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Contributions to the relative trace formula of Guo-Jacquet

Par **Huajie LI**

Thèse de doctorat de mathématiques
Dirigée par **Pierre-Henri CHAUDOUARD**

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Title, abstract and key words (in English and French)

Title: Contributions to the relative trace formula of Guo-Jacquet

Abstract: We establish global and local trace formulae for infinitesimal symmetric spaces of Guo-Jacquet. We also prove several local results concerning the comparison of regular semi-simple terms which are noninvariant weighted orbital integrals. This thesis contains five chapters. In Chapter 1, we recall the motivations and state our main reults. Our work is inspired by a conjecture of Guo-Jacquet, which is an example in the relative Langlands programme, and analytic problems appearing in the relative trace formula approach. In Chapter 2, we establish an infinitesimal variant of Guo-Jacquet trace formula for the case of $(GL_{2n,D}, GL_{n,D} \times GL_{n,D})$. It is a kind of Poisson summation formula obtained by an analogue of Arthur's truncation. We describe regular semi-simple terms as explicit weighted orbital integrals. In Chapter 3, we estabilish a similar formula and have a similar description of regular semi-simple terms for the case of a central simple algebra containing a quadratic extension. Moreover, we state and prove the weighted fundamental lemma thanks to Labesse's work on the base change for GL_n. In Chapter 4, we establish an infinitesimal invariant local trace formula of Guo-Jacquet over a p-adic field by following works of Waldspurger and Arthur. During the proof, we also obtain an infinitesimal noninvariant local trace formula, Howe's finiteness for weighted orbital integrals and the representability of the Fourier transform of weighted orbital integrals. In Chapter 5, with the results in previous chapters, we adopt Waldspurger's strategy on the endoscopic transfer to prove some relations between Fourier transforms of invariant local weighted orbital integrals.

Key words: Guo-Jacquet trace formula, Arthur's truncation, weighted fundamental lemma, local trace formula, noninvariant transfer

Titre : Contributions à la formule des traces relative de Jacquet-Guo

Résumé : On établit des formules des traces globale et locale pour les espaces symétriques infinitésimaux de Jacquet-Guo. On prouve également quelques résultats locaux concernant la comparaison de termes semi-simples réguliers qui sont des intégrales orbitales pondérées non invariantes. Cette thèse contient cinq chapitres. Dans le chapitre 1, on rappelle les motivations et énonce nos principaux résultats. Notre travail s'inspire d'une conjecture de Jacquet-Guo, qui est un exemple dans le programme de Langlands relatif, et des problèmes analytiques apparaissant dans l'approche par la formule des traces relative. Dans le chapitre 2, on établit une variante infinitésimale de la formule des traces de Jacquet-Guo pour le cas de $(GL_{2n,D}, GL_{n,D} \times GL_{n,D})$. Elle est une sorte de formule sommatoire de Poisson obtenue par un analogue de la troncature d'Arthur. On décrit les termes semi-simples réguliers comme des intégrales orbitales pondérées explicites. Dans le chapitre 3, on établit une formule similaire et a une description similaire des termes semi-simples réguliers pour le cas d'une algèbre centrale simple contenant une extension quadratique. De plus, on énonce et prouve le lemme fondamental pondéré grâce aux travaux de Labesse sur le changement de base pour GL_n. Dans le chapitre 4, on établit une formule des traces locale invariante infinitésimale de Jacquet-Guo sur un corps p-adique en suivant les travaux de Waldspurger et Arthur. Au cours de la démonstration, on obtient également une formule des traces locale non invariante infinitésimale, la finitude de Howe pour les intégrales orbitales pondérées et la représentabilité de la transformée de Fourier des intégrales orbitales pondérées. Dans le chapitre 5, avec les résultats des

chapitres précédents, on adopte la stratégie de Waldspurger sur le transfert endoscopique pour prouver certaines relations entre transformées de Fourier des intégrales orbitales pondérées locales invariantes.

Mots-clefs : formule des traces de Jacquet-Guo, troncature d'Arthur, lemme fondamental pondéré, formule des traces locale, transfert non invariant

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Introduction en français

Ce projet de thèse s'inspire d'une conjecture de Jacquet-Guo [23] qui généralise le théorème connu de Waldspurger [50] reliant les périodes toriques aux valeurs centrales des fonctions L automorphes pour GL_2 en dimensions supérieures. C'est l'un des premiers exemples intéressants du programme de Langlands relatif initié par Jacquet et ses collaborateurs et systématiquement proposé par Sakellaridis-Venkatesh [46]. Un outil efficace pour attaquer ce genre de problèmes est la formule des traces relative introduite par Jacquet [29] pour réprouver le résultat de Waldspurger. Les objectifs de cette thèse sous la direction de Pierre-Henri Chaudouard incluent le développement de variantes infinitésimales des formules des traces de Jacquet-Guo globale et locale et l'établissement des résultats locaux de comparaison nécessaires pour une étude plus approfondie de cette conjecture. On espère que certaines méthodes ici seront également utiles dans d'autres formules des traces relatives.

1. La conjecture de Jacquet-Guo et l'approche de la formule des traces relative

Rappelons brièvement la conjecture de Jacquet-Guo proposée dans [23]. Soient E/F une extension quadratique de corps de nombres et η le caractère quadratique attaché de $\mathbb{A}^\times/F^\times$, où $\mathbb{A}$ est l'anneau des adèles de F. Soient $G := GL_{2n}$ et $H := GL_n \times GL_n$ son sous-groupe. Tous les groupes considérés sont définis sur F. Soit π une représentation automorphe cuspidale de $G(\mathbb{A})$ avec un caractère central trivial. On dit que π est H-distinguée si les deux formes linéaires (appelées "périodes") sur son espace

$$\mathcal{P}_H : \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)dh$$

et

$$\mathcal{P}_{H,\eta} : \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)\eta(\det(h))dh$$

sont non nulles, où Z est le centre de G. Cette propriété est directement liée à la non-annulation de certaines valeurs centrales de fontions L grâce au travail de Friedberg-Jacquet [22]. On doit également considérer une autre paire de groupes. Notons $X(E)$ l'ensemble des classes d'isomorphismes d'algèbres de quaternions D/F contenant E. Pour tout $D \in X(E)$, on note $G' = GL_{n,D}$ le groupe algébrique défini sur F dont le groupe de points sur F est $GL_n(D)$. Soit $H' = \mathrm{Res}_{E/F} GL_{n,E}$ son sous-groupe. Soit π' une représentation automorphe cuspidale de $G'(\mathbb{A})$ avec un caractère central trivial. On dit que π' est H'-distinguée si la forme linéaire sur son espace

$$\mathcal{P}_{H'} : \phi \mapsto \int_{H'(F)Z(\mathbb{A})\backslash H'(\mathbb{A})} \phi(h)dh$$

est non nulle, où le centre de G' s'identifie à Z. Une direction de la conjecture de Jacquet-Guo dit que si π' est H'-distinguée et correspond par la correspondance de Jacquet-Langlands à π, alors π est H-distinguée. On s'attend également à une réciproque au moins pour n impair. Pour $n = 1$, celles-ci étaient connues par Waldspurger [50] et réprouvées par Jacquet [29] via la formule des traces relative.

Maintenant, on décrit formellement l'approche par la formule des traces relative suivant Jacquet [29] à cette conjecture. Soit f_G une fonction lisse sur $G(\mathbb{A})$ à support compact. En tant qu'analogue de la formule des traces d'Arthur-Selberg, la formule des traces relative pour le cas de (G, H) indique grossièrement qu'il existe deux façons d'écrire l'intégrale (vue comme une distribution)

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \mathbb{K}_{f_G}(x, y)\eta(\det(x))dxdy, \tag{1.0.1}$$

où $G(\mathbb{A})^1$ est le sous-ensemble des éléments dans $G(\mathbb{A})$ dont les déterminants sont de valeur absolue 1 et $\mathbb{K}_{f_G}(x, y) := \sum_{\gamma\in G(F)} f_G(x^{-1}\gamma y)$. Le côté géométrique devrait être une somme d'intégrales orbitales relatives tandis que le côté spectral devrait être une expansion de périodes. On pourrait de même imaginer une autre formule pour le cas de (G', H'). Alors la comparaison des périodes sur les différents

groupes prédite par la conjecture de Jacquet-Guo est réduite à la comparaison d'intégrales orbitales relatives, pour laquelle on a plus d'outils tels que le lemme fondamental de Guo [23] et le transfert lisse de Zhang [58].

Cette approche a été adoptée par Feigon-Martin-Whitehouse [21] pour obtenir des résultats partiels. Cependant, on a négligé la difficulté analytique dans la discussion ci-dessus. C'est-à-dire qu'une obstruction de l'approche par la formule des traces relative à la conjecture de Jacquet-Guo est le problème suivant.

Problème 1.1. *L'intégrale double* (1.0.1) *et ses développements ne sont pas convergents.*

Ce problème explique certaines conditions locales restrictives dans les principaux résultats de [21] basés sur une simple formule des traces relative. Si l'on espère supprimer ces restrictions apparemment artificielles et obtenir des informations sur toutes les représentations cuspidales, ce problème ne peut être ignoré. Par conséquent, il est nécessaire d'établir des formules des traces relatives valides et générales pour les cas (G, H) et (G', H') plutôt que des formules formelles ou simples. Ensuite, on doit comparer les nouveaux termes apparaissant dans ces formules en dehors des intégrales orbitales relatives ordinaires.

Un tel problème existait également dans la formule des traces d'Arthur-Selberg classique et Arthur a introduit un processus de troncature dans [3] et [4] pour le résoudre. Dans le cadre relatif, Jacquet-Lapid-Rogawski nous a fourni le premier exemple de la "troncature mixte" dans [30], dont l'idée grossière est de définir une troncature compatible avec les différents groupes concernés. Ces travaux et d'autres donnent de bons exemples pour notre étude du problème 1.1.

On se concentrera sur une variante infinitésimale du problème 1.1 et la comparaison de nouveaux termes impliqués dans cette thèse. Cela signifie que l'on va travailler sur l'espace tangent d'un espace symétrique (appelé espace symétrique infinitésimal). Cela serait intéressant pour au moins deux raisons. D'une part, il est proche du côté géométrique de la formule des traces relative originale et nous suggère une façon d'aborder le problème original, mais le côté spectral de la formule des traces relative est remplacé par la transformée de Fourier du côté géométrique où l'analyse harmonique est plus simple. D'autre part, une variante infinitésimale des formules des traces de Jacquet-Guo est également utile dans la comparaison de formules des traces relatives entre deux cas ; par exemple, sa version simple a été utilisée par Zhang [58] pour montrer le transfert lisse.

2. Principaux résultats

2.1. Une variante infinitésimale des formules des traces de Jacquet-Guo. On résout d'abord une variante infinitésimale du problème 1.1 dans le chapitre 2 et le chapitre 3. C'est le principal résultat global de cette thèse.

2.1.1. ***Le cas de*** (G, H). On remarque que le cadre du chapitre 2 est un peu plus général pour inclure le cas de [57], mais on se concentrera sur la paire (G, H) définie ci-dessus dans l'introduction. Notons S l'espace symétrique G/H. La double intégrale (1.0.1) peut être formellement écrite comme une intégrale seule

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} K_{f_S}(x)\eta(\det(x))dx,$$

où $f_S(x) := \int_{H(\mathbb{A})\cap G(\mathbb{A})^1} f_G(xy)dy$ définit une fonction lisse sur $S(\mathbb{A})$ à support compact et $K_{f_S}(x) := \sum_{\gamma\in S(F)} f_S(x^{-1}\gamma x)$. Si l'on remplace S par l'espace tangent $\mathfrak{s} \simeq \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ en l'élément neutre, alors le problème 1.1 apparaît comme la divergence de l'intégrale

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_f(x)\eta(\det(x))dx, \tag{2.1.1}$$

où f est une fonction de Bruhat-Schwartz sur $\mathfrak{s}(\mathbb{A})$ et $k_f(x) := \sum_{\gamma\in\mathfrak{s}(F)} f(x^{-1}\gamma x)$.

On remplace $k_f(x)$ par une expression explicite $k_f^T(x)$ définie dans §4 du chapitre 2 pour rendre (2.1.1) absolument convergente, où $T \in \mathbb{R}^{2n}$ est un paramètre de troncature. Cette définition essentielle combine des idées de [28] [61] [17] pour la décomposition de $H(\mathbb{A})$ et [40] pour la décomposition de $\mathfrak{s}(\mathbb{A})$. Afin de décrire le développement géométrique, on définit une relation d'équivalence sur $\mathfrak{s}(F)$: deux éléments dans $\mathfrak{s}(F)$ sont équivalents si et seulement s'ils se trouvent dans la même fibre du quotient catégorique $\mathfrak{s}//H$. Notons $\mathcal{O}$ l'ensemble des classes d'équivalence. Pour tout $\mathfrak{o} \in \mathcal{O}$, on définit de même $k_{f,\mathfrak{o}}^T(x)$ en remplaçant $\mathfrak{s}(F)$ par $\mathfrak{o}$. Le principal résultat du chapitre 2 est le théorème suivant qui donne le développement géométrique de (2.1.1) avec $k_f(x)$ remplacé par $k_f^T(x)$.

Théorème 2.1 (voir le théorème 4.11 du chapitre 2). *Pour tout T appartenant à un cône convenable dans $\mathbb{R}^{2n}$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}}\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k^T_{f,\mathfrak{o}}(x)\eta(\det(x))dx$$

est absolument convergente.

Notons $J^T_{\mathfrak{o}}(\eta,f)$ l'intégrale (2.1.1) avec $k_f(x)$ remplacé par $k^T_{f,\mathfrak{o}}(x)$. On montre que c'est un polynôme-exponentielle en T (voir le corollaire 5.6 du chapitre 2). Notons $J_{\mathfrak{o}}(\eta,f)$ son terme constant. Dans le cadre infinitésimal, le développement géométrique de la transformée de Fourier de f (notée $\hat{f}$) joue le rôle du côté spectral original (cf. [13]). Alors la variante infinitésimale de la formule des traces de Jacquet-Guo pour le cas de (G,H) ci-dessous est déduite de la formule sommatoire de Poisson.

Théorème 2.2 (voir le théorème 7.1 du chapitre 2). *Pour toute fonction de Bruhat-Schwartz f sur $\mathfrak{s}(\mathbb{A})$, on a l'égalité*

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta,f)=\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta,\hat{f}).$$

De plus, pour la plupart des classes $\mathfrak{o}$ (c'est-à-dire les classes semi-simples régulières), on peut écrire $J_{\mathfrak{o}}(\eta,f)$ comme une intégrale orbitale pondérée explicite ; en particulier, pour les classes elliptiques $\mathfrak{o}$, on voit que $J_{\mathfrak{o}}(\eta,f)$ n'est rien d'autre que l'intégrale orbitale ordinaire. On dit qu'un sous-groupe parabolique semi-standard de G est ω-stable s'il contient $\omega:=\begin{pmatrix}0 & 1_n\\ 1_n & 0\end{pmatrix}$.

Théorème 2.3 (voir le théorème 9.2 du chapitre 2). *Soient $\mathfrak{o}\in\mathcal{O}$ une classe semi-simple régulière, P_1 un sous-groupe parabolique relativement standard ω-stable de G et $X_1\in\mathfrak{o}$ un élément elliptique relatif à P_1 (défini dans la section 9.2 du chapitre 2). Pour toute fonction de Bruhat-Schwartz f sur $\mathfrak{s}(\mathbb{A})$, on a*

$$J_{\mathfrak{o}}(\eta,f)=\mathrm{vol}([H_{X_1}])\cdot\int_{H_{X_1}(\mathbb{A})\backslash H(\mathbb{A})} f(x^{-1}X_1x)v_{P_1}(x)\eta(\det(x))dx,$$

où H_{X_1} est le centralisateur de X_1 dans H, $\mathrm{vol}([H_{X_1}])$ est son volume associé et $v_{P_1}(x)$ est le volume d'un enveloppe convexe.

Le poids $v_{P_1}(x)$ est exactement la restriction à $H(\mathbb{A})$ du poids d'Arthur dans [3] pour $G(\mathbb{A})$. Il est remarquable que notre intégrale orbitale pondérée a le même poids que celui de la formule des traces tordue (voir [39, p. 131]) pour l'espace tordu $(GL_n\times GL_n)\rtimes\sigma$, où $\sigma(x,y):=(y,x)$.

2.1.2. ***Le cas de*** (G',H'). On étudie encore un cadre plus général dans le chapitre 3 que la paire (G',H') ci-dessus. En fait, on considère une algèbre centrale simple $\mathfrak{g}'$ sur F contenant E et le centralisateur $\mathfrak{h}'$ de E dans $\mathfrak{g}'$. On définit G' et H' comme les groupes d'éléments inversibles respectivement dans $\mathfrak{g}'$ et $\mathfrak{h}'$. Ce cas plus général s'inspire de la conjecture locale de Prasad et Takloo-Bighash [44, Conjecture 1] et est nécessaire pour compléter la réciproque de la conjecture de Jacquet-Guo pour n pair.

Quitte à conjuguer (G',H') par $G'(F)$, la paire symétrique est réduite à l'un des deux cas ci-dessous (voir la proposition 3.7 du chapitre 3). Ces deux cas peuvent être traités de la même manière.

Cas I: $(G',H')\simeq(GL_{n,D},\mathrm{Res}_{E/F}GL_{\frac{n}{2},D\otimes_F E})$, où D est une algèbre à division centrale sur F contenant E, et $D':=\mathrm{Cent}_D(E)$ est le centralisateur de E dans D.

Cas II: $(G',H')\simeq(GL_{n,D},\mathrm{Res}_{E/F}GL_{\frac{n}{2},D\otimes_F E})$, où D est une algèbre à division centrale sur F telle qu'il n'y a pas de plongement $E\to D$ en tant que F-algèbres.

Remarquons que (G',H') est identique à (G,H) après le changement de base à une clôture algébrique de F contenant E. On a des définitions et des résultats similaires pour le cas de (G',H') comme suit. Notons $H'(\mathbb{A})^1$ le sous-ensemble des éléments dans $H'(\mathbb{A})$ dont les normes réduites sont de valeur absolue 1. On note $\mathcal{O}'$ l'ensemble des classes d'équivalence sur $\mathfrak{s}'(F)$ définies par le quotient catégorique $\mathfrak{s}'//H'$. Soient f' une fonction de Bruhat-Schwartz sur $\mathfrak{s}'(\mathbb{A})$ et $\mathfrak{o}\in\mathcal{O}'$. Pour un paramètre de troncature $T\in\mathbb{R}^n$ et $x\in H'(x)$, on définit un noyau tronqué $k^T_{f',\mathfrak{o}}(x)$ dans §4 du chapitre 3.

Théorème 2.4 (voir le théorème 4.2 du chapitre 3). *Pour tout T appartenant à un cône convenable dans $\mathbb{R}^n$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}'}\int_{H'(F)\backslash H'(\mathbb{A})^1} k^T_{f',\mathfrak{o}}(x)dx$$

est absolument convergente.

Pour toute $\mathfrak{o} \in \mathcal{O}'$, on note $J_{\mathfrak{o}}^T(f')$ l'intégrale correspondante dans la dernière somme. On prouve que c'est un polynôme en T (voir le corollaire 5.3 du chapitre 3). On note $J_{\mathfrak{o}}(f')$ son terme constant. Notons $\hat{f}'$ la transformée de Fourier de f'. Voici la variante infinitésimale de la formule des traces de Jacquet-Guo pour le cas de (G', H').

THÉORÈME 2.5 (voir le théorème 7.1 du chapitre 3). *Pour toute fonction de Bruhat-Schwartz f' sur $\mathfrak{s}'(\mathbb{A})$, on a l'égalité*

$$\sum_{\mathfrak{o} \in \mathcal{O}'} J_{\mathfrak{o}}(f') = \sum_{\mathfrak{o} \in \mathcal{O}'} J_{\mathfrak{o}}(\hat{f}').$$

Comme précédemment, on peut écrire les termes semi-simples réguliers comme des intégrales orbitales pondérées explicites avec les mêmes poids que ceux d'Arthur dans [3]. Pour $(G', H') = (GL_{n,D}, \mathrm{Res}_{E/F} GL_{n,E})$, on obtient les mêmes poids apparaissant dans la formule des traces tordue (voir [39, p. 131]) pour l'espace tordu $(\mathrm{Res}_{E/F} GL_{n,E}) \rtimes \sigma'$, où σ' est la conjugaison galoisienne non triviale.

THÉORÈME 2.6 (voir le théorème 9.2 du chapitre 3). *Soient $\mathfrak{o} \in \mathcal{O}'$ une classe semi-simple régulière, P_1' un sous-groupe parabolique standard de H' et $Y_1 \in \mathfrak{o}$ un élément elliptique par rapport à P_1' (défini dans la section 9 du chapitre 3). Pour toute fonction de Bruhat-Schwartz f' sur $\mathfrak{s}'(\mathbb{A})$, on a*

$$J_{\mathfrak{o}}(f') = \mathrm{vol}([H'_{Y_1}]) \cdot \int_{H'_{Y_1}(\mathbb{A}) \backslash H'(\mathbb{A})} f'(\mathrm{Ad}(x^{-1})(Y_1)) v_{P_1'}(x) dx,$$

où H'_{Y_1} est le centralisateur de Y_1 dans H', $\mathrm{vol}([H'_{Y_1}])$ est le volume associé à H'_{Y_1} et $v_{P_1'}(x)$ est le volume d'un envelope convexe.

Effectivement, ce cas est encore plus simple que celui de (G, H) ; par exemple, $J_{\mathfrak{o}}^T(f')$ est un polynôme pur en T au lieu d'un polynôme-exponentielle. Une des raisons de cette simplicité est que le tore central F-déployé maximal de H' est le même que celui de G'. De plus, il existe une bijection entre l'ensemble des sous-groupes paraboliques semi-standards de H' et l'ensemble des sous-groupes paraboliques relativement standards de G'. Cependant, il y a quelques problèmes de rationalité supplémentaires dans ce cas.

2.2. Quelques résultats locaux pour la comparaison de termes semi-simples réguliers. Les termes $J_{\mathfrak{o}}(\eta, f)$ and $J_{\mathfrak{o}}(f')$ ci-dessus associés aux classes semi-simples régulières mais non elliptiques $\mathfrak{o} \in \mathcal{O}$ et $\mathfrak{o} \in \mathcal{O}'$ respectivement, qui sont des intégrales orbitales pondérées globales, sont les premiers nouveaux termes que l'on doit étudier et comparer pour les applications des formules des traces de Jacquet-Guo. La principale difficulté pour étudier ces distributions globales par rapport à celles de [60] ou [61] est que on est confronté à des distributions non invariantes sous la conjugaison de $H(\mathbb{A})$ ou $H'(\mathbb{A})$. C'est proche de la situation de la formule des traces d'Arthur-Selberg classique. Alors que la procédure standard mais assez difficile est de rendre notre formule des traces invariante comme Arthur l'a fait dans [6] et [7], une autre manière proposée par Labesse dans [37] est de comparer directement les distributions non invariantes. On suivra cette dernière approche. Les intégrales orbitales pondérées locales correspondantes sont les principaux objets de §10 du chapitre 3, le chapitre 4 et le chapitre 5.

Soit E/F une extension quadratique de corps locaux non-archimédiens de caractéristique zéro. Soit η le caractère quadratique de $F/NE^\times$ attaché à E/F, où $NE^\times$ désigne la norme de $E^\times$. On définit des paires symétriques comme dans le cas global. Notons $\mathfrak{s}_{\mathrm{rs}}$ l'ensemble des éléments semi-simples réguliers dans $\mathfrak{s}$. On note $\mathcal{C}_c^\infty(\mathfrak{s}(F))$ (resp. $\mathcal{C}_c^\infty(\mathfrak{s}'(F))$) l'espace des fonctions localement constantes à support compact sur $\mathfrak{s}(F)$ (resp. $\mathfrak{s}'(F)$). Soit M un sous-groupe de Levi ω-stable de G, c'est-à-dire qu'il est un facteur de Levi d'un sous-groupe parabolique ω-stable. Soit $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Pour toute $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, on définit l'intégrale orbitale pondérée locale

$$J_M^G(\eta, X, f) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F) \backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) \eta(\det(x)) v_M^G(x) dx,$$

où $|D^{\mathfrak{s}}(X)|_F$ est le discriminant de Weyl et $v_M^G(x)$ est la fonction de poids locale correspondante. De même, soit M' un sous-groupe de Levi de H' et $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, où $\widetilde{M'}$ est le sous-groupe de Levi de G' tel que $\widetilde{M'} \cap H' = M'$. Pour toute $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, on définit l'intégrale orbitale pondérée locale

$$J_{M'}^{H'}(Y, f') := |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F) \backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y)) v_{M'}^{H'}(x) dx,$$

où $|D^{\mathfrak{s}'}(Y)|_F$ est le discriminant de Weyl et $v_{M'}^{H'}(x)$ est la fonction de poids locale correspondante.

Il y a une application injective $M' \mapsto M$ de l'ensemble des sous-groupes de Levi de H' dans l'ensemble des sous-groupes de Levi ω-stables de G. On supposera que G et G' ont la même dimension n^2. Comme $\mathfrak{s}//H \simeq \mathbf{A}^n \simeq \mathfrak{s}'//H'$, si $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ et $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$ ont la même image dans $\mathbf{A}^n$, on dit qu'ils ont des orbites associées. Plus généralement, fixons une paire de sous-groupes de Levi associés M et M' respectivement de G et H'. On a la notion d'orbites M-associées définies par blocs pour $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ et $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.

2.2.1. ***Le lemme fondamental pondéré.*** On obtient le premier résultat sur la comparaison non invariante des formules des traces de Jacquet-Guo dans §10 du chapitre 3. Dans la comparaison des côtés géométriques des formules des traces de Jacquet-Guo, un cas important est appelé le lemme fondamental. Il dit en gros que en presque toute place non ramifiée, certaines fonctions basiques sur $G(F)$ et $G'(F)$ devraient avoir des intégrales orbitales relatives locales associées sur des orbites associées. Guo [23] l'a prouvé pour les unités d'algèbres sphériques de Hecke à l'aide du lemme fondamental de changement de base pour les algèbres de Hecke sphériques complètes pour GL_n connu par Kottwitz [36, lemme 8.8] et Arthur-Clozel [10, théorème 4.5 du chapitre 1]. Une version infinitésimale [58, lemme 5.18] a été utilisée par Zhang pour prouver le transfert lisse des intégrales orbitales ordinaires pour les formules des traces de Jacquet-Guo en suivant la même philosophie du travail de Waldspurger [52] sur le transfert endoscopique.

Inspiré de [37, définition III.3.2], on définit dans §10 du chapitre 3 la notion de paires "fortement associées" de fonctions localement constantes à support compact sur $\mathfrak{s}(F)$ et $\mathfrak{s}'(F)$. En gros, deux fonctions sont fortement associées si elles ont des intégrales orbitales pondérées associées sur des orbites associées. En presque toute place non ramifiée, (G', H') est isomorphe à $(GL_{2n}, \mathrm{Res}_{E/F} GL_{n,E})$ et $\mathfrak{s}'(F) \simeq \mathfrak{gl}_n(E)$. Notons $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) l'anneau des entiers de F (resp. E). Soit f_0 et f'_0 les fonctions caractéristiques respectivement de $\mathfrak{s}(\mathcal{O}_F) \simeq (\mathfrak{gl}_n \oplus \mathfrak{gl}_n)(\mathcal{O}_F)$ et $\mathfrak{s}'(\mathcal{O}_F) \simeq \mathfrak{gl}_n(\mathcal{O}_E)$. Parce que l'on obtient les mêmes poids pour les cas de Jacquet-Guo que ceux des formules des traces tordues, on peut réduire la version pondérée de [58, lemme 5.18] ci-dessous au travail de Labesse [37] sur le changement de base pour GL_n. On montre qu'en presque toute place non ramifiée v, f_0 et f'_0 sont fortement associées.

Théorème 2.7 (voir le théorème 10.9 du chapitre 3 pour un énoncé précis et général). *Soient M et M' une paire de sous-groupes de Levi associés respectivement de G et H'. On a*

(1) si $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ et $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ ont des orbites M-associées, alors

$$\kappa(X) J_M^G(\eta, X, f_0) = J_{M'}^{H'}(Y, f'_0),$$

où $\kappa(X)$ est un facteur de transfert;

(2) si $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ satisfait $\det(AB) \notin NE^\times$, alors

$$J_M^G(\eta, X, f_0) = 0.$$

2.2.2. ***Une formule des traces locale infinitésimale.*** Notre prochain objectif est de prouver le transfert non invariant des intégrales orbitales pondérées en utilisant le lemme fondamental pondéré suivant la stratégie de [14] et [15] sur le changement de base stable. Pour y parvenir, on doit préparer quelques résultats sur l'analyse harmonique locale comme dans [51]. On prouve une variante infinitésimale des formules des traces locales invariantes pour le cas de Jacquet-Guo dans le chapitre 4 suivant [51] et [8].

On se concentre ici sur le cas de (G, H) pour l'illustration. On définit l'intégrale orbitale pondérée locale (H, η)-invariante $I_M^G(\eta, X, f)$ par le processus standard d'Arthur à partir de l'intégrale orbitale pondérée locale non invariante ci-dessus $J_M^G(\eta, X, f)$. Pour $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, on définit (voir §3.2 du chapitre 4 pour les notations)

$$I^G(\eta, f, f') := \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} I_M^G(\eta, X, \hat{f}) I_G^G(\eta, X, f') dX.$$

Le principal résultat du chapitre 4 est la formule des traces locale invariante suivante.

Théorème 2.8 (voir le théorème 9.1 du chapitre 4). *On a l'égalité*

$$I^G(\eta, f, f') = I^G(\eta, f', f).$$

Elle est déduite d'une formule des traces locale non invariante (voir le théorème 5.3 du chapitre 4), qui est essentiellement une conséquence de la formule de Plancherel combinée avec un processus de

troncature. Certains autres résultats locaux, y compris la finitude de Howe pour les intégrales orbitales pondérées et la représentabilité de la transformée de Fourier des intégrales orbitales pondérées, sont obtenus lors de la démonstration.

Proposition 2.9 (voir le corollaire 6.6 du chapitre 4). *Soient r un sous-groupe compact ouvert de $\mathfrak{s}(F)$, M un sous-groupe de Levi ω-stable de G et $\sigma \subseteq (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Notons $\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ le sous-espace de $\mathcal{C}_c^\infty(\mathfrak{s}(F))$ constitué par les fonctions invariantes par translation de r. Supposons qu'il existe un sous-ensemble compact $\sigma_0 \subseteq (\mathfrak{m} \cap \mathfrak{s})(F)$ tel que $\sigma \subseteq \mathrm{Ad}((M \cap H)(F))(\sigma_0)$. Alors il existe un sous-ensemble fini $\{X_i : i \in I\} \subseteq \sigma$ et un sous-ensemble fini $\{f_i : i \in I\} \subseteq \mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ tels que pour tout $X \in \sigma$ et toute $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$, on a l'égalité*

$$J_M^G(\eta, X, f) = \sum_{i \in I} J_M^G(\eta, X_i, f) J_M^G(\eta, X, f_i).$$

Proposition 2.10 (voir la proposition 7.2 du chapitre 4). *Soient M un sous-groupe de Levi ω-stable de G et $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Alors il existe une fonction localement constante $\hat{j}_M^G(\eta, X, \cdot)$ sur $\mathfrak{s}_{\mathrm{rs}}(F)$ telle que*

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), J_M^G(\eta, X, \hat{f}) = \int_{\mathfrak{s}(F)} f(Y) \hat{j}_M^G(\eta, X, Y) |D^{\mathfrak{s}}(Y)|_F^{-1/2} dY.$$

On obtient également une propriété d'annulation à "l'infini" analogue à [14, proposition 2.2].

Proposition 2.11 (voir la proposition 10.1 du chapitre 4). *Soit $M \neq G$ un sous-groupe de Levi ω-stable de G. Soient $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ et $Y \in \mathfrak{s}_{\mathrm{rs}}(F)$. Alors il existe $N \in \mathbb{N}$ tel que si $\lambda \in F^\times$ satisfait $v_F(\lambda) < -N$, on a*

$$\hat{i}_M^G(\eta, \lambda X, Y) = 0,$$

où $\hat{i}_M^G(\eta, X, \cdot)$ est l'analogue (H, η)-invariant de $\hat{j}_M^G(\eta, X, \cdot)$.

Ce sont des résultats d'intérêt indépendant. Des analogues de tous les résultats ci-dessus sont obtenus pour le cas de (G', H').

2.2.3. ***Certaines identités entre transformées de Fourier des intégrales orbitales pondérées.*** Comme mentionné ci-dessus, la prochaine étape de notre recherche est de comparer les termes semi-simples réguliers dans les côtés géométriques des formules des traces de Jacquet-Guo comme dans [14] et [15]. Cela devrait servir d'exemple de comparaison non invariante dans le contexte relatif. On obtient des relations entre transformées de Fourier des intégrales orbitales pondérées dans le chapitre 5 qui généralisent certains principaux résultats de [58] et sont des analogues pour les formules des traces de Jacquet-Guo de [14].

Soient M un sous-groupe de Levi ω-stable de G et $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Notons $\hat{J}_M^G(\eta, X, \cdot)$ (resp. $\hat{I}_M^G(\eta, X, \cdot)$) la transformée de Fourier de la distribution $J_M^G(\eta, X, \cdot)$ (resp. $I_M^G(\eta, X, \cdot)$). Soit $\hat{j}_M^G(\eta, X, \cdot)$ (resp. $\hat{i}_M^G(\eta, X, \cdot)$) la fonction localement constante sur $\mathfrak{s}_{\mathrm{rs}}(F)$ représentant $\hat{J}_M^G(\eta, X, \cdot)$ (resp. $\hat{I}_M^G(\eta, X, \cdot)$). Pour un sous-groupe de Levi M' de H' et $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, on obtient de même les fonctions localement constantes $\hat{j}_{M'}^{H'}(Y, \cdot)$ et $\hat{i}_{M'}^{H'}(Y, \cdot)$ sur $\mathfrak{s}'_{\mathrm{rs}}(F)$.

Les fonctions $\hat{j}_M^G(\eta, X, \cdot)$ sont décomposées comme leurs analogues invariants $\hat{i}_M^G(\eta, X, \cdot)$ et les fonctions de poids v_M^G. La décomposition des fonctions $\hat{j}_{M'}^{H'}(Y, \cdot)$ est similaire. Afin d'obtenir des relations entre $\hat{j}_M^G(\eta, X, \cdot)$ et $\hat{j}_{M'}^{H'}(Y, \cdot)$, qui fait partie de la comparaison non invariante, on se concentrera sur les relations entre $\hat{i}_M^G(\eta, X, \cdot)$ et $\hat{i}_{M'}^{H'}(Y, \cdot)$ dans le chapitre 5.

Fixons une paire de sous-groupes de Levi associés M' et M respectivement de H' et G. Pour $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, on note $\eta(X) := \eta(\det(AB))$. Notre principal résultat dans le chapitre 5 est le suivant.

Théorème 2.12 (voir le corollaire 5.6 et la proposition 5.9 du chapitre 5). *1) Supposons que $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ et $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ ont des orbites M-associées. Supposons que $U \in \mathfrak{s}_{\mathrm{rs}}(F)$ et $V \in \mathfrak{s}'_{\mathrm{rs}}(F)$ ont des orbites associées. Alors on a l'égalité*

$$\gamma_\psi(\mathfrak{h}(F))^{-1} \kappa(X) \kappa(U) \hat{i}_M^G(\eta, X, U) = \gamma_\psi(\mathfrak{h}'(F))^{-1} \hat{i}_{M'}^{H'}(Y, V),$$

où $\gamma_\psi(\mathfrak{h}(F))$ et $\gamma_\psi(\mathfrak{h}'(F))$ sont des constantes de Weil (voir la section 2.2 du chapitre 5).

2) Soient $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ et $U \in \mathfrak{s}_{\mathrm{rs}}(F)$. Si $\eta(X) \neq \eta(U)$, alors

$$\hat{i}_M^G(\eta, X, U) = 0.$$

Comme dans [58], on utilise la méthode globale de Waldspurger sur le transfert endoscopique [52] pour montrer 1) et une méthode locale pour montrer 2). Pour montrer 1), on définit une notion d'intégrales orbitales pondérées associées (voir la définition 5.2 du chapitre 5) et prouve que cette propriété commute sous certaines restrictions avec la transformée de Fourier (voir le théorème 5.3 du chapitre 5) . Sa preuve peut être considérée comme une application de presque tous les résultats des chapitres précédents. Ensuite, on peut extraire les relations entre $\hat{\imath}_M^G(\eta, X, \cdot)$ et $\hat{\imath}_{M'}^{H'}(Y, \cdot)$ à l'aide du lemme de Labesse [37, lemme 1.7.1]. Ces étapes sont proches de celles de [14]. Cependant, alors que le lemme fondamental pondéré pour les formes intérieures est tautologique dans *loc. cit.*, la condition d'annulation du lemme fondamental pondéré est ici plus subtile. On utilise la cohomologie galoisienne abélianisée (voir [38]) pour résoudre quelques difficultés techniques (voir §4.3-4.4 et la preuve de la proposition 11.2 du chapitre 5).

CHAPTER 1

Introduction

This PhD project is inspired by a conjecture of Guo-Jacquet [23] which generalises Waldspurger's well-known theorem [50] relating toric periods to central values of automorphic L-functions for GL_2 to higher ranks. It is one of the first interesting examples in the relative Langlands programme initiated by Jacquet and his collaborators and systematically proposed by Sakellaridis-Venkatesh [46]. An effective tool for attacking such kind of problems is the relative trace formula which was first introduced by Jacquet [29] to reprove Waldspurger's result. The aims of this thesis under the supervision of Pierre-Henri Chaudouard include developing infinitesimal variants of global and local Guo-Jacquet trace formulae and establishing necessary local results of comparison for further study of this conjecture. We expect that some methods here would also be useful in other relative trace formulae.

1. Guo-Jacquet conjecture and the relative trace formula approach

Let us briefly recall the Guo-Jacquet conjecture proposed in [23]. Let E/F be a quadratic extension of number fields and η the quadratic character of $\mathbb{A}^\times/F^\times$ attached to it, where $\mathbb{A}$ denotes the ring of adèles of F. Let $G := GL_{2n}$ and let $H := GL_n \times GL_n$ be its subgroup. All the groups considered are defined over F. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. We say that π is H-distinguished if the two linear forms (called "periods") on it

$$\mathcal{P}_H : \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)dh$$

and

$$\mathcal{P}_{H,\eta} : \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)\eta(\det(h))dh$$

are both non-zero, where Z denotes the centre of G. This property is directly related to the non-vanishing of some central L-values by Friedberg-Jacquet's work [22]. We also need to consider another pair of groups. Denote by $X(E)$ the set of isomorphic classes of quaternion algebras D/F in which E embeds. For any $D \in X(E)$, let $G' = GL_{n,D}$ be the algebraic group defined over F whose F-points are $GL_n(D)$ and let $H' = \mathrm{Res}_{E/F}GL_{n,E}$ be its subgroup. Let π' be a cuspidal automorphic representation of $G'(\mathbb{A})$ with trivial central character. We say that π' is H'-distinguished if the linear form on it

$$\mathcal{P}_{H'} : \phi \mapsto \int_{H'(F)Z(\mathbb{A})\backslash H'(\mathbb{A})} \phi(h)dh$$

is not zero, where we identify the centre of G' with Z. One direction of the Guo-Jacquet conjecture says that if π' is H'-distinguished and π is deduced from π' by the Jacquet-Langlands correspondence, then π is H-distinguished. One may also expect a converse at least when n is odd. For $n = 1$, these were known by Waldspurger [50] and reproved by Jacquet [29] via relative trace formulae.

Now we formally describe the relative trace formula approach following Jacquet [29] to this conjecture. Let f_G be a smooth function on $G(\mathbb{A})$ with compact support. As an analogue of Arthur-Selberg's trace formula, the relative trace formula for the case of (G, H) roughly says that there are two ways to write the integral (viewed as a distribution)

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \mathbb{K}_{f_G}(x, y)\eta(\det(x))dxdy, \tag{1.0.1}$$

where $G(\mathbb{A})^1$ denotes the subset of elements in $G(\mathbb{A})$ whose determinants are of absolute value 1 and $\mathbb{K}_{f_G}(x, y) := \sum_{\gamma\in G(F)} f_G(x^{-1}\gamma y)$. The geometric side is expected to be a sum of relative orbital integrals while the spectral side should be an expansion of periods. Similarly, one could imagine another formula for the case of (G', H'). Then the comparison of periods of different pairs of groups predicted by the Guo-Jacquet conjecture is reduced to the comparison of relative orbital integrals, for which we have more tools such as Guo's fundamental lemma [23] and Zhang's smooth transfer [58].

This approach has been adopted by Feigon-Martin-Whitehouse [21] to obtain some partial results. However, we have neglected analytic difficulty in the above discussion. That is to say, an obstruction of the relative trace formula approach to Guo-Jacquet conjecture is the following problem.

PROBLEM 1.1. *The double integral* (1.0.1) *is not convergent and neither are two ways of its expansions.*

The above problem accounts for some restrictive local conditions in the main results of [21] based on a simple relative trace formula. If one hopes to remove these seemingly artificial restriction and obtain information about all cuspidal representations, this problem can not be ignored. Therefore, it is necessary to establish valid and general relative trace formulae for the cases of (G, H) and (G', H') rather than formal or simple ones. Then we are supposed to compare new terms appearing in these formulae apart from ordinary relative orbital integrals.

Such a problem also existed in the classical Arthur-Selberg trace formula and Arthur introduced a truncation process in [3] and [4] to solve it. In the relative setting, Jacquet-Lapid-Rogawski povided us with the first example of the so-called "mixed truncation" in [30], whose rough idea is defining a truncation compatible with different groups concerned. These work and others set good examples for our study of Problem 1.1.

We shall focus on an infinitesimal variant of Problem 1.1 and the comparison of new terms involved in this thesis. It means that we shall work on the tangent space of a symmetric space (called an infinitesimal symmetric space). This would be of interest for at least two reasons. For one thing, it is close to the geometric side of the original relative trace formula and suggests us a way to tackle the original problem, but the spectral side of the relative trace formula is replaced by the Fourier transform of the geometric side where the harmonic analysis is simpler. For another, an infinitesimal variant of Guo-Jacquet trace formulae is also useful in the comparison of relative trace formulae between two cases; for example, its simple version has been used by Zhang [58] to prove the smooth transfer.

2. Main results

2.1. An infinitesimal variant of Guo-Jacquet trace formulae. We first solve an infinitesimal variant of Problem 1.1 in Chapter 2 and Chapter 3. It is the main global result in this thesis.

2.1.1. ***The case of*** (G, H). We remark that the setting in Chapter 2 is a bit more general to include the case in [57], but we shall focus on the pair (G, H) defined above in the introduction. Denote by S the symmetric space G/H. Notice that the double integral (1.0.1) can be formally written as a single integral

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} K_{f_S}(x)\eta(\det(x))dx,$$

where $f_S(x) := \int_{H(\mathbb{A})\cap G(\mathbb{A})^1} f_G(xy)dy$ defines a smooth function on $S(\mathbb{A})$ with compact support and $K_{f_S}(x) := \sum_{\gamma\in S(F)} f_S(x^{-1}\gamma x)$. If one replaces S with the tangent space $\mathfrak{s} \simeq \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ at the neutral element, then Problem 1.1 appears as the divergence of the integral

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_f(x)\eta(\det(x))dx, \tag{2.1.1}$$

where f is a Bruhat-Schwartz function on $\mathfrak{s}(\mathbb{A})$ and $k_f(x) := \sum_{\gamma\in\mathfrak{s}(F)} f(x^{-1}\gamma x)$.

We replace $k_f(x)$ with some explicit $k_f^T(x)$ defined in §4 in Chapter 2 to make (2.1.1) absolutely convergent, where $T \in \mathbb{R}^{2n}$ is a truncation parameter. This key definition combines ideas from [28][61][17] for the decomposition of $H(\mathbb{A})$ and [40] for the decomposition of $\mathfrak{s}(\mathbb{A})$. To describe the geometric expansion, we define a relation of equivalence on $\mathfrak{s}(F)$: two element in $\mathfrak{s}(F)$ are equivalent if and only if they lie in the same fibre of the categorical quotient $\mathfrak{s}//H$. Denote by $\mathcal{O}$ the set of classes of equivalence. For all $\mathfrak{o} \in \mathcal{O}$, we define $k_{f,\mathfrak{o}}^T(x)$ similarly by replacing $\mathfrak{s}(F)$ with $\mathfrak{o}$. The main result of Chapter 2 is the following theorem which gives the geometric expansion of (2.1.1) with $k_f(x)$ replaced by $k_f^T(x)$.

THEOREM 2.1 (see Theorem 4.11 in Chapter 2). *For T in a suitable cone in $\mathbb{R}^{2n}$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_{f,\mathfrak{o}}^T(x)\eta(\det(x))dx$$

is absolutely convergent.

Denote by $J_{\mathfrak{o}}^T(\eta, f)$ the integral (2.1.1) with $k_f(x)$ replaced by $k_{f,\mathfrak{o}}^T(x)$. We prove that it is an exponential polynomial in T (see Corollary 5.6 in Chapter 2). Denote by $J_{\mathfrak{o}}(\eta, f)$ its constant term. In the infinitesimal setting, the geometric expansion of the Fourier transform of f (denoted by $\hat{f}$) plays the role of the original spectral side (cf. [13]). Then the infinitesimal variant of Guo-Jacquet trace formula for the case of (G, H) below is deduced from the Poisson summation formula.

THEOREM 2.2 (see Theorem 7.1 in Chapter 2). *For all Bruhat-Schwartz function f on $\mathfrak{s}(\mathbb{A})$, we have the equality*

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta, f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta, \hat{f}).$$

Additionally, for most (namely regular semi-simple) $\mathfrak{o}$, we can write $J_{\mathfrak{o}}(\eta, f)$ as an explicit weighted orbital integral; in particular, for elliptic $\mathfrak{o}$, we see that $J_{\mathfrak{o}}(\eta, f)$ is nothing but the ordinary orbital integral. We say a semi-standard parabolic subgroup of G is ω-stable if it contains $\omega := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$.

THEOREM 2.3 (see Theorem 9.2 in Chapter 2). *Let $\mathfrak{o} \in \mathcal{O}$ be a regular semi-simple class, P_1 an ω-stable relatively standard parabolic subgroup of G and $X_1 \in \mathfrak{o}$ an elliptic element relative to P_1 (defined in Section 9.2 in Chapter 2). For a Bruhat-Schwartz function f on $\mathfrak{s}(\mathbb{A})$, we have*

$$J_{\mathfrak{o}}(\eta, f) = \mathrm{vol}([H_{X_1}]) \cdot \int_{H_{X_1}(\mathbb{A})\backslash H(\mathbb{A})} f(x^{-1}X_1x)v_{P_1}(x)\eta(\det(x))dx,$$

where H_{X_1} denotes the centraliser of X_1 in H, $\mathrm{vol}([H_{X_1}])$ is its associated volume and $v_{P_1}(x)$ is the volume of some convex hull.

The weight $v_{P_1}(x)$ is exactly the restriction to $H(\mathbb{A})$ of Arthur's weight in [3] for $G(\mathbb{A})$. It is interesting that our weighted orbital integral shares the same weight as in the twisted trace formula (see [39, p. 131]) for the twisted space $(GL_n \times GL_n) \rtimes \sigma$, where $\sigma(x, y) := (y, x)$.

2.1.2. ***The case of*** (G', H'). Again we study a more general setting in Chapter 3 than the pair (G', H') above. Actually we deal with a central simple algebra $\mathfrak{g}'$ over F containing E and the centraliser $\mathfrak{h}'$ of E in $\mathfrak{g}'$. We define G' and H' as the groups of invertible elements in $\mathfrak{g}'$ and $\mathfrak{h}'$ respectively. This more general case is suggested by the related local conjecture of Prasad and Takloo-Bighash [44, Conjecture 1] and is necessary for completing the converse direction of Guo-Jacquet conjecture for n even.

Up to conjugation by $G'(F)$, the symmetric pair (G', H') is reduced to one of the two cases below (see Proposition 3.7 in Chapter 3). These two cases can be treated similarly.

Case I: $(G', H') \simeq (GL_{n,D}, \mathrm{Res}_{E/F}GL_{n,D'})$, where D is a central division algebra over F containing E, and $D' := \mathrm{Cent}_D(E)$ is the centraliser of E in D.

Case II: $(G', H') \simeq (GL_{n,D}, \mathrm{Res}_{E/F}GL_{\frac{n}{2},D\otimes_F E})$, where D is a central division algebra over F such that there is no embedding $E \to D$ as F-algebras.

Notice that (G', H') is the same as (G, H) after the base change to an algebraic closure of F containing E. We have similar construction and results for the case of (G', H') as follows. Denote by $H'(\mathbb{A})^1$ the subset of elements in $H'(\mathbb{A})$ whose reduced norms are of absolute value 1. Denote by $\mathcal{O}'$ the set of classes of equivalence on $\mathfrak{s}'(F)$ defined by the categorical quotient $\mathfrak{s}'//H'$. Let f' be a Bruhat-Schwartz function on $\mathfrak{s}'(\mathbb{A})$ and $\mathfrak{o} \in \mathcal{O}'$. For a truncation parameter $T \in \mathbb{R}^n$ and $x \in H'(x)$, we define some truncated kernel $k_{f',\mathfrak{o}}^T(x)$ in §4 in Chapter 3.

THEOREM 2.4 (see Theorem 4.2 in Chapter 3). *For T in a suitable cone in $\mathbb{R}^n$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}'} \int_{H'(F)\backslash H'(\mathbb{A})^1} k_{f',\mathfrak{o}}^T(x)dx$$

is absolutely convergent.

For $\mathfrak{o} \in \mathcal{O}'$, denote by $J_{\mathfrak{o}}^T(f')$ the summand of the last sum. We prove that it is a polynomial in T (see Corollary 5.3 in Chapter 3). Denote by $J_{\mathfrak{o}}(f')$ its constant term. Denote by $\hat{f}'$ the Fourier transform of f'. Here is the infinitesimal variant of Guo-Jacquet trace formula for the case of (G', H').

THEOREM 2.5 (see Theorem 7.1 in Chapter 3). *For all Bruhat-Schwartz function f' on $\mathfrak{s}'(\mathbb{A})$, we have the equality*

$$\sum_{\mathfrak{o}\in\mathcal{O}'} J_{\mathfrak{o}}(f') = \sum_{\mathfrak{o}\in\mathcal{O}'} J_{\mathfrak{o}}(\hat{f}').$$

As before, we can write the regular semi-simple terms as explicit weighted orbital integrals with the same weights as Arthur's in [3]. For $(G', H') = (GL_{n,D}, \mathrm{Res}_{E/F} GL_{n,E})$, we obtain the same weights appearing in the twisted trace formula (see [39, p. 131]) for the twisted space $(\mathrm{Res}_{E/F} GL_{n,E}) \rtimes \sigma'$, where σ' is the nontrivial Galois conjugation.

THEOREM 2.6 (see Theorem 9.2 in Chapter 3). *Let $\mathfrak{o} \in \mathcal{O}'$ be a regular semi-simple class, P_1' a standard parabolic subgroup of H' and $Y_1 \in \mathfrak{o}$ an elliptic element with respect to P_1' (defined in Section 9 in Chapter 3). For all Bruhat-Schwartz function f' on $\mathfrak{s}'(\mathbb{A})$, we have*

$$J_{\mathfrak{o}}(f') = \mathrm{vol}([H'_{Y_1}]) \cdot \int_{H'_{Y_1}(\mathbb{A}) \backslash H'(\mathbb{A})} f'(\mathrm{Ad}(x^{-1})(Y_1)) v_{P'_1}(x) dx,$$

where H'_{Y_1} denotes the centraliser of Y_1 in H', $\mathrm{vol}([H'_{Y_1}])$ is the volume associated to H'_{Y_1} and $v_{P'_1}(x)$ is the volume of some convex hull.

In fact, this case is even simpler than that of (G, H) in some sense; for example, $J_{\mathfrak{o}}^T(f')$ is a pure polynomial in T instead of an exponential polynomial. One reason for the simplicity is that the maximal F-split torus in the centre of H' is the same as that of G'. Moreover, there is a bijection between the set of semi-standard parabolic subgroups of H' and the set of so-called relatively standard parabolic subgroups of G'. However, there are some additional rationality issues in this case.

2.2. Some local results for comparison of regular semi-simple terms. The terms $J_{\mathfrak{o}}(\eta, f)$ and $J_{\mathfrak{o}}(f')$ above associated to regular semi-simple but not elliptic classes $\mathfrak{o} \in \mathcal{O}$ and $\mathfrak{o} \in \mathcal{O}'$ respectively, which are global weighted orbital integrals, are the first new terms that we need to study and compare for the application of Guo-Jacquet trace formulae. The main difficulty to study these global distributions compared to those in [60] or [61] is that we are facing noninvariant distributions under the conjugation of $H(\mathbb{A})$ or $H'(\mathbb{A})$. This is close to the situation of the classical Arthur-Selberg trace formula. While the standard but quite difficult procedure is to make our trace formula invariant as Arthur did in [6] and [7], a different way suggested by Labesse in [37] is to compare directly noninvariant distributions. We shall follow the latter approach. Their corresponding local weighted orbital integrals are the main objects in §10 in Chapter 3, Chapter 4 and Chapter 5.

Let E/F be a quadratic extension of non-archimedean local fields of characteristic zero. Let η be the quadratic character of $F/NE^\times$ attached to E/F, where $NE^\times$ denotes the norm of $E^\times$. We define symmetric pairs as in the global case. Denote by $\mathfrak{s}_{\mathrm{rs}}$ the set of regular semi-simple elements in $\mathfrak{s}$. Denote by $\mathcal{C}_c^\infty(\mathfrak{s}(F))$ (resp. $\mathcal{C}_c^\infty(\mathfrak{s}'(F))$) the space of locally constant and compactly supported functions on $\mathfrak{s}(F)$ (resp. $\mathfrak{s}'(F)$). Let M be an ω-stable Levi subgroup of G, i.e., it is a Levi factor of some ω-stable parabolic subgroup. Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define the local weighted orbital integral

$$J_M^G(\eta, X, f) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F) \backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) \eta(\det(x)) v_M^G(x) dx,$$

where $|D^{\mathfrak{s}}(X)|_F$ is the Weyl discriminant and $v_M^G(x)$ is the corresponding local weight function. Similarly, let M' be a Levi subgroup of H' and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, where $\widetilde{M'}$ is the Levi subgroup of G' such that $\widetilde{M'} \cap H' = M'$. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define the local weighted orbital integral

$$J_{M'}^{H'}(Y, f') := |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F) \backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y)) v_{M'}^{H'}(x) dx,$$

where $|D^{\mathfrak{s}'}(Y)|_F$ is the Weyl discriminant and $v_{M'}^{H'}(x)$ is the corresponding local weight function.

There is an injection $M' \mapsto M$ from the set of Levi subgroups of H' into the set of ω-stable Levi subgroups of G. We shall suppose that G and G' have the same dimension n^2. Since $\mathfrak{s}//H \simeq \mathbf{A}^n \simeq \mathfrak{s}'//H'$, if $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$ have the same image in $\mathbf{A}^n$, we say that they have matching orbits. More generally, fix a pair of matching Levi subgroups M and M' of G and H' respectively. We have the notion of M-matching orbits defined by blocks for $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.

2.2.1. ***The weighted fundamental lemma.*** We provide the first evidence of noninvariant comparison of Guo-Jacquet trace formulae in §10 in Chapter 3. In the comparison of geometric sides of Guo-Jacquet trace formulae, an important case is the so-called fundamental lemma. It roughly says that at almost all unramified places, some basic functions on $G(F)$ and $G'(F)$ should have associated local relative orbital integrals on matching orbits. Guo [23] proved it for the units of spherical Hecke algebras with the help of the base change fundamental lemma for the full spherical Hecke algebras for GL_n known

by Kottwitz [36, Lemma 8.8] and Arthur-Clozel [10, Theorem 4.5 in Chapter 1]. An infinitesimal version [58, Lemma 5.18] was used by Zhang to prove the smooth transfer of ordinary orbital integrals for Guo-Jacquet trace formulae following the same philosophy of Waldspurger's work [52] on the endoscopic transfer.

Inspired by [37, Definition III.3.2], we define in §10 in Chapter 3 the notion of "strongly associated" pairs of locally constant and compactly supported functions on $\mathfrak{s}(F)$ and $\mathfrak{s}'(F)$. Roughly speaking, two functions are strongly associated if they have associated weighted orbital integrals on matching orbits. For almost all unramified places, (G', H') is isomorphic to $(GL_{2n}, \mathrm{Res}_{E/F}GL_{n,E})$ and $\mathfrak{s}'(F) \simeq \mathfrak{gl}_n(E)$. Denote by $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) the ring of integers in F (resp. E). Let f_0 and f_0' be the characteristc functions on $\mathfrak{s}(\mathcal{O}_F) \simeq (\mathfrak{gl}_n \oplus \mathfrak{gl}_n)(\mathcal{O}_F)$ and $\mathfrak{s}'(\mathcal{O}_F) \simeq \mathfrak{gl}_n(\mathcal{O}_E)$ respectively. Because we get the same weights for the cases of Guo-Jacquet as those in twisted trace formulae, we are able to reduce the following weighted version of [58, Lemma 5.18] to Labesse's work [37] on the base change for GL_n. We show that for almost all unramified places v, f_0 and f_0' are strongly associated.

Theorem 2.7 (see Theorem 10.9 in Chapter 3 for a precise and general statement). *Let M and M' be a pair of matching Levi subgroups of G and H' respectively. We have*

(1) if $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits, then

$$\kappa(X) J_M^G(\eta, X, f_0) = J_{M'}^{H'}(Y, f_0'),$$

where $\kappa(X)$ is a transfer factor;

(2) if $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ satisfies $\det(AB) \notin NE^\times$, then

$$J_M^G(\eta, X, f_0) = 0.$$

2.2.2. ***An infinitesimal local trace formula.*** Our next goal is to prove the noninvariant transfer of weighted orbital integrals using the weighted fundamental lemma following the strategy of [14] and [15] on the stable base change. To achieve this, we need to prepare some results of local harmonic analysis as in [51]. We prove an infinitesimal variant of invariant local trace formulae for the case of Guo-Jacquet in Chapter 4 following [51] and [8].

We focus on the case of (G, H) here for illustration. We define the (H, η)-invariant local weighted orbital integral $I_M^G(\eta, X, f)$ by Arthur's standard process from the above noninvariant local weighted orbital integral $J_M^G(\eta, X, f)$. For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define (see §3.2 in Chapter 4 for notations)

$$I^G(\eta, f, f') := \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} I_M^G(\eta, X, \hat{f}) I_G^G(\eta, X, f') dX.$$

The main result in Chapter 4 is the following invariant local trace formula.

Theorem 2.8 (see Theorem 9.1 in Chapter 4). *We have the equality*

$$I^G(\eta, f, f') = I^G(\eta, f', f).$$

It is deduced from a noninvariant local trace formula (see Theorem 5.3 in Chapter 4), which is essentially a consequence of the Plancherel formula combined with a truncation process. Some other local results including Howe's finiteness for weighted orbital integrals and representability of the Fourier transform of weighted orbital integrals are given during the proof.

Proposition 2.9 (see Corollary 6.6 in Chapter 4). *Let r be an open compact subgroup of $\mathfrak{s}(F)$, M be an ω-stable Levi subgroup of G and $\sigma \subseteq (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Denote by $\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ the subspace of $\mathcal{C}_c^\infty(\mathfrak{s}(F))$ consisting of the functions invariant by translation of r. Suppose that there exists a compact subset $\sigma_0 \subseteq (\mathfrak{m} \cap \mathfrak{s})(F)$ such that $\sigma \subseteq \mathrm{Ad}((M \cap H)(F))(\sigma_0)$. Then there exists a finite subset $\{X_i : i \in I\} \subseteq \sigma$ and a finite subset $\{f_i : i \in I\} \subseteq \mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ such that for all $X \in \sigma$ and all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$, we have the equality*

$$J_M^G(\eta, X, f) = \sum_{i \in I} J_M^G(\eta, X_i, f) J_M^G(\eta, X, f_i).$$

Proposition 2.10 (see Proposition 7.2 in Chapter 4). *Let M be an ω-stable Levi subgroup of G and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Then there exists a locally constant function $\hat{\jmath}_M^G(\eta, X, \cdot)$ on $\mathfrak{s}_{\mathrm{rs}}(F)$ such that*

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), J_M^G(\eta, X, \hat{f}) = \int_{\mathfrak{s}(F)} f(Y) \hat{\jmath}_M^G(\eta, X, Y) |D^{\mathfrak{s}}(Y)|_F^{-1/2} dY.$$

We also obtain a vanishing property at "infinity" analogous to [14, Proposition 2.2].

Proposition 2.11 (see Proposition 10.1 in Chapter 4). *Let $M \neq G$ be an ω-stable Levi subgroup of G. Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in \mathfrak{s}_{\mathrm{rs}}(F)$. Then there exists $N \in \mathbb{N}$ such that if $\lambda \in F^\times$ satisfies $v_F(\lambda) < -N$, we have*

$$\hat{i}_M^G(\eta, \lambda X, Y) = 0,$$

where $\hat{i}_M^G(\eta, X, \cdot)$ is the (H, η)-invariant analogue of $\hat{j}_M^G(\eta, X, \cdot)$.

These results are of independent interest. Analogues of all the results above are obtained for the case of (G', H').

2.2.3. ***Certain identities between Fourier transforms of weighted orbital integrals.*** As mentioned above, the next step of our research is to compare regular semi-simple terms in the geometric sides of Guo-Jaquect trace formulae as in [14] and [15]. This should serve as an example of noninvariant comparison in the relative context. We obtain some relations between Fourier transforms of weighted orbital integrals in Chapter 5 which generalise some of the main results in [58] and are analogues for Guo-Jacquet trace formulae of [14].

Let M be an ω-stable Levi subgroup of G and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Denote by $\hat{J}_M^G(\eta, X, \cdot)$ (resp. $\hat{I}_M^G(\eta, X, \cdot)$) the Fourier transform of the distribution $J_M^G(\eta, X, \cdot)$ (resp. $I_M^G(\eta, X, \cdot)$). Let $\hat{j}_M^G(\eta, X, \cdot)$ (resp. $\hat{i}_M^G(\eta, X, \cdot)$) be the locally constant function on $\mathfrak{s}_{\mathrm{rs}}(F)$ representing $\hat{J}_M^G(\eta, X, \cdot)$ (resp. $\hat{I}_M^G(\eta, X, \cdot)$). For a Levi subgroup M' of H' and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, we similarly obtain locally constant functions $\hat{j}_{M'}^{H'}(Y, \cdot)$ and $\hat{i}_{M'}^{H'}(Y, \cdot)$ on $\mathfrak{s}'_{\mathrm{rs}}(F)$.

The functions $\hat{j}_M^G(\eta, X, \cdot)$ is decomposed as their invariant analogues $\hat{i}_M^G(\eta, X, \cdot)$ and weight functions v_M^G. The decomposition for the functions $\hat{j}_{M'}^{H'}(Y, \cdot)$ is similar. In order to obtain relations between $\hat{j}_M^G(\eta, X, \cdot)$ and $\hat{j}_{M'}^{H'}(Y, \cdot)$, which is part of the noninvariant comparison, we shall focus on the relations between $\hat{i}_M^G(\eta, X, \cdot)$ and $\hat{i}_{M'}^{H'}(Y, \cdot)$ in Chapter 5.

Fix a pair of matching Levi subgroups M' and M of H' and G respectively. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, we denote $\eta(X) := \eta(\det(AB))$. Our main result in Chapter 5 is as follows.

Theorem 2.12 (see Corollary 5.6 and Proposition 5.9 in Chapter 5). *1) Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits. Let $U \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $V \in \mathfrak{s}'_{\mathrm{rs}}(F)$ have matching orbits. Then we have the equality*

$$\gamma_\psi(\mathfrak{h}(F))^{-1}\kappa(X)\kappa(U)\hat{i}_M^G(\eta, X, U) = \gamma_\psi(\mathfrak{h}'(F))^{-1}\hat{i}_{M'}^{H'}(Y, V),$$

where $\gamma_\psi(\mathfrak{h}(F))$ and $\gamma_\psi(\mathfrak{h}'(F))$ are Weil constants (see Section 2.2 in Chapter 5).

2) Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in \mathfrak{s}_{\mathrm{rs}}(F)$. If $\eta(X) \neq \eta(U)$, then

$$\hat{i}_M^G(\eta, X, U) = 0.$$

As in [58], we use Waldspurger's global method on the endoscopic transfer [52] to show 1) and a local method to show 2). To show 1), we define a notion of matching weighted orbital integrals (see Definition 5.2 in Chapter 5) and prove that this property commutes with Fourier transform under some restriction (see Theorem 5.3 in Chapter 5). Its proof can be viewed as an application of almost all results in previous chapters. Then we may extract the relations between $\hat{i}_M^G(\eta, X, \cdot)$ and $\hat{i}_{M'}^{H'}(Y, \cdot)$ with the help of Labesse's lemma [37, Lemma 1.7.1]. These steps are close to those in [14]. However, while the weighted fundamental lemma for inner forms is tautological in *loc. cit.*, the vanishing condition of the weighted fundamental lemma here is more subtle. We use abelian Galois cohomology (see [38]) to deal with some technical difficulties (see §4.3-4.4 and the proof of Proposition 11.2 in Chapter 5).

CHAPTER 2

An infinitesimal variant of Guo-Jacquet trace formula: the case of $(GL_{2n,D}, GL_{n,D} \times GL_{n,D})$

We establish an infinitesimal variant of Guo-Jacquet trace formula for the case of $(GL_{2n,D}, GL_{n,D} \times GL_{n,D})$. It is a kind of Poisson summation formula obtained by an analogue of Arthur's truncation process. It consists in the equality of the sums of two types of distributions which are non-equivariant in general: one type is associated to rational points in the categorical quotient, while the other type is the Fourier transform of the first type. For regular semi-simple points in the categorical quotient, we obtain weighted orbital integrals.

1. Introduction

The Guo-Jacquet conjecture proposed in [23] is a possible generalisation in higher dimensions of Waldspurger's well-known theorem on central values of automorphic L-functions for GL_2. We briefly recall it as follows. Let E/F be a quadratic extension of number fields and η the quadratic character of $\mathbb{A}^\times/F^\times$ attached to it, where $\mathbb{A}$ denotes the ring of adèles of F. Consider the group $G = GL_{2n}$ and its subgroup $H = GL_n \times GL_n$ defined over F. Let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. We say that π is H-distinguished if the two linear forms (called "periods") on it

$$\mathcal{P}_H : \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)dh$$

and

$$\mathcal{P}_{H,\eta} : \phi \mapsto \int_{H(F)Z(\mathbb{A})\backslash H(\mathbb{A})} \phi(h)\eta(\det(h))dh$$

are both non-zero, where Z denotes the centre of G. This property is directly connected with the non-vanishing of some central L-values (see Friedberg-Jacquet's work [22]). We also need to deal with another pair of groups. Let $X(E)$ denote the set of isomorphic classes of quaternion algebras D/F in which E embeds. For any $D \in X(E)$, let $G_D = GL_{n,D}$ be the algebraic group defined over F whose F-points are $GL_n(D)$ and $H_D = Res_{E/F}GL_{n,E}$ be its subgroup. Let π_D be a cuspidal automorphic representation of $G_D(\mathbb{A})$ with trivial central character. We say that π_D is H_D-distinguished if the linear form on it

$$\mathcal{P}_{H_D} : \phi \mapsto \int_{H_D(F)Z(\mathbb{A})\backslash H_D(\mathbb{A})} \phi(h)dh,$$

is not zero, where we identify the centre of G_D with Z. One part of the Guo-Jacquet conjecture says that if π_D is H_D-distinguished and π is deduced from π_D by the Jacquet-Langlands correspondence, then π is H-distinguished. We can also expect a converse at least when n is odd. For $n = 1$, these were known by Waldspurger [50] and reproved by Jacquet [29].

Now we formally describe the approach of relative trace formulae following Jacquet [29]. This was adopted by Feigon-Martin-Whitehouse [21] to obtain some partial results. Let f^G be a smooth function on $G(\mathbb{A})$ with compact support. As an analogue of Arthur-Selberg trace formula, the relative trace formula for the case (G, H) roughly says that there are two ways to write the integral (viewed as a distribution)

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \mathbb{K}_{f^G}(x, y)\eta(\det(x))dxdy,$$

where $G(\mathbb{A})^1$ denotes the elements in $G(\mathbb{A})$ with absolute-value-1 determinant and $\mathbb{K}_{f^G}(x, y) = \sum_{\gamma\in G(F)} f^G(x^{-1}\gamma y)$. The geometric side is expected to be a sum of relative (weighted) orbital integrals while the spectral side should be an expansion of periods. Similarly there is also another formula for the

case of (G_D, H_D). Then the comparison of periods of different pairs of groups predicted by the Guo-Jacquet conjecture is reduced to the comparison of relative (weighted) orbital integrals, for which there are already some works such as Guo's fundamental lemma [23] and Zhang's transfer [58].

However, we have neglected analytic difficulty in the above discussion. That is to say, the double integral above is not convergent and neither are two ways of its expansions. This is the reason why some restrictive local conditions are needed in the main results of [21] though they seem kind of artificial. The aim of this article is to solve this kind of problem at the level of Lie algebras for the case of (G, H). Denote by S the symmetric space G/H. Notice that such double integral can be formally written as a single integral

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} K_{f^S}(x)\eta(\det(x))dx,$$

where $f^S(x) = \int_{H(\mathbb{A})\cap G(\mathbb{A})^1} f^G(xy)dy$ defines a smooth function on $S(\mathbb{A})$ with compact support and $K_{f^S}(x) = \sum_{\gamma\in S(F)} f^S(x^{-1}\gamma x)$. Replacing S by its tangent space $\mathfrak{s} \simeq \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ at the neutral element, we are faced with the divergence of the integral

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_f(x)\eta(\det(x))dx,$$

where f is a Bruhat-Schwartz function on $\mathfrak{s}(\mathbb{A})$ and $k_f(x) = \sum_{\gamma\in\mathfrak{s}(F)} f(x^{-1}\gamma x)$.

Our main results can be described as follows.

First of all, as in [3], we replace $k_f(x)$ with some explicit $k_f^T(x)$ (see its definition in (4.0.1) and (4.2.1)) to make the last integral absolutely convergent, where $T \in \mathbb{R}^{2n}$ is a truncation parameter. Moreover, there is a relation of equivalence on $\mathfrak{s}(F)$ defined by the categorical quotient $\mathfrak{s}//H$; we denote by $\mathcal{O}$ the set of classes of equivalence. For each class $\mathfrak{o} \in \mathcal{O}$, we define $k_{f,\mathfrak{o}}^T(x)$ and its integral similarly by replacing $\mathfrak{s}(F)$ with $\mathfrak{o}$. Then we have

$$k_f^T(x) = \sum_{\mathfrak{o}\in\mathcal{O}} k_{f,\mathfrak{o}}^T(x),$$

and prove the following theorem which gives the geometric expansion of

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_f^T(x)\eta(\det(x))dx.$$

Theorem 1.1 (see Theorem 4.11). *For T in a suitable cone in $\mathbb{R}^{2n}$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_{f,\mathfrak{o}}^T(x)\eta(\det(x))dx$$

is absolutely convergent.

Moreover, we see that each summand in the geometric expansion is a sum of products of polynomials and exponential functions in T. In fact, most (namely regular semi-simple) terms are simply polynomial distributions.

Theorem 1.2 (see Corollary 5.6). *For T in a suitable cone in $\mathbb{R}^{2n}$ and each $\mathfrak{o} \in \mathcal{O}$,*

$$J_{\mathfrak{o}}^T(\eta, f) := \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_{f,\mathfrak{o}}^T(x)\eta(\det(x))dx$$

is an exponential polynomial in T. In particular, if $\mathfrak{o}$ is regular semisimple, it is a polynomial in T.

This property allows us to take the constant term $J_{\mathfrak{o}}(\eta, f)$ of $J_{\mathfrak{o}}^T(\eta, f)$ to eliminate the truncation parameter. In the infinitesimal setting, the geometric expansion of the Fourier transform of f plays the role of the original spectral side (cf. [13]). Our infinitesimal variant of Guo-Jacquet trace formula equating the geometric developments of f and its Fourier transform (defined by (3.5.2) and denoted by $\hat{f}$) is the following, which essentially comes from the Poisson summation formula.

Theorem 1.3 (see Theorem 7.1). *For a Bruhat-Schwartz function f on $\mathfrak{s}(\mathbb{A})$, we have the equality*

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta, f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(\eta, \hat{f}).$$

Such a formula should be of interest for at least two reasons. For one thing, it is close to but easier than its analogue for the symmetric space. For another, a simplified version of this formula (see [58, Theorem 8.4 and p. 1875]) has been used in Zhang's proof of the smooth transfer.

Unfortunately, the distributions $J_{\mathfrak{o}}(\eta,\cdot)$ on $\mathfrak{s}(\mathbb{A})$ that we obtained are non-equivariant under the conjugation of $H(\mathbb{A})\cap G(\mathbb{A})^1$ in general, which is close to the situation in [3] and different from that in [61]. In fact, we have the following formula of non-equivariance.

Proposition 1.4 (see Corollary 6.2). *For a Bruhat-Schwartz function f on $\mathfrak{s}(\mathbb{A})$ and $y\in H(\mathbb{A})\cap G(\mathbb{A})^1$, we denote $f^y(x):=f(yxy^{-1})$. Then*

$$J_{\mathfrak{o}}(\eta,f^y)=\eta(\det(y))\sum_{Q}J_{\mathfrak{o}}^{Q}(\eta,f_{Q,y}^{\eta}),$$

where the sum on Q runs over all ω-stable relatively standard parabolic subgroups of G (defined in Section 5.2). Here $J_{\mathfrak{o}}^{Q}(\eta,\cdot)$ is an analogue of $J_{\mathfrak{o}}(\eta,\cdot)$ with G replaced by Q, and $f_{Q,y}^{\eta}$ is defined by (6.0.1) with $s=0$.

Nevertheless, we can write regular semi-simple terms as explicit weighted orbital integrals whose weights are the restriction to $H(\mathbb{A})$ of Arthur's in [3] for $G(\mathbb{A})$.

Theorem 1.5 (see Theorem 9.2). *Let $\mathfrak{o}\in\mathcal{O}$ be a regular semi-simple class, P_1 an ω-stable relatively standard parabolic subgroup of G and $X_1\in\mathfrak{o}$ an elliptic element relative to P_1 (defined in Section 9.2). For a Bruhat-Schwartz function f on $\mathfrak{s}(\mathbb{A})$, we have*

$$J_{\mathfrak{o}}(\eta,f)=\mathrm{vol}([H_{X_1}])\cdot\int_{H_{X_1}(\mathbb{A})\backslash H(\mathbb{A})}f(x^{-1}X_1x)v_{P_1}(x)\eta(\det(x))dx,$$

where H_{X_1} denotes the centraliser of X_1 in H, $\mathrm{vol}([H_{X_1}])$ is its associated volume and $v_{P_1}(x)$ is the volume of some convex hull.

This paper is organised in the following way. Section 2 and 3 are devoted to standard notation in Arthur's work on trace formulae and characterisation of $\mathcal{O}$ in the specific symmetric pair that we consider respectively. We define the truncated kernel $k_{f,\mathfrak{o}}^{T}(x)$ and prove its integrability in Section 4. This key definition is partly inspired by [28] [61] [17] (for the decomposition of groups) and [40] (for the decomposition of linear spaces) apart from Arthur's pioneering work [3] and its Lie algebra variant [13]. Section 5 is about the quantitive behaviour of the distributions that we got with respect to the truncation parameter T. In Section 6, we study their variance under the conjugation of $H(\mathbb{A})\cap G(\mathbb{A})^1$. In Section 7, the infinitesimal Guo-Jacquet trace formula for the case of $(GL_{2n},GL_n\times GL_n)$ is given. Section 8 and 9 aim to express the regular semi-simple distribution as weighted orbital integrals.

Here are two final remarks. Firstly, actually we study the more general symmetric pair $(GL_{p+q,D}, GL_{p,D}\times GL_{q,D})$ instead of $(GL_{2n},GL_n\times GL_n)$ and add an extra term $|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^{s}$ to the integrand in most of this article. Not only do we prefer more general results (including the case considered in [57] for instance) or possible applications (cf. [41] for the study of the first derivative of L-functions), but the study of the case where $p=q$ and $s=0$ also yields consideration on a more general setting. A simple reason for this comes from the structure of the intersection of H and semi-standard Levi subgroups of G. Secondly, there are some similarities between our case and the twisted trace formula (cf. [39]) for $(GL_n\times GL_n)\rtimes\sigma$ where σ exchanges two copies of GL_n. In fact, we obtain the same weights for regular semi-simple orbits. However, we shall see that more parabolic subgroups will be needed to define the truncation here. We shall return to its discussion at the end of this paper.

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2. Notation

2.1. Roots and weights. Let F be a number field and G a reductive group defined over F. Denote by Z_G the centre of G. Fix a minimal Levi F-subgroup M_0 of G. All the following groups are assumed to be defined over F without further mention. We call a parabolic subgroup P of G semi-standard if $M_0\subseteq P$. For any semi-standard parabolic subgroup P of G, we usually write M_P for the Levi factor

containing M_0 and N_P the unipotent radical. Denote by A_P the maximal F-split torus in the centre of M_P. Let $X(M_P)_F$ be the group of characters of M_P defined over F. Then define

$$\mathfrak{a}_P := \mathrm{Hom}_{\mathbb{Z}}(X(M_P)_F, \mathbb{R})$$

and its dual space

$$\mathfrak{a}_P^* := X(M_P)_F \otimes_{\mathbb{Z}} \mathbb{R},$$

which are both $\mathbb{R}$-linear spaces of dimention $\dim(A_P)$. Notice that the restriction $X(M_P)_F \hookrightarrow X(A_P)_F$ induces an isomorphism

$$\mathfrak{a}_P^* \simeq X(A_P)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

Suppose that $P_1 \subseteq P_2$ are a pair of semi-standard parabolic subgroups of G. The restriction $X(M_{P_2})_F \hookrightarrow X(M_{P_1})_F$ induces $\mathfrak{a}_{P_2}^* \hookrightarrow \mathfrak{a}_{P_1}^*$ and its dual map $\mathfrak{a}_{P_1} \twoheadrightarrow \mathfrak{a}_{P_2}$. Denote by $\mathfrak{a}_{P_1}^{P_2}$ the kernel of the latter map $\mathfrak{a}_{P_1} \twoheadrightarrow \mathfrak{a}_{P_2}$. The restriction $X(A_{P_1})_F \twoheadrightarrow X(A_{P_2})_F$ induces $\mathfrak{a}_{P_1}^* \twoheadrightarrow \mathfrak{a}_{P_2}^*$ and its dual map $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{a}_{P_1}$. The latter map $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{a}_{P_1}$ provides a section of the previous map $\mathfrak{a}_{P_1} \twoheadrightarrow \mathfrak{a}_{P_2}$. Thus we have decompositions

$$\mathfrak{a}_{P_1} = \mathfrak{a}_{P_2} \oplus \mathfrak{a}_{P_1}^{P_2}$$

and

$$\mathfrak{a}_{P_1}^* = \mathfrak{a}_{P_2}^* \oplus (\mathfrak{a}_{P_1}^{P_2})^*.$$

When P_1 is a minimal semi-standard parabolic subgroup, since $\mathfrak{a}_{P_1}$ (resp. A_{P_1}) and $\mathfrak{a}_{P_1}^{P_2}$ are independent of the choice of P_1, we write them as $\mathfrak{a}_0$ (resp. A_0) and $\mathfrak{a}_0^{P_2}$ respectively.

For a pair of semi-standard parabolic subgroups $P_1 \subseteq P_2$ of G, write $\Delta_{P_1}^{P_2}$ for the set of simple roots for the action of A_{P_1} on $N_{P_1}^{P_2} := N_{P_1} \cap M_{P_2}$. Notice that $\Delta_{P_1}^{P_2}$ is a basis of $(\mathfrak{a}_{P_1}^{P_2})^*$. Let

$$(\widehat{\Delta}_{P_1}^{P_2})^\vee := \{\varpi_\alpha^\vee : \alpha \in \Delta_{P_1}^{P_2}\}$$

be the basis of $\mathfrak{a}_{P_1}^{P_2}$ dual to $\Delta_{P_1}^{P_2}$. If B is a minimal semi-standard parabolic subgroup contained in P_1, one has the coroot $\beta^\vee$ associated to any $\beta \in \Delta_B^{P_2}$. For every $\alpha \in \Delta_{P_1}^{P_2}$, let $\alpha^\vee$ be the projection of $\beta^\vee$ to $\mathfrak{a}_{P_1}^{P_2}$, where $\beta \in \Delta_B^{P_2}$ whose restriction to $\mathfrak{a}_{P_1}^{P_2}$ is α. Such $\alpha^\vee$ is independent of the choice of B. Define

$$(\Delta_{P_1}^{P_2})^\vee := \{\alpha^\vee : \alpha \in \Delta_{P_1}^{P_2}\},$$

which is a basis of $\mathfrak{a}_{P_1}^{P_2}$. Denote by

$$\widehat{\Delta}_{P_1}^{P_2} := \{\varpi_\alpha : \alpha \in \Delta_{P_1}^{P_2}\}$$

the basis of $(\mathfrak{a}_{P_1}^{P_2})^*$ dual to $(\Delta_{P_1}^{P_2})^\vee$.

For a semi-standard parabolic subgroup P of G, set

$$\mathfrak{a}_P^+ := \{T \in \mathfrak{a}_P : \alpha(T) > 0, \alpha \in \Delta_P^G\}.$$

For $P_1 \subseteq P_2$ as above, define $\tau_{P_1}^{P_2}$ and $\widehat{\tau}_{P_1}^{P_2}$ as the characteristic functions of

$$\{T \in \mathfrak{a}_0 : \alpha(T) > 0, \alpha \in \Delta_{P_1}^{P_2}\}$$

and

$$\{T \in \mathfrak{a}_0 : \varpi(T) > 0, \varpi \in \widehat{\Delta}_{P_1}^{P_2}\}$$

respectively.

2.2. The functions H_P and F^P. Let $\mathbb{A}$ be the ring of adèles of F and $|\cdot|_{\mathbb{A}}$ the product of normalised local absolute values on the group of idèles $\mathbb{A}^*$. Fix a maximal compact subgroup K of $G(\mathbb{A})$ that is admissible relative to M_0 in the sense of [5, p. 9]. In this paper, we choose the standard maximal compact subgroup for inner forms of GL_n (see [54, p. 191 and 199] for example). More concretely, suppose that $G(F) = GL_n(D)$, where D is a central division algebra over F. For every place v of F, fix an isomorphism $D \otimes_F F_v \simeq \mathfrak{gl}_{r_v}(D_v)$, where D_v is a central division algebra over F_v. Under this isomorphism, the completion at v of $G(F)$ is $G_v \simeq GL_{n_v}(D_v)$, where $n_v = nr_v$. For v a finite place of F, let $K_v \simeq GL_{n_v}(\mathcal{O}_{D_v})$, where $\mathcal{O}_{D_v}$ is the ring of integers of D_v; for v an infinite place of F, we choose K_v to be the orthogonal group, unitary group and compact symplectic group (see [25, Chapter 1.2.8] for example) for $G_v \simeq GL_{n_v}(\mathbb{R})$, $GL_{n_v}(\mathbb{C})$ and $GL_{n_v}(\mathbb{H})$ respectively; let $K := \prod_v K_v$. Suppose that P is a semi-standard parabolic subgroup of G. If $m \in M_P(\mathbb{A})$, define $H_P(m) \in \mathfrak{a}_P$ by

$$\langle H_P(m), \chi\rangle = \log(|\chi(m)|_{\mathbb{A}}), \chi \in X(M_P)_F.$$

Write $M_P(\mathbb{A})^1$ for the kernel of H_P and A_P^∞ for the neutral component for the topology of $\mathbb{R}$-manifolds of the group of $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}}A_P$. Then any element $x \in G(\mathbb{A})$ can be written as $x = nmak$, where $n \in N_P(\mathbb{A})$, $m \in M_P(\mathbb{A})^1$, $a \in A_P^\infty$ and $k \in K$. We can define a continuous map $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ by setting $H_P(x) := H_P(a)$ with respect to this decomposition. Notice that H_P induces an isomorphism from A_P^∞ to $\mathfrak{a}_P$. If $P \subseteq Q$ are a pair of semi-standard parabolic subgroups, write

$$A_P^{Q,\infty} := A_P^\infty \cap M_Q(\mathbb{A})^1.$$

Then H_P also induces an isomorphism from $A_P^{Q,\infty}$ to $\mathfrak{a}_P^Q$.

Denote by Ω^G the Weyl group of (G, A_0). In the cases to be considered in this paper, for every $s \in \Omega^G$, we can always choose one representative $\omega_s \in G(F) \cap K$ such that ω_s normalises A_0. In fact, we are dealing with the case of $G = GL_n$ or its inner forms, thus we can choose Ω^G to be the group of permutation matrices. For an F-subgroup H of G and $s \in \Omega^G$, we usually write $sH := \omega_s H \omega_s^{-1}$. Let P_1 and P_2 be a pair of semi-standard parabolic subgroups of G. Denote by $\Omega^G(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ the set (perhaps empty) of distinct isomorphisms from $\mathfrak{a}_{P_1}$ to $\mathfrak{a}_{P_2}$ obtained by restriction of elements in Ω^G. Denote by $\Omega^G(\mathfrak{a}_{P_1}; P_2)$ the subset (perhaps empty) of double classes in $\Omega^{M_{P_2}} \backslash \Omega^G / \Omega^{M_{P_1}}$ of elements $s \in \Omega^G$ such that $s(\mathfrak{a}_{P_1}) \supseteq \mathfrak{a}_{P_2}$. Suppose additionally that P_1 and P_2 contain a common minimal semi-standard parabolic subgroup P_0 of G. We can talk about positive roots with respect to P_0. By [39, Lemme 1.3.6], all $s \in \Omega^G(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ admits a unique representative (still denoted by s) in Ω^G such that $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_{P_0}^{P_2}$. By [39, Lemme 1.3.7], all $s \in \Omega^G(\mathfrak{a}_{P_1}; P_2)$ admits a unique representative (still denoted by s) in Ω^G such that $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_{P_0}^{P_2}$.

From the reduction theory (see [3, p. 941]), we know that there exists a real number $t_0 < 0$ and a compact subset $\omega_B \subseteq N_B(\mathbb{A})M_0(\mathbb{A})^1$ for each minimal semi-standard parabolic subgroup B of G such that for any semi-standard parabolic subgroup P of G containing B, we have

$$G(\mathbb{A}) = P(F)\mathfrak{S}_B^P(\omega_B, t_0).$$

Here the Siegel set $\mathfrak{S}_B^P(\omega_B, t_0)$ is defined by

$$\mathfrak{S}_B^P(\omega_B, t_0) := \omega_B A_B^\infty(P, t_0)K,$$

where

$$A_B^\infty(P, t_0) := \{a \in A_B^\infty : \alpha(H_B(a)) > t_0, \alpha \in \Delta_B^P\}.$$

We shall fix such t_0 and ω_B. Additionally, we are authorised to assume that $\omega_{sB} = \omega_s \omega_B \omega_s^{-1}$ for $s \in \Omega^G$. Moreover, we require that $(M_P(\mathbb{A}) \cap \omega_B, M_P(\mathbb{A}) \cap K, B \cap M_P, t_0)$ will play the role of (ω_B, K, B, t_0) for any semi-standard parabolic subgroup P of G containing B.

