



Milnor-Witt sheaves and modules

Niels Feld

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THÈSE

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l'information, Informatique**

Faisceaux et modules de Milnor-Witt Milnor-Witt sheaves and modules

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*"You gotta do it everyday, that's the hard part.
But it does get easier."
A jogging baboon.*

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Introduction (in French)

Approche historique

Modules de cycles de Rost Dans les années quatre-vingt-dix, Rost développa la théorie des modules de cycles [Ros96] qui fut utilisée dans la démonstration de la conjecture de Milnor par Voevodsky (voir [SV00] ou [Voe11]). En effet, une étape cruciale dans la preuve était la construction du motif associé à certaines formes quadratiques [Ros98]. Dans [Ros96], un module de cycles M sur un corps parfait k est la donnée d'un groupe abélien \mathbb{Z} -gradué $M(E)$ pour toute extension de corps finiment engendrée E/k , équipé de morphismes de restrictions, de corestriction, d'une structure de module au-dessus de la K-théorie de Milnor et de morphismes de résidus ∂ . Ces données sont soumises à certains axiomes $(r1a), \dots, (r3e)$ de sorte que la K-théorie de Milnor représente un module de cycles de Rost. À partir de définitions formelles, Rost produisit une théorie générale qui englobait les précédentes (par exemple, la K-théorie de Quillen et la cohomologie étale). De plus, la théorie de Rost est fondamentalement liée à la théorie des motifs : un module de cycles peut être réalisé géométriquement. Cela peut être illustré par le théorème suivant.

Theorem 1 (voir [Dé03]). *Soit k un corps parfait. La catégorie des modules de cycles de Rost sur k est équivalente au cœur de la catégorie des motifs de Voevodsky $\mathbf{DM}(k, \mathbb{Z})$ munie de sa t -structure homotopique.*

Théorie \mathbb{A}^1 -homotopique L'idée de la théorie \mathbb{A}^1 -homotopique, de Morel et Voevodsky, était d'appliquer les techniques de topologie algébrique à l'étude des schémas (la droite affine \mathbb{A}^1 jouant alors le rôle de l'intervalle unité $[0, 1]$). Cette idée engendra de nombreux résultats importants et de nouvelles catégories, comme $\mathbf{DM}(k, \mathbb{Z})$ mentionnée plus haut et $\mathbf{SH}(k)$, la catégorie homotopique stable [MV99].

K-théorie de Milnor-Witt Dans son étude de $\mathbf{SH}(k)$, Morel (avec l'aide de Hopkins) définit pour un corps E la K-théorie de Milnor-Witt $\mathbf{K}_*^{\text{MW}}(E)$ (cf [Mor12, Definition 3.1]). Ce groupe abélien \mathbb{Z} -gradué se comporte en degrés positifs comme les groupes de K-théorie de Milnor $\mathbf{K}_n^M(E)$, et en degrés négatifs comme les groupes de Witt $W(E)$ et de Grothendieck-Witt $\text{GW}(E)$. La K-théorie de Milnor-Witt fut utilisée pour résoudre certains problèmes de scindages pour des modules projectifs. Par exemple, généralisant des idées de la théorie des groupes de Chow, on peut utiliser le complexe de Rost-Schmid pour définir les groupes de Chow-Witt $\widetilde{\text{CH}}^*(X)$ d'un k -schéma lisse X (cf 2.5.2.1), et la classe d'Euler d'un fibré vectoriel de rang r au-dessus de X (avec la donnée d'une trivialisation de son déterminant) comme élément de $\widetilde{\text{CH}}^r(X)$. Lorsque $X = \text{Spec } A$ est une variété affine lisse de dimension 2 et $r = 2$, Barge et Morel ont démontré dans [BM00]

que la classe d'Euler associée à un module projectif P de rang 2 sur A est nulle si, et seulement si, $P \simeq P' \oplus A$ pour un certain module projectif P' (ce résultat a été ensuite étendu en dimension 3 [FS09], et en dimension 4 et plus [Mor12]).

Faisceaux homotopiques avec transferts de Voevodsky Afin de construire la catégorie dérivée des motifs $\mathbf{DM}(k, \mathbb{Z})$, Voevodsky introduisit les faisceaux homotopiques (avec transferts) qui sont des faisceaux Nisnevich homotopiquement invariant avec transferts. Un exemple important est celui du faisceau des unités \mathbb{G}_m . Voevodsky démontra que tout faisceau homotopique avec transferts F possède une résolution de Gersten, sous-entendant que F est déterminée en un certain sens par les données de ses fibres en tout points. Cette affirmation fut détaillée dans la thèse de Déglise : le cœur de $\mathbf{DM}(k, \mathbb{Z})$ pour la t-structure homotopique peut être représentée par la catégorie des modules de cycles de Rost sur k .

Modules homotopiques orientés Le point de vue de Morel sur le cœur de $\mathbf{DM}(k, \mathbb{Z})$ est illustré par la catégorie des modules homotopiques orientés. On rappelle qu'un module homotopique est un faisceau gradué Nisnevich strictement \mathbb{A}^1 -invariant muni d'une donnée de stabilité, défini sur la catégorie des schémas lisses (cf la définition 3.4.1.2) ; il est dit *orienté* lorsque l'application de Hopf η agit trivialement. Le théorème de Déglise montre que les modules homotopiques orientés forment une sous-catégorie pleine de la catégorie des modules homotopiques qui est équivalente à la catégorie des modules de cycles de Rost. La conjecture naturelle de Morel [Mor12, Remark 2.49] était qu'il existe une présentation du cœur de la catégorie homotopique stable $\mathbf{SH}(k)$ (ou de manière équivalente, la catégorie des modules homotopiques) en termes de modules de cycles non-orientés à définir.

Travail présent

Modules de cycles de Milnor-Witt

Dans le chapitre 2 (voir aussi [Fel18]), on développe une théorie qui étudie des complexes de cycles généraux $C^*(X, M, \mathcal{V}_X)$ et leurs groupes de (co)homologie $A^*(X, M, \mathcal{V}_X)$ (appelés *groupes de Chow-Witt à coefficients*) dans un contexte quadratique¹. Les systèmes de coefficients M pour ces complexes sont nommés *modules de cycles de Milnor-Witt*. L'exemple principal de tel module de cycles est donné par la K-théorie de Milnor-Witt (voir le théorème 2.3.2.13) ; d'autres exemples peuvent être déduit du théorème 3.4.0.4, de l'exemple 2.3.2.7 ou bien du théorème 2.8.2.5. Une différence majeure avec la théorie de Rost est que la graduation à considérer n'est pas \mathbb{Z} mais la catégorie des fibrés virtuels (ou, de manière équivalente, la catégorie des espaces vectoriels virtuels), où un fibré virtuel sur un corps est (grossièrement) représenté par la donnée d'un entier relatif et d'un fibré en droites \mathcal{L} (cf la sous-section 1.9.1).

¹Ou plutôt : *bilinéaire*, puisque l'on travaille aussi avec des corps de caractéristique 2.

Intuitivement, un module de cycles de Milnor-Witt est donné par un groupe abélien gradué équipé de données supplémentaires (restrictions, corestrictions, action de \mathbb{K}^{MW} et flèches de résidus). Le plus difficile fut de trouver les bons axiomes singeant ceux de Rost et prenant en compte les tortillons (*twists*) apparaissant naturellement dans ce contexte non-orienté (les fibrés vectoriels virtuels jouent un rôle primordial). Une différence notable avec la définition de Rost [Ros96, Definition 1.1] est que l'on a seulement besoin de l'axiome affaibli (R1c) : on ne considère pas de multiplicités (on montrera plus tard des formules avec multiplicités découlant de nos axiomes). De plus, on a eu besoin de rajouter un axiome dans nos définitions (voir (R4a) et la sous-section 2.8.2).

Suivant [Ros96, §3], on définit les opérations nécessaires pour la suite : poussés-en-avant, tirés-en-arrière, multiplication par les unités et la flèche motivique de Hopf η , et les flèches de bord. Comme nos règles ne manipulent pas de multiplicités, on ne définit le tiré-en-arrière que si l'application est lisse. On démontre les théorèmes de fonctorialité usuels et montre comment calculer ces groupes de cohomologie dans certains cas.

On peut résumer nos résultats comme suit :

Theorem 2 (voir la sous-section 2.4.1 et la sous-section 2.4.2). *Pour tout schéma X sur un corps parfait, tout fibré virtuel \mathcal{V}_X et tout module de cycles de Milnor-Witt M , il y a un complexe $C^*(X, M, \mathcal{V}_X)$ équipé de tirés-en-arrière, poussés-en-avant, d'une action de la K -théorie de Milnor-Witt et de flèches de résidus satisfaisant les propriétés fonctorielles usuelles.*

Ce formalisme généralise la thèse de Schmid [Sch98] et précise le travail de Morel [Mor12].

On présente un exemple ; considérons la suite de localisation :

$$\widetilde{\mathrm{CH}}_p(Z) \rightarrow \widetilde{\mathrm{CH}}_p(X) \rightarrow \widetilde{\mathrm{CH}}_p(X \setminus Z),$$

où $Z \subset X$ est une sous-variété fermée. Les modules de cycles de Milnor-Witt peuvent être utilisés afin d'étendre cette suite exacte à gauche et à droite, c'est fait dans la sous-section 2.5.1 (voir aussi [Fas08, Corollaire 10.4.9]).

Dans la sous-section 2.6.1, on démontre un résultat similaire à la conjecture de Gersten : les groupes de cohomologie $A^p(X, M, \mathcal{V}_X)$ sont triviaux lorsque X est un schéma semi-local lisse (sur le corps parfait k), \mathcal{V}_X un fibré virtuel sur X et $p > 0$. La preuve est similaire à celle de Rost (cf [Ros96, Theorem 6.1]) et suit les idées classiques de Gabber et Panin.

De plus, on prouve la propriété d'invariance par homotopie tant attendue, plaçant ainsi notre travail dans la théorie \mathbb{A}^1 -homotopique.

Theorem 3 (voir le théorème 2.6.2.4). *Soit X un schéma, V un fibré vectoriel sur X , $\pi : V \rightarrow X$ la projection canonique et \mathcal{V}_X un fibré virtuel sur X . Alors, pour tout entier $q \in \mathbb{Z}$, le morphisme canonique*

$$\pi^* : A^q(X, M, \mathcal{V}_X) \rightarrow A^q(V, M, -\mathcal{T}_{V/X} + \mathcal{V}_V)$$

est un isomorphisme.

Notre démonstration de ce théorème ne suit pas les traces laissées par Rost. En effet, dans son papier original [Ros96, §7], Rost construit un module de cycles $A^q[\rho, M]$ pour tout module de cycles M et tout morphisme de schémas $\rho : Q \rightarrow B$, puis utilise une suite spectrale impliquant ce nouveau module de cycles afin de démontrer le théorème. On ne pouvait facilement étendre cette technique à notre contexte. Néanmoins, suivant certaines

idées de Déglise (voir [Dé14b, §2]), on définit une suite spectrale de coniveau qui nous aide à ramener le problème au cas connu (voir (H)). On remarque qu’il est aussi possible de prouver le résultat en explicitant un inverse homotopique au niveau du complexe (comme l’a fait Rost dans [Ros96, §8] ; voir aussi le théorème 1.7.2.3).

Dans [Ros96, §12], Rost définit des tirés-en-arrière (ou *morphismes de Gysin*) pour des applications $f : X \rightarrow Y$ entre schémas lisses. Dans la section 2.7, on fait de même mais pour un ensemble plus vaste de flèches : les morphismes quasi-projectifs localement d’intersection complète. On prouve des théorèmes de fonctorialité et de changement de base qui s’avèreront utiles plus tard pour l’étude des multiplicités.

Dans la sous-section 2.8.1, on construit des produits dans notre contexte. Utilisant les morphismes de Gysin, on définit alors un produit d’intersection sur les groupes d’homologie de schémas lisses. En particulier, on retrouve le produit d’intersection déjà existant pour les groupes de Chow-Witt (voir [Fas19, §3.4]). On utilisera ces notions pour étudier les multiplicités quadratiques de notre théorie. On remarque néanmoins que l’on ne définit pas encore de produit tensoriel entre modules de Milnor-Witt (mais cela sera fait grâce au théorème 3.4.0.4).

Modules homotopiques de Morel

Dans le chapitre 3 (voir aussi [Fel19]), on démontre que les modules de cycles de Milnor-Witt sont proches de la théorie \mathbb{A}^1 -homotopique de Morel : ils peuvent être réalisés géométriquement comme éléments de la catégorie homotopique stable. Précisément, on prouve le théorème suivant.

Theorem 4 (Théorème 3.4.0.4). *Soit k un corps parfait. La catégorie des modules de cycles de Milnor-Witt est équivalente au cœur de la catégorie homotopique stable de Morel-Voevodsky (équipée de sa t -structure homotopique) :*

$$\mathfrak{M}_k^{MW} \simeq \mathbf{SH}(k)^{\heartsuit}.$$

Afin de démontrer ce théorème, on étudie la théorie cohomologique associée à un spectre motivique. Cette notion est naturellement duale à la théorie bivariable développée dans [DJK18] et rappelée dans la section 3.2 (voir le théorème 3.2.3.1). Un spectre motivique \mathbb{E} conduit à un foncteur $\hat{\mathbb{E}}$ de la catégorie des corps finiment engendrés sur k vers la catégorie des groupes abéliens gradués (attention au fait que la graduation n’est pas \mathbb{Z} mais la catégorie des espaces vectoriels virtuels). On prouve que le foncteur $\hat{\mathbb{E}}$ est un prémodule de cycles de Milnor-Witt (voir la définition 2.3.1.1). En effet, la plupart des axiomes découlent immédiatement de la théorie générale [DJK18]. De plus, dans le théorème 3.2.2.2, on montre un théorème de ramification qui peut être appliqué afin d’obtenir la règle (R3a). De plus, on vérifie les axiomes (FD) et (C) de sorte que $\hat{\mathbb{E}}$ est un module de cycles de Milnor-Witt. Ces deux axiomes proviennent de l’étude d’une suite spectrale définie dans la section 3.3 (voir le théorème 3.3.3.2) ; une autre preuve – plus élémentaire – pourrait aussi découler du théorème 2.3.2.12, mais cette méthode s’appuie fortement sur le fait que le corps de base est parfait.

Dans la section 3.4, on construit un module homotopique pour tout module de cycles de Milnor-Witt et on démontre que le cœur de la catégorie homotopique stable est équivalent à la catégorie des modules de cycles de Milnor-Witt.

Ce résultat généralise la thèse de Déglise (voir le théorème 3.5.2.3) et répond affirmativement à une ancienne conjecture de Morel (voir [Mor12, Remark 2.49]). On note

que l'on a déjà une description du cœur $\mathbf{SH}(k)^\heartsuit$ en termes de modules homotopiques (qui sont des faisceaux Nisnevich gradués \mathbb{A}^1 -invariant munis d'une structure additionnelle, voir [Dé11, §1]). Les modules de cycles ont l'avantage d'être défini seulement pour des corps, au lieu des habituels schémas lisses. Un corollaire important du théorème 3.4.0.4 est le résultat suivant (démontré indépendamment par Ananyevskiy et Neshitov dans [AN18, Theorem 8.12]):

Theorem 5 (Théorème 3.5.2.2). *Le cœur de la catégorie homotopique stable de Morel-Voevodsky est équivalente au cœur de la catégorie des MW-motifs (toutes les deux équipées de leur t -structure homotopique respective):*

$$\mathbf{SH}(k)^\heartsuit \simeq \widetilde{\mathbf{DM}}(k)^\heartsuit.$$

MW-Faisceaux homotopiques et transferts généralisés

Dans [Mor12], Morel étudie les faisceaux Nisnevich invariant par homotopie afin d'obtenir des outils en \mathbb{A}^1 -homotopie similaire à la théorie des faisceaux avec transferts de Voevodsky. Le résultat fondamental est qu'un faisceau (non-ramifié) est principalement caractérisé par ses sections sur les corps (voir la sous-section 1.5.1). Un des théorèmes principaux de [Mor12] est l'équivalence entre la notion de faisceaux en groupes abéliens *fortement* \mathbb{A}^1 -invariant et *strictement* \mathbb{A}^1 -invariant (voir le théorème 1.7.2.7). Afin de prouver cela, Morel définit des transferts géométriques sur la contraction M_{-1} d'un faisceau homotopique (i.e. un faisceau Nisnevich en groupes abéliens fortement \mathbb{A}^1 -invariant). La définition est une adaptation de celle donnée par Bass et Tate pour la K-théorie de Milnor [BT73]. Morel démontre que les transferts sont fonctoriels (i.e. ne dépendent pas du choix des générateurs) pour toute 2-contraction M_{-2} d'un faisceau homotopique et conjecture que le résultat est vrai pour M_{-1} (voir la conjecture 4.4.1.13 ou [Mor12, Remark 4.31]).

La notion de *faisceaux avec transferts généralisés* fut premièrement définie dans [Mor11, Definition 5.7] afin de formaliser les différentes structures apparaissant sur certains faisceaux homotopiques. Dans le chapitre 4 (voir aussi [Fel20a]), on donne une définition légèrement modifiée de *faisceaux avec transferts généralisés* qui prend en compte les tortillons (*twists*) provenant des fibrés en droites usuels. Suivant [Mor12, Chapter 5], on définit le complexe de Rost-Schmid associé à un tel faisceau homotopique et étudie les poussés-en-avant, tirés-en-arrière, l'action de GW et les flèches de résidus (voir aussi la section 1.7). De plus, on démontre le théorème suivant.

Theorem 6 (Théorème 4.3.2.3). *Soit $M \in \mathbf{HI}^{\text{gtr}}(k)$ un faisceau homotopique avec transferts généralisés. Le préfaisceau $\tilde{\Gamma}_*(M)$ en groupes abéliens défini par*

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M \otimes (\omega_{X/k})^\vee)$$

pour tout schéma lisse X/k , est un MW-faisceau homotopique canoniquement isomorphe à M .

Dans la section 4.4, on rappelle la construction des transferts de Bass-Tate sur un faisceau homotopique contracté M_{-1} et démontre que cela définit une structure de transferts généralisés.

Theorem 7 (Théorème 4.4.1.20). *Soit $M \in \mathbf{HI}(k)$ un faisceau homotopique. Alors :*

1. Si 2 est inversible, alors le faisceau homotopique contracté rationnel $M_{-1, \mathbb{Q}}$ est un faisceau homotopique avec transferts généralisés.
2. Si la conjecture 4.4.1.13 est vraie, alors le faisceau homotopique contracté M_{-1} est un faisceau homotopique avec transferts généralisés.

En particulier, on obtient la formule d'intersection suivante :

Theorem 8 (Théorème 4.4.1.16). Soient $M \in \mathbf{HI}(k)$ un faisceau homotopique, $\varphi : E \rightarrow F$ et $\psi : E \rightarrow L$ avec φ fini. Notons R l'anneau $F \otimes_E L$. Pour tout $\mathfrak{p} \in \text{Spec } R$, soit $\varphi_{\mathfrak{p}} : L \rightarrow R/\mathfrak{p}$ et $\psi_{\mathfrak{p}} : F \rightarrow R/\mathfrak{p}$ les morphismes induits par φ et ψ . On a :

$$M_{-1}(\psi) \circ \text{Tr}_{\varphi} = \sum_{\mathfrak{p} \in \text{Spec } R} e_{\mathfrak{p}, \varepsilon} \text{Tr}_{\varphi_{\mathfrak{p}}} \circ M_{-1}(\psi_{\mathfrak{p}})$$

où $e_{\mathfrak{p}, \varepsilon} = \sum_{i=1}^{e_{\mathfrak{p}}} \langle -1 \rangle^{i-1}$ est la forme quadratique associée à la longueur $e_{\mathfrak{p}}$ de l'anneau localisé $R_{(\mathfrak{p})}$.

Généralisant des idées de Voevodsky, Calmès et Fasel ont introduit la catégorie monoïdale symétrique additive $\widetilde{\text{Cor}}_k$ des k -schémas lisses avec pour flèches les *correspondances finies de Milnor-Witt* (voir [BCD⁺20, Chapter 2]). Dans la section 4.5, on rappelle les définitions de base de cette théorie et on démontre que tout faisceau homotopique avec MW-transferts possède une structure de faisceau avec transferts généralisés. Précisément, on montre que les deux notions coïncident :

Theorem 9 (Théorème 4.5.2.5). Il existe une paire des foncteurs

$$\mathbf{HI}^{\text{MW}}(k) \begin{array}{c} \xrightarrow{\tilde{\Gamma}^*} \\ \xleftarrow{\tilde{\Gamma}^*} \end{array} \mathbf{HI}^{\text{gtr}}(k)$$

qui forme une équivalence entre la catégorie des faisceaux homotopiques avec MW-transferts et celle des faisceaux homotopiques avec transferts généralisés.

Dans la section 4.6, on démontre le théorème suivant qui caractérise l'image essentielle du foncteur $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ oubliant les MW-transferts.

Theorem 10 (Théorème 4.6.1.6). Soit $M \in \mathbf{HI}(k)$ un faisceau homotopique. Les assertions suivantes sont équivalentes :

- (i) Il existe $M' \in \mathbf{HI}(k)$ satisfaisant la conjecture 4.4.1.13 et tel que $M \simeq M'_{-1}$.
- (ii) Il existe une structure de transferts généralisés sur M .
- (iii) Il existe une structure de MW-transferts sur M .
- (iv) Il existe $M'' \in \mathbf{HI}(k)$ tel que $M \simeq M''_{-2}$.

Ce résultat est lié à la conjecture de conservativité de [BY18] et nous permet de démontrer les théorèmes suivants.

Theorem 11 (Corollaire 4.6.2.3). Soit $d > 0$ un entier naturel. La conjecture de Bachmann-Yakerson est vraie (intégralement) pour $d \geq 2$ et rationnellement pour $d \geq 1$. Autrement dit, le foncteur canonique

$$\mathbf{SH}^{S^1}(k)(2) \rightarrow \mathbf{SH}(k)$$

est conservatif sur les objets inférieurement bornés², le foncteur canonique

$$\mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$$

est conservatif sur les objets rationnels inférieurement bornés, et le foncteur canonique

$$\mathbf{HI}(k, \mathbb{Q})(1) \rightarrow \mathbf{HI}^{\mathrm{fr}}(k, \mathbb{Q})$$

est une équivalence de catégories.

De plus, soit \mathcal{X} un espace motivique pointé. Alors le morphisme canonique

$$\pi_0 \Omega_{\mathbb{P}^1}^d \Sigma_{\mathbb{P}^1}^d \mathcal{X} \rightarrow \pi_0 \Omega_{\mathbb{P}^1}^{d+1} \Sigma_{\mathbb{P}^1}^{d+1} \mathcal{X}$$

est un isomorphisme pour $d = 2$.

Theorem 12 (Corollaire 4.6.2.4). *La catégorie des faisceaux homotopiques avec transferts généralisés, la catégorie des MW-faisceaux homotopiques et celle des faisceaux homotopiques avec transferts encadrés sont équivalentes :*

$$\mathbf{HI}^{\mathrm{gtr}}(k) \simeq \mathbf{HI}^{\mathrm{MW}}(k) \simeq \mathbf{HI}^{\mathrm{fr}}(k).$$

Transferts pour la K-théorie de Milnor-Witt

Dans [Mor12, Chapter 3], Morel introduit la K-théorie de Milnor-Witt d'un corps. Suivant des idées de Bass et Tate [BT73], on peut définir des morphismes de transferts géométriques.

$$\mathrm{Tr}_{x_1, \dots, x_r/E} = \mathrm{Tr}_{x_r/E(x_1, \dots, x_{r-1})} \circ \dots \circ \mathrm{Tr}_{x_2/E(x_1)} \circ \mathrm{Tr}_{x_1/E} : \mathbf{K}_*^{\mathrm{MW}}(E(x_1, \dots, x_r)) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(E)$$

sur \mathbf{K}^{MW} pour les extensions finies $E(x_1, \dots, x_r)/E$. Rappelons que Morel montre dans [Mor12, Chapter 4] que tout faisceau homotopique de la forme M_{-2} admet de tels transferts et qu'ils sont fonctoriels. En particulier, cela s'applique à la K-théorie de Milnor-Witt.

Au chapitre 5 (voir aussi [Fel20b]), on donne une preuve alternative de ce résultat.

Theorem 13 (Théorème 5.2.2.4). *Soit $E(x_1, \dots, x_r)/E$ une extension finie. Le morphisme $\mathrm{Tr}_{x_1, \dots, x_r/E} : \mathbf{K}_*^{\mathrm{MW}}(E(x_1, \dots, x_r)) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(E)$ ne dépend pas du choix d'un système de générateurs (x_1, \dots, x_r) .*

L'idée fondamentale est de se réduire au cas des corps p -primaires puis d'étudier les transferts à la main, comme Kato l'a originellement fait pour la K-théorie de Milnor (voir [GS17]). Plus élémentaire, cette démonstration ne s'applique pas en tout généralité au cas des faisceaux homotopiques contractés M_{-1} . Néanmoins, on obtient comme corollaire la réduction suivante de la conjecture de Morel [Fel20a, Conjecture 4.1.13].

Theorem 14 (Théorème 5.2.2.13). *Afin de démontrer la conjecture de Morel 4.4.1.13, il suffit de considérer le cas des corps p -primaires (où p est un nombre premier).*

²On parle aussi d'objets connectifs.

Travail futur

Théorie de l'intersection quadratique Dans 2.4.1.6, on définit un tiré-en-arrière f^* entre des complexes de modules de cycles de Milnor-Witt, où f est un morphisme lisse. Une question naturelle est : comment étendre cette définition à des morphismes plus généraux ? Dans [Ros96], Rost réussit à définir de tels tirés-en-arrière pour tout morphisme plat. Une telle généralisation à notre contexte quadratique impliquerait une étude attentive des multiplicités (qui doivent être des formes quadratiques, pas seulement des entiers naturels). On a déjà obtenu certains résultats pointant dans la bonne direction (voir par exemple le théorème 3.2.2.2 ou le théorème 4.4.1.16).

Schéma de base général Dans le futur, on étudiera la théorie des modules de cycles de Milnor-Witt au-dessus d'un schéma de base plus général S et démontrera le théorème 3.4.0.4 dans ce contexte général. Une telle tâche fut déjà envisagée pour le cas orienté des modules de cycles de Rost. En effet, on sait que la catégorie $\mathbf{DM}(k)$ des motifs de Voevodsky est équipée d'une t-structure dont le cœur peut être identifié avec les modules de cycles de Rost. Dans [Ayo07], Ayoub introduisit une version relative de cette t-structure au-dessus d'un schéma S , appelée la *t-structure perverse*, et conjectura que le cœur de cette t-structure devrait pouvoir être décrit pour les modules de cycles de Rost. Une réponse partielle fut apportée par Déglise dans [Dé14a] et on espère généraliser cela aux modules de cycles de Milnor-Witt. En supposant un tel travail accompli, on pourrait dès à présent énoncer les corollaires de la section 3.5 (notamment le théorème 3.5.2.2 dont la preuve originelle a besoin du fait que le corps de base soit parfait).

Structure monoïdale L'équivalence de catégories démontrée dans le théorème 3.4.0.4 implique que la catégorie des modules de cycles de Milnor-Witt est une catégorie de Grothendieck équipée d'un produit tensoriel. On pourrait étudier plus attentivement cette structure monoïdale et en donner une description plus fine en termes de générateurs et relations. On devrait alors pouvoir obtenir une belle interprétation de la notion d'*accouplement* (voir la définition 2.3.1.21).

Conjecture de Morel L'auteur travaille depuis un moment sur la conjecture de Morel 4.4.1.13 et a l'intention de poursuivre cette quête dans le futur. Le fait qu'un faisceau homotopique contracté M_{-1} admette des transferts fonctoriels implique non seulement la conjecture de Bachmann-Yakerson pour $d = 1$ (voir le théorème 4.6.2.2), mais entraîne aussi un certain nombre d'améliorations de la théorie \mathbb{A}^1 -homotopique (e.g. certaines propriétés du complexe de Rost-Schmid, voir [Mor12, §5]).

Conjecture de Bachmann-Yakerson Dans [BY18], les auteurs ont construit (pour un faisceau homotopique fortement \mathbb{A}^1 -invariant M , un k -schéma lisse X et un entier naturel d) un complexe de cycles $C^*(X, M, q)$ et l'utilisèrent pour étudier la conjecture 4.6.2.1. Il serait intéressant d'approfondir les liens entre ce complexe et celui que l'on a défini dans la sous-section 2.4.1.

De plus, on appliquera la conjecture de conservativité de Bachmann et Yakerson à l'étude de certains points d'intersection entre la théorie \mathbb{A}^1 -homotopique et la géométrie algébrique affine. Par exemple, suivant [AØ19, Conjecture 5.3.11 et Remark 5.3.12], on devrait pouvoir obtenir :

Theorem 15. *Soit X un schéma lisse et $x \in X$ un point fermé. Si $\Sigma_{\mathbb{P}^1}^\infty(X, x) \simeq *$ dans $\mathbf{SH}(k)$, alors $\Sigma_{\mathbb{P}^1}^2(X, x)$ est \mathbb{A}^1 -contractile.*

En particulier, cela s'applique lorsque X est la 3-variété de Koras-Russel de première ou seconde espèce (voir [AØ19, Theorem 5.3.9] et [DF18b, HKØ15] pour des résultats similaires). Cela devrait contribuer à une meilleure compréhension de la conjecture de Zariski (voir [AØ19, Question 5.1.1]).

Introduction (in English)

Historical approach

Rost's cycle modules Back in the nineties, Rost developed the theory of cycle modules [Ros96] which was used in the famous proof of Milnor conjecture by Voevodsky (see [SV00] or [Voe11]). Indeed, a crucial step in the proof was the construction of the motive associated to some quadratic forms [Ros98]. In [Ros96], a cycle module M over a perfect field k is the data of a \mathbb{Z} -graded abelian group $M(E)$ for every finitely generated field extension E/k , equipped with restriction maps, corestriction maps, a Milnor K-theory module action and residue maps ∂ . These data are subject to some axioms $(r1a), \dots, (r3e)$ so that Milnor K-theory groups form a Rost cycle module. From the formal definitions, Rost produced a general theory that encompassed previous ones (Quillen K-theory and étale cohomology, for instance). Moreover, Rost's theory is fundamentally linked with the theory of motives: cycle modules can be realized geometrically. This can be illustrated by the following theorem.

Theorem 1 (see [Dé03]). *Let k be a perfect field. The category of Rost cycle modules over k is equivalent to the heart of the category of Voevodsky's motives $\mathbf{DM}(k, \mathbb{Z})$ with respect to the homotopy t -structure.*

\mathbb{A}^1 -homotopy theory The idea of \mathbb{A}^1 -homotopy theory, due to Morel and Voevodsky, was to apply techniques from algebraic topology to the study of schemes (the affine line \mathbb{A}^1 playing the role of the unit interval $[0, 1]$). This idea gave rise to many important results and new categories, such as $\mathbf{DM}(k, \mathbb{Z})$ mentioned above and $\mathbf{SH}(k)$, the stable homotopy category [MV99].

Milnor-Witt K-theory In his study of $\mathbf{SH}(k)$, Morel (in joint work with Hopkins) defined for a field E the Milnor-Witt K-theory $\mathbf{K}_*^{\mathrm{MW}}(E)$ (see [Mor12, Definition 3.1]). This \mathbb{Z} -graded abelian group behaves in positive degrees like Milnor K-theory groups $\mathbf{K}_n^M(E)$, and in non-positive degrees like Grothendieck-Witt and Witt groups of quadratic forms, $\mathrm{GW}(E)$ and $\mathrm{W}(E)$. The Milnor-Witt K-theory was used for solving some splitting problems for projective modules. For instance, generalizing ideas from the theory of Chow groups, one can use the Rost-Schmid complex of Morel to define the Chow-Witt groups $\widetilde{\mathrm{CH}}^*(X)$ for a smooth k -scheme X (recalled in 2.5.2.1), and the Euler class of a vector bundle of rank r over X (with a given trivialization of its determinant) as an element in $\widetilde{\mathrm{CH}}^r(X)$. When $X = \mathrm{Spec} A$ is a 2-dimensional smooth affine variety and $r = 2$, Barge and Morel proved in [BM00] that the Euler class associated to a projective module P of rank 2 over A vanishes if and only if $P \simeq P' \oplus A$ for some projective module P' .

Homotopy sheaves with Voevodsky transfers In order to construct the derived category of motives $\mathbf{DM}(k, \mathbb{Z})$, Voevodsky introduced the so-called homotopy sheaves (with transfers) which are homotopy invariant Nisnevich sheaves with transfers. One important example is given by \mathbb{G}_m , the sheaf of global units. Voevodsky proved that any homotopy sheaf F has a Gersten resolution, implying that F is determined in some sense by the data of its fibers in every function fields. This statement was made more precise in Déglise’s thesis: the heart of $\mathbf{DM}(k, \mathbb{Z})$ with respect to its homotopy t-structure has a presentation given by the category of Rost cycle modules over k .

Oriented homotopy modules Morel’s point of view on the heart of $\mathbf{DM}(k, \mathbb{Z})$ is given by the category of oriented homotopy modules. We recall that a homotopy module is a strictly \mathbb{A}^1 -invariant Nisnevich sheaf with an additional structure defined over the category of smooth schemes (see Definition 3.4.1.2); it is called *oriented* when the Hopf map η acts on it by 0. Déglise’s theorem proves that oriented homotopy modules form a full subcategory of the category of homotopy modules which is equivalent to the category of Rost cycle modules. Morel’s natural conjecture [Mor12, Remark 2.49] was that there is a presentation of the heart of the stable homotopy category $\mathbf{SH}(k)$ (or equivalently, the category of homotopy modules) in terms of some non-oriented version of cycle modules.

Current work

Milnor-Witt cycle modules

In chapter 2 (see also [Fel18]), we develop a conjectured³ theory that studies general cycle complexes $C^*(X, M, \mathcal{V}_X)$ and their (co)homology groups $A^*(X, M, \mathcal{V}_X)$ (called *Chow-Witt groups with coefficients*) in a quadratic⁴ setting. The general coefficient systems M for these complexes are called Milnor-Witt cycle modules. The main example of such a cycle module is given by the Milnor-Witt K-theory (see Theorem 2.3.2.13); other examples can be deduced from Theorem 3.4.0.4, Example 2.3.2.7 or Theorem 2.8.2.5 (e.g. the representability of hermitian K-theory in $\mathbf{SH}(k)$ will lead to a MW-cycle module, associated with hermitian K-theory). A major difference with Rost’s theory is that the grading to be considered is not \mathbb{Z} but the category of virtual bundles (or, equivalently, the category of virtual vector spaces), where a virtual bundle \mathcal{V} is, roughly speaking, the data of an integer n and a line bundle \mathcal{L} (see Subsection 1.9.1).

Intuitively, Milnor-Witt cycle modules are given by (twisted) graded abelian groups equipped with extra data (restriction, corestriction, \underline{K}^{MW} -action and residue maps). The difficult part was to find good axioms mimicking Rost’s and taking into account the twists naturally arising in the non-oriented setting (virtual vector bundles play a major role). One important difference with the construction of Rost cycle modules [Ros96, Definition 1.1] is that we only need a weakened rule (R1c): we do not consider multiplicities (we will show in future works that formulas involving multiplicities do hold as a consequence

³See [Ros96, Remark 2.6] and [Mor12, Remark 5.37].

⁴Or rather: symmetric bilinear, since we allow fields of characteristic 2.

of our axioms). Moreover, we had to add another rule in our definition in order to link our theory with the classical one later (see (R4a) and Section 2.8.2).

Following [Ros96, §3], we define the necessary operations needed later on: pushforwards, pullbacks, multiplication with units and the motivic Hopf map η , and boundary maps. Since our rules do not handle multiplicities, we first define pullbacks only for (essentially) smooth morphisms. We prove the usual functoriality theorems and show how to compute the (co)homology groups in special cases.

We can summarize the results as follows:

Theorem 2 (See Subsection 2.4.1 and Subsection 2.4.2). *For any scheme X (over a perfect field k), any virtual bundle \mathcal{V}_X and any Milnor-Witt cycle module M , there is a complex $C^*(X, M, \mathcal{V}_X)$ equipped with pushforwards, pullbacks, a Milnor-Witt K -theory action and residue maps satisfying the standard functoriality properties.*

This formalism generalizes Schmid's thesis [Sch98] and provides more details to Morel's work [Mor12], regarding twists by virtual vector bundles.

An example of an application is as follows. Consider the localization sequence:

$$\widetilde{\mathrm{CH}}_p(Z) \rightarrow \widetilde{\mathrm{CH}}_p(X) \rightarrow \widetilde{\mathrm{CH}}_p(X \setminus Z),$$

where $Z \subset X$ is a closed subvariety. Milnor-Witt cycle modules can be used to extend this exact sequence on the left and right, this is done in Subsection 2.5.1.

In Subsection 2.6.1, we prove a result similar to the Gersten Conjecture: the cohomology groups $A^p(X, M, \mathcal{V}_X)$ are trivial when X is a smooth semi-local scheme, \mathcal{V}_X a virtual bundle over X and $p > 0$. The proof is akin to Rost's Theorem 6.1 [Ros96] and follows classical ideas of Gabber and Panin.

Moreover, we prove the much expected homotopy invariance property, framing our work in \mathbb{A}^1 -homotopy theory:

Theorem 3 (see Theorem 2.6.2.4). *Let X be a scheme, V a vector bundle over X , $\pi : V \rightarrow X$ the canonical projection and \mathcal{V}_X a virtual vector bundle over X . Then, for every $q \in \mathbb{Z}$, the canonical morphism*

$$\pi^* : A^q(X, M, \mathcal{V}_X) \rightarrow A^q(V, M, -\mathcal{T}_{V/X} + \mathcal{V}_V)$$

is an isomorphism.

Our proof of this theorem does not follow Rost's ideas. Indeed, in his original paper [Ros96, §7], Rost constructs a cycle module $A^q[\rho, M]$ from any cycle module M and any scheme morphism $\rho : Q \rightarrow B$, then uses a spectral sequence involving this new cycle module to prove the theorem. We could not easily extend the construction to our setting for some reasons related to twists. However, following ideas of Déglise (see [Dé14b, §2]), we define a coniveau spectral sequence that helps us to reduce to the known case (see (H)). Note that we could also prove the result by giving a homotopy inverse on the complex level (as Rost did in [Ros96, §8]; see also Theorem 1.7.2.3).

In [Ros96, §12], Rost defines pullback maps f^* (or *Gysin morphisms*) for morphisms $f : X \rightarrow Y$ between smooth schemes. In Section 2.7, we do the same but for a more general set of maps, the classical quasi-projective local complete intersection (lci) morphisms, between possibly singular schemes. We prove functoriality and base change theorems which will be useful later when considering multiplicities.

In Subsection 2.8.1, we construct products in our setting. Using the Gysin morphisms, we define then an intersection product on the homology groups of smooth schemes. In particular, we recover the intersection product already defined for Chow-Witt groups (see [Fas19, §3.4]). We will use these notions to study the multiplicities naturally arising. Note however that we do not define a tensor product between Milnor-Witt cycle modules (but it can be obtained from Theorem 3.4.0.4 below).

Morel homotopy modules

In chapter 3 (see also [Fel19]), we prove that Milnor-Witt cycle modules are closely related to Morel's \mathbb{A}^1 -homotopy theory: they can be realized geometrically as elements of the stable homotopy category. Precisely, we prove the following theorem.

Theorem 4 (Theorem 3.4.0.4). *Let k be a perfect field. The category of Milnor-Witt cycle modules is equivalent to the heart of Morel-Voevodsky stable homotopy category (equipped with the homotopy t -structure):*

$$\mathfrak{M}_k^{MW} \simeq \mathbf{SH}(k)^\heartsuit.$$

In order to prove this theorem, we study the cohomology theory associated with a motivic spectrum. This notion is naturally dual to the bivariant theory developed in [DJK18] and recalled in Section 3.2 (see Theorem 3.2.3.1). A motivic spectrum \mathbb{E} leads to a functor $\hat{\mathbb{E}}$ from the category of finitely generated fields over k to the category of graded abelian groups (be careful that the grading is not \mathbb{Z} but is given by the category of virtual vector spaces). We prove that the functor $\hat{\mathbb{E}}$ is a Milnor-Witt cycle premodule (see Definition 2.3.1.1). Indeed, most axioms are immediate consequences of the general theory [DJK18]. Moreover, in Theorem 3.2.2.2 we prove a ramification theorem of independent interest that can be applied to prove rule (R3a). Furthermore, we check axioms (FD) and (C) so that $\hat{\mathbb{E}}$ is a Milnor-Witt cycle module. These two axioms follows from the study of a spectral sequence defined in Section 3.3 (see Theorem 3.3.3.2); another – more elementary – proof may result from an adaptation of [Ros96, Theorem 2.3] to the context of Milnor-Witt cycle modules but this method would rely heavily on the fact that the base field is perfect (see the proof of Theorem 1.7.2.3 for more details).

In Section 3.4, we construct a homotopy module for any Milnor-Witt cycle module and proceed to prove that the heart of the stable homotopy category (which is known to be equivalent to the category of homotopy module according to [Mor03]) is equivalent to the category of Milnor-Witt cycle modules.

This result generalizes Déglise's thesis (see Theorem 3.5.2.3) and answers affirmatively an old conjecture of Morel (see [Mor12, Remark 2.49]). Note that we already have a description of the heart $\mathbf{SH}(k)^\heartsuit$ in terms of homotopy modules (which are strictly \mathbb{A}^1 -invariant Nisnevich sheaves with an additional structure defined over the category of smooth schemes, see [Mor03, §5.2] or [Dé11, §1]). Cycle modules may be easier to work with since they are defined over fields. An important corollary of Theorem 3.4.0.4 is the following result (which was proved independently by Ananyevskiy and Neshitov in [AN18, Theorem 8.12]):

Theorem 5 (Theorem 3.5.2.2). *The heart of Morel-Voevodsky stable homotopy category is equivalent to the heart of the category of MW-motives [DF18a] (both equipped with their respective homotopy t -structures):*

$$\mathbf{SH}(k)^\heartsuit \simeq \widetilde{\mathbf{DM}}(k)^\heartsuit.$$

MW-homotopy sheaves and generalized transfers

In [Mor12], Morel studied homotopy invariant Nisnevich sheaves in order to provide computational tools in \mathbb{A}^1 -homotopy analogous to Voevodsky's theory of sheaves with transfers. The most basic result is that (unramified) sheaves are characterized by their sections on fields and some extra data (see Subsection 1.5.1). One of the main theorem of [Mor12] is the equivalence between the notions of *strongly* \mathbb{A}^1 -invariant and *strictly* \mathbb{A}^1 -invariant for sheaves of abelian groups (see *loc. cit.* Theorem 1.16). In order to prove this, Morel defined geometric transfers on the contraction M_{-1} of a homotopy sheaf (i.e. a strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups). The definition is an adaptation of the original one of Bass and Tate for Milnor K-theory [BT73]. Morel proved that the transfers were functorial (i.e. they do not depend on the choice of generators) for any two-fold contraction M_{-2} of a homotopy sheaf and conjectured that the result should hold for M_{-1} (see Conjecture 4.4.1.13 or [Mor12, Remark 4.31]).

The notion of *sheaves with generalized transfers* was first defined in [Mor11, Definition 5.7] as a way to formalize the different structures naturally arising on some homotopy sheaves. In Section 4.3, we give a slightly modified definition of *sheaves with generalized transfers* which takes into account twists by the usual line bundles. Following [Mor12, Chapter 5], we define the Rost-Schmid complex associated to such homotopy sheaves and study the usual pushforward maps f_* , pullback maps g^* , GW-action $\langle a \rangle$ and residue maps ∂ (see also Section 1.7 for more details). Moreover, we prove the following theorem.

Theorem 6 (see Theorem 4.3.2.3). *Let $M \in \mathbf{HI}^{\text{gr}}(k)$ be a homotopy sheaf with generalized transfers. The presheaf $\tilde{\Gamma}_*(M)$ of abelian groups, defined by*

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M \otimes (\omega_{X/k})^\vee)$$

for any smooth scheme X/k , is a MW-homotopy sheaf canonically isomorphic to M as presheaves.

In Section 4.4, we recall the construction of the Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} and prove that this defines a structure of generalized transfers:

Theorem 7 (see Theorem 4.4.1.20). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Then:*

1. *Assume that 2 is invertible. The rational contracted homotopy sheaf $M_{-1, \mathbb{Q}}$ is a homotopy sheaf with generalized transfers.*
2. *Assuming Conjecture 4.4.1.13, the contracted homotopy sheaf M_{-1} is a homotopy sheaf with generalized transfers.*

In particular, we obtain the following intersection multiplicity formula which was left open in [Fel18]:

Theorem 8 (see Theorem 4.4.1.16). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Consider $\varphi : E \rightarrow F$ and $\psi : E \rightarrow L$ with φ finite. Let R be the ring $F \otimes_E L$. For each $\mathfrak{p} \in \text{Spec } R$, let $\varphi_{\mathfrak{p}} : L \rightarrow R/\mathfrak{p}$ and $\psi_{\mathfrak{p}} : F \rightarrow R/\mathfrak{p}$ be the morphisms induced by φ and ψ . One has*

$$M_{-1}(\psi) \circ \text{Tr}_{\varphi} = \sum_{\mathfrak{p} \in \text{Spec } R} e_{\mathfrak{p}, \varepsilon} \text{Tr}_{\varphi_{\mathfrak{p}}} \circ M_{-1}(\psi_{\mathfrak{p}})$$

where $e_{p,\varepsilon} = \sum_{i=1}^{e_p} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the length e_p of the localized ring $R_{(p)}$.

Generalizing ideas of Voevodsky, Calmès and Fasel introduced the additive symmetric monoidal category $\widehat{\text{Cor}}_k$ of smooth k -schemes with morphisms given by the so-called *finite Milnor-Witt correspondences* (see [BCD⁺20, Chapter 2]). In Section 1.8, we recall the basic definitions regarding this theory and prove that any homotopy sheaf with MW-transfers has a structure of sheaf with generalized transfers. More precisely, we show that the two notions coincide:

Theorem 9 (Theorem 4.5.2.5). *There is a pair of functors*

$$\mathbf{HI}^{\text{MW}}(k) \begin{array}{c} \xrightarrow{\tilde{\Gamma}^*} \\ \xleftarrow{\tilde{\Gamma}^*} \end{array} \mathbf{HI}^{\text{gtr}}(k)$$

that forms an equivalence between the category of homotopy sheaves with MW-transfers and the category of homotopy sheaves with generalized transfers.

In Section 4.6, we prove the following theorem that characterizes the essential image of the functor $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers.

Theorem 10 (Theorem 4.6.1.6). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The following assertions are equivalent:*

- (i) *There exists $M' \in \mathbf{HI}(k)$ satisfying Conjecture 4.4.1.13 and such that $M \simeq M'_{-1}$.*
- (ii) *There exists a structure of generalized transfers on M .*
- (iii) *There exists a structure of MW-transfers on M .*
- (iv) *There exists $M'' \in \mathbf{HI}(k)$ such that $M \simeq M''_{-2}$.*

This result is linked with the conservativity conjecture from [BY18] and allows us to prove the following theorems.

Theorem 11 (Corollary 4.6.2.3). *Let $d > 0$ be a natural number. The Bachmann-Yakerson conjecture holds (integrally) for $d \geq 2$ and rationally for $d \geq 1$: namely, the canonical functor*

$$\mathbf{SH}^{S^1}(k)(2) \rightarrow \mathbf{SH}(k)$$

is conservative on bounded below objects⁵, the canonical functor

$$\mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$$

is conservative on rational bounded below objects, and the canonical functor

$$\mathbf{HI}(k, \mathbb{Q})(1) \rightarrow \mathbf{HI}^{\text{fr}}(k, \mathbb{Q})$$

is an equivalence of abelian categories.

Moreover, let \mathcal{X} be a pointed motivic space. Then the canonical map

⁵Also known as *connective* objects.

$$\pi_0 \Omega_{\mathbb{P}^1}^d \Sigma_{\mathbb{P}^1}^d \mathcal{X} \rightarrow \pi_0 \Omega_{\mathbb{P}^1}^{d+1} \Sigma_{\mathbb{P}^1}^{d+1} \mathcal{X}$$

is an isomorphism for $d = 2$.

Theorem 12 (Corollary 4.6.2.4). *The category of homotopy sheaves with generalized transfers, the category of MW-homotopy sheaves and the category of homotopy sheaves with framed transfers are equivalent:*

$$\mathbf{HI}^{\text{gr}}(k) \simeq \mathbf{HI}^{\text{MW}}(k) \simeq \mathbf{HI}^{\text{fr}}(k).$$

Transfers for Milnor-Witt K-theory

In [Mor12, Chapter 3], Morel introduced the Milnor-Witt K-theory of a field. Following ideas of Bass and Tate [BT73], one can define geometric transfer maps

$$\text{Tr}_{x_1, \dots, x_r/E} = \text{Tr}_{x_r/E(x_1, \dots, x_{r-1})} \circ \dots \circ \text{Tr}_{x_2/E(x_1)} \circ \text{Tr}_{x_1/E} : \mathbf{K}_*^{\text{MW}}(E(x_1, \dots, x_r)) \rightarrow \mathbf{K}_*^{\text{MW}}(E)$$

on \mathbf{K}^{MW} for finite extensions $E(x_1, \dots, x_r)/E$. Morel proved in [Mor12, Chapter 4] that any homotopy sheaves of the form M_{-2} admit such transfers and that they are functorial. In particular, this applies for Milnor-Witt K-theory.

In Chapter 5 (see also [Fel20b]), we give an alternative proof of this result:

Theorem 13 (Theorem 5.2.2.4). *The maps $\text{Tr}_{x_1, \dots, x_r/E} : \mathbf{K}_*^{\text{MW}}(E(x_1, \dots, x_r)) \rightarrow \mathbf{K}_*^{\text{MW}}(E)$ do not depend on the choice of the generating system (x_1, \dots, x_r) .*

The idea is to reduce to the case of p -primary fields then study the transfers manually, as Kato originally did for Milnor K-theory (see [GS17]). More elementary, the proof does not apply in full generality to the case of contracted homotopy sheaves M_{-1} . However, we obtain as a corollary a reduction of Morel's conjecture [Fel20a, Conjecture 4.1.13].

Theorem 14 (Theorem 5.2.2.13). *In order to prove Morel's Conjecture 4.4.1.13, it suffices to consider the case of p -primary fields (where p is a prime number).*

Future work

Quadratic intersection theory In 2.4.1.6, we have defined a pullback map f^* between complexes of Milnor-Witt cycle modules, where f is a *smooth* morphism. A natural question is : how to extend this definition to more general morphisms? In [Ros96], Rost managed to define such pullbacks for any *flat* morphism. Such a generalization in our quadratic context would require a thorough study of new multiplicity elements (which ought to be quadratic forms, not just natural numbers). We already have obtained promising results that point in the right direction (see e.g. Theorem 3.2.2.2 or Theorem 4.4.1.16).

General base scheme In future work, we will study the theory of Milnor-Witt cycle modules over a more general base scheme S and prove Theorem 3.4.0.4 in this more general setting. Such task was already conjectured for the oriented case of Rost cycle modules over the perfect base field k . Indeed, we know that the category $\mathbf{DM}(k)$ of

Voevodsky motives comes equipped with a t-structure whose heart can be identified with Rost cycle modules. In [Ayo07], Ayoub introduced a relative version of this t-structure over a scheme S , called the *perverse homotopy t-structure*, and conjectured that the heart of this t-structure should be described by Rost cycle modules. A partial answer was given by Déglise in [Dé14a] and we hope to generalize it for Milnor-Witt cycle modules. Assuming such task accomplished, we may already claim the corollaries stated in Section 3.5 to be true (including Theorem 3.5.2.2 whose original proof [AN18] needs the base field to be perfect).

Monoidal structure The equivalence of categories that we proved in Theorem 3.4.0.4 implies that the category of Milnor-Witt cycle modules is a Grothendieck category equipped with a tensor product. We could study more deeply this monoidal structure and give a sharper description in terms of generators and relations. In this way, we should obtain a nice interpretation of the notion of *pairing* (see Definition 2.3.1.21).

Morel conjecture We have been working on Conjecture 4.4.1.13 for quite some time already and intend to pursue this quest with great attention in the future. The fact that a contracted homotopy sheaf M_{-1} admits functorial transfers not only implies Bachmann-Yakerson conjecture for $d = 1$ (see Theorem 4.6.2.2), but also could yield to a number of improvements in \mathbb{A}^1 -homotopy theory (e.g. some properties of the Rost-Schmid complex, see [Mor12, §5]).

Bachmann-Yakerson conjecture In [BY18], the authors construct (for a strictly homotopy invariant sheaf M , a smooth k -scheme X and a natural number q) a cycle complex $C^*(X, M, q)$ and used it to examine Conjecture 4.6.2.1. It could be interesting to deepen the relations between this complex and the one defined in Subsection 2.4.1.

Moreover, we will apply the conservative conjecture of Bachmann and Yakerson to study some intersection points between \mathbb{A}^1 -homotopy theory and affine algebraic geometry. For instance, following [AØ19, Conjecture 5.3.11 and Remark 5.3.12], one should obtain:

Theorem 15. *Let X be a smooth scheme and $x \in X$ a closed point. If $\Sigma_{\mathbb{P}^1}^\infty(X, x) \simeq *$ in $\mathbf{SH}(k)$, then $\Sigma_{\mathbb{P}^1}^2(X, x)$ is \mathbb{A}^1 -contractible.*

In particular, this applies when X is a Koras-Russel threefold of the first or second kind (see [AØ19, Theorem 5.3.9] and [DF18b, HKØ15] for similar results). This should improve our understanding of Zariski conjecture (see [AØ19, Question 5.1.1]).

1

Preliminary results

We summarize the basic results needed to understand our thesis. After some remarks about set theory, we recall the construction of the usual unstable, stable and derived homotopy categories. In order to stray away from the classical expositions, we use the language of ∞ -categories which also enables us to state some fundamental universal properties. We then define the category of homotopy sheaves and its primary element given by Milnor-Witt K-theory. Moreover, we present the construction of transfers on contracted homotopy sheaves and use these maps to study the Rost-Schmid complex. We end with a description of Milnor-Witt and framed correspondences which will be needed in chapter 4.

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1.1 Introduction

1.1.1 Notation and conventions

Throughout the chapter, we fix a (commutative) **perfect field** k . When dealing with MW- and framed correspondences, we assume moreover that k is infinite of characteristic not 2. We need these assumptions in order to apply the cancellation theorem [BCD⁺20, Chapter 4] but we believe these restrictions could be lifted.

We denote by **Grp** and **Ab** the categories of (abelian) groups.

We consider only schemes that are essentially of finite type over k . All schemes and morphisms of schemes are defined over k . The category of smooth k -schemes of finite type is denoted by **Sm** $_k$ and is endowed with the Nisnevich topology (thus, *sheaf* always means *sheaf for the Nisnevich topology*).

If X is a scheme and n a natural number, we denote by $X_{(n)}$ (resp. $X^{(n)}$) the set of point of dimension n (resp. codimension n).

By a field E over k , we mean a k -finitely generated field E . Since k is perfect, notice that $\text{Spec } E$ is essentially smooth over $S = \text{Spec } k$. We denote by \mathcal{F}_k the category of such fields.

Let F/E be a field extension, we denote by $\Omega_{F/E}$ the F -vector space of (Kähler) differentials. Dually, let X/S be a scheme morphism, we denote by $\mathcal{T}_{X/S}$ the sheaf of modules of differentials (or rather, its virtual representant, see Section 1.9.1).

Let $f : X \rightarrow Y$ be a morphism of schemes. Denote by \mathcal{L}_f (or $\mathcal{L}_{X/Y}$) the virtual vector bundle over Y associated with the cotangent complex of f , and by ω_f (or $\omega_{X/Y}$) its determinant. Recall that if $p : X \rightarrow Y$ is a smooth morphism, then \mathcal{L}_p is (isomorphic to) $\mathcal{T}_p = \Omega_{X/Y}$ the space of (Kähler) differentials. If $i : Z \rightarrow X$ is a regular closed immersion, then \mathcal{L}_i is the oppositve of the normal cone $-\mathcal{N}_Z X$. If f is the composite $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ with p and i as previously (in other words, if f is lci projective), then \mathcal{L}_f is isomorphic to the virtual tangent bundle $i^* \mathcal{T}_{\mathbb{P}_X^n/X} - \mathcal{N}_Y(\mathbb{P}_X^n)$ (see also [Fel18, Section 9]).

Let X be a scheme and $x \in X$ a point, we denote by $\mathcal{L}_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ and ω_x its determinant. Similarly, let v a discrete valuation on a field, we denote by ω_v the line bundle $(\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee$.

Let E be a field (over k) and v a valuation on E . We will always assume that v is **discrete**. We denote by \mathcal{O}_v its valuation ring, by \mathfrak{m}_v its maximal ideal and by $\kappa(v)$ its residue class field. We consider only valuations of geometric type, that is we assume: $k \subset \mathcal{O}_v$, the residue field $\kappa(v)$ is finitely generated over k and satisfies $\text{tr. deg}_k(\kappa(v)) + 1 = \text{tr. deg}_k(E)$.

Let E be a field. We denote by $\text{GW}(E)$ the Grothendieck-Witt ring of symmetric bilinear forms on E . For any $a \in E^*$, we denote by $\langle a \rangle$ the class of the symmetric bilinear form on E defined by $(X, Y) \mapsto aXY$ and, for any natural number n , we put $n_\varepsilon = \sum_{i=1}^n \langle -1 \rangle^{i-1}$.

1.1.2 Set theoretic remarks

The whole thesis takes place in the axiomatic ZFCU of Zermelo-Fraenkel with the axiom of choice and the axioms of Grothendieck universes. Precisely, we work with Bourbaki's set theory [Bou06] and consider the following definition:

Definition 1.1.2.1 ([AGV⁺72], Exposé I). A *universe* is a non-empty set \mathbb{U} such that

($\mathcal{U}1$) If $x \in \mathbb{U}$ and $y \in x$, then $y \in \mathbb{U}$.

($\mathcal{U}2$) If $x, y \in \mathbb{U}$, then $\{x, y\} \in \mathbb{U}$.

($\mathcal{U}3$) If $x \in \mathbb{U}$, then $\mathfrak{P}(x) \in \mathbb{U}$.

($\mathcal{U}4$) If $(x_i, i \in I)$ is a family of elements of \mathbb{U} and $I \in \mathbb{U}$, then $\cup_{i \in I} x_i \in \mathbb{U}$.

Remark 1.1.2.2. Let \mathbb{U} be a universe. The previous definition yields:

1. If $x \in \mathbb{U}$, then $\{x\} \in \mathbb{U}$.
2. If x is a subset of $y \in \mathbb{U}$, then $x \in \mathbb{U}$.
3. If $x, y \in \mathbb{U}$, then the couple $(x, y) = \{x, \{x, y\}\}$ (Kuratowski's definition) is an element of \mathbb{U} .
4. If $x, y \in \mathbb{U}$, then the union $x \cup y$ and the product $x \times y$ are elements of \mathbb{U} .
5. If $(x_i, i \in I \in \mathbb{U})$ is a family of elements in \mathbb{U} , then the product $\prod_{i \in I} x_i$ is an element of \mathbb{U} .
6. If $x \in \mathbb{U}$, then $\text{Card}(x) < \text{Card}(\mathbb{U})$. In particular, we do not have the relation $\mathbb{U} \in \mathbb{U}$.

The only known universe is the enumerable set of symbols of the form $\{\{\emptyset\}, \{\{\emptyset\}, \emptyset\}\}$, etc... That is why we assume that a universe contains an element of infinite cardinal and we add moreover the following axiom to our theory:

($\mathcal{U}A$) For any set x , there exists a universe \mathbb{U} such that $x \in \mathbb{U}$.

The notion of universes is a convenient¹ way to play with categories. For instance, we denote by $\mathbb{U}\mathbf{Set}$ the category whose objects are elements of \mathbb{U} . By definition, we say that a set is \mathbb{U} -small if it is isomorphic to an element of \mathbb{U} and we say that a category \mathcal{C} is a \mathbb{U} -category (or is locally \mathbb{U} -small) if, for any objects $x, y \in \mathcal{C}$, the set $\text{Hom}_{\mathcal{C}}(x, y)$ is \mathbb{U} -small.

Example 1.1.2.3. Let \mathcal{C}, \mathcal{D} be two categories and $\text{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from \mathcal{C} to \mathcal{D} . We have:

1. If \mathcal{C} and \mathcal{D} are elements of a universe \mathbb{U} (resp. are \mathbb{U} -small), then the category $\text{Fun}(\mathcal{C}, \mathcal{D})$ is an element of \mathbb{U} (resp. is \mathbb{U} -small).
2. If \mathcal{C} is \mathbb{U} -small and \mathcal{D} is a \mathbb{U} -category, then $\text{Fun}(\mathcal{C}, \mathcal{D})$ is a \mathbb{U} -category.

Remark 1.1.2.4. Let \mathcal{D} be a category with the two following properties:

(C1) The set $\text{ob}(\mathcal{D})$ is an element of \mathbb{U} .

(C2) For any objects x, y in \mathcal{D} , the set $\text{Hom}_{\mathcal{D}}(x, y)$ is in \mathbb{U} .

Let \mathcal{C} be a category in \mathbb{U} . Then the category $\text{Fun}(\mathcal{C}, \mathcal{D})$ does not satisfy the conditions (C1) and (C2) in general (e.g. take $\mathcal{C} = \mathbb{U}\mathbf{Set}$). This explains the definition of \mathbb{U} -category, instead of a more restrictive definition using the conditions (C1) and (C2).

¹However, one could certainly work with ZFC alone.

For technical reason, we also add the following axiom to our theory:

($\mathcal{U}B$) Let $R\{x\}$ be a relation and \mathbb{U} a universe. If there exists an element $y \in \mathbb{U}$ such that $R\{y\}$, then $\tau_x R\{x\} \in \mathbb{U}$.

Here *relation* could also be translated by *well-formed formula*, and τ denotes the Hilbert operator as defined in [Bou06, Chapitre I]. We illustrate this axiom with the following construction of a Yoneda embedding.

Definition 1.1.2.5 ([AGV⁺72], Exposé I, §1.3). Let \mathbb{U} be a universe and let \mathcal{C} be a \mathbb{U} -category. We denote by $\text{PSh}_{\mathbb{U}}(\mathcal{C})$ (or abusively, $\text{PSh}(\mathcal{C})$) the category of presheaves from \mathcal{C} to $\mathbb{U}\mathbf{Set}$.

Let x be an object of \mathcal{C} . We call \mathbb{U} -*functor represented by x* the functor $h_{\mathbb{U}}(x) : \mathcal{C}^{op} \rightarrow \mathbb{U}\mathbf{Set}$ defined as follows. Let y be an object of \mathcal{C} .

a) If $\text{Hom}_{\mathcal{C}}(y, x)$ is an element of \mathbb{U} , then put

$$h_{\mathbb{U}}(x)(y) = \text{Hom}_{\mathcal{C}}(y, x).$$

b) If $\text{Hom}_{\mathcal{C}}(y, x)$ is not an element of \mathbb{U} , we let $R(Z, x, y)$ be the relation «The set Z is such that there is an isomorphism $\text{Hom}_{\mathcal{C}}(y, x) \simeq Z$ ». We thus put

$$h_{\mathbb{U}}(x)(y) = \tau_Z R(Z)$$

which is an element of \mathbb{U} according to axiom ($\mathbb{U}B$).

Let $R'(u, x, y)$ the relation « u is a bijection

$$u : \text{Hom}_{\mathcal{C}}(y, x) \simeq h_{\mathbb{U}}(y)(x) \text{ »}.$$

We put

$$\varphi(y, x) = \tau_u R'(u).$$

In both cases (a) and (b), we have a canonical isomorphism

$$\varphi(y, x) : \text{Hom}_{\mathcal{C}}(y, x) \rightarrow h_{\mathbb{U}}(x)(y).$$

Let $u : y \rightarrow y'$ be an arrow of \mathcal{C} and consider the induced map:

$$\text{Hom}_{\mathcal{C}}(u, x) : \text{Hom}_{\mathcal{C}}(y', x) \rightarrow \text{Hom}_{\mathcal{C}}(y, x).$$

We put

$$h_{\mathbb{U}}(x)(u) = \varphi(y, x) \circ \text{Hom}_{\mathcal{C}}(u, x) \circ \varphi(y', x)^{-1}.$$

We can check that $h_{\mathbb{U}}(x)$ is a functor $\mathcal{C}^{op} \rightarrow \mathbb{U}\mathbf{Set}$.

Similarly, we can use the canonical isomorphisms φ to define, for any map $f : x \rightarrow x'$, a morphism of functors:

$$h_{\mathbb{U}}(f) : h_{\mathbb{U}}(x) \rightarrow h_{\mathbb{U}}(x')$$

and we may check that this defines a functor

$$h_{\mathbb{U}} : \mathcal{C} \rightarrow \mathbf{PSh}_{\mathbb{U}}(\mathcal{C})$$

which is also fully faithful.

Fix \mathbb{U} a universe. Expanding the previous definitions, we say that a mathematical object \mathcal{C} is \mathbb{U} -small if all the data defining \mathcal{C} are given by \mathbb{U} -small sets. For instance, a simplicial set X is \mathbb{U} -small if all its level sets X_i are isomorphic to elements in \mathbb{U} . A mathematical object \mathcal{C} is *essentially \mathbb{U} -small* if it is equivalent (in a sense to be specified) to a \mathbb{U} -small object.

Consider another universe \mathbb{V} such that $\mathbb{U} \in \mathbb{V}$. It is clear that any \mathbb{U} -small object is also \mathbb{V} -small. With a convenient choice for \mathbb{V} , the set of all \mathbb{U} -small ∞ -categories forms a \mathbb{V} -small ∞ -category denoted by $\mathbf{Cat}_{\infty}^{\mathbb{U}}$ (see [Lur09, Chapter 3] for more details). We have a canonical inclusion $\hat{\Delta}_{\mathbb{U}} \subset \hat{\Delta}_{\mathbb{V}}$ between the categories of \mathbb{U} -small and \mathbb{V} -small simplicial sets, which is compatible with the Joyal model structure. Again, if \mathbb{W} is another universe such that $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$, then we have an inclusion of \mathbb{W} -small categories $\mathbf{Cat}_{\infty}^{\mathbb{U}} \subset \mathbf{Cat}_{\infty}^{\mathbb{V}}$ and we say that a \mathbb{V} -small ∞ -category is *essentially \mathbb{U} -small* if it is weakly equivalent in $\hat{\Delta}_{\mathbb{V}}$ to a \mathbb{U} -small ∞ -category.

Throughout the text, we fix once and for all three universes $\mathbb{U} \in \mathbb{V} \in \mathbb{W}$. For simplicity, we use the adjectives *small*, *big* and *very big* instead of \mathbb{U} -small, \mathbb{V} -small and \mathbb{W} -small, respectively. For instance, we write \mathbf{Cat}_{∞} (resp. $\mathbf{Cat}_{\infty}^{big}$) to denote the ∞ -category of small (resp. big) ∞ -categories.

1.2 Stable and unstable homotopy ∞ -categories

1.2.1 Unstable homotopy category

We follow the work of Robalo and define the stable and unstable homotopy categories in the context of ∞ -categories (see [Rob15, §2]).

1.2.1.1. We use the language of higher category theory, particularly the model of *quasicategories* as introduced by Boardman, Vogt [BV73] and developed by Joyal and Lurie. We generally follow the terminological and notational conventions of Lurie [Lur09, Lur17]. The word ∞ -category will always mean $(\infty, 1)$ -category or *quasicategory*. Recall that the Joyal's model structure is a combinatorial, left proper, cartesian closed model structure in the category of simplicial sets $\hat{\Delta}$: the cofibrations are the monomorphisms, and the fibrant objects are the quasicategories, that is the simplicial sets \mathcal{C} with the lifting property

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{f} & \mathcal{C} \\ \downarrow & \nearrow \text{dashed} & \\ \Delta[n] & & \end{array}$$

for any inclusion of an inner-horn $\Lambda^k[n] \subset \Delta[n]$ with $0 < k < n$ and any map f .

1.2.1.2. Recall the classical construction of the original category of spaces using techniques from model category theory. Start with the category of smooth schemes \mathbf{Sm}_k , add formally (homotopy) colimits by considering simplicial presheaves $\Delta\mathbf{Psh}(\mathbf{Sm}_k)$ with the

projective model structure, use Bousfield localization with respect to the set of the hypercovers associated to the Nisnevich topology in order to obtain a category $L_{Nis}(\Delta\mathbf{Psh}(\mathbf{Sm}_k))$ and, finally, localize with respect to the set of all projection maps $X \times \mathbb{A}^1 \rightarrow X$ and force the affine line \mathbb{A}^1 to be contractible:

$$\mathcal{M}_{\mathbb{A}^1} = L_{\mathbb{A}^1} L_{Nis}(\Delta\mathbf{Psh}(\mathbf{Sm}_k)).$$

1.2.1.3. We want to define an ∞ -category $\mathbf{H}(k)$ that would generalize the previous one. Again, we start with the category of smooth schemes \mathbf{Sm}_k and consider its nerve $N(\mathbf{Sm}_k)$. The Nisnevich topology yields a structure of ∞ -topos (see [Lur09, Definition 6.2.2.1]). By definition, this topology is generated by the pre-topology whose covering families of a k -scheme X are the set of étale maps $\{f_i : U_i \rightarrow X\}_{i \in I}$ such that, for any $x \in X$, there exist $i \in I$ and $u_i \in U_i$ such that f_i induces an isomorphism between the residual fields $k(x) \simeq k(u_i)$. This topology is also generated by the following *elementary squares*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

where the square is Cartesian in \mathbf{Sm}_k , and where j is an open immersion and p is an étale map such that $p^{-1}(X - U)_{red} \rightarrow (X - U)_{red}$ is an isomorphism.

We then add formally colimits to the nerve $N(\mathbf{Sm}_k)$ by considering the ∞ -category of presheaves $\mathbf{PSh}^{big}(N(\mathbf{Sm}_k))$. Note that this ∞ -category is the underlying ∞ -category of the model category of simplicial presheaves on \mathbf{Sm}_k endowed with the projective model structure. Moreover, we have a Yoneda embedding

$$j : N(\mathbf{Sm}_k) \rightarrow \mathbf{PSh}^{big}(N(\mathbf{Sm}_k))$$

that allows us to identify a smooth scheme X with its image $j(X)$.

We restrict to presheaves that are sheaves with respect to the Nisnevich topology. Precisely, a presheaf $F \in \mathbf{PSh}^{big}(N(\mathbf{Sm}_k))$ is a sheaf if, and only if, it maps elementary Nisnevich squares to pullback squares. In particular, $j(X)$ is always a sheaf for any smooth scheme X .

Denote by $\mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k)$ the full subcategory of such sheaves. The canonical inclusion $\mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k) \subset \mathbf{PSh}(N(\mathbf{Sm}_k))$ has an exact left adjoint (see [Lur09, 5.5.4.15 and 6.2.2.7]) and gives a typical example of a presentable localization of a presentable ∞ -category.

We consider the hypercompletion of the ∞ -topos $\mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k)$ which is a presentable localization of $\mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k)$ (see [Lur09, Section 6.5.2]). In our setting, this step is superfluous since the hypercompletion is canonically isomorphic to $\mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k)$ (because $\mathrm{Spec} k$ is a Noetherian scheme of finite Krull dimension, see [Lur04, 2.2.5]).

For the final step, we restrict ourselves to the sheaves in $\mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k)$ satisfying \mathbb{A}^1 -invariance, that is to say, the canonical map $F(X) \rightarrow F(X \times \mathbb{A}^1)$ is equivalence for any smooth scheme X . We have a localization functor

$$l_{\mathbb{A}^1} : \mathbf{Sh}_{Nis}^{big}(\mathbf{Sm}_k) \rightarrow \mathbf{H}(k)$$

where $\mathbf{H}(k)$ is our unstable homotopy ∞ -category. It is presentable and satisfies the following universal property.

Theorem 1.2.1.4 (Robalo). *Let $L : N(\mathbf{Sm}_k) \rightarrow \mathbf{H}(k)$ be the composition of localizations*

$$N(\mathbf{Sm}_k) \rightarrow \mathrm{PSh}^{big}(N(\mathbf{Sm}_k)) \rightarrow \mathrm{Sh}_{Nis}^{big}(\mathbf{Sm}_k) \rightarrow \mathrm{Sh}_{Nis}^{big}(\mathbf{Sm}_k)^{hyp} \rightarrow \mathbf{H}(k).$$

Then, for any ∞ -category \mathcal{D} with all \mathbb{V} -small colimits, the maps induced by composition with L

$$\mathrm{Fun}^L(\mathbf{H}(k), \mathcal{D}) \rightarrow \mathrm{Fun}(N(\mathbf{Sm}_k), \mathcal{D})$$

is fully faithful and its essential image is the full subcategory of $\mathrm{Fun}(N(\mathbf{Sm}_k), \mathcal{D})$ spanned by those functors satisfying Nisnevich descent and \mathbb{A}^1 -invariance. The left-side denotes the full subcategory of $\mathrm{Fun}(\mathbf{H}(k), \mathcal{D})$ spanned by the colimit preserving maps.

1.2.1.5. For completion's sake, we present another construction of the unstable homotopy ∞ -category, following ideas from Bachmann and Hoyois [BH17]. Both perspectives are equivalent (see [Bra18, §4]).

Let \mathcal{C} be a small ∞ -category with finite coproducts. We denote by $\mathrm{PSh}_\Sigma(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ the full subcategory of presheaves which map finite coproducts to finite products (this is known as the *nonabelian derived ∞ -category of \mathcal{C}*). The canonical inclusion $\mathrm{PSh}_\Sigma(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ has an adjoint denoted by L_Σ . We consider the topology on \mathcal{C} generated by finite coproduct decompositions and we denote by $\mathrm{Sh}_\sqcup(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ the corresponding ∞ -topos of sheaves.

Definition 1.2.1.6. An ∞ -category is called *extensive* if it admits finite coproducts, binary coproducts are disjoint and finite coproduct decompositions are stable under pullbacks.

Proposition 1.2.1.7 ([BH17], Lemma 2.4). *Assume \mathcal{C} is a small extensive ∞ -category. Then $\mathrm{PSh}_\Sigma(\mathcal{C}) \simeq \mathrm{Sh}_\sqcup(\mathcal{C})$.*

In our case, the category \mathbf{Sm}_k of (finitely-presented) smooth k -schemes is extensive (since $\mathrm{Spec} k$ is quasi-compact quasi-separated). The fact that coproduct decompositions are Nisnevich coverings implies that

$$\mathrm{Sh}_{Nis}(\mathbf{Sm}_k) \subset \mathrm{PSh}_\Sigma(\mathbf{Sm}_k)$$

thanks to the previous proposition. If we denote by $\mathrm{PSh}_{\mathbb{A}^1}(\mathbf{Sm}_k) \subset \mathrm{PSh}(\mathbf{Sm}_k)$ the full subcategory of \mathbb{A}^1 -invariant presheaves, we can define the unstable homotopy ∞ -category as

$$\mathbf{H}(k) = \mathrm{Sh}_{Nis}(\mathbf{Sm}_k) \cap \mathrm{PSh}(\mathbf{Sm}_k).$$

Remark 1.2.1.8. We conclude with the fact that a careful examination of the previous constructions shows that $\mathbf{H}(k)$ is indeed the underlying ∞ -category of the category defined by Morel and Voevodsky.

