{t}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \frac{\beta - \alpha}{2\pi} \right| \leq \frac{C}{t} (\beta - \alpha).$$

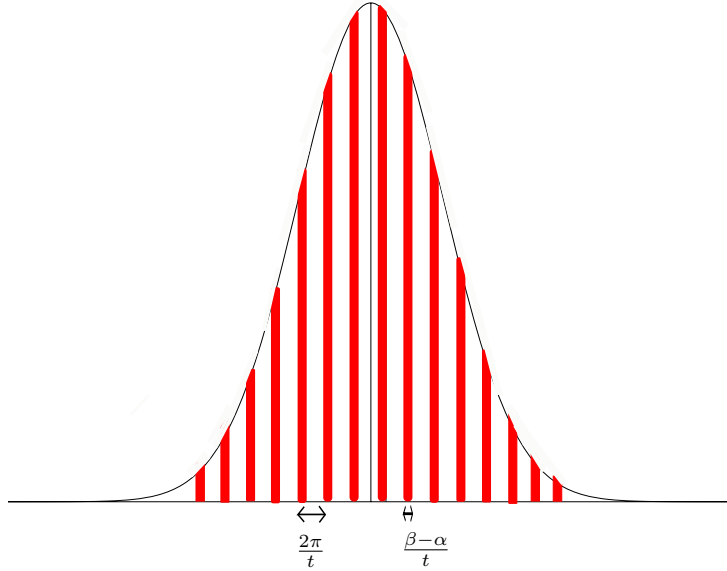


FIGURE 3. As  $t$  goes to infinity, the red area converges to  $\frac{\beta - \alpha}{2\pi}$  with speed  $O(\frac{\beta - \alpha}{t})$ .

PROOF. By mean value inequality, if  $|x - y| \leq 1$ , then

$$\left| e^{-\frac{x^2}{2}} - e^{-\frac{y^2}{2}} \right| \leq |x - y| f(y)$$

for the  $L^1$  function

$$f(y) := \sup_{|u-y| \leq 1} |u| e^{-\frac{u^2}{2}}$$

Let us then write for  $u \in [\frac{\alpha}{t}, \frac{\beta}{t}]$ ,  $u_k := u + \frac{2k\pi}{t}$  and  $I_k := [u_k, u_{k+1}]$ . We have just seen that for  $t \geq 2\pi$ , for all  $y \in I_k$ ,

$$\left| e^{-\frac{u_k^2}{2}} - e^{-\frac{y^2}{2}} \right| \leq \frac{C}{t} f(y).$$

Integrating over  $y \in I_k$  of length  $\frac{2\pi}{t}$  and summing over  $k \in \mathbb{Z}$  yields

$$\left| \frac{2\pi}{t} \sum_{k \in \mathbb{Z}} e^{-\frac{u_k^2}{2}} - \sqrt{2\pi} \right| \leq \frac{C}{t}$$

(The value of the constant  $C$  changes at each line, but it depends neither on  $t$ , nor on  $\alpha, \beta$ .) Averaging over  $u \in [\frac{\alpha}{t}, \frac{\beta}{t}]$  gives

$$\left| \frac{2\pi}{\beta - \alpha} \sum_{k \in \mathbb{Z}} \int_{\frac{\alpha}{t}}^{\frac{\beta}{t}} \exp\left(-\frac{(u - 2k\pi)^2}{2}\right) du - \sqrt{2\pi} \right| \leq \frac{C}{t}.$$

Consequently,

$$\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbb{1}_{\frac{\alpha+2k\pi}{t} \leq x \leq \frac{\beta+2k\pi}{t}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \frac{\beta - \alpha}{2\pi} \right| \leq \frac{C}{t}(\beta - \alpha)$$

□

PROOF OF LEMMA 3.5. Let us denote by  $E$  the expectation

$$E := \mathbb{E} \left[ \prod_{m|n} \left( \prod_{O \in \mathcal{P}_m} \lambda_O \mathbb{1}_{A_O} \left( e^{i\xi(X_O^n + Y_O^n)} \right) \right)^m \right].$$

If we write respectively  $\mathbb{P}_X$ ,  $\mathbb{P}_Y$  and  $\mathbb{P}_{X,Y}$  the probability laws of the variables  $(\xi X_O^n)_{O \in \text{Per}(n)}$ ,  $(\xi Y_O^n)_{O \in \text{Per}(n)}$  and  $(\xi X_O^n)_{O \in \text{Per}(n)} \cup (\xi Y_O^n)_{O \in \text{Per}(n)}$  respectively, then condition (C) of Definition (3.2) implies

$$(3.14) \quad d\mathbb{P}_{X,Y}((x_O)_{O \in \text{Per}(n)}, (y_O)_{O \in \text{Per}(n)}) = \prod_{m|n} \prod_{O \in \mathcal{P}_m} e^{-\frac{x_O^2}{2\sigma_{n,\xi}^2}} \frac{dx_O}{\sigma_{n,\xi} \sqrt{2\pi}} \otimes d\mathbb{P}_Y((y_O)_{O \in \text{Per}(n)}).$$

with the variance  $\sigma_{n,\xi}^2 := \xi^2 \frac{n^2}{m} K_n(0)$ . We have

$$(3.15) \quad E = \int_{\mathbb{R}^{2\#\text{Per}(n)}} \prod_{O \in \text{Per}(n)} \left( \sum_{k \in \mathbb{Z}} \lambda_O^m \mathbb{1}_{\alpha_O + 2k\pi, \beta_O + 2k\pi}((x_O + y_O)) \right) d\mathbb{P}_{X,Y}((x_O)_{O \in \text{Per}(n)}, (y_O)_{O \in \text{Per}(n)}).$$

Thus, writing  $u_O = \frac{x_O}{\sigma_{n,\xi}}$  for  $O \in \mathcal{P}_m$ ,

(3.16)

$$E = \int_{\mathbb{R}^{\#\text{Per}(n)}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} \left( \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \lambda_O^m \mathbb{1} \right]_{\frac{\alpha_O - y_O + 2k\pi}{\sigma_{n,\xi}}, \frac{\beta_O - y_O + 2k\pi}{\sigma_{n,\xi}}} \left[ (u_O) e^{-\frac{u_O^2}{2}} \frac{du_O}{\sqrt{2\pi}} \right] d\mathbb{P}_Y((y_O)_{O \in \text{Per}(n)}).$$

Let us write for  $O \in \text{Per}(n)$

$$I_O = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \lambda_O^m \mathbb{1} \right]_{\frac{\alpha_O - y_O + 2k\pi}{\sigma_{n,\xi}}, \frac{\beta_O - y_O + 2k\pi}{\sigma_{n,\xi}}} \left[ (u_O) e^{-\frac{u_O^2}{2}} \frac{du_O}{\sqrt{2\pi}} \right].$$

Lemma 2.73 yields

$$I_O = \lambda_O^m \frac{\beta_O - \alpha_O}{2\pi} (1 + \epsilon_O),$$

where

$$\begin{aligned} \exists C > 0, |\epsilon_O| &\leq \frac{C}{\sigma_{n,\xi}} \\ &\leq \frac{C}{\xi \sqrt{n K_n(0)}}. \end{aligned}$$

Let us remark that for every finite family  $\{x_k\}_k \subset \mathbb{R}$ , the expansion of the product and factorization after triangular inequality give

$$\left| \prod_k (1 + x_k) - 1 \right| \leq \prod_k (1 + |x_k|) - 1.$$

Thus,

$$\begin{aligned} \left| \frac{\prod_{m|n} \prod_{O \in \mathcal{P}_m} I_O}{\prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_O^m \left( \frac{\beta_O - \alpha_O}{2\pi} \right)} - 1 \right| &= \left| \prod_{m|n} \prod_{O \in \mathcal{P}_m} (1 + \epsilon_O) - 1 \right| \\ &\leq \left( \left( 1 + \frac{C}{\xi (n K_n(0))^{1/2}} \right)^{\#\text{Per}(n)} - 1 \right). \end{aligned}$$

From Lemma 2.A.1 we have  $\#\text{Per}(n) \leq l^n$ .

Using hypothesis (1.12) we can bound the prefactor:

$$\begin{aligned} \left( 1 + \frac{C}{\xi (n K_n(0))^{1/2}} \right)^{l^n} - 1 &\stackrel{(3.2), (3.1)}{=} \left( 1 + \frac{C M^{n(k + \frac{1}{2} + \frac{\epsilon}{2})}}{\xi \sqrt{n}} \right)^{l^n} - 1 \\ &\leq \exp(l^n \frac{C M^{n(k + \frac{1}{2} + \frac{\epsilon}{2})}}{\xi \sqrt{n}}) - 1 \\ &\leq C' \frac{l^n M^{n(k + \frac{1}{2} + \frac{\epsilon}{2})}}{\xi \sqrt{n}} \end{aligned}$$

for some  $C' > 0$  for  $n$  and  $\xi$  large enough and satisfying (1.12) since

$$(3.17) \quad l^n \frac{C M^{n(k + \frac{1}{2} + \frac{\epsilon}{2})}}{\xi \sqrt{n}} \stackrel{(1.12)}{\leq} C \xi^{c-1} n^{-\frac{1}{2}} \rightarrow 0.$$

□

**3.4. End of proof.** We can now easily extend the lemma 3.5 from characteristic functions to step functions.

COROLLARY 3.8. *Assume that  $n$  and  $\xi$  satisfy (1.12). For any families  $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  and  $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  of real random variables satisfying condition (C) of Definition (3.2), there exists  $C > 0$  such that, if  $(f_{n,O})_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  is a family of step functions  $S^1 \rightarrow \mathbb{R}$ , then*

$$(3.18) \quad \left| \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} f_{n,O}^m(e^{i\xi(X_O^n + Y_O^n)}) \right] - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right| \leq C \xi^{c-1} n^{-\frac{1}{2}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int |f_{n,O}^m| d\text{Leb}.$$

PROOF. Let us write each  $f_{n,O}$  as

$$f_{n,O} = \sum_{q=1}^{p_{n,O}} \lambda_{n,O,q} \mathbb{1}_{A_{n,O,q}},$$

where the  $\lambda_{n,O,q}$  are complex numbers and the  $A_{n,O,q}$ ,  $1 \leq q \leq p_{n,O}$  are disjoint intervals. We develop (3.18), we use Lemma 3.5 and factorize the result:

$$\begin{aligned} E &:= \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} f_{n,O}^m(e^{i\xi(X_O^n + Y_O^n)}) \right] \\ &= \sum_{(q_O) \in \prod_{m|n} \prod_{O \in \mathcal{P}_m} \{1, \dots, p_{n,O}\}} \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O}^m \mathbb{1}_{A_{n,O,q_O}}(e^{i\xi(X_O^n + Y_O^n)}) \right]. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left| E - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right| \\ &\leq \sum_{(q_O) \in \prod_{m|n} \prod_{O \in \mathcal{P}_m} \{1, \dots, p_{n,O}\}} \left| \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O}^m \mathbb{1}_{A_{n,O,q_O}}(e^{i\xi(X_O^n + Y_O^n)}) \right] - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O}^m \text{Leb}(A_{n,O,q_O}) \right| \\ &\leq C \xi^{c-1} n^{-\frac{1}{2}} \sum_{(q_O) \in \prod_{m|n} \prod_{O \in \mathcal{P}_m} \{1, \dots, p_{n,O}\}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} |\lambda_{n,O,q_O}|^m \text{Leb}(A_{n,O,q_O}) \end{aligned}$$

from the previous lemma.

Hence,

$$\left| E - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right| \leq C \xi^{c-1} n^{-\frac{1}{2}} \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int |f_{n,O}|^m d\text{Leb}.$$

□

We can use this result in order to estimate the characteristic function of  $\text{Tr}^b(\mathcal{L}_{\xi,\tau}^n)$ , using remark (3.4).

**COROLLARY 3.9.** *Assume that  $n$  and  $\xi$  satisfy (1.12). Let  $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  and  $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  be two families of real random variables satisfying condition (C) of Definition (3.2). There exists  $C > 0$  such that for all  $(\mu_O, \nu_O)_{O \in \text{Per}(n)} \in \mathbb{R}^{2\#\text{Per}(n)}$ ,*

$$\left| \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} e^{im\mu_O \cos(\xi(X_O^n + Y_O^n)) + im\nu_O \sin(\xi(X_O^n + Y_O^n))} \right] - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i(m\mu_O \cos \theta + m\nu_O \sin \theta)} \frac{d\theta}{2\pi} \right| \leq C \xi^{c-1} n^{-\frac{1}{2}}.$$

**PROOF.** Let  $C$  be the constant from corollary 3.8. For  $O \in \text{Per}(n)$ , let  $f_O$  be the function defined on  $S^1$  by

$$f_O(e^{i\theta}) = e^{i(\mu_O \cos \theta + \nu_O \sin \theta)}.$$

Each  $f_O$  is bounded by 1, we can consequently find for each  $O \in \text{Per}(n)$  a family  $(f_{j,O})_j$  of step functions uniformly bounded by 1 converging pointwise towards  $f_O$ . We have for  $n$  fixed, by dominated convergence

$$E_j := \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} f_{j,O}^m(e^{i\xi(X_O^n + Y_O^n)}) \right] \xrightarrow{j \rightarrow \infty} E := \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \mathcal{P}_m} f_O^m(e^{i\xi(X_O^n + Y_O^n)}) \right]$$

as well as

$$I_j := \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} f_{j,O}^m(e^{i\theta}) \frac{d\theta}{2\pi} \xrightarrow{j \rightarrow \infty} I := \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} f_O^m(e^{i\theta}) \frac{d\theta}{2\pi}.$$

It is thus possible to find an integer  $j_0$  such that both

$$|E - E_{j_0}| \leq \xi^{c-1} n^{-\frac{1}{2}}$$

and

$$|I - I_{j_0}| \leq \xi^{c-1} n^{-\frac{1}{2}}$$

hold.