Let $B \subseteq P$ and t_0 be as above. For $T \in \mathfrak{a}_0$, define the truncated Siegel set

$$\mathfrak{S}_B^P(\omega_B, t_0, T) := \omega_B A_B^\infty(P, t_0, T)K,$$

where

$$A_B^\infty(P, t_0, T) := \{a \in A_B^\infty(P, t_0) : \varpi(H_B(a) - T) \leq 0, \varpi \in \widehat{\Delta}_B^P\}.$$

Denote by $F_B^P(\cdot, T)$ the characteristic function of the projection of $\mathfrak{S}_B^P(\omega_B, t_0, T)$ to $P(F)\backslash G(\mathbb{A})$.

2.3. Bruhat-Schwartz functions and Haar measures. Write $\mathfrak{g}$ for the Lie algebra of G. For an F-linear subspace $\mathfrak{s}$ of $\mathfrak{g}$, denote by $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ the Bruhat-Schwartz space of $\mathfrak{s}(\mathbb{A})$, namely the $\mathbb{C}$-linear space of functions on $\mathfrak{s}(\mathbb{A})$ generated by $f_\infty \otimes \chi^\infty$, where f_∞ is a Schwartz function on $\mathfrak{s}(F \otimes_{\mathbb{Q}} \mathbb{R})$ and χ^∞ is the characteristic function of an open compact subgroup of $\mathfrak{s}(\mathbb{A}^\infty)$, where we denote by $\mathbb{A}^\infty$ the ring of finite adèles of F.

Let P be a semi-standard parabolic subgroup of G. For every connected subgroup V of N_P (resp. every subspace $\mathfrak{h}$ of $\mathfrak{g}$), choose the unique Haar measure on $V(\mathbb{A})$ (resp. on $\mathfrak{h}(\mathbb{A})$) such that $\mathrm{vol}(V(F)\backslash V(\mathbb{A})) = 1$ (resp. $\mathrm{vol}(\mathfrak{h}(F)\backslash\mathfrak{h}(\mathbb{A})) = 1$). We also take the Haar measure on K such that $\mathrm{vol}(K) = 1$.

Fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_0$ invariant by the group Ω^G and Haar measures on all subspaces of $\mathfrak{a}_0$ compatible with this norm. If $P \subseteq Q$ are a pair of semi-standard parabolic subgroups, we obtain the Haar measures on A_P^∞ and $A_P^{Q,\infty}$ via the isomorphism H_P.

Denote by $\rho_P \in (\mathfrak{a}_P^G)^*$ the half of the sum of weights (with multiplicities) for the action of A_P on $\mathfrak{n}_P$. We choose compatible Haar measures on $G(\mathbb{A})$ and its subgroups by requiring that for any $f \in L^1(G(\mathbb{A}))$,

$$\begin{aligned}\int_{G(\mathbb{A})} f(x)dx &= \int_{N_P(\mathbb{A})}\int_{M_P(\mathbb{A})}\int_K f(nmk)e^{-2\rho_P(H_P(m))}dndmdk \\ &= \int_{N_P(\mathbb{A})}\int_{M_P(\mathbb{A})^1}\int_{A_P^\infty}\int_K f(nmak)e^{-2\rho_P(H_P(a))}dndmdadk.\end{aligned}$$

3. The symmetric pair

Let F be a number field and D a central division algebra over F. Let d be the degree of D, i.e., $\dim_F(D) = d^2$. Denote by $GL_{n,D}$ the reductive group over F whose F-points are $GL_n(D)$. For $x \in GL_n(D)$, we write $\mathrm{Nrd}(x)$ for its reduced norm, $\mathrm{Trd}(x)$ for its reduced trace and Prd_x for its reduced characteristic polynomial. For $x \in GL_p(D) \times GL_q(D)$, denote by x_1 (resp. x_2) its projection to the first (resp. second) component. Until further notice, we shall work in a more general setting than that of Guo-Jacquet for later use, i.e., we shall study the case of $(GL_{p+q,D}, GL_{p,D} \times GL_{q,D})$ and add an additional term $|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s$ in the integral of the modified kernel.

3.1. Groups and linear spaces. Let $G := GL_{p+q,D}$ and $H := GL_{p,D} \times GL_{q,D}$ its subgroup by diagonal embedding. Define an involution θ on G by $\theta(g) = \epsilon g \epsilon^{-1}$, where $\epsilon = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$. Thus $H = G^\theta$, where G^θ denotes the θ-invariant subgroup of G.

Define an anti-involution ι on G by $\iota(g) = \theta(g^{-1})$. Denote by G^ι the ι-invariant subvariety of G. There is a symmetrization map

$$s : G \to G^\iota, s(g) := g\iota(g),$$

by which one can regard the symmetric space $S := G/H$ as a subvariety of G^ι. We see that $H \times H$ acts on G by left and right translation and that H acts on G^ι by conjugation.

Let $\mathfrak{g} := Lie(G)$ and $\mathfrak{h} := Lie(H)$. Denote by $d\theta$ the differential of θ. Thus $\mathfrak{h} = \{X \in \mathfrak{g} : (d\theta)(X) = X\}$. Let $\mathfrak{s}$ be the tangent space of S at the neutral element. We shall always view $\mathfrak{s}$ as a subspace of $\mathfrak{g}$. Then $\mathfrak{s} = \{X \in \mathfrak{g} : (d\theta)(X) = -X\}$ and $\mathfrak{s}(F) = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A \in \mathrm{Mat}_{p\times q}(D), B \in \mathrm{Mat}_{q\times p}(D) \right\} \simeq \mathrm{Mat}_{p\times q}(D) \oplus \mathrm{Mat}_{q\times p}(D)$. There is an $H(F)$-action on $\mathfrak{s}(F)$ by conjugation, i.e., $(h_1, h_2) \cdot (A, B) = (h_1 A h_2^{-1}, h_2 A h_1^{-1})$.

3.2. Semi-simple elements. We say that an element $X \in \mathfrak{s}$ is semi-simple if the orbit $H \cdot X$ is Zariski closed in $\mathfrak{s}$. By a regular element $X \in \mathfrak{s}$, we mean that the stabiliser H_X has minimal dimension.

Proposition 3.1. *An element X of $\mathfrak{s}(F)$ is semi-simple if and only if it is $H(F)$-conjugate to an element of the form*

$$X(A) := \begin{pmatrix} 0 & 0 & 1_m & 0 \\ 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $A \in GL_m(D)$ being semi-simple in the usual sense. More precisely, the set of $H(F)$-conjugacy classes of semi-simple elements of $\mathfrak{s}(F)$ is bijective to the set of pairs $(m, \{A\})$ where $0 \le m \le \min\{p, q\}$ is an integer and $\{A\}$ is a semi-simple conjugacy class in $GL_m(D)$. Moreover, $X(A)$ is regular semi-simple if and only if $m = \min\{p, q\}$ and A is regular semi-simple in $GL_{\min\{p,q\}}(D)$ in the usual sense.

Proof. The case $D = F$ is [31, Proposition 2.1 and Lemma 2.1] while the case $p = q$ is [57, Proposition 5.2]. This proposition is nothing but a slightly more general one combining both cases, whose proofs are similar and still work here. □

Proposition 3.2. *If $p \le q$, an element $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}$ is regular semi-simple if and only if Prd_{AB} is separable and $\mathrm{Prd}_{AB}(0) \neq 0$. If $p > q$, an element $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}$ is regular semi-simple if and only if Prd_{BA} is separable and $\mathrm{Prd}_{BA}(0) \neq 0$.*

Proof. We only consider the case $p \leq q$ since the other case can be deduced by symmetry. We may study the proposition over an algebraic closure $\overline{F}$ of F.

Suppose that Prd_{AB} is separable and $\mathrm{Prd}_{AB}(0) \neq 0$. Let $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = X_s + X_n$ be the Jordan decomposition in $\mathfrak{g}$, where X_s is semi-simple, X_n is nilpotent and $X_s X_n = X_n X_s$. By the uniqueness of the Jordan decomposition, we see that $X_s, X_n \in \mathfrak{s}$. From Proposition 3.1, up to conjugation by H, we may suppose that $X_s = \begin{pmatrix} 0 & 1_{dp} & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $C \in GL_{dp}(\overline{F})$ is semi-simple. Since $X_s X_n = X_n X_s$, simple computation (cf. [31, Lemma 2.1]) shows that $X_n = \begin{pmatrix} 0 & D & 0 \\ DC & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $D \in \mathfrak{gl}_{dp}(\overline{F})$ and $DC = CD$. But X_n is nilpotent, which forces D to be zero because C is invertible. Then $X_n = 0$ and $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = X_s$. We deduce that Prd_C is separable and that $\mathrm{Prd}_C(0) \neq 0$. By linear algebra, C is regular semi-simple in $GL_{dp}(\overline{F})$. Hence, $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ is regular semi-simple by Proposition 3.1.

The other direction is a direct consequence of Proposition 3.1. □

3.3. Invariants. Denote by $\mathfrak{c}$ the affine space $\mathbf{A}^{d\min\{p,q\}}$. Define a morphism $\pi : \mathfrak{s} \to \mathfrak{c}$ by mapping $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}$ to the coefficients of the reduced characteristic polynomial of AB. It is constant on H-orbits. Denote by $\mathfrak{c}_{rs}$ the subset of $(c_i)_{0 \leq i \leq d\min\{p,q\}-1} \in \mathfrak{c}$ such that the polynomial

$$P(\lambda) := \lambda^{d\min\{p,q\}} + \sum_{i=0}^{d\min\{p,q\}-1} c_i \lambda^i$$

is separable and $c_0 \neq 0$. It is a principal Zariski open subset of $\mathfrak{c}$. Denote by $\mathfrak{c}^\times$ the subset of $(c_i)_{0 \leq i \leq d\min\{p,q\}-1} \in \mathfrak{c}$ such that $c_0 \neq 0$. Then $\mathfrak{c}_{rs} \subseteq \mathfrak{c}^\times$.

Proposition 3.3. *The pair $(\mathfrak{c}, \pi)$ defines a categorical quotient of $\mathfrak{s}$ by H over F.*

Proof. It suffices to consider the case $p \leq q$ since the case $p > q$ can be obtained by symmetry.

We first extend the base field to an algebraic closure $\overline{F}$ of F. Then $H_{\overline{F}} \simeq GL_{dp,\overline{F}} \times GL_{dq,\overline{F}}$ and $\mathfrak{s}_{\overline{F}} \simeq \mathrm{Mat}_{dp \times dq,\overline{F}} \oplus \mathrm{Mat}_{dq \times dp,\overline{F}}$. For $(c_i)_{0 \leq i \leq dp-1} \in \mathfrak{c}_{\overline{F}}$, denote by $A((c_i)_{0 \leq i \leq dp-1}) \in GL_{pd}$ its companion matrix

$$A((c_i)_{0 \leq i \leq dp-1}) := \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \ddots & \vdots & -c_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -c_{dp-1} \end{pmatrix}.$$

Define a morphism $\mathfrak{c}_{\overline{F}} \to \mathfrak{s}_{\overline{F}}$ by mapping $(c_i)_{0 \leq i \leq dp-1}$ to

$$\begin{pmatrix} 0 & 1_{dp} & 0 \\ A((c_i)_{0 \leq i \leq dp-1}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is a section of π, so π is surjective. By Propositions 3.2 and 3.1, the fibre of any point in the non-empty open subset $\mathfrak{c}_{\overline{F},rs} \subseteq \mathfrak{c}_{\overline{F}}$ contains exactly one closed orbit. We may use Igusa's criterion (see [43, Theorem 4.13] and Remark 3.4 below) to show that the pair $(\mathfrak{c}_{\overline{F}}, \pi)$ defines a categorical quotient of $\mathfrak{s}_{\overline{F}}$ by $H_{\overline{F}}$.

The morphism $\pi : \mathfrak{s} \to \mathfrak{c}$ defined over F factors through the categorical quotient $\mathrm{Spec}(F[\mathfrak{s}]^H)$ of $\mathfrak{s}$ by H over F. This induces a dual morphism $F[\mathfrak{c}] \to F[\mathfrak{s}]^H$ of F-algebras. We have shown that after the base change to $\overline{F}$, it is an isomorphism of $\overline{F}$-algebras. By Galois descent, we deduce that the morphism $F[\mathfrak{c}] \to F[\mathfrak{s}]^H$ is an isomorphism of F-algebras, i.e., the pair $(\mathfrak{c}, \pi)$ defines a categorical quotient of $\mathfrak{s}$ by H over F. □

Remark 3.4. We notice that $\mathfrak{c}_{\overline{F}}$ can be of dimension 1 (when $D = F$ and $\min\{p,q\} = 1$) in the proof of Proposition 3.3 above, so the first condition in [43, Theorem 4.13] may not be satisfied. However, as is evident from the proof of Igusa's criterion, this condition can be replaced with the surjectivity of π.

The categorical quotient $(\mathfrak{c}, \pi)$ defines a relation of equivalence on $\mathfrak{s}(F)$, where two elements are in the same class if and only if they have the same image under π. We denote by $\mathcal{O}$ the set of equivalent classes for this relation. By Proposition 3.1, two semi-simple elements of $\mathfrak{s}(F)$ belong to the same class of $\mathcal{O}$ if and only if they are conjugate by $H(F)$. Denote by $\mathcal{O}_{rs}$ the subset of $\mathcal{O}$ with images in $\mathfrak{c}_{rs}$. By Proposition 3.2, each class in $\mathcal{O}_{rs}$ is a regular semi-simple $H(F)$-orbit in $\mathfrak{s}(F)$. Denote by $\mathcal{O}^\times$ the subset of $\mathcal{O}$ with images in $\mathfrak{c}^\times$. Then $\mathcal{O}_{rs} \subseteq \mathcal{O}^\times$.

3.4. Relatively standard parabolic subgroups. Fix $\widetilde{P}_0$ a minimal parabolic subgroup of H defined over F and M_0 a Levi factor of $\widetilde{P}_0$ defined over F. Then M_0 is also a Levi subgroup of G defined over F. For a semi-standard parabolic subgroup P of G (namely $M_0 \subseteq P$), we say that P is "relatively standard" if $\widetilde{P}_0 \subseteq P$, i.e., $P \cap H$ is a standard parabolic subgroup of H (namely $\widetilde{P}_0 \subseteq P \cap H$). We shall suppose that $\omega_{\widetilde{P}_0} \subseteq \omega_B$ for all relatively standard minimal parabolic subgroup B of G. Denote by K the standard maximal compact subgroup of $G(\mathbb{A})$ and by $K_H := H(\mathbb{A}) \cap K$ the maximal compact subgroup of $H(\mathbb{A})$. Up to conjugation by $G(F)$, we may assume that M_0 is the subgroup of diagonal matrices in G and that $\widetilde{P}_0$ is the product of groups of upper triangular matrices.

We can describe the embedding $H \hookrightarrow G$ via D-bimodules. Let $V := \langle e_1, \cdots, e_p \rangle_D$ (resp. $W := \langle f_1, \cdots, f_q \rangle_D$) be the free D-bimodule generated by the basis $\{e_1, \cdots, e_p\}$ (resp. $\{f_1, \cdots, f_q\}$). Set $GL(V)$ to be the group of F-linear automorphisms on V, which acts on V on the left. Denote by $GL(V)_D$ the subgroup of $GL(V)$ which respects the right D-module structure on V. Put $G := GL(V \oplus W)_D$ and $H := GL(V)_D \times GL(W)_D$. Then M_0 is the stabiliser in G (or in H) of the D-lines $\langle e_i \rangle_D, 1 \leq i \leq p$ and $\langle f_i \rangle_D, 1 \leq i \leq q$. Suppose that $\widetilde{P}_0$ is the direct product of the stabiliser in $GL(V)_D$ of the flag

$$0 \subsetneq \langle e_1 \rangle_D \subsetneq \langle e_1, e_2 \rangle_D \subsetneq \cdots \subsetneq \langle e_1, \cdots, e_p \rangle_D =: V$$

and the stabiliser in $GL(W)_D$ of the flag

$$0 \subsetneq \langle f_1 \rangle_D \subsetneq \langle f_1, f_2 \rangle_D \subsetneq \cdots \subsetneq \langle f_1, \cdots, f_q \rangle_D =: W.$$

A relative standard parabolic subgroup P of G can be interpretated as the stabiliser in G of the flag

$$\begin{aligned} 0 &\subsetneq \langle e_1, \cdots, e_{p_1}, f_1, \cdots, f_{q_1} \rangle_D \subsetneq \langle e_1, \cdots, e_{p_1}, f_1, \cdots, f_{q_1}, e_{p_1+1}, \cdots, e_{p_1+p_2}, f_{q_1+1}, \cdots, f_{q_1+q_2} \rangle_D \\ &\subsetneq \cdots \subsetneq \langle e_1, \cdots, e_{p_1}, f_1, \cdots, f_{q_1}, \cdots, e_{p-p_l+1}, \cdots, e_p, f_{q-q_l+1}, \cdots, f_q \rangle_D =: V \oplus W, \end{aligned}$$

where $\sum_{i=1}^{l} p_i = p$, $\sum_{i=1}^{l} q_i = q$ and we allow p_i or q_i to be zero. In particular, we have

$$M_P \simeq GL_{p_1+q_1,D} \times \cdots \times GL_{p_l+q_l,D}$$

and

$$M_{P_H} \simeq GL_{p_1,D} \times \cdots \times GL_{p_l,D} \times GL_{q_1,D} \times \cdots \times GL_{q_l,D}.$$

Proposition 3.5. *Let P be a relative standard parabolic subgroup of G. For all $X \in (\mathfrak{m}_P \cap \mathfrak{s})(F)$ and $U \in (\mathfrak{n}_P \cap \mathfrak{s})(F)$, we have*

$$\pi(X) = \pi(X + U).$$

Proof. It is a consequence of [40, Lemma 2.1]. We can also give a direct proof as follows. Let $\overline{F}$ be an algebraic closure of F. For $A \in \mathrm{Mat}_{dp \times dq}(\overline{F})$ and $B \in \mathrm{Mat}_{dq \times dp}(\overline{F})$, we see that

$$\det\left(\lambda I_{d(p+q)} - \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right) = \lambda^{d(q-p)} \det(\lambda^2 I_{dp} - AB).$$

Then for any $X \in \mathfrak{s}(F)$, $\pi(X)$ is determined by the coefficients of the reduced characteristic polynomial of X regarded as an element of $\mathfrak{g}(F)$. The proposition follows from the easy fact: for $X \in \mathfrak{m}_P(F)$ and $U \in \mathfrak{n}_P(F)$, the reduced characteristic polynomial of $X + U$ is equal to that of X. □

Corollary 3.6. *Let P be a relative standard parabolic subgroup of G and $\mathfrak{o} \in \mathcal{O}$. For all subsets $S_1 \subseteq (\mathfrak{m}_P \cap \mathfrak{s})(F)$ and $S_2 \subseteq (\mathfrak{n}_P \cap \mathfrak{s})(F)$, we have $\mathfrak{o} \cap (S_1 \oplus S_2) = (\mathfrak{o} \cap S_1) \oplus S_2$.*

3.5. Fourier transform. Fix a nontrivial unitary character Ψ of $\mathbb{A}/F$. Let $\langle \cdot, \cdot \rangle$ be the non-degenerate $H(\mathbb{A})$-invariant bilinear form on $\mathfrak{s}(\mathbb{A})$ defined by

$$\forall X_1, X_2 \in \mathfrak{s}(\mathbb{A}), \langle X_1, X_2 \rangle := \mathrm{Trd}(X_1 X_2). \tag{3.5.1}$$

For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, its Fourier transform $\hat{f} \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ is defined by

$$\forall \widehat{X} \in \mathfrak{s}(\mathbb{A}), \hat{f}(\widehat{X}) := \int_{\mathfrak{s}(\mathbb{A})} f(X) \Psi(\langle X, \widehat{X} \rangle) dX. \tag{3.5.2}$$

4. Integrability of the modified kernel

Fix a minimal semi-standard parabolic subgroup P_0 of G. For any semi-standard parabolic subgroup P of G and $T \in \mathfrak{a}_0$, denote by T_P the projection of sT in $\mathfrak{a}_P$, where s is any element in Ω^G such that $sP_0 \subseteq P$. Notice that this definition is independent of the choice of s. For a semi-standard parabolic subgroup P of G, $x \in H(\mathbb{A})$ and $T \in \mathfrak{a}_0$, define

$$F^P(x,T) := F^P_{sP_0}(x, T_{sP_0}),$$

where s is any element in Ω^G such that $sP_0 \subseteq P$. Note that this definition is independent of the choice of s since we choose all $\omega_s \in G(F) \cap K$. In fact, for any minimal semi-standard parabolic subgroup $B \subseteq P$, $F^{sP}_{sB}(x,T) = F^P_B(\omega_s^{-1}x, s^{-1}T)$ and $F^P_B(\cdot, T)$ is left $M_P(F)$-invariant.

Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, P be a relatively standard parabolic subgroup of G and $\mathfrak{o} \in \mathcal{O}$. Write $P_H := P \cap H$. For $x \in M_{P_H}(F)N_{P_H}(\mathbb{A})\backslash H(\mathbb{A})$, define

$$k_{f,P,\mathfrak{o}}(x) := \sum_{X \in \mathfrak{m}_P(F)\cap \mathfrak{o}} \int_{(\mathfrak{n}_P \cap \mathfrak{s})(\mathbb{A})} f(x^{-1}(X+U)x)dU,$$

and for $x \in H(F)\backslash H(\mathbb{A})$, define

$$k^T_{f,\mathfrak{o}}(x) := \sum_{\{P : \widetilde{P}_0 \subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P_H(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x) - T_P) \cdot k_{f,P,\mathfrak{o}}(\delta x). \tag{4.0.1}$$

From [3, Lemma 5.1], we know that the sum over $\delta \in P_H(F)\backslash H(F)$ is finite.

4.1. Reduction theory. There is a $T_+ \in \mathfrak{a}^+_{P_0}$ such that [3, Lemma 6.4] holds for $T \in T_+ + \mathfrak{a}^+_{P_0}$. We shall fix such a T_+ and say that such T is sufficiently regular.

Lemma 4.1. *For all relatively standard parabolic subgroup Q of G, sufficiently regular T and $x \in H(\mathbb{A})$, we have*

$$\sum_{\{P : \widetilde{P}_0 \subseteq P \subseteq Q\}} \sum_{\delta \in P_H(F)\backslash Q_H(F)} F^P(\delta x, T)\tau^Q_P(H_P(\delta x) - T_P) = 1.$$

This is an analogue of [61, Proposition 2.3] whose proof relies on [28, (2.5) in p. 674] (cf. Lemma 4.8 below). It is essentially a restricted form to H from [3, Lemma 6.4] for G. We can give a proof close to the steps in an early version of [61], which reflects that a main complexity of the truncation here arises from the fact that none of the Siegel sets of H is contained in any Siegel set of G, as mentioned in [28]. However, we shall adopt alternatively the point of view in [17] to give a more conceptual proof here, which might be useful in other relative trace formulae as well.

First we introduce a variant (see [17, §1.5]) of some concepts and results in [17, §2] without reproducing proofs. We say that a semi-standard parabolic subgroup Q of G is standard if $P_0 \subseteq Q$. For $P \subseteq Q$ a pair of standard parabolic subgroups of G, denote by ρ^Q_P the half of the sum of weights (with multiplicities) for the action of A_P on $\mathfrak{n}_P \cap \mathfrak{m}_Q$. We denote by $\overline{\mathfrak{a}^+_{P_0}}$ the closure of $\mathfrak{a}^+_{P_0}$ in $\mathfrak{a}_0$.

Definition 4.2. *For $g \in G(\mathbb{A})$, Q a standard parabolic subgroup of G and $T \in \overline{\mathfrak{a}^+_{P_0}}$, we define the degree of T-instability of g with respect to Q by the following formula*

$$\deg^Q_{i,T}(g) := \max_{(P,\delta)} \langle \rho^Q_P, H_P(\delta g) - T\rangle$$

where (P,δ) runs over the pairs of a standard parabolic subgroup $P \subseteq Q$ and an element $\delta \in P(F)\backslash Q(F)$.

From [3, Lemma 5.1], we know that the supremum of $\langle \rho^Q_P, H_P(\delta g) - T\rangle$ in the definition is finite and attainable.

Lemma 4.3 (cf. [17, Lemme 2.2.1]). *Let $g \in G(\mathbb{A})$, Q be a standard parabolic subgroup of G and $T \in \overline{\mathfrak{a}^+_{P_0}}$. The following two conditions are equivalent:*

(1) $\deg^Q_{i,T}(g) \leq 0$;

(2) for all parabolic subgroup $P \subseteq Q$, all $\delta \in P(F)\backslash Q(F)$ and all $\varpi \in \widehat{\Delta}^Q_P$, we have $\langle \varpi, H_P(\delta g) - T\rangle \leq 0$.

Definition 4.4. *Let $g \in G(\mathbb{A})$ and $T \in \overline{\mathfrak{a}^+_{P_0}}$. We say that a pair (P,δ) of a standard parabolic subgroup $P \subseteq G$ and an element $\delta \in P(F)\backslash G(F)$ is T-canonical for g if it satisfies the following two conditions:*

(1) $\langle \rho_P^G, H_P(\delta g) - T\rangle = \deg_{i,T}^G(g)$;

(2) for any parabolic subgroup $Q \supseteq P$ *such that* $\langle \rho_Q^G, H_Q(\delta g) - T\rangle = \deg_{i,T}^G(g)$, *we have* $Q = P$.

LEMMA 4.5 (cf. [17, Lemme 2.3.2]). *Let* $g \in G(\mathbb{A})$ *and* $T \in \overline{\mathfrak{a}_{P_0}^+}$. *Then* (P, δ) *is a* T*-canonical pair for* g *if and only if it satisfies the following two conditions:*

(1) $\deg_{i,T}^P(\delta g) \leq 0$;

(2) for any $\alpha \in \Delta_P^G$, *we have* $\langle \alpha, H_P(\delta g) - T\rangle > 0$.

PROPOSITION 4.6 (cf. [17, Proposition 2.4.1]). *Let* $g \in G(\mathbb{A})$ *and* $T \in \overline{\mathfrak{a}_{P_0}^+}$. *Then there exists a unique* T*-canonical pair for* g.

Let $T \in \mathfrak{a}_0$ and Q be a standard parabolic subgroup of G. Define $\widetilde{F}^Q(\cdot, T)$ as the characteristic function of $g \in G(\mathbb{A})$ such that $\deg_{i,T}^Q(g) \leq 0$.

PROPOSITION 4.7 (cf. [17, Proposition 2.5.1]). *For* $g \in G(\mathbb{A})$, Q *a standard parabolic subgroup of* G *and* $T \in \overline{\mathfrak{a}_{P_0}^+}$, *we have*

(1)
$$\sum_{\{P: P_0 \subseteq P \subseteq Q\}} \sum_{\delta \in P(F)\backslash Q(F)} \widetilde{F}^P(\delta g, T)\tau_P^Q(H_P(\delta g) - T_P) = 1;$$

(2)
$$\widetilde{F}^Q(g, T) = \sum_{\{P: P_0 \subseteq P \subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta \in P(F)\backslash Q(F)} \widehat{\tau}_P^Q(H_P(\delta g) - T_P).$$

Since we have similar formulae for $F^Q(\cdot, T)$ for sufficiently regular T (see [3, Lemma 6.4]), we know that $\widetilde{F}^Q(\cdot, T) = F_{P_0}^Q(\cdot, T)$ for such T. Now we can return to the proof of Lemma 4.1.

PROOF OF LEMMA 4.1. It is noticeable that the identity is reduced to its analogues for semi-standard Levi factors of Q, which is a product of $GL_{p_i+q_i,D}$ whose intersection with H is $GL_{p_i,D} \times GL_{q_i,D}$. By induction on the rank of G, it suffices to prove the identity for $Q = G$.

For a standard parabolic subgroup P of G, fix a set of representatives $\Omega_{P,G}$ in $\{s \in \Omega^G | \widetilde{P}_0 \subseteq s^{-1}P\}$ for the relation $s_1 \sim s_2$ if and only if $s_2 s_1^{-1} \in \Omega^{M_P}$. We can rewrite the equality in the lemma as
$$\sum_{\{P: P_0 \subseteq P\}} \sum_{s \in \Omega_{P,G}} \sum_{\delta \in (s^{-1}P)_H(F)\backslash H(F)} F_{P_0}^P(\omega_s \delta x, T_{P_0})\tau_P^G(H_P(\omega_s \delta x) - T_P) = 1.$$
In fact, this follows from
$$F^{s^{-1}P}(\delta x, T) = F_{s^{-1}P_0}^{s^{-1}P}(\delta x, T_{s^{-1}P_0}) = F_{P_0}^P(\omega_s \delta x, T_{P_0})$$
and
$$\tau_{s^{-1}P}^G(H_{s^{-1}P}(\delta x) - T_{s^{-1}P}) = \tau_P^G(H_P(\omega_s \delta x) - T_P).$$
Combining the double sums over s and δ, we claim that the equality above is equivalent to
$$\sum_{\{P: P_0 \subseteq P\}} \sum_{\delta \in P(F)\backslash P(F)\Omega^G H(F)} F_{P_0}^P(\delta x, T_{P_0})\tau_P^G(H_P(\delta x) - T_P) = 1.$$
In fact, for any $s \in \Omega_{P,G}$, consider the map
$$(s^{-1}P)_H(F)\backslash H(F) \to P(F)\backslash P(F)\Omega^G H(F), \delta \mapsto \omega_s \delta.$$
Firstly, it is well-defined: if $\delta_1 = \omega_s^{-1} p \omega_s \delta_2$ with $p \in P(F)$, then $\omega_s \delta_1 = p\omega_s \delta_2$. Secondly, it is injective: if $\omega_s \delta_1 = p\omega_s \delta_2$ with $p \in P(F)$, then $\delta_1 = \omega_s^{-1} p \omega_s \delta_2$ with $\omega_s^{-1} p \omega_s = \delta_1 \delta_2^{-1} \in (s^{-1}P)_H(F)$. Thirdly, for $s_1 \neq s_2$ in $\Omega_{P,G}$, we have $\omega_{s_1}\delta_1 \neq p\omega_{s_2}\delta_2$ with $p \in P(F)$: otherwise, $s_1^{-1}P = (\delta_2\delta_1^{-1})^{-1}(s_2^{-1}P)(\delta_2\delta_1^{-1})$ with $\delta_2\delta_1^{-1} \in H(F)$, so $(s_1^{-1}P)_H = (\delta_2\delta_1^{-1})^{-1}(s_2^{-1}P)_H(\delta_2\delta_1^{-1})$, and then $\delta_2\delta_1^{-1} \in (s_1^{-1}P)_H(F) = (s_2^{-1}P)_H(F)$ for both of $(s_1^{-1}P)_H$ and $(s_2^{-1}P)_H$ are standard parabolic subgroups of H, which implies $s_1^{-1}P = s_2^{-1}P$ contradicting $s_1 \neq s_2$. Fourthly, any $\widetilde{s} \in \Omega^G$ appears in the image of the map for some $s \in \Omega_{P,G}$: since $(\widetilde{s}^{-1}P)_H$ is a semi-standard parabolic subgroup of H, there exists an $s_0 \in \Omega^H$ such that $\widetilde{P}_0 \subseteq s_0^{-1}((\widetilde{s}^{-1}P)_H) = (s_0^{-1}(\widetilde{s}^{-1}P))_H = ((\widetilde{s}s_0)^{-1}P)_H$, i.e., $\widetilde{s}s_0 \in \Omega_{P,G}$. To sum up, we finish the argument of the claim.

It suffices to prove an analogue of the last equality by replacing $F_{P_0}^P$ with $\widetilde{F}^P$ for $T \in \overline{\mathfrak{a}_{P_0}^+}$, as they are identical for sufficiently regular T. That is to say, for $x \in H(\mathbb{A}) = G^\theta(\mathbb{A})$, if (P, δ) is the unique

T-canonical pair for x, we need to prove that $\delta \in P(F)\backslash P(F)\Omega^G H(F)$. Recall that $\theta(g) = \epsilon g \epsilon^{-1}$ for $g \in G(\mathbb{A})$, where $\epsilon = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$. Since $\epsilon \in M_0(F) \cap K$, from Lemma 4.5, we deduce that (P, δ) is the unique T-canonical pair for $g \in G(\mathbb{A})$ if and only if $(P, \theta(\delta))$ is the unique T-canonical pair for $\theta(g)$. In particular, if (P, δ) is the unique T-canonical pair for $x \in H(\mathbb{A})$, we have $\delta = \theta(\delta)$. Denote by δ_0 a representative of $\delta \in P(F)\backslash G(F)$. Then $\delta_0 \epsilon \delta_0^{-1} \in P(F)$.

Suppose that $\delta_0 \epsilon \delta_0^{-1} = mu$, where $m \in M_P(F)$ and $u \in N_P(F)$. Both of mu and m are semi-simple in $G(F)$ (in the classical sense) for $(mu)^2 = m^2 = 1$. Applying [3, Lemma 2.1] to the characteristic function of the singleton $\{u\}$, one obtains that mu is $N_P(F)$-conjugate to mu' for some $u' \in N_P(F)$ such that $mu' = u'm$. Since both of mu' and m are semi-simple in $G(F)$, by the uniqueness of Jordan decomposition, we have $u' = 1$, i.e., $\delta_0 \epsilon \delta_0^{-1}$ is $N_P(F)$-conjugate to m. By linear algebra, m is $M_P(F)$-conjugate to a diagonal matrix with entries $\{\pm 1\}$ with expected multiplicities p and q respectively. In sum, $\delta_0 \epsilon \delta_0^{-1}$ is $P(F)$-conjugate to $\omega_s \epsilon \omega_s^{-1}$ for some $s \in \Omega^G$. Suppose that $p_0 \in P(F)$ satisfies $\delta_0 \epsilon \delta_0^{-1} = p_0(\omega_s \epsilon \omega_s^{-1}) p_0^{-1}$. Then $\omega_s^{-1} p_0^{-1} \delta_0 \in G^\theta(F) = H(F)$, i.e., $\delta = P(F)\delta_0 \in P(F)\backslash P(F)\Omega^G H(F)$. □

Lemma 4.8. *Let P be a relatively standard parabolic subgroup of G. For any $a \in A^\infty_{\widetilde{P}_0}(P_H, t_0)$, there exists a relatively standard minimal parabolic subgroup $B \subseteq P$ such that $a \in A^\infty_B(P, t_0)$.*

Proof. This is an analogue of [28, (2.5) in p. 674]. By induction on $\dim(A_P)$, it suffices to prove this assertion for $P = G$.

Let $a \in A^\infty_{\widetilde{P}_0}(H, t_0)$. Then $a = \mathrm{diag}(a_1, \cdots, a_{p+q})$, where $\frac{a_i}{a_{i+1}} > e^{t_0}$ for $1 \le i \le p-1$ and $p+1 \le i \le p+q-1$. In the definition of Siegel sets, we suppose that $t_0 < 0$, so $0 < e^{t_0} < 1$. Note that $A^\infty_{\widetilde{P}_0}(G, t_0) = \{\mathrm{diag}(b_1, \cdots, b_{p+q}) | \frac{b_i}{b_{i+1}} > e^{t_0}, \forall 1 \le i \le p+q-1\}$. Thus we need to show that there exists a permutation $s \in \Omega^G$ such that $s \cdot a = \mathrm{diag}(a_{s^{-1}(1)}, \cdots, a_{s^{-1}(p+q)})$ satisfies the following two conditions:

(1) $s(i) < s(i+1)$ for $1 \le i \le p-1$ and $p+1 \le i \le p+q-1$;

(2) $\frac{a_{s^{-1}(i)}}{a_{s^{-1}(i+1)}} > e^{t_0}$ for $1 \le i \le p+q-1$.

Firstly, we show that one can move a_{p+1} to its left hand side in $(a_1, \cdots, a_{p+q})$ such that both the first $p+1$ elements and the last $q-1$ ones in the new sequence are in "good" order (which means that the quotient of any consecutive pairs is $> e^{t_0}$), while keeping the original relative orders among $(a_1, \cdots, a_p)$ and among $(a_{p+1}, \cdots, a_{p+q})$. If $\frac{a_p}{a_{p+1}} > e^{t_0}$, we are already done (one can take $s = 1$). In general, write

$$i_1 := \max\left\{0, \max\left\{1 \le i \le p \,\middle|\, \frac{a_i}{a_{p+1}} > e^{t_0}\right\}\right\}.$$

When $1 \le i_1 \le p-1$, since $e^{t_0} < 1$, $\frac{a_{i_1+1}}{a_{p+1}} \le e^{t_0}$ implies $\frac{a_{p+1}}{a_{i_1+1}} \ge e^{-t_0} > 1$; there is an $s \in \Omega^G$ such that $s \cdot a = \mathrm{diag}(a_1, \cdots, a_{i_1}, a_{p+1}, a_{i_1+1}, \cdots, a_p, a_{p+2}, \cdots, a_{p+q})$. When $i_1 = 0$, which implies $a_{p+1} > a_1$, there is an $s \in \Omega^G$ such that $s \cdot a = \mathrm{diag}(a_{p+1}, a_1, \cdots, a_p, a_{p+2}, \cdots, a_{p+q})$.

Secondly, we consider moving a_{p+2} as before. One should check that a_{p+2} will not exceed the new place of a_{p+1}, which results from the fact that $\frac{a_{p+1}}{a_{p+2}} > e^{t_0}$. Thus one can move a_{p+1} and a_{p+2} to their left hand side in $(a_1, \cdots, a_{p+q})$ such that both the first $p+2$ elements and the last $q-2$ ones in the new sequence are in "good" order, while still keeping the original relative orders among $(a_1, \cdots, a_p)$ and among $(a_{p+1}, \cdots, a_{p+q})$.

We can finish the argument of our claim by induction on q. □

Proposition 4.9. *Let B be a minimal semi-standard parabolic subgroup of G. Let P be a parabolic subgroup of G containing B. Suppose that T is sufficiently regular. If $m \in \omega_B \cap M_P(\mathbb{A}), a \in A^\infty_B(P, t_0)$ and $k \in K \cap M_P(\mathbb{A})$ satisfy $F^P_B(mak, T_B) = 1$, then $a \in A^\infty_B(P, t_0, T_B)$.*

Proof. It results from Lemma 4.3, since $\widetilde{F}^P_B(\cdot, T) = F^P_B(\cdot, T)$ for sufficiently regular T. Here we write $\widetilde{F}^P_B(\cdot, T)$ for $\widetilde{F}^P(\cdot, T)$ when B plays the role of P_0. □

For a relatively standard parabolic subgroup P of G, denote by $\mathcal{P}(\widetilde{P}_0, P)$ the set of relatively standard minimal parabolic subgroups of G contained in P. For $B \in \mathcal{P}(\widetilde{P}_0, P)$, write

$$A^{G,\infty}_B(P, t_0) := A^\infty_B(P, t_0) \cap G(\mathbb{A})^1$$

and

$$\forall T \in \mathfrak{a}_0, A^{G,\infty}_B(P, t_0, T) := A^\infty_B(P, t_0, T) \cap G(\mathbb{A})^1.$$

Corollary 4.10. *Let P be a relatively standard parabolic subgroup of G. For sufficiently regular T, the following subset of $M_{P_H}(\mathbb{A}) \cap G(\mathbb{A})^1$*

$$\bigcup_{B \in \mathcal{P}(\widetilde{P}_0, P)} (\omega_{\widetilde{P}_0} \cap M_{P_H}(\mathbb{A})) \cdot (A^{\infty}_{\widetilde{P}_0}(P_H, t_0) \cap A^{G,\infty}_B(P, t_0, T_B)) \cdot (K_H \cap M_{P_H}(\mathbb{A}))$$

projects surjectively on $\{m \in M_{P_H}(F) \backslash M_{P_H}(\mathbb{A}) \cap G(\mathbb{A})^1 | F^P(m, T) = 1\}$.

Proof. This is an analogue of [61, Corollaire 2.5]. By Lemma 4.8, the following subset of $M_{P_H}(\mathbb{A}) \cap G(\mathbb{A})^1$

$$\bigcup_{B \in \mathcal{P}(\widetilde{P}_0, P)} (\omega_{\widetilde{P}_0} \cap M_{P_H}(\mathbb{A})) \cdot (A^{\infty}_{\widetilde{P}_0}(P_H, t_0) \cap A^{G,\infty}_B(P, t_0)) \cdot (K_H \cap M_{P_H}(\mathbb{A}))$$

projects surjectively on $M_{P_H}(F) \backslash M_{P_H}(\mathbb{A}) \cap G(\mathbb{A})^1$. Recall that $\omega_{\widetilde{P}_0} \subseteq \omega_B$ for all $B \in \mathcal{P}(\widetilde{P}_0, P)$ and that $K_H \subseteq K$ by our choices (see Section 3.4). Therefore, the statement to be proved follows from Proposition 4.9. □

4.2. Integrability.

Theorem 4.11. *For all sufficiently regular T and all $s \in \mathbb{R}$,*

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{H(F) \backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} |k^T_{f,\mathfrak{o}}(x)| |\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx < \infty,$$

where we write $x = (x_1, x_2) \in GL_{p,D}(\mathbb{A}) \times GL_{q,D}(\mathbb{A})$.

Proof. Let $P_1 \subseteq P_2$ be a pair of relatively standard parabolic subgroups of G. Following [3, §6], for $T_1 \in \mathfrak{a}_{P_1}$, we define the characteristic function

$$\sigma^{P_2}_{P_1}(T_1) := \sum_{\{Q : P_2 \subseteq Q\}} (-1)^{\dim(A_{P_2}/A_Q)} \tau^Q_{P_1}(T_1) \widehat{\tau}^G_Q(T_1),$$

and recall that for $P \supseteq P_1$ a relatively standard parabolic subgroup of G,

$$\tau^P_{P_1}(T_1) \widehat{\tau}^G_P(T_1) = \sum_{\{P_2 : P \subseteq P_2\}} \sigma^{P_2}_{P_1}(T_1).$$

Denote $P_{1,H} := P_1 \cap H$. For $x \in P_{1,H}(F) \backslash H(\mathbb{A})$, we put

$$\chi^T_{P_1,P_2}(x) := F^{P_1}(x, T) \sigma^{P_2}_{P_1}(H_{P_1}(x) - T_{P_1}),$$

and

$$k_{P_1,P_2,\mathfrak{o}}(x) := \sum_{\{P : P_1 \subseteq P \subseteq P_2\}} (-1)^{\dim(A_P/A_G)} k_{f,P,\mathfrak{o}}(x).$$

Using Lemma 4.1 and the left invariance of H_P and $k_{f,P,\mathfrak{o}}$ by $P_H(F)$, we have

$$k^T_{f,\mathfrak{o}}(x) = \sum_{\{P_1,P_2 : \widetilde{P}_0 \subseteq P_1 \subseteq P_2\}} \sum_{\delta \in P_{1,H}(F) \backslash H(F)} \chi^T_{P_1,P_2}(\delta x) k_{P_1,P_2,\mathfrak{o}}(\delta x).$$

Thus

$$\begin{aligned} &\sum_{\mathfrak{o} \in \mathcal{O}} \int_{H(F) \backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} |k^T_{f,\mathfrak{o}}(x)| |\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx \\ \leq &\sum_{\mathfrak{o} \in \mathcal{O}} \sum_{\{P_1,P_2 : \widetilde{P}_0 \subseteq P_1 \subseteq P_2\}} \int_{P_{1,H}(F) \backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) |k_{P_1,P_2,\mathfrak{o}}(x)| |\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx. \end{aligned}$$

It suffices to prove that for any pair of relatively standard parabolic subgroups $P_1 \subseteq P_2$ of G,

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{P_{1,H}(F) \backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) |k_{P_1,P_2,\mathfrak{o}}(x)| |\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx < \infty.$$

If $P_1 = P_2 \neq G$, by [3, Lemma 6.1], we have $\sigma^{P_2}_{P_1} = 0$ and then $\chi^T_{P_1,P_2} = 0$, so the integration is zero. If $P_1 = P_2 = G$, by Corollary 4.10, every $x \in H(F) \backslash H(\mathbb{A}) \cap G(\mathbb{A})^1$ with $F^G(x, T) = 1$ has a representative in the compact subset

$$\bigcup_{B \in \mathcal{P}(\widetilde{P}_0, G)} \omega_{\widetilde{P}_0} \cdot A^{G,\infty}_B(G, t_0, T_B) \cdot K_H,$$

so the integral is bounded by an integral of a continuous function over a compact subset and thus convergent. Therefore, we reduce ourselves to proving the following proposition. □

PROPOSITION 4.12. *Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, $s \in \mathbb{R}$ and $P_1 \subsetneq P_2$ be two relatively standard parabolic subgroups of G. Fix any two positive real numbers ϵ_0 and N. Then there exists a constant C such that*

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x)|k_{P_1,P_2,\mathfrak{o}}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx \leq Ce^{-N\|T\|}$$

for all sufficiently regular T satisfying $\alpha(T) \geq \epsilon_0 \parallel T \parallel$ for any $\alpha \in \Delta^G_{P_0}$.

For $x \in H(F)\backslash H(\mathbb{A})$, define

$$k_{f,G}(x) := \sum_{\mathfrak{o}\in\mathcal{O}} k_{f,G,\mathfrak{o}}(x) = \sum_{X\in\mathfrak{s}(F)} f(x^{-1}Xx)$$

and

$$k^T_f(x) := \sum_{\mathfrak{o}\in\mathcal{O}} k^T_{f,\mathfrak{o}}(x). \tag{4.2.1}$$

COROLLARY 4.13. *Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $s \in \mathbb{R}$. Fix any two positive real numbers ϵ_0 and N. Then there exists a constant C such that*

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} |k^T_f(x) - F^G(x,T)k_{f,G}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx \leq Ce^{-N\|T\|}$$

for all sufficiently regular T satisfying $\alpha(T) \geq \epsilon_0 \parallel T \parallel$ for any $\alpha \in \Delta^G_{P_0}$.

PROOF OF PROPOSITION 4.12. Let P be a relatively standard parabolic subgroup of G such that $P_1 \subseteq P \subseteq P_2$. For any $X \in \mathfrak{m}_P(F) \cap \mathfrak{o}$, there exists a unique relatively standard parabolic subgroup R of G such that $P_1 \subseteq R \subseteq P$ and $X \in (\mathfrak{m}_P(F) \cap \mathfrak{r}(F) \cap \mathfrak{o}) - \left(\bigcup_{P_1\subseteq Q\subsetneq R} \mathfrak{m}_P(F) \cap \mathfrak{q}(F) \cap \mathfrak{o}\right)$. Write

$$\widetilde{\mathfrak{m}}^R_{P_1} := \mathfrak{m}_R - \left(\bigcup_{\{Q:P_1\subseteq Q\subsetneq R\}} \mathfrak{m}_R \cap \mathfrak{q}\right)$$

and

$$\mathfrak{n}^P_R := \mathfrak{n}_R \cap \mathfrak{m}_P.$$

By Corollary 3.6, we have

$$(\mathfrak{m}_P(F) \cap \mathfrak{r}(F) \cap \mathfrak{o}) - \left(\bigcup_{P_1\subseteq Q\subsetneq R} \mathfrak{m}_P(F) \cap \mathfrak{q}(F) \cap \mathfrak{o}\right) = (\widetilde{\mathfrak{m}}^R_{P_1}(F) \cap \mathfrak{o}) \oplus ((\mathfrak{n}^P_R \cap \mathfrak{s})(F)).$$

Hence

$$\begin{aligned} k_{f,P,\mathfrak{o}}(x) &= \sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \int_{\mathfrak{n}_P\cap\mathfrak{s}(\mathbb{A})} f(x^{-1}(X+U)x)dU \\ &= \sum_{\{R:P_1\subseteq R\subseteq P\}} \sum_{\xi\in\widetilde{\mathfrak{m}}^R_{P_1}(F)\cap\mathfrak{o}} \sum_{X\in(\mathfrak{n}^P_R\cap\mathfrak{s})(F)} \int_{(n_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+X+U)x)dU. \end{aligned}$$

Denote by $\overline{P}$ the parabolic subgroup of G opposite to P and write

$$\overline{\mathfrak{n}}^P_R := \mathfrak{n}_{\overline{R}} \cap \mathfrak{m}_P.$$

Note that the restriction of $\langle\cdot,\cdot\rangle$ (defined in (3.5.1)) to $((\mathfrak{n}^P_R\cap\mathfrak{s})(\mathbb{A}))\times((\overline{\mathfrak{n}}^P_R\cap\mathfrak{s})(\mathbb{A}))$ is also non-degenerate. For any $\xi \in (\mathfrak{m}_R \cap \mathfrak{s})(\mathbb{A})$, applying the Poisson summation formula to the Bruhat-Schwartz function $\int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+\cdot+U)x)dU$, we get

$$\sum_{X\in(\mathfrak{n}^P_R\cap\mathfrak{s})(F)} \int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+X+U)x)dU = \sum_{\widehat{X}\in(\overline{\mathfrak{n}}^P_R\cap\mathfrak{s})(F)} \Phi^{x,R}_\xi(\widehat{X}),$$

where the partial Fourier transform $\Phi_\xi^{x,R}$ of $\int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+\cdot+U)x)dU$ is defined by

$$\forall \widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(\mathbb{A}), \Phi_\xi^{x,R}(\widehat{X}) := \int_{(\mathfrak{n}_R^P\cap\mathfrak{s})(\mathbb{A})}\left(\int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+X+U)x)dU\right)\Psi(\langle X,\widehat{X}\rangle)dX.$$

Since $\langle U,\widehat{X}\rangle=0$ for $U\in(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})$ and $\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(\mathbb{A})$, as well as $\mathfrak{n}_R=\mathfrak{n}_P\oplus\mathfrak{n}_R^P$, we have

$$\forall \widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(\mathbb{A}), \Phi_\xi^{x,R}(\widehat{X}) = \int_{(\mathfrak{n}_R\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+U)x)\Psi(\langle U,\widehat{X}\rangle)dU,$$

whose expression is actually independent of P.

To sum up,

$$k_{f,P,\mathfrak{o}}(x) = \sum_{\{R:P_1\subseteq R\subseteq P\}}\sum_{\xi\in\widetilde{\mathfrak{m}}_{P_1}^R(F)\cap\mathfrak{o}}\sum_{\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X}).$$

Hence

$$\begin{aligned}
k_{P_1,P_2,\mathfrak{o}}(x) &= \sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_G)}k_{f,P,\mathfrak{o}}(x)\\
&= \sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_G)}\left(\sum_{\{R:P_1\subseteq R\subseteq P\}}\sum_{\xi\in\widetilde{\mathfrak{m}}_{P_1}^R(F)\cap\mathfrak{o}}\sum_{\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X})\right)\\
&= \sum_{\{R:P_1\subseteq R\subseteq P_2\}}\sum_{\xi\in\widetilde{\mathfrak{m}}_{P_1}^R(F)\cap\mathfrak{o}}\left(\sum_{\{P:R\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_G)}\sum_{\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X})\right)
\end{aligned}$$

For a relatively standard parabolic subgroup P_3 of G containing R, we write

$$(\overline{\mathfrak{n}}_R^{P_3})' := \overline{\mathfrak{n}}_R^{P_3} - \left(\bigcup_{\{Q:R\subseteq Q\subsetneq P_3\}}\overline{\mathfrak{n}}_R^Q\right).$$

Then

$$\sum_{\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X}) = \sum_{\{P_3:R\subseteq P_3\subseteq P\}}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_3})'\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X}).$$

We have

$$\begin{aligned}
&\sum_{\{P:R\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_G)}\sum_{\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X})\\
=&\sum_{\{P:R\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_G)}\sum_{\{P_3:R\subseteq P_3\subseteq P\}}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_3})'\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X})\\
=&(-1)^{\dim(A_{P_2}/A_G)}\sum_{\{P_3:R\subseteq P_3\subseteq P_2\}}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_3})'\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X})\sum_{\{P:P_3\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_{P_2})}.
\end{aligned}$$

From [3, Proposition 1.1], we know that

$$\sum_{\{P:P_3\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_{P_2})} = \begin{cases}1, & \text{if } P_3=P_2;\\ 0, & \text{otherwise.}\end{cases}$$

We obtain

$$\sum_{\{P:R\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_G)}\sum_{\widehat{X}\in(\overline{\mathfrak{n}}_R^P\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X}) = (-1)^{\dim(A_{P_2}/A_G)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X}).$$

Thus

$$k_{P_1,P_2,\mathfrak{o}}(x) = (-1)^{\dim(A_{P_2}/A_G)}\sum_{\{R:P_1\subseteq R\subseteq P_2\}}\sum_{\xi\in\widetilde{\mathfrak{m}}_{P_1}^R(F)\cap\mathfrak{o}}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(F)}\Phi_\xi^{x,R}(\widehat{X}).$$

Now

$$\sum_{\mathfrak{o}\in\mathcal{O}}\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)|k_{P_1,P_2,\mathfrak{o}}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$

$$\leq\sum_{\mathfrak{o}\in\mathcal{O}}\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\left(\sum_{\{R:P_1\subseteq R\subseteq P_2\}}\sum_{\xi\in\widetilde{\mathfrak{m}}^R_{P_1}(F)\cap\mathfrak{o}}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{x,R}_\xi(\widehat{X})|\right)|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$

$$=\sum_{\{R:P_1\subseteq R\subseteq P_2\}}\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\xi\in(\widetilde{\mathfrak{m}}^R_{P_1}\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{x,R}_\xi(\widehat{X})||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx.$$

We reduce ourselves to bounding

$$\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\xi\in(\widetilde{\mathfrak{m}}^R_{P_1}\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{x,R}_\xi(\widehat{X})||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx \tag{4.2.2}$$

for any fixed relatively standard parabolic subgroup R of G such that $P_1\subseteq R\subseteq P_2$.

By the Iwasawa decomposition and our choice of measures, we have

$$\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\xi\in(\widetilde{\mathfrak{m}}^R_{P_1}\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{x,R}_\xi(\widehat{X})||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$

$$=\int_{K_H}\int_{M_{P_{1,H}}(F)\backslash M_{P_{1,H}}(\mathbb{A})\cap G(\mathbb{A})^1}\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}F^{P_1}(m_1,T)\sigma^{P_2}_{P_1}(H_{P_1}(m_1)-T_{P_1})$$

$$\cdot\sum_{\xi\in(\widetilde{\mathfrak{m}}^R_{P_1}\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{n_1m_1k,R}_\xi(\widehat{X})|e^{-2\rho_{P_{1,H}}(H_{P_{1,H}}(m_1))}|\mathrm{Nrd}(m_{1,1})|^s_{\mathbb{A}}dn_1dm_1dk,$$

where we write $m_1=(m_{1,1},m_{1,2})\in GL_{p,D}(\mathbb{A})\times GL_{q,D}(\mathbb{A})$.

By Corollary 4.10, the following subset of $M_{P_{1,H}}(\mathbb{A})\cap G(\mathbb{A})^1$

$$\bigcup_{B\in\mathcal{P}(\widetilde{P}_0,P_1)}(\omega_{\widetilde{P}_0}\cap M_{P_{1,H}}(\mathbb{A}))\cdot(A^\infty_{\widetilde{P}_0}(P_{1,H},t_0)\cap A^{G,\infty}_B(P_1,t_0,T_B))\cdot(K_H\cap M_{P_{1,H}}(\mathbb{A}))$$

projects surjectively on $\{m_1\in M_{P_{1,H}}(F)\backslash M_{P_{1,H}}(\mathbb{A})\cap G(\mathbb{A})^1|F^{P_1}(m_1,T)=1\}$. Hence

$$\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\xi\in(\widetilde{\mathfrak{m}}^R_{P_1}\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{x,R}_\xi(\widehat{X})||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$

$$\leq c_1\sum_{B\in\mathcal{P}(\widetilde{P}_0,P_1)}\int_{K_H}\int_{[cpt\subseteq M_{\widetilde{P}_0}(\mathbb{A})^1]}\int_{A^{G,\infty}_B(P_1,t_0,T_B)}\int_{[cpt\subseteq N^{P_{2,H}}_{\widetilde{P}_0}(\mathbb{A})]}\int_{[cpt\subseteq N_{P_{2,H}}(\mathbb{A})]}\sigma^{P_2}_{P_1}(H_{P_1}(a)-T_{P_1})$$

$$\cdot\sum_{\xi\in(\widetilde{\mathfrak{m}}^R_{P_1}\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}^{P_2}_R)'\cap\mathfrak{s})(F)}|\Phi^{n_2namk,R}_\xi(\widehat{X})|e^{-2\rho_{\widetilde{P}_0}(H_B(a))}|\mathrm{Nrd}(a_1)|^s_{\mathbb{A}}dn_2dndadmdk,$$

where $c_1=\mathrm{vol}(K_H\cap M_{P_{1,H}}(\mathbb{A}))$ is a constant independent of T, and all the compact subsets in the integrals are independent of T. (We use the notation $[cpt\subseteq *]$ for denoting a compact subset in $*$.)

Lemma 4.14. *Let $x\in H(\mathbb{A}),\xi\in(\mathfrak{m}_R\cap\mathfrak{s})(\mathbb{A})$ and $\widehat{X}\in(\overline{\mathfrak{n}}_R\cap\mathfrak{s})(\mathbb{A})$. Let $R\subseteq P_2$ be a pair of relatively standard parabolic subgroups of G. For $n_2\in N_{P_{2,H}}(\mathbb{A})$, we have*

$$\Phi^{n_2x,R}_\xi(\widehat{X})=\Phi^{x,R}_\xi(\widehat{X}).$$

Proof of Lemma 4.14. Let $U_2:=n_2^{-1}\xi n_2-\xi$. Then

$$\Phi^{n_2x,R}_\xi(\widehat{X})=\int_{(\mathfrak{n}_R\cap\mathfrak{s})(\mathbb{A})}f(x^{-1}n_2^{-1}(\xi+U)n_2x)\Psi(\langle U,\widehat{X}\rangle)dU$$

$$=\int_{(\mathfrak{n}_R\cap\mathfrak{s})(\mathbb{A})}f(x^{-1}(\xi+U_2+n_2^{-1}Un_2)x)\Psi(\langle U,\widehat{X}\rangle)dU.$$

Since both U_2 and $n_2^{-1}Un_2-U$ belong to $(\mathfrak{n}_{P_2}\cap\mathfrak{s})(\mathbb{A})$, we have

$$\langle U_2+n_2^{-1}Un_2-U,\widehat{X}\rangle=0,$$

so

$$\Phi_{\xi}^{n_2x,R}(\widehat{X}) = \int_{(\mathfrak{n}_R\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(\xi+U_2+n_2^{-1}Un_2)x)\Psi(\langle U_2+n_2^{-1}Un_2,\widehat{X}\rangle)dU.$$

Because the change of variables $U_2+n_2^{-1}Un_2\mapsto U$ does not change the Haar measure, we obtain

$$\Phi_{\xi}^{n_2x,R}(\widehat{X}) = \Phi_{\xi}^{x,R}(\widehat{X}).$$

□

Using Lemma 4.14, we get

$$\Phi_{\xi}^{n_2namk,R}(\widehat{X}) = \Phi_{\xi}^{namk,R}(\widehat{X}) = \Phi_{\xi}^{aa^{-1}namk,R}(\widehat{X}).$$

By change of variables $a^{-1}Ua\mapsto U$, using the fact that

$$\langle U,\widehat{X}\rangle = \langle a^{-1}Ua, a^{-1}\widehat{X}a\rangle,$$

we have

$$\Phi_{\xi}^{n_2namk,R}(\widehat{X}) = e^{2\rho_{R,+}(H_B(a))}\Phi_{a^{-1}\xi a}^{a^{-1}namk,R}(a^{-1}\widehat{X}a),$$

where we denote by $\rho_{R,+}$ the half of the sum of weights (with multiplicities) for the action of A_0 on $\mathfrak{n}_R\cap\mathfrak{s}$. From the reduction theory (see [3, p. 944]), we know that for a satisfying $\sigma_{P_1}^{P_2}(H_{P_1}(a)-T_{P_1})\neq 0$, $a^{-1}na$ belongs to a compact subset independent of T. In sum,

$$\begin{aligned}
&\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi_{P_1,P_2}^T(x)\sum_{\xi\in(\widetilde{\mathfrak{m}}_{P_1}^R\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(F)}|\Phi_{\xi}^{x,R}(\widehat{X})||\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx\\
\leq & c_2\sum_{B\in\mathcal{P}(\widetilde{P}_0,P_1)}\sup_{y\in\Gamma}\int_{A_B^{G,\infty}(P_1,t_0,T_B)}e^{(2\rho_{R,+}-2\rho_{\widetilde{P}_0})(H_B(a))}\sigma_{P_1}^{P_2}(H_{P_1}(a)-T_{P_1})\\
&\cdot\sum_{\xi\in(\widetilde{\mathfrak{m}}_{P_1}^R\cap\mathfrak{s})(F)}\sum_{\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(F)}|\Phi_{a^{-1}\xi a}^{y,R}(a^{-1}\widehat{X}a)||\mathrm{Nrd}(a_1)|_{\mathbb{A}}^s da,
\end{aligned}$$

where c_2 is a constant independent of T, and Γ is a compact subset independent of T.

Denote by $\mathcal{O}_F$ the ring of integers of F. Fix an F-basis for each weight space for the action of A_0 on $\mathfrak{s}(F)$. Then we are authorised to talk about $\mathcal{O}_F$-points of such a weight space. Since the function $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ is compactly supported on finite places, there exists a positive integer N_1 independent of T such that the sums over $\xi\in(\widetilde{\mathfrak{m}}_{P_1}^R\cap\mathfrak{s})(F)$ and $\widehat{X}\in((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(F)$ can be restricted to lattices $\frac{1}{N_1}(\widetilde{\mathfrak{m}}_{P_1}^R\cap\mathfrak{s})(\mathcal{O}_F)$ and $\frac{1}{N_1}((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(\mathcal{O}_F)$ respectively. In fact, N_1 can be made explicit as in [13, §1.9] by replacing $\mathfrak{m}_R$ and $\mathfrak{n}_R$ in *loc. cit.* with $\mathfrak{m}_R\cap\mathfrak{s}$ and $\mathfrak{n}_R\cap\mathfrak{s}$ respectively.

Fix a Euclidean norm $\|\cdot\|$ on the $\mathbb{R}$-linear space $\mathfrak{s}(F\otimes_{\mathbb{Q}}\mathbb{R})$. Consider a sufficiently large integer $k>0$ to be described precisely at the end of the proof. There exists an integer $m\geq 0$, a real number $k_\alpha\geq 0$ for each $\alpha\in\Delta_B^{P_2}$, and a real number $c_3>0$ satisfying the following conditions (cf. [13, (4.10) in p. 372]):

(1) if $R=P_2$, $m=0$;
(2) for all $\alpha\in\Delta_B^{P_2}-\Delta_B^R$, $k_\alpha\geq k$;
(3) for all $a\in A_B^\infty$,

$$\sum_{\widehat{X}\in\frac{1}{N_1}((\overline{\mathfrak{n}}_R^{P_2})'\cap\mathfrak{s})(\mathcal{O}_F)}\|a^{-1}\widehat{X}a\|^{-m}\leq c_3\prod_{\alpha\in\Delta_B^{P_2}}e^{-k_\alpha\alpha(H_B(a))}. \tag{4.2.3}$$

We can choose a multi-index $\overrightarrow{i}$ whose sum of components is m. We extend the differential operator $\partial^{\overrightarrow{i}}$ on $\mathfrak{s}(F\otimes\mathbb{R})$ to $\mathfrak{s}(\mathbb{A})$ by defining $\partial^{\overrightarrow{i}}(f_\infty\otimes\chi^\infty):=(\partial^{\overrightarrow{i}}f_\infty)\otimes\chi^\infty$ (see Section 2.3). Write

$$\Phi_{\xi}^{x,R,\overrightarrow{i}}(\widehat{X}) := \int_{(\mathfrak{n}_R\cap\mathfrak{s})(\mathbb{A})}(\partial^{\overrightarrow{i}}f)(x^{-1}(\xi+U)x)\Psi(\langle U,\widehat{X}\rangle)dU.$$

Invoking integration by parts, for $\widehat{X}\neq 0$, we get

$$|\Phi_{a^{-1}\xi a}^{y,R}(a^{-1}\widehat{X}a)| = c_4(y)\|a^{-1}\widehat{X}a\|^{-m}|\Phi_{a^{-1}\xi a}^{y,R,\overrightarrow{i}}(a^{-1}\widehat{X}a)|,$$

where $c_4(y)$ is a continuous function of y.

Denote by $\Phi(A_B,\mathfrak{m}_R\cap\mathfrak{s})$ the set of weights of A_B in $\mathfrak{m}_R\cap\mathfrak{s}$. For any $\mu\in\Phi(A_B,\mathfrak{m}_R\cap\mathfrak{s})$, let $\mathfrak{m}_\mu$ be the corresponding weight space. From [53, §41], we know that there exists a function $\phi_\mu\in\mathcal{S}(\mathfrak{m}_\mu(\mathbb{A}))$ for each

$\mu \in \Phi(A_B, \mathfrak{m}_R \cap \mathfrak{s})$ and a function $\phi_{\mathfrak{n}_R \cap \mathfrak{s}} \in \mathcal{S}((\mathfrak{n}_R \cap \mathfrak{s})(\mathbb{A}))$ such that for all $\xi + U \in (\mathfrak{m}_R \cap \mathfrak{s})(\mathbb{A}) \oplus (\mathfrak{n}_R \cap \mathfrak{s})(\mathbb{A})$ and $y \in \Gamma$,

$$|(\partial^{\overrightarrow{i}} f)(y^{-1}(\xi + U)y)| \leq \left(\prod_{\mu \in \Phi(A_B, \mathfrak{m}_R \cap \mathfrak{s})} \phi_\mu(\xi_\mu) \right) \phi_{\mathfrak{n}_R \cap \mathfrak{s}}(U),$$

where ξ_μ denotes the projection of ξ to $\mathfrak{m}_\mu(\mathbb{A})$.

Now

$$\begin{aligned}
&\sum_{\xi \in (\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(F)} \sum_{\widehat{X} \in ((\overline{\mathfrak{n}}_R^{P_2})' \cap \mathfrak{s})(F)} |\Phi^{y,R}_{a^{-1}\xi a}(a^{-1}\widehat{X}a)| \\
=& \sum_{\xi \in \frac{1}{N_1}(\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(\mathcal{O}_F)} \sum_{\widehat{X} \in \frac{1}{N_1}((\overline{\mathfrak{n}}_R^{P_2})' \cap \mathfrak{s})(\mathcal{O}_F)} |\Phi^{y,R}_{a^{-1}\xi a}(a^{-1}\widehat{X}a)| \\
=& \sum_{\xi \in \frac{1}{N_1}(\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(\mathcal{O}_F)} \sum_{\widehat{X} \in \frac{1}{N_1}((\overline{\mathfrak{n}}_R^{P_2})' \cap \mathfrak{s})(\mathcal{O}_F)} c_4(y) \|a^{-1}\widehat{X}a\|^{-m} |\Phi^{y,R,\overrightarrow{i}}_{a^{-1}\xi a}(a^{-1}\widehat{X}a)| \\
\leq& c_5 \sum_{\xi \in \frac{1}{N_1}(\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(\mathcal{O}_F)} \left(\prod_{\mu \in \Phi(A_B, \mathfrak{m}_R \cap \mathfrak{s})} \phi_\mu(\mu(a)^{-1}\xi_\mu) \right) \cdot \sum_{\widehat{X} \in \frac{1}{N_1}((\overline{\mathfrak{n}}_R^{P_2})' \cap \mathfrak{s})(\mathcal{O}_F)} \|a^{-1}\widehat{X}a\|^{-m} \\
\leq& c_5 c_3 \sum_{\xi \in \frac{1}{N_1}(\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(\mathcal{O}_F)} \left(\prod_{\mu \in \Phi(A_B, \mathfrak{m}_R \cap \mathfrak{s})} \phi_\mu(\mu(a)^{-1}\xi_\mu) \right) \cdot \prod_{\alpha \in \Delta_B^{P_2}} e^{-k_\alpha \alpha(H_B(a))},
\end{aligned}$$

where $c_5 := \sup_{y \in \Gamma} c_4(y) \int_{(\mathfrak{n}_R \cap \mathfrak{s})(\mathbb{A})} \phi_{\mathfrak{n}_R \cap \mathfrak{s}}(U) dU$, and we have used (4.2.3) in the last inequality. Thus

$$\begin{aligned}
&\int_{P_{1,H}(F) \backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) \sum_{\xi \in (\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(F)} \sum_{\widehat{X} \in ((\overline{\mathfrak{n}}_R^{P_2})' \cap \mathfrak{s})(F)} |\Phi^{x,R}_\xi(\widehat{X})| |\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx \\
\leq& c_2 c_5 c_3 \sum_{B \in \mathcal{P}(\widetilde{P}_0, P_1)} \int_{A_B^{G,\infty}(P_1, t_0, T_B)} e^{(2\rho_{R,+} - 2\rho_{\widetilde{P}_0})(H_B(a))} \sigma_{P_1}^{P_2}(H_{P_1}(a) - T_{P_1}) \\
&\cdot \sum_{\xi \in \frac{1}{N_1}(\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s})(\mathcal{O}_F)} \left(\prod_{\mu \in \Phi(A_B, \mathfrak{m}_R \cap \mathfrak{s})} \phi_\mu(\mu(a)^{-1}\xi_\mu) \right) \cdot \prod_{\alpha \in \Delta_B^{P_2}} e^{-k_\alpha \alpha(H_B(a))} |\mathrm{Nrd}(a_1)|^s_{\mathbb{A}} da.
\end{aligned}$$

From [13, p. 375], we know that for all $a \in A_B^{G,\infty}(P_1, t_0, T_B)$ satisfying $\sigma_{P_1}^{P_2}(H_{P_1}(a) - T_{P_1}) \neq 0$ and $\alpha \in \Delta_B^{P_2}$, we have $\alpha(H_B(a)) > t_0$. Denote by $\Sigma_B^{\mathfrak{m}_R \cap \mathfrak{s}}$ the positive weights of $\mathfrak{m}_R \cap \mathfrak{s}$ under the action of A_B. Consider the subsets S of $\Sigma_B^{\mathfrak{m}_R \cap \mathfrak{s}}$ with the following property: for all $\alpha \in \Delta_B^R - \Delta_B^{P_1}$, there exists $\mu \in S$ such that its α-coordinate is > 0. Then

$$\begin{aligned}
&\sum_{\xi \in \frac{1}{N_1}\widetilde{\mathfrak{m}}_{P_1}^R \cap \mathfrak{s}(\mathcal{O}_F)} \left(\prod_{\mu \in \Phi(A_B, \mathfrak{m}_R \cap \mathfrak{s})} \phi_\mu(\mu(a)^{-1}\xi_\mu) \right) \\
\leq& \sum_S \left[\prod_{\mu \in S} \left(\sum_{\xi_- \in \frac{1}{N_1}\mathfrak{m}_{-\mu}(\mathcal{O}_F) - \{0\}} \phi_{-\mu}(\mu(a)\xi_-) \right) \right] \left[\prod_{\mu \in \Sigma_B^{\mathfrak{m}_R \cap \mathfrak{s}}} \left(\sum_{\xi_+ \in \frac{1}{N_1}\mathfrak{m}_\mu(\mathcal{O}_F)} \phi_\mu(\mu(a^{-1})\xi_+) \right) \right] \\
&\cdot \left[\sum_{\xi_0 \in \frac{1}{N_1}\mathfrak{m}_0(\mathcal{O}_F)} \phi_0(\xi_0) \right].
\end{aligned}$$

As in [13, p. 373], for the first and third factors, we also have

$$\prod_{\mu \in S} \left(\sum_{\xi_- \in \frac{1}{N_1}\mathfrak{m}_{-\mu}(\mathcal{O}_F) - \{0\}} \phi_{-\mu}(\mu(a)\xi_-) \right) \leq c_6 \prod_{\alpha \in \Delta_B^R - \Delta_B^{P_1}} e^{-k\alpha(H_B(a))}$$

and

$$\sum_{\xi_0 \in \frac{1}{N_1}\mathfrak{m}_0(\mathcal{O}_F)} \phi_0(\xi_0) \leq c_7,$$

where c_6 and c_7 are constants independent of T and a. One may note that our bound for the second factor is slightly different from [13, (4.14) in p. 373]. Actually we get

$$\prod_{\mu \in \Sigma_B^{\mathfrak{m}_R \cap \mathfrak{s}}} \left(\sum_{\xi_+ \in \frac{1}{N_1}\mathfrak{m}_\mu(\mathcal{O}_F)} \phi_\mu(\mu(a^{-1})\xi_+) \right) \leq c_8 e^{(2\rho_{B,+} - 2\rho_{R,+})(H_B(a))},$$

where c_8 is a constant independent of T. However, we claim that this discrepancy will be unimportant when we follow the rest of the proof of [13, p. 375], as mentioned in [61, end of Theorem 3.7]. In fact, it suffices to add a factor

$$e^{(2\rho_{B,+} - 2\rho_{\tilde{P}_0})(H_B(a))} |\mathrm{Nrd}(a_1)|_{\mathbb{A}}^s = e^{(2\rho_{B,+} - 2\rho_{\tilde{P}_0})(H_B(a))} \left(\frac{|\mathrm{Nrd}(a_1)|_{\mathbb{A}}^{1/p}}{|\mathrm{Nrd}(a_2)|_{\mathbb{A}}^{1/q}} \right)^{\frac{pq}{p+q}s}$$

in the form of $\prod_{\alpha \in \Delta_B^G} e^{c_\alpha \alpha(H_B(a))}$ to [13, (4.17) in p. 375], where c_α are constant coefficients. This factor only results in an extra factor $e^{c_9\|T\| + c_{10}t}$ to the integral in [13, (4.18) in p. 375], where c_9 and c_{10} are constant coefficients (here we have used [3, Corollary 6.2]). Since we can choose sufficiently large k, it does not matter. Hence we complete the argument of our claim and conclude. □

5. Exponential polynomial distributions

Let T be sufficiently regular, $\mathfrak{o} \in \mathcal{O}$ and η be the quadratic character of $\mathbb{A}^\times/F^\times$ attached to a quadratic field extension E/F. For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $s \in \mathbb{C}$, define

$$J_{\mathfrak{o}}^{G,T}(\eta, s, f) := \int_{H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} k_{f,\mathfrak{o}}^T(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx \tag{5.0.1}$$

and

$$J^{G,T}(\eta, s, f) := \int_{H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} k_f^T(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx,$$

where $k_{f,\mathfrak{o}}^T(x)$ and $k_f^T(x)$ are defined by (4.0.1) and (4.2.1) respectively, and we write $x = (x_1, x_2) \in GL_{p,D}(\mathbb{A}) \times GL_{q,D}(\mathbb{A})$. From Theorem 4.11, we know that $J_{\mathfrak{o}}^{G,T}(\eta, s, \cdot)$ and $J^{G,T}(\eta, s, \cdot)$ are well-defined distributions on $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and that

$$J^{G,T}(\eta, s, f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^{G,T}(\eta, s, f),$$

which is an analogue of the geometric side of Arthur's trace formula.

5.1. A generalised case in the product form. Let Q be a relatively standard parabolic subgroup of G. Then

$$M_Q \simeq GL_{p_1+q_1,D} \times \cdots \times GL_{p_l+q_l,D}$$

and

$$M_{Q_H} \simeq GL_{p_1,D} \times \cdots \times GL_{p_l,D} \times GL_{q_1,D} \times \cdots \times GL_{q_l,D},$$

where $\sum_{i=1}^l p_i = p$, $\sum_{i=1}^l q_i = q$ and we allow p_i or q_i to be zero. The tangent space of M_Q/M_{Q_H} at the neutral element is

$$\mathfrak{m}_Q \cap \mathfrak{s} \simeq \bigoplus_{\{1 \leq i \leq l | p_i q_i \neq 0\}} \begin{pmatrix} 0 & \mathrm{Mat}_{p_i \times q_i, D} \\ \mathrm{Mat}_{q_i \times p_i, D} & 0 \end{pmatrix}.$$

The conjugate action of $M_{Q_H}(F)$ on $(\mathfrak{m}_Q \cap \mathfrak{s})(F)$ can be described as follows:

(1) if $p_i q_i \neq 0$, $\begin{pmatrix} GL_{p_i}(D) & \\ & GL_{q_i}(D) \end{pmatrix}$ acts on $\begin{pmatrix} 0 & \mathrm{Mat}_{p_i \times q_i}(D) \\ \mathrm{Mat}_{q_i \times p_i}(D) & 0 \end{pmatrix}$ by conjugation;

(2) if $p_i q_i = 0$, $\begin{pmatrix} GL_{p_i}(D) & \\ & GL_{q_i}(D) \end{pmatrix}$ acts on 0 (viewed as a 0-dimensional vector space) trivially.

We may generalise integrability in last section to the product setting here whose proof is similar.

Define a relation of equivalence on $(\mathfrak{m}_Q \cap \mathfrak{s})(F)$ which is similar to that on $\mathfrak{s}(F)$ on each component. We denote by $\mathcal{O}^{\mathfrak{m}_Q \cap \mathfrak{s}}$ the set of equivalent classes for this relation. For $\mathfrak{o} \in \mathcal{O}$, the intersection $\mathfrak{o} \cap \mathfrak{m}_Q(F)$ is a finite (perhaps empty) union of classes $\mathfrak{o}_1, \cdots, \mathfrak{o}_t \in \mathcal{O}^{\mathfrak{m}_Q \cap \mathfrak{s}}$. Fix the minimal parabolic subgroup $\widetilde{P}_0' := \widetilde{P}_0 \cap M_{Q_H} = \widetilde{P}_0 \cap M_Q$ of M_{Q_H} and its Levi factor M_0. We say that a parabolic subgroup P' of M_Q is semi-standard (resp. relatively standard) if $M_0 \subseteq P'$ (resp. $\widetilde{P}_0' \subseteq P'$). Notice that there exists a bijection from the set of semi-standard (resp. relatively standard) parabolic subgroups of G contained in Q to the set of semi-standard (resp. relatively standard) parabolic subgroups of M_Q given by $P \mapsto P \cap M_Q$, whose inverse is given by $P' \mapsto P' N_Q$.

Choose $\varsigma_Q \in \Omega^G$ (not unique) such that $\varsigma_Q P_0 \subseteq Q$. Fix the minimal semi-standard parabolic subgroup $P_0' := (\varsigma_Q P_0) \cap M_Q$ of M_Q depending on the choice of ς_Q. For any semi-standard parabolic subgroup P' of M_Q and $T \in \mathfrak{a}_0$, denote by $T_{P'}$ the projection of sT in $\mathfrak{a}_{P'}$, where $s \in \Omega^{M_Q}$ such that $sP_0' \subseteq P'$. For $s \in \Omega^{M_Q}$ and a semi-standard parabolic subgroup $P \subseteq Q$ of G, we see that $sP_0' \subseteq P \cap M_Q$ if and only if $s\varsigma_Q P_0 \subseteq P$. Then $(\varsigma_Q T)_{P \cap M_Q} = T_P$ which is independent of the choice of ς_Q. This is also the reason why we introduce ς_Q. If $T \in \mathfrak{a}_{P_0}^+$ is sufficiently regular with respect to $P_0 \subseteq G$, then $\varsigma_Q T \in \mathfrak{a}_{P_0'}^+$ is sufficiently regular with respect to $P_0' \subseteq M_Q$.

Let $f' \in \mathcal{S}((\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}))$, P' be a relatively standard parabolic subgroup of M_Q and $1 \leq j \leq t$. Write $P_H' := P' \cap M_{Q_H} = P' \cap H$. For $x \in M_{P_H'}(F) N_{P_H'}(\mathbb{A}) \backslash M_{Q_H}(\mathbb{A})$, define

$$k_{f',P',\mathfrak{o}_j}^{M_Q}(x) := \sum_{X \in \mathfrak{m}_{P'}(F) \cap \mathfrak{o}_j} \int_{(\mathfrak{n}_{P'} \cap \mathfrak{s})(\mathbb{A})} f'(x^{-1}(X+U)x) dU. \tag{5.1.1}$$

For $T \in \mathfrak{a}_0$ and $x \in M_{Q_H}(F) \backslash M_{Q_H}(\mathbb{A})$, define

$$k_{f',\mathfrak{o}_j}^{Q,T}(x) := \sum_{\{P' : \widetilde{P}_0' \subseteq P'\}} (-1)^{\dim(A_{P'}/A_{M_Q})} \sum_{\delta \in P_H'(F) \backslash M_{Q_H}(F)} \widehat{\tau}_{P'}^{M_Q}(H_{P'}(\delta x) - T_{P'}) \cdot k_{f',P',\mathfrak{o}_j}^{M_Q}(\delta x).$$

For sufficiently regular $T \in \mathfrak{a}_{P_0}^+$ and $\{s_i\}_{1 \leq i \leq l} \in \mathbb{C}^l$, define

$$J_{\mathfrak{o}_j}^{Q,T}(\eta, \{s_i\}, f') := \int_{M_{Q_H}(F) \backslash M_{Q_H}(\mathbb{A}) \cap M_Q(\mathbb{A})^1} k_{f',\mathfrak{o}_j}^{Q,\varsigma_Q T}(x) \eta(\mathrm{Nrd}(x)) \prod_{1 \leq i \leq l} |\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}^{s_i} dx,$$

where we write $x = (x_1, ..., x_l) \in GL_{p_1+q_1,D}(\mathbb{A}) \times \cdots \times GL_{p_l+q_l,D}(\mathbb{A})$ and $x_i = (x_{i,1}, x_{i,2}) \in GL_{p_i,D}(\mathbb{A}) \times GL_{q_i,D}(\mathbb{A})$. As explained above, $k_{f',\mathfrak{o}_j}^{Q,\varsigma_Q T}$ and $J_{\mathfrak{o}_j}^{Q,T}$ are independent of the choice of ς_Q. Then we have well-defined distributions $J_{\mathfrak{o}_j}^{Q,T}(\eta, \{s_i\}, \cdot)$ on $\mathcal{S}((\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}))$. It only depends on the projection of $\varsigma_Q T$ to $\mathfrak{a}_{\varsigma_Q P_0}^Q$ and does not depend on T_Q. Now we define

$$J_{\mathfrak{o}}^{Q,T} := \sum_{j=1}^{t} J_{\mathfrak{o}_j}^{Q,T} \tag{5.1.2}$$

and

$$J^{Q,T} := \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^{Q,T}.$$

For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, define $f_Q^\eta \in \mathcal{S}((\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}))$ by

$$\forall X \in (\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}), f_Q^\eta(X) := \int_{K_H} \int_{(\mathfrak{n}_Q \cap \mathfrak{s})(\mathbb{A})} f(k^{-1}(X+V)k) \eta(\mathrm{Nrd}(k)) dV dk. \tag{5.1.3}$$

5.2. ω-stable parabolic subgroups. In our case, we can embed G into $\mathfrak{g}$ in the standard way. For any linear subspace $\mathfrak{v}$ of $\mathfrak{g}$, we denote by $\mathfrak{v}^\times$ the intersection of $\mathfrak{v}$ and G in $\mathfrak{g}$. Assume that $p = q$. Let us denote $n := p = q$. Then $\mathfrak{s}^\times(F)$ is the union of classes in $\mathcal{O}^\times$. Let $\omega := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in G(F)$. By the notation in Section 3.4, ω is the element in G exchanging e_i and f_i for all $1 \leq i \leq n$. Then $\omega \widetilde{P}_0 \omega^{-1} = \widetilde{P}_0$. We say that a semi-standard parabolic subgroup Q of G is "ω-stable" if $\omega Q \omega^{-1} = Q$. By Chevalley's theorem, this condition is equivalent to $\omega \in Q$. For a relatively standard parabolic subgroup Q of G, we

see that Q is ω-stable if and only if $p_i = q_i$ for any $1 \le i \le l$; an illustrating example for $l = 2$ looks like

$$Q = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ * & * & * & * \\ 0 & * & 0 & * \end{pmatrix}^{\times}.$$

Notice that there is a bijection $P_n \mapsto \begin{pmatrix} \mathfrak{p}_n & \mathfrak{p}_n \\ \mathfrak{p}_n & \mathfrak{p}_n \end{pmatrix}^{\times}$ from the set of standard parabolic subgroups in $GL_{n,D}$ (namely containing the group of upper triangular matrices) to the set of ω-stable relatively standard parabolic subgroups in G. For $Q \subseteq R$ a pair of relatively standard parabolic subgroups of G, one sees that "Q is ω-stable" implies "R is ω-stable", but "R is ω-stable" does not imply "Q is ω-stable". For any relative standard parabolic subgroup Q of G, define

$$\overline{Q}^{\omega\text{-st}} := \bigcap_{\{R: Q \subseteq R, \omega R \omega^{-1} = R\}} R,$$

which is the minimal ω-stable parabolic subgroup of G containing Q.

Denote by $\rho_{Q,+}$ the half of the sum of weights (with multiplicities) for the action of A_0 on $\mathfrak{n}_Q \cap \mathfrak{s}$. We see that $\rho_{Q,+} = \rho_Q - \rho_{Q_H}$ and that for $Q \subseteq R$ a pair of relatively standard parabolic subgroup of G, the restriction of $(2\rho_{Q,+} - 2\rho_{Q_H})\big|_{\mathfrak{a}_Q}$ to $\mathfrak{a}_R$ equals $(2\rho_{R,+} - 2\rho_{R_H})\big|_{\mathfrak{a}_R}$.

Proposition 5.1. *Assume that $p = q = n$. Let $\mathfrak{o} \in \mathcal{O}$. The following three conditions are equivalent:*

(1) $\mathfrak{o} \in \mathcal{O}^{\times}$;

(2) for all relatively standard parabolic subgroup Q of G, if $\mathfrak{o} \cap \mathfrak{q}(F) \neq \emptyset$, then Q is ω-stable;

(3) for all relatively standard parabolic subgroup Q of G, if $\mathfrak{o} \cap \mathfrak{m}_Q(F) \neq \emptyset$, then Q is ω-stable.

Proof. The direction (2)⇒(3) is trivial. We actually have (2)⇔(3) from Proposition 3.5.

Next, we prove the direction (1)⇒(2). We assume that $\mathfrak{o} \in \mathcal{O}^{\times}$ and that $\mathfrak{o} \cap \mathfrak{q}(F) \neq \emptyset$ for some relatively standard parabolic subgroup Q of G. If Q is not ω-stable, let k be the minimal integer such that $1 \le k \le l-1$ and that

$$\sum_{1 \le i \le k} p_i - \sum_{1 \le i \le k} q_i \neq 0.$$

Without loss of generality, we may assume that

$$\sum_{1 \le i \le k} p_i - \sum_{1 \le i \le k} q_i < 0.$$

Let $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{o} \cap \mathfrak{q}(F)$. Then $A \in \mathfrak{gl}_n(D)$ is in the form of $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the size of the zero matrix in the lower left corner is at least $\left(\sum_{k+1 \le i \le l} p_i\right) \times \left(1 + \sum_{k+1 \le i \le l} p_i\right)$. Therefore, A is not invertible, which contradicts with $\mathfrak{o} \in \mathcal{O}^{\times}$. This establishes (1)⇒(2).