1.2.2 Monoidal structure on the pointed unstable homotopy ∞ -category

The unstable homotopy ∞ -category $\mathbf{H}(k)$ previously defined is presentable hence admits a final object $*$ and the ∞ -category of pointed objects $\mathbf{H}(k)_*$ is also presentable (see [Lur09, 5.5.2.10]). Since the forgetful functor $\mathbf{H}(k)_* \rightarrow \mathbf{H}(k)$ commutes with limits, the Adjoint Functor Theorem (see [Lur09, 5.5.2.9]) shows that there is a left adjoint

$$\begin{aligned} (-)_+ : \mathbf{H}(k) &\rightarrow \mathbf{H}(k)_* \\ X &\mapsto X_+ = X \sqcup *. \end{aligned}$$

We want to extend the cartesian product in $\mathbf{H}(k)$ to a symmetric monoidal structure in $\mathbf{H}(k)_*$. This follows from more general considerations:

Proposition 1.2.2.1 (Lurie, Robalo). *The formula $\mathcal{C} \mapsto \mathcal{C}_*$ defines a monoidal left adjoint to the inclusion $\mathcal{P}r_*^L \subset \mathcal{P}r^L$ where $\mathcal{P}r^L$ is the ∞ -category of presentable ∞ -categories with functors that commute with colimits. In particular, for any presentable symmetric monoidal ∞ -category \mathcal{C}^\otimes , there exists a pointed presentable symmetric monoidal ∞ -category $\mathcal{C}_*^{\wedge(\otimes)}$ whose underlying ∞ -category is \mathcal{C}_* , together with a monoidal functor $\mathcal{C}^\otimes \rightarrow \mathcal{C}_*^{\wedge(\otimes)}$ extending the pointing map $\mathcal{C} \rightarrow \mathcal{C}_*$, and satisfying the following universal property:*

for any pointed presentable symmetric monoidal ∞ -category \mathcal{D}^\otimes , the composition

$$\mathrm{Fun}^{\otimes, L}(\mathcal{C}_*^{\wedge(\otimes)}, \mathcal{D}^\otimes) \rightarrow \mathrm{Fun}^{\otimes, L}(\mathcal{C}^\otimes, \mathcal{D}^\otimes)$$

is an equivalence.

Proof. The ∞ -category \mathcal{S} is the unit for the symmetric monoidal structure of the ∞ -category $\mathcal{P}r^L$. The general theory proves that the functor $\mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{S}_*$ is a monoidal left adjoint of the canonical inclusion $\mathcal{P}r_*^L \subset \mathcal{P}r^L$, where \mathcal{S} is the ∞ -category of spaces (see [Lur09, Definition 1.2.16.1]). We conclude from the fact that, for any presentable ∞ -category \mathcal{C} , there is an equivalence $\mathcal{C}_* \simeq \mathcal{C} \otimes \mathcal{S}_*$ (see [Lur17, 4.8.1.20]) so that the pointed map $\mathcal{C} \rightarrow \mathcal{C}_*$ is equivalent to the product map $\mathrm{Id}_{\mathcal{C}} \otimes (-)_+ : \mathcal{C} \otimes \mathcal{S} \rightarrow \mathcal{C} \otimes \mathcal{S}_*$. \square

The symmetric monoidal structure $\mathcal{C}_*^{\wedge(\otimes)}$ is called the *smash product induced by \mathcal{C}^\otimes* .

We can link the previous results with the classical approach. Recall that the category $\mathcal{M}_{\mathbb{A}^1}$ encoding the \mathbb{A}^1 -homotopy theory of Morel-Voevodsky is a combinatorial simplicial model category. Moreover, this category is cartesian closed with a cofibrant final object $*$. This yields to a canonical model structure on the pointed category $(\mathcal{M}_{\mathbb{A}^1})_*$ which is again combinatorial and simplicial, and comes equipped with a left-Quillen functor $(-)_+ : \mathcal{M}_{\mathbb{A}^1} \rightarrow (\mathcal{M}_{\mathbb{A}^1})_*$, $X \mapsto X \sqcup *$. We can define a structure of symmetric monoidal model category via the usual definition of the smash product, given by the formula

$$(X, x) \wedge (Y, y) = ((X, x) \times (Y, y)) / ((X, x) \vee (Y, y)).$$

One can prove that $\mathbf{H}(k)_*^{\wedge}$ is the underlying symmetric monoidal ∞ -category of the classical construction of Morel-Voevodsky, that is to say: there is an equivalence of presentable symmetric monoidal ∞ -categories

$$\mathbf{H}(k)_*^{\wedge} \rightarrow N_{\Delta}^{\otimes}((((\mathcal{M}_{\mathbb{A}^1})_*)^{cf})^{\wedge}).$$

1.3 Stable homotopy ∞ -category and derived homotopy ∞ -category

1.3.1 Stable homotopy ∞ -category

We conclude this section with the construction of the stable homotopy category $\mathbf{SH}(k)$. Recall that our motivic setting includes not one but two different spheres: the topological

circle $S^1 = \Delta[1]/\partial\Delta[1]$ (pointed by the image of $\partial\Delta[1]$) and the algebraic circle $\mathbb{G}_m = (\mathbb{A}^1 - \{0\}, 1)$. We stabilize with respect to both spheres, that is to say we invert the product $S^1 \wedge \mathbb{G}_m$ which is known to be equivalent to (\mathbb{P}^1, ∞) .

Definition 1.3.1.1. The stable motivic \mathbb{A}^1 -homotopy ∞ -category over k is the underlying ∞ -category of the presentable symmetric monoidal ∞ -category $\mathbf{SH}(k)^\otimes$ defined by the formula

$$\mathbf{SH}(k)^\otimes = \mathbf{H}(k)_*^\wedge[(\mathbb{P}^1, \infty)^{-1}],$$

where the formal inversion of (\mathbb{P}^1, ∞) is defined in [Rob13, Definition 4.8].

Remark 1.3.1.2. One can prove that $\mathbf{SH}(k)^\otimes$ recovers the classical definition which considers the combinatorial simplicial symmetric monoidal model category $Sp^\Sigma((\mathcal{M}_{\mathbb{A}^1})_*, (\mathbb{P}^1, \infty))$. Moreover, we have a canonical monoidal equivalences of stable presentable symmetric monoidal ∞ -categories:

$$\begin{aligned} \mathbf{SH}(k)^\otimes &\simeq (\mathbf{H}(k)_*^\wedge)[(\mathbb{G}_m \wedge S^1)^{-1}] \simeq (\mathbf{H}(k)_*^\wedge)[((\mathbb{P}^1, \infty) \wedge S^1)^{-1}] \simeq \\ &((\mathbf{H}(k)_*^\wedge[(S^1)^{-1}])(\mathbb{P}^1, \infty)^{-1}). \end{aligned}$$

We end with the fundamental property that characterizes the stable motivic ∞ -category:

Proposition 1.3.1.3 ([Rob13], Cor. 5.11). *The composition of monoidal functors*

$$N(\mathbf{Sm}_k)^\times \rightarrow \mathcal{P}^{big}(N(\mathbf{Sm}_k))^\times \rightarrow \mathbf{H}(k)^\times \rightarrow \mathbf{H}(k)_*^\wedge \rightarrow \mathbf{H}(k)_*^\wedge[(S^1)^{-1}] \rightarrow \mathbf{SH}(k)^\otimes$$

satisfies the following universal property: for any pointed presentable symmetric monoidal ∞ -category \mathcal{D}^\otimes , the composition map

$$\mathrm{Fun}^{\otimes, L}(\mathbf{SH}(k)^\otimes, \mathcal{D}^\otimes) \rightarrow \mathrm{Fun}^\otimes(N(\mathbf{Sm}_k)^\times, \mathcal{D}^\otimes)$$

is fully faithful and its image consists of those monoidal functors $N(\mathbf{Sm}_k)^\times \rightarrow \mathcal{D}^\otimes$ whose underlying functor satisfy Nisnevich descent, \mathbb{A}^1 -invariance and such that the cofiber of the image of the point at ∞ , $S \xrightarrow{\infty} \mathbb{P}^1$ is an invertible object in \mathcal{D}^\otimes . Moreover, any pointed presentable symmetric monoidal ∞ -category \mathcal{D}^\otimes admitting such a monoidal functor is necessarily stable.

1.3.2 Derived homotopy ∞ -category

Recall that presentable ∞ -categories can be localized, a theory that parallels the notion of Bousfield localization of model categories (see [Lur09, 5.5.4]). We will need further a condition that ensure that the localization of a presentable ∞ -category equipped with a monoidal structure admits a canonical monoidal structure. This is provided by [Lur17, Prop. 2.2.1.9]) as explained after the following definition.

Definition 1.3.2.1. Let \mathcal{C} be a presentable monoidal ∞ -category, and \mathfrak{S} a set of maps in \mathcal{C} . We say that \mathfrak{S} is \otimes -compatible if, for every S -equivalence f and every object M , the map $f \otimes M$ is an \mathfrak{S} -equivalence.

Proposition 2.2.1.9 of [Lur17] shows that in that case, there exists a canonical monoidal structure on the localized ∞ -category $\mathcal{C}[\mathfrak{S}^{-1}]$ such that the localization functor

$$L_S : \mathcal{C} \rightarrow \mathcal{C}[\mathfrak{S}^{-1}]$$

is compatible with the monoidal structure (i.e. is part of a morphism of operads over the commutative operad).

The \mathbb{A}^1 -derived ∞ -category satisfies a universal property which was previously stated in terms of model categories before (see [CD19, §5]).

Proposition 1.3.2.2. *Let S be a scheme and R a coefficient ring. There exists an initial R -linear ∞ -category $\mathbf{D}_{\mathbb{A}^1}(S, R)$ equipped with a monoidal ∞ -functor*

$$M : \mathbf{Sm}_S \rightarrow \mathbf{D}_{\mathbb{A}^1}(S, R)$$

such that:

1. the map $p_* : M(\mathbb{A}_S^1) \rightarrow M(S)$ induced by the projection is an equivalence;
2. the cokernel $\mathbb{L}_S(1)[2]$ of the split map $M(\{\infty\}) \rightarrow M(\mathbb{P}_S^1)$ is \otimes -invertible;
3. for any Nisnevich hyper-cover $p : W_\bullet \rightarrow X$ of a smooth S -scheme X , the induced map

$$p_* : M(W_\bullet) \rightarrow M(X)$$

is an equivalence, where we have used the canonical left Kan extension of the functor M to the ∞ -category of simplicial schemes using the fact that $\mathbf{D}_{\mathbb{A}^1}(S, R)$ admits colimits.

Proof. The existence of the category follow from the classical model theoretic construction after Morel and Voevodsky (see [CD19, §5]) and the associated (monoidal) ∞ -category. However, it is now easier to use the ∞ -categorical construction, and necessary to get the universal property. The following material is folklore (see also [Dre18]). We include it for completeness. We follow the approach of [Rob15]. We start with the ∞ -category of ∞ -functors

$$\mathrm{PSh}(S, R) := \mathrm{PSh}(\mathbf{Sm}_S, D(R))$$

equipped with its natural monoidal structure (see [Lur17, 4.8.1.12, 4.8.1.13]). The Yoneda embedding gives us the functor:

$$M : \mathbf{Sm}_S \rightarrow \mathrm{PSh}(S, R),$$

that we extend to simplicial smooth schemes

$$M : \Delta^{op} \mathbf{Sm}_S \rightarrow \mathrm{PSh}(S, R)$$

using the existence of colimit in $\mathrm{PSh}(S, R)$. Then we localize the (presentable) monoidal ∞ -category $\mathrm{PSh}(S, R)$ with respect to the set of maps \mathfrak{S} of the form:

1. $M(\mathbb{A}_X^1) \rightarrow M(X)$;
2. $M(W_\bullet) \rightarrow M(X)$ for any Nisnevich hyper-cover $W_\bullet \rightarrow X$ of a smooth S -scheme X .

The set \mathfrak{S} is \otimes -compatible (see 1.3.2.1, as follows from the classical theory), so the resulting localized ∞ -category admits a monoidal structure. We denote the latter by $\mathbf{D}_{\mathbb{A}^1}^{eff}(S, R)$ and call it the *effective localized \mathbb{A}^1 -derived Nisnevich ∞ -category with R -coefficients*.

Finally, we define $\mathbf{D}_{\mathbb{A}^1}(S, R)$ as the formal inversion of the Tate twist $\mathbb{L}_S(1)$ in the presentable monoidal ∞ -category $\mathbf{D}_{\mathbb{A}^1}^{eff}(S, R)$ as defined in [Rob15, Cor. 4.25]. The universal property of the composite monoidal functor

$$M : \mathbf{Sm}_S \rightarrow \mathbf{D}_{\mathbb{A}^1}^{eff}(S, R) \rightarrow \mathbf{D}_{\mathbb{A}^1}(S, R)$$

follows from the construction as in [Rob13, Cor. 5.11]. \square

Remark 1.3.2.3. By construction, the ∞ -category $\mathbf{D}_{\mathbb{A}^1}^{eff}(S, R)$ is equivalent to the ∞ -category associated with the \mathbb{A}^1 -localization of the Nisnevich-local model category structure on the category of complexes $C(\mathbf{Sm}_S, R)$ of (pre)sheaves of R -modules on \mathbf{Sm}_S as constructed in [CD19, §5].

Then, taking into account [Rob13, Th. 4.29], this implies that the ∞ -category $\mathbf{D}_{\mathbb{A}^1}(S, R)$ is equivalent to the ∞ -category associated with the \mathbb{P}^1 -stable, Nisnevich-local and \mathbb{A}^1 -local model structure on the category $C(\mathbf{Sm}_S, R)$.

Remark 1.3.2.4. In practice, we have $S = \text{Spec } k$ and $R = \mathbb{Z}$, and denote by $\mathbf{D}_{\mathbb{A}^1}(k) = \mathbf{D}_{\mathbb{A}^1}(S, R)$.

Remark 1.3.2.5. Replacing the category \mathbf{Sm}_k of smooth schemes by the category $\widetilde{\text{Cor}}_k$ in the previous construction, we obtain an ∞ -category which corresponds to the Milnor-Witt derived homotopy category $\widetilde{\mathbf{DM}}_k$ defined in [BCD⁺20, Ch. 3].

1.4 Milnor-Witt K-theory

1.4.1 Definitions

We describe the Milnor-Witt K-theory, as defined by Morel (see [Mor12, §3] or [Fas19, §1.1]).

Definition 1.4.1.1. Let E be a field. The Milnor-Witt K-theory algebra of E is defined to be the quotient of the free \mathbb{Z} -graded algebra generated by the symbols $[a]$ of degree 1 for any $a \in E^\times$ and a symbol η in degree -1 by the following relations:

- $[a][1 - a] = 0$ for any $a \in E^\times \setminus \{1\}$.
- $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in E^\times$.
- $\eta[a] = [a]\eta$ for any $a \in E^\times$.
- $\eta(\eta[-1] + 2) = 0$.

The relations being homogeneous, the resultant algebra is \mathbb{Z} -graded. We denote it by $\mathbf{K}_*^{\text{MW}}(E)$.

1.4.1.2. NOTATION. We will use the following notations.

- $[a_1, \dots, a_n] = [a_1] \dots [a_n]$ for any $a_1, \dots, a_n \in E^\times$.
- $\langle a \rangle = 1 + \eta[a]$ for any $a \in E^\times$.
- $\varepsilon = -\langle -1 \rangle$.
- $n_\varepsilon = \sum_{i=1}^n \langle (-1)^{i-1} \rangle$ for any $n \geq 0$, and $n_\varepsilon = \varepsilon(-n)_\varepsilon$ if $n < 0$.

1.4.1.3. **TWISTED MILNOR-WITT K-THEORY.** Let E be a field and \mathcal{V}_E a virtual vector bundle over E with rank n and determinant \mathcal{L}_E (see Subsection 1.9.1). The group E^\times of invertible elements of E acts naturally on \mathcal{L}_E^\times , the set of non-zero elements in \mathcal{L}_E ; hence the free abelian group $\mathbb{Z}[\mathcal{L}_E^\times]$ is a $\mathbb{Z}[E^\times]$ -module. Define

$$\underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) = \mathbf{K}_n^{MW}(E) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\mathcal{L}_E^\times],$$

where the action of E^\times on $\mathbf{K}_n^{MW}(E)$ is given by $u \mapsto \langle u \rangle$.

By abuse of notation, we might denote by $\boldsymbol{\eta}$ the element $(\boldsymbol{\eta} \otimes 1) \in \underline{\mathbf{K}}^{MW}(E, -\mathbb{A}^1)$ and by $[u]$ the element $([u] \otimes 1) \in \underline{\mathbf{K}}^{MW}(E, \mathbb{A}^1)$ (where $u \in E^\times$).

Let \mathcal{V}_E and \mathcal{W}_E be two virtual bundles over E . The product of the Milnor-Witt K-theory groups induces a product

$$\begin{aligned} \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) \otimes \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E) &\rightarrow \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E + \mathcal{W}_E) \\ (x \otimes l, x' \otimes l') &\mapsto (xx') \otimes (l \wedge l'), \end{aligned}$$

so that $\underline{\mathbf{K}}^{MW}(E, -)$ is a lax monoidal functor from the category of virtual bundles over E to the category of abelian groups.

1.4.1.4. **RESIDUE MORPHISMS** (see [Mor12, Theorem 3.15]) Let E be a field endowed with a discrete valuation v . We choose a uniformizing parameter π . As in the classical Milnor K-theory, we can define a residue morphism

$$\partial_v^\pi : \mathbf{K}_*^{MW}(E) \rightarrow \mathbf{K}_{*-1}^{MW}(\kappa(v))$$

characterized by the fact that it commutes with the multiplication by $\boldsymbol{\eta}$ and satisfies the following two properties:

- $\partial_v^\pi([\pi, a_1, \dots, a_n]) = [\overline{a_1}, \dots, \overline{a_n}]$ for any $a_1, \dots, a_n \in \mathcal{O}_v^\times$.
- $\partial_v^\pi([a_1, \dots, a_n]) = 0$ for any $a_1, \dots, a_n \in \mathcal{O}_v^\times$.

This morphism does depend on the choice of π . Indeed, if we consider another uniformizer π' and write $\pi' = u\pi$ where u is a unit, then we have $\partial_v^{\pi'}(x) = \langle u \rangle \partial_v^\pi(x)$ for any $x \in \mathbf{K}_*^{MW}(E)$. Nevertheless, by making a choice of isomorphism $\mathcal{N}_v \simeq \mathbb{A}_{\kappa(v)}^1$ (where $\mathcal{N}_v = \mathfrak{m}_v / \mathfrak{m}_v^2$ is the normal cone of v), we can define a twisted residue morphism that does not depend on π :

$$\begin{aligned} \partial_v : \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) &\rightarrow \underline{\mathbf{K}}^{MW}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \\ x \otimes l &\mapsto \partial_v^\pi(x) \otimes (\bar{\pi}^* \wedge l) \end{aligned}$$

where \mathcal{V} is a virtual vector bundle over \mathcal{O}_v , $\bar{\pi}$ is the class of π modulo \mathfrak{m}_v and $\bar{\pi}^*$ its canonical associated linear form (see also Example 1.9.1.3).

We have the following proposition (see [Mor12, Lemma 3.19]).

Proposition 1.4.1.5. *Let $j : E \subset F$ be a field extension, and let w be a discrete valuation on F which restricts to a discrete valuation v on E with ramification index e . Let $i : \kappa(v) \rightarrow \kappa(w)$ be the field extension of the residue fields. Let \mathcal{V} be a virtual vector bundle over \mathcal{O}_v . Then, we have a commutative diagram*

$$\begin{array}{ccc} \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) & \xrightarrow{\partial_v} & \underline{\mathbf{K}}^{MW}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \\ j_* \downarrow & & \downarrow e_\pi \cdot i_* \\ \underline{\mathbf{K}}^{MW}(F, \mathcal{V}_F) & \xrightarrow{\partial_w} & \underline{\mathbf{K}}^{MW}(\kappa(w), -\mathcal{N}_w + \mathcal{V}_{\kappa(w)}) \end{array}$$

where the right vertical map $e_\varepsilon \cdot i_*$ is defined by

$$x \otimes \bar{\rho}^* \otimes l \mapsto e_\varepsilon i(x) \otimes \bar{\pi}^* \otimes l$$

where π is a uniformizer for w and $\rho = \pi^e$ is a uniformizer for v .

1.4.1.6. SPECIALIZATION MAP. (see [Mor12, Lemma 3.16]) Keeping the previous notations, we also have a specialization map

$$s_v^\pi : \mathbf{K}_*^{\text{MW}}(E) \rightarrow \mathbf{K}_*^{\text{MW}}(\kappa(v))$$

commuting with the multiplication by η and satisfying

$$s_v^\pi([\pi^{m_1} a_1, \dots, \pi^{m_n} a_n]) = [\bar{a}_1, \dots, \bar{a}_n]$$

for any $a_1, \dots, a_n \in \mathcal{O}_v^\times$ and any integers m_1, \dots, m_n . If π' is another uniformizer, write $\pi' = \pi u$ where u is a unit. Then, for any $x \in \mathbf{K}_*^{\text{MW}}(E)$, we have $s_v^{\pi'}(x) = s_v^\pi(x) - [\bar{u}] \partial_v^{\pi'}(x)$.

It is related with the residue map as follows:

$$s_v^\pi(\alpha) = \langle -1 \rangle \partial_v^\pi([- \pi] \cdot \alpha)$$

for any $\alpha \in \mathbf{K}_*^{\text{MW}}(E)$. One can define a twisted specialization map (also denoted by s_v^π) as follows:

$$\begin{aligned} s_v^\pi : \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) &\rightarrow \underline{\mathbf{K}}^{MW}(\kappa(v), \mathcal{V}_{\kappa(v)}) \\ x \otimes l &\mapsto s_v^\pi(x) \otimes l. \end{aligned}$$

It satisfies:

$$s_v^\pi = \Theta_\pi \circ \partial_v \circ \gamma_{[-\pi]}$$

where $\gamma_{[-\pi]} : \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) \rightarrow \underline{\mathbf{K}}^{MW}(E, \mathbb{A}_E^1 + \mathcal{V}_E)$ is the multiplication by $([-\pi] \otimes 1)$ and where Θ_π is induced by the composite of two isomorphisms $-\mathcal{N}_v + \mathbb{A}_E^1 \simeq \mathbb{A}_E^1 + (-\mathcal{N}_v) \simeq 0$ of virtual vector bundles (the first one is the canonical switch isomorphism; the second one comes from the choice of uniformizer π).

1.4.2 Homotopy invariance and transfers

1.4.2.1. THE HOMOTOPY EXACT SEQUENCE. Let E be a field with a virtual vector bundle \mathcal{V}_E , and let $E(t)$ be the function field of E in the variable t . Any point x of codimension 1 in \mathbb{A}_E^1 corresponds to a maximal ideal \mathfrak{m}_x in $E[t]$, generated by a unique irreducible monic polynomial π_x which is a uniformizing parameter of the \mathfrak{m}_x -adic valuation v_x on $E(t)$. Additionally, the valuation v_∞ at ∞ is given by $v_\infty(f/g) = \deg(g) - \deg(f)$ and we can choose (as in [Mor12, §4.2]) the rational function $-1/t$ as uniformizing parameter. For any $a \in E(t)^\times$, there is only a finite number of x such that $v_x(a) \neq 0$. Furthermore, since the Milnor-Witt K-theory of E is generated by elements of degree 1 (and η in degree -1), it follows that the total residue map

$$d : \underline{\mathbf{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \rightarrow \bigoplus_{x \in (\mathbb{A}_E^1)(1)} \underline{\mathbf{K}}^{MW}(E(x), \Omega_{E(x)/k} + \mathcal{V}_{E(x)})$$

with $d = \sum_x \Theta_x \circ \partial_{v_x}$ is well-defined (where Θ_x comes from the canonical isomorphism $-\mathcal{N}_{v_x} + \Omega_{E(t)/k} \otimes_{E(t)} E(x) \simeq \Omega_{E(x)/k}$ of virtual vector bundles). The following theorem (see [Mor12, Theorem 3.24] for a non-twisted statement) is one of the most fundamental results of Milnor-Witt K-theory. Let

$$i_* : \underline{\mathbb{K}}^{MW}(E, \mathbb{A}_E^1 + \Omega_{E/k} + \mathcal{V}_E) \rightarrow \underline{\mathbb{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)})$$

be the morphism induced by the field extension $E \subset E(t)$ and the canonical isomorphism $\mathbb{A}_{E(t)}^1 + \Omega_{E(t)/k} \otimes_E E(t) \simeq \Omega_{E(t)}$ of virtual vector bundles.

Theorem 1.4.2.2 (Homotopy invariance). *With the previous notations, the following sequence is split exact*

$$0 \rightarrow \underline{\mathbb{K}}^{MW}(E, \mathbb{A}_E^1 + \Omega_{E/k} + \mathcal{V}_E) \xrightarrow{i_*} \underline{\mathbb{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \xrightarrow{d} \bigoplus_{x \in (\mathbb{A}_E^1)^{(1)}} \underline{\mathbb{K}}^{MW}(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_{\kappa(x)}) \rightarrow 0$$

Proof. See [Mor12, Theorem 3.24], [Fas19, Proposition 1.20]. \square

1.4.2.3. TRANSFER MAPS. Let $\varphi : E \rightarrow F$ be a monogenous finite field extension with \mathcal{V}_E a virtual vector bundle over E and choose $x \in F$ such that $F = E(x)$. The homotopy exact sequence implies that for any $\beta \in \underline{\mathbb{K}}^{MW}(F, \Omega_{F/k} + \mathcal{V}_F)$ there exists $\gamma \in \underline{\mathbb{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)})$ with the property that $d(\gamma) = \beta$. Now the valuation at ∞ yields a morphism

$$\partial_\infty^{-1/t} : \underline{\mathbb{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \rightarrow \underline{\mathbb{K}}^{MW}(E, \Omega_{E/k} + \mathcal{V}_E)$$

which vanishes on the image of i_* . We denote by $\varphi^*(\beta)$ or by $\text{Tr}_E^F(\beta)$ the element $-\partial_\infty^{-1/t}(\gamma)$; it does not depend on the choice of γ . This defines a group morphism

$$\varphi^* : \underline{\mathbb{K}}^{MW}(F, \Omega_{F/k} + \mathcal{V}_F) \rightarrow \underline{\mathbb{K}}^{MW}(E, \Omega_{E/k} + \mathcal{V}_E)$$

called the *transfer map* and also denoted by Tr_E^F . The following result completely characterizes the transfer maps.

Lemma 1.4.2.4. *Keeping the previous notations, let*

$$d : \underline{\mathbb{K}}^{MW}(E(t), \Omega_{F(t)/k} + \mathcal{V}_{E(t)}) \rightarrow \left(\bigoplus_x \underline{\mathbb{K}}^{MW}(E(x), \Omega_{E(x)/k} + \mathcal{V}_{E(x)}) \right) \oplus \underline{\mathbb{K}}^{MW}(E, \Omega_{E/k} + \mathcal{V}_E)$$

be the total twisted residue morphism (where x runs through the set of monic irreducible polynomials in $E(t)$). Then, the transfer maps $\text{Tr}_E^{E(x)}$ are the unique morphisms such that $\sum_x (\text{Tr}_E^{E(x)} \circ d_x) + d_\infty = 0$.

Proof. See [Mor12, §4.2]. \square

Let $\varphi : E \rightarrow F$ be a finite field extension. We can find a factorization

$$E = F_0 \subset F_1 \subset \cdots \subset F_n = F$$

such that F_i/F_{i-1} is monogenous for any $i = 1, \dots, n$. We set $\text{Tr}_E^F = \text{Tr}_E^{F_1} \circ \cdots \circ \text{Tr}_{F_{n-1}}^F$. Morel proved (see [Mor12, Theorem 4.27]) that the definition does not depend on the choice of the factorization (thanks to the twists).

The pullback Tr_E^F is functorial by definition. It satisfies the usual projection formula and the following base change theorem.

Proposition 1.4.2.5 (Base change). *Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ and $\psi : (E, \mathcal{V}_E) \rightarrow (L, \mathcal{V}_L)$ with φ finite and ψ separable. Let R be the ring $F \otimes_E L$. For each $p \in \text{Spec } R$, let $\varphi_p : (L, \mathcal{V}_L) \rightarrow (R/p, \mathcal{V}_{R/p})$ and $\psi_p : (F, \mathcal{V}_F) \rightarrow (R/p, \mathcal{V}_{R/p})$ be the morphisms induced by φ and ψ . One has :*

$$\psi_* \circ \varphi^* = \sum_{p \in \operatorname{Spec} R} (\varphi_p)^* \circ \Theta_p \circ (\psi_p)_*$$

where Θ_p is induced by the canonical isomorphism $\Omega_{F/E} \otimes_F (R/p) \simeq \Omega_{(R/p)/L}$ (since ψ is separable).

Proof. This follows from general base change theorem on the Milnor-Witt K-theory (see [Fas08, Corollaire 12.3.7]). \square

Finally, transfer maps are compatible with residue maps in the following sense.

Proposition 1.4.2.6. *Let $E \rightarrow F$ be a finite extension of fields over k , let v be a valuation on E and let \mathcal{V} be a virtual \mathcal{O}_v -module. For each extension w of v , we denote by $\varphi_w : (\kappa(v), \mathcal{V}_{\kappa(v)}) \rightarrow (\kappa(w), \mathcal{V}_{\kappa(w)})$ the morphism induced by $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$. We have:*

$$\partial_v \circ \varphi^* = \sum_w \varphi_w^* \circ \partial_w.$$

Proof. This follows from a more general result of Morel: transfer maps induce morphisms of the corresponding Rost-Schmid complexes [Mor12, §5].² \square

1.5 Homotopy sheaves

1.5.1 Unramified sheaves

In this subsection, we summarize [Mor12, Chapter 2] and recall the basic results concerning unramified sheaves.

Definition 1.5.1.1. 1. A sheaf of sets \mathcal{S} on \mathbf{Sm}_k is said to be \mathbb{A}^1 -invariant if for any $X \in \mathbf{Sm}_k$, the map

$$\mathcal{S}(X) \rightarrow \mathcal{S}(\mathbb{A}_X^1)$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection.

2. A sheaf of groups \mathcal{G} on \mathbf{Sm}_k is said to be *strongly* \mathbb{A}^1 -invariant if, for any $X \in \mathbf{Sm}_k$, the map

$$H_{Nis}^i(X, \mathcal{G}) \rightarrow H_{Nis}^i(\mathbb{A}^1 \times X, \mathcal{G})$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection for $i \in \{0, 1\}$.

3. A sheaf of abelian groups M on \mathbf{Sm}_k is said to be *strictly* \mathbb{A}^1 -invariant if, for any $X \in \mathbf{Sm}_k$, the map

$$H_{Nis}^i(X, M) \rightarrow H_{Nis}^i(\mathbb{A}^1 \times X, M)$$

²The reader looking for a more elementary proof of the proposition may follow the ideas of [Sus80].

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection for $i \in \mathbb{N}$.

Remark 1.5.1.2. In the sequel, we work with M a sheaf of groups. We could give more general definitions for sheaves of sets but, in practice, we need only the case of sheaves of abelian groups. In that case, we recall that a strongly \mathbb{A}^1 -invariant sheaf of abelian groups is necessarily strictly \mathbb{A}^1 -invariant (see [Mor12, Corollary 5.45]).

Definition 1.5.1.3. An unramified presheaf of groups M on \mathbf{Sm}_k is a presheaf of groups M such that the following holds:

- (0) For any smooth scheme $X \in \mathbf{Sm}_k$ with irreducible components X_α ($\alpha \in X^{(0)}$), the canonical map $M(X) \rightarrow \prod_{\alpha \in X^{(0)}} M(X_\alpha)$ is an isomorphism.
- (1) For any smooth scheme $X \in \mathbf{Sm}_k$ and any open subscheme $U \subset X$ everywhere dense in X , the restriction map $M(X) \rightarrow M(U)$ is injective.
- (2) For any smooth scheme $X \in \mathbf{Sm}_k$, irreducible with function field F , the injective map $M(X) \rightarrow \bigcap_{x \in X^{(1)}} M(\mathcal{O}_{X,x})$ is an isomorphism (the intersection being computed in $M(F)$).

Example 1.5.1.4. Homotopy modules with transfers [Dé11] and Rost cycle modules [Ros96] define unramified sheaves. In characteristic not 2, the sheaf associated to the presheaf of Witt groups $X \rightarrow W(X)$ is unramified.

We may give an explicit description of unramified sheaves on \mathbf{Sm}_k in terms of their sections on fields $F \in \mathcal{F}_k$ and some extra structure. We will say that a function $M : \mathcal{F}_k \rightarrow \mathbf{Grp}$ is *continuous* if $M(F)$ is the filtering colimit of the groups $M(F_\alpha)$ where F_α runs over the subfields of F of finite type over k .

Definition 1.5.1.5 ([Mor12], Definition 2.6). An unramified \mathcal{F}_k -datum consists of:

uD1 A continuous functor $M : \mathcal{F}_k \rightarrow \mathbf{Grp}$.

uD2 For any field $F \in \mathcal{F}_k$ and any discrete valuation v on F , a subgroup

$$M(\mathcal{O}_v) \subset M(F).$$

uD3 For any field $F \in \mathcal{F}_k$ and any valuation v on F , a map $s_v : M(\mathcal{O}_v) \rightarrow M(\kappa(v))$, called the specialization map associated to v .

The previous data should satisfy the following axioms:

uA1 If $\iota : E \subset F$ is a separable extension in \mathcal{F}_k and w is a valuation on F which restrict to a discrete valuation v on E with ramification index 1, then the arrow $M(\iota)$ maps $M(\mathcal{O}_v)$ into $M(\mathcal{O}_w)$. Moreover, if the induced extension $\bar{\iota} : \kappa(v) \rightarrow \kappa(w)$ is an isomorphism, then the following square

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_w) \\ \downarrow & & \downarrow \\ M(E) & \longrightarrow & M(F) \end{array}$$

is cartesian.

uA2 Let $X \in \mathbf{Sm}_k$ be an irreducible smooth scheme with function field F . If $x \in M(F)$, then x lies in all but a finite number of $M(\mathcal{O}_x)$ where x runs over the set $X^{(1)}$ of points of codimension 1.

uA3(i) If $\iota : E \subset F$ is an extension in \mathcal{F}_k and w is a discrete valuation on F which restricts to a discrete valuation v on E , then $M(\iota)$ maps $M(\mathcal{O}_v)$ into $M(\mathcal{O}_w)$ and the following diagram

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_w) \\ \downarrow & & \downarrow \\ M(\kappa(v)) & \longrightarrow & M(\kappa(w)) \end{array}$$

is commutative.

uA3(ii) If $\iota : E \subset F$ is an extension in \mathcal{F}_k and w a discrete valuation on F which restricts to zero on E , then the map $M(\iota) : M(E) \rightarrow M(F)$ has its image contained in $M(\mathcal{O}_w)$. Moreover, if $\bar{\iota} : E \subset \kappa(w)$ denotes the induced extension, the composition $M(E) \longrightarrow M(\mathcal{O}_v) \xrightarrow{s_v} M(\kappa(w))$ is equal to $M(\bar{\iota})$.

uA4(i) For any smooth scheme $X \in \mathbf{Sm}_k$ local of dimension 2 with closed point $z \in X^{(2)}$, and for any point $y_0 \in X^{(1)}$ such that the reduced closed scheme \bar{y}_0 is k -smooth, then $s_{y_0} : M(\mathcal{O}_{y_0}) \rightarrow M(\kappa(y_0))$ maps $\bigcap_{y \in X^{(1)}} M(\mathcal{O}_y)$ into $M(\mathcal{O}_{\bar{y}_0, z}) \subset M(\kappa(y_0))$.

uA4(ii) The composition

$$\bigcap_{y \in X^{(1)}} M(\mathcal{O}_y) \rightarrow M(\mathcal{O}_{\bar{y}_0, z}) \rightarrow M(\kappa(z))$$

does not depend on the choice of y_0 such that $\bar{y}_0 \in \mathbf{Sm}_k$.

Example 1.5.1.6. For any integer n , the functor $\mathbf{K}^{\text{MW}} : \mathcal{F}_k \rightarrow \mathbf{Grp}$ defined in Section 1.4 is an unramified \mathcal{F}_k -datum.

1.5.1.7. An unramified sheaf M defines in an obvious way an unramified \mathcal{F}_k -datum. Indeed, taking the evaluation³ on the field extensions of k yields a restriction functor:

$$M : \mathcal{F}_k \rightarrow \mathbf{Grp}, F \mapsto M(F)$$

such that, for any field F with valuation v , we have an $M(\mathcal{O}_v) \subset M(F)$ and a specialization map $s_v : M(\mathcal{O}_v) \rightarrow M(\kappa(v))$ (obtained by choosing smooth models over k for the closed immersion $\text{Spec } \kappa(v) \rightarrow \text{Spec } \mathcal{O}_v$). We claim that this satisfies axioms **uA1**, ..., **uA4(ii)**.

Reciprocally, given an unramified \mathcal{F}_k -datum M and $X \in \mathbf{Sm}_k$ an irreducible smooth scheme with function field F , we define the subset $M(X) \subset M(F)$ as the intersection $\bigcap_{x \in X^{(1)}} M(\mathcal{O}_x) \subset M(F)$. We extend the definition for any X so that property **(0)** is satisfied. Using the fact that any map $f : Y \rightarrow X$ between smooth schemes is the composition $Y \hookrightarrow Y \times_k X \twoheadrightarrow X$ of closed immersion followed by a smooth projection, one can define an unramified sheaf $M : \mathbf{Sm}_k \rightarrow \mathbf{Grp}$. In short, we have the following theorem.

³See Subsection 1.9.2.

Theorem 1.5.1.8 ([Mor12], Theorem 2.11). *The two functors described above define an equivalence between the category of unramified sheaves on \mathbf{Sm}_k and that of unramified \mathcal{F}_k -data.*

Example 1.5.1.9. As a corollary of the previous theorem, we obtain a definition of the (unramified) sheaf of Milnor-Witt K-theory \mathbf{K}_n^{MW} for any integer n .

1.5.1.10. From now on, we will not distinguish between the notion of unramified sheaves on \mathbf{Sm}_k and that of unramified \mathcal{F}_k -datum. In the remaining subsection, we fix M an unramified sheaf of groups on \mathbf{Sm}_k and explain how it is related to strongly \mathbb{A}^1 -invariant sheaves.

1.5.1.11. NOTATION. If $\varphi : E \rightarrow F$ is an extension of fields, the map

$$M(\varphi) : M(E) \rightarrow M(F)$$

is also denoted by res_φ , $\text{res}_{F/E}$ or φ_* .

1.5.1.12. Let $F \in \mathcal{F}_k$ be a field and v a valuation on F . We define the pointed set

$$H_v^1(\mathcal{O}_v, M) = M(F)/M(\mathcal{O}_v).$$

This is a left $M(F)$ -set. Moreover, for any point y of codimension 1 in $X \in \mathbf{Sm}_k$, we set $H_y^1(X, M) = H_y^1(\mathcal{O}_{X,y}, M)$. By axiom [uA2](#), if X is irreducible with function field F , the induced left action of $M(F)$ on $\prod_{y \in X^{(1)}} H_y^1(X, M)$ preserves the weak-product

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \subset \prod_{y \in X^{(1)}} H_y^1(X, M)$$

where the weak-product $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ means the set of families for which all but a finite number of terms are the base point of $H_y^1(X, M)$. By definition and axiom [uA2](#), the isotropy subgroup of this action of $M(F)$ on the base point of $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ is exactly $M(X) = \cap_{y \in X^{(1)}} M(\mathcal{O}_{X,y})$. We summarize this property by saying that the diagram

$$1 \rightarrow M(X) \rightarrow M(F) \rightrightarrows \prod'_{y \in X^{(1)}} H_y^1(X, M)$$

is exact.

Definition 1.5.1.13. For any point z of codimension 2 in a smooth scheme X , we denote by $H_z^2(X, M)$ the orbit set of $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ under the left action of $M(F)$ where $F \in \mathcal{F}_k$ is the function field of $X_{(z)}$.

1.5.1.14. For an irreducible essentially smooth scheme X with function field F , we define the boundary $M(F)$ -equivariant map

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod_{z \in X^{(2)}} H_z^2(X, M)$$

by collecting together the compositions

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod'_{y \in X_{(z)}^{(1)}} H_y^1(X, M) \rightarrow H_z^2(X, M)$$

for each $z \in X^{(2)}$.

It is not clear in general whether or not the image of the boundary map is always contained in the weak product $\prod'_{z \in X^{(2)}} H_z^2(X, M)$. For this reason we introduce the following axiom:

uA2' For any irreducible essentially smooth scheme X , the image of the boundary map

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod_{z \in X^{(2)}} H_z^2(X, M)$$

is contained in the weak product $\prod'_{z \in X^{(2)}} H_z^2(X, M)$.

1.5.1.15. From now on we assume that M satisfies **uA2'**. For any smooth scheme X irreducible with function field F , we have a complex $C^*(X, M)$

$$1 \rightarrow M(X) \rightarrow M^{(0)}(X) \Rightarrow M^{(1)}(X) \rightarrow M^{(2)}(X)$$

where $M^{(0)}(X) = \prod'_{x \in X^{(0)}} M(\kappa(x)) = \prod_{x \in X^{(0)}} M(\kappa(x))$, $M^{(1)}(X) = \prod'_{y \in X^{(1)}} H_y^1(X, M)$ and $M^{(2)}(X) = \prod'_{z \in X^{(2)}} H_z^2(X, M)$. By construction, this complex is exact (in an obvious sense, see [Mor12, Definition 2.20]) for any (essentially) smooth local scheme of dimension ≤ 2 .

Definition 1.5.1.16. A strongly unramified \mathcal{F}_k -data is an unramified \mathcal{F}_k -data M satisfying **uA2'** and the following axioms:

uA5(i) For any separable finite extension $\iota : E \subset F$ in \mathcal{F}_k , any discrete valuation w on F which restricts to a discrete valuation v on E with ramification index 1, and such that the induced extension $\bar{\iota} : \kappa(v) \rightarrow \kappa(w)$ is an isomorphism, the commutative square of groups

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(E) \\ \downarrow & & \downarrow \\ M(\mathcal{O}_w) & \longrightarrow & M(F) \end{array}$$

induces a bijection $H_w^1(\mathcal{O}_w, M) \simeq H_v^1(\mathcal{O}_v, M)$.

uA5(ii) For any étale morphism $X' \rightarrow X$ between smooth local k -schemes of dimension 2, with closed point respectively z' and z , inducing an isomorphism on the residue fields $\kappa(z) \simeq \kappa(z')$, the pointed map

$$H_z^2(X, M) \rightarrow H_{z'}^2(X', M)$$

has trivial kernel.

uA6 For any localization U of a smooth k -scheme at some point u of codimension ≤ 1 , the complex:

$$1 \rightarrow M(\mathbb{A}_U^1) \rightarrow M^{(0)}(\mathbb{A}_U^1) \Rightarrow M^{(1)}(\mathbb{A}_U^1) \rightarrow M^{(2)}(\mathbb{A}_U^1)$$

is exact. Moreover, the morphism $M(U) \rightarrow M(\mathbb{A}_U^1)$ is an isomorphism.

Theorem 1.5.1.17 ([Mor12], Theorem 2.27). *There is an equivalence between the category of strongly \mathbb{A}^1 -invariant sheaves of groups on \mathbf{Sm}_k and that of strongly unramified \mathcal{F}_k -data of groups on \mathbf{Sm}_k .*

Definition 1.5.1.18. A strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups is called a *homotopy sheaf*. We denote by $\mathbf{HI}(k)$ the category of homotopy sheaves and natural transformations of sheaves.

Example 1.5.1.19. As a corollary of the previous theorem, we obtain that the sheaf of Milnor-Witt K-theory \mathbf{K}_n^{MW} is a homotopy sheaf for any integer n .

1.5.1.20. MONOIDAL STRUCTURE. Recall that there is a canonical adjunction of categories

$$\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k) \begin{array}{c} \xrightarrow{\mathcal{O}} \\ \xleftarrow{\pi_{\mathbb{A}^1}} \end{array} D(\text{Sh}(\mathbf{Sm}_k))$$

where $\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k)$ the effective \mathbb{A}^1 -derived category and $D(\text{Sh}(\mathbf{Sm}_k))$ is the derived category of complexes of sheaves over \mathbf{Sm}_k (see Subsection 1.3.2 or [CD19, §5]). Thanks to Morel's \mathbb{A}^1 -localization theorem, we can prove that there is a unique t-structure on $\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k)$ such that the forgetful functor \mathcal{O} is t-exact and that the category of homotopy sheaves $\mathbf{HI}(k)$ is equivalent to the heart $(\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k))^{\heartsuit}$ for this t-structure (in particular, $\mathbf{HI}(k)$ is a Grothendieck category). Since the canonical tensor product $\otimes_{\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}}$ is right t-exact, it induces a monoidal structure on $\mathbf{HI}(k)$. Precisely, if $F, G \in \mathbf{HI}(k)$ are two homotopy sheaves, then their tensor product is

$$F \otimes_{\mathbf{HI}} G = H_0^{\mathbb{A}^1}(F \otimes_{\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}} G)$$

where $H_0^{\mathbb{A}^1}$ is the homology object in degree 0 for the homotopy t-structure.

Example 1.5.1.21. For any integers n and m , we have a canonical isomorphism

$$\mathbf{K}_n^{\text{MW}} \otimes_{\mathbf{HI}} \mathbf{K}_m^{\text{MW}} \simeq \mathbf{K}_{n+m}^{\text{MW}}.$$

In particular $\text{GW} = \mathbf{K}_0^{\text{MW}}$ is a commutative monoid.

1.5.2 Contracted homotopy sheaves

1.5.2.1. In this section, we fix $M \in \mathbf{HI}(k)$ a homotopy sheaf. Recall that the contraction M_{-1} is by definition the sheaf of abelian groups

$$X \mapsto \ker(M(\mathbb{G}_m \times X) \rightarrow M(X)).$$

Equivalently, we have

$$M_{-1} = \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M)$$

where $\underline{\mathcal{H}om}$ is the internal hom-object of $\mathbf{HI}(k)$. According to [Mor12, Lemma 2.32], the sheaf M_{-1} is also a homotopy sheaf and is called a *contracted homotopy sheaf*.

Example 1.5.2.2. For any integer n , we have a canonical isomorphism $(\mathbf{K}_n^{\text{MW}})_{-1} = \mathbf{K}_{n-1}^{\text{MW}}$.

1.5.2.3. Any unit map $a : \mathbb{G}_m \rightarrow \mathbb{G}_m$ induces a morphism

$$\langle a \rangle : \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M) \rightarrow \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M)$$

and thus M_{-1} has a module structure over the commutative monoid GW (see also [Mor12, Lemma 3.49]). Moreover, we have a bilinear pairing

$$\begin{aligned} \mathbf{K}_1^{\text{MW}} \times M_{-1} &\rightarrow M \\ ([u], \mu) &\mapsto [u] \cdot \mu, \end{aligned}$$

where \mathbf{K}_1^{MW} is defined in 1.5.1.19.

1.5.2.4. Let $N \in \mathbf{HI}(k)$ be another homotopy sheaf and assume N is equipped with a GW-module structure. Let $X \in \mathbf{Sm}_k$ be a smooth scheme and \mathcal{L} a vector line bundle on X . As usual, we define the twist of N by \mathcal{L} :

$$N\{\mathcal{L}\} = N \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[\mathcal{L}^\times]$$

as the sheaf associated to the presheaf on the Zariski site X_{Zar} :

$$U \mapsto N(U) \otimes_{\mathbb{Z}[\mathcal{O}_X(U)^\times]} \mathbb{Z}[\mathcal{L}_U^\times]$$

where \mathcal{L}_U^\times is the set of isomorphisms between \mathcal{O}_U and \mathcal{L}_U (which may be empty). We put $N(X, \mathcal{L}) = \Gamma(X, N\{\mathcal{L}\})$ and remark that, if X is affine, then $N(X, \mathcal{L}) = N(X) \otimes_{\mathbb{Z}[\mathcal{O}_X(X)^\times]} \mathbb{Z}[\mathcal{L}_X^\times]$. In particular, this definition applies to the sheaf $N = M_{-1}$.

1.5.2.5. COHOMOLOGY WITH SUPPORT EXACT SEQUENCE. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. For any closed immersion $i : Z \rightarrow X$ of smooth schemes over k , with complementary open immersion $j : U \rightarrow X$, there exists a canonical cohomology with support exact sequence of the form:

$$\Gamma_Z(X, M) \xrightarrow{i_*} M(X) \xrightarrow{j^*} M(U) \xrightarrow{\partial} H_Z^1(X, M) \longrightarrow \dots$$

1.5.2.6. The purity isomorphism (more precisely: Axiom uA5(i) and [Mor12, Lemma 3.50]) implies that for any discrete valuation v on a field $F \in \mathcal{F}_k$, one has a canonical bijection

$$H_v^1(\mathcal{O}_v, M) \simeq M_{-1}(\kappa(v), \omega_v)$$

and thus obtain a residue map

$$\partial_v : M(F) \rightarrow M_{-1}(\kappa(v), \omega_v).$$

1.6 Geometric and cohomological transfers

1.6.1 Geometric and cohomological transfers on M_{-n}

Let M be a strongly \mathbb{A}^1 -invariant sheaf of groups. Following [Mor12, §4.2], we prove that M_{-1} admits canonical geometric transfer morphisms for finite monogenous fields extensions in \mathcal{F}_k and that, for sheaves of the form M_{-n} (with $n \geq 2$), these transfers can be extended in a functorial way to canonical transfer maps for any finite extension in \mathcal{F}_k . Transfers for monogenous extensions are called *geometric transfers* and the construction strictly follows the ideas of Bass and Tate (that is why such transfers are also called *Bass-Tate transfers* in [Fel20a] and chapter 4).

1.6.1.1. Let X be an essentially smooth k -scheme of dimension 2. Let $i : Y \subset X$ be an irreducible closed subscheme of codimension 1 which is essentially k -smooth and let y be its generic point. Let τ be a trivialization of the normal bundle \mathcal{N}_i . The isomorphism from 1.5.2.5

$$\Theta_\tau : M_{-1}(\kappa(y)) \simeq H_y^1(X, M)$$

induces for each closed point $z \in Y$ an isomorphism

$$\varphi_\tau : H_z^1(Y, M_{-1}) \simeq H_z^2(X, M),$$

according to [Mor12, §4.1]. Let $\mu \in (\mathcal{N}_z/\mathcal{N}_i)^\times$ where \mathcal{N}_z is the maximal ideal corresponding to z in the discrete valuation ring $\mathcal{O}_{Y,z}$. We get a further isomorphism

$$\Theta_\mu : M_{-2}(\kappa(z)) \simeq H_z^1(Y, M_{-1}),$$

according to [Mor12, Corollary 2.35]. Thus, for any pair (τ, μ) as above, we can define a canonical isomorphism

$$\Phi_{\tau, \mu} = \varphi_\tau \circ \Theta_\mu : M_{-2}(\kappa(z)) \simeq H_z^2(X, M).$$

1.6.1.2. We know that M_{-1} is also strongly \mathbb{A}^1 -invariant and that, for any field $K \in \mathcal{F}_k$, we have a short exact sequence

$$0 \rightarrow M(K) \subset M(K(X)) \rightarrow \bigoplus_{z \in (\mathbb{A}_K^1)^{(1)}} H_z^1(\mathbb{A}_K^1, M) \rightarrow 0.$$

For any closed point $z \in (\mathbb{A}_K^1)^{(1)}$, the minimal polynomial P_z defines a uniformizing element and thus a canonical isomorphism (as in 1.6.1.1)

$$\Theta_{P_z} : M_{-1}(\kappa(z)) \simeq H_z^1(\mathbb{A}_K^1, M)$$

depending only on the class of P_z in $\mathfrak{m}_z/\mathfrak{m}_z^2$ where \mathfrak{m} is the maximal ideal in the local ring of \mathbb{A}_K^1 at z . Hence the previous exact sequence is

$$0 \rightarrow M(K) \subset M(K(X)) \rightarrow \bigoplus_{z \in (\mathbb{A}_K^1)^{(1)}} M_{-1}(\kappa(z)) \rightarrow 0.$$

Likewise for the valuation at infinity of $K(X)$, we have a map

$$\partial_\infty^{-1/X} : M(K(X)) \rightarrow M_{-1}(K)$$

which is zero on $M(K) \subset M(K(X))$. Using the previous exact sequence and the map $-\partial_\infty^{-1/X}$, we obtain a morphism

$$\bigoplus_{z \in (\mathbb{A}_K^1)^{(1)}} M_{-1}(\kappa(z)) \rightarrow M_{-1}(K).$$

In other words, given a monogenous extension $K \subset L$ and a generator, we have a map

$$\tau_{z/K} : M_{-1}(L) \rightarrow M_{-1}(K)$$

which is called the *geometric transfer*.

Remark 1.6.1.3. The minus sign in front of $\partial_\infty^{-1/X}$ is there to guaranty the reciprocity formula

$$\sum_{z \in \mathbb{P}_K^1} \tau_{z,K} \circ \partial_z^{P_z} = 0 : M(K(X)) \rightarrow M_{-1}(K)$$

where $\tau_{z,K}$ for $z = \infty$ is the isomorphism $\Theta_{P_\infty} : M_{-1}(K) \simeq H_\infty^1(\mathbb{P}_K^1, M) \simeq H^1(\mathbb{P}_K^1, M)$ which we also denote by γ . Note that γ is equal to $\Theta_{X-z} : M_{-1}(K) \simeq H^1(\mathbb{P}_K^1, M)$ according to [Mor12, Lemma 4.25].

Remark 1.6.1.4. Another way to define the transfer map is to write $\tau_{z,K}$ as the composite

$$M_{-1}(L) \xrightarrow[\simeq]{\Theta_{P_z}} H_z^1(\mathbb{P}_K^1, M) \longrightarrow H^1(\mathbb{P}_K^1, M) \xrightarrow[\simeq]{\gamma^{-1}} M_{-1}(K)$$

1.6.1.5. THE COHOMOLOGICAL TRANSFER. Given a field K of characteristic exponent p and a monic irreducible polynomial $P \in K[X]$, we may write canonically P as $P_0(X^{p^i})$ where P_0 is a monic irreducible separable polynomial and $i \in \mathbb{N}$ a natural number (which is equal to 0 in characteristic 0). This corresponds to the canonical factorization of $K \subset L = K[X]/P$ as $K \subset L_{sep} \subset L$ where $L_{sep} = K[x^{p^i}]/K$ is the separable closure of K in L , and L/L_{sep} is purely inseparable. The natural number i is such that the inseparable degree of x over K is $[L : L_{sep}] = p^i$. The polynomial P_0 is then the minimal polynomial of x^{p^i} over K . Notice that its derivative P'_0 is nonzero and such that $P'_0(x^{p^i}) \in L_{sep}^\times$.

Definition 1.6.1.6. Keeping the previous notations, we set

$$\omega_0(x) := P'_0(x^{p^i}) \in L^\times$$

and the composed morphism

$$\mathrm{Tr}_{x,K} : M_{-1}(L) \xrightarrow[\simeq]{\langle \omega_0(x) \rangle} M_{-1}(L) \xrightarrow{\tau_{x,K}} M_{-1}(K)$$

is called the *cohomological transfer* for the monogenous extension L/K with generator x , where the first arrow is defined thanks to 1.5.2.3.

The following theorem is the main result concerning the functoriality of transfer maps. Its proof is postponed to the next subsection.

Theorem 1.6.1.7 (Morel). *Let $K \subset L$ be a finite extension of fields in \mathcal{F}_k . Assume one of the following assumptions holds:*

- (a) *The characteristic $\mathrm{char}(k)$ of k is not 2.*
- (b) *The extension $K \subset L$ is separable.*
- (c) *The structure of GW-module on M_{-2} is trivial.*

Choose an increasing sequence

$$K = L_0 \subset L_1 \subset \cdots \subset L_r = L$$

for which any of the intermediate extensions $L_{i-1} \subset L_i$ is monogenous and choose, for each i , a generator x_i of L_i over L_{i-1} . Then the composition of each cohomological transfers previously constructed

$$M_{-2}(L) \xrightarrow{\mathrm{Tr}_{x_r/K}} M_{-2}(L_{r-1}) \longrightarrow \cdots \longrightarrow M_{-2}(L_1) \xrightarrow{\mathrm{Tr}_{x_1/K}} M_{-2}(K)$$

only depends on the extension $K \subset L$.

Assuming the previous theorem, we can give the following definition.

Definition 1.6.1.8. Let $K \subset L$ be a finite extension between fields in \mathcal{F}_k . Assume one of the assumptions (a), (b) or (c) holds, then the morphism

$$M_{-2}(L) \rightarrow M_{-2}(K)$$

obtained from Theorem 1.6.1.7 and any choice of increasing sequence as in the statement is called the *cohomological transfer morphism* for the extension $K \subset L$ and is denoted by

$$\mathrm{Tr}_{L/K} : M_{-2}(L) \rightarrow M_{-2}(K).$$

In [Mor12, §5.1], Morel explains how the previous transfers can be defined in all characteristic (i.e. avoiding assumption (a)). The idea is to twist $M_{-2}(K)$ by the line bundle $\omega_{K/k} = \Lambda^{\max}(\Omega_{K/k})$ in order to make the construction functorial.

Theorem 1.6.1.9 ([Mor12], Definition 5.4, Lemma 5.5). *Let $K \subset L$ be a finite extension of fields in \mathcal{F}_k and $n \geq 2$ a natural number. There is a canonical morphism (constructed in any characteristic) of the form*

$$\mathrm{Tr}_{L/K} : M_{-n}(L, \omega_{L/k}) \rightarrow M_{-n}(K, \omega_{K/k})$$

which is called *absolute transfer morphism* and is *functorial* in the sense that for any composable finite extensions $K \subset E \subset L$, the composition

$$M_{-n}(L, \omega_{L/k}) \xrightarrow{\mathrm{Tr}_{L/E}} M_{-n}(E, \omega_{E/k}) \xrightarrow{\mathrm{Tr}_{E/K}} M_{-n}(K, \omega_{K/k})$$

equals $\mathrm{Tr}_{L/K}$.

Remark 1.6.1.10. 1. Theorem 1.6.1.9 is conjectured to be also true if we replace M_{-2} by M_{-1} (see [Mor12, Remark 4.31], or Conjecture 4.4.1.13).

2. When M is the sheaf \mathbf{K}_n^M of unramified Milnor K-theory in degree n , which is a strongly \mathbb{A}^1 -invariant sheaf with the property that $(\mathbf{K}_{n+1}^M)_{-1} \simeq \mathbf{K}_n^M$. It is clear that the $\mathbf{K}_0^{\mathrm{MW}} = \mathrm{GW}$ -module structure is the trivial one (i.e. η acts by 0) and that the previous transfers

$$\tau_{x/K} : \mathbf{K}_n^M(L) \rightarrow \mathbf{K}_n^M(K)$$

obtained by the above method for a finite extension $K \subset L$ with generator x is the same as the one constructed by Bass and Tate [BT73]. Theorem 1.6.1.7 thus also reproves in a new way the theorem of Kato that these transfers are independent of any choices of an increasing sequence as in Theorem 1.6.1.7.

1.6.2 Functoriality of transfers

In this subsection, we sketch the proof of Theorem 1.6.1.7 (see [Mor12, §4.3] for more details). We keep the previous notations (in particular, we denote by p the exponential characteristic of k).

Lemma 1.6.2.1. *Let $K \subset L$ be a finite extension of degree n and let $x \in L$ be a generator of L over K . For any unit $u \in K^\times$, the following diagram*

$$\begin{array}{ccc} M_{-1}(L) & \xrightarrow{\tau_{x/K}} & M_{-1}(K) \\ \simeq \downarrow \langle u^n \rangle & & \simeq \downarrow \langle u \rangle \\ M_{-1}(L) & \xrightarrow{\tau_{(u.x)/K}} & M_{-1}(K) \end{array}$$

is commutative.

For odd exponential characteristic p or for $p = 2$ and $K \subset L$ separable, the following diagram

$$\begin{array}{ccc} M_{-1}(L) & \xrightarrow{\text{Tr}_{x/K}} & M_{-1}(K) \\ \simeq \downarrow \langle u \rangle & & \simeq \downarrow \langle u \rangle \\ M_{-1}(L) & \xrightarrow{\text{Tr}_{(u.x)/K}} & M_{-1}(K) \end{array}$$

is commutative. For $p = 2$ and a purely inseparable extension, the following diagram

$$\begin{array}{ccc} M_{-1}(L) & \xrightarrow{\text{Tr}_{x/K}} & M_{-1}(K) \\ \parallel & & \simeq \downarrow \langle u \rangle \\ M_{-1}(L) & \xrightarrow{\text{Tr}_{(u.x)/K}} & M_{-1}(K) \end{array}$$

is commutative.

Remark 1.6.2.2. In the case where $[L : K] = 1$, we have $\text{Tr}_{x/K} = \tau_{x/K} : M_{-2}(L) \rightarrow M_{-2}(K)$ is nothing but the inverse to the canonical isomorphism $M_{-2}(K) \simeq M_{-2}(L)$. In particular, it does not depend on x .

1.6.2.3. We now prove Theorem 1.6.1.7 by induction on the length r of an increasing sequence of subfields. It suffices to prove the case $r = 2$ since it implies the case $r \geq 3$ (by induction) and the case $r = 1$ (apply the case $r = 2$ to the extension $K \subset L$ with generators x and y of L over K).

For the remaining of the proof of Theorem 1.6.1.7, we fix $K \subset L$ a finite extension of fields in \mathcal{F}_k and (x, y) generators of L/K . Set $E = K[x] \subset L$ and $F = K[y] \subset L$. We thus have a commutative square of monogenous extensions

$$\begin{array}{ccc} K & \longrightarrow & F \\ \downarrow & & \downarrow \\ E & \longrightarrow & L \end{array}$$

and we want to prove that the following diagram

$$\begin{array}{ccc} M_{-2}(L) & \xrightarrow{\text{Tr}_{x/F}} & M_{-2}(F) \\ \text{Tr}_{y/E} \downarrow & & \downarrow \text{Tr}_{y/K} \\ M_{-2}(E) & \xrightarrow{\text{Tr}_{x/K}} & M_{-2}(K) \end{array}$$

commutes. Denote by $(\infty, \infty) \in (\mathbb{P}^1)_K^2$ the K -rational point given by the points at ∞ . Associated to this point is the pair of functions $(-1/X, -1/Y)$ which is a system of local parameters. We get a canonical isomorphism

$$\gamma_{X,Y} : M_{-2}(K) \xrightarrow[\simeq]{\Phi_{-1/X, -1/Y}} H_{\infty, \infty}^2((\mathbb{P}^1)_K^2, M) \xrightarrow[\simeq]{} H^2((\mathbb{P}^1)_K^2, M)$$

where the right isomorphism is given by [Mor12, Corollary 4.15]. Note that if we interchange $-1/X$ and $-1/Y$, then $\gamma_{Y,X} = \gamma_{X,Y} \circ \langle -1 \rangle$ according to Theorem [Mor12, 4.19].

Denote by P (resp. Q) the minimal polynomial of x (resp. y) over K . We may write $P(X) = P_0(X^{p^i})$ where $P_0 \in K[X]$ is irreducible and separable, and analogously $Q(Y) = Q_0(Y^{p^j})$ for y . Moreover, we denote by $z = (x, y)$ the closed point of $\mathbb{A}_K^2 \subset (\mathbb{P}^1)_K^2$ corresponding to L with the generators (x, y) in this order; we denote by \mathfrak{m}_z the maximal ideal at z . The following lemma is the most important step in the proof of Theorem 1.6.1.7.

Lemma 1.6.2.4 ([Mor12], Lemma 4.35). *Assume that $(P, Q) \in \mathfrak{m}_z$ is a regular system of parameters.*

1. *We set $\omega(P) = \omega_0(x)P(X) = P'_0(X^{p^i})P(X)$ and $\omega(Q) = \omega_0(x)Q(X) = Q'_0(X^{p^j})Q(X)$. Then $(\omega(P), \omega(Q))$ is also a regular system of parameters in \mathfrak{m}_z . Let P_F (resp. Q_E) be the minimal polynomial of x (resp. y) over F (resp. E). Then, one has:*

$$\Phi_{\omega(P), \omega(Q_E)} = \Phi_{\omega(P), \omega(Q)} = \Phi_{\omega(P_F), \omega(Q)},$$

where we use the notation of 1.6.1.1.

2. *The diagram*

$$\begin{array}{ccc} M_{-2}(L) & \xrightarrow{\text{Tr}_{x/F}} & M_{-2}(F) \\ \text{Tr}_{y/E} \downarrow & & \downarrow \text{Tr}_{y/K} \\ M_{-2}(E) & \xrightarrow{\text{Tr}_{x/K}} & M_{-2}(K) \end{array}$$

is commutative.

Remark 1.6.2.5. For instance $(P, Q) \in \mathfrak{m}_z$ is a regular system of parameters if one of x or y is separable over K .

Proof. 1. It suffices to prove the first equality. In $E[Y]$, we have the factorization $Q(Y) = Q_E(Y)R(Y)$. Since (P, Q) is a regular system of generators, the polynomial $R(Y)$ is prime to $Q_E(Y)$, that is to say Q_E has multiplicity 1 in Q . Recall that $Q(Y) = Q_0(Y^{p^j})$ in $K[Y]$ with Q_0 irreducible and separable. Write $Q_{E,0}(Y)$ for the minimal polynomial of y^{p^j} over E so that we have in $E[Y]$ the decomposition $Q_0(Y) = Q_{E,0}(Y)R_0(Y)$ with $R_0(Y)$ prime to $Q_{E,0}$. We can see that $Q(Y) = Q_{E,0}(Y^{p^j})R_0(Y^{p^j}) = Q_E(Y)R_0(Y^{p^j})$ and $Q'_0(Y^{p^j}) = Q'_{E,0}(Y^{p^j})R_0(Y^{p^j}) + Q_E(Y)R'_0(Y^{p^j})$ hence

$$\Phi_{\omega(P),\omega(Q)} = \Phi_{\omega(P),u^2\omega(Q_E)},$$

where $u = \overline{R_0(Y^{p^j})} \in E^\times$. According to [Mor12, Corollary 4.23], one gets

$$\Phi_{\omega(P),u^2\omega(Q_E)} = \Phi_{\omega(P),\omega(Q_E)}.$$

2. Denote by $\text{Tr}_{(x,y)/K}$ the composite

$$M_{-2}(L) \xrightarrow[\simeq]{\Phi_{\omega(P),\omega(Q)}} H_z^2((\mathbb{P}^1)_K^2, M) \longrightarrow H^2((\mathbb{P}^1)_K^2, M) \xrightarrow[\simeq]{\gamma_{X,Y}^{-1}} M_{-2}(K).$$

We claim that this map is equal to the composite

$$M_{-2}(L) \xrightarrow{\text{Tr}_{y/E}} M_{-2}(E) \xrightarrow{\text{Tr}_{-x/K}} M_{-2}(K).$$

This fact boils down to proving that the diagram

$$\begin{array}{ccccc}
 M_{-2}(L) & \xrightarrow[\simeq]{\Theta_{\omega_{Q_E}}} & H_{(x,y)}^1(x \times \mathbb{P}^1, M_{-1}) & \xrightarrow{\Phi_{\omega_P}} & H_{(x,y)}^2((\mathbb{P}^1)_K^2, M) \\
 \downarrow \text{Tr}_{y,E} & & \downarrow & & \downarrow \\
 M_{-2}(E) & \xrightarrow[\simeq]{\gamma} & H^1(x \times \mathbb{P}^1, M_{-1}) & & H_{x \times \mathbb{P}^1}^2((\mathbb{P}^1)_K^2, M) \\
 \swarrow & \searrow \gamma & \swarrow & \searrow & \downarrow \\
 M_{-2}(E) & \xrightarrow[\simeq]{\gamma} & H_{(x,\infty)}^1(x \times \mathbb{P}^1, M_{-1}) & \xrightarrow[\simeq]{\Phi_{\omega_P}} & H_{(x,\infty)}^2((\mathbb{P}^1)_K^2, M) \\
 \downarrow \simeq \langle -1 \rangle & & \downarrow \simeq & \searrow \Phi_{\frac{-1}{Y}} & \downarrow \\
 M_{-2}(E) & \xrightarrow[\simeq]{\Theta_{\omega_P}} & H_{(x,\infty)}^1(\mathbb{P}^1 \times \infty, M) & & H^2((\mathbb{P}^1)_K^2, M) \\
 \downarrow \text{Tr}_{x/K} & & \downarrow & & \swarrow \\
 M_{-2}(K) & \xrightarrow[\simeq]{\gamma} & H^1(\mathbb{P}^1 \times \infty, M) & \xrightarrow[\simeq]{\Phi_{\frac{-1}{Y}}} & H_{\mathbb{P}^1 \times \infty}^2((\mathbb{P}^1)_K^2, M)
 \end{array}$$

is commutative, where γ is defined in 1.6.1.3.

Now, since $\gamma_{X,Y} = \gamma_{Y,X} \circ \langle -1 \rangle$ and $\Phi_{\omega(Q),\omega(P)} = \Phi_{\omega(P),\omega(Q)} \circ \langle -1 \rangle$ (see [Mor12, Corollary 4.23]), we have

$$\text{Tr}_{-x/K} \circ \text{Tr}_{y/E} = \langle -1 \rangle \circ \text{Tr}_{-y/K} \circ \text{Tr}_{x/F} \circ \langle -1 \rangle = \text{Tr}_{y/K} \circ \text{Tr}_{-x/F}$$

which conclude the proof. \square

The previous lemma and an induction imply the following result.

Corollary 1.6.2.6 ([Mor12], Corollary 4.37). *Assume $K \subset L$ is separable. Then Theorem 1.6.1.7 holds in characteristic 0 or in case (b).*

Corollary 1.6.2.7. *Assume that E and F are linearly disjoint, that is to say $E \otimes_K F \rightarrow L$ is an isomorphism (in particular, this is true if x is separable over K and y is purely inseparable). Then Theorem 1.6.1.7 holds.*

We consider again the assumption of 1.6.2.3 and prove Theorem 1.6.1.7 in characteristic p . For this, we need the following lemmas whose proofs are similar (albeit more technical) to Lemma 1.6.2.4.

Lemma 1.6.2.8. *Assume that $y = x^{p^m}$ for some integer m . Then the following diagram*

$$\begin{array}{ccc} M_{-2}(L) & \xlongequal{\quad} & M_{-2}(L) \\ \tau_{-x/F} \downarrow & & \downarrow \tau_{-x/K} \\ M_{-2}(E) & \xrightarrow{\tau_{y/K}} & M_{-2}(K) \end{array}$$

is commutative. In particular, for p odd, the diagram

$$\begin{array}{ccc} M_{-2}(L) & \xlongequal{\quad} & M_{-2}(L) \\ \text{Tr}_{-\omega_0(x), x/F} \downarrow & & \downarrow \text{Tr}_{-x/K} \\ M_{-2}(E) & \xrightarrow{\text{Tr}_{y/K}} & M_{-2}(K) \end{array}$$

is commutative.

Lemma 1.6.2.9. *If $K \subset L = E = F$ is purely inseparable, and ($\text{char}(k) \neq 2$ or the GW-module structure on M_{-2} is trivial), then*

$$\tau_{x/K} = \tau_{y/K} : M_{-2}(L) \rightarrow M_{-2}(K).$$

Lemma 1.6.2.10. *Assume $\text{char}(k) = 2$ and let $K \subset L$ be purely inseparable and monogenous generated by x . Let $R \in K[X]$ be such that $R(x)$ is a generator of L/K (that is to say $R' \neq 0 \in K[X]$). Then one has*

$$\tau_{y/K} = \tau_{x/K} \circ \langle R'(x) \rangle : M_{-2}(L) \rightarrow M_{-2}(K).$$

Lemma 1.6.2.11 ([Mor12], Cor. 4.43). *Assume that L is purely inseparable over K (and so are E and F). Then Theorem 1.6.1.7 holds if the characteristic is not 2 or the GW-structure is trivial.*

Proof. Let $K' \subset L$ be the subfield $E \cap F$. We check that if $p^{i'} = [L : F]$ is the degree of x over F and $p^{j'} = [L : E]$ that of y over E , then $\alpha = x^{p^{i'}} \in K'$ and $\beta = y^{p^{j'}} \in K'$ are both generators of K' and that the extensions $K' \subset E$ and $K' \subset F$ are linearly disjoint. By Lemma 1.6.2.9, we see that $\tau_{\alpha/K} = \tau_{\beta/K}$. We conclude thanks to Corollary 1.6.2.7 and Lemma 1.6.2.8. \square

Proof of Theorem 1.6.1.7. The end of the proof follows from the consideration of the different separable closures

$$\begin{array}{ccccc} E & \longrightarrow & E.F_{\text{sep}} & \longrightarrow & L \\ \uparrow & & \uparrow & & \uparrow \\ E_{\text{sep}} & \longrightarrow & E_{\text{sep}}.F_{\text{sep}} & \longrightarrow & E_{\text{sep}}.F \\ \uparrow & & \uparrow & & \uparrow \\ K & \longrightarrow & F_{\text{sep}} & \longrightarrow & F \end{array}$$

and the use of the previous results. □

1.7 The Rost-Schmid complex

In this section, we follow [Mor12, §5] and recall some basic results about the Rost-Schmid complex. Unless stated otherwise, we denote by M a strongly \mathbb{A}^1 -invariant sheaf of abelian groups and by X an essentially smooth k -scheme.

1.7.1 Definitions

The Rost-Schmid complex is constructed along the line of the complex of X with coefficients in a Rost cycle module [Ros96] and its extension to Witt groups [Sch98].

Recall that for any field $F \in \mathcal{F}_k$, L an F -vector space of dimension 1 and $n \geq 1$ a natural number, we set

$$M_{-n}(F, L) = M_{-n}(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[L^\times]$$

where L^\times is the set of nonzero elements of L endowed with its free transitive action of F^\times through scalar multiplication, and where the action of F^\times on $M_{-n}(F)$ is given by $u \mapsto \langle u \rangle$. By convention, we also put $M_{-0}(F, \mathbb{A}_F^1) = M(F)$. Note that the twist does not distinguish squares in the sense that we have a canonical isomorphism

$$M_{-n}(F, L) \simeq M_{-n}(F, L^\vee)$$

where L^\vee denotes the dual of L as an F -vector space.

Now let z be a point of codimension n in an essentially smooth k -scheme X . The $\kappa(z)$ -vector space $\mathfrak{m}_z/\mathfrak{m}_z^2$ has dimension n and its dual $T_z X$ is called the *tangent space* of X at the point z . We denote by

$$\omega_{z/X} = \Lambda_{\kappa(z)}^{\max}(T_z X)$$

its maximal exterior power.

Definition 1.7.1.1. Let X be an essentially smooth k -scheme and n an integer. Let $X^{(n)}$ be a point of codimension n . We put

$$C^n(X, M)_z = M_{-n}(\kappa(z), \omega_{z/X})$$

and

$$C^n(X, M) = \bigoplus_{z \in X^{(n)}} M_{-n}(\kappa(z), \omega_{z/X}).$$

Remark 1.7.1.2. Note that the twist $\omega_{z/X}$ is trivial when $n = 0$: this justifies the twist by $\omega_{z/X}$ instead of $\omega_{z/k}$ in this case.

1.7.1.3. The transition maps d are defined as follows. If X is normal with generic point ξ , then for any $x \in X^{(1)}$ the local ring of X at x is a valuation ring so that we have a map $\partial_x : M_{-n}(\kappa(\xi), \omega_{\xi/X}) \rightarrow M_{-n-1}(\kappa(x), \omega_{x/X})$ for any n thanks to [Mor12, Lemma 5.10].

Now suppose X is a scheme essentially of finite type over k and let x, y be two points in X . We define a map

$$\partial_y^x : M_{-n}(\kappa(x), \omega_{\kappa(x)/k}) \rightarrow M_{-n-1}(\kappa(y), \omega_{\kappa(y)/k})$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z^{(1)}$, then put $\partial_y^x = 0$. If $y \in Z^{(1)}$, let $\tilde{Z} \rightarrow Z$ be the normalization and put

$$\partial_y^x = \sum_{z|y} \text{Tr}_{\kappa(z)/\kappa(y)} \circ \partial_z^x$$

with z running through the finitely many points of \tilde{Z} lying over y .

Definition 1.7.1.4. Let X be an essentially smooth k -scheme. The Rost-Schmid complex of X with coefficients in M is the graded abelian group $\{C^n(X, M)\}_{n \in \mathbb{Z}}$ endowed with the boundary map

$$d = \sum_{x,y} \partial_y^x : C^*(X, M) \rightarrow C^{*+1}(X, M).$$

The fact that the Rost-Schmid complex is indeed a complex is postponed to the next subsection. For now, it is only a quasi-complex in the following sense.

Definition 1.7.1.5. A pair (C^*, ∂) consisting of a graded abelian groups C^* and a morphism $\partial : C^* \rightarrow C^{*+1}$ of graded abelian groups of degree $+1$ is called a *quasi-complex*.