From corollary 3.8, we know that for all  $n \in \mathbb{N}$

$$|E_{j_0} - I_{j_0}| \leq C \xi^{c-1} n^{-\frac{1}{2}} \sup |f_{j_0}|.$$

Thus,

$$\begin{aligned} |E - I| &\leq |E - E_{j_0}| + |E_{j_0} - I_{j_0}| + |I - I_{j_0}| \\ &\leq (C + 2) \xi^{c-1} n^{-\frac{1}{2}}. \end{aligned}$$



□

We can now prove the final proposition :

PROPOSITION 3.10. *Let  $(X_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  and  $(Y_O^n)_{\substack{n \geq 1 \\ O \in \text{Per}(n)}}$  be two families of real random variables satisfying condition (C) of Definition (3.2). If condition (1.12) is satisfied then we have the following convergence in law*

$$(3.19) \quad T_{n,\xi} := A_n \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi(X_O^n + Y_O^n)}}{e^{J_O^n} - 1} \xrightarrow{n, \xi \rightarrow \infty} \mathcal{N}_{\mathbb{C}}(0, 1),$$

with the amplitude  $A_n$  defined in (1.9) by

$$(3.20) \quad A_n = \left( \sum_{m|n} m^2 \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}}.$$

PROOF. Let us fix two real numbers  $\xi_1$  and  $\xi_2$  and let  $\phi_n$  be the characteristic function of  $T_{n,\xi}$ :

$$\phi_n(\xi_1, \xi_2) := \mathbb{E} \left[ \exp \left( i A_n \left( \xi_1 \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{\cos(\xi(X_O^n + Y_O^n))}{e^{J_O^n} - 1} + \xi_2 \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{\sin(\xi(X_O^n + Y_O^n))}{e^{J_O^n} - 1} \right) \right) \right].$$

We compute the limit of  $\phi_n(\xi_1, \xi_2)$  as  $n$  goes to infinity. Corollary (3.9) yields

$$(3.21) \quad \left| \phi_n(\xi_1, \xi_2) - \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i \frac{mA_n}{e^{J_O^n} - 1} (\xi_1 \cos \theta + \xi_2 \sin \theta)} \frac{d\theta}{2\pi} \right| \leq C \xi^{c-1} n^{-\frac{1}{2}} \rightarrow 0$$

under the assumption (1.12).

Let

$$\psi(\xi_1, \xi_2) := \int_0^{2\pi} e^{i(\xi_1 \cos \theta + \xi_2 \sin \theta)} \frac{d\theta}{2\pi}.$$

We have the following Taylor's expansion in 0:

$$\psi(\xi_1, \xi_2) = 1 - \frac{1}{4}(\xi_1^2 + \xi_2^2) + o(\xi_1^2 + \xi_2^2).$$

In order to apply this to equation (3.21), we need to check that

LEMMA 3.11.

$$n A_n \sup_{O \in \text{Per}(n)} \frac{1}{e^{J_O^n} - 1} \xrightarrow{n \rightarrow \infty} 0.$$

PROOF. See appendix 2.E.3

□

We can now state that

$$\begin{aligned}
& \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i \frac{mA_n}{e^{J_O}-1} (\xi_1 \cos \theta + \xi_2 \sin \theta)} \frac{d\theta}{2\pi} = \prod_{m|n} \prod_{O \in \mathcal{P}_m} \psi \left( \xi_1 \frac{mA_n}{e^{J_O}-1}, \xi_2 \frac{mA_n}{e^{J_O}-1} \right) \\
& = \prod_{m|n} \prod_{O \in \mathcal{P}_m} \left( 1 - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^{J_O}-1)^2} + o \left( \frac{(mA_n)^2}{(e^{J_O}-1)^2} \right) \right) \\
& = \exp \left( \sum_{m|n} \sum_{O \in \mathcal{P}_m} \log \left( 1 - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^{J_O}-1)^2} + o \left( \frac{(mA_n)^2}{(e^{J_O}-1)^2} \right) \right) \right) \\
& = \exp \left( \sum_{m|n} \sum_{O \in \mathcal{P}_m} -\frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^{J_O}-1)^2} + o \left( \frac{(mA_n)^2}{(e^{J_O}-1)^2} \right) \right) \\
& \stackrel{(3.20)}{=} e^{-\frac{\xi_1^2 + \xi_2^2}{4} + o(1)},
\end{aligned}$$

We deduce that

$$\phi_n(\xi_1, \xi_2) \xrightarrow[n \rightarrow \infty]{} e^{-\frac{\xi_1^2 + \xi_2^2}{4}}$$

which is the characteristic function of a Gaussian variable of law  $\mathcal{N}_{\mathbb{C}}(0, 1)$ .  $\square$

#### 4. Discussion

In this paper we have considered a model where the roof function  $\tau$  is random. However, the numerical experiments suggest a far stronger result: for a fixed function  $\tau_0$  and a semiclassical parameter  $\xi$  chosen according to a uniform random distribution in a small window at high frequencies, the result seems to remain true, as shown in the following figures for  $\tau_0(x) = \sin(2\pi x)$ . The moduli also seem to become uniform.

It would be interesting to understand what informations about the Ruelle resonances can be recovered from the convergence (1.11). We know from the Weyl law from [AFW17] established in a similar context that the number of resonances of  $\mathcal{L}_{\xi, \tau}$  outside the essential spectral radius, for a given  $\tau$ , are of order  $O(\xi)$ . A complete characterization would thus require a knowledge of the traces of  $\mathcal{L}_{\xi, \tau}^n$  up to times of order  $O(\xi)$ , while we only have information for  $n = O(\log \xi)$ .

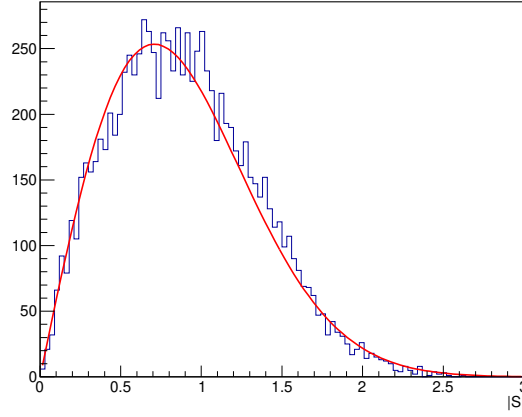


FIGURE 4. Histogram of  $S = \left| A_n \text{Tr}^b \left( \mathcal{L}_{\tau_0, \xi}^n \right) \right|$  for  $\tau_0 = \sin(2\pi \cdot)$ , and a sample of  $10^4$  random values of  $\xi$  uniformly distributed in  $[\xi_0, \xi_0 + 10]$  with  $\xi_0 = 2 \cdot 10^6$  and  $n = 11$  corresponding to a fraction of the Ehrenfest time  $C_e := n \frac{\log 2}{\log \xi_0} = 0.5$ . It is well fitted by the red curve  $S \mapsto C S \exp(-S^2)$ .

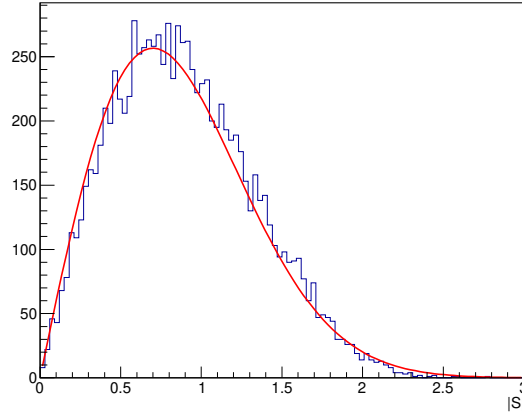


FIGURE 5. Histogram of  $S = \left| A_n \text{Tr}^b \left( \mathcal{L}_{\tau_0, \xi}^n \right) \right|$  for a sample of  $10^4$  random values of  $\xi$  uniformly distributed in  $[\xi_0, \xi_0 + 10]$  with  $\xi_0 = 2000$  and  $n = 11$  giving  $C_e = 1.0$  (that is, at the Ehrenfest time:  $n = \frac{\log \xi_0}{\log 2}$ ). The red curve corresponds to  $S \mapsto C S \exp(-S^2)$ .

### Appendix Appendix 2.A Proof of lemma 2.A.1

LEMMA 2.A.1. *For every integer  $n$ ,  $E^n$  has  $l^n - 1$  fixed points. The distance between two distinct periodic points is bounded from below by  $\frac{1}{M^n - 1}$ .*

PROOF.  $E$  is topologically conjugated to the linear expanding map of same degree  $x \mapsto lx \pmod{1}$ , (see [KH97], p.73). Thus  $E^n$  has  $l^n - 1$  fixed points. Let  $\tilde{E} : \mathbb{R} \rightarrow \mathbb{R}$  be a lift of  $E$ ,  $x \neq y$  be two fixed points of  $E^n$  and  $\tilde{x}, \tilde{y} \in \mathbb{R}$  be representatives of  $x$  and  $y$  respectively. Note that

$$d(x, y) = \inf |\tilde{x} - \tilde{y}|$$

where the infimum is taken over all couples of representatives  $(\tilde{x}, \tilde{y})$ . Since  $E^n(x) = x$  and  $E^n(y) = y$ ,  $\tilde{E}^n(\tilde{y}) - \tilde{E}^n(\tilde{x}) - (\tilde{y} - \tilde{x})$  is an integer, different from 0 because  $\tilde{E}^n$  is expanding. Thus,

$$\left| \tilde{E}^n(\tilde{y}) - \tilde{E}^n(\tilde{x}) - (\tilde{y} - \tilde{x}) \right| \geq 1,$$

that is

$$\left| \int_{\tilde{x}}^{\tilde{y}} \left( (\tilde{E}^n)'(t) - 1 \right) dt \right| \geq 1$$

Finally,

$$|\tilde{y} - \tilde{x}|(M^n - 1) \geq 1.$$

Taking the infimum gives the result.  $\square$

### Appendix Appendix 2.B Proof of lemma 1.3

Let us recall the following classical estimate:

LEMMA 2.B.1. *If  $(X_p)_{p \in \mathbb{Z}}$  is a family of independent centered Gaussian random variables of variance 1, then, almost surely,*

$$\forall \delta > 0, X_p = o(p^\delta).$$

PROOF. Let  $\delta > 0$ . Let us use Borel-Cantelli lemma:

$$\forall p \in \mathbb{Z}, \mathbb{P}(|X_p| > p^\delta) = \int_{|x| > p^\delta} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

Now, we have the upper bound

$$p^\delta \int_{p^\delta}^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \leq \int_{p^\delta}^{+\infty} x e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{e^{-\frac{p^{2\delta}}{2}}}{\sqrt{2\pi}}.$$

Thus,

$$\forall p \in \mathbb{Z}^*, \mathbb{P}(|X_p| > p^\delta) \leq \frac{2}{\sqrt{2\pi} p^\delta} e^{-\frac{p^{2\delta}}{2}}.$$

Consequently,

$$\sum_p \mathbb{P}(|X_p| > p^\delta) < \infty$$

and by Borel-Cantelli, almost surely,

$$\#\{p \in \mathbb{Z}, |X_p| > p^\delta\} < \infty.$$

$\square$

With this in mind, we can see that if a real random function  $\tau$  has random Fourier coefficients  $(c_p)_{p \in \mathbb{Z}}$ , pairwise independent (for non-negative values of  $p$ ), with variance

$$\sigma_p^2 := \mathbb{E}[|c_p|^2] = O\left(\frac{1}{p^{2k+2+\eta}}\right),$$

for some  $\eta > 0$ , then by the previous lemma, almost surely, for all  $\delta > 0$ ,

$$\frac{c_p}{\sigma_p} = o(p^\delta),$$

and thus for  $\delta = \frac{\eta}{2}$ ,

$$c_p = o\left(\frac{1}{p^{k+1+\frac{\eta}{2}}}\right) \text{ a.s.}$$

As a consequence,

$$\sum_p c_p (2i\pi p)^k e^{2i\pi p x}$$

converges normally and thus  $\tau$  is almost surely  $\mathcal{C}^k$ .

## Appendix Appendix 2.C Ruelle resonances and Flat trace

**2.C.1 Ruelle spectrum.** If  $\tau \in \mathcal{C}^k(\mathbb{T})$ , the operator  $\mathcal{L}_{\xi, \tau}$  can be extended to distributions  $(\mathcal{C}^k(\mathbb{T}))'$  by duality. We will denote  $H^s(\mathbb{T})$  the Sobolev space of order  $s \in \mathbb{R}$ .

**THEOREM 2.C.1** ([Rue86], [Bal18] Thm 2.15 and Lemma 2.16). *Let  $k \geq 1$ . If  $\tau$  belongs to  $\mathcal{C}^k$ , then for every  $0 \leq s < k$ ,  $\mathcal{L}_{\xi, \tau} : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$  is bounded and its essential spectral radius  $r_{\text{ess}}$  satisfies*

$$r_{\text{ess}} \leq \frac{e^{\text{Pr}(-\frac{1}{2}J)}}{m^s},$$

where  $m = \inf E'$ ,  $J(x) = \log E'(x)$  and  $\text{Pr}(-\frac{1}{2}J)$  is defined in 2.E.1.

The discrete set of eigenvalues of finite multiplicities outside a given disk of radius  $r \geq \frac{e^{\text{Pr}(-\frac{1}{2}J)}}{m^s}$ , and the associated eigenspaces remain the same in every space  $H^{-s'}(\mathbb{T})$  for  $s' \geq s$ . This can be deduced for example from the fact that these spectral elements give the asymptotic behaviour of the correlation functions: for any smooth functions  $f, g$  on  $\mathbb{T}$ , for any  $s$  large enough, if  $\mathcal{L}_{\xi, \tau} : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$  has no eigenvalue of modulus  $r$ ,

$$(2.C.1) \quad \int \mathcal{L}_{\xi, \tau}^n f \cdot g = \sum_{\substack{\lambda \in \sigma(\mathcal{L}_{\xi, \tau}) \\ |\lambda| > r}} \int \mathcal{L}_{\xi, \tau}^n (\Pi_\lambda f) \cdot g + O_{n \rightarrow \infty}(r^n),$$

where  $\Pi_\lambda$  is the spectral projector associated to  $\lambda$ . We are interested in the statistical properties of these eigenvalues, called Ruelle-Pollicott spectrum or Ruelle resonances, when  $\tau$  is a random function. One way to get informations about the spectrum of such operators is using a trace formula. Although  $\mathcal{L}_{\xi, \tau}$  is not trace-class, we can give a certain sense to the trace of  $\mathcal{L}_{\xi, \tau}$ .