Finally, we prove the direction (3)⇒(1). We assume (3). Suppose that $\mathfrak{o} \notin \mathcal{O}^{\times}$. Let $P(\lambda) := \mathrm{Prd}_{AB}(\lambda)$, where $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ is any element in $\mathfrak{o}$. By [56, Proposition 5], $P(\lambda) = \lambda^d R(\lambda)$, where $R(\lambda) = \mathrm{Prd}_C(\lambda)$ for some $C \in GL_{n-1}(D)$. Let Q be the relative standard parabolic subgroup of G containing elements of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}^{\times}$, where the size of the zero matrix in the lower left corner is $1 \times (n-1)$. Then $\begin{pmatrix} 0 & 0 & 1_{n-1} & 0 \\ 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{o} \cap \mathfrak{m}_Q(F)$, which contradicts with (3). This shows (3)⇒(1). □

Lemma 5.2. *Assume that $p = q = n$. Let Q be a relatively standard parabolic subgroup of G. For all $\varpi^{\vee} \in \widehat{\Delta}_Q^{\vee}$, we have $(2\rho_{Q,+} - 2\rho_{Q_H})(\varpi^{\vee}) \ge 0$. Moreover, $2\rho_{Q,+} - 2\rho_{Q_H}$ viewed as an element of $(\mathfrak{a}_Q^G)^*$ is zero if and only if Q is ω-stable.*

Proof. We use the notation in Section 3.4. Put $e_i^* \in \mathfrak{a}_0^*$ (resp. $f_i^* \in \mathfrak{a}_0^*$) to be the character of the action of A_0 on e_i (resp. f_i). Write $e_i^{\vee} \in \mathfrak{a}_0$ (resp. $f_i^{\vee} \in \mathfrak{a}_0$) to be the dual basis, i.e., $e_i^*(e_j^{\vee}) = \delta_{ij}$

(resp. $f_i^*(f_j^\vee)=\delta_{ij}$) for $1\le i,j\le n$. A basis of $\mathfrak{a}_Q$ is given by $h_i^\vee := e^\vee_{p_1+\cdots+p_{i-1}+1}+\cdots+e^\vee_{p_1+\cdots+p_i}+f^\vee_{q_1+\cdots+q_{i-1}+1}+\cdots+f^\vee_{q_1+\cdots+q_i}$ for $1\le i\le l$. Write $h_i^*\in(\mathfrak{a}_Q)^*$ to be the dual basis. Denote

$$\varpi_k^\vee := \frac{\sum_{i=k+1}^{l}(p_i+q_i)}{2n}(h_1^\vee+\cdots+h_k^\vee)-\frac{\sum_{i=1}^{k}(p_i+q_i)}{2n}(h_{k+1}^\vee+\cdots+h_l^\vee).$$

Recall that

$$(\widehat{\Delta}_Q^G)^\vee=\{\varpi_k^\vee|1\le k\le l-1\}$$

is a basis of $\mathfrak{a}_Q^G$. We can also see that

$$2\rho_{Q,+}\big|_{\mathfrak{a}_Q}=\dim_F(D)\sum_{1\le i<j\le l}(p_iq_j+q_ip_j)(h_i^*-h_j^*)$$

and that

$$2\rho_{Q_H}\big|_{\mathfrak{a}_Q}=\dim_F(D)\sum_{1\le i<j\le l}(p_ip_j+q_iq_j)(h_i^*-h_j^*),$$

so

$$(2\rho_{Q,+}-2\rho_{Q_H})\big|_{\mathfrak{a}_Q}=\dim_F(D)\sum_{1\le i<j\le l}(p_i-q_i)(q_j-p_j)(h_i^*-h_j^*).$$

Since $\sum_{i=1}^{l}p_i=\sum_{i=1}^{l}q_i=n$, we have

$$(h_i^*-h_j^*)(\varpi_k^\vee)=\begin{cases}0, & \text{if } k+1\le i<j\le l \text{ or } 1\le i<j\le k;\\ 1, & \text{if } 1\le i\le k \text{ and } k+1\le j\le l.\end{cases}$$

Then

$$\begin{aligned}(2\rho_{Q,+}-2\rho_{Q_H})(\varpi_k^\vee)&=\dim_F(D)\sum_{\substack{1\le i\le k\\ k+1\le j\le l}}(p_i-q_i)(q_j-p_j)\\ &=\dim_F(D)\left(\sum_{1\le i\le k}p_i-\sum_{1\le i\le k}q_i\right)\left(\sum_{k+1\le j\le l}q_j-\sum_{k+1\le j\le l}p_j\right)\\ &=\dim_F(D)\left(\sum_{1\le i\le k}p_i-\sum_{1\le i\le k}q_i\right)^2\ge 0.\end{aligned}$$

It is clear that $(2\rho_{Q,+}-2\rho_{Q_H})(\varpi_k^\vee)=0$ for all $1\le k\le l-1$ if and only if $p_i=q_i$ for all $1\le i\le l$. □

5.3. Exponential polynomials. Let $T_1,T_2\in\mathfrak{a}_0$. Following [5, §2], define the function $\Gamma_P(T_1,T_2)$ inductively on $\dim(A_P/A_G)$ by setting

$$\widehat{\tau}_P^G(T_1-T_2)=\sum_{\{Q:P\subseteq Q\}}(-1)^{\dim(A_Q/A_G)}\widehat{\tau}_P^Q(T_1)\Gamma_Q(T_1,T_2)$$

for any relatively standard parabolic subgroup P of G. This definition can be explicitly given by [5, (2.1) in p. 13] and only depends on the projections of T_1,T_2 onto $\mathfrak{a}_P^G$. For $T=(t_1,...,t_{p+q})\in\mathfrak{a}_0$, we denote $\Sigma_1(T):=t_1+...+t_p$. If we use the notation in Section 3.4 and put $e_i^*\in\mathfrak{a}_0^*$ (resp. $f_i^*\in\mathfrak{a}_0^*$) to be the character of the action of A_0 on e_i (resp. f_i), it is equivalent to say that $\Sigma_1=\sum_{1\le i\le p}e_i^*$. For $T_2\in\mathfrak{a}_Q$ and $s\in\mathbb{C}$, write

$$p_{Q,s}(T_2):=\int_{\mathfrak{a}_Q^G}e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_1)}\Gamma_Q(T_1,T_2)dT_1. \tag{5.3.1}$$

When $p=q=n, s=0$ and Q is ω-stable, it is reduced to

$$p_{Q,0}(T_2)=\int_{\mathfrak{a}_Q^G}\Gamma_Q(T_1,T_2)dT_1$$

by Lemma 5.2.

For $Q \subseteq R$ a pair of relatively standard parabolic subgroups of G, denote by $\mathbb{Z}(\widehat{\Delta}_Q^R)^\vee$ the lattice generated by $(\widehat{\Delta}_Q^R)^\vee$ in $\mathfrak{a}_Q^R$ and by $\mathbb{Z}(\Delta_R^G)^\vee$ the lattice generated by $(\Delta_R^G)^\vee$ in $\mathfrak{a}_R^G$. Following [5, §2], for $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^* := \mathfrak{a}_Q^* \otimes_{\mathbb{R}} \mathbb{C}$, define

$$\widehat{\theta}_Q^R(\lambda) := \mathrm{vol}(\mathfrak{a}_Q^R/\mathbb{Z}(\widehat{\Delta}_Q^R)^\vee)^{-1} \prod_{\varpi^\vee \in (\widehat{\Delta}_Q^R)^\vee} \lambda(\varpi^\vee)$$

and

$$\theta_R^G(\lambda) := \mathrm{vol}(\mathfrak{a}_R^G/\mathbb{Z}(\Delta_R^G)^\vee)^{-1} \prod_{\alpha^\vee \in (\Delta_R^G)^\vee} \lambda(\alpha^\vee).$$

Proposition 5.3. *Let Q be a relatively standard parabolic subgroup of G, $T_2 \in \mathfrak{a}_Q$ and $s \in \mathbb{C}$. The function $T_1 \mapsto \Gamma_Q(T_1, T_2)$ is compactly supported on $\mathfrak{a}_Q^G$. Moreover, the function $T_2 \mapsto p_{Q,s}(T_2)$ is an exponential polynomial in T_2; more precisely, there exists a polynomial $p_{Q,R,s}$ (not necessarily unique) on $\mathfrak{a}_R^G$ of degree $\leq \dim(A_Q/A_G)$ for each relatively standard parabolic subgroup R containing Q such that*

$$p_{Q,s}(T_2) = \sum_{\{R:Q\subseteq R\}} e^{(2\rho_{R,+} - 2\rho_{R_H} + s\Sigma_1)(T_{2,R}^G)} p_{Q,R,s}(T_{2,R}^G),$$

where we write $T_{2,R}^G$ for the projection of $T_2 \in \mathfrak{a}_Q$ in $\mathfrak{a}_R^G$ via the decomposition $\mathfrak{a}_Q = \mathfrak{a}_Q^R \oplus \mathfrak{a}_R^G \oplus \mathfrak{a}_G$. When $p = q = n$ and $s = 0$, the purely polynomial term of $p_{Q,0}(T_2)$ is given by

$$\sum_{\{R:Q\subseteq R, \omega R \omega^{-1} = R\}} p_{Q,R,0}(T_{2,R}^G),$$

which is a homogeneous polynomial in T_2 of degree $\dim(A_{\overline{Q}^{\omega\text{-}st}}/A_G)$; in particular, if Q is ω-stable, then $p_{Q,0}(T_2)$ is a homogeneous polynomial in T_2 of degree $\dim(A_Q/A_G)$.

Proof. The first statement is [5, Lemmas 2.1]. First let us prove the second one.

From [5, Lemma 2.2], we know that the integral

$$\int_{\mathfrak{a}_Q^G} e^{\lambda(T_1)} \Gamma_Q(T_1, T_2) dT_1$$

is an entire function in $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*$, and its value is given by

$$\sum_{\{R:Q\subseteq R\}} (-1)^{\dim(A_Q/A_R)} e^{\lambda(T_{2,R}^G)} \widehat{\theta}_Q^R(\lambda)^{-1} \theta_R^G(\lambda)^{-1}$$

when the latter expression makes sense.

Fix $\varepsilon \in \mathfrak{a}_{Q,\mathbb{C}}^*$ such that $\widehat{\theta}_Q^R(\varepsilon) \neq 0$ and $\theta_R^G(\varepsilon) \neq 0$ for all relatively standard parabolic subgroups R containing Q. Then for $t \in \mathbb{R}^\times$ whose absolute value is small enough, we also have $\widehat{\theta}_Q^R(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon) \neq 0$ and $\theta_R^G(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon) \neq 0$ for all relatively standard parabolic subgroups R containing Q. Let $\lambda = 2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon$ in the formula above, and we obtain

$$\begin{aligned} p_Q(T_2) = \lim_{t\mapsto 0} \sum_{\{R:Q\subseteq R\}} & (-1)^{\dim(A_Q/A_R)} e^{(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon)(T_{2,R}^G)} \widehat{\theta}_Q^R(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon)^{-1} \\ & \cdot \theta_R^G(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon)^{-1}. \end{aligned}$$

Since the restriction of $2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1$ to $\mathfrak{a}_R$ equals $2\rho_{R,+} - 2\rho_{R_H} + s\Sigma_1$, we get

$$e^{(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1)(T_{2,R}^G)} = e^{(2\rho_{R,+} - 2\rho_{R_H} + s\Sigma_1)(T_{2,R}^G)}.$$

We can put $p_{Q,R,s}(T_{2,R}^G)$ to be the constant term of the Laurent series development around $t = 0$ of

$$t \mapsto (-1)^{\dim(A_Q/A_R)} e^{(t\varepsilon)(T_{2,R}^G)} \widehat{\theta}_Q^R(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon)^{-1} \theta_R^G(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1 + t\varepsilon)^{-1}.$$

Then $p_{Q,R,s}(T_{2,R}^G)$ is a polynomial in $T_{2,R}^G$ of degree $\leq \dim(A_Q/A_G)$. Hence we prove the existence in the second statement.

Now let $p = q = n$ and $s = 0$. From Lemma 5.2, we know that the purely polynomial term of $p_{Q,0}$ is given by

$$\sum_{\{R:Q\subseteq R, \omega R \omega^{-1} = R\}} p_{Q,R,0}(T_{2,R}^G).$$

Next we compute the degree of $p_{Q,R,0}$ that we chose above for each ω-stable parabolic subgroup R containing Q. Denote

$$N_1 := \sharp\{\varpi^\vee \in (\widehat{\Delta}_Q^R)^\vee : (2\rho_{Q,+} - 2\rho_{Q_H})(\varpi^\vee) = 0\}$$

and

$$N_2 := \sharp\{\alpha^\vee \in (\Delta_R^G)^\vee : (2\rho_{Q,+} - 2\rho_{Q_H})(\alpha^\vee) = 0\},$$

where $\sharp$ means the cardinality of a finite set. Then

$$\deg(p_{Q,R,0}) = N_1 + N_2.$$

Recall that both of $(\widehat{\Delta}_R^G)^\vee$ and $(\Delta_R^G)^\vee$ are bases of $\mathfrak{a}_R^G$. Since R is relatively standard and ω-stable, by Lemma 5.2, we have

$$N_2 = \dim(A_R/A_G).$$

Keep the notation as in the proof of Lemma 5.2 for Q. Since R is relatively standard and ω-stable, by Lemma 5.2, we may suppose that R is the stabiliser in G of the flag

$$0 \subsetneq \langle e_1, \cdots, e_{r_1}, f_1, \cdots, f_{r_1}\rangle_D \subsetneq \langle e_1, \cdots, e_{r_1+r_2}, f_1, \cdots, f_{r_1+r_2}\rangle_D \subsetneq \ldots \subsetneq \langle e_1, \cdots, e_{r_1+\cdots+r_{l'}}, f_1, \cdots, f_{r_1+\cdots+r_{l'}}\rangle_D.$$

The fact that $Q \subseteq R$ tells us that both of the partitions $(p_1, \cdots, p_l)$ and $(q_1, \cdots, q_l)$ are refinements of the partition $(r_1, \cdots, r_{l'})$ of n, and that every r_i is divided into the same number of segments in these two refinements. Then

$$\begin{aligned}(\widehat{\Delta}_Q^R)^\vee &= \left\{\text{projection of } \varpi_k^\vee \in (\widehat{\Delta}_Q^G)^\vee \text{ to } \mathfrak{a}_Q^R \middle| 1 \le k \le l-1, \sum_{i=1}^k (p_i + q_i) \neq \sum_{i=1}^j 2r_i \forall 1 \le j \le l'-1\right\} \\ &= \left\{\text{projection of } \varpi_k^\vee \in (\widehat{\Delta}_Q^G)^\vee \text{ to } \mathfrak{a}_Q^R \middle| 1 \le k \le l-1, \nexists 1 \le j \le l'-1 s.t. \sum_{i=1}^k p_i = \sum_{i=1}^k q_i = \sum_{i=1}^j r_i\right\}.\end{aligned}$$

Because the restriction of $2\rho_{Q,+} - 2\rho_{Q_H}$ to $\mathfrak{a}_R$ equals $2\rho_{R,+} - 2\rho_{R_H}$ and R is relatively standard and ω-stable, by Lemma 5.2, we do not need the projection, i.e.,

$$(2\rho_{Q,+} - 2\rho_{Q_H})\left(\text{projection of } \varpi_k^\vee \in (\widehat{\Delta}_Q^G)^\vee \text{ to } \mathfrak{a}_Q^R\right) = (2\rho_{Q,+} - 2\rho_{Q_H})\left(\varpi_k^\vee \in (\widehat{\Delta}_Q^G)^\vee\right).$$

From the proof of Lemma 5.2, for any $1 \le k \le l$, we have $(2\rho_{Q,+} - 2\rho_{Q_H})(\varpi_k^\vee) = 0$ if and only if $\sum_{i=1}^k p_i = \sum_{i=1}^k q_i$. We can also see that $\overline{Q}^{\omega\text{-st}}$ is the ω-stable parabolic subgroup R containing Q with maximal $l' := \dim(A_R)$. To sum up, we have

$$N_1 = \dim(A_{\overline{Q}^{\omega\text{-st}}}/A_R).$$

Hence for each ω-stable parabolic subgroup R containing Q,

$$\deg(p_{Q,R,0}) = N_1 + N_2 = \dim(A_{\overline{Q}^{\omega\text{-st}}}/A_R) + \dim(A_R/A_G) = \dim(A_{\overline{Q}^{\omega\text{-st}}}/A_G).$$

The assertion about the particular case where Q is ω-stable is [5, Lemma 2.2] combined with Lemma 5.2; it can also be read from the results above that we have proved. □

5.4. Quantitive behaviour in T. For a relatively standard parabolic subgroup Q of G, let $\{s_i^Q\}_{1\le i\le l} \in \mathbb{Z}^l$ be the explicit constants determined by

$$\forall x \in M_{Q_H}(\mathbb{A}) \cap M_Q(\mathbb{A})^1, \prod_{1\le i\le l} |\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}^{s_i^Q} = e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(x))}, \tag{5.4.1}$$

where we write $x = (x_1, ..., x_l) \in GL_{p_1+q_1,D}(\mathbb{A}) \times \cdots \times GL_{p_l+q_l,D}(\mathbb{A})$ and $x_i = (x_{i,1}, x_{i,2}) \in GL_{p_i,D}(\mathbb{A}) \times GL_{q_i,D}(\mathbb{A})$. If $p_iq_i = 0$ for some $1 \le i \le l$, we shall take $|\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}^{s_i^Q} = 1$ and $s_i^Q = 0$ by convention. Then such constants are unique.

Proposition 5.4. *Let Q be a relatively standard parabolic subgroup of G. If $p_iq_i \neq 0$ for some $1 \le i \le l$, then*

$$s_i^Q = 2d\left(\sum_{k<i}(p_k - q_k) + \sum_{k>i}(q_k - p_k)\right).$$

When $p = q = n$, if Q is ω-stable, then $s_i^Q = 0$ for all $1 \le i \le l$.

Proof. Assume that $p_iq_i \neq 0$ for some $1 \le i \le l$. Let $x \in M_{Q_H}(\mathbb{A})$. We have

(1) the contribution of $x_{i,1}$ to $e^{2\rho_{Q,+}(H_{Q_H}(x))}$ is the $d\Big(\sum_{k>i} q_k - \sum_{k<i} q_k\Big)$-th power of $|\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}$;

(2) the contribution of $x_{i,1}$ to $e^{2\rho_{Q_H}(H_{Q_H}(x))}$ is the $d\Big(\sum_{k>i} p_k - \sum_{k<i} p_k\Big)$-th power of $|\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}$;

(3) the contribution of $x_{i,2}$ to $e^{2\rho_{Q,+}(H_{Q_H}(x))}$ is the $d\Big(\sum_{k>i} p_k - \sum_{k<i} p_k\Big)$-th power of $|\mathrm{Nrd}(x_{i,2})|_{\mathbb{A}}$;

(4) the contribution of $x_{i,2}$ to $e^{2\rho_{Q_H}(H_{Q_H}(x))}$ is the $d\Big(\sum_{k>i} q_k - \sum_{k<i} q_k\Big)$-th power of $|\mathrm{Nrd}(x_{i,2})|_{\mathbb{A}}$.

In sum, the contribution of x_i to $e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(x))}$ is the product of the $d\Big(\sum_{k<i}(p_k-q_k)+\sum_{k>i}(q_k-p_k)\Big)$-th power of $|\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}$ and the $d\Big(\sum_{k<i}(q_k-p_k)+\sum_{k>i}(p_k-q_k)\Big)$-th power of $|\mathrm{Nrd}(x_{i,2})|_{\mathbb{A}}$.

Now let $x \in M_{Q_H}(\mathbb{A}) \cap M_Q(\mathbb{A})^1$. Then $|\mathrm{Nrd}(x_{i,1})\mathrm{Nrd}(x_{i,2})|_{\mathbb{A}} = |\mathrm{Nrd}(x_i)|_{\mathbb{A}} = 1$. Therefore, the contribution of x_i to $e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(x))}$ is the $2d\Big(\sum_{k<i}(p_k-q_k)+\sum_{k>i}(q_k-p_k)\Big)$-th power of $|\mathrm{Nrd}(x_{i,1})|_{\mathbb{A}}$. We have proved the first statement.

The second statement is nothing but a special case of the first one, since we have $p_k = q_k$ for $1 \le k \le l$ in this case. □

THEOREM 5.5. *Let T' be sufficiently regular, $\mathfrak{o} \in \mathcal{O}$ and $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$. Then for all sufficiently regular T and $s \in \mathbb{C}$, we have*

$$J_{\mathfrak{o}}^{G,T}(\eta, s, f) = \sum_{\{Q:\widetilde{P}_0 \subseteq Q\}} p_{Q,s}(T_Q - T'_Q) e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)((T')_Q^G)} J_{\mathfrak{o}}^{Q,T'}(\eta, \{s_i^Q + s\}, f_Q^\eta),$$

where we write $(T')_Q^G$ for the projection of $T'_Q \in \mathfrak{a}_Q$ in $\mathfrak{a}_Q^G$ via the decomposition $\mathfrak{a}_Q = \mathfrak{a}_Q^G \oplus \mathfrak{a}_G$, the distributions $J_{\mathfrak{o}}^{G,T}$ and $J_{\mathfrak{o}}^{Q,T'}$ are defined by the formulae (5.0.1) and (5.1.2) respectively, and f_Q^η and $p_{Q,s}$ are defined by the formulae (5.1.3) and (5.3.1) respectively.

COROLLARY 5.6. *Let $\mathfrak{o} \in \mathcal{O}$, $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $s \in \mathbb{C}$. Then $J_{\mathfrak{o}}^{G,T}(\eta, s, f)$ and $J^{G,T}(\eta, s, f)$ are exponential polynomials in T for sufficiently regular T, so we can extend them to all $T \in \mathfrak{a}_0$. When $p = q = n$ and $s = 0$, their purely polynomial terms have degree $\le n-1$; in particular, if $\mathfrak{o} \in \mathcal{O}^\times$ (e.g., $\mathfrak{o} \in \mathcal{O}_{rs}$), $J_{\mathfrak{o}}^{G,T}(\eta, 0, f)$ is a polynomial in T of degree $\le n-1$ for sufficiently regular T.*

PROOF OF COROLLARY 5.6. It results from Theorem 5.5, Propositions 5.3 and 5.1. □

REMARK 5.7. We may extend our result to the product form in Section 5.1 by similar argument. Let Q be a relatively standard parabolic subgroup of G. Let $\mathfrak{o} \in \mathcal{O}$, $f' \in \mathcal{S}((\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}))$ and $\{s_i\}_{1\le i\le l} \in \mathbb{C}^l$. For sufficiently regular $T \in \mathfrak{a}_{P_0}^+$, $J_{\mathfrak{o}}^{Q,T}(\eta, \{s_i\}, f')$ and $J^{Q,T}(\eta, \{s_i\}, f')$ are exponential polynomials in T independent of T_Q, so we can extend them to all $T \in \mathfrak{a}_0$.

PROOF OF THEOREM 5.5. Let P be a relatively standard parabolic subgroup of G, $\delta \in P_H(F)\backslash H(F)$ and $x \in H(\mathbb{A}) \cap G(\mathbb{A})^1$. Substituting $T_1 = H_P(\delta x) - T'_P$ and $T_2 = T_P - T'_P$ in the definition of $\Gamma_P(T_1, T_2)$, we get

$$\widehat{\tau}_P^G(H_P(\delta x) - T_P) = \sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_G)} \widehat{\tau}_P^Q(H_P(\delta x) - T'_P)\Gamma_Q(H_P(\delta x) - T'_P, T_P - T'_P).$$

Then

$$\begin{aligned}
J_{\mathfrak{o}}^{G,T}(\eta, s, f) &= \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \left(\sum_{\{P:\widetilde{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P_H(F)\backslash H(F)} \widehat{\tau}_P^G(H_P(\delta x) - T_P) \cdot k_{f,P,\mathfrak{o}}(\delta x) \right) \\
&\quad \cdot \eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx \\
&= \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \sum_{\{P:\widetilde{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P_H(F)\backslash H(F)} \\
&\quad \left(\sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_G)} \widehat{\tau}_P^Q(H_P(\delta x) - T'_P)\Gamma_Q(H_P(\delta x) - T'_P, T_P - T'_P) \right) k_{f,P,\mathfrak{o}}(\delta x) \\
&\quad \cdot \eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx.
\end{aligned}$$

Exchanging the order of two sums over P and Q, and decomposing the sum over $P_H(F)\backslash H(F)$ into two sums over $P_H(F)\backslash Q_H(F)$ and $Q_H(F)\backslash H(F)$, we have

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in P_H(F)\backslash Q_H(F)}\sum_{\delta'\in Q_H(F)\backslash H(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta\delta'x)-T'_P)\Gamma_Q(H_P(\delta\delta'x)-T'_P,T_P-T'_P)k_{f,P,\mathfrak{o}}(\delta\delta'x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx.$$

Combining the integral over $H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1$ and the sum over $Q_H(F)\backslash H(F)$ into the integral over $Q_H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1$, and using the fact that

$$P_H(F)\backslash Q_H(F)\simeq (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F),$$

we obtain

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{Q_H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta x)-T'_P)\Gamma_Q(H_P(\delta x)-T'_P,T_P-T'_P)k_{f,P,\mathfrak{o}}(\delta x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx.$$

By the Iwasawa decomposition and our choice of measures, we have

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{K_H}\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\int_{A_Q^{G,\infty}}\int_{N_{Q_H}(F)\backslash N_{Q_H}(\mathbb{A})}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}$$
$$\sum_{\delta\in (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\widehat{\tau}_P^Q(H_P(\delta namk)-T'_P)\Gamma_Q(H_P(\delta namk)-T'_P,T_P-T'_P)$$
$$\cdot\, k_{f,P,\mathfrak{o}}(\delta namk)\eta(\mathrm{Nrd}(mk))|\mathrm{Nrd}(a_1m_1)|_{\mathbb{A}}^s e^{-2\rho_{Q_H}(H_{Q_H}(am))}dndadmdk.$$

Notice that

$$\widehat{\tau}_P^Q(H_P(\delta namk)-T'_P)=\widehat{\tau}_P^Q(H_P(\delta m)+H_P(a)-T'_P)=\widehat{\tau}_P^Q(H_P(\delta m)-T'_P),$$

and that

$$\Gamma_Q(H_P(\delta namk)-T'_P,T_P-T'_P)=\Gamma_Q(H_Q(\delta namk)-T'_Q,T_Q-T'_Q)=\Gamma_Q(H_Q(a)-T'_Q,T_Q-T'_Q).$$

In addition, by change of variables, we see that

$$\begin{aligned}k_{f,P,\mathfrak{o}}(\delta namk)&=\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})}f((\delta namk)^{-1}(X+U)\delta namk)dU\\&=\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})}f((\delta a^{-1}namk)^{-1}(X+a^{-1}Ua)\delta a^{-1}namk)dU\\&=\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})}f((\delta a^{-1}namk)^{-1}(X+U)\delta a^{-1}namk)e^{2\rho_{Q,+}(H_Q(a))}dU\\&=e^{2\rho_{Q,+}(H_Q(a))}k_{f,P,\mathfrak{o}}(\delta a^{-1}namk).\end{aligned}$$

Since $\delta a^{-1}na\delta^{-1}\in N_{Q_H}(\mathbb{A})\subseteq N_{P_H}(\mathbb{A})$ and $k_{f,P,\mathfrak{o}}$ is left invariant by $N_{P_H}(\mathbb{A})$, we deduce that

$$k_{f,P,\mathfrak{o}}(\delta namk)=e^{2\rho_{Q,+}(H_Q(a))}k_{f,P,\mathfrak{o}}(\delta mk).$$

In sum, the integrand in $J_{\mathfrak{o}}^{G,T}(\eta,s,f)$ is independent of $n\in N_{Q_H}(F)\backslash N_{Q_H}(\mathbb{A})$. We can choose the Haar measure such that $\mathrm{vol}(N_{Q_H}(F)\backslash N_{Q_H}(\mathbb{A}))=1$. Then

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\left(\int_{A_Q^{G,\infty}}|\mathrm{Nrd}(a_1)|_{\mathbb{A}}^s e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_Q(a))}\Gamma_Q(H_Q(a)-T'_Q,T_Q-T'_Q)da\right)$$
$$\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta m)-T'_P)\left(\int_{K_H}k_{f,P,\mathfrak{o}}(\delta mk)\eta(\mathrm{Nrd}(k))dk\right)\eta(\mathrm{Nrd}(m))|\mathrm{Nrd}(m_1)|_{\mathbb{A}}^s e^{-2\rho_{Q_H}(H_{Q_H}(m))}dm.$$

By the definition of the Haar measure on $A_Q^{G,\infty}$, we have

$$\begin{aligned}
&\int_{A_Q^{G,\infty}} |\mathrm{Nrd}(a_1)|_{\mathbb{A}}^s e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_Q(a))}\Gamma_Q(H_Q(a)-T'_Q,T_Q-T'_Q)da\\
:=&\int_{\mathfrak{a}_Q^G} e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_1)}\Gamma_Q(T_1-T'_Q,T_Q-T'_Q)dT_1\\
=&e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)((T')_Q^G)}\int_{\mathfrak{a}_Q^G} e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_1)}\Gamma_Q(T_1,T_Q-T'_Q)dT_1\\
=&e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)((T')_Q^G)}p_{Q,s}(T_Q-T'_Q).
\end{aligned}$$

Since $\mathfrak{n}_P=\mathfrak{n}_P^Q\oplus\mathfrak{n}_Q$, by change of variables, we see that

$$\begin{aligned}
k_{f,P,\mathfrak{o}}(\delta mk)=&\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P^Q\cap\mathfrak{s})(\mathbb{A})}dU\int_{(\mathfrak{n}_Q\cap\mathfrak{s})(\mathbb{A})}f((\delta mk)^{-1}(X+U+V)\delta mk)dV\\
=&\,e^{2\rho_{Q,+}(H_{Q_H}(m))}\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P^Q\cap\mathfrak{s})(\mathbb{A})}dU\int_{(\mathfrak{n}_Q\cap\mathfrak{s})(\mathbb{A})}f(k^{-1}((\delta m)^{-1}(X+U)\delta m+V)k)dV,
\end{aligned}$$

so we can write

$$\begin{aligned}
\int_{K_H}k_{f,P,\mathfrak{o}}(\delta mk)\eta(\mathrm{Nrd}(k))dk=&\,e^{2\rho_{Q,+}(H_{Q_H}(m))}\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P^Q\cap\mathfrak{s})(\mathbb{A})}f_Q^\eta((\delta m)^{-1}(X+U)\delta m)dU\\
=&\,e^{2\rho_{Q,+}(H_{Q_H}(m))}\sum_{j=1}^t k^{M_Q}_{f_Q^\eta,P\cap M_Q,\mathfrak{o}_j}(\delta m)
\end{aligned}$$

by (5.1.1). Now we can draw our conclusion by noting that

$$\begin{aligned}
J_{\mathfrak{o}}^{Q,T'}(\eta,\{s_i^Q+s\},f_Q^\eta)=&\sum_{j=1}^t\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_{P\cap M_Q}/A_{M_Q})}\\
&\sum_{\delta\in((P\cap M_Q)(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\widehat{\tau}^{M_Q}_{P\cap M_Q}(H_{P\cap M_Q}(\delta m)-(\varsigma_QT')_{P\cap M_Q})\\
&\cdot k^{M_Q}_{f_Q^\eta,P\cap M_Q,\mathfrak{o}_j}(\delta m)\eta(\mathrm{Nrd}(m))|\mathrm{Nrd}(m_1)|_{\mathbb{A}}^s e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(m))}dm\\
=&\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in(P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\\
&\widehat{\tau}_P^Q(H_P(\delta m)-T'_P)\left(\sum_{j=1}^t k^{M_Q}_{f_Q^\eta,P\cap M_Q,\mathfrak{o}_j}(\delta m)\right)\eta(\mathrm{Nrd}(m))|\mathrm{Nrd}(m_1)|_{\mathbb{A}}^s\\
&\cdot e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(m))}dm.
\end{aligned}$$

□

5.5. Independence of constant terms. Let $J_{\mathfrak{o}}^G(\eta,s,f)$ and $J^G(\eta,s,f)$ be the constant terms of $J_{\mathfrak{o}}^{G,T}(\eta,s,f)$ and $J^{G,T}(\eta,s,f)$ respectively. We fix a common minimal Levi subgroup M_0 of H and G.

Firstly, the distributions $J_{\mathfrak{o}}^G(\eta,s,f)$ and $J^G(\eta,s,f)$ are independent of the choice of the relatively standard minimal parabolic subgroup P_0 of G at the very beginning of last section. In fact, let P'_0 be another relatively standard minimal parabolic subgroup of G and $\sigma\in\Omega^G$ such that $P'_0=\sigma P_0$. Denote by $J_{P'_0,\mathfrak{o}}^{G,T}(\eta,s,f)$ and $J_{P'_0,\mathfrak{o}}^G(\eta,s,f)$ the distributions obtained starting from P'_0. Then if $T\in\mathfrak{a}_{P'_0}$, we have $J_{P'_0,\mathfrak{o}}^{G,T}(\eta,s,f)=J_{\mathfrak{o}}^{G,\sigma^{-1}T}(\eta,s,f)$, so $J_{P'_0,\mathfrak{o}}^G(\eta,s,f)=J_{\mathfrak{o}}^G(\eta,s,f)$.

Secondly, the distributions $J_{\mathfrak{o}}^G(\eta,s,f)$ and $J^G(\eta,s,f)$ are independent of the choice of the minimal parabolic subgroup $\widetilde{P}_0$ of H. In fact, let $\widetilde{P}'_0$ be another minimal parabolic subgroup of H and $\sigma\in\Omega^H$ such that $\widetilde{P}'_0=\sigma^{-1}\widetilde{P}_0$. Put $P'_0:=\sigma^{-1}P_0$. Denote by $J_{\widetilde{P}'_0,\mathfrak{o}}^{G,T}(\eta,s,f)$ and $J_{\widetilde{P}'_0,\mathfrak{o}}^G(\eta,s,f)$ the distributions obtained starting from $\widetilde{P}'_0$ and P'_0. We can apply the argument of [13, Proposition 4.6] after some minor modifications here to prove that $J_{\mathfrak{o}}^{G,T}(\eta,s,f)=J_{\widetilde{P}'_0,\mathfrak{o}}^{G,\sigma^{-1}T}(\eta,s,f)$, so $J_{\mathfrak{o}}^G(\eta,s,f)=J_{\widetilde{P}'_0,\mathfrak{o}}^G(\eta,s,f)$.

6. Non-equivariance

Let Q be a relatively standard parabolic subgroup of G, $s \in \mathbb{R}$ and $y \in H(\mathbb{A}) \cap G(\mathbb{A})^1$. For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, define $f^{\eta}_{Q,s,y} \in \mathcal{S}((\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}))$ by

$$\forall X \in (\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}), f^{\eta}_{Q,s,y}(X) := \int_{K_H} \int_{(\mathfrak{n}_Q \cap \mathfrak{s})(\mathbb{A})} f(k^{-1}(X+V)k)\eta(\mathrm{Nrd}(k))p_{Q,s}(-H_Q(ky))dV dk, \tag{6.0.1}$$

where $p_{Q,s}$ is defined by the formula (5.3.1).

Proposition 6.1. *For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $y \in H(\mathbb{A}) \cap G(\mathbb{A})^1$, we denote $f^y(x) := f(yxy^{-1})$. Then for all sufficiently regular T, $\mathfrak{o} \in \mathcal{O}$ and $s \in \mathbb{R}$, we have*

$$J^{G,T}_{\mathfrak{o}}(\eta, s, f^y) = \eta(\mathrm{Nrd}(y))|\mathrm{Nrd}(y_1)|^s_{\mathbb{A}} \sum_{\{Q:\widetilde{P}_0 \subseteq Q\}} e^{(2\rho_{Q,+} - 2\rho_{Q_H} + s\Sigma_1)(T^G_Q)} J^{Q,T}_{\mathfrak{o}}(\eta, \{s^Q_i + s\}, f^{\eta}_{Q,s,y}),$$

where $J^{G,T}_{\mathfrak{o}}$ and $J^{Q,T}_{\mathfrak{o}}$ are defined by the formulae (5.0.1) and (5.1.2) respectively, $\{s^Q_i\}_{1 \le i \le l} \in \mathbb{Z}^l$ are the explicit constants determined by (5.4.1), and we write T^G_Q for the projection of $T_Q \in \mathfrak{a}_Q$ in $\mathfrak{a}^G_Q$ via the decomposition $\mathfrak{a}_Q = \mathfrak{a}^G_Q \oplus \mathfrak{a}_G$.

For $\mathfrak{o} \in \mathcal{O}$ and $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ (resp. $f' \in \mathcal{S}((\mathfrak{m}_Q \cap \mathfrak{s})(\mathbb{A}))$), thanks to Corollary 5.6 (resp. Remark 5.7), we may take the constant term $J^G_{\mathfrak{o}}(\eta, s, f)$ of $J^{G,T}_{\mathfrak{o}}(\eta, s, f)$ (resp. $J^Q_{\mathfrak{o}}(\eta, \{s_i\}, f')$ of $J^{Q,T}_{\mathfrak{o}}(\eta, \{s_i\}, f')$) for $s \in \mathbb{C}$ (resp. $\{s_i\}_{1 \le i \le l} \in \mathbb{C}^l$). When $s = 0$ (resp. $s_i = 0$ for all $1 \le i \le l$), denote $J^G_{\mathfrak{o}}(\eta, f) := J^G_{\mathfrak{o}}(\eta, 0, f)$ (resp. $J^Q_{\mathfrak{o}}(\eta, f') := J^Q_{\mathfrak{o}}(\eta, \{0\}, f')$).

Corollary 6.2. *Assume that $p = q = n$. Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A})), y \in H(\mathbb{A}) \cap G(\mathbb{A})^1$ and $\mathfrak{o} \in \mathcal{O}$. We have*

$$J^G_{\mathfrak{o}}(\eta, f^y) = \eta(\mathrm{Nrd}(y)) \sum_{\{Q:\widetilde{P}_0 \subseteq Q, \omega Q \omega^{-1} = Q\}} J^Q_{\mathfrak{o}}(\eta, f^{\eta}_{Q,0,y}).$$

Proof of Corollary 6.2. We apply Proposition 6.1 to the case $s = 0$ and consider the constant terms of both sides. Because $J^{Q,T}_{\mathfrak{o}}$ is independent of T_Q, by Lemma 5.2, only ω-stable Q contribute to the purely polynomial term. Then we apply Proposition 5.4 to the case $p = q = n$ to conclude. □

Proof of Proposition 6.1. By definition,

$$J^{G,T}_{\mathfrak{o}}(\eta, s, f^y) = \int_{H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} \left(\sum_{\{P:\widetilde{P}_0 \subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P_H(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x) - T_P) \cdot k_{f^y,P,\mathfrak{o}}(\delta x) \right) \cdot \eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx,$$

where

$$k_{f^y,P,\mathfrak{o}}(\delta x) = \sum_{X \in \mathfrak{m}_P(F) \cap \mathfrak{o}} \int_{(\mathfrak{n}_P \cap \mathfrak{s})(\mathbb{A})} f(y(\delta x)^{-1}(X+U)\delta x y^{-1}) dU = k_{f,P,\mathfrak{o}}(\delta x y^{-1}).$$

By change of variables, we have

$$J^{G,T}_{\mathfrak{o}}(\eta, s, f^y) = \int_{H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} \left(\sum_{\{P:\widetilde{P}_0 \subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P_H(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta xy) - T_P) \cdot k_{f,P,\mathfrak{o}}(\delta x) \right) \cdot \eta(\mathrm{Nrd}(xy))|\mathrm{Nrd}(x_1 y_1)|^s_{\mathbb{A}} dx.$$

For $x \in H(\mathbb{A})$ and P a relatively standard parabolic subgroup of G, let $k_P(x)$ be an element in K_H such that $x k_P(x)^{-1} \in P_H(\mathbb{A})$. Then

$$\widehat{\tau}^G_P(H_P(\delta xy) - T_P) = \widehat{\tau}^G_P(H_P(\delta x) - T_P + H_P(k_P(\delta x)y)).$$

Substituting $T_1 = H_P(\delta x) - T_P$ and $T_2 = -H_P(k_P(\delta x)y)$ in the definition of $\Gamma_P(T_1, T_2)$, we get

$$\widehat{\tau}^G_P(H_P(\delta xy) - T_P) = \sum_{\{Q:P \subseteq Q\}} (-1)^{\dim(A_Q/A_G)} \widehat{\tau}^Q_P(H_P(\delta x) - T_P) \Gamma_Q(H_P(\delta x) - T_P, -H_P(k_P(\delta x)y)).$$

Thus

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)=\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\}}(-1)^{\dim(A_P/A_G)}\sum_{\delta\in P_H(F)\backslash H(F)}$$
$$\left(\sum_{\{Q:P\subseteq Q\}}(-1)^{\dim(A_Q/A_G)}\widehat{\tau}_P^Q(H_P(\delta x)-T_P)\Gamma_Q(H_P(\delta x)-T_P,-H_P(k_P(\delta x)y))\right)$$
$$\cdot\, k_{f,P,\mathfrak{o}}(\delta x)\eta(\mathrm{Nrd}(xy))|\mathrm{Nrd}(x_1y_1)|_{\mathbb{A}}^s dx,$$

Exchanging the order of two sums over P and Q, and decomposing the sum over $P_H(F)\backslash H(F)$ into two sums over $P_H(F)\backslash Q_H(F)$ and $Q_H(F)\backslash H(F)$, we obtain

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in P_H(F)\backslash Q_H(F)}\sum_{\delta'\in Q_H(F)\backslash H(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta\delta' x)-T_P)\Gamma_Q(H_P(\delta\delta' x)-T_P,-H_P(k_P(\delta\delta' x)y))k_{f,P,\mathfrak{o}}(\delta\delta' x)\eta(\mathrm{Nrd}(xy))$$
$$\cdot\,|\mathrm{Nrd}(x_1y_1)|_{\mathbb{A}}^s dx.$$

Combining the integral over $H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1$ and the sum over $Q_H(F)\backslash H(F)$ into the integral over $Q_H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1$, and using the fact that

$$P_H(F)\backslash Q_H(F)\simeq (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F),$$

we have

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{Q_H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta x)-T_P)\Gamma_Q(H_P(\delta x)-T_P,-H_P(k_P(\delta x)y))k_{f,P,\mathfrak{o}}(\delta x)\eta(\mathrm{Nrd}(xy))|\mathrm{Nrd}(x_1y_1)|_{\mathbb{A}}^s dx.$$

By the Iwasawa decomposition and our choice of measures, we get

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{K_H}\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\int_{A_Q^{G,\infty}}\int_{N_{Q_H}(F)\backslash N_{Q_H}(\mathbb{A})}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}$$
$$\sum_{\delta\in (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\widehat{\tau}_P^Q(H_P(\delta namk)-T_P)\Gamma_Q(H_P(\delta namk)-T_P,-H_P(k_P(\delta namk)y))$$
$$\cdot\, k_{f,P,\mathfrak{o}}(\delta namk)\eta(\mathrm{Nrd}(mky))|\mathrm{Nrd}(a_1m_1y_1)|_{\mathbb{A}}^s e^{-2\rho_{Q_H}(H_{Q_H}(am))}dndadmdk.$$

As in the proof of Theorem 5.5, we see that

$$\widehat{\tau}_P^Q(H_P(\delta namk)-T_P)=\widehat{\tau}_P^Q(H_P(\delta m)-T_P),$$

and that

$$k_{f,P,\mathfrak{o}}(\delta namk)=e^{2\rho_{Q,+}(H_Q(a))}k_{f,P,\mathfrak{o}}(\delta mk).$$

In addition,

$$\begin{aligned}\Gamma_Q(H_P(\delta namk)-T_P,-H_P(k_P(\delta namk)y))&=\Gamma_Q(H_Q(\delta namk)-T_Q,-H_Q(k_P(\delta namk)y))\\&=\Gamma_Q(H_Q(a)-T_Q,-H_Q(k_Q(\delta namk)y))\\&=\Gamma_Q(H_Q(a)-T_Q,-H_Q(ky)).\end{aligned}$$

To sum up, the integrand in $J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)$ is independent of $n\in N_{Q_H}(F)\backslash N_{Q_H}(\mathbb{A})$. We can choose the Haar measure such that $\mathrm{vol}(N_{Q_H}(F)\backslash N_{Q_H}(\mathbb{A}))=1$. Then

$$J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)=\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}\int_{K_H}\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\int_{A_Q^{G,\infty}}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}$$
$$\sum_{\delta\in (P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\widehat{\tau}_P^Q(H_P(\delta m)-T_P)\Gamma_Q(H_Q(a)-T_Q,-H_Q(ky))$$
$$\cdot\, e^{2\rho_{Q,+}(H_Q(a))}k_{f,P,\mathfrak{o}}(\delta mk)\eta(\mathrm{Nrd}(mky))|\mathrm{Nrd}(a_1m_1y_1)|_{\mathbb{A}}^s e^{-2\rho_{Q_H}(H_{Q_H}(am))}dadmdk.$$

First, let us compute the integral on $A_Q^{G,\infty}$, which is

$$\begin{aligned}
&\int_{A_Q^{G,\infty}} |\mathrm{Nrd}(a_1)|_{\mathbb{A}}^s e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_Q(a))}\Gamma_Q(H_Q(a)-T_Q,-H_Q(ky))da\\
:=&\int_{\mathfrak{a}_Q^G} e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_1)}\Gamma_Q(T_1-T_Q,-H_Q(ky))dT_1\\
=&e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_Q^G)}\int_{\mathfrak{a}_Q^G} e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_1)}\Gamma_Q(T_1,-H_Q(ky))dT_1\\
=&e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_Q^G)}p_{Q,s}(-H_Q(ky)).
\end{aligned}$$

Next, we consider the integral on K_H, which is

$$\int_{K_H} k_{f,P,\mathfrak{o}}(\delta mk)\eta(\mathrm{Nrd}(k))p_{Q,s}(-H_Q(ky))dk.$$

As in the proof of Theorem 5.5, we see that

$$k_{f,P,\mathfrak{o}}(\delta mk)=e^{2\rho_{Q,+}(H_{Q_H}(m))}\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P^Q\cap\mathfrak{s})(\mathbb{A})}dU\int_{(\mathfrak{n}_Q\cap\mathfrak{s})(\mathbb{A})}f(k^{-1}((\delta m)^{-1}(X+U)\delta m+V)k)dV,$$

so we can write

$$\begin{aligned}
&\int_{K_H} k_{f,P,\mathfrak{o}}(\delta mk)\eta(\mathrm{Nrd}(k))p_{Q,s}(-H_Q(ky))dk\\
=&e^{2\rho_{Q,+}(H_{Q_H}(m))}\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_P^Q\cap\mathfrak{s})(\mathbb{A})} f_{Q,s,y}^{\eta}((\delta m)^{-1}(X+U)\delta m)dU\\
=&e^{2\rho_{Q,+}(H_{Q_H}(m))}\sum_{j=1}^{t} k^{M_Q}_{f^{\eta}_{Q,s,y},P\cap M_Q,\mathfrak{o}_j}(\delta m)
\end{aligned}$$

by (5.1.1). Hence

$$\begin{aligned}
J_{\mathfrak{o}}^{G,T}(\eta,s,f^y)=&\eta(\mathrm{Nrd}(y))|\mathrm{Nrd}(y_1)|_{\mathbb{A}}^s\sum_{\{Q:\widetilde{P}_0\subseteq Q\}}e^{(2\rho_{Q,+}-2\rho_{Q_H}+s\Sigma_1)(T_Q^G)}\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\\
&\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in(P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\widehat{\tau}_P^Q(H_P(\delta m)-T_P)\\
&\left(\sum_{j=1}^{t}k^{M_Q}_{f^{\eta}_{Q,s,y},P\cap M_Q,\mathfrak{o}_j}(\delta m)\right)\eta(\mathrm{Nrd}(m))|\mathrm{Nrd}(m_1)|_{\mathbb{A}}^s e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(m))}dm.
\end{aligned}$$

As in the proof of Theorem 5.5, we notice that

$$\begin{aligned}
J_{\mathfrak{o}}^{Q,T}(\eta,\{s_i^Q+s\},f^{\eta}_{Q,s,y})=&\int_{M_{Q_H}(F)\backslash M_{Q_H}(\mathbb{A})\cap M_Q(\mathbb{A})^1}\sum_{\{P:\widetilde{P}_0\subseteq P\subseteq Q\}}(-1)^{\dim(A_P/A_Q)}\sum_{\delta\in(P_H(F)\cap M_{Q_H}(F))\backslash M_{Q_H}(F)}\\
&\widehat{\tau}_P^Q(H_P(\delta m)-T_P)\left(\sum_{j=1}^{t}k^{M_Q}_{f^{\eta}_{Q,s,y},P\cap M_Q,\mathfrak{o}_j}(\delta m)\right)\eta(\mathrm{Nrd}(m))|\mathrm{Nrd}(m_1)|_{\mathbb{A}}^s\\
&\cdot e^{(2\rho_{Q,+}-2\rho_{Q_H})(H_{Q_H}(m))}dm.
\end{aligned}$$

Then we finish the proof. □

7. An infinitesimal trace formula for $\mathrm{Mat}_{p\times q,D}\oplus\mathrm{Mat}_{q\times p,D}//GL_{p,D}\times GL_{q,D}$

THEOREM 7.1. *For $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $s\in\mathbb{R}$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}}J_{\mathfrak{o}}^G(\eta,s,f)=\sum_{\mathfrak{o}\in\mathcal{O}}J_{\mathfrak{o}}^G(\eta,s,\hat{f}),$$

where $\hat{f}$ is the Fourier transform of f defined by (3.5.2), and $J_{\mathfrak{o}}^G(\eta,s,\cdot)$ denotes the constant term of $J_{\mathfrak{o}}^{G,T}(\eta,s,\cdot)$.

Proof. From the Poisson summation formula, we know that for any $x \in H(\mathbb{A})$,

$$\sum_{X \in \mathfrak{s}(F)} f(x^{-1}Xx) = \sum_{X \in \mathfrak{s}(F)} \hat{f}(x^{-1}Xx),$$

i.e.,

$$k_{f,G}(x) = k_{\hat{f},G}(x).$$

Using Corollary 4.13, for all sufficiently regular T satisfying $\alpha(T) \geq \epsilon_0 \parallel T \parallel$ for any $\alpha \in \Delta_{P_0}$, we have

$$\left| J^{G,T}(\eta, s, f) - \int_{H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} F^G(x,T) k_{f,G}(x) \eta(\mathrm{Nrd}(x)) |\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx \right| \leq C_1 e^{-N\|T\|}$$

and

$$\left| J^{G,T}(\eta, s, \hat{f}) - \int_{H(F)\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1} F^G(x,T) k_{\hat{f},G}(x) \eta(\mathrm{Nrd}(x)) |\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx \right| \leq C_2 e^{-N\|T\|}.$$

Thus

$$|J^{G,T}(\eta, s, f) - J^{G,T}(\eta, s, \hat{f})| \leq (C_1 + C_2) e^{-N\|T\|}.$$

By Corollary 5.6, we know that both of $J^{G,T}(\eta, s, f)$ and $J^{G,T}(\eta, s, \hat{f})$ are exponential polynomials in T. Because we can choose N to be large enough, we deduce that

$$J^{G,T}(\eta, s, f) = J^{G,T}(\eta, s, \hat{f}).$$

Since

$$J^{G,T}(\eta, s, f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^{G,T}(\eta, s, f)$$

and

$$J^{G,T}(\eta, s, \hat{f}) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^{G,T}(\eta, s, \hat{f}),$$

we obtain

$$\sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^{G,T}(\eta, s, f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^{G,T}(\eta, s, \hat{f}).$$

We may conclude by taking the constant terms of both sides. □

8. The second modified kernel

In this section and the next, we shall focus on the case where $p = q = n$ in order to get better description for distributions associated to regular semi-simple orbits. We shall change our notation by denoting $G := GL_{2n,D}$ and $H := GL_{n,D} \times GL_{n,D}$ without further mention.

Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, P be a relatively standard parabolic subgroup of G and $\mathfrak{o} \in \mathcal{O}_{rs}$ (see Section 3.3). For $x \in P_H(F)\backslash H(\mathbb{A})$, define

$$j_{f,P,\mathfrak{o}}(x) := \sum_{X \in \mathfrak{m}_P(F) \cap \mathfrak{o}} \sum_{n \in N_{P_H}(F)} f((nx)^{-1}Xnx).$$

Let $T \in \mathfrak{a}_0$. For $x \in H(F)\backslash H(\mathbb{A})$, define

$$j_{f,\mathfrak{o}}^T(x) := \sum_{\{P : \widetilde{P}_0 \subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P_H(F)\backslash H(F)} \widehat{\tau}_P^G(H_P(\delta x) - T_P) \cdot j_{f,P,\mathfrak{o}}(\delta x).$$

From [3, Lemma 5.1], we know that the sum over $\delta \in P_H(F)\backslash H(F)$ is finite. Recall that since $\mathfrak{o} \in \mathcal{O}_{rs} \subseteq \mathcal{O}^\times$, if $\mathfrak{m}_P(F) \cap \mathfrak{o} \neq \emptyset$, then P is ω-stable by Proposition 5.1. Thus the above definitions only involve the relatively standard parabolic subgroups that are ω-stable.

Lemma 8.1. *Let P be a relatively standard parabolic subgroup of G and $\mathfrak{o} \in \mathcal{O}_{rs}$. For $X \in \mathfrak{m}_P(F) \cap \mathfrak{o}$, the map*

$$N_{P_H} \to \mathfrak{n}_P \cap \mathfrak{s}, n \mapsto n^{-1}Xn - X$$

is an F-isomorphism of algebraic varieties and preserves the Haar measures on $\mathbb{A}$-points.

Proof. Since P is relatively standard and ω-stable, we can suppose

$$P = \begin{pmatrix} \mathfrak{p}_{n,D} & \mathfrak{p}_{n,D} \\ \mathfrak{p}_{n,D} & \mathfrak{p}_{n,D} \end{pmatrix}^{\times},$$

where

$$P_{n,D} = \begin{pmatrix} GL_{n_1,D} & \mathrm{Mat}_{n_1\times n_2,D} & \cdots & \mathrm{Mat}_{n_1\times n_l,D} \\ & GL_{n_2,D} & \cdots & \mathrm{Mat}_{n_2\times n_l,D} \\ & & \ddots & \vdots \\ & & & GL_{n_l,D} \end{pmatrix}.$$

Then we have

$$\mathfrak{m}_P \cap \mathfrak{s} = \begin{pmatrix} & \mathfrak{m}_{P_{n,D}} \\ \mathfrak{m}_{P_{n,D}} & \end{pmatrix}, N_{P_H} = \begin{pmatrix} N_{P_{n,D}} & \\ & N_{P_{n,D}} \end{pmatrix}, \mathfrak{n}_P \cap \mathfrak{s} = \begin{pmatrix} & \mathfrak{n}_{P_{n,D}} \\ \mathfrak{n}_{P_{n,D}} & \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} & & & A_1 & & \\ & & & & \ddots & \\ & & & & & A_l \\ B_1 & & & & & \\ & \ddots & & & & \\ & & B_l & & & \end{pmatrix} \in \mathfrak{m}_P(F) \cap \mathfrak{o},$$

where $A_i, B_i \in GL_{n_i}(D)$ for $1 \le i \le l$, and

$$n = \begin{pmatrix} 1 & C_{12} & \cdots & C_{1l} & & & & \\ & 1 & \cdots & C_{2l} & & & & \\ & & \ddots & \vdots & & & & \\ & & & 1 & & & & \\ & & & & 1 & D_{12} & \cdots & D_{1l} \\ & & & & & 1 & \cdots & D_{2l} \\ & & & & & & \ddots & \vdots \\ & & & & & & & 1 \end{pmatrix} \in N_{P_H},$$

where $C_{ij}, D_{ij} \in \mathrm{Mat}_{n_i\times n_j,D}$ for $1 \le i < j \le l$. Then

$$Xn - nX = \begin{pmatrix} & & & & 0 & A_1D_{12} - C_{12}A_2 & \cdots & A_1D_{1l} - C_{1l}A_l \\ & & & & & 0 & \cdots & A_2D_{2l} - C_{2l}A_l \\ & & & & & & \ddots & \vdots \\ & & & & & & & 0 \\ 0 & B_1C_{12} - D_{12}B_2 & \cdots & B_1C_{1l} - D_{1l}B_l & & & & \\ & 0 & \cdots & B_2C_{2l} - D_{2l}B_l & & & & \\ & & \ddots & \vdots & & & & \\ & & & 0 & & & & \end{pmatrix} \in \mathfrak{n}_P \cap \mathfrak{s}.$$

We claim that the morphism of F-affine spaces

$$\begin{aligned} \mathrm{Mat}_{n_i\times n_j,D} \oplus \mathrm{Mat}_{n_i\times n_j,D} &\to \mathrm{Mat}_{n_i\times n_j,D} \oplus \mathrm{Mat}_{n_i\times n_j,D} \\ (C_{ij}, D_{ij}) &\mapsto (A_iD_{ij} - C_{ij}A_j, B_iC_{ij} - D_{ij}B_j) \end{aligned}$$

induces an F-linear isomorphism on F-points. In fact, since it gives an F-linear map between finite dimensional linear spaces of the same dimension, we only need to prove that this map is injective under base change to an algebraic closure of F. Then without loss of generality, it suffices to consider the case where $D = F$. If $A_iD_{ij} - C_{ij}A_j = B_iC_{ij} - D_{ij}B_j = 0$, then $C_{ij}A_jB_j = A_iD_{ij}B_j = A_iB_iC_{ij}$ and $D_{ij}B_jA_j = B_iC_{ij}A_j = B_iA_iD_{ij}$. Since X is regular semi-simple, A_iB_i and A_jB_j (resp. B_iA_i and B_jA_j) have no common eigenvalue. By the classical theory of Sylvester equation, we know that $C_{ij} = D_{ij} = 0$ and conclude.

From this claim, we know that the map

$$N_{P_H} \to \mathfrak{n}_P \cap \mathfrak{s}, n \mapsto Xn - nX$$

is an F-isomorphism of algebraic varieties and preserves the Haar measures on $\mathbb{A}$-points. Notice that $n^{-1}Xn - X = n^{-1}(Xn - nX)$. It is not hard to check that here n^{-1} functions as some translation $A_iD_{ij} - C_{ij}A_j \mapsto A_iD_{ij} - C_{ij}A_j +$ (a polynomial of $C_{i'j'}$and $D_{i'j'}, i' > i, j' \leq j$ or $i' \geq i, j' < j$), so an analogous assertion still holds for the map $n \mapsto n^{-1}Xn - X$. $\square$

THEOREM 8.2. *For all sufficiently regular T, all $s \in \mathbb{R}$ and $\mathfrak{o} \in \mathcal{O}_{rs}$,*

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} |j^T_{f,\mathfrak{o}}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx < \infty,$$

where we write $x = (x_1, x_2) \in GL_{n,D}(\mathbb{A}) \times GL_{n,D}(\mathbb{A})$. Moreover, for $s \in \mathbb{C}$,

$$J^{G,T}_{\mathfrak{o}}(\eta, s, f) = \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx.$$

PROOF. As in the proof of Theorem 4.11, using the left invariance of $j_{f,P,\mathfrak{o}}$ by $P_H(F)$, we reduce ourselves to proving

$$\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x)|j_{P_1,P_2,\mathfrak{o}}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}} dx < \infty,$$

where $P_1 \subsetneq P_2$ are a pair of relatively standard parabolic subgroups of G and for $x \in P_{1,H}(F)\backslash H(\mathbb{A})$, we put

$$j_{P_1,P_2,\mathfrak{o}}(x) := \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)} j_{f,P,\mathfrak{o}}(x).$$

In addition,

$$j_{f,P,\mathfrak{o}}(x) = \sum_{\{R:P_1\subseteq R\subseteq P\}} \sum_{\xi\in\widetilde{m}^R_{P_1}(F)\cap\mathfrak{o}} \sum_{X\in(\mathfrak{n}^P_R\cap\mathfrak{s})(F)} \sum_{n\in N_{P_H}(F)} f((nx)^{-1}(\xi + X)nx).$$

Applying Lemma 8.1, we get

$$\begin{aligned} j_{f,P,\mathfrak{o}}(x) &= \sum_{\{R:P_1\subseteq R\subseteq P\}} \sum_{\xi\in\widetilde{m}^R_{P_1}(F)\cap\mathfrak{o}} \sum_{X\in(\mathfrak{n}^P_R\cap\mathfrak{s})(F)} \sum_{u\in(\mathfrak{n}_P\cap\mathfrak{s})(F)} f(x^{-1}(\xi + X + u)x) \\ &= \sum_{\{R:P_1\subseteq R\subseteq P\}} \sum_{\xi\in\widetilde{m}^R_{P_1}(F)\cap\mathfrak{o}} \sum_{X\in(\mathfrak{n}_R\cap\mathfrak{s})(F)} f(x^{-1}(\xi + X)x). \end{aligned}$$

Hence

$$\begin{aligned} j_{P_1,P_2,\mathfrak{o}}(x) &= \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)} \left(\sum_{\{R:P_1\subseteq R\subseteq P\}} \sum_{\xi\in\widetilde{m}^R_{P_1}(F)\cap\mathfrak{o}} \sum_{X\in(\mathfrak{n}_R\cap\mathfrak{s})(F)} f(x^{-1}(\xi + X)x) \right) \\ &= \sum_{\{R:P_1\subseteq R\subseteq P_2\}} \sum_{\xi\in\widetilde{m}^R_{P_1}(F)\cap\mathfrak{o}} \left(\sum_{\{P:R\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)} \right) \sum_{X\in(\mathfrak{n}_R\cap\mathfrak{s})(F)} f(x^{-1}(\xi + X)x). \end{aligned}$$

By [3, Proposition 1.1], we have

$$j_{P_1,P_2,\mathfrak{o}}(x) = (-1)^{\dim(A_{P_2}/A_G)} \sum_{\xi\in\widetilde{m}^{P_2}_{P_1}(F)\cap\mathfrak{o}} \sum_{X\in(\mathfrak{n}_{P_2}\cap\mathfrak{s})(F)} f(x^{-1}(\xi + X)x).$$

Applying Lemma 8.1 again, we obtain

$$j_{P_1,P_2,\mathfrak{o}}(x) = (-1)^{\dim(A_{P_2}/A_G)} \sum_{\xi\in\widetilde{m}^{P_2}_{P_1}(F)\cap\mathfrak{o}} \sum_{n_2\in N_{P_{2,H}}(F)} f((n_2x)^{-1}\xi n_2x),$$

where we denote $P_{2,H} := P_2 \cap H$.

Decomposing the integral over $x \in P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1$ into double integrals $n_1 \in N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})$ and $y \in M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1$, and using the fact that $\chi^T_{P_1,P_2}(x)$ is left invariant under

$N_{P_{1,H}}(\mathbb{A})$, we have

$$\begin{aligned}
&\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x)|j_{P_1,P_2,\mathfrak{o}}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx \\
=&\int_{M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \chi^T_{P_1,P_2}(n_1y) \\
&\cdot\left|\sum_{\xi\in\widetilde{m}^{P_2}_{P_1}(F)\cap\mathfrak{o}}\sum_{n_2\in N_{P_{2,H}}(F)} f((n_2n_1y)^{-1}\xi n_2n_1y)\right||\mathrm{Nrd}(y_1)|^s_{\mathbb{A}}dn_1dy \\
\leq&\int_{M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(y)\sum_{\xi\in\widetilde{m}^{P_2}_{P_1}(F)\cap\mathfrak{o}} \\
&\left(\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}\sum_{n_2\in N_{P_{2,H}}(F)}|f((n_2n_1y)^{-1}\xi n_2n_1y)|dn_1\right)|\mathrm{Nrd}(y_1)|^s_{\mathbb{A}}dy.
\end{aligned}$$

Since $P_{1,H}\subseteq P_{2,H}$ and $\mathrm{vol}(N_{P_{2,H}}(F)\backslash N_{P_{2,H}}(\mathbb{A}))=1$, we see that

$$\begin{aligned}
&\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}\sum_{n_2\in N_{P_{2,H}}(F)}|f((n_2n_1y)^{-1}\xi n_2n_1y)|dn_1 \\
=&\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}\int_{N_{P_{2,H}}(F)\backslash N_{P_{2,H}}(\mathbb{A})}\sum_{n_2\in N_{P_{2,H}}(F)}|f((n_2nn_1y)^{-1}\xi n_2nn_1y)|dndn_1 \\
=&\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}\int_{N_{P_{2,H}}(\mathbb{A})}|f((nn_1y)^{-1}\xi nn_1y)|dndn_1 \\
=&\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}\int_{(\mathfrak{n}_{P_2}\cap\mathfrak{s})(\mathbb{A})}|f((n_1y)^{-1}(\xi+U)n_1y)|dUdn_1,
\end{aligned}$$

where we have applied Lemma 8.1 in the last equality. Therefore

$$\begin{aligned}
&\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)|j_{P_1,P_2,\mathfrak{o}}(x)||\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx \\
\leq&\int_{M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(y)\sum_{\xi\in\widetilde{m}^{P_2}_{P_1}(F)\cap\mathfrak{o}} \\
&\left(\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})}\int_{(\mathfrak{n}_{P_2}\cap\mathfrak{s})(\mathbb{A})}|f((n_1y)^{-1}(\xi+U)n_1y)|dUdn_1\right)|\mathrm{Nrd}(y_1)|^s_{\mathbb{A}}dy \\
=&\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\xi\in\widetilde{m}^{P_2}_{P_1}(F)\cap\mathfrak{o}}\left(\int_{(\mathfrak{n}_{P_2}\cap\mathfrak{s})(\mathbb{A})}|f(x^{-1}(\xi+U)x)|dU\right)|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx,
\end{aligned}$$

whose convergence results from that of the formula (4.2.2) when $R=P_2$.

Now we begin to prove the second statement. From the first statement, now we have the right to write

$$\begin{aligned}
&\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx \\
=&\sum_{\{P_1,P_2:\widetilde{P}_0\subseteq P_1\subseteq P_2\}}\int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)j_{P_1,P_2,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx,
\end{aligned}$$

where

$$j_{P_1,P_2,\mathfrak{o}}(x) = \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)} j_{f,P,\mathfrak{o}}(x)$$
$$= \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)} \left(\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \sum_{n\in N_{P_H}(F)} f((nx)^{-1}Xnx) \right).$$

Decompose the integral over $x \in P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1$ into double integrals over $n_1 \in N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})$ and $y \in M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A}) \cap G(\mathbb{A})^1$. Since $N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})$ is compact, by Lemma 8.1 and [53, §41],

$$\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \sum_{n\in N_{P_H}(F)} |f((nn_1y)^{-1}Xnn_1y)| = \sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \sum_{u\in(\mathfrak{n}_P\cap\mathfrak{s})(F)} |f((n_1y)^{-1}(X+u)n_1y)|$$

is bounded on $n_1 \in N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})$. Then using the fact that $\chi^T_{P_1,P_2}(x)$ is left invariant under $N_{P_{1,H}}(\mathbb{A})$, we have

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$
$$= \sum_{\{P_1,P_2:\widetilde{P}_0\subseteq P_1\subseteq P_2\}} \int_{M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(y) \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)}$$
$$\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \left(\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \sum_{n\in N_{P_H}(F)} f((nn_1y)^{-1}Xnn_1y)dn_1 \right) \eta(\mathrm{Nrd}(y))|\mathrm{Nrd}(y_1)|^s_{\mathbb{A}}dy.$$

Since $P_{1,H} \subseteq P_H$ and $\mathrm{vol}(N_{P_H}(F)\backslash N_{P_H}(\mathbb{A})) = 1$, we see that

$$\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \sum_{n\in N_{P_H}(F)} f((nn_1y)^{-1}Xnn_1y)dn_1$$
$$= \int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \int_{N_{P_H}(F)\backslash N_{P_H}(\mathbb{A})} \sum_{n\in N_{P_H}(F)} f((nn_2n_1y)^{-1}Xnn_2n_1y)dn_2dn_1$$
$$= \int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \int_{N_{P_H}(\mathbb{A})} f((nn_1y)^{-1}Xnn_1y)dndn_1$$
$$= \int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f((n_1y)^{-1}(X+U)n_1y)dUdn_1,$$

where we have applied Lemma 8.1 in the last equality. Therefore

$$\int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$
$$= \sum_{\{P_1,P_2:\widetilde{P}_0\subseteq P_1\subseteq P_2\}} \int_{M_{P_{1,H}}(F)N_{P_{1,H}}(\mathbb{A})\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(y) \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)}$$
$$\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \left(\int_{N_{P_{1,H}}(F)\backslash N_{P_{1,H}}(\mathbb{A})} \int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f((n_1y)^{-1}(X+U)n_1y)dUdn_1 \right) \eta(\mathrm{Nrd}(y))|\mathrm{Nrd}(y_1)|^s_{\mathbb{A}}dy$$
$$= \sum_{\{P_1,P_2:\widetilde{P}_0\subseteq P_1\subseteq P_2\}} \int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_G)}$$
$$\cdot \left(\sum_{X\in\mathfrak{m}_P(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_P\cap\mathfrak{s})(\mathbb{A})} f(x^{-1}(X+U)x)dU \right) \eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx$$
$$= \sum_{\{P_1,P_2:\widetilde{P}_0\subseteq P_1\subseteq P_2\}} \int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi^T_{P_1,P_2}(x)k_{P_1,P_2,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|^s_{\mathbb{A}}dx.$$

From Theorem 4.11, we are authorised to write

$$\begin{aligned} J_{\mathfrak{o}}^{G,T}(\eta,s,f) &= \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} k_{f,\mathfrak{o}}^T(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx \\ &= \sum_{\{P_1,P_2:\widetilde{P}_0\subseteq P_1\subseteq P_2\}} \int_{P_{1,H}(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} \chi_{P_1,P_2}^T(x)k_{P_1,P_2,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s dx, \end{aligned}$$

which completes the proof. □

9. Weighted orbital integrals

As in the last section, we shall assume that $p = q = n$ in the following discussion. Moreover, we shall suppose that $s = 0$ in the orbital integral for convenience, since $|\mathrm{Nrd}(x_1)|_{\mathbb{A}}^s$ is not invariant under the translation by A_G^∞. Recall that for $\mathfrak{o} \in \mathcal{O}$ and $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, we denote by $J_{\mathfrak{o}}^G(\eta, f)$ the constant term of $J_{\mathfrak{o}}^{G,T}(\eta, 0, f)$.

9.1. Weyl groups. From Section 5.5, we may choose P_0 to be the stabiliser in G of the flag

$$0 \subsetneq \langle e_1\rangle_D \subsetneq \langle e_1, f_1\rangle_D \subsetneq \langle e_1, f_1, e_2\rangle_D \subsetneq \langle e_1, f_1, e_2, f_2\rangle_D \subsetneq \cdots \subsetneq \langle e_1, f_1 \cdots, e_n, f_n\rangle_D = V \oplus W$$

by the notation in Section 3.4. Then all ω-stable relatively standard parabolic subgroups of G contain P_0. Denote by $\mathscr{P}_0$ the stabiliser in G of the flag

$$0 \subsetneq \langle e_1, f_1\rangle_D \subsetneq \langle e_1, f_1, e_2, f_2\rangle_D \subsetneq \cdots \subsetneq \langle e_1, f_1 \cdots, e_n, f_n\rangle_D = V \oplus W.$$

It is the minimal ω-stable relatively standard parabolic subgroup of G. A parabolic subgroup P of G is relatively standard and ω-stable if and only if $\mathscr{P}_0 \subseteq P$. Let $\mathscr{P}_{0,n}$ be the group of upper triangular matrices in $GL_{n,D}$. We can talk about positive roots for G, H and $GL_{n,D}$ with respect to $P_0, \widetilde{P}_0$ and $\mathscr{P}_{0,n}$ respectively.

Lemma 9.1. *Let* $P_1 = \begin{pmatrix} \mathfrak{p}_{1,n} & \mathfrak{p}_{1,n} \\ \mathfrak{p}_{1,n} & \mathfrak{p}_{1,n} \end{pmatrix}^\times$ *and* $P_2 = \begin{pmatrix} \mathfrak{p}_{2,n} & \mathfrak{p}_{2,n} \\ \mathfrak{p}_{2,n} & \mathfrak{p}_{2,n} \end{pmatrix}^\times$ *be a pair of ω-stable relatively standard parabolic subgroups of G, where $P_{1,n}$ and $P_{2,n}$ are standard parabolic subgroups of $GL_{n,D}$.*

1) The map $s_n \mapsto s = \begin{pmatrix} s_n & \\ & s_n \end{pmatrix}$ *induces a bijection from*

a) the set of representatives s_n of $\Omega^{GL_{n,D}}(\mathfrak{a}_{P_{1,n}}, \mathfrak{a}_{P_{2,n}})$ in $\Omega^{GL_{n,D}}$ such that $s_n^{-1}\alpha > 0$ for all $\alpha \in \Delta_{\mathscr{P}_{0,n}}^{P_{2,n}}$

to

b) the set of representatives s of $\Omega^G(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ in Ω^G such that $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_{P_0}^{P_2}$.

2) The map $s_n \mapsto s = \begin{pmatrix} s_n & \\ & s_n \end{pmatrix}$ *induces a bijection from*

a) the set of representatives s_n of $\Omega^{GL_{n,D}}(\mathfrak{a}_{P_{1,n}}; P_{2,n})$ in $\Omega^{GL_{n,D}}$ such that $s_n(\mathfrak{a}_{P_{1,n}}) \supseteq \mathfrak{a}_{P_{2,n}}$ and $s_n^{-1}\alpha > 0$ for all $\alpha \in \Delta_{\mathscr{P}_{0,n}}^{P_{2,n}}$

to

b) the set of representatives s of $\Omega^G(\mathfrak{a}_{P_1}; P_2)$ in Ω^G such that $s(\mathfrak{a}_{P_1}) \supseteq \mathfrak{a}_{P_2}$ and $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_{P_0}^{P_2}$.

Proof. Suppose that $P_{1,n}$ and $P_{2,n}$ correspond to the partitions $(n_1, \cdots, n_l)$ and $(n'_1, \cdots, n'_{l'})$ respectively of n. Then P_1 and P_2 correspond to the partitions $(2n_1, \cdots, 2n_l)$ and $(2n'_1, \cdots, 2n'_{l'})$ respectively of $2n$. For an integer $m > 0$, denote by S_m the symmetric group of degree m.

1) From [9, p. 33], the set $\Omega^{GL_{n,D}}(\mathfrak{a}_{P_{1,n}}, \mathfrak{a}_{P_{2,n}})$ is empty unless $l = l'$, in which case

$$\Omega^{GL_{n,D}}(\mathfrak{a}_{P_{1,n}}, \mathfrak{a}_{P_{2,n}}) \simeq \{s_n \in S_l : n'_i = n_{s_n(i)}, 1 \le i \le l\}. \tag{9.1.1}$$

Similarly, the set $\Omega^G(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ is empty unless $l = l'$, in which case

$$\Omega^G(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2}) \simeq \{s \in S_l : 2n'_i = 2n_{s(i)}, 1 \le i \le l\}. \tag{9.1.2}$$

The map in the lemma is induced by the obvious bijection between the right hand sides of (9.1.1) and (9.1.2).

2) From [9, p. 59], the set a) is identified with the set of $s_n \in S_l \subseteq S_n$ such that $(n_{s_n(1)}, \cdots, n_{s_n(l)})$ is finer than $(n'_1, \cdots, n'_{l'})$, and such that $s_n^{-1}(i) < s_n^{-1}(i+1)$ for any $1 \le i \le n-1$ that is not of the form

$n'_1+\cdots+n'_k$ for some $1\leq k\leq l'$. Similarly, the set b) is identified with the set of $s\in S_l\subseteq S_{2n}$ such that $(2n_{s(1)},\cdots,2n_{s(l)})$ is finer than $(2n'_1,\cdots,2n'_{l'})$, and such that $s^{-1}(i)<s^{-1}(i+1)$ for any $1\leq i\leq 2n-1$ that is not of the form $2n'_1+\cdots+2n'_k$ for some $1\leq k\leq l'$. The map in the lemma is induced by the obvious bijection between these two sets. □

For P_1 and P_2 a pair of ω-stable relatively standard parabolic subgroups of G, denote by $\Omega^H(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$ the set (perhaps empty) of distinct isomorphisms from $\mathfrak{a}_{P_1}$ to $\mathfrak{a}_{P_2}$ obtained by restriction of elements in Ω^H. It is a subset of $\Omega^G(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$ *a priori*. However, since the image of the map in Lemma 9.1.1) is contained in Ω^H, we actually have $\Omega^H(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})=\Omega^G(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$ (cf. [39, Lemme 2.8.1]). Denote by $\Omega^H(\mathfrak{a}_{P_1};P_2)$ the set of $s\in\bigcup_{\mathfrak{a}_Q}\Omega^H(\mathfrak{a}_{P_1},\mathfrak{a}_Q)$ such that $s(\mathfrak{a}_{P_1})\supseteq\mathfrak{a}_{P_2}$ and $s^{-1}\alpha>0$ for each $\alpha\in\Delta_{Q_H}^{P_2\cap H}$, where the union takes over all $\mathfrak{a}_Q$ associated to some ω-stable relatively standard parabolic subgroup Q of G. Then $\Omega^H(\mathfrak{a}_{P_1};P_2)=\Omega^G(\mathfrak{a}_{P_1};P_2)$ by Lemma 9.1.2).

9.2. Regular semi-simple terms. Let $\mathfrak{o}\in\mathcal{O}_{rs}$ (see Section 3.3). It is possible to choose an element $X_1\in\mathfrak{o}$ and a relatively standard parabolic subgroup P_1 of G such that $X_1\in\mathfrak{m}_{P_1}(F)$ (thus P_1 is ω-stable by Proposition 5.1) but X_1 can not be $H(F)$-conjugate to an element in the Lie algebra of any relatively standard parabolic subgroup $R\subsetneq P_1$. We call such X_1 an elliptic element in $(\mathfrak{m}_{P_1}\cap\mathfrak{s})(F)$.

Let $P_1=\begin{pmatrix}\mathfrak{p}_{1,n} & \mathfrak{p}_{1,n}\\ \mathfrak{p}_{1,n} & \mathfrak{p}_{1,n}\end{pmatrix}^{\times}$ be an ω-stable relatively standard parabolic subgroup of G, where $P_{1,n}$ is a standard parabolic subgroup of $GL_{n,D}$. Let $X_1=\begin{pmatrix}0 & A_1\\ B_1 & 0\end{pmatrix}\in(\mathfrak{m}_{P_1}\cap\mathfrak{s})(F)$ be a regular semi-simple element in $\mathfrak{s}$. Then X_1 is elliptic in $(\mathfrak{m}_{P_1}\cap\mathfrak{s})(F)$ if and only if A_1B_1 is elliptic in $\mathfrak{m}_{P_{1,n}}(F)$ in the usual sense, i.e., $\mathrm{Prd}_{A_1B_1}$ is irreducible (see [56, Proposition 5] for example). Let H_{X_1} be the centraliser of X_1 in H. Then X_1 is elliptic in $(\mathfrak{m}_{P_1}\cap\mathfrak{s})(F)$ if and only if the maximal F-split torus in H_{X_1} is A_{P_1}.

THEOREM 9.2. *Let $\mathfrak{o}\in\mathcal{O}_{rs}, P_1$ be a relatively standard parabolic subgroup of G and $X_1\in\mathfrak{o}$ be an elliptic element in $(\mathfrak{m}_{P_1}\cap\mathfrak{s})(F)$. For $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$, we have*

$$J_{\mathfrak{o}}^G(\eta,f)=\mathrm{vol}(A_{P_1}^{\infty}H_{X_1}(F)\backslash H_{X_1}(\mathbb{A}))\cdot\int_{H_{X_1}(\mathbb{A})\backslash H(\mathbb{A})}f(x^{-1}X_1x)v_{P_1}(x)\eta(\mathrm{Nrd}(x))dx,$$

where $v_{P_1}(x)$ is left-invariant under $H_{X_1}(\mathbb{A})$ and equals the volume of the projection onto $\mathfrak{a}_{P_1}^G$ of the convex hull of $\{-H_Q(x)\}$, where Q takes over all semi-standard parabolic subgroups of G with $M_Q=M_{P_1}$.

PROOF. Consider a relatively standard parabolic subgroup P of G and $X\in\mathfrak{m}_P(F)\cap\mathfrak{o}$ (thus P is ω-stable by Proposition 5.1). There exists an ω-stable relatively standard parabolic subgroup $P_2\subseteq P$ and $X_2\in(\mathfrak{m}_{P_2}\cap\mathfrak{s})(F)$ in the form of $\begin{pmatrix} & 1\\ * & \end{pmatrix}$ such that X_2 is conjugate to X via an element in $M_{P_H}(F)$ and the maximal F-split torus in H_{X_2} is A_{P_2}. Then any element in $H(F)$ which conjugates X_1 and X_2 will conjugate A_{P_1} and A_{P_2}. It follows that there exists $s\in\Omega^H(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$ and $m\in M_{P_H}(F)$ such that

$$X=m\omega_sX_1\omega_s^{-1}m^{-1}.$$

Suppose that $P_3\subseteq P$ is another relatively standard parabolic subgroup, $s'\in\Omega^H(\mathfrak{a}_{P_1},\mathfrak{a}_{P_3})$ and $m'\in M_{P_H}(F)$ such that

$$X=m'\omega_{s'}X_1\omega_{s'}^{-1}m'^{-1}.$$

Then there is $\zeta\in H_X(F)$ such that

$$m'\omega_{s'}=\zeta m\omega_s.$$

Since $H_X\subseteq M_{P_H}$, we see that

$$\omega_{s'}=\xi\omega_s$$

for some $\xi\in M_{P_H}(F)$. In sum, for any given P a relatively standard parabolic subgroup of G and $X\in\mathfrak{m}_P(F)\cap\mathfrak{o}$, there is a unique $s\in\Omega^H(\mathfrak{a}_{P_1};P)$ such that $X=m\omega_sX_1\omega_s^{-1}m^{-1}$ for some $m\in M_{P_H}(F)$.

For $x \in P_H(F)\backslash H(\mathbb{A})$, we obtain

$$\begin{aligned} j_{f,P,\mathfrak{o}}(x) &= \sum_{X \in \mathfrak{m}_P(F)\cap\mathfrak{o}} \sum_{n \in N_{P_H}(F)} f((nx)^{-1}Xnx) \\ &= \sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \sum_{m \in M_{P_H,\omega_s X_1 \omega_s^{-1}}(F)\backslash M_{P_H}(F)} \sum_{n\in N_{P_H}(F)} f((mnx)^{-1}\omega_s X_1 \omega_s^{-1} mnx) \\ &= \sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \sum_{m \in M_{P_H,\omega_s X_1 \omega_s^{-1}}(F)\backslash P_H(F)} f((mx)^{-1}\omega_s X_1 \omega_s^{-1} mx), \end{aligned}$$

where $M_{P_H,\omega_s X_1\omega_s^{-1}}$ denotes the centraliser of $\omega_s X_1 \omega_s^{-1}$ in M_{P_H}. For $T \in \mathfrak{a}_0$ and $x \in H(F)\backslash H(\mathbb{A})$, we have

$$\begin{aligned} j^T_{f,\mathfrak{o}}(x) &= \sum_{\{P:\mathscr{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P_H(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x)-T_P)\cdot j_{f,P,\mathfrak{o}}(\delta x) \\ &= \sum_{\{P:\mathscr{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{\delta\in P_H(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x)-T_P) \\ &\quad \cdot \left(\sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \sum_{m\in M_{P_H,\omega_s X_1\omega_s^{-1}}(F)\backslash P_H(F)} f((m\delta x)^{-1}\omega_s X_1\omega_s^{-1} m\delta x) \right) \\ &= \sum_{\{P:\mathscr{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \sum_{\delta\in M_{P_H,\omega_s X_1\omega_s^{-1}}(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x)-T_P) \\ &\quad \cdot f((\delta x)^{-1}\omega_s X_1\omega_s^{-1}\delta x). \end{aligned}$$

Notice that the centraliser of $\omega_s X_1\omega_s^{-1}$ in H is actually contained in M_{P_H}. We deduce that

$$\begin{aligned} j^T_{f,\mathfrak{o}}(x) &= \sum_{\{P:\mathscr{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \sum_{\delta\in H_{\omega_s X_1\omega_s^{-1}}(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\delta x)-T_P)\cdot f((\delta x)^{-1}\omega_s X_1\omega_s^{-1}\delta x) \\ &= \sum_{\{P:\mathscr{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \sum_{\delta\in H_{X_1}(F)\backslash H(F)} \widehat{\tau}^G_P(H_P(\omega_s\delta x)-T_P)\cdot f((\delta x)^{-1}X_1\delta x). \end{aligned}$$

For $y \in H(\mathbb{A})$, write

$$\chi_T(y) := \sum_{\{P:\mathscr{P}_0\subseteq P\}} (-1)^{\dim(A_P/A_G)} \sum_{s\in\Omega^H(\mathfrak{a}_{P_1};P)} \widehat{\tau}^G_P(H_P(\omega_s y)-T_P).$$

Then

$$j^T_{f,\mathfrak{o}}(x) = \sum_{\delta\in H_{X_1}(F)\backslash H(F)} f((\delta x)^{-1}X_1\delta x)\cdot\chi_T(\delta x).$$

For sufficiently regular T, using Theorem 8.2 and the fact that $j^T_{f,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))$ is left invariant by A^∞_G, we have

$$\begin{aligned} J^{G,T}_{\mathfrak{o}}(\eta,0,f) &= \int_{H(F)\backslash H(\mathbb{A})\cap G(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)\eta(\mathrm{Nrd}(x))dx \\ &= \int_{A^\infty_G H(F)\backslash H(\mathbb{A})} \left(\sum_{\delta\in H_{X_1}(F)\backslash H(F)} f((\delta x)^{-1}X_1\delta x)\cdot\chi_T(\delta x) \right) \eta(\mathrm{Nrd}(x))dx. \end{aligned}$$

Hence,

$$J^{G,T}_{\mathfrak{o}}(\eta,0,f) = \mathrm{vol}(A^\infty_{P_1}H_{X_1}(F)\backslash H_{X_1}(\mathbb{A}))\cdot\int_{H_{X_1}(\mathbb{A})\backslash H(\mathbb{A})} f(x^{-1}X_1x)v_{P_1}(x,T)\eta(\mathrm{Nrd}(x))dx. \tag{9.2.1}$$

where

$$v_{P_1}(x,T) := \int_{A^\infty_G\backslash A^\infty_{P_1}} \chi_T(ax)da.$$

Here we have cheated by assuming that $v_{P_1}(x,T)$ is well-defined and left-invariant under $H_{X_1}(\mathbb{A})$ in the last equality, which is explained below along with its geometric interpretation.