Definition 1.7.1.6. Given two quasi-complexes (C^*, ∂) and (D^*, ∂) , a morphism $f : (C^*, \partial) \rightarrow (D^*, \partial)$ of quasi-complexes is a graded abelian groups of degree 0 which commutes with the differentials.

1.7.1.7. PULLBACK. Let $X' \rightarrow X$ be a smooth morphism between smooth k -schemes. Let $z \in X^{(n)}$ be a point of codimension n in X ; denote by $Z = \overline{\{z\}} \subset X$ its reduced closure and $Z' \subset X'$ its inverse image. The morphism $Z' \rightarrow Z$ being smooth, Z' is a finite disjoint sum of integral closed subschemes of codimension n in X' . We denote by z'_i its finitely many irreducible components. The map $\mathfrak{m}_z/\mathfrak{m}_z^2 \otimes_{\kappa(z)} \kappa(z'_i) \rightarrow \mathfrak{m}_{z'_i}/\mathfrak{m}_{z'_i}^2$ is an isomorphism, thus we get a canonical morphism for each such z :

$$f_z^* : M_{-n}(\kappa(z), \omega_{z/X}) \rightarrow \bigoplus_i M_{-n}(\kappa(z'_i), \omega_{z'_i/X'}).$$

By summing up the previous maps, we get a morphism of graded abelian groups of degree 0:

$$f^* : C^*(X, M) \rightarrow C^*(X', M).$$

It is straightforward to prove that this defines a morphism of quasi-complexes.

1.7.1.8. PUSHFORWARD. Let $f : X \rightarrow Y$ be a morphism between schemes essentially of finite type over k and assume that X is connected (if X is not connected, take the sum over each connected component). Let $d = \dim(Y) - \dim(X)$. We define

$$f_* : C^*(X, M) \rightarrow C^{*-d}(Y, M_{-d})$$

as follows. If $y = f(x)$ and $\kappa(x)$ is finite over $\kappa(y)$, then put $(f_*)_x^y = \text{Tr}_{\kappa(x)/\kappa(y)}$ where $\text{Tr}_{\kappa(x)/\kappa(y)}$ is the transfer map. Otherwise, put $(f_*)_x^y = 0$.

Remark 1.7.1.9. This definition does not always make sense in general because the existence of transfer maps is known only for M_{-n} ($n > 1$) or for monogenous extensions, and because a twist by ω_f might be necessary.

Theorem 1.7.1.10. *Let X be an essentially smooth scheme over k and M an homotopy sheaf. Then the Rost-Schmid complex $C^*(X, M_{-1})$ is a complex.*

Proof. Let $z \in X$ be a point of codimension n and let Y be an integral closed subscheme of codimension $n - 2$ with generic point y . We want to prove that the component of $\partial \circ \partial$ starting from the summand $M_{-n+1}(\kappa(y), \omega_{y/X})$ to $M_{-n-1}(\kappa(z), \omega_{z/X})$ is zero. We may reduce to the case where X is of finite type, affine and smooth over $\kappa(z)$ and z is a closed point of codimension n in X . By the Normalization lemma [Ser65, Théorème 2 p.57], there exists a finite morphism $X \rightarrow \mathbb{A}_{\kappa(z)}^n$ such that z maps to 0 (with same residue field) and such that the image of Y is a linear $\mathbb{A}_{\kappa(z)}^2 \subset \mathbb{A}_{\kappa(z)}^n$. Using the pushforward maps (see [Mor12, Corollary 5.30], we may reduce to the case $X = \mathbb{A}_{\kappa(z)}^n$, $Y = \mathbb{A}_{\kappa(z)}^2$ and $z = 0$. This follows from [Mor12, Corollary 5.29]. \square

Remark 1.7.1.11. This theorem should be true in full generality for $C^*(X, M)$ according to [Mor12, Theorem 5.31] but we were not able to understand the proof (Conjecture 4.4.1.13 seems to be needed). Fortunately, we won't need this.

1.7.2 Homotopy invariance and other fundamental properties

In this subsection, M still denotes a strongly \mathbb{A}^1 -invariant sheaf of abelian groups. We give some of the fundamental properties of the Rost-Schmid complex of an essentially smooth k -scheme X with coefficients in M , including the homotopy invariance.

1.7.2.1. BOUNDARY MAP OF A SMOOTH DIVISOR AND MULTIPLICATION BY A UNIT. As usual, we follow [Ros96, §3] and [Mor12, §5.3]. Given a closed immersion $i : Y \subset X$ of codimension 1 between smooth k -schemes, we define a *boundary map* of the form

$$\partial_Y : C^*(U, M) \rightarrow C^*(Y, M_{-1}\{\omega_i\})$$

with $U = X - Y$ as follows. Let $w \in U^{(n)}$ be a point of codimension n , $W \subset U$ be the corresponding integral closed subscheme and $\overline{W} \in X$ its closure in X . Let z_1, \dots, z_r be the irreducible components of the closed complement $\overline{W} - W$. These points also have codimension n in Y . We define ∂_Y to be the sum

$$\sum_i \partial_{z_i}^w : M_{-n}(\kappa(w), \omega_{w/X}) \rightarrow \bigoplus_i M_{-n-1}(\kappa(z_i), \omega_{z_i/X})$$

where we used the canonical isomorphism $\omega_{z_i/X} \simeq \omega_i \otimes \omega_{z_i/Y}$. Moreover, we can define a morphism of graded abelian groups

$$[u] \cdot : C^*(X, M_{-1}) \rightarrow C^*(X, M)$$

for any element $[u] \in \mathbf{K}_1^{\text{MW}}(X)$ thanks to the action of \mathbf{K}_1^{MW} on M_{-1} . This is not a morphism of complexes but instead satisfies the formula

$$\partial([u] \cdot m) = \varepsilon \cdot [u] \partial(m),$$

where we recall that $\varepsilon = -\langle -1 \rangle$.

We now state the analogue of [Ros96, Lemma 4.5].

Lemma 1.7.2.2. *Let $g : X \rightarrow Y$ be a smooth morphism of smooth k -schemes of relative dimension 1 and assume that $\sigma : Y \rightarrow X$ is a section of g with $t \in \mathcal{O}(X)$ a function defining $\sigma(Y)$ (that is to say, the parameter t generates the sheaf of ideal defining the inclusion $\sigma(Y) \subset X$). Write $U = X - \sigma(Y)$ and let $\tilde{g} : U \rightarrow Y$ be the restriction of g to U and let ∂_Y be the boundary map associated to $\sigma : Y \subset X$. Then*

$$\partial_Y \circ \tilde{g}^* = 0 \text{ and } \partial_Y \circ [t] \circ \tilde{g}^* = (\text{Id}_Y)_*.$$

Proof. See [Mor12, Lemma 5.36] (see also Lemma 2.4.2.5 for more details). \square

Theorem 1.7.2.3 (Homotopy invariance). *Let X be a smooth k -scheme. Then the morphism of complexes defined by the projection $\pi : \mathbb{A}_X^1 \rightarrow X$*

$$\pi^* : C^*(X, M_{-1}) \rightarrow C^*(\mathbb{A}_X^1, M_{-1})$$

induces an isomorphism on cohomology groups.

Proof. We only sketch the main idea; the details are left to the reader (see [Mor12, Theorem 5.38]). The trick is to use the formulas of [Ros96, §9.1], conveniently adapted, to define an explicit homotopy of complexes.

First, we remark that the result is already known in degree ≤ 1 because the Rost-Schmid complex coincides with the Gersten complex (see [Mor12, Corollary 5.29] and [Mor12, Remark 5.32]), thus we only need to prove the theorem in degree ≥ 2 . This fact is crucial for it allows us to make use of the transfers defined on M_{-n} (where $n \geq 2$ is a fixed natural number).

In the following, we use the notation of Rost

$$X \bullet \rightarrow Y$$

to denote maps of complexes

$$C^*(X, M_{-1}) \rightarrow C^*(Y, M_{-1})$$

which are sums of composites of the maps $f_*, g^*, [a], \partial$ defined previously. For instance, we define the composites

$$r : X \times \mathbb{A}^1 \bullet \xrightarrow{j^*} X \times (\mathbb{A}^1 - \{0\}) \bullet \xrightarrow{\langle -1 \rangle^n [-1/t]} X \times (\mathbb{A}^1 - \{0\}) \bullet \xrightarrow{\partial_\infty} X,$$

$$H : X \times \mathbb{A}^1 \bullet \xrightarrow{p_2^*} X \times (\mathbb{A}^1 \times \mathbb{A}^1 - \Delta) \bullet \xrightarrow{\langle -1 \rangle^n [s-t]} X \times (\mathbb{A}^1 \times \mathbb{A}^1 - \Delta) \bullet \xrightarrow{p_{1*}} X \times \mathbb{A}^1.$$

Here t is the coordinate of $\mathbb{A}^1 = \text{Spec} k[t]$ and s, t are the coordinates of $\mathbb{A}^1 \times \mathbb{A}^1 = \text{Spec} k[s] \times \text{Spec} k[t]$. Moreover $\Delta = \{s - t = 0\}$ is the diagonal, j is the standard inclusion, p_1 and p_2 are given by the standard projections and ∂_∞ is induced by $X = X \times \infty \subset X \times (\mathbb{P}^1 - \{0\})$. Notice the sign $\langle -1 \rangle^n$ which does not exist in [Ros96].

It is easy to check that r commutes with the differentials in the Rost-Schmid complex. Moreover, it is a section of $\pi^* : C^*(X, M_{-1}) \rightarrow C^*(Y, M_{-1})$ (i.e. $r \circ \pi^* = \text{Id}$) thanks to 1.7.2.2. In order to prove the result, it suffices to prove that

$$\delta(H) = \text{Id} - \pi^* \circ r$$

where $\delta(H) = \partial^{n-1} \circ H + H \circ \partial^{n+1}$. Note that there is no way to define H in degree 1 because there is no transfer morphism on M in general. Consider the decomposition

$$p_{1*} : X \bullet \xrightarrow{q_*} X \times \mathbb{A}^1 \times \mathbb{P}^1 \bullet \xrightarrow{\bar{p}_{1*}} X \times \mathbb{A}^1$$

where q is the inclusion and \bar{p}_1 is the projection. Since \bar{p}_{1*} is a morphism of complexes according to [Mor12, Corollary 5.33], we have

$$\delta(H) = \bar{p}_{1*} \circ \delta(q_*) \circ [s - t] \circ p_2^*.$$

Moreover,

$$\delta(q_*) = (i_\Delta)_* \circ \partial_\Delta + (i_\infty)_* \circ \partial_\infty$$

where $i_\Delta : X \times \Delta \rightarrow X \times \mathbb{A}^1 \times \mathbb{P}^1$, $i_\infty : X \times \mathbb{A}^1 \times \infty \rightarrow X \times \mathbb{A}^1 \times \mathbb{P}^1$ are the inclusions and $\partial_\Delta, \partial_\infty$ are the boundary maps for $X \times \Delta \rightarrow X \times \mathbb{A}^1 \times \mathbb{A}^1$, $X \times \mathbb{A}^1 \times \infty \rightarrow X \times (\mathbb{A}^1 \times \mathbb{P}^1 - \Delta)$, respectively.

Since $s - t$ is a parameter for Δ , one finds,

$$\bar{p}_{1*} \circ (i_\Delta)_* \circ \partial_\Delta \circ [s - t] \circ p_2^* = \text{Id}$$

according to Lemma 1.7.2.2.

Let $W = \mathbb{A}^1 \times \mathbb{P}^1 - (\Delta \cup \mathbb{A}^1 \times 0)$. Moreover, let \tilde{p}_2 be the restriction of p_2 to $U = X \times (W - \mathbb{A}^1 \times \infty)$ and let $\tilde{\partial}_\infty$ be the boundary map corresponding to the inclusion $X \times \mathbb{A}^1 \times \infty \rightarrow X \times W$. Then

$$\partial_\infty \circ [s - t] \circ p_2^* = \tilde{\partial}_\infty \circ [s - t] \circ \tilde{p}_2^*.$$

Since $(s - t)/(-t)$ is a unit on W with constant value 1 on $X \times \mathbb{A}^1 \times \infty$, one has

$$\tilde{\partial}_\infty \circ [s - t] \circ p_2^* = \tilde{\partial}_\infty \circ [-t] \circ \tilde{p}_2^*$$

and thus

$$\bar{p}_{1*} \circ (i_\infty)_* \circ \partial_\infty \circ [s - t] \circ p_2^* = -\pi^* \circ r.$$

Putting things together yields to the result. \square

Theorem 1.7.2.4 (Acyclicity for local essentially smooth schemes). *For any integer n and any localization X of a point of codimension $\leq n$ in a smooth k -scheme, the diagram of abelian groups*

$$0 \rightarrow M_{-1}(X) \rightarrow C^0(X, M_{-1}) \rightarrow \cdots \rightarrow C^{n-1}(X, M_{-1}) \rightarrow C^n(X, M_{-1}) \rightarrow 0$$

is an exact sequence.

Proof. See [Mor12, Theorem 5.41]. The idea is to proceed by induction on n . The result is known for $n \leq 2$ according to [Mor12, Corollary 5.29]. Let X be the localization at a point of codimension $n \geq 2$. Thanks to Gabber's presentation Lemma [Mor12, Lemma 6.6], we may assume without loss of generality that $X = \mathbb{A}_S^1$ where S is the localization at a point of codimension $n - 1$ in an affine space. By the inductive assumption, $C^*(S, M_{-1})$ is a resolution of $M_{-1}(S)$ hence the result (since $\pi^* : C^*(S, M_{-1}) \rightarrow C^*(\mathbb{A}_S^1, M_{-1})$ is a quasi-isomorphism according to 1.7.2.3). \square

Proposition 1.7.2.5. *Let X be a smooth scheme and n a natural number. The presheaf on the small Nisnevich site X_{Nis} given by $U \mapsto C^n(X, M_{-1})$ is a sheaf for the Nisnevich topology. This sheaf is moreover acyclic in the Zariski and Nisnevich topology in the sense that for any U :*

$$H_{\text{Zar}}^*(U, C^n(X, M_{-1})) = H_{\text{Nis}}^*(U, C^n(X, M_{-1})) = 0$$

if $ > 0$.*

Proof. See [Mor12, Lemma 5.42]. \square

Proposition 1.7.2.6. *For any essentially smooth k -scheme X , the Rost-Schmid complex is an acyclic resolution on X_{Nis} of M both in the Zariski and the Nisnevich topology. Consequently, for any strongly \mathbb{A}^1 -invariant sheaf of abelian groups M , one gets canonical isomorphisms*

$$H^*(C^*(X, M_{-1})) \simeq H_{Zar}^*(X, M_{-1}) \simeq H_{Nis}^*(X, M_{-1}).$$

Proof. This is a consequence of Theorem 1.7.2.4 and 1.7.2.5. \square

Theorem 1.7.2.7. *Let M be a sheaf of abelian groups on \mathbf{Sm}_k . Then the following conditions are equivalent:*

1. M_{-1} is strongly \mathbb{A}^1 -invariant;
2. M_{-1} is strictly \mathbb{A}^1 -invariant.

Proof. The implication (1) \Rightarrow (2) follows from Corollary 1.7.2.6 and Theorem 1.7.2.3. The converse is trivial. \square

1.8 Milnor-Witt and framed correspondences

1.8.1 Sheaves with MW-transfers

In this subsection, we recall the basic definition of sheaves with MW-transfers in order to fix the notations. We follow the presentation of [BCD⁺20, Chapter 2].

1.8.1.1. Let X and Y be smooth schemes over k and let $T \subset X \times Y$ be a closed subset. Any irreducible component of T maps to an irreducible component of X through the projection $X \times Y \rightarrow X$.

Definition 1.8.1.2. If, when T is endowed with its reduced structure, the projection map $T \rightarrow X$ is finite and surjective for every irreducible component of T , we say that T is an admissible subset of $X \times Y$. We denote by $\mathcal{A}(X, Y)$ the set of admissible subsets of $X \times Y$, partially ordered by inclusions.

1.8.1.3. If Y is equidimensional, $d = \dim Y$ and $p_Y : X \times Y \rightarrow Y$ is the projection, we define a covariant functor

$$\mathcal{A}(X, Y) \rightarrow \mathbf{Ab}$$

by associating to each admissible subset $T \in \mathcal{A}(X, Y)$ the group $\widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k}) = H_T^d(X \times Y, \mathbf{K}_d^{\mathrm{MW}}\{p_Y^* \omega_{Y/k}\})$ (see [Fas19, Definition 2.5]) and to each morphism $T' \subset T$ the extension of support morphism

$$\widetilde{\mathrm{CH}}_{T'}^d(X \times Y, p_Y^* \omega_{Y/k}) \rightarrow \widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k})$$

and, using that functor, we set

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \mathrm{colim}_{T \in \mathcal{A}(X, Y)} \widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k}).$$

If Y is not equidimensional, then $Y = \bigsqcup_j Y_j$ with each Y_j equidimensional and we set

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \prod_j \widetilde{\mathrm{Cor}}_k(X, Y_j).$$

By additivity of Chow-Witt groups, if $X = \bigsqcup_i X_i$ and $Y = \bigsqcup_j Y_j$ are the respective decompositions of X and Y in irreducible components, we have

$$\widetilde{\text{Cor}}_k(X, Y) = \prod_{i,j} \widetilde{\text{Cor}}_k(X_i, Y_j).$$

Remark 1.8.1.4. In the sequel, we will simply write ω_Y in place of $p_Y^* \omega_{Y/k}$.

Example 1.8.1.5. Let X be a smooth scheme of dimension d . Then

$$\widetilde{\text{Cor}}_k(\text{Spec}(k), X) = \bigoplus_{x \in X^{(d)}} \widetilde{\text{CH}}_{\{x\}}^d(X, \omega_X) = \bigoplus_{x \in X^{(d)}} \text{GW}(\kappa(x), \omega_{\kappa(x)/k}).$$

On the other hand, $\widetilde{\text{Cor}}_k(X, \text{Spec}(k)) = \widetilde{\text{CH}}^0(X) = \mathbf{K}_0^{\text{MW}}(X)$ (recall 1.5.1.19) for any smooth scheme X .

1.8.1.6. The group $\widetilde{\text{Cor}}_k(X, Y)$ admits an alternate description which is often useful. Let X and Y be smooth schemes, with Y equidimensional. For any closed subscheme $T \subset X \times Y$ of codimension $d = \dim Y$, we have an inclusion

$$\widetilde{\text{CH}}_T^d(X \times Y, \omega_{Y/k}) \subset \bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x))$$

and thus

$$\begin{aligned} \widetilde{\text{Cor}}_k(X, Y) &= \bigcup_{T \in \mathcal{A}(X, Y)} \widetilde{\text{CH}}_T^d(X \times Y, \omega_{Y/k}) \subset \\ &\bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x)). \end{aligned}$$

In general, the inclusion $\widetilde{\text{Cor}}_k(X, Y) \subset \bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x))$ is strict as shown by Example 1.8.1.5. As an immediate consequence of this description, we see that the map

$$\widetilde{\text{CH}}_T^d(X \times Y, \omega_Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$$

is injective for any $T \in \mathcal{A}(X, Y)$.

1.8.1.7 (Composition of finite MW-correspondences). Let X, Y and Z be smooth schemes of respective dimension d_X, d_Y and d_Z , with X and Y connected. Let $V \in \mathcal{A}(X, Y)$ and $\mathcal{A}(Y, Z)$ be admissible subsets. If $\beta \in \widetilde{\text{CH}}_V^{d_Y}(X \times Y, \omega_{Y/k})$ and $\alpha \in \widetilde{\text{CH}}_T^{d_Z}(Y \times Z, \omega_{Z/k})$ are two cycles, then the expression

$$\alpha \circ \beta = (q_{XY})_* [(q_{YZ})^* \beta \cdot (p_{XY})^* \alpha]$$

is well-defined and yields a composition

$$\circ : \widetilde{\text{Cor}}_k(X, Y) \times \widetilde{\text{Cor}}_k(Y, Z) \rightarrow \widetilde{\text{Cor}}_k(X, Z)$$

which is associative (see [BCD⁺20, Ch.2, §4.2]).

Definition 1.8.1.8. The category of finite MW-correspondences over k is by definition the category $\widetilde{\text{Cor}}_k$ whose objects are smooth schemes and whose morphisms are the elements of abelian groups $\widetilde{\text{Cor}}_k(X, Y)$.

Remark 1.8.1.9. The category $\widetilde{\text{Cor}}_k$ is a symmetric monoidal additive category (see [BCD⁺20, Chapter 2, Lemma 4.4.2]).

Definition 1.8.1.10. A presheaf with MW-transfers is a contravariant additive functor $\widetilde{\text{Cor}}_k \rightarrow \mathbf{Ab}$. A (Nisnevich) sheaf with MW-transfers is a presheaf with MW-transfers such that its restriction to \mathbf{Sm}_k via the graph functor is a Nisnevich sheaf. We denote by $\widetilde{\text{PSh}}(k)$ (resp. $\widetilde{\text{Sh}}(k)$) the category of presheaves (resp. sheaves) with MW-transfers and by $\mathbf{HI}^{\text{MW}}(k)$ the category of homotopy sheaves with MW-transfers (also called *MW-homotopy sheaves*).

Example 1.8.1.11. For any $j \in \mathbb{Z}$, the contravariant functor $X \mapsto \mathbf{K}_j^{\text{MW}}(X)$ is a presheaf on $\widetilde{\text{Cor}}_k$.

1.8.1.12. **PUSHFORWARDS.** Let X and Y be two smooth schemes of dimension d and let $f : X \rightarrow Y$ be a finite morphism such that any irreducible component of X surjects to the irreducible component of Y it maps to. Assume that we have an orientation (\mathcal{L}, ψ) of ω_f , that is an isomorphism $\psi : \mathcal{L} \otimes \mathcal{L} \rightarrow \omega_f$ of line bundles. We define a finite correspondence $\alpha(f, \mathcal{L}, \psi) \in \widetilde{\text{Cor}}_k(Y, X)$. Let $\gamma'_f : X \rightarrow Y \times X$ be the transpose of the graph of f . Since X is an admissible subset of $Y \times X$, we have a transfer map

$$(\gamma'_f)_* : \mathbf{K}_0^{\text{MW}}(X, \omega_f) \rightarrow \widetilde{\text{CH}}_X^d(Y \times X, \omega_{X/k}) \rightarrow \widetilde{\text{Cor}}_k(Y, X).$$

The map ψ yields an isomorphism $\mathbf{K}_0^{\text{MW}}(X) \rightarrow \mathbf{K}_0^{\text{MW}}(X, \omega_f)$. We define the finite MW-correspondence $\alpha(f, \mathcal{L}, \psi)$ as the image of $\langle 1 \rangle$ under the composite

$$\mathbf{K}_0^{\text{MW}}(X) \rightarrow \mathbf{K}_0^{\text{MW}}(X, \omega_f) \rightarrow \widetilde{\text{CH}}_X^d(Y \times X, \omega_{X/k}) \rightarrow \widetilde{\text{Cor}}_k(Y, X).$$

Now let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a homotopy sheaf with MW-transfers. Denote by $(M \otimes \omega_f)_X$ (resp. M_Y) the canonical (twisted) sheaf associated to M defined on the Zariski site X_{Zar} (resp. Y_{Zar}) introduced in 1.5.2.4 and define a natural transformation

$$f_* : f_*(M \otimes \omega_f)_X \rightarrow M_Y$$

by taking (as in 1.5.2.4) the sheafification of the natural transformation of presheaves

$$\begin{aligned} V \in Y_{\text{Zar}} &\mapsto (M(f^{-1}(V), \omega_{f|_{f^{-1}(V)}}) \rightarrow M(V)) \\ (\mu \otimes l) &\mapsto \alpha(f, \psi_l, L_l)^*(\mu) \end{aligned}$$

where (ψ_l, \mathcal{L}_l) is the orientation of $\omega_{f|_{f^{-1}(V)}}$ associated to $l \in \omega_{f|_{f^{-1}(V)}}^\times$. Taking global sections, this leads in particular to a map

$$M({}^t f) : M(X, \omega_f) \rightarrow M(Y)$$

for any finite morphism $f : X \rightarrow Y$.

We can check the following propositions.

Proposition 1.8.1.13. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ and consider two finite morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of smooth schemes. Then*

$$M({}^t(g \circ f)) = M({}^t g) \circ M({}^t f).$$

Proof. Keeping the previous notations, if (\mathcal{L}', ψ') is an orientation of ω_g , then $(\mathcal{L} \otimes f^* \mathcal{L}', \psi \otimes f^* \psi')$ is an orientation of $\omega_{g \circ f} = \omega_f \otimes f^* \omega_g$, and we have $\alpha(f, \mathcal{L}, \psi) \circ \alpha(g, \mathcal{L}', \psi') = \alpha(g \circ f, \mathcal{L} \otimes f^* \mathcal{L}', \psi \otimes f^* \psi')$. \square

Proposition 1.8.1.14. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a homotopy sheaf with MW-transfers. Let $i : Z \rightarrow X$ and $i' : T \rightarrow Y$ be two closed immersions and let $f : Y \rightarrow X$ be a finite morphism. The following diagram*

$$\begin{array}{ccc} M(Y - T, \omega_{(Y-T)/k}) & \xrightarrow{\partial} & M_{-1}(T, \omega_{T/k}) \\ M(i'f) \downarrow & & \downarrow M(i'f) \\ M(X - Z, \omega_{(X-Z)/k}) & \xrightarrow{\partial} & M_{-1}(Z, \omega_{Z/k}) \end{array}$$

is commutative.

1.8.1.15. **TENSOR PRODUCTS.** Let X_1, X_2, Y_1, Y_2 be smooth schemes over $\text{Spec } k$. Let $d_1 = \dim Y_1$ and $d_2 = \dim Y_2$. Let $\alpha_1 \in \widetilde{\text{CH}}_{T_1}^{d_1}(X_1 \times Y_1, \omega_{Y_1/k})$ and $\alpha_2 \in \widetilde{\text{CH}}_{T_2}^{d_2}(X_2 \times Y_2, \omega_{Y_2/k})$ for some admissible subsets $T_i \subset X_i \times Y_i$. The exterior product defined in [Fas08, §4] gives a cycle

$$(\alpha_1 \times \alpha_2) \in \widetilde{\text{CH}}_{T_1 \times T_2}^{d_1+d_2}(X_1 \times Y_1 \times X_2 \times Y_2, p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k})$$

where $p_{Y_i} : X_1 \times Y_1 \times X_2 \times Y_2 \rightarrow Y_i$ is the canonical projection to the corresponding factor. Let $\sigma : X_1 \times Y_1 \times X_2 \times Y_2 \rightarrow X_1 \times X_2 \times Y_1 \times Y_2$ be the transpose isomorphism. Applying σ_* , we get a cycle

$$\sigma_*(\alpha_1 \times \alpha_2) \in \widetilde{\text{CH}}_{\sigma(T_1 \times T_2)}^{d_1+d_2}(X_1 \times X_2 \times Y_1 \times Y_2, p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k}).$$

Since $p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k} = \omega_{Y_1 \times Y_2/k}$, it is straightforward to check that $\sigma(T_1 \times T_2)$ is finite and surjective over $X_1 \times X_2$. Thus $\sigma_*(\alpha_1 \times \alpha_2)$ defines a finite MW-correspondence between $X_1 \times X_2$ and $Y_1 \times Y_2$.

Definition 1.8.1.16. Let X_1, X_2, Y_1, Y_2 be smooth schemes over $\text{Spec } k$, and $\alpha_1 \in \widetilde{\text{Cor}}_k(X_1, Y_1)$ and $\alpha_2 \in \widetilde{\text{Cor}}_k(X_2, Y_2)$ two MW-correspondences. We define their tensor products as $X_1 \otimes X_2 = X_1 \times X_2$ and $\alpha_1 \otimes \alpha_2 = \sigma_*(\alpha_1 \times \alpha_2)$.

1.8.1.17. We denote by $\tilde{c}(X) : Y \mapsto \widetilde{\text{Cor}}_k(Y, X)$ the representable presheaf associated to a smooth scheme X (be careful that this is not a Nisnevich sheaf in general). The category of MW-presheaves is an abelian Grothendieck category with a unique symmetric monoidal structure such that the Yoneda embedding

$$\widetilde{\text{Cor}}_k \rightarrow \widetilde{\text{Sh}}(k), X \mapsto \tilde{a}\tilde{c}(X)$$

is symmetric monoidal (where \tilde{a} is the sheafification functor, see [BCD⁺20, Chapter 3, §1.2.7]). The tensor product is denoted by $\otimes_{\mathbf{HI}^{\text{MW}}}$ and commutes with colimits (hence the monoidal structure is closed, see [BCD⁺20, Chapter 3, §1.2.14]).

1.8.2 Framed correspondences

The aim of the section is to make a link between the category of linear framed correspondences (after Garkusha-Panin-Voevodsky) and the category of MW-presheaves. We follow the presentation of [BCD⁺20, Ch. 3, §3] and [GP14].

Definition 1.8.2.1. Let U be a smooth k -scheme and $Z \subset U$ be a closed subset of codimension n . A set of regular functions $\varphi_1, \dots, \varphi_n \in k[U]$ is called a framing of Z in U if Z coincides with the closed subset defined by the equations $\varphi_1 = \dots = \varphi_n = 0$.

Definition 1.8.2.2. Let X and Y be smooth k -schemes, and let $n \in \mathbb{N}$ be a natural number. An explicit framed correspondence $c = (U, \varphi, f)$ of level n from X to Y consists of the following data:

1. A closed subset $Z \subset \mathbb{A}_X^n$ which is finite over X (here, Z is endowed with its reduced structure).
2. An étale neighbourhood $\alpha : U \rightarrow \mathbb{A}_X^n$ of Z .
3. A framing $\varphi = (\varphi_1, \dots, \varphi_n)$ of Z in U .
4. A morphism $f : U \rightarrow Y$.

The closed subset Z is called the *support* of the explicit framed correspondence $c = (U, \varphi, f)$.

Remark 1.8.2.3. One could give an alternative approach to the above definition. A framed correspondence (U, φ, f) corresponds to a pair of morphisms $\varphi : U \rightarrow \mathbb{A}_k^n$ and $f : U \rightarrow Y$ yielding a unique morphism $\varphi : U \rightarrow \mathbb{A}_Y^n$. The closed subset $Z \subset U$ corresponds to the preimage of $Y \times \{0\} \subset Y \times \mathbb{A}_k^n = \mathbb{A}_Y^n$. This correspondence is unique.

Remark 1.8.2.4. Note that Z is not supposed to map surjectively onto a component of X . For instance, $Z = \emptyset$ is an explicit framed correspondence of level n , denoted by 0_n . If Z is non-empty, then an easy dimension count shows that $Z \subset \mathbb{A}_X^n \rightarrow X$ is indeed surjective onto a component of X .

Remark 1.8.2.5. Suppose that X is a smooth connected scheme. By definition, an explicit framed correspondence of level $n = 0$ is either a morphism of schemes $f : X \rightarrow Y$ or 0_0 .

Definition 1.8.2.6. Let $c = (U, \varphi, f)$ and $c' = (U', \varphi', f')$ be two explicit framed correspondences of level $n \geq 0$. Then, c and c' are said to be *equivalent* if they have the same support and there exists an open neighbourhood V of Z in $U \times_{\mathbb{A}_X^n} U'$ such that the diagrams

$$\begin{array}{ccc} U \times_{\mathbb{A}_X^n} U' & \longrightarrow & U' \\ \downarrow & & \downarrow f' \\ U & \xrightarrow{f} & Y \end{array}$$

and

$$\begin{array}{ccc} U \times_{\mathbb{A}_X^n} U' & \longrightarrow & U' \\ \downarrow & & \downarrow \varphi' \\ U & \xrightarrow{\varphi} & \mathbb{A}_k^n \end{array}$$

are both commutative when restricted to V . A *framed correspondence* of level n is an equivalence class of explicit framed correspondences of level n .

Definition 1.8.2.7. Let X and Y be smooth schemes and let $n \in \mathbb{N}$. We denote by $\text{Fr}_n(X, Y)$ the set of framed correspondences of level n from X to Y and by $\text{Fr}_*(X, Y)$ the set $\sqcup_n \text{Fr}_n(X, Y)$. Together with the composition of framed correspondences described in [GP14, §2], this defines a category whose objects are smooth schemes and morphisms are $\text{Fr}_*(-, -)$. We denote this category by $\text{Fr}_*(k)$ and refer to it as the *category of framed correspondences*.

We now pass to the linear version of the above category following [GP14, §7], starting with the following observation. Let X and Y be smooth schemes, and let $c_Z = (U, \varphi, f)$ be an explicit framed correspondence of level n from X to Y with support Z of the form $Z = Z_1 \sqcup Z_2$. let $U_1 = U - Z_2$ and $U_2 = U - Z_1$. For $i \in \{1, 2\}$, we get étale morphisms $\alpha_i : U_i \rightarrow X$ and morphisms $\varphi_i : U_i \rightarrow \mathbb{A}_k^n, f_i : U_i \rightarrow Y$ by precomposing the morphisms α, φ and f with the open immersion $U_i \rightarrow U$. Note that U_i is an étale neighbourhood of Z_i for $i \in \{1, 2\}$, and that $c_{Z_i} = (U_i, \varphi_i, f_i)$ are explicit framed correspondences of level n from X to Y with support Z_i .

Definition 1.8.2.8. Let X and Y be smooth schemes and let $n \in \mathbb{N}$. let

$$\mathbb{Z}F_n(X, Y) = \mathbb{Z}\text{Fr}_n(X, Y)/H$$

where H is the subgroup generated by elements of the form $c_Z - c_{Z_1} - c_{Z_2}$ where $Z = Z_1 \sqcup Z_2$ is as above and $\mathbb{Z}\text{Fr}_n(X, Y)$ is the free abelian group on $\text{Fr}_n(X, Y)$. The category $\mathbb{Z}F_*(k)$ of *linear framed correspondences* is the category whose objects are smooth schemes and whose morphisms are

$$\text{Hom}_{\mathbb{Z}F_*(k)}(X, Y) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}F_n(X, Y).$$

Remark 1.8.2.9. Note that there is an obvious functor $\iota : \text{Fr}_*(k) \rightarrow \mathbb{Z}F_*(k)$ with $\iota(0_n) = 0$ for any $n \in \mathbb{N}$.

We now compare the category of finite MW-correspondences with the above categories.

Let U be a smooth k -scheme and let $\varphi : U \rightarrow \mathbb{A}_k^n$ be a morphism corresponding to (nonzero) global sections $\varphi_i \in \mathcal{O}(U)$. Each section φ_i can be seen as an element of $k(U)^\times$ and defines then an element of $\mathbf{K}_1^{\text{MW}}(k(U))$. Let $|\varphi_i|$ be the support of φ_i (i.e. its vanishing locus), and let $Z = |\varphi_1| \cap \dots \cap |\varphi_n|$. Consider the residue map

$$d : \mathbf{K}_1^{\text{MW}}(k(U)) \rightarrow \bigoplus_{x \in U^{(1)}} \mathbf{K}_0^{\text{MW}}(k(x), \omega_{x/k}).$$

Then, $d(\varphi_i)$ defines an element supported on $|\varphi_i|$. As it is a boundary, it defines a cycle $Z(\varphi) \in H_{|\varphi_i|}^1(U, \mathbf{K}_1^{\text{MW}})$. Now, we can consider the intersection product

$$H_{|\varphi_1|}^1(U, \mathbf{K}_1^{\text{MW}}) \times \dots \times H_{|\varphi_n|}^1(U, \mathbf{K}_1^{\text{MW}}) \rightarrow H_Z^n(U, \mathbf{K}_n^{\text{MW}})$$

to get an element $Z(\varphi_1) \cdot \dots \cdot Z(\varphi_n)$ that we denote by $Z(\varphi)$.

Lemma 1.8.2.10. Any explicit framed correspondence $c = (U, \varphi, f)$ induces a finite MW-correspondence $\alpha(c)$ from X to Y . Moreover, two equivalent explicit framed correspondences c and c' induce the same finite MW-correspondence.

Proof. Let us start with the first assertion. If Z is empty, its image is defined to be zero. If c is of level 0, then it corresponds to a morphism of schemes, and we use the functor $\mathbf{Sm}_k \rightarrow \widehat{\text{Cor}}_k$ to define the image of c . We thus suppose that Z is non-empty (thus finite and surjective on some components of X) of level $n \geq 1$. Consider the following diagram

$$\begin{array}{ccc} & U & \xrightarrow{(\varphi, f)} \mathbb{A}_Y^n \\ & \searrow \alpha & \downarrow \\ Z & \longrightarrow & \mathbb{A}_X^n \\ & & \downarrow p_X \\ & & X \end{array}$$

defining an explicit framed correspondence (U, φ, f) of level n . The framing φ defines an element $Z(\varphi) \in H_Z^n(U, \mathbf{K}_n^{\text{MW}})$ as explained above. Now, α is étale and therefore induces an isomorphism $\alpha^* \omega_{\mathbb{A}_X^n/k} \simeq \omega_{U/k}$. Choosing the usual orientation for \mathbb{A}_k^n , we get an isomorphism $\mathcal{O}_{\mathbb{A}_X^n} \simeq \omega_{\mathbb{A}_X^n/k} \otimes (p_X)^* \omega_{X/k}^\vee$ and therefore an isomorphism

$$\mathcal{O}_U \simeq \alpha^*(\mathcal{O}_{\mathbb{A}_X^n}) \simeq \alpha^*(\omega_{\mathbb{A}_X^n/k} \otimes (p_X)^* \omega_{X/k}^\vee) \simeq \omega_{U/k} \otimes (p_X \alpha)^* \omega_{X/k}^\vee.$$

We can then see $Z(\varphi)$ as an element of the group $H_Z^n(U, \mathbf{K}_n^{\text{MW}}, \omega_{U/k} \otimes (p_X \alpha)^* \omega_{X/k}^\vee)$. Consider the map $(p_X \alpha, f) : U \rightarrow X \times Y$ and the image T of Z under the map of underlying topological spaces. It follows from [MVW06, Lemma 1.4] that T is closed, finite and surjective over some components of X . Moreover, the morphism $Z \rightarrow T$ is finite and it follows that we have a pushforward map

$$(p_X \alpha, f)_* : H_Z^n(U, \mathbf{K}_n^{\text{MW}}, \omega_{U/k} \otimes (p_X \alpha)^* \omega_{X/k}^\vee) \rightarrow H_T^n(X \times Y, \mathbf{K}_n^{\text{MW}}, \omega_{X \times Y/X})$$

yielding a finite Chow-Witt correspondence $\alpha(c) := (p_X \alpha, f)_*(Z(\varphi))$ between X and Y .

Suppose next that $c = (U, \varphi, f)$ and $c' = (U', \varphi', f')$ are two equivalent explicit framed correspondences of level n . Following the above construction, we obtain two cocycles $\tilde{\alpha}(c) \in H_Z^n(U, \mathbf{K}_n^{\text{MW}}, \omega_{U/k} \otimes (p_X \alpha)^* \omega_{X/k}^\vee)$ and $\tilde{\alpha}(c') \in H_Z^n(U', \mathbf{K}_n^{\text{MW}}, \omega_{U'/k} \otimes (p_X \alpha')^* \omega_{X/k}^\vee)$. Now, the pullbacks along the projections

$$\begin{array}{c} U \times_{\mathbb{A}_X^n} U' \xrightarrow{p_2} U' \\ \downarrow p_1 \\ U \end{array}$$

yield morphisms

$$p_1^* : H_Z^n(U, \mathbf{K}_n^{\text{MW}}, \omega_{U/k} \otimes (p_X \alpha)^* \omega_{X/k}^\vee) \simeq H_{p_1^{-1}(Z)}^n(U \times_{\mathbb{A}_X^n} U', \mathbf{K}_n^{\text{MW}}, \omega_{U \times_{\mathbb{A}_X^n} U'/k} \otimes (p_X \alpha p_1)^* \omega_{X/k}^\vee)$$

and

$$p_2^* : H_Z^n(U', \mathbf{K}_n^{\text{MW}}, \omega_{U'/k} \otimes (p_X \alpha')^* \omega_{X/k}^\vee) \simeq H_{p_2^{-1}(Z)}^n(U \times_{\mathbb{A}_X^n} U', \mathbf{K}_n^{\text{MW}}, \omega_{U \times_{\mathbb{A}_X^n} U'/k} \otimes (p_X \alpha p_2)^* \omega_{X/k}^\vee),$$

while the pullback along the open immersion $i : V \rightarrow U \times_{\mathbb{A}_X^n} U'$ induces morphisms

$$i^* : H_{p_1^{-1}(Z)}^n(U \times_{\mathbb{A}_X^n} U', \mathbf{K}_n^{\text{MW}}, \omega_{U \times_{\mathbb{A}_X^n} U'/k} \otimes (p_X \alpha p_1)^* \omega_{X/k}^\vee) \simeq H_Z^n(V, \mathbf{K}_n^{\text{MW}}, \omega_{V/k} \otimes (p_X \alpha p_1 i)^* \omega_{X/k}^\vee)$$

and

$$i^* : H_{p_2^{-1}(Z)}^n(U \times_{\mathbb{A}_X^n} U', \mathbf{K}_n^{\text{MW}}, \omega_{U \times_{\mathbb{A}_X^n} U'/k} \otimes (p_X \alpha p_2)^* \omega_{X/k}^\vee) \simeq H_Z^n(V, \mathbf{K}_n^{\text{MW}}, \omega_{V/k} \otimes (p_X \alpha p_2 i)^* \omega_{X/k}^\vee).$$

Note that $p_X \alpha p_2 = p_X \alpha p_1$ and that $i^* p_1^*(\tilde{\alpha}(c)) = i^* p_2^*(\tilde{\alpha}(c))$ by construction. Pushing forward along $V \rightarrow U \times_{\mathbb{A}_X^n} U' \rightarrow U \rightarrow X \times Y$, we get the result. \square

Proposition 1.8.2.11. *The assignment $c = (U, \varphi, f) \mapsto \alpha(c)$ made explicit in Lemma 1.8.2.10 defines functors $\alpha : \text{Fr}_*(k) \rightarrow \widetilde{\text{Cor}}_k$ and $\alpha' : \mathbb{Z}\text{F}_*(k) \rightarrow \widetilde{\text{Cor}}_k$ such that we have a commutative diagram of functors*

$$\begin{array}{ccccc} & & \text{Fr}_*(k) & & \\ & \nearrow & \downarrow \alpha & \searrow \iota & \\ \text{Sm}_k & & & & \mathbb{Z}\text{F}_*(k) \\ & \searrow \tilde{\gamma} & & \swarrow \alpha' & \\ & & \widetilde{\text{Cor}}_k & & \end{array}$$

Proof. For any smooth schemes X, Y and any natural number $n \geq 0$, we have a well-defined map $\alpha : \text{Fr}_n(X, Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$ and therefore a well-defined map $\mathbb{Z}\text{Fr}_n(X, Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$. Let $c = (U, \varphi, f)$ be an explicit framed correspondence of level n with support Z of the form $Z = Z_1 \sqcup Z_2$. Let $c_i = (U_i, \varphi_i, f_i)$ be the explicit framed correspondences with support Z_i obtained as in Definition 1.8.2.8. By construction, we get $\alpha(c) = \alpha(c_1) + \alpha(c_2)$ and it follows that $\alpha : \text{Fr}_n(X, Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$ induces a map $\alpha' : \mathbb{Z}\text{Fr}_n(X, Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$.

It remains to prove that the functors $\alpha : \text{Fr}_k \rightarrow \widetilde{\text{Cor}}_k$ and $\alpha' : \mathbb{Z}\text{F}_*(k) \rightarrow \widetilde{\text{Cor}}_k$ are well-defined, which amounts to prove that the respective compositions are preserved. Suppose that (U, φ, f) is an explicit framed correspondence of level n between X and Y , and (V, ψ, g) is an explicit framed correspondence of level m between Y and Z . We use the diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{pr_V} & V & \xrightarrow{\psi} & \mathbb{A}^m \\
 \downarrow & & \downarrow \beta & \searrow g & \\
 U \times \mathbb{A}^m & \xrightarrow{f \times \text{Id}} & Y \times \mathbb{A}^m & & Z \\
 \downarrow & & \downarrow p_Y & & \\
 U & \xrightarrow{f} & Y & & \\
 \downarrow p_X \alpha & & \searrow \varphi & & \mathbb{A}^n \\
 X & & & &
 \end{array}$$

(A curved arrow labeled pr_U goes from W to U)

in which all squares are cartesian. The composition (U, φ, f) with (V, ψ, g) is given by $(W, (\varphi \circ pr_U, \psi \circ pr_V), g \circ pr_V)$.

On the other hand, the map $(p_X \alpha, f) \circ pr_U : W \rightarrow X \times Y$ and $(p_Y \beta, g) \circ pr_V : W \rightarrow Y \times Z$ yield a morphism $\rho : W \rightarrow X \times Y \times Z$ and then a diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{pr_V} & V & & \\
 \downarrow \rho & & \downarrow (p_Y \beta, g) & & \\
 W & \xrightarrow{\rho} & X \times Y \times Z & \xrightarrow{p_{Y \times Z}} & Y \times Z \\
 \downarrow pr_U & & \downarrow p_{X \times Y} & & \downarrow \\
 U & \xrightarrow{(p_X \alpha, f)} & X \times Y & \longrightarrow & Y
 \end{array}$$

in which all squares are cartesian. By [BCD⁺20, Ch. 2, Prop. 3.1.1, Rem. 3.1.2], we have $(p_{X \times Y})^*(p_X \alpha, f)_* = \rho_*(pr_U)^*$ and $(p_{Y \times Z})^*(p_Y \beta, g)_* = \rho_*(pr_V)^*$. By definition of the pullback and the product, we have $(pr_U)^*(Z(\varphi)) = Z(\varphi \circ pr_U)$ and $(pr_V)^*(Z(\psi)) = Z(\psi \circ pr_V)$. It follows that

$$Z(\varphi \circ pr_U, \psi \circ pr_V) = (pr_U)^*(Z(\varphi)) \cdot (pr_V)^*(Z(\psi)).$$

Finally, we observe that there following diagram

$$\begin{array}{ccc}
 W & \xrightarrow{g \circ pr_V} & Z \\
 \parallel & & \uparrow \\
 W & \longrightarrow & X \times Y \times Z \xrightarrow{p_{X \times Z}} X \times Z \\
 \parallel & & \downarrow \\
 W & \xrightarrow{p_X \alpha \circ pr_U} & X
 \end{array}$$

is commutative, thus the composition is preserved. \square

Remark 1.8.2.12. The functor $\alpha' : \mathbb{Z}F_*(k) \rightarrow \widetilde{\text{Cor}}_k$ is additive but it is not faithful (see [BCD⁺20, Ch. 3, Example 2.1.11]). Thus we have a functor

$$\widetilde{\text{PSh}}(k) \rightarrow \text{PSh}^{\text{Fr}}(k)$$

from the category of MW-presheaves to the category of presheaves with framed transfers.

1.9 Miscellaneous

1.9.1 Virtual objects

In this section, we closely follow [Del87, §4] and show how to construct the category $\mathfrak{V}(\mathcal{A})$ of virtual objects from an exact category \mathcal{A} .

1.9.1.1. Recall that an exact category is an additive category \mathcal{A} equipped with a set of exact sequences $X \rightarrowtail Y \twoheadrightarrow Z$ satisfying various axioms (see [Buh10, Definition 2.1]). Our main example consists in the category of vector bundles over a scheme X (with locally split short exact sequences).

1.9.1.2. A (commutative) Picard category is a non-empty category \mathcal{P} where all arrows are isomorphisms, with a functor $+: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ following some associativity and commutativity constraints, and such that for any object P , the two functors $X \mapsto X + P$ and $X \mapsto P + X$ are auto-equivalences of \mathcal{P} .

One can show that there is a zero object 0 (unique up to unique isomorphism) and that any object X has an inverse $-X$ (unique up to unique isomorphism) such that $X + (-X) \simeq 0$. The commutativity condition gives switch isomorphisms $X + Y \simeq Y + X$ compatible with the associative data. Beware that these switch isomorphisms are not identities in general.

The functor $X \mapsto -X$ is compatible with $+$ as follows: the isomorphism $(X + Y) + ((-X) + (-Y)) \simeq (X + (-X)) + (Y + (-Y)) \simeq 0 + 0 \simeq 0$ is such that $(-X) + (-Y)$ is (isomorphic to) $-(X + Y)$. Nevertheless, beware that the diagram

$$\begin{array}{ccccc} ((-X) + X) + (-X) & \xrightarrow{\simeq} & (-X) + (X + (-X)) \\ \downarrow \simeq & & \downarrow \simeq \\ 0 + (-X) & \xrightarrow{\simeq} & (-X) & \xrightarrow{\simeq} & (-X) + 0 \end{array}$$

is not commutative.

Example 1.9.1.3 (Graded line bundles, see also [Fas19], §1.3). Let S be a scheme and $\mathcal{L}(S, \mathbb{Z})$ be the category of graded line bundles over S . An object of this category is a pair (a, L) where L is a line bundle (an invertible \mathcal{O}_X -module) and a is a locally constant integer (an element of $\Gamma(S, \mathbb{Z})$). A map $(a, L) \rightarrow (a', L')$ between two such objects is an isomorphism $L \rightarrow L'$ of line bundles with the data $a = a'$. By definition, we have $(a, L) + (a', L') = (a + a', L \otimes L')$. The associativity constraint is deduced from the one ruling the tensor product and the commutativity constraint $(a, L) + (a', L') \simeq (a', L') + (a, L)$ is given by $l \wedge l' \mapsto (-1)^{aa'} l' \wedge l$ (also called *Koszul sign convention*). The unit object 0

is $(0, \mathcal{O}_X)$. For any object (a, L) , there is an isomorphism $L \otimes L^\vee \simeq \mathcal{O}_X$ induced by the pairing $L \times L^\vee \rightarrow \mathcal{O}_X, (l, \varphi) \mapsto \varphi(l)$ (where L^\vee denotes the dual of L). Hence the following isomorphism:

$$(a, L) + (-a, L^\vee) \simeq (0, \mathcal{O}_X).$$

1.9.1.4. Let \mathcal{A} be an exact category. We construct the category $\mathfrak{V}(\mathcal{A})$ of virtual objects of \mathcal{A} . Denote by \mathcal{A}_{iso} the category with the same objects as \mathcal{A} but with morphisms the isomorphisms of \mathcal{A} . Let \mathcal{P} be a Picard category. Consider the functors $[-] : \mathcal{A}_{iso} \rightarrow \mathcal{P}$ with data (a), (b) and rules (c), (d), (e) described as follows:

(a) (additivity) For any short exact sequence $\Sigma : A' \rightarrow A \rightarrow A''$, we have a morphism $[\Sigma] : [A] \rightarrow [A'] + [A'']$, functorial in morphisms of short exact sequences.

(b) If 0 is a zero object in \mathcal{A} , then we have an isomorphism $[0] \rightarrow 0$.

(c) Let $\varphi : A \rightarrow B$ be an isomorphism in \mathcal{A} . Denote by Σ the short exact sequence $0 \rightarrow A \rightarrow B$ (resp. $A \rightarrow B \rightarrow 0$). Then $[\varphi]$ (resp. $[\varphi]^{-1}$) is the composite

$$[A] \xrightarrow{[\Sigma]} [0] + [B] \xrightarrow{(b)} [B]$$

(resp.

$$[B] \xrightarrow{[\Sigma]} [B] + [0] \xrightarrow{(b)} [A]).$$

(d) (associativity) Consider the following filtration $C \supset B \supset A \supset 0$. Then the diagram (with morphisms given by (a))

$$\begin{array}{ccc} [C] & \longrightarrow & [A] + [C/A] \\ \downarrow & & \downarrow \\ [B] + [C/B] & \longrightarrow & [A] + [B/A] + [C/B] \end{array}$$

is commutative.

(e) ((a) is additive) For any $A = A' \oplus A''$, the short exact sequences $\Sigma : A' \rightarrow A \rightarrow A''$ and $\Sigma' : A'' \rightarrow A \rightarrow A'$ give a commutative triangle

$$\begin{array}{ccc} [A'] + [A''] & \longrightarrow & [A''] + [A'] \\ & \searrow [\Sigma] \quad \swarrow [\Sigma'] & \\ & [A] & \end{array}$$

For any commutative Picard category \mathcal{P} , the functors $[-] : \mathcal{A} \rightarrow \mathcal{P}$ satisfying the conditions (a),..., (e) define a category denoted by $[[[-]]_{\mathcal{P}}$. One can prove that there is a (commutative) Picard category $\mathfrak{V}(\mathcal{A})$ with functors $[-]_{\mathcal{A}} : \mathcal{A}_{iso} \rightarrow \mathfrak{V}(\mathcal{A})$ satisfying the conditions (a),..., (e) which is universal in the sense that for any Picard category \mathcal{P} with functors $[-] : \mathcal{A}_{iso} \rightarrow \mathcal{P}$ satisfying (a),..., (e), the category $[[[-]]_{\mathcal{P}}$ is equivalent to the category of additive functors $\mathfrak{V}(\mathcal{A}) \rightarrow \mathcal{P}$. By definition, $\mathfrak{V}(\mathcal{A})$ is the category of virtual objects of \mathcal{A} .

1.9.1.5. Let $T : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between two exact categories (it respects short exact sequences). With the previous notations, the composition $[-] \circ T : \mathcal{A}_{iso} \rightarrow \mathfrak{V}(\mathcal{B})$ is a functor satisfying (a),..., (e) thus it induces an additive functor $\mathfrak{V}(\mathcal{A}) \rightarrow \mathfrak{V}(\mathcal{B})$.

1.9.1.6. Let S be a scheme and denote by $\text{Vect}(S)$ the category of vector bundle over S . We put $\mathfrak{V}(S) = \mathfrak{V}(\text{Vect}(S))$ (also denoted by $\underline{K}(S)$ in [Del87, §4]). Any scheme morphism $f : X \rightarrow S$ defines a pullback $f^* : \mathfrak{V}(S) \rightarrow \mathfrak{V}(X)$.

1.9.1.7. Let S be a scheme and V be a vector bundle over S . The rank $\text{rk}(V)$ of V is a locally constant integer on S . The determinant $\det(V)$ of V is the line bundle $\Lambda^{\text{rk}(V)}(V)$. We also call determinant of V (and write $\text{Det}(V)$) the graded line bundle $(\text{rk}(V), \det(V))$. For any short exact sequence

$$V' \rightarrowtail V \twoheadrightarrow V'',$$

we have an isomorphism

$$\det(V') \otimes \det(V'') \rightarrow \det(V)$$

illustrated by the symbols

$$(e'_1 \wedge \cdots \wedge e'_n) \otimes (e''_1 \wedge \cdots \wedge e''_m) \mapsto e'_1 \wedge \cdots \wedge e'_n \wedge e''_1 \wedge \cdots \wedge e''_m.$$

This induces an isomorphism between the associated line bundles which satisfies conditions (a),..., (e). The universal property gives a factorization of $\text{Det} : \text{Vect}(S) \rightarrow \mathcal{L}(S, \mathbb{Z})$ through a functor $\mathfrak{V}(S) \rightarrow \mathcal{L}(S, \mathbb{Z})$ also denoted by Det .

Note in particular that for a vector bundle $V = V' \oplus V''$, the two short exact sequences give the following commutative diagram:

$$\begin{array}{ccc} \det(V') \otimes \det(V'') & \xrightarrow{(-1)^{nm}} & \det(V'') \otimes \det(V') \\ & \searrow \simeq & \swarrow \simeq \\ & \det(V) & \end{array}$$

where n, m are the rank of V', V'' , respectively.

If \mathcal{V} is a virtual vector bundle over S (i.e. an object of $\mathfrak{V}(S)$), then we can write the graded line bundle $\text{Det}(\mathcal{V})$ as $(\text{rk}(\mathcal{V}), \det(\mathcal{V}))$ so that we can define the rank and the determinant of \mathcal{V} in an obvious way.

In this thesis, we are mostly interested in the case where $S = \text{Spec}(A)$ is a local scheme. Then the functor $\text{Det} : \mathfrak{V}(S) \rightarrow \mathcal{L}(S, \mathbb{Z})$ is an equivalence and we will identify these two categories.

1.9.2 Smooth models

Definition 1.9.2.1. Let E be a finitely generated field over the perfect field k . A *smooth model* of E is an affine smooth scheme $X = \text{Spec} A$ of finite type such that A is a sub- k -algebra of E , with function field E .

Remark 1.9.2.2. Since k is perfect, such a smooth model always exists.

Definition 1.9.2.3. Let E/k and L/k be two extensions and $\varphi : E \rightarrow L$ a morphism such that the extension L/E is finite. We call *k-model of L/E* any triplet $((X, x), (Y, y), f : Y \rightarrow X)$ such that (X, x) is a model of E/k , (Y, y) is a model of L/k and f is a dominant finite morphism making the following diagram commutative:

$$\begin{array}{ccc} \mathrm{Spec} L & \xrightarrow{\mathrm{Spec} \varphi} & \mathrm{Spec} E \\ y \downarrow & & \downarrow x \\ Y & \xrightarrow{f} & X \end{array}$$

where the vertical maps are induced by the points x and y .

Fortunately, such models always exist as the following lemma shows.

Lemma 1.9.2.4. *Let E/k be an extension and E/L a finite extension of fields. Then there exists a k -model of L/E .*

Proof. Let X' be a model of E and consider \tilde{X}' the normalization of X' in L/E . Then \tilde{X}' is of finite type over k , \tilde{X}' is equipped with an L -point that induces an isomorphism $\kappa(\tilde{X}') \rightarrow L$, and we have a canonical finite morphism $f : \tilde{X}' \rightarrow X'$ which is compatible with the canonical points of these schemes. Moreover, \tilde{X}' is generically smooth. Let U be an open smooth subscheme of \tilde{X}' , we consider $X = X' - f(\tilde{X}' - U)$, $Y = f^{-1}(X)$ and $f|_Y : Y \rightarrow X$. We conclude thanks to Lemma 1.9.2.5 below. \square

Lemma 1.9.2.5. *Let $f : Y \rightarrow X$ be an equidimensional finite morphism of schemes. If U is a dense open subscheme of Y , then the open subscheme $f^{-1}(X - f(Y - U))$ is dense containing U .*

Proof. The set $Z = Y - U$ is closed of codimension greater than 1 (since U is dense). In particular, $f(Z)$ is closed of codimension greater than 1 in X (since f is finite). Thus $X - f(Z)$ is open in X and dense in $f(Y)$. Hence the result. \square

The following lemma is sometimes used to prove that some constructions do not depend on the choice of models.

Lemma 1.9.2.6. *Let E/k be an extension and L/E a finite extension. Consider $f : Y \rightarrow X$ and $f' : Y' \rightarrow X'$ two k -models of L/E . Hence there is a k -model $f'' : Y'' \rightarrow X''$ of L/E such that the diagram*

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ Y'' & \xrightarrow{f''} & X'' \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

is commutative and compatible with the base points.

Proof. Without loss a generality, we study the affine case and put $X = \mathrm{Spec} A, X' = \mathrm{Spec} A', Y = \mathrm{Spec} B, Y' = \mathrm{Spec} B'$. Since $Y \rightarrow X$ and $Y' \rightarrow X'$ are models of L/E , the canonical isomorphisms

$$\begin{array}{ccc} \kappa(X) & \longrightarrow & \kappa(Y) \\ \sim \downarrow & & \downarrow \sim \\ E & \longrightarrow & L \\ \sim \uparrow & & \uparrow \sim \\ \kappa(X') & \longrightarrow & \kappa(Y') \end{array}$$

are compatible. We may assume that $A, A' \subset E$ and $B, B' \subset L$. Consider the sub- k -algebra of finite type $A'' = k[A \cup A'] \subset E$ and $B'' = k[B \cup B'] \subset L$. Any element of $B \cup B'$ is integral over A'' , thus B'' is integral over A'' . In particular, the normal closure of A'' in L contains B'' , hence B and B' . Denote by Y'' its spectrum and $X'' = \text{Spec } B''$ and assume these schemes are smooth. Hence we have a dominant finite morphism $Y'' \rightarrow X''$ and the result. \square

2

Milnor-Witt cycle modules

We generalize Rost's theory of cycle modules [Ros96] using the Milnor-Witt K-theory instead of the classical Milnor K-theory. We obtain a (quadratic) setting to study general cycle complexes and their (co)homology groups. The standard constructions are developed: proper pushforward, (essentially) smooth pullback, long exact sequences, spectral sequences and products, as well as the homotopy invariance property; in addition, Gysin morphisms for lci morphisms are constructed. We prove an adjunction theorem linking our theory to Rost's. This work extends Schmid's thesis [Sch98].

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2.1 Introduction

2.1.1 Historical approach

Back in the nineties, Rost developed the theory of cycle modules [Ros96] which was used in the famous proof of Milnor conjecture by Voevodsky (see [SV00] or [Voe11]). Indeed, a crucial step in the proof was the construction of the motive associated to some quadratic forms [Ros98]. In [Ros96], a cycle module M over a perfect field k is the data of a \mathbb{Z} -graded abelian group $M(E)$ for every finitely generated field extension E/k , equipped with restriction maps, (finite) corestriction maps, a Milnor K-theory module action and residue maps ∂ . These data are subject to some axioms $(r1a), \dots, (r3e)$ so that Milnor K-theory groups form a Rost cycle module. From the formal definitions, Rost produced a general theory that encompassed previous ones (Quillen K-theory and étale cohomology, for instance). Moreover, Rost's theory is fundamentally linked with the theory of motives: cycle modules can be realized geometrically. This can be illustrated by the following theorem.

Theorem 1 (see [Dé03]). *Let k be a perfect field. The category of Rost cycle modules over k is equivalent to the heart of the category of Voevodsky's motives $\mathbf{DM}(k, \mathbb{Z})$ with respect to the homotopy t -structure.*

The idea of \mathbb{A}^1 -homotopy theory, due to Morel and Voevodsky, was to apply techniques from algebraic topology to the study of schemes (the affine line \mathbb{A}^1 playing the role of the unit interval $[0, 1]$). This idea gave rise to many important results and new categories, such as $\mathbf{DM}(k, \mathbb{Z})$ mentioned above and $\mathbf{SH}(k)$, the stable homotopy category [MV99].

In his study of $\mathbf{SH}(k)$, Morel (in joint work with Hopkins) defined for a field E the Milnor-Witt K-theory $\mathbf{K}_*^{\mathrm{MW}}(E)$ (see [Mor12, Definition 3.1]). This \mathbb{Z} -graded abelian group behaves in positive degrees like Milnor K-theory groups $\mathbf{K}_n^M(E)$, and in non-positive degrees like Grothendieck-Witt and Witt groups of quadratic forms $\mathrm{GW}(E)$ and $\mathrm{W}(E)$. The Milnor-Witt K-theory was used for solving some splitting problems for projective modules. For instance, generalizing ideas from the theory of Chow groups, one can use the Rost-Schmid complex of Morel to define the Chow-Witt groups $\widetilde{\mathrm{CH}}^*(X)$ for a smooth k -scheme X (recalled in 2.5.2.1), and the Euler class of a vector bundle of rank r over X (with a given trivialization of its determinant) as an element in $\widetilde{\mathrm{CH}}^r(X)$. When $X = \mathrm{Spec} A$ is a 2-dimensional smooth affine variety and $r = 2$, Barge and Morel proved in [BM00] that the Euler class associated to a projective module P of rank 2 over A vanishes if and only if $P \simeq P' \oplus A$ for some projective module P' .

2.1.2 Current work

In this chapter, we develop a conjectured¹ theory that studies general cycle complexes $C^*(X, M, \mathcal{V}_X)$ and their (co)homology groups $A^*(X, M, \mathcal{V}_X)$ (called *Chow-Witt groups with coefficients*) in a quadratic setting. The general coefficient systems M for these complexes are called Milnor-Witt cycle modules. The main example of such a cycle module is given by the Milnor-Witt K-theory (see Theorem 2.3.2.13); other examples can be deduced from

¹See [Ros96, Remark 2.6] and [Mor12, Remark 5.37].

Claim 3.4.0.4, Example 2.3.2.7 or Theorem 2.8.2.5 (e.g. the representability of hermitian K-theory in $\mathbf{SH}(k)$) will lead to a MW-cycle module, associated with hermitian K-theory). A major difference with Rost's theory is that the grading to be considered is not \mathbb{Z} but the category of virtual bundles (or, equivalently, the category of virtual vector spaces), where a virtual bundle \mathcal{V} is, roughly speaking, the data of an integer n and a line bundle \mathcal{L} (see Appendix 1.9.1).

Intuitively, Milnor-Witt cycle modules are given by (twisted) graded abelian groups equipped with extra data (restriction, corestriction, \underline{K}^{MW} -action and residue maps). The difficult part was to find good axioms mimicking Rost's and taking into account the twists naturally arising in the non-oriented setting (virtual vector bundles play a major role). One important difference with the construction of Rost cycle modules [Ros96, Definition 1.1] is that we only need a weakened rule (R1c): we do not consider multiplicities (we will show in future work that formulas involving multiplicities do hold as a consequence of our axioms). Moreover, we had to add another rule in our definition in order to link our theory with the classical one later (see (R4a) and Section 2.8.2).

Following [Ros96, §3], we define the necessary operations needed later on: pushforwards, pullbacks, multiplication with units and the motivic Hopf map η , and boundary maps. Since our rules do not handle multiplicities, we first define pullbacks only for (essentially) smooth morphisms. We prove the usual functoriality theorems and show how to compute the (co)homology groups in special cases.

We can summarize the results as follows:

Theorem 2 (See Section 2.4.1 and Section 2.4.2). *For any scheme X , any virtual bundle \mathcal{V}_X and any Milnor-Witt cycle module M , there is a complex $C^*(X, M, \mathcal{V}_X)$ equipped with pushforwards, pullbacks, a Milnor-Witt K-theory action and residue maps satisfying the standard functoriality properties.*

This formalism generalizes Schmid's thesis [Sch98] and provides more details to Morel's work [Mor12], regarding twists by virtual vector bundles.

An example of an application is as follows. Consider the localization sequence:

$$\widetilde{\mathrm{CH}}_p(Z) \rightarrow \widetilde{\mathrm{CH}}_p(X) \rightarrow \widetilde{\mathrm{CH}}_p(X \setminus Z),$$

where $Z \subset X$ is a closed subvariety. Milnor-Witt cycle modules can be used to extend this exact sequence on the left and right, this is done in Section 2.5.1.

In Section 2.6.1, we prove a result similar to the Gersten Conjecture: the cohomology groups $A^p(X, M, \mathcal{V}_X)$ are trivial when X is a smooth semi-local scheme, \mathcal{V}_X a virtual bundle over X and $p > 0$. The proof is akin to Rost's Theorem 6.1 [Ros96] and follows the classical ideas of Gabber and Panin.

Moreover, we prove the much expected homotopy invariance property, framing our work in \mathbb{A}^1 -homotopy theory:

Theorem 3 (see Theorem 2.6.2.4). *Let X be a scheme, V a vector bundle over X , $\pi : V \rightarrow X$ the canonical projection and \mathcal{V}_X a virtual vector bundle over X . Then, for every $q \in \mathbb{Z}$, the canonical morphism*

$$\pi^* : A^q(X, M, \mathcal{V}_X) \rightarrow A^q(V, M, -\mathcal{T}_{V/X} + \mathcal{V}_V)$$

is an isomorphism.

Our proof of this theorem does not follow Rost's ideas. Indeed, in his original paper [Ros96, §7], Rost constructs a cycle module $A^q[\rho, M]$ from any cycle module M and any scheme morphism $\rho : Q \rightarrow B$, then uses a spectral sequence involving this new cycle module to prove the theorem. We could not easily extend the construction to our setting for some reasons related to twists. However, following ideas of Déglise (see [Dé14b, §2]), we define a coniveau spectral sequence that helps us to reduce to the known case (see (H)). Note that we could also prove the result by giving a homotopy inverse on the complex level (as Rost did in [Ros96, §8]; see also [Mor12, Theorem 4.38]).

In [Ros96, §12], Rost defines pullback maps f^* (or *Gysin morphisms*) for morphisms $f : X \rightarrow Y$ between smooth schemes. In Section 2.7, we do the same but for a more general set of maps, the classical local complete intersection (lci) morphisms, between possibly singular schemes. We prove functoriality and base change theorems which will prove to be useful later when considering multiplicities.

In Section 2.8.1, we construct products in our setting. Using the Gysin morphisms, we define then an intersection product on the homology groups of smooth schemes. In particular, we recover the intersection product already defined for Chow-Witt groups (see [Fas19, §3.4]). We will use these notions to study the multiplicities naturally arising. Note however that we do not define a tensor product between Milnor-Witt cycle modules (but it can be obtained from Theorem 3.4.0.4).

Outline of the chapter

In Section 1.4, we recall some known properties of the Milnor-Witt K-theory (residue maps, specialization maps, transfer maps, homotopy invariance). Our main references are the original work of Morel [Mor12, §2] and also work of Calmès-Fasel (see [Fas19]). This will constitute our central example of a Milnor-Witt cycle module.

In Section 2.3.1 and Section 2.3.2, we define the main objects of our theory: Milnor-Witt cycle modules. These are Milnor-Witt cycle premodules satisfying two axioms (FD) and (C) which allow the associated complexes to have well-defined differentials.

In Section 2.4.1 and Section 2.4.2, we define the basic operations needed further on: pushforwards, pullbacks, multiplication by units and the motivic Hopf map η , and boundary maps. We prove the basic compatibility properties for these operations.

In Section 2.5.1, we define homology groups called *Chow-Witt groups with coefficients* and describe how they are related to the classical Chow-Witt groups. We show how to compute these groups in some special cases in Section 2.6.1 and Section 2.6.2.

In Section 2.7, we define Gysin pullbacks for regular closed immersions and lci morphisms. We then proceed to construct products on the level of complexes in Section 2.8.1.

In Section 2.8.2, we define a pair of functors between our category of Milnor-Witt cycle modules and Rost's category of classical cycle modules. We can infer from Theorem 3.4.0.4 that this is an adjunction but, for completeness, we give an elementary proof of this fact (conditionally to Theorem 2.3.1.10 whose proof is postponed to Chapter 4).

Notation

Throughout the chapter, we fix a (commutative) field k and we assume moreover that k is perfect (of arbitrary characteristic). We consider only schemes that are noetherian and essentially of finite type² over k . All schemes and morphisms of schemes are defined over k .

By a field E over k , we mean a *finitely generated extension of fields* E/k .

We denote by \mathbb{F}_k the category of fields over k (with obvious morphisms).

If A is a commutative ring, denote by $\mathbb{V}(A)$ the category of projective A -modules of finite type and $\mathfrak{V}(A)$ the category of virtual projective A -modules of finite type (see [Del87, §4] which uses the notation $\underline{K}(A)$; see also Appendix 1.9.1 for more details). Recall that there is a *contravariant* equivalence functor from $\mathbb{V}(A)$ to the category of vector bundles over $X = \operatorname{Spec} A$ (we follow the conventions of [GD71, §9]). We will sometimes go from one category to the other without mentioning this functor.

Let $f : X \rightarrow S$ be a morphism of schemes and \mathcal{V}_S be a virtual bundle over S . We denote by \mathcal{V}_X or by $f^*\mathcal{V}_S$ or by $\mathcal{V}_S \times_S X$ the pullback of \mathcal{V}_S along f .

Now let \mathfrak{F}_k be the category whose objects are couples (E, \mathcal{V}_E) where E is a field over k and $\mathcal{V}_E \in \mathfrak{V}(E)$ is a virtual vector space (of finite dimension over E). A morphism $(E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ is the data of a morphism $E \rightarrow F$ of fields over k and an isomorphism $\mathcal{V}_E \otimes_E F \simeq \mathcal{V}_F$ of virtual F -vector spaces³.

A morphism $(E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ in \mathfrak{F}_k is said to be *finite* (resp. *separable*) if the field extension F/E is finite (resp. separable).

Let F/E be a field extension, we denote by $\Omega_{F/E}$ the F -vector space of relative (Kähler) differentials. We use the same notation to denote its canonical image in the category of virtual vector spaces. Dually, let X/S be a scheme morphism, we denote by $\mathcal{T}_{X/S}$ the sheaf of modules of differentials.

Let E be a field (over k) and v a (discrete) valuation on E . We denote by \mathcal{O}_v its valuation ring, by \mathfrak{m}_v its maximal ideal and by $\kappa(v)$ its residue field. We consider only valuations on E of *geometric type*, that is we assume: $k \subset \mathcal{O}_v$, the residue field $\kappa(v)$ is finitely generated over k and satisfies $\operatorname{tr. deg}_k(\kappa(v)) + 1 = \operatorname{tr. deg}_k(E)$ (in particular, E cannot be k).

Let E be a field and v be a valuation on E . We denote by \mathcal{N}_v the $\kappa(v)$ -vector space $\mathfrak{m}_v/\mathfrak{m}_v^2$ and call it the *normal bundle* of v .

For E a field (resp. X a scheme), denote by \mathbb{A}_E^1 (resp. \mathbb{A}_X^1) the virtual vector space of dimension 1 over E (resp. the virtual affine space of rank one over X).

²That is, isomorphic to a limit of finite type schemes with affine étale transition maps.

³This category satisfies a universal property over \mathbb{F}_k (one could generalize [Del87, §4] for fibred categories).

2.2 Milnor-Witt K-theory

2.2.1 Definitions

We describe the Milnor-Witt K-theory, as defined by Morel (see [Mor12, §3] or [Fas19, §1.1]).

Definition 2.2.1.1. Let E be a field. The Milnor-Witt K-theory algebra of E is defined to be the quotient of the free \mathbb{Z} -graded algebra generated by the symbols $[a]$ of degree 1 for any $a \in E^\times$ and a symbol η in degree -1 by the following relations:

- $[a][1-a] = 0$ for any $a \in E^\times \setminus \{1\}$.
- $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in E^\times$.
- $\eta[a] = [a]\eta$ for any $a \in E^\times$.
- $\eta(\eta[-1] + 2) = 0$.

The relations being homogeneous, the resultant algebra is \mathbb{Z} -graded. We denote it by $\mathbf{K}_*^{\text{MW}}(E)$.

2.2.1.2. NOTATION. We will use the following notations.

- $[a_1, \dots, a_n] = [a_1] \dots [a_n]$ for any $a_1, \dots, a_n \in E^\times$.
- $\langle a \rangle = 1 + \eta[a]$ for any $a \in E^\times$.
- $\varepsilon = -\langle -1 \rangle$.
- $n_\varepsilon = \sum_{i=1}^n \langle (-1)^{i-1} \rangle$ for any $n \geq 0$, and $n_\varepsilon = \varepsilon(-n)_\varepsilon$ if $n < 0$.

2.2.1.3. TWISTED MILNOR-WITT K-THEORY. Let E be a field and \mathcal{V}_E a virtual vector bundle over E with rank n and determinant \mathcal{L}_E (see Subsection 1.9.1). The group E^\times of invertible elements of E acts naturally on \mathcal{L}_E^\times , the set of non-zero elements in \mathcal{L}_E ; hence the free abelian group $\mathbb{Z}[\mathcal{L}_E^\times]$ is a $\mathbb{Z}[E^\times]$ -module. Define

$$\underline{\mathbf{K}}^{\text{MW}}(E, \mathcal{V}_E) = \mathbf{K}_n^{\text{MW}}(E) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\mathcal{L}_E^\times],$$

where the action of E^\times on $\mathbf{K}_n^{\text{MW}}(E)$ is given by $u \mapsto \langle u \rangle$.

By abuse of notation, we might denote by η the element $(\eta \otimes 1) \in \underline{\mathbf{K}}^{\text{MW}}(E, -\mathbb{A}^1)$ and by $[u]$ the element $([u] \otimes 1) \in \underline{\mathbf{K}}^{\text{MW}}(E, \mathbb{A}^1)$ (where $u \in E^\times$).

Let \mathcal{V}_E and \mathcal{W}_E be two virtual bundles over E . The product of the Milnor-Witt K-theory groups induces a product

$$\begin{aligned} \underline{\mathbf{K}}^{\text{MW}}(E, \mathcal{V}_E) \otimes \underline{\mathbf{K}}^{\text{MW}}(E, \mathcal{W}_E) &\rightarrow \underline{\mathbf{K}}^{\text{MW}}(E, \mathcal{V}_E + \mathcal{W}_E) \\ (x \otimes l, x' \otimes l') &\mapsto (xx') \otimes (l \wedge l'), \end{aligned}$$

so that $\underline{\mathbf{K}}^{\text{MW}}(E, -)$ is a lax monoidal functor from the category of virtual bundles over E to the category of abelian groups.