**2.C.2 Flat trace.** This section is an adaptation of section 3.2.2 in [Bal18]. In order to motivate the definition of flat trace, let us first recall the following fact:

LEMMA 2.C.2. *Let  $m > \frac{1}{2}$ . (Then the Dirac distributions belong to  $H^{-m}(\mathbb{T})$ ). If  $T : H^{-m}(\mathbb{T}) \rightarrow H^m(\mathbb{T})$  is a bounded operator, then it has a continuous Schwartz kernel  $K$  and*

$$K(x, y) = \langle \delta_x, T\delta_y \rangle.$$

*If moreover  $T$  is class-trace, then*

$$\mathrm{Tr} \, T = \int_{\mathbb{T}} K(x, x) dx.$$

Let  $\rho$  be a smooth compactly supported function such that  $\int_{\mathbb{R}} \rho = 1$ . For  $\epsilon > 0$  and  $y \in \mathbb{T}$  we write

$$\rho_{\epsilon, y}(t) = \frac{1}{\epsilon} \rho\left(\frac{t - y}{\epsilon}\right).$$

Periodizing this function gives rise to a smooth function  $\rho_{\epsilon, y}$  on  $\mathbb{T}$  satisfying

$$\rho_{\epsilon, y} \xrightarrow{\epsilon \rightarrow 0} \delta_y$$

as distributions.

DEFINITION 2.C.3. Let  $m \geq 0$  and  $T : H^{-m}(\mathbb{T}) \rightarrow H^{-m}(\mathbb{T})$  be a bounded operator extending to a continuous operator  $(\mathcal{C}^0(\mathbb{T}))' \rightarrow (\mathcal{C}^0(\mathbb{T}))'$ . Then the formula

$$K_{\epsilon}(x, y) := \langle \rho_{\epsilon, x}, T\delta_y \rangle$$

defines for every  $\epsilon > 0$  a continuous function on  $\mathbb{T}^2$ . Let

$$\mathrm{Tr}_{\epsilon}^b(T) := \int_{\mathbb{T}} K_{\epsilon}(x, x) dx.$$

We say that  $T$  admits a flat trace  $\mathrm{Tr}^b(T)$  if  $\mathrm{Tr}_{\epsilon}^b(T) \rightarrow \mathrm{Tr}^b(T)$  as  $\epsilon$  goes to zero, independently of the choice of the mollifying function  $\rho$ .

Note that, for any  $n \in \mathbb{N}^*$ ,  $\xi \in \mathbb{R}$ ,  $\tau \in \mathcal{C}^0(\mathbb{T})$ , the transfer operator  $\mathcal{L}_{\xi, \tau}^n : (\mathcal{C}^0(\mathbb{T}))' \rightarrow (\mathcal{C}^0(\mathbb{T}))'$  is bounded.

LEMMA 2.C.4 (Trace formula, [AB67], [G<sup>+</sup>77]). *Let  $\tau \in \mathcal{C}^k(\mathbb{T})$ ,  $k \geq 0$ . For any integer  $n \geq 1$ ,  $\mathcal{L}_{\xi, \tau}^n$  has a flat trace*

$$(2.C.2) \quad \mathrm{Tr}^b \mathcal{L}_{\xi, \tau}^n = \sum_{x, E^n(x)=x} \frac{e^{i\xi \tau_x^n}}{(E^n)'(x) - 1}$$

PROOF.

$$\mathrm{Tr}_{\epsilon}^b(\mathcal{L}_{\xi, \tau}^n) = \int_{\mathbb{T}} \langle \rho_{\epsilon, x}, \mathcal{L}_{\xi, \tau}^n \delta_x \rangle dx.$$

By definition of the action of  $\mathcal{L}_{\xi, \tau}^n$  on distributions,

$$\langle \rho_{\epsilon, x}, \mathcal{L}_{\xi, \tau}^n \delta_x \rangle = (\mathcal{L}_{\xi, \tau}^n)^* \rho_{\epsilon, x}(x),$$

where  $(\mathcal{L}_{\xi, \tau}^n)^*$  is the  $L^2$ -adjoint of  $\mathcal{L}_{\xi, \tau}^n$ . Let us recall that, if  $\phi : \mathbb{T} \rightarrow \mathbb{T}$  is a local diffeomorphism, for every continuous functions  $u, v$  on  $\mathbb{T}$ ,

$$(2.C.3) \quad \int u(\phi(y))v(y)dy = \int u(x) \sum_{\phi(y)=x} \frac{v(y)}{|\phi'(y)|} dx.$$

Thus,

$$(\mathcal{L}_{\xi,\tau}^n)^* v(x) = \sum_{E^n(y)=x} \frac{v(y) e^{i\xi\tau_y^n}}{(E^n)'(y)}.$$

Therefore

$$\begin{aligned} \mathrm{Tr}_\epsilon^b(\mathcal{L}_{\xi,\tau}^n) &= \int_{\mathbb{T}} (\mathcal{L}_{\xi,\tau}^n)^* \rho_{\epsilon,x}(x) dx \\ &= \int_{\mathbb{T}} \sum_{E^n(y)=x} \frac{\rho_{\epsilon,0}(y - E^n(y)) e^{i\xi\tau_y^n}}{(E^n)'(y)} dx \\ &= \int_{\mathbb{T}} \rho_{\epsilon,0}(y - E^n(y)) e^{i\xi\tau_y^n} dy \end{aligned}$$

by the change of variables  $x = E^n(y)$ . Now, since  $E$  is expansive,  $y \mapsto y - E^n(y)$  is a local diffeomorphism, so applying (2.C.3) once again gives

$$\begin{aligned} \mathrm{Tr}_\epsilon^b(\mathcal{L}_{\xi,\tau}^n) &= \int_{\mathbb{T}} \rho_{\epsilon,0}(z) \sum_{y-E^n(y)=z} \frac{e^{i\xi\tau_y^n}}{(E^n)'(y) - 1} dz \\ &\xrightarrow{\epsilon \rightarrow 0} \sum_{E^n(y)=y} \frac{e^{i\xi\tau_y^n}}{(E^n)'(y) - 1}. \end{aligned}$$

□

If  $E$  and  $\tau$  are analytic, it is well known that  $\mathcal{L}$  is trace-class and that  $\mathrm{Tr}^b(\mathcal{L}_{\xi,\tau}) = \mathrm{Tr}(\mathcal{L}_{\xi,\tau})$  (see for instance [Jéz17]). In the smooth setting however the decay of the Ruelle-Pollicott spectrum can be arbitrarily slow ([Jéz17], Proposition 1.10). The flat trace is however related to the Ruelle-Pollicott spectrum defined above in the following way (This is a consequence of Thm 3.5 in [Bal18] and Thm 2.4 in [Jéz17]):

**PROPOSITION 2.C.5.** *Assume that  $\tau \in \mathcal{C}^k(\mathbb{T})$  for some  $k \geq 1$ . Let  $\xi \in \mathbb{R}$ ,  $0 \leq s < k$ , and  $r > \frac{e^{\mathrm{Pr}(-\frac{1}{2}J)}}{m^s}$  be such that  $\mathcal{L}_{\xi,\tau} : H^{-s}(\mathbb{T}) \rightarrow H^{-s}(\mathbb{T})$  has no eigenvalue of modulus  $r$ , then*

$$(2.C.4) \quad \exists C > 0, \forall n \in \mathbb{N}, \left| \mathrm{Tr}^b \mathcal{L}_{\xi,\tau}^n - \sum_{\substack{\lambda \in \sigma(\mathcal{L}_{\xi,\tau}) \\ |\lambda| > r}} \lambda^n \right| \leq C r^n,$$

where the eigenvalues are counted with multiplicity.

## Appendix Appendix 2.D Proof of lemma 1.9

**PROOF.** Let  $X, X', Y$ , and  $Y'$  be as in the statement of the lemma real random variables such that  $e^{iX}, e^{iX'}$  are uniform on  $S^1$  and so that  $X$  and  $X'$  are both independent of all three other random variables. Let us write  $\mathbb{P}_Z$  the law of a random variable  $Z$ . To show that  $e^{i(X+Y)}$  and  $e^{i(X'+Y')}$  are independent and uniform on  $S^1$ , it suffices to show that for any continuous functions  $f, g : S^1 \rightarrow \mathbb{R}$ ,

$$\mathbb{E} \left[ f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] = \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}.$$

$$\mathbb{E} \left[ f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] = \int_{(S^1)^4} f(e^{i(x+y)}) g(e^{i(x'+y')}) d\mathbb{P}_{(X,Y,X',Y')}(x, y, x', y').$$

By hypothesis,

$$d\mathbb{P}_{(X,Y,X',Y')}(x, y, x', y') = \frac{dx}{2\pi} \frac{dx'}{2\pi} d\mathbb{P}_{(Y,Y')}(y, y').$$

Thus,

$$\begin{aligned} \mathbb{E} \left[ f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] &= \int_{(S^1)^2} \left( \int_0^{2\pi} \int_0^{2\pi} f(e^{i(x+y)}) g(e^{i(x'+y')}) \frac{dx}{2\pi} \frac{dx'}{2\pi} \right) d\mathbb{P}_{(Y,Y')}(y, y') \\ &\stackrel{\theta=x+y, \theta'=x'+y'}{=} \int_{(S^1)^2} \left( \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} \right) d\mathbb{P}_{(Y,Y')}(y, y') \\ &= \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}. \end{aligned}$$

□

## Appendix Appendix 2.E Topological pressure

### 2.E.1 Definition.

DEFINITION 2.E.1. Let  $\phi : \mathbb{T} \rightarrow \mathbb{R}$  be a Hölder-continuous function. The limit

$$(2.E.1) \quad \text{Pr}(\phi) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{E^n(x)=x} e^{\phi_x^n} \right)$$

exists and is called the topological pressure of  $\phi$  (see [KH97] Proposition 20.3.3 p.630).

In other words

$$(2.E.2) \quad \sum_{E^n(x)=x} e^{\phi_x^n} = e^{n\text{Pr}(\phi)+o(n)}.$$

The particular case  $\phi = 0$  gives the topological entropy  $\text{Pr}(0) = h_{\text{top}}$ .

REMARK 2.E.2. Note that the expression  $e^{n\text{Pr}(\phi)+o(n)}$  describes a large class of sequences, since for instance for any  $k \in \mathbb{N}$ ,

$$n^k e^{n\text{Pr}(\phi)} = e^{n\text{Pr}(\phi)+o(n)}.$$

**2.E.2 Variational principle.** Another definition of the pressure is given by the variational principle. Let us denote by  $h(\mu)$  the entropy of a measure  $\mu$  invariant under  $E$  (see [KH97] section 4.3 for a definition of entropy). For the next theorem, see [KH97], sections 20.2 and 20.3. The last sentence comes from Proposition 20.3.10.

THEOREM 2.E.3 (Variational principle). *Let  $\phi : \mathbb{T} \rightarrow \mathbb{R}$  be a Hölder function.*

$$\text{Pr}(\phi) = \sup_{\mu \text{ } E\text{-invariant}} \left( \int \phi \, d\mu + h(\mu) \right).$$



This supremum, taken over the invariant **probability** measures, is moreover attained for a unique  $E$ -invariant measure  $\mu$ , called equilibrium measure. In addition, if we note  $J = \log E'$  and  $\mu_\beta$  the equilibrium measure of  $-\beta J$ ,  $\beta \mapsto \mu_\beta$  is one-to-one.

COROLLARY 2.E.4. *The function*

$$(2.E.3) \quad F : \begin{cases} \mathbb{R}_+^* & \longrightarrow \mathbb{R} \\ \beta & \longmapsto \frac{1}{\beta} \Pr(-\beta J) \end{cases}$$

is strictly decreasing.

PROOF. Let  $\beta' > \beta > 0$ . By the previous theorem, with the same notations,

$$\int -\beta J \, d\mu_\beta + h(\mu_\beta) > \int -\beta J \, d\mu_{\beta'} + h(\mu_{\beta'})$$

and thus

$$F(\beta) = \int -J \, d\mu_\beta + \frac{h(\mu_\beta)}{\beta} > \int -J \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta} \geq \int -J \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta'} = F(\beta').$$

□

**2.E.3 Proof of Lemma 3.11.** Let  $\phi : \mathbb{T} \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  function. Let as before  $\phi_x^n$  be the Birkhoff sum (1.3). By subadditivity of the sequence  $(\inf_{x \in \mathbb{T}} \phi_x^n)_n$  and Fekete's Lemma we can define the following quantity:

DEFINITION 2.E.5. Let us define

$$(2.E.4) \quad \phi_{\min} := \lim_{n \rightarrow \infty} \inf_{x \in \mathbb{T}} \frac{1}{n} \phi_x^n.$$

LEMMA 2.E.6. *The infimum in (2.E.4) can be taken over periodic points:*

$$(2.E.5) \quad \phi_{\min} = \lim_{n \rightarrow \infty} \inf_{x, E^n(x)=x} \frac{1}{n} \phi_x^n.$$

PROOF. By lifting the expanding map to  $\mathbb{R}$ , we easily see that  $E$  has at least a fixed point  $x_0$ . This point has  $l^n$  preimages by  $E^n$ , defining  $l^n - 1$  intervals  $I_k^n$  such that for all  $1 \leq k \leq l^n - 1$

$$E^n : I_k^n \rightarrow \mathbb{T} \setminus \{x_0\}$$

is a diffeomorphism. Thus, there exists  $C > 0$  such that for all  $k$ , if  $x, y \in \overline{I_k^n}$ ,

$$\forall 0 \leq j \leq n, d(E^j(x), E^j(y)) \leq \frac{C}{m^{n-j}},$$

with  $m = \inf |E'| > 1$ . Each  $\overline{I_k^n}$  contains moreover a periodic point  $y_{k,n}$  of period  $n$  given by  $E^n(y_{k,n}) = y_{k,n} + k$ . Hence let  $n \in \mathbb{N}$ , let  $x_n \in \mathbb{T}$  be such that

$$\phi_{x_n}^n = \inf_{x \in \mathbb{T}} \phi_x^n,$$

and suppose that  $x_n \in \overline{I_k^n}$ . We have

$$\begin{aligned} \left| \phi_{y_{k,n}}^n - \phi_{x_n}^n \right| &= \left| \sum_{j=0}^{n-1} \phi(E^j(x_n)) - \phi(E^j(y_{k,n})) \right| \\ &\leq C \max |\phi'| \sum_{k=0}^{\infty} \frac{1}{m^k} \end{aligned}$$

is bounded independently of  $n$ . Consequently

$$\lim_{n \rightarrow \infty} \inf_{x, E^n(x)=x} \frac{1}{n} \phi_x^n = \phi_{\min}.$$