Let Q be a parabolic subgroup of G containing P_0. Since $\mathscr{P}_0 \subseteq P_1$, by the charaterisation in [9, p. 59], $\Omega^G(\mathfrak{a}_{P_1};Q)$ is empty unless $\mathscr{P}_0 \subseteq Q$, in which case we have $\Omega^G(\mathfrak{a}_{P_1};Q) = \Omega^H(\mathfrak{a}_{P_1};Q)$ by Lemma 9.1.2). Therefore, we have

$$\chi_T(y) = \sum_{\{Q:P_0\subseteq Q\}} (-1)^{\dim(A_Q/A_G)} \sum_{s\in\Omega^G(\mathfrak{a}_{P_1};Q)} \widehat{\tau}_Q^G(H_Q(\omega_s y) - T_Q).$$

Compared to [3, p. 951], $v_{P_1}(x,T)$ is nothing but the restriction to $H(\mathbb{A})$ of Arthur's weight for $G(\mathbb{A})$. It showed in [2, Corollary 3.3] that the integral over a can be taken over a compact subset. From [2, Corollary 3.5], $v_{P_1}(x,T)$ equals the volume of the projection onto $\mathfrak{a}_{P_1}^G$ of the convex hull of $\{T_Q - H_Q(x)\}$, where Q takes over all semi-standard parabolic subgroups of G with $M_Q = M_{P_1}$. For $y \in H_{X_1}(\mathbb{A}) \subseteq M_{P_1\cap H}(\mathbb{A})$, the convex hull associated to $v_{P_1}(yx,T)$ is a translation of that associated to $v_{P_1}(x,T)$, so they have the same volume, i.e., $v_{P_1}(yx,T) = v_{P_1}(x,T)$. By taking constant terms of both sides of (9.2.1), we obtain the theorem. □

Remark 9.3. As mentioned in the proof of Theorem 9.2, the weights we get for regular semi-simple orbits are the restriction to $H(\mathbb{A})$ of Arthur's weights (see [3, p. 951]) for $G(\mathbb{A})$. They are also the same as those (see [39, p. 131]) appearing in the twisted trace formula for $(GL_{n,D} \times GL_{n,D}) \rtimes \sigma$, where σ acts on $GL_{n,D} \times GL_{n,D}$ by $\sigma(x,y) := (y,x)$. For P_n a standard parabolic subgroup of $GL_{n,D}$ and $P = \begin{pmatrix} \mathfrak{p}_n & \mathfrak{p}_n \\ \mathfrak{p}_n & \mathfrak{p}_n \end{pmatrix}^\times$ an ω-stable relatively standard parabolic subgroup of G, we may identify $\mathfrak{a}_P$ with the σ-invariant subspace of $\mathfrak{a}_{P_n\times P_n}$. The ω-stable relatively standard parabolic subgroups of G here play the role of the σ-stable standard parabolic subgroups of $GL_{n,D} \times GL_{n,D}$, which correspond to the standard parabolic subsets of $(GL_{n,D} \times GL_{n,D}) \rtimes \sigma$ in the sense of [39, §2.7]. However, we need more (namely relatively standard) parabolic subgroups in our truncation to deal with $\mathfrak{o} \notin \mathcal{O}^\times$.

CHAPTER 3

An infinitesimal variant of Guo-Jacquet trace formula: the case of a central simple algebra containing a quadratic extension

We establish an infinitesimal variant of Guo-Jacquet trace formula for the case of a central simple algebra over a number field F containing a quadratic field extension E/F. It is an equality between a sum of geometric distributions on the tangent space of some symmetric space and its Fourier transform. To prove this, we need to define an analogue of Arthur's truncation and then use the Poisson summation formula. We describe the terms attached to regular semi-simple orbits as explicit weighted orbital integrals. To compare them to those for another case studied in our previous work, we state and prove the weighted fundamental lemma at the infinitesimal level by using Labesse's work on the base change for GL_n.

1. Introduction

Guo and Jacquet have proposed a conjecture [23] in order to generalise Waldspurger's famous result [50], which relates toric periods and central values of automorphic L-functions for GL_2, to higher ranks. The approach of relative trace formulae makes it possible to reduce the conjectural comparison of periods (related to the spectral side) to the comparison of (weighted) orbital integrals (related to the geometric side) on different symmetric spaces. This approach was first adopted by Jacquet [29] to reprove Waldspurger's theorem. For higher ranks, Feigon-Martin-Whitehouse [21] obtained some partial results using a simple form of relative trace formulae. For the comparison of local orbital integrals, Guo reduced the fundamental lemma [23] to that of the base change for GL_n and Zhang proved the smooth transfer [58] by global methods.

However, an obstruction in the approach is the divergence of sums of integrals in both sides of relative trace formulae. Such a problem has already existed in the classical Arthur-Selberg trace formula and Arthur introduced a truncation process [3][4] to tackle it (see also [13] for its Lie algebra variant). We start working at the infinitesimal level (namely the tangent space of a symmetric space) for a couple of reasons. Firstly, our truncation for the tangent space is expected to be adapted to a truncation for the symmetric space. Secondly, infinitesimal trace formulae should be useful for the proof of results on the transfer (see Zhang's work [58] on the ordinary orbital integrals).

Guo-Jacquet trace formulae concern two symmetric pairs. The first one is (G', H'), where $G' := GL_{2n}$ and $H' := GL_n \times GL_n$ are reductive groups over a number field F and H' embeds diagonally in G'. Let $\mathfrak{s}' \simeq \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ be the tangent space at the neutral element of the symmetric space G'/H'. We have established an infinitesimal trace formula in Chapter 2 for the action of H' on $\mathfrak{s}'$ by conjugation. The second one denoted by (G, H) is the main object in this paper. Before introducing it, we remark that we shall work in a more general setting than the original one. The reason is that the converse direction of Guo-Jacquet conjecture was originally proposed only for n odd. In our searching for an analogue for n even, the related local conjecture of Prasad and Takloo-Bighash [44, Conjecture 1] suggests that we should consider more inner forms of G'. Some recent progress on this local conjecture has been made by Xue [55] with the help of a simple form of global relative trace formulae.

Let E/F be a quadratic extension of number fields. Suppose that $E = F(\alpha)$, where $\alpha \in E$ and $\alpha^2 \in F$. Let $\mathfrak{g}$ be a central simple algebra over F containing E. Write $\mathfrak{h}$ to be the centralizer of α in $\mathfrak{g}$. Denote by G and H the groups of invertible elements in $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Both of them are viewed as reductive groups over F. Let $\mathfrak{s} := \{X \in \mathfrak{g} : \mathrm{Ad}(\alpha)(X) = -X\}$, where Ad denotes the adjoint action of G on $\mathfrak{g}$. It is the tangent space at the neutral element of the symmetric space G/H. The main global result in this paper is an infinitesimal trace formula for the action of H on $\mathfrak{s}$ by conjugation.

Denote by $\mathbb{A}$ the ring of adèles of F and by $H(\mathbb{A})^1$ the subset of elements in $H(\mathbb{A})$ with absolute-value-1 reduced norm. We define a relation of equivalence on $\mathfrak{s}(F)$: two elements of $\mathfrak{s}(F)$ are equivalent if and only if they lie in the same fibre of the categorical quotient $\mathfrak{s}//H$. Denote by $\mathcal{O}$ the set of classes

of equivalence. Let f be a Bruhat-Schwartz function on $\mathfrak{s}(\mathbb{A})$. For each $\mathfrak{o} \in \mathcal{O}$ and $x \in H(F)\backslash H(\mathbb{A})$, define

$$k_{f,\mathfrak{o}}(x) := \sum_{Y \in \mathfrak{o}} f(\mathrm{Ad}(x^{-1})(Y)).$$

As mentioned, we are facing the problem that

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{H(F)\backslash H(\mathbb{A})^1} k_{f,\mathfrak{o}}(x)dx$$

is divergent. We define the truncation $k^T_{f,\mathfrak{o}}(x)$ (see (4.0.1)) which is an analogue of Arthur's truncation in [3], where T is a truncation parameter in some cone $T_+ + \mathfrak{a}^+_{P_0}$ of the coroot space of H, such that the following theorem holds.

THEOREM 1.1 (see Theorem 4.2). *For all $T \in T_+ + \mathfrak{a}^+_{P_0}$,*

$$\sum_{\mathfrak{o} \in \mathcal{O}} \int_{H(F)\backslash H(\mathbb{A})^1} k^T_{f,\mathfrak{o}}(x)dx$$

is absolutely convergent.

We also know the behaviour of each term (viewed as a distribution) with respect to the truncation parameter. It is even simpler than that in the case of (G', H') (cf. Theorem 1.2 in Chapter 2).

THEOREM 1.2 (see Corollary 5.3). *For all $T \in T_+ + \mathfrak{a}^+_{P_0}$ and $\mathfrak{o} \in \mathcal{O}$,*

$$J^T_{\mathfrak{o}}(f) := \int_{H(F)\backslash H(\mathbb{A})^1} k^T_{f,\mathfrak{o}}(x)dx$$

is a polynomial in T.

Now we can take the constant term of each term to eliminate the truncation parameter T. Denote by $J_{\mathfrak{o}}(f)$ the constant term of $J^T_{\mathfrak{o}}(f)$. These distributions are not invariant by $H(\mathbb{A})^1$ (see Proposition 6.1), but we can write the regular semi-simple terms as explicit weighted orbital integrals with the same weights as Arthur's in [3].

THEOREM 1.3 (see Theorem 9.2). *Let $\mathfrak{o} \in \mathcal{O}$ be a class associated to regular semi-simple orbits, P_1 a standard parabolic subgroup of H and $Y_1 \in \mathfrak{o}$ an elliptic element with respect to P_1 (see the precise definition in Section 9). Denote by H_{Y_1} the centralizer of Y_1 in H. We have*

$$J_{\mathfrak{o}}(f) = \mathrm{vol}([H_{Y_1}]) \cdot \int_{H_{Y_1}(\mathbb{A})\backslash H(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(Y_1))v_{P_1}(x)dx,$$

where $\mathrm{vol}([H_{Y_1}])$ is the volume associated to H_{Y_1} and $v_{P_1}(x)$ is the volume of some convex hull.

Thanks to the truncation, we solve the divergence issue in the following infinitesimal trace formula. It is a consequence of the Poisson summation formula.

THEOREM 1.4 (see Theorem 7.1). *We have the equality*

$$\sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(\hat{f}),$$

where $\hat{f}$ (see (3.3.1)) is the Fourier transform of f.

Notice that the symmetric pairs (G, H) and (G', H') are the same after the base change to an algebraic closure of F containing E. In fact, the truncation and most proofs of the global results above are simpler than those in Chapter 2 in some sense. The simplicity results from the equality $H(\mathbb{A})^1 = H(\mathbb{A}) \cap G(\mathbb{A})^1$ here, where $G(\mathbb{A})^1$ denotes the subset of elements in $G(\mathbb{A})$ with absolute-value-1 reduced norm. Moreover, there is a bijection between the set of standard parabolic subgroups in H and the set of semi-standard parabolic subgroups in G whose intersection with H is a standard parabolic subgroup in H. One may consult Section 3.4 for more details. However, there are still some rationality issues. We shall give sufficient details and self-contained proofs here for completeness.

At the end of this paper, we hope to provide some new evidence of noninvariant comparison of Guo-Jacquet trace formulae. We shall turn to the local setting with F denoting a local field. In the comparison of geometric sides, an important case is the so-called fundamental lemma. It roughly says that some basic functions for two symmetric pairs should have associated local orbital integrals on matching orbits at

almost all unramified places. Guo [23] proved it for the units of spherical Hecke algebras for Guo-Jacquet trace formulae with the help of the base change fundamental lemma for the full spherical Hecke algebras for GL_n known by Kottwitz [36, Lemma 8.8] and Arthur-Clozel [10, Theorem 4.5 in Chapter 1]. An infinitesimal version [58, Lemma 5.18] was used by Zhang to prove the smooth transfer for Guo-Jacquet trace formulae following the same philosophy of Waldspurger's work [52] on the endoscopic transfer. We would like to generalise [58, Lemma 5.18] in the weighted context.

For almost all unramified places, (G, H) is isomorphic to $(GL_{2n}, \mathrm{Res}_{E/F}GL_{n,E})$ and $\mathfrak{s}(F) \simeq \mathfrak{gl}_n(E)$. Denote by $\mathcal{O}_F$ (resp. $\mathcal{O}_E$) the ring of integers in F (resp. E). For f and f' a pair of locally constant and compactly supported complex functions on $\mathfrak{s}(F)$ and $\mathfrak{s}'(F)$ respectively, we define the notion of being "strongly associated" (see the precise definition in Definition 10.4) inspired by [37, Definition III.3.2]. Roughly speaking, f and f' are said to be strongly associated if their local weighted orbital integrals are equal at matching orbits. Let f_0 and f'_0 be the characteristic functions of $\mathfrak{s}(\mathcal{O}_F) \simeq \mathfrak{gl}_n(\mathcal{O}_E)$ and $\mathfrak{s}'(\mathcal{O}_F) \simeq (\mathfrak{gl}_n \oplus \mathfrak{gl}_n)(\mathcal{O}_F)$ respectively. Because the weighted orbital integrals that we got share the same weights with those in twisted trace formulae (see Remark 9.3 and Remark 9.3 in Chapter 2), we are able to show the following result by using Labesse's work on the base change weighted fundamental lemma for the full spherical Hecke algebras for GL_n.

THEOREM 1.5 (see Theorem 8.1). *For almost all unramified places, f_0 and f'_0 are strongly associated.*

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2. Notation

We shall use F to denote a number field in this article except for the last section where F denotes a non-archimedean local field of characteristic 0.

2.1. Roots and weights. Let F be a number field or a non-archimedean local field of characteristic 0. Suppose that H is a reductive group defined over F. Fix a minimal Levi F-subgroup M_0 of H. All the following groups are assumed to be defined over F without further mention. We call a parabolic subgroup or a Levi subgroup of H semi-standard if it contains M_0. Fix a minimal semi-standard parabolic subgroup P_0 of H. We call a parabolic subgroup P of H standard if $P_0 \subseteq P$. For any semi-standard parabolic subgroup P of H, we usually write M_P for the Levi factor containing M_0 and N_P the unipotent radical. Denote by A_P the maximal F-split torus in the centre of M_P. Let $X(M_P)_F$ be the group of characters of M_P defined over F. Then define

$$\mathfrak{a}_P := \mathrm{Hom}_{\mathbb{Z}}(X(M_P)_F, \mathbb{R})$$

and its dual space

$$\mathfrak{a}_P^* := X(M_P)_F \otimes_{\mathbb{Z}} \mathbb{R},$$

which are both $\mathbb{R}$-linear spaces of dimention $\dim(A_P)$. Notice that the restriction $X(M_P)_F \hookrightarrow X(A_P)_F$ induces an isomorphism

$$\mathfrak{a}_P^* \simeq X(A_P)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

Suppose that $P_1 \subseteq P_2$ are a pair of standard parabolic subgroups of H. The restriction $X(M_{P_2})_F \hookrightarrow X(M_{P_1})_F$ induces $\mathfrak{a}_{P_2}^* \hookrightarrow \mathfrak{a}_{P_1}^*$ and its dual map $\mathfrak{a}_{P_1} \twoheadrightarrow \mathfrak{a}_{P_2}$. Denote by $\mathfrak{a}_{P_1}^{P_2}$ the kernel of the latter map $\mathfrak{a}_{P_1} \twoheadrightarrow \mathfrak{a}_{P_2}$. The restriction $X(A_{P_1})_F \twoheadrightarrow X(A_{P_2})_F$ induces $\mathfrak{a}_{P_1}^* \twoheadrightarrow \mathfrak{a}_{P_2}^*$ and its dual map $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{a}_{P_1}$. The latter map $\mathfrak{a}_{P_2} \hookrightarrow \mathfrak{a}_{P_1}$ provides a section of the previous map $\mathfrak{a}_{P_1} \twoheadrightarrow \mathfrak{a}_{P_2}$. Thus we have decompositions

$$\mathfrak{a}_{P_1} = \mathfrak{a}_{P_2} \oplus \mathfrak{a}_{P_1}^{P_2}$$

and

$$\mathfrak{a}_{P_1}^* = \mathfrak{a}_{P_2}^* \oplus (\mathfrak{a}_{P_1}^{P_2})^*.$$

When $P_1 = P_0$, we write $\mathfrak{a}_{P_1}$, A_{P_1} and $\mathfrak{a}_{P_1}^{P_2}$ as $\mathfrak{a}_0$, A_0 and $\mathfrak{a}_0^{P_2}$ respectively.

For a pair of standard parabolic subgroups $P_1 \subseteq P_2$ of H, write $\Delta_{P_1}^{P_2}$ for the set of simple roots for the action of A_{P_1} on $N_{P_1}^{P_2} := N_{P_1} \cap M_{P_2}$. Notice that $\Delta_{P_1}^{P_2}$ is a basis of $(\mathfrak{a}_{P_1}^{P_2})^*$. Let

$$(\widehat{\Delta}_{P_1}^{P_2})^\vee := \{\varpi_\alpha^\vee : \alpha \in \Delta_{P_1}^{P_2}\}$$

be the basis of $\mathfrak{a}_{P_1}^{P_2}$ dual to $\Delta_{P_1}^{P_2}$. One has the coroot $\beta^\vee$ associated to any $\beta \in \Delta_{P_0}^{P_2}$. For every $\alpha \in \Delta_{P_1}^{P_2}$, let $\alpha^\vee$ be the projection of $\beta^\vee$ to $\mathfrak{a}_{P_1}^{P_2}$, where $\beta \in \Delta_{P_0}^{P_2}$ whose restriction to $\mathfrak{a}_{P_1}^{P_2}$ is α. Define

$$(\Delta_{P_1}^{P_2})^\vee := \{\alpha^\vee : \alpha \in \Delta_{P_1}^{P_2}\},$$

which is a basis of $\mathfrak{a}_{P_1}^{P_2}$. Denote by

$$\widehat{\Delta}_{P_1}^{P_2} := \{\varpi_\alpha : \alpha \in \Delta_{P_1}^{P_2}\}$$

the basis of $(\mathfrak{a}_{P_1}^{P_2})^*$ dual to $(\Delta_{P_1}^{P_2})^\vee$.

For a standard parabolic subgroup P of H, set

$$\mathfrak{a}_P^+ := \{T \in \mathfrak{a}_P : \alpha(T) > 0, \alpha \in \Delta_P^H\}.$$

For $P_1 \subseteq P_2$ as above, define $\tau_{P_1}^{P_2}$ and $\widehat{\tau}_{P_1}^{P_2}$ as the characteristic functions of

$$\{T \in \mathfrak{a}_0 : \alpha(T) > 0, \alpha \in \Delta_{P_1}^{P_2}\}$$

and

$$\{T \in \mathfrak{a}_0 : \varpi(T) > 0, \varpi \in \widehat{\Delta}_{P_1}^{P_2}\}$$

respectively.

2.2. The functions H_P and F^P. Let F be a number field. Let $\mathbb{A}$ be the ring of adèles of F and let $|\cdot|_{\mathbb{A}}$ be the product of normalised local absolute values on the group of idèles $\mathbb{A}^*$. Fix a maximal compact subgroup K of $H(\mathbb{A})$ that is admissible relative to M_0 in the sense of [5, p. 9]. In this paper, we choose the standard maximal compact subgroup when $G(F) = GL_n(D)$, where D is a central division algebra over a finite field extension E of F. That is to say, $K := \prod_v K_v$ where at every non-archimedean place v of E, K_v is the group of automorphism of some lattice (see [54, p. 191]) and at every archimedean place, K_v is the unitary group with respect to some hermitian form (see [54, p. 199]). Suppose that P is a standard parabolic subgroup of H. Let H_P be the homomorphism $M_P(\mathbb{A}) \to \mathfrak{a}_P$ given by

$$\forall m \in M_P(\mathbb{A}), \langle H_P(m), \chi\rangle = \log(|\chi(m)|_{\mathbb{A}}), \chi \in X(M_P)_F.$$

Write $M_P(\mathbb{A})^1$ for the kernel of H_P and A_P^∞ for the neutral component for the topology of $\mathbb{R}$-manifolds of the group of $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in $\mathrm{Res}_{F/\mathbb{Q}} A_P$. Then any element $x \in H(\mathbb{A})$ can be written as $x = nmak$, where $n \in N_P(\mathbb{A})$, $m \in M_P(\mathbb{A})^1$, $a \in A_P^\infty$ and $k \in K$. We can define a continuous map $H_P : H(\mathbb{A}) \to \mathfrak{a}_P$ by setting $H_P(x) := H_P(a)$ with respect to this decomposition. Notice that H_P induces an isomorphism from A_P^∞ to $\mathfrak{a}_P$. If $P \subseteq Q$ are a pair of semi-standard parabolic subgroups, write

$$A_P^{Q,\infty} := A_P^\infty \cap M_Q(\mathbb{A})^1.$$

Then H_P also induces an isomorphism from $A_P^{Q,\infty}$ to $\mathfrak{a}_P^Q$.

Denote by Ω^H the Weyl group of (H, A_0). In the cases to be considered in this paper, for every $s \in \Omega^H$, we can always choose one representative $\omega_s \in H(F) \cap K$ such that ω_s normalises A_0. In fact, we are dealing with the restriction of scalars of inner forms of GL_n, thus we can choose Ω^H to be permutation matrices.

From the reduction theory (see [3, p. 941]), we know that there exists a real number $t_0 < 0$ and a compact subset $\omega_{P_0} \subseteq N_{P_0}(\mathbb{A})M_0(\mathbb{A})^1$ such that for any standard parabolic subgroup P of H, we have

$$H(\mathbb{A}) = P(F)\mathfrak{S}_{P_0}^P(\omega_{P_0}, t_0).$$

Here the Siegel set $\mathfrak{S}_{P_0}^P(\omega_{P_0}, t_0)$ is defined by

$$\mathfrak{S}_{P_0}^P(\omega_{P_0}, t_0) := \omega_{P_0} A_{P_0}^\infty(P, t_0)K,$$

where

$$A_{P_0}^\infty(P, t_0) := \{a \in A_{P_0}^\infty : \alpha(H_{P_0}(a)) > t_0, \alpha \in \Delta_{P_0}^P\}.$$

We shall fix such t_0 and ω_{P_0}. Moreover, we require that $(M_P(\mathbb{A}) \cap \omega_{P_0}, M_P(\mathbb{A}) \cap K, P_0 \cap M_P, t_0)$ will play the role of $(\omega_{P_0}, K, P_0, t_0)$ for any standard parabolic subgroup P of H.

Let t_0 be as above. For $T \in \mathfrak{a}_0$, define the truncated Siegel set

$$\mathfrak{S}_{P_0}^P(\omega_{P_0}, t_0, T) := \omega_{P_0} A_{P_0}^\infty(P, t_0, T)K,$$

where

$$A_{P_0}^\infty(P, t_0, T) := \{a \in A_{P_0}^\infty(P, t_0) : \varpi(H_{P_0}(a) - T) \leq 0, \varpi \in \widehat{\Delta}_{P_0}^P\}.$$

Denote by $F_{P_0}^P(\cdot, T)$ the characteristic function of the projection of $\mathfrak{S}_{P_0}^P(\omega_{P_0}, t_0, T)$ to $P(F)\backslash H(\mathbb{A})$.

2.3. Bruhat-Schwartz functions and Haar measures. Let F be a number field. Write $\mathfrak{h}$ for the Lie algebra of H. For an F-linear subspace $\mathfrak{s}$ of $\mathfrak{h}$, denote by $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ the Bruhat-Schwartz space of $\mathfrak{s}(\mathbb{A})$, namely the $\mathbb{C}$-linear space of functions on $\mathfrak{s}(\mathbb{A})$ generated by $f_\infty \otimes \chi^\infty$, where f_∞ is a Schwartz function on $\mathfrak{s}(F \otimes_{\mathbb{Q}} \mathbb{R})$ and χ^∞ is the characteristic function of an open compact subgroup of $\mathfrak{s}(\mathbb{A}^\infty)$, where $\mathbb{A}^\infty$ denotes the ring of finite adèles of F.

Let P be a standard parabolic subgroup of H. For every connected subgroup V of N_P (resp. every subspace $\mathfrak{v}$ of $\mathfrak{h}$), choose the unique Haar measure on $V(\mathbb{A})$ (resp. on $\mathfrak{v}(\mathbb{A})$) such that $\mathrm{vol}(V(F)\backslash V(\mathbb{A})) = 1$ (resp. $\mathrm{vol}(\mathfrak{v}(F)\backslash\mathfrak{v}(\mathbb{A})) = 1$). We also take the Haar measure on K such that $\mathrm{vol}(K) = 1$.

Fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_0$ invariant by the group Ω^H and Haar measures on subspaces of $\mathfrak{a}_0$ compatible with this norm. If $P \subseteq Q$ are a pair of standard parabolic subgroups, we obtain the Haar measures on A_P^∞ and $A_P^{Q,\infty}$ via the isomorphism H_P.

Denote by $\rho_P \in (\mathfrak{a}_P^H)^*$ the half of the sum of weights (with multiplicities) for the action of A_P on $\mathfrak{n}_P$. We choose compatible Haar measures on $H(\mathbb{A})$ and its subgroups by requiring that for any $f \in L^1(H(\mathbb{A}))$,

$$\begin{aligned}\int_{H(\mathbb{A})} f(x)dx &= \int_{N_P(\mathbb{A})}\int_{M_P(\mathbb{A})}\int_K f(nmk)e^{-2\rho_P(H_P(m))}dndmdk\\ &= \int_{N_P(\mathbb{A})}\int_{M_P(\mathbb{A})^1}\int_{A_P^\infty}\int_K f(nmak)e^{-2\rho_P(H_P(a))}dndmdadk.\end{aligned}$$

3. The symmetric pair

3.1. Groups and linear spaces. Let F be a number field and E a quadratic extension of F. Let $\mathfrak{g}$ be a central simple algebra over F with a fixed embedding $E \to \mathfrak{g}$ as F-algebras. Denote by $\mathfrak{h} := \mathrm{Cent}_{\mathfrak{g}}(E)$ the centralizer of E in $\mathfrak{g}$. Then by the the double centralizer theorem (see [42, Theorem 3.1 in Chapter IV] for example), $\mathfrak{h}(F)$ is a central simple algebra over E. Write $G := \mathfrak{g}^\times$ and $H := \mathfrak{h}^\times$ for the group of invertible elements. They are considered as algebraic groups over F with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

Let $\alpha \in E$ such that $\alpha^2 \in F$ and that $E = F(\alpha)$. Denote by Ad the adjoint action of G on $\mathfrak{g}$. Define an involution θ on $\mathfrak{g}$ by $\theta(X) := \mathrm{Ad}(\alpha)(X)$. Then $H = G^\theta$, where G^θ denotes the θ-invariant subgroup of G. Thus $S := G/H$ is a symmetric space. Define an anti-involution on G by $\iota(g) := \theta(g^{-1})$. Denote by G^ι the ι-invariant subvariety of G. Then there is a symmetrization map $s : G \to G^\iota$ defined by $s(g) := g\iota(g)$.

Lemma 3.1. *The symmetrization map s induces a bijection $S(F) \simeq G^\iota(F)$.*

Remark 3.2. For the special case $(G, H) = (GL_{n,D}, \mathrm{Res}_{E/F}GL_{n,E})$, where D is a quaternion algebra over F containing E, this result is included in [24, p. 282].

Proof of Lemma 3.1. Since $H^1(F, H) = 1$, we have $S(F) = G(F)/H(F)$. For $g \in G(F)$, let $s_0(g) := s(g)\alpha = \mathrm{Ad}(g)(\alpha)$. Let $G_0 := G^\iota\alpha = \{g \in G : g^2 = \alpha^2\}$. It suffices to prove that the map $s_0 : G(F) \to G_0(F)$ is surjective. Let $g \in G_0(F)$. Its minimal polynomial in $\mathfrak{g}(F)$ is $\lambda^2 - \alpha^2$, which is irreducible over F. Therefore, its reduced characteristic polynomial in $\mathfrak{g}(F)$ must be $(\lambda^2 - \alpha^2)^m$, where $\dim_F(\mathfrak{g}(F)) = (2m)^2$. We deduce that all elements in $G_0(F)$ are conjugate by $G(F)$ (see [56, Theorem 9] for example). Since $\alpha \in G_0(F)$, we draw our conclusion. □

One may consider the left and right translation of $H \times H$ on G and the conjugation of H on S. Denote by $\mathfrak{s}$ the tangent space of S at the neutral element. We shall always view $\mathfrak{s}$ as a subspace in $\mathfrak{g}$. Then $\mathfrak{s} = \{X \in \mathfrak{g} : \theta(X) = -X\}$ and H acts on $\mathfrak{s}$ by conjugation.

3.2. Semi-simple elements. We say that an element Y of $\mathfrak{s}$ is semi-simple if the orbit $\mathrm{Ad}(H)(Y)$ is Zariski closed in $\mathfrak{s}$. By a regular element Y of $\mathfrak{s}$, we mean that the centralizer H_Y of Y in H has minimal dimension.

Proposition 3.3. *The map $Y \mapsto Y^2$ from $\mathfrak{s}(F)$ to $\mathfrak{h}(F)$ induces an injection from the set of $H(F)$-conjugacy classes of semi-simple elements in $\mathfrak{s}(F)$ to the set of conjugacy classes of semi-simple elements in $\mathfrak{h}(F)$.*

Remark 3.4. In the special case $(G, H) = (GL_{n,D}, \mathrm{Res}_{E/F}GL_{n,E})$, where D is a quaternion algebra over F containing E, this map plays the role of the norm map (see [10, §1 in Chapter 1]).

Proof of Proposition 3.3. Let $\chi_{\mathfrak{g},F}(X)$ be the reduced characteristic polynomial of $X \in \mathfrak{g}(F)$ and $\chi_{\mathfrak{h},E}(X^*)$ the reduced characteristic polynomial of $X^* \in \mathfrak{h}(F)$ (viewed as a central simple algebra over E). After the base change to an algebraic closure of F containing E, the embedding $\mathfrak{h} \subseteq \mathfrak{g}$ is identical to the diagonal embedding $\mathfrak{h}' := \mathfrak{gl}_m \oplus \mathfrak{gl}_m \subseteq \mathfrak{g}' := \mathfrak{gl}_{2m}$ and $\mathfrak{s} \subseteq \mathfrak{g}$ becomes $\mathfrak{s}' := \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A, B \in \mathfrak{gl}_m \right\} \subseteq \mathfrak{g}'$, where m denotes the degree of $\mathfrak{h}(F)$ (viewed as a central simple algebra over E). Let H' be the group of invertible elements in $\mathfrak{h}'$, which is viewed as an algebraic group $GL_m \times GL_m$ over F. Since

$$\det\left(\lambda I_{2m} - \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}\right) = \det(\lambda^2 I_m - AB),$$

we see that for $Y \in \mathfrak{s}(F) \subseteq \mathfrak{g}(F)$,

$$\chi_{\mathfrak{g},F}(Y)(\lambda) = \chi_{\mathfrak{h},E}(Y^2)(\lambda^2),$$

which implies that $\chi_{\mathfrak{h},E}(Y^2)$ is actually defined over F.

It is known that the semi-simple conjugacy classes in $\mathfrak{h}(F)$ are uniquely determined by $\chi_{\mathfrak{h},E}$ (see [56, Theorem 9] for example). Thus it suffices to prove that the semi-simple $H(F)$-conjugacy classes in $\mathfrak{s}(F)$ are uniquely determined by $\chi_{\mathfrak{g},F}$. From [31, Proposition 2.1], we know that the semi-simple H-conjugacy classes in $\mathfrak{s}(F)$ are uniquely determined by $\chi_{\mathfrak{g},F}$. Therefore, we reduce ourselves to proving that each semi-simple H-conjugacy class in $\mathfrak{s}(F)$ contains a unique $H(F)$-conjugacy class.

For a semi-simple element $Y \in \mathfrak{s}(F)$, the $H(F)$-orbits in $\mathrm{Ad}(H)(Y)$ are parametrized by

$$\ker[H^1(F, H_Y) \to H^1(F, H)] = H^1(F, H_Y),$$

where H_Y is the centralizer of Y in H. By [47, Exercice 2 in p. 160], we obtain

$$H^1(F, H_Y) = 1,$$

which completes our proof. □

3.3. Invariants. Denote by $\mathfrak{c}$ the affine space $\mathbf{A}^m$, where m denotes the degree of $\mathfrak{h}(F)$ (viewed as a central simple algebra over E). Define a morphism $\pi : \mathfrak{s} \to \mathfrak{c}$ which is contant on H-orbits by mapping $Y \in \mathfrak{s}$ to the coefficients of the reduced polynomial of $Y \in \mathfrak{g}$. In fact, we see that the coefficients in odd degrees vanish for $Y \in \mathfrak{s}$ from the proof of Proposition 3.3. On F-points, alternatively, π is given by mapping $Y \in \mathfrak{s}(F)$ to the coefficients of the reduced polynomial of $Y^2 \in \mathfrak{h}(F)$ (viewed as a central simple algebra over E).

Proposition 3.5. *The pair $(\mathfrak{c}, \pi)$ defines a categorical quotient of $\mathfrak{s}$ by H over F.*

Proof. By the proof of Proposition 3.3 in Chapter 2, after the base change to an algebraic closure $\overline{F}$ of F containing E, the pair $(\mathfrak{c}_{\overline{F}}, \pi_{\overline{F}})$ defines a categorical quotient of $\mathfrak{s}_{\overline{F}}$ by $H_{\overline{F}}$. That is to say, we have an isomorphism of $\overline{F}$-algebras $\overline{F}[\mathfrak{c}] \simeq \overline{F}[\mathfrak{s}]^H$ dual to $\pi_{\overline{F}}$. But this isomorphism is obtained from the base change of a morphism of F-algebras $F[\mathfrak{c}] \to F[\mathfrak{s}]^H$ dual to π. By Galois descent, the latter morphism is necessarily an isomorphism of F-algebras. Then the pair $(\mathfrak{c}, \pi)$ defines a categorical quotient of $\mathfrak{s}$ by H over F. □

Remark 3.6. The morphism π is surjective as a morphism of algebraic varieties (see the proof of Proposition 3.3 in Chapter 2) but not surjective on the level of F-points.

We define a relation of equivalence on $\mathfrak{s}(F)$ using the categorical quotient $(\mathfrak{c}, \pi)$, where two elements are in the same class if and only if they have the same image under π. We denote by $\mathcal{O}$ the set of equivalent classes for this relation. From the proof of Proposition 3.3, we see that two semi-simple elements of $\mathfrak{s}(F)$ belong to the same class of $\mathcal{O}$ if and only if they are conjugate by $H(F)$. Denote by $\mathcal{O}_{rs}$ the subset of $\mathcal{O}$ formed by $Y \in \mathfrak{s}(F)$ such that $\chi_{\mathfrak{h},E}(Y^2)$ is separable and $\chi_{\mathfrak{h},E}(Y^2)(0) \neq 0$, where $\chi_{\mathfrak{h},E}$ denotes the reduced polynomial of an element in $\mathfrak{h}(F)$ (viewed as a central simple algebra over E). By Proposition 3.2 in Chapter 2 and the base change to an algebraic closure of F containing E, we see that each class in $\mathcal{O}_{rs}$ is a regular semi-simple $H(F)$-orbit in $\mathfrak{s}(F)$.

3.4. Explicit description of $H \hookrightarrow G$. First of all, we would like to describe the symmetric pair (G, H) in a more explicit way. By the Noether-Skolem theorem (see [42, Theorem 2.10 of Chapter IV] for example), the embedding $E \to \mathfrak{g}(F)$ is unique up to conjugation by an element of $G(F)$. From the Wedderburn-Artin theorem, we know that G is isomorphic to $GL_{n,D}$, which denotes the reductive group over F whose F-points are $GL_n(D)$, for some central division algebra D over F. We recall that n is called the capacity of $\mathfrak{g}(F)$ and we denote it by $\mathrm{capa}(\mathfrak{g}(F))$. Let d be the degree of D, i.e., $\dim_F(D) = d^2$. Since there is an embedding $E \to \mathfrak{g}(F)$ as F-algebras, we know that nd is even.

Proposition 3.7. *Up to conjugation by $G(F)$, the embedding $H \hookrightarrow G$ is reduced to one of the two cases below.*

***Case I**: if there is an embedding $E \to D$ as F-algebras, then the embedding $H \hookrightarrow G$ is isomorphic to $\mathrm{Res}_{E/F} GL_{n,D'} \hookrightarrow GL_{n,D}$ up to conjugation by $G(F)$. Here $D' := \mathrm{Cent}_D(E)$ denotes the centralizer of E in D and is a central division algebra over E.*

***Case II**: if there is no embedding $E \to D$ as F-algebras, then the embedding $H \hookrightarrow G$ is isomorphic to $\mathrm{Res}_{E/F} GL_{\frac{n}{2}, D\otimes_F E} \hookrightarrow GL_{n,D}$ up to conjugation by $G(F)$. Here $D \otimes_F E$ is a central division algebra over E.*

Proof. **Case I**: there is an embedding $E \to D$ as F-algebras. This case is a direct consequence of the Noether-Skolem theorem. By the double centralizer theorem, we know that D' is a central division algebra over E.

Case II: there is no embedding $E \to D$ as F-algebras. By [48, Theorem 1.1.2], when nd is even, there is an embedding $E \to \mathfrak{g}(F)$ as F-algebras if and only if $n \cdot \mathrm{capa}(D \otimes_F E)$ is even, where $\mathrm{capa}(D \otimes_F E)$ denotes the capacity of the central simple algebra $D \otimes_F E$ over E (see [42, Proposition 2.15 in Chapter IV] for example). Additionally, from [48, Theorem 1.1.3], we show that $\mathrm{capa}(D \otimes_F E) \leq [E : F] = 2$. In this case, there are two possibilities.

(1) d is even. By [48, Theorem 1.1.2], $\mathrm{capa}(D \otimes_F E)$ is odd, so $\mathrm{capa}(D \otimes_F E) = 1$. Since $n \cdot \mathrm{capa}(D \otimes_F E)$ is even, we know that n is even.

(2) d is odd. Since nd is even, we see that n is even. Besides, from [48, Theorem 1.1.3], we also deduce that $\mathrm{capa}(D \otimes_F E) = 1$.

In sum, we have shown that n is even and that $D \otimes_F E$ is a central division algebra over E. The tensor of $\mathfrak{gl}_{\frac{n}{2},D}$ and a fixed embedding $\mathrm{Res}_{E/F}\mathfrak{gl}_{1,E} \to \mathfrak{gl}_2$ gives the indicated way to embed $\mathfrak{h}$ to $\mathfrak{g}$. By the Noether-Skolem theorem, such an embedding is unique up to conjugation by $G(F)$. □

Next, we describe the correspondence of some parabolic subgroups in H and G in both cases.

Case I: $(G, H) = (GL_{n,D}, \mathrm{Res}_{E/F} GL_{n,D'})$, where $D' := \mathrm{Cent}_D(E)$. We denote by $M_0 \simeq (\mathrm{Res}_{E/F}\mathbb{G}_{m,D'})^n$ the subgroup of diagonal elements in H, which is a minimal Levi F-subgroup of H, and by $M_{\widetilde{0}} \simeq (\mathbb{G}_{m,D})^n$ the subgroup of diagonal elements in G, which is a minimal Levi F-subgroup of G. We also fix P_0 the subgroup of upper triangular elements in H, which is a minimal parabolic F-subgroup of H. There is a bijection $P \mapsto \widetilde{P}$ between the set of standard parabolic subgroups P (namely $P_0 \subseteq P$) in H and the set of semi-standard parabolic subgroups $\widetilde{P}$ (namely $M_{\widetilde{0}} \subseteq \widetilde{P}$) in G which contain P_0. In this case, the latter is exactly the set of standard parabolic subgroups (namely containing $\widetilde{P_0}$ the subgroup of upper triangular elements in G) of G. We shall always write $\widetilde{P}$ for the image of P under this bijection. Notice that $P = \widetilde{P} \cap H$ and that we can identify A_P with $A_{\widetilde{P}}$.

Case II: $(G, H) = (GL_{n,D}, \mathrm{Res}_{E/F} GL_{\frac{n}{2}, D\otimes_F E})$. We denote by $M_0 \simeq (\mathrm{Res}_{E/F}\mathbb{G}_{m,D\otimes_F E})^{\frac{n}{2}}$ the subgroup of diagonal elements in H, which is a minimal Levi F-subgroup of H, and by $M_{\widetilde{0}} \simeq (\mathbb{G}_{m,D})^n$ the subgroup of diagonal elements in G, which is a minimal Levi F-subgroup of G. We also fix P_0 the subgroup of upper triangular elements in H, which is a minimal parabolic F-subgroup of H. There is a bijection $P \mapsto \widetilde{P}$ between the set of standard parabolic subgroups P (namely $P_0 \subseteq P$) in H and the set of semi-standard parabolic subgroups $\widetilde{P}$ (namely $M_{\widetilde{0}} \subseteq \widetilde{P}$) in G which contain P_0. In this case, the latter is a subset of the set of standard parabolic subgroups (namely containing the subgroup of upper triangular elements in G) of G. We shall always write $\widetilde{P}$ for the image of P under this bijection. Notice that $P = \widetilde{P} \cap H$ and that we can identify A_P with $A_{\widetilde{P}}$.

Proposition 3.8. *Let P be a standard parabolic subgroup of H. For all $Y \in (\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s})(F)$ and $U \in (\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(F)$, we have*

$$\pi(Y) = \pi(Y + U).$$

Proof. It is obvious, since the reduced characteristic polynomial of $Y+U\in\mathfrak{g}$ is equal to that of $Y\in\mathfrak{g}$. □

Corollary 3.9. *Let P be a standard parabolic subgroup of H and $\mathfrak{o}\in\mathcal{O}$. For all subsets $S_1\subseteq(\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s})(F)$ and $S_2\subseteq(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(F)$, we have $\mathfrak{o}\cap(S_1\oplus S_2)=(\mathfrak{o}\cap S_1)\oplus S_2$.*

Let P be a standard parabolic subgroup of H. We denote by $\Phi(A_0,\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s})$ (resp. $\Phi(A_0,\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})$) the set of weights of A_0 in $\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s}$ (resp. $\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s}$). We also denote by $\Phi(A_0,\mathfrak{m}_P)$ (resp. $\Phi(A_0,\mathfrak{n}_P)$) the set of weights of A_0 in $\mathfrak{m}_P$ (resp. $\mathfrak{n}_P$).

Proposition 3.10. *For any standard parabolic subgroup P of H, we have*

$$\Phi(A_0,\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s})=\Phi(A_0,\mathfrak{m}_P)$$

and

$$\Phi(A_0,\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})=\Phi(A_0,\mathfrak{n}_P).$$

Moreover, each weight of A_0 has the same multiplicity in $\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s}$ (resp. $\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s}$) and $\mathfrak{m}_P$ (resp. $\mathfrak{n}_P$).

Proof. It is obvious for both of **Case I** and **Case II** described above. □

For P a standard parabolic subgroup of H, let $\rho_{P,\mathfrak{s}}$ (resp. ρ_P) denote the half of the sum of weights (with multiplicities) of A_0 in $\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s}$ (resp. $\mathfrak{n}_P$).

Corollary 3.11. *For any standard parabolic subgroup P of H, we have*

$$\rho_{P,\mathfrak{s}}=\rho_P.$$

At the end of this subsection, we point out a non-canonical F-linear isomorphism between $\mathfrak{h}$ and $\mathfrak{s}$ which will be useful for some technical problems. We have chosen an element $\alpha\in E$ in Section 3.1. Let $\tau\in D^\times$ in **Case I** (resp. $\tau\in GL_2(D)$ in **Case II**) be an element such that $\mathrm{Ad}(\alpha)(\tau)=-\tau$.

Proposition 3.12. *There is a non-canonical isomorphism induced by multiplication by τ between $\mathfrak{h}$ and $\mathfrak{s}$ as free D'-modules (resp. $D\otimes_F E$-modules), i.e.,*

$$\mathfrak{s}=\mathfrak{h}\tau=\tau\mathfrak{h}.$$

Moreover, for any standard parabolic subgroup P of H, we have

$$\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s}=\mathfrak{m}_P\tau=\tau\mathfrak{m}_P$$

and

$$\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s}=\mathfrak{n}_P\tau=\tau\mathfrak{n}_P.$$

Proof. It is obvious for both of **Case I** and **Case II** described above. □

3.5. Fourier transform. Fix a nontrivial unitary character Ψ of $\mathbb{A}/F$. Let $\langle\cdot,\cdot\rangle$ be the $H(\mathbb{A})$-invariant bilinear form on $\mathfrak{s}(\mathbb{A})$ defined by

$$\forall Y_1,Y_2\in\mathfrak{s}(\mathbb{A}),\langle Y_1,Y_2\rangle:=\mathrm{Trd}_{\mathfrak{g},F}(Y_1Y_2),\tag{3.5.1}$$

where $\mathrm{Trd}_{\mathfrak{g},F}(Y_1Y_2)$ denotes the reduced trace of $Y_1Y_2\in\mathfrak{g}(\mathbb{A})$. It is non-degenerate, which can be seen after the base change to an algebraic closure of F. For $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$, its Fourier transform $\hat{f}$ is defined by

$$\forall\widehat{Y}\in\mathfrak{s}(\mathbb{A}),\hat{f}(\widehat{Y}):=\int_{\mathfrak{s}(\mathbb{A})}f(Y)\Psi(\langle Y,\widehat{Y}\rangle)dY.\tag{3.5.2}$$

4. Integrability of the modified kernel

Let $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$, P be a standard parabolic subgroup of H and $\mathfrak{o}\in\mathcal{O}$. For $x\in M_P(F)N_P(\mathbb{A})\backslash H(\mathbb{A})$, define

$$k_{f,P,\mathfrak{o}}(x):=\sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})}f(\mathrm{Ad}(x^{-1})(Y+U))dU.$$

For $T\in\mathfrak{a}_0$ and $x\in H(F)\backslash H(\mathbb{A})$, define

$$k^T_{f,\mathfrak{o}}(x):=\sum_{\{P:P_0\subseteq P\}}(-1)^{\dim(A_P/A_H)}\sum_{\delta\in P(F)\backslash H(F)}\widehat{\tau}^H_P(H_P(\delta x)-T)\cdot k_{f,P,\mathfrak{o}}(\delta x).\tag{4.0.1}$$

By [3, Lemma 5.1], we know that the sum over $\delta\in P(F)\backslash H(F)$ is finite.

Lemma 4.1. *There is a $T_+ \in \mathfrak{a}_{P_0}^+$ such that for all standard parabolic subgroup P of H, $T \in T_+ + \mathfrak{a}_{P_0}^+$ and $x \in H(\mathbb{A})$, we have*

$$\sum_{\{P_1:P_0\subseteq P_1\subseteq P\}} \sum_{\delta_1\in P_1(F)\backslash P(F)} F^{P_1}(\delta_1 x, T)\tau_{P_1}^P(H_{P_1}(\delta_1 x) - T) = 1.$$

Proof. This is [3, Lemma 6.4]. □

We shall fix such a T_+.

Theorem 4.2. *For all $T \in T_+ + \mathfrak{a}_{P_0}^+$,*

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{H(F)\backslash H(\mathbb{A})^1} |k_{f,\mathfrak{o}}^T(x)|dx < \infty.$$

Proof. Let $P_1 \subseteq P_2$ be a pair of standard parabolic subgroups of H. As in [3, §6], for $T_1 \in \mathfrak{a}_{P_1}$, define the characteristic function

$$\sigma_{P_1}^{P_2}(T_1) := \sum_{\{Q:P_2\subseteq Q\}} (-1)^{\dim(A_{P_2}/A_Q)}\tau_{P_1}^Q(T_1)\widehat{\tau}_Q^H(T_1).$$

Recall that for $P \supseteq P_1$ a standard parabolic subgroup of H, we have

$$\tau_{P_1}^P(T_1)\widehat{\tau}_P^H(T_1) = \sum_{\{P_2:P\subseteq P_2\}} \sigma_{P_1}^{P_2}(T_1).$$

For $x \in P_1(F)\backslash H(\mathbb{A})$, we write

$$\chi_{P_1,P_2}^T(x) := F^{P_1}(x,T)\sigma_{P_1}^{P_2}(H_{P_1}(x) - T)$$

and

$$k_{P_1,P_2,\mathfrak{o}}(x) := \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} k_{f,P,\mathfrak{o}}(x).$$

By Lemma 4.1 and the left invariance of H_P and $k_{f,P,\mathfrak{o}}$ by $P(F)$, we obtain

$$k_{f,\mathfrak{o}}^T(x) = \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}} \sum_{\delta\in P_1(F)\backslash H(F)} \chi_{P_1,P_2}^T(\delta x) k_{P_1,P_2,\mathfrak{o}}(\delta x).$$

Thus

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{H(F)\backslash H(\mathbb{A})^1} |k_{f,\mathfrak{o}}^T(x)|dx \le \sum_{\mathfrak{o}\in\mathcal{O}} \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x)|k_{P_1,P_2,\mathfrak{o}}(x)|dx.$$

Then we only need to show that for any pair of standard parabolic subgroups $P_1 \subseteq P_2$ of H,

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x)|k_{P_1,P_2,\mathfrak{o}}(x)|dx < \infty.$$

If $P_1 = P_2 \neq H$, by [3, Lemma 6.1], we have $\sigma_{P_1}^{P_2} = 0$ and then $\chi_{P_1,P_2}^T = 0$, so the integration vanishes. If $P_1 = P_2 = H$, since $F^H(\cdot, T)$ is a characteristic function with compact support in $H(F)\backslash H(\mathbb{A})^1$, the integration is convergent. Hence, we reduce ourselves to proving the following proposition. □

Proposition 4.3. *Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $P_1 \subsetneq P_2$ be a pair of standard parabolic subgroups of H. Suppose that ϵ_0 and N are two arbitrary but fixed positive real numbers. Then there exists a constant C such that*

$$\sum_{\mathfrak{o}\in\mathcal{O}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x)|k_{P_1,P_2,\mathfrak{o}}(x)|dx \le Ce^{-N\|T\|}$$

for all $T \in T_+ + \mathfrak{a}_{P_0}^+$ satisfying $\alpha(T) \ge \epsilon_0 \parallel T \parallel$ for any $\alpha \in \Delta_{P_0}^H$.

For $x \in H(F)\backslash H(\mathbb{A})$, define

$$k_{f,H}(x) := \sum_{\mathfrak{o}\in\mathcal{O}} k_{f,H,\mathfrak{o}}(x) = \sum_{Y\in\mathfrak{s}(F)} f(\mathrm{Ad}(x^{-1})(Y))$$

and

$$k_f^T(x) := \sum_{\mathfrak{o}\in\mathcal{O}} k_{f,\mathfrak{o}}^T(x). \tag{4.0.2}$$

Corollary 4.4. *Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$. For two arbitrary but fixed positive real number ϵ_0 and N, there exists a constant C such that*

$$\int_{H(F)\backslash H(\mathbb{A})^1} |k_f^T(x) - F^H(x,T)k_{f,H}(x)|dx \leq Ce^{-N\|T\|}$$

for all $T \in T_+ + \mathfrak{a}_{P_0}^+$ satisfying $\alpha(T) \geq \epsilon_0 \parallel T \parallel$ for any $\alpha \in \Delta_{P_0}^H$.

Proof of Proposition 4.3. Let P be any standard parabolic subgroup of H such that $P_1 \subseteq P \subseteq P_2$. For any $Y \in \mathfrak{m}_{\widetilde{P}}(F) \cap \mathfrak{o}$, there is a unique standard parabolic subgroup R of H such that $P_1 \subseteq R \subseteq P$ and $Y \in (\mathfrak{m}_{\widetilde{P}}(F) \cap \widetilde{\mathfrak{r}}(F) \cap \mathfrak{o}) - \left(\bigcup_{P_1 \subseteq Q \subsetneq R} \mathfrak{m}_{\widetilde{P}}(F) \cap \widetilde{\mathfrak{q}}(F) \cap \mathfrak{o}\right)$. We denote

$$\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}} := \mathfrak{m}_{\widetilde{R}} - \left(\bigcup_{\{Q:P_1 \subseteq Q \subsetneq R\}} \mathfrak{m}_{\widetilde{R}} \cap \widetilde{\mathfrak{q}}\right)$$

and

$$\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} := \mathfrak{n}_{\widetilde{R}} \cap \mathfrak{m}_{\widetilde{P}}.$$

From Corollary 3.9, we get

$$(\mathfrak{m}_{\widetilde{P}}(F) \cap \widetilde{\mathfrak{r}}(F) \cap \mathfrak{o}) - \left(\bigcup_{P_1 \subseteq Q \subsetneq R} \mathfrak{m}_{\widetilde{P}}(F) \cap \widetilde{\mathfrak{q}}(F) \cap \mathfrak{o}\right) = (\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F) \cap \mathfrak{o}) \oplus ((\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)).$$

Thus

$$\begin{aligned} k_{f,P,\mathfrak{o}}(x) &= \sum_{Y \in \mathfrak{m}_{\widetilde{P}}(F) \cap \mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(Y+U))dU \\ &= \sum_{\{R:P_1 \subseteq R \subseteq P\}} \sum_{\xi \in \widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F) \cap \mathfrak{o}} \sum_{Y \in (\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)} \int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi + Y + U))dU. \end{aligned}$$

We write $\overline{\widetilde{P}}$ for the parabolic subgroup of G opposite to $\widetilde{P}$ and

$$\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} := \mathfrak{n}_{\overline{\widetilde{R}}} \cap \mathfrak{m}_{\widetilde{P}}.$$

Notice that the restriction of $\langle \cdot, \cdot \rangle$ (see (3.5.1)) to $((\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})) \times ((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A}))$ is also non-degenerate. For any $\xi \in (\mathfrak{m}_{\widetilde{R}} \cap \mathfrak{s})(\mathbb{A})$, applying the Poisson summation formula to the Bruhat-Schwartz function $\int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi + \cdot + U))dU$, we have

$$\sum_{Y \in (\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)} \int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi + Y + U))dU = \sum_{\widehat{Y} \in (\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}),$$

where the partial Fourier transform $\Phi_\xi^{x,R}$ of $\int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi + \cdot + U))dU$ is defined by

$$\forall \widehat{Y} \in (\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A}), \Phi_\xi^{x,R}(\widehat{Y}) := \int_{(\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} \left(\int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi + Y + U))dU \right) \Psi(\langle Y, \widehat{Y} \rangle)dY.$$

Since $\langle U, \widehat{Y} \rangle = 0$ for $U \in (\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})$ and $\widehat{Y} \in (\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})$, as well as $\mathfrak{n}_{\widetilde{R}} = \mathfrak{n}_{\widetilde{P}} \oplus \mathfrak{n}_{\widetilde{R}}^{\widetilde{P}}$, we have

$$\forall \widehat{Y} \in (\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A}), \Phi_\xi^{x,R}(\widehat{Y}) = \int_{(\mathfrak{n}_{\widetilde{R}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi + U))\Psi(\langle U, \widehat{Y} \rangle)dU,$$

which is actually independent of P.

In sum,

$$k_{f,P,\mathfrak{o}}(x) = \sum_{\{R:P_1 \subseteq R \subseteq P\}} \sum_{\xi \in \widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F) \cap \mathfrak{o}} \sum_{\widehat{Y} \in (\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}).$$

Then we have

$$\begin{aligned}
k_{P_1,P_2,\mathfrak{o}}(x) &= \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} k_{f,P,\mathfrak{o}}(x) \\
&= \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \left(\sum_{\{R:P_1\subseteq R\subseteq P\}} \sum_{\xi\in\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{\widehat{Y}\in(\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}}\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}) \right) \\
&= \sum_{\{R:P_1\subseteq R\subseteq P_2\}} \sum_{\xi\in\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \left(\sum_{\{P:R\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \sum_{\widehat{Y}\in(\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}}\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}) \right).
\end{aligned}$$

For P_3 a standard parabolic subgroup of H containing R, denote

$$(\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_3}})' := \overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_3}} - \left(\bigcup_{\{Q:R\subseteq Q\subsetneq P_3\}} \overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{Q}} \right).$$

We write

$$\begin{aligned}
&\sum_{\{P:R\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \sum_{\widehat{Y}\in(\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P}}\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}) \\
=& \sum_{\{P:R\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \sum_{\{P_3:R\subseteq P_3\subseteq P\}} \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_3}})'\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}) \\
=&(-1)^{\dim(A_{P_2}/A_H)} \sum_{\{P_3:R\subseteq P_3\subseteq P_2\}} \left(\sum_{\{P:P_3\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_{P_2})} \right) \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_3}})'\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}) \\
=&(-1)^{\dim(A_{P_2}/A_H)} \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}),
\end{aligned}$$

where we have used [3, Proposition 1.1] in the last equality. Then

$$k_{P_1,P_2,\mathfrak{o}}(x) = (-1)^{\dim(A_{P_2}/A_H)} \sum_{\{R:P_1\subseteq R\subseteq P_2\}} \sum_{\xi\in\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} \Phi_\xi^{x,R}(\widehat{Y}).$$

Now we get

$$\begin{aligned}
&\sum_{\mathfrak{o}\in\mathcal{O}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) |k_{P_1,P_2,\mathfrak{o}}(x)| dx \\
\leq& \sum_{\mathfrak{o}\in\mathcal{O}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) \left(\sum_{\{R:P_1\subseteq R\subseteq P_2\}} \sum_{\xi\in\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{x,R}(\widehat{Y})| \right) dx \\
=& \sum_{\{R:P_1\subseteq R\subseteq P_2\}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) \sum_{\xi\in(\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{x,R}(\widehat{Y})| dx.
\end{aligned}$$

Thus it suffices to bound

$$\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) \sum_{\xi\in(\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{x,R}(\widehat{Y})| dx \tag{4.0.3}$$

for any fixed standard parabolic subgroup R of H such that $P_1 \subseteq R \subseteq P_2$.

Denote $A_{P_1}^{H,\infty} := A_{P_1}^{\infty} \cap H(\mathbb{A})^1$. By Iwasawa decomposition and our choice of measures, we have

$$\begin{aligned}
&\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) \sum_{\xi\in(\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\widetilde{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{x,R}(\widehat{Y})|dx \\
=&\int_K \int_{A_{P_1}^{H,\infty}} \int_{M_{P_1}(F)\backslash M_{P_1}(\mathbb{A})^1} \int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} F^{P_1}(m_1,T)\sigma_{P_1}^{P_2}(H_{P_1}(a_1)-T) \\
&\cdot \sum_{\xi\in(\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\widetilde{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{n_1m_1a_1k,R}(\widehat{Y})|e^{-2\rho_{P_1}(H_{P_0}(a_1))}dn_1dm_1da_1dk.
\end{aligned}$$

Because only those m_1 satisfying $F^{P_1}(m_1,T)\neq 0$ contribute to the integration, we can restrict the integration over those having representatives in $(N_{P_0}(\mathbb{A})M_{P_0}(\mathbb{A})^1A_{P_0}^{P_1,\infty}(t_0,T)K)\cap M_{P_1}(\mathbb{A})^1$, where $A_{P_0}^{P_1,\infty}(t_0,T) := A_{P_0}^{\infty}(P_1,t_0,T)\cap M_{P_1}(\mathbb{A})^1$. Then

$$\begin{aligned}
&\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) \sum_{\xi\in(\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\widetilde{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{x,R}(\widehat{Y})|dx \\
\leq& c_1\int_K \int_{[cpt\subseteq M_{P_0}(\mathbb{A})^1]} \int_{A_{P_0}^{P_1,\infty}(t_0,T)} \int_{A_{P_1}^{H,\infty}} \int_{[cpt\subseteq N_{P_0}^{P_2}(\mathbb{A})]} \int_{[cpt\subseteq N_{P_2}(\mathbb{A})]} \sigma_{P_1}^{P_2}(H_{P_1}(a_1)-T) \\
&\cdot \sum_{\xi\in(\widetilde{\mathfrak{m}}_{\widetilde{P_1}}^{\widetilde{R}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\widetilde{\mathfrak{n}}_{\widetilde{R}}^{\widetilde{P_2}})'\cap\mathfrak{s})(F)} |\Phi_\xi^{n_2na_1amk,R}(\widehat{Y})|e^{-2\rho_{P_0}(H_{P_0}(a_1a))}dn_2dnda_1dadmdk,
\end{aligned}$$

where $c_1 := \mathrm{vol}(K\cap M_{P_1}(\mathbb{A})^1)$ is a constant independent of T. Here we use the notation $[cpt\subseteq *]$ for denoting a compact subset in $*$ independent of T.

We claim that for $n_2\in N_{P_2}(\mathbb{A})$,

$$\Phi_\xi^{n_2x,R}(\widehat{Y}) = \Phi_\xi^{x,R}(\widehat{Y}).$$

In fact, let $U_2 := \mathrm{Ad}(n_2^{-1})(\xi)-\xi$. Then

$$\begin{aligned}
\Phi_\xi^{n_2x,R}(\widehat{Y}) &= \int_{(\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(n_2x)^{-1}(\xi+U))\Psi(\langle U,\widehat{Y}\rangle)dU \\
&= \int_{(\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi+U_2+\mathrm{Ad}(n_2^{-1})(U)))\Psi(\langle U,\widehat{Y}\rangle)dU.
\end{aligned}$$

As both of U_2 and $\mathrm{Ad}(n_2^{-1})(U)-U$ belong to $(\mathfrak{n}_{\widetilde{P_2}}\cap\mathfrak{s})(\mathbb{A})$, we get

$$\langle U_2+\mathrm{Ad}(n_2^{-1})(U)-U,\widehat{Y}\rangle = 0.$$

Hence

$$\Phi_\xi^{n_2x,R}(\widehat{Y}) = \int_{(\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(\xi+U_2+\mathrm{Ad}(n_2^{-1})(U)))\Psi(\langle U_2+\mathrm{Ad}(n_2^{-1})(U),\widehat{Y}\rangle)dU.$$

Since the change of variables $U_2+\mathrm{Ad}(n_2^{-1})(U)\mapsto U$ does not change the Haar measure, we proved our claim.

By this claim, we have

$$\Phi_\xi^{n_2na_1amk,R}(\widehat{Y}) = \Phi_\xi^{na_1amk,R}(\widehat{Y}) = \Phi_\xi^{(a_1a)(a_1a)^{-1}n(a_1a)mk,R}(\widehat{Y}).$$

Applying change of variables $\mathrm{Ad}(a_1a)^{-1}(U)\mapsto U$ and the fact that

$$\langle U,\widehat{Y}\rangle = \langle \mathrm{Ad}(a_1a)^{-1}(U),\mathrm{Ad}(a_1a)^{-1}(\widehat{Y})\rangle,$$

we deduce that

$$\Phi_\xi^{n_2na_1amk,R}(\widehat{Y}) = e^{2\rho_{R,\mathfrak{s}}(H_{P_0}(a_1a))}\Phi_{\mathrm{Ad}(a_1a)^{-1}(\xi)}^{(a_1a)^{-1}n(a_1a)mk,R}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y})).$$

Recall that $\rho_{R,\mathfrak{s}} = \rho_R$ by Corollary 3.11. From the reduction theory (see [3, p. 944]), for a_1 satisfying $\sigma_{P_1}^{P_2}(H_{P_1}(a_1)-T)\neq 0$, we know that $\mathrm{Ad}(a_1a)^{-1}(n)$ belongs to a compact subset independent of T. To

sum up,

$$\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) \sum_{\xi\in(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\widetilde{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(F)} |\Phi^{x,R}_\xi(\widehat{Y})|dx$$

$$\leq c_2 \sup_{y\in\Gamma} \int_{A^{P_1,\infty}_{P_0}(t_0,T)} \int_{A^{H,\infty}_{P_1}} e^{(2\rho_R-2\rho_{P_0})(H_{P_0}(a_1a))}\sigma^{P_2}_{P_1}(H_{P_1}(a_1)-T)$$

$$\cdot \sum_{\xi\in(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(F)} \sum_{\widehat{Y}\in((\widetilde{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(F)} |\Phi^{y,R}_{\mathrm{Ad}(a_1a)^{-1}(\xi)}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y}))|da_1da,$$

where c_2 is a constant independent of T, and Γ is a compact subset independent of T.

Let $\mathcal{O}_F$ denote the ring of integers of F. We fix an F-basis for each weight space for the action of A_0 on $\mathfrak{s}(F)$. Then $\mathcal{O}_F$-points of such a weight space make sense. Since the $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ is compactly supported on finite places, there exists a positive integer N_1 independent of T such that the sums over $\xi\in(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(F)$ and $\widehat{Y}\in((\widetilde{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(F)$ can be restricted to lattices $\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)$ and $\frac{1}{N_1}((\widetilde{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(\mathcal{O}_F)$ respectively, which can be explicit as in [13, §1.9] (we need to replace $\mathfrak{m}_R$ and $\mathfrak{n}_R$ in *loc. cit.* by $\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}$ and $\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s}$ respectively).

Fix a Euclidean norm $\|\cdot\|$ on the $\mathbb{R}$-linear space $\mathfrak{s}(F\otimes_{\mathbb{Q}}\mathbb{R})$. Consider a sufficiently large positive integer k to be precise. Thanks to Proposition 3.10, as in [13, (4.10) in p. 372], there exists an integer $m\geq 0$, a real number $k_\alpha\geq 0$ for each $\alpha\in\Delta^{P_2}_{P_0}$, and a real number $c_3>0$ satisfying the following conditions:

(1) if $R=P_2$, $m=0$;

(2) for all $\alpha\in\Delta^{P_2}_{P_0}-\Delta^R_{P_0}, k_\alpha\geq k$;

(3) for all $a\in A^\infty_{P_0}$,

$$\sum_{\widehat{Y}\in\frac{1}{N_1}((\widetilde{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(\mathcal{O}_F)} \|\mathrm{Ad}(a^{-1})(\widehat{Y})\|^{-m} \leq c_3 \prod_{\alpha\in\Delta^{P_2}_{P_0}} e^{-k_\alpha\alpha(H_{P_0}(a))}. \tag{4.0.4}$$

We fix such data.

For a multi-index $\overrightarrow{i}$, denote by $\partial^{\overrightarrow{i}}$ the corresponding differential operator on $\mathfrak{s}(F\otimes\mathbb{R})$. It can be extended to $\mathfrak{s}(\mathbb{A})$ by $\partial^{\overrightarrow{i}}(f_\infty\otimes\chi^\infty):=(\partial^{\overrightarrow{i}}f_\infty)\otimes\chi^\infty$, where we use the notation in Section 2.3. Choose a multi-index $\overrightarrow{i}$ whose sum of components is m. Denote

$$\Phi^{x,R,\overrightarrow{i}}_\xi(\widehat{Y}):=\int_{(\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})}(\partial^{\overrightarrow{i}}f)(\mathrm{Ad}(x^{-1})(\xi+U))\Psi(\langle U,\widehat{Y}\rangle)dU.$$

Using integration by parts, for $\widehat{Y}\neq 0$, we get

$$|\Phi^{y,R}_{\mathrm{Ad}(a_1a)^{-1}(\xi)}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y}))| = c_4(y)\|\mathrm{Ad}(a_1a)^{-1}(\widehat{Y})\|^{-m}|\Phi^{y,R,\overrightarrow{i}}_{\mathrm{Ad}(a_1a)^{-1}(\xi)}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y}))|,$$

where $c_4(y)$ is a continuous function of y.

For $\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})$ (refer to Section 3.4 for the notation), denote by $(\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})_\mu$ the corresponding weight space. From [53, §41], there exists a function $\phi_\mu\in\mathcal{S}((\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})_\mu(\mathbb{A}))$ for each $\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})$ and a function $\phi_{\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s}}\in\mathcal{S}((\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A}))$ such that for all $\xi+U\in(\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})\oplus(\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})$ and $y\in\Gamma$,

$$|(\partial^{\overrightarrow{i}}f)(\mathrm{Ad}(y^{-1})(\xi+U))|\leq\left(\prod_{\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})}\phi_\mu(\xi_\mu)\right)\phi_{\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s}}(U),$$

where ξ_μ denotes the projection to $(\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})_\mu(\mathbb{A})$ of ξ.

In sum, we deduce that

$$
\begin{aligned}
&\sum_{\xi\in(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(F)}\ \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(F)} |\Phi^{y,R}_{\mathrm{Ad}(a_1a)^{-1}(\xi)}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y}))| \\
=&\sum_{\xi\in\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)}\ \sum_{\widehat{Y}\in\frac{1}{N_1}((\overline{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(\mathcal{O}_F)} |\Phi^{y,R}_{\mathrm{Ad}(a_1a)^{-1}(\xi)}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y}))| \\
=&\sum_{\xi\in\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)}\ \sum_{\widehat{Y}\in\frac{1}{N_1}((\overline{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(\mathcal{O}_F)} c_4(y)\|\mathrm{Ad}(a_1a)^{-1}(\widehat{Y})\|^{-m}|\Phi^{y,R,\overrightarrow{i}}_{\mathrm{Ad}(a_1a)^{-1}(\xi)}(\mathrm{Ad}(a_1a)^{-1}(\widehat{Y}))| \\
\leq& c_5\sum_{\xi\in\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)}\left(\prod_{\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})}\phi_\mu(\mu(a_1a)^{-1}\xi_\mu)\right)\cdot\sum_{\widehat{Y}\in\frac{1}{N_1}((\overline{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(\mathcal{O}_F)}\|\mathrm{Ad}(a_1a)^{-1}(\widehat{Y})\|^{-m} \\
\leq& c_5c_3\sum_{\xi\in\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)}\left(\prod_{\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})}\phi_\mu(\mu(a_1a)^{-1}\xi_\mu)\right)\cdot\prod_{\alpha\in\Delta^{P_2}_{P_0}}e^{-k_\alpha\alpha(H_{P_0}(a_1a))},
\end{aligned}
$$

where $c_5 := \sup_{y\in\Gamma} c_4(y)\int_{(\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s})(\mathbb{A})}\phi_{\mathfrak{n}_{\widetilde{R}}\cap\mathfrak{s}}(U)dU$; in the last inequality, we have used (4.0.4). Thus

$$
\begin{aligned}
&\int_{P_1(F)\backslash H(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\xi\in(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(F)}\ \sum_{\widehat{Y}\in((\overline{\mathfrak{n}}^{\widetilde{P_2}}_{\widetilde{R}})'\cap\mathfrak{s})(F)}|\Phi^{x,R}_\xi(\widehat{Y})|dx \\
\leq& c_2c_5c_3\int_{A^{P_1,\infty}_{P_0}(t_0,T)}\int_{A^{H,\infty}_{P_1}}e^{(2\rho_R-2\rho_{P_0})(H_{P_0}(a_1a))}\sigma^{P_2}_{P_1}(H_{P_1}(a_1)-T) \\
&\cdot\sum_{\xi\in\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)}\left(\prod_{\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})}\phi_\mu(\mu(a_1a)^{-1}\xi_\mu)\right)\cdot\prod_{\alpha\in\Delta^{P_2}_{P_0}}e^{-k_\alpha\alpha(H_{P_0}(a_1a))}da_1da.
\end{aligned}
$$

From [13, p. 375], when $\sigma^{P_2}_{P_1}(H_{P_1}(a_1)-T)\neq 0$, we have $\alpha(H_{P_0}(a_1a))>t_0$ for all $\alpha\in\Delta^{P_2}_{P_0}$. Denote by $\Sigma^{\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}}_{P_0}$ the positive weights of $\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}$ under the action of A_0. Consider the subsets S of $\Sigma^{\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}}_{P_0}$ with the following property: for all $\alpha\in\Delta^R_{P_0}-\Delta^{P_1}_{P_0}$, there exists $\mu\in S$ such that its α-coordinate is >0. Then

$$
\begin{aligned}
&\sum_{\xi\in\frac{1}{N_1}(\widetilde{\mathfrak{m}}^{\widetilde{R}}_{\widetilde{P_1}}\cap\mathfrak{s})(\mathcal{O}_F)}\left(\prod_{\mu\in\Phi(A_0,\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s})}\phi_\mu(\mu(a_1a)^{-1}\xi_\mu)\right) \\
\leq&\sum_S\left[\prod_{\mu\in S}\left(\sum_{\xi_-\in\frac{1}{N_1}\mathfrak{m}_{-\mu}(\mathcal{O}_F)-\{0\}}\phi_{-\mu}(\mu(a_1a)\xi_-)\right)\right]\left[\prod_{\mu\in\Sigma^{\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}}_{P_0}}\left(\sum_{\xi_+\in\frac{1}{N_1}\mathfrak{m}_\mu(\mathcal{O}_F)}\phi_\mu(\mu(a_1a)^{-1}\xi_+)\right)\right] \\
&\cdot\left[\sum_{\xi_0\in\frac{1}{N_1}\mathfrak{m}_0(\mathcal{O}_F)}\phi_0(\xi_0)\right].
\end{aligned}
$$

Denote by $\Sigma^{\mathfrak{m}_R}_{P_0}$ the positive weights of $\mathfrak{m}_R$ under the action of A_0. From Proposition 3.10, we know that $\Sigma^{\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}}_{P_0}=\Sigma^{\mathfrak{m}_R}_{P_0}$ and that each weight has the same multiplicity in $\mathfrak{m}_{\widetilde{R}}\cap\mathfrak{s}$ and $\mathfrak{m}_R$. From now on, we are in exactly the same situation as in [13, p. 373] and able to borrow the rest of its proof to conclude. □

5. Polynomial distributions

Let $T\in T_+ + \mathfrak{a}^+_{P_0}$ and $\mathfrak{o}\in\mathcal{O}$. For $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$, define

$$
J^{H,T}_{\mathfrak{o}}(f) := \int_{H(F)\backslash H(\mathbb{A})^1} k^T_{f,\mathfrak{o}}(x)dx \tag{5.0.1}
$$

and

$$J^{H,T}(f) := \int_{H(F)\backslash H(\mathbb{A})^1} k_f^T(x)dx, \tag{5.0.2}$$

where $k_{f,\mathfrak{o}}^T(x)$ and $k_f^T(x)$ are defined by (4.0.1) and (4.0.2) respectively. From Theorem 4.2, we know that $J_{\mathfrak{o}}^{H,T}$ and $J^{H,T}$ are well-defined distributions on $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$. We also have

$$J^{H,T}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H,T}(f),$$

which is an analogue of the geometric side of Arthur's trace formula. In this section, we shall prove that $J_{\mathfrak{o}}^{H,T}(f)$ and $J^{H,T}(f)$ can be extended to polynomials in $T \in \mathfrak{a}_{P_0}$ (see Corollary 5.3 below), whose constant terms will be denoted by $J_{\mathfrak{o}}^{H}(f)$ and $J^{H}(f)$ respectively.

Let us begin with a generalisation of our results in last section. Let Q be a standard parabolic subgroup of H. Recall the two cases studied in Section 3.4. In **Case I**, we have

$$M_Q \simeq \mathrm{Res}_{E/F}GL_{n_1,D'} \times \cdots \times \mathrm{Res}_{E/F}GL_{n_l,D'}$$

and

$$M_{\widetilde{Q}} \simeq GL_{n_1,D} \times \cdots \times GL_{n_l,D},$$

where $\sum_{i=1}^{l} n_i = n$. In **Case II**, we have

$$M_Q \simeq \mathrm{Res}_{E/F}GL_{\frac{n_1}{2},D\otimes_F E} \times \cdots \times \mathrm{Res}_{E/F}GL_{\frac{n_l}{2},D\otimes_F E}$$

and

$$M_{\widetilde{Q}} \simeq GL_{n_1,D} \times \cdots \times GL_{n_l,D},$$

where n_i is even for all $1 \le i \le l$ and $\sum_{i=1}^{l} n_i = n$. In either case of the two, the tangent space of $M_{\widetilde{Q}}/M_Q$ is $\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s}$, on which M_Q acts by conjugation. We remark that our results in last section can be generalised to the product setting here, whose proofs are similar and will be omitted. Define a relation of equivalence on $(\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(F)$ which is similar to that on $\mathfrak{s}(F)$ on each component. We denote by $\mathcal{O}^{\mathfrak{m}_{\widetilde{Q}}\cap\mathfrak{s}}$ the set of equivalent classes for this relation. For $\mathfrak{o} \in \mathcal{O}$, the intersection $\mathfrak{o} \cap \mathfrak{m}_{\widetilde{Q}}(F)$ is a finite (perhaps empty) union of classes $\mathfrak{o}_1, \cdots, \mathfrak{o}_t \in \mathcal{O}^{\mathfrak{m}_{\widetilde{Q}}\cap\mathfrak{s}}$. Notice that there exists a bijection between the set of standard parabolic subgroups P of H contained in Q and the set of standard parabolic subgroups P^* of M_Q (namely $P_0 \cap M_Q \subseteq P^*$) given by $P \mapsto P \cap M_Q$, whose inverse is given by $P^* \mapsto P^* N_Q$. Let $f^* \in \mathcal{S}((\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}))$, P^* be a standard parabolic subgroup of M_Q and $1 \le j \le t$. For $x \in M_{P^*}(F)N_{P^*}(\mathbb{A})\backslash M_Q(\mathbb{A})$, define

$$k_{f^*,P^*,\mathfrak{o}_j}^{M_Q}(x) := \sum_{Y\in\mathfrak{m}_{\widetilde{P^*}}(F)\cap\mathfrak{o}_j} \int_{(\mathfrak{n}_{\widetilde{P^*}}\cap\mathfrak{s})(\mathbb{A})} f^*(\mathrm{Ad}(x^{-1})(Y+U))dU. \tag{5.0.3}$$

For $T \in \mathfrak{a}_0$ and $x \in M_Q(F)\backslash M_Q(\mathbb{A})$, define

$$k_{f^*,\mathfrak{o}_j}^{M_Q,T}(x) := \sum_{\{P^*:P_0\cap M_Q\subseteq P^*\}} (-1)^{\dim(A_{P^*}/A_{M_Q})} \sum_{\delta\in P^*(F)\backslash M_Q(F)} \widehat{\tau}_{P^*}^{M_Q}(H_{P^*}(\delta x) - T) \cdot k_{f^*,P^*,\mathfrak{o}_j}^{M_Q}(\delta x).$$

For $T \in T_+ + \mathfrak{a}_{P_0}^+$, define

$$J_{\mathfrak{o}_j}^{M_Q,T}(f^*) := \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} k_{f^*,\mathfrak{o}_j}^{M_Q,T}(x)dx.$$

Then we obtain a well-defined distribution $J_{\mathfrak{o}_j}^{M_Q,T}$ on $\mathcal{S}((\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}))$. Now we define

$$J_{\mathfrak{o}}^{M_Q,T} := \sum_{j=1}^{t} J_{\mathfrak{o}_j}^{M_Q,T} \tag{5.0.4}$$

and

$$J^{M_Q,T} := \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{M_Q,T}. \tag{5.0.5}$$

For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, define $f_Q \in \mathcal{S}((\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}))$ by

$$\forall Y \in (\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}), f_Q(Y) := \int_K \int_{(\mathfrak{n}_{\widetilde{Q}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(k^{-1})(Y+V))dVdk. \tag{5.0.6}$$

Let $T_1, T_2 \in \mathfrak{a}_{P_0}$. As in [5, §2], we define the function $\Gamma_P(T_1, T_2)$ inductively on $\dim(A_P/A_H)$ by setting

$$\widehat{\tau}_P^H(T_1 - T_2) = \sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_H)} \widehat{\tau}_P^Q(T_1)\Gamma_Q(T_1, T_2) \tag{5.0.7}$$

for any standard parabolic subgroup P of H. This definition can be explicitly given by [5, (2.1) in p. 13] and only depends on the projections of T_1, T_2 onto $\mathfrak{a}_P^H$.

LEMMA 5.1. *Let $T_2 \in \mathfrak{a}_{P_0}$ and Q be a standard parabolic subgroup of H. The function $T_1 \mapsto \Gamma_Q(T_1, T_2)$ is compactly supported on $\mathfrak{a}_Q^H$. Moreover, the function $T_2 \mapsto \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1, T_2)dT_1$ is a homogeneous polynomial in T_2 of degree $\dim(A_Q/A_H)$.*

PROOF. This is [5, Lemmas 2.1 and 2.2]. □

THEOREM 5.2. *Let $T' \in T_+ + \mathfrak{a}_{P_0}^+$, $\mathfrak{o} \in \mathcal{O}$ and $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$. Then for all $T \in T_+ + \mathfrak{a}_{P_0}^+$,*

$$J_{\mathfrak{o}}^{H,T}(f) = \sum_{\{Q:P_0\subseteq Q\}} J_{\mathfrak{o}}^{M_Q,T'}(f_Q) \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1, T - T')dT_1,$$

where $J_{\mathfrak{o}}^{H,T}, J_{\mathfrak{o}}^{M_Q,T'}$ and f_Q are defined by the formulae (5.0.1), (5.0.4) and (5.0.6) respectively.

COROLLARY 5.3. *Let $\mathfrak{o} \in \mathcal{O}$ and $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$. Then $J_{\mathfrak{o}}^{H,T}(f)$ and $J^{H,T}(f)$ (defined by (5.0.2)) are polynomials in T of degree $\leq n-1$ for $T \in T_+ + \mathfrak{a}_{P_0}^+$. Thus we can extend them to all $T \in \mathfrak{a}_{P_0}$.*

REMARK 5.4. We fix M_0 and $M_{\widetilde{0}}$ which are minimal Levi subgroups of H and G respectively. The distributions $J_{\mathfrak{o}}^H(f)$ and $J^H(f)$ (defined as constant terms of $J_{\mathfrak{o}}^{H,T}(f)$ and $J^{H,T}(f)$ respectively) are independent of the choice of the minimal parabolic subgroup $P_0 \supseteq M_0$ of H. In fact, the argument of [13, Proposition 4.6] after some minor modifications applies here because elements in Ω^H have representatives in $H(F) \cap K$ in our cases.

PROOF OF THEOREM 5.2. Let P be any standard parabolic subgroup of H, $\delta \in P(F)\backslash H(F)$ and $x \in H(\mathbb{A})^1$. By substituting $T_1 = H_P(\delta x) - T'$ and $T_2 = T - T'$ in (5.0.7), we have

$$\widehat{\tau}_P^H(H_P(\delta x) - T) = \sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_H)} \widehat{\tau}_P^Q(H_P(\delta x) - T')\Gamma_Q(H_P(\delta x) - T', T - T').$$

Then

$$\begin{aligned} J_{\mathfrak{o}}^{H,T}(f) &= \int_{H(F)\backslash H(\mathbb{A})^1} \left(\sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta\in P(F)\backslash H(F)} \widehat{\tau}_P^H(H_P(\delta x) - T) \cdot k_{f,P,\mathfrak{o}}(\delta x) \right) dx \\ &= \int_{H(F)\backslash H(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta\in P(F)\backslash H(F)} \\ &\quad \left(\sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_H)} \widehat{\tau}_P^Q(H_P(\delta x) - T')\Gamma_Q(H_P(\delta x) - T', T - T') \right) k_{f,P,\mathfrak{o}}(\delta x)dx. \end{aligned}$$

By exchanging the order of two sums over P and Q and decomposing the sum over $P(F)\backslash H(F)$ into two sums over $P(F)\backslash Q(F)$ and $Q(F)\backslash H(F)$, we obtain

$$\begin{aligned} J_{\mathfrak{o}}^{H,T}(f) = \sum_{\{Q:P_0\subseteq Q\}} \int_{H(F)\backslash H(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in P(F)\backslash Q(F)} \sum_{\delta'\in Q(F)\backslash H(F)} \\ \widehat{\tau}_P^Q(H_P(\delta\delta' x) - T')\Gamma_Q(H_P(\delta\delta' x) - T', T - T')k_{f,P,\mathfrak{o}}(\delta\delta' x)dx. \end{aligned}$$

Combining the integral over $H(F)\backslash H(\mathbb{A})^1$ and the sum over $Q(F)\backslash H(F)$ into the integral over $Q(F)\backslash H(\mathbb{A})^1$, and noticing that

$$P(F)\backslash Q(F) \simeq (P(F) \cap M_Q(F))\backslash M_Q(F),$$

we have

$$\begin{aligned} J_{\mathfrak{o}}^{H,T}(f) = \sum_{\{Q:P_0\subseteq Q\}} \int_{Q(F)\backslash H(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)} \\ \widehat{\tau}_P^Q(H_P(\delta x) - T')\Gamma_Q(H_P(\delta x) - T', T - T')k_{f,P,\mathfrak{o}}(\delta x)dx. \end{aligned}$$

By the Iwasawa decomposition and our choice of measures, we get

$$J_{\mathfrak{o}}^{H,T}(f) = \sum_{\{Q:P_0\subseteq Q\}} \int_K \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \int_{A_Q^{H,\infty}} \int_{N_Q(F)\backslash N_Q(\mathbb{A})} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)}$$
$$\sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)} \widehat{\tau}_P^Q(H_P(\delta namk)-T')\Gamma_Q(H_P(\delta namk)-T',T-T')$$
$$\cdot\, k_{f,P,\mathfrak{o}}(\delta namk)e^{-2\rho_Q(H_{P_0}(a))}dndadmdk.$$

We notice that

$$\widehat{\tau}_P^Q(H_P(\delta namk)-T') = \widehat{\tau}_P^Q(H_P(\delta m)+H_P(a)-T') = \widehat{\tau}_P^Q(H_P(\delta m)-T')$$

and that

$$\Gamma_Q(H_P(\delta namk)-T',T-T') = \Gamma_Q(H_Q(\delta namk)-T',T-T') = \Gamma_Q(H_Q(a)-T',T-T').$$

Additionally, using $A_Q = A_{\widetilde{Q}}$ and change of variables, we see that

$$\begin{aligned}
k_{f,P,\mathfrak{o}}(\delta namk) &= \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(\delta namk)^{-1}(Y+U))dU \\
&= \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(\delta a^{-1}namk)^{-1}(Y+a^{-1}Ua))dU \\
&= \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(\delta a^{-1}namk)^{-1}(Y+U))e^{2\rho_{Q,\mathfrak{s}}(H_{P_0}(a))}dU \\
&= e^{2\rho_{Q,\mathfrak{s}}(H_{P_0}(a))}k_{f,P,\mathfrak{o}}(\delta a^{-1}namk),
\end{aligned}$$

where $\rho_{Q,\mathfrak{s}}$ is defined in Section 3.4. Since $\delta a^{-1}na\delta^{-1}\in N_Q(\mathbb{A})\subseteq N_P(\mathbb{A})$ and $k_{f,P,\mathfrak{o}}$ is left invariant by $N_P(\mathbb{A})$, we deduce that

$$k_{f,P,\mathfrak{o}}(\delta namk) = e^{2\rho_{Q,\mathfrak{s}}(H_{P_0}(a))}k_{f,P,\mathfrak{o}}(\delta mk).$$

To sum up, the integrand in $J_{\mathfrak{o}}^{H,T}(f)$ is independent of $n\in N_Q(F)\backslash N_Q(\mathbb{A})$. Recall that we choose the Haar measure such that $\mathrm{vol}(N_Q(F)\backslash N_Q(\mathbb{A}))=1$. By Corollary 3.11, the factors $e^{-2\rho_Q(H_{P_0}(a))}$ and $e^{2\rho_{Q,\mathfrak{s}}(H_{P_0}(a))}$ cancel, and then

$$J_{\mathfrak{o}}^{H,T}(f) = \sum_{\{Q:P_0\subseteq Q\}} \left(\int_{A_Q^{H,\infty}} \Gamma_Q(H_Q(a)-T',T-T')da\right) \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}}$$
$$(-1)^{\dim(A_P/A_Q)} \sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)} \widehat{\tau}_P^Q(H_P(\delta m)-T')\left(\int_K k_{f,P,\mathfrak{o}}(\delta mk)dk\right)dm.$$

From the definition of the Haar measure on $A_Q^{H,\infty}$, we have

$$\begin{aligned}
\int_{A_Q^{H,\infty}} \Gamma_Q(H_Q(a)-T',T-T')da &:= \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1-T',T-T')dT_1 \\
&= \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1,T-T')dT_1.
\end{aligned}$$

Since $\mathfrak{n}_{\widetilde{P}} = \mathfrak{n}_{\widetilde{P}}^{\widetilde{Q}} \oplus \mathfrak{n}_{\widetilde{Q}}$, by change of variables, we deduce that

$$\begin{aligned}
k_{f,P,\mathfrak{o}}(\delta mk) &= \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}}^{\widetilde{Q}}\cap\mathfrak{s})(\mathbb{A})} dU \int_{(\mathfrak{n}_{\widetilde{Q}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(\delta mk)^{-1}(Y+U+V))dV \\
&= \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}}^{\widetilde{Q}}\cap\mathfrak{s})(\mathbb{A})} dU \int_{(\mathfrak{n}_{\widetilde{Q}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(k^{-1})(\mathrm{Ad}(\delta m)^{-1}(Y+U)+V))dV,
\end{aligned}$$

where we need to verify that the change of variables $V\mapsto \mathrm{Ad}(\delta m)(V)$ does not change the Haar measure. This can be shown by Proposition 3.12 in two steps: firstly, $\mathfrak{n}_{\widetilde{Q}}\cap\mathfrak{s} = \mathfrak{n}_Q\tau$ shows that $V\mapsto V' := (\delta m)V$

does not change any Haar measure; secondly, $\mathfrak{n}_{\widetilde{Q}} \cap \mathfrak{s} = \tau \mathfrak{n}_Q$ shows that $V' \mapsto V'(\delta m)^{-1}$ does not change any Haar measure. Then we can write

$$\begin{aligned}\int_K k_{f,P,\mathfrak{o}}(\delta m k)dk &= \sum_{Y \in \mathfrak{m}_{\widetilde{P}}(F)\cap \mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}}^{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A})} f_Q(\mathrm{Ad}(\delta m)^{-1}(Y+U))dU \\ &= \sum_{j=1}^{t} k^{M_Q}_{f_Q, P\cap M_Q, \mathfrak{o}_j}(\delta m)\end{aligned}$$

by (5.0.3). Now we can conclude by noting that

$$\begin{aligned}J_{\mathfrak{o}}^{M_Q,T'}(f_Q) =& \sum_{j=1}^{t} \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \sum_{\{P: P_0 \subseteq P \subseteq Q\}} (-1)^{\dim(A_{P\cap M_Q}/A_{M_Q})} \sum_{\delta \in (P(F)\cap M_Q(F))\backslash M_Q(F)} \\ & \widehat{\tau}^{M_Q}_{P\cap M_Q}(H_{P\cap M_Q}(\delta m) - T') k^{M_Q}_{f_Q, P\cap M_Q, \mathfrak{o}_j}(\delta m) dm \\ =& \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \sum_{\{P: P_0 \subseteq P \subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta \in (P(F)\cap M_Q(F))\backslash M_Q(F)} \\ & \widehat{\tau}^{Q}_{P}(H_P(\delta m) - T') \left(\sum_{j=1}^{t} k^{M_Q}_{f_Q, P\cap M_Q, \mathfrak{o}_j}(\delta m) \right) dm.\end{aligned}$$

□

6. Noninvariance

Let Q be a standard parabolic subgroup of H and $y \in H(\mathbb{A})^1$. For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, define $f_{Q,y} \in \mathcal{S}((\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}))$ by

$$\forall Y \in (\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}), f_{Q,y}(Y) := \int_K \int_{(\mathfrak{n}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(k^{-1})(Y+V)) p_Q(-H_Q(ky)) dV dk, \tag{6.0.1}$$

where for $T \in \mathfrak{a}_{P_0}$, we write

$$p_Q(T) := \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1, T) dT_1.$$

We can also extend our results in last section to the product setting by the same argument. Let $\mathfrak{o} \in \mathcal{O}$ and $f^* \in \mathcal{S}((\mathfrak{m}_{\widetilde{Q}} \cap \mathfrak{s})(\mathbb{A}))$. For $T \in T_+ + \mathfrak{a}_{P_0}^+$, $J_{\mathfrak{o}}^{M_Q,T}(f^*)$ and $J^{M_Q,T}(f^*)$ (defined by (5.0.4) and (5.0.5) respectively) are polynomials in T. Then we can extend them to all $T \in \mathfrak{a}_{P_0}$. Denote by $J_{\mathfrak{o}}^{M_Q}(f^*)$ the constant term of $J_{\mathfrak{o}}^{M_Q,T}(f^*)$.