2.2.1.4. RESIDUE MORPHISMS (see [Mor12, Theorem 3.15]) Let E be a field endowed with a discrete valuation v . We choose a uniformizing parameter π . As in the classical Milnor K-theory, we can define a residue morphism

$$\partial_v^\pi : \mathbf{K}_*^{\text{MW}}(E) \rightarrow \mathbf{K}_{*-1}^{\text{MW}}(\kappa(v))$$

characterized by the fact that it commutes with the multiplication by η and satisfies the following two properties:

- $\partial_v^\pi([\pi, a_1, \dots, a_n]) = [\overline{a_1}, \dots, \overline{a_n}]$ for any $a_1, \dots, a_n \in \mathcal{O}_v^\times$.
- $\partial_v^\pi([a_1, \dots, a_n]) = 0$ for any $a_1, \dots, a_n \in \mathcal{O}_v^\times$.

This morphism does depend on the choice of π . Indeed, if we consider another uniformizer π' and write $\pi' = u\pi$ where u is a unit, then we have $\partial_v^\pi(x) = \langle u \rangle \partial_v^{\pi'}(x)$ for any $x \in \mathbf{K}_*^{\text{MW}}(E)$. Nevertheless, by making a choice of isomorphism $\mathcal{N}_v \simeq \mathbb{A}_{\kappa(v)}^1$ (where $\mathcal{N}_v = \mathfrak{m}_v / \mathfrak{m}_v^2$ is the normal cone of v), we can define a twisted residue morphism that does not depend on π :

$$\begin{aligned} \partial_v : \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) &\rightarrow \underline{\mathbf{K}}^{MW}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \\ x \otimes l &\mapsto \partial_v^\pi(x) \otimes (\bar{\pi}^* \wedge l) \end{aligned}$$

where \mathcal{V} is a virtual vector bundle over \mathcal{O}_v , $\bar{\pi}$ is the class of π modulo \mathfrak{m}_v and $\bar{\pi}^*$ its canonical associated linear form (see also Example 1.9.1.3).

We have the following proposition (see [Mor12, Lemma 3.19]).

Proposition 2.2.1.5. *Let $j : E \subset F$ be a field extension, and let w be a discrete valuation on F which restricts to a discrete valuation v on E with ramification index e . Let $i : \kappa(v) \rightarrow \kappa(w)$ be the field extension of the residue fields. Let \mathcal{V} be a virtual vector bundle over \mathcal{O}_v . Then, we have a commutative diagram*

$$\begin{array}{ccc} \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) & \xrightarrow{\partial_v} & \underline{\mathbf{K}}^{MW}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \\ j_* \downarrow & & \downarrow e_\varepsilon \cdot i_* \\ \underline{\mathbf{K}}^{MW}(F, \mathcal{V}_F) & \xrightarrow{\partial_w} & \underline{\mathbf{K}}^{MW}(\kappa(w), -\mathcal{N}_w + \mathcal{V}_{\kappa(w)}) \end{array}$$

where the right vertical map $e_\varepsilon \cdot i_*$ is defined by

$$x \otimes \bar{\rho}^* \otimes l \mapsto e_\varepsilon i(x) \otimes \bar{\pi}^* \otimes l$$

where π is a uniformizer for w and $\rho = \pi^e$ is a uniformizer for v .

2.2.1.6. SPECIALIZATION MAP. (see [Mor12, Lemma 3.16]) Keeping the previous notations, we also have a specialization map

$$s_v^\pi : \mathbf{K}_*^{\text{MW}}(E) \rightarrow \mathbf{K}_*^{\text{MW}}(\kappa(v))$$

commuting with the multiplication by η and satisfying

$$s_v^\pi([\pi^{m_1} a_1, \dots, \pi^{m_n} a_n]) = [\overline{a_1}, \dots, \overline{a_n}]$$

for any $a_1, \dots, a_n \in \mathcal{O}_v^\times$ and any integers m_1, \dots, m_n . If π' is another uniformizer, write $\pi' = \pi u$ where u is a unit. Then, for any $x \in \mathbf{K}_*^{\text{MW}}(E)$, we have $s_v^{\pi'}(x) = s_v^\pi(x) - [\bar{u}] \partial_v^{\pi'}(x)$.

It is related with the residue map as follows:

$$s_v^\pi(\alpha) = \langle -1 \rangle \partial_v^\pi([- \pi] \cdot \alpha)$$

for any $\alpha \in \mathbf{K}_*^{MW}(E)$. One can define a twisted specialization map (also denoted by s_v^π) as follows:

$$\begin{aligned} s_v^\pi : \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) &\rightarrow \underline{\mathbf{K}}^{MW}(\kappa(v), \mathcal{V}_{\kappa(v)}) \\ x \otimes l &\mapsto s_v^\pi(x) \otimes l. \end{aligned}$$

It satisfies:

$$s_v^\pi = \Theta_\pi \circ \partial_v \circ \gamma_{[-\pi]}$$

where $\gamma_{[-\pi]} : \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E) \rightarrow \underline{\mathbf{K}}^{MW}(E, \mathbb{A}_E^1 + \mathcal{V}_E)$ is the multiplication by $([-\pi] \otimes 1)$ and where Θ_π is induced by the composite of two isomorphisms $-\mathcal{N}_v + \mathbb{A}_E^1 \simeq \mathbb{A}_E^1 + (-\mathcal{N}_v) \simeq 0$ of virtual vector bundles (the first one is the canonical switch isomorphism; the second one comes from the choice of uniformizer π).

2.2.2 Homotopy invariance and transfers

2.2.2.1. THE HOMOTOPY EXACT SEQUENCE. Let E be a field with a virtual vector bundle \mathcal{V}_E , and let $E(t)$ be the function field of E in the variable t . Any point x of codimension 1 in \mathbb{A}_E^1 corresponds to a maximal ideal \mathfrak{m}_x in $E[t]$, generated by a unique irreducible monic polynomial π_x which is a uniformizing parameter of the \mathfrak{m}_x -adic valuation v_x on $E(t)$. Additionally, the valuation v_∞ at ∞ is given by $v_\infty(f/g) = \deg(g) - \deg(f)$ and we can choose (as in [Mor12, §4.2]) the rational function $-1/t$ as uniformizing parameter. For any $a \in E(t)^\times$, there is only a finite number of x such that $v_x(a) \neq 0$. Furthermore, since the Milnor-Witt K-theory of E is generated by elements of degree 1 (and η in degree -1), it follows that the total residue map

$$d : \underline{\mathbf{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \rightarrow \bigoplus_{x \in (\mathbb{A}_E^1)^{(1)}} \underline{\mathbf{K}}^{MW}(E(x), \Omega_{E(x)/k} + \mathcal{V}_{E(x)})$$

with $d = \sum_x \Theta_x \circ \partial_{v_x}$ is well-defined (where Θ_x comes from the canonical isomorphism $-\mathcal{N}_{v_x} + \Omega_{E(t)/k} \otimes_{E(t)} E(x) \simeq \Omega_{E(x)/k}$ of virtual vector bundles). The following theorem (see [Mor12, Theorem 3.24] for a non-twisted statement) is one of the most fundamental results of Milnor-Witt K-theory. Let

$$i_* : \underline{\mathbf{K}}^{MW}(E, \mathbb{A}_E^1 + \Omega_{E/k} + \mathcal{V}_E) \rightarrow \underline{\mathbf{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)})$$

be the morphism induced by the field extension $E \subset E(t)$ and the canonical isomorphism $\mathbb{A}_{E(t)}^1 + \Omega_{E/k} \otimes_{E(t)} E(t) \simeq \Omega_{E(t)}$ of virtual vector bundles.

Theorem 2.2.2.2 (Homotopy invariance). *With the previous notations, the following sequence is split exact*

$$0 \rightarrow \underline{\mathbf{K}}^{MW}(E, \mathbb{A}_E^1 + \Omega_{E/k} + \mathcal{V}_E) \xrightarrow{i_*} \underline{\mathbf{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \xrightarrow{d} \bigoplus_{x \in (\mathbb{A}_E^1)^{(1)}} \underline{\mathbf{K}}^{MW}(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_{\kappa(x)}) \rightarrow 0$$

Proof. See [Mor12, Theorem 3.24], [Fas19, Proposition 1.20]. \square

2.2.2.3. TRANSFER MAPS. Let $\varphi : E \rightarrow F$ be a monogenous finite field extension with \mathcal{V}_E a virtual vector bundle over E and choose $x \in F$ such that $F = E(x)$. The homotopy exact sequence implies that for any $\beta \in \underline{\mathbf{K}}^{MW}(F, \Omega_{F/k} + \mathcal{V}_F)$ there exists $\gamma \in \underline{\mathbf{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)})$ with the property that $d(\gamma) = \beta$. Now the valuation at ∞ yields a morphism

$$\partial_\infty^{-1/t} : \underline{\mathbb{K}}^{MW}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \rightarrow \underline{\mathbb{K}}^{MW}(E, \Omega_{E/k} + \mathcal{V}_E)$$

which vanishes on the image of i_* . We denote by $\varphi^*(\beta)$ or by $\mathrm{Tr}_E^F(\beta)$ the element $-\partial^{-1/t}(\gamma)$; it does not depend on the choice of γ . This defines a group morphism

$$\varphi^* : \underline{\mathbb{K}}^{MW}(F, \Omega_{F/k} + \mathcal{V}_F) \rightarrow \underline{\mathbb{K}}^{MW}(E, \Omega_{E/k} + \mathcal{V}_E)$$

called the *transfer map* and also denoted by Tr_E^F . The following result completely characterizes the transfer maps.

Lemma 2.2.2.4. *Keeping the previous notations, let*

$$d : \underline{\mathbb{K}}^{MW}(E(t), \Omega_{F(t)/k} + \mathcal{V}_{E(t)}) \rightarrow (\oplus_x \underline{\mathbb{K}}^{MW}(E(x), \Omega_{E(x)/k} + \mathcal{V}_{E(x)})) \oplus \underline{\mathbb{K}}^{MW}(E, \Omega_{E/k} + \mathcal{V}_E)$$

be the total twisted residue morphism (where x runs through the set of monic irreducible polynomials in $E(t)$). Then, the transfer maps $\mathrm{Tr}_E^{E(x)}$ are the unique morphisms such that $\sum_x (\mathrm{Tr}_E^{E(x)} \circ d_x) + d_\infty = 0$.

Proof. See [Mor12, §4.2]. □

Let $\varphi : E \rightarrow F$ be a finite field extension. We can find a factorization

$$E = F_0 \subset F_1 \subset \cdots \subset F_n = F$$

such that F_i/F_{i-1} is monogenous for any $i = 1, \dots, n$. We set $\mathrm{Tr}_E^F = \mathrm{Tr}_{F_1}^{F_1} \circ \cdots \circ \mathrm{Tr}_{F_{n-1}}^{F_{n-1}}$. Morel proved (see [Mor12, Theorem 4.27]) that the definition does not depend on the choice of the factorization (thanks to the twists).

The pullback Tr_E^F is functorial by definition. It satisfies the usual projection formula and the following base change theorem.

Proposition 2.2.2.5 (Base change). *Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ and $\psi : (E, \mathcal{V}_E) \rightarrow (L, \mathcal{V}_L)$ with φ finite and ψ separable. Let R be the ring $F \otimes_E L$. For each $p \in \mathrm{Spec} R$, let $\varphi_p : (L, \mathcal{V}_L) \rightarrow (R/p, \mathcal{V}_{R/p})$ and $\psi_p : (F, \mathcal{V}_F) \rightarrow (R/p, \mathcal{V}_{R/p})$ be the morphisms induced by φ and ψ . One has :*

$$\psi_* \circ \varphi^* = \sum_{p \in \mathrm{Spec} R} (\varphi_p)^* \circ \Theta_p \circ (\psi_p)_*$$

where Θ_p is induced by the canonical isomorphism $\Omega_{F/E} \otimes_F (R/p) \simeq \Omega_{(R/p)/L}$ (since ψ is separable).

Proof. This follows from general base change theorem on the Milnor-Witt K-theory (see [Fas08, Corollaire 12.3.7]). □

Finally, transfer maps are compatible with residue maps in the following sense.

Proposition 2.2.2.6. *Let $E \rightarrow F$ be a finite extension of fields over k , let v be a valuation on E and let \mathcal{V} be a virtual \mathcal{O}_v -module. For each extension w of v , we denote by $\varphi_w : (\kappa(v), \mathcal{V}_{\kappa(v)}) \rightarrow (\kappa(w), \mathcal{V}_{\kappa(w)})$ the morphism induced by $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$. We have:*

$$\partial_v \circ \varphi^* = \sum_w \varphi_w^* \circ \partial_w.$$

Proof. This follows from a more general result of Morel: transfer maps induce morphisms of the corresponding Rost-Schmid complexes [Mor12, §5].⁴ □

⁴The reader looking for a more elementary proof of the proposition may follow the ideas of [Sus80].

2.3 Milnor-Witt cycle modules

2.3.1 Milnor-Witt cycle premodules

We now define the main object of this chapter. See the subsequent remarks for more details.

Definition 2.3.1.1. A Milnor-Witt cycle premodule M (also written: MW-cycle premodule) is a functor from \mathfrak{F}_k to the category \mathbf{Ab} of abelian groups with the following data (D1), ..., (D4) and the following rules (R1a), ..., (R4a).

D1 Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k . The functor M gives a morphism $\varphi_* : M(E, \mathcal{V}_E) \rightarrow M(F, \mathcal{V}_F)$.

D2 Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k where the morphism $E \rightarrow F$ is *finite*. There is a morphism $\varphi^* : M(F, \Omega_{F/k} + \mathcal{V}_F) \rightarrow M(E, \Omega_{E/k} + \mathcal{V}_E)$.

D3 Let (E, \mathcal{V}_E) and (E, \mathcal{W}_E) be two objects of \mathfrak{F}_k . For any element x of $\underline{K}^{MW}(E, \mathcal{W}_E)$, there is a morphism

$$\gamma_x : M(E, \mathcal{V}_E) \rightarrow M(E, \mathcal{W}_E + \mathcal{V}_E)$$

so that the functor $M(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ is a left module over the lax monoidal functor $\underline{K}^{MW}(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ (see [Yet03, Definition 39]; see also remarks below).

D4 Let E be a field over k , let v be a valuation on E and let \mathcal{V} be a virtual projective \mathcal{O}_v -module of finite type. Denote by $\mathcal{V}_E = \mathcal{V} \otimes_{\mathcal{O}_v} E$ and $\mathcal{V}_{\kappa(v)} = \mathcal{V} \otimes_{\mathcal{O}_v} \kappa(v)$. There is a morphism

$$\partial_v : M(E, \mathcal{V}_E) \rightarrow M(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}).$$

R1a Let φ and ψ be two composable morphisms in \mathfrak{F}_k . One has

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

R1b Let φ and ψ be two composable finite morphisms in \mathfrak{F}_k . One has

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

R1c Consider $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ and $\psi : (E, \mathcal{V}_E) \rightarrow (L, \mathcal{V}_L)$ with φ finite and ψ separable. Let R be the ring $F \otimes_E L$. For each $p \in \text{Spec } R$, let $\varphi_p : (L, \mathcal{V}_L) \rightarrow (R/p, \mathcal{V}_{R/p})$ and $\psi_p : (F, \mathcal{V}_F) \rightarrow (R/p, \mathcal{V}_{R/p})$ be the morphisms induced by φ and ψ . One has

$$\psi_* \circ \varphi^* = \sum_{p \in \text{Spec } R} (\varphi_p)^* \circ (\psi_p)_*.$$

R2 Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k , let x be in $\underline{K}^{MW}(E, \mathcal{W}_E)$ and y be in $\underline{K}^{MW}(F, \Omega_{F/k} + \mathcal{W}'_F)$ where (E, \mathcal{W}_E) and (F, \mathcal{W}'_F) are two objects of \mathfrak{F}_k .

R2a We have $\varphi_* \circ \gamma_x = \gamma_{\varphi_*(x)} \circ \varphi_*$.

R2b Suppose φ finite. We have $\varphi^* \circ \gamma_{\varphi_*(x)} = \gamma_x \circ \varphi^*$.

R2c Suppose φ finite. We have $\varphi^* \circ \gamma_y \circ \varphi_* = \gamma_{\varphi^*(y)}$.

R3a Let $E \rightarrow F$ be a field extension and w be a valuation on F which restricts to a non trivial valuation v on E with ramification e . Let \mathcal{V} be a virtual \mathcal{O}_v -module so that we have a morphism $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ which induces a morphism $\overline{\varphi} : (\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \rightarrow (\kappa(w), -\mathcal{N}_w + \mathcal{V}_{\kappa(w)})$. We have

$$\partial_w \circ \varphi_* = \gamma_{e_e} \circ \overline{\varphi}_* \circ \partial_v.$$

R3b Let $E \rightarrow F$ be a finite extension of fields over k , let v be a valuation on E and let \mathcal{V} be a \mathcal{O}_v -module. For each extension w of v , we denote by $\varphi_w : (\kappa(v), \mathcal{V}_{\kappa(v)}) \rightarrow (\kappa(w), \mathcal{V}_{\kappa(w)})$ the morphism induced by $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$. We have

$$\partial_v \circ \varphi^* = \sum_w (\varphi_w)^* \circ \partial_w.$$

R3c Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k and let w be a valuation on F which restricts to the trivial valuation on E . Then

$$\partial_w \circ \varphi_* = 0.$$

R3d Let φ and w be as in (R3c), and let $\overline{\varphi} : (E, \mathcal{V}_E) \rightarrow (\kappa(w), \mathcal{V}_{\kappa(w)})$ be the induced morphism. For any uniformizer π of v , we have

$$\partial_w \circ \gamma_{[-\pi]} \circ \varphi_* = \overline{\varphi}_*.$$

R3e Let E be a field over k , v be a valuation on E and u be a unit of v . Then

$$\begin{aligned} \partial_v \circ \gamma_{[u]} &= \gamma_{\varepsilon[\overline{u}]} \circ \partial_v \text{ and} \\ \partial_v \circ \gamma_\eta &= \gamma_\eta \circ \partial_v. \end{aligned}$$

R4a Let $(E, \mathcal{V}_E) \in \mathfrak{F}_k$ and let Θ be an endomorphism of (E, \mathcal{V}_E) (that is, an automorphism of \mathcal{V}_E). Denote by Δ the canonical map⁵ from the group of automorphisms of \mathcal{V}_E to the group $\mathbf{K}^{\text{MW}}(E, 0)$. Then

$$\Theta_* = \gamma_{\Delta(\Theta)} : M(E, \mathcal{V}_E) \rightarrow M(E, \mathcal{V}_E).$$

Here is a series of remarks about our definition.

2.3.1.2. One could expand the definition and work with coefficients in an arbitrary abelian category instead of **Ab**.

2.3.1.3. The maps φ_* and φ^* are respectively called the restriction and corestriction morphisms. We also use the notations $\varphi_* = \text{res}_{F/E}$ and $\varphi^* = \text{cores}_{F/E}$.

The map ∂_v is called the residue map.

2.3.1.4. (D1) and (R1a) are redundant, given the fact that M is a (covariant) functor.

⁵See 2.3.1.17 for more details.

2.3.1.5. Intuitively, the data (D3) state that we have a left $\bigoplus_{\mathcal{W}_E \in \mathfrak{V}(E)} \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$ -module structure on the abelian group

$$\bigoplus_{\mathcal{V}_E \in \mathfrak{V}(E)} M(E, \mathcal{V}_E).$$

This statement is inaccurate for the following reasons. First, one should avoid the use of infinite direct sums (our theory ought to work if we replace **Ab** by any abelian category). Moreover, we have implied that

$$\bigoplus_{\mathcal{W}_E \in \mathfrak{V}(E)} \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$$

is a ring which is, rigorously speaking, not true. Among other reasons, the multiplicative structure on this abelian group does not have an identity element because we do not have strict equality between a virtual vector bundle \mathcal{W}_E and $0 + \mathcal{W}_E$ (there is only a canonical isomorphism). Instead, we should say that $\underline{\mathbf{K}}^{MW}(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ is a lax monoidal functor (see [Yet03, Definition 5]) and that $M(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ is a left module over $\underline{\mathbf{K}}^{MW}(E, -)$ (see [Yet03, Definition 39]).

Finally, we add that the functor $\underline{\mathbf{K}}^{MW}(E, -)$ is skewed or antisymmetric in the sense that for any virtual vector bundles \mathcal{V}_E and \mathcal{W}_E of rank m, n (respectively), and for any $x \in \underline{\mathbf{K}}^{MW}(E, \mathcal{V}_E)$ and $y \in \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$, we have $xy = (-1)^{nm} \Theta(yx)$ where Θ is induced by the canonical switch isomorphism $\mathcal{W}_E + \mathcal{V}_E \simeq \mathcal{V}_E + \mathcal{W}_E$. Intuitively, the switch isomorphism Θ corresponds to multiplication by $\langle -1 \rangle^{nm} = (-\varepsilon)^{nm}$ hence we say that the abelian group equipped with the obvious multiplication maps

$$\bigoplus_{\mathcal{W}_E \in \mathfrak{V}(E)} \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$$

is ε -commutative.

2.3.1.6. A MW-cycle premodule M can be equipped with a right $\underline{\mathbf{K}}^{MW}$ -module action as follows. For $x \in \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$, define

$$\rho_x : M(E, \mathcal{V}_E) \rightarrow M(E, \mathcal{V}_E + \mathcal{W}_E)$$

to be the composite $(-1)^{nm} \Theta \circ \gamma_x$ where Θ is induced by the canonical switch isomorphism $\mathcal{V}_E + \mathcal{W}_E \simeq \mathcal{W}_E + \mathcal{V}_E$ and where m, n are the ranks of $\mathcal{W}_E, \mathcal{V}_E$, respectively.

2.3.1.7. With the notations of (D4), note that we have a short exact sequence of $\kappa(v)$ -vector space

$$0 \rightarrow \mathcal{N}_v \rightarrow \Omega_{\mathcal{O}_v/k} \otimes_{\mathcal{O}_v} \kappa(v) \rightarrow \Omega_{\kappa(v)/k} \rightarrow 0$$

and that we have a canonical isomorphism

$$\Omega_{\mathcal{O}_v/k} \otimes_{\mathcal{O}_v} \kappa(v) \simeq \Omega_{E/k} \otimes_{\mathcal{O}_v} \kappa(v)$$

so that the data (D4) is equivalent to having a map

$$\delta_v : M(E, \Omega_{E/k} + \mathcal{V}_E) \rightarrow M(\kappa(v), \Omega_{\kappa(v)/k} + \mathcal{V}_{\kappa(v)}).$$

2.3.1.8. We assume that (R1b) also contains the rule $\varphi^* = \text{Id}$ when φ is the identity map.

2.3.1.9. In (R1c), the fact that the extension is separable means that there are no multiplicities to consider. This rule is weaker than the corresponding one used by Rost (see [Ros96, Definition 1.1.(R1c)]) but it is sufficient for our needs. Indeed, we can prove the following theorem (see also Theorem 4.4.1.16 and Theorem 3.2.3.9).

Theorem 2.3.1.10 (Strong R1c). *Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ and $\psi : (E, \mathcal{V}_E) \rightarrow (L, \mathcal{V}_L)$ with φ finite. Let R be the ring $F \otimes_E L$. For each $p \in \operatorname{Spec} R$, let $\varphi_p : (L, \mathcal{V}_L) \rightarrow (R/p, \mathcal{V}_{R/p})$ and $\psi_p : (F, \mathcal{V}_F) \rightarrow (R/p, \mathcal{V}_{R/p})$ be the morphisms induced by φ and ψ . One has*

$$\psi_* \circ \varphi^* = \sum_{p \in \operatorname{Spec} R} m_p \cdot (\varphi_p)^* \circ (\psi_p)_*$$

where m_p are some quadratic forms.

One important property is that the rank of m_p is the usual multiplicity as defined in [Ros96, §1] (that is, the length of the localized ring $R_{(p)}$).

2.3.1.11. In the rules (R2a), (R2b) and (R2c), $\varphi_*(x)$ and $\varphi^*(y)$ are defined as in 1.4.2.3. Moreover, these three rules are sometimes called *projection formulae* and could be illustrated with the following identities:

$$\begin{aligned} \varphi_*(x \cdot \rho) &= \varphi_*(x) \cdot \varphi_*(\rho), \\ \varphi^*(\varphi_*(x) \cdot \mu) &= x \cdot \varphi^*(\mu), \\ \varphi^*(y \cdot \varphi_*(\rho)) &= \varphi^*(y) \cdot \rho. \end{aligned}$$

Rigorously, we should write (R2b) as $\Theta \circ \varphi^* \circ \Theta' \circ \gamma_{\varphi_*(x)} = \gamma_x \circ \varphi^*$ where Θ, Θ' are induced by the canonical isomorphisms $\mathcal{W}_F + \Omega_{F/k} \simeq \Omega_{F/k} + \mathcal{W}_F$ and $\Omega_{E/k} + \mathcal{W}_E \simeq \mathcal{W}_E + \Omega_{E/k}$, respectively.

2.3.1.12. With the notation of (R3a), notice that we do have an isomorphism $\mathcal{N}_v \otimes_{\kappa(v)} \kappa(w) \simeq \mathcal{N}_w$ of virtual vector spaces defined as in 1.4.1.5. Moreover, we consider e_ε to be an element of $\underline{K}^{MW}(\kappa(w), 0)$. We may show that we could weaken this rule by considering only unramified extensions.

2.3.1.13. The rule (R3b) could be understood as

$$\delta_v \circ \varphi^* = \sum_w (\varphi_w)^* \circ \delta_w.$$

2.3.1.14. In the rule (R3d), the element $[-\pi]$ is considered to be in $\underline{K}^{MW}(F, \mathbb{A}_F^1)$ so that the map $\partial_w \circ \gamma_{[-\pi]} \circ \varphi_*$ takes value in $M(\kappa(w), -\mathcal{N}_w + \mathbb{A}_{\kappa(w)}^1 + \mathcal{V}_{\kappa(w)})$ (not in $M(\kappa(w), \mathcal{V}_{\kappa(w)})$, as needed). In order to fix the issue, we make (as in the previous Section) a choice of isomorphism between $\mathbb{A}_{\kappa(w)}^1$ and \mathcal{N}_w using the uniformizer π so that we can consider the composite of the following two⁶ isomorphisms $-\mathcal{N}_w + \mathbb{A}_{\kappa(w)}^1 \simeq \mathbb{A}_{\kappa(w)}^1 + (-\mathcal{N}_w) \simeq 0$. Thanks to the data (D1), this induces an isomorphism:

$$\Theta_\pi : M(\kappa(w), -\mathcal{N}_w + \mathbb{A}_{\kappa(w)}^1 + \mathcal{V}_{\kappa(w)}) \rightarrow M(\kappa(w), \mathcal{V}_{\kappa(w)}).$$

The attentive reader might be pleased to see the rule (R3d) stated as:

$$\Theta_\pi \circ \partial_w \circ \gamma_{[-\pi]} \circ \varphi_* = \overline{\varphi}_*.$$

Actually, we want more flexibility regarding the grading so that we may worry less about hidden isomorphisms. Indeed, keeping the previous notations, consider an isomorphism $\mathcal{W} \simeq \mathbb{A}_{\mathcal{O}_w}^1 + \mathcal{V}$ of virtual \mathcal{O}_w -modules. It induces (via (D1)) two group isomorphisms:

⁶Beware of the switch isomorphism $-\mathcal{N}_w + \mathbb{A}_{\kappa(w)}^1 \simeq \mathbb{A}_{\kappa(w)}^1 + (-\mathcal{N}_w)$ which corresponds (in some sense) to multiplication by $\langle -1 \rangle$.

$$\Theta_{\kappa(w)} : M(\kappa(w), -\mathcal{N}_v + \mathcal{W}_{\kappa(w)}) \rightarrow M(\kappa(w), -\mathcal{N}_v + \mathbb{A}_{\kappa(w)}^1 + \mathcal{V}_{\kappa(w)}) \text{ and}$$

$$\Theta_F : M(F, \mathcal{W}_F) \rightarrow M(F, \mathbb{A}_F^1 + \mathcal{V}_F).$$

The rule (R3d) should state that we have the identity:

$$\Theta_\pi \circ \Theta_{\kappa(w)}^{-1} \circ \partial_w \circ \Theta_F \circ \gamma_{[-\pi]} \circ \varphi_* = \overline{\varphi}_*.$$

In this chapter, we will encounter a number of similar cases where equality holds only up to a canonical isomorphism. It will always be an isomorphism Θ coming from the data (D1) and an isomorphism between virtual spaces.

Definition 2.3.1.15. With the previous notations, we denote by s_w^π the morphism

$$\Theta_\pi \circ \partial_w \circ \gamma_{[-\pi]} : M(F, \mathcal{V}_F) \rightarrow M(\kappa(w), \mathcal{V}_{\kappa(w)})$$

and we call it the *specialization map*.

2.3.1.16. A more rigorous way to state (R3e) would be to write:

$$\begin{aligned} \partial_v \circ \gamma_{[u] \otimes 1} &= -\Theta \circ \gamma_{[\bar{u}] \otimes 1} \circ \partial_v \text{ and} \\ \partial_v \circ \gamma_{\eta \otimes 1} &= -\Theta' \circ \gamma_{\eta \otimes 1} \circ \partial_v, \end{aligned}$$

where Θ (resp. Θ') is induced by the isomorphism $\mathbb{A}_F^1 + (-\mathcal{N}_v) \simeq -\mathcal{N}_v + \mathbb{A}_F^1$ (resp. $-\mathbb{A}_F^1 + (-\mathcal{N}_v) \simeq -\mathcal{N}_v + (-\mathbb{A}_F^1)$) of virtual vector bundles. As in the previous remark concerning (R3d), we actually want to be flexible regarding the grading and allow isomorphisms of virtual vector bundles in the axioms.

2.3.1.17. The rule (R4a) does not appear in the classical theory of [Ros96], obviously. We mainly need this rule to prove the adjunction theorem (see 2.8.2.5). Keeping the notations of this rule, we note that the automorphism $\Theta : \mathcal{V}_E \rightarrow \mathcal{V}_E$ of virtual vector spaces induces an automorphism $\det(\Theta) : \det(\mathcal{V}_E) \rightarrow \det(\mathcal{V}_E)$ (of dimension 1 vector spaces) which corresponds to a unit $\delta_\Theta \in F^\times$; by definition $\Delta(\Theta) = \langle \delta_\Theta \rangle$ (recall the notation defined in 1.4.1.2).

2.3.1.18. As in [Ros96, §1.1], the rule (R3e) implies:

R3f Let (E, \mathcal{V}_E) and (E, \mathcal{W}_E) be two objects in \mathfrak{F}_k , let v be a valuation on E and π be a uniformizer of v . For any $x \in \underline{\mathbb{K}}^{MW}(E, \mathcal{W}_E)$ and $\rho \in M(E, \mathcal{V}_E)$, we have

$$\begin{aligned} \partial_v(x \cdot \rho) &= \partial_v(x) \cdot s_v^\pi(\rho) + (-1)^n \Theta_s(s_v^\pi(x) \cdot \partial_v(\rho)) + (-1)^n \Theta_\pi([-1] \cdot \partial_v(x) \cdot \partial_v(\rho)), \\ s_v^\pi(x \cdot \rho) &= s_v^\pi(x) \cdot s_v^\pi(\rho), \end{aligned}$$

where $n = \text{rk}(\mathcal{W}_E)$, Θ_s is induced by the switch isomorphism $\mathcal{W}_E + (-\mathcal{N}_v) \simeq -\mathcal{N}_v + \mathcal{W}_E$ and Θ_π is induced by the composite isomorphism

$$\begin{aligned} (\mathbb{A}_{\kappa(v)}^1 + (-\mathcal{N}_v)) + \mathcal{W}_{\kappa(v)} + (-\mathcal{N}_v) + \mathcal{V}_{\kappa(v)} &\simeq 0 + \mathcal{W}_{\kappa(v)} + (-\mathcal{N}_v) + \mathcal{V}_{\kappa(v)} \simeq \\ &= -\mathcal{N}_v + \mathcal{W}_{\kappa(v)} + \mathcal{V}_{\kappa(v)}. \end{aligned}$$

2.3.1.19. With the previous notations, if π' is another uniformizer of v , put $\pi' = u\pi$. Then (by (R3e) and (R4a))

$$s_v^{\pi'}(x) = s_v^\pi(x) - \Theta_{\pi'}([\bar{u}] \cdot \partial_v(x)),$$

where $\Theta_{\pi'}$ is induced by the isomorphism $\mathbb{A}_{\kappa(v)}^1 + (-\mathcal{N}_v) \simeq 0$ defined by π' .

The following theorem follows from the results of Section 1.4.

Theorem 2.3.1.20. *The functor $\underline{\mathbf{K}}^{MW}$ is a MW-cycle premodule.*

Proof. The Milnor-Witt K-theory is indeed equipped with the data (D1) (see 1.4.1.3), (D2) (see 1.4.2.3), (D3) (see 1.4.1.3) and (D4) (see 1.4.1.4). Rule (R1a) and rule (R1b) are obvious from the definitions. Rule (R1c) is Theorem 1.4.2.5. The projection formulae (R2a), (R2b) and (R2c) are straightforward computations. Rule (R3a) is Theorem 1.4.1.5. Rule (R3b) is Theorem 1.4.2.6. Rule (R3c), rule (R3d) and rule (R3e) are straightforward computations (see [Mor12, Theorem 3.15 and Proposition 3.17]).

We prove Rule (R4a). Let $(E, \mathcal{V}_E) \in \mathfrak{F}_k$ and let Θ be an endomorphism of (E, \mathcal{V}_E) . The map Θ induces a morphism of dimension 1 vector spaces

$$\begin{aligned} \det(\mathcal{V}_E) &\rightarrow \det(\mathcal{V}_E) \\ l &\mapsto \delta_{\Theta} \cdot l \end{aligned}$$

where $\delta_{\Theta} \in E^{\times}$. We need to prove that

$$\Theta_* = \gamma_{\Delta(\Theta)} : \mathbf{K}_n^{MW}(E) \otimes_{\mathbb{Z}[E^{\times}]} \det(\mathcal{V}_E) \rightarrow \mathbf{K}_n^{MW}(E) \otimes_{\mathbb{Z}[E^{\times}]} \det(\mathcal{V}_E)$$

where n is the rank of \mathcal{V}_E and $\Delta(\Theta)$ is the quadratic form $\langle \delta_{\Theta} \rangle$. Let $x \in \mathbf{K}_n^{MW}(E)$ and $l \in \det(\mathcal{V}_E)$, we have $\Theta_*(x \otimes l) = x \otimes (\delta_{\Theta} \cdot l) = (x \cdot \langle \delta_{\Theta} \rangle) \otimes l = \gamma_{\Delta(\Theta)}(x \otimes l)$, hence the result. \square

Definition 2.3.1.21. A pairing $M \times M' \rightarrow M''$ of MW-cycle premodules over k is given by bilinear maps for each $(E, \mathcal{V}_E), (E, \mathcal{W}_E)$ in \mathfrak{F}_k

$$\begin{aligned} M(E, \mathcal{V}_E) \times M'(E, \mathcal{W}_E) &\rightarrow M''(E, \mathcal{V}_E + \mathcal{W}_E) \\ (\rho, \mu) &\rightarrow \rho \cdot \mu \end{aligned}$$

which respect the $\underline{\mathbf{K}}^{MW}$ -module structure and which have the properties P1, P2, P3 stated below.

P1 For $x \in \underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E), \rho \in M(E, \mathcal{V}_E), \mu \in M'(E, \mathcal{V}_E')$,

P1a $(x \cdot \rho) \cdot \mu = x \cdot (\rho \cdot \mu)$,

P1b $(\rho \cdot x) \cdot \mu = \rho \cdot (x \cdot \mu)$.

P2 Let $\varphi : E \rightarrow F$ (finite in P2b, P2c), $\lambda \in M(E, \mathcal{V}_E)$, $v \in M(F, \Omega_{F/k} + \mathcal{W}_F)$, $\rho \in M'(E, \mathcal{V}_E')$, $\mu \in M'(F, \Omega_{F/k} + \mathcal{W}_F')$,

P2a $\varphi_*(\lambda \cdot \rho) = \varphi_*(\lambda) \cdot \varphi_*(\rho)$,

P2b $\Theta(\varphi^*(\Theta'(\varphi_*(\lambda) \cdot \mu))) = \lambda \cdot \varphi^*(\mu)$ where Θ, Θ' are induced by the canonical isomorphisms $\mathcal{W}_F + \Omega_{F/k} \simeq \Omega_{F/k} + \mathcal{W}_F$ and $\Omega_{E/k} + \mathcal{W}_E \simeq \mathcal{W}_E + \Omega_{E/k}$, respectively,

P2c $\varphi^*(v \cdot \varphi_*(\rho)) = \varphi^*(v) \cdot \rho$.

P3 For a valuation v on E , $x \in M(E, \mathcal{V}_E)$, $\rho \in M'(E, \mathcal{W}_E)$ and a uniformizer π of v , one has

$$\partial_v(x \cdot \rho) = \partial_v(x) \cdot s_v^\pi(\rho) + (-1)^n \Theta_s(s_v^\pi(x) \cdot \partial_v(\rho)) + (-1)^n \Theta_\pi([-1] \cdot \partial_v(x) \cdot \partial_v(\rho))$$

where n is the rank of \mathcal{V}_E , where Θ_s is induced by the switch isomorphism

$$\mathcal{W}_E + (-\mathcal{N}_v) \simeq -\mathcal{N}_v + \mathcal{W}_E$$

and where Θ_π is induced by the composite isomorphism

$$\begin{aligned} (\mathbb{A}_{\kappa(v)}^1 + (-\mathcal{N}_v)) + \mathcal{W}_{\kappa(v)} + (-\mathcal{N}_v) + \mathcal{V}_{\kappa(v)} &\simeq 0 + \mathcal{W}_{\kappa(v)} + (-\mathcal{N}_v) + \mathcal{V}_{\kappa(v)} \simeq \\ &-\mathcal{N}_v + \mathcal{W}_{\kappa(v)} + \mathcal{V}_{\kappa(v)}. \end{aligned}$$

A ring structure on a MW-cycle premodule M is a pairing $M \times M \rightarrow M$ which induces on $\bigoplus_{\mathcal{V}_E \in \mathfrak{V}(E)} M(E, \mathcal{V}_E)$ an associative and ε -commutative ring structure (see also 2.3.1.5).

Example 2.3.1.22. By definition, a Milnor-Witt cycle premodule M comes equipped with a pairing $\underline{\mathbb{K}}^{MW} \times M \rightarrow M$. When $M = \underline{\mathbb{K}}^{MW}$, this defines a ring structure on M .

2.3.2 Milnor-Witt cycle modules

2.3.2.1. Throughout this section, M denotes a Milnor-Witt cycle premodule over k .

Let X be a scheme over k and \mathcal{V}_X be a virtual bundle over X .

For x in X the virtual bundle $\mathcal{V}_X \times_X \text{Spec } \kappa(x)$ over $\text{Spec } \kappa(x)$ corresponds to a virtual vector space \mathcal{V}_x over $\kappa(x)$ via the equivalence between vector bundles over $\text{Spec } \kappa(x)$ and $\kappa(x)$ -vector spaces.

Throughout this chapter, we write

$$M(x, \mathcal{V}_X) = M(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_x).$$

If X is irreducible, we write ξ_X or ξ for its generic point.

If X is normal, then for any $x \in X^{(1)}$ the local ring of X at x is a valuation ring so that (D4) gives us a map $\partial_x : M(\xi, \mathcal{V}_X) \rightarrow M(x, \mathcal{V}_X)$.

Now suppose X is an arbitrary scheme over k and let x, y be two points in X . We define a map

$$\partial_y^x : M(x, \mathcal{V}_X) \rightarrow M(y, \mathcal{V}_X)$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z^{(1)}$, then put $\partial_y^x = 0$. If $y \in Z^{(1)}$, let $\tilde{Z} \rightarrow Z$ be the normalization and put

$$\partial_y^x = \sum_{z|y} \text{cores}_{\kappa(z)/\kappa(y)} \circ \partial_z$$

with z running through the finitely many points of \tilde{Z} lying over y .

Definition 2.3.2.2. A Milnor-Witt cycle module M over k is a Milnor-Witt cycle premodule M which satisfies the following conditions (FD) and (C).

(FD) FINITE SUPPORT OF DIVISORS. Let X be a normal scheme, \mathcal{V}_X be a virtual vector bundle over X and ρ be an element of $M(\xi_X, \mathcal{V}_X)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

(C) CLOSEDNESS. Let X be integral and local of dimension 2 and \mathcal{V}_X be a virtual bundle over X . Then

$$0 = \sum_{x \in X^{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : M(\xi_X, \mathcal{V}_X) \rightarrow M(x_0, \mathcal{V}_X)$$

where ξ is the generic point and x_0 the closed point of X .

2.3.2.3. Of course (C) makes sense only under presence of (FD) which guarantees finiteness in the sum. More generally, note that if (FD) holds, then for any scheme X , any virtual bundle \mathcal{V}_X over X , any $x \in X$ and any $\rho \in M(x, \mathcal{V}_X)$ one has $\partial_y^x(\rho) = 0$ for all but finitely many $y \in X$.

2.3.2.4. If X is integral and (FD) holds for X , we put

$$d = (\partial_x^\xi)_{x \in X^{(1)}} : M(\xi, \mathcal{V}_X) \rightarrow \bigoplus_{x \in X^{(1)}} M(x, \mathcal{V}_X).$$

Definition 2.3.2.5. A morphism $\omega : M \rightarrow M'$ of Milnor-Witt cycle modules over k is a natural transformation which commutes⁷ with the data (D1), ..., (D4) (in other words, it is a left map in the sense of [Yet03, Definition 40]).

Remark 2.3.2.6. We denote by $\mathfrak{M}_k^{\text{MW}}$ the category of Milnor-Witt cycle modules (where arrows are given by morphisms of MW-cycle modules). This is an abelian category.

Example 2.3.2.7. Let M be a Milnor-Witt cycle module. For any integer n , we define a Milnor-Witt cycle module $M\{n\}$ by

$$M\{n\}(E, \mathcal{V}_E) = M(E, n \cdot \mathbb{A}_E^1 + \mathcal{V}_E)$$

for any (finitely generated) field E/k and any virtual bundle \mathcal{V}_E over E .

Now, we define a Milnor-Witt cycle module $M[\eta^{-1}]$ as follows. Let I be the category of finite ordinal numbers (objects are natural numbers n and arrows are given by the relation $n \leq m$). We consider the functor $F_M : I \rightarrow \mathfrak{M}_k^{\text{MW}}$ that takes a natural number n to $M\{-n\}$ and an arrow $n \leq m$ to the multiplication by η^{m-n} . By definition, the Milnor-Witt cycle module $M[\eta^{-1}]$ is the colimit of the diagram F_M . In particular, for $M = \underline{\mathbf{K}}^{\text{MW}}$ and for any field E , we obtain the following group isomorphism which respects the multiplication in some obvious sense:

$$\bigoplus_{n \in \mathbb{Z}} \underline{\mathbf{K}}^{\text{MW}}[\eta^{-1}](E, n \cdot \mathbb{A}_E^1) \simeq \mathbf{K}_*^{\text{MW}}(E)[\eta^{-1}].$$

The ring $\mathbf{K}_*^{\text{MW}}(E)[\eta^{-1}]$ is known to be isomorphic to $W(E)[\eta^{\pm 1}]$.

2.3.2.8. In the following, let F be a field over k , \mathcal{V}_F be a virtual bundle over $\text{Spec } F$ and $\mathbb{A}_F^1 = \text{Spec } F[t]$ be the affine line over $\text{Spec } F$ with function field $F(t)$.

Proposition 2.3.2.9. *Let M be a Milnor-Witt cycle module over k . With the previous notations, the following properties hold.*

(H) HOMOTOPY PROPERTY FOR \mathbb{A}^1 . *We have a short exact sequence*

⁷In particular, it commutes with multiplication by η (up to canonical isomorphisms). This fact will be important for Theorem 2.8.2.5.

$$0 \rightarrow M(F, \mathbb{A}_F^1 + \Omega_{F/k} + \mathcal{V}_F) \xrightarrow{\text{res}} M(F(t), \Omega_{F(t)/k} + \mathcal{V}_{F(t)}) \xrightarrow{d} \bigoplus_{x \in (\mathbb{A}_F^1)^{(1)}} M(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_{\kappa(x)}) \rightarrow 0$$

where the map d is defined in 2.3.2.4.

(RC) RECIPROCITY FOR CURVES. Let X be a proper curve over F and \mathcal{V}_F a virtual bundle⁸ over $\text{Spec } F$. Then

$$M(\xi_X, \mathcal{V}_X) \xrightarrow{d} \bigoplus_{x \in X^{(1)}} M(x, \mathcal{V}_X) \xrightarrow{c} M(F, \Omega_{F/k} + \mathcal{V}_F)$$

is a complex, that is $c \circ d = 0$ (where $c = \sum_x \text{cores}_{\kappa(x)/F}$).

2.3.2.10. Axiom **(FD)** enables one to write down the differential d of the soon-to-be-defined complex $C_*(X, M, \mathcal{V}_X)$, axiom **(C)** guarantees that $d \circ d = 0$, property **(H)** yields the homotopy invariance of the Chow groups $A_*(X, M, \mathcal{V}_X)$ and finally **(RC)** is needed to establish proper pushforward.

2.3.2.11. For an integral scheme X and \mathcal{V}_X a virtual bundle over X , we put

$$A^0(X, M, \mathcal{V}_X) = \ker d = \bigcap_{x \in X^{(1)}} \ker \partial_x^\xi \subset M(\xi_X, \mathcal{V}_X).$$

Theorem 2.3.2.12. Let M be a Milnor-Witt cycle premodule over a perfect field k . Then M is a cycle module if and only if the following properties **(FDL)** and **(WR)** hold for all fields F over k and all virtual F -vector space \mathcal{V}_F .

(FDL) FINITE SUPPORT OF DIVISORS ON THE LINE. Let $\rho \in M(F(t), \mathcal{V}_{F(t)})$. Then $\partial_v(\rho) = 0$ for all but finitely many valuations v of $F(t)$ over F .

(WR) WEAK RECIPROCITY. Let ∂_∞ be the residue map for the valuation of $F(t)/F$ at infinity. Then

$$\partial_\infty(A^0(\mathbb{A}_F^1, M, \mathcal{V}_{\mathbb{A}_F^1})) = 0.$$

The proofs of Proposition 2.3.2.9 and Theorem 2.3.2.12 are almost the same as Rost's (see [Ros96, Theorem 2.3]). Indeed, we can check that all involved twists match (up to canonical isomorphisms) and, of course, we use the identity $[ab] = [a] + [b] + \eta[a][b]$ every time Rost use $[ab] = [a] + [b]$ (this causes no harm to the proof since η commutes with our data).

Theorem 2.3.2.13. The Milnor-Witt K-theory \underline{K}^{MW} is a MW-cycle module.

Proof. We already know that it is a Milnor-Witt cycle premodule. For the two remaining axioms, it suffices to prove **(FD)** and **(H)** which are true (see Section §1.4.2.1 and Theorem 1.4.2.2; see also [Mor12, Theorem 3.24]). \square

⁸We also write \mathcal{V}_F for the corresponding virtual F -vector space.

2.4 The five basic maps

The purpose of this section is to introduce the cycle complexes and each operation on them needed further on. Note that the five basic maps defined below are analogous to those of Rost (see [Ros96, §3]); they are the basic foundations for the construction of more refined maps such as Gysin morphisms (see Section 2.7).

2.4.1 Definitions

2.4.1.1. Let M and N be two Milnor-Witt cycle modules over k , let X and Y be two schemes, let \mathcal{V}_X and \mathcal{V}_Y be two virtual bundles over X and Y respectively, and let $U \subset X$ and $V \subset Y$ be subsets. Given a morphism

$$\alpha : \bigoplus_{x \in U} M(x, \mathcal{V}_X) \rightarrow \bigoplus_{y \in V} M(y, \mathcal{V}_Y),$$

we write $\alpha_y^x : M(x, \mathcal{V}_X) \rightarrow M(y, \mathcal{V}_Y)$ for the components of α .

2.4.1.2. MILNOR-WITT CYCLE COMPLEXES. Let M be a Milnor-Witt cycle module, let X be a scheme, \mathcal{V}_X be a virtual bundle over X and p be an integer. Put $X_{(p)}$ the set of p -dimensional points of X . Define

$$C_p(X, M, \mathcal{V}_X) = \bigoplus_{x \in X_{(p)}} M(x, \mathcal{V}_X)$$

and

$$d : C_p(X, M, \mathcal{V}_X) \rightarrow C_{p-1}(X, M, \mathcal{V}_X)$$

where $d_y^x = \partial_y^x$ as in 3.3.1.1. This definition makes sense by axiom (FD).

Proposition 2.4.1.3. *With the previous notations, we have $d \circ d = 0$.*

Proof. Same as in [Ros96, §3.3]. Axiom (C) is needed. \square

Definition 2.4.1.4. The complex $(C_p(X, M, \mathcal{V}_X), d)_{p \geq 0}$ is called the *Milnor-Witt complex of cycles on X with coefficients in M* .

2.4.1.5. PUSHFORWARD. Let $f : X \rightarrow Y$ be a k -morphism of schemes, let \mathcal{V}_Y be a virtual bundle over Y and denote by \mathcal{V}_X its pullback along f . Define

$$f_* : C_p(X, M, \mathcal{V}_X) \rightarrow C_p(Y, M, \mathcal{V}_Y)$$

as follows. If $y = f(x)$ and if $\kappa(x)$ is finite over $\kappa(y)$, then $(f_*)_y^x = \text{cores}_{\kappa(x)/\kappa(y)}$. Otherwise, $(f_*)_y^x = 0$.

2.4.1.6. PULLBACK. Let $f : X \rightarrow Y$ be an *essentially smooth* morphism of schemes. Let \mathcal{V}_Y a virtual bundle over Y and \mathcal{V}_X be its pullback along f . Suppose X connected and denote by s the relative dimension of f . Define

$$f^* : C_p(Y, M, \mathcal{V}_Y) \rightarrow C_{p+s}(X, M, -\mathcal{T}_{X/Y} + \mathcal{V}_X)$$

as follows. If $f(x) = y$, then $(f^*)_x^y = \Theta \circ \text{res}_{\kappa(x)/\kappa(y)}$, where Θ is the canonical isomorphism induced by $\mathcal{T}_{\text{Spec } \kappa(x)/\text{Spec } \kappa(y)} \simeq \mathcal{T}_{X/Y} \times_X \text{Spec } \kappa(x)$. Otherwise, $(f^*)_x^y = 0$. If X is not connected, take the sum over each connected component.

Remark 2.4.1.7. The fact that the morphism f is (essentially) smooth implies that there are no multiplicities to consider. We do not consider the case of flat morphisms in this chapter (this can be done after studying the multiplicities mentioned in Theorem 2.3.1.10). However, we define Gysin morphisms for lci projective morphisms in Section 2.7.

2.4.1.8. MULTIPLICATION WITH UNITS. Let a_1, \dots, a_n be global units in \mathcal{O}_X^* , let \mathcal{V}_X be a virtual bundle. Define

$$[a_1, \dots, a_n] : C_p(X, M, \mathcal{V}_X) \rightarrow C_p(X, M, n \cdot \mathbb{A}_X^1 + \mathcal{V}_X)$$

as follows. Let x be in $X_{(p)}$ and $\rho \in M(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_x)$. We consider⁹ $\langle -1 \rangle^{np} [a_1(x), \dots, a_n(x)]$ as an element of $\underline{\mathbb{K}}^{MW}(\kappa(x), n \cdot \mathbb{A}_{\kappa(x)}^1)$. If $x = y$, then put $[a_1, \dots, a_n]_y^x(\rho) = \Theta(\langle -1 \rangle^{np} [a_1(x), \dots, a_n(x)] \cdot \rho)$ where Θ is induced by the canonical isomorphism $n \cdot \mathbb{A}_{\kappa(x)}^1 + \Omega_{\kappa(x)/k} \simeq \Omega_{\kappa(x)/k} + n \cdot \mathbb{A}_{\kappa(x)}^1$. Otherwise, put $[a_1, \dots, a_n]_y^x(\rho) = 0$.

2.4.1.9. MULTIPLICATION WITH η . Define

$$\eta : C_p(X, M, \mathcal{V}_X) \rightarrow C_p(X, M, -\mathbb{A}_X^1 + \mathcal{V}_X)$$

as follows. If $x = y$, then $\eta_y^x(\rho) = \gamma_\eta(\rho)$. Otherwise, $\eta_y^x(\rho) = 0$.

2.4.1.10. BOUNDARY MAPS. Let X be a scheme of finite type over k with a virtual bundle \mathcal{V}_X , let $i : Z \rightarrow X$ be a closed immersion and let $j : U = X \setminus Z \rightarrow X$ be the inclusion of the open complement. We will refer to (Z, i, X, j, U) as a boundary triple and define

$$\partial = \partial_Z^U : C_p(U, M, \mathcal{V}_U) \rightarrow C_{p-1}(Z, M, \mathcal{V}_Z)$$

by taking ∂_y^x to be as the definition in 3.3.1.1 with respect to X . The map ∂_Z^U is called the boundary map associated to the boundary triple, or just the boundary map for the closed immersion $i : Z \rightarrow X$.

2.4.1.11. GENERALIZED CORRESPONDENCES. We will use the notation

$$\alpha : [X, \mathcal{V}_X] \bullet \rightarrow [Y, \mathcal{V}_Y]$$

or simply

$$\alpha : X \bullet \rightarrow Y$$

to denote maps of complexes which are sums of composites of the five basic maps f_* , g^* , $[a]$, η , ∂ for schemes over k . Unlike Rost in [Ros96, §3], we look at these morphisms up to quasi-isomorphisms so that a morphism $\alpha : X \bullet \rightarrow Y$ may be a weak inverse of a well-defined morphism of complexes.

2.4.2 Compatibilities

In the following, we establish the basic compatibilities for the maps considered in the last section. Fix M a Milnor-Witt cycle module.

Proposition 2.4.2.1. *1. Let $f : X \rightarrow Y$ and $f' : Y \rightarrow Z$ be two morphisms of schemes. Then*

⁹Instead of multiplying by $\langle -1 \rangle$ and using (R4a), we could make a canonical choice of isomorphism with determinant (-1) and use (D1) (by (R4a), any choice will work).

$$(f' \circ f)_* = f'_* \circ f_*.$$

2. Let $g : Y \rightarrow X$ and $g' : Z \rightarrow Y$ be two essentially smooth morphisms. Then (up to the canonical isomorphism given by $\mathcal{T}_{Z/X} \simeq \mathcal{T}_{Z/Y} + (g')^* \mathcal{T}_{Y/Z}$):

$$(g \circ g')^* = g'^* \circ g^*.$$

3. Consider a pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

with f, f', g, g' as previously. Then

$$g^* \circ f_* = \Theta \circ f'_* \circ g'^*$$

where Θ is the canonical isomorphism induced by $\mathcal{T}_{U/Z} \simeq \mathcal{T}_{Y/X} \times_Y U$.

Proof. 1. This is clear from the definition and by (R1b).

2. The claim is trivial by (R1a) (again, there are no multiplicities).

3. This reduces to the rule (R1c) (see [Ros96, Proposition 4.1]).

□

Proposition 2.4.2.2. Let $f : Y \rightarrow X$ be a morphism of schemes. If a is a unit on X , then

$$f_* \circ [\tilde{f}^*(a)] = [a] \circ f_*$$

where $\tilde{f}^* : \mathcal{O}_X^* \rightarrow \mathcal{O}_Y^*$ is induced by f .

Proof. This comes from (R2b).

□

Proposition 2.4.2.3. Let a be a unit on a scheme X .

1. Let $g : Y \rightarrow X$ be an essentially smooth morphism. One has

$$g^* \circ [a] = [\tilde{g}^*(a)] \circ g^*.$$

2. Let (Z, i, X, j, U) be a boundary triple. One has

$$\partial_Z^U \circ [\tilde{j}^*(a)] = \varepsilon[\tilde{i}^*(a)] \circ \partial_Z^U.$$

Moreover,

$$\partial_Z^U \circ \boldsymbol{\eta} = \boldsymbol{\eta} \circ \partial_Z^U.$$

Proof. The first result comes from (R2a), the second from (R2b) and (R3e).

□

Proposition 2.4.2.4. *Let $h : X \rightarrow X'$ be a morphism of schemes. Let $Z' \hookrightarrow X'$ be a closed immersion. Consider the induced diagram given by $U' = X' \setminus Z'$ and pullback:*

$$\begin{array}{ccccc} Z & \hookrightarrow & X & \longleftarrow & U \\ \downarrow f & & \downarrow h & & \downarrow g \\ Z' & \hookrightarrow & X' & \longleftarrow & U' \end{array}$$

1. *If h is proper, then*

$$f_* \circ \partial_Z^U = \partial_{Z'}^{U'} \circ g_*.$$

2. *If h is essentially smooth, then*

$$f^* \circ \partial_{Z'}^{U'} = \partial_Z^U \circ g^*.$$

Proof. This will follow from Proposition 2.4.2.6. □

Lemma 2.4.2.5. *Let $g : Y \rightarrow X$ be a smooth morphism of schemes of finite type over a field of constant fiber dimension 1, let $\sigma : X \rightarrow Y$ be a section of g and let $t \in \mathcal{O}_Y$ be a global parameter defining the subscheme $\sigma(X)$. Moreover, let $\tilde{g} : U \rightarrow X$ be the restriction of g where $U = Y \setminus \sigma(X)$ and let ∂ be the boundary map associated to σ . Then*

$$\Theta \circ \partial \circ [t] \circ \tilde{g}^* = (\text{id}_X)_* \text{ and } \partial \circ \tilde{g}^* = 0,$$

with Θ the canonical isomorphism given by $\mathcal{T}_{U/X} \simeq \mathbb{A}_U^1$.

Proof. In order to simplify the notations, we identify X with $\sigma(X)$ through σ and we forget the canonical isomorphism Θ . By definition, the map $\partial \circ [t] \circ \tilde{g}^*$ is

$$\sum_{x \in X} \sum_{y \in g^{-1}(x) \setminus \{x\}} \partial_x^y \circ [t] \circ \text{res}_{\kappa(y)/\kappa(x)} : \bigoplus_{x \in X} M(x, \mathcal{V}_X) \rightarrow \bigoplus_{x \in X} M(x, \mathcal{V}_X).$$

Fix $x \in X$ and let $y \in g^{-1}(x) \setminus \{x\}$. If $y = \xi_x$ is the generic point of $g^{-1}(x)$, then we have

$$\partial_x^{\xi_x} \circ [t] \circ \text{res}_{\kappa(\xi_x)/\kappa(x)} = \text{res}_{\kappa(x)/\kappa(x)} = \text{Id}$$

according to (R3d). Otherwise

$$\partial_x^y \circ [t] \circ \text{res}_{\kappa(y)/\kappa(x)} = \varepsilon \cdot [t] \cdot \partial_x^y \circ \text{res}_{\kappa(y)/\kappa(x)} = 0$$

according to (R3e) and (R3c). The second equality is proved in a similar fashion. □

Proposition 2.4.2.6. 1. *Let $f : X \rightarrow Y$ be a proper morphism of schemes. Then*

$$d_Y \circ f_* = f_* \circ d_X.$$

2. *Let $g : Y \rightarrow X$ be an essentially smooth morphism. Then*

$$g^* \circ d_X = d_Y \circ g^*.$$

3. *Let a be a unit on X . Then*

$$d_X \circ [a] = \varepsilon[a] \circ d_X.$$

Moreover,

$$d_X \circ \eta = \eta \circ d_X.$$

4. Let (Z, i, X, j, U) be a boundary triple. Then

$$d_Z \circ \partial_Z^U = -\partial_Z^U \circ d_U.$$

Proof. Same as [Ros96, Proposition 4.6]. The first assertion comes from Proposition 2.4.2.1.1 and (R3b), the second from (R3c), Proposition 2.4.2.1.3, Proposition 2.4.2.6.1 and (R3a) (note that the proof is actually much easier in our case since there are no multiplicities to consider). The third assertion follows from the definitions and Proposition 2.4.2.3.2, the fourth from the fact that $d \circ d = 0$. \square

Remark 2.4.2.7. Note the quadratic form ε playing the role of (-1) in the third formula but not in the fourth.

2.5 Milnor-Witt cycle complexes and Chow-Witt groups

2.5.1 Milnor-Witt cycle complexes

Let M be a Milnor-Witt cycle module, let X be a scheme with a virtual bundle \mathcal{V}_X and p an integer. We have defined (see Definition 2.4.1.4)

$$C_p(X, M, \mathcal{V}_X) = \bigoplus_{x \in X^{(p)}} M(x, \mathcal{V}_X)$$

with differential

$$d = d_X : C_p(X, M, \mathcal{V}_X) \rightarrow C_{p-1}(X, M, \mathcal{V}_X).$$

In the same way, we can define

$$C^p(X, M, \mathcal{V}_X) = \bigoplus_{x \in X^{(p)}} M(x, \mathcal{V}_X)$$

with differential

$$d = d_X : C^p(X, M, \mathcal{V}_X) \rightarrow C^{p+1}(X, M, \mathcal{V}_X)$$

and show that this gives us another complex.

Definition 2.5.1.1. The *Chow-Witt group of p -dimensional cycles with coefficients in M* is defined as the p -th homology group of the complex $C_*(X, M, \mathcal{V}_X)$ and denoted by $A_p(X, M, \mathcal{V}_X)$. Similarly, we define a cohomology group $A^p(X, M, \mathcal{V}_X)$ with $C^*(X, M, \mathcal{V}_X)$.

2.5.1.2. According to the previous section (see Proposition 2.4.2.6), the morphisms f_* for f proper, g^* for g essentially smooth, multiplication by $[a_1, \dots, a_n]$ or η , ∂_Y^U (anti)commute with the differentials.

2.5.1.3. Let (Z, i, X, j, U) be a boundary triple. We can split the complex $C_*(X, M, \mathcal{V}_X)$ as

$$C_*(X, M, \mathcal{V}_X) = C_*(Z, M, \mathcal{V}_Z) \oplus C_*(U, M, \mathcal{V}_U)$$

so that there is a long exact sequence

$$\dots \xrightarrow{\partial} A_p(Z, M, \mathcal{V}_Z) \xrightarrow{i_*} A_p(X, M, \mathcal{V}_X) \xrightarrow{j^*} A_p(U, M, \mathcal{V}_U) \xrightarrow{\partial} A_{p-1}(Z, M, \mathcal{V}_Z) \xrightarrow{i_*} \dots$$

2.5.2 Classical Chow-Witt groups.

2.5.2.1. Recall some definitions about the classical theory of Chow-Witt groups (see [Fas08]). We note that the results below are true in any characteristic.

Let X be a smooth scheme and let $p, r \in \mathbb{Z}$. For $x \in X^{(p)}$, \mathcal{N}_x is a virtual $\kappa(x)$ -vector space of dimension p . Denote $\Lambda_x = \bigwedge^p \mathcal{N}_x$ its determinant and Λ_x^* its dual.

The Rost-Schmid complex in Milnor-Witt K-theory $C(X, \mathbf{K}_r^{\text{MW}})$ is the complex

$$\dots \rightarrow \bigoplus_{x \in X^{(p)}} \mathbf{K}_{r-p}^{\text{MW}}(\kappa(x), p, \Lambda_x^*) \rightarrow \bigoplus_{x \in X^{(p+1)}} \mathbf{K}_{r-p-1}^{\text{MW}}(\kappa(x), p+1, \Lambda_x^*) \rightarrow \dots$$

where $\mathbf{K}_{r-p}^{\text{MW}}(\kappa(x), p, \Lambda_x^*) = \underline{\mathbf{K}}^{\text{MW}}(\kappa(x), \Lambda_x^* + (r-p-1) \cdot \mathbb{A}_{\kappa(x)}^1)$ (see also [Mor12, Definition 5.7]). Denote by $\widetilde{\text{CH}}^p(X)_r$ its p -th cohomology group. By definition, the classical Chow-Witt group is $\widetilde{\text{CH}}^p(X)_p$ and is simply denoted by $\widetilde{\text{CH}}^p(X)$. It is related with our previous constructions as follows.

Proposition 2.5.2.2. *Let X be a smooth scheme and let $p, r \in \mathbb{Z}$. Then we have a canonical isomorphism*

$$A^p(X, \underline{\mathbf{K}}^{\text{MW}}, -\mathcal{T}_{X/k} + r \cdot \mathbb{A}_X^1) \simeq \widetilde{\text{CH}}^p(X)_r.$$

In particular with $r = p$,

$$A^p(X, \underline{\mathbf{K}}^{\text{MW}}, -\mathcal{T}_{X/k} + p \cdot \mathbb{A}_X^1) \simeq \widetilde{\text{CH}}^p(X)$$

Proof. Let x be in $X^{(p)}$. We have the following canonical isomorphisms

$$\begin{aligned} \mathbf{K}_{r-p}^{\text{MW}}(\kappa(x), p, \Lambda_x^*) &\simeq \underline{\mathbf{K}}^{\text{MW}}(\kappa(x), \Lambda_x^* + (r-p-1) \cdot \mathbb{A}_{\kappa(x)}^1), \\ &\simeq \underline{\mathbf{K}}^{\text{MW}}(\kappa(x), -\Lambda_x + 2 \cdot \mathbb{A}_{\kappa(x)} + (r-p-1) \cdot \mathbb{A}_{\kappa(x)}^1), \\ &\simeq \underline{\mathbf{K}}^{\text{MW}}(\kappa(x), -(\Lambda_x + (p-1) \cdot \mathbb{A}_{\kappa(x)}) + r \cdot \mathbb{A}_{\kappa(x)}^1), \\ &\simeq \underline{\mathbf{K}}^{\text{MW}}(\kappa(x), -\mathcal{N}_x + r \cdot \mathbb{A}_{\kappa(x)}^1), \\ &\simeq \underline{\mathbf{K}}^{\text{MW}}(\kappa(x), \Omega_{\kappa(x)/k} - \mathcal{T}_{X/k} \times_X \kappa(x) + r \cdot \mathbb{A}_{\kappa(x)}^1), \\ &\simeq \underline{\mathbf{K}}^{\text{MW}}(x, -\mathcal{T}_{X/k} + r \cdot \mathbb{A}_X^1), \end{aligned}$$

and so

$$\bigoplus_{x \in X^{(p)}} \mathbf{K}_{r-p}^{\text{MW}}(\kappa(x), p, \Lambda_x^*) \simeq C^p(X, \underline{\mathbf{K}}^{\text{MW}}, -\mathcal{T}_{X/k} + r \cdot \mathbb{A}_X^1).$$

By definition, the differentials agree, hence the result. \square

2.6 Acyclicity and homotopy invariance

2.6.1 Acyclicity for smooth local rings

Fix M a MW-cycle module over k . We follow [Ros96, §6]. We prove that the cohomology of the complex associated to M can be computed using the Zariski sheaf given by

$$U \mapsto A^0(U, M).$$

Theorem 2.6.1.1. *Let X be an essentially smooth and semi-local scheme, and let \mathcal{V}_X a virtual bundle over X . Then*

$$A^p(X, M, \mathcal{V}_X) = 0$$

for $p > 0$.

The proof is postponed till the end of the section; we will need the following lemmas.

Let V be a vector space over k and let $\mathbb{A}(V)$ be the associated affine space. For a linear subspace W of V let

$$\begin{aligned} \pi_W : \mathbb{A}(V) &\rightarrow \mathbb{A}(V/W), \\ \pi_W(v) &= v + W \end{aligned}$$

be the projection.

Lemma 2.6.1.2. *Let $X \subset \mathbb{A}(V)$ be an equidimensional closed subscheme of dimension d and let Y be a closed subscheme with $\dim Y < d$. Moreover, let $S \subset Y$ be a finite subset such that X is smooth in S . Then for a generic $(d-1)$ -codimensional linear subspace W of V the following conditions hold.*

- The restriction $\pi_W|_Y : Y \rightarrow \mathbb{A}(V/W)$ is finite.
- The restriction $\pi_W|_X : X \rightarrow \mathbb{A}(V/W)$ is locally around S smooth.

Proof. See [Ros96, Proposition 6.2]. □

The existence of the generic space W is not guaranteed over finite base fields. Note that if X is semi-local, then $X \times_k \operatorname{Spec}(k(t))$ is also semi-local and essentially smooth over k . The following lemma shows that we can assume k to be infinite.

Lemma 2.6.1.3. *Let X be a scheme over k and let $g : X_{k(t)} \rightarrow X$ be the (smooth) base change. Then*

$$g^* : A^*(X, M, \mathcal{V}_X) \rightarrow A^*(X_{k(t)}, M, -\mathcal{T}_{X_{k(t)}/X} + \mathcal{V}_{X_{k(t)}})$$

is injective.

Proof. (see also 2.7.2.2) Let $U \subset \mathbb{A}_k^1$ be an open set containing 0 and write $X^U = X \times_k (U \setminus \{0\})$. Define s_U^* as the composition

$$\begin{array}{ccc} C_{p+1}(X^U, M, -\mathcal{T}_{X^U/X} + \mathcal{V}_{X^U}) & \xrightarrow{[t]} & C_{p+1}(X^U, M, \mathcal{V}_{X^U}) \\ s_U^* \downarrow & & \downarrow \partial^U \\ C_p(X, M, \mathcal{V}_X) & \xleftarrow{\cong} & C_p(X \times_k \{0\}, M, \mathcal{V}_{X \times_k \{0\}}). \end{array}$$

This defines a morphism on the homology groups, also denoted by s_U^* . Consider the map $g_U^* : A_p(X, M, \mathcal{V}_X) \rightarrow A_{p+1}(X^U, M, -\mathcal{T}_{X^U/X} + \mathcal{V}_{X^U})$ (induced by the canonical projection $g_U : X^U \rightarrow X$) so that we have $g^* = \operatorname{colim}_U g_U^*$ where the limit is taken over the open sets $U \subset \mathbb{A}_k^1$ containing 0. Define likewise $s^* = \operatorname{colim}_U s_U^*$. Since $s_U^* \circ g_U^* = \operatorname{id}$ for all U (by (R3d), see also the proof of Lemma 2.4.2.5), we see that s^* is a section of g^* . \square

Proposition 2.6.1.4. *Let X be a smooth scheme over a field and let $Y \subset X$ be a closed subscheme of codimension ≥ 1 . Then for any finite subset $S \subset Y$ there is an open neighbourhood X' of S in X such that the map*

$$i_* : A_*(Y \cap X', M, \mathcal{V}_{Y \cap X'}) \rightarrow A_*(X', M, \mathcal{V}_{X'})$$

is the trivial map. Here $i : Y \cap X' \rightarrow X'$ is the inclusion.

Proof. Same as in [Ros96, Proposition 6.4]. This uses Lemma 2.4.2.5. \square

Proof of Theorem 2.6.1.1. We may assume that X is connected. Put $d = \dim X$. Consider pairs (U, S) where U is a smooth d -dimensional scheme of finite type over k and $S \subset U$ is a finite subset such that X is the localization of U in S . Then

$$C^*(X, M, \mathcal{V}_X) = \operatorname{colim}_{(U, S)} C^p(U, M, \mathcal{V}_U).$$

Moreover,

$$C^p(U, M, \mathcal{V}_U) = C_{d-p}(U, M, \mathcal{V}_U) = \operatorname{colim}_Y C_{d-p}(Y, M, \mathcal{V}_Y)$$

where Y runs over the closed p -codimensional subsets of U . Hence

$$A^p(X, M, \mathcal{V}_X) = \operatorname{colim}_{(U, S)} A^p(U, M, \mathcal{V}_U) = \operatorname{colim}_{(U, S)} \operatorname{colim}_Y A_{d-p}(Y, M, \mathcal{V}_Y).$$

Finally, Proposition 2.6.1.4 tells that the map $A_{d-p}(Y, M, \mathcal{V}_Y) \rightarrow A^p(U, M, \mathcal{V}_U) \rightarrow A^p(U', M, \mathcal{V}_{U'})$ is trivial for $U' \subset U$ small enough. \square

Corollary 2.6.1.5. *Let X be a smooth scheme over k , let \mathcal{V}_X be a virtual bundle over X and let \mathcal{M}_X be the Zariski sheaf on X given by*

$$U \mapsto A^0(U, M, \mathcal{V}_U) \subset M(\xi_X, \mathcal{V}_X)$$

for open subsets U of X . There are natural isomorphisms

$$A^p(X, M, \mathcal{V}_X) \rightarrow H_{\text{Zar}}^p(X, \mathcal{M}_X).$$

Proof. Same as [Ros96, Corollary 6.5]. \square

2.6.2 Homotopy invariance

Following [Dé14b, §2], we define a coniveau spectral sequence that will help us reduce the homotopy invariance property to the known case (H).

Proposition 2.6.2.1. *Let F be a field, $X = \operatorname{Spec} F$ its spectrum, \mathbb{A}_X^n the affine vector bundle of rank n over X , $\pi : \mathbb{A}_X^n \rightarrow X$ the canonical projection and \mathcal{V}_X a virtual vector bundle over X . Then, for every $q \in \mathbb{Z}$, the canonical morphism*

$$\pi^* : A^q(X, M, \mathcal{V}_X) \rightarrow A^q(\mathbb{A}_X^n, M, -\mathcal{T}_{\mathbb{A}_X^n/X} + \mathcal{V}_{\mathbb{A}_X^n})$$

is an isomorphism.

Proof. Same as [Sch98, Satz 6.1.1] or [Fas08, Théorème 11.2.4]. If $q \neq 0$, then the map is trivial. We assume $q = 0$. By induction, we can reduce to the case $n = 1$. In this case, we have $\mathbb{A}_X^1 = \text{Spec } F[t]$ for some parameter t . By definition, $A^0(X, M, \mathcal{V}_X) = M(F, \Omega_{F/k} + \mathcal{V}_F)$ and $A^0(\mathbb{A}_X^n, M, -\mathcal{T}_{\mathbb{A}_X^1/X} + \mathcal{V}_{\mathbb{A}_X^n})$ is the cokernel of the map

$$d : M(F(t), \Omega_{F(t)/k} - \mathbb{A}_{F(t)}^1 + \mathcal{V}_{F(t)}) \rightarrow \bigoplus_{x \in (\mathbb{A}_F^1)^{(1)}} M(\kappa(x), \Omega_{\kappa(x)/k} - \mathbb{A}_{\kappa(x)}^1 + \mathcal{V}_{\kappa(x)})$$

defined in 2.3.2.4. Thus, the result follows from Proposition 2.3.2.9, (H). \square

Let X be a scheme with a virtual bundle \mathcal{V}_X , let V be a scheme and let $\pi : V \rightarrow X$ be an essentially smooth morphism.

A flag on X is a decreasing sequence $(Z^p)_{p \in \mathbb{Z}}$ of closed subschemes of X such that $\text{codim}_X(Z^p) \geq p$ where, by convention, $Z^p = X$ if $p < 0$ and $Z^p = \emptyset$ if $p > \dim X$. The set of flags of X is denoted by $\text{Flag}(X)$ and it is ordered by the inclusion termwise so that it becomes a filtrant set.

Let $\mathfrak{Z} = (Z^p)_{p \in \mathbb{Z}}$ be a flag of X and define $\pi^* \mathfrak{Z} = (\pi^* Z^p)_{p \in \mathbb{Z}}$ a flag over V by $\pi^* Z^p = V \times_X Z^p$. For $p, q \in \mathbb{Z}$, define

$$\begin{aligned} D_{\mathfrak{Z}}^{p,q} &= A^{p+q-1}(V - \pi^* Z^p, M, \mathcal{V}_{(X - \pi^* Z^p)}), \\ E_{\mathfrak{Z}}^{p,q} &= A^q(\pi^* Z^p - \pi^* Z^{p+1}, M, \mathcal{V}_{(\pi^* Z^p - \pi^* Z^{p+1})}). \end{aligned}$$

We have a long exact sequence

$$\dots \longrightarrow D_{\mathfrak{Z}}^{p+1,q-1} \xrightarrow{j_p^*} D_{\mathfrak{Z}}^{p,q} \xrightarrow{\partial_p} E_{\mathfrak{Z}}^{p,q} \xrightarrow{i_{p,*}} D_{\mathfrak{Z}}^{p+1,q} \longrightarrow \dots$$

so that $(D_{\mathfrak{Z}}^{p,q}, E_{\mathfrak{Z}}^{p,q})_{p,q \in \mathbb{Z}}$ is an exact couple where j_p^* and $i_{p,*}$ are induced by the canonical immersions. By the general theory (see [McC01, Chapter 3]), this defines a spectral sequence that converges to $A^{p+q}(V, M, \mathcal{V}_V)$ because the $E_1^{p,q}$ -term is bounded (since the dimension of V is finite).