□

LEMMA 2.E.7.

$$F(\beta) \xrightarrow{\beta \rightarrow +\infty} -\phi_{\min}.$$

PROOF. Let  $\beta > 0$ . Let us write

$$F_n(\beta) = \frac{1}{n\beta} \log \left( \sum_{E^n(x)=x} e^{-\beta \phi_x^n} \right),$$

so that

$$F(\beta) \stackrel{(2.E.1, 2.E.3)}{=} \lim_{n \rightarrow \infty} F_n(\beta).$$

Let  $\epsilon > 0$ . By definition of  $\phi_{\min}$ , for  $n$  large enough,

$$\forall x \in \text{Per}(n), \phi_x^n \geq n(\phi_{\min} - \epsilon)$$

and

$$\exists x \in \text{Per}(n), \phi_x^n \leq n(\phi_{\min} + \epsilon).$$

Thus,

$$e^{-\beta n(\phi_{\min} + \epsilon)} \leq \sum_{E^n(x)=x} e^{-\beta \phi_x^n} \leq l^n e^{-\beta n(\phi_{\min} - \epsilon)}$$

and consequently

$$-\phi_{\min} - \epsilon \leq F_n(\beta) \leq \frac{\log l}{\beta} - \phi_{\min} + \epsilon.$$

Hence, letting  $\epsilon \rightarrow 0$ , we get

$$-\phi_{\min} \leq F(\beta) \leq \frac{\log l}{\beta} - \phi_{\min}.$$

When  $\beta$  goes to infinity, the result follows. □

PROOF OF LEMMA 3.11. Now we take  $\phi = J = \log(E')$ . By the definition of  $J_{\min}$

$$\inf_{O \in \text{Per}(n)} J_O^n = nJ_{\min} + o(n),$$

thus

$$(2.E.6) \quad \sup_{O \in \text{Per}(n)} \frac{1}{e^{J_O^n} - 1} = e^{-nJ_{\min} + o(n)} = e^{n \lim_{\beta \rightarrow \infty} F(\beta) + o(n)}.$$

We have

(2.E.7)

$$\begin{aligned} \frac{1}{\sqrt{n}} \left( \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}} &\leq A_n \stackrel{(1.9)}{=} \left( \sum_{m|n} m^2 \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}} \\ &\leq \left( \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} \right)^{-\frac{1}{2}}. \end{aligned}$$

Since

$$\begin{aligned}
 \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{J_O^n} - 1)^2} &= \sum_{E^n(x)=x} \frac{1}{(e^{J_x^n} - 1)^2} \\
 &= \sum_{E^n(x)=x} e^{-2J_x^n} \left(1 + O\left(e^{-J_x^n}\right)\right) \\
 &= \left( \sum_{E^n(x)=x} e^{-2J_x^n} \right) (1 + o(1)) \\
 &= e^{n\Pr(-2J)+o(n)},
 \end{aligned}$$

Eq.(2.E.7) gives

$$\frac{1}{\sqrt{n}} e^{-\frac{n}{2}\Pr(-2J)+o(n)} \leq A_n \leq e^{-\frac{n}{2}\Pr(-2J)+o(n)}$$

hence from Remark 2.E.2

$$nA_n = e^{-\frac{n}{2}\Pr(-2J)+o(n)} = e^{-nF(2)+o(n)}.$$

Finally,

$$nA_n \sup_{O \in \text{Per}(n)} \frac{1}{e^{J_O} - 1} \stackrel{(2.E.6)}{=} e^{n(\lim_{\infty} F - F(2)+o(n))} \rightarrow 0$$

from Corollary 2.E.4. □

## CHAPTER 3

# Flat traces for the skew product of an Anosov diffeomorphism

## 1. Introduction

This paper follows [Gos20a] and extends its results to the case of an Anosov diffeomorphism on a compact Riemannian manifold.

The main object of study are the flat traces of the transfer operator. This operator acts by pulling back functions and its spectral properties are linked to the dynamical correlations. For Anosov diffeomorphisms, the statistical properties have been studied since the late 1960's, with help of Markov partitions and symbolic dynamics [Bow75]. The construction of spaces, in the Anosov framework, in which the transfer operator is quasicompact was first achieved later by Blank, Keller and Liverani [BKL02].

For Anosov flows or partially hyperbolic diffeomorphisms, the neutral direction adds a substantial difficulty to the study. In this setting, Dolgopyat [Dol98] showed exponential decay of correlations for the geodesic flow on negatively curved surfaces, and Liverani [Liv04] generalized this result to all  $C^4$  contact Anosov flows, by constructing anisotropic Banach spaces in which the generating vector field has a spectral gap and resolvent. Tsujii [Tsu10] extended this method and showed quasicompactness of the transfer operator itself, with an explicit bound on the essential spectral radius, for contact Anosov flows, in some Hilbert spaces. Butterley and Liverani [BL07] then constructed Banach spaces to study the spectrum of general Anosov flows. Weich and Bonthouneau [BW17] on their side constructed outside the scope of compact manifolds appropriate spaces for geodesic flow on negatively curved manifolds with a finite number of cusps. Dyatlov and Guillarmou [DG16] did it for open hyperbolic systems.

A simple example of Anosov flow is the suspension of an Anosov diffeomorphism, or the suspension semi-flow of an expanding map. Pollicott [POL99] showed exponential decay of correlations in this setting under a weak condition and Tsujii constructed suitable spaces for the transfer operator and gave an upper bound on its essential spectral radius in [Tsu08].

Here we consider a close model, namely a  $\mathbb{R}$ -extension of an Anosov diffeomorphism on a compact Riemannian manifold, for which Dolgopyat [Dol02] has shown generic rapid decay of correlations. In a series of papers, de Simoi, Liverani, Poquet and Volk [DSL16] [DSL18] [DSL17] studied statistical properties of fast-slow dynamical systems. Their model generalizes  $\mathbb{T}$  extensions of circle expanding maps.

We investigate a small random perturbation of the roof function and show that the flat traces (1.5) of the iterates of the transfer operator (restricted to a given frequency  $\xi$  in the fiber direction) satisfy a central limit theorem in a semiclassical regime linking time  $n$  and frequency  $\xi$  (theorem 1.5). We obtain convergence towards a Gaussian law up to a constant times the Ehrenfest time, this constant being a decreasing function of the regularity of the random function. The principle is the same as in [Gos20a]: We show pointwise convergence of the characteristic function, by decomposing for each time  $n$  the roof function as the sum of a random function that decorrelates at a scale corresponding to the minimal distance between

periodic points of period  $n$ , and an other function that plays no role if the frequency is large enough.

Naud [Nau16] found in the case of circle extensions of some analytic Anosov maps of the torus lower bounds on the first eigenvalue, both in the deterministic and random settings. He makes use of the fact that with positive probability, there is a lower bound on the modulus of the trace (of same order as the scaling  $A_n$  from (1.14)), and takes advantage of the fact that the operator is trace class. This is not the case in our setting and we don't know whether information on the Ruelle-Pollicott spectrum can be recovered from our estimation of the flat traces.

**1.1. Model.** Let  $M$  be a smooth Riemannian manifold of dimension  $d$  and  $T : M \rightarrow M$  be a transitive Anosov diffeomorphism. This means that the tangent bundle admits a splitting  $TM = E^u \oplus E^s$  such that

$$(1) \quad dT_x(E^i(x)) = E^i(f(x)), \quad i \in \{u, s\},$$

$$(2) \quad \exists 0 < \lambda < 1, \exists C > 0, \forall n \in \mathbb{N}, \begin{cases} \forall v \in E^u, \|dT^{-n} \cdot v\| & \leq C\lambda^n \|v\|, \\ \forall v \in E^s, \|dT^n \cdot v\| & \leq C\lambda^n \|v\| \end{cases}$$

and that  $T$  has a dense orbit. We will be interested, given  $k \geq 0$  and a  $\mathcal{C}^k$  function  $\tau$ , in the skew-product

$$(1.1) \quad F : \begin{cases} M \times \mathbb{R} & \longrightarrow M \times \mathbb{R} \\ (x, y) & \longmapsto (T(x), y + \tau(x)) \end{cases}.$$

**1.2. Ruelle spectrum.** To the map  $F$  can be associated a transfer operator  $\mathcal{L}_\tau$  acting on  $\mathcal{C}^k(M \times \mathbb{R})$  by composition:

$$(1.2) \quad \mathcal{L}_\tau v = v \circ F.$$

Fourier analysis with respect to  $y$  leads to the introduction of the family of operators on  $\mathcal{C}^k(M)$  indexed by  $\xi \in \mathbb{R}$

$$(1.3) \quad \mathcal{L}_{\xi, \tau} u = e^{i\xi\tau} u \circ T.$$

Indeed, if  $v$  is a Fourier mode with respect to  $y$ , that is  $v(x, y) = u(x)e^{i\xi y}$  for some  $u \in \mathcal{C}^k(M)$ ,  $\in \mathbb{R}$ ,

$$(1.4) \quad \mathcal{L}_\tau v(x, y) = \mathcal{L}_{\xi, \tau} u(x)e^{i\xi y}.$$

These operators can be extended to distributions by duality. They have their essential spectral radius bounded by explicit constants in appropriate spaces. The operators are not trace class, but we can define a generalization of their trace, called flat trace, which has a connexion with their spectrum. See Appendix 3.D for a brief discussion about this. The flat trace is the main object studied in this paper, we express a central limit theorem for a small random perturbation of a given function  $\tau$  in the limit of large times  $n$  and frequencies  $\xi$  in Theorem 1.5. Its expression involves periodic points and is given by

$$(1.5) \quad \mathrm{Tr}^\flat(\mathcal{L}_{\xi, \tau}^n) = \sum_{x, T^n(x)=x} \frac{e^{i\xi \tau_x^n}}{|\det(1 - d(T^n)_x)|},$$

where  $\tau_x^n$  denotes the Birkhoff sum

$$(1.6) \quad \tau_x^n := \tau(x) + \tau(T(x)) + \cdots + \tau(T^{n-1}(x)).$$

### 1.3. Eigenfunction Gaussian random fields.

DEFINITION 1.1. We will call centered Gaussian field on  $M$  a random distribution of the form

$$(1.7) \quad f = \sum_{j \geq 0} c_j \zeta_j \phi_j,$$

where the  $c_j \geq 0$  grow at most polynomially with  $j$ ,  $\zeta_j$  are i.i.d centered Gaussian random variables of variance 1 and  $(\phi_j)_j$  is a Hilbert basis of eigenfunctions of the Laplace-Beltrami operator:

$$(1.8) \quad \Delta \phi_j = \lambda_j \phi_j,$$

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots.$$

This sum is in general understood in the sense of distributions, Proposition 1.4 thereafter expresses a link between the growth of  $(c_j)_{j \geq 0}$  and the regularity of the field. We will only be interested in at least continuous fields in what follows.

EXAMPLE 1.2. If  $c_j = 1$  for all  $j$ , the random field  $W$  is called white noise:

$$(1.9) \quad W = \sum_{j \geq 0} \zeta_j \phi_j.$$

It is a random distribution, almost surely not in  $L^2(M)$ .

On  $M$ , we have a notion of Sobolev spaces:

DEFINITION 1.3. Let  $s \in \mathbb{R}$ . The Sobolev space  $H^s(M)$  is defined by

$$(1.10) \quad H^s(M) = (1 + \Delta)^{-s/2} L^2(M).$$

PROPOSITION 1.4. Assume that

$$(1.11) \quad c_j = O(j^{-\alpha})$$

for some  $\alpha \in \mathbb{R}$ . Then, almost surely, the centered Gaussian field

$$(1.12) \quad f := \sum_{j \geq 0} c_j \zeta_j \phi_j \in H^s(M)$$

for every  $s < d(\alpha - \frac{1}{2})$ . Thus,

$$(1.13) \quad f \in \mathcal{C}^k(M)$$

for every  $k < d(\alpha - 1)$  (where  $\mathcal{C}^k(M)$  is understood as  $(\mathcal{C}^{-k}(M))'$  for negative  $k$ .)

PROOF. See appendix 3.C. □

**1.4. Result.** If  $x$  is a periodic point of  $T$ , we write its primitive period  $m_x$ . Let us define the amplitudes  $A_n$  by

$$(1.14) \quad A_n := \left( \sum_{T^n(x)=x} \frac{m_x}{|\det(1 - dT_x^n)|^2} \right)^{-\frac{1}{2}}.$$

Let also

$$(1.15) \quad \Lambda^\pm := \lim_{n \rightarrow \infty} \max_{x \in M} \|dT_x^{\pm n}\|^{\frac{1}{n}}$$

and

$$(1.16) \quad \Lambda := \max(\Lambda^\pm).$$

Let  $h_{\text{top}} \stackrel{(2.31)}{\leq} \frac{d}{2} \log \Lambda$  be the topological entropy of the map  $T$  (see Definition 3.1.3 in [KH97]).

**THEOREM 1.5.** *Let us fix any  $\tau_0 \in \mathcal{C}^0(M)$  and  $\varepsilon > 0$ . Let*

$$(1.17) \quad \delta\tau = \sum_{j \geq 0} c_j \zeta_j \phi_j$$

*be a centered Gaussian field with  $\zeta_j$  i.i.d.  $\mathcal{N}(0, 1)$  such that*

$$(1.18) \quad \frac{1}{C} j^{-\alpha} \leq c_j \leq C j^{-\beta}$$

*for some constants  $C > 0, \alpha > \beta > 1$ . By Proposition 1.4, the condition involving  $\beta$  ensures that  $\delta\tau$  is almost surely continuous, and that we can define the flat trace of  $\mathcal{L}_{\xi, \tau}^n$  for  $\tau := \tau_0 + \varepsilon \delta\tau$ . Then, for any  $0 < c < 1$ , we have the following convergence in law*

$$(1.19) \quad A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \longrightarrow \mathcal{N}_{\mathbb{C}}(0, 1)$$

*as  $n$  and  $\xi$  go to infinity under the relation*

$$(1.20) \quad n \leq c \frac{\log \xi}{h_{\text{top}} + \frac{d}{2}(\alpha - \frac{1}{2}) \log \Lambda}.$$

**REMARK 1.6.**  $\beta$  plays no other role than to make sure that the flat trace is well defined.

**REMARK 1.7.** if for some reason we want to impose a certain regularity on the function  $\delta\tau$  using proposition 1.4, we need to take  $\alpha$  large enough, and the larger it is, the more restrictive condition 1.20 imposed on the time  $n$  is.