Proposition 6.1. *For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $y \in H(\mathbb{A})^1$, we denote $f^y(x) := f(\mathrm{Ad}(y)(x))$. For $\mathfrak{o} \in \mathcal{O}$, we have*

$$J_{\mathfrak{o}}^{H}(f^y) = \sum_{\{Q: P_0 \subseteq Q\}} J_{\mathfrak{o}}^{M_Q}(f_{Q,y}).$$

Proof. Let $T \in T_+ + \mathfrak{a}_{P_0}^+$. By definition,

$$J_{\mathfrak{o}}^{H,T}(f^y) = \int_{H(F)\backslash H(\mathbb{A})^1} \left(\sum_{\{P: P_0 \subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta \in P(F)\backslash H(F)} \widehat{\tau}_P^H(H_P(\delta x) - T) \cdot k_{f^y,P,\mathfrak{o}}(\delta x) \right) dx,$$

where

$$k_{f^y,P,\mathfrak{o}}(\delta x) = \sum_{Y \in \mathfrak{m}_{\widetilde{P}}(F)\cap \mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(y)\mathrm{Ad}(\delta x)^{-1}(Y+U))dU = k_{f,P,\mathfrak{o}}(\delta x y^{-1}).$$

Invoking change of variables, we get

$$J_{\mathfrak{o}}^{H,T}(f^y) = \int_{H(F)\backslash H(\mathbb{A})^1} \left(\sum_{\{P: P_0 \subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta \in P(F)\backslash H(F)} \widehat{\tau}_P^H(H_P(\delta x y) - T) \cdot k_{f,P,\mathfrak{o}}(\delta x) \right) dx.$$

For $x \in H(\mathbb{A})$ and P a standard parabolic subgroup of H, let $k_P(x)$ be an element in K satisfying $xk_P(x)^{-1} \in P(\mathbb{A})$. Then

$$\widehat{\tau}_P^H(H_P(\delta xy) - T) = \widehat{\tau}_P^H(H_P(\delta x) - T + H_P(k_P(\delta x)y)).$$

By substituting $T_1 = H_P(\delta x) - T$ and $T_2 = -H_P(k_P(\delta x)y)$ in (5.0.7), we get

$$\widehat{\tau}_P^H(H_P(\delta xy) - T) = \sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_H)} \widehat{\tau}_P^Q(H_P(\delta x) - T)\Gamma_Q(H_P(\delta x) - T, -H_P(k_P(\delta x)y)).$$

Then

$$J_{\mathfrak{o}}^{H,T}(f^y) = \int_{H(F)\backslash H(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta\in P(F)\backslash H(F)}$$
$$\left(\sum_{\{Q:P\subseteq Q\}} (-1)^{\dim(A_Q/A_H)} \widehat{\tau}_P^Q(H_P(\delta x) - T)\Gamma_Q(H_P(\delta x) - T, -H_P(k_P(\delta x)y))\right) \cdot k_{f,P,\mathfrak{o}}(\delta x)dx,$$

By exchanging the order of two sums over P and Q, and decomposing the sum over $P(F)\backslash H(F)$ into two sums over $P(F)\backslash Q(F)$ and $Q(F)\backslash H(F)$, we deduce that

$$J_{\mathfrak{o}}^{H,T}(f^y) = \sum_{\{Q:P_0\subseteq Q\}} \int_{H(F)\backslash H(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in P(F)\backslash Q(F)} \sum_{\delta'\in Q(F)\backslash H(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta\delta' x) - T)\Gamma_Q(H_P(\delta\delta' x) - T, -H_P(k_P(\delta\delta' x)y))k_{f,P,\mathfrak{o}}(\delta\delta' x)dx.$$

By combining the integral over $H(F)\backslash H(\mathbb{A})^1$ and the sum over $Q(F)\backslash H(F)$ into the integral over $Q(F)\backslash H(\mathbb{A})^1$ and using the fact that

$$P(F)\backslash Q(F) \simeq (P(F)\cap M_Q(F))\backslash M_Q(F),$$

we have

$$J_{\mathfrak{o}}^{H,T}(f^y) = \sum_{\{Q:P_0\subseteq Q\}} \int_{Q(F)\backslash H(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta x) - T)\Gamma_Q(H_P(\delta x) - T, -H_P(k_P(\delta x)y))k_{f,P,\mathfrak{o}}(\delta x)dx.$$

By the Iwasawa decomposition and our choice of measures, we get

$$J_{\mathfrak{o}}^{H,T}(f^y) = \sum_{\{Q:P_0\subseteq Q\}} \int_K \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \int_{A_Q^{H,\infty}} \int_{N_Q(F)\backslash N_Q(\mathbb{A})} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)}$$
$$\sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)} \widehat{\tau}_P^Q(H_P(\delta namk) - T)\Gamma_Q(H_P(\delta namk) - T, -H_P(k_P(\delta namk)y))$$
$$\cdot\, k_{f,P,\mathfrak{o}}(\delta namk)e^{-2\rho_Q(H_{P_0}(a))}dndadmdk.$$

As in the proof of Theorem 5.2, we see that

$$\widehat{\tau}_P^Q(H_P(\delta namk) - T) = \widehat{\tau}_P^Q(H_P(\delta m) - T),$$

and that

$$k_{f,P,\mathfrak{o}}(\delta namk) = e^{2\rho_Q(H_{P_0}(a))}k_{f,P,\mathfrak{o}}(\delta mk).$$

Additionally,

$$\begin{aligned}\Gamma_Q(H_P(\delta namk) - T, -H_P(k_P(\delta namk)y)) &= \Gamma_Q(H_Q(\delta namk) - T, -H_Q(k_P(\delta namk)y))\\ &= \Gamma_Q(H_Q(a) - T, -H_Q(k_Q(\delta namk)y))\\ &= \Gamma_Q(H_Q(a) - T, -H_Q(ky)).\end{aligned}$$

In sum, the integrand in $J_{\mathfrak{o}}^{H,T}(f^y)$ is independent of $n \in N_Q(F)\backslash N_Q(\mathbb{A})$. Recall that we choose the Haar measure such that $\mathrm{vol}(N_Q(F)\backslash N_Q(\mathbb{A})) = 1$. Then

$$J_{\mathfrak{o}}^{H,T}(f^y) = \sum_{\{Q:P_0\subseteq Q\}} \int_K \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \int_{A_Q^{H,\infty}} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)}$$
$$\widehat{\tau}_P^Q(H_P(\delta m) - T)\Gamma_Q(H_Q(a) - T, -H_Q(ky))k_{f,P,\mathfrak{o}}(\delta mk)dadmdk.$$

First, let us consider the integral on $A_Q^{H,\infty}$, which is

$$\begin{aligned}\int_{A_Q^{H,\infty}} \Gamma_Q(H_Q(a)-T,-H_Q(ky))da :=& \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1-T,-H_Q(ky))dT_1\\ =& \int_{\mathfrak{a}_Q^H} \Gamma_Q(T_1,-H_Q(ky))dT_1\\ =& p_Q(-H_Q(ky)).\end{aligned}$$

Next, we compute the integral on K, which is

$$\int_K k_{f,P,\mathfrak{o}}(\delta mk)p_Q(-H_Q(ky))dk.$$

As in the proof of Theorem 5.2, we see that

$$k_{f,P,\mathfrak{o}}(\delta mk) = \sum_{Y\in\mathfrak{m}_{\tilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\tilde{P}}^{\tilde{Q}}\cap\mathfrak{s})(\mathbb{A})} dU \int_{(\mathfrak{n}_{\tilde{Q}}\cap\mathfrak{s})(\mathbb{A})} f(\mathrm{Ad}(k^{-1})(\mathrm{Ad}(\delta m)^{-1}(Y+U)+V))dV,$$

so we can write

$$\begin{aligned}\int_K k_{f,P,\mathfrak{o}}(\delta mk)p_Q(-H_Q(ky))dk =& \sum_{Y\in\mathfrak{m}_{\tilde{P}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\tilde{P}}^{\tilde{Q}}\cap\mathfrak{s})(\mathbb{A})} f_{Q,y}(\mathrm{Ad}(\delta m)^{-1}(Y+U))dU\\ =& \sum_{j=1}^{t} k^{M_Q}_{f_{Q,y},P\cap M_Q,\mathfrak{o}_j}(\delta m)\end{aligned}$$

by (5.0.3). Therefore, we obtain

$$\begin{aligned}J_{\mathfrak{o}}^{H,T}(f^y) = \sum_{\{Q:P_0\subseteq Q\}} \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)}\\ \widehat{\tau}_P^Q(H_P(\delta m)-T)\left(\sum_{j=1}^{t} k^{M_Q}_{f_{Q,y},P\cap M_Q,\mathfrak{o}_j}(\delta m)\right)dm.\end{aligned}$$

As in the proof of Theorem 5.2, we notice that

$$\begin{aligned}J_{\mathfrak{o}}^{M_Q,T}(f_{Q,y}) = \int_{M_Q(F)\backslash M_Q(\mathbb{A})^1} \sum_{\{P:P_0\subseteq P\subseteq Q\}} (-1)^{\dim(A_P/A_Q)} \sum_{\delta\in(P(F)\cap M_Q(F))\backslash M_Q(F)}\\ \widehat{\tau}_P^Q(H_P(\delta m)-T)\left(\sum_{j=1}^{t} k^{M_Q}_{f_{Q,y},P\cap M_Q,\mathfrak{o}_j}(\delta m)\right)dm.\end{aligned}$$

Thus we deduce that

$$J_{\mathfrak{o}}^{H,T}(f^y) = \sum_{\{Q:P_0\subseteq Q\}} J_{\mathfrak{o}}^{M_Q,T}(f_{Q,y}).$$

We may conclude by taking the constant terms of both sides. □

7. An infinitesimal trace formula for $\mathfrak{s}//H$

Recall that for $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$, we have defined its Fourier transform $\hat{f}\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ by (3.3.1) and denoted the constant term of $J_{\mathfrak{o}}^{H,T}(f)$ by $J_{\mathfrak{o}}^{H}(f)$.

THEOREM 7.1. *For $f\in\mathcal{S}(\mathfrak{s}(\mathbb{A}))$, we have the equality,*

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H}(\hat{f}).$$

PROOF. Applying the Poisson summation formula, for any $x\in H(\mathbb{A})$, we have

$$\sum_{Y\in\mathfrak{s}(F)} f(\mathrm{Ad}(x^{-1})(Y)) = \sum_{Y\in\mathfrak{s}(F)} \hat{f}(\mathrm{Ad}(x^{-1})(Y)),$$

i.e.,

$$k_{f,H}(x) = k_{\hat{f},H}(x).$$

By Corollary 4.4, for all $T \in T_+ + \mathfrak{a}_{P_0}^+$ satisfying $\alpha(T) \geq \epsilon_0 \parallel T \parallel$ for any $\alpha \in \Delta_{P_0}$, we get

$$|J^{H,T}(f) - \int_{H(F)\backslash H(\mathbb{A})^1} F^H(x,T)k_{f,H}(x)dx| \leq C_1 e^{-N\|T\|}$$

and

$$|J^{H,T}(\hat{f}) - \int_{H(F)\backslash H(\mathbb{A})^1} F^H(x,T)k_{\hat{f},H}(x)dx| \leq C_2 e^{-N\|T\|}.$$

Hence

$$|J^{H,T}(f) - J^{H,T}(\hat{f})| \leq (C_1 + C_2)e^{-N\|T\|}.$$

By Corollary 5.3, we know that both of $J^{H,T}(f)$ and $J^{H,T}(\hat{f})$ are polynomials in T, so we deduce that

$$J^{H,T}(f) = J^{H,T}(\hat{f}).$$

From

$$J^{H,T}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H,T}(f)$$

and

$$J^{H,T}(\hat{f}) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H,T}(\hat{f}),$$

we obtain

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H,T}(f) = \sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}^{H,T}(\hat{f}).$$

We can draw the conclusion by taking the constant terms of both sides. □

8. The second modified kernel

Let $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, P be a standard parabolic subgroup of H and $\mathfrak{o} \in \mathcal{O}_{rs}$ (see Section 3.3). For $x \in P(F)\backslash H(\mathbb{A})$, define

$$j_{f,P,\mathfrak{o}}(x) := \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \sum_{n\in N_P(F)} f(\mathrm{Ad}(nx)^{-1}(Y)).$$

For $T \in \mathfrak{a}_0$ and $x \in H(F)\backslash H(\mathbb{A})$, define

$$j_{f,\mathfrak{o}}^T(x) := \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta\in P(F)\backslash H(F)} \widehat{\tau}_P^H(H_P(\delta x) - T)\cdot j_{f,P,\mathfrak{o}}(\delta x).$$

By [3, Lemma 5.1], we know that the sum over $\delta \in P(F)\backslash H(F)$ is finite.

Lemma 8.1. *Let P be a standard parabolic subgroup of H and $\mathfrak{o} \in \mathcal{O}_{rs}$. For $Y \in \mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}$, the map*

$$N_P \to \mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s}, n \mapsto \mathrm{Ad}(n^{-1})(Y) - Y$$

is an F-isomorphism of algebraic varieties and preserves the Haar measures on $\mathbb{A}$-points.

Proof. Recall that there are two cases considered in Section 3.4. First let us focus on **Case I**. In this case, we can suppose

$$P = \begin{pmatrix} \mathrm{Res}_{E/F}GL_{n_1,D'} & \mathrm{Res}_{E/F}Mat_{n_1\times n_2,D'} & \cdots & \mathrm{Res}_{E/F}Mat_{n_1\times n_l,D'} \\ & \mathrm{Res}_{E/F}GL_{n_2,D'} & \cdots & \mathrm{Res}_{E/F}Mat_{n_2\times n_l,D'} \\ & & \ddots & \vdots \\ & & & \mathrm{Res}_{E/F}GL_{n_l,D'} \end{pmatrix}.$$

Then

$$\widetilde{P} = \begin{pmatrix} GL_{n_1,D} & Mat_{n_1\times n_2,D} & \cdots & Mat_{n_1\times n_l,D} \\ & GL_{n_2,D} & \cdots & Mat_{n_2\times n_l,D} \\ & & \ddots & \vdots \\ & & & GL_{n_l,D} \end{pmatrix}.$$

We have chosen an element $\tau \in D^\times$ in Section 3.4. Recall also Proposition 3.12.

Let

$$Y = \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_l \end{pmatrix} \in \mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o},$$

where $Y_i \in GL_{n_i}(D')\tau$ for $1 \le i \le l$, and

$$n = \begin{pmatrix} 1 & n_{12} & \cdots & n_{1l} \\ & 1 & \cdots & n_{2l} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \in N_P,$$

where $n_{ij} \in \mathrm{Res}_{E/F} Mat_{n_i \times n_j, D'}$ for $1 \le i < j \le l$. Then

$$Yn - nY = \begin{pmatrix} 0 & Y_1 n_{12} - n_{12} Y_2 & \cdots & Y_1 n_{1l} - n_{1l} Y_l \\ & 0 & \cdots & Y_2 n_{2l} - n_{2l} Y_l \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix} \in \mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s}.$$

Now we claim that the morphism of F-affine spaces

$$\begin{aligned} \mathrm{Res}_{E/F} Mat_{n_i \times n_j, D'} &\to (\mathrm{Res}_{E/F} Mat_{n_i \times n_j, D'})\tau \\ n_{ij} &\mapsto Y_i n_{ij} - n_{ij} Y_j \end{aligned}$$

induces an F-linear isomorphism on F-points. In fact, since it gives an F-linear map between finite dimensional linear spaces of the same dimension, we only need to prove that this map is injective. If $Y_i n_{ij} - n_{ij} Y_j = 0$, then $Y_i^2 n_{ij} = Y_i n_{ij} Y_j = n_{ij} Y_j^2$. We view this as an equation of matrices with entries in D' or its base change to an algebraic closure of E. Since Y is regular semi-simple, Y^2 is regular semi-simple in $\mathfrak{h}(F)$ (viewed as a central simple algebra over E), so Y_i^2 and Y_j^2 have no common eigenvalue. By the classical theory of Sylvester equation, we know that $n_{ij} = 0$ and conclude.

Using this claim, we see that the map

$$N_P \to \mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s}, n \mapsto Yn - nY$$

is an F-isomorphism of algebraic varieties and preserves the Haar measure on $\mathbb{A}$-points. Notice that $\mathrm{Ad}(n^{-1})(Y) - Y = n^{-1}(Yn - nY)$. It is easy to see that here n^{-1} functions as some translation $Y_i n_{ij} - n_{ij} Y_j \mapsto Y_i n_{ij} - n_{ij} Y_j + \sum_{k>j} (\text{a polynomial of } n_{i'j'}, i' > i, j' \le j \text{ or } i' \ge i, j' < j) \cdot (Y_k n_{kj} - n_{kj} Y_j)$, so an analogous assertion still holds for the map $n \mapsto \mathrm{Ad}(n^{-1})(Y) - Y$.

Next let us turn to **Case II** whose proof is close to the first one. In this case, we may suppose

$$P = \begin{pmatrix} \mathrm{Res}_{E/F} GL_{\frac{n_1}{2}, D \otimes_F E} & \mathrm{Res}_{E/F} Mat_{\frac{n_1}{2} \times \frac{n_2}{2}, D \otimes_F E} & \cdots & \mathrm{Res}_{E/F} Mat_{\frac{n_1}{2} \times \frac{n_l}{2}, D \otimes_F E} \\ & \mathrm{Res}_{E/F} GL_{\frac{n_2}{2}, D \otimes_F E} & \cdots & \mathrm{Res}_{E/F} Mat_{\frac{n_2}{2} \times \frac{n_l}{2}, D \otimes_F E} \\ & & \ddots & \vdots \\ & & & \mathrm{Res}_{E/F} GL_{\frac{n_l}{2}, D \otimes_F E} \end{pmatrix}.$$

Then

$$\widetilde{P} = \begin{pmatrix} GL_{n_1, D} & Mat_{n_1 \times n_2, D} & \cdots & Mat_{n_1 \times n_l, D} \\ & GL_{n_2, D} & \cdots & Mat_{n_2 \times n_l, D} \\ & & \ddots & \vdots \\ & & & GL_{n_l, D} \end{pmatrix}.$$

We have chosen an element $\tau \in GL_2(D)$ in Section 3.4. Recall again Proposition 3.12.

Let

$$Y = \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_l \end{pmatrix} \in \mathfrak{m}_{\widetilde{P}}(F) \cap \mathfrak{o},$$

where $Y_i \in GL_{\frac{n_i}{2}}(D \otimes_F E)\tau$ for $1 \le i \le l$, and

$$n = \begin{pmatrix} 1 & n_{12} & \cdots & n_{1l} \\ & 1 & \cdots & n_{2l} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \in N_P,$$

where $n_{ij} \in \mathrm{Res}_{E/F} Mat_{\frac{n_i}{2}\times\frac{n_j}{2}, D\otimes_F E}$ for $1 \le i < j \le l$. Then

$$Yn - nY = \begin{pmatrix} 0 & Y_1 n_{12} - n_{12} Y_2 & \cdots & Y_1 n_{1l} - n_{1l} Y_l \\ & 0 & \cdots & Y_2 n_{2l} - n_{2l} Y_l \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix} \in \mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s}.$$

As in the proof of the first case, we show that the morphism of F-affine spaces

$$\mathrm{Res}_{E/F} Mat_{\frac{n_i}{2}\times\frac{n_j}{2}, D\otimes_F E} \to (\mathrm{Res}_{E/F} Mat_{\frac{n_i}{2}\times\frac{n_j}{2}, D\otimes_F E})\tau$$
$$n_{ij} \mapsto Y_i n_{ij} - n_{ij} Y_j$$

induces an F-linear isomorphism on F-points. This implies that the map

$$N_P \to \mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s}, n \mapsto Yn - nY$$

is an F-isomorphism of algebraic varieties and preserves the Haar measure on $\mathbb{A}$-points. By an argument similar to that in the first case, we deduce that an analogous assertion is still true for the map $n \mapsto \mathrm{Ad}(n^{-1})(Y) - Y$. □

THEOREM 8.2. *For all $T \in T_+ + \mathfrak{a}_{P_0}^+$ and $\mathfrak{o} \in \mathcal{O}_{rs}$, we have*

$$\int_{H(F)\backslash H(\mathbb{A})^1} |j_{f,\mathfrak{o}}^T(x)| dx < \infty$$

and

$$J_{\mathfrak{o}}^{H,T}(f) = \int_{H(F)\backslash H(\mathbb{A})^1} j_{f,\mathfrak{o}}^T(x) dx.$$

PROOF. As in the proof of Theorem 4.2, by the left invariance of $j_{f,P,\mathfrak{o}}$ by $P(F)$, we reduce the first statement to

$$\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi_{P_1,P_2}^T(x) |j_{P_1,P_2,\mathfrak{o}}(x)| dx < \infty,$$

where $P_1 \subsetneq P_2$ are a pair of standard parabolic subgroups of H and for $x \in P_1(F)\backslash H(\mathbb{A})$, we put

$$j_{P_1,P_2,\mathfrak{o}}(x) := \sum_{\{P: P_1 \subseteq P \subseteq P_2\}} (-1)^{\dim(A_P/A_H)} j_{f,P,\mathfrak{o}}(x).$$

Additionally,

$$j_{f,P,\mathfrak{o}}(x) = \sum_{\{R: P_1 \subseteq R \subseteq P\}} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{Y \in (\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)} \sum_{n \in N_P(F)} f(\mathrm{Ad}(nx)^{-1}(\xi + Y)).$$

From Lemma 8.1, we have

$$\begin{aligned} j_{f,P,\mathfrak{o}}(x) &= \sum_{\{R: P_1 \subseteq R \subseteq P\}} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{Y \in (\mathfrak{n}_{\widetilde{R}}^{\widetilde{P}} \cap \mathfrak{s})(F)} \sum_{u \in (\mathfrak{n}_{\widetilde{P}} \cap \mathfrak{s})(F)} f(\mathrm{Ad}(x^{-1})(\xi + Y + u)) \\ &= \sum_{\{R: P_1 \subseteq R \subseteq P\}} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{Y \in (\mathfrak{n}_{\widetilde{R}} \cap \mathfrak{s})(F)} f(\mathrm{Ad}(x^{-1})(\xi + Y)). \end{aligned}$$

Hence

$$\begin{aligned} j_{P_1,P_2,\mathfrak{o}}(x) &= \sum_{\{P: P_1 \subseteq P \subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \left(\sum_{\{R: P_1 \subseteq R \subseteq P\}} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \sum_{Y \in (\mathfrak{n}_{\widetilde{R}} \cap \mathfrak{s})(F)} f(\mathrm{Ad}(x^{-1})(\xi + Y)) \right) \\ &= \sum_{\{R: P_1 \subseteq R \subseteq P_2\}} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{R}}(F)\cap\mathfrak{o}} \left(\sum_{\{P: R \subseteq P \subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \right) \sum_{Y \in (\mathfrak{n}_{\widetilde{R}} \cap \mathfrak{s})(F)} f(\mathrm{Ad}(x^{-1})(\xi + Y)). \end{aligned}$$

Using [3, Proposition 1.1], we get

$$j_{P_1,P_2,\mathfrak{o}}(x) = (-1)^{\dim(A_{P_2}/A_H)} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{P_2}}(F)\cap\mathfrak{o}} \sum_{Y \in (\mathfrak{n}_{\widetilde{P_2}} \cap \mathfrak{s})(F)} f(\mathrm{Ad}(x^{-1})(\xi + Y)).$$

Applying Lemma 8.1 again, we obtain

$$j_{P_1,P_2,\mathfrak{o}}(x) = (-1)^{\dim(A_{P_2}/A_H)} \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{P_2}}(F)\cap\mathfrak{o}} \sum_{n_2 \in N_{P_2}(F)} f(\mathrm{Ad}(n_2x)^{-1}(\xi)).$$

Now we decompose the integral over $x \in P_1(F)\backslash H(\mathbb{A})^1$ into double integrals $n_1 \in N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})$ and $y \in M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1$ and use the fact that $\chi^T_{P_1,P_2}(x)$ is left invariant under $N_{P_1}(\mathbb{A})$. We have

$$\begin{aligned}
&\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(x)|j_{P_1,P_2,\mathfrak{o}}(x)|dx \\
&= \int_{M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1} \int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \chi^T_{P_1,P_2}(n_1y) \left| \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{P_2}}(F)\cap\mathfrak{o}} \sum_{n_2 \in N_{P_2}(F)} f(\mathrm{Ad}(n_2n_1y)^{-1}(\xi)) \right| dn_1dy \\
&\leq \int_{M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(y) \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{P_2}}(F)\cap\mathfrak{o}} \left(\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \sum_{n_2 \in N_{P_2}(F)} |f(\mathrm{Ad}(n_2n_1y)^{-1}(\xi))|dn_1 \right) dy.
\end{aligned}$$

Since $P_1 \subseteq P_2$ and $\mathrm{vol}(N_{P_2}(F)\backslash N_{P_2}(\mathbb{A})) = 1$, we see that

$$\begin{aligned}
&\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \sum_{n_2 \in N_{P_2}(F)} |f(\mathrm{Ad}(n_2n_1y)^{-1}(\xi))|dn_1 \\
&= \int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \int_{N_{P_2}(F)\backslash N_{P_2}(\mathbb{A})} \sum_{n_2 \in N_{P_2}(F)} |f(\mathrm{Ad}(n_2nn_1y)^{-1}(\xi))|dndn_1 \\
&= \int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \int_{N_{P_2}(\mathbb{A})} |f(\mathrm{Ad}(nn_1y)^{-1}(\xi))|dndn_1 \\
&= \int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \int_{(\mathfrak{n}_{\widetilde{P_2}}\cap\mathfrak{s})(\mathbb{A})} |f(\mathrm{Ad}(n_1y)^{-1}(\xi+U))|dUdn_1,
\end{aligned}$$

where we have applied Lemma 8.1 in the last equality. Hence

$$\begin{aligned}
&\int_{P_1(F)\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(x)|j_{P_1,P_2,\mathfrak{o}}(x)|dx \\
&\leq \int_{M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(y) \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{P_2}}(F)\cap\mathfrak{o}} \left(\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})} \int_{(\mathfrak{n}_{\widetilde{P_2}}\cap\mathfrak{s})(\mathbb{A})} |f(\mathrm{Ad}(n_1y)^{-1}(\xi+U))|dUdn_1 \right) dy \\
&= \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) \sum_{\xi \in \widetilde{m}_{\widetilde{P_1}}^{\widetilde{P_2}}(F)\cap\mathfrak{o}} \int_{(\mathfrak{n}_{\widetilde{P_2}}\cap\mathfrak{s})(\mathbb{A})} |f(\mathrm{Ad}(x^{-1})(\xi+U))|dUdx,
\end{aligned}$$

whose convergence comes from that of the formula (4.0.3) when $R = P_2$.

Next we begin to prove the second statement. From the first statement, now we are authorised to write

$$\int_{H(F)\backslash H(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)dx = \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}} \int_{P_1(F)\backslash H(\mathbb{A})^1} \chi^T_{P_1,P_2}(x) j_{P_1,P_2,\mathfrak{o}}(x)dx,$$

where

$$\begin{aligned}
j_{P_1,P_2,\mathfrak{o}}(x) &= \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} j_{f,P,\mathfrak{o}}(x) \\
&= \sum_{\{P:P_1\subseteq P\subseteq P_2\}} (-1)^{\dim(A_P/A_H)} \left(\sum_{Y \in \mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}} \sum_{n \in N_P(F)} f(\mathrm{Ad}(nx)^{-1}(Y)) \right).
\end{aligned}$$

Decompose the integral over $x \in P_1(F)\backslash H(\mathbb{A})^1$ into double integrals over $n_1 \in N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})$ and $y \in M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1$. Since $N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})$ is compact, from Lemma 8.1 and [53, §41], we know

that

$$\sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}}\sum_{n\in N_P(F)}|f(\mathrm{Ad}(nn_1y)^{-1}(Y))| = \sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}}\sum_{u\in(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(F)}|f(\mathrm{Ad}(n_1y)^{-1}(Y+u))|$$

is bounded on $n_1\in N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})$. Then using the fact that $\chi^T_{P_1,P_2}(x)$ is left invariant under $N_{P_1}(\mathbb{A})$, we have

$$\int_{H(F)\backslash H(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)dx = \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}}\int_{M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1}\chi^T_{P_1,P_2}(y)\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_H)}$$
$$\sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}}\left(\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})}\sum_{n\in N_P(F)}f(\mathrm{Ad}(nn_1y)^{-1}(Y))dn_1\right)dy.$$

Since $P_1\subseteq P$ and $\mathrm{vol}(N_P(F)\backslash N_P(\mathbb{A}))=1$, we see that

$$\begin{aligned}
&\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})}\sum_{n\in N_P(F)}f(\mathrm{Ad}(nn_1y)^{-1}(Y))dn_1\\
&=\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})}\int_{N_P(F)\backslash N_P(\mathbb{A})}\sum_{n\in N_P(F)}f(\mathrm{Ad}(nn_2n_1y)^{-1}(Y))dn_2dn_1\\
&=\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})}\int_{N_P(\mathbb{A})}f(\mathrm{Ad}(nn_1y)^{-1}(Y))dndn_1\\
&=\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})}\int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})}f(\mathrm{Ad}(n_1y)^{-1}(Y+U))dUdn_1,
\end{aligned}$$

where we have applied Lemma 8.1 in the last equality. Therefore, we have

$$\begin{aligned}
\int_{H(F)\backslash H(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)dx &= \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}}\int_{M_{P_1}(F)N_{P_1}(\mathbb{A})\backslash H(\mathbb{A})^1}\chi^T_{P_1,P_2}(y)\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_H)}\\
&\quad\sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}}\left(\int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A})}\int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})}f(\mathrm{Ad}(n_1y)^{-1}(Y+U))dUdn_1\right)dy\\
&= \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}}\int_{P_1(F)\backslash H(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)\sum_{\{P:P_1\subseteq P\subseteq P_2\}}(-1)^{\dim(A_P/A_H)}\\
&\quad\cdot\left(\sum_{Y\in\mathfrak{m}_{\widetilde{P}}(F)\cap\mathfrak{o}}\int_{(\mathfrak{n}_{\widetilde{P}}\cap\mathfrak{s})(\mathbb{A})}f(\mathrm{Ad}(x^{-1})(Y+U))dU\right)dx\\
&= \sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}}\int_{P_1(F)\backslash H'(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)k_{P_1,P_2,\mathfrak{o}}(x)dx.
\end{aligned}$$

Thanks to Theorem 4.2, we are able to write

$$J^{H,T}_{\mathfrak{o}}(f)=\int_{H(F)\backslash H(\mathbb{A})^1}k^T_{f,\mathfrak{o}}(x)dx=\sum_{\{P_1,P_2:P_0\subseteq P_1\subseteq P_2\}}\int_{P_1(F)\backslash H(\mathbb{A})^1}\chi^T_{P_1,P_2}(x)k_{P_1,P_2,\mathfrak{o}}(x)dx,$$

which completes our proof. □

9. Weighted orbital integrals

Let $\mathfrak{o}\in\mathcal{O}_{rs}$ (see Section 3.3). There is an element $Y_1\in\mathfrak{o}$ and a standard parabolic subgroup P_1 of H such that $Y_1\in\mathfrak{m}_{\widetilde{P_1}}(F)$ but Y_1 can not be $M_{P_1}(F)$-conjugate to an element in $\widetilde{R}$ (or equivalently in $M_{\widetilde{R}}$ by Proposition 3.3) for any standard parabolic subgroup $R\subsetneq P_1$. We call such Y_1 an elliptic element in $(\mathfrak{m}_{\widetilde{P_1}}\cap\mathfrak{s})(F)$. For P_1 and P_2 a pair of standard parabolic subgroups of H, denote by $\Omega^H(\mathfrak{a}_{P_1},\mathfrak{a}_{P_2})$ the set (perhaps empty) of distinct isomorphisms from $\mathfrak{a}_{P_1}$ to $\mathfrak{a}_{P_2}$ obtained by restriction of elements in Ω^H.

Lemma 9.1. *Let P be a standard parabolic subgroup of H. Let $Y\in(\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s})(F)$ be a regular semi-simple element in $\mathfrak{s}$. Then Y is an elliptic element in $(\mathfrak{m}_{\widetilde{P}}\cap\mathfrak{s})(F)$ if and only if the maximal F-split torus in H_Y is A_P.*

Proof. It is evident that $H_Y \subseteq H_{Y^2}$. From $Y \in (\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s})(F)$, one knows that $A_P \subseteq H_Y$. Since Y is regular semi-simple in $\mathfrak{s}$, one deduces that $Y^2 \in \mathfrak{m}_P(F)$ is regular semi-simple in $\mathfrak{h}(F)$ (viewed as a central simple algebra over E). Thus $H_{Y^2} \subseteq M_P$.

On the one hand, suppose that Y is an elliptic element in $(\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s})(F)$. If the maximal F-split torus in H_Y is not A_P, then there exists a F-split torus A_* such that $A_P \subsetneq A_* \subseteq H_Y$. For $A_* \subseteq H_Y \subseteq H_{Y^2} \subseteq M_P$, the centralizer $\mathrm{Cent}_{M_P}(A_*)$ of A_* in M_P is a Levi subgroup of M_P. There exists $m \in M_P(F)$ such that $\mathrm{Ad}(m)(\mathrm{Cent}_{M_P}(A_*)) = M_{R_*}$ for some standard parabolic subgroup R_* of M_P. Then $\mathrm{Ad}(m)(A_*) \subseteq A_{R_*}$ and the centralizer of $\mathrm{Ad}(m)(A_*)$ in M_P is M_{R_*}. Let R be the unique standard parabolic subgroup of H such that $R \subseteq P$ and that $R \cap M_P = R_*$. Then $A_R = A_{R_*}$ and $M_R = M_{R_*}$. Since τ (see Section 3.4) commutes with A_0, by Proposition 3.12, the centralizer of $\mathrm{Ad}(m)(A_*)$ in $\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s}$ is $\mathfrak{m}_{\widetilde{R}} \cap \mathfrak{s}$. From $A_* \subseteq H_Y$, one obtains $\mathrm{Ad}(m)(A_*) \subseteq H_{\mathrm{Ad}(m)(Y)}$. Since $\mathrm{Ad}(m)(Y) \in (\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s})(F)$, we deduce that $\mathrm{Ad}(m)(Y) \in (\mathfrak{m}_{\widetilde{R}} \cap \mathfrak{s})(F)$. Because $A_P \subsetneq A_*$ and $\mathrm{Ad}(m)(A_*) \subseteq A_R$, we have $A_P \subsetneq A_R$ and thus $R \subsetneq P$. That is to say, Y is not an elliptic element in $(\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s})(F)$. It is a contradiction. This proves one direction.

On the other hand, suppose that the maximal F-split torus in H_Y is A_P. If Y is not an elliptic element in $(\mathfrak{m}_{\widetilde{P}} \cap \mathfrak{s})(F)$, there exists $m \in M_P(F)$ such that $\mathrm{Ad}(m)(Y) \in (\mathfrak{m}_{\widetilde{R}} \cap \mathfrak{s})(F)$ for some standard parabolic subgroup $R \subsetneq P$. Then $A_R \subseteq H_{\mathrm{Ad}(m)(Y)}$, i.e., $\mathrm{Ad}(m^{-1})(A_R) \subseteq H_Y$. For $R \subsetneq P$, one sees that $A_P \subsetneq \mathrm{Ad}(m^{-1})(A_R)$. That is to say, $\mathrm{Ad}(m^{-1})(A_R)$ is a strictly larger split torus than A_P in H_Y. It contradicts our hypothesis. This proves the other direction. □

Theorem 9.2. *Let $\mathfrak{o} \in \mathcal{O}_{rs}$, P_1 be a standard parabolic subgroup of H and $Y_1 \in \mathfrak{o}$ be an elliptic element in $(\mathfrak{m}_{\widetilde{P_1}} \cap \mathfrak{s})(F)$. For $f \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, we have*

$$J_{\mathfrak{o}}^H(f) = \mathrm{vol}(A_{P_1}^{\infty} H_{Y_1}(F)\backslash H_{Y_1}(\mathbb{A})) \cdot \int_{H_{Y_1}(\mathbb{A})\backslash H(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(Y_1)) v_{P_1}(x) dx,$$

where $v_{P_1}(x)$ is left-invariant under $H_{Y_1}(\mathbb{A})$ and is equal to the volume of the projection onto $\mathfrak{a}_{P_1}^H$ of the convex hull of $\{-H_Q(x)\}$, where Q takes over all semi-standard parabolic subgroups of H with $M_Q = M_{P_1}$.

Remark 9.3. The weights that we obtain for regular semi-simple orbits are the same as Arthur's in [3, p. 951]. These weights are also the same as those (see [39, p. 131]) appearing in the twisted trace formula for $H \rtimes \sigma$, where σ acts on H by $\mathrm{Ad}(\tau)$ (see Section 3.4 for the choice of τ). Notice that the action σ stabilises P_0 and M_0. All standard parabolic subgroups P of H are σ-stable and σ fixes $\mathfrak{a}_P = \mathfrak{a}_{\widetilde{P}}$.

Proof of Theorem 9.2. Let P be any standard parabolic subgroup of H and $Y \in \mathfrak{m}_{\widetilde{P}}(F) \cap \mathfrak{o}$. There exists a standard parabolic subgroup $P_2 \subseteq P$ and Y_2 an elliptic element in $(\mathfrak{m}_{\widetilde{P_2}} \cap \mathfrak{s})(F)$ such that Y_2 is $M_P(F)$-conjugate to Y. By Lemma 9.1, the maximal F-split torus in H_{Y_2} is A_{P_2}. Any element in $H(F)$ which conjugates Y_1 and Y_2 will conjugate A_{P_1} and A_{P_2}. It follows that there exists $s \in \Omega^H(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ and $m \in M_P(F)$ such that

$$Y = \mathrm{Ad}(m\omega_s)(Y_1).$$

Suppose that $P_3 \subseteq P$ is another standard parabolic subgroup, $s' \in \Omega^H(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_3})$ and $m' \in M_P(F)$ such that

$$Y = \mathrm{Ad}(m'\omega_{s'})(Y_1).$$

Then there is $\zeta \in H_Y(F)$ such that

$$m'\omega_{s'} = \zeta m \omega_s.$$

From $H_Y \subseteq M_P$, we see that

$$\omega_{s'} = \xi \omega_s$$

for some $\xi \in M_P(F)$. Denote by $\Omega^H(\mathfrak{a}_{P_1}; P)$ the set of $s \in \bigcup_{\mathfrak{a}_{P_2}} \Omega^H(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ satisfying $\mathfrak{a}_P \subseteq s\mathfrak{a}_{P_1}$ and $s^{-1}\alpha > 0$ for each $\alpha \in \Delta_{P_2}^P$. In sum, for any given P a standard parabolic subgroup of H and $Y \in \mathfrak{m}_{\widetilde{P}}(F) \cap \mathfrak{o}$, there is a unique $s \in \Omega^H(\mathfrak{a}_{P_1}; P)$ such that $Y = \mathrm{Ad}(m\omega_s)(Y_1)$ for some $m \in M_P(F)$.

For $x \in P(F)\backslash H(\mathbb{A})$, we have

$$\begin{aligned} j_{f,P,\mathfrak{o}}(x) &= \sum_{Y \in \mathfrak{m}_{\bar{P}}(F)\cap \mathfrak{o}} \sum_{n \in N_P(F)} f(\mathrm{Ad}(nx)^{-1}(Y)) \\ &= \sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \sum_{m \in M_{P,\mathrm{Ad}(\omega_s)(Y_1)}(F)\backslash M_P(F)} \sum_{n \in N_P(F)} f(\mathrm{Ad}(mnx)^{-1}\mathrm{Ad}(\omega_s)(Y_1)) \\ &= \sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \sum_{m \in M_{P,\mathrm{Ad}(\omega_s)(Y_1)}(F)\backslash P(F)} f(\mathrm{Ad}(mx)^{-1}\mathrm{Ad}(\omega_s)(Y_1)), \end{aligned}$$

where $M_{P,\mathrm{Ad}(\omega_s)(Y_1)}$ denotes the centralizer of $\mathrm{Ad}(\omega_s)(Y_1)$ in M_P. Then for $T \in \mathfrak{a}_0$ and $x \in H(F)\backslash H(\mathbb{A})$, we deduce that

$$\begin{aligned} j^T_{f,\mathfrak{o}}(x) &= \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta \in P(F)\backslash H(F)} \widehat{\tau}^H_P(H_P(\delta x)-T)\cdot j_{f,P,\mathfrak{o}}(\delta x) \\ &= \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{\delta \in P(F)\backslash H(F)} \widehat{\tau}^H_P(H_P(\delta x)-T) \\ &\quad \cdot \left(\sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \sum_{m \in M_{P,\mathrm{Ad}(\omega_s)(Y_1)}(F)\backslash P(F)} f(\mathrm{Ad}(m\delta x)^{-1}\mathrm{Ad}(\omega_s)(Y_1)) \right) \\ &= \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \sum_{\delta \in M_{P,\mathrm{Ad}(\omega_s)(Y_1)}(F)\backslash H(F)} \widehat{\tau}^H_P(H_P(\delta x)-T) \\ &\quad \cdot f(\mathrm{Ad}(\delta x)^{-1}\mathrm{Ad}(\omega_s)(Y_1)). \end{aligned}$$

Notice that the centralizer of $\mathrm{Ad}(\omega_s)(Y_1)$ in H is actually contained in M_P, so

$$\begin{aligned} j^T_{f,\mathfrak{o}}(x) &= \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \sum_{\delta \in H_{\omega_s Y_1 \omega_s^{-1}}(F)\backslash H(F)} \widehat{\tau}^H_P(H_P(\delta x)-T)\cdot f(\mathrm{Ad}(\delta x)^{-1}\mathrm{Ad}(\omega_s)(Y_1)) \\ &= \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \sum_{\delta \in H_{Y_1}(F)\backslash H(F)} \widehat{\tau}^H_P(H_P(\omega_s\delta x)-T)\cdot f(\mathrm{Ad}(\delta x)^{-1}(Y_1)). \end{aligned}$$

For $y \in H(\mathbb{A})$, we write

$$\chi_T(y) := \sum_{\{P:P_0\subseteq P\}} (-1)^{\dim(A_P/A_H)} \sum_{s \in \Omega^H(\mathfrak{a}_{P_1};P)} \widehat{\tau}^H_P(H_P(\omega_s y)-T).$$

Then

$$j^T_{f,\mathfrak{o}}(x) = \sum_{\delta \in H_{Y_1}(F)\backslash H(F)} f(\mathrm{Ad}(\delta x)^{-1}(Y_1))\cdot \chi_T(\delta x).$$

For $T \in T_+ + \mathfrak{a}^+_{P_0}$, applying Theorem 8.2 and the fact that $j^T_{f,\mathfrak{o}}(x)$ is left invariant by A^∞_H, we have

$$\begin{aligned} J^{H,T}_{\mathfrak{o}}(f) &= \int_{H(F)\backslash H(\mathbb{A})^1} j^T_{f,\mathfrak{o}}(x)dx \\ &= \int_{A^\infty_H H(F)\backslash H(\mathbb{A})} \left(\sum_{\delta \in H_{Y_1}(F)\backslash H(F)} f(\mathrm{Ad}(\delta x)^{-1}(Y_1))\cdot \chi_T(\delta x) \right) dx. \end{aligned}$$

Then we obtain

$$J^{H,T}_{\mathfrak{o}}(f) = \mathrm{vol}(A^\infty_{P_1} H_{Y_1}(F)\backslash H_{Y_1}(\mathbb{A}))\cdot \int_{H_{Y_1}(\mathbb{A})\backslash H(\mathbb{A})} f(\mathrm{Ad}(x^{-1})(Y_1))v_{P_1}(x,T)dx, \tag{9.0.1}$$

where

$$v_{P_1}(x,T) := \int_{A^\infty_H\backslash A^\infty_{P_1}} \chi_T(ax)da.$$

Here we have used the fact that $v_{P_1}(x,T)$ is well-defined and left-invariant under $H_{Y_1}(\mathbb{A}) \subseteq M_P(\mathbb{A})$. Moreover, $v_{P_1}(x,T)$ is equal to the volume of the projection onto $\mathfrak{a}^H_{P_1}$ of the convex hull of $\{T_Q - H_Q(x)\}$, where T_Q denotes the projection of sT in $\mathfrak{a}_Q$ for any $s \in \Omega^H$ satisfying $sP_0 \subseteq Q$, and Q takes over all

semi-standard parabolic subgroups of H with $M_Q = M_{P_1}$. These properties follow from [3, p. 951]. We have also assumed the finiteness of $\mathrm{vol}(A_{P_1}^{\infty} H_{Y_1}(F)\backslash H_{Y_1}(\mathbb{A}))$, which results from Lemma 9.1.

In the end, we may conclude by taking contant terms of both sides of (9.0.1). □

10. The weighted fundamental lemma

In this section, we turn to the local setting and change the notation by letting F be a non-archimedean local field of characteristic 0.

10.1. (H, M)-families associated to local weighted orbital integrals. Suppose that H is a reductive group defined over F. Fix a maximal compact subgroup K of $H(F)$ which is admissible relative to M_0 in the sense of [5, p. 9]. Again we choose the standard one for an inner of GL_n. For a parabolic subgroup P of H and $x \in H(F)$, we have $H(F) = P(F)K$ by Iwasawa decomposition and define $H_P(x)$ as in Section 2.2 by replacing $|\cdot|_{\mathbb{A}}$ with the normalised absolute value $|\cdot|_F$ on F. Suppose that M is a Levi subgroup of H containing M_0. Let $\mathcal{P}(M)$ be the set of parabolic subgroups of H with Levi component M. According to [5, p. 40-41],

$$v_P(\lambda, x) := e^{-\lambda(H_P(x))}, \forall \lambda \in i\mathfrak{a}_M^*, P \in \mathcal{P}(M),$$

is an (H, M)-family in the sense of [5, p. 36]. Let $P \in \mathcal{P}(M)$ and Q be a parabolic subgroup of H containing P. Define

$$\theta_P^Q(\lambda) := \mathrm{vol}(\mathfrak{a}_P^Q/\mathbb{Z}(\Delta_P^Q)^\vee)^{-1} \prod_{\alpha^\vee \in (\Delta_P^Q)^\vee} \lambda(\alpha^\vee),$$

where $\mathbb{Z}(\Delta_P^Q)^\vee$ denotes the lattice in $\mathfrak{a}_P^Q$ generated by $(\Delta_P^Q)^\vee$. Then we obtain a function

$$v_M^Q(x) := \lim_{\lambda \to 0} \sum_{\{P \in \mathcal{P}(M): P \subseteq Q\}} v_P(\lambda, x)\theta_P^Q(\lambda)^{-1}, \forall x \in H(F).$$

10.2. Matching of orbits. Assume that F has odd residue characteristic and that E is an unramified quadratic extension over F. Let $G := GL_{2n}$ and $H := \mathrm{Res}_{E/F} GL_{n,E}$ be the centralizer of $E^\times$ in G. Let $H' := GL_n \times GL_n$ be the subgroup of G by diagonal embedding. Denote by $\mathcal{O}_F$ the ring of integers of F. All of G, H and H' are regarded as group schemes over $\mathcal{O}_F$. Let $\mathfrak{s}$ (resp. $\mathfrak{s}'$) be the tangent space at the neutral element of G/H (resp. G/H'), which is viewed as a subspace of $\mathfrak{g}$. Here we can and shall identify $\mathfrak{s}(F) \simeq \mathfrak{h}(F)$, on which $H(F)$ acts by twisted conjugation, i.e., $h \cdot Y = hY\overline{h}^{-1}$, where $\overline{h}$ denotes the nontrivial Galois conjugate of $h \in \mathfrak{h}(F)$. In fact, we can write concretely $G(F) \simeq \left\{ \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} : A, B \in GL_n(E) \right\}$, $H(F) \simeq \left\{ \begin{pmatrix} A & \\ & \overline{A} \end{pmatrix} : A \in GL_n(E) \right\}$ and $\mathfrak{s}(F) \simeq \left\{ \begin{pmatrix} 0 & B \\ \overline{B} & 0 \end{pmatrix} : B \in \mathfrak{gl}_n(E) \right\}$, and choose $\tau = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ in this form. Additionally, if we write $G(F)$ in the usual form of invertible $2n \times 2n$-matrices over F and $H'(F) = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} : A, B \in GL_n(F) \right\}$, then $\mathfrak{s}'(F) = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A, B \in \mathfrak{gl}_n(F) \right\} \simeq \mathfrak{gl}_n(F) \oplus \mathfrak{gl}_n(F)$. We see that $H'(F)$ acts on $\mathfrak{s}'(F)$ by conjugation, i.e., $(x_1, x_2) \cdot (A, B) = (x_1 A x_2^{-1}, x_2 B x_1^{-1})$.

Recall [10, Lemma 1.1 of Chapter 1] that the norm map $Y \mapsto Y\overline{Y}$ induces an injection from the set of twisted conjugacy classes in $GL_n(E)$ to the set of conjugacy classes in $GL_n(F)$, whose image is denoted by $N(GL_n(E))$; in particular, we write $NE^\times$ for $N(GL_1(E))$. We have the notions of regular semisimple elements in $\mathfrak{s}(F)$ and $\mathfrak{s}'(F)$ (whose sets are denoted by $\mathfrak{s}_{rs}(F)$ and $\mathfrak{s}'_{rs}(F)$ respectively) as before, which are explicitly described as follows.

Proposition 10.1. *1) An element Y of $\mathfrak{s}(F)$ is regular semi-simple if and only if $Y\overline{Y}$ belongs to $GL_n(E)$ and is regular semi-simple. The map $Y \mapsto Y\overline{Y}$ from $\mathfrak{s}(F)$ to $GL_n(E)$ induces an injection from the set of $H(F)$-conjugacy classes of regular semi-simple elements in $\mathfrak{s}(F)$ into the set of regular semi-simple conjugacy classes in $GL_n(F)$.*

2) An element X of $\mathfrak{s}'(F)$ is regular semi-simple if and only if it is $H'(F)$-conjugate to an element of the form

$$X(A) := \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix}$$

with $A \in GL_n(F)$ being regular semi-simple. The map $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mapsto AB$ from $\mathfrak{s}'(F)$ to $GL_n(F)$ induces a bijection between the set of $H'(F)$-conjugacy classes of regular semi-simple elements in $\mathfrak{s}'(F)$ and the set of regular semi-simple conjugacy classes in $GL_n(F)$.

Proof. 1) is contained in [24, Lemma 2.1], while 2) is proved in [31, Proposition 2.1 and Lemma 2.1]. □

To sum up, the composition of the map in 1) and the inverse of the map in 2) above induces an injection from the set of $H(F)$-orbits in $\mathfrak{s}_{rs}(F)$ into the set of $H'(F)$-orbits in $\mathfrak{s}'_{rs}(F)$. We shall say that $Y \in \mathfrak{s}_{rs}(F)$ and $X \in \mathfrak{s}'_{rs}(F)$ have matching orbits if their orbits are matched under this injection. Alternatively, this can be canonically characterized by an identification of categorical quotients $\mathfrak{s}//H \simeq \mathfrak{s}'//H'$ (see Proposition 3.5 and Proposition 3.3 in Chapter 2). With our identification $\mathfrak{s}(F) \simeq \mathfrak{h}(F)$, we see that $Y \in \mathfrak{s}_{rs}(F)$ and $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}'_{rs}(F)$ have matching orbits if and only if the characteristic polynomial of $Y\overline{Y} \in GL_n(E)$ equals that of $AB \in GL_n(F)$.

10.3. Matching of Levi subgroups involved. We recall some terminology in §3.4 and §5.2 in Chapter 2. The subgroup of diagonal matrices in G is a common minimal Levi subgroup of G and H'. We also fix a minimal semi-standard parabolic subgroup of H' to be the group of products of upper triangular matrices. We say that a semi-standard parabolic subgroup of G is "relatively standard" if its intersection with H' is a standard parabolic subgroup of H'. Let $\omega := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. We say that a semi-standard parabolic subgroup P of G is "ω-stable" if $\mathrm{Ad}(\omega)(P) = P$. Recall that if the Lie algebra of a relatively standard parabolic subgroup P of G has non-empty intersection with $\mathfrak{s}'_{rs}$, then P must be ω-stable (see Proposition 5.1 in Chapter 2).

We shall say that a semi-standard Levi subgroup M' of G is "ω-stable" if $M' = M_{P'}$ for some ω-stable parabolic subgroup P' of G. We should remark that this condition is stronger than $\mathrm{Ad}(\omega)(M') = M'$: for example, the minimal Levi subgroup of diagonal matrices in G is not considered to be ω-stable in our sense. Here ω-stable Levi subgroups of G play the role of semi-standard Levi subsets of $(GL_n \times GL_n) \rtimes \sigma'$ in the sense of [37, §I.1], where σ' exchanges two copies of GL_n. For any linear subspace $\mathfrak{v}$ of $\mathfrak{g}$, we denote by $\mathfrak{v}^\times$ the intersection of $\mathfrak{v}$ and G in $\mathfrak{g}$. Notice that there is a bijection between the set of semi-standard Levi subgroups of GL_n and the set of semi-standard Levi subgroups of H (resp. the set of ω-stable Levi subgroups of G) induced by $M_n \mapsto M = \mathrm{Res}_{E/F} M_{n,E}$ (resp. $M_n \mapsto M' = \begin{pmatrix} \mathfrak{m}_n & \mathfrak{m}_n \\ \mathfrak{m}_n & \mathfrak{m}_n \end{pmatrix}^\times$); here $\mathfrak{m}_n$ denotes the Lie algebra of M_n. We shall use the notations M_n, M, M' to denote the corresponding semi-standard or ω-stable Levi subgroups of different groups under these bijections after fixing one of the three. We also have bijections among semi-standard or ω-stable parabolic subgroups (denoted by Q_n, Q, Q') of different groups containing these Levi subgroups.

Let M' be an ω-stable Levi subgroup of G. We shall say that $Y \in \mathfrak{m}(F) \cap \mathfrak{s}_{rs}(F)$ and $X \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ have M'-matching orbits if in each pair of blocks of $\mathfrak{m}$ and $\mathfrak{m}'$, their components have matching orbits.

10.4. Transfer factor. Let η be the quadratic character of $F^\times/NE^\times$ attached to the quadratic field extension E/F. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}'_{rs}(F)$, define a transfer factor (see [58, Definition 5.8])

$$\kappa(X) := \eta(\det(A)),$$

which satisfies $\kappa(\mathrm{Ad}(x^{-1})(X)) = \eta(\det(x))\kappa(X)$ for any $x \in H'(F)$.

10.5. Transfer of weighted orbital integrals. Fix the Haar measures on $H(F)$ and $H'(F)$ such that $\mathrm{vol}(H(\mathcal{O}_F)) = \mathrm{vol}(H'(\mathcal{O}_F)) = 1$. For a locally compact and totally disconnected space X, denote by $\mathcal{C}_c^\infty(X)$ the $\mathbb{C}$-linear space of locally constant and compactly supported functions on X.

Definition 10.2. *1) Let M be a semi-standard Levi subgroup of H and Q a parabolic subgroup of H containing M. For $Y \in \mathfrak{m}(F) \cap \mathfrak{s}_{rs}(F)$ and $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define the weighted orbital integral of f at Y by*

$$J_M^Q(Y, f) := \int_{H_Y(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(Y)) v_M^Q(x) dx.$$

2) Let M' be an ω-stable Levi subgroup of G and Q' a parabolic subgroup of G containing M' (thus Q' is ω-stable). For $X \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define the weighted η-orbital integral of f' at X by

$$J_{M'}^{\eta,Q'}(X, f') := \int_{H'_X(F)\backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(X))\eta(\det(x))v_{M'}^{Q'}(x)dx.$$

Remark 10.3. 1) v_M^Q is a local analogue of the weight that we got in Theorem 9.2. By Remark 9.3, it is the same as $v_{(\mathrm{Res}_{E/F}M_{n,E})\rtimes\sigma}^{(\mathrm{Res}_{E/F}Q_{n,E})\rtimes\sigma}$ in [37, §I.3], where σ is the nontrivial Galois conjugation.

2) $v_{M'}^{Q'}$ is a local analogue of the weight that we got in Theorem 9.2 in Chapter 2. By Remark 9.3 in Chapter 2, it is the same as $v_{(M_n\times M_n)\rtimes\sigma'}^{(Q_n\times Q_n)\rtimes\sigma'}$ in [37, §I.3], where σ' exchanges two copies.

If $Y \in \mathfrak{s}_{rs}(F)$ and $X \in \mathfrak{s}'_{rs}(F)$ have matching orbits, their centralizers H_Y and H'_X are canonically isomorphic. We shall fix compatible Haar measures on them.

Definition 10.4. *For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we say that f and f' are strongly associated if for all ω-stable Levi subgroup M' of G and all parabolic subgroup Q' of G containing M' (thus Q' is ω-stable), we have*

(1) if $Y \in \mathfrak{m}(F) \cap \mathfrak{s}_{rs}(F)$ and $X \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ have M'-matching orbits, then

$$\kappa(X)J_{M'}^{\eta,Q'}(X, f') = J_M^Q(Y, f);$$

(2) if $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ satisfies $\xi(AB) \notin NE^\times$ for some $\xi \in X(M_{Q_n})_F$, then

$$J_{M'}^{\eta,Q'}(X, f') = 0.$$

We remark that this definition is inspired by [37, Definition III.3.2] on the base change for GL_n. The following result (cf. [37, Remark III.3.2.(i)]) shows that to check the vanishing statement (2) in the above definition, it suffices to check it for all ω-stable Levi subgroup M' of G such that X is an elliptic element in $\mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ (i.e. $A_{M'}$ is the maximal F-split torus in H'_X).

Proposition 10.5. *Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$. The following two conditions are equivalent:*

1) for all ω-stable Levi subgroup M' of G and all parabolic subgroup Q' of G containing M', if $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ satisfies $\xi(AB) \notin NE^\times$ for some $\xi \in X(M_{Q_n})_F$, then

$$J_{M'}^{\eta,Q'}(X, f') = 0;$$

2) for all ω-stable Levi subgroup M' of G and all parabolic subgroup Q' of G containing M', if $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ is an elliptic element in $\mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ and satisfies $\xi(AB) \notin NE^\times$ for some $\xi \in X(M_{Q_n})_F$, then

$$J_{M'}^{\eta,Q'}(X, f') = 0.$$

Proof. The direction 1)⇒2) is trivial. Now we assume 2) and prove 1).

Let $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$ satisfy $\xi(AB) \notin NE^\times$ for some $\xi \in X(M_{Q_n})_F$. There is an ω-stable Levi subgroup M'_* of G contained in M' and an element $y \in M'(F) \cap H'(F)$ such that $X_* := \mathrm{Ad}(y)(X)$ is an elliptic element in $\mathfrak{m}'_*(F) \cap \mathfrak{s}'_{rs}(F)$. We have

$$J_{M'}^{\eta,Q'}(X_*, f') = \eta(\det(y))J_{M'}^{\eta,Q'}(X, f'). \tag{10.5.1}$$

Suppose that $X_* = \begin{pmatrix} 0 & A_* \\ B_* & 0 \end{pmatrix}$. Then $\xi(A_*B_*) \notin NE^\times$ for the above $\xi \in X(M_{Q_n})_F$.

By the descent formula for (G, M')-families (see [37, Lemma I.1.2]), we have

$$v_{M'}^{Q'} = \sum_{L' \in \mathscr{L}^{Q'}(M'_*)} d_{M'_*}^{Q'}(M', L')v_{M'_*}^{Q'_{L'}},$$

where $\mathscr{L}^{Q'}(M'_*)$ denotes the set of Levi subgroups of G contained in Q' and containing M'_* (thus L' is ω-stable), $Q'_{L'}$ is some parabolic subgroup of G with Levi factor L' (thus $Q'_{L'}$ is ω-stable), and

$d^{Q'}_{M'_*}(M', L') \in \mathbb{R}_{\geq 0}$ is defined in [6, p. 356]. Thus

$$J^{\eta,Q'}_{M'}(X_*, f') = \sum_{L' \in \mathscr{L}^{Q'}(M'_*)} d^{Q'}_{M'_*}(M', L') J^{\eta,Q'_{L'}}_{M'_*}(X_*, f'). \tag{10.5.2}$$

For all $L' \in \mathscr{L}^{Q'}(M'_*)$, let $\xi_L \in X(L_n)_F$ be the image of ξ under the restriction $X(M_{Q_n})_F \hookrightarrow X(L_n)_F$. Then $\xi_L(A_*B_*) \notin NE^\times$. By our assumption 2), we have

$$J^{\eta,Q'_{L'}}_{M'_*}(X_*, f') = 0.$$

Then by (10.5.1) and (10.5.2), we obtain

$$J^{\eta,Q'}_{M'}(X, f') = \eta(\det(y))^{-1} J^{\eta,Q'}_{M'}(X_*, f') = 0,$$

which shows 1). □

The proposition below (cf. [37, Lemma III.3.3]) shows that strongly associated functions are smooth transfers of each other in the sense of [58, Definition 5.10.(ii)].

Proposition 10.6. *If $f' \in \mathcal{C}^\infty_c(\mathfrak{s}'(F))$ satisfies the conditions in Proposition 10.5, then for $X \in \mathfrak{s}'_{rs}(F)$ with no matching orbit in $\mathfrak{s}_{rs}(F)$, we have*

$$J^{\eta,G'}_{G'}(X, f') = 0.$$

To prove this proposition, we recall two basic facts.

Lemma 10.7. *Suppose that $\sum_{j=1}^{l} n_j = n$. Let $A = (A_1, ..., A_l) \in GL_{n_1}(F) \times \cdots \times GL_{n_l}(F)$ be a regular semi-simple element in $GL_n(F)$. Then $A \in N(GL_n(E))$ if and only if $A_j \in N(GL_{n_j}(E))$ for all $1 \leq j \leq l$.*

Proof. This is known, but we include its proof here for completeness (cf. [36, Lemma 8.8]). For $A \in N(GL_n(E))$, there exists $B \in GL_n(E)$ such that $A = B\overline{B}$. Since $A \in GL_n(F)$, we have $B\overline{B} = \overline{B}B$, which implies that $AB = BA$. But A is regular semi-simple in $GL_n(E)$. Thus $B \in GL_{n_1}(E) \times \cdots \times GL_{n_l}(E)$. We write $B = (B_1, ..., B_l)$ with $B_j \in GL_{n_j}(E)$ for all $1 \leq j \leq l$. Then we obtain $A_j = B_j\overline{B_j} \in N(GL_{n_j}(E))$ for all $1 \leq j \leq l$. This shows one direction. The other direction is trivial. □

Lemma 10.8. *Let $A \in GL_n(F)$ be an elliptic regular element. Then $A \in N(GL_n(E))$ if and only if $\det(A) \in NE^\times$.*

Proof. This is a special case of [10, Lemma 1.4 in Chapter 1]. □

Proof of Proposition 10.6. Up to conjugation by $H(F)$, it suffices to consider $X = \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix}$ with A an elliptic regular element in $M_n(F)$ for some semi-standard Levi subgroup M_n of GL_n. Then by Lemmas 10.7 and 10.8, X has no matching orbit in $\mathfrak{s}_{rs}(F)$ if and only if $\xi(A) \notin NE^\times$ for some $\xi \in X(M_n)_F$. Let Q' be a parabolic subgroup of G with Levi factor M'. We have

$$J^{\eta,G'}_{G'}(X, f') = J^{\eta,Q'}_{M'}(X, f').$$

But $J^{\eta,Q'}_{M'}(X, f')$ vanishes for $f' \in \mathcal{C}^\infty_c(\mathfrak{s}'(F))$ satisfying the conditions in Proposition 10.5. Then we finish the proof. □

10.6. The weighted fundamental lemma. Let $f_0 \in \mathcal{C}^\infty_c(\mathfrak{s}(F))$ (resp. $f'_0 \in \mathcal{C}^\infty_c(\mathfrak{s}'(F))$) be the characteristic function of $\mathfrak{s}(\mathcal{O}_F) \simeq \mathfrak{gl}_n(\mathcal{O}_E)$ (resp. of $\mathfrak{s}'(\mathcal{O}_F) \simeq (\mathfrak{gl}_n \oplus \mathfrak{gl}_n)(\mathcal{O}_F)$).

Theorem 10.9. *The functions f_0 and f'_0 are strongly associated.*

The rest of the section is devoted to the proof of this theorem with the help of (split and unramified) base changes for GL_n. Suppose that M' is an ω-stable Levi subgroup of G and that Q' is a parabolic subgroup of G containing M' (thus Q' is ω-stable). For $x = (x_{i,j}) \in \mathfrak{gl}_n(E)$, let $|x| := \max_{i,j} |x_{i,j}\overline{x_{i,j}}|_F^{1/2}$.

10.6.1. *Split base change.* Let $A \in GL_n(F)$ be regular semi-simple and denote $v := |\det(A)|_F$. We shall define $\Phi_v \in \mathcal{C}_c^\infty(GL_n(F) \times GL_n(F))$ and $\Psi_v \in \mathcal{C}_c^\infty(GL_n(F))$ as in the proof of [58, Lemma 5.18]. Let Φ_v be the characteristic function of the subset of $(x_1, x_2) \in GL_n(F) \times GL_n(F)$ satisfying $|x_1| \leq 1$, $|x_2| \leq 1$ and $|\det(x_1x_2)|_F = v$. Let Ψ_v be the function on $GL_n(F)$ defined by

$$\Psi_v(g) := \int_{GL_n(F)} \Phi_v(x_2, x_2^{-1}g)\eta(\det(x_2))dx_2.$$

We also define $\Theta_v(x_1, x_2) := \Phi_v(x_1, x_2)\eta(\det(x_1)) \in \mathcal{C}_c^\infty(GL_n(F) \times GL_n(F))$.

We shall consider the action σ' on $GL_n \times GL_n$ which exchanges two copies. Denote by $(GL_n \times GL_n)_{(1_n,A),\sigma'}$ the twisted (by σ') centralizer of $GL_n \times GL_n$ at $(1_n, A)$ and by $GL_n(F)_A$ the centralizer of $GL_n(F)$ at A. Recall the (split) base change homomorphism (see [10, §5 of Chapter 1] for example)

$$bc_{F\times F/F} : \mathcal{H}(GL_n(F) \times GL_n(F), GL_n(\mathcal{O}_F) \times GL_n(\mathcal{O}_F)) \to \mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$$

defined by the convolution product, where $\mathcal{H}(GL_n(F)\times GL_n(F), GL_n(\mathcal{O}_F)\times GL_n(\mathcal{O}_F))$ and $\mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$ denote the corresponding spherical Hecke algebras. Notice that $\Phi_v, \Theta_v \in \mathcal{H}(GL_n(F) \times GL_n(F), GL_n(\mathcal{O}_F) \times GL_n(\mathcal{O}_F))$ and that $\Psi_v \in \mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$.

Lemma 10.10. *We have*

$$\Psi_v = bc_{F\times F/F}(\Theta_v).$$

Proof. Let $\psi_n := bc_{F\times F/F}(\Theta_v)$. Via the Satake isomorphism, it suffices to prove that ψ_n and Ψ_v have the same orbital integrals at any regular element in the diagonal torus $A_n(F)$ of $GL_n(F)$. Let $a \in A_n(F)$ be a regular element in $GL_n(F)$. From [10, §5 in Chapter 1], we know that the orbital integral of ψ_n at a is equal to the twisted (by σ') orbital integral of Θ_v at $(1_n, a)$. By change of variables (cf. the proof of [58, Lemma 5.18]), the latter is

$$\begin{aligned}
&\int_{(GL_n\times GL_n)_{(1_n,a),\sigma'}(F)\backslash GL_n(F)\times GL_n(F)} \Phi_v(x_1^{-1}x_2, x_2^{-1}ax_1)\eta(\det(x_1^{-1}x_2))dx_1dx_2\\
=&\int_{(GL_n(F)_a\backslash GL_n(F))\times GL_n(F)} \Phi_v(x_2, x_2^{-1}x_1^{-1}ax_1)\eta(\det(x_2))dx_2dx_1\\
=&\int_{GL_n(F)_a\backslash GL_n(F)} \Psi_v(\mathrm{Ad}(x_1^{-1})(a))dx_1,
\end{aligned}$$

which is the orbital integral of Ψ_v at a. This completes the proof. □

Suppose additionally that A belongs to the Levi subgroup $M_n(F)$. The twisted (by σ') weighted orbital integral of $\Theta_v \in \mathcal{C}_c^\infty(GL_n(F) \times GL_n(F))$ at $(1_n, A)$ is defined by

$$J^{(Q_n\times Q_n)\rtimes\sigma'}_{(M_n\times M_n)\rtimes\sigma'}((1_n, A), \Theta_v) := \int_{(GL_n\times GL_n)_{(1,A),\sigma'}(F)\backslash GL_n(F)\times GL_n(F)} \Theta_v(x^{-1}(1_n, A)\sigma'(x))v^{(Q_n\times Q_n)\rtimes\sigma'}_{(M_n\times M_n)\rtimes\sigma'}(x)dx.$$

The weighted orbital integral of $\Psi_v \in \mathcal{C}_c^\infty(GL_n(F))$ at A is defined by

$$J^{Q_n}_{M_n}(A, \Psi_v) := \int_{GL_n(F)_A\backslash GL_n(F)} \Psi_v(\mathrm{Ad}(x^{-1})(A))v^{Q_n}_{M_n}(x)dx.$$

Corollary 10.11. *For $A \in M_n(F)$ which is regular semi-simple in $GL_n(F)$, we have*

$$J^{(Q_n\times Q_n)\rtimes\sigma'}_{(M_n\times M_n)\rtimes\sigma'}((1_n, A), \Theta_v) = J^{Q_n}_{M_n}(A, \Psi_v).$$

Proof. It results from Lemma 10.10 and [37, Theorem IV.5.2] for the (split) base change $F \times F/F$. □

Let $X = \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix} \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$. Then $\kappa(X) = 1$. By Remark 10.3.1), since $\eta(\det(x_1x_2)) = \eta(\det(x_1^{-1}x_2))$ for $(x_1, x_2) \in GL_n(F) \times GL_n(F)$, we see that

$$\kappa(X)J^{\eta,Q'}_{M'}(X, f'_0) = J^{(Q_n\times Q_n)\rtimes\sigma'}_{(M_n\times M_n)\rtimes\sigma'}((1_n, A), \Theta_v). \tag{10.6.1}$$

10.6.2. *Unramified base change.* Let $B \in GL_n(E)$ be such that $B\overline{B}$ is regular semi-simple and denote $w := |\det(B\overline{B})|_F$. We shall define $\Xi_w \in \mathcal{C}_c^\infty(GL_n(E))$ as in the proof of [58, Lemma 5.18]. Let Ξ_w be the characteristic function of the subset of $x \in GL_n(E)$ satisfying $|x| \leq 1$ and $|\det(x\overline{x})|_F = w$.

We shall consider the nontrivial Galois conjugation σ on $\mathrm{Res}_{E/F}GL_{n,E}$. Denote by $GL_n(E)_{B,\sigma}$ the twisted (by σ) centralizer of $GL_n(E)$ at B. Recall the (unramified) base change homomorphism (see [10, §4.2 of Chapter 1] for example)

$$bc_{E/F} : \mathcal{H}(GL_n(E), GL_n(\mathcal{O}_E)) \to \mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$$

described via the Satake transform by $f(z) \mapsto f(z^2)$, where $\mathcal{H}(GL_n(E), GL_n(\mathcal{O}_E))$ denotes the corresponding spherical Hecke algebra. Note that $\Xi_w \in \mathcal{H}(GL_n(E), GL_n(\mathcal{O}_E))$.

Lemma 10.12. *We have*

$$\Psi_w = bc_{E/F}(\Xi_w).$$

Proof. This is essentially included in [23, Corollary 3.7]. Via the Satake isomorphism, it suffices to prove that $bc_{E/F}(\Xi_w)$ and Ψ_v have the same orbital integrals at any regular element in the diagonal torus $A_n(F)$ of $GL_n(F)$. From [10, Theorem 4.5 in Chapter 1], we reduce ourselves to comparing the twisted (by σ) orbital integral of Ξ_w at $\beta \in A_n(E)$ such that $\beta\overline{\beta}$ belongs to $A_n(F)$ and is regular with the orbital integral of Ψ_v at regular elements in $A_n(F)$. The former is computed in [23, the first case in p. 139], while the latter is computed in [23, the first case in p. 137]. □

Suppose additionally that B belongs to the Levi subgroup $M_n(E)$. The twisted (by σ) weighted orbital integral of $\Xi_w \in \mathcal{C}_c^\infty(GL_n(E))$ at B is defined by

$$J^{(\mathrm{Res}_{E/F}Q_{n,E})\rtimes\sigma}_{(\mathrm{Res}_{E/F}M_{n,E})\rtimes\sigma}(B, \Xi_w) := \int_{GL_n(E)_{B,\sigma}\backslash GL_n(E)} \Xi_w(x^{-1}B\sigma(x))v^{(\mathrm{Res}_{E/F}Q_{n,E})\rtimes\sigma}_{(\mathrm{Res}_{E/F}M_{n,E})\rtimes\sigma}(x)dx.$$

Corollary 10.13. *For $B \in M_n(E)$ such that $A = B\overline{B}$ belongs to $M_n(F)$ and is regular semi-simple in $GL_n(F)$, we have*

$$J^{(\mathrm{Res}_{E/F}Q_{n,E})\rtimes\sigma}_{(\mathrm{Res}_{E/F}M_{n,E})\rtimes\sigma}(B, \Xi_w) = J^{Q_n}_{M_n}(A, \Psi_v).$$

Proof. It results from Lemma 10.12 and [37, Theorem IV.5.2] for the (unramified) base change E/F. □

Let $Y = B \in \mathfrak{m}(F) \cap \mathfrak{s}_{rs}(F)$. By Remark 10.3.2), we have

$$J^Q_M(Y, f_0) = J^{(\mathrm{Res}_{E/F}Q_{n,E})\rtimes\sigma}_{(\mathrm{Res}_{E/F}M_{n,E})\rtimes\sigma}(B, \Xi_w). \tag{10.6.2}$$

10.6.3. *A reduction formula.* We fix Haar measures on $M_{Q'}(F) \cap H'(F)$ and $N_{Q'}(F) \cap H'(F)$ such that $\mathrm{vol}(M_{Q'}(F) \cap H'(\mathcal{O}_F)) = \mathrm{vol}(N_{Q'}(F) \cap H'(\mathcal{O}_F)) = 1$. Then for $f^{H'} \in \mathcal{C}_c^\infty(H'(F))$, we have (see [12, §4.1])

$$\int_{H'(F)} f^{H'}(x)dx = \int_{M_{Q'}(F)\cap H'(F)} \int_{N_{Q'}(F)\cap H'(F)} \int_{H'(\mathcal{O}_F)} f^{H'}(mnk)dkdndm.$$

We choose the Haar measure on $\mathfrak{n}_{Q'}(F) \cap \mathfrak{h}'(F)$ compatible with that on $N_{Q'}(F) \cap H'(F)$ under the exponential map. We choose the same Haar measure on four copies of $\mathfrak{n}_{Q_n}(F)$ in $\mathfrak{n}_{Q'}(F) = \begin{pmatrix} \mathfrak{n}_{Q_n}(F) & \mathfrak{n}_{Q_n}(F) \\ \mathfrak{n}_{Q_n}(F) & \mathfrak{n}_{Q_n}(F \end{pmatrix}$. Then $\mathrm{vol}(\mathfrak{n}_{Q'}(F) \cap \mathfrak{s}'(\mathcal{O}_F)) = 1$.

Let $X \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$. We may define a distribution $J^{\eta, M_{Q'}}_{M'}(X, \cdot)$ on $\mathcal{C}_c^\infty(\mathfrak{m}_{Q'}(F) \cap \mathfrak{s}'(F))$ as in Definition 10.2.2). It appears as a product of distributions in the form of $J^{\eta, G'}_{M'}(X, \cdot)$ in lower ranks. As in [58, §3.2], we define the Weyl discriminant factor by

$$|D^{\mathfrak{m}_{Q'}\cap\mathfrak{s}'}(X)|_F := |\det(\mathrm{ad}(X)|_{\mathfrak{m}_{Q'}/\mathfrak{m}_{Q',X}})|_F^{1/2} > 0,$$

where $\mathfrak{m}_{Q',X}$ denotes the centralizer of X in $\mathfrak{m}_{Q'}$. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ which is invariant under $\mathrm{Ad}(H'(\mathcal{O}_F))$, we define its constant term $f'_{Q'} \in \mathcal{C}_c^\infty(\mathfrak{m}_{Q'}(F) \cap \mathfrak{s}'(F))$ by

$$f'_{Q'}(Z) := \int_{\mathfrak{n}_{Q'}(F)\cap\mathfrak{s}'(F)} f(Z + U)dU, \forall Z \in \mathfrak{m}_{Q'}(F) \cap \mathfrak{s}'(F).$$

Let $f_0^{M_{Q'}} \in \mathcal{C}_c^\infty(\mathfrak{m}_{Q'}(F) \cap \mathfrak{s}'(F))$ be the characteristic function of $\mathfrak{m}_{Q'}(F) \cap \mathfrak{s}'(\mathcal{O}_F)$. Then $(f'_0)_{Q'} = f_0^{M_{Q'}}$.

Proposition 10.14. *Let $X \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$. For all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ which is invariant under $\mathrm{Ad}(H'(\mathcal{O}_F))$, we have*

$$J_{M'}^{\eta,Q'}(X,f') = |D^{\mathfrak{s}'}(X)|_F^{-1/2}|D^{\mathfrak{m}_{Q'}\cap\mathfrak{s}'}(X)|_F^{1/2} J_{M'}^{\eta,M_{Q'}}(X,f'_{Q'}).$$

Proof. We apply the change of variables $x = mnk$ to $x \in H'(F)$, where $m \in M_{Q'}(F) \cap H'(F)$, $n \in N_{Q'}(F) \cap H'(F)$ and $k \in H'(\mathcal{O}_F)$. Notice that $v_{M'}^{Q'}(x) = v_{M'}^{M_{Q'}}(m)$. Since E/F is an unramified extension, the restriction of $\eta(\det(\cdot))$ on $H'(\mathcal{O}_F)$ is trivial. Recall that $\mathrm{vol}(H'(\mathcal{O}_F)) = 1$ and that $H'_X \subseteq M' \cap H'$ for $X \in \mathfrak{m}'(F) \cap \mathfrak{s}'_{rs}(F)$. We deduce that

$$J_{M'}^{\eta,Q'}(X,f') = \int_{H'_X(F)\backslash M_{Q'}(F)\cap H'(F)} \int_{N_{Q'}(F)\cap H'(F)} f'(\mathrm{Ad}(mn)^{-1}(X))\eta(\det(m))v_{M'}^{M_{Q'}}(m)dndm.$$

By Lemma 8.1 in Chapter 2, for $Z := \mathrm{Ad}(m^{-1})(X) \in \mathfrak{m}_{Q'}(F) \cap \mathfrak{s}'_{rs}(F)$, the map

$$N_{Q'}(F) \cap H'(F) \to \mathfrak{n}_{Q'}(F) \cap \mathfrak{s}'(F), n \mapsto \mathrm{Ad}(n^{-1})(Z) - Z$$

is an isomorphism of F-analytic varieties. From the proof of [58, Proposition 6.3.(ii)], its Jacobian is

$$c(X) := |D^{\mathfrak{s}'}(X)|_F^{1/2}|D^{\mathfrak{m}_{Q'}\cap\mathfrak{s}'}(X)|_F^{-1/2} > 0.$$

Then

$$\begin{aligned} J_{M'}^{\eta,Q'}(X,f') =& c(X)^{-1} \int_{H'_X(F)\backslash M_{Q'}(F)\cap H'(F)} \int_{\mathfrak{n}_{Q'}(F)\cap\mathfrak{s}'(F)} f'(\mathrm{Ad}(m^{-1})(X)+U)\eta(\det(m))v_{M'}^{M_{Q'}}(m)dUdm \\ =& c(X)^{-1} \int_{H'_X(F)\backslash M_{Q'}(F)\cap H'(F)} f'_Q(\mathrm{Ad}(m^{-1})(X))\eta(\det(m))v_{M'}^{M_{Q'}}(m)dm \\ =& c(X)^{-1} J_{M'}^{\eta,M_{Q'}}(X,f'_{Q'}). \end{aligned}$$

□

10.6.4. *End of the proof.*

Lemma 10.15. *For $v \notin |NE^\times|_F$, we have*

$$\Psi_v = 0.$$

Proof. This is essentially included the proof of [23, Proposition 3.7 and Corollary 3.7]. In fact, our assertion is equivalent to [23, the first line in p. 138] since E/F is unramified. But we shall also give a direct proof as follows.