For $p, q \in \mathbb{Z}$, denote by

$$\begin{aligned} D_{1,\pi}^{p,q} &= \text{colim}_{\mathfrak{Z} \in \text{Flag}(X)} D_{\mathfrak{Z}}^{p,q}, \\ E_{1,\pi}^{p,q} &= \text{colim}_{\mathfrak{Z} \in \text{Flag}(X)} E_{\mathfrak{Z}}^{p,q} \end{aligned}$$

where the colimit is taken over the flags \mathfrak{Z} of X (see Proposition 2.4.2.4 for functoriality). Since filtered direct limits are exact in the derived category of abelian groups, the previous spectral sequences give the following theorem.

Theorem 2.6.2.2. *We have the convergent spectral sequence*

$$E_{1,\pi}^{p,q} \Rightarrow A^{p+q}(V, M, \mathcal{V}_V).$$

We need to compute this spectral sequence. This is done in the following theorem.

Theorem 2.6.2.3. *For $p, q \in \mathbb{Z}$, we have a canonical isomorphism*

$$E_{1,\pi}^{p,q} \simeq \bigoplus_{x \in X^{(p)}} A^q(V_x, M, \mathcal{V}_{V_x}).$$

Proof. (see [D  14b,  4] for a similar result in the oriented case) Denote by \mathcal{J}_p the set of pairs (Z, Z') where Z is a reduced closed subscheme of X of codimension p and $Z' \subset Z$ is a closed subset containing the singular locus of Z . Notice that any such pair (Z, Z') can be (in a functorial way) extended into a flag of X . Moreover, for any x in X , consider $\overline{\{x\}}$ the reduced closure of x in X and $\mathfrak{F}(x)$ be the set of closed subschemes Z' of $\overline{\{x\}}$ containing its singular locus. The following equalities are all canonical isomorphisms:

$$\begin{aligned} E_{1,\pi}^{p,q} &\simeq \operatorname{colim}_{\mathfrak{Z} \in \operatorname{Flag}(X)} A^q(V \times_X (Z^p - Z^{p+1}), M, \mathcal{V}_{(V \times_X (Z^p - Z^{p+1}))}) \\ &\simeq \operatorname{colim}_{(Z, Z') \in \mathcal{J}_p} A^q(V \times_X (Z - Z'), M, \mathcal{V}_{(V \times_X (Z - Z'))}) \\ &\simeq \bigoplus_{x \in X^{(p)}} \operatorname{colim}_{Z' \in \mathfrak{F}(x)} A^q(V \times_X (\overline{\{x\}} - Z'), M, \mathcal{V}_{V \times_X (\overline{\{x\}} - Z')}) \\ &\simeq \bigoplus_{x \in X^{(p)}} A^q(V_x, M, \mathcal{V}_{V_x}). \end{aligned}$$

□

Theorem 2.6.2.4 (Homotopy Invariance). *Let X be a scheme, V a vector bundle of rank n over X , $\pi : V \rightarrow X$ the canonical projection and \mathcal{V}_X a virtual vector bundle over X . Then, for every $q \in \mathbb{Z}$, the canonical morphism*

$$\pi^* : A^q(X, M, \mathcal{V}_X) \rightarrow A^q(V, M, -\mathcal{T}_{V/X} + \mathcal{V}_V)$$

is an isomorphism.

Proof. From a noetherian induction and the localization sequence 2.5.1.3, we can reduce to the case where $V = \mathbb{A}_X^n$ is the affine vector bundle of rank n . With the previous notations, Theorem 2.6.2.2 gives the spectral sequence

$$E_{1,\pi}^{p,q} \Rightarrow A^{p+q}(V, M, -\mathcal{T}_{V/X} + \mathcal{V}_V)$$

where $E_{1,\pi}^{p,q}$ is (abusively) defined as previously, but twisted accordingly. By Theorem 2.6.2.3, the page $E_{1,\pi}^{p,q}$ is isomorphic to

$$\bigoplus_{x \in X^{(p)}} A^q(V_x, M, -\mathcal{T}_{V_x/X_x} + \mathcal{V}_{V_x}).$$

According to Proposition 2.6.2.1, this last expression is isomorphic (via the map π) to $\bigoplus_{x \in X^{(p)}} A^q(\operatorname{Spec} \kappa(x), M, \mathcal{V}_x)$ (in other words, the theorem is true when X is the spectrum of a field). Using again 2.6.2.3, this group is isomorphic to $E_{1, \operatorname{Id}_X}^{p,q}$, which converge to $A^{p+q}(X, M, \mathcal{V}_X)$. By Proposition 2.4.2.4, the map π induces a morphism of exact couples $(D_{1, \operatorname{Id}_X}^{p,q}, E_{1, \operatorname{Id}_X}^{p,q}) \rightarrow (D_{1,\pi}^{p,q}, E_{1,\pi}^{p,q})$ hence we have compatible isomorphisms on the pages which induce the pullback

$$\pi^* : A^q(X, M, \mathcal{V}_X) \rightarrow A^q(V, M, -\mathcal{T}_{V/X} + \mathcal{V}_V)$$

(see also [Wei94, 5.2.12]).

□

2.7 Gysin morphisms

2.7.1 Gysin morphisms for regular embeddings

We define Gysin morphisms for regular closed immersions and prove functoriality theorems. As always, the main tool is the deformation to the normal cone.

Let $i : Z \rightarrow X$ be a closed immersion, and let \mathcal{V}_X be a virtual bundle over the scheme X . Let t be a parameter such that $\mathbb{A}_k = \text{Spec} k[t]$, and let $q : X \times_k (\mathbb{A}_k^1 \setminus \{0\}) \rightarrow X$ be the canonical projection. Denote by $D = D_Z X$ the deformation space such that $D = U \sqcup N_Z X$ where $U = X \times_k (\mathbb{A}_k^1 \setminus \{0\})$ (see [Ros96, §10] for more).

Consider the morphism

$$J(X, Z) = J_{Z/X} : C_*(X, M, \mathcal{V}_X) \rightarrow C_*(N_Z X, M, \mathcal{V}_{N_Z X})$$

defined by the composition:

$$\begin{array}{ccc} C_p(X, M, \mathcal{V}_X) & \xrightarrow{q^*} & C_{p+1}(U, M, -\mathcal{T}_{U/X} + \mathcal{V}_U) \\ \downarrow J_{Z/X} & & \downarrow [t] \\ C_p(N_Z X, M, \mathcal{V}_{N_Z X}) & \xleftarrow{\partial} & C_{p+1}(U, M, \mathcal{V}_U) \end{array}$$

where the multiplication with t is twisted by the isomorphism $\mathcal{T}_{U/X} \simeq \mathbb{A}_U^1$ (which only depends on t) and ∂ is the boundary map as in 2.4.1.10. This defines a morphism (also denoted by $J(X, Z)$ or $J_{Z/X}$) by passing to homology.

Assume moreover that $i : Z \rightarrow X$ is regular of codimension m , the map $\pi : N_Z X \rightarrow Z$ is a vector bundle over Z of dimension m . By homotopy invariance (Theorem 2.6.2.4), we have an isomorphism

$$\pi^* : A_p(Z, M, \mathcal{N}_Z X + \mathcal{V}_Z) \rightarrow A_{p+m}(N_Z X, M, \mathcal{V}_{N_Z X})$$

where $\mathcal{N}_Z X$ is the virtual vector bundle associated to $N_Z X$ (we have used the canonical isomorphism $\pi^*(\mathcal{N}_Z X) = \mathcal{T}_{N_Z X/Z}$). Denote by $r_{Z/X} = (\pi^*)^{-1}$ its inverse.

Definition 2.7.1.1. With the previous notations, we define the map

$$i^* : A_p(X, M, \mathcal{V}_X) \rightarrow A_{p-m}(Z, M, \mathcal{N}_Z X + \mathcal{V}_Z)$$

by putting $i^* = r_{Z/X} \circ J_{Z/X}$ and call it the *Gysin morphism of i* .

The following lemmas are needed to prove functoriality of the previous construction (see Theorem 2.7.1.6).

Lemma 2.7.1.2. *Let $i : Z \rightarrow X$ be a regular closed immersion and $g : V \rightarrow X$ be an essentially smooth morphism. Denote by $N(g)$ the projection from $N(V, V \times_X Z) = N_Z(X) \times_X V$ to $N_Z X$. Then*

$$\Theta \circ J(V, V \times_X Z) \circ g^* = N(g)^* \circ J(X, Z),$$

where Θ comes from the canonical isomorphism $\mathcal{T}_{N(V, V \times_X Z)/N(X, Y)} \simeq \mathcal{T}_{V/X} \times_V N(V, Y \times_X V)$.

Proof. See [Ros96, Lemma 11.3]. This follows from our Proposition 2.4.2.1.2, Proposition 2.4.2.4.2 and Proposition 2.4.2.3.2. \square

Lemma 2.7.1.3. *Let $Z \rightarrow X$ be a closed immersion and let $p : X \rightarrow Y$ be essentially smooth. Let \mathcal{V}_Y be a virtual vector bundle over Y . Suppose that the composite*

$$q : N_Z X \rightarrow Z \rightarrow X \rightarrow Y$$

is essentially smooth of same relative dimension as p . Then

$$\Theta \circ J(X, Z) \circ p^* = q^*$$

where Θ comes from the canonical isomorphism $\mathcal{T}_{N_Z X/Y} \simeq \mathcal{T}_{V/Y} \times_X N_Z X$.

Proof. Same as [Ros96, Lemma 11.4] except that Rost only needs the composite morphism

$$f : D(X, Z) \longrightarrow X \times_k \mathbb{A}_k^1 \xrightarrow{p \times \text{Id}} Y \times_k \mathbb{A}_k^1$$

to be flat. We need moreover (in order to use our Proposition 2.4.2.5) the fact that f is essentially smooth which is true because it is flat and its fibers are essentially smooth. \square

Lemma 2.7.1.4. *Let $\rho : T \rightarrow T'$ be a morphism, let $T'_1, T'_2 \subset T'$ be closed subschemes and let $T_i = T \times_{T'} T'_i$ for $i = 1, 2$. Let $\mathcal{V}_{T'}$ be a virtual bundle vector bundle over T' .*

Put $T_3 = T \setminus (T_1 \cup T_2)$, $T_0 = T_1 \subset T_2$, $\tilde{T}_1 = T_1 \setminus T_0$, $\tilde{T}_2 = T_2 \setminus T_0$ and let $\partial_1^3, \partial_0^1, \partial_2^3, \partial_0^2$ be the boundary maps for the closed immersions

$$\tilde{T}_1 \rightarrow T \setminus T_2, T_0 \rightarrow T_1, \tilde{T}_2 \rightarrow T \setminus T_1, T_0 \rightarrow T_2,$$

respectively. Then

$$0 = \partial_0^1 \circ \partial_1^3 + \partial_0^2 \circ \partial_2^3 : [T_3, \mathcal{V}_{T_3}] \bullet \rightarrow [T_0, \mathcal{V}_{T_0}]$$

at the homology level (recall the notation introduced in 2.4.1.11).

Proof. Corresponding to the set theoretic decomposition of T we have

$$A_*(T, M, \mathcal{V}_T) = A_*(T_0, M, \mathcal{V}_{T_0}) \oplus A_*(\tilde{T}_1, M, \mathcal{V}_{\tilde{T}_1}) \oplus A_*(\tilde{T}_2, M, \mathcal{V}_{\tilde{T}_2}) \oplus A_*(T_3, M, \mathcal{V}_{T_3}).$$

Then $d_T \circ d_T = 0$ gives the result. \square

Lemma 2.7.1.5. *Let $T = \overline{D} = \overline{D}(X, Y, Z)$, $T_1 = \overline{D}(\{0\} \times \mathbb{A}_k^1)$, $T_2 = \overline{D}(\mathbb{A}_k^1 \times \{0\})$ where \overline{D} is the double deformation cone (see [Ros96, §10.5]). We keep the notations of Lemma 2.7.1.4. Then $T_3 = X \times_k (\mathbb{A}_k^1 \setminus \{0\}) \times (\mathbb{A}_k^1 \setminus \{0\})$ and $T_0 = \overline{D}(\{0, 0\})$. Let $\pi : T_3 \rightarrow X$ be the projection and let t, s be the coordinates of $\mathbb{A}_k^2 = \text{Spec} k[t, s]$, so that $T_1 = \{t = 0\}$, $T_2 = \{s = 0\}$.*

Let $Z \rightarrow Y \rightarrow X$ be regular closed immersions. Then (up to canonical isomorphisms Θ and Θ')

$$\begin{aligned} \partial_0^1 \circ \partial_1^3 \circ [t, s] \circ \pi^* &= \Theta \circ J(N_Y X, N_Y X|Z) \circ J(X, Y), \\ \partial_0^2 \circ \partial_2^3 \circ [s, t] \circ \pi^* &= \Theta' \circ J(N_Z X, N_Z Y) \circ J(X, Z). \end{aligned}$$

Proof. Same as [Ros96, Lemma 11.7] except one has to be careful with the twists (this uses Lemma 2.7.1.2). \square

Theorem 2.7.1.6. *Let $l : Z \rightarrow Y$ and $i : Y \rightarrow X$ be regular closed immersions of respective codimension n and m . Then $i \circ l$ is a regular closed immersion of codimension $m + n$ and (up to a canonical isomorphism)*

$$(i \circ l)^* = l^* \circ i^* \text{ as morphism } A_p(X, M, \mathcal{V}_X) \rightarrow A_{p-m-n}(Z, M, \mathcal{N}_Z X + \mathcal{V}_Z).$$

Proof. The first assertion follows from [Gro67, 19.1.5.(iii)] which also gives the canonical isomorphism of virtual vector spaces $\mathcal{N}_Z X = \mathcal{N}_Z Y + \mathcal{N}_Y X \times_Y Z$. The second assertion follows from 2.7.1.2, 2.7.1.4, 2.7.1.5 and 2.7.1.3 as in [Ros96, Theorem 13.1]. \square

We conclude with one interesting property.

Proposition 2.7.1.7 (Base change for regular closed immersions). *Consider the cartesian square*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow g & & \downarrow f \\ Z' & \xrightarrow{i'} & X' \end{array}$$

where i, i' are regular closed immersions and f, g are proper morphisms. Suppose moreover that we have a canonical isomorphism of virtual vector spaces $\mathcal{N}_{Z'} X' \simeq \mathcal{N}_Z X \times_X X'$. Then (up to the canonical isomorphism Θ induced by the previous isomorphism)

$$\Theta \circ g_* \circ i^* = i'^* \circ f_*.$$

Proof. It suffices to prove that the following diagram is commutative (recall the notation introduced in 2.4.1.11):

$$\begin{array}{ccccccccc} X \bullet & \xrightarrow{q^*} & X \times (\mathbb{A}_k^1 \setminus \{0\}) \bullet & \xrightarrow{[t]} & X \times (\mathbb{A}_k^1 \setminus \{0\}) \bullet & \xrightarrow{\partial} & \mathcal{N}_Z X \bullet & \xrightarrow{\simeq} & Z \bullet \\ \downarrow f_* & (1) & \downarrow (f \times \text{Id})_* & (2) & \downarrow (f \times \text{Id})_* & (3) & \downarrow h_* & (4) & \downarrow g_* \\ X' \bullet & \xrightarrow{q'^*} & X' \times (\mathbb{A}_k^1 \setminus \{0\}) \bullet & \xrightarrow{[t]} & X' \times (\mathbb{A}_k^1 \setminus \{0\}) \bullet & \xrightarrow{\partial'} & \mathcal{N}_{Z'} X' \bullet & \xrightarrow{\simeq} & Z' \bullet \end{array}$$

with obvious notations (see the definition of Gysin morphisms). The (cartesian) squares (1) and (4) commute by the base change theorem for essentially smooth morphisms (see Proposition 2.4.2.1.3). The squares (2) and (3) commute by Proposition 2.4.2.2.1 and Proposition 2.4.2.4.1, respectively. \square

2.7.2 Gysin morphisms for lci projective morphisms

We define Gysin morphisms for lci projective morphisms¹⁰ and prove functoriality theorems (see [Dé08a, §5] for similar results in the classical oriented case). One could also define Gysin morphisms for morphisms between two essentially smooth schemes as in [Ros96, §12].

Lemma 2.7.2.1. *Consider a regular closed immersion $i : Z \rightarrow X$ and a natural number n . Consider the pullback square*

¹⁰By definition, a map $f : Y \rightarrow X$ between two schemes is called a *local complete intersection projective* morphism (or *lci projective*) if there exist a natural number n and a factorization $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ of f into a regular closed immersion followed by the canonical projection.

$$\begin{array}{ccc} \mathbb{P}_Z^n & \xrightarrow{l} & \mathbb{P}_X^n \\ \downarrow q & & \downarrow p \\ Z & \xrightarrow{i} & X. \end{array}$$

Then $l^* \circ p^* = q^* \circ i^*$ (up to the canonical isomorphism induced by $\mathcal{T}_q - q^* \mathcal{N}_i \simeq l^* \mathcal{T}_p - \mathcal{N}_l$).

Proof. This follows from the definitions and Lemma 2.7.1.2. \square

Lemma 2.7.2.2. Consider a natural number n and an essentially smooth scheme X . Let $p : \mathbb{P}_X^n \rightarrow X$ be the canonical projection. Then for any section $s : X \rightarrow \mathbb{P}_X^n$ of p , we have $s^* p^* = \text{Id}$ (up to a canonical isomorphism).

Proof. We can assume that $X = \text{Spec} k$, then apply rule (R3d) (see also the proof of Lemma 2.4.2.5). \square

Lemma 2.7.2.3. Consider the following commutative diagram:

$$\begin{array}{ccc} & \mathbb{P}_X^n & \\ i \nearrow & & \searrow p \\ Y & & X \\ i' \searrow & & \nearrow q \\ & \mathbb{P}_X^m & \end{array}$$

where i, i' are regular closed immersions and p, q are the canonical projection. Then $i^* \circ p^* = i'^* \circ q^*$ (up to the canonical isomorphism induced by $\mathcal{N}_i - (i')^* \mathcal{T}_q \simeq \mathcal{N}_i - i^* \mathcal{T}_p$).

Proof. Let us introduce the following morphisms:

$$\begin{array}{ccccc} & & \mathbb{P}_X^n & & \\ & i \nearrow & & \searrow p & \\ Y & \xrightarrow{v} \mathbb{P}_X^n \times_X \mathbb{P}_X^m & & & X \\ & \searrow i' & & \nearrow q & \\ & & \mathbb{P}_X^m & & \end{array}$$

$\begin{array}{ccc} & q' \nearrow & \\ & & \mathbb{P}_X^n \\ & p' \searrow & \end{array}$

Applying Proposition 2.4.2.1.2, we are reduced to prove $i^* = v^* q'^*$ and $i'^* = v^* p'^*$. In other words, we are reduced to the case $m = 0$ and $q = \text{Id}_X$.

In this case, we introduce the following morphisms:

$$\begin{array}{ccc} Y & \xrightarrow{i} & \mathbb{P}_X^n \\ \downarrow s & & \downarrow p \\ \mathbb{P}_Y^n & \xrightarrow{l} & \mathbb{P}_X^n \\ \downarrow q & & \downarrow p \\ Y & \xrightarrow{i'} & X. \end{array}$$

Then the lemma follows from Lemma 2.7.2.1, Lemma 2.7.2.2 and Theorem 2.7.1.6. \square

Let X and Y be two schemes and let $f : Y \rightarrow X$ be a projective lci morphism. Consider an arbitrary factorization $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ of f into a regular closed immersion followed by the canonical projection. By the preceding lemma, the composition i^*p^* does not depend on the chosen factorization.

We recall the definition of \mathcal{T}_f , the virtual tangent bundle of f (see [Fas19, §3]; see also [Ful98, B.7.6]). Consider a factorization $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ as before and define $\mathcal{T}_{pi} = i^* \mathcal{T}_{\mathbb{P}_X^n/X} - \mathcal{N}_Y(\mathbb{P}_X^n)$ the relative (virtual) bundle with respect to the factorization $f = pi$. Consider next a commutative diagram

$$\begin{array}{ccc} & \mathbb{P}_X^n & \\ i \nearrow & \downarrow h & \searrow p \\ Y & & X \\ i' \searrow & \downarrow q & \nearrow \\ & \mathbb{P}_X^m & \end{array}$$

where $pi = qi' = f$ are two factorizations of f and where h is smooth. By taking the pullback, we can construct a third factorization $f = \pi(i \times i')$ such that we have canonical isomorphisms of virtual vector bundles

$$\begin{array}{ccc} & \mathcal{T}_{\pi(i \times i')} & \\ & \swarrow \quad \searrow & \\ \mathcal{T}_{pi} & & \mathcal{T}_{qi'}. \end{array}$$

Hence we can define \mathcal{T}_f as the limit (over all factorizations) of \mathcal{T}_{pi} .

Definition 2.7.2.4. Keeping the previous notations, we define the Gysin morphism associated to f as the morphism

$$f^* = \Theta \circ i^* \circ p^* : A^*(X, M, \mathcal{V}_X) \rightarrow A^*(Y, M, -\mathcal{T}_f + \mathcal{V}_Y)$$

where Θ is the canonical isomorphism coming from $\mathcal{T}_f \simeq \mathcal{T}_{pi}$.

Proposition 2.7.2.5. Consider projective morphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$. Then (up to the canonical isomorphism induced by $\mathcal{T}_{fg} \simeq \mathcal{T}_g + g^* \mathcal{T}_f$):

$$g^* \circ f^* = (f \circ g)^*.$$

Proof. (see also [Dé08a, Proposition 5.14] for the classical oriented case)

We choose a factorization $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ (resp. $Z \xrightarrow{j} \mathbb{P}_X^m \xrightarrow{q} X$) of f (resp. fg) and we introduce the diagram

$$\begin{array}{ccccc} & & \mathbb{P}_X^m & & \\ & & \uparrow p' & & \\ & & \mathbb{P}_X^n \times_X \mathbb{P}_X^m & & \\ & i' \nearrow & & \searrow q' & \\ \mathbb{P}_Y^m & & & & \mathbb{P}_X^n \\ & \searrow q'' & & \nearrow i & \\ Z & \xrightarrow{k} & Y & \xrightarrow{f} & X \\ & \xrightarrow{g} & & & \end{array}$$

in which p' is deduced from p by base change, and so on for q' and q'' . Then, by using the factorization given in the preceding diagram, the proposition follows from 2.7.2.1, 2.7.1.6, 2.7.2.3 and 2.4.2.1.2. \square

Now consider a cartesian square of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

with f proper and g lci. Suppose moreover that the square is *tor-independent*, that is for any $x \in X$, $y' \in Y'$ with $y = f(x) = g(y')$ and for any $i > 0$ we have

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.$$

From the fact that cotangent complex is stable under derived base change, it follows that there is a canonical isomorphism $f'^* \mathcal{T}_g \simeq \mathcal{T}_{g'}$ so that the following proposition makes sense.

Proposition 2.7.2.6 (Base change for lci morphisms). *Keeping the previous assumptions, we have (up to the canonical isomorphism induced by the previous isomorphism):*

$$f'_* \circ g'^* = g^* \circ f_*.$$

Proof. It suffices to consider the case where g is the projection of a projective bundle or a regular closed immersion. It follows from Proposition 2.4.2.1 in the first case and from Proposition 2.7.1.7 in the second. \square

2.8 Other constructions

2.8.1 Products

2.8.1.1. CROSS PRODUCTS. Let $M \times M' \rightarrow M''$ be a pairing of MW-cycle modules over k . Let Y and Z be two essentially smooth schemes over k equipped with virtual vector bundles \mathcal{V}_Y and \mathcal{W}_Z . We define the cross product

$$\times : C_p(Y, M, \mathcal{V}_Y) \times C_q(Z, M', \mathcal{W}_Z) \rightarrow C_{p+q}(Y \times Z, M'', \mathcal{V}_{Y \times Z} + \mathcal{W}_{Y \times Z})$$

as follows. For $y \in Y$, let $Z_y = \mathrm{Spec} \kappa(y) \times Z$, let $\pi_y : Z_y \rightarrow Z$ be the projection and let $i_y : Z_y \rightarrow Y \times Z$ be the inclusion. For $z \in Z$ we understand similarly $Y_z, \pi_z : Y_z \rightarrow Y$ and $i_z : Y_z \rightarrow Y \times Z$. We give the following two equivalent definitions:

$$\begin{aligned} \rho \times \mu &= \sum_{y \in Y(p)} (i_y)_* (\rho_y \cdot \pi_y^* (\mu)), \\ \rho \times \mu &= \sum_{z \in Z(q)} (i_z)_* (\pi_z^* (\rho) \cdot \mu). \end{aligned}$$

We give more details on this definition. The map

$$(i_y)_* : C_q(Z_y, M, \mathcal{V}_{Z_y} + \mathcal{W}_{Z_y}) \rightarrow C_{p+q}(Y \times Z, M, \mathcal{V}_{Y \times Z} + \mathcal{W}_{Y \times Z})$$

is by definition the inclusion corresponding to $Z_{y(q)} \subset (Y \times Z)_{(p+q)}$. Let $y \in Y$ and denote by ρ_y the y -component of ρ in $M(\kappa(y), \Omega_{\kappa(y)/k} + \mathcal{V}_y)$. Write $\pi_y^*(\mu) = \sum_{u \in Z_{y(q)}} (\pi_y^*(\mu))_u$. By definition,

$$\rho_y \cdot \pi_y^*(\mu) = \sum_{u \in Z_{y(q)}} \Theta_u(\text{res}_{\kappa(u)/\kappa(y)}(\rho_y) \cdot (\pi_y^*(\mu))_u),$$

where Θ_u is the canonical isomorphism induced by $\mathcal{T}_{Z_y/Z} \times \text{Spec } \kappa(u) \simeq \mathcal{T}_{\kappa(y)/k} \times \text{Spec } \kappa(u)$.

To check equality of the two definitions, consider the u -component for $u \in Y \times Z$. Let y, z be the images of u under the projections $Y \times Z \rightarrow Y, Z$. The u -components are either trivial or given by

$$(\rho \times \mu)_u = \text{res}_{\kappa(u)/\kappa(y)}(\rho_y) \cdot \text{res}_{\kappa(u)/\kappa(z)}(\mu_z).$$

2.8.1.2. ASSOCIATIVITY. Consider four pairings

$$\begin{aligned} M \times M' &\rightarrow N, \\ M' \times M'' &\rightarrow N', \\ N \times M'' &\rightarrow N'', \\ M \times N' &\rightarrow N''. \end{aligned}$$

For X another scheme and $v \in C_r(X, M, \mathcal{V}_X)$, we have the identity

$$v \times (\rho \times \mu) = (v \times \rho) \times \mu.$$

Indeed, consider the u -component for $u \in X \times Y \times Z$. Let x, y, z be the images of u in X, Y, Z , respectively. The u -component is either trivial or given by

$$(v \times \rho \times \mu)_u = \text{res}_{\kappa(u)/\kappa(x)}(v_x) \cdot \text{res}_{\kappa(u)/\kappa(y)}(\rho_y) \cdot \text{res}_{\kappa(u)/\kappa(z)}(\mu_z)$$

where the right hand of the equality makes sense if we assume the pairings to be compatible in some obvious sense.

2.8.1.3. COMMUTATIVITY. Let M be a MW-cycle module with a ring structure. Let $\tau : Y \times Z \rightarrow Z \times Y$ be the interchange of factors. For $\rho \in C_p(Y, M, \mathcal{V}_Y)$ and $\mu \in C_q(Z, M, \mathcal{W}_Z)$ one has

$$\tau_*(\rho \times \mu) = (-1)^{mn} \Theta(\mu \times \rho) \in C_{p+q}(Z \times Y, M, \mathcal{V}_{Z \times Y} + \mathcal{W}_{Z \times Y})$$

where m and n are the rank of \mathcal{V}_Y and \mathcal{W}_Z (respectively) and where Θ is induced by the switch isomorphism $\mathcal{W}_{Z \times Y} + \mathcal{V}_{Z \times Y} \simeq \mathcal{V}_{Z \times Y} + \mathcal{W}_{Z \times Y}$.

This is immediate from the definitions.

2.8.1.4. CHAIN RULE. For $\rho \in C_p(Y, M, \mathcal{V}_Y)$ and $\mu \in C_q(Z, M, \mathcal{W}_Z)$ one has

$$d(\rho \times \mu) = d(\rho) \times \mu + \varepsilon^n \rho \times d(\mu)$$

where $n = \text{rk } \mathcal{V}_Y$. The proof comes from (R3a), (R3c), (R3d) and P3 (see also [Ros96, §14.4] for more details).

2.8.1.5. INTERSECTION. For X smooth, the product induces a map

$$A^p(X, M, \mathcal{V}_X) \times A^q(X, M, \mathcal{W}_X) \rightarrow A^{p+q}(X \times X, M, \mathcal{V}_{X \times X} + \mathcal{W}_{X \times X}).$$

By composing with the Gysin morphism

$$\Delta^* : A^{p+q}(X \times X, M, \mathcal{V}_{X \times X} + \mathcal{W}_{X \times X}) \rightarrow A^{p+q}(X, M, -\mathcal{T}_\Delta + \mathcal{V}_X + \mathcal{W}_X)$$

induced by the diagonal $\Delta : X \rightarrow X \times X$, we obtain the map

$$A^p(X, M, \mathcal{V}_X) \times A^q(X, M, \mathcal{W}_X) \rightarrow A^{p+q}(X, M, -\mathcal{T}_\Delta + \mathcal{V}_X + \mathcal{W}_X).$$

The preceding considerations and the functoriality of the Gysin maps prove the following theorem.

Theorem 2.8.1.6. *If M is a MW-cycle module with a ring structure and X a smooth scheme over k , the pairing turns $\bigoplus_{\mathcal{V}_X \in \mathfrak{V}(X)} A^*(X, M, \mathcal{V}_X)$ into a graded commutative associative algebra over $\bigoplus_{\mathcal{V}_X \in \mathfrak{V}(X)} A^*(X, \underline{\mathbf{K}}^{\text{MW}}, \mathcal{V}_X)$ (see also 2.3.1.5 for another formulation).*

In particular, we obtain a product on $\widetilde{\text{CH}}(X)$ which coincides with the intersection product (defined in [Fas19, §3.4], see also [Fas08]). Indeed, our construction of Gysin morphisms follows the classical one (using deformation to the normal cone) and our cross products correspond to the one already defined for the Milnor-Witt K-theory (see [Fas19, §3]).

2.8.2 Adjunction between MW-cycle modules and Rost cycle modules

2.8.2.1. Denote by \mathbf{F}_k the category whose objects are couple (E, n) where E is a (finitely generated) field over k and n an integer, and where a morphism $(E, m) \rightarrow (F, n)$ is the data of a field extension $E \rightarrow F$ and the identity $m = n$.

Consider M a classical cycle module (*à la Rost*, see [Ros96, §1]). By definition, M is a functor from \mathbf{F}_k to the category of abelian groups with some data $(d1), \dots, (d4)$ and rules $(r1a), \dots, (r3e), (fd)$ and (c) (we use small letters to designate Rost's axioms so that $(d1)$ corresponds to (D1), etc.). Equivalently, M is a functor from the category of finitely generated fields to the category of \mathbb{Z} -graded abelian groups satisfying the same conditions.

We define a Milnor-Witt cycle module $\Gamma_*(M)$ as follows. Let (E, \mathcal{V}_E) be in \mathfrak{F}_k and put

$$\Gamma_*(M)(E, \mathcal{V}_E) = M(E, \text{rk } \mathcal{V}_E).$$

The data (D1), ..., (D4) are defined in an obvious way (for (D2) notice that $\text{rk } \Omega_{F/k} = \text{rk } \Omega_{E/k}$ if F/E is finite ; for (D3) the generator η acts trivially).

Moreover, we can check that this defines a fully faithful exact functor

$$\Gamma_* : \mathfrak{M}_k^{\text{M}} \rightarrow \mathfrak{M}_k^{\text{MW}}.$$

where $\mathfrak{M}_k^{\text{M}}$ (resp. $\mathfrak{M}_k^{\text{MW}}$) is the category of Rost cycle modules (resp. Milnor-Witt cycle modules).

Remark 2.8.2.2. Let \mathcal{V}_X be a virtual vector bundle of rank n over a scheme X and let p be an integer. With the notations of Section 2.5.1 and the one in [Ros96, §5], we have

$$C_p(X, \Gamma_*(M), \mathcal{V}_X) = C_p(X, M, n).$$

In order to define the adjoint functor, we need to following lemma.

Lemma 2.8.2.3. *Let (E, \mathcal{V}_E) be in \mathfrak{F}_k and let $\Theta : \mathcal{V}_E \rightarrow n \cdot \mathbb{A}_E^1$, $\Theta' : \mathcal{V}_E \rightarrow n \cdot \mathbb{A}_E^1$ be two trivializations. The induced isomorphisms*

$$\begin{aligned}\Theta_* : M(E, \mathcal{V}_E) &\rightarrow M(E, n \cdot \mathbb{A}_E^1), \\ \Theta'_* : M(E, \mathcal{V}_E) &\rightarrow M(E, n \cdot \mathbb{A}_E^1)\end{aligned}$$

are related as follows:

$$(\Theta - \Theta')_*(M(E, \mathcal{V}_E)) \subset \boldsymbol{\eta}[u] \cdot M(E, n \cdot \mathbb{A}_E^1),$$

for some $u \in E^\times$. Hence, they are equal modulo $\boldsymbol{\eta}$.

Proof. By (R1a), write $(\Theta - \Theta')_* = ((\Theta \circ \Theta'^{-1})_* - \text{Id}) \circ \Theta'_*$. The rule (R4a) gives an element $u \in E^\times$ such that

$$(\Theta \circ \Theta'^{-1})_* = \gamma_{\langle u \rangle}.$$

Since $\boldsymbol{\eta}[u] = \langle u \rangle - 1$, the result follows. □

2.8.2.4. Let M be a Milnor-Witt cycle module. We want to define a cycle module *à la Rost* $\Gamma^*(M)$. If E is a field over k and n an integer, denote by $\mathcal{M}_n(E) = M(E, n \cdot \mathbb{A}_E^1)$ and consider the graded group

$$\mathcal{M}(E) = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n(E).$$

This is in fact a module over the ring $\mathbf{K}_*^{\text{MW}}(E)$. Now consider $\mathcal{J}(E) = \mathcal{M}(E)\boldsymbol{\eta}$ the left sub- $\mathbf{K}_*^{\text{MW}}(E)$ -module generated by $\boldsymbol{\eta}$. Define $\Gamma^*(M)$ as follows:

$$\Gamma^*(M)(E) = \mathcal{M}(E) / \mathcal{J}(E).$$

This is a \mathbb{Z} -graded abelian group.

- (d1) Let $\varphi : E \rightarrow F$ be a morphism of fields. The collection of maps

$$\mathcal{M}_n(E) \rightarrow \mathcal{M}_n(F)$$

given by (D1) defines a map

$$\varphi_* : \Gamma^*(M)(E) \rightarrow \Gamma^*(M)(F)$$

of degree 0.

- (d2) Let $\varphi : E \rightarrow F$ be a finite morphism of fields. Denote by $r = \text{rk } \Omega_{F/k} = \text{rk } \Omega_{E/k}$. For $n \in \mathbb{Z}$, We define a map φ_Θ^* as in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{M}_n(F) & \xrightarrow{\varphi_\Theta^*} & \mathcal{M}_n(E) \\ \Theta \downarrow \simeq & & \simeq \downarrow \Theta' \\ M(F, \Omega_{F/k} + (n-r) \cdot \mathbb{A}_F^1) & \xrightarrow{\varphi^*} & M(E, \Omega_{E/k} + (n-r) \cdot \mathbb{A}_E^1) \end{array}$$

where Θ and Θ' are isomorphisms induced by $\Omega_{F/k} \simeq r \cdot \mathbb{A}_F^1$ and $\Omega_{E/k} \simeq r \cdot \mathbb{A}_E^1$, respectively. Hence (by (R2a) and (R2b)) we have a morphism of graded groups

$$\varphi_{\Theta, \Theta'}^* : \Gamma^*(M)(F) \rightarrow \Gamma^*(M)(E)$$

of degree 0. By Lemma 2.8.2.3, this morphism does not depend on Θ, Θ' .

- (d3) We have an obvious action of $\mathbf{K}_*^{\text{MW}}(E)$ on $\Gamma^*(M)$ where η acts trivially.
- (d4) Let E be a field with a valuation v . Let $\Theta : \mathcal{N}_v \rightarrow \mathbb{A}_{\kappa(v)}^1$ be an isomorphism of virtual $\kappa(v)$ -vector spaces. For $n \in \mathbb{Z}$, this defines a map ∂_Θ as in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{M}_n(E) & \xrightarrow{\partial_\Theta} & \mathcal{M}_{n-1}(\kappa(v)) \\ \parallel & & \uparrow \simeq \\ M(E, n \cdot \mathbb{A}_F^1) & \xrightarrow{\partial} & M(E, -\mathcal{N}_v + n \cdot \mathbb{A}_E^1). \end{array}$$

Hence (by (R3e)) we have a morphism of graded groups

$$\partial_\Theta : \Gamma^*(M)(E) \rightarrow \Gamma^*(M)(\kappa(v))$$

of degree (-1) . By Lemma 2.8.2.3, this morphism does not depend on Θ .

Since ε modulo η is (-1) , we easily check that $\Gamma^*(M)$ satisfies all axioms of Rost's cycle modules, except for (R1c) which can be deduced from Theorem 2.3.1.10.

Moreover, we can check that this defines a functor

$$\Gamma^* : \mathfrak{M}_k^{\text{MW}} \rightarrow \mathfrak{M}_k^{\text{M}}.$$

We now prove the following adjunction theorem (hence our theory extends Rost's).

Theorem 2.8.2.5 (Adjunction Theorem). *The two previously defined functors form an adjunction $\Gamma^* \dashv \Gamma_*$ between the category of Milnor-Witt cycle modules and the category of classical cycle modules:*

$$\Gamma^* : \mathfrak{M}_k^{\text{MW}} \rightleftarrows \mathfrak{M}_k^{\text{M}} : \Gamma_*.$$

Proof. Let M be a Milnor-Witt cycle module and N a Rost cycle module. We define a map

$$\Phi : \text{Hom}_{\mathfrak{M}_k^{\text{M}}}(\Gamma^*(M), N) \rightarrow \text{Hom}_{\mathfrak{M}_k^{\text{MW}}}(M, \Gamma_*(N)).$$

Let E be a field. By definition, $\Gamma^*(M)(E)$ is a \mathbb{Z} -graded abelian group; we write:

$$\Gamma^*(M)(E) = \bigoplus_{n \in \mathbb{Z}} \Gamma^*(M)(E, n).$$

We consider $\Gamma^*(M)$ as a functor $(E, n) \mapsto \Gamma^*(M)(E, n)$ with data $(d1), \dots, (d4)$ and rules $(r1a), \dots, (r3e), (fd)$ and (c) . Let α be a morphism from $\Gamma^*(M)$ to N . By definition, we have (for any field E and any integer n) maps

$$\alpha_{(E, n)} : \Gamma^*(M)(E, n) \rightarrow N(E, n)$$

compatible (in an obvious sense) with data $(d1), \dots, (d4)$ (see [Ros96, Definition 1.3]).

Let E be a field and \mathcal{V}_E be a virtual vector bundle of rank n over E . We want to define a map:

$$\Phi(\alpha)_{(E, \mathcal{V}_E)} : M(E, \mathcal{V}_E) \rightarrow \Gamma_*(N)(E, \mathcal{V}_E).$$

Consider $\Theta : \mathcal{V}_E \simeq n \cdot \mathbb{A}_E^1$ a trivialisation. We have a map $\Phi(\alpha)_E^\Theta$ defined as the composite:

$$\begin{array}{ccccc} M(E, \mathcal{V}_E) & \xrightarrow{\Theta} & M(E, n \cdot \mathbb{A}_E^1) & \xrightarrow{\iota_n} & \bigoplus_{m \in \mathbb{Z}} M(E, m \cdot \mathbb{A}_E^1) \\ \downarrow \Phi(\alpha)_{(E, \mathcal{V}_E)}^\Theta & & & & \downarrow \text{mod } \eta \\ \Gamma_*(N)(E, \mathcal{V}_E) & \xleftarrow{\alpha_{(E, n)}} & \Gamma^*(M)(E, n) & \xleftarrow{p_n} & \Gamma^*(M)(E) \end{array}$$

where ι_n and p_n are the canonical maps (recall that $\Gamma_*(N)(E, \mathcal{V}_E) = N(E, n)$ by definition). By Lemma 2.8.2.3, the map $\Phi(\alpha)_{(E, \mathcal{V}_E)}^\Theta$ does not depend on Θ and is denoted by $\Phi(\alpha)_{(E, \mathcal{V}_E)}$. In order to prove that $\Phi(\alpha) : (E, \mathcal{V}_E) \mapsto \Phi(\alpha)_{(E, \mathcal{V}_E)}$ is a morphism of Milnor-Witt cycle modules, we have to check that it is compatible with (D1), (D2), (D3) and (D4).

We prove compatibility with (D1). Let $\varphi : E \rightarrow F$ be an extension of fields and \mathcal{V}_E a virtual vector bundle of rank n over E . We want to prove that the following diagram (1) is commutative:

$$\begin{array}{ccccccc} M(E, \mathcal{V}_E) & \xrightarrow{\Theta} & M(E, n \cdot \mathbb{A}_E^1) & \xrightarrow{\iota_n} & \bigoplus_{m \in \mathbb{Z}} M(E, m \cdot \mathbb{A}_E^1) & & \\ \downarrow \varphi_* & \searrow \Phi(\alpha)_{(E, \mathcal{V}_E)} & \downarrow \varphi_* & & \downarrow \varphi_* & \searrow \text{mod } \eta & \\ \Gamma_*(N)(E, \mathcal{V}_E) & \xleftarrow{\alpha_{(E, n)}} & \Gamma^*(M)(E, n) & \xleftarrow{p_n} & \Gamma^*(M)(E) & & \\ \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \varphi_* & & \\ M(F, \mathcal{V}_F) & \xrightarrow{\Theta} & M(F, n \cdot \mathbb{A}_F^1) & \xrightarrow{\iota_n} & \bigoplus_{m \in \mathbb{Z}} M(F, m \cdot \mathbb{A}_F^1) & & \\ \downarrow \varphi_* & \searrow \Phi(\alpha)_{(F, \mathcal{V}_F)} & \downarrow \varphi_* & & \downarrow \varphi_* & \searrow \text{mod } \eta & \\ \Gamma_*(N)(F, \mathcal{V}_F) & \xleftarrow{\alpha_{(F, n)}} & \Gamma^*(M)(F, n) & \xleftarrow{p_n} & \Gamma^*(M)(F) & & \end{array}$$

(1)

where the maps are defined as before. The left back square is commutative according to (R2a) and (R4a). The left front square is commutative since α is compatible with (d1). The remaining squares are commutative by definition. Hence the square (1) is commutative.

We prove compatibility with (D2). Let $\psi : F \rightarrow E$ be a finite extension of fields and \mathcal{V}_F a virtual vector bundle of rank n over F . Denote by r the rank of $\Omega_{E/k}$. We want to prove that the following diagram (2) is commutative:

$$\begin{array}{ccccccc}
 M(E, \Omega_E/k + \mathcal{V}_E) & \xrightarrow{\Theta} & M(E, (r+n) \cdot \mathbb{A}_E^1) & \xrightarrow{l_{(r+n)}} & \bigoplus_{m \in \mathbb{Z}} M(E, m \cdot \mathbb{A}_E^1) & \xrightarrow{\text{mod } \eta} & \Gamma^*(M)(E) \\
 \downarrow \psi^* & \searrow \Phi(\alpha)_{(E, \Omega_E/k + \mathcal{V}_E)} & \downarrow \psi^* & & \downarrow \psi^* & & \downarrow \psi^* \\
 & & \Gamma_*(N)(E, \Omega_E/k + \mathcal{V}_E) & \xleftarrow{\alpha_{(E, r+n)}} & \Gamma^*(M)(E, (r+n)) & \xleftarrow{p_{(r+n)}} & \Gamma^*(M)(E) \\
 & & \downarrow \psi^* & & \downarrow \psi^* & & \downarrow \psi^* \\
 M(F, \Omega_F/k + \mathcal{V}_F) & \xrightarrow{\Theta} & M(F, (r+n) \cdot \mathbb{A}_F^1) & \xrightarrow{l_{(r+n)}} & \bigoplus_{m \in \mathbb{Z}} M(F, m \cdot \mathbb{A}_F^1) & \xrightarrow{\text{mod } \eta} & \Gamma^*(M)(F) \\
 \downarrow \psi^* & \searrow \Phi(\alpha)_{(F, \Omega_F/k + \mathcal{V}_F)} & \downarrow \psi^* & & \downarrow \psi^* & & \downarrow \psi^* \\
 & & \Gamma_*(N)(F, \Omega_F/k + \mathcal{V}_F) & \xleftarrow{\alpha_{(F, r+n)}} & \Gamma^*(M)(F, (r+n)) & \xleftarrow{p_{(r+n)}} & \Gamma^*(M)(F)
 \end{array}$$

where the maps are defined as before. The left back square is commutative according to (R2b) and (R4a). The left front square is commutative since α is compatible with (d2). The remaining squares are commutative by definition. Hence the square (2) is commutative.

We prove compatibility with (D3). Let E be a field and \mathcal{V}_E a virtual vector space of dimension n over E . Let $u \in E^\times$ be a unit and consider multiplication with $[u] \in \underline{K}^{MW}(E, \mathbb{A}^1)$. We want to prove that the following diagram (3) is commutative:

$$\begin{array}{ccccccc}
 M(E, \mathcal{V}_E) & \xrightarrow{\Theta} & M(E, n \cdot \mathbb{A}_E^1) & \xrightarrow{l_n} & \bigoplus_{m \in \mathbb{Z}} M(E, m \cdot \mathbb{A}_E^1) & \xrightarrow{\text{mod } \eta} & \Gamma^*(M)(E) \\
 \downarrow \gamma_{[u]} & \searrow \Phi(\alpha)_{(E, \mathcal{V}_E)} & \downarrow \gamma_{[u]} & & \downarrow \gamma_{[u]} & & \downarrow \gamma_{[u]} \\
 & & \Gamma_*(N)(E, \mathcal{V}_E) & \xleftarrow{\alpha_{(E, n)}} & \Gamma^*(M)(E, n) & \xleftarrow{p_n} & \Gamma^*(M)(E) \\
 & & \downarrow \gamma_{[u]} & & \downarrow \gamma_{[u]} & & \downarrow \gamma_{[u]} \\
 M(E, \mathbb{A}_E^1 + \mathcal{V}_E) & \xrightarrow{\Theta} & M(E, (n+1) \cdot \mathbb{A}_E^1) & \xrightarrow{l_{n+1}} & \bigoplus_{m \in \mathbb{Z}} M(E, m \cdot \mathbb{A}_E^1) & \xrightarrow{\text{mod } \eta} & \Gamma^*(M)(E) \\
 \downarrow \gamma_{[u]} & \searrow \Phi(\alpha)_{(E, \mathbb{A}_E^1 + \mathcal{V}_E)} & \downarrow \gamma_{[u]} & & \downarrow \gamma_{[u]} & & \downarrow \gamma_{[u]} \\
 & & \Gamma_*(N)(E, \mathbb{A}_E^1 + \mathcal{V}_E) & \xleftarrow{\alpha_{(E, n+1)}} & \Gamma^*(M)(E, n+1) & \xleftarrow{p_{n+1}} & \Gamma^*(M)(E)
 \end{array}$$

where the maps are defined as before. The left back square is commutative according to (R4a). The left front square is commutative since α is compatible with (d3). The remaining squares are commutative by definition. Hence the square (3) is commutative. The same proof works for the generator η , hence $\Phi(\alpha)$ is compatible with (D3).

We prove compatibility with (D4). Let E be a field over k , let v be a valuation on E and let \mathcal{V} be a virtual projective \mathcal{O}_v -module of finite type. Denote by $\mathcal{V}_E = \mathcal{V} \otimes_{\mathcal{O}_v} E$ and $\mathcal{V}_{\kappa(v)} = \mathcal{V} \otimes_{\mathcal{O}_v} \kappa(v)$. We want to prove that the following diagram (4) is commutative:

$$\begin{array}{ccccccc}
 M(E, \mathcal{V}_E) & \xrightarrow{\Theta} & M(E, n \cdot \mathbb{A}_E^1) & \xrightarrow{l_n} & \bigoplus_{m \in \mathbb{Z}} M(E, m \cdot \mathbb{A}_E^1) & \xrightarrow{\text{mod } \eta} & \Gamma^*(M)(E) \\
 \downarrow \partial_v & \searrow \Phi(\alpha)_{(E, \mathcal{V}_E)} & \downarrow \partial_v & & \downarrow \partial_v & & \downarrow \partial_v \\
 & & \Gamma_*(N)(E, \mathcal{V}_E) & \xleftarrow{\alpha_{(E, n)}} & \Gamma^*(M)(E, n) & \xleftarrow{p_n} & \Gamma^*(M)(E) \\
 & & \downarrow \partial_v & & \downarrow \partial_v & & \downarrow \partial_v \\
 M(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) & \xrightarrow{\Theta} & M(\kappa(v), (n-1) \cdot \mathbb{A}_{\kappa(v)}^1) & \xrightarrow{l_{n-1}} & \bigoplus_{m \in \mathbb{Z}} M(\kappa(v), m \cdot \mathbb{A}_{\kappa(v)}^1) & \xrightarrow{\text{mod } \eta} & \Gamma^*(M)(\kappa(v)) \\
 \downarrow \partial_v & \searrow \Phi(\alpha)_{(F, \mathcal{V}_F)} & \downarrow \partial_v & & \downarrow \partial_v & & \downarrow \partial_v \\
 & & \Gamma_*(N)(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) & \xleftarrow{\alpha_{(\kappa(v), n-1)}} & \Gamma^*(M)(\kappa(v), n-1) & \xleftarrow{p_{n-1}} & \Gamma^*(M)(\kappa(v))
 \end{array}$$

where the maps are defined as before. The left back square is commutative according to (R4a). The left front square is commutative since α is compatible with (d4). The remaining squares are commutative by definition. Hence the square (4) is commutative.

Thus, we have defined a map

$$\begin{aligned}\Phi : \operatorname{Hom}_{\mathfrak{M}_k^M}(\Gamma^*(M), N) &\rightarrow \operatorname{Hom}_{\mathfrak{M}_k^{\text{MW}}}(M, \Gamma_*(N)) \\ \alpha &\mapsto \Phi(\alpha)\end{aligned}$$

which is natural in M and N .

Let M be a Milnor-Witt cycle module and N a Rost cycle module, we now define a map

$$\Psi : \operatorname{Hom}_{\mathfrak{M}_k^{\text{MW}}}(M, \Gamma_*(N)) \rightarrow \operatorname{Hom}_{\mathfrak{M}_k^M}(\Gamma^*(M), N).$$

Let β be an element of $\operatorname{Hom}_{\mathfrak{M}_k^{\text{MW}}}(M, \Gamma_*(N))$. Let E be a field. By definition, we have maps¹¹

$$\beta_{(E,n)} : M(E, \mathcal{V}_E) \rightarrow N(E, n)$$

for any virtual vector space \mathcal{V}_E of rank n over E . These maps are compatible with data (D1), (D2), (D3) and (D4) according to Definition 2.3.2.5. Thus, we can define a map

$$\bigoplus_{n \in \mathbb{Z}} \beta_{(E,n)} : \bigoplus_{n \in \mathbb{Z}} M(E, n \cdot \mathbb{A}_E^1) \rightarrow \bigoplus_{n \in \mathbb{Z}} N(E, n)$$

which can be factorized as

$$\Psi(\beta)_E : \Gamma^*(M(E)) \rightarrow N(E) = \bigoplus_{n \in \mathbb{Z}} N(E, n)$$

since β commutes with the $\underline{K}^{\text{MW}}$ -action and η acts trivially on the Milnor-Witt cycle module $\Gamma_*(N)$ by definition. Since β commutes with data (D1), (D2), (D3) and (D4), we can prove that the map $\Psi(\beta) : E \mapsto \Psi(\beta)_E$ is a morphism of Rost cycle modules.

Thus, we have defined a map

$$\begin{aligned}\Psi : \operatorname{Hom}_{\mathfrak{M}_k^{\text{MW}}}(M, \Gamma_*(N)) &\rightarrow \operatorname{Hom}_{\mathfrak{M}_k^M}(\Gamma^*(M), N) \\ \beta &\mapsto \Psi(\beta)\end{aligned}$$

which is natural in M and N .

Since the map Ψ is inverse to Φ , we conclude that the two functors Γ^* and Γ_* form an adjunction $\Gamma^* \dashv \Gamma_*$ between the category of Milnor-Witt cycle modules and the category of classical cycle modules:

$$\Gamma^* : \mathfrak{M}_k^{\text{MW}} \rightleftarrows \mathfrak{M}_k^M : \Gamma_*.$$

□

¹¹The fact that β commutes with (D1) implies that these maps depend only on the rank n of \mathcal{V}_E .

3

Homotopy modules and Milnor-Witt cycle modules

We study the cohomology theory and the canonical Milnor-Witt cycle module associated to a motivic spectrum. We prove that the heart of Morel-Voevodsky stable homotopy category over a perfect field (equipped with its homotopy t -structure) is equivalent to the category of Milnor-Witt cycle modules, thus generalising Déglise’s thesis [Dé11]. As a corollary, we recover a theorem of Ananyevskiy and Neshitov [AN18] and we prove that Chow-Witt groups in degree zero are birational invariants.

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3.1 Introduction

In the fundamental paper [Ros96], Rost introduced the notion of a cycle module. The idea was to find a good axiomatization of the main properties encountered in the study of Milnor K-theory, Quillen K-theory or Galois cohomology. According to [Ros96], a cycle module M over a perfect field k is the data of a \mathbb{Z} -graded abelian group $M(E)$ for every finitely generated field extension E/k , equipped with restriction maps, corestriction maps, a Milnor K-theory module action and residue maps ∂ . Moreover, these data are subject to certain compatibility relations $(r1a), \dots, (r3e), (fd)$ and (c) . The theory results in the construction of Gersten type complexes whose cohomology groups are called *Chow groups with coefficients* and can be used, for instance, to extend to the left the localization sequence of Chow groups associated with a closed embedding.

In order to construct the derived category of motives $\mathbf{DM}(k, \mathbb{Z})$, Voevodsky introduced the so-called homotopy sheaves (with transfers) which are homotopy invariant Nisnevich sheaves with transfers. One important example is given by \mathbb{G}_m , the sheaf of global units. Voevodsky proved that any homotopy sheaf F has a Gersten resolution, implying that F is determined in some sense by the data of its fibers in every function fields. This statement was made more precise in Déglise's thesis: the heart of $\mathbf{DM}(k, \mathbb{Z})$ with respect to its homotopy t-structure has a presentation given by the category of Rost cycle modules over k .

Morel's point of view on the heart of $\mathbf{DM}(k, \mathbb{Z})$ is given by the category of oriented homotopy modules. We recall that a homotopy module is a strictly \mathbb{A}^1 -invariant Nisnevich sheaf with an additional structure defined over the category of smooth schemes (see Definition 3.4.1.2); it is called *oriented* when the Hopf map η acts on it by 0. Déglise's theorem proves that oriented homotopy modules form a subcategory of the category of homotopy modules which is equivalent to the category of Rost cycle modules. Morel's natural conjecture [Mor12, Remark 2.49] was that there is a presentation of the heart of the stable homotopy category $\mathbf{SH}(k)$ (or equivalently, the category of homotopy modules) in terms of some non-oriented version of cycle modules.

3.1.1 Current work

In chapter 2, we introduced the theory of Milnor-Witt cycle modules, generalising the work of Rost [Ros96] on cycle modules and Schmid's thesis [Sch98]. Indeed, we have studied general cycle complexes $C^*(X, M, \mathcal{V}_X)$ and their (co)homology groups $A^*(X, M, \mathcal{V}_X)$ (called *Chow-Witt groups with coefficients*) in a quadratic setting over a perfect base field of any characteristic. The general coefficient systems M for these complexes are called Milnor-Witt cycle modules. The main example of such a cycle module is given by Milnor-Witt K-theory (see Theorem 2.3.2.13); other examples will be deduced from Theorem 3.4.0.4 or Theorem 3.5.2.4 (e.g. the representability of hermitian K-theory in $\mathbf{SH}(k)$ will lead to a MW-cycle module, associated with hermitian K-theory). A major difference with Rost's theory is that the grading to be considered is not \mathbb{Z} but the category of virtual bundles (or, equivalently, the category of virtual vector spaces), where a virtual bundle \mathcal{V} is, roughly speaking, the data of an integer n and a line bundle \mathcal{L} (see Subsection 1.9.1). Intuitively, Milnor-Witt cycle modules are given by (twisted) graded abelian groups equipped with extra data (restriction, corestriction, \underline{K}^{MW} -action and residue maps).

For any scheme X , any virtual bundle \mathcal{V}_X and any Milnor-Witt cycle module M , we have proved that there exists a complex $C^*(X, M, \mathcal{V}_X)$ equipped with pushforwards,

pullbacks, a Milnor-Witt K-theory action and residue maps satisfying standard functoriality properties. A fundamental theorem is that the associated cohomology groups $A^*(X, M, \mathcal{V}_X)$ satisfy the homotopy invariance property (see Theorem 2.6.2.4).

In this chapter, we prove that Milnor-Witt cycle modules are closely related to Morel's \mathbb{A}^1 -homotopy theory: they can be realized geometrically as elements of the stable homotopy category. Precisely, we prove the following theorem.

Theorem 1 (Theorem 3.4.0.4). *Let k be a perfect field. The category of Milnor-Witt cycle modules is equivalent to the heart of Morel-Voevodsky stable homotopy category (equipped with the homotopy t -structure):*

$$\mathfrak{M}_k^{MW} \simeq \mathbf{SH}(k)^{\heartsuit}.$$

In order to prove this theorem, we study the cohomology theory associated with a motivic spectrum. This notion is naturally dual to the bivariant theory developed in [DJK18] and recalled in Section 3.2 (see Theorem 3.2.3.1). A motivic spectrum \mathbb{E} leads to a functor $\hat{\mathbb{E}}$ from the category of finitely generated fields over k to the category of graded abelian groups (be careful that the grading is not \mathbb{Z} but is given by the category of virtual vector spaces). We prove that the functor $\hat{\mathbb{E}}$ is a Milnor-Witt cycle premodule (see Definition 2.3.1.1). Indeed, most axioms are immediate consequences of the general theory [DJK18]. Moreover, in Theorem 3.2.2.2 we prove a ramification theorem of independent interest that can be applied to prove rule (R3a). Furthermore, we check axioms (FD) and (C) so that $\hat{\mathbb{E}}$ is a Milnor-Witt cycle module. These two axioms follow from the study of a spectral sequence defined in Section 3.3 (see Theorem 3.3.3.2); another – more elementary – proof may result from an adaptation of [Ros96, Theorem 2.3] to the context of Milnor-Witt cycle modules but this method would rely heavily on the fact that the base field is perfect.

In Section 3.4, we construct a homotopy module for any Milnor-Witt cycle module and proceed to prove that the heart of the stable homotopy category (which is known to be equivalent to the category of homotopy module) is equivalent to the category of Milnor-Witt cycle modules.

This result generalizes Déglise's thesis (see Theorem 3.5.2.3) and answers affirmatively an old conjecture of Morel (see [Mor12, Remark 2.49]). An important corollary is the following result (which was proved independently by Ananyevskiy and Neshitov in [AN18, Theorem 8.12]):

Theorem 2 (Theorem 3.5.2.2). *The heart of Morel-Voevodsky stable homotopy category is equivalent to the heart of the category of MW-motives [DF18a] (both equipped with their respective homotopy t -structures):*

$$\mathbf{SH}(k)^{\heartsuit} \simeq \widetilde{\mathbf{DM}}(k)^{\heartsuit}.$$

3.1.2 Outline of the chapter

In Section 3.2, we follow [DJK18] and define the bivariant theory associated to a motivic spectrum. We extend the main results for the associated cohomology theory. We study the basic properties of fundamental classes and prove a ramification formula.

In Section 3.3, we recall the theory of Milnor-Witt cycle modules developed in chapter 2. For any motivic spectrum, we then construct a Milnor-Witt cycle modules in a functorial way.

The heart of the chapter is Section 3.4 where we define a homotopy module for any Milnor-Witt cycle module and prove our main theorem: the heart of the stable homotopy category (which is known to be equivalent to the category of homotopy module according to [Mor03]) is equivalent to the category of Milnor-Witt cycle modules.

Finally in Section 3.5, we give some corollaries of the main theorem. In particular, we show that the heart of stable homotopy category is equivalent to the heart of the category of MW-motives [DF18a]. We also prove that Chow-Witt groups and Morel homotopy modules are birational invariants of smooth projective k -schemes.

3.1.3 Notation

Throughout the chapter, we fix a (commutative) field k and we assume moreover that k is perfect (of arbitrary characteristic). We consider only schemes that are noetherian and essentially of finite type¹ over k . All schemes and morphisms of schemes are defined over k .

We denote by $S = \operatorname{Spec} k$ the spectrum of k .

By a field E over k , we mean a k -finitely generated field E . Since k is perfect, notice that $\operatorname{Spec} E$ is essentially smooth over S .

Let $f : X \rightarrow S$ be a morphism of schemes and \mathcal{V}_S be a virtual bundle over S . We denote by \mathcal{V}_X or by $f^*\mathcal{V}_S$ or by $\mathcal{V}_S \times_S X$ the pullback of \mathcal{V}_S by f .

Let $f : X \rightarrow Y$ be a morphism of schemes. Denote by \mathcal{L}_f or by $\mathcal{L}_{X/Y}$ the virtual vector bundle over Y associated with the cotangent complex of f . If $p : X \rightarrow Y$ is a smooth morphism, then \mathcal{L}_p is (isomorphic to) $\mathcal{T}_p = \Omega_{X/Y}$ the space of (Kähler) differentials. If $i : Z \rightarrow X$ is a regular closed immersion, then \mathcal{L}_i is the normal cone $-\mathcal{N}_Z X$. If f is the composite $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ with p and i as previously (in other words, if f is lci projective), then \mathcal{L}_f is isomorphic to the virtual tangent bundle $i^* \mathcal{T}_{\mathbb{P}_X^n/X} - \mathcal{N}_Y(\mathbb{P}_X^n)$ (see also [Fel18, Section 9]).

Let E be a field (over k) and v a (discrete) valuation on E . We denote by \mathcal{O}_v its valuation ring, by \mathfrak{m}_v its maximal ideal and by $\kappa(v)$ its residue class field. We consider only valuations of geometric type, that is we assume: $k \subset \mathcal{O}_v$, the residue field $\kappa(v)$ is finitely generated over k and satisfies $\operatorname{tr. deg}_k(\kappa(v)) + 1 = \operatorname{tr. deg}_k(E)$.

For E a field (resp. X a scheme), we denote by $\langle n \rangle$ the virtual space \mathbb{A}_E^n (resp. \mathbb{A}_X^n).

3.2 Bivariant theory

3.2.1 Recollection and notations

In this subsection, we recall some results from [DJK18, §2]. Let S be a base scheme. Denote by $\mathbb{V}(S)$ the Picard groupoid of virtual vector bundles on S (see Subsection 1.9.1). If \mathcal{V}_S is a virtual vector bundle over S , we denote by $\operatorname{Th}_S(\mathcal{V}_S)$ its associated Thom space (this is an \otimes -invertible motivic spectra over S , see [CD19, Remark 2.4.15]).

¹That is, isomorphic to a limit of finite type schemes with affine étale transition maps.

Definition 3.2.1.1. Let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum. Given a separated morphism of finite type $p : X \rightarrow S$, an integer $n \in \mathbb{Z}$ and a virtual bundle $\mathcal{V}_X \in \mathbb{V}(X)$, we define the *bivariant theory* of X/S in bidegree (n, \mathcal{V}_X) , with coefficients in \mathbb{E} , as the abelian group:

$$\mathbb{E}_n(X/S, \mathcal{V}_X) = [\mathrm{Th}_X(\mathcal{V}_X)[n], p^!(\mathbb{E})] = [p_! \mathrm{Th}_X(\mathcal{V}_X)[n], \mathbb{E}].$$

The *cohomology theory* represented by \mathbb{E} is defined by the formula:

$$\mathbb{E}^n(X, \mathcal{V}_X) = \mathbb{E}_{-n}(X/X, -\mathcal{V}_X) = [\mathbb{1}_X, \mathbb{E}_X \otimes \mathrm{Th}_X(\mathcal{V}_X)[n]]$$

for any scheme X over S and any pair $(n, \mathcal{V}_X) \in \mathbb{Z} \times \mathbb{V}(X)$.

In the special case where $\mathbb{E} = \mathbb{1}$ is the sphere spectrum, we will use the notation

$$H_n(X/S, \mathcal{V}_X) = \mathbb{1}_n(X/S, \mathcal{V}_X) = [\mathrm{Th}_X(\mathcal{V}_X)[n], p^!(\mathbb{1}_S)]$$

and we will refer to this simply as the *bivariant \mathbb{A}^1 -theory*.

Similarly, we set $H^n(X, \mathcal{V}_X) = \mathbb{1}^n(X, \mathcal{V}_X)$ and refer to this as the *\mathbb{A}^1 -cohomology*.

3.2.1.2. BASE CHANGE. For any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ q \downarrow & \Delta & \downarrow p \\ T & \xrightarrow{f} & S, \end{array}$$

one gets a map

$$\Delta^* : \mathbb{E}_n(X/S, \mathcal{V}_X) \rightarrow \mathbb{E}_n(Y/T, \mathcal{V}_T)$$

by applying the functor $g^* : \mathbf{SH}(X) \rightarrow \mathbf{SH}(Y)$ and using the exchange transformation $\mathrm{Ex}^{*,!} : g^* p^! \rightarrow q^! f^*$ associated with the square Δ .

3.2.1.3. COVARIANCE FOR PROPER MORPHISMS. Let $f : Y \rightarrow X$ be a proper morphism. We have a map

$$f_* : \mathbb{E}_n(Y, \mathcal{V}_Y) \rightarrow \mathbb{E}_n(X, \mathcal{V}_X)$$

coming from the unit map $f_! f^! \rightarrow \mathrm{Id}$ and the fact that $f_! = f_*$ since f is proper.

3.2.1.4. CONTRAVARIANCE FOR ÉTALE MORPHISMS. Let $f : Y \rightarrow X$ be an étale morphism, we have a map

$$f^* : \mathbb{E}_n(X/S, \mathcal{V}_X) \rightarrow \mathbb{E}_n(Y/S, \mathcal{V}_Y)$$

obtained by applying the functor $f^! : \mathbf{SH}(X) \rightarrow \mathbf{SH}(S)$ and using the purity isomorphism $f^! = f^*$ as f is étale.

3.2.1.5. PRODUCTS. Consider a multiplication map $\mu : \mathbb{E} \otimes \mathbb{E}' \rightarrow \mathbb{E}''$ between motivic spectra. For any s-schemes $Y \xrightarrow{q} X \xrightarrow{p} S$, any integers n, m and any virtual vector bundles \mathcal{W}_Y/Y and \mathcal{V}_X/X , there is a multiplication map

$$\mathbb{E}_m(Y/X, \mathcal{W}_Y) \otimes \mathbb{E}'_n(X/S, \mathcal{V}_X) \rightarrow \mathbb{E}''_{m+n}(Y/S, \mathcal{W}_Y + \mathcal{V}_X).$$

Definition 3.2.1.6. Let $X \rightarrow S$ be a separated morphism of finite type.

- A *fundamental class*² of f is an element

$$\eta_f \in H(X/S, \mathcal{V}_f)$$

for a given virtual vector bundle \mathcal{V}_f over X .

- Let \mathcal{C} be a subcategory of the category of (quasi-compact and quasi-separated) schemes. A *system of fundamental classes* for \mathcal{C} is the data, for each morphism $f : X \rightarrow Y$ in \mathcal{C} , of a virtual bundle $\mathcal{V}_f \in \mathbb{V}(X)$ and an orientation $\eta_f^\mathcal{C} \in H_0(f, \mathcal{V}_f)$ such that the following relations hold:

1. *Normalisation.* If $f = \text{Id}_S$, then $\mathcal{V}_f = 0$ and the orientation $\eta_f^\mathcal{C} \in H_0(\text{Id}_S, 0)$ is given by the identity $\text{Id} : \mathbb{1}_S \rightarrow \mathbb{1}_S$.
2. *Associativity formula.* For any composable morphisms f and g in \mathcal{C} , one has an isomorphism:

$$\mathcal{V}_{f \circ g} \simeq \mathcal{V}_g + g^* \mathcal{V}_f$$

and, modulo this identification, the following relation holds:

$$\eta_g^\mathcal{C} \cdot \eta_f^\mathcal{C} = \eta_{f \circ g}^\mathcal{C}$$

- Suppose the category \mathcal{C} admits fibred products. We say that a system of fundamental classes $(\eta_f^\mathcal{C})_f$ is *stable under transverse base change* if it satisfies the following condition: for any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{g} & T \\ q \downarrow & \Delta & \downarrow p \\ X & \xrightarrow{f} & S. \end{array}$$

such that f and g are in \mathcal{C} and p is transverse to f , then one has $\mathcal{V}_g = q^* \mathcal{V}_f$ and the following formula holds in $H_0(g, \mathcal{V}_g)$: $\Delta^*(\eta_f^\mathcal{C}) = \eta_g^\mathcal{C}$.

In [DJK18, Theorem 3.3.2], the authors prove the following theorem.

Theorem 3.2.1.7. *There exists a unique system of fundamental classes $\eta_f \in H_0(X/S, \mathcal{L}_f)$ associated with the class of quasi-projective lci morphisms f such that:*

1. *For any smooth separated morphism of finite type p , the class η_p agrees with the fundamental class defined in [DJK18, Example 2.3.9] thanks to the purity isomorphism.*
2. *For any regular closed immersion $i : Z \rightarrow X$, the class η_i agrees with the fundamental class defined in [DJK18, Theorem 3.2.21] by deformation to the normal cone.*

Thanks to this system of fundamental classes, we can define Gysin morphisms as follows.

²Also called *orientation* in [DJK18, Definition 2.3.2].

3.2.1.8. CONTRAVARIANCE FOR LCI MORPHISMS. Let $f : Y \rightarrow X$ be an lci quasi-projective morphism $f : Y/S \rightarrow X/S$ of separated morphisms of finite type with virtual cotangent bundle \mathcal{L}_f . For any motivic spectrum $\mathbb{E} \in \mathbf{SH}(S)$, for any integer n and any virtual bundle \mathcal{V}_X over X , there exists a Gysin morphism:

$$\begin{aligned} f^* : \mathbb{E}_n(X/S, \mathcal{V}_X) &\rightarrow \mathbb{E}_n(Y/S, \mathcal{L}_f + \mathcal{V}_Y) \\ x &\mapsto \eta_f \cdot x \end{aligned}$$

using the product defined in 3.2.1.5. These Gysin morphisms satisfy the following formulas:

1. *Functoriality:* For any suitable morphisms f and g , one has $(fg)^* = g^* f^*$.
2. *Base change:* For any cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{g} & T \\ q \downarrow & \Delta & \downarrow p \\ X & \xrightarrow{f} & S. \end{array}$$

such that f is quasi-projective lci and transverse to p , one has $f^* p_* = q_* g^*$.

3.2.1.9. LOCALIZATION LONG EXACT SEQUENCE. Another essential property is the following long exact sequence following from the usual localization triangle in the six functors formalism in \mathbf{SH} .

Indeed, let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum. For any closed immersion $i : Z \rightarrow X$ of separated schemes over S , with (quasi-compact) complementary open immersion $j : U \rightarrow X$, there exists a canonical localization long exact sequence of the form:

$$\dots \rightarrow \mathbb{E}_n(Z/S, \mathcal{V}_Z) \xrightarrow{i_*} \mathbb{E}_n(X/S, \mathcal{V}_X) \xrightarrow{j^*} \mathbb{E}_n(U/S, \mathcal{V}_U) \xrightarrow{\partial_i} \mathbb{E}_{n-1}(Z/S, \mathcal{V}_Z) \rightarrow \dots$$

We will need the following properties of localizations long exact sequences.

Proposition 3.2.1.10. *Let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum and consider the following commutative square*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ q \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

of closed immersions of separated schemes over S . For \mathcal{V}_X a virtual vector bundle over X , we have the following diagram

$$\begin{array}{ccccccc} \mathbb{E}_n(T/S, \mathcal{V}_T) & \xrightarrow{k_*} & \mathbb{E}_n(Y/S, \mathcal{V}_Y) & \xrightarrow{k'^*} & \mathbb{E}_n(Y-T, \mathcal{V}_{Y-T}) & \xrightarrow{\partial_k} & \mathbb{E}_{n-1}(T/S, \mathcal{V}_T) \\ q_* \downarrow & & p_* \downarrow & & \tilde{p}_* \downarrow & & q_* \downarrow \\ \mathbb{E}_n(Z/S, \mathcal{V}_Z) & \xrightarrow{i_*} & \mathbb{E}_n(X/S, \mathcal{V}_X) & \xrightarrow{i'^*} & \mathbb{E}_n(X-Z, \mathcal{V}_{X-Z}) & \xrightarrow{\partial_i} & \mathbb{E}_{n+1}(Z/S, \mathcal{V}_Z) \\ q'^* \downarrow & & p'_* \downarrow & & \tilde{p}'_* \downarrow & & q'^* \downarrow \\ \mathbb{E}_n(Z-T/S, \mathcal{V}_{Z-T}) & \xrightarrow{\tilde{i}_*} & \mathbb{E}_n(X-Y/S, \mathcal{V}_{X-Y}) & \xrightarrow{\tilde{i}'^*} & \mathbb{E}_n(X-(Z \cup Y)/S, \mathcal{V}_{X-(Z \cup Y)}) & \xrightarrow{\partial_{\tilde{i}}} & \mathbb{E}_{n-1}(Z-T/S, \mathcal{V}_{Z-T}) \\ \partial_k \downarrow & & \partial_p \downarrow & & \partial_{\tilde{p}} \downarrow & (*) & \partial_{\tilde{k}} \downarrow \\ \mathbb{E}_{n-1}(T/S, \mathcal{V}_T) & \xrightarrow{k_*} & \mathbb{E}_{n-1}(Y/S, \mathcal{V}_Y) & \xrightarrow{k'^*} & \mathbb{E}_{n-1}(Y-T/S, \mathcal{V}_{Y-T}) & \xrightarrow{\partial_k} & \mathbb{E}_{n-2}(T/S, \mathcal{V}_T) \end{array}$$

with obvious maps. Each squares of this diagram is commutative except for $(*)$ which is anti-commutative.

Proof. See [DJK18, Proposition 2.2.11]. \square

Proposition 3.2.1.11 (Base change for lci morphisms). *Consider a cartesian square of schemes*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

with f proper, and g (quasi-projective) lci. Suppose moreover that the square is tor-independent, that is for any $x \in X$, $y' \in Y'$ with $y = f(x) = g(y')$ and for any $i > 0$ we have

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.$$

Up to the canonical isomorphism $f'^* \mathcal{L}_g \simeq \mathcal{L}_{g'}$, we have:

$$f'_* \circ g'^* = g^* \circ f_*.$$

Proof. See [DJK18, Proposition 4.1.3]. \square

The bivariant theory of a spectrum satisfies some \mathbb{A}^1 -homotopy invariance property.

Proposition 3.2.1.12. *Let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum. Let X be an s -scheme over S and let $p : V \rightarrow X$ be a vector bundle with virtual tangent bundle $\mathcal{L}_p = p^* \langle V \rangle$. Then the Gysin morphism*

$$p^* : \mathbb{E}_n(X/S, \mathcal{V}_X) \rightarrow \mathbb{E}_n(V/S, \mathcal{L}_p + \mathcal{V}_V)$$

is an isomorphism for any integer n and any virtual bundle \mathcal{V}_X over X .