**REMARK 1.8.** In [Gos20a], we obtained instead of (1.20) the condition

$$(1.21) \quad n \leq c \frac{\log \xi}{\log l + (k + \frac{1}{2} + \frac{\delta}{2}) \log M}.$$

In this setting,  $k + \frac{1}{2} + \frac{\delta}{2}$  was the analog of  $\alpha$ , and  $\log l$  the topological entropy.  $M$  was analogous to  $\Lambda^{\frac{d}{2}}$ . The dimension was 1 and the exponent  $\frac{1}{2}$  comes from the fact that  $T$  is invertible, while  $E$  was not and had therefore possibly denser



periodic points.  $M = \sup E'$  could have been refined as  $\lim_n (\sup(E')^n)^{\frac{1}{n}}$  to match the definition of  $\Lambda$ . With this in mind, the bound of [Gos20a] translates to

$$(1.22) \quad n \leq c \frac{\log \xi}{h_{\text{top}} + \frac{d}{2} \alpha \log \Lambda}.$$

We obtain here a slightly better bound, with  $\alpha - \frac{1}{2}$  instead of  $\alpha$ , due to the fact that our proof directly deals with the multivariate Gaussian probability density in a space of dimension approximately  $e^{nh_{\text{top}}}$  instead of reducing to the unidimensional variables.

## 2. Proof

**REMARK 2.1.** Let us first remark that the convergence in law stated in theorem 1.5 only involves the law of  $\delta\tau$ . Therefore, we will abusively name  $\delta\tau$  another field that has the same law.

**2.1. Sketch of proof.** Let us choose  $\alpha > 1$ ,  $\eta > 0$  and  $\delta\tau$  as in the statement of Theorem 1.5. We will show that condition (1.18) allows us to construct a centered Gaussian field with the same law as  $\delta\tau$ , as a sum of independent Gaussian fields

$$(2.1) \quad \delta\tau = \delta\tau_0 + \sum_{j \geq 1} \delta\tau_j,$$

so that the covariances  $\mathbb{E}[\delta\tau_j(x)\delta\tau_j(y)]$  become very small at a distance greater than  $\Lambda'^{-\frac{j}{2}}$ , for some  $\Lambda' > \Lambda$  that we will choose small enough (*i.e.* close enough to  $\Lambda$ ), which is smaller than the minimal distance between two periodic points of large period  $j$ , as we know from Lemma 3.A.1. Here we have used the abusive notation described in the preliminary remark 2.1. Therefore, the phases appearing in the trace formula (1.5) will behave as independent random variables on  $S^1$ , almost uniform when  $\xi$  is large enough, that is, under the condition (1.20).

**2.2. Construction of the fields  $\delta\tau_j$ .** Let  $\chi \in \mathcal{S}(\mathbb{R})$  be a positive Schwartz<sup>1</sup> function such that  $\chi(0) = 0$ , normalized so that

$$(2.2) \quad \int \chi^2 \, dx = (2\pi)^d.$$

Let us define the family of operators

$$(2.3) \quad P_h := \chi(h^2\Delta).$$

**PROPOSITION 2.2** ([Zwo12] Theorem 14.9 p.358 and Theorem 9.6 p.209).  *$P_h$  is a  $h$ -pseudodifferential operator and its Schwartz kernel  $K_h$  is a smooth function satisfying*

$$(2.4) \quad \forall N > 0, \exists C_N > 0, \forall x \neq y, |K_h(x, y)| \leq \frac{C_N h^N}{d(x, y)^N}.$$

---

<sup>1</sup>Here  $\chi$  plays the role of the Fourier transform of  $K_{\text{init}}$  in the previous chapter. We will see that the kernel of  $\chi(h^2\Delta)$  determines the correlations between the values at different points. I do not know if we can manage to get as previously both following important points: that this kernel vanishes outside a tubular neighbourhood of the diagonal of size  $h$ , and that  $\chi$  is non-negative. We can assume  $\chi$  to be compactly supported, but this does not improve the estimate (2.4)

The distance  $d$  is here the Riemannian distance.

Let  $W_j$  be a family of independent white noises (defined in (1.9)) independent of  $\delta\tau$ . Let  $\gamma > 0$  and  $\tilde{\Lambda} > \Lambda$  to be chosen small enough later and

$$(2.5) \quad h_j = \tilde{\Lambda}^{-\frac{j}{2}}$$

$P_{h_j}$  is a positive selfadjoint operator, so we can define

$$(2.6) \quad \delta\tau_j := h_j^{d\alpha+\gamma} \sqrt{P_{h_j}} W_j.$$

In other terms, there exist i.i.d. random variables  $\zeta_{j,k}$  of law  $\mathcal{N}(0, 1)$  such that

$$(2.7) \quad \delta\tau_j = h_j^{d\alpha+\gamma} \sum_{k \geq 1} \sqrt{\chi(h_j^2 \lambda_k)} \zeta_{j,k} \phi_k.$$

LEMMA 2.3. *Let  $K_j$  be the Schwartz kernel of the operator  $h_j^{2(d\alpha+\gamma)} P_{h_j}$ . Then,*

$$(2.8) \quad \mathbb{E}[\delta\tau_j(x) \delta\tau_j(y)] = K_j(x, y).$$

Moreover, on the diagonal, [Zwo12, Theorem 14.10 p.361]

$$(2.9) \quad K_j(x, x) \underset{(2.2)}{=} h_j^{d(2\alpha-1)+2\gamma} (1 + O(h_j)).$$

PROOF. Indeed, on one hand

$$(2.10) \quad \begin{aligned} \mathbb{E}[\delta\tau_j(x) \delta\tau_j(y)] &= h_j^{2(d\alpha+\gamma)} \sum_{k, k' \geq 0} \sqrt{\chi(h_j^2 \lambda_k)} \sqrt{\chi(h_j^2 \lambda_{k'})} \mathbb{E}[\zeta_{j,k} \zeta_{j,k'}] \phi_k(x) \phi_{k'}(y) \\ &= h_j^{2(d\alpha+\gamma)} \sum_{k \geq 0} \chi(h_j^2 \lambda_k) \phi_k(x) \phi_k(y), \end{aligned}$$

and on the other hand, since

$$(2.11) \quad P_{h_j} \phi_k = \chi(h_j^2 \lambda_k) \phi_k,$$

we have for any  $u, v \in L^2(M)$

$$(2.12) \quad \begin{aligned} \langle v, P_{h_j} u \rangle_{L^2(M)} &= \int \bar{v}(x) \sum_{k \geq 0} (P_{h_j} \phi_k)(x) \int u(y) \phi_k(y) \, dy \, dx \\ &= \int \left( \sum_{k \geq 0} \chi(h_j^2 \lambda_k) \phi_k(x) \phi_k(y) \right) \bar{v}(x) u(y) \, dx \, dy. \end{aligned}$$

□

Let us choose  $\Lambda < \Lambda' < \tilde{\Lambda}$  (recall that  $\tilde{\Lambda}$  is involved in the definition of  $h_j$  in (2.5)). As a consequence of Lemma 3.A.1, Proposition 2.2, and the definition of  $h_j$  we have the following decay:

$\forall N > 0, \exists C_N > 0, \forall x \neq y \in M, \forall j$  large enough,

$$\begin{aligned}
 (2.13) \quad (T^j x = x \text{ and } T^j y = y) &\implies |K_j(x, y)| \leq C_N \frac{h_j^{N+2(d\alpha+\gamma)}}{d(x, y)^N} \\
 &\leq C_N \frac{h_j^N}{d(x, y)^N} \\
 &\leq \frac{C_N}{C^N} \left( \frac{\Lambda'}{\Lambda} \right)^{\frac{N}{2}j}
 \end{aligned}$$

for some  $C > 0$ . Thus,

$$(2.14) \quad \forall k > 0, \exists C_k > 0, \forall x \neq y \in M, (T^j x = x \text{ and } T^j y = y) \implies |K_j(x, y)| \leq C_k e^{-kj}.$$

LEMMA 2.4. *Let us write as in Theorem 1.5*

$$(2.15) \quad \delta\tau = \sum_{k \geq 0} c_k \zeta_k \phi_k.$$

We have

$$(2.16) \quad \sum_{j \geq 1} \delta\tau_j = \sum_{k \geq 0} c'_k \zeta'_k \phi_k,$$

where  $(\zeta'_k)_k$  is a family of i.i.d. random variables of law  $\mathcal{N}(0, 1)$  independent of the variables  $\zeta_k$ , and

$$(2.17) \quad c'_k = O(c_k).$$

PROOF.

$$(2.18) \quad \sum_j \delta\tau_j \stackrel{(2.6)}{=} \sum_{k \geq 1} \left( \sum_{j \geq 1} h_j^{d\alpha+\gamma} \sqrt{\chi(h_j^2 \lambda_k) \zeta_{j,k}} \right) \phi_k.$$

Since every variables are independent from each other, the

$$(2.19) \quad \left( \sum_{j \geq 1} h_j^{d\alpha+\gamma} \sqrt{\chi(h_j^2 \lambda_k) \zeta_{j,k}} \right)_k$$

are independent Gaussian variables of variances

$$(2.20) \quad c_k'^2 := \sum_{j \geq 1} h_j^{2d\alpha+2\gamma} \chi(h_j^2 \lambda_k).$$

Now, since  $\chi \in \mathcal{S}(\mathbb{R})$ ,

$$(2.21) \quad \exists C > 0, \forall j, k, \chi(h_j^2 \lambda_k) \leq \frac{C}{h_j^{2d\alpha} \lambda_k^{d\alpha}}.$$

Also,

$$(2.22) \quad \sum_{j \geq 1} h_j^{2\gamma} < \infty$$

so the variances satisfy

$$(2.23) \quad c_k'^2 = \sum_{j \geq 1} h_j^{2d\alpha+2\gamma} \chi(h_j^2 \lambda_k) = O(\lambda_k^{-d\alpha})$$

By Weyl's law [Zwo12, (14.3.21) p.362] there exists a constant  $C$  depending on  $M$  such that

$$(2.24) \quad \lambda_k^d \sim Ck^2$$

so

$$(2.25) \quad c_k'^2 = O(k^{-2\alpha}).$$

□

COROLLARY 2.5. *Consequently, up to the multiplication of each  $\delta\tau_j$  by the same constant, condition (1.18) allows us to define the field*

$$(2.26) \quad \delta\tau_0 := \sum_{k \geq 1} \sqrt{c_k^2 - c_k'^2} \zeta_k'' \phi_k,$$

for i.i.d. random variables  $\zeta_k''$  of law  $\mathcal{N}(0, 1)$  independent of the variables  $\zeta_k$  and  $\zeta_k'$ . According to Remark 2.1, we will abusively write

$$(2.27) \quad \delta\tau = \sum_{j \geq 0} \delta\tau_j.$$

DEFINITION 2.6. We will use the following notations for periodic orbits in this paper:  $\text{Per}(n)$  will be the set of periodic orbits of period  $n$ , while  $\mathcal{P}_m$  will be the set of periodic orbits of primitive period  $m$ . This way, we have a disjoint union

$$(2.28) \quad \text{Per}(n) = \coprod_{m|n} \mathcal{P}_m.$$

If  $O \in \text{Per}(n)$ , then the Birkhoff sums  $f_x^n = \sum_{k=0}^{n-1} f(T^k x)$  do not depend on the point  $x \in O$  and will be written  $f_O^n$ . Similarly,  $\det(1 - dT_O^n)$  will denote the Jacobian  $\det(1 - d(T^n)_x)$  for any  $x \in O$ .