Let $g \in GL_n(F)$. By the change of variables $x_2 = gx^{-1}$, we obtain

$$\Psi_v(g) = \int_{GL_n(F)} \Phi_v(x_2, x_2^{-1}g)\eta(\det(x_2))dx_2 = \eta(\det(g)) \int_{GL_n(F)} \Phi_v(gx^{-1},x)\eta(\det(x))dx.$$

For all $x_1, x_2 \in GL_n(F)$, we notice that

$$\Phi_v(x_1,x_2) = \Phi_v(x_2^t, x_1^t),$$

where the transpose of $x \in GL_n(F)$ is denoted by x^t. Therefore, we have

$$\int_{GL_n(F)} \Phi_v(gx^{-1},x)\eta(\det(x))dx = \int_{GL_n(F)} \Phi_v(x^t,(x^t)^{-1}g^t)\eta(\det(x))dx.$$

By the change of variables $x^t \mapsto x$, we see that the last integral is equal to $\Psi_v(g^t)$. Thus

$$\Psi_v(g) = \eta(\det(g))\Psi_v(g^t).$$

Because $\Psi_v \in \mathcal{H}(GL_n(F), GL_n(\mathcal{O}_F))$, by Cartan decomposition, we have

$$\Psi_v(g^t) = \Psi_v(g).$$

Then

$$\Psi_v(g) = \eta(\det(g))\Psi_v(g). \tag{10.6.3}$$

Suppose that $v \notin |NE^\times|_F$. We see from the definition that $\Psi_v(g) = 0$ unless $|\det(g)|_F = v$, in which case we have $\det(g) \notin NE^\times$ since E/F is unramified. Thus $\eta(\det(g)) = -1$ in this case, which implies that $\Psi_v(g) = 0$ by (10.6.3). □

Proof of Theorem 8.1. For (1) in Definition 10.4, it suffices to consider $X = \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix}$ and $Y = B$, where $B \in M_n(E)$ is such that $A = B\overline{B}$ belongs to $M_n(F)$ and is regular semi-simple in $GL_n(F)$. By Corollaries 10.11 and 10.13, we obtain

$$J^{(Q_n \times Q_n)\rtimes\sigma'}_{(M_n \times M_n)\rtimes\sigma'}((1_n, A), \Theta_v) = J^{(\mathrm{Res}_{E/F} Q_{n,E})\rtimes\sigma}_{(\mathrm{Res}_{E/F} M_{n,E})\rtimes\sigma}(B, \Xi_w). \tag{10.6.4}$$

Combining the formulas (10.6.1), (10.6.2) and (10.6.4), we obtain

$$\kappa(X) J^{\eta, Q'}_{M'}(X, f'_0) = J^{Q}_{M}(Y, f_0).$$

For (2) in Definition 10.4, it suffices to consider $X = \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix}$ with $A \in M_n(F)$ being regular semi-simple in $GL_n(F)$ such that $\xi(A) \notin NE^\times$ for some $\xi \in X(M_{Q_n})_F$. We still have Corollary 10.11. For the case $Q' = G$, we conclude by Lemma 10.15. We now consider a general Q'. Applying the reduction formula (Proposition 10.14) to f'_0, we may write

$$J^{\eta, Q'}_{M'}(X, f'_0) = |D^{\mathfrak{s}'}(X)|_F^{-1/2} |D^{\mathfrak{m}_{Q'} \cap \mathfrak{s}'}(X)|_F^{1/2} J^{\eta, M_{Q'}}_{M'}(X, f_0^{M_{Q'}}). \tag{10.6.5}$$

Suppose that

$$M_{Q'} \simeq GL_{2n_1} \times \cdots \times GL_{2n_l}$$

and that

$$M' \simeq M'_1 \times \cdots \times M'_l,$$

where $\sum_{i=1}^{l} n_i = n$ and M'_i is an ω-stable Levi subgroup of GL_{2n_i} for $1 \le i \le l$. We have

$$f_0^{M_{Q'}} = f'_{0,1} \otimes \cdots \otimes f'_{0,l}$$

and

$$X = (X_1, \cdots, X_l),$$

where $f'_{0,i}$ (resp. X_i) is an analogue of f'_0 (resp. X) when n is replaced by n_i for $1 \le i \le l$. Then

$$J^{\eta, M_{Q'}}_{M'}(X, f_0^{M_{Q'}}) = \prod_{i=1}^{l} J^{\eta, GL_{2n_i}}_{M'_i}(X_i, f'_{0,i}).$$

Our condition on A and the special case $Q' = G$ above tell us that at least one factor $J^{\eta, GL_{2n_i}}_{M'_i}(X_i, f'_{0,i})$ in the above product vanishes. Thus $J^{\eta, Q'}_{M'}(X, f'_0) = 0$ by (10.6.5). $\square$

CHAPTER 4

A local trace formula for p-adic infinitesimal symmetric spaces: the case of Guo-Jacquet

We establish an invariant local trace formula for the tangent space of some symmetric spaces over a non-archimedean local field of characteristic zero. These symmetric spaces are studied in Guo-Jacquet trace formulae and our methods are inspired by works of Waldspurger and Arthur. Some other results are given during the proof including a noninvariant local trace formula, Howe's finiteness for weighted orbital integrals and the representability of the Fourier transform of weighted orbital integrals. These local results are prepared for the comparison of regular semi-simple terms, which are weighted orbital integrals, of an infinitesimal variant of Guo-Jacquet trace formulae.

1. Introduction

The Guo-Jacquet trace formula [23] is a promising tool to generalise Waldspurger's result [50] on the relation between toric periods and central values of automorphic L-functions for GL_2 to higher ranks. It is inspired by Jacquet's new proof [29] of Waldspurger's theorem. Although such a formula has not been established in full generality, its simple form was used by Feigon-Martin-Whitehouse [21] to obtain some evidence for the conjecture of Guo-Jacquet. For applications, one needs to compare geometric sides of Guo-Jacquet trace formulae for different symmetric pairs. Some local results on the comparison of relative orbital integrals include Guo's fundamental lemma [23] and Zhang's smooth transfer [58].

In order to study the Guo-Jacquet trace formula and its comparison, one may begin with an infinitesimal variant. That is to say, we replace a symmetric space by its tangent space (called an infinitesimal symmetric space). Such a variant should share some similarities with the geometric side of Guo-Jacquet trace formula. It is simpler than the original formula because spectral objects are replaced by the Fourier transform of geometric objects (cf. [51] and [13]). Moreover, by the method of descent dating back to Harish-Chandra's works, the comparison at the infinitesimal level should imply the comparison of geometric sides of original formulae (see [58] on the transfer of orbital integrals).

An infinitesimal variant of Guo-Jacquet trace formulae has been established in Chapter 2 and Chapter 3 via an analogue of Arthur's truncation process in [3] (see also [13] for its Lie algebra variant). We actually consider more general cases suggested by [57] and [44]. Most (namely regular semi-simple) terms appearing in these formulae can be written as explicit weighted orbital integrals on infinitesimal symmetric spaces over a number field (see Theorem 9.2 in Chapter 2 and Theorem 9.2 in Chapter 3). They are noninvariant analogues of ordinary orbital integrals (which can be compared locally thanks to [23] and [58]) and should be the next objects to be compared. As the first evidence, the weighted fundamental lemma has been proved in Theorem 10.9 in Chapter 3 thanks to Labesse's work [37] on the base change for GL_n.

The same philosophy of Waldspurger's work [52] on the endoscopic transfer has been followed by Zhang [58] to prove the transfer of local orbital integrals on infinitesimal symmetric spaces of Guo-Jacquet. A simple form of the local trace formula [58, Lemma 6.5], Howe's finiteness for orbital integrals [45, Theorem 6.1] and representability of the Fourier transform of orbital integrals [58, Theorem 6.1] apart from the fundamental lemma [58, Lemma 5.18] at the infinitesimal level have been used in Zhang's proof. It is expected that such a strategy should be extended to the weighted context. In fact, some successful attempts have been made in [14] and [15] on the stable base change. We would like to follow these ideas in the comparison of local weighted orbital integrals on infinitesimal symmetric spaces of Guo-Jacquet. However, further study in noninvariant local harmonic analysis on infinitesimal symmetric spaces is needed to achieve our goal. This paper aims to prepare some essential ingredients such as a noninvariant local trace formula, Howe's finiteness for weighted orbital integrals, representability of the Fourier transform of weighted orbital integrals and an invariant local trace formula. Our methods are mainly inspired by the works of Waldspurger's [51] and Arthur's [8].

Let E/F be a quadratic field extension of non-archimedean local fields of characteristic zero. Denote by D a central division algebra over F and by $GL_{n,D}$ the reductive group over F whose F-points are $GL_n(D)$. We study two generalised cases of Guo-Jacquet trace formulae. The first case is (G, H), where $G := GL_{2n,D}$ and $H := GL_{n,D} \times GL_{n,D}$ denotes its diagonal subgroup by diagonal embedding. Denote by $\mathfrak{s}$ the tangent space of the symmetric space G/H at the neutral element, on which H acts by conjugation. The second case is (G', H'), where G' is the group of invertible elements in a central simple algebra $\mathfrak{g}'$ over F containing E, and H' is the centraliser of $E^\times$ in G'. Denote by $\mathfrak{s}'$ the corresponding infinitesimal symmetric space. Notice that (G, H) and (G', H') are the same symmetric pair after a base change to an algebraic closure of F containing E. In the rest of the introduction and this paper, we shall focus on results in the first case and provide complete proofs. The second case is similar in statements and proofs, so we shall only state main results, point out additional ingredients and sketch necessary steps for later use.

To explain the main theorems of this paper, we first introduce some notations. Denote by η the quadratic character of $F^\times/NE^\times$ attached to E/F, where $NE^\times$ denotes the norm of $E^\times$. Let M be an ω-stable Levi subgroup of G (see Section 3.2). Let Q be a parabolic subgroup of G containing M. Suppose that $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, where $\mathfrak{m}$ is the Lie algebra of M and $\mathfrak{s}_{\mathrm{rs}}$ denotes the subset of regular semi-simple elements in $\mathfrak{s}$ (see Section 3.1). This paper is organised in the following way.

In Section 2, we fix some notations of local harmonic analysis and recall some facts of Arthur's (G, M)-families, most of which can be found in [51, §I-II].

In Section 3, we prepare some properties of infinitesimal symmetric spaces. Some of them are stated for a general symmetric pair and most of them are relative avatars of classical works of Harish-Chandra [26]. Preliminaries on symmetric pairs can be found in [45] and [1].

In Section 4, we define local weighted orbital integrals $J_M^Q(\eta, X, \cdot)$ by (4.1.1) for the action of H on $\mathfrak{s}$ and study their properties. They are distributions on $\mathfrak{s}(F)$ and local analogues of the global weighted orbital integrals obtained in Theorem 9.2 in Chapter 2.

In Section 5, we establish the noninvariant local trace formula which results from the Plancherel formula and an analogue of Arthur's truncation process in [8]. Let $\mathcal{C}_c^\infty(\mathfrak{s}(F))$ be the space of locally constant, compactly supported, complex-valued functions on $\mathfrak{s}(F)$. For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define its Fourier transform $\hat{f}$ by (3.2.1). For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define $J^G(\eta, f, f')$ by (5.1.2).

Theorem 1.1 (see Theorem 5.3). *For all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$J^G(\eta, f, \hat{f}') = J^G(\eta, \hat{f}, f').$$

We can not deduce it via the exponential map as in [51, §V] for lack of a local trace formula for symmetric spaces. One needs to return to the proof of [8] instead.

In Section 6, we show Howe's finiteness for weighted orbital integrals on $\mathfrak{s}(F)$ (see Proposition 6.1). The proof originates from Howe's seminal work [27] which is extended to weighted orbital integrals on Lie algebras by [51]. We modify the argument in [51, §IV] to make it apply to our case.

In Section 7, we show that the distribution on $\mathfrak{s}(F)$ defined by $f \mapsto J_M^G(\eta, X, \hat{f})$ is represented by a locally integrable function on $\mathfrak{s}(F)$ (see Proposition 7.2). Its proof is similar to that in [51, §V] and makes use of the noninvariant trace formula and Howe's finiteness for weighted orbital integrals.

In Section 8, we modify weighted orbital integrals to obtain invariant distributions $I_M^G(\eta, X, \cdot)$ on $\mathfrak{s}(F)$ by (8.1.1) and (8.1.7). The method is close to Arthur's standard one, but it is simpler here since there is no spectral object involved, which is also a feature of [51].

In Section 9, we establish the invariant local trace formula which is deduced from the noninvariant one. For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define $I^G(\eta, f, f')$ by (9.1.1).

Theorem 1.2 (see Theorem 9.1). *For all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$I^G(\eta, f, f') = I^G(\eta, f', f).$$

For its proof, we mainly consult [51, §VII].

In Section 10, we prove a vanishing property at "infinity" of the function on $\mathfrak{s}(F)$ representing the Fourier transform of $I_M^G(\eta, X, \cdot)$ with $M \neq G$ (see Proposition 10.1). It is an analogue of [14] and serves as a complement of the limit formula in [58, §7.1].

In the end, we remark that although we concentrate on the case of Guo-Jacquet here, many results in this paper might be extended to other symmetric pairs, which can be seen from their proofs.

2. Notation and preliminaries

2.1. Fields. Let F be a non-archimedean local field of characteristic zero. Denote by $|\cdot|_F$ (resp. $v_F(\cdot)$) the normalised absolute value (resp. the valuation) on F and by $\mathcal{O}_F$ the ring of integers of F. Fix a uniformiser ϖ of $\mathcal{O}_F$. Let q be the cardinality of the residue field of $\mathcal{O}_F$.

2.2. Groups and the map H_P. Let G be a reductive group defined over F. All algebraic groups and algebraic varieties are assumed to be defined over F in this article. Fix a Levi subgroup M_0 of a minimal parabolic subgroup of G.

Denote by A_G the maximal F-split torus in the centre of G. Define

$$\mathfrak{a}_G := \mathrm{Hom}_{\mathbb{Z}}(X(G)_F, \mathbb{R}),$$

where $X(G)_F$ is the group of F-rational characters of G. Define the homomorphism $H_G : G(F) \to \mathfrak{a}_G$ by

$$\langle H_G(x), \chi\rangle = \log(|\chi(x)|_F)$$

for all $x \in G(F)$ and $\chi \in X(G)_F$. Set $\mathfrak{a}_{G,F} := H_G(G(F))$, which is a lattice in $\mathfrak{a}_G$.

Fix a maximal compact subgroup $K = K_G$ of $G(F)$ which is admissible relative to M_0 in the sense of [5, p. 9]. In this paper, when $G(F) = GL_n(D)$ with D being a central division algebra over F, we choose the standard maximal compact subgroup $K = GL_n(\mathcal{O}_D)$ with $\mathcal{O}_D$ being the ring of integers of D (see [54, p. 191] for example). Set $W_0^G := \mathrm{Norm}_{G(F)}(M_0)/M_0(F)$ to be the Weyl group of (G, M_0), where $\mathrm{Norm}_{G(F)}(M_0)$ denotes the normaliser of M_0 in $G(F)$. It is known that any element in W_0^G admits a representative in K.

By a Levi subgroup of G, we mean a group M containing M_0 which is the Levi component of some parabolic subgroup of G. For such a group M, set $K_M := M(F) \cap K$. Then the triplet (M, K_M, M_0) satisfies the same hypotheses as (G, K, M_0). Denote by $\mathscr{F}^G(M)$, $\mathscr{P}^G(M)$ and $\mathscr{L}^G(M)$ the set of parabolic subgroups of G containing M, parabolic subgroups of G with Levi factor M and Levi subgroups of G containing M respectively.

For $P \in \mathscr{F}^G(M_0)$, let M_P be the Levi component containing M_0 and N_P the unipotent radical. Denote $A_P := A_{M_P}$ and $\mathfrak{a}_P := \mathfrak{a}_{M_P}$ whose dual $\mathbb{R}$-linear space is denoted by $\mathfrak{a}_P^*$. Define a map $H_P : G(F) \to \mathfrak{a}_{M_P}$ by

$$H_P(mnk) = H_{M_P}(m)$$

for all $m \in M_P(F)$, $n \in N_P(F)$ and $k \in K$. Let $\overline{P} \in \mathscr{P}^G(M_P)$ be the parabolic subgroup opposite to P.

For $P \subseteq Q$ a pair of parabolic subgroups in $\mathscr{F}^G(M_0)$, the restriction $X(M_Q)_F \hookrightarrow X(M_P)_F$ induces a pair of dual maps $\mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q$ and $\mathfrak{a}_Q^* \hookrightarrow \mathfrak{a}_P^*$. Let $\mathfrak{a}_P^Q$ be the kernel of the former map $\mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q$. Set Δ_P^Q to be the set of simple roots for the action of A_P on $P \cap M_Q$. Denote by $(\Delta_P^Q)^\vee$ the set of "coroots" as in [9, p. 26]. Then $(\Delta_P^Q)^\vee$ is a basis of the $\mathbb{R}$-linear space $\mathfrak{a}_P^Q$.

2.3. Heights. We fix a height function $\|\cdot\| : G(F) \to \mathbb{R}$ as in [8, §4]. It satisfies the following properties:

(1) $\|x\| \geq 1, \forall x \in G(F)$;

(2) $\|xy\| \leq \|x\|\|y\|, \forall x, y \in G(F)$;

(3) there exists $c > 0$ and $N \in \mathbb{N}$ such that $\|x^{-1}\| \leq c\|x\|^N, \forall x \in G(F)$.

If $P \in \mathscr{F}^G(M_0)$, for any $x \in G(F)$, we can choose $m_P(x) \in M_P(F), n_P(x) \in N_P(F)$ and $k_P(x) \in K$ such that $x = m_P(x)n_P(x)k_P(x)$. Then

(4) there exists $c > 0$ and $N \in \mathbb{N}$ such that $\|m_P(x)\| + \|n_P(x)\| \leq c\|x\|^N$.

We also fix a Euclidean norm (still denoted by $\|\cdot\|$) on the $\mathbb{R}$-linear space $\mathfrak{a}_{M_0}$ which is invariant under the action of W_0^G on $\mathfrak{a}_{M_0}$. Then

(5) there exist $c_1, c_2 > 0$ such that

$$c_1(1 + \log\|y\|) \leq 1 + \|H_{M_0}(y)\| \leq c_2(1 + \log\|y\|), \forall y \in M_0(F).$$

In addition, we require that $\|\cdot\|$ is a norm on $G(F)$ in the sense of [35, §18.2]. This is possible. For example, for $G = GL_n$, by writing $(g, g^{-1}) = (g_{ij}, h_{ij})_{1\leq i,j\leq n}$, one may define $\|g\| := \sup_{i,j}\{|g_{ij}|_F, |h_{ij}|_F\}$ for $g \in G(F)$. Since $\{g_{ij}, h_{ij}\}_{1\leq i,j\leq n}$ is a set of generators for the ring of regular functions of G (viewed as an affine variety over F), this defines a norm in the sense of [35, §18.2] on $GL_n(F)$. For general G, one can choose an closed embedding $G \to GL_n$ over F and define the norm on $G(F)$ by the pull-back of the norm on $GL_n(F)$. By [35, Proposition 18.1.(2)], this defines a norm in the sense of [35, §18.2] on $G(F)$.

2.4. Functions and distributions. Let $\mathfrak{g} := \mathrm{Lie}(G)$. More generally, we shall use a minuscule Fraktur letter to denote the Lie algebra of its corresponding algebraic group. Denote by Ad the adjoint action of G on itself or $\mathfrak{g}$. The adjoint action of $\mathfrak{g}$ on itself is denoted by ad.

For a locally compact and totally disconnected topological space X (e.g. $G(F)$ or $\mathfrak{g}(F)$), denote by $\mathcal{C}_c^\infty(X)$ the space of locally constant, compactly supported, complex-valued functions on X. For $f \in \mathcal{C}_c^\infty(X)$, denote by $\mathrm{Supp}(f)$ its support. Denote by $\mathcal{C}_c^\infty(X)^*$ the space of distributions on X, i.e., the linear dual of $\mathcal{C}_c^\infty(X)$.

Suppose that $G(F)$ acts on such an X. Then $G(F)$ acts on $\mathcal{C}_c^\infty(X)$ (or more generally the space of complex functions on X) by

$$g \cdot f(x) := f(g^{-1} \cdot x), \forall g \in G(F), f \in \mathcal{C}_c^\infty(X), x \in X.$$

Moreover, $G(F)$ acts on $\mathcal{C}_c^\infty(X)^*$ by

$$g \cdot d(f) := d(g^{-1} \cdot f), \forall g \in G(F), d \in \mathcal{C}_c^\infty(X)^*, f \in \mathcal{C}_c^\infty(X).$$

Let $\eta : G(F) \to \mathbb{C}^\times$ be a locally constant character. We say a function $f \in \mathcal{C}_c^\infty(X)$ (resp. a distribution $d \in \mathcal{C}_c^\infty(X)^*$) is η-invariant if $g \cdot f = \eta(g) f$ (resp. $g \cdot d = \eta(g) d$) for all $g \in G(F)$. For trivial η, we simply say that such a function (resp. distribution) is invariant.

2.5. Haar measures. Fix the Haar measure on K such that $\mathrm{vol}(K) = 1$. Following [51, §I.4], for all $P \in \mathscr{F}^G(M_0)$, we fix a Haar measure on $N_P(F)$ such that

$$\int_{N_P(F)} \exp(2\rho_{\overline{P}}(H_{\overline{P}}(n)))dn = 1,$$

where $\rho_{\overline{P}}$ is the half of the sum of roots (with multiplicity) associated to the parabolic subgroup $\overline{P}$ opposite to P. From [8, p. 12], for all $M \in \mathscr{L}^G(M_0)$, there are compatible Haar measures on $G(F)$ and $M(F)$ such that for all $P \in \mathscr{P}^G(M)$ and $f \in \mathcal{C}_c^\infty(G(F))$, we have

$$\int_{G(F)} f(x)dx = \int_{M(F)\times N_P(F)\times K} f(mnk)dkdndm.$$

We shall fix such measures.

For a F-split torus T, we choose the Haar measure on $T(F)$ such that the maximal compact subgroup of $T(F)$ is of volume 1. For a general torus T, we choose the Haar measure on T such that the induced measure on $T(F)/A_T(F)$ satisfies $\mathrm{vol}(T(F)/A_T(F)) = 1$.

Notice that if M_0 is a torus, we have associated to it two measures. However, it will be clear which one should be used according to the context.

Fix open neighbourhoods $V_{\mathfrak{g}}$ of 0 in $\mathfrak{g}$ and V_G of 1 in G such that the exponential map induces a homeomorphism between them. Choose the unique Haar measure on $\mathfrak{g}$ such that the exponential map $V_{\mathfrak{g}} \to V_G$ preserves the measures. Similarly, we obtain Haar measures on Lie algebras of subgroups of G.

From the fixed Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_{M_0}$, we deduce measures on $\mathfrak{a}_{M_0}$ and its subspaces.

2.6. (G, M)-families. Following [5, p. 15], we define

$$\theta_P^Q(\lambda) := \mathrm{vol}(\mathfrak{a}_P^Q/\mathbb{Z}(\Delta_P^Q)^\vee)^{-1} \prod_{\alpha^\vee \in (\Delta_P^Q)^\vee} \lambda(\alpha^\vee), \forall \lambda \in i\mathfrak{a}_P^*,$$

where $\mathbb{Z}(\Delta_P^Q)^\vee$ denotes the lattice in $\mathfrak{a}_P^Q$ generated by $(\Delta_P^Q)^\vee$.

Suppose that $M \in \mathscr{L}^G(M_0)$ and that $Q \in \mathscr{F}^G(M)$. Let $(c_P)_{P\in\mathscr{P}^G(M)}$ be a (G, M)-family in the sense of [5, p. 36]. By [5, Lemma 6.2], we can define

$$c_M^Q := \lim_{\lambda\to 0} \sum_{\{P\in\mathscr{P}^G(M):P\subseteq Q\}} c_P(\lambda)\theta_P^Q(\lambda)^{-1}.$$

We sometimes write $c_M := c_M^G$ if $Q = G$.

An important example is following. According to [5, p. 40-41], for $x \in G(F)$,

$$v_P(\lambda, x) := e^{-\lambda(H_P(x))}, \forall \lambda \in i\mathfrak{a}_M^*, P \in \mathscr{P}^G(M),$$

is a (G, M)-family (denoted by $(v_P(x))_{P\in\mathscr{P}^G(M)}$). Then we obtain a function

$$v_M^Q(x) := \lim_{\lambda\to 0} \sum_{\{P\in\mathscr{P}^G(M):P\subseteq Q\}} v_P(\lambda, x)\theta_P^Q(\lambda)^{-1}, \forall x \in G(F).$$

For a smooth function $c_P(\lambda)$ on $i\mathfrak{a}_P^*$, we can associate to it a smooth function $c'_P(\lambda)$ on $i\mathfrak{a}_P^*$ as in [5, (6.3) in §6]. Denote by c'_P the value of $c'_P(\lambda)$ at $\lambda = 0$. Let $(c_P)_{P\in\mathscr{P}^G(M)}$ and $(d_P)_{P\in\mathscr{P}^G(M)}$ be two (G, M)-families, we define their product $((cd)_P)_{P\in\mathscr{P}^G(M)}$ in the obvious way and have the following product formula (see [5, Lemma 6.3])

$$(cd)_M = \sum_{Q\in\mathscr{F}^G(M)} c'_Q d_M^Q. \tag{2.6.1}$$

2.7. The maps d_M^G and s. Suppose that $M \in \mathscr{L}^G(M_0)$. As in [6, p. 356], we define a map

$$d_M^G : \mathscr{L}^G(M) \times \mathscr{L}^G(M) \to \mathbb{R}_{\geq 0}$$

such that for all $(L_1, L_2) \in \mathscr{L}^G(M) \times \mathscr{L}^G(M)$,

(1) $d_M^G(G, M) = d_M^G(M, G) = 1$;
(2) $d_M^G(L_1, L_2) = d_M^G(L_2, L_1)$;
(3) $d_M^G(L_1, L_2) \neq 0$ if and only if $\mathfrak{a}_M^G = \mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2}$.

Following [51, §II.4], we also choose a map (not unique)

$$s : \mathscr{L}^G(M) \times \mathscr{L}^G(M) \to \mathscr{F}^G(M) \times \mathscr{F}^G(M)$$

such that for all $(L_1, L_2) \in \mathscr{L}^G(M) \times \mathscr{L}^G(M)$,

(4) $s(L_1, L_2) \in \mathscr{P}^G(L_1) \times \mathscr{P}^G(L_2)$;
(5) if $s(L_1, L_2) = (Q_1, Q_2)$, then $s(L_2, L_1) = (\overline{Q_2}, \overline{Q_1})$;
(6) (splitting formula) if $(c_P)_{P\in\mathscr{P}^G(M)}$ and $(d_P)_{P\in\mathscr{P}^G(M)}$ are (G, M)-families, we have the equality

$$(cd)_M = \sum_{L_1,L_2\in\mathscr{L}^G(M)} d_M^G(L_1, L_2) c_M^{Q_1} c_M^{Q_2},$$

where $(Q_1, Q_2) := s(L_1, L_2)$;

(7) (descent formula) if $(c_P)_{P\in\mathscr{P}^G(M)}$ is a (G, M)-family and $L \in \mathscr{L}^G(M)$, we have the equality

$$c_L = \sum_{L'\in\mathscr{L}^G(M)} d_M^G(L, L') c_M^{Q'},$$

where Q' denotes the second component of $s(L, L')$.

3. Symmetric pairs

3.1. General cases. Following [1, Definition 7.1.1], by a symmetric pair, we mean a triple (G, H, θ) where $H \subseteq G$ are a pair of reductive groups, and θ is an involution on G such that H is the subgroup of fixed points of θ.

Suppose that (G, H, θ) is a symmetric pair. Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Write $d\theta$ for the differential of θ. Then $\mathfrak{h} = \{X \in \mathfrak{g} : (d\theta)(X) = X\}$. Let $\mathfrak{s}$ be the tangent space at the neutral element of the symmetric space $S := G/H$. We shall always view $\mathfrak{s}$ as a subspace of $\mathfrak{g}$. Then $\mathfrak{s} = \{X \in \mathfrak{g} : (d\theta)(X) = -X\}$ and H acts on $\mathfrak{s}$ by restriction of the adjoint action.

We say an element $X \in \mathfrak{s}$ is semi-simple if the orbit $\mathrm{Ad}(H)(X)$ is Zariski closed in $\mathfrak{s}$. From [45, Fact A, p. 108-109], we know that $X \in \mathfrak{s}(F)$ is semi-simple if and only if $\mathrm{Ad}(H(F))(X)$ is closed in $\mathfrak{s}(F)$ in the analytic topology. By a regular element $X \in \mathfrak{s}$, we mean that the centraliser H_X of X in H has minimal dimension. Denote by $\mathfrak{s}_{\mathrm{rs}}$ the principal Zariski open subset (see [45, end of p. 107]) of $\mathfrak{s}$ consisting of regular semi-simple elements in $\mathfrak{s}$.

By a Cartan subspace of $\mathfrak{s}$, we mean a maximal abelian subspace for the Lie bracket $\mathfrak{c} \subseteq \mathfrak{s}$ defined over F consisting of semi-simple elements. For such $\mathfrak{c}$, denote by $\mathfrak{c}_{\mathrm{reg}}$ the subset of regular elements in $\mathfrak{c}$. Denote by $T_{\mathfrak{c}}$ the centraliser of $\mathfrak{c}$ in H, which is a torus. Set $\mathfrak{t}_{\mathfrak{c}} := \mathrm{Lie}(T_{\mathfrak{c}})$.

Following [58, p. 1828], for $X \in \mathfrak{c}_{\mathrm{reg}}(F)$, where $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{s}$, we define the Weyl discriminant factor

$$|D^{\mathfrak{s}}(X)|_F := |\det(\mathrm{ad}(X)|_{\mathfrak{h}/\mathfrak{t}_{\mathfrak{c}}\oplus\mathfrak{s}/\mathfrak{c}})|_F^{1/2}. \tag{3.1.1}$$

For a Cartan subspace $\mathfrak{c} \subseteq \mathfrak{s}$, set $W(H, \mathfrak{c}) := \mathrm{Norm}_{H(F)}(\mathfrak{c})/T_{\mathfrak{c}}(F)$ to be its Weyl group, where $\mathrm{Norm}_{H(F)}(\mathfrak{c})$ denotes the normaliser of $\mathfrak{c}$ in $H(F)$. Fix a set $\mathscr{T}(\mathfrak{s})$ of representatives for $H(F)$-conjugacy

classes of Cartan subspaces in $\mathfrak{s}$, which is a finite set by [45, p. 105]. Then we have the Weyl integration formula (see [45, p. 106])

$$\int_{\mathfrak{s}(F)} f(X)dX = \sum_{\mathfrak{c}\in\mathscr{T}(\mathfrak{s})} |W(H,\mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F \int_{T_{\mathfrak{c}}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))dxdX \tag{3.1.2}$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. Recall that the adjoint action induces a local isomorphism $\beta : (T_{\mathfrak{c}}(F)\backslash H(F)) \times \mathfrak{c}_{\mathrm{reg}}(F) \to \mathfrak{s}_{\mathrm{rs}}(F)$ of F-analytic manifolds, whose image is open in $\mathfrak{s}_{\mathrm{rs}}(F)$. Here we should use compatible Haar measures on $\mathfrak{s}(F)$ and $\mathfrak{c}_{\mathrm{reg}}(F)$, i.e., we require that β should preserve the measures. For particular cases to be considered, we shall fix Haar measures on $\mathfrak{s}(F)$ in the following sections. Notice that we shall not use the Haar measure on $\mathfrak{c}_{\mathrm{reg}}(F)$ obtained via the exponential map.

The lemma below makes the definition of Fourier transform on $\mathfrak{s}(F)$ possible.

Lemma 3.1. *Let (G, H, θ) be a symmetric pair. Then there exists a G-invariant θ-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. In particular, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ is an orthogonal direct sum with respect to $\langle\cdot,\cdot\rangle$, and the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{h}$ or $\mathfrak{s}$ is non-degenerate.*

Proof. This is [1, Lemma 7.1.9]. □

The following lemma is a special case of [59, Lemma 3.10], which is an analogue of Harish-Chandra's compactness lemma [26, Lemma 25].

Lemma 3.2. *Let $\sigma_{\mathfrak{s}}$ be a compact subset of $\mathfrak{s}(F)$. Suppose that $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{s}$. Let $\sigma_{\mathfrak{c}}$ be a compact subset of $\mathfrak{c}_{\mathrm{reg}}(F)$. Then*

$$\{x \in T_{\mathfrak{c}}(F)\backslash H(F) : \mathrm{Ad}(x^{-1})(\sigma_{\mathfrak{c}}) \cap \sigma_{\mathfrak{s}} \neq \emptyset\}$$

is relatively compact in $T_{\mathfrak{c}}(F)\backslash H(F)$.

Proof. Choose an arbitrary $X \in \sigma_{\mathfrak{c}}(F)$. We have $H_X = T_{\mathfrak{c}}$. Let $N^{\mathfrak{s}}_{\mathrm{Ad}(H)(X),X}$ be the normal space (see [1, Notation 2.3.3]) to $\mathrm{Ad}(H)(X)$ in $\mathfrak{s}$ at the point X. Let $\mathfrak{s}_X$ be the centraliser of X in $\mathfrak{s}$. By [1, Proposition 7.2.1], since $X \in \mathfrak{s}$ is semi-simple, one has $N^{\mathfrak{s}}_{\mathrm{Ad}(H)(X),X} \simeq \mathfrak{s}_X$ as H_X-spaces. Note that since $X \in \mathfrak{c}_{\mathrm{reg}}$, $\mathfrak{c}_{\mathrm{reg}} \subseteq \mathfrak{s}_X$ is an étale Luna slice at X in the sense of [1, Theorem A.2.3]. Thus we can apply [59, Lemma 3.10]. □

The next lemma is an analogue of [26, Lemma 28].

Lemma 3.3. *Let $\sigma \subseteq \mathfrak{s}(F)$ be a compact subset. Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{s}$. Then $\mathfrak{c}(F) \cap \mathrm{Cl}(\mathrm{Ad}(H(F))(\sigma))$ is relatively compact in $\mathfrak{c}(F)$, where* Cl *denotes the closure of a subset in $\mathfrak{s}(F)$.*

Proof. This is [58, Lemma 6.12], whose proof relying on the Chevalley restriction theorem for symmetric spaces [49, Theorem 7 in §4.4] applies to an arbitrary symmetric pair. □

The following lemma is an analogue of [51, Lemme III.4].

Lemma 3.4. *Let $\sigma \subseteq \mathfrak{s}(F)$ be a compact subset. Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{s}$ and $T_{\mathfrak{c}}$ the centraliser of $\mathfrak{c}$ in H. Then there exists $c_\sigma > 0$ such that for all $x \in H(F)$ and $X \in \mathfrak{c}_{\mathrm{reg}}(F)$ satisfying $\mathrm{Ad}(x^{-1})(X) \in \sigma$, we have*

$$\inf_{\tau\in T_{\mathfrak{c}}(F)} \log\|\tau x\| \leq c_\sigma \sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}.$$

Proof. Let $\|\cdot\|_{T_{\mathfrak{c}}\backslash H}$ be any norm on $(T_{\mathfrak{c}}\backslash H)(F)$ in the sense of [35, §18.2]. Applying the argument of [35, Lemma 20.3] to the finite morphism

$$\beta : (T_{\mathfrak{c}}\backslash H) \times \mathfrak{c}_{\mathrm{reg}} \to \mathfrak{s}_{\mathrm{rs}}$$

of affine algebraic varieties defined by $\beta(x, X) := \mathrm{Ad}(x^{-1})(X)$, we show the inequality

$$\log\|x\|_{T_{\mathfrak{c}}\backslash H} \leq c_\sigma \sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}.$$

By [35, Proposition 18.3], the quotient $H \to T_{\mathfrak{c}}\backslash H$ has the norm descent property in the sense of [35, §18.6]. That is to say, the restriction of $\|\cdot\|_{T_{\mathfrak{c}}\backslash H}$ to $T_{\mathfrak{c}}(F)\backslash H(F)$ is equivalent to the abstract norm $\inf_{\tau\in T_{\mathfrak{c}}(F)} \|\tau\cdot\|$ on $T_{\mathfrak{c}}(F)\backslash H(F)$. □

The lemma below is an analogue of [26, Lemma 44].

Lemma 3.5. *There exists $\varepsilon > 0$ such that the function $|D^{\mathfrak{s}}(X)|_F^{-\varepsilon}$ is locally integrable on $\mathfrak{c}(F)$ for any Cartan subspace $\mathfrak{c}$ of $\mathfrak{s}$.*

Proof. See [59, Lemma 4.3]. □

Corollary 3.6. *For any $r \geq 0$, the function $\sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^r$ is locally integrable on $\mathfrak{c}(F)$ for any Cartan subspace $\mathfrak{c}$ of $\mathfrak{s}$.*

Proof. We have the elementary fact (cf. the proof of [35, Corollary 20.2]): for $\varepsilon > 0$ and $r \geq 0$, there exist $c > 0$ such that

$$\sup\{1, \log y\}^r \leq cy^{\varepsilon} + 1, \forall y > 0.$$

Then it suffices to apply Lemma 3.5. □

We say an element $X \in \mathfrak{s}(F)$ is nilpotent if $0 \in \mathrm{Cl}(\mathrm{Ad}(H(F))(X))$, where Cl denotes the closure of a subset in $\mathfrak{s}(F)$. From [1, Lemmas 2.3.12 and 7.3.8], we know that $X \in \mathfrak{s}(F)$ is nilpotent if and only if it is a nilpotent element in $\mathfrak{g}$. Denote by $\mathcal{N}^{\mathfrak{s}}$ the set of nilpotent elements in $\mathfrak{s}(F)$, which is a cone. The following lemma is an analogue of Jacobson-Morozov theorem.

Lemma 3.7. *Let (G, H, θ) be a symmetric pair and $X \in \mathcal{N}^{\mathfrak{s}}$. Then there exists a group homomorphism $\varphi : SL_2(F) \to G(F)$ such that*

$$d\varphi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X, d\varphi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{s}(F) \text{ and } \varphi\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in H(F), \forall t \in F^{\times}.$$

Proof. This is [1, Lemma 7.1.11]. □

Let $X \in \mathfrak{s}_{\mathrm{rs}}(F)$. The orbital integral of X is the distribution I_X on $\mathfrak{s}(F)$ defined by

$$\forall f \in \mathcal{C}_c^{\infty}(\mathfrak{s}(F)), I_X(f) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))dx. \tag{3.1.3}$$

The next lemma is an analogue of Harish-Chandra's submersion principle [26, Theorem 11].

Lemma 3.8. *Let $I : \mathfrak{s}_{\mathrm{rs}}(F) \to \mathbb{C}$ be a function. The following conditions are equivalent:*

(1) I is locally constant, invariant by the adjoint action of $H(F)$ and of support included in $\mathrm{Ad}(H(F))(\sigma)$ with $\sigma \subseteq \mathfrak{s}_{\mathrm{rs}}(F)$ a compact subset;

(2) there exists $f \in \mathcal{C}_c^{\infty}(\mathfrak{s}_{\mathrm{rs}}(F))$ such that

$$\forall X \in \mathfrak{s}_{\mathrm{rs}}(F), I(X) = I_X(f).$$

Proof. For $\mathfrak{c} \in \mathscr{T}(\mathfrak{s})$, apply the argument of [16, Lemme 6.1] to the morphism

$$\Phi_{\mathfrak{c}} : (T_{\mathfrak{c}}(F)\backslash H(F)) \times \mathfrak{c}_{\mathrm{reg}}(F) \to \mathfrak{s}_{\mathrm{rs}}(F)$$

defined by $\Phi_{\mathfrak{c}}(x, X) := \mathrm{Ad}(x^{-1})(X)$. Then glue the results for all $\mathfrak{c} \in \mathscr{T}(\mathfrak{s})$ together. □

3.2. The case of (G, H). Let D be a central division algebra over F. Denote by $GL_{n,D}$ the reductive group over F whose F-points are $GL_n(D)$. Let $G := GL_{2n,D}$ and $H := GL_{n,D} \times GL_{n,D}$ denotes its subgroup by diagonal embedding. Then H is the subgroup of fixed points of the involution $\mathrm{Ad}(\epsilon)$ on G, where $\epsilon := \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}$. Here we can embed G into $\mathfrak{g}$ in the standard way. For a linear subspace $\mathfrak{v} \subseteq \mathfrak{g}$, we write $\mathfrak{v}^{\times} := \mathfrak{v} \cap G$. Recall that $\mathfrak{s}_{\mathrm{rs}} \subseteq \mathfrak{s}^{\times}$ in our case.

Lemma 3.9. *Let P be a parabolic subgroup of G. Then $P \cap H$ is a parabolic subgroup of H if and only if $\epsilon \in P$. Moreover, if ϵ belongs to a Levi factor M of P, then $M \cap H$ is a Levi factor of $P \cap H$.*

Proof. One may consider all the groups over an algebraic closure of F. We first suppose that $P \cap H$ is a parabolic subgroup of H. Then $\epsilon \in \mathrm{Cent}(H) \subseteq P \cap H$, where $\mathrm{Cent}(H)$ denotes the centre of H. This establishes one direction.

We now suppose that $\epsilon \in P$. Denote by N the unipotent radical of P and let M be a Levi factor of P. By the argument in the last paragraph of the proof of Lemma 4.1 in Chapter 2, we show that ϵ is N-conjugate to an element in M with the help of [3, Lemma 2.1] (actually we need its variant over a local field for the characteristic function of a singleton here, whose proof is similar). Then replacing M by its N-conjugate if necessary, we may assume that $\epsilon \in M$.

Let $G = GL(V)$ for a vector space $V = \oplus_{1 \leq i \leq r} V_i$. Suppose that

$$P = \{g \in G : g(V_1 \oplus ... \oplus V_i) \subseteq V_1 \oplus ... \oplus V_i, \forall 1 \leq i \leq r\}$$

and that

$$M = \{g \in G : g(V_i) \subseteq V_i, \forall 1 \leq i \leq r\}.$$

Since $\epsilon \in M$, we have $\epsilon(V_i) \subseteq V_i$ for all $1 \leq i \leq r$. Let V_i^+ (resp. V_i^-) be the $+1$(resp. -1)-eigenspace of V_i under the action of ϵ. For $\epsilon^2 = 1$, we have $V_i = V_i^+ \oplus V_i^-$. Let $V^+ := \oplus_{1\leq i\leq r} V_i^+$ and $V^- := \oplus_{1\leq i\leq r} V_i^-$. Then

$$H = \{g \in G : g(V^+) \subseteq V^+, g(V^-) \subseteq V^-\}.$$

Hence,

$$P \cap H = \{g \in G : g(V_1^+ \oplus ... \oplus V_i^+) \subseteq V_1^+ \oplus ... \oplus V_i^+, g(V_1^- \oplus ... \oplus V_i^-) \subseteq V_1^- \oplus ... \oplus V_i^-, \forall 1 \leq i \leq r\}.$$

It means exactly that $P \cap H$ is a parabolic subgroup of H and proves the other direction. Morover,

$$M \cap H = \{g \in G : g(V_i^+) \subseteq V_i^+, g(V_i^-) \subseteq V_i^-, \forall 1 \leq i \leq r\}.$$

That is to say, $M \cap H$ is a Levi factor of $P \cap H$. □

Let M_0 be the group of diagonal matrices in G. Set $\omega := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. For $P \in \mathscr{F}^G(M_0)$, we say that P is "ω-stable" if $\omega \in P$. Denote by $\mathscr{F}^{G,\omega}(M_0)$ the subset of ω-stable parabolic subgroups in $\mathscr{F}^G(M_0)$. For $M \in \mathscr{L}^G(M_0)$, we say that M is "ω-stable" if $M = M_P$ for some $P \in \mathscr{F}^{G,\omega}(M_0)$. This condition is stronger than $\mathrm{Ad}(\omega)(M) = M$; for example, M_0 is not considered to be ω-stable in our sense. Denote by $\mathscr{L}^{G,\omega}(M_0)$ the subset of ω-stable Levi subgroups in $\mathscr{L}^G(M_0)$. Let A_n be the group of diagonal matrices in GL_n. Recall that there is a bijection between $\mathscr{L}^{GL_n}(A_n)$ and $\mathscr{L}^{G,\omega}(M_0)$ induced by $M_n \mapsto M = \begin{pmatrix} \mathfrak{m}_{n,D} & \mathfrak{m}_{n,D} \\ \mathfrak{m}_{n,D} & \mathfrak{m}_{n,D} \end{pmatrix}^\times$. We shall always use the notation M_n to denote the preimage of M under this bijection. Notice that if $M \in \mathscr{L}^{G,\omega}(M_0)$ and $Q \in \mathscr{F}^G(M)$, then $Q \in \mathscr{F}^{G,\omega}(M_0)$.

Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$. We say an element $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ is M-elliptic if A_M is the maximal F-split torus in H_X. Denote by $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ the set of M-elliptic elements in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Write $M_H := M \cap H$. Denote by $\Gamma_{\mathrm{ell}}((\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F))$ the set of $M_H(F)$-conjugacy classes in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. We say a Cartan subspace $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ is M-elliptic if $A_{T_\mathfrak{c}} = A_M$. Since $(M_H, \mathfrak{m} \cap \mathfrak{s})$ appears as the product of some copies of the form $(H, \mathfrak{s})$ in lower dimensions, we define $W(M_H, \mathfrak{c})$ and $\mathscr{T}(\mathfrak{m} \cap \mathfrak{s})$ as in Section 3.1. Denote by $\mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})$ the subset of M-elliptic Cartan subspaces in $\mathscr{T}(\mathfrak{m} \cap \mathfrak{s})$.

Lemma 3.10. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $\{X\} \in \Gamma_{\mathrm{ell}}((\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F))$.*

1) Let $M' \in \mathscr{L}^{G,\omega}(M_0)$ and $\{X'\} \in \Gamma_{\mathrm{ell}}((\mathfrak{m}' \cap \mathfrak{s}_{\mathrm{rs}})(F))$ be such that X' is $H(F)$-conjugate to X. Then there exists

$$w \in \left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{GL_{n,D}} \right\},$$

where $W_0^{GL_{n,D}}$ denotes the Weyl group of $(GL_{n,D}, A_{n,D})$, such that

$$(\mathrm{Ad}(w)(M), \{\mathrm{Ad}(w)(X)\}) = (M', \{X'\}).$$

2) The cardinality of

$$\{(M', \{X'\}) : M' \in \mathscr{L}^{G,\omega}(M_0), \{X'\} \in \Gamma_{\mathrm{ell}}((\mathfrak{m}' \cap \mathfrak{s}_{\mathrm{rs}})(F)), X' \text{ is } H(F)\text{-conjugate to } X\}$$

is

$$|W_0^{GL_n}||W_0^{M_n}|^{-1},$$

where $W_0^{GL_n}$ (resp. $W_0^{M_n}$) denotes the Weyl group of (GL_n, A_n) (resp. (M_n, A_n)).

Proof. 1) Let $x \in H(F)$ be such that $\mathrm{Ad}(x)(X) = X'$. Then $\mathrm{Ad}(x)(H_X) = H_{X'}$. Since $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ and $X' \in (\mathfrak{m}' \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we have $\mathrm{Ad}(x)(A_M) = A_{M'}$ and thus $\mathrm{Ad}(x)(M) = M'$. As $x \in H(F)$, we have $\mathrm{Ad}(x)(M_H) = M'_H$. We see that $\mathrm{Ad}(x)(A_{M_0}) \subseteq M'_H$ is a maximal F-split torus, so there exists $m' \in M'_H(F)$ such that $\mathrm{Ad}({m'}^{-1}x)(A_{M_0}) = A_{M_0}$. That is to say, $w' := {m'}^{-1}x \in \mathrm{Norm}_{H(F)}(A_{M_0}) = \mathrm{Norm}_{H(F)}(M_0)$, where $\mathrm{Norm}_{H(F)}(A_{M_0})$ denotes the normaliser of A_{M_0} in $H(F)$. Now $\mathrm{Ad}(x)(A_M) = A_{M'}$ implies that $\mathrm{Ad}(w')(A_M) = A_{M'}$. Because $M, M' \in \mathscr{L}^{G,\omega}(M_0)$, it is shown in §9.1 in Chapter 2 that any isomorphism $A_M \to A_{M'}$ induced by W_0^H can be given by $\left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{GL_{n,D}} \right\}$. Hence, there exists $w \in \left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{GL_{n,D}} \right\}$ such that $w^{-1}w' \in \mathrm{Cent}_{W_0^H}(A_M) = W_0^{M_H}$, where $\mathrm{Cent}_{W_0^H}(A_M)$ denotes the centraliser of A_M in W_0^H. We can check that such a w satisfies the condition in the lemma.

2) By 1), the group $\left\{\begin{pmatrix}\omega_n & \\ & \omega_n\end{pmatrix} : \omega_n \in W_0^{GL_{n,D}}\right\}$ acts transitively on this set. Let

$$w \in \left\{\begin{pmatrix}\omega_n & \\ & \omega_n\end{pmatrix} : \omega_n \in W_0^{GL_{n,D}}\right\}$$

be such that

$$(\mathrm{Ad}(w)(M), \{\mathrm{Ad}(w)(X)\}) = (M, \{X\}).$$

Then $w \in M_H(F)$. Thus the condition on w is equivalent to

$$w \in \left\{\begin{pmatrix}\omega_n & \\ & \omega_n\end{pmatrix} : \omega_n \in W_0^{M_{n,D}}\right\},$$

where $W_0^{M_{n,D}}$ denotes the Weyl group of $(M_{n,D}, A_{n,D})$. We see that the cardinality of the set in the lemma is $|W_0^{GL_{n,D}}||W_0^{M_{n,D}}|^{-1}$ or $|W_0^{GL_n}||W_0^{M_n}|^{-1}$. □

Proposition 3.11. *For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$\int_{\mathfrak{s}(F)} f(X)dX = \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F \int_{A_M(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))dxdX.$$

Proof. Recall that any $H(F)$-conjugacy class in $\mathfrak{s}_{\mathrm{rs}}(F)$ is the image of a class $\{X\} \in \Gamma_{\mathrm{ell}}((\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F))$ for some $M \in \mathscr{L}^{G,\omega}(M_0)$ in our case. By Lemma 3.10, the Weyl integration formula (3.1.2) can be written as the above equality (cf. [8, p. 16-17] and [51, (3) in §I.3]). □

Recall that there is a bijection between $\mathscr{F}^{GL_n}(A_n)$ and $\mathscr{F}^{G,\omega}(M_0)$ induced by $P_n \mapsto P = \begin{pmatrix}\mathfrak{p}_{n,D} & \mathfrak{p}_{n,D} \\ \mathfrak{p}_{n,D} & \mathfrak{p}_{n,D}\end{pmatrix}^\times$. We shall always use the notation P_n to denote the preimage of P under this bijection. Following [58, p. 1846], we shall fix the Haar measures on some subspaces of $\mathfrak{s}(F)$ as follows. Let $P \in \mathscr{F}^{G,\omega}(M_0)$. Then we have $\mathfrak{m}_P = \begin{pmatrix}\mathfrak{m}_{n,D} & \mathfrak{m}_{n,D} \\ \mathfrak{m}_{n,D} & \mathfrak{m}_{n,D}\end{pmatrix}$ and $\mathfrak{n}_P = \begin{pmatrix}\mathfrak{n}_{n,D} & \mathfrak{n}_{n,D} \\ \mathfrak{n}_{n,D} & \mathfrak{n}_{n,D}\end{pmatrix}$, where we denote $M_n := M_{P_n}$ and $N_n := N_{P_n}$. We have fixed the Haar measures on $\mathfrak{m}_n(D)$ and $\mathfrak{n}_n(D)$ in Section 2.5. We shall choose the same Haar measure for any of the four copies in $\mathfrak{m}_P(F)$ or $\mathfrak{n}_P(F)$ under these identifications. In particular, we obtain the Haar measures on $(\mathfrak{m}_P \cap \mathfrak{s})(F)$ and $(\mathfrak{n}_P \cap \mathfrak{s})(F)$.

Lemma 3.12. *Let $Q \in \mathscr{F}^{G,\omega}(M_0)$. For $Y \in (\mathfrak{m}_Q \cap \mathfrak{s}_{\mathrm{rs}})(F)$, the map*

$$N_{Q_H}(F) \to (\mathfrak{n}_Q \cap \mathfrak{s})(F), n \mapsto \mathrm{Ad}(n^{-1})(Y) - Y$$

is an isomorphism of F-analytic manifolds whose Jacobian is $|D^{\mathfrak{s}}(Y)|_F^{1/2}|D^{\mathfrak{m}_Q\cap\mathfrak{s}}(Y)|_F^{-1/2}$.

Proof. See Lemma 8.1 in Chapter 2 for the isomorphism and the proof of [58, Proposition 6.3.(ii)] for the Jacobian. □

Fix a continuous and nontrivial unitary character $\Psi : F \to \mathbb{C}^\times$. Let $\langle\cdot,\cdot\rangle$ be the non-degenerate symmetric bilinear form on $\mathfrak{g}(F)$ defined by

$$\langle X, Y\rangle := \mathrm{Trd}(XY), \forall X, Y \in \mathfrak{g}(F),$$

where Trd denotes the reduced trace on $\mathfrak{g}(F)$. It is invariant by the adjoint action of $G(F)$ and $\mathrm{Ad}(\epsilon)$. For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, define its normalised Fourier transform $\hat{f} \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ by

$$\forall X \in \mathfrak{s}(F), \hat{f}(X) := c_\Psi(\mathfrak{s}(F)) \int_{\mathfrak{s}(F)} f(Y)\Psi(\langle X, Y\rangle)dY, \tag{3.2.1}$$

where $c_\Psi(\mathfrak{s}(F))$ is the unique constant such that $\hat{\hat{f}}(X) = f(-X)$ for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and all $X \in \mathfrak{s}(F)$. For any $M \in \mathscr{L}^{G,\omega}(M_0)$, the restriction of $\langle\cdot,\cdot\rangle$ on $\mathfrak{m}\cap\mathfrak{s}$ is non-degenerate. Then we can define similarly the normalised Fourier transform of $f \in \mathcal{C}_c^\infty((\mathfrak{m} \cap \mathfrak{s})(F))$.

Suppose that $P \in \mathscr{F}^{G,\omega}(M_0)$. Let η be the quadratic character of $F^\times$ attached to a quadratic extension E/F. Denote by Nrd the reduced norm on $G(F)$. For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define a function (parabolic descent) $f_P^\eta \in \mathcal{C}_c^\infty((\mathfrak{m}_P \cap \mathfrak{s})(F))$ by

$$f_P^\eta(Z) := \int_{K_H \times (\mathfrak{n}_P\cap\mathfrak{s})(F)} f(\mathrm{Ad}(k^{-1})(Z + U))\eta(\mathrm{Nrd}(k))dUdk \tag{3.2.2}$$

for all $Z \in (\mathfrak{m}_P \cap \mathfrak{s})(F)$. We show that $(\hat{f})^\eta_P = (f^\eta_P)\hat{}$, so we shall denote it by $\hat{f}^\eta_P$ without confusion. In fact, the integral on $(\mathfrak{n}_P \cap \mathfrak{s})(F)$ and the Fourier transform commute by our choices of Haar measures (see [51, §I.7]); the commutativity of the integral on K_H and the Fourier transform results from the $H(F)$-invariance of $\langle \cdot, \cdot \rangle$.

The following result is an analogue of [26, Theorem 13].

PROPOSITION 3.13. *Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. Then*

$$\sup_{X \in \mathfrak{s}_{\mathrm{rs}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} |f(\mathrm{Ad}(x^{-1})(X))| dx < +\infty.$$

PROOF. It is proved in [58, Theorem 6.11] (see also [57, p. 77]) that for any fixed Cartan subspace $\mathfrak{c}$ of $\mathfrak{s}$,

$$\sup_{X \in \mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} |f(\mathrm{Ad}(x^{-1})(X))| dx < +\infty.$$

Since $\mathscr{T}(\mathfrak{s})$ is a finite set and the orbital integral is constant on any $H(F)$-orbit, we obtain a uniform bound for all $X \in \mathfrak{s}_{\mathrm{rs}}(F)$. □

The lemma below is an analogue of [26, Theorem 15].

LEMMA 3.14. *There exists $\varepsilon > 0$ such that the function $|D^{\mathfrak{s}}(X)|_F^{-\frac{1}{2}-\varepsilon}$ is locally integrable on $\mathfrak{s}(F)$.*

PROOF. Choose $\varepsilon > 0$ verifying the condition of Lemma 3.5. Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $f \geq 0$. By the Weyl integration formula (3.1.2), we have

$$\int_{\mathfrak{s}(F)} |D^{\mathfrak{s}}(X)|_F^{-\frac{1}{2}-\varepsilon} f(X) dX = \sum_{\mathfrak{c} \in \mathscr{T}(\mathfrak{s})} |W(H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{\frac{1}{2}-\varepsilon} \int_{A_{T_{\mathfrak{c}}}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) dx dX.$$

The convergence of the right hand side results from Proposition 3.13 and Lemmas 3.3 and 3.5. □

COROLLARY 3.15. *For any $r \geq 0$, the function $|D^{\mathfrak{s}}(X)|_F^{-\frac{1}{2}} \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^r$ is locally integrable on $\mathfrak{s}(F)$.*

PROOF. It is the same as the proof of Corollary 3.6. □

3.3. The case of (G', H'). Let E be a quadratic extension of F. Let $\mathfrak{g}'$ be a central simple algebra over F with a fixed embedding of F-algebras $E \hookrightarrow \mathfrak{g}'$. Let $\mathfrak{h}' := \mathrm{Cent}_{\mathfrak{g}'}(E)$ be the centraliser of E in $\mathfrak{g}'$. By the the double centraliser theorem (see [42, Theorem 3.1 in Chapter IV] for example), $\mathfrak{h}'(F)$ is a central simple algebra over E. Denote by $G' := {\mathfrak{g}'}^\times$ (resp. $H' := {\mathfrak{h}'}^\times$) the group of invertible elements in $\mathfrak{g}'$ (resp. $\mathfrak{h}'$), which is considered as an algebraic group over F with Lie algebra $\mathfrak{g}'$ (resp. $\mathfrak{h}'$). Let $\alpha \in E \backslash F$ such that $\alpha^2 \in F$. Then $E = F(\alpha)$ and H' is the subgroup of fixed points of the involution $\mathrm{Ad}(\alpha)$ on G'. Denote by $\mathfrak{s}'$ the corresponding tangent space of G'/H'. For a linear subspace $\mathfrak{v}' \subseteq \mathfrak{g}'$, we write ${\mathfrak{v}'}^\times := \mathfrak{v}' \cap G'$. Then we see that $\mathfrak{s}'_{\mathrm{rs}} \subseteq {\mathfrak{s}'}^\times$ via a base change to an algebraic closure of F containing E.

LEMMA 3.16 (cf. Lemma 3.9). *Let $\widetilde{P'}$ be a parabolic subgroup of G'. Then $\widetilde{P'} \cap H'$ is a parabolic subgroup of H' if and only if $\alpha \in \widetilde{P'}$. Moreover, if α belongs to a Levi factor $\widetilde{M'}$ of $\widetilde{P'}$, then $\widetilde{M'} \cap H'$ is a Levi factor of $\widetilde{P'} \cap H'$.*

By the Wedderburn-Artin theorem, G' is isomorphic to $GL_{n,D}$ for some positive integer n and some central division algebra D over F. Since E embeds into $\mathfrak{g}'(F)$, we see that $n \deg(D)$ is even, where $\deg(D)$ denotes the degree of D. From the Noether-Skolem theorem (see [42, Theorem 2.10 of Chapter IV] for example), up to conjugation by $G'(F)$, the emdedding $H' \hookrightarrow G'$ is reduced to one of the two cases below (see [18, §2.1 and §3.1] and §3.4 in Chapter 3).

Case I: if $\deg(D)$ is even, then $(G', H') = (GL_{n,D}, \mathrm{Res}_{E/F} GL_{n,D'})$, where $D' := \mathrm{Cent}_D(E)$ denoting the centraliser of E in D is a central division algebra over E of degree $\frac{\deg(D)}{2}$. Let $M'_0 \simeq (\mathrm{Res}_{E/F}\mathbb{G}_{m,D'})^n$ (resp. $M'_{\widetilde{0}} \simeq (\mathbb{G}_{m,D})^n$) be the subgroup of diagonal elements in H' (resp. G'). Recall that there is a bijection $M' \mapsto \widetilde{M'}$ between $\mathscr{L}^{H'}(M'_0)$ and $\mathscr{L}^{G'}(M'_{\widetilde{0}})$. We shall always write $\widetilde{M'}$ for the image of M' under this bijection. Notice that $M' = \widetilde{M'} \cap H'$ and that we can identify $A_{M'}$ with $A_{\widetilde{M'}}$.

Case II: if $\deg(D)$ is odd, then $(G', H') = (GL_{n,D}, \mathrm{Res}_{E/F} GL_{\frac{n}{2}, D \otimes_F E})$, where $D \otimes_F E$ is a central division algebra over E of degree $\deg(D)$. Let $M'_0 \simeq (\mathrm{Res}_{E/F}\mathbb{G}_{m,D\otimes_F E})^{\frac{n}{2}}$ (resp. $M'_{\widetilde{0}} \simeq (\mathbb{G}_{m,D})^n$) be

the subgroup of diagonal elements in H' (resp. G'). Denote by $\mathscr{L}^{G'}(M'_{\widetilde{0}}, M'_0)$ the subset of elements in $\mathscr{L}^{G'}(M'_{\widetilde{0}})$ containing M'_0. Recall that there is a bijection $M' \mapsto \widetilde{M'}$ between $\mathscr{L}^{H'}(M'_0)$ and $\mathscr{L}^{G'}(M'_{\widetilde{0}}, M'_0)$. We shall always write $\widetilde{M'}$ for the image of M' under this bijection. Notice that $M' = \widetilde{M'} \cap H'$ and that we can identify $A_{M'}$ with $A_{\widetilde{M'}}$.

Suppose that $M' \in \mathscr{L}^{H'}(M'_0)$. We say an element $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ is M'-elliptic if $A_{M'}$ is the maximal F-split torus in H'_Y. Denote by $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$ the set of M'-elliptic elements in $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Denote by $\Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F))$ the set of $M'(F)$-conjugacy classes in $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$. We say a Cartan subspace $\mathfrak{c}' \subseteq \widetilde{\mathfrak{m}'} \cap \mathfrak{s}'$ is M'-elliptic if $A_{T_{\mathfrak{c}'}} = A_{M'}$. Since $(M', \widetilde{\mathfrak{m}'} \cap \mathfrak{s}')$ appears as the product of some copies of the form $(H', \mathfrak{s}')$ in lower dimensions, we define $W(M', \mathfrak{c}')$ and $\mathscr{T}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')$ as in Section 3.1. Denote by $\mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')$ the subset of M'-elliptic Cartan subspaces in $\mathscr{T}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')$.

Lemma 3.17 (cf. Lemma 3.10). *Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $\{Y'\} \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F))$.*

1) Let $M \in \mathscr{L}^{H'}(M'_0)$ and $\{Y\} \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{m}} \cap \mathfrak{s}'_{\mathrm{rs}})(F))$ be such that Y is $H'(F)$-conjugate to Y'. Then there exists $w \in W_0^{H'}$ such that

$$(\mathrm{Ad}(w)(M'), \{\mathrm{Ad}(w)(Y')\}) = (M, \{Y\}).$$

2) The cardinality of

$$\{(M, \{Y\}) : M \in \mathscr{L}^{H'}(M'_0), \{Y\} \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{m}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)), Y \text{ is } H'(F)\text{-conjugate to } Y'\}$$

is

$$|W_0^{H'}||W_0^{M'}|^{-1}.$$

Proposition 3.18. *For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we have the equality*

$$\int_{\mathfrak{s}'(F)} f'(Y)dY = \sum_{M' \in \mathscr{L}^{H'}(M'_0)} |W_0^{M'}||W_0^{H'}|^{-1} \sum_{\mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')} |W(M', \mathfrak{c}')|^{-1} \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} |D^{\mathfrak{s}'}(Y)|_F \int_{A_{M'}(F)\backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y))dxdY.$$

Proof. Recall that any $H'(F)$-conjugacy class in $\mathfrak{s}'_{\mathrm{rs}}(F)$ is the image of a class $\{Y\} \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F))$ for some $M' \in \mathscr{L}^{H'}(M'_0)$ in our case. By Lemma 3.17, the Weyl integration formula (3.1.2) can be written as the above equality (cf. [8, p. 16-17] and [51, (3) in §I.3]). □

Recall there is a bijection $P' \mapsto \widetilde{P'}$ between $\mathscr{F}^{H'}(M'_0)$ and $\mathscr{F}^{G'}(\widetilde{M'_0})$ in both of **Case I** and **Case II**. We shall always write $\widetilde{P'}$ for the image of P' under this bijection. Let $\tau \in D^\times$ in **Case I** (resp. $\tau \in GL_2(D)$ in **Case II**) be an element such that $\mathrm{Ad}(\alpha)(\tau) = -\tau$. Let $P' \in \mathscr{F}^{H'}(M'_0)$. Then we have $\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}' = \mathfrak{m}_{P'}\tau = \tau\mathfrak{m}_{P'}$ and $\mathfrak{n}_{\widetilde{P'}} \cap \mathfrak{s}' = \mathfrak{n}_{P'}\tau = \tau\mathfrak{n}_{P'}$ by Proposition 3.12 in Chapter 3. We have fixed the Haar measures on $\mathfrak{m}_{P'}(F)$ and $\mathfrak{n}_{P'}(F)$ in Section 2.5. We shall choose the same Haar measures on $(\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}')(F)$ and $(\mathfrak{n}_{\widetilde{P'}} \cap \mathfrak{s}')(F)$ using above identifications induced by τ. Such Haar measures depend on the choice of τ.

Lemma 3.19. *Let $Q' \in \mathscr{F}^{H'}(M'_0)$. For $X \in (\mathfrak{m}_{\widetilde{Q'}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, the map*

$$N_{Q'}(F) \to (\mathfrak{n}_{\widetilde{Q'}} \cap \mathfrak{s}')(F), n \mapsto \mathrm{Ad}(n^{-1})(X) - X$$

is an isomorphism of F-analytic manifolds whose Jacobian is $|D^{\mathfrak{s}'}(X)|_F^{1/2}|D^{\mathfrak{m}_{\widetilde{Q'}} \cap \mathfrak{s}'}(X)|_F^{-1/2}$.

Proof. See Lemma 8.1 in Chapter 3 for the isomorphism. The computation of its Jacobian is close to the proof of [58, Proposition 6.3.(ii)]. □

Fix a continuous and nontrivial unitary character $\Psi : F \to \mathbb{C}^\times$. Let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form on $\mathfrak{g}'(F)$ defined by

$$\langle Y, X \rangle := \mathrm{Trd}(YX), \forall Y, X \in \mathfrak{g}'(F),$$

where Trd denotes the reduced trace on $\mathfrak{g}'(F)$. It is non-degenerate, which can be seen after the base change to an algebraic closure of F. It is also invariant by the adjoint action of $G'(F)$ and $\mathrm{Ad}(\alpha)$. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, define its normalised Fourier transform $\hat{f}' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ by

$$\forall Y \in \mathfrak{s}'(F), \hat{f}'(Y) := c_\Psi(\mathfrak{s}'(F)) \int_{\mathfrak{s}'(F)} f'(X)\Psi(\langle Y, X \rangle)dX, \tag{3.3.1}$$

where $c_\Psi(\mathfrak{s}'(F))$ is the unique constant such that $\hat{\hat{f}}'(Y) = f'(-Y)$ for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ and all $Y \in \mathfrak{s}'(F)$. For any $M' \in \mathscr{L}^{H'}(M_0')$, the restriction of $\langle\cdot,\cdot\rangle$ on $\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'$ is non-degenerate. Then we can define similarly the normalised Fourier transform of $f' \in \mathcal{C}_c^\infty((\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')(F))$.

Suppose that $P' \in \mathscr{F}^{H'}(M_0')$. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define a function (parabolic descent) $f'_{P'} \in \mathcal{C}_c^\infty((\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}')(F))$ by

$$f'_{P'}(Z) := \int_{K_{H'} \times (\mathfrak{n}_{\widetilde{P'}} \cap \mathfrak{s}')(F)} f'(\mathrm{Ad}(k^{-1})(Z+U))dUdk \tag{3.3.2}$$

for all $Z \in (\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}')(F)$. By our choices of Haar measures (see [51, §I.7]) and the $H'(F)$-invariance of $\langle\cdot,\cdot\rangle$, we show that $(\hat{f}')_{P'} = (f'_{P'})\hat{}$, which will be denoted by $\hat{f}'_{P'}$ without confusion.

Proposition 3.20. *Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$. Then*

$$\sup_{Y \in \mathfrak{s}'_{\mathrm{rs}}(F)} |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F)\backslash H'(F)} |f'(\mathrm{Ad}(x^{-1})(Y))|dx < +\infty.$$

Corollary 3.21 (cf. Corollary 3.15). *For any $r \geq 0$, the function $|D^{\mathfrak{s}'}(Y)|_F^{-\frac{1}{2}} \sup\{1, -\log|D^{\mathfrak{s}'}(Y)|_F\}^r$ is locally integrable on $\mathfrak{s}'(F)$.*

The rest of this section is devoted to the proof of Proposition 3.20. We shall follow the main steps in [58, §6.3], which is similar to the proof of [26, Theorem 13], and only point out some additional ingredients. Let n is the F-rank of G'. Denote $G'_n := G', H'_n := H'$ and $\mathfrak{s}'_n := \mathfrak{s}'$. Recall that the F-rank of H'_n is n in **Case I** (resp. $\frac{n}{2}$ in **Case II**). We shall use induction on n. For $n = 1$ in **Case I** (resp. $n = 2$ in **Case II**), the proposition is evident since $H'_Y(F)\backslash H'(F)$ is compact in our case.

The following description of semi-simple elements and descendants (see [1, Definition 7.2.2]) is a generalisation of [24, Lemma 2.1] (see also [58, Proposition 4.7]).

Proposition 3.22. *1) An element Y of $\mathfrak{s}'(F)$ is semi-simple if and only if it is $H'(F)$-conjugate to an element of the form*

$$Y(B) := \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

*with $B \in {\mathfrak{s}'_m}^\times(F)$ being semi-simple with respect to the H'_m-action. More precisely, the set of $H'(F)$-conjugacy classes of semi-simple elements in $\mathfrak{s}'(F)$ is bijective to the set of pairs $(m, \{B\})$ where $0 \leq m \leq n$ is an integer in **Case I** (resp. an even number in **Case II**) and $\{B\}$ is a semi-simple $H'_m(F)$-conjugacy class in ${\mathfrak{s}'_m}^\times(F)$. Moreover, $Y(B)$ is regular semi-simple if and only if $m = n$ and B is regular semi-simple in ${\mathfrak{s}'}^\times(F)$.*

2) Let $Y = Y(B) \in \mathfrak{s}'(F)$ be semi-simple. Then the descendant $(H'_Y, \mathfrak{s}'_Y)$ (as a representation) is isomorphic to

$$(H'_{m,B}, \mathfrak{s}'_{m,B}) \times (H'_{n-m}, \mathfrak{s}'_{n-m}),$$

where $H'_{m.B}$ (resp. $\mathfrak{s}'_{m,B}$) denotes the centraliser of B in H'_m (resp. $\mathfrak{s}'_m$).

Proof. 1) By the base change to an algebraic closure of F containing E, we see from [31, Proposition 2.1] that an element $Y \in \mathfrak{s}'(F)$ which is $H'(F)$-conjugate to $Y(B)$ in the proposition is semi-simple. Now we suppose that $Y \in \mathfrak{s}'(F)$ is semi-simple. Since $Y^2 \in \mathfrak{h}'(F)$, up to $H'(F)$ conjugation, we may suppose that $Y^2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ with $A \in {\mathfrak{h}'_m}^\times(F)$ being semi-simple in the usual sense. From $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} Y = Y \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, we deduce that $Y = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ for some $B \in \mathfrak{s}'_m(F)$ such that $AB = BA$ and some $C \in \mathfrak{s}'_{n-m}(F)$. As $Y^2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, we have $B \in {\mathfrak{s}'_m}^\times(F)$. Because Y is semi-simple, it is shown in [31, p. 71] that Y and Y^2 have the same rank over an algebraic closure of F containing E. Then $C = 0$. We can also see from [31, Proposition 2.1] that B is semi-simple with respect to the H'_m-action after the base change. We have established the first statement.

For the second statement, it suffices to notice that two such elements $Y(B_1)$ with $B_1 \in {\mathfrak{s}'_{m_1}}^\times(F)$ and $Y(B_2)$ with $B_2 \in {\mathfrak{s}'_{m_2}}^\times(F)$ in the proposition are $H'(F)$-conjugate if and only if $m_1 = m_2$ (denoted by m) and B_1 and B_2 are $H'_m(F)$-conjugate.

The third statement follows from the base change or 2).

2) It can be shown by direct calculation. □

Fix a Cartan subspace $\mathfrak{c}'$ of $\mathfrak{s}'$. The lemma below is an analogue of [26, Lemma 29].

LEMMA 3.23. *Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F) - \mathcal{N}^{\mathfrak{s}'})$. Then*

$$\sup_{Y \in \mathfrak{c}'_{\mathrm{reg}}(F)} |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F)\backslash H'(F)} |f'(\mathrm{Ad}(x^{-1})(Y))| dx < +\infty.$$

PROOF. We may apply the argument of [58, Lemma 6.14] relying on Lemma 3.3 and [59, Proposition 3.11], which is an analogue of Harish-Chandra's semi-simple descent for orbital integrals [35, Lemma 16.1]. By Proposition 3.22.2), it suffices to prove the boundedness of orbital integrals for

$$(H'_{m,B}, \mathfrak{s}'_{m,B}) \times (H'_{n-m}, \mathfrak{s}'_{n-m})$$

with $B \in {\mathfrak{s}'_m}^\times(F)$ being semi-simple with respect to the H'_m-action and $0 < m \leq n$. Since there exists an $H'_{m,B}$-equivariant linear isomorphism $\mathfrak{s}'_{m,B} \to \mathfrak{h}'_{m,B}$ induced by $Z \mapsto ZB$, the first factor $(H'_{m,B}, \mathfrak{s}'_{m,B})$ is covered by Harish-Chandra's work [26, Theorem 13] on classical orbital integrals on Lie algebras. Then we conclude by applying the induction hypothesis to the second factor $(H'_{n-m}, \mathfrak{s}'_{n-m})$. □

Consider $X_0 \in \mathcal{N}^{\mathfrak{s}'}$. By the Jacobson-Morozov theorem for symmetric spaces (Lemma 3.7), there exists a group homomorphism $\varphi : SL_2(F) \to G'(F)$ such that

$$X_0 = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y_0 := d\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{s}'(F) \text{ and } d := d\varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h}'(F).$$

Write $r' := \dim \mathfrak{s}'_{Y_0}$ and $m' := \frac{1}{2}\mathrm{Tr}(\mathrm{ad}(-d)|_{\mathfrak{s}'_{Y_0}})$, where $\mathfrak{s}'_{Y_0}$ denotes the centraliser of Y_0 in $\mathfrak{s}'$.

LEMMA 3.24. *We have*
1) $r' \geq \frac{1}{2}\sqrt{\dim \mathfrak{g}'}$;
2) $r' + m' > \frac{1}{4}\dim \mathfrak{g}' + \frac{1}{4}\sqrt{\dim \mathfrak{g}'}$.

PROOF. It suffices to check these relations after a base change to an algebraic closure of F containing E. Then the lemma is exactly [58, Proposition 4.4]. □

Let $\mathfrak{s}'_{\mathrm{bdd}}$ be the set of $X \in \mathfrak{s}'(F)$ such that there exists an open neighbourhood σ' of X in $\mathfrak{s}'(F)$ satisfying

$$\sup_{Y \in \mathfrak{c}'_{\mathrm{reg}}(F)} |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F)\backslash H'(F)} |f'(\mathrm{Ad}(x^{-1})(Y))| dx < +\infty$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ with $\mathrm{Supp}(f') \subseteq \sigma'$. The next lemma is an analogue of [26, Lemma 38].

LEMMA 3.25. *We have $\mathcal{N}^{\mathfrak{s}'} - \{0\} \subseteq \mathfrak{s}'_{\mathrm{bdd}}$.*

PROOF. We may apply the argument of [58, Lemma 6.16] thanks to Lemma 3.24. □

PROOF OF PROPOSITION 3.20. We may use the argument in [26, §VI.7] to show that $0 \in \mathfrak{s}'_{\mathrm{bdd}}$. Then the proposition follows from Lemmas 3.23 and 3.25. □

4. Weighted orbital integrals

4.1. The case of (G, H). Let E/F be a quadratic field extension and η the quadratic character of $F^\times/NE^\times$ attached to it, where $NE^\times$ denotes the norm of $E^\times$. For $x \in H(F)$, which is viewed as an element in $G(F)$, we denote by $\mathrm{Nrd}(x)$ its reduced norm. Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$ and that $Q \in \mathscr{F}^G(M)$. For all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we define the weighted orbital integral

$$J_M^Q(\eta, X, f) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))\eta(\mathrm{Nrd}(x))v_M^Q(x)dx. \tag{4.1.1}$$

Since $v_M^Q(x)$ is left-invariant by $M(F)$ and we have $H_X \subseteq M_H$ for $X \in \mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}}$, we see that $v_M^Q(x)$ is left-invariant by $H_X(F)$. This integral is absolutely convergent since the orbit $\mathrm{Ad}(H(F))(X)$ is closed in $\mathfrak{s}(F)$, which ensures that the integrand is a compactly supported (and locally constant) function on the homogeneous space.

Notice that for $x \in M_H(F)$, we have $J_M^Q(\eta, \mathrm{Ad}(x^{-1})(X), f) = \eta(\mathrm{Nrd}(x))J_M^Q(\eta, X, f)$. Sometimes it is convenient to introduce a transfer factor as in [58, Definition 5.7]: for $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, define

$$\kappa(X) := \eta(\mathrm{Nrd}(A)), \tag{4.1.2}$$

where $\mathrm{Nrd}(A)$ denotes the reduced norm of $A \in GL_n(D)$. Then we have $\kappa(\mathrm{Ad}(x^{-1})(X)) = \eta(\mathrm{Nrd}(x))\kappa(X)$, and thus the function $\kappa(\cdot)J_M^Q(\eta, \cdot, f)$ is constant on $\mathrm{Ad}(M_H(F))(X)$.

Though we mainly consider $M \in \mathscr{L}^{G,\omega}(M_0)$, it is unharmful to extend our definition by (4.1.1) to all Levi subgroups of the form $M = \mathrm{Ad}(w)(L)$, where $L \in \mathscr{L}^{G,\omega}(M_0)$ and $w \in W_0^H$.

One may also extend in the obvious way the definition (4.1.1) of weighted orbital integrals to the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$, since it appears as the product of some copies of the form $(G, H, \mathrm{Ad}(\epsilon))$ in lower dimensions.

Proposition 4.1. *Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$ and that $Q \in \mathscr{F}^G(M)$.*

1) For $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ fixed, the support of the distribution $J_M^Q(\eta, X, \cdot)$ is contained in the closed orbit $\mathrm{Ad}(H(F))(X)$.

2) For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ fixed, the function $J_M^Q(\eta, \cdot, f)$ is locally constant on $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. If $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ is a Cartan subspace, the restriction of this function to $\mathfrak{c}_{\mathrm{reg}}(F)$ vanishes outside a compact subset of $\mathfrak{c}(F)$.

3) If $w \in \mathrm{Norm}_{H(F)}(M_0)$, $x \in M_H(F)$ and $k \in K_H$, we have the equality

$$J^G_{\mathrm{Ad}(w)(M)}(\eta, \mathrm{Ad}(wx)(X), \mathrm{Ad}(k)(f)) = \eta(\mathrm{Nrd}(wxk))J_M^G(\eta, X, f)$$

for all $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$.

4) For $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality

$$J_M^Q(\eta, X, f) = J_M^{M_Q}(\eta, X, f_Q^\eta),$$

where $f_Q^\eta \in \mathcal{C}_c^\infty((\mathfrak{m}_Q \cap \mathfrak{s})(F))$ is defined by (3.2.2).

5) (Descent formula) If $L \in \mathscr{L}^{G,\omega}(M_0)$, $L \subseteq M$ and $X \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we have

$$J_M^G(\eta, X, f) = \sum_{L' \in \mathscr{L}^G(L)} d_L^G(M, L') J_L^{L'}(\eta, X, f_{Q'}^\eta)$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, where Q' denotes the second component of $s(M, L')$ (see Section 2.7).

6) (Non-equivariance) For $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, $y \in H(F)$ and $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality

$$J_M^G(\eta, X, \mathrm{Ad}(y^{-1})(f)) = \eta(\mathrm{Nrd}(y)) \sum_{Q \in \mathscr{F}^G(M)} J_M^{M_Q}(\eta, X, f_{Q,y}^\eta),$$

where $f_{Q,y}^\eta \in \mathcal{C}_c^\infty((\mathfrak{m}_Q \cap \mathfrak{s})(F))$ is defined by

$$f_{Q,y}^\eta(Z) := \int_{K_H \times (\mathfrak{n}_Q \cap \mathfrak{s})(F)} f(\mathrm{Ad}(k^{-1})(Z + U))\eta(\mathrm{Nrd}(k))v_Q'(ky)dUdk, \forall Z \in (\mathfrak{m}_Q \cap \mathfrak{s})(F). \tag{4.1.3}$$

Proof. 1) This is obvious from the definition.

2) Let $Y \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Let $\mathfrak{c}$ be the centraliser of Y in $\mathfrak{s}$. Then $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ is a Cartan subspace and $Y \in \mathfrak{c}_{\mathrm{reg}}(F)$. Since $\mathrm{Ad}(M_H(F))(\mathfrak{c}_{\mathrm{reg}}(F))$ is an open subset of $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ (see [45, p. 105]), in order to prove the first statement, it suffices to find a neighbourhood U of Y in $\mathfrak{c}_{\mathrm{reg}}(F)$ on which the function $\kappa(\cdot)J_M^Q(\eta, \cdot, f)$ is constant. We shall follow the proof of [35, Theorem 17.11]. Consider the function ϕ on $\mathfrak{c}_{\mathrm{reg}}(F) \times (T_{\mathfrak{c}}(F)\backslash H(F))$ defined by $\phi(X, x) := (\kappa f)(\mathrm{Ad}(x^{-1})(X))$. Then ϕ is locally constant but usually not compactly supported. However, now choosing a compact neighbourhood $\sigma_{\mathfrak{c}}$ of Y in $\mathfrak{c}_{\mathrm{reg}}(F)$, we see from Harish-Chandra's compactness lemma for symmetric spaces (Lemma 3.2) applied to $\sigma_{\mathfrak{s}} := \mathrm{Supp}(f)$ that the restriction of ϕ to $\sigma_{\mathfrak{c}} \times (T_{\mathfrak{c}}(F)\backslash H(F))$ is compactly supported. By [35, Lemma 2.1], there exists an open neighbourhood U of Y in $\sigma_{\mathfrak{c}}$ such that $\phi(X, x) = \phi(Y, x)$ for all $X \in U$ and $x \in T_{\mathfrak{c}}(F)\backslash H(F)$. It follows that the function $\kappa(\cdot)J_M^Q(\eta, \cdot, f)$ is constant on U.

The second statement is a corollary of Lemma 3.3.

3) The effect of $\mathrm{Ad}(w)$ is a consequence of our choice of Haar measures. The effect of $\mathrm{Ad}(x)$ results from the left-invariance of $v_M^G(x)$ by $M_H(F)$. The effect of $\mathrm{Ad}(k)$ comes from the right-invariance of $v_M^G(x)$ by K_H. One should keep in mind the effect of $\eta(\mathrm{Nrd}(x))$ in every step.

4) Write $Q_H := Q \cap H \in \mathscr{F}^H(M_0)$. One sees that $M_{Q_H} = M_Q \cap H$ and that $N_{Q_H} = N_Q \cap H$. Applying the change of variables $x = mnk$ with $m \in M_{Q_H}(F)$, $n \in N_{Q_H}(F)$ and $k \in K_H$ in (4.1.1), since $v_M^Q(x) = v_M^{M_Q}(m)$, we have

$$J_M^Q(\eta, X, f) = |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{(M_{Q_H,X}(F)\backslash M_{Q_H}(F)) \times N_{Q_H}(F) \times K_H} f(\mathrm{Ad}(mnk)^{-1}(X))\eta(\mathrm{Nrd}(mk))v_M^{M_Q}(m)dkdndm.$$

Applying Lemma 3.12 to $Y = \mathrm{Ad}(m^{-1})(X)$, we deduce

$$\begin{aligned}
J_M^Q(\eta, X, f) =& |D^{\mathfrak{m}_Q\cap\mathfrak{s}}(X)|_F^{1/2} \int_{(M_{Q_H,X}(F)\backslash M_{Q_H}(F))\times K_H\times(\mathfrak{n}_Q\cap\mathfrak{s})(F)} f(\mathrm{Ad}(k^{-1})(\mathrm{Ad}(m^{-1})(X)+U)) \\
& \eta(\mathrm{Nrd}(mk))v_M^{M_Q}(m)dUdkdm \\
=& |D^{\mathfrak{m}_Q\cap\mathfrak{s}}(X)|_F^{1/2} \int_{M_{Q_H,X}(F)\backslash M_{Q_H}(F)} f_Q^\eta(\mathrm{Ad}(m^{-1})(X))\eta(\mathrm{Nrd}(m))v_M^{M_Q}(m)dm \\
=& J_M^{M_Q}(\eta, X, f_Q^\eta).
\end{aligned}$$

5) It follows from (7) in Section 2.7 and 4).