Proof. See [DJK18, Lemma 2.4.4] when \mathbb{E} is the sphere spectrum $\mathbb{1}$. The general case follows from the definitions. \square

Definition 3.2.1.13. Keeping the notations of 3.2.1.12, we define the *Thom isomorphism*

$$\Phi_{V/X} : \mathbb{E}_n(V/S, \mathcal{V}_V) \rightarrow \mathbb{E}_n(X/S, \mathcal{V} - \langle V \rangle),$$

associated with V/X , as the inverse of the Gysin morphism $p^* : \mathbb{E}_n(X/S, \mathcal{V}_X - \langle E \rangle) \rightarrow \mathbb{E}_n(V/S, \mathcal{V}_V)$

Remark 3.2.1.14. The Thom isomorphism is compatible with base change and direct sums (see [DJK18, Remark 2.4.6]).

3.2.2 Ramification formula

3.2.2.1. Consider a topologically cartesian square

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ q \downarrow & \Delta & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

where k and i are regular closed immersions of codimension 1 and T is a reduced connected scheme. We have $T = (Z \times_X Y)_{\text{red}}$. Denote by \mathfrak{I}' the ideal of the immersion $Z \times_X Y \rightarrow Y$, by \mathfrak{I} the ideal of k and by \mathfrak{J} the ideal of i . Assume moreover that the square is ramified with ramification index e in the sense that there exists a nonzero natural number e such that $\mathfrak{I}' = \mathfrak{I}^e$. We consider the morphism of deformation spaces $v^{(e)} : D_T Y \rightarrow D_Z X$ defined as the spectrum of the composite

$$\bigoplus_{n \in \mathbb{Z}} \mathfrak{I}^n \cdot t^{-n} \longrightarrow \bigoplus_{n \in \mathbb{Z}} (\mathfrak{I}^e)^n \cdot t^{-n} \longrightarrow \bigoplus_{m \in \mathbb{Z}} \mathfrak{I}^m \cdot t^{-m}$$

$$x \cdot t^{-n} \longmapsto \tilde{f}(x) \cdot t^{-n} \longmapsto \tilde{f}(x) \cdot t^{-en}$$

where the first map is induced by the morphism $\tilde{f} : \mathfrak{I} \rightarrow \mathfrak{I}'$ defined via f and where the second map takes the parameter t to its power t^e .

The map $v^{(e)}$ factors naturally making the following diagram commutative:

$$\begin{array}{ccccc} N_T Y & \hookrightarrow & D_T Y & \longleftarrow & \mathbb{G}_m Y \\ \downarrow & & \downarrow & & \parallel \\ q^* N_Z X & \hookrightarrow & q^* D_Z X & \longleftarrow & \mathbb{G}_m Y \\ \downarrow & & \downarrow & & \downarrow \\ N_Z X & \hookrightarrow & D_Z X & \longleftarrow & \mathbb{G}_m X. \end{array}$$

Hence, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{E}_{n+1}(\mathbb{G}_m Y/Y, *) & \xrightarrow{\partial_{T/Y}} & \mathbb{E}_n(N_T Y/Y, *) & \xrightarrow[\sim]{\Phi_{N_T Y/T}} & \mathbb{E}_n(T/Y, -\mathcal{N}_T Y + *) \\ \parallel & (1) & \downarrow & (3) & \downarrow v_*^{(e)} \\ \mathbb{E}_{n+1}(\mathbb{G}_m Y/Y, *) & \longrightarrow & \mathbb{E}_n(q^* N_Z X/Y, *) & \xrightarrow[\sim]{\Phi_{q^* N_Z X/T}} & \mathbb{E}_n(T/Y, -q^* \mathcal{N}_Z X + *) \\ \Delta^* \uparrow & (2) & \Delta^* \uparrow & (4) & \Delta^* \uparrow \\ \mathbb{E}_{n+1}(\mathbb{G}_m X/X, *) & \xrightarrow{\partial_{Z/X}} & \mathbb{E}_n(N_Z X/X, *) & \xrightarrow[\sim]{\Phi_{N_Z X/Z}} & \mathbb{E}_n(Z/X, -\mathcal{N}_Z X + *) \end{array}$$

where the arrows Δ^* denote the obvious maps induced by the corresponding squares. Square (1) (resp. square (2)) is commutative because of the naturality of localization long exact sequences with respect to the proper covariance (resp. base change). The map $v_*^{(e)}$ is defined so that the square (3) commutes. Square (4) commutes by compatibility of Thom isomorphisms with respect to base change (3.2.1.14).

From this, we deduce the formula

$$\Delta^*(\eta_i) = v_*^{(e)}(\eta_k).$$

The map $v^{(e)}$ induces a morphism of Thom spaces

$$\mathrm{Th}(N_T Y) \rightarrow \mathrm{Th}(q^* N_Z X)$$

which corresponds after delooping to the quadratic form e_ε in $\mathbf{K}^{\mathrm{MW}}(\kappa(v))$ (see also [Caz08]). We have proved the following theorem:

Theorem 3.2.2.2. *Keeping the previous notations, the following holds in $\mathbb{E}_n(T/Y, N_T Y)$*

$$\Delta^*(\eta_i) = e_\varepsilon \cdot \eta_k.$$

Corollary 3.2.2.3. *Consider the topologically cartesian square*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ q \downarrow & \Delta & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

where k and i are regular closed immersions of codimension 1 and T is a reduced connected scheme. Assume moreover that the square is ramified with ramification index e as before. Then we have the following ramification formula:

$$p^* i_* = e_\varepsilon k_* q^*$$

Remark 3.2.2.4. One could say that the ramification index e_ε represents the defect of transversality.

3.2.3 Application to cohomology

For a spectrum $\mathbb{E} \in \mathbf{SH}(S)$, a natural number $n \in \mathbb{N}$, a morphism $p : X \rightarrow S$ of schemes and \mathcal{V}_X a virtual vector bundle over X , we define the cohomology group

$$\mathbb{E}^n(X, \mathcal{V}_X) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathbb{E}_X \otimes \mathrm{Th}_X(\mathcal{V}_X)[n])$$

where $\mathbb{E}_X = p^* \mathbb{E}_S$.

It is dual to the bivariant theory $\mathbb{E}_*(-, *)$ defined previously. Indeed, we have the following theorem:

Theorem 3.2.3.1. *Let $f : X \rightarrow S$ be an essentially³ smooth scheme and $\mathbb{E} \in \mathbf{SH}(S)$ a motivic spectrum. Then for any integer n and any virtual bundle \mathcal{V}_X over X , there is a canonical isomorphism*

$$\mathbb{E}^n(X, \mathcal{V}_X) \simeq \mathbb{E}_{-n}(X/S, \mathcal{L}_f - \mathcal{V}_X)$$

which is contravariantly natural in X with respect to étale morphisms.

Example 3.2.3.2. A crucial example follows from the work of Morel: if \mathbb{E} is the unit sphere $\mathbb{1}$ and $X = \mathrm{Spec} E$ is the spectrum of a field, then the group $H^n(X, \langle n \rangle)$ is isomorphic to the Milnor-Witt theory $\mathbf{K}_n^{\mathrm{MW}}(E)$.

³In [DJK18], the authors worked only with separated S -schemes of finite type but we can extend in a canonical way most of the results for separated S -schemes *essentially* of finite type (see also [ADN19, §2.1.1]).

In the following, we give the usual properties of the bivariant cohomology theory. The proofs follow directly by duality and the preceding results. In practice (since our base field k is perfect) we work mainly with essentially smooth schemes, hence we could also apply Theorem 3.2.3.1.

3.2.3.3. CONTRAVARIANCE. Let $f : Y \rightarrow X$ be a morphism of schemes and \mathcal{V}_X be a virtual bundle over X . There exists a pullback map

$$f^* : \mathbb{E}^n(X, \mathcal{V}_X) \rightarrow \mathbb{E}^n(Y, \mathcal{V}_Y).$$

3.2.3.4. COVARIANCE. Let $f : Y \rightarrow X$ be an lci projective map. There exists a Gysin morphism

$$f_* : \mathbb{E}^n(Y, \mathcal{L}_f + \mathcal{V}_Y) \rightarrow \mathbb{E}^n(X, \mathcal{V}_X).$$

As always, the definition follows from general considerations using the six functors formalism. For instance, assume f smooth. By adjunction, the set $\mathrm{Hom}_{\mathbf{SH}(Y)}(\mathbb{1}_Y, \mathbb{E}_Y \otimes \mathrm{Th}(\mathcal{L}_f))$ is in bijection with $\mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, f_*(\mathbb{E}_Y \otimes \mathrm{Th}(\mathcal{L}_f)))$. The purity isomorphism and the fact that $f_* = f_!$ (since f is proper) lead a bijection with the set $\mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, f_! f^! \mathbb{E}_X)$. Any element of this set can be composed with the counit map $f_! f^! \rightarrow \mathrm{Id}$ so that we obtain an element in $\mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathbb{E}_X)$.

More generally, the group $\mathbb{E}^r(Y, \mathcal{L}_f + \mathcal{V}_Y)$ is isomorphic to $[f^*(\mathbb{1}_X), f^*(\mathbb{E}_X \otimes \mathrm{Th}(\mathcal{V}_X)[r]) \otimes \mathrm{Th}(\mathcal{L}_f)]_Y$ which is, by adjunction, isomorphic to $[\mathbb{1}_X, f_*(f^*(\mathbb{E}_X \otimes \mathrm{Th}(\mathcal{V}_X)[r]) \otimes \mathrm{Th}(\mathcal{L}_f))]_X$. We can then compose with the trace map $\mathrm{Tr}_f : f_* \Sigma^{\mathcal{L}_f} f^* \rightarrow \mathrm{Id}$ defined in [DJK18, §2.5.3] in order to obtain a map whose target is the group $\mathbb{E}^r(X, \mathcal{V}_X)$.

3.2.3.5. MILNOR-WITT ACTION. Any motivic spectrum \mathbb{E} equipped with a unit isomorphism $\mathbb{1} \otimes \mathbb{E} \rightarrow \mathbb{E}$ defines an action by composition on the left

$$\gamma : H^m(X, \mathcal{W}_X) \otimes \mathbb{E}^n(X, \mathcal{V}_X) \rightarrow \mathbb{E}^{m+n}(X, \mathcal{W}_X + \mathcal{V}_X).$$

3.2.3.6. LOCALIZATION LONG EXACT SEQUENCE. Another essential property is the following long exact sequence deduced from the usual localization triangle of the six functors formalism in \mathbf{SH} .

Indeed, let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum. For any closed immersion $i : Z \rightarrow X$ of separated schemes over S , with (quasi-compact) complementary open immersion $j : U \rightarrow X$, there exists a canonical localization long exact sequence of the form:

$$\dots \rightarrow \mathbb{E}^n(Z, \mathcal{L}_Z + \mathcal{V}_Z) \xrightarrow{i_*} \mathbb{E}^n(X, \mathcal{L}_X + \mathcal{V}_X) \xrightarrow{j^*} \mathbb{E}^n(U, \mathcal{L}_U + \mathcal{V}_U) \xrightarrow{\partial_i} \mathbb{E}^{n+1}(Z, \mathcal{L}_Z + \mathcal{V}_Z) \rightarrow \dots$$

where the residue map ∂_i can be defined as the following composition

$$\begin{array}{ccc} \mathbb{E}^n(X - Z, \mathcal{V}_{X-Z}) & \xlongequal{\quad} & \mathbb{E}_{-n}(X - Z/X - Z, -\mathcal{V}_{X-Z}) \xrightarrow{\sim} \mathbb{E}_{-n}(X - Z/X, -\mathcal{V}_{X-Z}) \\ \downarrow \partial_i & & \downarrow \partial \\ \mathbb{E}^{n+1}(Z, \mathcal{L}_i + \mathcal{V}_Z) & \xlongequal{\quad} & \mathbb{E}_{-n-1}(Z/Z, -\mathcal{L}_i - \mathcal{V}_Z) \xrightarrow[\sim]{-\times \eta_i} \mathbb{E}_{-n-1}(Z/X, -\mathcal{V}_Z). \end{array}$$

Proposition 3.2.3.7. *Let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum and consider the following commutative square*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ q \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

of closed immersions of separated schemes over S . For \mathcal{V}_X a virtual vector bundle over X , we have the following diagram

$$\begin{array}{ccccccc} \mathbb{E}^n(T, *) & \xrightarrow{k_*} & \mathbb{E}^n(Y, *) & \xrightarrow{k'^*} & \mathbb{E}^n(Y - T, *) & \xrightarrow{\partial_k} & \mathbb{E}^{n+1}(T, *) \\ q_* \downarrow & & \downarrow p_* & & \downarrow \tilde{p}_* & & \downarrow q_* \\ \mathbb{E}^n(Z, *) & \xrightarrow{i_*} & \mathbb{E}^n(X, *) & \xrightarrow{i'^*} & \mathbb{E}^n(X - Z, *) & \xrightarrow{\partial_i} & \mathbb{E}^{n+1}(Z, *) \\ q'^* \downarrow & & \downarrow p'_* & & \downarrow \tilde{p}'_* & & \downarrow q'^* \\ \mathbb{E}^n(Z - T, *) & \xrightarrow{\tilde{i}_*} & \mathbb{E}^n(X - Y, *) & \xrightarrow{\tilde{i}'_*} & \mathbb{E}^n(X - (Z \cup Y), *) & \xrightarrow{\partial_{\tilde{i}}} & \mathbb{E}^{n+1}(Z - T, *) \\ \downarrow \partial_k & & \downarrow \partial_p & & \downarrow \partial_{\tilde{p}} & (*) & \downarrow \partial_{\tilde{k}} \\ \mathbb{E}^{n+1}(T, *) & \xrightarrow{k_*} & \mathbb{E}^{n+1}(Y, *) & \xrightarrow{k'^*} & \mathbb{E}^{n+1}(Y - T, *) & \xrightarrow{\partial_k} & \mathbb{E}^{n+2}(T, *) \end{array}$$

with obvious maps. Each square of this diagram is commutative except for $(*)$ which is anti-commutative.

Proof. This follows from Proposition 3.2.1.10. \square

Proposition 3.2.3.8. Let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic spectrum. Let $i : Z \rightarrow X$ a closed immersion with complementary open immersion $j : U \rightarrow X$. Let $x \in H^m(Y, \mathcal{V}_Y)$. Then the following diagram is commutative:

$$\begin{array}{ccccccc} \mathbb{E}^n(Z, \mathcal{L}_i + \mathcal{W}_Z) & \xrightarrow{i_*} & \mathbb{E}^n(X, \mathcal{W}_X) & \xrightarrow{j^*} & \mathbb{E}^n(U, \mathcal{W}_U) & \xrightarrow{\partial_{Z,X}} & \mathbb{E}^{n-1}(Z, \mathcal{L}_i + \mathcal{W}_Z) \\ \downarrow \varepsilon \circ \gamma_{i^*(x)} & & \downarrow \gamma_x & & \downarrow \gamma_{j^*(x)} & & \downarrow \varepsilon \circ \gamma_{i^*(x)} \\ \mathbb{E}^{n+m}(Z, \mathcal{L}_i + \mathcal{V}_Z + \mathcal{W}_Z) & \xrightarrow{i_*} & \mathbb{E}^{n+m}(X, \mathcal{V}_X + \mathcal{W}_X) & \xrightarrow{j^*} & \mathbb{E}^{n+m}(U, \mathcal{V}_U + \mathcal{W}_U) & \xrightarrow{\partial_{Z,X}} & \mathbb{E}^{n+m-1}(X, \mathcal{L}_i + \mathcal{V}_Z + \mathcal{W}_Z) \end{array}$$

where γ is the multiplication map defined in 3.2.3.5 (see also 3.2.1.5) and where ε is the isomorphism induced by the switch isomorphism $\mathcal{L}_i + \mathcal{V}_Z \simeq \mathcal{V}_Z + \mathcal{L}_i$.

Proof. This follows from [DJK18, Proposition 2.2.12]. \square

Proposition 3.2.3.9 (Base change for lci morphisms). Consider a cartesian square of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

with f proper, and g (quasi-projective) lci. Suppose moreover that the square is tor-independent, that is for any $x \in X$, $y' \in Y'$ with $y = f(x) = g(y')$ and for any $i > 0$ we have

$$\mathrm{Tor}_i^{\mathcal{O}_{Y,y}}(\mathcal{O}_{X,x}, \mathcal{O}_{Y',y'}) = 0.$$

Up to the canonical isomorphism $f'^* \mathcal{L}_g \simeq \mathcal{L}_{g'}$, we have:

$$f'_* \circ g'^* = g^* \circ f_*.$$

Proof. See Proposition 3.2.1.2. □

We will need the following proposition:

Proposition 3.2.3.10. *Let $v : Z \rightarrow X$ a closed immersion of smooth schemes. Consider the canonical decomposition $Z = \sqcup_{i \in I} Z_i$ and $X = \sqcup_{j \in J} X_j$ into connected components. Denote by $\hat{Z}_j = Z \times_X X_j$. For any $i \in I$, let $j \in J$ be the unique element such that $Z_i \subset X_j$ and denote by $v_i^j : Z_i \rightarrow Z_j$ the induced immersion. Consider the complement Z'_i such that $\hat{Z}_i = Z_i \sqcup Z'_i$. The following diagram is commutative:*

$$\begin{array}{ccccc} \mathbb{E}^{n-1}(X - Z, *) & \xrightarrow{\partial_{X,Z}} & \mathbb{E}^n(Z, *) & \xrightarrow{v_*} & \mathbb{E}^n(X, *) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \bigoplus_{j \in J} \mathbb{E}^{n-1}(X_j - \hat{Z}_j, *) & \xrightarrow{(\partial_{ij})_{i \in I, j \in J}} & \bigoplus_{i \in I} \mathbb{E}^n(Z_i, *) & \xrightarrow{(v_{ij})_{i \in I, j \in J}} & \bigoplus_{j \in J} \mathbb{E}^n(X_j, *) \end{array}$$

where the vertical maps are the canonical isomorphisms and where, for any $(i, j) \in I \times J$, if $Z_i \subset X_j$, then $v_{ij} = (v_i^j)_*$ and $\partial_{ij} = \partial_{X_j - Z'_i, Z_i}$; otherwise $v_{ij} = 0$ and $\partial_{ij} = 0$.

Proof. Straightforward. □

3.3 From homotopy modules to Milnor-Witt cycle modules

3.3.1 Recollection on Milnor-Witt cycle modules

We denote by \mathfrak{F}_k the category whose objects are the couple (E, \mathcal{V}_E) where E is a field over k and $\mathcal{V}_E \in \mathfrak{V}(E)$ is a virtual vector space (of finite dimension over F). A morphism $(E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ is the data of a morphism $E \rightarrow F$ of fields over k and an isomorphism $\mathcal{V}_E \otimes_E F \simeq \mathcal{V}_F$ of virtual F -vector spaces.

A morphism $(E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ in \mathfrak{F}_k is said to be finite (resp. separable) if the field extension F/E is finite (resp. separable).

We recall (see Definition 2.3.1.1) that a Milnor-Witt cycle modules M over k is a functor from \mathfrak{F}_k to the category **Ab** of abelian groups equipped with data

- (D1) (restriction maps) Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k . The functor M gives a morphism $\varphi_* : M(E, \mathcal{V}_E) \rightarrow M(F, \mathcal{V}_F)$,
- (D2) (corestriction maps) Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k where the morphism $E \rightarrow F$ is finite. There is a morphism $\varphi^* : M(F, \Omega_{F/k} + \mathcal{V}_F) \rightarrow M(E, \Omega_{E/k} + \mathcal{V}_E)$,

(D3) (Milnor-Witt K-theory action) Let (E, \mathcal{V}_E) and (E, \mathcal{W}_E) be two objects of \mathfrak{F}_k . For any element x of $\underline{K}^{MW}(E, \mathcal{W}_E)$, there is a morphism

$$\gamma_x : M(E, \mathcal{V}_E) \rightarrow M(E, \mathcal{W}_E + \mathcal{V}_E)$$

so that the functor $M(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ is a left module over the lax monoidal functor $\underline{K}^{MW}(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ (see [Yet03, Definition 39]),

(D4) (residue maps) Let E be a field over k , let v be a valuation on E and let \mathcal{V} be a virtual projective \mathcal{O}_v -module of finite type. Denote by $\mathcal{V}_E = \mathcal{V} \otimes_{\mathcal{O}_v} E$ and $\mathcal{V}_{\kappa(v)} = \mathcal{V} \otimes_{\mathcal{O}_v} \kappa(v)$. There is a morphism

$$\partial_v : M(E, \mathcal{V}_E) \rightarrow M(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}),$$

and satisfying rules

(R1a) (functoriality of restriction maps),

(R1b) (functoriality of corestriction maps),

(R1c) (base change property),

(R2a), (R2b), (R2c) (projection formulae),

(R3a) (ramification formula),

(R3b), (R3c), (R3d), (R3e) (compatibility between residue maps and the first three data),

(R4a) (compatibility with orientations).

Moreover, a Milnor-Witt cycle module M satisfies axioms (FD) (finite support of divisors) and (C) (closedness) that enable us to define a complex $(C_p(X, M, \mathcal{V}_X), d_p)_{p \in \mathbb{Z}}$ for any scheme X and virtual bundle \mathcal{V}_X over X where

$$C_p(X, M, \mathcal{V}_X) = \bigoplus_{x \in X_{(p)}} M(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_x)$$

and the differential $d_p = (\partial_y^x)_{(x,y) \in X_{(p)} \times X_{(p-1)}}$ is defined as follows (see Subsection 2.3.2).

3.3.1.1. First, for x a point of X , denote by

$$M(x, \mathcal{V}_X) = M(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_x).$$

If X is normal, then for any $x \in X^{(1)}$ the local ring of X at x is a valuation ring so that (D4) gives us a map $\partial_x : M(\xi, \mathcal{V}_X) \rightarrow M(x, \mathcal{V}_X)$ where ξ is the generic point of X .

If X is any scheme, let x, y be any points in X . We define a map

$$\partial_y^x : M(x, \mathcal{V}_X) \rightarrow M(y, \mathcal{V}_X)$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z$, then put $\partial_y^x = 0$. If $y \in Z$, let $\tilde{Z} \rightarrow Z$ be the normalization and put

$$\partial_y^x = \sum_{z|y} \text{cores}_{\kappa(z)/\kappa(y)} \circ \partial_z$$

with z running through the finitely many points of \tilde{Z} lying over y .

3.3.1.2. The complex $(C_p(X, M, \mathcal{V}_X), d)_{p \geq 0}$ is called the *Milnor-Witt complex of cycles on X with coefficients in M* and we denote by $A_p(X, M, \mathcal{V}_X)$ the associated homology groups (called *Chow-Witt groups with coefficients in M*). We can define five basic maps on the complex level (see Subsection 2.4.1):

Pushforward Let $f : X \rightarrow Y$ be a k -morphism of schemes, let \mathcal{V}_Y be a virtual bundle over the scheme Y . The data (D2) induces a map

$$f_* : C_p(X, M, \mathcal{V}_X) \rightarrow C_p(Y, M, \mathcal{V}_Y).$$

Pullback Let $g : X \rightarrow Y$ be an essentially smooth morphism of schemes. Let \mathcal{V}_Y a virtual bundle over Y . Suppose X connected (if X is not connected, take the sum over each connected component) and denote by s the relative dimension of g . The data (D1) induces a map

$$g^* : C_p(Y, M, \mathcal{V}_Y) \rightarrow C_{p+s}(X, M, -\mathcal{L}_{X/Y} + \mathcal{V}_X).$$

Multiplication with units Let X be a scheme of finite type over k with a virtual bundle \mathcal{V}_X . Let a_1, \dots, a_n be global units in \mathcal{O}_X^* . The data (D3) induces a map

$$[a_1, \dots, a_n] : C_p(X, M, \mathcal{V}_X) \rightarrow C_p(X, M, \langle n \rangle + \mathcal{V}_X).$$

Multiplication with η Let X be a scheme of finite type over k with a virtual bundle \mathcal{V}_X . The Hopf map η and the data (D3) induces a map

$$\eta : C_p(X, M, \mathcal{V}_X) \rightarrow C_p(X, M, -\mathbb{A}_X^1 + \mathcal{V}_X).$$

Boundary map Let X be a scheme of finite type over k with a virtual bundle \mathcal{V}_X , let $i : Z \rightarrow X$ be a closed immersion and let $j : U = X \setminus Z \rightarrow X$ be the inclusion of the open complement. The data (D4) induces (as in 3.3.1.1) a map

$$\partial = \partial_Z^U : C_p(U, M, \mathcal{V}_U) \rightarrow C_{p-1}(Z, M, \mathcal{V}_Z).$$

These maps satisfy the usual compatibility properties (see Subsection 2.4.2). In particular, they induce maps $f_*, g^*, [u], \eta, \partial_Z^U$ on the homology groups $A_*(X, M, *)$.

We end this subsection with a lemma illustrating the importance of the rule (R4a). This will be useful in Section 3.4.

Lemma 3.3.1.3. *Let M be a Milnor-Witt cycle module over k . For any field E/k and any virtual vector bundle \mathcal{V}_E over E , we have a canonical isomorphism*

$$M(E, \mathcal{V}_E) \simeq M(E, \langle n \rangle) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\det(\mathcal{V}_E)^\times]$$

where n is the rank of \mathcal{V}_E .

Proof. Any element $u \in \det(\mathcal{V}_E)^\times$ defines an isomorphism

$$\Theta_u : M(E, \mathcal{V}_E) \simeq M(E, \langle n \rangle) \otimes_{\mathbb{Z}[E^\times]} \mathbb{Z}[\det(\mathcal{V}_E)^\times]$$

in an obvious way thanks to (D1). One can check that this map does not depend on the choice of u according to rule (R4a). □

Note that this lemma is true for Milnor-Witt K-theory \underline{K}^{MW} by definition.

3.3.2 Cycle premodule structure

Let $\mathbb{E} \in \mathbf{SH}(S)$ be a motivic ring spectrum. For any field E and any virtual vector space \mathcal{V}_E of rank r over E , we put

$$\hat{\mathbb{E}}(E, \mathcal{V}_E) = \mathbb{E}^{-r}(X, \mathcal{V}_X) = \mathrm{Hom}_{\mathbf{SH}(X)}(\mathbb{1}_X, \mathbb{E}_X \otimes \mathrm{Th}_X(\mathcal{V}_X)[-r]),$$

where $X = \mathrm{Spec} E$ (recall Definition 3.2.1.1). We prove that this defines a functor $\hat{\mathbb{E}} : \mathfrak{F}_k \rightarrow \mathbf{Ab}$ which is a Milnor-Witt cycle module. Indeed we have the following data:

- (D1) Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k . The cohomology theory $\mathbb{E}^*(-, *)$ being contravariant (see 3.2.3.3), we obtain a map $\varphi_* : \hat{\mathbb{E}}(E, \mathcal{V}_E) \rightarrow \hat{\mathbb{E}}(F, \mathcal{V}_F)$.
- (D2) Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k where the morphism $E \rightarrow F$ is *finite*. The (twisted) covariance described in 3.2.3.4 leads to a morphism $\varphi^* : \hat{\mathbb{E}}(F, \Omega_{F/k} + \mathcal{V}_F) \rightarrow \hat{\mathbb{E}}(E, \Omega_{E/k} + \mathcal{V}_E)$.
- (D3) Let (E, \mathcal{V}_E) and (E, \mathcal{W}_E) be two objects of \mathfrak{F}_k . For any element x of $\underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$, there is a morphism

$$\gamma_x : \hat{\mathbb{E}}(E, \mathcal{V}_E) \rightarrow \hat{\mathbb{E}}(E, \mathcal{W}_E + \mathcal{V}_E)$$

given by composition on the left by x (as in 3.2.3.5) since we can identify $\underline{\mathbf{K}}^{MW}(E, \mathcal{W}_E)$ with $\hat{\mathbb{1}}(E, \mathcal{W}_E)$ (see Example 3.2.3.2). We can check that the functor $\hat{\mathbb{E}}(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ is then a left module over the lax monoidal functor $\underline{\mathbf{K}}^{MW}(E, -) : \mathfrak{V}(E) \rightarrow \mathbf{Ab}$ (see [Yet03] Definition 39).

- (D4) Let E be a field over k , let v be a valuation on E and let \mathcal{V} be a virtual projective \mathcal{O}_v -module of finite type. As before, denote by $\mathcal{V}_E = \mathcal{V} \otimes_{\mathcal{O}_v} E$ and $\mathcal{V}_{\kappa(v)} = \mathcal{V} \otimes_{\mathcal{O}_v} \kappa(v)$. There is a morphism

$$\partial_v : \hat{\mathbb{E}}(E, \mathcal{V}_E) \rightarrow \hat{\mathbb{E}}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)})$$

given by the long exact sequence 3.2.3.6 where the closed immersion is

$$\mathrm{Spec} \kappa(v) \hookrightarrow \mathrm{Spec} \mathcal{O}_v.$$

It is clear that the data (D1) and (D2) are functorial so that the two following rules hold:

- (R1a) Let φ and ψ be two composable morphisms in \mathfrak{F}_k . One has

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

- (R1b) Let φ and ψ be two composable finite morphisms in \mathfrak{F}_k . One has

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

The base change theorem 3.2.3.9 leads to the following rule (R1c):

(R1c) Consider $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ and $\psi : (E, \mathcal{V}_E) \rightarrow (L, \mathcal{V}_L)$ with φ finite and ψ separable. Let R be the ring $F \otimes_E L$. For each $p \in \text{Spec } R$, let $\varphi_p : (L, \mathcal{V}_L) \rightarrow (R/p, \mathcal{V}_{R/p})$ and $\psi_p : (F, \mathcal{V}_F) \rightarrow (R/p, \mathcal{V}_{R/p})$ be the morphisms induced by φ and ψ . One has

$$\psi_* \circ \varphi^* = \sum_{p \in \text{Spec } R} (\varphi_p)^* \circ (\psi_p)_*.$$

The general formalism of Fulton-McPherson gives the usual projection formulas (see also [Dé17, 1.2.8]):

(R2) Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k , let x be in $\underline{K}^{MW}(E, \mathcal{W}_E)$ and y be in $\underline{K}^{MW}(F, \Omega_{F/k} + \mathcal{W}'_F)$ where (E, \mathcal{W}_E) and (F, \mathcal{W}'_F) are two objects of \mathfrak{F}_k .

(R2a) We have $\varphi_* \circ \gamma_x = \gamma_{\varphi_*(x)} \circ \varphi_*$.

(R2b) Suppose φ finite. We have $\varphi^* \circ \gamma_{\varphi_*(x)} = \gamma_x \circ \varphi^*$.

(R2c) Suppose φ finite. We have $\varphi^* \circ \gamma_y \circ \varphi_* = \gamma_{\varphi^*(y)}$.

We now prove the remaining rules.

(R3a) Let $E \rightarrow F$ be a field extension and w be a valuation on F which restricts to a non trivial valuation v on E with ramification index e . Let \mathcal{V} be a virtual \mathcal{O}_v -module so that we have a morphism $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ which induces a morphism

$$\overline{\varphi} : (\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \rightarrow (\kappa(w), -\mathcal{N}_w + \mathcal{V}_{\kappa(w)}).$$

We have

$$\partial_w \circ \varphi_* = \gamma_{e_e} \circ \overline{\varphi}_* \circ \partial_v.$$

Proof. We apply Theorem 3.2.2.2 to the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_v & \xrightarrow{\quad} & \mathcal{O}_w \\ \downarrow & & \downarrow \\ \kappa(v) & \xrightarrow{\quad} & \kappa(v) \otimes_{\mathcal{O}_v} \mathcal{O}_w = \mathcal{O}_w / \mathfrak{m}_w^e \\ & \searrow & \downarrow \\ & & \kappa(w). \end{array}$$

□

(R3b) Let $\varphi : E \rightarrow F$ be a finite morphism of fields, let v be a valuation over E and let \mathcal{V} be a virtual vector bundle over \mathcal{O}_v . For each extension w of v , we denote by $\varphi_w : \kappa(v) \rightarrow \kappa(w)$ the map induced by φ . We have

$$\partial_v \circ \varphi^* = \sum_w \varphi_w^* \circ \partial_w.$$

Proof. There exists a semilocal ring A over \mathcal{O}_v such that the set of maximal ideals consists of the ideals \mathfrak{m}_w where w is an extension of v .

Denote by $T = \bigoplus_{w|v} \text{Spec } \kappa(w)$, $Y = \text{Spec } A$, $Z = \text{Spec } \kappa(v)$ and $X = \text{Spec } \mathcal{O}_v$ so that we have the following commutative diagram formed by two cartesian squares:

$$\begin{array}{ccccc} T & \xrightarrow{k} & Y & \xleftarrow{k'} & Y - T \\ g \downarrow & & \downarrow f & & \downarrow h \\ Z & \xrightarrow{i} & X & \xleftarrow{i'} & X - Z \end{array}$$

where k, i are the canonical closed immersions with complementary open immersions k', i' respectively and where f, g, h are the canonical maps.

According to Proposition 3.2.3.7, this leads to the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{E}^r(T, \mathcal{L}_T + \mathcal{V}_T) & \xrightarrow{k_*} & \mathbb{E}^r(Y, \mathcal{L}_Y + \mathcal{V}_Y) & \xrightarrow{k'} & \mathbb{E}^r(Y - T, \mathcal{L}_{Y-T} + \mathcal{V}_{Y-T}) & \xrightarrow{\partial} & \mathbb{E}^{r+1}(T, \mathcal{L}_T + \mathcal{V}_T) \\ g_* \downarrow & & \downarrow f_* & & \downarrow h_* & (*) & \downarrow g_* \\ \mathbb{E}^r(Z, \mathcal{L}_Z + \mathcal{V}_Z) & \xrightarrow{i_*} & \mathbb{E}^r(X, \mathcal{L}_X + \mathcal{V}_X) & \xrightarrow{i'} & \mathbb{E}^r(X - Z, \mathcal{L}_{X-Z} + \mathcal{V}_{X-Z}) & \xrightarrow{\partial} & \mathbb{E}^{r+1}(Z, \mathcal{L}_Z + \mathcal{V}_Z) \end{array}$$

where r is the rank of \mathcal{V}_X . The rule (R3b) follows from the commutativity of the square (*).

□

(R3c) Let $\varphi : (E, \mathcal{V}_E) \rightarrow (F, \mathcal{V}_F)$ be a morphism in \mathfrak{F}_k and let w a valuation on F which restricts to the trivial valuation on E . Then

$$\partial_w \circ \varphi_* = 0.$$

Proof. Consider the closed inclusion $i : Z \rightarrow X$ and its open complementary map $j : U \rightarrow X$ where $X = \text{Spec } \mathcal{O}_v$, $Z = \text{Spec } \kappa(w)$ and $U = \text{Spec } F$. According to the long exact sequence 3.2.3.6, the composite

$$\mathbb{E}^r(X, \mathcal{V}_X) \xrightarrow{j^*} \mathbb{E}^r(U, \mathcal{V}_U) \xrightarrow{\partial_{Z,X}} \mathbb{E}^{r+1}(Z, \mathcal{L}_{Z/X} + \mathcal{V}_Z)$$

is zero. The result follows from the fact that the map $\text{Spec } E \rightarrow \text{Spec } F$ factors through j since w restricts to the trivial valuation on E .

□

(R3d) Let φ and w be as in (R3c), and let $\bar{\varphi} : (E, \mathcal{V}_E) \rightarrow (\kappa(w), \mathcal{V}_{\kappa(w)})$ be the induced morphism. For any prime π of v , we have

$$\partial_w \circ \gamma_{[-\pi]} \circ \varphi_* = \bar{\varphi}_*.$$

Proof. Denote by $Z = \text{Spec } \kappa(w)$, $U = \text{Spec } F$, $X = \text{Spec } \mathcal{O}_w$ and $Y = \text{Spec } E$ and consider the induced maps as defined in the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\ & \searrow \bar{f} & \downarrow f & \swarrow f & \\ & & Y & & \end{array}$$

We want to prove that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{E}^{-r}(Y, \mathcal{V}_Y) & \xrightarrow{f^*} & \mathbb{E}^{-r-1}(U, \mathcal{V}_U) \\ \bar{f} \downarrow & & \downarrow \gamma_{[-\pi]} \\ \mathbb{E}^{-r}(Z, \mathcal{V}_Z) & \xleftarrow{\partial} & \mathbb{E}^{-r}(U, \mathbb{A}_U^1 + \mathcal{V}_U) \end{array}$$

where we use the isomorphism $\mathcal{N}_Z X \simeq \mathbb{A}_Z^1$ defined by the choice of prime π . We can split this diagram into the following one:

$$\begin{array}{ccccc} & & \mathbb{E}^{-r}(Y, \mathcal{V}_Y) & & \\ & \swarrow \tilde{f}^* & \downarrow h^* & \searrow f^* & \\ \mathbb{E}^{-r}(X, \mathcal{V}_X) & \xleftarrow{q^*} & \mathbb{E}^{-r}(D_Z X - N_Z X, \mathcal{V}) & \xrightarrow{d^*} & \mathbb{E}^{-r}(U, \mathcal{V}_U) \\ \downarrow i^* & & \downarrow \gamma_{[-\pi]} & (2) & \downarrow \gamma_{[-\pi]} \\ & (1) & \mathbb{E}^{-r-1}(D_Z X - N_Z X, \mathbb{A}_U^1 + \mathcal{V}) & \xrightarrow{d^*} & \mathbb{E}^{-r-1}(U, \mathbb{A}_U^1 + \mathcal{V}_U) \\ & & \downarrow \partial & (3) & \downarrow \partial \\ \mathbb{E}^{-r}(Z, \mathcal{V}_Z) & \xleftarrow[p^*]{\simeq} & \mathbb{E}^{-r}(N_Z X, \mathcal{V}) & \xrightarrow[p^*]{\simeq} & \mathbb{E}^{-r}(Z, \mathcal{V}) \end{array}$$

where the morphisms $d : U \rightarrow (D_Z X - N_Z X)$, $q : D_Z X - N_Z X \simeq \mathbb{G}_m \times X \rightarrow X$ and $p : N_Z X \rightarrow Z$ are the canonical maps used in the deformation to the normal cone.

We can check that the square (1), (2) and (3) are commutative (same proof as [Dé08b, Proposition 2.6.5]), hence the whole diagram is commutative by functoriality of the pull-back maps. □

(R3e) Let E be a field over k , v be a valuation on E and u be a unit of v . Then

$$\begin{aligned} \partial_v \circ \gamma_{[u]} &= \gamma_{\varepsilon[\bar{u}]} \circ \partial_v \text{ and} \\ \partial_v \circ \gamma_{\boldsymbol{\eta}} &= \gamma_{\boldsymbol{\eta}} \circ \partial_v. \end{aligned}$$

Proof. This follows from Proposition 3.2.3.8 since $\varepsilon \boldsymbol{\eta} = \boldsymbol{\eta}$ (where $\varepsilon = -\langle -1 \rangle$). □

(R4a) Let $(E, \mathcal{V}_E) \in \mathfrak{F}_k$ and let Θ be an automorphism of \mathcal{V} . Denote by Δ the canonical map from the group of automorphism of \mathcal{V}_E to the group $\mathbf{K}^{\text{MW}}(E, 0)$. Then

$$\Theta_* = \gamma_{\Delta(\Theta)} : \hat{\mathbb{E}}(E, \mathcal{V}_E) \rightarrow \hat{\mathbb{E}}(E, \mathcal{V}_E).$$

Proof. One reduce to the case where $\mathbb{E} = \mathbb{1}$. In this case, $\hat{\mathbb{E}}$ is (isomorphic to) the Milnor-Witt K-theory $\underline{\mathbf{K}}^{\text{MW}}$ (see Example 3.2.3.2), hence the result. □

We have proved that $\hat{\mathbb{E}}$ is a Milnor-Witt cycle premodule. In the following subsection, we prove that it satisfies axioms (FD) and (C).

3.3.3 Cycle module structure

Put $S = \operatorname{Spec}(k)$. Let $f : X \rightarrow S$ be a scheme and \mathcal{V}_X be a virtual vector bundle over X . Let $\mathbb{E}_S \in \mathbf{SH}(S)$ be a motivic spectrum. Recall that we denote by \mathbb{E}_X the spectrum $f^*(\mathbb{E}_S)$. The purpose of this subsection is to prove that the Milnor-Witt cycle premodule $\hat{\mathbb{E}}$ is in fact a cycle module. Roughly speaking, this means that the graded group $C^*(X, \hat{\mathbb{E}}, *)$ forms a complex.

Consider a flag $\mathfrak{Z} = (Z_p)_{p \in \mathbb{Z}}$ over X , that is a sequence a closed subschemes of X such that

$$\emptyset \subset Z_1 \subset Z_2 \subset \cdots \subset Z_n \subset X$$

where $\dim Z_p \leq p$.

For an integer $p \in \mathbb{Z}$, put $U_p = X - Z_p$ and $T_p = Z_p - Z_{p-1}$. Consider the canonical maps $j_p : U_p \subset U_{p-1}$ and $i_p : T_p \subset U_{p-1}$.

For $p, q \in \mathbb{Z}$, denote by

$$E_{p,q}^{1,\mathfrak{Z}} = \mathbb{E}^{q-p}(T_p, \mathcal{L}_{T_p} + \mathcal{V}_{T_p}) = [\mathbb{1}_{T_p}, \mathbb{E}_{T_p} \otimes \operatorname{Th}_{T_p}(\mathcal{L}_{T_p} + \mathcal{V}_{T_p})[q-p]]_{T_p}$$

and

$$D_{p,q}^{1,\mathfrak{Z}} = \mathbb{E}^{q-p-1}(U_p, \mathcal{L}_{U_p} + \mathcal{V}_{U_p}) = [\mathbb{1}_{U_p}, \mathbb{E}_{U_p} \otimes \operatorname{Th}_{U_p}(\mathcal{L}_{U_p} + \mathcal{V}_{U_p})[q-p-1]]_{U_p}.$$

According to 3.2.3.6, we have a long exact sequence

$$\cdots \longrightarrow D_{p-1,q+1}^{1,\mathfrak{Z}} \xrightarrow{j_p^*} D_{p,q}^{1,\mathfrak{Z}} \xrightarrow{\partial_p} E_{p,q}^{1,\mathfrak{Z}} \xrightarrow{i_{p,*}} D_{p-1,q}^{1,\mathfrak{Z}} \longrightarrow \cdots$$

so that $(D_{p,q}^{1,\mathfrak{Z}}, E_{p,q}^{1,\mathfrak{Z}})_{p,q \in \mathbb{Z}}$ is an exact couple. By the general theory (see [McC01], Chapter 3), this defines a spectral sequence. In particular, we have canonical differential maps d which are well-defined and satisfying $d \circ d = 0$. Moreover, we can prove that this spectral sequence converges to $\mathbb{E}^{p+q}(X, \mathcal{L}_{X/S} + \mathcal{V}_X)$ (because the $E_{p,q}^1$ -term is bounded) but we do not need this fact.

For $p, q \in \mathbb{Z}$, denote by

$$\begin{aligned} D_{p,q}^{1,X} &= \operatorname{colim}_{\mathfrak{Z} \in \operatorname{Flag}(X)^{op}} D_{p,q}^{1,\mathfrak{Z}}, \\ E_{p,q}^{1,X} &= \operatorname{colim}_{\mathfrak{Z} \in \operatorname{Flag}(X)^{op}} E_{p,q}^{1,\mathfrak{Z}} \end{aligned}$$

where the colimit is taken over the flags \mathfrak{Z} of X . Since the colimit is filtered, we get an exact couple and a spectral sequence.

We need to compute this spectral sequence. This is done in the following.

Theorem 3.3.3.1. *For $p, q \in \mathbb{Z}$, we have a canonical isomorphism*

$$E_{p,q}^{1,X} \simeq \bigoplus_{x \in X_{(p)}} [\mathbb{1}_{\kappa(x)}, \mathbb{E}_{\kappa(x)} \otimes \operatorname{Th}_{\kappa(x)}(\mathcal{L}_{\kappa(x)} + \mathcal{V}_{\kappa(x)})[q-p]]_{\kappa(x)}.$$

In particular, if r is the rank of \mathcal{V}_X , then

$$E_{p,-r}^{1,X} \simeq C_p(X, \hat{\mathbb{E}}, \mathcal{V}_X).$$

Proof. The proof is the same as Theorem 2.6.2.3. □

Theorem 3.3.3.2. *Assume X is a smooth scheme. Keeping the previous notations, the following diagram is commutative:*

$$\begin{array}{ccc} E_{p,q}^{1,X} & \xrightarrow{d_{p,q}} & E_{p-1,q}^{1,X} \\ \downarrow & & \downarrow \\ \mathbb{E}^{q-p}(\mathrm{Spec}(\kappa(y), \mathcal{L}_y + \mathcal{V}_y)) & \xrightarrow{\partial_y^x} & \mathbb{E}^{q-p+1}(\mathrm{Spec} \kappa(x), \mathcal{L}_x + \mathcal{V}_x) \end{array}$$

where $d_{p,q}$ is the differential canonically associated to the spectral sequence and where the vertical maps are the canonical projections associated to isomorphism of Theorem 3.3.3.1.

Proof. (see also [Dé12, Proposition 1.15])

By definition, $d_{p,q}$ is the colimit of arrows

$$\mathbb{E}^{q-p}(Z - Y, \mathcal{L}_{Z-Y} + \mathcal{V}_{Z-Y}) \xrightarrow{i_{p*}} \mathbb{E}^{q-p}(X - Y, \mathcal{L}_{X-Y} + \mathcal{V}_{X-Y}) \xrightarrow{\partial_{p-1}} \mathbb{E}^{q-p+1}(Y - W, \mathcal{L}_{Y-W} + \mathcal{V}_{Y-W})$$

where $W \subset Y \subset Z$ are large enough closed subschemes with $\dim_X(Z) = p$, $\dim_X(Y) = p - 1$ and $\dim_X(W) \leq p - 2$. In the following, we consider W, Y, Z as above. For simplification, we replace X by $X - W$ so that we can remove any subset of X if its dimension is $\leq p - 2$.

Enlarging Y , we may assume that Y contains Z_{sing} the singular locus of Z . Since the singular locus of Y has dimension strictly lesser than $p - 1$, we may assume that Y is smooth. In short, we study the composite:

$$\mathbb{E}^{q-p}(Z - Y, \mathcal{L}_{Z-Y} + \mathcal{V}_{Z-Y}) \xrightarrow{i_{Y*}} \mathbb{E}^{q-p}(X - Y, \mathcal{L}_{X-Y} + \mathcal{V}_{X-Y}) \xrightarrow{\partial_p} \mathbb{E}^{q-p+1}(Y, \mathcal{L}_Y + \mathcal{V}_Y)$$

where $i_Y : Z - Y \rightarrow X - Y$ is the restriction of the canonical closed immersion $Z \rightarrow X$.

We denote by Y_y (resp. Z_x) the irreducible component of Y (resp. Z) containing y (resp. x). We may decompose Y as $Y = Y_y \sqcup Y'_y$ with $Y'_y = Y \setminus Y_y$ since Y is smooth. Denote by $\hat{Y}_x = Y \times_Z Z_x$ so that $Z_x - \hat{Y}_x$ is a connected component of the (smooth) scheme $Z - Y$. Denote by $i_x : Z_x - \hat{Y}_x \rightarrow X - Y$ the canonical inclusion. According to Proposition 3.2.3.10, we get the following commutative diagram

$$\begin{array}{ccccc} \mathbb{E}^{q-p}(Z - Y, \mathcal{L}_{Z-Y} + \mathcal{V}_{Z-Y}) & \xrightarrow{i_{Y*}} & \mathbb{E}^{q-p}(X - Y, \mathcal{L}_{X-Y} + \mathcal{V}_{X-Y}) & \xrightarrow{\partial_{X,Y}} & \mathbb{E}^{q-p+1}(Y, \mathcal{L}_Y + \mathcal{V}_Y) \\ \downarrow & & \parallel & & \downarrow \\ \mathbb{E}^{q-p}(Z_x - Y_x, \mathcal{L}_{Z_x - Y_x} + \mathcal{V}_{Z_x - Y_x}) & \xrightarrow{i_{x*}} & \mathbb{E}^{q-p}(X - Y, \mathcal{L}_{X-Y} + \mathcal{V}_{X-Y}) & \xrightarrow{\partial_{X-Y_y, Y_y}} & \mathbb{E}^{q-p+1}(Y_y, \mathcal{L}_{Y_y} + \mathcal{V}_{Y_y}) \\ & & \searrow & \nearrow & \\ & & \partial_{Y,y}^{Z,x} & & \end{array}$$

where the vertical maps are the canonical projections. The theorem is equivalent to proving that the differential ∂_y^x defined in 3.3.1.1 is the colimit of the maps $\partial_{Y,y}^{Z,x}$ defined in the above diagram.

Assume that y is not a specialization of x , that is $y \notin \overline{\{x\}}$. Then $\dim_X(Y_x \cap Z_x) \leq p - 2$ hence (reducing X to $X - (Y_x \cap Z_x)$) we may assume that $Y_x \cap Z_x = \emptyset$. Thus $Y_x \cap (Z_x - \hat{Y}_x) = \emptyset$ and we get the following cartesian square of closed immersions

$$\begin{array}{ccc} \emptyset & \longrightarrow & Y_y \\ \downarrow & & \downarrow \\ Z_x - \hat{Y}_x & \longrightarrow & X - Y'_y \end{array}$$

which spawns the equality $\partial_{X-Y'_y, Y_y} \circ i_{x*}$ by naturality of the residue maps (see Proposition 3.2.3.7). This proves the proposition in this case.

Assume that y is a specialization of x so that $Y_y \subset Z_x$ and $Y_y \subset \hat{Y}_x$. For simplification, we assume that $Z = Z_x$, that is Z is irreducible with generic point x . Consider the normalization $f : \tilde{Z} \rightarrow Z$ of Z . The singular locus \tilde{Z}_{sing} is of codimension greater than 1 in \tilde{Z} hence $f(\tilde{Z}_{\text{sing}})$ is of dimension strictly lesser than $p-1$ in X and (reducing X) we may assume that \tilde{Z} is smooth.

Denote by \tilde{Y} (resp. $\tilde{Y}_y, \tilde{Y}'_y$) the reduced inverse image of Y (resp. Y_y, Y'_y) along f . Reducing X again, we may assume that \tilde{Y}_y is smooth and $\tilde{Y}_y \cap \tilde{Y}'_y = \emptyset$. We can also assume that every connected component of \tilde{Y}_y dominates Y_y (by reducing X , we can remove the non-dominant connected components). From this, we see that the map $g_y : \tilde{Y}_y \rightarrow Y_y$ induced by f is finite and equidimensional. Consider the following topologically cartesian square:

$$\begin{array}{ccc} \tilde{Y}_y & \xrightarrow{\tilde{\sigma}} & \tilde{Z} - \tilde{Y}'_y \\ \downarrow g_y & & \downarrow \\ Y_y & \xrightarrow{\sigma} & X - Y'_y \end{array}$$

where σ and $\tilde{\sigma}$ are the canonical closed immersions and the right vertical map is induced by the composite

$$\tilde{Z} \xrightarrow{f} Z \xrightarrow{i} X.$$

By taking complements of $\tilde{\sigma}$ and σ , it induces the map

$$\tilde{Z} - \tilde{Y} \xrightarrow{h} Z - Y \xrightarrow{i} X - Y.$$

By naturality of residues with respect to Gysin morphisms and by functoriality of the Gysin morphisms, we get the commutative diagram

$$\begin{array}{ccccc} \mathbb{E}^{q-p}(Z - Y, \mathcal{L}_{Z-Y} + \mathcal{V}_{Z-Y}) & \xrightarrow{i_{x*}} & \mathbb{E}^{q-p}(X - Z, \mathcal{L}_{X-Y} + \mathcal{V}_{X-Y}) & \xrightarrow{\partial_{X-Y'_y, Y_y}} & \mathbb{E}^{q-p+1}(Y_y, \mathcal{L}_{Y_y} + \mathcal{V}_{Y_y}) \\ \uparrow h_* & & & & \downarrow \\ \mathbb{E}^{q-p}(\tilde{Z} - \tilde{Y}, \mathcal{L}_{\tilde{Z}-\tilde{Y}} + \mathcal{V}_{\tilde{Z}-\tilde{Y}}) & \xrightarrow{\partial_{\tilde{Z}-\tilde{Y}'_y, \tilde{Y}_y}} & \mathbb{E}^{q-p}(\tilde{Y}_y, \mathcal{L}_{\tilde{Y}_y} + \mathcal{V}_{\tilde{Y}_y}) & \xrightarrow{g_{y*}} & \mathbb{E}^{q-p+1}(Y_y, \mathcal{L}_{Y_y} + \mathcal{V}_{Y_y}). \end{array}$$

For any $t \in f^{-1}(y)$, there exists a unique connected component \tilde{Y}_t in the (smooth) scheme \tilde{Y}_y so that $\tilde{Y}_y = \sqcup_{t \in f^{-1}(y)} \tilde{Y}_t$. Note that \tilde{Y}_t is also a connected component of \tilde{Y} . Denote by $\tilde{Z}_t = \tilde{Z} - (\tilde{Y} - \tilde{Y}_t)$; this is an open subscheme of \tilde{Z} containing \tilde{Y}_t and $\tilde{Z}_t - \tilde{Y}_t = \tilde{Z} - \tilde{Y}$. According the Proposition 3.2.3.10, we have the following commutative diagram

$$\begin{array}{ccccc}
 \mathbb{E}^{q-p}(\tilde{Z} - \tilde{Y}, \mathcal{L}_{\tilde{Z}-\tilde{Y}} + \mathcal{V}_{\tilde{Z}-\tilde{Y}}) & \xrightarrow{\partial_{\tilde{Z}-\tilde{Y}', \tilde{Y}_y}} & \mathbb{E}^{q-p+1}(\tilde{Y}_y, \mathcal{L}_{\tilde{Y}_y} + \mathcal{V}_{\tilde{Y}_y}) & \xrightarrow{g_{y*}} & \mathbb{E}^{q-p+1}(Y_y, \mathcal{L}_{Y_y} + \mathcal{V}_{Y_y}) \\
 \parallel & & \downarrow \simeq & & \parallel \\
 \mathbb{E}^{q-p}(\tilde{Z} - \tilde{Y}, \mathcal{L}_{\tilde{Z}-\tilde{Y}} + \mathcal{V}_{\tilde{Z}-\tilde{Y}}) & \xrightarrow{\sum_t \partial_{\tilde{Z}_t, \tilde{Y}_t}} & \bigoplus_{t \in f^{-1}(y)} \mathbb{E}^{q-p+1}(\tilde{Y}_t, \mathcal{L}_{\tilde{Y}_t} + \mathcal{V}_{\tilde{Y}_t}) & \xrightarrow{\sum_t g_{t*}} & \mathbb{E}^{q-p+1}(Y_y, \mathcal{L}_{Y_y} + \mathcal{V}_{Y_y})
 \end{array}$$

where the middle vertical map is the canonical isomorphism.

We can now identify ∂_y^x with the formal colimit of $\tilde{\partial}_{Y,y}^{Z,x}$ for Y, W . In view of 3.3.1.1, this is justified because:

- h is birational and $\tilde{Z} - \tilde{Y}$ is smooth with function field $\kappa(x)$.
- The closed pair $(\tilde{Z}_t, \tilde{Y}_t)$ is smooth of codimension 1 and the local ring of $\mathcal{O}_{\tilde{Z}_t, \tilde{Y}_t}$ is isomorphic (through h) to the valuation ring \mathcal{O}_{v_t} corresponding to the valuation v_t on $\kappa(x)$ considered in 3.3.1.1.

□

From Theorem 3.3.3.1 and Theorem 3.3.3.2, we deduce that the differentials coincide so that $C_*(X, \hat{\mathbb{E}}, \mathcal{V}_X)$ is a complex when the scheme X is smooth. We use this to prove that the premodule $\hat{\mathbb{E}}$ is a Milnor-Witt cycle module:

(FD) FINITE SUPPORT OF DIVISORS. Let X be a normal scheme, \mathcal{V}_X be a virtual vector bundle over X and ρ be an element of $M(\xi_X, \mathcal{V}_X)$. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in X^{(1)}$.

Proof. We can assume without loss of generality that X is affine of finite type. Then there exist a virtual vector bundle $\mathcal{V}_{\mathbb{A}_k^r}$ over \mathbb{A}_k^r and a closed immersion $i : X \rightarrow \mathbb{A}_k^r$ for some $r \geq 0$ which induces an inclusion

$$C^*(X, \hat{\mathbb{E}}, \mathcal{V}_X) \subset C^*(\mathbb{A}_k^r, \hat{\mathbb{E}}, \mathcal{V}_{\mathbb{A}_k^r})$$

compatible with the differentials thanks to the previous theorem. Hence (FD) holds.

□

(C) CLOSEDNESS. Let X be integral and local of dimension 2 and \mathcal{V}_X be a virtual bundle over X . Then

$$0 = \sum_{x \in X^{(1)}} \partial_{x_0}^x \circ \partial_x^\xi : M(\xi_X, \mathcal{V}_X) \rightarrow M(x_0, \mathcal{V}_X)$$

where ξ is the generic point and x_0 the closed point of X .

Proof. According to (FD), the differentials d of $C^*(X, \hat{\mathbb{E}}, \mathcal{V}_X)$ are well-defined. We want to prove that $d \circ d = 0$. Again, we can assume X to be affine of finite type over k . Then there exist a virtual vector bundle $\mathcal{V}_{\mathbb{A}_k^r}$ over \mathbb{A}_k^r and a closed immersion $i : X \rightarrow \mathbb{A}_k^r$ for some $r \geq 0$ which induces an inclusion

$$C^*(X, \hat{\mathbb{E}}, \mathcal{V}_X) \subset C^*(\mathbb{A}_k^r, \hat{\mathbb{E}}, \mathcal{V}_{\mathbb{A}_k^r})$$

compatible with the differentials. Hence (C) holds.

□

Since our constructions are natural in the motivic spectrum \mathbb{E} , we can conclude:

Theorem 3.3.3.3. *Consider $S = \mathrm{Spec} k$ the spectrum of a perfect field. The map $\mathbb{E} \mapsto \hat{\mathbb{E}}$ defines a functor from the category $\mathbf{SH}(S)$ of motivic spectrum to the category $\mathfrak{M}_k^{\mathrm{MW}}$ of Milnor-Witt cycle modules over k .*

3.4 An equivalence of categories

The purpose of this section is to prove the following theorem.

Theorem 3.4.0.4. *Let k be a perfect field. The functor of Theorem 3.3.3.3 induces an equivalence between the category of Milnor-Witt cycle modules and the heart of Morel-Voevodsky stable homotopy category (equipped with the homotopy t -structure):*

$$\mathfrak{M}_k^{\mathrm{MW}} \simeq \mathbf{SH}(k)^{\heartsuit}.$$

3.4.1 Associated homotopy module

We recall some facts about the heart of the stable homotopy category (see [Mor03, §5.2] or [Dé11, §1]).

Definition 3.4.1.1. Let M be an abelian Nisnevich sheaf on \mathbf{Sm}_k . We denote by $M_{-1}(X)$ the kernel of the morphism $M(X \times \mathbb{G}_m) \rightarrow M(X)$ induced by the unit section of \mathbb{G}_m . We say that M is strictly homotopy invariant if the Nisnevich cohomology sheaf $H_{\mathrm{Nis}}^*(-, M)$ is homotopy invariant.

Definition 3.4.1.2. A *homotopy module* is a pair (M_*, ω_*) where M_* is a \mathbb{Z} -graded abelian Nisnevich sheaf on \mathbf{Sm}_k which is strictly homotopy invariant and $\omega_n : M_{n-1} \rightarrow (M_n)_{-1}$ is an isomorphism (called *desuspension map*). A morphism of homotopy modules is a homogeneous natural transformation of \mathbb{Z} -graded sheaves compatible with the given isomorphisms. We denote by $\mathbf{HM}(k)$ the category of homotopy modules over k .

3.4.1.3. For any spectrum \mathbb{E} , the spectrum $\pi_0(\mathbb{E})$ has a canonical structure of a homotopy module. Moreover, the functor $\pi_0 : \mathbb{E} \mapsto \pi_0(\mathbb{E})$ induces an equivalence of categories between the heart of $\mathbf{SH}(k)$ for the homotopy t -structure and the category $\mathbf{HM}(k)$ (see [Mor03]). We denote its inverse by

$$H : \mathbf{HM}(k) \rightarrow \mathbf{SH}(k)^{\heartsuit}.$$

We continue with two lemmas of independent interest.

Lemma 3.4.1.4. *Let $g : Y \rightarrow X$ be a smooth morphism of schemes of finite type over k of constant fiber dimension l , let σ be a section of g , let \mathcal{V}_X be a virtual vector bundle over X and let $t \in \mathcal{O}_Y$ be a global parameter defining the subscheme $\sigma(X)$.*

Then $\sigma_ : C_*(X, M, \mathcal{V}_X) \rightarrow C_*(Y, M, \mathcal{V}_Y)$ is zero on homology.*

Proof. Consider the open subscheme $j : U = Y \setminus \sigma(X) \rightarrow Y$ and let $\tilde{g} = g \circ j$ the restriction of g . Let ∂ be the boundary map associated to σ . According to Lemma 2.4.2.5, we have $\sigma_* = \sigma_* \circ \partial \circ [t] \circ \tilde{g}^* = d \circ j_* \circ [t] \circ \tilde{g}^*$. □

With the same proof, we have a slightly more general result:

Lemma 3.4.1.5. *Let $g : Y \rightarrow X$, $\sigma : X \rightarrow Y$ and \mathcal{V}_X as previously. Let $i : Z \rightarrow X$ be a closed immersion and consider $\bar{Z} = g^{-1}(Z)$ the pullback along g . The induced map $\bar{\sigma} : Z \rightarrow \bar{Z}$ is such that the pushforward $\bar{\sigma}_* : C_*(Z, M, \mathcal{V}_Z) \rightarrow C_*(\bar{Z}, M, \mathcal{V}_{\bar{Z}})$ is zero on homology.*

Remark 3.4.1.6. This result may be compared to [FS08, Corollary 3.5].

Fix M a Milnor-Witt cycle module over k . We associate to M a homotopy module F^M , that is a homotopy invariant Nisnevich sheaf of \mathbb{Z} -graded abelian groups equipped with desuspension isomorphisms. Indeed, let X be a smooth scheme over S . For any integer n , we put

$$F_n^M(X) = A^0(X, M, -\Omega_{X/k} + \langle n \rangle).$$

This defines a presheaf $F^M = (F_n^M)_{n \in \mathbb{Z}}$ of graded abelian groups satisfying the homotopy invariance property (see Theorem 2.6.2.4).

Denote by $s_1 : X = \{1\} \times X \rightarrow \mathbb{G}_m X$ the induced closed immersion. We have $(F_n^M)_{-1}(X) = \ker s_1^*$. By homotopy invariance, we have also $F_n^M(\mathbb{A}_X^1) \simeq F_n^M(X)$ hence $(F_n^M)_{-1}(X) = \operatorname{coker} j^*$ where j is the open immersion $\mathbb{G}_m X \rightarrow \mathbb{A}_X^1$.

As usual, we get the following long exact sequence:

$$0 \longrightarrow F_n^M(X) \xrightarrow{j^*} F_n^M(\mathbb{G}_m X) \xrightarrow{\partial} F_{n-1}^M(X) \xrightarrow{i_*} A^1(\mathbb{A}_X^1, M, -\Omega_{\mathbb{A}_X^1/k} + \langle n \rangle).$$

Thus we see that ∂ induces a map $(F_n^M)_{-1}(X) \rightarrow F_{n-1}^M(X)$ which is an isomorphism because i_* is zero (according to Lemma 3.4.1.5).

We prove that F^M is a Nisnevich sheaf. We start with the complex $C_*(-, M, *)$ and consider a Nisnevich square

$$\begin{array}{ccc} U_V & \xrightarrow{j} & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

where i is open and p étale. Denote by $Z = (X - U)_{\text{red}}$ so that we have the decomposition

$$C_*(X, M, *) = C_*(U, M, *) \oplus C_*(Z, M, *)$$

and

$$C_*(V, M, *) = C_*(U_V, M, *) \oplus C_*(Z_V, M, *).$$

By assumption the induced map $p : Z_V \rightarrow Z$ is an isomorphism, hence the canonical map $p_* : C_*(Z_V, M, *) \rightarrow C_*(Z, M, *)$ is an isomorphism. Hence we can see that the image of the Nisnevich square by $C_*(-, M, *)$ is cocartesian. This proves that $C_*(-, M, *)$ is a Nisnevich sheaf and so is F^M .

We have proved the following theorem.

Theorem 3.4.1.7. *Let M be a Milnor-Witt cycle module over k . The graded presheaf F^M of abelian groups, defined by*

$$F_n^M(X) = A^0(X, M, -\Omega_{X/k} + \langle n \rangle)$$

for any smooth scheme X/S and any integer n , is a homotopy module.

3.4.2 First isomorphism

In order to prove Theorem 3.4.0.4, we construct two natural transformations and prove that they are isomorphisms. We start with the first isomorphism:

Let M be a Milnor-Witt cycle module. Since the category of homotopy modules is equivalent to the heart of the stable homotopy category $\mathbf{SH}(S)$ (see 3.4.1.3), Theorem 3.4.1.7 implies that there is an object \mathbb{M} of $\mathbf{SH}(S)^\heartsuit$ equipped with isomorphisms

$$\alpha_X : \mathbb{M}^{-n}(X, \langle n \rangle) \rightarrow F_n^M(X)$$

for any irreducible smooth scheme X of dimension d and any integer n (we recall that $F_n^M(X) = A^0(X, M, -\Omega_{X/k} + \langle n \rangle)$). The maps α are compatible with the right-way maps (contravariance) and the desuspension functor $(-)_-1$ in the sense that the following diagrams commute

$$\begin{array}{ccc} \mathbb{M}^{-n}(X, \langle n \rangle) & \xrightarrow{f^*} & \mathbb{M}^{-n}(Y, \langle n \rangle) \\ \downarrow \alpha_X & & \downarrow \alpha_Y \\ A^0(X, M, -\Omega_{X/k} + \langle n \rangle) & \xrightarrow{f^*} & A^0(Y, M, -\Omega_{Y/k} + \langle n \rangle) \end{array}$$

for any morphism $f : Y \rightarrow X$ of smooth schemes and

$$\begin{array}{ccc} \mathbb{M}^{-n}(X, \langle n-1 \rangle) & \xrightarrow{\omega_n} & (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1} \\ \downarrow \alpha_X & & \downarrow (\alpha_Y)_{-1} \\ A^0(X, M, -\Omega_{X/k} + \langle n-1 \rangle) & \xrightarrow{\omega'_n} & (A^0(X, M, -\Omega_{X/k} + \langle n \rangle))_{-1} \end{array}$$

where ω_n and ω'_n are the structural desuspension maps associated the two homotopy modules for any integer n .

Fix E/k a field and n an integer. Using the previous isomorphism α_X with $X = \text{Spec } A$ a smooth model of E and taking the limit over all such X , we obtain an isomorphism of abelian groups

$$\alpha_E : \hat{\mathbb{M}}(E, \langle n \rangle) \rightarrow M(E, \langle n \rangle).$$

According to 3.3.1.3, this also defines in a canonical way an isomorphism

$$\alpha_E : \hat{\mathbb{M}}(E, \mathcal{V}_E) \rightarrow M(E, \mathcal{V}_E)$$

for any virtual vector bundles \mathcal{V}_E over E .

We want to prove that this defines a morphism of Milnor-Witt cycle modules. It suffices to prove that α_E is natural in the data (D1), (D2), (D3) and (D4) (see Definition 2.3.2.5).

(D1) For any morphism $f : Y \rightarrow X$ of smooth schemes, the maps α are compatible with right-way (pullbacks) morphisms thus the following diagram is commutative

$$\begin{array}{ccc} \hat{\mathbb{M}}(E, \mathcal{V}_E) & \xrightarrow{\text{res}_{F/E}} & \hat{\mathbb{M}}(F, \mathcal{V}_F) \\ \downarrow \alpha_E & & \downarrow \alpha_F \\ M(E, \mathcal{V}_E) & \xrightarrow{\text{res}_{F/E}} & M(F, \mathcal{V}_F) \end{array}$$

where F/E is a field extension and \mathcal{V}_E is a virtual vector bundle over E .

(D4) Let Z be a smooth scheme over S . Since the maps α commute with the functor $(-)_-1$, we have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{M}^{-n}(\mathbb{A}_Z^1, \langle n \rangle) & \xrightarrow{j^*} & \mathbb{M}^{-n}(\mathbb{G}_m Z, \langle n \rangle) & \xrightarrow{\partial} & \mathbb{M}^{-n}(Z, \langle n \rangle) \\ \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\ A^0(\mathbb{A}_Z^1, -\Omega_{\mathbb{A}_Z^1/S} + \langle n \rangle) & \xrightarrow{j^*} & A^0(\mathbb{G}_m Z, -\Omega_{\mathbb{G}_m Z/S} + \langle n \rangle) & \xrightarrow{\partial} & A^0(Z, -\Omega_{Z/S} + \langle n \rangle) \end{array}$$

where $j : \mathbb{G}_m Z \rightarrow \mathbb{A}_Z^1$ is the open immersion complementary to the zero section $i : Z \rightarrow \mathbb{A}_Z^1$.

By deformation to the normal cone, we have the same commutative diagram when $j : \mathbb{G}_m Z \rightarrow \mathbb{A}_Z^1$ is replaced by an open immersion $j : X - Z \rightarrow X$ associated with a regular immersion $i : Z \rightarrow X$ of codimension 1. In particular, when $X = \text{Spec } \mathcal{O}_v$ is the spectrum of a valuation ring and $Z = \text{Spec } \kappa(v)$, we find that the maps α are compatible with the residue maps:

$$\begin{array}{ccc} M(E, \mathcal{V}_E) & \xrightarrow{\partial_v} & M(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \\ \alpha_E \downarrow & & \downarrow \alpha_{\kappa(v)} \\ \hat{\mathbb{M}}(E, \mathcal{V}_E) & \xrightarrow{\partial_v} & \hat{\mathbb{M}}(\kappa(v), -\mathcal{N}_v + \mathcal{V}_{\kappa(v)}) \end{array}$$

is a commutative square.

(D2) Let E be a field. The homotopy invariance property **(H)** states that the following sequence is split exact:

$$\textbf{(H)} \quad 0 \longrightarrow \hat{\mathbb{M}}(E, \mathbb{A}_E^1 + \Omega_{E/k} + \mathcal{V}_E) \xrightarrow{\text{res}_{E(t)/E}} \hat{\mathbb{M}}(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)})$$

$$\xrightarrow{d} \bigoplus_{x \in (\mathbb{A}_E^1)^{(1)}} \hat{\mathbb{M}}(\kappa(x), \Omega_{\kappa(x)/k} + \mathcal{V}_{\kappa(x)}) \longrightarrow 0$$

where $d = \sum_{x \in (\mathbb{A}_E^1)^{(1)}} \partial_x$ and where \mathcal{V}_E is a virtual vector bundle over E (this is true for any Milnor-Witt cycle module hence in particular for $\hat{\mathbb{M}}$).

We can use this property **(H)** and the data **(D1)** to characterize the data **(D2)**. Indeed, let F/E be a finite field extension. Assume F/E is monogenous, thus $F = E(x)$ where x corresponds to a point in $(\mathbb{A}_F^1)^{(1)}$. For any $\beta \in M(F, \Omega_{F/k} + \mathcal{V}_F)$ there exists $\gamma \in M(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)})$ with the property that $d(\gamma) = \beta$. Now the valuation at ∞ yields a morphism

$$\partial_\infty : M(E(t), \Omega_{E(t)/k} + \mathcal{V}_{E(t)}) \rightarrow M(E, \Omega_{E/k} + \mathcal{V}_E)$$

which vanishes on the image of $\text{res}_{E(t)/E}$. The element $-\partial(\gamma)$ does not depend on the choice of γ and is in fact equal to $\text{cores}_{F/E}(\beta)$. Using this characterization, we see that **(D2)** commutes with the maps α since they commute with **(D4)**.

(D3) In order to prove that the maps α commute with the \underline{K}^{MW} -action on the left, it suffices to do it for any generator $[u]$ (where u is a unit) and the Hopf map η .

Let E be a field over k and u be a unit of E . Denote by X the essentially smooth scheme $\text{Spec } E$. The unit defines a map $u : X \rightarrow \mathbb{G}_m X$ which induces a map

$$u^* : \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle) \rightarrow \mathbb{M}^{-n}(X, \langle n \rangle)$$

for any integer n . Moreover, we consider the canonical maps

$$\omega_n : \mathbb{M}^{-(n-1)}(X, \langle -1 + n \rangle) \rightarrow (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1}$$

and

$$v_n : (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1} \subset \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle).$$

Now consider the canonical morphism

$$\eta^* : \mathbb{M}^{-n}(X, \langle n \rangle) \rightarrow \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle)$$

induced by the Hopf map and the canonical projection

$$\pi_n : \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle) \rightarrow (\mathbb{M}^{-n}(X, \langle n \rangle))_{-1}.$$

One can check that the data D3 satisfies

$$\gamma_{[u]} = u^* v_n \omega_n : \mathbb{M}^{-(n-1)}(X, \langle -1 + n \rangle) \rightarrow \mathbb{M}^{-n}(X, \langle n \rangle).$$

and

$$\gamma_\eta = \omega_n \pi_n \eta^* : \mathbb{M}^{-n}(X, \langle n \rangle) \rightarrow \mathbb{M}^{-(n-1)}(X, \langle -1 + n \rangle)$$

We have the same description for the Milnor-Witt cycle module M . Since the maps α commute with pullbacks and transition maps ω_n , we see that they also commute with the \underline{K}^{MW} -action.

3.4.3 Second isomorphism

Let $\mathbb{M} \in \mathbf{SH}(k)^\heartsuit$. Let X be a smooth scheme over k and let x be a generic point of X . For any integer n , we have a canonical map $\mathbb{M}^{-n}(X, \langle n \rangle) \rightarrow \mathbb{M}^{-n}(\kappa(x), \langle n \rangle) = \hat{\mathbb{M}}(\kappa(x), \langle n \rangle)$. Thus we have a map $\beta_X : \mathbb{M}^{-n}(X, \langle n \rangle) \rightarrow C^0(X, \hat{\mathbb{M}}, -\Omega_{X/k} + \langle n \rangle)$ which factors through $A^0(X, \hat{\mathbb{M}}, -\Omega_{X/k} + \langle n \rangle)$. We want to prove that the arrow

$$\beta : \mathbb{M} \rightarrow A^0(-, \hat{\mathbb{M}}, *)$$

is an isomorphism of homotopy modules.

We prove that β_X are natural in X (with respect to Gysin morphisms). If $p : Y \rightarrow X$ is a smooth map of smooth schemes, it is clear by the definition of pullbacks for Chow-Witt groups with coefficients in M (see Subsection 2.4.1) that p^* commutes with β .

Now consider a regular closed immersion $i : Z \rightarrow X$ of smooth schemes. Recall that (by Definition 2.7.1.1) the Gysin morphism i^* (for Chow-Witt groups with coefficients in M) makes the following diagram commutative

$$\begin{array}{ccccc}
 A^0(X, M, \langle n \rangle) & \xrightarrow{q^*} & A^0(\mathbb{G}_m X, M, -\mathcal{L}_q + \langle n \rangle) & \xrightarrow{[t]} & A^0(\mathbb{G}_m X, M, \langle n \rangle) \\
 \parallel & & & & \downarrow \partial \\
 A^0(X, M, \langle n \rangle) & \xrightarrow{i^*} & A^0(Z, M, -\mathcal{L}_i + \langle n \rangle) & \xrightarrow[\simeq]{\pi^*} & A^0(N_Z X, M, \langle n \rangle)
 \end{array}$$

where $q : \mathbb{G}_m X \rightarrow X$ is the canonical projection and t is a parameter such that $\mathbb{A}_k^1 = \text{Spec } k[t]$ and where $\mathcal{L}_i = -\mathcal{N}_Z X$.