PROPOSITION 2.7 ([KH97, Theorem 18.5.5 p.585]). *There exists  $C > 0$  such that*

$$(2.29) \quad \frac{1}{C} e^{nh_{\text{top}}} \leq \#\text{Per}(n) \leq C e^{nh_{\text{top}}}.$$

REMARK 2.8. Note that, since we assume the map  $T$  to be transitive, by the Closing Lemma [KH97, Theorem 6.4.15 p.269] it has periodic orbits of arbitrary large period. Therefore, necessarily,  $h_{\text{top}} > 0$ . Thus,

$$(2.30) \quad \#\text{Per}(n) \xrightarrow{n \rightarrow \infty} \infty.$$

REMARK 2.9. Let us notice that Lemma 3.A.1 implies

$$(2.31) \quad h_{\text{top}} \leq \frac{d}{2} \log \Lambda.$$

PROOF. Let indeed  $\Lambda' > \Lambda$ .  $M$  can be covered by  $O(\Lambda'^{\frac{nd}{2}})$  balls of radius  $\Lambda'^{-\frac{n}{2}}$ . The constraint of Lemma 3.A.1 implies that each ball of radius  $\Lambda'^{-\frac{n}{2}}$  contains a bounded number of points of  $\text{Per}(n)$ . (2.29) then implies

$$(2.32) \quad e^{nh_{\text{top}}} \leq C\Lambda'^{\frac{nd}{2}}$$

for every  $\Lambda' > \Lambda$ . □

Recall that

$$(2.33) \quad \tau = \tau_0 + \varepsilon \delta \tau \stackrel{(2.27)}{=} \tau_0 + \varepsilon \sum_{j \geq 0} \delta \tau_j.$$

LEMMA 2.10. *We can write*

$$(2.34) \quad \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) = \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\varepsilon \xi(X_O^n + Y_O^n)}}{|\det(1 - dT_O^n)|},$$

where

(1) *for all  $n \in \mathbb{N}$ ,  $(X_O^n)_{O \in \text{Per}(n)}$  is a Gaussian random vector such that*

$$(2.35) \quad \exists C > 0, \forall n \in \mathbb{N}, \forall O \in \text{Per}(n), \frac{1}{C} n h_n^{d(2\alpha-1)+2\gamma} \leq \sigma_O^2 := \mathbb{E}[(X_O^n)^2] \leq C n^2 h_n^{d(2\alpha-1)+2\gamma}$$

and

$$(2.36) \quad \forall \alpha > 0, \exists C_\alpha, \forall n \in \mathbb{N}, \forall O \neq O' \in \text{Per}(n), |\mathbb{E}[X_O^n X_{O'}^n]| \leq C_\alpha e^{-\alpha n}.$$

(2) *For every integer  $n$  the random variable  $(Y_O^n)_{O \in \text{Per}(n)}$  is independent of  $(X_O^n)_{O \in \text{Per}(n)}$ .*

PROOF OF LEMMA 2.10. Using (1.5), and the fact that the Birkhoff sums and differentials  $dT_x^n$  only depend on the orbit, one can pack the terms

$$(2.37) \quad \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) = \sum_{m|n} m \sum_{O \in \mathcal{P}_m} \frac{e^{i\xi \tau_O^n}}{|\det(1 - dT_O^n)|}$$

Now, we isolate the term  $\delta \tau_n$  in (2.33) (where  $n$  is the time appearing in the expression  $\text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$ ) and set

$$(2.38) \quad X_O^n := (\delta \tau_n)_O^n$$

and

$$(2.39) \quad Y_O^n := \left( \frac{\tau_0}{\varepsilon} + \sum_{j \neq n} \tau_j \right)_O^n.$$

The independence of the family  $(\zeta_{j,n})_{n \geq 0} \cup (\zeta_k'')_{k \geq 0}$  involved in (2.7) and Corollary 2.5 gives the independence between the families  $(X_O^n)_O$  and  $(Y_O^n)_O$ . Then, for an orbit  $O \in \mathcal{P}_m \subset \text{Per}(n)$ ,

$$\begin{aligned}
(2.40) \quad X_O^n &= (\delta\tau_n)_O^n \\
&= \frac{n}{m} \sum_{x \in O} \delta\tau_n(x).
\end{aligned}$$

Thus  $X_O^n$  is a Gaussian random variable and

$$(2.41) \quad (X_O^n)^2 = \frac{n^2}{m^2} \left( \sum_{x \in O} \delta\tau_n(x)^2 + \sum_{\substack{x, y \in O \\ x \neq y}} \delta\tau_n(x) \delta\tau_n(y) \right).$$

So

$$\begin{aligned}
(2.42) \quad \mathbb{E}[(X_O^n)^2] &= \frac{n^2}{m^2} \left( \sum_{x \in O} K_n(x, x) + \sum_{\substack{x, y \in O \\ x \neq y}} K_n(x, y) \right) \\
&\stackrel{(2.9), (2.14)}{=} \frac{n^2}{m} h_n^{d(2\alpha-1)+2\gamma} (1 + O(h_n)) + O(e^{-\alpha n})
\end{aligned}$$

for every  $\alpha > 0$ . Similarly

$$\begin{aligned}
(2.43) \quad \mathbb{E}[X_O^n X_{O'}^n] &= \mathbb{E} \left[ \sum_{x \in O} \delta\tau_n(x) \sum_{y \in O'} \delta\tau_n(y) \right] \\
&= \sum_{x \in O, y \in O'} K_n(x, y) \\
&= O(e^{-\alpha n})
\end{aligned}$$

for every  $\alpha > 0$ .

This gives the expressions (2.35) and (2.36).  $\square$

By Levy's theorem, in order to get the convergence in law of  $A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$  towards  $\mathcal{N}_{\mathbb{C}}(0, 1)$ , it is sufficient to show that the characteristic function

$$(2.44) \quad \mathbb{E} \left[ e^{i \langle (\mu, \nu), A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \rangle_{\mathbb{R}^2}} \right]$$

converges pointwise towards  $e^{-\frac{\mu^2 + \nu^2}{4}}$ .

PROPOSITION 2.11. *The characteristic function of the rescaled flat traces*

$$(2.45) \quad E(\mu, \nu) := \mathbb{E} \left[ \exp \left( i \left( \mu \text{Re} \left( A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \right) + \nu \text{Im} \left( A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n) \right) \right) \right) \right]$$

satisfies

$$(2.46) \quad E(\mu, \nu) \sim \prod_{m|n} \prod_{O \in \mathcal{P}_m} \int \exp \left( i \frac{mA_n}{|\det(1 - dT_O^n)|} (\mu \cos x + \nu \sin x) \right) dx$$

under condition (1.20).

PROOF. Let  $n \in \mathbb{N}$  and  $m|n$ . Let us write

$$(2.47) \quad N := \#\text{Per}(n) \underset{(2.29)}{\asymp} e^{nh_{\text{top}}} \underset{(2.31)}{=} O\left(\Lambda^{\frac{nd}{2}}\right).$$

Let us write for  $O \in \mathcal{P}_m$  and  $t \in \mathbb{R}$

$$(2.48) \quad f_O(t) := \exp\left(i \frac{mA_n}{|\det(1 - dT_O^n)|} (\mu \cos(t) + \nu \sin(t))\right)$$

and for  $x \in \mathbb{R}^N$

$$(2.49) \quad g_n(x) := \frac{e^{-\frac{1}{2}\langle x, \Sigma_n^{-1}x \rangle}}{\sqrt{(2\pi)^N \det(\Sigma_n)}},$$

where  $\Sigma_n$  is the covariance matrix of  $(X_O^n)_{O \in \text{Per}(n)}$ : From Lemma 2.10,

$$(2.50) \quad \Sigma_n = \text{Diag}(\sigma_O^2)_{O \in \text{Per}(n)} + R_n,$$

$R_n$  being a matrix with entries uniformly  $O(e^{-kn})$  for every  $k > 0$ . The proposition will quickly lead to the pointwise convergence of the characteristic function, and its proof is a consequence of the following technical lemmas:

LEMMA 2.12. For  $k \in \mathbb{Z}^N$ , let

$$(2.51) \quad H_k := \prod_{O \in \text{Per}(n)} \left[ \frac{2\pi}{\xi} k_O, \frac{2\pi}{\xi} (k_O + 1) \right].$$

$$(2.52) \quad \left| \frac{E(\mu, \nu)}{\prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt} - 1 \right| \leq 4\pi \frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\|$$

LEMMA 2.13.  $\frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\| \rightarrow 0$  under condition (1.20).

PROOF OF LEMMA 2.12. Using Lemma 2.10, we see that

$$(2.53) \quad E(\mu, \nu) = \int_{\mathbb{R}^{\text{Per}(n)}} \left( \int_{\mathbb{R}^{\text{Per}(n)}} \left( \prod_{O \in \text{Per}(n)} f_O(\xi(x_O + y_O)) \right) g_n(x) dx \right) d\mathbb{P}_Y(y)$$

Let us write for a given  $y = (y_O)_{O \in \text{Per}(n)}$

$$(2.54) \quad E_y(\mu, \nu) = \int_{\mathbb{R}^{\text{Per}(n)}} \left( \prod_{O \in \text{Per}(n)} f_O(\xi(x_O + y_O)) \right) g_n(x) dx$$

Since, for fixed  $y$  the functions  $x_O \mapsto f_O(\xi(x_O + y_O))$  are fast oscillating periodic functions, of period  $\frac{2\pi}{\xi}$ , while the Gaussian factor is almost constant at this scale, we approximate the integral by splitting the space into hypercubes  $H_k$  of side length  $\frac{2\pi}{\xi}$ .

Each  $H_k$  has diameter  $2\pi \frac{\sqrt{N}}{\xi}$ . Let us notice that  $\int g_n = 1$  and that for every  $k \in \mathbb{Z}^N$  and  $y = (y_O)_{O \in \text{Per}(n)}$ ,

$$(2.55) \quad \int_{H_k} \left( \prod_{O \in \text{Per}(n)} f_O(\xi(x_O + y_O)) \right) dx = \frac{1}{\xi^N} \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt.$$

For any fixed  $y = (y_O)_{O \in \text{Per}(n)} \in \mathbb{R}^{\text{Per}(n)}$ , by periodicity,

$$\begin{aligned}
 (2.56) \quad & \left| E_y(\mu, \nu) - \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt \right| = \left| E_y(\mu, \nu) - \left( \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt \right) \int g_n \right| \\
 & \leq \left| E_y(\mu, \nu) - \left( \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt \right) \left( \frac{2\pi}{\xi} \right)^N \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} g_n \left( 2\pi \frac{k}{\xi} \right) \right| \\
 & \quad + \left| \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt \right| \left| \left( \frac{2\pi}{\xi} \right)^N \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} g_n \left( 2\pi \frac{k}{\xi} \right) - \int g_n \right|
 \end{aligned}$$

Splitting the integral in (2.54) into a sum of integrals over the  $H_k$ , and using (2.55) allows us to bound the first term of the right hand-side of (2.56) by

$$\begin{aligned}
 (2.57) \quad & \left| \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} \int_{H_k} \left( \prod_{O \in \text{Per}(n)} f_O(\xi(x_O + y_O)) \right) \left( g_n(x) - g_n \left( 2\pi \frac{k}{\xi} \right) \right) dx \right| \\
 & \leq \left| \prod_{O \in \text{Per}(n)} \int_0^{2\pi} f_O(t) dt \right| 2\pi \frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^N} \frac{\sup_{H_k} \|\nabla g_n\|}{\xi^N}
 \end{aligned}$$

by mean-value inequality. Likewise, in the second term,

$$\begin{aligned}
 (2.58) \quad & \left| \left( \frac{2\pi}{\xi} \right)^N \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} g_n \left( 2\pi \frac{k}{\xi} \right) - \int g_n \right| = \left| \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} \int_{H_k} \left( g_n \left( 2\pi \frac{k}{\xi} \right) - g_n(x) \right) dx \right| \\
 & \leq 2\pi \frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^{\text{Per}(n)}} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\|.
 \end{aligned}$$

This ends the proof of Lemma 2.12.  $\square$

PROOF OF LEMMA 2.13.

$$(2.59) \quad \|\nabla g_n(x)\| = \frac{\|\Sigma_n^{-1}x\| e^{-\frac{1}{2}\|\sqrt{\Sigma_n^{-1}}x\|^2}}{\sqrt{(2\pi)^N \det(\Sigma_n)}} \leq \|\sqrt{\Sigma_n^{-1}}\| \frac{\|\sqrt{\Sigma_n^{-1}}x\| e^{-\frac{1}{2}\|\sqrt{\Sigma_n^{-1}}x\|^2}}{\sqrt{(2\pi)^N \det(\Sigma_n)}}.$$

Since  $\|\nabla g_n\|$  is a function of  $\|\sqrt{\Sigma_n^{-1}}x\|$ , we will pack the terms of the sum

$$(2.60) \quad \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\|$$

by level sets of  $\|\Sigma_n^{-1}\|$ . Let us first observe that, by (2.50)

$$(2.61) \quad \sqrt{\Sigma_n^{-1}} = \text{Diag} \left( \frac{1}{\sigma_O} \right)_{O \in \text{Per}(n)} + O(e^{-kn})$$



for every  $k$  so, by Lemma 2.10

$$(2.62) \quad \begin{aligned} \|\sqrt{\Sigma_n^{-1}}\| &\stackrel{(2.35)}{=} O(h_n^{-d(\alpha-\frac{1}{2})-\gamma}) \\ &\stackrel{(2.5)}{=} O(\tilde{\Lambda}^{\frac{n}{2}(d(\alpha-\frac{1}{2})+\gamma)}). \end{aligned}$$

Provided that  $\tilde{\Lambda} > \Lambda$  and  $\gamma > 0$  are chosen small enough with respect to  $\frac{1}{c}$  in (1.20), writing  $1 + \delta := \frac{1}{c}$ , we have

$$(2.63) \quad \begin{aligned} \xi &\stackrel{(1.20)}{\geq} e^{\frac{n}{c} h_{\text{top}} \Lambda^{\frac{nd}{2c}(\alpha-\frac{1}{2})}} \\ &\geq e^{(1+\delta)n h_{\text{top}} \tilde{\Lambda}^{\frac{n}{2}(d(\alpha-\frac{1}{2})+\gamma)}} \\ &\stackrel{(2.62), (2.29)}{\geq} C N^{1+\delta} \|\sqrt{\Sigma_n^{-1}}\|. \end{aligned}$$

So

$$(2.64) \quad \exists \delta > 0, \|\sqrt{\Sigma_n^{-1}}\| = o\left(\frac{\xi}{N^{1+\delta}}\right).$$

Let us now fix such a  $\delta$  and write for  $j \geq 0$

$$(2.65) \quad C_j := \left\{ x \in \mathbb{R}^N, \frac{j}{N^{\frac{1+\delta}{2}}} < \|\sqrt{\Sigma_n^{-1}}x\| \leq \frac{j+1}{N^{\frac{1+\delta}{2}}} \right\}.$$

The diameter of the cubes  $H_k$  is then very small compared to the distance between  $C_j$  and  $C_{j+2}$  for every  $j$  (see Figure 1): Indeed,

$$(2.66) \quad x \in C_j \implies \|\sqrt{\Sigma_n^{-1}}x\| \leq \frac{j+1}{N^{\frac{1+\delta}{2}}}$$

and

$$(2.67) \quad y \in C_{j+2} \implies \|\sqrt{\Sigma_n^{-1}}y\| \geq \frac{j+2}{N^{\frac{1+\delta}{2}}}.$$

So by triangular inequality

$$(2.68) \quad \frac{1}{N^{\frac{1+\delta}{2}}} \leq \|\sqrt{\Sigma_n^{-1}}y\| - \|\sqrt{\Sigma_n^{-1}}x\| \leq \|\sqrt{\Sigma_n^{-1}}(x-y)\| \leq \|\sqrt{\Sigma_n^{-1}}\| \|x-y\|.$$

Thus by (2.64), for all  $j, k$ ,

$$(2.69) \quad d(C_{j-2}, C_j) \geq \frac{1}{N^{\frac{1+\delta}{2}} \|\sqrt{\Sigma_n^{-1}}x\|} \geq C \frac{N^{\frac{1+\delta}{2}}}{\xi} \gg 2\pi \frac{\sqrt{N}}{\xi} = \text{Diam}(H_k).$$

Consequently, we have

$$(2.70) \quad \{H_k, H_k \cap C_j \neq \emptyset\} \subset C_{j-1} \cup C_j \cup C_{j+1}.$$

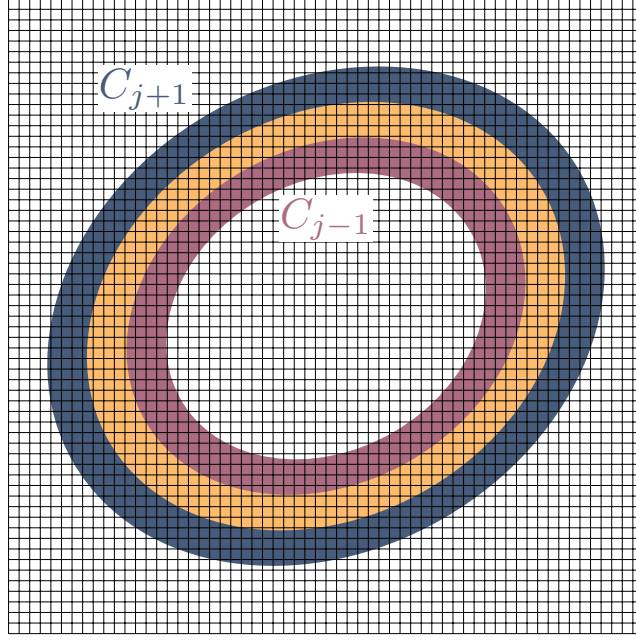


FIGURE 1. The hypercubes  $H_k$ , pictured as a grid, have small diameter, relatively to the distance between the annuli  $C_j$ ,  $C_{j+2}$  (or  $C_j$  and  $C_{j-2}$ ). Consequently, the supremum  $\sup_{H_k} \|\nabla g_n\|$  in (2.60) can be replaced by a supremum over  $C_{j-1} \cap C_j \cap C_{j+1}$  if  $H_k$  intersects  $C_j$ .