6) By the change of variables, we see that

$$J_M^G(\eta, X, \mathrm{Ad}(y^{-1})(f)) = |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))\eta(\mathrm{Nrd}(xy))v_M(xy)dx.$$

For $x \in H(F)$ and $Q \in \mathscr{F}^{G,\omega}(M_0)$, let $k_Q(x)$ be an element in K_H such that $xk_Q(x)^{-1} \in Q_H(F)$. It follows from the product formula (2.6.1) that (see the proof of [5, Lemma 8.2])

$$v_M(xy) = \sum_{Q\in\mathscr{F}^G(M)} v_M^Q(x)v_Q'(k_Q(x)y).$$

As in 4), we write

$$\begin{aligned}
J_M^G(\eta, X, \mathrm{Ad}(y^{-1})(f)) =& \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))\eta(\mathrm{Nrd}(x))v_M^Q(x) \\
& v_Q'(k_Q(x)y)dx \\
=& \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{(M_{Q_H,X}(F)\backslash M_{Q_H}(F))\times N_{Q_H}(F)\times K_H} \\
& f(\mathrm{Ad}(mnk)^{-1}(X))\eta(\mathrm{Nrd}(mk))v_M^{M_Q}(m)v_Q'(ky)dkdndm.
\end{aligned}$$

Applying again Lemma 3.12 to $Y = \mathrm{Ad}(m^{-1})(X)$, we obtain

$$\begin{aligned}
J_M^G(\eta, X, \mathrm{Ad}(y^{-1})(f)) =& \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} |D^{\mathfrak{m}_Q\cap\mathfrak{s}}(X)|_F^{1/2} \int_{(M_{Q_H,X}(F)\backslash M_{Q_H}(F))\times K_H\times(\mathfrak{n}_Q\cap\mathfrak{s})(F)} \\
& f(\mathrm{Ad}(k^{-1})(\mathrm{Ad}(m^{-1})(X)+U))\eta(\mathrm{Nrd}(mk))v_M^{M_Q}(m)v_Q'(ky)dUdkdm \\
=& \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} |D^{\mathfrak{m}_Q\cap\mathfrak{s}}(X)|_F^{1/2} \int_{M_{Q_H,X}(F)\backslash M_{Q_H}(F)} f_{Q,y}^\eta(\mathrm{Ad}(m^{-1})(X)) \\
& \eta(\mathrm{Nrd}(m))v_M^{M_Q}(m)dm \\
=& \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} J_M^{M_Q}(\eta, X, f_{Q,y}^\eta).
\end{aligned}$$

□

Lemma 4.2. *Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$ and that $Q \in \mathscr{F}^G(M)$. Let $\sigma \subseteq \mathfrak{s}(F)$ be a compact subset. There exists $c > 0$ and $N \in \mathbb{N}$ such that if $x \in H(F)$ and $X \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$ satisfy $\mathrm{Ad}(x^{-1})(X) \in \sigma$, then*

$$|v_M^Q(x)| \le c\sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^N.$$

Proof. It is shown in the proof of [51, Lemme III.5] that there exists $c_1 > 0$ and $N \in \mathbb{N}$ such that for all $x \in G(F)$,

$$|v_M^Q(x)| \le c_1(1+\log\|x\|)^N.$$

Suppose that $x \in H(F)$ and $X \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$ satisfy $\mathrm{Ad}(x^{-1})(X) \in \sigma$. If we replace x by yx and X by $\mathrm{Ad}(y)(X)$, where $y \in M_H(F)$, the two sides in the inequality to be proved remain unchanged. Since $\mathscr{T}(\mathfrak{m}\cap\mathfrak{s})$ is a finite set, we may fix a Cartan subspace $\mathfrak{c} \subseteq \mathfrak{m}\cap\mathfrak{s}$ and suppose that $X \in \mathfrak{c}_{\mathrm{reg}}(F)$. Let $\tau \in T_{\mathfrak{c}}(F)$ be such that

$$\|\tau x\| = \inf_{\tau'\in T_{\mathfrak{c}}(F)} \|\tau' x\|.$$

Then

$$|v_M^Q(x)| = |v_M^Q(\tau x)| \leq c_1(1+\log\|\tau x\|)^N = c_1(1+\inf_{\tau'\in T_{\mathfrak{c}}(F)}\log\|\tau' x\|)^N.$$

Now it suffices to apply Lemma 3.4. □

COROLLARY 4.3. *Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$ and that $Q \in \mathscr{F}^G(M)$. Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. There exists $c > 0$ and $N \in \mathbb{N}$ such that for all $X \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$, we have*

$$|J_M^{M_Q}(\eta, X, f_Q^\eta)| \leq c\sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^N.$$

PROOF. By Proposition 4.1.4) and Lemma 4.2 applied to $\sigma = \mathrm{Supp}(f)$, we see that

$$\begin{aligned}|J_M^{M_Q}(\eta, X, f_Q^\eta)| \leq & |D^{\mathfrak{s}}(X)|_F^{1/2}\int_{H_X(F)\backslash H(F)}|f(\mathrm{Ad}(x^{-1})(X))v_M^Q(x)|dx\\ \leq & c\sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^N|D^{\mathfrak{s}}(X)|_F^{1/2}\int_{H_X(F)\backslash H(F)}|f(\mathrm{Ad}(x^{-1})(X))|dx.\end{aligned}$$

Now we draw our conclusion by Proposition 3.13. □

4.2. The case of (G', H'). Suppose that $M' \in \mathscr{L}^{H'}(M_0')$ and that $Q' \in \mathscr{F}^{H'}(M')$. For all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ and $Y \in (\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$, we define the weighted orbital integral

$$J_{M'}^{Q'}(Y, f') := |D^{\mathfrak{s}'}(Y)|_F^{1/2}\int_{H_Y'(F)\backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y))v_{M'}^{Q'}(x)dx. \tag{4.2.1}$$

By the base change to an algebraic closure of F containing E, we see that $H_Y' \subseteq H_{Y^2}' \subseteq M'$ for $Y \in \widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}'$. Then $v_{M'}^{Q'}(x)$ is left-invariant by $H_Y'(F)$. This integral is absolutely convergent since the orbit $\mathrm{Ad}(H'(F))(Y)$ is closed in $\mathfrak{s}'(F)$. Notice that for $x \in M'(F)$, we have $J_{M'}^{Q'}(\mathrm{Ad}(x^{-1})(Y), f') = J_{M'}^{Q'}(Y, f')$, i.e., the function $J_{M'}^{Q'}(\cdot, f')$ is constant on $\mathrm{Ad}(M'(F))(Y)$. One may extend in the obvious way the definition (4.2.1) to the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M_0')$, since it appears as the product of some copies of the form $(G', H', \mathrm{Ad}(\alpha))$ in lower dimensions.

PROPOSITION 4.4. *Suppose that $M' \in \mathscr{L}^{H'}(M_0')$ and that $Q' \in \mathscr{F}^{H'}(M')$.*

1) For $Y \in (\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$ fixed, the support of the distribution $J_{M'}^{Q'}(Y, \cdot)$ is contained in the closed orbit $\mathrm{Ad}(H'(F))(Y)$.

2) For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ fixed, the function $J_{M'}^{Q'}(\cdot, f')$ is locally constant on $(\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$. If $\mathfrak{c}' \subseteq \widetilde{\mathfrak{m}'}\cap\mathfrak{s}'$ is a Cartan subspace, the restriction of this function to $\mathfrak{c}_{\mathrm{reg}}'(F)$ vanishes outside a compact subset of $\mathfrak{c}'(F)$.

3) If $w \in \mathrm{Norm}_{H'(F)}(M_0')$, $x \in M'(F)$ and $k \in K_{H'}$, we have the equality

$$J_{\mathrm{Ad}(w)(M')}^{H'}(\mathrm{Ad}(wx)(Y), \mathrm{Ad}(k)(f')) = J_{M'}^{H'}(Y, f')$$

for all $Y \in (\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$.

4) For $Y \in (\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we have the equality

$$J_{M'}^{Q'}(Y, f') = J_{M'}^{M_{Q'}}(Y, f_{Q'}'),$$

where $f_{Q'}' \in \mathcal{C}_c^\infty((\mathfrak{m}_{\widetilde{Q'}}\cap\mathfrak{s}')(F))$ is defined by (3.3.2).

5) (Descent formula) If $L' \in \mathscr{L}^{H'}(M_0')$, $L' \subseteq M'$ and $Y \in (\widetilde{\mathfrak{l}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$, we have

$$J_{M'}^{H'}(Y, f') = \sum_{L\in\mathscr{L}^{H'}(L')} d_{L'}^{H'}(M', L)J_{L'}^L(Y, f_Q')$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, where Q denotes the second component of $s(M', L)$ (see Section 2.7).

6) (Noninvariance) For $Y \in (\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$, $y \in H'(F)$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we have the equality

$$J_{M'}^{H'}(Y, \mathrm{Ad}(y^{-1})(f')) = \sum_{Q'\in\mathscr{F}^{H'}(M')} J_{M'}^{M_{Q'}}(Y, f_{Q',y}'),$$

where $f_{Q',y}' \in \mathcal{C}_c^\infty((\mathfrak{m}_{\widetilde{Q'}}\cap\mathfrak{s}')(F))$ is defined by

$$f_{Q',y}'(Z) := \int_{K_{H'}\times(\mathfrak{n}_{\widetilde{Q'}}\cap\mathfrak{s}')(F)} f'(\mathrm{Ad}(k^{-1})(Z+U))v_{Q'}'(ky)dUdk, \forall Z \in (\mathfrak{m}_{\widetilde{Q'}}\cap\mathfrak{s}')(F). \tag{4.2.2}$$

Proof. It is almost the same as the proof of Proposition 4.1, except that one needs to use Lemma 3.19 to show 4) and 6). □

Lemma 4.5 (cf. Lemma 4.2). *Suppose that $M' \in \mathscr{L}^{H'}(M'_0)$ and that $Q' \in \mathscr{F}^{H'}(M')$. Let $\sigma' \subseteq \mathfrak{s}'(F)$ be a compact subset. There exists $c > 0$ and $N \in \mathbb{N}$ such that if $x \in H'(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ satisfy $\mathrm{Ad}(x^{-1})(Y) \in \sigma'$, then*

$$|v^{Q'}_{M'}(x)| \leq c \sup\{1, -\log |D^{\mathfrak{s}'}(Y)|_F\}^N.$$

Corollary 4.6. *Suppose that $M' \in \mathscr{L}^{H'}(M'_0)$ and that $Q' \in \mathscr{F}^{H'}(M')$. Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$. There exists $c > 0$ and $N \in \mathbb{N}$ such that for all $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, we have*

$$|J^{M_{Q'}}_{M'}(Y, f'_{Q'})| \leq c \sup\{1, -\log |D^{\mathfrak{s}'}(Y)|_F\}^N.$$

Proof. We may apply the argument of Corollary 4.3 with the help of Proposition 4.4.4), Lemma 4.5 and Proposition 3.20. □

5. The noninvariant trace formula

5.1. The case of (G, H). Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$. For $x, y \in G(F)$, we define a (G, M)-family $(v_P(x, y))_{P \in \mathscr{P}^G(M)}$ as in [8, (12.1) in §12] by

$$v_P(\lambda, x, y) := e^{-\lambda(H_P(y) - H_{\overline{P}}(x))}, \forall \lambda \in i\mathfrak{a}^*_M, P \in \mathscr{P}^G(M),$$

where $\overline{P} \in \mathscr{P}^G(M)$ is the parabolic subgroup opposite to P. Let E/F be a quadratic field extension and η the quadratic character of $F^\times/NE^\times$ attached to it. For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we define

(5.1.1)

$$J^G_M(\eta, X, f, f') := |D^{\mathfrak{s}}(X)|_F \int_{(A_M(F)\backslash H(F))^2} f(\mathrm{Ad}(x^{-1})(X)) f'(\mathrm{Ad}(y^{-1})(X)) \eta(\mathrm{Nrd}(x^{-1}y)) v_M(x, y) dx dy.$$

Proposition 5.1. *Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$ and that $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$.*

1) The integral (5.1.1) is absolutely convergent.

2) The function $J^G_M(\eta, \cdot, f, f')$ is locally constant on $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$.

3) If $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ is an M-elliptic Cartan subspace, the restriction of $J^G_M(\eta, \cdot, f, f')$ to $\mathfrak{c}_{\mathrm{reg}}(F)$ vanishes outside a compact subset of $\mathfrak{c}(F)$.

4) If $w \in \mathrm{Norm}_{H(F)}(M_0)$, $x \in M_H(F)$ and $k, k' \in K_H$, we have the equality

$$J^G_{\mathrm{Ad}(w)(M)}(\eta, \mathrm{Ad}(wx)(X), \mathrm{Ad}(k)(f), \mathrm{Ad}(k')(f')) = \eta(\mathrm{Nrd}(kk')) J^G_M(\eta, X, f, f')$$

for all $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$.

5) There exists $c > 0$ and $N \in \mathbb{N}$ such that for all $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we have

$$|J^G_M(\eta, X, f, f')| \leq c \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N.$$

6) For all $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we have

$$J^G_M(\eta, X, f, f') = \sum_{L_1, L_2 \in \mathscr{L}^G(M)} d^G_M(L_1, L_2) J^{L_1}_M(\eta, X, f^\eta_{\overline{Q_1}}) J^{L_2}_M(\eta, X, f'^\eta_{Q_2}),$$

where $(Q_1, Q_2) := s(L_1, L_2)$ (see Section 2.7).

Proof. The statements 1)-4) can be proved in the same way as the proof of analogous properties for (4.1.1) in Section 4. Notice that the $\eta(\mathrm{Nrd}(\cdot))$-invariant effects coming from x and y may sometimes cancel.

For $x \in G(F)$, we define a (G, M)-family $(\overline{v}_P(x))_{P \in \mathscr{P}^G(M)}$ by

$$\overline{v}_P(\lambda, x) := e^{\lambda(H_{\overline{P}}(x))}, \forall \lambda \in i\mathfrak{a}^*_M, P \in \mathscr{P}^G(M).$$

Then $v_P(x, y) = \overline{v}_P(x) v_P(y)$ as the product of (G, M)-families. Notice that for all $Q \in \mathscr{P}^G(M)$ and $x \in G(F)$, we have

$$\overline{v}^Q_M(x) = v^{\overline{Q}}_M(x).$$

The statement 6) is a consequence of the splitting formula of (G, M)-families ((6) in Section 2.7) and Proposition 4.1.4). It together with Corollary 4.3 implies the statement 5). □

For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define

$$J^G(\eta, f, f') := \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta, X, f, f')dX. \tag{5.1.2}$$

This expression is absolutely convergent by Proposition 5.1.5) and Corollary 3.6.

Remark 5.2. We have the equality

$$J^G(\eta, f, f') = J^G(\eta, f', f).$$

It results from the fact that for all $M \in \mathscr{L}^{G,\omega}(M_0)$ and all $x, y \in G(F)$, we have $v_M(x, y) = v_M(y, x)$.

Again, one may extend in the obvious way the definitions (5.1.1) and (5.1.2) to the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$, since it appears as the product of some copies of the form $(G, H, \mathrm{Ad}(\epsilon))$ in lower dimensions.

Theorem 5.3 (Noninvariant trace formula). *For all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$J^G(\eta, f, \hat{f}') = J^G(\eta, \hat{f}, f').$$

The rest of this section is devoted to the proof of Theorem 5.3.

Fix $P_0 \in \mathscr{P}^G(M_0)$. Denote

$$\overline{\mathfrak{a}_{P_0}^+} := \{T \in \mathfrak{a}_{M_0} : \alpha(T) \geq 0, \forall \alpha \in \Delta_{P_0}^G\}.$$

For $T \in \overline{\mathfrak{a}_{P_0}^+}$, write

$$d(T) := \inf_{\alpha \in \Delta_{P_0}^G} \alpha(T),$$

which is invariant under the translation by $\mathfrak{a}_G$. Set $R_0 := (\mathfrak{a}_{M_0,F} + \mathfrak{a}_G)/\mathfrak{a}_G$, which is a lattice in $\mathfrak{a}_{M_0}/\mathfrak{a}_G$. For $T \in R_0 \cap (\overline{\mathfrak{a}_{P_0}^+}/\mathfrak{a}_G)$, we define a function $u(\cdot, T)$ on $A_G(F)\backslash G(F)$ as in [8, p. 21], which is the characteristic function of certain compact subset. To be precise, let $C_{M_0}(T)$ be the convex hull in $\mathfrak{a}_{M_0}/\mathfrak{a}_G$ of

$$\{T_B : B \in \mathscr{P}^G(M_0)\},$$

where T_B denotes the unique W_0^G-translate of T which lies in $\overline{\mathfrak{a}_B^+}$. Then $u(x, T)$ is defined as the characteristic function of the set of points

$$x = k_1 m k_2, m \in A_G(F)\backslash M_0(F), k_1, k_2 \in K_G,$$

in $A_G(F)\backslash G(F)$ such that $H_{M_0}(m)$ lies in $C_{M_0}(T)$.

Let $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. For $x \in H(F)$, we define

$$k(x, f, f') := \int_{\mathfrak{s}(F)} f(X) f'(\mathrm{Ad}(x^{-1})(X))dX.$$

For $T \in R_0 \cap (\overline{\mathfrak{a}_{P_0}^+}/\mathfrak{a}_G)$, we define

$$K^T(\eta, f, f') := \int_{A_G(F)\backslash H(F)} k(x, f, f')\eta(\mathrm{Nrd}(x))u(x, T)dx.$$

Since $A_G(F)\backslash H(F)$ is a closed subgroup of $A_G(F)\backslash G(F)$, the restriction of $u(x, T)$ to $A_G(F)\backslash H(F)$ is also compactly supported, and the above integral is absolutely convergent.

By the Weyl integration formula (Proposition 3.11), we obtain the geometric expansion

$$K^T(\eta, f, f') = \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} K^T(\eta, X, f, f')dX,$$

where

$$K^T(\eta, X, f, f') := |D^{\mathfrak{s}}(X)|_F \int_{A_G(F)\backslash H(F)} \int_{A_M(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) f'(\mathrm{Ad}(xy)^{-1}(X))\eta(\mathrm{Nrd}(y)) u(y, T)dxdy.$$

By the change of variables $xy \mapsto y$, we can write
(5.1.3)
$$K^T(\eta, X, f, f') = |D^{\mathfrak{s}}(X)|_F \int_{(A_M(F)\backslash H(F))^2} f(\mathrm{Ad}(x^{-1})(X)) f'(\mathrm{Ad}(y^{-1})(X)) \eta(\mathrm{Nrd}(x^{-1}y)) u_M(x, y, T) dx dy,$$
where
$$u_M(x, y, T) := \int_{A_G(F)\backslash A_M(F)} u(x^{-1}ay, T) da$$
is defined as in [8, p. 21].

For $x, y \in G(F)$ and $T \in R_0 \cap (\overline{\mathfrak{a}_{P_0}^+}/\mathfrak{a}_G)$, we define the second weight function $v_M(x, y, T)$ as in [8, p. 30], which is left-invariant under the multiplication of $A_M(F)$ on x or y. To be precise, let $\lambda \in \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$ be a point whose real part $\lambda_{\mathbb{R}} \in \mathfrak{a}_M^*$ is in general position. For $P \in \mathscr{P}^G(M)$, set
$$\Delta_P^\lambda := \{\alpha \in \Delta_P^G : \lambda_{\mathbb{R}}(\alpha^\vee) < 0\},$$
where $\alpha^\vee$ is the "coroot" associated to $\alpha \in \Delta_P^G$ (see [9, p. 26]). Denote by φ_P^λ the characteristic function of the set of $T' \in \mathfrak{a}_M$ such that $\varpi_\alpha(T') > 0$ for each $\alpha \in \Delta_P^\lambda$ and that $\varpi_\alpha(T') \leq 0$ for each $\alpha \in \Delta_P^G - \Delta_P^\lambda$, where $\{\varpi_\alpha : \alpha \in \Delta_P^G\}$ is the basis of $(\mathfrak{a}_P^G)^*$ which is dual to $\{\alpha^\vee : \alpha \in \Delta_P^G\}$. Let
$$Y_P(x, y, T) := T_P + H_P(x) - H_{\overline{P}}(y), \forall P \in \mathscr{P}^G(M).$$
The set $\mathscr{Y}_M(x, y, T) = \{Y_P(x, y, T) : P \in \mathscr{P}^G(M)\}$ is a (G, M)-orthogonal set in the sense of [8, p. 19]. Define
$$\sigma_M(T', \mathscr{Y}_M(x, y, T)) := \sum_{P \in \mathscr{P}^G(M)} (-1)^{|\Delta_P^\lambda|} \varphi_P^\lambda(T' - Y_P(x, y, T)), \forall T' \in \mathfrak{a}_M/\mathfrak{a}_G.$$
The function $\sigma_M(\cdot, \mathscr{Y}_M(x, y, T))$ is known to be compactly supported (see [8, p. 22]). Then $v_M(x, y, T)$ is defined as the integral
$$v_M(x, y, T) := \int_{A_G(F)\backslash A_M(F)} \sigma_M(H_M(a), \mathscr{Y}_M(x, y, T)) da.$$
Now, we define the corresponding weighted orbital integral
(5.1.4)
$$J^T(\eta, X, f, f') := |D^{\mathfrak{s}}(X)|_F \int_{(A_M(F)\backslash H(F))^2} f(\mathrm{Ad}(x^{-1})(X)) f'(\mathrm{Ad}(y^{-1})(X)) \eta(\mathrm{Nrd}(x^{-1}y)) v_M(x, y, T) dx dy.$$

Let $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ be an M-elliptic Cartan subspace. For $\varepsilon > 0$ and $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}_{P_0}^+}$ with large $\|T\|$, consider the domain near the singular set
$$\mathfrak{c}(\varepsilon, T) := \{X \in \mathfrak{c}_{\mathrm{reg}}(F) : |D^{\mathfrak{s}}(X)| \leq e^{-\varepsilon\|T\|}\}.$$

Lemma 5.4. *Fix an arbitary constant $\varepsilon_0 > 0$. Fix a constant $\varepsilon' > 0$ satisfying the condition of Lemma 3.5. Let $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ be an M-elliptic Cartan subspace. Given $\varepsilon > 0$, there exists $c > 0$ such that for any $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}_{P_0}^+}$ with $\|T\| \geq \varepsilon_0$,*
$$\int_{\mathfrak{c}(\varepsilon,T)} (|K^T(\eta, X, f, f')| + |J^T(\eta, X, f, f')|) dX \leq c e^{-\frac{\varepsilon'\varepsilon\|T\|}{2}}.$$

Proof. It is shown in [8, (4.8) in p. 31] that there exist $c_1, d_1 > 0$ such that for all $x, y \in G(F)$ and $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}_{P_0}^+}$ with $\|T\| \geq \varepsilon_0$,
$$u_M(x, y, T) \leq c_1(\|T\| + \log\|x\| + \log\|y\|)^{d_1}.$$

For any $a_1, a_2 \in A_M(F)$, we deduce that
$$u_M(x, y, T) = u_M(a_1x, a_2y, T) \leq c_1(\|T\| + \log\|a_1x\| + \log\|a_2y\|)^{d_1}.$$
Since $T_{\mathfrak{c}}(F)/A_M(F)$ is compact, there exists $c_2 > 0$ such that
$$u_M(x, y, T) \leq c_2(\|T\| + \inf_{\tau_1 \in T_{\mathfrak{c}}(F)} \log\|\tau_1 x\| + \inf_{\tau_2 \in T_{\mathfrak{c}}(F)} \log\|\tau_2 y\|)^{d_1}.$$

Now let $x, y \in H(F)$ and $X \in \mathfrak{c}(\varepsilon, T)$, and assume that
$$f(\mathrm{Ad}(x^{-1})(X)) f'(\mathrm{Ad}(y^{-1})(X)) \neq 0.$$

Let $\sigma \subseteq \mathfrak{s}(F)$ be a compact subset containing $\mathrm{Supp}(f) \cup \mathrm{Supp}(f')$. From Lemma 3.4, there exists $c_\sigma > 0$ such that

$$\inf_{\tau_1 \in T_{\mathfrak{c}}(F)} \log \|\tau_1 x\|, \inf_{\tau_2 \in T_{\mathfrak{c}}(F)} \log \|\tau_2 y\| \leq c_\sigma \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}.$$

Therefore, there exists $c'_\sigma > 0$ such that

$$u_M(x, y, T) \leq c'_\sigma (\|T\| - \log |D^{\mathfrak{s}}(X)|_F)^{d_1}. \tag{5.1.5}$$

By Proposition 3.13, there exists $c_3 > 0$ such that

$$|D^{\mathfrak{s}}(X)|_F^{1/2} \int_{A_M(F)\backslash H(F)} |f(x^{-1}Xx)| dx \leq c_3 \tag{5.1.6}$$

and

$$|D^{\mathfrak{s}}(X)|_F^{1/2} \int_{A_M(F)\backslash H(F)} |f'(y^{-1}Xy)| dy \leq c_3 \tag{5.1.7}$$

for all $X \in \mathfrak{c}_{\mathrm{reg}}(F)$.

Putting the estimates (5.1.5), (5.1.6) and (5.1.7) into the definition (5.1.3) of $K^T(\eta, X, f, f')$, we obtain the inequality

$$|K^T(\eta, X, f, f')| \leq c'_\sigma c_3^2 (\|T\| - \log |D^{\mathfrak{s}}(X)|_F)^{d_1}.$$

By Lemma 3.5, for any subset B of $\mathfrak{c}_{\mathrm{reg}}(F)$ which is relatively compact in $\mathfrak{c}(F)$, there exists $c_B > 0$ such that

$$\int_B |D^{\mathfrak{s}}(X)|_F^{-\varepsilon'} dX \leq c_B.$$

We deduce that for $m \in \mathbb{Z}$,

$$\mathrm{vol}(B \cap \{X \in \mathfrak{c}_{\mathrm{reg}}(F) : |D^{\mathfrak{s}}(X)|_F = q^{-\frac{m}{2}}\}) \leq c_B q^{-\frac{\varepsilon' m}{2}}. \tag{5.1.8}$$

We claim that for any B as above, there exists $c'_B > 0$ such that

$$\int_{B \cap \mathfrak{c}(\varepsilon, T)} (\|T\| - \log |D^{\mathfrak{s}}(X)|_F)^{d_1} dX \leq c'_B e^{-\frac{\varepsilon' \varepsilon \|T\|}{2}}.$$

This is an analogue of the exercise in [8, p. 32] and we include here a proof for completeness. For $X \in \mathfrak{c}(\varepsilon, T)$, we have

$$\|T\| \leq -\frac{1}{\varepsilon} \log |D^{\mathfrak{s}}(X)|_F.$$

Therefore,

$$\int_{B \cap \mathfrak{c}(\varepsilon, T)} (\|T\| - \log |D^{\mathfrak{s}}(X)|_F)^{d_1} dX \leq \left(1 + \frac{1}{\varepsilon}\right)^{d_1} \int_{B \cap \mathfrak{c}(\varepsilon, T)} (-\log |D^{\mathfrak{s}}(X)|_F)^{d_1} dX.$$

Since

$$B \cap \mathfrak{c}(\varepsilon, T) = \coprod_{m \geq \frac{2\varepsilon \|T\|}{\log q}} (B \cap \{X \in \mathfrak{c}_{\mathrm{reg}}(F) : |D^{\mathfrak{s}}(X)|_F = q^{-\frac{m}{2}}\}),$$

we have

$$\int_{B \cap \mathfrak{c}(\varepsilon, T)} (-\log |D^{\mathfrak{s}}(X)|_F)^{d_1} dX = \sum_{m \geq \frac{2\varepsilon \|T\|}{\log q}} \left(\frac{m \log q}{2}\right)^{d_1} \mathrm{vol}(B \cap \{X \in \mathfrak{c}_{\mathrm{reg}}(F) : |D^{\mathfrak{s}}(X)|_F = q^{-\frac{m}{2}}\}).$$

Applying (5.1.8), we obtain

$$\int_{B \cap \mathfrak{c}(\varepsilon, T)} (-\log |D^{\mathfrak{s}}(X)|_F)^{d_1} dX \leq \sum_{m \geq \frac{2\varepsilon \|T\|}{\log q}} \left(\frac{m \log q}{2}\right)^{d_1} c_B q^{-\frac{\varepsilon' m}{2}}.$$

Now we can confirm our claim by noting the basic fact: for $d > 0$ and $a > 1$, there exists $c_{d,a} > 0$ such that

$$\sum_{m \geq x} \frac{m^d}{a^m} \leq c_{d,a} a^{-\frac{x}{2}}, \forall x \geq 0.$$

Taking

$$B = \{X \in \mathfrak{c}_{\mathrm{reg}}(F) : K^T(\eta, X, f, f') \neq 0\},$$

we see that

$$\int_{\mathfrak{c}(\varepsilon,T)} |K^T(\eta, X, f, f')| dX \leq c'_\sigma c_3^2 c'_B e^{-\frac{\varepsilon'\varepsilon\|T\|}{2}}.$$

This is half of the lemma.

It is proved in [8, p. 32] that there exist $c_4, d_2 > 0$ such that for all $x, y \in G(F)$ and $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}_{P_0}^+}$ with $\|T\| \geq \varepsilon_0$,

$$v_M(x, y, T) \leq c_4(\|T\| + \log\|x\| + \log\|y\|)^{d_2}.$$

By the same argument as before, we obtain

$$\int_{\mathfrak{c}(\varepsilon,T)} |J^T(\eta, X, f, f')| dX \leq c_5 e^{-\frac{\varepsilon'\varepsilon\|T\|}{2}}$$

for some $c_5 > 0$. This establishes the other half of the lemma. □

LEMMA 5.5. *Suppose that $\delta > 0$. Then there exist $c, \varepsilon_1, \varepsilon_2 > 0$ such that*

$$|u_M(x, y, T) - v_M(x, y, T)| \leq ce^{-\varepsilon_1\|T\|}$$

for all $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}_{P_0}^+}$ with $d(T) \geq \delta\|T\|$, and all $x, y \in \{x \in G(F) : \|x\| \leq e^{\varepsilon_2\|T\|}\}$.

PROOF. This is Arthur's main geometric lemma [8, Lemma 4.4]. □

LEMMA 5.6. *Suppose that $\delta > 0$. Let $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ be an M-elliptic Cartan subspace. Then there exist $c, \varepsilon > 0$ such that*

$$\int_{\mathfrak{c}_{\mathrm{reg}}(F)} |K^T(\eta, X, f, f') - J^T(\eta, X, f, f')| dX \leq ce^{-\varepsilon\|T\|}$$

for all $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}_{P_0}^+}$ with sufficiently large $\|T\|$ and $d(T) \geq \delta\|T\|$.

PROOF. Fix $\varepsilon_2 > 0$ to be the constant given by Lemma 5.5. Let $x, y \in H(F)$ and $X \in \mathfrak{c}_{\mathrm{reg}}(F) - \mathfrak{c}(\frac{\varepsilon_2}{2}, T)$, and assume that

$$f(\mathrm{Ad}(x^{-1})(X))f'(\mathrm{Ad}(y^{-1})(X)) \neq 0.$$

Let $\sigma \subseteq \mathfrak{s}(F)$ be a compact subset containing $\mathrm{Supp}(f) \cup \mathrm{Supp}(f')$. From Lemma 3.4, there exists $c_\sigma > 0$ such that

$$\inf_{\tau_1 \in T_{\mathfrak{c}}(F)} \|\tau_1 x\|, \inf_{\tau_2 \in T_{\mathfrak{c}}(F)} \|\tau_2 y\| \leq c_\sigma \sup\{1, |D^{\mathfrak{s}}(X)|_F^{-1}\}.$$

Since $X \in \mathfrak{c}_{\mathrm{reg}}(F) - \mathfrak{c}(\frac{\varepsilon_2}{2}, T)$, we have

$$\sup\{1, |D^{\mathfrak{s}}(X)|_F^{-1}\} \leq \sup\{1, e^{\frac{\varepsilon_2\|T\|}{2}}\} = e^{\frac{\varepsilon_2\|T\|}{2}}.$$

Then, multiplying x and y by elements in $T_{\mathfrak{c}}(F)$ if necessary, and taking $\|T\| \geq \frac{2\log c_\sigma}{\varepsilon_2}$, we can assume that

$$\|x\|, \|y\| \leq e^{\varepsilon_2\|T\|}.$$

It follows from Lemma 5.5 that

$$|u_M(x, y, T) - v_M(x, y, T)| \leq ce^{-\varepsilon_1\|T\|}.$$

By the definitions (see (5.1.3) and (5.1.4)) of $K^T(\eta, X, f, f')$ and $J^T(\eta, X, f, f')$, we obtain that

$$\int_{\mathfrak{c}_{\mathrm{reg}}(F) - \mathfrak{c}(\frac{\varepsilon_2}{2}, T)} |K^T(\eta, X, f, f') - J^T(\eta, X, f, f')| dX \leq c_1 e^{-\varepsilon_1\|T\|},$$

where

$$c_1 := c \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F \int_{(A_M(F)\backslash H(F))^2} |f(\mathrm{Ad}(x^{-1})(X))f'(\mathrm{Ad}(y^{-1})(X))| dx dy dX$$

is finite by Proposition 3.13 and Lemma 3.3.

One can draw the conclusion by combining this with Lemma 5.4. □

Define

$$J^T(\eta, f, f') := \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J^T(\eta, X, f, f') dX,$$

where $J^T(\eta, X, f, f')$ is defined by (5.1.4).

Proposition 5.7. *Suppose that $\delta > 0$. Then there exist $c, \varepsilon > 0$ such that*

$$|K^T(\eta, f, f') - J^T(\eta, f, f')| \leq ce^{-\varepsilon\|T\|}$$

for all $T \in \mathfrak{a}_{M_0,F} \cap \overline{\mathfrak{a}^+_{P_0}}$ with sufficiently large $\|T\|$ and $d(T) \geq \delta\|T\|$.

Proof. Apply Lemma 5.6. □

It is proved in [8, (6.5) in p. 46] that $v_M(x, y, T)$ is an exponential polynomial in $T \in R_0 \cap (\overline{\mathfrak{a}^+_{P_0}}/\mathfrak{a}_G)$. Denote by $\tilde{v}_M(x, y)$ the constant term of $v_M(x, y, T)$ as in [8, (6.6) in p. 46]. Then for $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, $J^T(\eta, X, f, f')$ is also an exponential polynomial in $T \in R_0 \cap (\overline{\mathfrak{a}^+_{P_0}}/\mathfrak{a}_G)$ whose constant term is given by

(5.1.9)

$$\tilde{J}_M(\eta, X, f, f') := |D^{\mathfrak{s}}(X)|_F \int_{(A_M(F)\backslash H(F))^2} f(\mathrm{Ad}(x^{-1})(X))f'(\mathrm{Ad}(y^{-1})(X))\eta(\mathrm{Nrd}(x^{-1}y))\tilde{v}_M(x, y)dxdy.$$

Thus for $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, $J^T(\eta, f, f')$ is still an exponential polynomial in $T \in R_0 \cap (\overline{\mathfrak{a}^+_{P_0}}/\mathfrak{a}_G)$ whose constant term is given by

$$\tilde{J}(\eta, f, f') := \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} \tilde{J}_M(\eta, X, f, f')dX. \tag{5.1.10}$$

Corollary 5.8. *For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$\tilde{J}(\eta, f, \hat{f}') = \tilde{J}(\eta, \hat{f}, f').$$

Proof. By the Plancherel formula, for $x \in H(F)$, we have

$$k(x, f, \hat{f}') = k(x, \hat{f}, f').$$

Then for all $T \in R_0 \cap (\overline{\mathfrak{a}^+_{P_0}}/\mathfrak{a}_G)$,

$$K^T(\eta, f, \hat{f}') = K^T(\eta, \hat{f}, f').$$

Finally, apply Proposition 5.7 to conclude. □

Lemma 5.9. *For all $Q \in \mathscr{F}^{G,\omega}(M_0)$, there exists a constant c'_Q such that for all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$\tilde{J}(\eta, f, f') = \sum_{Q \in \mathscr{F}^{G,\omega}(M_0)} |W_0^{M_{Q_n}}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_Q/A_G)} J^{M_Q}(\eta, f^\eta_{\overline{Q}}, f'^\eta_Q)c'_Q,$$

where $J^{M_Q}(\eta, f^\eta_{\overline{Q}}, f'^\eta_Q)$ is defined by (5.1.2).

Proof. Suppose that $M \in \mathscr{L}^{G,\omega}(M_0)$. It is shown in [8, p. 92] that

$$\tilde{v}_M(x, y) = (-1)^{\dim(A_M/A_G)} \sum_{Q \in \mathscr{F}^G(M)} v_M^Q(x, y)c'_Q,$$

where c'_Q is a constant for each $Q \in \mathscr{F}^{G,\omega}(M_0)$.

Now substitude this in the definition (5.1.9) of $\tilde{J}_M(\eta, X, f, f')$. Note that

$$v_M^Q(m_1n_1k_1, m_2n_2k_2) = v_M^{M_Q}(m_1, m_2)$$

for $m_1 \in M_{\overline{Q}_H}(F), n_1 \in N_{\overline{Q}_H}(F), m_2 \in M_{Q_H}(F), n_2 \in N_{Q_H}(F), k_1, k_2 \in K_H$. By the same argument as the proof of Proposition 4.1.4), one shows that

$$|D^{\mathfrak{s}}(X)|_F \int_{(A_M(F)\backslash H(F))^2} f(\mathrm{Ad}(x^{-1})(X))f'(\mathrm{Ad}(y^{-1})(X))\eta(\mathrm{Nrd}(x^{-1}y))v_M^Q(x, y)dxdy = J_M^{M_Q}(\eta, X, f^\eta_{\overline{Q}}, f'^\eta_Q),$$

where $J_M^{M_Q}(\eta, X, f^\eta_{\overline{Q}}, f'^\eta_Q)$ is defined by (5.1.1). Therefore, we have

$$\tilde{J}_M(\eta, X, f, f') = (-1)^{\dim(A_M/A_G)} \sum_{Q \in \mathscr{F}^G(M)} J_M^{M_Q}(\eta, X, f^\eta_{\overline{Q}}, f'^\eta_Q)c'_Q.$$

Then the lemma follows from the definition (5.1.10) of $\tilde{J}(\eta, f, f')$. □

Proof of Theorem 5.3. Using Lemma 5.9 and Corollary 5.8, we can prove the theorem by induction on the dimension of G. □

5.2. The case of (G', H'). Suppose that $M' \in \mathscr{L}^{H'}(M'_0)$. For $x, y \in H'(F)$, we define an (H', M')-family $(v_{P'}(x,y))_{P' \in \mathscr{P}^{H'}(M')}$ by

$$v_{P'}(\lambda, x, y) := e^{-\lambda(H_{P'}(y) - H_{\overline{P'}}(x))}, \forall \lambda \in i\mathfrak{a}^*_{M'}, P' \in \mathscr{P}^{H'}(M'),$$

where $\overline{P'} \in \mathscr{P}^{H'}(M')$ is the parabolic subgroup opposite to P'. For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we define

$$J^{H'}_{M'}(Y, f, f') := |D^{\mathfrak{s}'}(Y)|_F \int_{(A_{M'}(F)\backslash H'(F))^2} f(\mathrm{Ad}(x^{-1})(Y)) f'(\mathrm{Ad}(y^{-1})(Y)) v_{M'}(x,y) dx dy. \tag{5.2.1}$$

Proposition 5.10. *Suppose that $M' \in \mathscr{L}^{H'}(M'_0)$ and that $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$.*

1) The integral (5.2.1) is absolutely convergent.

2) The function $J^{H'}_{M'}(\cdot, f, f')$ is locally constant on $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$.

3) If $\mathfrak{c}' \subseteq \widetilde{\mathfrak{m}'} \cap \mathfrak{s}'$ is an M'-elliptic Cartan subspace, the restriction of $J^{H'}_{M'}(\cdot, f, f')$ to $\mathfrak{c}'_{\mathrm{reg}}(F)$ vanishes outside a compact subset of $\mathfrak{c}'(F)$.

4) If $w \in \mathrm{Norm}_{H'(F)}(M'_0)$, $x \in M'(F)$ and $k, k' \in K_{H'}$, we have the equality

$$J^{H'}_{\mathrm{Ad}(w)(M')}(\mathrm{Ad}(wx)(Y), \mathrm{Ad}(k)(f), \mathrm{Ad}(k')(f')) = J^{H'}_{M'}(Y, f, f')$$

for all $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$.

5) There exists $c > 0$ and $N \in \mathbb{N}$ such that for all $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we have

$$|J^{H'}_{M'}(Y, f, f')| \leq c \sup\{1, -\log |D^{\mathfrak{s}'}(Y)|_F\}^N.$$

6) For all $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$, we have

$$J^{H'}_{M'}(Y, f, f') = \sum_{L'_1, L'_2 \in \mathscr{L}^{H'}(M')} d^{H'}_{M'}(L'_1, L'_2) J^{L'_1}_{M'}(Y, f_{\overline{Q'_1}}) J^{L'_2}_{M'}(Y, f'_{Q'_2}),$$

where $(Q'_1, Q'_2) := s(L'_1, L'_2)$ (see Section 2.7).

Proof. It is almost the same as the proof of Proposition 5.1, except that one needs to use Proposition 4.4.4) and Corollary 4.6 to show 6) and 5). □

For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define

$$J^{H'}(f, f') := \sum_{M' \in \mathscr{L}^{H'}(M'_0)} |W_0^{H'}||W_0^{M'}|^{-1}(-1)^{\dim(A_{M'}/A_{H'})} \sum_{\mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')} |W(M', \mathfrak{c}')|^{-1} \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J^{H'}_{M'}(Y, f, f') dY. \tag{5.2.2}$$

This expression is absolutely convergent by Proposition 5.10.5) and Corollary 3.6. One may extend in the obvious way the definitions (5.2.1) and (5.2.2) to the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M'_0)$.

Remark 5.11. We have the equality

$$J^{H'}(f, f') = J^{H'}(f', f).$$

It results from the fact that for all $M' \in \mathscr{L}^{H'}(M'_0)$ and all $x, y \in H'(F)$, we have $v_{M'}(x,y) = v_{M'}(y,x)$.

Theorem 5.12 (Noninvariant trace formula). *For all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we have the equality*

$$J^{H'}(f, \hat{f}') = J^{H'}(\hat{f}, f').$$

Proof. We may simply copy the proof of Theorem 5.3 here with obvious modifications. Especially, one needs to use Proposition 3.20 to show analogues of Lemmas 5.4 and 5.6 for the case of (G', H'). □

6. Howe's finiteness for weighted orbital integrals

6.1. The case of (G,H). For an open compact subgroup r of $\mathfrak{s}(F)$, denote by $\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ the subspace of $\mathcal{C}_c^\infty(\mathfrak{s}(F))$ consisting of the functions invariant by translation of r. Let $i_r^{\mathfrak{s}*}:\mathcal{C}_c^\infty(\mathfrak{s}(F))^*\to\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)^*$ be the dual map of the natural injection $\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)\hookrightarrow\mathcal{C}_c^\infty(\mathfrak{s}(F))$.

For any set σ, denote by $\mathbb{C}[\sigma]$ the $\mathbb{C}$-linear space of maps from σ to $\mathbb{C}$ with finite support. For $M\in\mathscr{L}^{G,\omega}(M_0)$ and $\sigma\subseteq(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$, we define the linear map

$$\delta_M^G:\mathbb{C}[\sigma]\to\mathcal{C}_c^\infty(\mathfrak{s}(F))^*,$$
$$(z_X)_{X\in\sigma}\mapsto\sum_{X\in\sigma}z_X J_M^G(\eta,X,\cdot),$$

where $z_X\in\mathbb{C}$ is the coordinate at $X\in\sigma$.

Once again, one may extend in the obvious way the definitions of $i_r^{\mathfrak{s}*}$ and δ_M^G to the symmetric pair $(M,M_H,\mathrm{Ad}(\epsilon))$, where $M\in\mathscr{L}^{G,\omega}(M_0)$, since it appears as the product of some copies of the form $(G,H,\mathrm{Ad}(\epsilon))$ in lower dimensions.

Proposition 6.1 (Howe's finiteness). *Let r be an open compact subgroup of $\mathfrak{s}(F)$, $M\in\mathscr{L}^{G,\omega}(M_0)$ and $\sigma\subseteq(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$. Suppose that there exists a compact subset $\sigma_0\subseteq(\mathfrak{m}\cap\mathfrak{s})(F)$ such that $\sigma\subseteq\mathrm{Ad}(M_H(F))(\sigma_0)$. Then the image of the linear map*

$$i_r^{\mathfrak{s}*}\circ\delta_M^G:\mathbb{C}[\sigma]\to\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)^*$$

is of finite dimension.

Remark 6.2. For $M=G$, Proposition 6.1 is essentially included in a more general result [45, Theorem 6.1] in the context of θ-groups (in the sense of [49, p. 467]).

The rest of this section is devoted to the proof of Proposition 6.1. We shall follow the main steps in [51, §IV.2-6]. We may and shall suppose that σ is relatively compact in $(\mathfrak{m}\cap\mathfrak{s})(F)$. The proposition will be proved by induction on the dimension of G.

Recall that we have chosen the standard maximal compact subgroup $K=GL_{2n}(\mathcal{O}_D)$ of $G(F)=GL_{2n}(D)$. Let $k=\mathfrak{g}(\mathcal{O}_F):=\mathfrak{gl}_{2n}(\mathcal{O}_D)$, which is an $\mathcal{O}_F$-lattice in $\mathfrak{g}(F)=\mathfrak{gl}_{2n}(D)$ and is stable under the adjoint action of K. Since $H\in\mathscr{L}^G(M_0)$, we have set $K_H=K\cap H(F)=GL_n(\mathcal{O}_D)\times GL_n(\mathcal{O}_D)$. Let $\mathfrak{h}(\mathcal{O}_F):=k\cap\mathfrak{h}(F)$ and $\mathfrak{s}(\mathcal{O}_F):=k\cap\mathfrak{s}(F)$. Then we see that $k=\mathfrak{h}(\mathcal{O}_F)\oplus\mathfrak{s}(\mathcal{O}_F)$ and that $\mathfrak{s}(\mathcal{O}_F)$ is stable under the adjoint action of K_H. For all $P\in\mathscr{F}^G(M_0)$, we fix $a_P\in A_P(F)$ such that $|\alpha(a_P)|_F<1,\forall\alpha\in\Delta_P^G$.

Recall that we denote by $\mathcal{N}^{\mathfrak{s}}$ the set of nilpotent elements in $\mathfrak{s}(F)$ and fix a uniformiser ϖ of $\mathcal{O}_F$. Let $X\in\mathcal{N}^{\mathfrak{s}}\cap(k-\varpi k)$. By the Jacobson-Morozov theorem for symmetric spaces (Lemma 3.7), there exists a group homomorphism $\varphi:SL_2(F)\to G(F)$ such that

$$d\varphi\begin{pmatrix}0&1\\0&0\end{pmatrix}=X\text{ and }a^\varphi:=\varphi\begin{pmatrix}\varpi&\\&\varpi^{-1}\end{pmatrix}\in H(F).$$

We define the parabolic subgroup P^X of G as in [51, §IV.3]. More concretely, set

$$\mathfrak{g}[i]:=\{Y\in\mathfrak{g}:\mathrm{Ad}(a^\varphi)(Y)=\varpi^iY,\forall i\in\mathbb{Z}\}$$

and

$$\mathfrak{p}^X:=\bigoplus_{i\geq0}\mathfrak{g}[i];$$

then let

$$P^X:=\{x\in G:\mathrm{Ad}(x)(\mathfrak{p}^X)=\mathfrak{p}^X\}. \tag{6.1.1}$$

Note that P^X is independent of the choice of φ by [11, Proposition 5.7.1]. Since a^φ commutes with ϵ, one has $\epsilon\in P^X$. By Lemma 3.9, $P^X\cap H$ is a parabolic subgroup of H. Then there exists an element $x\in K_H$ such that $P':=\mathrm{Ad}(x)(P^X)\in\mathscr{F}^G(M_0)$. We shall fix such an x. Let $a^X:=\mathrm{Ad}(x^{-1})(a_{P'})\in H(F)$. Note that a^X depends on the choice of x, but this is unimportant. By [51, (3) in §IV.3], we have

$$\mathrm{Ad}(a^X)(X)\in(\varpi k)\cap\mathfrak{s}(F)=\varpi\mathfrak{s}(\mathcal{O}_F). \tag{6.1.2}$$

Lemma 6.3. *There exists an integer $h\in\mathbb{N}$ such that for all $Y\in\varpi^h\mathfrak{s}(\mathcal{O}_F)$, all integer $l\geq h$ and all $Z\in\mathrm{Ad}(a^X)^{-1}(\varpi^l\mathfrak{s}(\mathcal{O}_F))$, there exists $\gamma\in K_H$ with $\eta(\mathrm{Nrd}(\gamma))=1$ such that*

$$\mathrm{Ad}(\gamma)(X+Y+Z)\in X+Y+\varpi^l\mathfrak{s}(\mathcal{O}_F).$$

Proof. We may choose h following the proof of [51, Lemme IV.3]. The point is that we require $\gamma \in H(F)$ and $\eta(\mathrm{Nrd}(\gamma)) = 1$ here.

Set

$$\mathfrak{n}^X := \bigoplus_{i \geq 1} \mathfrak{g}[i].$$

Since a^φ commutes with ϵ, one has $\mathfrak{g}[i] = (\mathfrak{g}[i] \cap \mathfrak{h}) \oplus (\mathfrak{g}[i] \cap \mathfrak{s})$.

By the intersection of $\mathfrak{s}(\mathcal{O}_F)$ with [51, (4) in §IV.3], we have

$$\mathrm{Ad}(a^X)^{-1}(\mathfrak{s}(\mathcal{O}_F)) \subseteq \mathfrak{s}(\mathcal{O}_F) + \mathrm{Ad}(a^X)^{-1}(\mathfrak{s}(\mathcal{O}_F)) \cap \mathfrak{n}^X. \tag{6.1.3}$$

Let $c, c', c'', h \in \mathbb{N}$ be as in the proof of [51, Lemme IV.3], except that we require additionally $\exp(\varpi^l k) \subseteq \{x \in K : \eta(\mathrm{Nrd}(x)) = 1\}, \forall l \geq c'$. Then

$$\mathfrak{n}^X \cap \mathrm{Ad}(a^X)^{-1}(\mathfrak{s}(\mathcal{O}_F)) \subseteq \mathrm{ad}(X)(\varpi^{-c}\mathfrak{h}(\mathcal{O}_F)). \tag{6.1.4}$$

Let Y, l and Z be as in the statement. Thanks to (6.1.3), we can write $Z = Z_1 + Z_2$, where $Z_1 \in \varpi^l \mathfrak{s}(\mathcal{O}_F)$ and $Z_2 \in \mathrm{Ad}(a^X)^{-1}(\varpi^l \mathfrak{s}(\mathcal{O}_F)) \cap \mathfrak{n}^X$. Because of (6.1.4), we can choose $Z' \in \varpi^{l-c}\mathfrak{h}(\mathcal{O}_F)$ such that $Z_2 = \mathrm{ad}(X)(Z')$. Since $Z' \in \varpi^{c'}\mathfrak{s}(\mathcal{O}_F)$ from the hypothesis $l \geq h \geq c + c'$, we can define $\gamma := \exp(Z')$ as in the proof of [51, Lemme IV.3]. Then we have $\gamma \in K_H$ and $\eta(\mathrm{Nrd}(\gamma)) = 1$. Such a γ verifies [51, Lemme IV.3] and thus our statement. □

For $X \in \mathcal{N}^{\mathfrak{s}} \cap (k - \varpi k)$, we shall fix an integer h^X such that

(1) h^X verifies the condition of Lemma 6.3;

(2) $h^X \geq 1$;

(3) $\mathrm{Ad}(a^X)(\varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)) \subseteq \varpi\mathfrak{s}(\mathcal{O}_F)$.

Denote by $\mathcal{N}^G$ the set of nilpotent elements in $\mathfrak{g}(F)$. Let $\mathbb{P}\mathfrak{g}(F)$ (resp. $\mathbb{P}\mathfrak{s}(F)$) be the projective space associated to $\mathfrak{g}(F)$ (resp. $\mathfrak{s}(F)$) and $\pi : \mathfrak{g}(F) - \{0\} \to \mathbb{P}\mathfrak{g}(F)$ the natural projection. Since $\pi(\mathcal{N}^G - \{0\})$ is compact and $\mathcal{N}^{\mathfrak{s}} - \{0\}$ is a closed subset of $\mathcal{N}^G - \{0\}$, we know that $\pi(\mathcal{N}^{\mathfrak{s}} - \{0\})$ is compact. One also sees that

$$\pi(\mathcal{N}^{\mathfrak{s}} \cap (k - \varpi k)) = \pi(\mathcal{N}^{\mathfrak{s}} - \{0\}).$$

We can and shall choose a finite set $\mathcal{N}_0 \subseteq \mathcal{N}^{\mathfrak{s}} \cap (k - \varpi k)$ such that

$$\bigcup_{X \in \mathcal{N}_0} \pi(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F))$$

is an open neighbourhood of $\pi(\mathcal{N}^{\mathfrak{s}} - \{0\}) \subseteq \mathbb{P}\mathfrak{s}(F)$.

Lemma 6.4. *There exists an integer $c \in \mathbb{N}$ such that for all $d \in \delta_M^G(\mathbb{C}[\sigma])$ and all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ satisfying $d(f) \neq 0$, we have*

$$\mathrm{Supp}(f) \cap [\varpi^{-c}\mathfrak{s}(\mathcal{O}_F) \cup \bigcup_{X \in \mathcal{N}_0} F^\times(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F))] \neq \emptyset.$$

Proof. Recall that σ is assumed to be relatively compact in $(\mathfrak{m} \cap \mathfrak{s})(F)$. Fix an open compact neighbourhood σ' of σ in $\mathfrak{s}(F)$. Fix $P_0 \in \mathscr{P}^H(M_0)$ and set

$$A_{P_0}^+ := \{a \in A_{M_0}(F) : |\alpha(a)|_F \geq 1, \forall \alpha \in \Delta_{P_0}^H\}.$$

Similarly, for all $B \in \mathscr{P}^G(M_0)$, set

$$A_B^+ := \{a \in A_{M_0}(F) : |\alpha(a)|_F \geq 1, \forall \alpha \in \Delta_B^G\}.$$

We see from the argument of Lemma 4.8 in Chapter 2 that

$$A_{P_0}^+ \subseteq \bigcup_{\{B \in \mathscr{P}^G(M_0) : P_0 \subseteq B\}} A_B^+.$$

By the Cartan decomposition, there exists a compact subset $\Gamma \subseteq H(F)$ such that $H(F) = K_H A_{P_0}^+ \Gamma$. Fix such a Γ. Then

$$H(F) \subseteq \bigcup_{\{B \in \mathscr{P}^G(M_0) : P_0 \subseteq B\}} K_H A_B^+ \Gamma.$$

Fix $c' \in \mathbb{N}$ such that $\mathrm{Ad}(\Gamma)(\sigma') \subseteq \varpi^{-c'}\mathfrak{s}(\mathcal{O}_F)$. Since $\mathrm{Ad}(A_B^+)(\mathfrak{s}(\mathcal{O}_F)) \subseteq \mathfrak{s}(\mathcal{O}_F) + (\mathfrak{n}_B \cap \mathfrak{s})(F)$ and $(\mathfrak{n}_B \cap \mathfrak{s})(F) \subseteq \mathcal{N}^{\mathfrak{s}}$, we obtain

$$\mathrm{Ad}(H(F))(\sigma') \subseteq \varpi^{-c'}\mathfrak{s}(\mathcal{O}_F) + \mathcal{N}^{\mathfrak{s}}.$$

Choose an integer c such that

$$c \geq c' + \sup_{X \in \mathcal{N}_0} h^X.$$

Let d and f be as in the statement. It is evident that

$$\mathrm{Supp}(f) \cap \mathrm{Ad}(H(F))(\sigma') \neq \emptyset.$$

Suppose that $X_1 \in \varpi^{-c'}\mathfrak{s}(\mathcal{O}_F), X_2 \in \mathcal{N}^{\mathfrak{s}}$ satisfy $X_1 + X_2 \in \mathrm{Supp}(f)$. If $X_2 \in \varpi^{-c}\mathfrak{s}(\mathcal{O}_F)$, we are done. If not, let $X \in \mathcal{N}_0$ be such that $\pi(X_2) \in \pi(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F))$. Then there exists $\lambda \in F^\times$ such that $X_2 \in \lambda(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F))$. Since $X_2 \notin \varpi^{-c}\mathfrak{s}(\mathcal{O}_F)$, we have $v_F(\lambda) < -c$ and thus $v_F(\lambda) + h^X < -c'$. Therefore, $X_1 \in \lambda\varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)$ and then $X_1 + X_2 \in \lambda(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)) \subseteq F^\times(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F))$. This is exactly what we want to prove. □

We shall fix an integer c verifying the condition in Lemma 6.4. We shall also fix another integer h such that

$$h \geq \sup_{X \in \mathcal{N}_0} h^X. \tag{6.1.5}$$

Denote

$$C := \{f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)/\varpi^h\mathfrak{s}(\mathcal{O}_F)) : \mathrm{Supp}(f) \subseteq \varpi^{-c}\mathfrak{s}(\mathcal{O}_F)\},$$

which is a $\mathbb{C}$-linear space of finite dimension. For all $L \in \mathscr{L}^G(M), L \neq G$, let $r(L) := \varpi^h(\mathfrak{l} \cap \mathfrak{s})(\mathcal{O}_F)$.

Lemma 6.5. *Let $\mathfrak{z} = (z_X)_{X \in \sigma} \in \mathbb{C}[\sigma]$. Suppose that*
1) ${i_{r(L)}^{\mathfrak{l}\cap\mathfrak{s}}}^* \circ \delta_M^L(\mathfrak{z}) = 0$ for all $L \in \mathscr{L}^G(M), L \neq G$;
2) $\delta_M^G(\mathfrak{z})(C) = 0$.
Then ${i_{\varpi^h\mathfrak{s}(\mathcal{O}_F)}^{\mathfrak{s}}}^* \circ \delta_M^G(\mathfrak{z}) = 0$.

Proof. Write $d := \delta_M^G(\mathfrak{z})$. It suffices to prove by induction on the integer $e \geq c$ the assertion $(A)_e$: for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)/\varpi^h\mathfrak{s}(\mathcal{O}_F))$ with $\mathrm{Supp}(f) \subseteq \varpi^{-e}\mathfrak{s}(\mathcal{O}_F)$, we have $d(f) = 0$.

If $e = c$, this is true by the hypothesis 2). Fix an $e > c$ and suppose that $(A)_{e-1}$ is true. For all open compact subset $s \subseteq \mathfrak{g}(F)$, denote by 1_s its characteristic function. It suffices to prove that for all $Y \in \varpi^{-e}\mathfrak{s}(\mathcal{O}_F) - \varpi^{-e+1}\mathfrak{s}(\mathcal{O}_F)$, we have $d(1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) = 0$.

Suppose that $Y \in \varpi^{-e}\mathfrak{s}(\mathcal{O}_F) - \varpi^{-e+1}\mathfrak{s}(\mathcal{O}_F)$. The hypothesis $e > c$ implies $\varpi^{-c}\mathfrak{s}(\mathcal{O}_F) \subseteq \varpi^{-e+1}\mathfrak{s}(\mathcal{O}_F)$ and thus $Y \notin \varpi^{-c}\mathfrak{s}(\mathcal{O}_F)$. Suppose on the contary that $d(1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) \neq 0$. By Lemma 6.4, there exists $X \in \mathcal{N}_0$ and $\lambda \in F^\times$ such that $(Y + \varpi^h\mathfrak{s}(\mathcal{O}_F)) \cap \lambda(X + \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)) \neq \emptyset$. Fix such X and λ. Since $v_F(Y) = -e$ and $v_F(X) = 0$, we have $v_F(\lambda) = -e$. As $h \geq h^X \geq h^X - e$ (see (6.1.5)), we obtain $\varpi^h\mathfrak{s}(\mathcal{O}_F) \subseteq \lambda\varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)$ and then $Y \in \lambda(X+\varpi^{h^X}\mathfrak{s}(\mathcal{O}_F))$. Let $Y' \in \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)$ such that $Y = \lambda(X+Y')$.

Let $Z \in \mathrm{Ad}(a^X)^{-1}(\varpi^h\mathfrak{s}(\mathcal{O}_F))$. Since $h + e \geq h^X$, we can apply Lemma 6.3 to $X, Y', \lambda^{-1}Z$ and $l := h + e$. Then there exists $\gamma \in K_H$ with $\eta(\mathrm{Nrd}(\gamma)) = 1$ such that

$$\mathrm{Ad}(\gamma)(X + Y' + \lambda^{-1}Z) \in X + Y' + \varpi^{h+e}\mathfrak{s}(\mathcal{O}_F).$$

From $v_F(\lambda) = -e$, we deduce that

$$\mathrm{Ad}(\gamma)(\lambda(X + Y') + Z) \in \lambda(X + Y') + \varpi^h\mathfrak{s}(\mathcal{O}_F),$$

i.e.,

$$\mathrm{Ad}(\gamma)(Y + Z) \in Y + \varpi^h\mathfrak{s}(\mathcal{O}_F).$$

Since $\gamma \in K_H$, this is equivalent to

$$\mathrm{Ad}(\gamma)(Y + Z + \varpi^h\mathfrak{s}(\mathcal{O}_F)) = Y + \varpi^h\mathfrak{s}(\mathcal{O}_F)$$

or

$$\mathrm{Ad}(\gamma)(1_{Y+Z+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) = 1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}.$$

By Proposition 4.1.3), we obtain

$$d(1_{Y+Z+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) = \eta(\mathrm{Nrd}(\gamma))d(1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}).$$

Because $\eta(\mathrm{Nrd}(\gamma)) = 1$, we have

$$d(1_{Y+Z+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) = d(1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}).$$

Now, by the sum over $Z \in \left(\mathrm{Ad}(a^X)^{-1}(\varpi^h\mathfrak{s}(\mathcal{O}_F)) + \varpi^h\mathfrak{s}(\mathcal{O}_F)\right)/\varpi^h\mathfrak{s}(\mathcal{O}_F)$ (a finite set), we get

$$d(1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) = [k' : \mathfrak{s}(\mathcal{O}_F)]^{-1}d(1_{Y+\varpi^h k'}), \tag{6.1.6}$$

where

$$k' := \mathrm{Ad}(a^X)^{-1}(\mathfrak{s}(\mathcal{O}_F)) + \mathfrak{s}(\mathcal{O}_F). \tag{6.1.7}$$

By Proposition 4.1.6), we have the equality

$$d\left(\mathrm{Ad}(a^X)(1_{Y+\varpi^h k'}) - \eta(\mathrm{Nrd}(a^X))1_{Y+\varpi^h k'}\right) = \eta(\mathrm{Nrd}(a^X)) \sum_{Q\in\mathscr{F}^G(M), Q\neq G} \delta_M^{M_Q}(\mathfrak{z})\left((1_{Y+\varpi^h k'})^{\eta}_{Q,(a^X)^{-1}}\right).$$

From (4.1.3), it is clear that $(1_{Y+\varpi^h k'})^{\eta}_{Q,(a^X)^{-1}}$ is invariant by translation of $r(M_Q)$. By the hypothesis 1), we have $\delta_M^{M_Q}(\mathfrak{z})\left((1_{Y+\varpi^h k'})^{\eta}_{Q,(a^X)^{-1}}\right) = 0$ for all $Q \in \mathscr{F}^G(M), Q \neq G$, so

$$\eta(\mathrm{Nrd}(a^X))d(1_{Y+\varpi^h k'}) = d\left(\mathrm{Ad}(a^X)(1_{Y+\varpi^h k'})\right). \tag{6.1.8}$$

We see easily that

$$\mathrm{Ad}(a^X)(1_{Y+\varpi^h k'}) = 1_s,$$

where

$$s := \mathrm{Ad}(a^X)(Y) + \varpi^h \mathrm{Ad}(a^X)(k'). \tag{6.1.9}$$

Recall $Y = \lambda(X + Y')$ above. As $X \in \mathcal{N}_0$, by (6.1.2), we have $\mathrm{Ad}(a^X)(X) \in \varpi\mathfrak{s}(\mathcal{O}_F)$. Since $Y' \in \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F)$, by the hypothesis (3) on h^X, we have $\mathrm{Ad}(a^X)(Y') \in \varpi\mathfrak{s}(\mathcal{O}_F)$. For $v_F(\lambda) = -e$, we obtain

$$\mathrm{Ad}(a^X)(Y) \in \varpi^{-e+1}\mathfrak{s}(\mathcal{O}_F). \tag{6.1.10}$$

We see from (6.1.7) that

$$\mathrm{Ad}(a^X)(k') = \mathfrak{s}(\mathcal{O}_F) + \mathrm{Ad}(a^X)(\mathfrak{s}(\mathcal{O}_F)).$$

By (6.1.5) and the hypothesis (3) on h^X, we have $\varpi^h\mathrm{Ad}(a^X)(\mathfrak{s}(\mathcal{O}_F)) \subseteq \varpi^{h^X}\mathrm{Ad}(a^X)(\mathfrak{s}(\mathcal{O}_F)) \subseteq \varpi\mathfrak{s}(\mathcal{O}_F)$. Then by (6.1.5) and the hypothesis (2) on h^X, we have

$$\varpi^h\mathrm{Ad}(a^X)(k') = \varpi^h\mathfrak{s}(\mathcal{O}_F) + \varpi^h\mathrm{Ad}(a^X)(\mathfrak{s}(\mathcal{O}_F)) \subseteq \varpi^{h^X}\mathfrak{s}(\mathcal{O}_F) + \varpi\mathfrak{s}(\mathcal{O}_F) = \varpi\mathfrak{s}(\mathcal{O}_F). \tag{6.1.11}$$

From (6.1.9), (6.1.10) and (6.1.11), we see that

$$\mathrm{Supp}(1_s) \subseteq \varpi^{-e+1}\mathfrak{s}(\mathcal{O}_F).$$

Since $\mathfrak{s}(\mathcal{O}_F) \subseteq \mathrm{Ad}(a^X)(k')$, we know that 1_s is invariant by translation of $\varpi^h\mathfrak{s}(\mathcal{O}_F)$. Using the induction hypothesis $(A)_{e-1}$, we have

$$d(1_s) = 0.$$

Thanks to (6.1.6) and (6.1.8), we obtain

$$d(1_{Y+\varpi^h\mathfrak{s}(\mathcal{O}_F)}) = 0.$$

This proves $(A)_e$ and thus the lemma. □

Proof of Proposition 6.1. We use induction on the dimension of G. Suppose that for all $L \in \mathscr{L}^G(M), L \neq G$, and all open compact subgroup r_L of $(\mathfrak{l}\cap\mathfrak{s})(F)$, the image of the linear map

$$i_{r_L}^{\mathfrak{l}\cap\mathfrak{s}\,*} \circ \delta_M^L : \mathbb{C}[\sigma] \to \mathcal{C}_c^\infty((\mathfrak{l}\cap\mathfrak{s})(F)/r_L)^*$$

is of finite dimension. This is actually a product form of the proposition in lower dimensions. Now we would like to prove the proposition. The argument below is also valid for the case $G = M$.

Enlarge h in (6.1.5) if necessary such that $r \supseteq \varpi^h\mathfrak{s}(\mathcal{O}_F)$. We shall prove that the image of $i^{\mathfrak{s}}_{\varpi^h\mathfrak{s}(\mathcal{O}_F)}{}^* \circ \delta_M^G$ is of finite dimension. Admit this for the moment. Since $i_r^{\mathfrak{s}\,*}$ factorises by $i^{\mathfrak{s}}_{\varpi^h\mathfrak{s}(\mathcal{O}_F)}{}^*$, the image of $i_r^{\mathfrak{s}\,*} \circ \delta_M^G$ is also of finite dimension. Then we finish the proof.

Let $\mathcal{K}_1$ be the kernel of the linear map

$$\bigoplus_{L\in\mathscr{L}^G(M), L\neq G} i_{r(L)}^{\mathfrak{l}\cap\mathfrak{s}}{}^* \circ \delta_M^L : \mathbb{C}[\sigma] \to \bigoplus_{L\in\mathscr{L}^G(M), L\neq G} \mathcal{C}_c^\infty((\mathfrak{l}\cap\mathfrak{s})(F)/r(L))^*,$$

whose image is of finite dimension by our induction hypothesis applied to $r_L := r(L)$ for all $L \in \mathscr{L}^G(M), L \neq G$. Hence, to prove that $i^{\mathfrak{s}}_{\varpi^h\mathfrak{s}(\mathcal{O}_F)}{}^* \circ \delta_M^G(\mathbb{C}[\sigma])$ is of finite dimension, it suffices to prove that $i^{\mathfrak{s}}_{\varpi^h\mathfrak{s}(\mathcal{O}_F)}{}^* \circ \delta_M^G(\mathcal{K}_1)$ is of finite dimension.

Consider the composition of the linear maps

$$d_1 := i^{\mathfrak{s}}_{\varpi^h\mathfrak{s}(\mathcal{O}_F)}{}^* \circ \delta_M^G\big|_{\mathcal{K}_1} : \mathcal{K}_1 \to \mathcal{C}_c^\infty(\mathfrak{s}(F)/\varpi^h\mathfrak{s}(\mathcal{O}_F))^*$$

and

$$\mathrm{Res}:\mathcal{C}_c^\infty(\mathfrak{s}(F)/\varpi^h\mathfrak{s}(\mathcal{O}_F))^*\to C^*.$$

The latter map is the natural restriction. Lemma 6.5 says that

$$\ker(\mathrm{Res}\circ d_1)=\ker(d_1),$$

which is denoted by $\mathcal{K}_2$. Then

$$d_1(\mathcal{K}_1)\simeq\mathcal{K}_1/\mathcal{K}_2\simeq\mathrm{Res}\circ d_1(\mathcal{K}_1)\subseteq C^*.$$

Since C is of finite dimension, we see that $d_1(\mathcal{K}_1)$ is of finite dimension. □

Corollary 6.6. *Let r be an open compact subgroup of $\mathfrak{s}(F)$, $M\in\mathscr{L}^{G,\omega}(M_0)$ and $\sigma\subseteq(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$. Suppose that there exists a compact subset $\sigma_0\subseteq(\mathfrak{m}\cap\mathfrak{s})(F)$ such that $\sigma\subseteq\mathrm{Ad}(M_H(F))(\sigma_0)$. Then there exists a finite subset $\{X_i:i\in I\}\subseteq\sigma$ and a finite subset $\{f_i:i\in I\}\subseteq\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ such that for all $X\in\sigma$ and all $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$, we have the equality*

$$J_M^G(\eta,X,f)=\sum_{i\in I}J_M^G(\eta,X_i,f)J_M^G(\eta,X,f_i).$$

Proof. By Proposition 6.1, there exists a finite subset $\{X_i:i\in I\}\subseteq\sigma$ such that $\{i_r^{\mathfrak{s}*}\circ\delta_M^G(X_i):i\in I\}$ is a basis of $i_r^{\mathfrak{s}*}\circ\delta_M^G(\mathbb{C}[\sigma])$. By linear algebra, there exists a finite subset $\{f_i:i\in I\}\subseteq\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)$ such that $i_r^{\mathfrak{s}*}\circ\delta_M^G(X_i)(f_j)=\delta_{ij},\forall i,j\in I$, where δ_{ij} denotes the Kronecker delta function. Choose such $\{X_i:i\in I\}$ and $\{f_i:i\in I\}$.

Then, for all $X\in\sigma$, there exists $\lambda_i,\forall i\in I$ such that

$$J_M^G(\eta,X,\cdot)=\sum_{i\in I}\lambda_iJ_M^G(\eta,X_i,\cdot)\in\mathcal{C}_c^\infty(\mathfrak{s}(F)/r)^*.$$

Hence, for all $i\in I$,

$$J_M^G(\eta,X,f_i)=\sum_{j\in I}\lambda_jJ_M^G(\eta,X_j,f_i)=\sum_{j\in I}\lambda_j\delta_{ji}=\lambda_i.$$

We have finished the proof. □

6.2. The case of (G',H'). For an open compact subgroup r' of $\mathfrak{s}'(F)$, denote by $\mathcal{C}_c^\infty(\mathfrak{s}'(F)/r')$ the subspace of $\mathcal{C}_c^\infty(\mathfrak{s}'(F))$ consisting of the functions invariant by translation of r'. Let $i_{r'}^{\mathfrak{s}'*}:\mathcal{C}_c^\infty(\mathfrak{s}'(F))^*\to\mathcal{C}_c^\infty(\mathfrak{s}'(F)/r')^*$ be the dual map of the natural injection $\mathcal{C}_c^\infty(\mathfrak{s}'(F)/r')\hookrightarrow\mathcal{C}_c^\infty(\mathfrak{s}'(F))$.

For any set σ', denote by $\mathbb{C}[\sigma']$ the $\mathbb{C}$-linear space with a basis σ'. For $M'\in\mathscr{L}^{H'}(M_0')$ and $\sigma'\subseteq(\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$, we define the linear map

$$\begin{aligned}\delta_{M'}^{H'}:\mathbb{C}[\sigma']&\to\mathcal{C}_c^\infty(\mathfrak{s}'(F))^*,\\(z_Y)_{Y\in\sigma'}&\mapsto\sum_{Y\in\sigma'}z_YJ_{M'}^{H'}(Y,\cdot),\end{aligned}$$

where $z_Y\in\mathbb{C}$ is the coordinate at $Y\in\sigma'$.

One may extend in the obvious way the definitions of $i_{r'}^{\mathfrak{s}'*}$ and $\delta_{M'}^{H'}$ to the symmetric pair $(\widetilde{M'},M',\mathrm{Ad}(\alpha))$, where $M'\in\mathscr{L}^{H'}(M_0')$.

Proposition 6.7 (Howe's finiteness). *Let r' be an open compact subgroup of $\mathfrak{s}'(F)$, $M'\in\mathscr{L}^{H'}(M_0')$ and $\sigma'\subseteq(\widetilde{\mathfrak{m}'}\cap\mathfrak{s}_{\mathrm{rs}}')(F)$. Suppose that there exists a compact subset $\sigma_0'\subseteq(\widetilde{\mathfrak{m}'}\cap\mathfrak{s}')(F)$ such that $\sigma'\subseteq\mathrm{Ad}(M'(F))(\sigma_0')$. Then the image of the linear map*

$$i_{r'}^{\mathfrak{s}'*}\circ\delta_{M'}^{H'}:\mathbb{C}[\sigma']\to\mathcal{C}_c^\infty(\mathfrak{s}'(F)/r')^*$$

is of finite dimension.

The rest of this section is devoted to the proof of Proposition 6.7. It is similar to the proof of Proposition 6.1 and we only point out some additional argument. Recall that we have chosen the standard maximal compact subgroup $K_{H'}=GL_n(\mathcal{O}_{D'})$ of $H'(F)=GL_n(D')$ in **Case I** (resp. $K_{H'}=GL_{\frac{n}{2}}(\mathcal{O}_{D\otimes_FE})$ of $H'(F)=GL_{\frac{n}{2}}(D\otimes_FE)$ in **Case II**). Snice $\mathrm{Ad}(\tau)(D')=D'$ in **Case I** (resp. $\mathrm{Ad}(\tau)(D\otimes_FE)=D\otimes_FE$ in **Case II**), we deduce that $\mathrm{Ad}(\tau)(\mathcal{O}_{D'})=\mathcal{O}_{D'}$ in **Case I** (resp. $\mathrm{Ad}(\tau)(\mathcal{O}_{D\otimes_FE})=\mathcal{O}_{D\otimes_FE}$ in **Case II**). Thus $\mathrm{Ad}(\tau)(K_{H'})=K_{H'}$. Let $\mathfrak{h}'(\mathcal{O}_F):=\mathfrak{gl}_n(\mathcal{O}_{D'})$ in **Case I** (resp. $\mathfrak{h}'(\mathcal{O}_F):=\mathfrak{gl}_{\frac{n}{2}}(\mathcal{O}_{D\otimes_FE})$ in **Case II**). Let $\mathfrak{s}'(\mathcal{O}_F):=\mathfrak{h}'(\mathcal{O}_F)\tau=\tau\mathfrak{h}'(\mathcal{O}_F)$ be an $\mathcal{O}_F$-lattice in $\mathfrak{s}'(F)$ (see Section 3.3 for the choice of

τ). Let $k' := \mathfrak{h}'(\mathcal{O}_F) \oplus \mathfrak{s}'(\mathcal{O}_F)$, whose decomposition is stable under the adjoint action of $K_{H'}$ because $\mathrm{Ad}(\tau)(K_{H'}) = K_{H'}$. For all $P' \in \mathscr{F}^{H'}(M'_0)$, we fix $a_{P'} \in A_{P'}(F)$ such that $|\alpha(a_{P'})|_F < 1, \forall \alpha \in \Delta^{H'}_{P'}$.

Starting from $X \in \mathcal{N}^{\mathfrak{s}'} \cap (\mathfrak{s}'(\mathcal{O}_F) - \varpi\mathfrak{s}'(\mathcal{O}_F))$, we obtain a group homomorphism $\varphi : SL_2(F) \to G'(F)$ by the Jacobson-Morozov theorem for symmetric spaces (Lemma 3.7). Denote $a^\varphi := \varphi \begin{pmatrix} \varpi & \\ & \varpi^{-1} \end{pmatrix} \in H'(F)$. Define P^X as in (6.1.1), which contains the centraliser $\mathrm{Cent}_{G'}(a^\varphi)$ of a^φ in G' as a Levi factor. Since a^φ commutes with α, by Lemma 3.16, $P^X \cap H'$ is a parabolic subgroup of H', which contains the centraliser $\mathrm{Cent}_{H'}(a^\varphi)$ of a^φ in H' as a Levi factor. We want to show that there exists $x \in K_{H'}$ such that $\mathrm{Ad}(x)(P^X \cap H') \in \mathscr{F}^{H'}(M'_0)$ and that $\mathrm{Ad}(x)(P^X) \in \mathscr{F}^{G'}(M'_0)$.

Lemma 6.8. *For $Y \in \mathcal{N}^{\mathfrak{s}'}$, there exists $x \in H'(F)$ such that $\mathrm{Ad}(x)(Y)$ is in the Jordan normal form, i.e., diagonal block matrices with entries in $D'\tau$ in **Case I** (resp. $(D \otimes_F E)\tau$ in **Case II**) whose blocks are of the form*

$$\begin{pmatrix} 0 & \tau & & \\ & \ddots & \ddots & \\ & & \ddots & \tau \\ & & & 0 \end{pmatrix}.$$

Proof. It can be proved in the same way as [24, Lemmas 2.2 and 2.3] by linear algebra over a division ring. □

Thanks to Lemma 6.8, we can construct explicitly the above morphism φ (see [19, p. 184]). If X is in the Jordan normal form, by *loc. cit.*, we may choose $a^\varphi \in A_{L'}(F)$ for some $L' \in \mathscr{L}^{H'}(M'_0)$ such that $\mathrm{Cent}_{H'}(a^\varphi) = L'$ and that $\mathrm{Cent}_{G'}(a^\varphi) = \widetilde{L'}$. For a general X as above, by Lemma 6.8, there exists $y \in H'(F)$ such that $\mathrm{Ad}(y)(a^\varphi) \in A_{L'}(F)$ for some $L' \in \mathscr{L}^{H'}(M'_0)$ satisfying $\mathrm{Cent}_{H'}(\mathrm{Ad}(y)(a^\varphi)) = L'$ and $\mathrm{Cent}_{G'}(\mathrm{Ad}(y)(a^\varphi)) = \widetilde{L'}$. Let $x \in K_{H'}$ be such that $x^{-1}y \in (P^X \cap H')(F)$. Then $\mathrm{Ad}(x)(P^X \cap H') = \mathrm{Ad}(y)(P^X \cap H')$ contains L' as a Levi factor and $\mathrm{Ad}(x)(P^X) = \mathrm{Ad}(y)(P^X)$ contains $\widetilde{L'}$ as a Levi factor. Furthermore, since $\mathrm{Ad}(x)(P^X) \cap H' = \mathrm{Ad}(x)(P^X \cap H')$, we see that $\mathrm{Ad}(x)(P^X \cap H')$ and $\mathrm{Ad}(x)(P^X)$ are associated under the bijection $P' \mapsto \widetilde{P'}$ between $\mathscr{F}^{H'}(M'_0)$ and $\mathscr{F}^{G'}(M'_0)$.

Fix $x \in K_{H'}$ as above and denote $P' := \mathrm{Ad}(x)(P^X \cap H') \in \mathscr{F}^{H'}(M'_0)$. Then $\widetilde{P'} = \mathrm{Ad}(x)(P^X)$. Put $a^X := \mathrm{Ad}(x^{-1})(a_{P'}) \in H'(F)$. By the argument of [51, (3) in §IV.3], we show that

$$\mathrm{Ad}(a^X)(X) \in \varpi\mathfrak{s}'(\mathcal{O}_F). \tag{6.2.1}$$

Proof of Proposition 6.7. We may apply the argument of Proposition 6.7 with obvious modifications. Especially, one needs to use Proposition 4.4 and (6.2.1) to show an analogue of Lemma 6.5 for the case of (G', H'). Additionally, to prove an analogue of Lemma 6.3 for this case, one may resort to the argument rather than the consequence of some steps in the proof of [51, Lemme IV.3] since our definition of k' is different from $\mathfrak{g}'(\mathcal{O}_F) := \mathfrak{gl}_n(\mathcal{O}_D)$. However, there is no essential difficulty with our preparation above and we omit details here. □

Corollary 6.9 (cf. Corollary 6.6). *Let r' be an open compact subgroup of $\mathfrak{s}'(F)$, $M' \in \mathscr{L}^{H'}(M'_0)$ and $\sigma' \subseteq (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Suppose that there exists a compact subset $\sigma'_0 \subseteq (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')(F)$ such that $\sigma' \subseteq \mathrm{Ad}(M'(F))(\sigma'_0)$. Then there exists a finite subset $\{Y_i : i \in I\} \subseteq \sigma'$ and a finite subset $\{f_i : i \in I\} \subseteq \mathcal{C}^\infty_c(\mathfrak{s}'(F)/r')$ such that for all $Y \in \sigma'$ and all $f' \in \mathcal{C}^\infty_c(\mathfrak{s}'(F)/r')$, we have the equality*

$$J^{H'}_{M'}(Y, f') = \sum_{i \in I} J^{H'}_{M'}(Y_i, f') J^{H'}_{M'}(Y, f'_i).$$

7. Representability of the Fourier transform of weighted orbital integrals

7.1. The case of (G, H). Following [51, §V.6], we denote by $\mathscr{E}^{\mathfrak{s}}$ the space of functions $e : \mathfrak{s}_{\mathrm{rs}}(F) \to \mathbb{C}$ such that

(1) e is locally constant;

(2) for all open compact subset r of $\mathfrak{s}(F)$, there exists $c > 0$ and $N \in \mathbb{N}$ such that for all $X \in r \cap \mathfrak{s}_{\mathrm{rs}}(F)$, one has the inequality

$$|e(X)| \leq c \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N.$$

If $e \in \mathscr{E}^{\mathfrak{s}}$, the function $X \mapsto |D^{\mathfrak{s}}(X)|_F^{-1/2} e(X)$ is locally integrable on $\mathfrak{s}(F)$ thanks to Corollary 3.15. It defines then a distribution on $\mathfrak{s}(F)$:

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \int_{\mathfrak{s}(F)} f(X)e(X)|D^{\mathfrak{s}}(X)|_F^{-1/2} dX. \tag{7.1.1}$$

Denote by $\mathscr{D}^{\mathfrak{s}}$ the space of distributions obtained in this way. The map $\mathscr{E}^{\mathfrak{s}} \to \mathscr{D}^{\mathfrak{s}}$ defined above is an isomorphism. For $d \in \mathscr{D}^{\mathfrak{s}}$, we shall always denote by e_d its preimage in $\mathscr{E}^{\mathfrak{s}}$.

Notice that the notion $\mathscr{E}^{\mathfrak{s}}$ can be defined for any symmetric pair, and that the definition $\mathscr{D}^{\mathfrak{s}}$ can at least be extended to the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$, since it appears as the product of some copies of the form $(G, H, \mathrm{Ad}(\epsilon))$ in lower dimensions.

If $d \in \mathscr{D}^{\mathfrak{s}}$ is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $H(F)$, then so is $e_d \in \mathscr{E}^{\mathfrak{s}}$ and by the Weyl integration formula (Proposition 3.11), we have the equality

$$d(f) = \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_G^G(\eta, X, f) e_d(X) dX \tag{7.1.2}$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, where $J_G^G(\eta, X, f)$ is defined by (4.1.1).

Remark 7.1 (Glueing). Let $d \in \mathcal{C}_c^\infty(\mathfrak{s}(F))^*$ and $(r_i)_{i \in I}$ be a family of open compact subsets of $\mathfrak{s}(F)$ such that $\bigcup_{i \in I} r_i = \mathfrak{s}(F)$. Suppose that for all $i \in I$, there exists $d_i \in \mathscr{D}^{\mathfrak{s}}$ such that $d(f) = d_i(f)$ for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $\mathrm{Supp}(f) \subseteq r_i$. Then $d \in \mathscr{D}^{\mathfrak{s}}$. Refer to [51, Remarque V.6] for the details.

Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Denote by $\hat{J}_M^G(\eta, X, \cdot)$ the distribution on $\mathfrak{s}(F)$ defined by

$$\hat{J}_M^G(\eta, X, f) := J_M^G(\eta, X, \hat{f})$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, where the right hand side is defined by (4.1.1). We also have a similar definition for the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$. The main result of this section is the following.

Proposition 7.2 (Representability). *Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Then the distribution $\hat{J}_M^G(\eta, X, \cdot) \in \mathscr{D}^{\mathfrak{s}}$.*

Remark 7.3. For $M = G$, Proposition 7.2 is essentially [58, Theorem 6.1.(i)] (see also [57, Theorem 6.2]).

The rest of this section is devoted to the proof of Proposition 7.2. We shall follow the main steps in [51, §V.7-10].

Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{s}$. Recall that $T_{\mathfrak{c}}$ denotes the centraliser of $\mathfrak{c}$ in H. Suppose that $e_0 : (T_{\mathfrak{c}}(F) \backslash H(F)) \times \mathfrak{c}_{\mathrm{reg}}(F) \to \mathbb{C}$ is a function such that

(1) e_0 is locally constant;

(2) for all open compact subset r of $\mathfrak{s}(F)$, there exists $c > 0$ and $N \in \mathbb{N}$ such that for all $x \in T_{\mathfrak{c}}(F) \backslash H(F)$ and $X \in \mathfrak{c}_{\mathrm{reg}}(F)$ satisfying $\mathrm{Ad}(x^{-1})(X) \in r$, one has the inequality

$$e_0(x, X) \leq c \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N.$$

Following [51, §V.7], for $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define

$$d_0(f) := \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F) \backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) e_0(x, X) dx dX. \tag{7.1.3}$$

Lemma 7.4. *Let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{s}$. Suppose that e_0 satisfies the above hypotheses. Then the integral (7.1.3) is absolutely convergent. Moreover, the distribution $d_0 \in \mathscr{D}^{\mathfrak{s}}$.*

Proof. We define a function $e' : \mathfrak{s}_{\mathrm{rs}}(F) \to \mathbb{C}$ by

$$e'(X) := \sum_{\{x \in T_{\mathfrak{c}}(F) \backslash H(F) : \mathrm{Ad}(x)(X) \in \mathfrak{c}(F)\}} e_0(x, \mathrm{Ad}(x)(X)) \tag{7.1.4}$$

for all $X \in \mathfrak{s}_{\mathrm{rs}}(F)$. If $X \notin \mathrm{Ad}(H(F))(\mathfrak{c}_{\mathrm{reg}}(F))$, then $e'(X) = 0$. If $X \in \mathrm{Ad}(H(F))(\mathfrak{c}_{\mathrm{reg}}(F))$, then the sum in (7.1.4) is actually over the finite set $W(H, \mathfrak{c})y$, where $y \in H(F)$ is any element such that $\mathrm{Ad}(y)(X) \in \mathfrak{c}(F)$. Hence, e' is well-defined.

Additionally, one may check that $e' \in \mathscr{E}^{\mathfrak{s}}$ from the hypotheses on e_0. Let $d' \in \mathscr{D}^{\mathfrak{s}}$ be the distribution associated to e' by (7.1.1). For all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, by the Weyl integration formula (3.1.2), we have

$$d'(f) = |W(H,\mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) e'(\mathrm{Ad}(x^{-1})(X)) dx dX. \tag{7.1.5}$$

Since

$$e'(\mathrm{Ad}(x^{-1})(X)) = \sum_{w \in W(H,\mathfrak{c})} e_0(wx, \mathrm{Ad}(w)(X))$$

for all $x \in T_{\mathfrak{c}}(F)\backslash H(F)$ and all $X \in \mathfrak{c}_{\mathrm{reg}}(F)$, we deduce that

$$d'(f) = |W(H,\mathfrak{c})|^{-1} \sum_{w \in W(H,\mathfrak{c})} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) e_0(wx, \mathrm{Ad}(w)(X)) dx dX.$$

Applying the change of variables $X \mapsto \mathrm{Ad}(w^{-1})(X)$ and $x \mapsto w^{-1}x$, which does not modify the Haar measures, we obtain

$$d'(f) = d_0(f)$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. That is to say, $d_0 = d' \in \mathscr{D}^{\mathfrak{s}}$.

Note that in the argument above, we have used the convergence of an analogue of (7.1.5) with e_0 and f replaced by their absolute values. It also results in the absolute convergence of (7.1.3). □

Corollary 7.5 (Parabolic induction). *Let $M \in \mathscr{L}^{G,\omega}(M_0)$, $P \in \mathscr{P}^G(M)$ and $d \in \mathscr{D}^{\mathfrak{m}\cap\mathfrak{s}}$. Then the distribution on $\mathfrak{s}(F)$ defined by $\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto d(f_P^\eta)$ belongs to $\mathscr{D}^{\mathfrak{s}}$, where f_P^η is defined by (3.2.2).*

Proof. Applying the Weyl integration formula (3.1.2) to $d(f_P^\eta)$, we see that it suffices to fix a Cartan subspace $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ and prove that the distribution on $\mathfrak{s}(F)$ defined by

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash M_H(F)} f_P^\eta(\mathrm{Ad}(x^{-1})(X)) e_d(\mathrm{Ad}(x^{-1})(X)) dx dX \tag{7.1.6}$$

belongs to $\mathscr{D}^{\mathfrak{s}}$. Recall that $e_d \in \mathscr{E}^{\mathfrak{m}\cap\mathfrak{s}}$ is associated to d by (7.1.1).