Similarly, the Gysin morphism i^* (for the cohomology theory \mathbb{M}) makes the following diagram commutative

$$\begin{array}{ccccc}
 \mathbb{M}^{-n}(X, \langle n \rangle) & \xrightarrow{q^*} & \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle) & \xrightarrow{[t]} & \mathbb{M}^{-n-1}(\mathbb{G}_m X, \langle n+1 \rangle) \\
 \parallel & & & & \downarrow \partial \\
 \mathbb{M}^{-n}(X, \langle n \rangle) & \xrightarrow{i^*} & \mathbb{M}^{-n}(Z, \langle n \rangle) & \xrightarrow[\simeq]{\pi^*} & \mathbb{M}^{-n}(N_Z X, \langle n \rangle)
 \end{array}$$

where q, π and t are defined as previously (the proof is the same as [D 08b, Proposition 2.6.5]). Putting things together, we see that the maps β commute with i^* hence with any pullbacks (of lci morphisms).

Moreover, we prove that β is compatible with the desuspension maps $\omega_n : \mathbb{M}_{n-1} \simeq (\mathbb{M}_n)_{-1}$ defining the homotopy modules \mathbb{M} and $A^0(-, \hat{\mathbb{M}}, *)$. Let X be an irreducible smooth scheme, we have the following diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{M}^{-n}(\mathbb{A}_X^1, \langle n \rangle) & \xrightarrow{j^*} & \mathbb{M}^{-n}(\mathbb{G}_m X, \langle n \rangle) & \xrightarrow{\partial_{\mathbb{M}}} & \mathbb{M}^{-n+1}(X, \langle n-1 \rangle) \longrightarrow \dots \\
 & & \downarrow \beta & (1) & \downarrow \beta & (2) & \downarrow \beta \\
 0 & \longrightarrow & A^0(\mathbb{A}_X^1, \hat{\mathbb{M}}, *) & \xrightarrow{j^*} & A^0(\mathbb{G}_m X, \hat{\mathbb{M}}, *) & \xrightarrow{\partial} & A^0(X, \hat{\mathbb{M}}, * - 1) \longrightarrow \dots
 \end{array}$$

We have already seen that the square (1) commutes. The map ∂ is defined in 3.3.1.2 using the data (D4) of the cycle module $\hat{\mathbb{M}}$ which corresponds to the map $\partial_{\mathbb{M}}$. Hence the square (2) commutes. According to Definition 3.4.1.1, the desuspension map $\omega_n : \mathbb{M}_{n-1} \simeq (\mathbb{M}_n)_{-1}$ is induced by $\partial_{\mathbb{M}}$. Thus β is a morphism of homotopy modules.

Finally, when X is the spectrum of a field, the map β_X is an isomorphism and so β is an isomorphism of homotopy modules.

Putting the second isomorphism β with the first isomorphism α of Subsection 3.4.2, we have proved Theorem 3.4.0.4.

3.5 Applications

3.5.1 Hermitian K-theory and Witt groups

We assume that the characteristic of k is different from 2.

In [Ati66], Atiyah studied the topological K-theory of $\mathbb{Z}/2$ -bundles on spaces with involution, expanding what we knew about real topological K-theory. The algebraic analogue is called *Hermitian K-theory* and was first introduced by Karoubi (see e.g.

[Kar80b, Kar80a]). A natural question was to translate this notion into the work of Morel and Voevodsky.

In [Hor05], Hornbostel proved that hermitian K-theory is representable in the stable homotopy category of Morel and Voevodsky. Precisely, there is a motivic $(8, 4)$ -periodic spectrum representing hermitian K-theory over the field k .

Moreover, the theory of quadratic forms was studied by Balmer. In particular, he introduced a graded 4-periodic generalization W_B^* of Witt groups (with the classical Witt groups standing in degree 0, see [Bal00, Bal01] for more details). Similarly, Hornbostel proved that there is a spectrum whose homotopy groups coincide with the groups W_B^* .

Thus, according to Theorem 3.4.0.4, we have the following theorem.

Theorem 3.5.1.1. *There exist Milnor-Witt cycle modules \underline{KO} and \underline{K}^W respectively associated to Hermitian K-theory and Balmer Witt groups in a canonical way.*

3.5.2 Monoidal structure, adjunction and equivalences of categories

Recall that a Grothendieck category is an abelian category \mathcal{C} with (infinite) coproducts (hence, all colimits) such that filtered colimits of exact sequences are exact and admitting a generator, that is, an object $G \in \mathcal{C}$ such that the functor $\mathrm{Hom}_{\mathcal{C}}(G, -)$ is faithful.

Thanks to Theorem 3.4.0.4, we can transpose known properties from the category of homotopy modules to the category of Milnor-Witt cycle modules:

Theorem 3.5.2.1. *The category $\mathfrak{M}_k^{\mathrm{MW}}$ of Milnor-Witt cycle modules is a Grothendieck category with products. Moreover, there is a canonical symmetric closed monoidal structure on $\mathfrak{M}_k^{\mathrm{MW}}$ such that the unit element is the cycle module $\underline{K}^{\mathrm{MW}}$. In addition, the monoidal tensor product commutes with the shifting functor defined in Example 2.3.2.7.*

Using the theory of framed correspondences, Ananyevskiy and Neshitov constructed Milnor-Witt transfers on homotopy modules, proving this way that the hearts of the homotopy t-structures on the stable \mathbb{A}^1 -derived category and the category of Milnor-Witt motives are equivalent [AN18]. Assuming k to be an infinite perfect field of characteristic not two, their proof relies on the work of Garkusha and Panin [GP14, GP18]. Similarly, one could give another proof of this fact:

Theorem 3.5.2.2. *The category of Milnor-Witt cycle modules is equivalent to the heart of the category of MW-motives (equipped with the homotopy t-structure):*

$$\mathfrak{M}_k^{\mathrm{MW}} \simeq \widetilde{\mathbf{DM}}(k)^{\heartsuit}.$$

In particular, the heart of Morel-Voevodsky stable homotopy category is equivalent to the heart of the category of MW-motives [DF18a] (both equipped with their respective homotopy t-structures):

$$\mathbf{SH}(k)^{\heartsuit} \simeq \widetilde{\mathbf{DM}}(k)^{\heartsuit}.$$

Proof. Let M be a Milnor-Witt cycle module. It corresponds to a homotopy module F^M according to the previous section. Since we have an action of the Milnor-Witt K-theory on F^M , we can prove (as in Theorem 4.3.2.2) that F^M has in fact MW-transfers (see 4.6.1.3 for the definition of homotopy modules with MW-transfers). We can then proceed as in the proof of Theorem 3.4.0.4 to prove the first equivalence of categories. The second equivalence follows from Theorem 3.4.0.4. \square

In his thesis [Dé03], Déglise studied the category of homotopy modules with transfers which is known to be equivalent to the heart of the category of Voevodsky's motives $\mathbf{DM}(k, \mathbb{Z})$ (with respect to the homotopy t-structure). Déglise's main theorem was that this category can be described with Rost's theory of cycle modules. This fact could be rediscovered thanks to our previous results:

Theorem 3.5.2.3 (Déglise). *Let k be a perfect field. The category of Rost cycle modules over k is equivalent to the heart of the category of Voevodsky's motives $\mathbf{DM}(k, \mathbb{Z})$ with respect to the homotopy t-structure:*

$$\mathfrak{M}_k^{\mathbf{M}} \simeq \mathbf{DM}(k, \mathbb{Z})^{\heartsuit}.$$

Proof. One can see that the category of Rost cycle modules is equivalent to the full subcategory of $\mathfrak{M}_k^{\mathbf{MW}}$ of Milnor-Witt cycle modules with trivial action of the generator η . Thanks to Theorem 3.5.2.2, this subcategory is equivalent to the full subcategory of $\widetilde{\mathbf{DM}}(k)^{\heartsuit}$ of homotopy modules with transfers and with trivial action of the Hopf map η . This last category is equivalent to $\mathbf{DM}(k, \mathbb{Z})^{\heartsuit}$. \square

Adjunction between MW-cycle modules and Rost cycle modules Consider M a classical cycle module (*à la* Rost, see [Ros96, §1]). Recall that we may define a Milnor-Witt cycle module $\Gamma_*(M)$ as follows. Let (E, \mathcal{V}_E) be in \mathfrak{F}_k and put

$$\Gamma_*(M)(E, \mathcal{V}_E) = M(E, \mathrm{rk} \mathcal{V}_E).$$

We can check that this defines a fully faithful exact functor

$$\Gamma_* : \mathfrak{M}_k^{\mathbf{M}} \rightarrow \mathfrak{M}_k^{\mathbf{MW}}.$$

where $\mathfrak{M}_k^{\mathbf{M}}$ (resp. $\mathfrak{M}_k^{\mathbf{MW}}$) is the category of Rost cycle modules (resp. Milnor-Witt cycle modules). This definition leads to the following theorem:

Theorem 3.5.2.4 (Adjunction Theorem). *There is an adjunction between the category of Milnor-Witt cycle modules and the category of classical cycle modules:*

$$\mathfrak{M}_k^{\mathbf{MW}} \rightleftarrows \mathfrak{M}_k^{\mathbf{M}}.$$

Proof. We gave an elementary proof of this result in Theorem 2.8.2.5 with an explicit description of the adjoint functors. For a second proof, combine Theorem 3.4.0.4 and Theorem 3.5.2.3. \square

3.5.3 Birational invariance

Studying unramified cohomology groups with $\mathbb{Z}/2$ -coefficients, one can see that an elliptic curve is not birational to the projective line. More generally, étale cohomology and K-theory are a source of such birational invariants (see [Col92] for more details).

Rost proved in [Ros96, Corollary 12.10] that, if X is a proper smooth variety over k and M a cycle module, then the group $A^0(X, M)$ is a birational invariant of X . A natural question is to extend this for Milnor-Witt modules, hoping that the quadratic nature of our theory will lead to more refined (birational) invariants and thus sharper theorems. Rost's proof heavily depends on the existence of pullback maps for flat morphisms. Unfortunately, such pullback maps remain to be constructed in our setting. Nevertheless, a different method yields the expected result:

Theorem 3.5.3.1. *Let X be a proper smooth integral scheme over k , let \mathcal{V}_k a virtual vector bundle over k and let M be a Milnor-Witt cycle module. Then the group $A^0(X, M, -\Omega_{X/k} + \mathcal{V}_X)$ is a birational invariant of X .*

Proof. (see also [Voi19, Lemma 1.3]). Denote by $F^M(X) = A^0(X, M, -\Omega_{X/k} + \mathcal{V}_X)$. This defines a contravariant functor that satisfies:

1. If $U \subset X$ is a Zariski open set, then the map $F^M(X) \rightarrow F^M(U)$ is injective,
2. If $U \subset X$ is a Zariski open set such that $\text{codim}_X(X \setminus U) \geq 2$, then the map $F^M(X) \rightarrow F^M(U)$ is an isomorphism.

Indeed, these properties follow from the localization long exact sequence 2.5.1.3.

Now let $\Phi : X \dashrightarrow Y$ be a birational map between smooth and proper integral schemes over k . Then there is an open set $U \subset X$ such that $\text{codim}_X(X \setminus U) \geq 2$ and Φ_U is an morphism. Then we have $F^M(X) \simeq F^M(U)$ and, by functoriality, a morphism $\Phi_U^* : F^M(Y) \rightarrow F^M(U)$, hence a morphism $\Phi_* : F^M(Y) \rightarrow F^M(X)$. Replacing Φ by Φ^{-1} , we get $\Phi_V^{-1} : F^M(X) \rightarrow F^M(V)$ for some Zariski open set V of Y such that $F^M(Y) \simeq F^M(V)$. Let $U' \subset U$ be defined as $\Phi_V^{-1}(V)$. Then $\Phi^{-1} \circ \Phi$ is the identity on U' , hence $(\Phi^{-1})_* \circ \Phi_* : F^M(X) \rightarrow F^M(X)$ is the identity. Since $F^M(X) \rightarrow F^M(U')$ is injective, we can conclude that Φ_* is an isomorphism. \square

As a corollary, we have the following theorem.

Theorem 3.5.3.2. *Let X be a proper smooth integral scheme over k . Then the Chow-Witt groups $\widetilde{\text{CH}}^0(X)$ are birational invariants of X . Moreover, let $F_* \in \mathbf{HM}(k)$ be a homotopy module, then $F_*(X)$ is a birational invariant of X .*

Proof. See 2.5.2.1 for the definition of $\widetilde{\text{CH}}^0(X)$. The first statement follows from Theorem 3.5.3.1 applied with $M = \underline{\mathbf{K}}^{MW}$ (see Proposition 2.5.2.2). The second statement follows from Theorem 3.5.3.1 and Theorem 3.4.0.4. \square

4

Milnor-Witt homotopy sheaves and generalized transfers

We explore a conjecture of Morel about the Bass-Tate transfers defined on the contraction of a homotopy sheaf [Mor12] and prove that the conjecture is true with rational coefficients. Moreover, we study the relations between (contracted) homotopy sheaves, sheaves with Morel generalized transfers and MW-homotopy sheaves, and prove an equivalence of categories. As applications, we describe the essential image of the canonical functor that forgets MW-transfers and use these results to discuss the conservativity conjecture in \mathbb{A}^1 -homotopy due to Bachmann and Yakerson [BY18, Conjecture 1.1].

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4.1 Introduction

4.1.1 Current work

In [Mor12], Morel studied homotopy invariant Nisnevich sheaves in order to provide computational tools in \mathbb{A}^1 -homotopy analogous to Voevodsky's theory of sheaves with transfers. The most basic result is that (unramified) sheaves are characterized by their sections on fields and some extra data (see Subsection 1.5.1). One of the main theorem of [Mor12] is the equivalence between the notions of *strongly \mathbb{A}^1 -invariance* and *strictly \mathbb{A}^1 -invariance* for sheaves of abelian groups (see *loc. cit.* Theorem 1.16). In order to prove this, Morel defined geometric transfers on the contraction M_{-1} of a homotopy sheaf (i.e. a strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups). The definition is an adaptation of the original one of Bass and Tate for Milnor K-theory [BT73]. Morel proved that the transfers are functorial (i.e. they do not depend on the choice of generators) for any two-fold contraction M_{-2} of a homotopy sheaf and conjectured that the result should hold for M_{-1} (see Conjecture 4.4.1.13 or [Mor12, Remark 4.31]).

The notion of *sheaves with generalized transfers* was first defined in [Mor11, Definition 5.7] as a way to formalize the different structures naturally arising on some homotopy sheaves. In Section 4.3, we give a slightly modified definition of *sheaves with generalized transfers* which takes into account twists by the usual line bundles. Following [Mor12, Chapter 5], we define the Rost-Schmid complex associated to such homotopy sheaves and study the usual pushforward maps f_* , pullback maps g^* , GW-action $\langle a \rangle$ and residue maps ∂ . Moreover, we prove the following theorem.

Theorem 1 (see Theorem 4.3.2.3). *Let $M \in \mathbf{HI}^{\text{gr}}(k)$ be a homotopy sheaf with generalized transfers. The presheaf $\tilde{\Gamma}_*(M)$ of abelian groups, defined by*

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M \otimes (\omega_{X/k})^\vee)$$

for any smooth scheme X/k , is a MW-homotopy sheaf canonically isomorphic to M as presheaves.

In Section 4.4, we recall the construction of the Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} and prove that this defines a structure of generalized transfers:

Theorem 2 (see Theorem 4.4.1.20). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Then:*

1. *Assume that 2 is invertible. The rational contracted homotopy sheaf $M_{-1, \mathbb{Q}}$ is a homotopy sheaf with generalized transfers.*
2. *Assuming Conjecture 4.4.1.13, the contracted homotopy sheaf M_{-1} is a homotopy sheaf with generalized transfers.*

In particular, we obtain the following intersection multiplicity formula which was left open in [Fel18]:

Theorem 3 (see Theorem 4.4.1.16). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Consider morphisms of fields over k , $\varphi : E \rightarrow F$ and $\psi : E \rightarrow L$ with φ finite. Let R be the ring $F \otimes_E L$. For each $\mathfrak{p} \in \text{Spec } R$, let $\varphi_{\mathfrak{p}} : L \rightarrow R/\mathfrak{p}$ and $\psi_{\mathfrak{p}} : F \rightarrow R/\mathfrak{p}$ be the morphisms induced by φ and ψ . One has*

$$M_{-1}(\psi) \circ \mathrm{Tr}_\varphi = \sum_{\mathfrak{p} \in \mathrm{Spec} R} e_{\mathfrak{p}, \varepsilon} \mathrm{Tr}_{\varphi_{\mathfrak{p}}} \circ M_{-1}(\psi_{\mathfrak{p}})$$

where $e_{\mathfrak{p}, \varepsilon} = \sum_{i=1}^{e_{\mathfrak{p}}} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the length $e_{\mathfrak{p}}$ of the localized ring $R_{(\mathfrak{p})}$.

Generalizing ideas of Voevodsky, Calmès and Fasel introduced the additive symmetric monoidal category $\widetilde{\mathrm{Cor}}_k$ of smooth k -schemes with morphisms given by the so-called *finite Milnor-Witt correspondences* (see [BCD⁺20, Chapter 2]). In Section 4.5, we recall the basic definitions regarding this theory and prove that any homotopy sheaf with MW-transfers has a structure of a sheaf with generalized transfers. More precisely, we show that the two notions coincide:

Theorem 4 (Theorem 4.5.2.5). *There is a pair of functors*

$$\mathbf{HI}^{\mathrm{MW}}(k) \begin{array}{c} \xrightarrow{\tilde{\Gamma}^*} \\ \xleftarrow{\tilde{\Gamma}_*} \end{array} \mathbf{HI}^{\mathrm{gtr}}(k)$$

that forms an equivalence between the category of homotopy sheaves with MW-transfers and the category of homotopy sheaves with generalized transfers.

In Section 4.6, we prove the following theorem that characterizes the essential image of the functor $\tilde{\gamma}_* : \mathbf{HI}^{\mathrm{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers.

Theorem 5 (Theorem 4.6.1.6). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The following assertions are equivalent:*

- (i) *There exists $M' \in \mathbf{HI}(k)$ satisfying Conjecture 4.4.1.13 and such that $M \simeq M'_{-1}$.*
- (ii) *There exists a structure of generalized transfers on M .*
- (iii) *There exists a structure of MW-transfers on M .*
- (iv) *There exists $M'' \in \mathbf{HI}(k)$ such that $M \simeq M''_{-2}$.*

This result is linked with the conservativity conjecture from [BY18] and allows us to prove the following theorems.

Theorem 6 (Corollary 4.6.2.3). *Let $d > 0$ be a natural number. The Bachmann-Yakerson conjecture holds (integrally) for $d = 2$ and rationally for $d = 1$: namely, the canonical functor*

$$\mathbf{SH}^{S^1}(k)(2) \rightarrow \mathbf{SH}(k)$$

is conservative on bounded below objects¹, the canonical functor

$$\mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$$

is conservative on rational bounded below objects, and the canonical functor

$$\mathbf{HI}(k, \mathbb{Q})(1) \rightarrow \mathbf{HI}^{\mathrm{fr}}(k, \mathbb{Q})$$

¹Also known as *connective* objects.

is an equivalence of abelian categories.

Moreover, let \mathcal{X} be a pointed motivic space. Then the canonical map

$$\pi_0 \Omega_{\mathbb{P}^1}^d \Sigma_{\mathbb{P}^1}^d \mathcal{X} \rightarrow \pi_0 \Omega_{\mathbb{P}^1}^{d+1} \Sigma_{\mathbb{P}^1}^{d+1} \mathcal{X}$$

is an isomorphism for $d \geq 2$.

Theorem 7 (Corollary 4.6.2.4). *The category of homotopy sheaves with generalized transfers, the category of MW-homotopy sheaves and the category of homotopy sheaves with framed transfers are equivalent:*

$$\mathbf{HI}^{\text{gr}}(k) \simeq \mathbf{HI}^{\text{MW}}(k) \simeq \mathbf{HI}^{\text{fr}}(k).$$

In future work, we will apply the conservative conjecture of Bachmann and Yakerson to study some intersection points between \mathbb{A}^1 -homotopy theory and affine algebraic geometry. For instance, following [AØ19, Conjecture 5.3.11 and Remark 5.3.12], one should obtain:

Theorem 8. *Let X be a smooth scheme and $x \in X$ a closed point. If $\Sigma_{\mathbb{P}^1}^\infty(X, x) \simeq *$ in $\mathbf{SH}(k)$, then $\Sigma_{\mathbb{P}^1}^2(X, x)$ is \mathbb{A}^1 -contractible.*

In particular, this applies when X is a Koras-Russel threefold of the first or second kind (see [AØ19, Theorem 5.3.9] and [DF18b, HKØ15] for similar results).

4.1.2 Outline of the chapter

In Section 4.2, we recall the theory of unramified sheaves and how they are related to homotopy sheaves of abelian groups.

In Section 4.3, we define the notion of sheaves with generalized transfers and study the associated Rost-Schmid complex.

In Section 4.4, we define the Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} and prove the conjecture of Morel in the case of rational coefficients.

In Section 4.5, we recall the theory of sheaves with MW-transfers [BCD⁺20] and prove that it is equivalent to the notion of sheaves with generalized transfers.

In Section 4.6, we give some corollaries of Theorem 4.5.2.5. In particular, we characterize the essential image of the functor $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers and use the previous results to discuss the *conservativity conjecture* in \mathbb{A}^1 -homotopy due to Bachmann and Yakerson (see [BY18, Conjecture 1.1] and [Bac20]).

Notation

Throughout the chapter, we fix a (commutative) field k and we assume moreover that k is infinite perfect of characteristic not 2. We need these assumptions in order to apply the cancellation theorem [BCD⁺20, Chapter 4] but we believe these restrictions could be lifted.

We denote by **Grp** and **Ab** the categories of (abelian) groups.

We consider only schemes that are essentially of finite type over k . All schemes and morphisms of schemes are defined over k . The category of smooth k -schemes of finite type is denoted by **Sm_k** and is endowed with the Nisnevich topology (thus, *sheaf* always means *sheaf for the Nisnevich topology*).

If X is a scheme and n a natural number, we denote by $X_{(n)}$ (resp. $X^{(n)}$) the set of point of dimension n (resp. codimension n) of X .

By a field E over k , we mean a k -finitely generated field E . Since k is perfect, notice that $\text{Spec } E$ is essentially smooth over S . We denote by \mathcal{F}_k the category of such fields.

Let $f : X \rightarrow Y$ be a morphism of schemes. Denote by \mathcal{L}_f (or $\mathcal{L}_{X/Y}$) the virtual vector bundle over Y associated with the cotangent complex of f , and by ω_f (or $\omega_{X/Y}$) its determinant. Recall that if $p : X \rightarrow Y$ is a smooth morphism, then \mathcal{L}_p is (isomorphic to) $\mathcal{T}_p = \Omega_{X/Y}$ the space of (Kähler) differentials. If $i : Z \rightarrow X$ is a regular closed immersion, then \mathcal{L}_i is the normal cone $-\mathcal{N}_Z X$. If f is the composite $Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X$ with p and i as previously (in other words, if f is lci projective), then \mathcal{L}_f is isomorphic to the virtual tangent bundle $i^* \mathcal{T}_{\mathbb{P}_X^n/X} - \mathcal{N}_Y(\mathbb{P}_X^n)$ (see also Subsection 1.9.1).

Let X be a scheme and $x \in X$ a point, we denote by $\mathcal{L}_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ and ω_x its determinant. Similarly, let v a discrete valuation on a field, we denote by ω_v the line bundle $(\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee$.

Let E be a field (over k) and v a valuation on E . We will always assume that v is discrete. We denote by \mathcal{O}_v its valuation ring, by \mathfrak{m}_v its maximal ideal and by $\kappa(v)$ its residue class field. We consider only valuations of geometric type, that is we assume: $k \subset \mathcal{O}_v$, the residue field $\kappa(v)$ is finitely generated over k and satisfies $\text{tr. deg}_k(\kappa(v)) + 1 = \text{tr. deg}_k(E)$.

Let E be a field. We denote by $\text{GW}(E)$ the Grothendieck-Witt ring of symmetric bilinear forms on E . For any $a \in E^*$, we denote by $\langle a \rangle$ the class of the symmetric bilinear form on E defined by $(X, Y) \mapsto aXY$ and, for any natural number n , we put $n_\varepsilon = \sum_{i=1}^n \langle -1 \rangle^{i-1}$.

To any natural number n , we can associate an element in $\text{GW}(E)$ denoted by $n_\varepsilon = \sum_{i=1}^n \langle -1 \rangle^{i-1}$. Recall that, if n and m are two natural numbers, then $(nm)_\varepsilon = n_\varepsilon m_\varepsilon$.

4.2 Homotopy sheaves

4.2.1 Unramified sheaves

In this subsection, we summarize [Mor12, Chapter 2] and recall the basic results concerning unramified sheaves.

Definition 4.2.1.1. 1. A sheaf of sets \mathcal{S} on \mathbf{Sm}_k is said to be \mathbb{A}^1 -invariant if for any $X \in \mathbf{Sm}_k$, the map

$$\mathcal{S}(X) \rightarrow \mathcal{S}(\mathbb{A}_X^1)$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection.

2. A sheaf of groups \mathcal{G} on \mathbf{Sm}_k is said to be *strongly* \mathbb{A}^1 -invariant if, for any $X \in \mathbf{Sm}_k$, the map

$$H_{Nis}^i(X, \mathcal{G}) \rightarrow H_{Nis}^i(\mathbb{A}^1 \times X, \mathcal{G})$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection for $i \in \{0, 1\}$.

3. A sheaf of abelian groups M on \mathbf{Sm}_k is said to be *strictly \mathbb{A}^1 -invariant* if, for any $X \in \mathbf{Sm}_k$, the map

$$H_{Nis}^i(X, M) \rightarrow H_{Nis}^i(\mathbb{A}^1 \times X, M)$$

induced by the projection $\mathbb{A}^1 \times X \rightarrow X$, is a bijection for $i \in \mathbb{N}$.

Remark 4.2.1.2. In the sequel, we work with M a sheaf of groups. We could give more general definitions for sheaves of sets but, in practice, we need only the case of sheaves of abelian groups. In that case, we recall that a strongly \mathbb{A}^1 -invariant sheaf of abelian groups is necessarily strictly \mathbb{A}^1 -invariant (see Theorem 1.7.2.7).

Definition 4.2.1.3. An unramified presheaf of groups M on \mathbf{Sm}_k is a presheaf of groups M such that the following holds:

- (0) For any smooth scheme $X \in \mathbf{Sm}_k$ with irreducible components X_α ($\alpha \in X^{(0)}$), the canonical map $M(X) \rightarrow \prod_{\alpha \in X^{(0)}} M(X_\alpha)$ is an isomorphism.
- (1) For any smooth scheme $X \in \mathbf{Sm}_k$ and any open subscheme $U \subset X$ everywhere dense in X , the restriction map $M(X) \rightarrow M(U)$ is injective.
- (2) For any smooth scheme $X \in \mathbf{Sm}_k$, irreducible with function field F , the injective map $M(X) \rightarrow \bigcap_{x \in X^{(1)}} M(\mathcal{O}_{X,x})$ is an isomorphism (the intersection being computed in $M(F)$).

Example 4.2.1.4. Homotopy modules with transfers [Dé11] and Rost cycle modules [Ros96] define unramified sheaves. In characteristic not 2, the sheaf associated to the presheaf of Witt groups $X \rightarrow W(X)$ is unramified.

We may give an explicit description of unramified sheaves on \mathbf{Sm}_k in terms of their sections on fields $F \in \mathcal{F}_k$ and some extra structure. We will say that a functor $M : \mathcal{F}_k \rightarrow \mathbf{Grp}$ is *continuous* if $M(F)$ is the filtering colimit of the groups $M(F_\alpha)$ where F_α runs over the subfields of F of finite type over k .

Definition 4.2.1.5 ([Mor12], Definition 2.6). An unramified \mathcal{F}_k -datum consists of:

uD1 A continuous functor $M : \mathcal{F}_k \rightarrow \mathbf{Grp}$.

uD2 For any field $F \in \mathcal{F}_k$ and any discrete valuation v on F , a subgroup

$$M(\mathcal{O}_v) \subset M(F).$$

uD3 For any field $F \in \mathcal{F}_k$ and any valuation v on F , a map $s_v : M(\mathcal{O}_v) \rightarrow M(\kappa(v))$, called the specialization map associated to v .

The previous data should satisfy the following axioms:

uA1 If $\iota : E \subset F$ is a separable extension in \mathcal{F}_k and w is a valuation on F which restrict to a discrete valuation v on E with ramification index 1, then the arrow $M(\iota)$ maps $M(\mathcal{O}_v)$ into $M(\mathcal{O}_w)$. Moreover, if the induced extension $\bar{\iota} : \kappa(v) \rightarrow \kappa(w)$ is an isomorphism, then the following square

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_w) \\ \downarrow & & \downarrow \\ M(E) & \longrightarrow & M(F) \end{array}$$

is cartesian.

uA2 Let $X \in \mathbf{Sm}_k$ be an irreducible smooth scheme with function field F . If $x \in M(F)$, then x lies in all but a finite number of $M(\mathcal{O}_x)$ where x runs over the set $X^{(1)}$ of points of codimension 1.

uA3(i) If $\iota : E \subset F$ is an extension in \mathcal{F}_k and w is a discrete valuation on F which restricts to a discrete valuation v on E , then $M(\iota)$ maps $M(\mathcal{O}_v)$ into $M(\mathcal{O}_w)$ and the following diagram

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(\mathcal{O}_w) \\ \downarrow & & \downarrow \\ M(\kappa(v)) & \longrightarrow & M(\kappa(w)) \end{array}$$

is commutative.

uA3(ii) If $\iota : E \subset F$ is an extension in \mathcal{F}_k and w a discrete valuation on F which restricts to zero on E , then the map $M(\iota) : M(E) \rightarrow M(F)$ has its image contained in $M(\mathcal{O}_w)$. Moreover, if $\bar{\iota} : E \subset \kappa(w)$ denotes the induced extension, the composition $M(E) \longrightarrow M(\mathcal{O}_w) \xrightarrow{s_w} M(\kappa(w))$ is equal to $M(\bar{\iota})$.

uA4(i) For any smooth scheme $X \in \mathbf{Sm}_k$ local of dimension 2 with closed point $z \in X^{(2)}$, and for any point $y_0 \in X^{(1)}$ such that the reduced closed scheme \bar{y}_0 is k -smooth, then $s_{y_0} : M(\mathcal{O}_{y_0}) \rightarrow M(\kappa(y_0))$ maps $\bigcap_{y \in X^{(1)}} M(\mathcal{O}_y)$ into $M(\mathcal{O}_{\bar{y}_0, z}) \subset M(\kappa(y_0))$.

uA4(ii) The composition

$$\bigcap_{y \in X^{(1)}} M(\mathcal{O}_y) \rightarrow M(\mathcal{O}_{\bar{y}_0, z}) \rightarrow M(\kappa(z))$$

does not depend on the choice of y_0 such that $\bar{y}_0 \in \mathbf{Sm}_k$.

Example 4.2.1.6. For any integer n , the functor $\mathbf{K}^{\text{MW}} : \mathcal{F}_k \rightarrow \mathbf{Grp}$ defined in Section 1.4 is an unramified \mathcal{F}_k -datum.

4.2.1.7. An unramified sheaf M defines in an obvious way an unramified \mathcal{F}_k -datum. Indeed, taking the evaluation² on the field extensions of k yields a restriction functor:

$$M : \mathcal{F}_k \rightarrow \mathbf{Grp}, F \mapsto M(F)$$

²See Subsection 1.9.2 or the proof of [Mor12, Proposition 2.8] for more details.

such that, for any field F with valuation v , we have an $M(\mathcal{O}_v) \subset M(F)$ and a specialization map $s_v : M(\mathcal{O}_v) \rightarrow M(\kappa(v))$ (obtained by choosing smooth models over k for the closed immersion $\text{Spec } \kappa(v) \rightarrow \text{Spec } \mathcal{O}_v$). We claim that this satisfies axioms [uA1](#), \dots , [uA4\(ii\)](#).

Reciprocally, given an unramified \mathcal{F}_k -datum M and $X \in \mathbf{Sm}_k$ an irreducible smooth scheme with function field F , we define the subset $M(X) \subset M(F)$ as the intersection $\bigcap_{x \in X^{(1)}} M(\mathcal{O}_x) \subset M(F)$. We extend the definition for any X so that property [\(0\)](#) is satisfied. Using the fact that any map $f : Y \rightarrow X$ between smooth schemes is the composition $Y \hookrightarrow Y \times_k X \twoheadrightarrow X$ of closed immersion followed by a smooth projection, one can define an unramified sheaf $M : \mathbf{Sm}_k \rightarrow \mathbf{Grp}$. In short, we have the following theorem.

Theorem 4.2.1.8 ([\[Mor12\]](#), Theorem 2.11). *The two functors described above define an equivalence between the category of unramified sheaves on \mathbf{Sm}_k and that of unramified \mathcal{F}_k -data.*

Example 4.2.1.9. As a corollary of the previous theorem, we obtain a definition of the (unramified) sheaf of Milnor-Witt K-theory \mathbf{K}_n^{MW} for any integer n .

4.2.1.10. From now on, we will not distinguish between the notion of unramified sheaves on \mathbf{Sm}_k and that of unramified \mathcal{F}_k -datum. In the remaining subsection, we fix M an unramified sheaf of groups on \mathbf{Sm}_k and explain how it is related to strongly \mathbb{A}^1 -invariant sheaves.

4.2.1.11. NOTATION. If $\varphi : E \rightarrow F$ is an extension of fields, the map from [uD1](#)

$$M(\varphi) : M(E) \rightarrow M(F)$$

is also denoted by res_φ , $\text{res}_{F/E}$ or φ_* .

4.2.1.12. Let $F \in \mathcal{F}_k$ be a field and v a valuation on F . We define the pointed set

$$H_v^1(\mathcal{O}_v, M) = M(F)/M(\mathcal{O}_v).$$

This is a left $M(F)$ -set. Moreover, for any point y of codimension 1 in $X \in \mathbf{Sm}_k$, we set $H_y^1(X, M) = H_y^1(\mathcal{O}_{X,y}, M)$. By axiom [uA2](#), if X is irreducible with function field F , the induced left action of $M(F)$ on $\prod_{y \in X^{(1)}} H_y^1(X, M)$ preserves the weak-product

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \subset \prod_{y \in X^{(1)}} H_y^1(X, M)$$

where the weak-product $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ means the set of families for which all but a finite number of terms are the base point of $H_y^1(X, M)$. By definition and axiom [uA2](#), the isotropy subgroup of this action of $M(F)$ on the base point of $\prod'_{y \in X^{(1)}} H_y^1(X, M)$ is exactly $M(X) = \bigcap_{y \in X^{(1)}} M(\mathcal{O}_{X,y})$. We summarize this property by saying that the diagram

$$1 \rightarrow M(X) \rightarrow M(F) \rightrightarrows \prod'_{y \in X^{(1)}} H_y^1(X, M)$$

is exact.

Definition 4.2.1.13. For any point z of codimension 2 in a smooth scheme X , we denote by $H_z^2(X, M)$ the orbit set of $\prod'_{y \in X^{(1)}_{(z)}} H_y^1(X, M)$ under the left action of $M(F)$ where $F \in \mathcal{F}_k$ is the function field of $X_{(z)}$.

4.2.1.14. For an irreducible essentially smooth scheme X with function field F , we define the boundary $M(F)$ -equivariant map

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod_{z \in X^{(2)}} H_z^2(X, M)$$

by collecting together the compositions

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod'_{y \in X_{(z)}^{(1)}} H_y^1(X, M) \rightarrow H_z^2(X, M)$$

for each $z \in X^{(2)}$.

It is not clear in general whether or not the image of the boundary map is always contained in the weak product $\prod'_{z \in X^{(2)}} H_z^2(X, M)$. For this reason we introduce the following axiom:

uA2' For any irreducible essentially smooth scheme X , the image of the boundary map

$$\prod'_{y \in X^{(1)}} H_y^1(X, M) \rightarrow \prod_{z \in X^{(2)}} H_z^2(X, M)$$

is contained in the weak product $\prod'_{z \in X^{(2)}} H_z^2(X, M)$.

4.2.1.15. From now on we assume that M satisfies **uA2'**. For any smooth scheme X irreducible with function field F , we have a complex $C^*(X, M)$

$$1 \rightarrow M(X) \rightarrow M^{(0)}(X) \Rightarrow M^{(1)}(X) \rightarrow M^{(2)}(X)$$

where $M^{(0)}(X) = \prod'_{x \in X^{(0)}} M(\kappa(x)) = \prod_{x \in X^{(0)}} M(\kappa(x))$, $M^{(1)}(X) = \prod'_{y \in X^{(1)}} H_y^1(X, M)$ and $M^{(2)}(X) = \prod'_{z \in X^{(2)}} H_z^2(X, M)$. By construction, this complex is exact (in an obvious sense, see [Mor12, Definition 2.20]) for any (essentially) smooth local scheme of dimension ≤ 2 .

Definition 4.2.1.16. A strongly unramified \mathcal{F}_k -data is an unramified \mathcal{F}_k -data M satisfying **uA2'** and the following axioms:

uA5(i) For any separable finite extension $\iota : E \subset F$ in \mathcal{F}_k , any discrete valuation w on F which restricts to a discrete valuation v on E with ramification index 1, and such that the induced extension $\bar{\iota} : \kappa(v) \rightarrow \kappa(w)$ is an isomorphism, the commutative square of groups

$$\begin{array}{ccc} M(\mathcal{O}_v) & \longrightarrow & M(E) \\ \downarrow & & \downarrow \\ M(\mathcal{O}_w) & \longrightarrow & M(F) \end{array}$$

induces a bijection $H_w^1(\mathcal{O}_w, M) \simeq H_v^1(\mathcal{O}_v, M)$.

uA5(ii) For any étale morphism $X' \rightarrow X$ between smooth local k -schemes of dimension 2, with closed point respectively z' and z , inducing an isomorphism on the residue fields $\kappa(z) \simeq \kappa(z')$, the pointed map

$$H_z^2(X, M) \rightarrow H_{z'}^2(X', M)$$

has trivial kernel.

uA6 For any localization U of a smooth k -scheme at some point u of codimension ≤ 1 , the complex:

$$1 \rightarrow M(\mathbb{A}_U^1) \rightarrow M^{(0)}(\mathbb{A}_U^1) \Rightarrow M^{(1)}(\mathbb{A}_U^1) \rightarrow M^{(2)}(\mathbb{A}_U^1)$$

is exact. Moreover, the morphism $M(U) \rightarrow M(\mathbb{A}_U^1)$ is an isomorphism.

Theorem 4.2.1.17 ([Mor12], Theorem 2.27). *There is an equivalence between the category of strongly \mathbb{A}^1 -invariant sheaves of groups on \mathbf{Sm}_k and that of strongly unramified \mathcal{F}_k -data of groups on \mathbf{Sm}_k .*

Definition 4.2.1.18. A strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups is called a *homotopy sheaf*. We denote by $\mathbf{HI}(k)$ the category of homotopy sheaves and natural transformations of sheaves.

Example 4.2.1.19. As a corollary of the previous theorem, we obtain that the sheaf of Milnor-Witt K-theory \mathbf{K}_n^{MW} is a homotopy sheaf for any integer n .

4.2.1.20. MONOIDAL STRUCTURE. Recall that there is a canonical adjunction of categories

$$\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k) \xrightleftharpoons[\pi_{\mathbb{A}^1}]{\mathcal{O}} D(\text{Sh}(\mathbf{Sm}_k))$$

where $\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k)$ the effective \mathbb{A}^1 -derived category and $D(\text{Sh}(\mathbf{Sm}_k))$ is the derived category of complexes of sheaves over \mathbf{Sm}_k (see Subsection 1.3.2 or [CD19, §5]). Thanks to Morel's \mathbb{A}^1 -localization theorem, we can prove that there is a unique t-structure on $\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k)$ such that the forgetful functor \mathcal{O} is t-exact and that the category of homotopy sheaves $\mathbf{HI}(k)$ is equivalent to the heart $(\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k))^{\heartsuit}$ for this t-structure (in particular, $\mathbf{HI}(k)$ is a Grothendieck category). Since the canonical tensor product $\otimes_{\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}}$ is right t-exact, it induces a monoidal structure on $\mathbf{HI}(k)$. Precisely, if $F, G \in \mathbf{HI}(k)$ are two homotopy sheaves, then their tensor product is

$$F \otimes_{\mathbf{HI}} G = H_0^{\mathbb{A}^1}(F \otimes_{\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}} G)$$

where $H_0^{\mathbb{A}^1}$ is the homology object in degree 0 for the homotopy t-structure.

Example 4.2.1.21. For any integers n and m , we have a canonical isomorphism

$$\mathbf{K}_n^{\text{MW}} \otimes_{\mathbf{HI}} \mathbf{K}_m^{\text{MW}} \simeq \mathbf{K}_{n+m}^{\text{MW}}.$$

In particular $\text{GW} = \mathbf{K}_0^{\text{MW}}$ is a commutative monoid.

4.2.2 Contracted homotopy sheaves

4.2.2.1. In this section, we fix $M \in \mathbf{HI}(k)$ a homotopy sheaf. Recall that the contraction M_{-1} is by definition the sheaf of abelian groups

$$X \mapsto \ker(M(\mathbb{G}_m \times X) \rightarrow M(X)).$$

Equivalently, we have

$$M_{-1} = \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M)$$

where $\underline{\mathcal{H}om}$ is the internal hom-object of $\mathbf{HI}(k)$. According to [Mor12, Lemma 2.32], the sheaf M_{-1} is also a homotopy sheaf and is called a *contracted homotopy sheaf*.

Example 4.2.2.2. For any integer n , we have a canonical isomorphism $(\mathbf{K}_n^{\text{MW}})_{-1} = \mathbf{K}_{n-1}^{\text{MW}}$.

4.2.2.3. Any unit map $a : \mathbb{G}_m \rightarrow \mathbb{G}_m$ induces a morphism

$$\langle a \rangle : \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M) \rightarrow \underline{\mathcal{H}om}(\mathbf{K}_1^{\text{MW}}, M)$$

and thus M_{-1} has a module structure over the commutative monoid GW (see also [Mor12, Lemma 3.49]). Moreover, we have a bilinear pairing

$$\begin{aligned} \mathbf{K}_1^{\text{MW}} \times M_{-1} &\rightarrow M \\ ([u], \mu) &\mapsto [u] \cdot \mu, \end{aligned}$$

where \mathbf{K}_1^{MW} is defined in 4.2.1.19 (see also [Mor12, Lemma 3.48]).

4.2.2.4. Let $N \in \mathbf{HI}(k)$ be another homotopy sheaf and assume N is equipped with a GW -module structure. Let $X \in \mathbf{Sm}_k$ be a smooth scheme and \mathcal{L} a line bundle on X . We define the twist of N by \mathcal{L} denoted by $N\{\mathcal{L}\}$ or $N \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[\mathcal{L}^\times]$ as the sheaf associated to the presheaf on the Zariski site X_{Zar} :

$$U \mapsto N(U) \otimes_{\mathbb{Z}[\mathcal{O}_X(U)^\times]} \mathbb{Z}[\mathcal{L}_U^\times]$$

where \mathcal{L}_U^\times is the set of isomorphisms between \mathcal{O}_U and \mathcal{L}_U (which may be empty). We put $N(X, \mathcal{L}) = \Gamma(X, N\{\mathcal{L}\})$ and remark that, if X is affine, then $N(X, \mathcal{L}) = N(X) \otimes_{\mathbb{Z}[\mathcal{O}_X(X)^\times]} \mathbb{Z}[\mathcal{L}_X^\times]$. In particular, this definition applies to the sheaf $N = M_{-1}$.

4.2.2.5. **COHOMOLOGY WITH SUPPORT EXACT SEQUENCE.** Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. For any closed immersion $i : Z \rightarrow X$ of smooth schemes over k , with complementary open immersion $j : U \rightarrow X$, there exists a canonical cohomology with support exact sequence of the form:

$$\Gamma_Z(X, M) \xrightarrow{i_*} M(X) \xrightarrow{j^*} M(U) \xrightarrow{\partial} H_Z^1(X, M) \longrightarrow \dots$$

4.2.2.6. Recall that, by convention, we implicitly extend our sheaves to the set of essentially smooth schemes in a canonical way. The purity isomorphism (more precisely: Axiom uA5(i) and [Mor12, Lemma 3.50]) implies that for any discrete valuation v on a field $F \in \mathcal{F}_k$, one has a canonical bijection

$$H_v^1(\mathcal{O}_v, M) \simeq M_{-1}(\kappa(v), \omega_v)$$

and thus obtain a residue map

$$\partial_v : M(F) \rightarrow M_{-1}(\kappa(v), \omega_v).$$

Proposition 4.2.2.7. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf and consider the following commutative square*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ q \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X \end{array}$$

of closed immersions of separated schemes over k . We have the following diagram

$$\begin{array}{ccccccc}
 \Gamma_T(X, M) & \xrightarrow{k_*} & \Gamma_Y(X, M) & \xrightarrow{k'^*} & \Gamma_{Y-T}(X-Z, M) & \xrightarrow{\partial_k} & H_T^1(X, M) \\
 \downarrow q_* & & \downarrow p_* & & \downarrow \tilde{p}_* & & \downarrow q_* \\
 \Gamma_Z(X, M) & \xrightarrow{i_*} & M(X) & \xrightarrow{i'^*} & M(X-Z) & \xrightarrow{\partial_i} & H_Z^1(X, M) \\
 \downarrow q'^* & & \downarrow p'_* & & \downarrow \tilde{p}'^* & & \downarrow q'^* \\
 \Gamma_{Z-T}(X-Y, M) & \xrightarrow{\tilde{i}_*} & M(X-Y) & \xrightarrow{\tilde{i}'^*} & M(X-(Y \cup Z)) & \xrightarrow{\partial_{\tilde{i}}} & H_{Z-T}^1(X-T, M) \\
 \downarrow \partial_k & & \downarrow \partial_p & & \downarrow \partial_{\tilde{p}} & (*) & \downarrow \partial_{\tilde{k}} \\
 H_T^1(X, M) & \xrightarrow{k_*} & H_Y^1(X, M) & \xrightarrow{k'^*} & H_{Y-T}^1(X-Z, M) & \xrightarrow{\partial_k} & H_T^2(X, M)
 \end{array}$$

with obvious maps. Each squares of this diagram is commutative except for $(*)$ which is anti-commutative.

Proof. This is a classical consequence of the octahedron axiom. \square

4.3 Sheaves with generalized transfers

4.3.1 Morel's axioms

The following definition is a slightly improved version of Morel's definition in [Mor11, Definition 5.7]. It is directly inspired by Rost's theory of cycles modules. Lastly, it can be seen as an effective counter-part of our own axiomatic of MW-cycle modules.

Definition 4.3.1.1. Let M be a homotopy sheaf (i.e. a strongly \mathbb{A}^1 -invariant Nisnevich sheaf of abelian groups on \mathbf{Sm}_k). We say that M has a structure of generalized transfers if M has a structure of GW-modules and satisfies the following datum

eD2 For any finite extension $\varphi : E \rightarrow F$ in \mathcal{F}_k and any natural number $i \in \mathbb{N}$, there is a map

$$\mathrm{Tr}_\varphi = \mathrm{Tr}_{F/E} : M_{-i}(F, \omega_{F/k}) \rightarrow M_{-i}(E, \omega_{E/k})$$

called the transfer morphism from F to E .

In addition, this datum satisfies the following axioms:

eR1b $\mathrm{Tr}_{\mathrm{Id}_E} = \mathrm{Id}_{M(E)}$ and for any composable finite morphisms φ and ψ in \mathcal{F}_k , we have

$$\mathrm{Tr}_{\psi \circ \varphi} = \mathrm{Tr}_\varphi \circ \mathrm{Tr}_\psi.$$

eR1c Consider $\varphi : E \rightarrow F$ and $\psi : E \rightarrow L$ with φ finite. Let R be the ring $F \otimes_E L$. For each $\mathfrak{p} \in \mathrm{Spec} R$, let $\varphi_{\mathfrak{p}} : L \rightarrow R/\mathfrak{p}$ and $\psi_{\mathfrak{p}} : F \rightarrow R/\mathfrak{p}$ be the morphisms induced by φ and ψ . One has

$$M(\psi) \circ \mathrm{Tr}_\varphi = \sum_{\mathfrak{p} \in \mathrm{Spec} R} e_{\mathfrak{p}, \varepsilon} \mathrm{Tr}_{\varphi_{\mathfrak{p}}} \circ M(\psi_{\mathfrak{p}})$$

where $e_{p,\varepsilon} = \sum_{i=1}^{e_p} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the length e_p of the local artinian ring $R_{(\mathfrak{p})}$.

eR2 Let $\psi : E \rightarrow F$ be a finite extension of fields.

eR2b For $\langle a \rangle \in \text{GW}(E)$ and $\mu \in M(F, \omega_{F/k})$, one has $\text{Tr}_{F/E}(\langle \psi(a) \rangle \cdot \mu) = \langle a \rangle \cdot \text{Tr}_{F/E}(\mu)$.

eR2c For $\langle a \rangle \in \text{GW}(F, \omega_{F/k})$ and $\mu \in M(E)$, one has $\text{Tr}_{F/E}(\langle a \rangle \cdot \text{res}_{F/E}(\mu)) = \text{Tr}_{F/E}(\langle a \rangle) \cdot \mu$.

eR3b Let $i \in \mathbb{N}$ be a natural number, $\phi : E \rightarrow F$ be a finite extension of fields and let v be a valuation on E . For each extension w of v , we denote by $\phi_w : \kappa(v) \rightarrow \kappa(w)$ the induced morphism. We have

$$\partial_v \circ \text{Tr}_\phi = \sum_w \text{Tr}_{\phi_w} \circ \partial_w$$

where $\partial_v : M_{-i}(E, \omega_{E/k}) \rightarrow M_{-i-1}(\kappa(v), \omega_{\kappa(v)/k})$ and $\partial_w : M_{-i}(F, \omega_{F/k}) \rightarrow M_{-i-1}(\kappa(w), \omega_{\kappa(w)/k})$ are the residue maps defined in 1.5.2.6.

Remark 4.3.1.2. Our definition differs from Morel's in two ways. First, we have taken into account the twists naturally arising (this is not really important if one works in characteristic zero). Second, Axiom **A3** of [Mor11, Definition 5.7] is replaced by **eR3b** (these two axioms should be equivalent in some cases).

Remark 4.3.1.3. We know from 1.5.1.17 that homotopy sheaves can be understood as certain functors on function fields. Taking into account this fact, our axioms are indeed *effective* variants of that of Milnor-Witt cycle modules. In fact, we will see that they correspond to homotopy sheaves with Milnor-Witt transfers, as MW-cycle modules correspond to homotopy modules. (This explains our choice of numbering of the axioms.)

Remark 4.3.1.4. A homotopy sheaf with generalized transfers is a particular case of a sheaf with \mathbb{A}^1 -transfers as defined in [BY18, §5].

Example 4.3.1.5. 1. The Milnor-Witt sheaf \mathbf{K}_n^{MW} has a structure of generalized transfers ([Mor12, §4.2], Subsection 1.4, see also Theorem 4.4.1.16).

2. If M is a homotopy sheaf with generalized transfers, then so is M_{-1} .

Remark 4.3.1.6. In the next section, we will give conditions for a contracted homotopy sheaf M_{-1} to be equipped with a structure of generalized transfers.

Definition 4.3.1.7. A map between homotopy sheaves with generalized transfers is a natural transformation commuting with the transfers. We denote by $\mathbf{HI}^{\text{gr}}(k)$ the category of homotopy sheaves with generalized transfers over k .

4.3.1.8. **ROST-SCHMID COMPLEX.** Let $M \in \mathbf{HI}^{\text{gr}}(k)$ and let X be a scheme essentially of finite type over k . We define the Rost-Schmid complex as the graded abelian group defined for any $n \in \mathbb{N}$ by:

$$C^n(X, M) = \bigoplus_{x \in X^{(n)}} M_{-n}(\kappa(x), \omega_{\kappa(x)/k}).$$

The transition maps d are defined as follows. If X is normal with generic point ξ , then for any $x \in X^{(1)}$ the local ring of X at x is a valuation ring so that we have a map $\partial_x : M_{-n}(\kappa(\xi), \omega_{\kappa(\xi)/k}) \rightarrow M_{-n-1}(\kappa(x), \omega_{\kappa(x)/k})$ for any n .

Now suppose X is a scheme essentially of finite type over k and let x, y be two points in X . We define a map

$$\partial_y^x : M_{-n}(\kappa(x), \omega_{\kappa(x)/k}) \rightarrow M_{-n-1}(\kappa(y), \omega_{\kappa(y)/k})$$

as follows. Let $Z = \overline{\{x\}}$. If $y \notin Z^{(1)}$, then put $\partial_y^x = 0$. If $y \in Z^{(1)}$, let $\tilde{Z} \rightarrow Z$ be the normalization and put

$$\partial_y^x = \sum_{z|y} \text{Tr}_{\kappa(z)/\kappa(y)} \circ \partial_z$$

with z running through the finitely many points of \tilde{Z} lying over y .

Thus, we may define a differential map

$$d = \sum_{x,y} \partial_y^x : C^n(X, M) \rightarrow C^{n+1}(X, M)$$

which is well-defined according to the following proposition.

Proposition 4.3.1.9. *For any homotopy sheaf with generalized transfers $M \in \mathbf{HI}^{\text{gr}}(k)$ and any scheme essentially of finite type X , the Rost-Schmid complex $C^*(X, M)$ is a complex.*

Proof. Same proof as Theorem 1.7.1.10. \square

Definition 4.3.1.10. For any homotopy sheaf with generalized transfers $M \in \mathbf{HI}^{\text{gr}}(k)$ and any scheme essentially of finite type X , the cohomology groups associated to the Rost-Schmid complex are denoted by $A^*(X, M)$.

We now define the analogue of Rost's four basic maps [Ros96, §3] for the Rost-Schmid complex and prove that they are complex morphisms in some special cases.

4.3.1.11. **PULLBACK.** Let $M \in \mathbf{HI}^{\text{gr}}(k)$. Let $f : X \rightarrow Y$ be an essentially smooth morphism of schemes essentially of finite type. Define

$$f^* : C^*(Y, M) \rightarrow C^*(X, M\{\omega_{X/Y}^\vee\})$$

as follows. If $f(x) = y$, then $(f^*)_x^y = \Theta \circ \text{res}_{\kappa(x)/\kappa(y)}$, where Θ is the canonical isomorphism induced by $\omega_{\text{Spec } \kappa(x)/\text{Spec } \kappa(y)} \simeq \omega_{X/Y} \times_X \text{Spec } \kappa(x)$. Otherwise, $(f^*)_x^y = 0$. If X is not connected, take the sum over each connected component.

4.3.1.12. **PUSHFORWARD.** Let $M \in \mathbf{HI}^{\text{gr}}(k)$. Let $f : X \rightarrow Y$ be a morphism between schemes essentially of finite type over k and assume that X is connected (if X is not connected, take the sum over each connected component). Let $d = \dim(Y) - \dim(X)$. We define

$$f_* : C^*(X, M) \rightarrow C^{*-d}(Y, M_{-d})$$

as follows. If $y = f(x)$ and $\kappa(x)$ is finite over $\kappa(y)$, then put $(f_*)_x^y = \text{Tr}_{\kappa(x)/\kappa(y)}$ where $\text{Tr}_{\kappa(x)/\kappa(y)}$ is the transfer map eD2. Otherwise, put $(f_*)_x^y = 0$.

4.3.1.13. **GW-ACTION.** Let $M \in \mathbf{HI}^{\text{gr}}(k)$. Let X be a scheme essentially of finite type over k and $a \in \mathcal{O}_X^*$ a global unit. Define a morphism

$$\langle a \rangle : C^*(X, M) \rightarrow C^*(X, M)$$

as follows. Let $x, y \in X^{(p)}$ and $\rho \in M_{-*}(\kappa(x), \omega_{x/k})$. If $x = y$, then $\langle a \rangle_x^y(\rho) = \langle a(x) \rangle \cdot \rho$. Otherwise, $\langle a \rangle_x^y(\rho) = 0$.

4.3.1.14. **BOUNDARY MAPS.** Let $M \in \mathbf{HI}^{\text{gr}}(k)$. Let X be a scheme essentially of finite type over k , let $i : Z \rightarrow X$ be a closed immersion and let $j : U = X \setminus Z \rightarrow X$ be the inclusion of the open complement. We will refer to (Z, i, X, j, U) as a boundary triple and define

$$\partial = \partial_Z^U : C_p(U, M) \rightarrow C_{p-1}(Z, M)$$

by taking ∂_y^x to be as the definition in 4.3.1.8 with respect to X . The map ∂_Z^U is called the boundary map associated to the boundary triple, or just the boundary map for the closed immersion $i : Z \rightarrow X$.

We now fix $M \in \mathbf{HI}^{\text{gr}}(k)$ a homotopy sheaf with generalized transfers and study the morphisms defined on the Rost-Schmid complex with coefficients in M .

Proposition 4.3.1.15 (Functoriality and base change). *1. Let $f : X \rightarrow Y$ and $f' : Y \rightarrow Z$ be two morphisms of schemes essentially of finite type. Then*

$$(f' \circ f)_* = f'_* \circ f_*.$$

2. Let $g : Y \rightarrow X$ and $g' : Z \rightarrow Y$ be two essentially smooth morphisms. Then (up to the canonical isomorphism given by $\omega_{Z/X} \simeq \omega_{Z/Y} + (g')^ \omega_{Y/X}$):*

$$(g \circ g')^* = g'^* \circ g^*.$$

3. Consider a pullback square

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & X \end{array}$$

with f, f', g, g' as previously. Then

$$g^* \circ f_* = f'_* \circ g'^*$$

up to the canonical isomorphism induced by $\omega_{U/Z} \simeq \omega_{Y/X} \times_Y U$.

Proof. This follows as in Proposition 2.4.2.1 from eR1b and eR1c. □

Proposition 4.3.1.16. *(i) Let $f : X \rightarrow Y$ be a finite morphism of schemes essentially of finite type. Then*

$$d_Y \circ f_* = f_* \circ d_X.$$

(ii) Let $g : Y \rightarrow X$ be an essentially smooth morphism. Then

$$g^* \circ d_X = d_Y \circ g^*.$$

(iii) Let a be a unit on X . Then

$$d_X \circ \langle a \rangle = \langle a \rangle \circ d_X$$

(iv) Let (Z, i, X, j, U) be a boundary triple. Then

$$d_Z \circ \partial_Z^U = -\partial_Z^U \circ d_U.$$

Proof. As in Proposition 2.4.2.6, (ii), (iii) and (iv) follow easily from the definitions. The assertion (i) is nontrivial since our axioms are weaker than in the stable case studied in Section 2.4.2. Fortunately, Morel proved that the map

$$g_* : C_*(Y, N_{-1}) \rightarrow C_*(X, N_{-1})$$

is a morphism of complexes when N is a homotopy sheaf (see [Mor12, Corollary 5.30])³. The proof can be adapted almost verbatim if we replace the contracted homotopy sheaf N_{-1} by any sheaf with generalized transfers M (the use of [Mor12, Theorem 5.19] is replaced by eR3b). \square

Remark 4.3.1.17. According to the previous proposition, the morphisms f_* for f finite, g^* for g essentially smooth, multiplication by $\langle a \rangle$ commute with the differentials. We use the same notations to denote the induced morphisms on the cohomology groups $A^*(X, M)$.

4.3.2 MW-transfers structure

Definition 4.3.2.1. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. For any smooth scheme X , denote by

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M\{\omega_{X/k}^\vee\}).$$

This defines a presheaf $\tilde{\Gamma}_*(M)$.

Theorem 4.3.2.2. Let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. Then the contravariant functor $\tilde{\Gamma}_*(M) : X \mapsto \tilde{\Gamma}_*(M)(X)$ induces a presheaf on $\widetilde{\text{Cor}}_k$.⁴

Proof. Let X, Y be two smooth schemes (which may be assumed to be connected without loss of generality) and $T \subset X \times Y$ be an admissible subset (see Definition 1.8.1.2). Let $\beta \in \tilde{\Gamma}_*(M)(Y)$ and $\alpha \in \widetilde{\text{CH}}_T^{d_Y}(X \times Y, \omega_Y)$. We set

$$\alpha^*(\beta) = (p_X)_*(\alpha \cdot p_Y^*(\beta))$$

where $p_X : T \rightarrow X$ and $p_Y : X \times Y \rightarrow Y$ are the canonical morphisms, and where the product $\alpha \cdot p_Y^*(\beta)$ is defined by linearity thanks to 4.3.1.13 and Proposition 4.3.1.16(iii) (we do not have to define a more refined intersection product as in Subsection 2.8.1 since we work with generic points). We remark that the map $(p_X)_*$ is well-defined⁵ thanks to Proposition 4.3.1.16(i). This yields to an application α^* which is additive. We can see that this definition does not depend on the choice of T . Thus $\alpha \mapsto \alpha^*$ defines a map $\widetilde{\text{Cor}}_k(X, Y) \rightarrow \text{Hom}_{\mathbf{Ab}}(\tilde{\Gamma}_*(M)(Y), \tilde{\Gamma}_*(M)(X))$. It remains to check that this map preserves the respective compositions. Consider the diagram

³Note that the proof of *loc. cit.* depends on Conjecture 4.4.1.13 which is true in our case thanks to eR1b.

⁴See Subsection 1.8.1 for more details on MW-transfers.

⁵We have used Voevodsky's trick here: $X \times Y \mapsto X$ is not finite, but its restriction $p_X : T \rightarrow X$ is finite by assumption.

$$\begin{array}{c}
 X \times Z \xleftarrow{q_{XZ}} X \times Y \times Z \xrightarrow{q_{YZ}} Y \times Z \xrightarrow{q_Z} Z \\
 \quad \quad \quad \downarrow q_{XY} \quad \quad \quad \downarrow p_Y \\
 \quad \quad \quad X \times Y \xrightarrow{q_Y} Y \\
 \quad \quad \quad \downarrow p_X \\
 \quad \quad \quad X.
 \end{array}$$

$r_X: X \times Z \rightarrow X$, $r_Z: X \times Z \rightarrow Z$

Let $\alpha_1 \in \widetilde{\text{CH}}_{T_1}^{d_Y}(X \times Y, \omega_Y)$ and $\alpha_2 \in \widetilde{\text{CH}}_{T_2}^{d_Z}(X \times Z, \omega_Y)$ be two correspondences, with $T_1 \subset X \times Y$ and $T_2 \subset Y \times Z$ admissible. Moreover, let $\beta \in M(Z)$. By definition, we have

$$(\alpha_2 \circ \alpha_1)^*(\beta) = (r_X)_*[(q_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2) \cdot r_Z^* \beta].$$

Using the projection formula [eR2b](#), we have

$$\begin{aligned}
 (r_X)_*[(q_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2) \cdot r_Z^* \beta] &= (r_X)_*[(q_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2 \cdot q_{XY}^* r_Z^* \beta)] \\
 &= (p_X)_*(p_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2 \cdot q_{XZ}^* r_Z^* \beta).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \alpha_1^* \circ \alpha_2^*(\beta) &= \alpha_1^*((p_Y)_*(\alpha_2 \cdot q_Z^* \beta)) \\
 &= (p_X)_*(\alpha_1 \cdot q_Y^*(p_Y)_*(\alpha_2 \cdot q_Z^* \beta))
 \end{aligned}$$

By base change [4.3.1.15](#), $q_Y^*(p_Y)_* = (p_{XY})_* q_{YZ}^*$ and it follows (using the projection formula once again) that

$$\begin{aligned}
 \alpha_1^* \circ \alpha_2^*(\beta) &= (p_X)_*(\alpha_1 \cdot (p_{XY})_*(q_{YZ}^* \alpha_2 \cdot q_{YZ}^* q_Z^* \beta)) \\
 &= (p_X)_*(p_{XY})_*(p_{XY}^* \alpha_1 \cdot q_{YZ}^* \alpha_2 \cdot q_{XZ}^* r_Z^* \beta).
 \end{aligned}$$

Hence the result. □

We have proved the following theorem.

Theorem 4.3.2.3. *Let $M \in \mathbf{HI}^{\text{gr}}(k)$ be a homotopy sheaf with generalized transfers. The presheaf $\tilde{\Gamma}_*(M)$ of abelian groups, defined by*

$$\tilde{\Gamma}_*(M)(X) = A^0(X, M\{\omega_{X/k}^\vee\})$$

for any smooth scheme X/k , is a MW-homotopy sheaf (see Definition [1.8.1.10](#)) canonically isomorphic to M as (pre)sheaves.

Proof. This follows from Theorem [4.3.2.2](#) and the fact that the natural map $M \rightarrow \tilde{\Gamma}_*(M)$ agrees on function fields and the sheaves are unramified (see also Theorem [1.7.2.4](#)). □

4.3.2.4. A morphism of sheaves with generalized transfers commutes with the transfers $\mathrm{Tr}_{F/E}$ and the GW -action hence induces a natural transformation on the Rost-Schmid complex which commutes with the respective maps 4.3.1.12, 4.3.1.11 and 4.3.1.13 defined on the Rost-Schmid complex. After a careful examination of the proof of Theorem 4.3.2.2, we can thus define a functor

$$\begin{aligned}\tilde{\Gamma}_* : \mathbf{HI}^{\mathrm{gtr}}(k) &\rightarrow \mathbf{HI}^{\mathrm{MW}}(k) \\ M &\mapsto \tilde{\Gamma}_*(M)\end{aligned}$$

which is conservative.

We end this section with a lemma that will be useful later.

Lemma 4.3.2.5. *Let $M \in \mathbf{HI}^{\mathrm{gtr}}(k)$ be a homotopy sheaf with generalized transfers and let $\psi : E \rightarrow F$ be a finite extension of fields. For any finite model⁶ $f : Y \rightarrow X$ of ψ , we have defined in 4.3.1.12 a pushforward map*

$$f_* : A^0(Y, M) \rightarrow A^0(X, M).$$

The limit of all such maps over finite models $f : Y \rightarrow X$ defines a map

$$M(F, \omega_{F/k}) \rightarrow M(E, \omega_{E/k})$$

which is equal to the generalized transfer map $\mathrm{Tr}_{F/E}$.

Proof. This follows from the definitions. □

4.4 Morel's conjecture on Bass-Tate transfers

4.4.1 Bass-Tate transfers

4.4.1.1. Let M be a homotopy sheaf and M_{-1} its contraction. We recall the construction of the Bass-Tate transfer maps

$$\mathrm{tr}_\psi = \mathrm{tr}_{F/E} : M_{-1}(F, \omega_{F/k}) \rightarrow M_{-1}(E, \omega_{E/k})$$

defined for any finite map $\psi : E \rightarrow F$ of fields.

Theorem 4.4.1.2. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F be a field and $F(t)$ the field of rational fractions with coefficients in F in one variable t . We have a split short exact sequence*

$$0 \rightarrow M(F) \xrightarrow{\mathrm{res}} M(F(t)) \xrightarrow{d} \bigoplus_{x \in (\mathbb{A}_F^1)^{(1)}} M_{-1}(\kappa(x), \omega_x) \rightarrow 0$$

where $d = \bigoplus_{x \in (\mathbb{A}_F^1)^{(1)}} \partial_x$ is the usual differential (see 4.3.1.8).

⁶See Definition 1.9.2.3.

Proof. See [Mor12, Theorem 5.38]⁷. □

Definition 4.4.1.3 (Coresidue maps). Keeping the previous notations, the fact that the homotopy sequence is canonically split allows us to define *coresidue maps*

$$\rho_x : M_{-1}(\kappa(x), \omega_x) \rightarrow M(F(t))$$

for any closed points $x \in (\mathbb{A}_F^1)^{(1)}$, satisfying $\partial_x \circ \rho_x = \text{Id}_{\kappa(x)}$ and $\partial_y \circ \rho_x = 0$ for $x \neq y$ where $y \in (\mathbb{A}_F^1)^{(1)}$.

Definition 4.4.1.4 (Bass-Tate transfers). Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F be a field and $F(t)$ the field of rational fractions with coefficients in F in one variable t . For $x \in (\mathbb{A}_F^1)^{(1)}$, we define the Bass-Tate transfer

$$\text{tr}_{x/F} : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F, \omega_{F/k})$$

by the formula $\text{tr}_{x/F} = -\partial_\infty \circ \rho_x$.

Remark 4.4.1.5. There is also an equivalent definition of the Bass-Tate transfers that does not use the coresidue maps (see [Mor12, §4.2]).

Lemma 4.4.1.6. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\varphi : E \rightarrow F$ be a field extension which induces a morphism, and w be a valuation on F which restricts to a non trivial valuation v on E with ramification e . We have a commutative square*

$$\begin{array}{ccc} M(E) & \xrightarrow{\partial_v} & M_{-1}(\kappa(v), \omega_v) \\ \varphi_* \downarrow & & \downarrow e_\varepsilon \cdot \bar{\varphi}_* \\ M(F) & \xrightarrow{\partial_w} & M_{-1}(\kappa(w), \omega_w) \end{array}$$

where $\bar{\varphi} : \kappa(v) \rightarrow \kappa(w)$ is the induced map and $e_\varepsilon = \sum_{i=1}^e \langle -1 \rangle^{i-1}$.

Proof. See Subsection 3.3.2, (R3a). □

We now prove a base change formula (see also Theorem 2.3.1.10 for a similar result). The proof is similar to the original case where M corresponds to Milnor K-theory (see [BT73, §1], or [GS17, §7]).

Lemma 4.4.1.7. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F/E be a field extension and $x \in (\mathbb{A}_E^1)^{(1)}$ a closed point. Then the following diagram*

$$\begin{array}{ccc} M_{-1}(E(x), \omega_{E(x)/k}) & \xrightarrow{\text{tr}_{x/E}} & M_{-1}(E, \omega_{E/k}) \\ \oplus_y \text{res}_{F(y)/E(x)} \downarrow & & \downarrow \text{res}_{F/E} \\ \oplus_{y \mapsto x} M_{-1}(F(y), \omega_{F(y)/k}) & \xrightarrow{\sum_y e_{y,\varepsilon} \text{tr}_{y/F}} & M_{-1}(F, \omega_{F/k}) \end{array}$$

⁷In fact, Morel does not use twisted sheaves but chooses a canonical generator for each ω_x instead, which is equivalent.

is commutative, where the notation $y \mapsto x$ stands for the closed points of \mathbb{A}_F^1 lying above x , and $e_{y,\varepsilon} = \sum_{i=1}^{\varepsilon_y} \langle -1 \rangle^{i-1}$ is the quadratic form associated to the ramification index of the valuation v_y extending v_x to $F(t)$.

Proof. According to Lemma 4.4.1.6, the following diagram

$$\begin{array}{ccc} M(E(t)) & \xrightarrow{\partial_x} & M_{-1}(E(x), \omega_x) \\ \text{res}_{F(t)/E(t)} \downarrow & & \downarrow \oplus_y e_{y,\varepsilon} \text{res}_{F(y)/E(x)} \\ M(F(t)) & \xrightarrow{\oplus_y \partial_y} & \oplus_{y \mapsto x} M_{-1}(F(y), \omega_y) \end{array}$$

is commutative hence so does the diagram

$$\begin{array}{ccc} M(E(t)) & \xleftarrow{\rho_x} & M_{-1}(E(x), \omega_x) \\ \text{res}_{F(t)/E(t)} \downarrow & & \downarrow \oplus_y e_{y,\varepsilon} \text{res}_{F(y)/E(x)} \\ M(F(t)) & \xleftarrow{\oplus_y \rho_y} & \oplus_{y \mapsto x} M_{-1}(F(y), \omega_y) \end{array}$$

since the difference is killed by all ∂_y for $y \in (\mathbb{A}_F^1)^{(1)}$. Then, we conclude according to the definition of the Bass-Tate transfer maps 4.4.1.4. \square

Remark 4.4.1.8. The multiplicities e_y appearing in the previous lemma are equal to

$$[E(x) : E]_i / [F(y) : F]_i$$

where $[E(x) : E]_i$ is the inseparable degree.

Lemma 4.4.1.9. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\varphi : E \rightarrow F = E(x)$ be a simple extension. Then*

1. *For $\langle a \rangle \in \text{GW}(E)$ and $\mu \in M(F, \omega_{F/k})$, one has $\text{tr}_{x/E}(\langle \psi(a) \rangle \cdot \mu) = \langle a \rangle \cdot \text{tr}_{x/E}(\mu)$.*
2. *For $\langle a \rangle \in \text{GW}(F, \omega_{F/k})$ and $\mu \in M(E)$, one has $\text{tr}_{x/E}(\langle a \rangle \cdot \text{res}_{F/E}(\mu)) = \text{tr}_{x/E}(\langle a \rangle) \cdot \mu$.*

Proof. This follows by GW-linearity from the definitions (see [BY18, Lemma 5.24]). \square

Definition 4.4.1.10. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. We denote by $M_{\mathbb{Q}}$ the homotopy sheaf defined by

$$X \mapsto M(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and by $M_{-1,\mathbb{Q}}$ the homotopy sheaf $(M_{\mathbb{Q}})_{-1}$.

Corollary 4.4.1.11. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\varphi : E \rightarrow F = E(x)$ be a simple extension. The kernel of the restriction map $\text{res}_{F/E} : M_{-1}(E) \rightarrow M_{-1}(F)$ is killed by $\text{tr}_{x/E}(1)$. In particular: if the Hopf map η acts trivially on M_{-1} , then the restriction map $\text{res}_{F/E} : M_{-1,\mathbb{Q}}(E) \rightarrow M_{-1,\mathbb{Q}}(F)$ is injective.*

Proof. This follows from the previous projection formula and the fact that, if η acts trivially, then the action of $\text{tr}_{x/E}(1)$ on $M_{-1,\mathbb{Q}}(E)$ is the multiplication by a nonzero natural number. \square

Definition 4.4.1.12. Let $F = E(x_1, x_2, \dots, x_r)$ be a finite extension of a field E and consider the chain of subfields

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \dots, x_r) = F.$$

Define by induction:

$$\mathrm{tr}_{x_1, \dots, x_r/E} = \mathrm{tr}_{x_r/E(x_1, \dots, x_{r-1})} \circ \cdots \circ \mathrm{tr}_{x_2/E(x_1)} \circ \mathrm{tr}_{x_1/E}$$

Conjecture 4.4.1.13 (Morel conjecture). *Keeping the previous notations, the maps $\mathrm{tr}_{x_1, \dots, x_r/E} : M_{-1}(F, \omega_{F/k}) \rightarrow M_{-1}(E, \omega_{E/k})$ do not depend on the choice of the generating system (x_1, \dots, x_r) .*

Remark 4.4.1.14. This was claimed by Morel in [Mor12, Remark 4.31] and [Mor11, Remark 5.10] (see also [Bac20, Remark 4.3] for a similar conjecture).

Proposition 4.4.1.15 (Projection formulas). *Let $\varphi : E \rightarrow F = E(x_1, x_2, \dots, x_r)$ be a finite extension. Then*

1. *For $\langle a \rangle \in \mathrm{GW}(E)$ and $\mu \in M(F, \omega_{F/k})$, one has $\mathrm{tr}_{x_1, \dots, x_r/E}(\langle \psi(a) \rangle \cdot \mu) = \langle a \rangle \cdot \mathrm{tr}_{x_1, \dots, x_r/E}(\mu)$.*
2. *For $\langle a \rangle \in \mathrm{GW}(F, \omega_{F/k})$ and $\mu \in M(E)$, one has $\mathrm{tr}_{x_1, \dots, x_r/E}(\langle a \rangle \cdot \mathrm{res}_{F/E}(\mu)) = \mathrm{tr}_{x_1, \dots, x_r/E}(\langle a \rangle) \cdot \mu$.*

Proof. This is immediate by induction on r according to Lemma 4.4.1.9. \square

Theorem 4.4.1.16 (Strong R1c). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let E be a field, F/E a finite field extension and L/E an arbitrary field extension. Write $F = E(x_1, \dots, x_r)$ with $x_i \in F$, $R = F \otimes_E L$ and $\psi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$ the natural projection defined for any $\mathfrak{p} \in \mathrm{Spec}(R)$. Then the diagram*

$$\begin{array}{ccc} M_{-1}(F, \omega_{F/k}) & \xrightarrow{\mathrm{tr}_{x_1, \dots, x_r/E}} & M_{-1}(E, \omega_{E/k}) \\ \oplus_{\mathfrak{p}} \mathrm{res}_{(R/\mathfrak{p})/F} \downarrow & & \downarrow \mathrm{res}_{L/E} \\ \oplus_{\mathfrak{p} \in \mathrm{Spec}(R)} M_{-1}(R/\mathfrak{p}, \omega_{(R/\mathfrak{p})/k}) & \xrightarrow{\sum_{\mathfrak{p}} (m_{\mathfrak{p}})_{\varepsilon} \mathrm{tr}_{\psi_{\mathfrak{p}}(a_1), \dots, \psi_{\mathfrak{p}}(a_r)/L}} & M_{-1}(L, \omega_{L/k}) \end{array}$$

is commutative where $(m_{\mathfrak{p}})_{\varepsilon}$ is the quadratic form associated to the length of the localized ring $R_{(\mathfrak{p})}$ (see Notation 4.1.2).