Thus,

$$\begin{aligned}
 (2.71) \quad \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\| &\leq \sum_{j \in \mathbb{N}} \sum_{k, H_k \cap C_j \neq \emptyset} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\| \\
 &\leq \sum_{j \in \mathbb{N}} \#\{k, H_k \cap C_j \neq \emptyset\} \left( \frac{2\pi}{\xi} \right)^N \sup_{C_{j-1} \cup C_j \cup C_{j+1}} \|\nabla g_n\|.
 \end{aligned}$$

We deduce from (2.70) that

$$\begin{aligned}
 (2.72) \quad \#\{k, H_k \cap C_j \neq \emptyset\} \left( \frac{2\pi}{\xi} \right)^N &= \#\{k, H_k \cap C_j \neq \emptyset\} \cdot \text{Vol}(H_k) \\
 &\leq \text{Vol} \left( \bigcup_{l \leq j+1} C_l \right) \\
 &= \text{Vol} \left\{ x, \|\sqrt{\Sigma_n^{-1}} x\| \leq \frac{j+2}{N^{\frac{1+\delta}{2}}} \right\} \\
 &= \sqrt{\det \Sigma_n} \left( \frac{j+2}{N^{\frac{1+\delta}{2}}} \right)^N \text{Vol}(B(0, 1)).
 \end{aligned}$$

And we also have

$$(2.73) \quad \sup_{C_{j-1} \cup C_j \cup C_{j+1}} \|\nabla g_n\| \stackrel{(2.64), (2.59)}{\leq} \frac{1}{N^{\frac{1}{2}+\delta}} \frac{\xi}{\sqrt{N}} \frac{j+2}{N^{\frac{1+\delta}{2}}} \frac{e^{-\frac{1}{2} \frac{(j-1)^2}{N^{1+\delta}}}}{\sqrt{(2\pi)^N \det(\Sigma_n)}}$$

We know moreover (see for instance [B<sup>+</sup>97] p.5) that the unit ball of dimension  $N$  has a volume equivalent to

$$(2.74) \quad \frac{1}{\sqrt{\pi}} \frac{(2\pi e)^{N/2}}{N^{\frac{N+1}{2}}}.$$

Finally, putting together (2.71), (2.72), (2.73), and (2.74), we have the following upper bound for the sum

$$(2.75) \quad \begin{aligned} & \frac{\sqrt{N}}{\xi} \sum_{k \in \mathbb{Z}^N} \left( \frac{2\pi}{\xi} \right)^N \sup_{H_k} \|\nabla g_n\| \\ & \stackrel{(2.71), (2.72)}{\leq} \frac{\sqrt{N}}{\xi} \text{Vol} B(0, 1) \sqrt{\det \Sigma_n} \sum_{j \geq 0} \left( \frac{j+2}{N^{\frac{1+\delta}{2}}} \right)^N \sup_{C_{j-1} \cup C_j \cup C_{j+1}} \|\nabla g_n\| \\ & \stackrel{(2.73)}{\leq} \frac{1}{N^{\frac{1}{2}+\delta}} \text{Vol} B(0, 1) \sum_{j \geq 0} \left( \frac{j+2}{N^{\frac{1+\delta}{2}}} \right)^{N+1} \frac{e^{-\frac{1}{2} \frac{(j-1)^2}{N^{1+\delta}}}}{(2\pi)^{N/2}} \\ & \stackrel{(2.74)}{\leq} C \frac{e^{N/2}}{N^{(N+1)(\frac{1+\delta}{2})}} \frac{1}{N^{\frac{1}{2}+\delta}} \frac{1}{N^{\frac{N+1}{2}}} \sum_{j \geq 0} (j+2)^{N+1} e^{-\frac{1}{2} \frac{(j-1)^2}{N^{1+\delta}}} \\ & = C \frac{e^{N/2}}{N^{(N+1)(1+\frac{\delta}{2})}} \frac{1}{N^{\frac{1}{2}+\delta}} \sum_{j \geq 0} (j+2)^{N+1} e^{-\frac{1}{2} \frac{(j-1)^2}{N^{1+\delta}}}. \end{aligned}$$

To conclude, we will now show that

$$(2.76) \quad \frac{e^{N/2}}{N^{(N+1)(1+\frac{\delta}{2})}} \sum_{j \geq 0} (j+2)^{N+1} e^{-\frac{1}{2} \frac{(j-1)^2}{N^{1+\delta}}} = O\left(N^{\frac{1}{2} + \frac{2\delta}{3}}\right) :$$

LEMMA 2.14. *Let*

$$(2.77) \quad f_N : \begin{cases} \mathbb{R} & \longrightarrow \mathbb{R} \\ x & \longmapsto \frac{e^{N/2}}{N^{(N+1)(1+\frac{\delta}{2})}} (x+2)^{N+1} e^{-\frac{(x-1)^2}{2N^{1+\delta}}} \end{cases}.$$

*It is first increasing then decreasing and admits a maximum  $O(1)$  at*

$$(2.78) \quad x_0 \sim N^{1+\frac{\delta}{2}}$$

*and decays at a scale  $N^{\frac{1}{2} + \frac{2}{3}\delta}$ : If we write for  $k \geq -1$ ,*

$$(2.79) \quad x_k = x_0 + kN^{\frac{1}{2} + \frac{2}{3}\delta},$$

*we have*

$$(2.80) \quad f_N(x_k) = O(e^{k - \frac{1}{2} k^2 N^{\frac{\delta}{3}}})$$

*as  $N$  goes to infinity.*

PROOF. Differentiating the logarithm of  $f_N$  tells us that the function is increasing, then decreasing, and that the maximum is attained for  $x_0$  such that

$$(2.81) \quad x_0^2 + x_0 - 2 = N^{2+\delta} + N^{1+\delta}.$$

This shows that

$$(2.82) \quad x_0 = N^{1+\frac{\delta}{2}} + \frac{1}{2}N^{\frac{\delta}{2}}(1 + o(1)),$$

so

$$(2.83) \quad x_0 = N^{1+\frac{\delta}{2}} + O(N^{\frac{\delta}{2}}) = N^{1+\frac{\delta}{2}}(1 + O(N^{-1}))$$

and hence

$$(2.84) \quad \begin{aligned} \max \log f_N &= \frac{N}{2} - (N+1)(1 + \frac{\delta}{2}) \log N \\ &\quad + (N+1) \log \left( N^{1+\frac{\delta}{2}}(1 + O(N^{-1})) \right) - \frac{1}{2} \frac{\left( N^{1+\frac{\delta}{2}} + O(N^{\frac{\delta}{2}}) \right)^2}{N^{1+\delta}} = O(1) \end{aligned}$$

(2.85)

$$x_k = x_0 + kN^{\frac{1}{2}+\frac{2}{3}\delta} = N^{1+\frac{\delta}{2}} + kN^{\frac{1}{2}+\frac{2}{3}\delta} + O(N^{\frac{\delta}{2}}) = N^{1+\frac{\delta}{2}}(1 + kN^{-\frac{1}{2}+\frac{\delta}{6}} + O(N^{-1})).$$

The remainder does not depend on  $k$ . Therefore, for  $N \geq 1$

$$(2.86) \quad \begin{aligned} \log f_N(x_k) &= \frac{N}{2} - (N+1)(1 + \frac{\delta}{2}) \log N \\ &\quad + (N+1) \log \left( N^{1+\frac{\delta}{2}}(1 + kN^{-\frac{1}{2}+\frac{\delta}{6}} + O(N^{-1})) \right) - \frac{1}{2} \frac{\left( N^{1+\frac{\delta}{2}} + kN^{\frac{1}{2}+\frac{2}{3}\delta} + O(N^{\frac{\delta}{2}}) \right)^2}{N^{1+\delta}} \\ &\leq k(N+1)N^{-\frac{1}{2}+\frac{\delta}{6}} + O(1) - kN^{\frac{1}{2}+\frac{\delta}{6}} - \frac{1}{2}k^2N^{\frac{\delta}{3}} + O(1) \\ &= -\frac{1}{2}k^2N^{\frac{\delta}{3}} + k + O(1). \end{aligned}$$

Consequently,

$$(2.87) \quad \begin{aligned} \sum_{j \geq 0} f_N(j) &= \sum_{j < x_{-1}} f_N(j) + \sum_{x_{-1} \leq j < x_1} f_N(j) + \sum_{k=1}^{+\infty} \sum_{x_k \leq j < x_{k+1}} f_N(j) \\ &\leq f_N(x_{-1})O(N^{1+\delta}) + O(N^{\frac{1}{2}+\frac{2}{3}\delta}) + O(N^{\frac{1}{2}+\frac{2}{3}\delta}) \\ &= O(N^{\frac{1}{2}+\frac{2}{3}\delta}). \end{aligned}$$

□

Together with (2.75), this concludes the proof of Lemma 2.13, and hence the proof of Proposition 2.11. □

□

In order to conclude the proof, we need the following lemma, whose proof is postponed to Appendix 3.B.

LEMMA 2.15.

$$(2.88) \quad nA_n \sup_{O \in \text{Per}(n)} \frac{1}{|\det(1 - dT_O^n)|} \xrightarrow{n \rightarrow \infty} 0.$$

Since

$$(2.89) \quad \int_{\mathbb{R}} e^{i(\mu \cos u + \nu \sin u)} du = 1 - \frac{\mu^2 + \nu^2}{4} + o(\mu^2 + \nu^2),$$

we obtain for a given  $(\mu, \nu) \in \mathbb{R}^2$

$$(2.90) \quad \begin{aligned} \prod_{O \in \text{Per}(n)} \int_{\mathbb{R}} f_O(u) du &= \prod_{m|n} \prod_{O \in \mathcal{P}_m} \left( 1 - \frac{\mu^2 + \nu^2}{4} \frac{m^2 A_n^2}{|\det(1 - dT_O^n)|^2} + o\left(\frac{m^2 A_n^2}{|\det(1 - dT_O^n)|^2}\right) \right) \\ &= \sum_{m|n} \sum_{O \in \mathcal{P}_m} \log \left( 1 - \frac{\mu^2 + \nu^2}{4} \frac{m^2 A_n^2}{|\det(1 - dT_O^n)|^2} + o\left(\frac{m^2 A_n^2}{|\det(1 - dT_O^n)|^2}\right) \right) \\ &= e^{-\frac{\mu^2 + \nu^2}{4} \sum_{m|n} \sum_{O \in \mathcal{P}_m} \frac{m^2 A_n^2}{|\det(1 - dT_O^n)|^2} + o\left(\sum_{m|n} \sum_{O \in \mathcal{P}_m} \frac{m^2 A_n^2}{|\det(1 - dT_O^n)|^2}\right)} \\ &\stackrel{(1.14)}{=} e^{-\frac{\mu^2 + \nu^2}{4} + o(1)}. \end{aligned}$$

Levy's theorem then implies the convergence in law of  $A_n \text{Tr}^b(\mathcal{L}_{\xi, \tau}^n)$  towards a complex Gaussian law  $\mathcal{N}_{\mathbb{C}}(0, 1)$ .

### Appendix Appendix 3.A Upper bound for the distance between periodic points of a given period

Note that for a given period, there is a finite number of periodic points, whose mutual distance is bounded below: Then,

LEMMA 3.A.1.