Proof. We prove the theorem by induction. For $r = 1$, this is Lemma 4.4.1.7. Write $E(x_1) \otimes_E L = \prod_j R_j$ for some Artin local L -algebras R_j , and decompose the finite dimensional L -algebra $F \otimes_{E(x_1)} R_j$ as $F \otimes_{E(x_1)} R_j = \prod_i R_{ij}$ for some local L -algebras R_{ij} . We have $F \otimes_E L \simeq \prod_{i,j} R_{ij}$. Denote by L_j (resp. L_{ij}) the residue fields of the Artin local L -algebras R_j (resp. R_{ij}), and m_j (resp. m_{ij}) for their geometric multiplicity. We can conclude as the following diagram commutes

$$\begin{array}{ccccc} M_{-1}(F, \omega_{F/k}) & \xrightarrow{\mathrm{tr}_{x_1, \dots, x_r/E}} & M_{-1}(E(x_1), \omega_{E(x_1)/k}) & \xrightarrow{\mathrm{tr}_{x_1/E}} & M_{-1}(E, \omega_{E/k}) \\ \oplus_{ij} \mathrm{res}_{L_{ij}/F} \downarrow & & \downarrow \oplus_{ij} \mathrm{res}_{L_j/E(x_1)} & & \downarrow \mathrm{res}_{L/E} \\ \oplus_{ij} M_{-1}(L_{ij}, \omega_{L_{ij}/k}) & \xrightarrow{\sum_{ij} (m_{ij} m_j^{-1})_{\varepsilon} \mathrm{tr}_{\psi_{ij}(x_1), \dots, \psi_{ij}(x_r)/L_j}} & \oplus_j M_{-1}(L_j, \omega_{L_j/k}) & \xrightarrow{\sum_j (m_j)_{\varepsilon} \mathrm{tr}_{\psi_j(x_1)/L}} & M_{-1}(L, \omega_{L/k}) \end{array}$$

since both squares are commutative by the inductive hypothesis and the multiplicity formula $(mn)_{\varepsilon} = m_{\varepsilon} n_{\varepsilon}$ for any natural numbers m, n . \square

Proposition 4.4.1.17. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $E \rightarrow F$ be a field extension. If the Hopf map η acts trivially on M_{-1} , then the restriction map $\text{res}_{F/E} : M_{-1, \mathbb{Q}}(E) \rightarrow M_{-1, \mathbb{Q}}(F)$ is injective.*

Proof. We may assume that F/E is finitely generated. By induction on the number of generators of F over E , we may assume that $F = E(x)$ is generated by a single element $x \in F$. If x is algebraic over E , we know from Corollary 4.4.1.11 that $\text{res}_{F/E}$ is killed by $\text{tr}_{x/E}(1)$. Since the Hopf map acts trivially, the action of $\text{tr}_{x/E}(1)$ is given by the multiplication by $[F : E]$ and we obtain the result. If x is transcendental over E , then we know (thanks to Theorem 4.4.1.2) that $M_{-1}(E)$ is a direct summand in $M_{-1}(F)$. \square

Corollary 4.4.1.18. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let F/E be a finite field extension, and let x_1, \dots, x_r and y_1, \dots, y_s be two generating system for F/E . If the Hopf map η acts trivially on M_{-1} , then*

$$\text{tr}_{x_1, \dots, x_r/E} = \text{tr}_{y_1, \dots, y_s/E}$$

seen as morphism from $M_{-1, \mathbb{Q}}(F)$ to $M_{-1, \mathbb{Q}}(E)$.

Proof. We apply Theorem 4.4.1.16 two times with $F = E(x_1, \dots, x_r)$ and $F = E(y_1, \dots, y_s)$, and with $L = \bar{E}$ an algebraic closure of E . Hence $\text{res}_{L/E} \circ \text{tr}_{x_1, \dots, x_r/E} = \text{res}_{L/E} \circ \text{tr}_{y_1, \dots, y_s/E}$ and we end with Proposition 4.4.1.17. \square

Theorem 4.4.1.19. *Assume that 2 is invertible. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The sheaf $M_{-1, \mathbb{Q}}$ has functorial Bass-Tate transfer maps (i.e. Conjecture 4.4.1.13 holds for $M_{\mathbb{Q}}$).*

Proof. The sheaf $M_{-1, \mathbb{Q}}$ splits into two sheaves $M_{-1, \mathbb{Q}}^+$ and $M_{-1, \mathbb{Q}}^-$. On one hand, the Hopf map η acts trivially on $M_{-1, \mathbb{Q}}^+$ hence there is a structure of functorial transfers thanks to Corollary 4.4.1.18. On the other hand, the Hopf map η is invertible on the minus part, leading to an isomorphism between $M_{-1, \mathbb{Q}}^-$ and $(M_{-1, \mathbb{Q}}^-)_{-1}$. The latter has a structure of functorial transfers according to Section 1.6. Hence the result. \square

We summarize the previous results in the following theorem.

Theorem 4.4.1.20. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Then:*

1. *Assume that 2 is invertible. The rational contracted homotopy sheaf $M_{-1, \mathbb{Q}}$ is a homotopy sheaf with generalized transfers.*
2. *Assuming Conjecture 4.4.1.13, the contracted homotopy sheaf M_{-1} is a homotopy sheaf with generalized transfers.*

Proof. The GW-action is defined in 1.5.2.3, functoriality of transfers eR1b follows from Corollary 4.4.1.18 or Conjecture 4.4.1.13, the base change eR1c is Theorem 4.4.1.16, the projection formulas eR2 are proved in Proposition 4.4.1.15 and the compatibility axiom eR3b can be deduced from [Mor12, Theorem 5.19]. \square

4.4.2 Unicity of transfers

The goal of this subsection is to prove Theorem 4.4.2.3 which asserts that the structure of Bass-Tate transfer maps on a contracted homotopy sheaf M_{-1} is, in some sense, unique.

Lemma 4.4.2.1 (eR3c and eR3d). *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. Let $\iota : E \rightarrow F$ be an extension of fields and w a valuation on F that restricts to the trivial valuation on E . Then the composition*

$$M(E) \xrightarrow{\iota^*} M(F) \xrightarrow{\partial_w} M_{-1}(\kappa(w), \omega_w)$$

is zero. Moreover, let $\bar{\iota} : E \rightarrow \kappa(w)$ be the induced map. For any prime π of w , the following diagram

$$\begin{array}{ccc} M_{-1}(E) & \xrightarrow{\iota^*} & M_{-1}(F) \\ \bar{\iota}^* \downarrow & & \downarrow [\pi] \\ M_{-1}(\kappa(w)) & \xleftarrow{\Theta \circ \partial_w} & M(F) \end{array}$$

is commutative (where Θ is the canonical isomorphism induced by the trivialization of ω_w through the choice of π).

Proof. The first identity is deduced from the long exact sequence 1.5.2.5. The commutative square follows from [BY18, Corollary 5.23]. \square

Lemma 4.4.2.2 (eR3e). *Let E be a field over k with a valuation v and let u be a unit of v . Then*

$$\partial_v \circ [u] = \varepsilon[u] \circ \partial_v$$

where ε is $-\langle -1 \rangle$ as usual.

Proof. See [Mor12, Lem 5.10]. \square

Theorem 4.4.2.3. *Let $M \in \mathbf{HI}^{\text{gr}}(k)$ be a homotopy sheaf with generalized transfers. We have for any simple extension $\psi : F \rightarrow F(x)$:*

(a) *A generalized transfer map $\text{Tr}_\psi : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F, \omega_{F/k})$ induced by the structure of a homotopy sheaf with generalized transfers on M (see also Remark 4.3.1.5).*

(b) *A Bass-Tate transfer map $\text{tr}_{x/F} : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M_{-1}(F, \omega_{F/k})$ defined in 4.4.1.4.*

Then the two transfer maps coincide : $\text{Tr}_\psi = \text{tr}_{x/F}$.

Proof. Fix a field F and $F(t)$ the field of rational fractions with coefficients in F in one variable t , and fix a simple extension $\psi : F \rightarrow F(x)$. Consider the canonical inclusion $\iota : F(x) \rightarrow F(x)(t)$ and define

$$\Phi^x : M_{-1}(F(x), \omega_{F(x)/k}) \rightarrow M(F(t), \omega_{F(t)/k})$$

as the composite $\Phi^x = \text{Tr}_{F(x)(t)/F(t)} \circ [t - \iota(x)] \circ \iota^*$ where Tr denotes the generalized transfers of point (a). A combination of eR3b, Lemma 4.4.2.1 and Lemma 4.4.2.2 shows that

$$\begin{aligned} \partial_x \circ \Phi^x &= \text{Id}, \\ -\partial_\infty \circ \Phi^x &= \text{Tr}_{F(x)/F}, \end{aligned}$$

which is exactly the definition of the Bass-Tate transfers $\text{tr}_{F(x)/F}$. \square

4.5 MW-homotopy sheaves

4.5.1 Sheaves with MW-transfers

In this subsection, we recall the basic definition of sheaves with MW-transfers in order to fix the notations. We follow the presentation of [BCD⁺20, Chapter 2].

4.5.1.1. Let X and Y be smooth schemes over k and let $T \subset X \times Y$ be a closed subset. Any irreducible component of T maps to an irreducible component of X through the projection $X \times Y \rightarrow X$.

Definition 4.5.1.2. If, when T is endowed with its reduced structure, the projection map $T \rightarrow X$ is finite and surjective for every irreducible component of T , we say that T is an admissible subset of $X \times Y$. We denote by $\mathcal{A}(X, Y)$ the set of admissible subsets of $X \times Y$, partially ordered by inclusions.

4.5.1.3. If Y is equidimensional, $d = \dim Y$ and $p_Y : X \times Y \rightarrow Y$ is the projection, we define a covariant functor

$$\mathcal{A}(X, Y) \rightarrow \mathbf{Ab}$$

by associating to each admissible subset $T \in \mathcal{A}(X, Y)$ the group $\widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k}) = H_T^d(X \times Y, \mathbf{K}_d^{\mathrm{MW}}\{p_Y^* \omega_{Y/k}\})$ (see [Fas19, Definition 2.5]) and to each morphism $T' \subset T$ the extension of support morphism

$$\widetilde{\mathrm{CH}}_{T'}^d(X \times Y, p_Y^* \omega_{Y/k}) \rightarrow \widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k})$$

and, using that functor, we set

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \mathrm{colim}_{T \in \mathcal{A}(X, Y)} \widetilde{\mathrm{CH}}_T^d(X \times Y, p_Y^* \omega_{Y/k}).$$

If Y is not equidimensional, then $Y = \bigsqcup_j Y_j$ with each Y_j equidimensional and we set

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \prod_j \widetilde{\mathrm{Cor}}_k(X, Y_j).$$

By additivity of Chow-Witt groups, if $X = \bigsqcup_i X_i$ and $Y = \bigsqcup_j Y_j$ are the respective decompositions of X and Y in irreducible components, we have

$$\widetilde{\mathrm{Cor}}_k(X, Y) = \prod_{i,j} \widetilde{\mathrm{Cor}}_k(X_i, Y_j).$$

Remark 4.5.1.4. In the sequel, we will simply write ω_Y in place of $p_Y^* \omega_{Y/k}$.

Example 4.5.1.5. Let X be a smooth scheme of dimension d . Then

$$\widetilde{\mathrm{Cor}}_k(\mathrm{Spec}(k), X) = \bigoplus_{x \in X^{(d)}} \widetilde{\mathrm{CH}}_{\{x\}}^d(X, \omega_X) = \bigoplus_{x \in X^{(d)}} \mathrm{GW}(\kappa(x), \omega_{\kappa(x)/k}).$$

On the other hand, $\widetilde{\mathrm{Cor}}_k(X, \mathrm{Spec}(k)) = \widetilde{\mathrm{CH}}^0(X) = \mathbf{K}_0^{\mathrm{MW}}(X)$ (recall 1.5.1.19) for any smooth scheme X .

4.5.1.6. The group $\widetilde{\mathrm{Cor}}_k(X, Y)$ admits an alternate description which is often useful. Let X and Y be smooth schemes, with Y equidimensional. For any closed subscheme $T \subset X \times Y$ of codimension $d = \dim Y$, we have an inclusion

$$\widetilde{\mathrm{CH}}_T^d(X \times Y, \omega_{Y/k}) \subset \bigoplus_{x \in (X \times Y)^{(d)}} \mathbf{K}_0^{\mathrm{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x))$$

and thus

$$\begin{aligned} \widetilde{\text{Cor}}_k(X, Y) &= \bigcup_{T \in \mathcal{A}(X, Y)} \widetilde{\text{CH}}_T^d(X \times Y, \omega_{Y/k}) \subset \\ &\quad \bigoplus_{x \in (X \times Y)^{(d)} } \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x)). \end{aligned}$$

In general, the inclusion $\widetilde{\text{Cor}}_k(X, Y) \subset \bigoplus_{x \in (X \times Y)^{(d)} } \mathbf{K}_0^{\text{MW}}(\kappa(x), \det(\omega_x \otimes (\omega_{Y/k})_x))$ is strict as shown by Example 1.8.1.5. As an immediate consequence of this description, we see that the map

$$\widetilde{\text{CH}}_T^d(X \times Y, \omega_Y) \rightarrow \widetilde{\text{Cor}}_k(X, Y)$$

is injective for any $T \in \mathcal{A}(X, Y)$.

4.5.1.7 (Composition of finite MW-correspondences). Let X, Y and Z be smooth schemes of respective dimension d_X, d_Y and d_Z , with X and Y connected. Let $V \in \mathcal{A}(X, Y)$ and $\mathcal{A}(Y, Z)$ be admissible subsets. If $\beta \in \widetilde{\text{CH}}_V^{d_Y}(X \times Y, \omega_{Y/k})$ and $\alpha \in \widetilde{\text{CH}}_T^{d_Z}(Y \times Z, \omega_{Z/k})$ are two cycles, then the expression

$$\alpha \circ \beta = (q_{XY})_*[(q_{YZ})^* \beta \cdot (p_{XY})^* \alpha]$$

is well-defined and yields a composition

$$\circ : \widetilde{\text{Cor}}_k(X, Y) \times \widetilde{\text{Cor}}_k(Y, Z) \rightarrow \widetilde{\text{Cor}}_k(X, Z)$$

which is associative (see [BCD⁺20, Ch.2, §4.2]).

Definition 4.5.1.8. The category of finite MW-correspondences over k is by definition the category $\widetilde{\text{Cor}}_k$ whose objects are smooth schemes and whose morphisms are the elements of abelian groups $\widetilde{\text{Cor}}_k(X, Y)$.

Remark 4.5.1.9. The category $\widetilde{\text{Cor}}_k$ is a symmetric monoidal additive category (see [BCD⁺20, Chapter 2, Lemma 4.4.2]).

Definition 4.5.1.10. A presheaf with MW-transfers is a contravariant additive functor $\widetilde{\text{Cor}}_k \rightarrow \mathbf{Ab}$. A (Nisnevich) sheaf with MW-transfers is a presheaf with MW-transfers such that its restriction to \mathbf{Sm}_k via the graph functor is a Nisnevich sheaf. We denote by $\widetilde{\text{PSh}}(k)$ (resp. $\widetilde{\text{Sh}}(k)$) the category of presheaves (resp. sheaves) with MW-transfers and by $\mathbf{HI}^{\text{MW}}(k)$ the category of homotopy sheaves with MW-transfers (also called *MW-homotopy sheaves*).

Example 4.5.1.11. For any $j \in \mathbb{Z}$, the contravariant functor $X \mapsto \mathbf{K}_j^{\text{MW}}(X)$ is a presheaf on $\widetilde{\text{Cor}}_k$.

4.5.1.12. PUSHFORWARDS. Let X and Y be two smooth schemes of dimension d and let $f : X \rightarrow Y$ be a finite morphism such that any irreducible component of X surjects to the irreducible component of Y it maps to. Assume that we have an orientation (\mathcal{L}, ψ) of ω_f , that is an isomorphism $\psi : \mathcal{L} \otimes \mathcal{L} \rightarrow \omega_f$ of line bundles. We define a finite correspondence $\alpha(f, \mathcal{L}, \psi) \in \widetilde{\text{Cor}}_k(Y, X)$. Let $\gamma_f' : X \rightarrow Y \times X$ be the transpose of the graph of f . Since X is an admissible subset of $Y \times X$, we have a transfer map

$$(\gamma_f')_* : \mathbf{K}_0^{\text{MW}}(X, \omega_f) \rightarrow \widetilde{\text{CH}}_X^d(Y \times X, \omega_{X/k}) \rightarrow \widetilde{\text{Cor}}_k(Y, X).$$

The map ψ yields an isomorphism $\mathbf{K}_0^{\text{MW}}(X) \rightarrow \mathbf{K}_0^{\text{MW}}(X, \omega_f)$. We define the finite MW-correspondence $\alpha(f, \mathcal{L}, \psi)$ as the image of $\langle 1 \rangle$ under the composite

$$\mathbf{K}_0^{\text{MW}}(X) \rightarrow \mathbf{K}_0^{\text{MW}}(X, \omega_f) \rightarrow \widetilde{\text{CH}}_X^d(Y \times X, \omega_{X/k}) \rightarrow \widetilde{\text{Cor}}_k(Y, X).$$

Now let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a homotopy sheaf with MW-transfers. Denote by $(M \otimes \omega_f)_X$ (resp. M_Y) the canonical (twisted) sheaf associated to M defined on the Zariski site X_{Zar} (resp. Y_{Zar}) introduced in 1.5.2.4 and define a natural transformation

$$f_* : f_*(M \otimes \omega_f)_X \rightarrow M_Y$$

by taking (as in 1.5.2.4) the sheafification of the natural transformation of presheaves

$$\begin{aligned} V \in Y_{\text{Zar}} &\mapsto (M(f^{-1}(V), \omega_{f|f^{-1}(V)}}) \rightarrow M(V)) \\ (\mu \otimes l) &\mapsto \alpha(f, \psi_l, L_l)^*(\mu) \end{aligned}$$

where (ψ_l, \mathcal{L}_l) is the orientation of $\omega_{f|f^{-1}(V)}$ associated to $l \in \omega_{f|f^{-1}(V)}^\times$. Taking global sections, this leads in particular to a map

$$M({}^t f) : M(X, \omega_f) \rightarrow M(Y)$$

for any finite morphism $f : X \rightarrow Y$.

We can check the following propositions.

Proposition 4.5.1.13. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ and consider two finite morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of smooth schemes. Then*

$$M({}^t(g \circ f)) = M({}^t g) \circ M({}^t f).$$

Proof. Keeping the previous notations, if (\mathcal{L}', ψ') is an orientation of ω_g , then $(\mathcal{L} \otimes f^* \mathcal{L}', \psi \otimes f^* \psi')$ is an orientation of $\omega_{g \circ f} = \omega_f \otimes f^* \omega_g$, and we have $\alpha(f, \mathcal{L}, \psi) \circ \alpha(g, \mathcal{L}', \psi') = \alpha(g \circ f, \mathcal{L} \otimes f^* \mathcal{L}', \psi \otimes f^* \psi')$. \square

Proposition 4.5.1.14. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a homotopy sheaf with MW-transfers. Let $i : Z \rightarrow X$ and $i' : T \rightarrow Y$ be two closed immersions and let $f : Y \rightarrow X$ be a finite morphism. The following diagram*

$$\begin{array}{ccc} M(Y - T, \omega_{(Y-T)/k}) & \xrightarrow{\partial_{i'}} & M_{-1}(T, \omega_{T/k}) \\ M({}^t f) \downarrow & & \downarrow M({}^t f) \\ M(X - Z, \omega_{(X-Z)/k}) & \xrightarrow{\partial_i} & M_{-1}(Z, \omega_{Z/k}) \end{array}$$

is commutative.

Proof. This follows from the fact that MW-transfers act on cohomology with support and the localization long exact sequence is functorial. \square

4.5.1.15. TENSOR PRODUCTS. Let X_1, X_2, Y_1, Y_2 be smooth schemes over $\text{Spec } k$. Let $d_1 = \dim Y_1$ and $d_2 = \dim Y_2$. Let $\alpha_1 \in \widetilde{\text{CH}}_{T_1}^{d_1}(X_1 \times Y_1, \omega_{Y_1/k})$ and $\alpha_2 \in \widetilde{\text{CH}}_{T_2}^{d_2}(X_2 \times Y_2, \omega_{Y_2/k})$ for some admissible subsets $T_i \subset X_i \times Y_i$. The exterior product defined in [Fas08, §4] gives a cycle

$$(\alpha_1 \times \alpha_2) \in \widetilde{\mathrm{CH}}_{T_1 \times T_2}^{d_1+d_2}(X_1 \times Y_1 \times X_2 \times Y_2, p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k})$$

where $p_{Y_i} : X_1 \times Y_1 \times X_2 \times Y_2 \rightarrow Y_i$ is the canonical projection to the corresponding factor. Let $\sigma : X_1 \times Y_1 \times X_2 \times Y_2 \rightarrow X_1 \times X_2 \times Y_1 \times Y_2$ be the transpose isomorphism. Applying σ_* , we get a cycle

$$\sigma_*(\alpha_1 \times \alpha_2) \in \widetilde{\mathrm{CH}}_{\sigma(T_1 \times T_2)}^{d_1+d_2}(X_1 \times X_2 \times Y_1 \times Y_2, p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k}).$$

Since $p_{Y_1}^* \omega_{Y_1/k} \otimes p_{Y_2}^* \omega_{Y_2/k} = \omega_{Y_1 \times Y_2/k}$, it is straightforward to check that $\sigma(T_1 \times T_2)$ is finite and surjective over $X_1 \times X_2$. Thus $\sigma_*(\alpha_1 \times \alpha_2)$ defines a finite MW-correspondence between $X_1 \times X_2$ and $Y_1 \times Y_2$.

Definition 4.5.1.16. Let X_1, X_2, Y_1, Y_2 be smooth schemes over $\mathrm{Spec} k$, and $\alpha_1 \in \widetilde{\mathrm{Cor}}_k(X_1, Y_1)$ and $\alpha_2 \in \widetilde{\mathrm{Cor}}_k(X_2, Y_2)$ two MW-correspondences. We define their tensor products as $X_1 \otimes X_2 = X_1 \times X_2$ and $\alpha_1 \otimes \alpha_2 = \sigma_*(\alpha_1 \times \alpha_2)$.

4.5.1.17. We denote by $\tilde{c}(X) : Y \mapsto \widetilde{\mathrm{Cor}}_k(Y, X)$ the representable presheaf associated to a smooth scheme X (be careful that this is not a Nisnevich sheaf in general). The category of MW-presheaves is an abelian Grothendieck category with a unique symmetric monoidal structure such that the Yoneda embedding

$$\widetilde{\mathrm{Cor}}_k \rightarrow \widetilde{\mathrm{Sh}}(k), X \mapsto \tilde{a}\tilde{c}(X)$$

is symmetric monoidal (where \tilde{a} is the sheafification functor, see [BCD⁺20, Chapter 3, §1.2.7]). The tensor product is denoted by $\otimes_{\mathbf{H}^{\mathrm{MW}}}$ and commutes with colimits (hence the monoidal structure is closed, see [BCD⁺20, Chapter 3, §1.2.14]).

4.5.2 Structure of generalized transfers

In this section, we study the category of MW-homotopy sheaves. For any MW-homotopy sheaf we construct a canonical structure of generalized transfers (see Definition 4.3.1.1).

4.5.2.1. Let $M \in \mathbf{H}^{\mathrm{MW}}(k)$ be a homotopy sheaf with MW-transfers. We denote by $\tilde{\Gamma}^*(M)$ the homotopy sheaf M equipped with its structure of GW-module coming from its structure of MW-transfers, and we define generalized transfers as follows. Let $\psi : E \rightarrow F$ be a finite extension of fields. Consider a smooth model⁸ (X, x) (resp. (Y, y)) of E/k (resp. F/k) such that ψ corresponds to a map $Y_y \rightarrow X_x$. We may assume that this map is induced by a finite morphism $f : Y \rightarrow X$. We consider the pushforward on the MW-homotopy sheaf $\tilde{\Gamma}^*(M)$ with respect to the finite morphism f defined in 1.8.1.12 and take the limit over all model of F/E so that we obtain a morphism

$$\psi^* : \tilde{\Gamma}^*(M)(F, \omega_{F/k}) \rightarrow \tilde{\Gamma}^*(M)(E, \omega_{E/k})$$

of abelian groups. This defines a homotopy sheaf $\tilde{\Gamma}^*(M)$ canonically isomorphic to M (as presheaves) and equipped with a structure of transfers eD2.

Theorem 4.5.2.2. *Keeping the previous notations, the transfer maps ψ^* defines a structure of generalized transfers (see Definition 4.3.1.1) on the homotopy sheaf $\tilde{\Gamma}^*(M)$.*

⁸See Definition 1.9.2.1.

Proof. The functoriality property [eR1b](#) results from Proposition [4.5.1.13](#).

According to Proposition [4.4.1.9](#) and Theorem [4.4.1.16](#), the projection formulas [eR2](#) and the base change rule [eR1c](#) are true for any contracted homotopy sheaf hence we only have to prove that $\tilde{\Gamma}^*(M)$ is a contracted homotopy sheaf:

Lemma 4.5.2.3. *Let $M \in \mathbf{HI}^{\text{MW}}(k)$ a MW-homotopy sheaf. Then there is a canonical isomorphism*

$$\tilde{\Gamma}^*(M) \simeq (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}$$

of homotopy sheaves which is compatible with the generalized transfers structure in the sense that the diagram

$$\begin{array}{ccc} \tilde{\Gamma}^*(M)(E(x)) & \xrightarrow{\psi^*} & \tilde{\Gamma}^*(M)(E) \\ \downarrow \simeq & & \downarrow \simeq \\ (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}(E(x)) & \xrightarrow{\text{tr}_{x/E}} & (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}(E) \end{array}$$

is commutative for any simple extension $\psi : E \rightarrow E(x)$ of fields, where $\text{tr}_{x/E}$ is the Bass-Tate transfer map defined in [4.4.1.4](#).

Proof. The isomorphism $\tilde{\Gamma}^*(M) \simeq (\mathbb{G}_m \otimes_{\mathbf{HI}^{\text{MW}}} \tilde{\Gamma}^*(M))_{-1}$ is an equivalent reformulation of the cancellation theorem [[FØ17](#), Theorem 4.0.1]. The second assertion is a corollary of Theorem [4.4.2.3](#). \square

Still denoting by $M \in \mathbf{HI}^{\text{MW}}(k)$ a MW-homotopy sheaf, we need to check that $\tilde{\Gamma}^*(M)$ satisfies [eR3b](#) where $M \in \mathbf{HI}^{\text{MW}}(k)$, which can be deduced from the definitions and Proposition [1.8.1.14](#). This concludes the proof of Theorem [4.5.2.2](#). \square

4.5.2.4. As in [4.3.2.4](#), we see that $\tilde{\Gamma}^*$ defines a functor

$$\begin{aligned} \tilde{\Gamma}^* : \mathbf{HI}^{\text{MW}}(k) &\rightarrow \mathbf{HI}^{\text{gtr}}(k) \\ M &\mapsto \tilde{\Gamma}^*(M) \end{aligned}$$

which is conservative.

Theorem 4.5.2.5. *Keeping the notations of [4.3.2.4](#) and [4.5.2.4](#), the functors*

$$\mathbf{HI}^{\text{MW}}(k) \begin{array}{c} \xrightarrow{\tilde{\Gamma}^*} \\ \xleftarrow{\tilde{\Gamma}_*} \end{array} \mathbf{HI}^{\text{gtr}}(k)$$

form an equivalence of categories.

Proof. First, let $M \in \mathbf{HI}^{\text{gtr}}(k)$ be a homotopy sheaf with generalized transfers. For any smooth scheme X , we have a canonical isomorphism

$$a_X : \tilde{\Gamma}^* \tilde{\Gamma}_*(M)(X) \rightarrow M(X)$$

which is compatible with pullback maps and the GW-action. Compatibility with the generalized transfers [eD2](#) results from Lemma [4.3.2.5](#).

Second, let $M \in \mathbf{HI}^{\text{MW}}(k)$ be a MW-homotopy sheaf. For any smooth scheme X , we have a canonical isomorphism

$$b_X : M \rightarrow \tilde{\Gamma}_* \tilde{\Gamma}^*(M)(X)$$

which is compatible with (smooth) pullbacks and the GW-action. Since pushforward p_* of a finite map $p : Y \rightarrow X$ is locally given by the multiplication by the correspondence $\alpha(p, \psi_l, L_l)$ of 1.8.1.12, we see that b commutes with p_* . Thus b commutes with the multiplication by any cycle $\alpha \in \widetilde{\text{CH}}_T^{dy}(X \times Y, \omega_{Y/k})$ (where X, Y are two smooth schemes and $T \in X \times Y$ is an admissible subset) thanks to the identity

$$\alpha^*(\beta) = (p_X)_*(\alpha \cdot p_Y^*(\beta))$$

where $p_X : T \rightarrow X \times Y$ and $p_Y : X \times Y \rightarrow Y$ are the canonical morphisms. □

4.6 Applications

4.6.1 Infinite suspensions of homotopy sheaves

In order to fix notations, we recall that we have the following commutative diagram of categories :

$$\begin{array}{ccccc} \mathbf{H}_\bullet(k) & \xrightleftharpoons[\Omega_{S^1}^\infty]{\Sigma_{S^1}^\infty} & \mathbf{SH}^{S^1}(k) & \xrightleftharpoons[\Omega_{\mathbb{G}_m}^\infty]{\Sigma_{\mathbb{G}_m}^\infty} & \mathbf{SH}(k) \\ & & \uparrow K \quad \downarrow N & & \uparrow K \quad \downarrow N \\ & & \mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k) & \xrightleftharpoons[\Omega^\infty]{\Sigma^\infty} & \mathbf{D}_{\mathbb{A}^1}(k) \end{array}$$

where $\mathbf{H}_\bullet(k)$ is the pointed unstable homotopy category, $\mathbf{SH}^{S^1}(k)$ (resp. $\mathbf{SH}(k)$) is the category obtain after S^1 -stabilization (resp. \mathbb{P}^1 -stabilization) and $\mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k)$ (resp. $\mathbf{D}_{\mathbb{A}^1}(k)$) the (resp. stable) effective \mathbb{A}^1 -derived category (see [CD19, §5]); all of these triangulated categories are equipped with Morel's homotopy t-structure. Since Ω^∞ and K are t-exact, we have an equivalence (of additive symmetric monoidal categories) $\mathbf{D}_{\mathbb{A}^1}(k)^\heartsuit \simeq \mathbf{SH}(k)^\heartsuit$ and an adjunction on the respective hearts (see also [BD17, §4]):

$$\mathbf{HI}(k) \xrightleftharpoons[\omega^\infty]{\sigma^\infty} \mathbf{HM}(k).$$

Lemma 4.6.1.1. *With the previous notations, if the functor Σ^∞ is fully faithful, then so is σ^∞ .*

Proof. For any sheaf $M \in \mathbf{HI}(k)$, we have $\omega^\infty(M) = \Omega^\infty(M) = H_0\Omega^\infty(M)$ and $\sigma^\infty(M) = \tau_{\geq 0}\Sigma^\infty(M) = H_0\Sigma^\infty(M)$ hence the arrow

$$\text{Id} \rightarrow \omega^\infty \sigma^\infty = H_0(\text{Id} \rightarrow \Omega^\infty \Sigma^\infty)$$

is an isomorphism if Σ^∞ is fully faithful. □

Remark 4.6.1.2. The converse is not true in general (except maybe for connective spectra).

4.6.1.3. We also have the following commutative diagram of categories

$$\begin{array}{ccc}
 \mathbf{D}_{\mathbb{A}^1}^{\text{eff}}(k) & \xrightarrow{\Sigma^\infty} & \mathbf{D}_{\mathbb{A}^1}(k) \\
 \downarrow & & \downarrow \\
 \widetilde{\mathbf{DM}}^{\text{eff}}(k) & \xrightarrow{\Sigma_{\text{MW}}^\infty} & \widetilde{\mathbf{DM}}(k)
 \end{array}$$

where $\widetilde{\mathbf{DM}}^{\text{eff}}(k)$ and $\widetilde{\mathbf{DM}}(k)$ are the categories of (effective) MW-motivic complexes (see [BCD⁺20, Chapter 3]). Looking at the respective hearts, we thus obtain the following commutative diagram

$$\begin{array}{ccc}
 \mathbf{HI}(k) & \xrightleftharpoons[\omega^\infty]{\sigma^\infty} & \mathbf{HM}(k) \\
 \tilde{\gamma}_* \uparrow \downarrow \tilde{\gamma}^* & & \gamma_* \uparrow \downarrow \gamma^* \\
 \mathbf{HI}^{\text{MW}}(k) & \xrightleftharpoons[\omega_{\text{MW}}^\infty]{\sigma_{\text{MW}}^\infty} & \mathbf{HM}^{\text{MW}}(k)
 \end{array}$$

where we recall that $\mathbf{HM}^{\text{MW}}(k)$ denotes the category of Milnor-Witt homotopy modules, i.e. the category of pairs (M_*, ω_*) where M_* is a \mathbb{Z} -graded homotopy invariant sheaf with MW-transfers and $\omega_i : M_i \rightarrow (M_{i+1})_{-1}$ are isomorphisms of sheaves with MW-transfers.

Finally, we recall the two following well-known theorems in order to motivate Theorem 4.6.1.6.

Theorem 4.6.1.4. *With the previous notations, the adjunction*

$$\mathbf{HM}(k) \xrightleftharpoons[\gamma_*]{\gamma^*} \mathbf{HM}^{\text{MW}}(k).$$

is an equivalence of categories.

Proof. See Theorem 3.5.2.2. □

Theorem 4.6.1.5. *With the previous notations, the functor $\sigma_{\text{MW}}^\infty : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HM}^{\text{MW}}(k)$ is fully faithful.*

In particular, if $M \in \mathbf{HI}^{\text{MW}}(k)$ is a sheaf with MW-transfers, then for any natural number n , there exists a sheaf with MW-transfers $N \in \mathbf{HI}^{\text{MW}}(k)$ such that $M \simeq N_{-n}$.

Proof. Use Lemma 4.6.1.1 and [BCD⁺20, Chapter 3, Cor. 3.3.9]. □

The previous theorems suggest that similar results should hold for the functor $\tilde{\gamma}_* : \mathbf{HI}^{\text{MW}}(k) \rightarrow \mathbf{HI}(k)$ that forgets MW-transfers. This functor is clearly faithful and conservative but cannot be full according to the following counterexample due to Bachmann:

Consider the constant sheaf \mathbb{Z} , which admits MW-transfers. For any MW-homotopy sheaf F , the set of maps $\text{Hom}_{\mathbf{HI}^{\text{MW}}}(\mathbb{Z}, F)$ injects into the subset of $F(k)$ given by the annihilator of the fundamental ideal \mathbf{I} of $\text{GW}(k)$ acting on $F(k)$ (since $\mathbb{Z} = \text{GW}/\mathbf{I}$ and maps with MW-transfers from GW to F are given by $F(k)$). On the other hand, the set of maps $\text{Hom}_{\mathbf{HI}}(\mathbb{Z}, F)$ is given by all of $F(k)$.

However, we can still characterize its essential image thanks to the following theorem.

Theorem 4.6.1.6. *Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. The following assertions are equivalent:*

- (i) *There exists $M' \in \mathbf{HI}(k)$ satisfying Conjecture 4.4.1.13 and such that $M \simeq M'_{-1}$.*
- (ii) *There exists a structure of generalized transfers on M .*
- (iii) *There exists a structure of MW-transfers on M .*
- (iv) *There exists $M'' \in \mathbf{HI}(k)$ such that $M \simeq M''_{-2}$.*

Proof. (i) \Rightarrow (ii) See Theorem 4.4.1.20.

(ii) \Rightarrow (iii) See Theorem 4.5.2.5.

(iii) \Rightarrow (iv) Assume $M \in \mathbf{HI}^{\text{MW}}(k)$. Put $M_* = \sigma_{\text{MW}}^\infty(M) \in \mathbf{HM}^{\text{MW}}(k)$ so that we have $\omega_{\text{MW}}^\infty(M_*) = M_0 \simeq (M_2)_{-2}$. Since $\sigma_{\text{MW}}^\infty$ is fully faithful, the map $M \rightarrow \omega_{\text{MW}}^\infty \sigma_{\text{MW}}^\infty(M)$ is an isomorphism thus $M \simeq M''_{-2}$ with $M'' = M_2$.

(iv) \Rightarrow (i) Straightforward. \square

Remark 4.6.1.7. Keeping the previous notations, we remark that the equivalence (i) \Leftrightarrow (ii) was conjectured by Morel in [Mor11, Remark 5.10].

4.6.2 Towards conservativity of \mathbb{G}_m -stabilization

We end with a discussion about a conjecture introduced in [BY18]. In the classical theory of topological spaces, the functor $\text{Spc}_* \rightarrow \mathfrak{D}(\mathbf{Ab})$, sending a space to its singular chain complex, is conservative on (at least) simply connected spaces. We would like to study a similar question in the motivic context: up to which extent is the functor

$$\Sigma_{\mathbb{G}_m}^\infty : \mathbf{SH}^{S^1}(k) \rightarrow \mathbf{SH}(k)$$

conservative? The conjecture of Bachmann and Yakerson relied on the hope that this is true after \mathbb{G}_m -suspension. Precisely, for any natural number n , denote by $\mathbf{SH}^{S^1}(k)(n)$ the localizing subcategory of $\mathbf{SH}^{S^1}(k)$ generated by $\Sigma_{\mathbb{G}_m}^n \Sigma_{S^1}^\infty X_+$ where $X \in \mathbf{Sm}_k$. The fact that the functor $\Sigma_{\mathbb{G}_m}^\infty : \mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$ is conservative on bounded below objects reduces to proving the following statement (see [BY18, Conjecture 1.1]).

Conjecture 4.6.2.1 (Bachmann-Yakerson). *If $d \geq 1$ is a natural number, then the canonical functor*

$$\Sigma_{\mathbb{G}_m}^{\infty-d\heartsuit} : \mathbf{SH}^{S^1}(k)(d)^\heartsuit \rightarrow \mathbf{SH}(k)^{\text{eff}\heartsuit}$$

is an equivalence of abelian categories.

Recall that $\mathbf{SH}(k)^{\text{eff}}$ denotes the localizing subcategory generated by the image of $\mathbf{SH}^{S^1}(k)$ in $\mathbf{SH}(k)$ under $\Sigma_{\mathbb{G}_m}^\infty$ and the hearts are taken with respect to homotopy t-structures on these categories. As a reformulation of the conjecture, we remark that the category $\mathbf{SH}^{S^1}(k)^\heartsuit$ is equivalent to the category of homotopy sheaves $\mathbf{HI}(k)$ and that the category $\mathbf{SH}(k)^{\text{eff}\heartsuit}$ is equivalent to the category $\mathbf{HI}^{\text{fr}}(k)$ of homotopy sheaves with *framed transfers* (see [BY18, Theorem 5.14]). We have the following theorem.

Theorem 4.6.2.2. *Let $d > 0$ be a natural number. If $d > 1$ or Conjecture 4.4.1.13 holds, then for any homotopy sheaf $M \in \mathbf{HI}(k)$, the Bass-Tate transfers on M_{-d} extend to framed transfers and the canonical functor*

$$\Sigma_{\mathbb{G}_m}^{\infty-d\heartsuit} : \mathbf{SH}^{S^1}(k)(d)^{\heartsuit} \rightarrow \mathbf{SH}(k)^{eff^{\heartsuit}}$$

is an equivalence of abelian categories.

Proof. Let $M \in \mathbf{HI}(k)$ be a homotopy sheaf. If $d > 1$ (resp. if Conjecture 4.4.1.13 holds), then M_{-d} (resp. M_{-1}) has a structure of generalized transfers hence of Milnor-Witt transfers according to Subsection 4.5.2. Thus it has a structure of framed transfers (see Remark 1.8.2.12) and the first result holds. The second one is [Bac20, Theorem 4.5]. \square

As an application of our theorem 4.4.1.20, we obtain:

Corollary 4.6.2.3. *Let $d > 0$ be a natural number. The Bachmann-Yakerson conjecture holds (integrally) for $d = 2$ and rationally for $d = 1$: namely, the canonical functor*

$$\mathbf{SH}^{S^1}(k)(2) \rightarrow \mathbf{SH}(k)$$

is conservative on bounded below objects, the canonical functor

$$\mathbf{SH}^{S^1}(k)(1) \rightarrow \mathbf{SH}(k)$$

is conservative on rational bounded below objects, and the canonical functor

$$\mathbf{HI}(k, \mathbb{Q})(1) \rightarrow \mathbf{HI}^{\text{fr}}(k, \mathbb{Q})$$

is an equivalence of abelian categories.

Moreover, let \mathcal{X} be a pointed motivic space. Then the canonical map

$$\pi_0 \Omega_{\mathbb{P}^1}^d \Sigma_{\mathbb{P}^1}^d \mathcal{X} \rightarrow \pi_0 \Omega_{\mathbb{P}^1}^{d+1} \Sigma_{\mathbb{P}^1}^{d+1} \mathcal{X}$$

is an isomorphism for $d = 2$.

Proof. The last result follows as in the proof of [Bac20, Theorem 1.1]. \square

Moreover, we can solve (integrally) a question left open after the work of [BCD⁺20] and [GP14]:

Corollary 4.6.2.4. *The category of homotopy sheaves with generalized transfers, the category of MW-homotopy sheaves and the category of homotopy sheaves with framed transfers are equivalent:*

$$\mathbf{HI}^{\text{gtr}}(k) \simeq \mathbf{HI}^{\text{MW}}(k) \simeq \mathbf{HI}^{\text{fr}}(k).$$

Proof. The first equivalence is our Theorem 4.5.2.5; the second one is due to Bachmann (combine [Bac18, Proposition 29] and [BY18, Theorem 5.14]). \square

5

Transfers on Milnor-Witt K-theory

We give an elementary proof of the fact that Milnor-Witt K-theory has functorial transfers. We apply the results to study Conjecture 4.4.1.13.

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5.1 Introduction

5.1.1 Current work

In [Mor12, Chapter 3], Morel introduced the Milnor-Witt K-theory of a field. Following ideas of Bass and Tate [BT73], one can define geometric transfer maps

$$\mathrm{Tr}_{x_1, \dots, x_r/E} = \mathrm{Tr}_{x_r/E(x_1, \dots, x_{r-1})} \circ \dots \circ \mathrm{Tr}_{x_2/E(x_1)} \circ \mathrm{Tr}_{x_1/E} : \mathbf{K}_*^{\mathrm{MW}}(E(x_1, \dots, x_r)) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(E)$$

on \mathbf{K}^{MW} for finite extensions $E(x_1, \dots, x_r)/E$. Morel proved in [Mor12, Chapter 4] that any homotopy sheaves of the form M_{-2} admit such transfers and that they are functorial. In particular, this applies to Milnor-Witt K-theory.

In this chapter (see also [Fel20b]), we give an alternative proof of this result:

Theorem 1 (Theorem 5.2.2.4). *The maps $\mathrm{Tr}_{x_1, \dots, x_r/E} : \mathbf{K}_*^{\mathrm{MW}}(E(x_1, \dots, x_r)) \rightarrow \mathbf{K}_*^{\mathrm{MW}}(E)$ do not depend on the choice of the generating system (x_1, \dots, x_r) .*

The idea is to reduce to the case of p -primary fields (see Definition 5.2.1.3) then study the transfers manually, as Kato originally did for Milnor K-theory (see [GS17] for a modern exposition). More elementary, the proof does not apply in full generality to the case of contracted homotopy sheaves M_{-1} . However, we obtain as a corollary a reduction of Morel's conjecture [Fel20a, Conjecture 4.1.13].

Theorem 2 (Theorem 5.2.2.13). *In order to prove Morel's Conjecture 4.4.1.13, it suffices to consider the case of p -primary fields (where p is a prime number).*

5.1.2 Outline of the chapter

In Subsection 5.2.1, we recall the basic properties of some fields that we call p -primary fields. For p a prime number, a p -primary field has no nontrivial finite extension prime to p (see Definition 5.2.1.3).

In Subsection 5.2.2, we prove that Milnor-Witt K-theory admit transfer maps that are functorial. The proof is similar to the original proof of Kato for Milnor K-theory (see [GS17]): we reduce to the case of p -primary fields then study the transfers manually.

5.2 Transfers on Milnor-Witt K-theory

5.2.1 On p -primary fields

We recall some facts about fields (See [Sha82, §1] and [BT73, Section 5]). Let E be a field and p a prime number. Fix a separable closure E_s of E and consider the set of all sub-extensions of E_s that contain E and that can be realized as a union of finite prime-to- p extensions of E . Zorn's lemma implies that this set contains a maximal element $E_{\langle p \rangle}$ for the inclusion.

Proposition 5.2.1.1. *If F is a finite extension of E contained in $E_{\langle p \rangle}$, then its degree $[F : E]$ is prime to p .*

Proof. Write $F = E(x_1, \dots, x_r)$ with $x_i \in F$. Each x_i is contained in a prime-to- p extension of E hence has a degree prime to p . \square

Proposition 5.2.1.2. *If F is a finite extension of $E_{\langle p \rangle}$, then its degree $[F : E_{\langle p \rangle}]$ is equal to p^n for some natural number n .*

Proof. Let x be any element in F and denote by P_x its irreducible polynomial over $E_{\langle p \rangle}$. We prove that its degree is a power of p . All the coefficients lie in a finite prime-to- p extension of E . If the degree of x over $E_{\langle p \rangle}$ is prime to p , then $E_{\langle p \rangle}(x)$, which is a nontrivial extension of $E_{\langle p \rangle}$, contradicts the maximality of $E_{\langle p \rangle}$. Write $p^n m$ the degree of x over $E_{\langle p \rangle}$ with $n, m \geq 1$ and $(n, p) = 1$. Let F_N be the normal closure of F in E_s ; it is a Galois extension of $E_{\langle p \rangle}$ whose degree over $E_{\langle p \rangle}$ is divisible by $p^n m$. If $n \neq 1$, then a Sylow p -subgroup $S(p)$ of $\text{Gal}(F_N/E_{\langle p \rangle})$ is a nontrivial proper subgroup and the fixed field $F_N^{S(p)}$ is a nontrivial prime-to- p extension of $E_{\langle p \rangle}$, which is absurd. Thus $n = 1$ and the result follows. \square

The previous result leads to the following definition.

Definition 5.2.1.3. A field that has no nontrivial finite extension prime to p is called p -primary.

Proposition 5.2.1.4. *Let F be a nontrivial finite extension of $E_{\langle p \rangle}$ contained in E_s and let p^n be the degree $[F : E_{\langle p \rangle}]$. Then there is a tower of fields*

$$E_{\langle p \rangle} = F_1 \subset F_2 \subset \dots \subset F_n = F$$

such that $[F_i : F_{i-1}] = p$.

Proof. We prove the result by induction on n . We need to find a subfield K of F whose degree over $E_{\langle p \rangle}$ is p^{n-1} . The group $G = \text{Gal}(E_s/E_{\langle p \rangle})$ is a pro- p -group since all finite extensions of $E_{\langle p \rangle}$ contained in E_s are p -power extensions. Galois theory implies that E is the fixed subfield of a subgroup H of G with $[G : H] = p^n$. We will find a subgroup H_1 , such that $H \subset H_1 \subset G$ and $[G : H_1] = p^{n-1}$. Letting $K = E_s^{H_1}$, we will get the desired subfield K .

The group H is subgroup of G of finite index hence is open. By the class equation, it also follows that H has only a finite number of conjugates in G . Let $H' = \bigcap_{x \in G} x^{-1} H x$, then H' is an open normal subgroup of G containing H . The group G/H' is a finite p -group containing H/H' . By the Sylow theorems, we can find H_1 , normal in G , with $H \subset H_1 \subset G$ and $[G : H_1] = p^{n-1}$. This ends to proof. \square

Similarly, we obtain the following result.

Lemma 5.2.1.5. *Let p be a prime number and E a p -primary field. Let F/E be a finite extension.*

1. *The field F inherits the property of having no nontrivial finite extension of degree prime to p .*
2. *If $F \neq E$, then there exists a subfield $E \subset F' \subset F$ such that F'/E is a normal extension of degree p .*

5.2.2 Transfers on Milnor-Witt K-theory

Recall the definition of transfers on Milnor-Witt K-theory, this follows from the original definition of Bass-Tate (see [BT73], see also [GS17]).

5.2.2.1. Let $\varphi : E \rightarrow F$ be a monogenous finite field extension and choose $x \in F$ such that $F = E(x)$. The homotopy exact sequence implies that for any $\beta \in \mathbf{K}_*^{\text{MW}}(F, \omega_{F/k})$ there exists $\gamma \in \mathbf{K}_*^{\text{MW}}(E(t), \omega_{E(t)/k})$ with the property that $d(\gamma) = \beta$. Now the valuation at ∞ yields a morphism

$$\partial_\infty : \mathbf{K}_{*+1}^{\text{MW}}(E(t), \omega_{E(t)/k}) \rightarrow \mathbf{K}_*^{\text{MW}}(E, \omega_{E/k})$$

which vanishes on the image of i_* . We denote by $\varphi^*(\beta)$ or by $\text{Tr}_{x/E}(\beta)$ the element $-\partial_\infty(\gamma)$; it does not depend on the choice of γ . This defines a group morphism

$$\text{Tr}_{x/E} : \mathbf{K}_*^{\text{MW}}(E(x), \omega_{F/k}) \rightarrow \mathbf{K}_*^{\text{MW}}(E, \omega_{E/k})$$

called the *transfer map* and also denoted by $\text{Tr}_{x/E}$. The following result completely characterizes the transfer maps.

Lemma 5.2.2.2. *Keeping the previous notations, let*

$$d = (\bigoplus_x d_x) \oplus d_\infty : \mathbf{K}_{*+1}^{\text{MW}}(E(t), \omega_{F(t)/k}) \rightarrow (\bigoplus_x \mathbf{K}_*^{\text{MW}}(E(x), \omega_{E(x)/k}) \oplus \mathbf{K}_*^{\text{MW}}(E, \omega_{E/k}))$$

be the total twisted residue morphism (where x runs through the set of monic irreducible polynomials in $E(t)$). Then, the transfer maps $\text{Tr}_{x/E}$ are the unique morphisms such that $\sum_x (\text{Tr}_{x/E} \circ d_x) + d_\infty = 0$.

Proof. Straightforward (see [Mor12, §4.2]). □

Definition 5.2.2.3. Let $F = E(x_1, x_2, \dots, x_r)$ be a finite extension of a field E and consider the chain of subfields

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \dots \subset E(x_1, \dots, x_r) = F.$$

Define by induction:

$$\text{Tr}_{x_1, \dots, x_r/E} = \text{Tr}_{x_r/E(x_1, \dots, x_{r-1})} \circ \dots \circ \text{Tr}_{x_2/E(x_1)} \circ \text{Tr}_{x_1/E}$$

We give an elementary proof of the fact that the definition does not depend on the choice of the factorization (see Theorem 1.6.1.7 or [Mor12, Theorem 4.27] for the original proof):

Theorem 5.2.2.4. *The maps $\text{Tr}_{x_1, \dots, x_r/E} : \mathbf{K}_*^{\text{MW}}(F) \rightarrow \mathbf{K}_*^{\text{MW}}(E)$ do not depend on the choice of the generating system (x_1, \dots, x_r) .*

We begin with a series of lemmas aimed at reducing the theorem to the case of p -primary fields.

Lemma 5.2.2.5. *Let $F = E(x)/E$ be a simple extension of degree n of characteristic zero fields and consider the transfer map $\text{Tr}_{F/E} : \text{GW}(F) \rightarrow \text{GW}(E)$. If n is odd, then*

$$\text{Tr}_{F/E}(1) = n_E.$$

If n is even, then

$$\mathrm{Tr}_{F/E}(1) = (n-1)\varepsilon + \langle -N_{F/E}(x) \rangle.$$

where $N_{F/E}(x)$ is the classical norm of $x \in F^\times$.

Proof. See [Lam05, VII.2.2]. \square

Lemma 5.2.2.6. *Let F/E be a finite extension of degree n of characteristic zero fields and consider the transfer map $\mathrm{Tr}_{F/E} : \mathrm{GW}(F) \rightarrow \mathrm{GW}(E)$. If n is odd, then*

$$\mathrm{Tr}_{F/E}(1) = n\varepsilon.$$

If n is even, then there exist $a_1, \dots, a_n \in E^\times$ such that

$$\mathrm{Tr}_{F/E}(1) = \sum_i \langle a_i \rangle.$$

Proof. When the extension is simple, this is Lemma 5.2.2.5. We conclude by induction on the number of generators. \square

Lemma 5.2.2.7. *Let E be a field of characteristic $p > 0$. Let $\alpha \in \mathrm{GW}(E)$ be an element in the kernel of the rank morphism $\mathrm{GW}(E) \rightarrow \mathbb{Z}$. Then α is nilpotent in $\mathrm{GW}(E)$.*

Proof. (See [MLZ16, Lemma B.4]) As the set of nilpotent elements in the commutative ring $\mathrm{GW}(E)$ is an ideal, we may assume $\alpha = \langle t \rangle - 1$ where $t \in E^\times$. We have $(1 + \alpha)^2 = \langle t^2 \rangle = 1$, so that $\alpha^2 = -2\alpha$. By induction, we get $\alpha^n = (-2)^{n-1}\alpha$ for $n \geq 1$: we have to show that α is annihilated by a power of two. If $p = 2$, $2\alpha = 0$ holds (see [Mor12, Lemma 3.9]), i.e. $\alpha^2 = 0$. Now we assume $p \geq 3$ so that there is no danger thinking in terms of usual quadratic forms. We first consider $\mu := \langle -1 \rangle - 1 \in \mathrm{GW}(\mathbb{F}_p)$. The quadratic form $-x^2 - y^2$ over \mathbb{F}_p represents 1 (see [Ser77, Proposition 4, §IV.1.7]) so that $\langle -1 \rangle + \langle -1 \rangle = \langle 1 \rangle + \langle 1 \rangle \in \mathrm{GW}(\mathbb{F}_p)$, i.e. $2\mu = 0 \in \mathrm{GW}(\mathbb{F}_p)$, which gives $\mu^2 = 0$. Let $t \in E^\times$ be any nonzero element in an extension E of \mathbb{F}_p . The quadratic form $q(x, y) := x^2 - y^2 = (x + y)(x - y)$ represents t (this is $q((1+t)/2, (1-t)/2)$), which easily implies that $\langle 1 \rangle + \langle -1 \rangle = \langle t \rangle + \langle -t \rangle$. This is equivalent to saying $(2 + \mu)\alpha = 0 \in \mathrm{GW}(E)$. It follows that $4\alpha = (2 - \mu)(2 + \mu)\alpha = 0$, and then $\alpha^3 = 0$. \square

Lemma 5.2.2.8. *Consider two finite extensions F/E and L/E of coprime degrees n and m , respectively. Let $x \in \mathbf{K}^{MW}(E)$ such that $\mathrm{res}_{F/E}(x) = 0 = \mathrm{res}_{L/E}(x)$. Then $x = 0$.*

Proof. Applying the transfer map to $\mathrm{res}_{F/E}(x)$ and $\mathrm{res}_{L/E}(x)$, we see that x is killed by $\mathrm{Tr}_{F/E}(1)$ and $\mathrm{Tr}_{L/E}(1)$. In characteristic zero, up to swapping n and m , we may assume that n is odd, hence $\mathrm{Tr}_{F/E}(1) = n\varepsilon$ and $\mathrm{Tr}_{L/E}(1) = \sum_i \langle a_i \rangle$ for some $a_1, \dots, a_m \in E^\times$. Write $n = 2r + 1$. There exist $a, b \in \mathbb{Z}$ such that $an + bm = r$ since n and m are coprime. Recall that the hyperbolic form $h = 1 + \langle -1 \rangle$ satisfies $\langle a_i \rangle h = h$ for any i (see [Mor12, Lemma 3.7]). Hence $rh = (an\varepsilon + b\sum_i \langle a_i \rangle)h$ and $1 = n\varepsilon - rh = (1 - ah)\mathrm{Tr}_{F/E}(1) - bh\mathrm{Tr}_{L/E}(1)$ kills x .

In characteristic $p > 0$, there exist two nilpotent α and α' in $\mathrm{GW}(E)$ such that $\mathrm{Tr}_{F/E}(1) = n + \alpha$ and $\mathrm{Tr}_{L/E}(1) = m + \alpha'$, according to Lemma 5.2.2.7. Hence for a natural number s large enough, the element x is killed by the coprime numbers n^{p^s} and m^{p^s} so that $x = 0$. \square

Lemma 5.2.2.9. *Let E be a field of characteristic $p > 0$. Let F_1, \dots, F_n be finite extensions of coprime degrees d_1, \dots, d_n . Let $\delta \in \mathbf{K}_*^{MW}(E)$ be an element such that $\mathrm{res}_{F_i/E}(\delta) = 0$ for any i . Then, δ is zero.*

Proof. In zero characteristic, this follows as in Lemma 5.2.2.8. Assume the characteristic of E to be nonzero. Let $1 \leq i \leq n$, the projection formula proves that δ is killed by $\text{Tr}_{F_i/E}(1)$. Thus, according to Lemma 5.2.2.7, there exist a nilpotent element α_i in $\text{GW}(E)$ such that $d_i + \alpha_i$ kills δ . Since the degrees d_i are coprime, a Bezout combination yields a nilpotent element α' in $\text{GW}(E)$ such that $1 + \alpha'$ kills δ . Finally, we can find a natural number n large enough such that $\delta = 1 + (\alpha')^{p^n} \cdot \delta = (1 + \alpha')^{p^n} \cdot \delta = 0$. \square

Theorem 5.2.2.10 (Strong R1c). *Let E be a field, F/E a finite field extension and L/E an arbitrary field extension. Write $F = E(x_1, \dots, x_r)$ with $x_i \in F$, $R = F \otimes_E L$ and $\psi_{\mathfrak{p}} : R \rightarrow R/\mathfrak{p}$ the natural projection defined for any $\mathfrak{p} \in \text{Spec}(R)$. Then the diagram*

$$\begin{array}{ccc} \mathbf{K}_*^{MW}(F, \omega_{F/k}) & \xrightarrow{\text{Tr}_{x_1, \dots, x_r/E}} & \mathbf{K}_*^{MW}(E, \omega_{E/k}) \\ \oplus_{\mathfrak{p}} \text{res}_{(R/\mathfrak{p})/F} \downarrow & & \downarrow \text{res}_{L/E} \\ \oplus_{\mathfrak{p} \in \text{Spec}(R)} \mathbf{K}_*^{MW}(R/\mathfrak{p}, \omega_{(R/\mathfrak{p})/k}) & \xrightarrow{\sum_{\mathfrak{p}} (m_{\mathfrak{p}})_{\varepsilon} \text{Tr}_{\psi_{\mathfrak{p}}(a_1), \dots, \psi_{\mathfrak{p}}(a_r)/L}} & \mathbf{K}_*^{MW}(L, \omega_{L/k}) \end{array}$$

is commutative where $m_{\mathfrak{p}}$ the length of the localized ring $R_{(\mathfrak{p})}$.

Proof. This follows by induction (see also Theorem 4.4.1.16). \square

Theorem 5.2.2.11. *Assume that Theorem 5.2.2.4 holds for all fields p -primary E for any prime number p . Then the theorem holds for any field E .*

Proof. Consider two decompositions

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \dots \subset E(x_1, \dots, x_r) = F.$$

and

$$E \subset E(y_1) \subset E(y_1, y_2) \subset \dots \subset E(y_1, \dots, y_s) = F.$$

of F . Let $\alpha \in M_{-1}(F)$ and denote by δ the element $\text{Tr}_{x_1, \dots, x_r/E}(\alpha) - \text{Tr}_{y_1, \dots, y_s/E}(\alpha)$. Fix p a prime number and let L be a maximal prime to p extension of E (L has no nontrivial finite extension of degree prime to p). With the notation of Theorem 5.2.2.10, the map $\sum_{\mathfrak{p}} (m_{\mathfrak{p}})_{\varepsilon} \text{Tr}_{\psi_{\mathfrak{p}}(x_1), \dots, \psi_{\mathfrak{p}}(x_r)/L}$ does not depend on the choice of x_i according to the assumption. Hence $\text{res}_{L/E}(\delta) = 0$ and we can find a finite extension L_p/E of degree prime to p such that $\text{res}_{L_p/E}(\delta) = 0$. Since this is true for all prime number p , we see that the assumption of Lemma 5.2.2.9 are satisfied. Thus $\delta = 0$ and the theorem is proved. \square

Remark 5.2.2.12. More generally, one may replace \mathbf{K}_*^{MW} by any contracted homotopy sheaf M_{-1} and apply the proof verbatim. In particular, we have the following simplification of Morel's conjecture [Fel20a, Conjecture 4.1.13].

Theorem 5.2.2.13. *In order to prove Morel's Conjecture 4.4.1.13, it suffices to consider the case of p -primary fields (where p is a prime number).*

Proposition 5.2.2.14 (Bass-Tate-Morel Lemma). *Let $F(x)$ be a monogenous extension of F . Then $\mathbf{K}_*^{MW}(F(x))$ is generated as a left $\mathbf{K}_*^{MW}(F)$ -module by elements of the form*

$$\eta^m \cdot [p_1(x), p_2(x), \dots, p_n(x)]$$

where the p_i are monic irreducible polynomials of $F[t]$ satisfying

$$\deg(p_1) < \deg(p_2) < \cdots < \deg(p_n) \leq d - 1$$

where d is the degree of the extension $F(x)/F$.

Proof. Straightforward computations (see also [Mor12, Lemma 3.25.1]). \square

Corollary 5.2.2.15. *Let F/E be a finite field extension and assume one of the following conditions holds:*

- F/E is a quadratic extension,
- F/E is a prime degree p extension and E has no nontrivial extension of degree prime to p .

Then $\mathbf{K}_^{MW}(F)$ is generated as a left $\mathbf{K}_*^{MW}(E)$ -module by F^\times .*

Proof. In both cases, the extension F/E is simple and the only monic irreducible polynomial in $E[t]$ of degree strictly smaller than $[F : E]$ are the polynomials of degree 1. We conclude by Proposition 5.2.2.14. \square

In the following, we fix a prime number p and E a p -primary field.

Proposition 5.2.2.16. *Let $F = E(x)$ be a monogenous extension of E of degree p . Then the transfers $\mathrm{Tr}_{x/E} : \mathbf{K}_*^{MW}(E(x), \omega_{E(x)/k}) \rightarrow \mathbf{K}_*^{MW}(E, \omega_{E/k})$ do not depend on the choice of x .*

Proof. According to Lemma 5.2.2.15, the group $\mathbf{K}_*^{MW}(F, \omega_{F/k})$ is generated by products of the form $\mathrm{res}_{F/E}(\alpha) \cdot [\beta]$ with $\alpha \in \mathbf{K}_*^{MW}(E, \omega_{E/k})$ and $\beta \in F^\times$. The projection formula yields

$$\mathrm{Tr}_{x/E}(\mathrm{res}_{F/E}(\alpha) \cdot [\beta]) = \alpha \cdot \mathrm{Tr}_{F/E}([\beta])$$

which does not depend on a x (see [Fas19, §1]). \square

5.2.2.17. We may now use the notation $\mathrm{Tr}_{F/E} : \mathbf{K}_*^{MW}(F, \omega_{F/k}) \rightarrow \mathbf{K}_*^{MW}(E, \omega_{E/k})$ for extensions of prime degree p .

Proposition 5.2.2.18. *Let F be a field complete with respect to a discrete valuation v , and F'/F a normal extension of degree p . Denote by v' the unique extension of v to F' . Then the diagram*

$$\begin{array}{ccc} \mathbf{K}_*^{MW}(F', \omega_{F'/k}) & \xrightarrow{\partial_{v'}} & \mathbf{K}_{*-1}^{MW}(\kappa(v'), \omega_{\kappa(v')}) \\ \mathrm{Tr}_{F'/F} \downarrow & & \downarrow \mathrm{Tr}_{\kappa(v')/\kappa(v)} \\ \mathbf{K}_*^{MW}(F) & \xrightarrow{\partial_v} & \mathbf{K}_{*-1}^{MW}(\kappa(v)) \end{array}$$

is commutative.

Proof. See [Mor12, Remark 5.20]. \square

Corollary 5.2.2.19. *Let F/E be a normal extension of degree p and let $x \in (\mathbb{A}_E^1)^{(1)}$. Then the diagram*

$$\begin{array}{ccc}
 \mathbf{K}_*^{\text{MW}}(F(t), \omega_{F(t)/k}) & \xrightarrow{\oplus \partial_y} & \bigoplus_{y \mapsto x} \mathbf{K}_{*-1}^{\text{MW}}(\kappa(y), \omega_{\kappa(y)/k}) \\
 \text{Tr}_{F(t)/E(t)} \downarrow & & \downarrow \Sigma \text{Tr}_{\kappa(y)/\kappa(x)} \\
 \mathbf{K}_*^{\text{MW}}(E(t), \omega_{E(t)/k}) & \xrightarrow{\partial_x} & \mathbf{K}_{*-1}^{\text{MW}}(\kappa(x), \omega_{\kappa(x)/k})
 \end{array}$$

is commutative, where $y \rightarrow x$ denotes the set of elements $y \in (\mathbb{A}_F^1)^{(1)}$ mapping to x through the canonical morphism.

Proof. Denote by \hat{E}_x (resp. \hat{F}_y) the completions of $E(t)$ (resp. $F(t)$) with respect to the valuations defined by x (resp. y). Consider the following diagram

$$\begin{array}{ccccc}
 \mathbf{K}_*^{\text{MW}}(F(t), \omega_{F(t)/k}) & \longrightarrow & \bigoplus_{y \rightarrow x} \mathbf{K}_*^{\text{MW}}(\hat{F}_y, \omega_{\hat{F}_y/k}) & \xrightarrow{\oplus \partial_y} & \bigoplus_{y \rightarrow x} \mathbf{K}_{*-1}^{\text{MW}}(\kappa(y), \omega_{\kappa(y)/k}) \\
 \text{Tr}_{F(t)/E(t)} \downarrow & & \downarrow \Sigma \text{Tr}_{\hat{F}_y/\hat{F}_x} & & \downarrow \Sigma \text{Tr}_{\kappa(y)/\kappa(x)} \\
 \mathbf{K}_*^{\text{MW}}(E(t), \omega_{E(t)/k}) & \longrightarrow & \mathbf{K}_*^{\text{MW}}(\hat{E}_x, \omega_{\hat{E}_x/k}) & \xrightarrow{\partial_x} & \mathbf{K}_{*-1}^{\text{MW}}(\kappa(x), \omega_{\kappa(x)/k}).
 \end{array}$$

The left-hand square is commutative according to Theorem 5.2.2.10. The right-hand square commute according to Proposition 5.2.2.18. Hence the corollary. \square

Lemma 5.2.2.20. *Let L/E be a normal extension of degree p , and let $E(a)/E$ be a monogenous finite extension. Assume that L and $E(a)$ are both subfields of some algebraic extension of E , and denote by $L(a)$ their composite. Then the following diagram*

$$\begin{array}{ccc}
 \mathbf{K}_*^{\text{MW}}(L(a), \omega_{L(a)/k}) & \xrightarrow{\text{Tr}_{a/L}} & \mathbf{K}_*^{\text{MW}}(L, \omega_{L/k}) \\
 \text{Tr}_{L(a)/E(a)} \downarrow & & \downarrow \text{Tr}_{L/E} \\
 \mathbf{K}_*^{\text{MW}}(E(a), \omega_{E(a)/k}) & \xrightarrow{\text{Tr}_{a/E}} & \mathbf{K}_*^{\text{MW}}(E, \omega_{E/k})
 \end{array}$$

is commutative.

Proof. Let x (resp y_0) be the closed point of \mathbb{A}_E^1 (resp. \mathbb{A}_L^1) defined by the minimal polynomial of a over E (resp. L). Given $\alpha \in \mathbf{K}_*^{\text{MW}}(L(a), \omega_{L(a)/k})$, we have $\text{Tr}_{a/L}(\alpha) = -\partial_\infty(\beta)$ for some $\beta \in \mathbf{K}_{*+1}^{\text{MW}}(L(t), \omega_{L(t)/k})$ satisfying $\partial_{y_0}(\beta) = \alpha$ and $\partial_\beta = 0$ for $y \neq y_0$. By Corollary 5.2.2.19

$$\partial_x(\text{Tr}_{L(t)/E(t)}(\beta)) = \sum_{y \mapsto x} \text{Tr}_{\kappa(y)/\kappa(x)}(\partial_y(\beta)) = \text{Tr}_{\kappa(y_0)/\kappa(x)}(\alpha),$$

and, similarly, $\partial_{x'}(\text{Tr}_{L(t)/E(t)}(\beta)) = 0$ for $x \neq x'$. Hence by definition of the transfer map $\text{Tr}_{a/E}$ we have

$$\text{Tr}_{a/E}(\text{Tr}_{L(a)/E(a)}(\alpha)) = -\partial_\infty(\text{Tr}_{L(t)/E(t)}(\beta)).$$

Moreover, since the only point of \mathbb{P}_L^1 above ∞ is ∞ , another application of Corollary 5.2.2.19 gives

$$\partial_\infty(\text{Tr}_{L(t)/E(t)}(\beta)) = \text{Tr}_{L/E}(\partial_\infty(\beta)).$$

Hence the result.

$$\text{Tr}_{a/E}(\text{Tr}_{L(a)/E(a)}(\alpha)) = -\text{Tr}_{L/E}(\partial_\infty(\beta)) = \text{Tr}_{L/E}(\text{Tr}_{a/L}(\alpha)).$$

\square

Proof of Theorem 5.2.2.4. We keep the previous notations. We already know that it suffices to treat the case when E has no nontrivial extension of degree prime to p . Let p^m be the degree of the extension F/E . We prove the result by induction on m . The case $m = 1$ follows from Proposition 5.2.2.16. Consider two decompositions

$$E \subset E(x_1) \subset E(x_1, x_2) \subset \cdots \subset E(x_1, \dots, x_r) = F.$$

and

$$E \subset E(y_1) \subset E(y_1, y_2) \subset \cdots \subset E(y_1, \dots, y_s) = F.$$

of F . By Lemma 5.2.1.5, the extension $E(x_1)/E$ contains a normal subfield $E(x'_1)$ of degree p over E . Applying Lemma 5.2.2.20 with $a = x_1$ and $L = E(x'_1)$ yields $\mathrm{Tr}_{x_1/E} = \mathrm{Tr}_{x'_1/E} \circ \mathrm{Tr}_{x_1/E(x'_1)}$. Hence, without loss of generality, we may assume that $x_1 = x'_1$ and, similarly, $[E(y_1) : E] = p$. Write F_0 for the composite of the fields $E(x_1)$ and $E(y_1)$ in F and write $F = F_0(z_1, \dots, z_t)$ with $z_i \in F$. The fields $E(x_1)$ and $E(y_1)$ have no nontrivial prime to p extension, thus we may conclude by the induction hypothesis that the triangles

$$\begin{array}{ccc} \mathbf{K}_*^{\mathrm{MW}}(F, \omega_{F/k}) & \xrightarrow{\mathrm{Tr}_{x_2, \dots, x_r/E(x_1)}} & \mathbf{K}_*^{\mathrm{MW}}(E(x_1), \omega_{E(x_1)/k}) \\ \mathrm{Tr}_{z_1, \dots, z_t/F_0} \downarrow & \nearrow \mathrm{Tr}_{F_0/E(x_1)} & \\ \mathbf{K}_*^{\mathrm{MW}}(F_0, \omega_{F_0/k}) & & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{K}_*^{\mathrm{MW}}(F, \omega_{F/k}) & \xrightarrow{\mathrm{Tr}_{y_2, \dots, y_s/E(y_1)}} & \mathbf{K}_*^{\mathrm{MW}}(E(y_1), \omega_{E(y_1)/k}) \\ \mathrm{Tr}_{z_1, \dots, z_t/F_0} \downarrow & \nearrow \mathrm{Tr}_{F_0/E(y_1)} & \\ \mathbf{K}_*^{\mathrm{MW}}(F_0, \omega_{F_0/k}) & & \end{array}$$

are commutative.

Moreover, Lemma 5.2.2.20 for $a = x_1$ and $L = E(y_1)$ implies that the following diagram

$$\begin{array}{ccc} \mathbf{K}_*^{\mathrm{MW}}(F_0, \omega_{F_0/k}) & \xrightarrow{\mathrm{Tr}_{F_0/E(x_1)}} & \mathbf{K}_*^{\mathrm{MW}}(E(x_1), \omega_{E(x_1)/k}) \\ \mathrm{Tr}_{F_0/E(y_1)} \downarrow & & \downarrow \mathrm{Tr}_{x_1/E} \\ \mathbf{K}_*^{\mathrm{MW}}(E(y_1), \omega_{E(y_1)/k}) & \xrightarrow{\mathrm{Tr}_{y_1/E}} & \mathbf{K}_*^{\mathrm{MW}}(E, \omega_{E/k}) \end{array}$$

is commutative. Putting everything together, we conclude that $\mathrm{Tr}_{x_1, \dots, x_r/E} = \mathrm{Tr}_{y_1, \dots, y_s/E}$. \square

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Milnor-Witt sheaves and modules

Résumé

On généralise la théorie des modules de cycles de Rost en utilisant la K-théorie de Milnor-Witt au lieu de la K-théorie de Milnor. On obtient un cadre (quadratique) pour étudier certains complexes de cycles et leurs groupes de (co)homologie. De plus, on démontre que le cœur de la catégorie homotopique stable de Morel-Voevodsky au-dessus d'un corps parfait (équipé de sa t-structure homotopique) est équivalente à la catégorie des modules de cycles de Milnor-Witt. Finalement, on explore une conjecture de Morel concernant les transferts de Bass-Tate définis sur la contraction d'un faisceau homotopique et démontre que la conjecture est vraie à coefficients rationnels. On étudie aussi les relations entre faisceaux homotopiques (contractés), faisceaux homotopiques avec transferts généralisés et MW-faisceaux homotopiques, et démontre une équivalence de catégories. Comme applications, on décrit l'image essentielle du foncteur canonique qui oublie les MW-transferts et utilise ces résultats pour discuter de la conjecture de conservativité en \mathbb{A}^1 -homotopie due à Bachmann et Yakerson.

Mots-clés : Modules de cycles, K-théorie de Milnor-Witt, groupes de Chow-Witt, \mathbb{A}^1 -homotopie, modules homotopiques, faisceaux homotopiques, transferts de Bass-Tate.

Abstract

We generalize Rost's theory of cycle modules using the Milnor-Witt K-theory instead of the classical Milnor K-theory. We obtain a (quadratic) setting to study general cycle complexes and their (co)homology groups. Moreover, we prove that the heart of Morel-Voevodsky stable homotopy category over a perfect field (equipped with its homotopy t-structure) is equivalent to the category of Milnor-Witt cycle modules. Finally, we explore a conjecture of Morel about the Bass-Tate transfers defined on the contraction of a homotopy sheaf and prove that the conjecture is true with rational coefficients. We also study the relations between (contracted) homotopy sheaves, sheaves with generalized transfers and MW-homotopy sheaves, and prove an equivalence of categories. As applications, we describe the essential image of the canonical functor that forgets MW-transfers and use these results to discuss the conservativity conjecture in \mathbb{A}^1 -homotopy due to Bachmann and Yakerson.

Keywords : Cycle modules, Milnor-Witt K-theory, Chow-Witt groups, \mathbb{A}^1 -homotopy, homotopy modules, homotopy sheaves, Bass-Tate transfers.