(3.A.1)

$$\forall \Lambda' > \Lambda, \exists C > 0, \forall n \in \mathbb{N}, \forall x \neq y \in M, (T^n x = x, T^n y = y) \implies d(x, y) \geq \frac{C}{\Lambda'^{\frac{n}{2}}}.$$

PROOF. Anosov diffeomorphisms are expansive (see [KH97], p.125):

$$(3.A.2) \quad \exists \delta > 0, \forall x, y \in M, (\forall k \in \mathbb{Z}, d(T^k(x), T^k(y)) \leq \delta) \implies x = y.$$

Let  $\Lambda' > \Lambda$ , and let  $n_0 \in \mathbb{N}$  such that

$$(3.A.3) \quad \forall |n| \geq n_0, \max_{x \in M} \|dT_x^n\| \leq \Lambda'^{|n|}.$$

Let  $L = \max_{x \in M} \|dT_x^{\pm 1}\| \geq \Lambda$ . For any  $x, y \in M$ ,

$$(3.A.4) \quad d(T^k x, T^k y) \leq \begin{cases} \Lambda'^{|k|} & \text{if } |k| \geq n_0 \\ L^{|k|} & \text{if } |k| < n_0 \end{cases}$$

Writing  $C = \left(\frac{L}{\Lambda'}\right)^{n_0}$ , we get for all  $n \in \mathbb{N}$ , for all periodic points  $x, y$  of period  $n$ , for all  $-n/2 \leq k \leq n/2$ ,

$$(3.A.5) \quad d(T^k x, T^k y) \leq C \Lambda'^{n/2} d(x, y).$$

Therefore, if  $d(x, y) \leq \frac{\delta}{C \Lambda'^{n/2}}$ , then for all  $-\frac{n}{2} \leq k \leq \frac{n}{2}$ , (thus for all  $k \in \mathbb{Z}$ )  $d(T^k x, T^k y) \leq \delta$  and  $x = y$ .  $\square$

### Appendix Appendix 3.B Proof of Lemma 2.15

#### 3.B.1 Topological pressure.

PROPOSITION 3.B.1 ([KH97, Proposition 20.3.3]). *Let  $\phi : M \rightarrow \mathbb{R}$  be a Hölder function. The topological pressure of  $\phi$  is given by*

$$(3.B.1) \quad \Pr(\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{x=T^n(x)} e^{\phi_x^n} \right).$$

**3.B.2 Variational principle.** The topological pressure of a Hölder function is given by a variational principle:

THEOREM 3.B.2 ([Bow75, Theorem 2.17]). *Let  $\phi : M \rightarrow \mathbb{R}$  be a Hölder function.*

$$(3.B.2) \quad \Pr(\phi) = \sup_{\mu} (h_{\mu}(T) + \int_M \phi \, d\mu)$$

where  $\mu$  goes through the  $T$ -invariant probability measures. This supremum is attained at a unique measure [KH97, Proposition 20.3.7] called equilibrium measure of  $\phi$ .

Let us write

$$(3.B.3) \quad J_u(x) := \log |\det dT|_{E^u(x)}|.$$

$J_u$  is said to be cohomologous to a constant if

$$(3.B.4) \quad \exists c \in \mathbb{R}, \exists h \in \mathcal{C}(M), J_u = c + h \circ T - h.$$

LEMMA 3.B.3 ([KH97, Proposition 20.3.10]). *Let  $\mu_\beta$  be the equilibrium measure of  $-\beta J_u$ ,  $\beta > 0$ . If  $J_u$  is not cohomologous to a constant, the map  $\beta \in \mathbb{R}_+ \mapsto \mu_\beta$  is injective.*

COROLLARY 3.B.4. *As a consequence, the function*

$$(3.B.5) \quad F : \begin{cases} \mathbb{R}_+^* & \longrightarrow \mathbb{R} \\ \beta & \longmapsto \frac{1}{\beta} \Pr(-\beta J_u) \end{cases}$$

*is strictly decreasing.*

PROOF. If  $J_u$  is cohomologous to a constant  $c$ , then by Proposition 2.7,

$$(3.B.6) \quad \sum_{T^n(x)=x} e^{-\beta(J_u)_x^n} \asymp e^{n(h_{\text{top}} - c\beta)}.$$

Hence,

$$(3.B.7) \quad \Pr(-\beta J_u) = h_{\text{top}} - c\beta,$$

with  $h_{\text{top}} > 0$ , which gives the wanted result. If  $J_u$  is not cohomologous to a constant, then the previous Lemma applies. Let  $\beta' > \beta > 0$ .

$$(3.B.8) \quad \int -\beta J_u \, d\mu_\beta = \Pr(-\beta J_u) > \int -\beta' J_u \, d\mu_{\beta'}$$

Consequently,

$$(3.B.9) \quad F(\beta) = \int -J_u \, d\beta + \frac{h(\mu_\beta)}{\beta} > \int -J_u \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta} \geq \int -J_u \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta'} = F(\beta').$$

□

$F$  is moreover bounded below, and admits consequently a limit as  $\beta$  goes to infinity.

**3.B.3 End of proof of Lemma 2.15.** The quantity  $\inf_{x \in M} (J_u)_x^n$  is sub-additive. We can consequently define

$$(3.B.10) \quad J_u^{\min} := \lim_{n \rightarrow \infty} \inf_{x \in M} \frac{1}{n} (J_u)_x^n.$$

LEMMA 3.B.5. *The infimum can be taken over periodic points of period  $n$ :*

$$(3.B.11) \quad J_u^{\min} = \lim_{n \rightarrow \infty} \inf_{T^n(x)=x} \frac{1}{n} (J_u)_x^n.$$

PROOF. For  $n \in \mathbb{N}$ , let  $x_n \in M$  be the point such that

$$(3.B.12) \quad (J_u)_{x_n}^n = \inf_{x \in M} (J_u)_x^n.$$

Let  $\varepsilon > 0$ . By the specification property [KH97, Theorem 18.3.9], there exists  $M \in \mathbb{N}$  and a periodic point  $x'_n$  of period  $n + M$  such that

$$(3.B.13) \quad \forall 0 \leq j \leq n-1, d(T^j(x_n), T^j(x'_n)) \leq \varepsilon.$$

Thus, for some  $C > 0$ ,

$$(3.B.14) \quad \frac{1}{n} \inf_{x \in M} (J_u)_x^n \leq \frac{1}{n} \inf_{T^{n+M}(x)=x} (J_u)_x^n \leq \frac{1}{n} \inf_{x \in M} (J_u)_x^n + C\varepsilon.$$

Then, we have

$$(3.B.15) \quad \begin{aligned} \frac{1}{n+M} \inf_{x \in M} (J_u)_x^{n+M} - C \frac{M}{n} &\leq \frac{1}{n+M} \inf_{T^{n+M}(x)=x} (J_u)_x^{n+M} \\ &\leq \frac{1}{n+M} \inf_{x \in M} (J_u)_x^{n+M} + C\varepsilon + C \frac{M}{n}. \end{aligned}$$

As a consequence,

$$(3.B.16) \quad \inf_{T^n(x)=x} \frac{1}{n} (J_u)_x^n \xrightarrow{n \rightarrow \infty} J_u^{\min}.$$

□

LEMMA 3.B.6.

$$(3.B.17) \quad F(\beta) \xrightarrow{\beta \rightarrow \infty} -J_u^{\min}$$

PROOF. Let

$$(3.B.18) \quad F_n(\beta) := \frac{1}{n\beta} \log \sum_{T^n(x)=x} e^{-\beta(J_u)_x^n} \xrightarrow{n \rightarrow \infty} F(\beta).$$

Let  $\varepsilon > 0$ . If  $n$  is large enough, for every periodic point  $x$  of period  $n$

$$(3.B.19) \quad (J_u)_x^n \geq n(J_u^{\min} - \varepsilon).$$

Moreover, there exists a periodic point of period  $n$  such that

$$(3.B.20) \quad (J_u)_x^n \leq n(J_u^{\min} + \varepsilon).$$

So

$$(3.B.21) \quad e^{-\beta n(J_u^{\min} + \varepsilon)} \leq \sum_{T^n(x)=x} e^{-\beta(J_u)_x^n} \leq e^{n h_{\text{top}}} e^{-\beta n(J_u^{\min} - \varepsilon)}.$$

Taking the logarithm and taking  $\varepsilon \rightarrow 0$  gives for every  $\beta > 0$

$$(3.B.22) \quad -J_u^{\min} \leq F(\beta) \leq \frac{h_{\text{top}}}{\beta} - J_u^{\min},$$

hence the result. □

The proof of Lemma 2.15 then derives from the following facts:

$$(3.B.23) \quad \sup_{O \in \text{Per}(n)} \frac{1}{|\det(1 - dT_O^n)|} = e^{-n J_u^{\min}} (1 + o(1)) = e^{n \lim_{\beta \rightarrow \infty} F(\beta) + o(n)}$$

and

$$(3.B.24) \quad A_n = e^{-n F(2) + o(n)}.$$

(3.B.23) is clear, and the definition (1.14) implies that

$$(3.B.25) \quad \frac{1}{\sqrt{n}} \left( \sum_{T^n(x)=x} e^{-n(J_u)_x^n} (1 + o(1)) \right)^{-\frac{1}{2}} \leq A_n \leq \left( \sum_{T^n(x)=x} e^{-n(J_u)_x^n} (1 + o(1)) \right)^{-\frac{1}{2}}.$$



### Appendix Appendix 3.C Proof of Proposition 1.4

These results are based on the fact that if  $\zeta_j$  are i.i.d. centered Gaussian variables of variance 1, then, almost surely, for every  $\varepsilon > 0$ ,

$$(3.C.1) \quad \zeta_j = o(j^\varepsilon).$$

It is a consequence of Borel-Cantelli Lemma:

$$(3.C.2) \quad \mathbb{P}(|\zeta_j| > j^\varepsilon) \leq \frac{2}{\sqrt{2\pi}} e^{-j^{2\varepsilon}/2},$$

it is thus summable. The second ingredient is Weyl's law (2.24): the asymptotics of the sequence of eigenvalues  $(\lambda_j)$  of the Laplace-Beltrami operator is given by

$$(3.C.3) \quad \lambda_j \sim C j^{\frac{2}{d}}$$

for some constant  $C$  depending only on  $M$  (and the metric).

PROOF OF PROPOSITION 1.4. If  $c_j = O(j^{-\alpha})$ , then for  $s < d(\alpha - \frac{1}{2})$ , or equivalently  $\alpha > \frac{s}{d} + \frac{1}{2}$ , almost surely,

$$(3.C.4) \quad |c_j \zeta_j|^2 \stackrel{(3.C.1)}{=} O(j^{-2s/d-1-\delta})$$

for some  $\delta > 0$ . Thus,

$$(3.C.5) \quad \sum_j |c_j \zeta_j|^2 j^{2\frac{s}{d}}$$

converges, and so does

$$(3.C.6) \quad \sum_j |c_j \zeta_j|^2 (1 + \lambda_j)^s$$

by Weyl's law (3.C.3). Consequently, almost surely,

$$(3.C.7) \quad (1 + \Delta)^{s/2} f = \sum_j c_j \zeta_j (1 + \lambda_j)^{\frac{s}{2}} \phi_j \in L^2.$$

In other terms,  $f$  belongs almost surely to  $H^s$ . The statement about  $\mathcal{C}^k(M)$  follows from the classical Sobolev embedding theorem:

THEOREM 3.C.1 ([Tay96, Proposition 3.3 p.282]). *For  $s > k + d/2$ ,*

$$(3.C.8) \quad H^s(M) \subset \mathcal{C}^k(M).$$

□

### Appendix Appendix 3.D Ruelle spectrum and flat trace

THEOREM 3.D.1 ([Bal18, Theorem 5.1]). *In our setting, if  $\tau$  is  $\mathcal{C}^k$ , there exists a Banach space  $B^k$  such that*

- $\mathcal{L}_{\xi, \tau} : B^k \longrightarrow B^k$  is a bounded operator,
- For any  $\alpha > \frac{k}{2}$ ,

$$(3.D.1) \quad \mathcal{C}^\alpha(M) \subset B^k \subset \mathcal{C}^\alpha(M)',$$

- The spectral radius  $r(\mathcal{L}_{\xi, \tau})$  of  $\mathcal{L}_{\xi, \tau}$  in  $B^k$  is

$$(3.D.2) \quad r(\mathcal{L}_{\xi, \tau}) \leq 1,$$

- The essential spectral radius  $r_{ess}(\mathcal{L}_{\xi,\tau})$  of  $\mathcal{L}_{\xi,\tau}$  in  $B^k$  satisfies

$$(3.D.3) \quad r_{ess}(\mathcal{L}_{\xi,\tau}) \leq m^k,$$

where  $m$  is the smallest constant satisfying the definition of Anosov diffeomorphism from Section 1.1.

This implies that the resolvent of the transfer operator admits a meromorphic extension as an operator  $\mathcal{C}^\infty(M) \rightarrow \mathcal{D}'(M)$  to the region

$$(3.D.4) \quad \{|z| > m^k\}$$

and that its poles in this domain are the eigenvalues of  $\mathcal{L}_{\xi,\tau}$  with same multiplicities and spectral projectors. These poles are called Ruelle-Pollicott resonances. For analytic Anosov maps, there are spaces in which the transfer operator is trace-class. There is however no hope for this to be true with less regularity. Yet, an analog of the trace can be defined, that coincides with the usual trace in the analytic case:

LEMMA 3.D.2 ([Bal18, Section 6.2]). *The operators  $\mathcal{L}_{\xi,\tau}^n, n \geq 1$  have integral kernel*

$$(3.D.5) \quad K_{\xi,\tau}^n(x, y) = e^{i\xi\tau_x^n} \delta(y - T^n(x)),$$

where

$$(3.D.6) \quad \tau_x^n := \sum_{j=0}^{n-1} \tau(T^j(x)).$$

Introducing a mollification, the distribution  $K_{\xi,\tau}^n$  can be integrated along the diagonal  $\{(x, x), x \in M\}$ . The resulting quantity is called flat trace of  $\mathcal{L}_{\xi,\tau}^n$  and can be expressed as a sum over periodic points:

$$(3.D.7) \quad \text{Tr}^b \mathcal{L}_{\xi,\tau}^n := \int_M K_{\xi,\tau}^n(x, x) dx = \sum_{x, T^n(x)=x} \frac{e^{i\xi\tau_x^n}}{|\det(1 - d(T^n)_x)|}.$$

This notion of trace is linked to the eigenvalues of the operator in the following way ([Bal18, Theorem 6.2], [Jéz17, Theorem 2.4]):

THEOREM 3.D.3. *Let  $\xi \in \mathbb{R}$ , let  $\varepsilon > 0$  be such that  $\mathcal{L}_{\xi,\tau}$  has no resonance of modulus  $\varepsilon$ , then*

$$(3.D.8) \quad \exists C_\xi > 0, \forall n \in \mathbb{N}, \left| \text{Tr}^b \mathcal{L}_{\xi,\tau}^n - \sum_{\substack{\lambda \in \text{Res}(\mathcal{L}_{\xi,\tau}) \\ |\lambda| > \varepsilon}} \lambda^n \right| \leq C_\xi \varepsilon^n.$$

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