Define a function $e_1 : H(F) \times \mathfrak{c}_{\mathrm{reg}}(F) \to \mathbb{C}$ by

$$e_1(x, X) := \eta(\mathrm{Nrd}(x)) \int_{M_H(F)\cap K} e_d(\mathrm{Ad}(m_P(x)k)^{-1}(X)) \eta(\mathrm{Nrd}(m_P(x)k)) dk \tag{7.1.7}$$

for all $x \in H(F)$ and all $X \in \mathfrak{c}_{\mathrm{reg}}(F)$, where $m_P(x) \in M_H(F)$ is any element such that $m_P(x)^{-1}x \in N_{P_H}(F)K_H$. Since $P_H(F) \cap K = (M_H(F) \cap K)(N_{P_H}(F) \cap K)$ (see [5, Property (iii), p. 9] for example), the integral (7.1.7) is independent of the choice of $m_P(x)$. The function e_1 is left $T_{\mathfrak{c}}(F)$-invariant on the first variable, so it induces a function (still denoted by e_1) : $T_{\mathfrak{c}}(F)\backslash H(F) \times \mathfrak{c}_{\mathrm{reg}}(F) \to \mathbb{C}$.

We shall check that e_1 verifies the hypotheses of Lemma 7.4. Firstly, e_1 is locally constant because e_d is locally constant and e_1 is right K_H-invariant on the first variable. Secondly, suppose that r is an open compact subset of $\mathfrak{s}(F)$. We fix an open compact subset $r_M \subseteq (\mathfrak{m} \cap \mathfrak{s})(F)$ such that if $X \in (\mathfrak{m}\cap\mathfrak{s})(F), U \in (\mathfrak{n}_P \cap \mathfrak{s})(F), k \in K_H$ satisfy $\mathrm{Ad}(k)(X+U) \in r$, then $X \in r_M$; this is possible for it suffices to let r_M contain the projection of $\mathrm{Ad}(K_H)(r)$ to $(\mathfrak{m} \cap \mathfrak{s})(F)$. Replacing r_M with $\mathrm{Ad}(M_H(F) \cap K)(r_M)$ if necessary, we may additionally assume that

$$\mathrm{Ad}(M_H(F) \cap K)(r_M) = r_M. \tag{7.1.8}$$

Since $e_d \in \mathscr{E}^{\mathfrak{m}\cap\mathfrak{s}}$, there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|e_d(X)| \leq c \sup\{1, -\log |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F\}^N$$

for all $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}(F)$. One sees from (3.1.1) that for all $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$,

$$|D^{\mathfrak{s}}(X)|_F |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F^{-1} = |\det(\mathrm{ad}(X)|_{\mathfrak{g}/\mathfrak{m}})|_F^{1/2}.$$

Hence, $|D^{\mathfrak{s}}(X)|_F |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F^{-1}$ is bounded for $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}(F)$. We deduce that there exists $c' > 0$ such that

$$|e_d(X)| \leq c' \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N \tag{7.1.9}$$

for all $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}(F)$. Now, suppose that $x \in T_{\mathfrak{c}}(F)\backslash H(F)$ and $X \in \mathfrak{c}_{\mathrm{reg}}(F)$ satisfy $\mathrm{Ad}(x^{-1})(X) \in r$. Write $x = mnk$ with $m \in M_H(F), n \in N_{P_H}(F)$ and $k \in K_H$. Then

$$\mathrm{Ad}(x^{-1})(X) = \mathrm{Ad}(k^{-1})(\mathrm{Ad}(m^{-1})(X) + U),$$

where $U := \mathrm{Ad}(n^{-1}m^{-1})(X) - \mathrm{Ad}(m^{-1})(X) \in (\mathfrak{n}_P \cap \mathfrak{s})(F)$. Thus $\mathrm{Ad}(m^{-1})(X) \in r_M$ by our assumption on r_M. Thanks to (7.1.8) and (7.1.9), we obtain

$$|e_1(x, X)| \leq c' \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N.$$

To sum up, e_1 verifies the hypotheses of Lemma 7.4.

Applying Lemma 7.4 to $\mathfrak{c}$ and e_1, we know that the distribution d_1 on $\mathfrak{s}(F)$ defined by

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), d_1(f) := \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) e_1(x, X) dx dX$$

belongs to $\mathscr{D}^{\mathfrak{s}}$. Note that $e_1(mnk, X) = \eta(\mathrm{Nrd}(k)) e_1(m, X)$ for $m \in M_H(F), n \in N_{P_H}(F), k \in K_H$. By the same argument as the proof of Proposition 4.1.4), one shows that

$$\begin{aligned} d_1(f) &= \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash M_H(F)} f_P^\eta(\mathrm{Ad}(m^{-1})(X)) e_1(m, X) dm dX \\ &= \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash M_H(F)} f_P^\eta(\mathrm{Ad}(m^{-1})(X)) \int_{M_H(F)\cap K} e_d(\mathrm{Ad}(mk)^{-1}(X)) \eta(\mathrm{Nrd}(k)) \\ &\quad dk dm dX. \end{aligned}$$

Note that for $k \in M_H(F) \cap K$, we have $\mathrm{Ad}(k^{-1}) f_P^\eta = \eta(\mathrm{Nrd}(k)) f_P^\eta$. By the change of variables $mk \mapsto m$, one can eliminate the integral over $M_H(F) \cap K$ and see that d_1 is the same as (7.1.6). □

Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $d \in \mathscr{D}^{\mathfrak{m}\cap\mathfrak{s}}$. Suppose that d is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $M_H(F)$. Following [51, §V.9], we define a distribution $\mathrm{Ind}_M^{G,w}(d)$ on $\mathfrak{s}(F)$ by

(7.1.10)

$$\mathrm{Ind}_M^{G,w}(d)(f) := \sum_{\{L \in \mathscr{L}^{G,\omega}(M_0) : L \subseteq M\}} |W_0^{L_n}||W_0^{M_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l}\cap\mathfrak{s})} |W(L_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta, X, f) e_d(X) dX$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, where $J_M^G(\eta, X, f)$ is defined by (4.1.1). In particular, if $M = G$ and $d \in \mathscr{D}^{\mathfrak{s}}$ is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $H(F)$, we have $\mathrm{Ind}_G^{G,w}(d) = d$ by (7.1.2).

Corollary 7.6. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $d \in \mathscr{D}^{\mathfrak{m}\cap\mathfrak{s}}$. Suppose that d is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $M_H(F)$. Then the integral (7.1.10) is absolutely convergent. Moreover, the distribution $\mathrm{Ind}_M^{G,w}(d) \in \mathscr{D}^{\mathfrak{s}}$.*

Remark 7.7. This corollary is unnecessary for the proof of Proposition 7.2 but useful in Section 8.1.

Proof of Corollary 7.6. It suffices to fix a Cartan subspace $\mathfrak{c} \subseteq \mathfrak{m} \cap \mathfrak{s}$ and prove the same assertion for the distribution on $\mathfrak{s}(F)$ defined by

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta, X, f) e_d(X) dX. \tag{7.1.11}$$

Define a function $e_2 : (T_{\mathfrak{c}}(F)\backslash H(F)) \times \mathfrak{c}_{\mathrm{reg}}(F) \to \mathbb{C}$ by

$$e_2(x, X) := \eta(\mathrm{Nrd}(x)) v_M^G(x) e_d(X) \tag{7.1.12}$$

for all $x \in T_{\mathfrak{c}}(F)\backslash H(F)$ and all $X \in \mathfrak{c}_{\mathrm{reg}}(F)$. It is locally constant. Note that $e_d(\mathrm{Ad}(m^{-1})(X)) = \eta(\mathrm{Nrd}(m)) e_d(X)$ for $x \in M_H(F)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ by our assumption on d. Thus we may use the same argument as in the proof of Corollary 7.5 to show the inequality

$$|e_d(X)| = |e_d(\mathrm{Ad}(m^{-1})(X))| \leq c' \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N.$$

Thanks to Lemma 4.2, one has a similar bound for $v_M^G(x)$. In sum, e_2 verifies the hypotheses of Lemma 7.4.

Applying Lemma 7.4 to $\mathfrak{c}$ and e_2, we know that the integral

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \int_{\mathfrak{c}_{\mathrm{reg}}(F)} |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{T_{\mathfrak{c}}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X)) e_2(x, X) dx dX$$

is absolutely convergent and defines a distribution in $\mathscr{D}^{\mathfrak{s}}$. This distribution is the same as (7.1.11). □

Remark 7.8. Instead of the hypothesis $e_d \in \mathscr{E}^{\mathfrak{m}\cap\mathfrak{s}}$, an analogue of Corollary 7.6 holds if one assumes that d is defined via (7.1.1) by a function $e_d : (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F) \to \mathbb{C}$ locally constant, $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $M_H(F)$, and such that for all open compact subset r of $(\mathfrak{m}\cap\mathfrak{s})(F)$, there exists $c > 0$ and $N \in \mathbb{N}$ such that for all $X \in r \cap \mathfrak{s}_{\mathrm{rs}}(F)$, one has the inequality

$$|e_d(X)| \leq c \sup\{1, -\log |D^{\mathfrak{s}}(X)|_F\}^N.$$

The proof is the same, except that we need not use the boundedness of $|D^{\mathfrak{s}}(X)|_F |D^{\mathfrak{m}\cap\mathfrak{s}}(X)|_F^{-1}$ for $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}(F)$ (see the proof of Corollary 7.5).

Lemma 7.9. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $\mathfrak{c} \subseteq \mathfrak{m}\cap\mathfrak{s}$ be an M-elliptic Cartan subspace.*

1) Let $M' \in \mathscr{L}^{G,\omega}(M_0), \mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}'\cap\mathfrak{s})$ and $x \in H(F)$ be such that $\mathrm{Ad}(x)(\mathfrak{c}) = \mathfrak{c}'$. Then there exists $m' \in M'_H(F)$ and $w \in \mathrm{Norm}_{H(F)}(M_0)$ such that $x = m'w$.

2) The cardinality of

$$\{(M', \mathfrak{c}') : M' \in \mathscr{L}^{G,\omega}(M_0), \mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}'\cap\mathfrak{s}), \mathfrak{c}' \text{ is } H(F)\text{-conjugate to } \mathfrak{c}\}$$

is

$$|W_0^{GL_n}||W_0^{M_n}|^{-1}|W(M_H, \mathfrak{c})||W(H, \mathfrak{c})|^{-1}.$$

Proof. 1) Since $\mathfrak{c} \subseteq \mathfrak{m}\cap\mathfrak{s}$ (resp. $\mathfrak{c}' \subseteq \mathfrak{m}'\cap\mathfrak{s}$) is M-elliptic (resp. M'-elliptic), we have $A_{T_{\mathfrak{c}}} = A_M$ (resp. $A_{T_{\mathfrak{c}'}} = A_{M'}$). From $\mathrm{Ad}(x)(\mathfrak{c}) = \mathfrak{c}'$, we obtain $\mathrm{Ad}(x)(A_{T_{\mathfrak{c}}}) = A_{T_{\mathfrak{c}'}}$, $\mathrm{Ad}(x)(A_M) = A_{M'}$ and $\mathrm{Ad}(x)(M) = M'$, which implies that $\mathrm{Ad}(x)(M_H) = M'_H$. Then 1) can be shown by the same argument as in the proof of Lemma 3.10.1).

2) We can and shall identify an M-elliptic (resp. M'-elliptic) Cartan subspace in $\mathfrak{m}\cap\mathfrak{s}$ (resp. $\mathfrak{m}'\cap\mathfrak{s}$) with its $M_H(F)$(resp. $M'_H(F)$)-conjugacy class. Then $\mathscr{T}_{\mathrm{ell}}(\mathfrak{m}'\cap\mathfrak{s})$ is identified to the set of $M'_H(F)$-conjugacy classes of M'-elliptic Cantan subspaces in $\mathfrak{m}'\cap\mathfrak{s}$. As in the proof of Lemma 3.10.1), we also see that the group $W_0^{H,\omega} := \left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{GL_{n,D}} \right\}$ acts transitively on the set of pairs in 2).

Firstly, let us count M' appearing in the pairs (cf. [35, p. 426]).

Since $M \in \mathscr{L}^{G,\omega}(M_0)$, for $w \in W_0^{H,\omega}$, we see that $w = \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} \in \mathrm{Norm}_{W_0^{H,\omega}}(M)$ if and only if $\omega_n \in \mathrm{Norm}_{W_0^{GL_{n,D}}}(M_{n,D})$, where $\mathrm{Norm}_{W_0^{H,\omega}}(M)$ (resp. $\mathrm{Norm}_{W_0^{GL_{n,D}}}(M_{n,D})$) denotes the normaliser of M (resp. $M_{n,D}$) in $W_0^{H,\omega}$ (resp. $W_0^{GL_{n,D}}$). Hence, the number of M' is

$$|W_0^{GL_{n,D}}||\mathrm{Norm}_{W_0^{GL_{n,D}}}(M_{n,D})|^{-1}.$$

Secondly, for such an M' fixed, we count $\mathfrak{c}'$ such that $(M', \mathfrak{c}')$ belongs to the set of pairs in 2) (cf. [35, Lemma 7.1]).

For $x \in H(F)$, we claim that $\mathrm{Ad}(x)(\mathfrak{c}') \subseteq \mathfrak{m}'\cap\mathfrak{s}$ if and only if $x \in \mathrm{Norm}_{H(F)}(M')$, which denotes the normaliser of M' in $H(F)$. On the one hand, suppose that $\mathrm{Ad}(x)(\mathfrak{c}') \subseteq \mathfrak{m}'\cap\mathfrak{s}$. Then $A_{M'} \subseteq \mathrm{Cent}_H(\mathfrak{m}'\cap\mathfrak{s}) \subseteq \mathrm{Ad}(x)(T_{\mathfrak{c}'})$, where $\mathrm{Cent}_H(\mathfrak{m}'\cap\mathfrak{s})$ denotes the centraliser of $\mathfrak{m}'\cap\mathfrak{s}$ in H. Since $\mathrm{Ad}(x)(A_{M'}) = \mathrm{Ad}(x)(A_{T_{\mathfrak{c}'}})$ is the maximal F-split torus in $\mathrm{Ad}(x)(T_{\mathfrak{c}'})$, we have $A_{M'} \subseteq \mathrm{Ad}(x)(A_{M'})$. By comparison of dimensions, we deduce that $\mathrm{Ad}(x)(A_{M'}) = A_{M'}$, so $x \in \mathrm{Norm}_{H(F)}(M')$. On the other hand, suppose that $x \in \mathrm{Norm}_{H(F)}(M')$. Since $x \in H(F)$, we have $\mathrm{Ad}(x)(\mathfrak{m}'\cap\mathfrak{s}) = (\mathfrak{m}'\cap\mathfrak{s})$. But $\mathfrak{c}' \subseteq \mathfrak{m}'\cap\mathfrak{s}$, so we obtain $\mathrm{Ad}(x)(\mathfrak{c}') \subseteq \mathfrak{m}'\cap\mathfrak{s}$. In sum, we have proved our claim.

From this claim, the number of $\mathfrak{c}'$ is

$$|M'_H(F)\backslash\mathrm{Norm}_{H(F)}(M')/\mathrm{Norm}_{H(F)}(\mathfrak{c}')|.$$

Since $M'_H(F)$ is a normal subgroup of $\mathrm{Norm}_{H(F)}(M')$, we know that the number of the double cosets is equal to

$$\begin{aligned} &|\mathrm{Norm}_{H(F)}(M')/M'_H(F)||\mathrm{Norm}_{H(F)}(\mathfrak{c}')/(\mathrm{Norm}_{H(F)}(\mathfrak{c}') \cap M'_H(F))|^{-1} \\ =&|\mathrm{Norm}_{H(F)}(M')/M'_H(F)||\mathrm{Norm}_{H(F)}(\mathfrak{c}')/\mathrm{Norm}_{M'_H(F)}(\mathfrak{c}')|^{-1}. \end{aligned}$$

For $x \in \mathrm{Norm}_{H(F)}(M')$, we have $\mathrm{Ad}(x)(A_{M'}) = A_{M'}$. Because $M' \in \mathscr{L}^{G,\omega}(M_0)$, there exists $w \in W_0^{H,\omega}$ such that $w^{-1}x \in \mathrm{Cent}_{H(F)}(A_{M'}) = M'_H(F)$, where $\mathrm{Cent}_{H(F)}(A_{M'})$ denotes the centraliser of $A_{M'}$ in $H(F)$. Since $x \in \mathrm{Norm}_{H(F)}(M')$, we have $w \in \mathrm{Norm}_{W_0^{H,\omega}}(M')$. That is to say,

$$\mathrm{Norm}_{H(F)}(M') = \mathrm{Norm}_{W_0^{H,\omega}}(M')M'_H(F).$$

Therefore,

$$\begin{aligned}|\mathrm{Norm}_{H(F)}(M')/M'_H(F)| =&|\mathrm{Norm}_{W_0^{H,\omega}}(M')||(\mathrm{Norm}_{W_0^{H,\omega}}(M')\cap M'_H(F))|^{-1}\\ =&|\mathrm{Norm}_{W_0^{GL_{n,D}}}(M'_{n,D})||(W_0^{H,\omega}\cap M'_H(F))|^{-1}\\ =&|\mathrm{Norm}_{W_0^{GL_{n,D}}}(M'_{n,D})||W_0^{M'_{n,D}}|^{-1}.\end{aligned}$$

Since M' and M are $W_0^{H,\omega}$-conjugate, $M'_{n,D}$ and $M_{n,D}$ are $W_0^{GL_{n,D}}$-conjugate. Hence,

$$|\mathrm{Norm}_{H(F)}(M')/M'_H(F)| = |\mathrm{Norm}_{W_0^{GL_{n,D}}}(M_{n,D})||W_0^{M_{n,D}}|^{-1}.$$

We also have

$$\begin{aligned}|\mathrm{Norm}_{H(F)}(\mathfrak{c}')/\mathrm{Norm}_{M'_H(F)}(\mathfrak{c}')| =&|\mathrm{Norm}_{H(F)}(\mathfrak{c}')/T_{\mathfrak{c}'}(F)||\mathrm{Norm}_{M'_H(F)}(\mathfrak{c}')/T_{\mathfrak{c}'}(F)|^{-1}\\ =&|W(H,\mathfrak{c}')||W(M'_H,\mathfrak{c}')|^{-1}.\end{aligned}$$

Since $(M',\mathfrak{c}')$ and $(M,\mathfrak{c})$ are $W_0^{H,\omega}$-conjugate, we obtain

$$|\mathrm{Norm}_{H(F)}(\mathfrak{c}')/\mathrm{Norm}_{M'_H(F)}(\mathfrak{c}')| = |W(H,\mathfrak{c})||W(M_H,\mathfrak{c})|^{-1}.$$

To sum up, the number of $\mathfrak{c}'$ is

$$|\mathrm{Norm}_{W_0^{GL_{n,D}}}(M_{n,D})||W_0^{M_{n,D}}|^{-1}|W(M_H,\mathfrak{c})||W(H,\mathfrak{c})|^{-1}.$$

Finally, combining the numbers of M' and $\mathfrak{c}'$, we obtain the number of pairs $(M',\mathfrak{c}')$ in 2). □

Proof of Proposition 7.2. First of all, suppose that $X\in(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. Let $\mathfrak{c}$ be the centraliser of X in $\mathfrak{s}$. Then $\mathfrak{c}\subseteq\mathfrak{m}\cap\mathfrak{s}$ is an M-elliptic Cartan subspace and $X\in\mathfrak{c}_{\mathrm{reg}}(F)$. Fix an open compact subgroup $r\subseteq\mathfrak{s}(F)$ and set $r^*:=\{Y\in\mathfrak{s}(F):\forall Z\in r,\Psi(\langle Y,Z\rangle)=1\}$, which is also an open compact subgroup of $\mathfrak{s}(F)$. For all $L\in\mathscr{L}^G(M)$, fix an open compact subgroup $r_L\subseteq(\mathfrak{l}\cap\mathfrak{s})(F)$ such that if $Q\in\mathscr{P}^G(L)$ and if $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ satisfies $\mathrm{Supp}(f)\subseteq r$, then $\mathrm{Supp}(f_Q^\eta)\subseteq r_L$, where f_Q^η is defined by (3.2.2). Define r_L^* in the same way as r^*.

There exists a neighbourhood σ of X in $\mathfrak{c}_{\mathrm{reg}}(F)$ such that for all $L\in\mathscr{L}^G(M)$ and all $f\in\mathcal{C}_c^\infty((\mathfrak{l}\cap\mathfrak{s})(F)/r_L^*)$, the function $J_M^L(\eta,\cdot,f)$ is constant on σ. In fact, for L and f fixed, this results from Proposition 4.1.2) (actually its product form is needed). It suffices to apply Howe's finiteness (the product form of Corollary 6.6) to each symmetric pair $(L,L_H,\mathrm{Ad}(\epsilon))$ and an arbitrary compact neighbourhood of X in $\mathfrak{c}_{\mathrm{reg}}(F)$, and then take the intersection of a finite number of neighbourhoods involved.

We shall fix a σ satisfying the above condition and such that if two elements of σ are $H(F)$-conjugate (or equivalently $W(H,\mathfrak{c})$-conjugate), then they are the same. The latter condition is achievable since the $W(H,\mathfrak{c})$-conjugates of an element in $\mathfrak{c}_{\mathrm{reg}}(F)$ form a finite subset, which is discrete. Consider the local isomorphism $\beta:(T_{\mathfrak{c}}(F)\backslash H(F))\times\mathfrak{c}_{\mathrm{reg}}(F)\to\mathfrak{s}_{\mathrm{rs}}(F)$ of F-analytic manifolds induced by the adjoint action. Its restriction to $(T_{\mathfrak{c}}(F)\backslash H(F))\times\sigma$ is injective. Choose a neighbourhood ε of 1 in $T_{\mathfrak{c}}(F)\backslash H(F)$ such that $\eta(\mathrm{Nrd}(\varepsilon))=1$. The set $\beta(\varepsilon,\sigma)$ is a neighbourhood of X in $\mathfrak{s}(F)$. Fix a function $f'\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ such that $\mathrm{Supp}(f')\subseteq\beta(\varepsilon,\sigma)$, $f'\geq 0$ and $f'(X)\neq 0$.

Let $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $\mathrm{Supp}(f)\subseteq r$. We shall calculate $J^G(\eta,\hat{f},f')$, which is defined by (5.1.2).

Consider $M'\in\mathscr{L}^{G,\omega}(M_0)$ and $\mathfrak{c}'\in\mathscr{T}_{\mathrm{ell}}(\mathfrak{m}'\cap\mathfrak{s})$. If $\mathfrak{c}'$ and $\mathfrak{c}$ are not $H(F)$-conjugate, by our choice of f', the function $J_{M'}^G(\eta,\cdot,\hat{f},f')$ vanishes on $\mathfrak{c}'_{\mathrm{reg}}(F)$. Now suppose that $\mathfrak{c}'$ and $\mathfrak{c}$ are $H(F)$-conjugate. Let $x\in H(F)$ be such that $\mathrm{Ad}(x)(\mathfrak{c})=\mathfrak{c}'$. By Lemma 7.9.1), there exists $m'\in M'_H(F)$ and $w\in\mathrm{Norm}_{H(F)}(M_0)$ such that $x=m'w$. By Proposition 5.1.4), for $X'\in\mathfrak{c}'_{\mathrm{reg}}(F)$, we have

$$\begin{aligned}J_{M'}^G(\eta,X',\hat{f},f') &= J_{M'}^G(\eta,\mathrm{Ad}(m'^{-1})(X'),\hat{f},f')\\ &= J_{\mathrm{Ad}(w)(M)}^G(\eta,\mathrm{Ad}(wx^{-1})(X'),\hat{f},f')\\ &= J_M^G(\eta,\mathrm{Ad}(x^{-1})(X'),\hat{f},f').\end{aligned}$$

From our choices of Haar measures, we obtain

$$\int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J_{M'}^G(\eta,X',\hat{f},f') = \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta,Y,\hat{f},f')dY.$$

By Lemma 7.9.2), the number of pairs $(M',\mathfrak{c}')$ with $\mathfrak{c}'$ being $H(F)$-conjugate to $\mathfrak{c}$ is

$$|W_0^{GL_n}||W_0^{M_n}|^{-1}|W(M_H,\mathfrak{c})||W(H,\mathfrak{c})|^{-1}.$$

We deduce that

$$J^G(\eta,\hat{f},f') = (-1)^{\dim(A_M/A_G)}|W(H,\mathfrak{c})|^{-1}\int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta,Y,\hat{f},f')dY.$$

It follows from our choice of f' that the support of the restriction to $\mathfrak{c}_{\mathrm{reg}}(F)$ of the function $J_M^G(\eta,\cdot,\hat{f},f')$ is contained in $\coprod_{w\in W(H,\mathfrak{c})}\mathrm{Ad}(w)(\sigma)$. Then

$$\int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta,Y,\hat{f},f')dY = \sum_{w\in W(H,\mathfrak{c})}\int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta,Y,\hat{f},f')1_{\mathrm{Ad}(w)(\sigma)}(Y)dY,$$

where $1_{\mathrm{Ad}(w)(\sigma)}$ denotes the characteristic function of $\mathrm{Ad}(w)(\sigma)$. By the change of variables $Y\mapsto \mathrm{Ad}(w)(Y)$, which does not modify the Haar measure, we have

$$\int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_M^G(\eta,Y,\hat{f},f')1_{\mathrm{Ad}(w)(\sigma)}(Y)dY = \int_\sigma J_M^G(\eta,\mathrm{Ad}(w)(Y),\hat{f},f')dY.$$

Since $w\in W(H,\mathfrak{c})$, we have shown above that

$$J_M^G(\eta,\mathrm{Ad}(w)(Y),\hat{f},f') = J_M^G(\eta,Y,\hat{f},f'),$$

which is independent of w. Therefore,

$$J^G(\eta,\hat{f},f') = (-1)^{\dim(A_M/A_G)}\int_\sigma J_M^G(\eta,Y,\hat{f},f')dY.$$

Let $Y\in\sigma$. Applying the splitting formula for $J_M^G(\eta,Y,\hat{f},f')$ (Proposition 5.1.6)), we have

$$J_M^G(\eta,Y,\hat{f},f') = \sum_{L_1,L_2\in\mathscr{L}^G(M)} d_M^G(L_1,L_2)J_M^{L_1}(\eta,Y,\hat{f}^\eta_{Q_1})J_M^{L_2}(\eta,Y,f'^\eta_{Q_2}).$$

For all $Q\in\mathscr{F}^G(M)$, since $\mathrm{Supp}(f_Q^\eta)\subseteq r_{M_Q}$, $\hat{f}_Q^\eta$ is invariant by translation of $r_{M_Q}^*$. In particular, $\hat{f}^\eta_{Q_1}$ is invariant by $r_{L_1}^*$. Then by our assumption on σ, $J_M^{L_1}(\eta,\cdot,\hat{f}^\eta_{Q_1})$ is constant on σ and thus equal to $J_M^{L_1}(\eta,X,\hat{f}^\eta_{Q_1})$. Therefore,

$$J^G(\eta,\hat{f},f') = \sum_{L_1,L_2\in\mathscr{L}^G(M)} c(L_1,L_2)J_M^{L_1}(\eta,X,\hat{f}^\eta_{Q_1}),$$

where

$$c(L_1,L_2) := d_M^G(L_1,L_2)(-1)^{\dim(A_M/A_G)}\int_\sigma J_M^{L_2}(\eta,Y,f'^\eta_{Q_2})dY.$$

We claim that $c(G,M)\neq 0$. In fact, from (1) and (4) in Section 2.7, we have

$$c(G,M) = (-1)^{\dim(A_M/A_G)}\int_\sigma J_M^M(\eta,Y,f'^\eta_{Q_2})dY,$$

where $M_{Q_2}=M$. By Proposition 4.1.4), we have

$$J_M^M(\eta,Y,f'^\eta_{Q_2}) = J_M^{Q_2}(\eta,Y,f').$$

Since $v_M^{Q_2}=1$, we obtain

$$J_M^{Q_2}(\eta,Y,f') = J_G^G(\eta,Y,f').$$

Hence,

$$c(G,M) = (-1)^{\dim(A_M/A_G)}\int_\sigma |D^{\mathfrak{s}}(Y)|_F^{1/2}\int_{H_Y(F)\backslash H(F)} f'(\mathrm{Ad}(x^{-1})(Y))\eta(\mathrm{Nrd}(x))dxdY.$$

If $\mathrm{Ad}(x^{-1})(Y)\in\mathrm{Supp}(f')\subseteq\beta(\varepsilon,\sigma)$, since the restriction of β to $(H_Y(F)\backslash H(F))\times\sigma$ is injective, we have $x\in\varepsilon$ and then $\eta(\mathrm{Nrd}(x))=1$. Since $f'\geq 0$ and $f'(X)\neq 0$, we deduce our claim. Now, because of (3) in Section 2.7, we have

$$\hat{J}_M^G(\eta,X,f) = J^G(\eta,\hat{f},f') - \sum_{L_1,L_2\in\mathscr{L}^G(M),L_1\neq G} c(L_1,L_2)c(G,M)^{-1}\hat{J}_M^{L_1}(\eta,X,f^\eta_{Q_1}) \tag{7.1.13}$$

for all $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $\mathrm{Supp}(f)\subseteq r$.

By induction on the dimension of G and parabolic induction (Corollary 7.5), one can suppose that for all $L_1 \in \mathscr{L}^G(M), L_1 \neq G$, the distribution on $\mathfrak{s}(F)$ defined by $\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \hat{J}_M^{L_1}(\eta, X, f^\eta_{\overline{Q_1}})$ belongs to $\mathscr{D}^\mathfrak{s}$. This is actually a product form of the induction hypothesis in lower dimensions.

We claim that the distribution on $\mathfrak{s}(F)$ defined by $\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto J^G(\eta, \hat{f}, f')$ belongs to $\mathscr{D}^\mathfrak{s}$ (cf. [51, (3) in §V.10]). In fact, thanks to the noninvariant trace formula (Theorem 5.3), one can replace $J^G(\eta, \hat{f}, f')$ with $J^G(\eta, f, \hat{f}')$. By its definition (5.1.2), it suffices to fix $M' \in \mathscr{L}^{G,\omega}(M_0), \mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}' \cap \mathfrak{s})$ and prove that the distribution on $\mathfrak{s}(F)$ defined by

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J^G_{M'}(\eta, Y, f, \hat{f}')dY$$

belongs to $\mathscr{D}^\mathfrak{s}$. By the splitting formula (Proposition 5.1.6)), it suffices to fix $L'_1, L'_2 \in \mathscr{L}^G(M')$ and prove that the distribution on $\mathfrak{s}(F)$ defined by

$$\forall f \in \mathcal{C}_c^\infty(\mathfrak{s}(F)), f \mapsto \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J^{L'_1}_{M'}(\eta, Y, f^\eta_{\overline{Q'_1}}) J^{L'_2}_{M'}(\eta, Y, \hat{f}'^\eta_{Q'_2})dY,$$

where $(Q'_1, Q'_2) := s(L'_1, L'_2)$, belongs to $\mathscr{D}^\mathfrak{s}$. By Proposition 4.1.4) and the definition (4.1.1), we have

$$J^{L'_1}_{M'}(\eta, Y, f^\eta_{\overline{Q'_1}}) = J^{\overline{Q'_1}}_{M'}(\eta, Y, f) = |D^\mathfrak{s}(Y)|_F^{1/2} \int_{H_Y(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(Y))\eta(\mathrm{Nrd}(x))v^{\overline{Q'_1}}_{M'}(x)dx.$$

Then

$$\begin{aligned}
&\int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J^{L'_1}_{M'}(\eta, Y, f^\eta_{\overline{Q'_1}}) J^{L'_2}_{M'}(\eta, Y, \hat{f}'^\eta_{Q'_2})dY \\
=&\int_{\mathfrak{c}'_{\mathrm{reg}}(F)} |D^\mathfrak{s}(Y)|_F^{1/2} \int_{T_{\mathfrak{c}'}(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(Y))\eta(\mathrm{Nrd}(x))v^{\overline{Q'_1}}_{M'}(x) J^{L'_2}_{M'}(\eta, Y, \hat{f}'^\eta_{Q'_2})dxdY.
\end{aligned}$$

Define a function $e_3 : (T_{\mathfrak{c}'}(F)\backslash H(F)) \times \mathfrak{c}'_{\mathrm{reg}}(F) \to \mathbb{C}$ by

$$e_3(x, Y) := \eta(\mathrm{Nrd}(x))v^{\overline{Q'_1}}_{M'}(x) J^{L'_2}_{M'}(\eta, Y, \hat{f}'^\eta_{Q'_2}).$$

It is locally constant by the product form of Proposition 4.1.2). Using Lemma 4.2 to dominate $v^{\overline{Q'_1}}_{M'}(x)$ and Corollary 4.3 to dominate $J^{L'_2}_{M'}(\eta, Y, \hat{f}'^\eta_{Q'_2})$, we check that e_3 verifies the hypotheses of Lemma 7.4, which implies our claim.

Now (7.1.13) shows that the distribution $\hat{J}^G_M(\eta, X, \cdot)$ conincides with some element in $\mathscr{D}^\mathfrak{s}$ for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $\mathrm{Supp}(f) \subseteq r$. By glueing (Remark 7.1), the distribution $\hat{J}^G_M(\eta, X, \cdot) \in \mathscr{D}^\mathfrak{s}$.

Finally, consider a general $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. There exists $x \in M_H(F)$ such that $Y := \mathrm{Ad}(x^{-1})(X) \in (\mathfrak{l} \cap \mathfrak{s})(F)_{\mathrm{ell}}$ for some $L \in \mathscr{L}^{G,\omega}(M_0), L \subseteq M$. Then $\hat{J}^G_M(\eta, X, \cdot) = \eta(\mathrm{Nrd}(x))\hat{J}^G_M(\eta, Y, \cdot)$. Applying the descent formula (Proposition 4.1.5)), the product form of the elliptic case that we have just proved (applied to $Y \in (\mathfrak{l} \cap \mathfrak{s})(F)_{\mathrm{ell}}$) and parabolic induction (Corollary 7.5), we deduce that the distribution $\hat{J}^G_M(\eta, Y, \cdot) \in \mathscr{D}^\mathfrak{s}$. Thus the distribution $\hat{J}^G_M(\eta, X, \cdot) \in \mathscr{D}^\mathfrak{s}$. □

7.2. The case of (G', H'). We define $\mathscr{E}^{\mathfrak{s}'}$ in the same way as the previous case. For $e \in \mathscr{E}^{\mathfrak{s}'}$, thanks to Corollary 3.21, it defines a distribution on $\mathfrak{s}'(F)$:

$$\forall f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F)), f' \mapsto \int_{\mathfrak{s}'(F)} f'(Y)e(Y)|D^{\mathfrak{s}'}(Y)|_F^{-1/2}dY. \tag{7.2.1}$$

Denote by $\mathscr{D}^{\mathfrak{s}'}$ the space of distributions obtained in this way. For $d \in \mathscr{D}^{\mathfrak{s}'}$, we shall always denote by $e_d \in \mathscr{E}^{\mathfrak{s}'}$ its preimage under the isomorphism $\mathscr{E}^{\mathfrak{s}'} \to \mathscr{D}^{\mathfrak{s}'}$ defined above. One may extend these definitions to the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M'_0)$. If $d \in \mathscr{D}^{\mathfrak{s}'}$ is invariant with respect to the adjoint action of $H'(F)$, then so is $e_d \in \mathscr{E}^{\mathfrak{s}'}$ and by the Weyl integration formula (Proposition 3.18), we have the equality

$$d(f') = \sum_{M' \in \mathscr{L}^{H'}(M'_0)} |W_0^{M'}||W_0^{H'}|^{-1} \sum_{\mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')} |W(M', \mathfrak{c}')|^{-1} \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J^{H'}_{H'}(Y, f')e_d(Y)dY \tag{7.2.2}$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, where $J^{H'}_{H'}(Y, f')$ is defined by (4.2.1).

Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Denote by $\hat{J}^{H'}_{M'}(Y, \cdot)$ the distribution on $\mathfrak{s}'(F)$ defined by

$$\hat{J}^{H'}_{M'}(Y, f') := J^{H'}_{M'}(Y, \hat{f}')$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, where the right hand side is defined by (4.2.1). One also has a similar definition for the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M_0')$.

Proposition 7.10 (Representability). *Let $M' \in \mathscr{L}^{H'}(M_0')$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Then the distribution $\hat{J}_{M'}^{H'}(Y, \cdot) \in \mathscr{D}^{\mathfrak{s}'}$.*

The rest of this section is devoted to the proof of Proposition 7.10. Although it is similar to the proof of Proposition 7.2, we shall sketch some steps for later use.

Let $\mathfrak{c}'$ be a Cartan subspace of $\mathfrak{s}'$. Recall that $T_{\mathfrak{c}'}$ denotes the centraliser of $\mathfrak{c}'$ in H'. Suppose that $e_0 : (T_{\mathfrak{c}'}(F)\backslash H'(F)) \times \mathfrak{c}'_{\mathrm{reg}}(F) \to \mathbb{C}$ is a function such that

(1) e_0 is locally constant;

(2) for all open compact subset r' of $\mathfrak{s}'(F)$, there exists $c > 0$ and $N \in \mathbb{N}$ such that for all $x \in T_{\mathfrak{c}'}(F)\backslash H'(F)$ and $Y \in \mathfrak{c}'_{\mathrm{reg}}(F)$ satisfying $\mathrm{Ad}(x^{-1})(Y) \in r'$, one has the inequality

$$e_0(x, Y) \leq c \sup\{1, -\log |D^{\mathfrak{s}'}(Y)|_F\}^N.$$

For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define

$$d_0(f') := \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{T_{\mathfrak{c}'}(F)\backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y)) e_0(x, Y) dx dY. \tag{7.2.3}$$

Lemma 7.11 (cf. Lemma 7.4). *Let $\mathfrak{c}'$ be a Cartan subspace of $\mathfrak{s}'$. Suppose that e_0 satisfies the above hypotheses. Then the integral (7.2.3) is absolutely convergent. Moreover, the distribution $d_0 \in \mathscr{D}^{\mathfrak{s}'}$.*

Corollary 7.12 (Parabolic induction). *Let $M' \in \mathscr{L}^{H'}(M_0')$, $P' \in \mathscr{P}^{H'}(M')$ and $d \in \mathscr{D}^{\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'}$. Then the distribution on $\mathfrak{s}'(F)$ defined by $\forall f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F)), f' \mapsto d(f'_{P'})$ belongs to $\mathscr{D}^{\mathfrak{s}'}$, where $f'_{P'}$ is defined by (3.3.2).*

Proof. We may apply the argument of Corollary 7.5 with the aid of Lemma 3.19. □

Let $M' \in \mathscr{L}^{H'}(M_0')$ and $d \in \mathscr{D}^{\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'}$. Suppose that d is invariant with respect to the adjoint action of $M'(F)$. We define a distribution $\mathrm{Ind}_{M'}^{H',w}(d)$ on $\mathfrak{s}'(F)$ by

$$\mathrm{Ind}_{M'}^{H',w}(d)(f') := \sum_{\{L' \in \mathscr{L}^{H'}(M_0') : L' \subseteq M'\}} |W_0^{L'}||W_0^{M'}|^{-1} \sum_{\mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{l}'} \cap \mathfrak{s}')} |W(L', \mathfrak{c}')|^{-1} \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} J_{M'}^{H'}(Y, f') e_d(Y) dY \tag{7.2.4}$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, where $J_{M'}^{H'}(Y, f')$ is defined by (4.2.1). In particular, if $M' = H'$ and $d \in \mathscr{D}^{\mathfrak{s}'}$ is invariant with respect to the adjoint action of $H'(F)$, we have $\mathrm{Ind}_{H'}^{H',w}(d) = d$ by (7.2.2).

Corollary 7.13. *Let $M' \in \mathscr{L}^{H'}(M_0')$ and $d \in \mathscr{D}^{\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'}$. Suppose that d is invariant with respect to the adjoint action of $M'(F)$. Then the integral (7.2.4) is absolutely convergent. Moreover, the distribution $\mathrm{Ind}_{M'}^{H',w}(d) \in \mathscr{D}^{\mathfrak{s}'}$.*

Remark 7.14. This corollary is unnecessary for the proof of Proposition 7.10 but useful in Section 8.2.

Proof of Corollary 7.13. We may apply the argument of Corollary 7.6 thanks to Lemmas 4.5 and 7.11. □

Lemma 7.15 (cf. Lemma 7.9). *Let $M' \in \mathscr{L}^{H'}(M_0')$ and $\mathfrak{c}' \subseteq \widetilde{\mathfrak{m}'} \cap \mathfrak{s}'$ be an M'-elliptic Cartan subspace.*

1) Let $M \in \mathscr{L}^{H'}(M_0'), \mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}} \cap \mathfrak{s}')$ and $x \in H'(F)$ be such that $\mathrm{Ad}(x)(\mathfrak{c}') = \mathfrak{c}$. Then there exists $m \in M(F)$ and $w \in \mathrm{Norm}_{H'(F)}(M_0')$ such that $x = mw$.

2) The cardinality of

$$\{(M, \mathfrak{c}) : M \in \mathscr{L}^{H'}(M_0'), \mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}} \cap \mathfrak{s}'), \mathfrak{c} \text{ is } H'(F)\text{-conjugate to } \mathfrak{c}'\}$$

is

$$|W_0^{H'}||W_0^{M'}|^{-1}|W(M', \mathfrak{c}')||W(H', \mathfrak{c}')|^{-1}.$$

Proof of Proposition 7.10. We may apply the argument of Proposition 7.2 with obvious modifications. One needs almost all results that we have prepared in this and previous sections, notably Howe's finiteness (Corollary 6.9) and the noninvariant trace formula (Theorem 5.12). □

8. Invariant weighted orbital integrals

8.1. The case of (G,H). Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$. We shall define a distribution $\hat{I}_M^G(\eta,X,\cdot) \in \mathscr{D}^{\mathfrak{s}}$ which is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $H(F)$ by induction on $\dim(G)$. Suppose that we have defined a distribution $\hat{I}_M^L(\eta,X,\cdot) \in \mathscr{D}^{\mathfrak{l}\cap\mathfrak{s}}$ which is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $L_H(F)$ for all $L \in \mathscr{L}^G(M), L \neq G$. This is actually a product form of the induction hypothesis in lower dimensions. Denote by $\hat{I}_M^{L,G,w}(\eta,X,\cdot)$ its image under $\mathrm{Ind}_L^{G,w}$ (see (7.1.10)). As in [51, (1) in §VI.1], for $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we set

$$\hat{I}_M^G(\eta,X,f) := \hat{J}_M^G(\eta,X,f) - \sum_{L\in\mathscr{L}^G(M),L\neq G} \hat{I}_M^{L,G,w}(\eta,X,f). \tag{8.1.1}$$

Proposition 8.1. *The distribution $\hat{I}_M^G(\eta,X,\cdot) \in \mathscr{D}^{\mathfrak{s}}$ and is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $H(F)$.*

Proof. The first statement results from the representability of $\hat{J}_M^G(\eta,X,\cdot)$ (Proposition 7.2), the induction hypothesis and Corollary 7.6. Now let us consider the second one.

Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $y \in H(F)$. By the $H(F)$-invariance of $\langle\cdot,\cdot\rangle$, we see that $(\mathrm{Ad}(y^{-1})(f))\hat{} = \mathrm{Ad}(y^{-1})(\hat{f})$. Applying Proposition 4.1.6), we have

$$\hat{J}_M^G(\eta,X,\mathrm{Ad}(y^{-1})(f)) = J_M^G(\eta,X,\mathrm{Ad}(y^{-1})(\hat{f})) = \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} J_M^{M_Q}(\eta,X,(\hat{f})_{Q,y}^\eta).$$

For all $Q \in \mathscr{F}^G(M)$, we show that

$$(\hat{f})_{Q,y}^\eta = (f_{Q,y}^\eta)\hat{}$$

by the same argument of an analogous property of (3.2.2). Then

$$\hat{J}_M^G(\eta,X,\mathrm{Ad}(y^{-1})(f)) = \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} \hat{J}_M^{M_Q}(\eta,X,f_{Q,y}^\eta). \tag{8.1.2}$$

Let $L \in \mathscr{L}^G(M), L \neq G$. Applying Proposition 4.1.6) again to $J_L^G(\eta,Y,\mathrm{Ad}(y^{-1})(f))$ in the integrand of the definition (7.1.10) of $\mathrm{Ind}_L^{G,w}$, we obtain

$$\hat{I}_M^{L,G,w}(\eta,X,\mathrm{Ad}(y^{-1})(f)) = \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(L)} \hat{I}_M^{L,M_Q,w}(\eta,X,f_{Q,y}^\eta), \tag{8.1.3}$$

where $\hat{I}_M^{L,M_Q,w}(\eta,X,\cdot) \in \mathscr{D}^{\mathfrak{m}_Q\cap\mathfrak{s}}$ denotes the image of $\hat{I}_M^L(\eta,X,\cdot)$ under $\mathrm{Ind}_L^{M_Q,w}$, which is defined by a product form of (7.1.10).

From (8.1.1), (8.1.2) and (8.1.3), we deduce that

$$\begin{aligned}\hat{I}_M^G(\eta,X,\mathrm{Ad}(y^{-1})(f)) =& \hat{J}_M^G(\eta,X,\mathrm{Ad}(y^{-1})(f)) - \sum_{L\in\mathscr{L}^G(M),L\neq G} \hat{I}_M^{L,G,w}(\eta,X,\mathrm{Ad}(y^{-1})(f)) \\ =& \eta(\mathrm{Nrd}(y)) \sum_{Q\in\mathscr{F}^G(M)} \left(\hat{J}_M^{M_Q}(\eta,X,f_{Q,y}^\eta) - \sum_{L\in\mathscr{L}^{M_Q}(M),L\neq G} \hat{I}_M^{L,M_Q,w}(\eta,X,f_{Q,y}^\eta) \right).\end{aligned}$$

Consider $Q \neq G$ first. By the induction hypothesis, $\hat{I}_M^{M_Q}(\eta,X,\cdot) \in \mathscr{D}^{\mathfrak{m}_Q\cap\mathfrak{s}}$ is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $M_{Q_H}(F)$, so $\hat{I}_M^{M_Q,M_Q,w}(\eta,X,\cdot) = \hat{I}_M^{M_Q}(\eta,X,\cdot)$. By the definition of $\hat{I}_M^{M_Q}(\eta,X,\cdot)$ (a product form of (8.1.1)), the term in brackets is zero. Thus it remains the term for $Q = G$. Note that $f_{G,y}^\eta = f_G^\eta$. By Proposition 4.1.4) applied to $Q = G$, we see that the term in brackets is exactly $\hat{I}_M^G(\eta,X,f)$ defined by (8.1.1). Therefore, we show that

$$\hat{I}_M^G(\eta,X,\mathrm{Ad}(y^{-1})(f)) = \eta(\mathrm{Nrd}(y))\hat{I}_M^G(\eta,X,f),$$

which is the second statement. □

Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$. Denote by $\hat{\imath}_M^G(\eta,X,\cdot)$ (resp. $\hat{\jmath}_M^G(\eta,X,\cdot)$) the element of $\mathscr{E}^{\mathfrak{s}}$ associated to $\hat{I}_M^G(\eta,X,\cdot)$ (resp. $\hat{J}_M^G(\eta,X,\cdot)$) $\in \mathscr{D}^{\mathfrak{s}}$ by (7.1.1). That is to say, for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$,

$$\hat{I}_M^G(\eta,X,f) = \int_{\mathfrak{s}(F)} f(Y)\hat{\imath}_M^G(\eta,X,Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY$$

and

$$\hat{j}_M^G(\eta, X, f) = \int_{\mathfrak{s}(F)} f(Y)\hat{j}_M^G(\eta, X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY.$$

We also have a similar definition for the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$.

LEMMA 8.2. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Let $L \in \mathscr{L}^{G,\omega}(M_0)$ and $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. Then $\hat{i}_M^G(\eta, X, Y) = \hat{j}_M^G(\eta, X, Y)$.*

PROOF. Let $L_2 \in \mathscr{L}^G(M), L_2 \neq G$, $L_1 \in \mathscr{L}^{G,\omega}(M_0), L_1 \subseteq L_2$ and $\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l}_1 \cap \mathfrak{s})$. We define a distribution $d_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, \cdot)$ on $\mathfrak{s}(F)$ by

$$d_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, f) := \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_{L_2}^G(\eta, Z, f)\hat{i}_M^{L_2}(\eta, X, Z)dZ \tag{8.1.4}$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. By a product form of Proposition 8.1, the distribution $\hat{I}_M^{L_2}(\eta, X, \cdot) \in \mathscr{D}^{\mathfrak{l}_2 \cap \mathfrak{s}}$ and is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $(L_2 \cap H)(F)$. We see from the proof of Corollary 7.6 that the integral (8.1.4) is absolutely convergent and the distribution $d_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, \cdot) \in \mathscr{D}^{\mathfrak{s}}$. We denote by $e_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, \cdot)$ its associated element in $\mathscr{E}^{\mathfrak{s}}$ by (7.1.1). From the definitions (8.1.1) and (7.1.10), we have

$$\begin{aligned}\hat{i}_M^G(\eta, X, \cdot) =& \hat{j}_M^G(\eta, X, \cdot) - \sum_{L_2 \in \mathscr{L}^G(M), L_2 \neq G} \sum_{\{L_1 \in \mathscr{L}^{G,\omega}(M_0): L_1 \subseteq L_2\}} |W_0^{L_1,n}||W_0^{L_2,n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l}_1 \cap \mathfrak{s})} \\ & |W(L_1 \cap H, \mathfrak{c})|^{-1} e_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, \cdot).\end{aligned}$$

To prove the lemma, it suffices to fix such a triple $(L_2, L_1, \mathfrak{c})$ and prove that $e_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, Y) = 0$. But (7.1.12) and (7.1.4) in the proofs of Corollary 7.6 and Lemma 7.4 respectively allow us to calculate $e_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, Y)$; explicitly, we have

$$e_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, Y) = \sum_{\{x \in T_{\mathfrak{c}}(F)\backslash H(F): \mathrm{Ad}(x)(Y) \in \mathfrak{c}(F)\}} \eta(\mathrm{Nrd}(x))v_{L_2}^G(x)\hat{i}_M^{L_2}(\eta, X, \mathrm{Ad}(x)(Y)). \tag{8.1.5}$$

Let $x \in T_{\mathfrak{c}}(F)\backslash H(F)$ such that $\mathrm{Ad}(x)(Y) \in \mathfrak{c}(F)$. As $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ and $\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l}_1 \cap \mathfrak{s})$, from the proof of Lemma 3.10.1), there exists $l_1 \in (L_1 \cap H)(F)$ and $w \in \mathrm{Norm}_{H(F)}(M_0)$ such that $x = l_1 w$. Since any element in W_0^H admits a representative in K_H, we can suppose that $w \in K_H$. Then $v_{L_2}^G(x) = v_{L_2}^G(1)$ since $L_1 \subseteq L_2$. But $v_{L_2}^G(1) = 0$ for $L_2 \neq G$. Thus $e_{L_1,L_2,\mathfrak{c}}^{G,w}(\eta, X, Y) = 0$ by (8.1.5). □

LEMMA 8.3. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$.*

1) The function $(X, Y) \mapsto \hat{i}_M^G(\eta, X, Y)$ is locally constant on $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$.

2) If $w \in \mathrm{Norm}_{H(F)}(M_0)$, $x \in M_H(F)$ and $y \in H(F)$, we have the equality

$$\hat{i}_{\mathrm{Ad}(w)(M)}^G(\eta, \mathrm{Ad}(wx)(X), \mathrm{Ad}(y)(Y)) = \eta(\mathrm{Nrd}(wxy))\hat{i}_M^G(\eta, X, Y)$$

for all $(X, Y) \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$.

3) If $\lambda \in F^\times$, we have the equality

$$\hat{i}_M^G(\eta, \lambda X, Y) = \hat{i}_M^G(\eta, X, \lambda Y)$$

for all $(X, Y) \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$.

4) Let $r_M \subseteq (\mathfrak{m} \cap \mathfrak{s})(F)$ and $r \subseteq \mathfrak{s}(F)$ be two compact subsets. Then there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|\hat{i}_M^G(\eta, X, Y)| \leq c \sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^N \sup\{1, -\log|D^{\mathfrak{s}}(Y)|_F\}^N$$

for all $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}$ and $Y \in r \cap \mathfrak{s}_{\mathrm{rs}}$.

5) Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $r_M \subseteq (\mathfrak{m} \cap \mathfrak{s})(F)$ be a compact subset. Then there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|\hat{I}_M^G(\eta, X, f)| \leq c \sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^N$$

for all $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}$.

PROOF. Let $r_M \subseteq (\mathfrak{m} \cap \mathfrak{s})(F)$ and $r \subseteq \mathfrak{s}(F)$ be two open compact subgroups. Set $r^* := \{Y \in \mathfrak{s}(F) : \forall Z \in r, \Psi(\langle Y, Z\rangle) = 1\}$, which is an open compact subgroup of $\mathfrak{s}(F)$. Notice that if $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ satisfies $\mathrm{Supp}(f) \subseteq r$, then $\hat{f} \in \mathcal{C}_c^\infty(\mathfrak{s}(F)/r^*)$. Applying Howe's finiteness (Corollary 6.6) to r^* and $r_M \cap \mathfrak{s}_{\mathrm{rs}}$, we

know that there exists a finite subset $\{X_i : i \in I\} \subseteq r_M \cap \mathfrak{s}_{\mathrm{rs}}$ and a finite subset $\{f_i : i \in I\} \subseteq \mathcal{C}_c^\infty(\mathfrak{s}(F)/r^*)$ such that for all $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}$ and all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $\mathrm{Supp}(f) \subseteq r$, we have

$$J_M^G(\eta, X, \hat{f}) = \sum_{i\in I} J_M^G(\eta, X_i, \hat{f}) J_M^G(\eta, X, f_i).$$

We deduce that

$$\hat{j}_M^G(\eta, X, Y) = \sum_{i\in I} \hat{j}_M^G(\eta, X_i, Y) J_M^G(\eta, X, f_i) \tag{8.1.6}$$

for all $X \in r_M \cap \mathfrak{s}_{\mathrm{rs}}$ and $Y \in r \cap \mathfrak{s}_{\mathrm{rs}}$.

1) The local constancy of $(X, Y) \mapsto \hat{j}_M^G(\eta, X, Y)$ on $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$ results from (8.1.6), Proposition 4.1.2) and $\hat{j}_M^G(\eta, X_i, \cdot) \in \mathscr{E}^{\mathfrak{s}}$ for $i \in I$. For $L \in \mathscr{L}^{G,\omega}(M_0)$, we deduce from Lemma 8.2 the local constancy of $(X, Y) \mapsto \hat{i}_M^G(\eta, X, Y)$ on $(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)\times(\mathfrak{l}\cap\mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. Let $(X, Y) \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)\times\mathfrak{s}_{\mathrm{rs}}(F)$. Choose $L \in \mathscr{L}^{G,\omega}(M_0)$ and $Y' \in (\mathfrak{l}\cap\mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ such that Y' is $H(F)$-conjugate to Y. Fix a neighbourhood $V_1 \times V_2$ of (X, Y') in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ such that $(X, Y) \mapsto \kappa(Y)\hat{i}_M^G(\eta, X, Y)$ is constant on $V_1 \times V_2$. Thanks to the $\eta(\mathrm{Nrd}(\cdot))$-invariance of $\hat{i}_M^G(\eta, X, \cdot)$ with respect to the adjoint action of $H(F)$ (Proposition 8.1), we know that $(X, Y) \mapsto \kappa(Y)\hat{i}_M^G(\eta, X, Y)$ is constant on $V_1 \times \mathrm{Ad}(H(F))(V_2)$ which is a neighbourhood of (X, Y) in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$. Since $\kappa(\cdot)$ is locally constant on $\mathfrak{s}_{\mathrm{rs}}(F)$, we show that $(X, Y) \mapsto \hat{i}_M^G(\eta, X, Y)$ is constant on a neighbourhood of (X, Y) in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$.

2) The effect of $\mathrm{Ad}(y)$ comes from Propostion 8.1. Then when considering the effects of $\mathrm{Ad}(w)$ and $\mathrm{Ad}(x)$, up to $H(F)$-conjugation, we may and shall suppose that $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ for some $L \in \mathscr{L}^{G,\omega}(M_0)$. That is to say, it suffices to prove the equality

$$\hat{i}^G_{\mathrm{Ad}(w)(M)}(\eta, \mathrm{Ad}(wx)(X), Y) = \eta(\mathrm{Nrd}(wx))\hat{i}_M^G(\eta, X, Y)$$

for all $(X, Y) \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. By Lemma 8.2, we may replace $\hat{i}^G_{\mathrm{Ad}(w)(M)}$ and $\hat{i}_M^G$ by $\hat{j}^G_{\mathrm{Ad}(w)(M)}$ and $\hat{j}_M^G$ respectively in the equality to be proved. Now the equality results from Proposition 4.1.3).

3) Let $\lambda \in F^\times, X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{reg}})(F)$ and $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. From the definition (3.1.1), we have

$$|D^{\mathfrak{s}}(\lambda X)|_F = |\lambda|_F^{(\dim(\mathfrak{g})-\mathrm{rank}(\mathfrak{g}))/2}|D^{\mathfrak{s}}(X)|_F,$$

where $\dim(\mathfrak{g})$ and $\mathrm{rank}(\mathfrak{g})$ denote the dimension and rank (over an algebraic closure of F) of $\mathfrak{g}$ respectively. Then we have

$$\begin{aligned}
&\int_{\mathfrak{s}(F)} f(Y)\hat{j}_M^G(\eta, \lambda X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY = \hat{J}_M^G(\eta, \lambda X, f) = J_M^G(\eta, \lambda X, \hat{f}) \\
=&|D^{\mathfrak{s}}(\lambda X)|_F^{1/2} \int_{H_{\lambda X}(F)\backslash H(F)} \hat{f}(\mathrm{Ad}(x^{-1})(\lambda X))\eta(\mathrm{Nrd}(x))v_M^Q(x)dx \\
=&|\lambda|_F^{(\dim(\mathfrak{g})-\mathrm{rank}(\mathfrak{g}))/4} J_M^G(\eta, X, \hat{f}(\lambda\cdot)).
\end{aligned}$$

But

$$\begin{aligned}
\hat{f}(\lambda\cdot) =&c_\Psi(\mathfrak{s}(F)) \int_{\mathfrak{s}(F)} f(Z)\Psi(\langle\lambda\cdot, Z\rangle)dZ = c_\Psi(\mathfrak{s}(F)) \int_{\mathfrak{s}(F)} f(Z)\Psi(\langle\cdot, \lambda Z\rangle)dZ \\
=&|\lambda|_F^{-\dim(\mathfrak{s})} c_\Psi(\mathfrak{s}(F)) \int_{\mathfrak{s}(F)} f(\lambda^{-1}Z)\Psi(\langle\cdot, Z\rangle)dZ \\
=&|\lambda|_F^{-\dim(\mathfrak{g})/2}(f(\lambda^{-1}\cdot))\hat{}\,.
\end{aligned}$$

Thus we have

$$\begin{aligned}
J_M^G(\eta, X, \hat{f}(\lambda\cdot)) =&|\lambda|_F^{-\dim(\mathfrak{g})/2}\hat{J}_M^G(\eta, X, f(\lambda^{-1}\cdot)) = |\lambda|_F^{-\dim(\mathfrak{g})/2} \int_{\mathfrak{s}(F)} f(\lambda^{-1}Y)\hat{j}_M^G(\eta, X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY \\
=& \int_{\mathfrak{s}(F)} f(Y)\hat{j}_M^G(\eta, X, \lambda Y)|D^{\mathfrak{s}}(\lambda Y)|_F^{-1/2}dY \\
=&|\lambda|_F^{(\mathrm{rank}(\mathfrak{g})-\dim(\mathfrak{g}))/4} \int_{\mathfrak{s}(F)} f(Y)\hat{j}_M^G(\eta, X, \lambda Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY.
\end{aligned}$$

Therefore, we deduce the equality

$$\hat{j}_M^G(\eta, \lambda X, Y) = \hat{j}_M^G(\eta, X, \lambda Y)$$

for all $\lambda \in F^\times$ and all $(X,Y) \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \times \mathfrak{s}_{\mathrm{rs}}(F)$. We obtain a similar equality for $\hat{i}_M^G$ thanks to Lemma 8.2 and the $\eta(\mathrm{Nrd}(\cdot))$-invariance of $\hat{i}_M^G(\eta, X, \cdot)$ with respect to the adjoint action of $H(F)$ (Proposition 8.1).

4) A similar bound for $(X,Y) \mapsto \hat{j}_M^G(\eta, X, Y)$ on $(r_M \cap \mathfrak{s}_{\mathrm{rs}}) \times (r \cap \mathfrak{s}_{\mathrm{rs}})$ results from (8.1.6), Corollary 4.3 (together with Proposition 4.1.4)) and $\hat{j}_M^G(\eta, X_i, \cdot) \in \mathscr{E}^{\mathfrak{s}}$ for $i \in I$. For $L \in \mathscr{L}^{G,\omega}(M_0)$, we deduce from Lemma 8.2 a similar bound of $(X,Y) \mapsto \hat{i}_M^G(\eta, X, Y)$ on $(r_M \cap \mathfrak{s}_{\mathrm{rs}}) \times (r \cap (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}})$. Let $(X,Y) \in (r_M \cap \mathfrak{s}_{\mathrm{rs}}) \times (r \cap \mathfrak{s}_{\mathrm{rs}})$. Thanks to the $\eta(\mathrm{Nrd}(\cdot))$-invariance of $\hat{i}_M^G(\eta, X, \cdot)$ with respect to the adjoint action of $H(F)$ (Proposition 8.1), if we replace Y by $\mathrm{Ad}(y)(Y)$, where $y \in H(F)$, the two sides in the inequality to be proved remain unchanged. Since any Cartan subspace in $\mathfrak{s}$ is $H(F)$-conjugate to an element in $\mathscr{T}_{\mathrm{ell}}(\mathfrak{l} \cap \mathfrak{s})$ for some $L \in \mathscr{L}^{G,\omega}(M_0)$, with the help of Lemma 3.3, it suffices to fix $L \in \mathscr{L}^{G,\omega}(M_0), \mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l} \cap \mathfrak{s})$ and $r_{\mathfrak{c}} \subseteq \mathfrak{c}(F)$ a compact subset, prove a similar bound for $(X,Y) \in (r_M \cap \mathfrak{s}_{\mathrm{rs}}) \times (r_{\mathfrak{c}} \cap \mathfrak{c}_{\mathrm{reg}})$, and then obtain a uniform bound for $(X,Y) \in (r_M \cap \mathfrak{s}_{\mathrm{rs}}) \times (r \cap \mathfrak{s}_{\mathrm{rs}})$ by the finiteness of $\mathscr{T}_{\mathrm{ell}}(\mathfrak{l} \cap \mathfrak{s})$. But this is what we have established.

5) It is a consequence of 4) applied to $r := \mathrm{Supp}(f)$ and Corollary 3.15. □

For $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we define a distribution $I_M^G(\eta, X, \cdot)$ on $\mathfrak{s}(F)$ by

$$I_M^G(\eta, X, \hat{f}) := \hat{I}_M^G(\eta, X, f) \tag{8.1.7}$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$.

Remark 8.4. For $M = G$, it is evident that $I_G^G(\eta, X, f) = J_G^G(\eta, X, f)$ for all $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$.

One may easily extend the definitions (8.1.1) and (8.1.7) to the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$, since it appears as the product of some copies of the form $(G, H, \mathrm{Ad}(\epsilon))$ in lower dimensions.

Lemma 8.5. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. The distribution $I_M^G(\eta, X, \cdot)$ on $\mathfrak{s}(F)$ is independent of the choice of the $H(F)$-invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}(F)$ or the continuous nontrivial unitary character Ψ of F.*

Proof. Suppose that $\langle \cdot, \cdot \rangle'$ is another bilinear form and that Ψ' is another character. Denote by $f \mapsto \tilde{f}$ the associated Fourier transform and by $\tilde{I}_M^G(\eta, X, \cdot)$ (resp. $\tilde{J}_M^G(\eta, X, \cdot)$) the associated analogue of $\hat{I}_M^G(\eta, X, \cdot)$ (resp. $\hat{J}_M^G(\eta, X, \cdot)$). Since $\tilde{\tilde{f}}(\cdot) = \hat{\hat{f}}(\cdot) = f(-\cdot)$ for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, it suffices to prove the equality

$$\tilde{I}_M^G(\eta, X, \tilde{f}) = \hat{I}_M^G(\eta, X, \hat{f})$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$.

Let τ' be the linear automorphism of $\mathfrak{s}(F)$ such that

$$\forall Y, Z \in \mathfrak{s}(F), \langle Y, Z \rangle' = \langle \tau'(Y), Z \rangle.$$

Let $a \in F^\times$ such that $\Psi'(\cdot) = \Psi(a\cdot)$. Set $\tau := a\tau'$. Then

$$\tilde{f}(\cdot) = \frac{c_{\Psi'}(\mathfrak{s}(F))}{c_{\Psi}(\mathfrak{s}(F))} \hat{f}(\tau(\cdot))$$

for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$. One may check that τ is an $H(F)$-equivariant linear automorphism of $\mathfrak{s}(F)$ thanks to $H(F)$-invariance of two bilinear forms. One also deduces that

$$\forall Y, Z \in \mathfrak{s}(F), \langle \tau(Y), Z \rangle = \langle Y, \tau(Z) \rangle$$

from the symmetry of two bilinear forms. Now for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and all $Y \in \mathfrak{s}(F)$, we have

$$\begin{aligned} f(-Y) = \tilde{\tilde{f}}(Y) &= c_{\Psi'}(\mathfrak{s}(F)) \int_{\mathfrak{s}(F)} \tilde{f}(Z) \Psi'(\langle Y, Z \rangle') dZ = c_{\Psi'}(\mathfrak{s}(F)) \int_{\mathfrak{s}(F)} \frac{c_{\Psi'}(\mathfrak{s}(F))}{c_{\Psi}(\mathfrak{s}(F))} \hat{f}(\tau(Z)) \Psi(\langle \tau(Y), Z \rangle) dZ \\ &= \frac{c_{\Psi'}(\mathfrak{s}(F))^2}{c_{\Psi}(\mathfrak{s}(F))} \int_{\mathfrak{s}(F)} \hat{f}(\tau(Z)) \Psi(\langle Y, \tau(Z) \rangle) dZ = \frac{c_{\Psi'}(\mathfrak{s}(F))^2}{c_{\Psi}(\mathfrak{s}(F)) |\det_{\mathfrak{s}(F)}(\tau)|_F} \int_{\mathfrak{s}(F)} \hat{f}(Z') \Psi(\langle Y, Z' \rangle) dZ' \\ &= \frac{c_{\Psi'}(\mathfrak{s}(F))^2}{c_{\Psi}(\mathfrak{s}(F))^2 |\det_{\mathfrak{s}(F)}(\tau)|_F} \hat{\hat{f}}(Y) = \frac{c_{\Psi'}(\mathfrak{s}(F))^2}{c_{\Psi}(\mathfrak{s}(F))^2 |\det_{\mathfrak{s}(F)}(\tau)|_F} f(-Y). \end{aligned}$$

Therefore, we obtain $\frac{c_{\Psi'}(\mathfrak{s}(F))}{c_{\Psi}(\mathfrak{s}(F))} = |\det_{\mathfrak{s}(F)}(\tau)|_F^{1/2}$. Then for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have

$$\tilde{f}(\cdot) = |\det_{\mathfrak{s}(F)}(\tau)|_F^{1/2} \hat{f}(\tau(\cdot)).$$

Denote by $\tilde{i}_M^G(\eta, X, \cdot)$ (resp. $\tilde{j}_M^G(\eta, X, \cdot)$) the element of $\mathscr{E}^{\mathfrak{s}}$ associated to $\tilde{I}_M^G(\eta, X, \cdot)$ (resp. $\tilde{J}_M^G(\eta, X, \cdot)$) $\in \mathscr{D}^{\mathfrak{s}}$ by (7.1.1). For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have

$$\begin{aligned}\tilde{I}_M^G(\eta, X, \tilde{f}) &= \int_{\mathfrak{s}(F)} \tilde{f}(Y)\tilde{i}_M^G(\eta, X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY \\ &= |\det_{\mathfrak{s}(F)}(\tau)|_F^{1/2} \int_{\mathfrak{s}(F)} \hat{f}(\tau(Y))\tilde{i}_M^G(\eta, X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}dY \\ &= |\det_{\mathfrak{s}(F)}(\tau)|_F^{-1/2} \int_{\mathfrak{s}(F)} \hat{f}(Y)\tilde{i}_M^G(\eta, X, \tau^{-1}(Y))|D^{\mathfrak{s}}(\tau^{-1}(Y))|_F^{-1/2}dY.\end{aligned}$$

We reduce ourselves to proving the equality

$$|\det_{\mathfrak{s}(F)}(\tau)|_F^{-1/2}\tilde{i}_M^G(\eta, X, \tau^{-1}(Y))|D^{\mathfrak{s}}(\tau^{-1}(Y))|_F^{-1/2} = \hat{i}_M^G(\eta, X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2} \tag{8.1.8}$$

for all $Y \in \mathfrak{s}_{\mathrm{rs}}(F)$. But we have the equality

$$\tilde{J}_M^G(\eta, X, \tilde{f}) = \hat{J}_M^G(\eta, X, \hat{f})$$

since both sides equal $J_M^G(\eta, X, f(-\cdot))$, which is defined by (4.1.1). The same computation as above shows that the equality (8.1.8) is true when one replaces $\tilde{i}_M^G$ and $\hat{i}_M^G$ with $\tilde{j}_M^G$ and $\hat{j}_M^G$ respectively. Recall that τ is $H(F)$-equivariant, so $H_Y = H_{\tau^{-1}(Y)}$ for $Y \in \mathfrak{s}_{\mathrm{rs}}(F)$. As a consequence, for $L \in \mathscr{L}^{G,\omega}(M_0)$, $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ if and only if $\tau^{-1}(Y) \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. One may conclude by Lemma 8.2 together with the $\eta(\mathrm{Nrd}(\cdot))$-invariance of $\hat{i}_M^G(\eta, X, \cdot)$ with respect to the adjoint action of $H(F)$ (Proposition 8.1). □

8.2. The case of (G', H'). Let $M' \in \mathscr{L}^{H'}(M_0')$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}_{\mathrm{rs}}')(F)$. We shall define a distribution $\hat{I}_{M'}^{H'}(Y, \cdot) \in \mathscr{D}^{\mathfrak{s}'}$ which is invariant with respect to the adjoint action of $H'(F)$ by induction on $\dim(H')$. Suppose that we have defined a distribution $\hat{I}_{M'}^{L'}(Y, \cdot) \in \mathscr{D}^{\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'}$ which is invariant with respect to the adjoint action of $L'(F)$ for all $L' \in \mathscr{L}^{H'}(M'), L' \neq H'$. This is actually a product form of the induction hypothesis in lower dimensions. Denote by $\hat{I}_{M'}^{L',H',w}(Y, \cdot)$ its image under $\mathrm{Ind}_{L'}^{H',w}$ (see (7.2.4)). For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we set

$$\hat{I}_{M'}^{H'}(Y, f') := \hat{J}_{M'}^{H'}(Y, f') - \sum_{L' \in \mathscr{L}^{H'}(M'), L' \neq H'} \hat{I}_{M'}^{L',H',w}(Y, f'). \tag{8.2.1}$$

Proposition 8.6. *The distribution $\hat{I}_{M'}^{H'}(Y, \cdot) \in \mathscr{D}^{\mathfrak{s}'}$ and is invariant with respect to the adjoint action of $H'(F)$.*

Proof. We may apply the argument of Proposition 8.1 thanks to the representability of $\hat{J}_{M'}^{H'}(Y, \cdot)$ (Proposition 7.10), Corollary 7.13 and Proposition 4.4.6). □

Let $M' \in \mathscr{L}^{H'}(M_0')$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}_{\mathrm{rs}}')(F)$. Denote by $\hat{i}_{M'}^{H'}(Y, \cdot)$ (resp. $\hat{j}_{M'}^{H'}(Y, \cdot)$) the element of $\mathscr{E}^{\mathfrak{s}'}$ associated to $\hat{I}_{M'}^{H'}(Y, \cdot)$ (resp. $\hat{J}_{M'}^{H'}(Y, \cdot)$) $\in \mathscr{D}^{\mathfrak{s}'}$ by (7.2.1). That is to say, for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$,

$$\hat{I}_{M'}^{H'}(Y, f') = \int_{\mathfrak{s}'(F)} f'(X)\hat{i}_{M'}^{H'}(Y, X)|D^{\mathfrak{s}'}(X)|_F^{-1/2}dX$$

and

$$\hat{J}_{M'}^{H'}(Y, f') = \int_{\mathfrak{s}'(F)} f'(X)\hat{j}_{M'}^{H'}(Y, X)|D^{\mathfrak{s}'}(X)|_F^{-1/2}dX.$$

One has a similar definition for the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M_0')$.

Lemma 8.7. *Let $M' \in \mathscr{L}^{H'}(M_0')$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}_{\mathrm{rs}}')(F)$. Let $L' \in \mathscr{L}^{H'}(M_0')$ and $X \in (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}_{\mathrm{rs}}')(F)_{\mathrm{ell}}$. Then $\hat{i}_{M'}^{H'}(Y, X) = \hat{j}_{M'}^{H'}(Y, X)$.*

Proof. We may apply the argument of Lemma 8.2 by using Proposition 8.6 and consulting the proofs of Corollary 7.13, Lemmas 7.11 and 3.17.1). □

Lemma 8.8. *Let $M' \in \mathscr{L}^{H'}(M_0')$.*
1) The function $(Y, X) \mapsto \hat{i}_{M'}^{H'}(Y, X)$ is locally constant on $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}_{\mathrm{rs}}')(F) \times \mathfrak{s}_{\mathrm{rs}}'(F)$.
2) If $w \in \mathrm{Norm}_{H'(F)}(M_0')$, $x \in M'(F)$ and $y \in H'(F)$, we have the equality

$$\hat{i}_{\mathrm{Ad}(w)(M')}^{H'}(\mathrm{Ad}(wx)(Y), \mathrm{Ad}(y)(X)) = \hat{i}_{M'}^{H'}(Y, X)$$

for all $(Y, X) \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}_{\mathrm{rs}}')(F) \times \mathfrak{s}_{\mathrm{rs}}'(F)$.

3) If $\lambda \in F^\times$, we have the equality

$$\hat{i}^{H'}_{M'}(\lambda Y, X) = \hat{i}^{H'}_{M'}(Y, \lambda X)$$

for all $(Y, X) \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F) \times \mathfrak{s}'_{\mathrm{rs}}(F)$.

4) Let $r'_{M'} \subseteq (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')(F)$ and $r' \subseteq \mathfrak{s}'(F)$ be two compact subsets. Then there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|\hat{i}^{H'}_{M'}(Y, X)| \leq c \sup\{1, -\log |D^{\mathfrak{s}'}(Y)|_F\}^N \sup\{1, -\log |D^{\mathfrak{s}'}(X)|_F\}^N$$

for all $Y \in r'_{M'} \cap \mathfrak{s}'_{\mathrm{rs}}$ and $X \in r' \cap \mathfrak{s}'_{\mathrm{rs}}$.

5) Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ and $r'_{M'} \subseteq (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')(F)$ be a compact subset. Then there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|\hat{I}^{H'}_{M'}(Y, f')| \leq c \sup\{1, -\log |D^{\mathfrak{s}'}(Y)|_F\}^N$$

for all $Y \in r'_{M'} \cap \mathfrak{s}'_{\mathrm{rs}}$.

Proof. It is almost the same as the proof of Lemma 8.3, except that one needs to use Howe's finiteness (Corollary 6.9), Proposition 8.6 and Lemma 8.7. We also need Proposition 4.4.2) for 1), Proposition 4.4.3) for 2), Corollary 4.6 for 4) and Corollary 3.21 for 5). □

For $M' \in \mathscr{L}^{H'}(M'_0)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, we define a distribution $I^{H'}_{M'}(Y, \cdot)$ on $\mathfrak{s}'(F)$ by

$$I^{H'}_{M'}(Y, \hat{f}') := \hat{I}^{H'}_{M'}(Y, f') \tag{8.2.2}$$

for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$. One may easily extend the definitions (8.2.1) and (8.2.2) to the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M'_0)$.

Remark 8.9. For $M' = H'$, it is evident that $I^{H'}_{H'}(Y, f') = J^{H'}_{H'}(Y, f')$ for all $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$.

Lemma 8.10. *Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. The distribution $I^{H'}_{M'}(Y, \cdot)$ on $\mathfrak{s}'(F)$ is independent of the choice of the $H'(F)$-invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{s}'(F)$ or the continuous nontrivial unitary character Ψ of F.*

Proof. We may apply the argument of Lemma 8.5 thanks to Proposition 8.6 and Lemma 8.7. □

9. The invariant trace formula

9.1. The case of (G, H). For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we define

$$\begin{aligned} I^G(\eta, f, f') := \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} \\ \hat{I}^G_M(\eta, X, f) I^G_G(\eta, X, f') dX. \end{aligned} \tag{9.1.1}$$

From Proposition 4.1.2), for any $\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})$, $I^G_G(\eta, \cdot, f')$ vanishes outside a compact subset of $\mathfrak{c}(F)$, so one may apply Lemma 8.3.5) to show that this expression is absolutely convergent with the help of Proposition 3.13 and Corollary 3.6.

Theorem 9.1 (Invariant trace formula). *For all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have the equality*

$$I^G(\eta, f, f') = I^G(\eta, f', f).$$

The rest of this section is devoted to the proof of Theorem 9.1. We shall follow the main steps in [51, §VII.2-3]. The theorem will be proved by induction on the dimension of G.

Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}_{\mathrm{rs}}(F))$ and $M \in \mathscr{L}^{G,\omega}(M_0)$. By Proposition 4.1.2) and 3), the function $\kappa(\cdot) J^G_M(\eta, \cdot, f) : (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F) \to \mathbb{C}$ is locally constant and invariant by the adjoint action of $M_H(F)$, where κ is defined by (4.1.2). Moreover, the support of its restriction to $\mathfrak{c}(F)$ for any $\mathfrak{c} \in \mathscr{T}(\mathfrak{m} \cap \mathfrak{s})$ is included in the compact subset $\mathfrak{c}_{\mathrm{reg}}(F) \cap \mathrm{Ad}(H(F))(\mathrm{Supp}(f))$. From Harish-Chandra's submersion principle (Lemma 3.8), there exists $f' \in \mathcal{C}_c^\infty((\mathfrak{m} \cap \mathfrak{s})_{\mathrm{rs}}(F))$ such that

$$\kappa(X) J^G_M(\eta, X, f) = |D^{\mathfrak{m} \cap \mathfrak{s}}(X)|_F^{1/2} \int_{M_{H,X}(F) \backslash M_H(F)} f'(\mathrm{Ad}(x^{-1})(X)) dx$$

for all $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Let $f'' := \kappa f' \in \mathcal{C}_c^\infty((\mathfrak{m} \cap \mathfrak{s})_{\mathrm{rs}}(F))$, where we extend the definition of κ to the product form. Then we have

$$J^G_M(\eta, X, f) = J^M_M(\eta, X, f''), \forall X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F). \tag{9.1.2}$$

We have shown that for $f \in \mathcal{C}_c^\infty(\mathfrak{s}_{\mathrm{rs}}(F))$ and $M \in \mathscr{L}^{G,\omega}(M_0)$, there exists a function $f'' \in \mathcal{C}_c^\infty((\mathfrak{m} \cap \mathfrak{s})_{\mathrm{rs}}(F))$ such that (9.1.2) holds. We shall fix such an f'' and denote it by $\phi_M^G(f)$.

As before, one may extend in the obvious way the definition (9.1.1) and the notation $\phi_M^G(f)$ to the symmetric pair $(M, M_H, \mathrm{Ad}(\epsilon))$, where $M \in \mathscr{L}^{G,\omega}(M_0)$, since it appears as the product of some copies of the form $(G, H, \mathrm{Ad}(\epsilon))$ in lower dimensions.

Lemma 9.2. *Let $M \in \mathscr{L}^{G,\omega}(M_0), X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ and $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}_{\mathrm{rs}}(F))$. Then we have the equality*

$$J_M^G(\eta, X, \hat{f}, f') = \sum_{L \in \mathscr{L}^G(M)} \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) \hat{I}_M^L(\eta, X, \phi_L^{L_1}(f_{\overline{Q_1}}^\eta)) I_L^L(\eta, X, \phi_L^{L_2}(f'^\eta_{Q_2})), \tag{9.1.3}$$

where $J_M^G(\eta, X, \hat{f}, f')$ is defined by (5.1.1), and $(Q_1, Q_2) := s(L_1, L_2)$ (see Section 2.7).

Proof. By definition,

$$J_M^G(\eta, X, \hat{f}, f') = |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{A_M(F)\backslash H(F)} f'(\mathrm{Ad}(y^{-1})(X)) \eta(\mathrm{Nrd}(y)) \varphi_1(y) dy,$$

where

$$\varphi_1(y) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{A_M(F)\backslash H(F)} \hat{f}(\mathrm{Ad}(x^{-1})(X)) \eta(\mathrm{Nrd}(x^{-1})) v_M(x, y) dx.$$

For $L \in \mathscr{L}^G(M)$ and $L_2 \in \mathscr{L}^G(L)$, since $Q_2 \in \mathscr{P}^G(L_2)$, by Proposition 4.1.4), we have

$$I_L^L(\eta, X, \phi_L^{L_2}(f'^\eta_{Q_2})) = J_L^{L_2}(\eta, X, f'^\eta_{Q_2}) = J_L^{Q_2}(\eta, X, f'). \tag{9.1.4}$$

Since $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, the right hand side of (9.1.3) is

$$|D^{\mathfrak{s}}(X)|_F^{1/2} \int_{A_M(F)\backslash H(F)} f'(\mathrm{Ad}(y^{-1})(X)) \eta(\mathrm{Nrd}(y)) \varphi_2(y) dy,$$

where

$$\varphi_2(y) := \sum_{L \in \mathscr{L}^G(M)} \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) \hat{I}_M^L(\eta, X, \phi_L^{L_1}(f_{\overline{Q_1}}^\eta)) v_L^{Q_2}(y).$$

It suffices to fix $y \in H(F)$ and prove that $\varphi_1(y) = \varphi_2(y)$.

Let $L \in \mathscr{L}^G(M)$ and

$$h_L := \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) \phi_L^{L_1}(f_{\overline{Q_1}}^\eta) v_L^{Q_2}(y).$$

Then

$$\varphi_2(y) = \sum_{L \in \mathscr{L}^G(M)} \hat{I}_M^L(\eta, X, h_L).$$

For $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we have

$$J_L^L(\eta, Y, h_L) = \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) J_L^L(\eta, Y, \phi_L^{L_1}(f_{\overline{Q_1}}^\eta)) v_L^{Q_2}(y).$$

For $L_1 \in \mathscr{L}^G(L)$, as in (9.1.4), we have

$$J_L^L(\eta, Y, \phi_L^{L_1}(f_{\overline{Q_1}}^\eta)) = J_L^{L_1}(\eta, Y, f_{\overline{Q_1}}^\eta) = J_L^{\overline{Q_1}}(\eta, Y, f).$$

Then

$$J_L^L(\eta, Y, h_L) = |D^{\mathfrak{s}}(Y)|_F^{1/2} \int_{H_Y(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(Y)) \eta(\mathrm{Nrd}(x)) h(x, y) dx,$$

where

$$h(x, y) := \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) v_L^{\overline{Q_1}}(x) v_L^{Q_2}(y).$$

It is shown in the proof of [51, Lemme VII.2] that

$$h(x, y) = v_L(x, y) = \sum_{Q \in \mathscr{F}^G(L)} v'_Q(y) v_L^{\overline{Q}}(x).$$

Thus

$$J_L^L(\eta, Y, h_L) = \sum_{Q \in \mathscr{F}^G(L)} v'_Q(y) J_L^{\overline{Q}}(\eta, Y, f).$$

As in (9.1.4), we have

$$J_L^{\overline{Q}}(\eta, Y, f) = J_L^{M_Q}(\eta, Y, f_{\overline{Q}}^{\eta}) = J_L^L(\eta, Y, \phi_L^{M_Q}(f_{\overline{Q}}^{\eta})).$$

Let

$$h'_L := \sum_{Q \in \mathscr{F}^G(L)} v'_Q(y) \phi_L^{M_Q}(f_{\overline{Q}}^{\eta}).$$

Then we obtain

$$J_L^L(\eta, Y, h_L) = J_L^L(\eta, Y, h'_L) \tag{9.1.5}$$

for all $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)$.

By a product form of Proposition 8.1, the distribution $d := \hat{I}_M^L(\eta, X, \cdot) \in \mathscr{D}^{\mathfrak{l} \cap \mathfrak{s}}$ and is $\eta(\mathrm{Nrd}(\cdot))$-invariant with respect to the adjoint action of $L_H(F)$. By a product form of (7.1.2), we deduce from (9.1.5) that

$$\hat{I}_M^L(\eta, X, h_L) = \hat{I}_M^L(\eta, X, h'_L).$$

Therefore,

$$\begin{aligned}\varphi_2(y) &= \sum_{L \in \mathscr{L}^G(M)} \hat{I}_M^L(\eta, X, h_L) = \sum_{L \in \mathscr{L}^G(M)} \hat{I}_M^L(\eta, X, h'_L) \\ &= \sum_{Q \in \mathscr{F}^G(M)} v'_Q(y) \sum_{L \in \mathscr{L}^{M_Q}(M)} \hat{I}_M^L(\eta, X, \phi_L^{M_Q}(f_{\overline{Q}}^{\eta})).\end{aligned}$$

By (7.1.10) and (7.1.2) (actually their product forms are needed), we have

$$\begin{aligned}&\hat{I}_M^{L, M_Q, w}(\eta, X, f_{\overline{Q}}^{\eta}) = \mathrm{Ind}_L^{M_Q, w}(d)(f_{\overline{Q}}^{\eta}) \\ =& \sum_{\{L' \in \mathscr{L}^{G,\omega}(M_0): L' \subseteq L\}} |W_0^{L'_n}||W_0^{L_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l}' \cap \mathfrak{s})} |W(L'_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_L^{M_Q}(\eta, Z, f_{\overline{Q}}^{\eta}) e_d(Z) dZ \\ =& \sum_{\{L' \in \mathscr{L}^{G,\omega}(M_0): L' \subseteq L\}} |W_0^{L'_n}||W_0^{L_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l}' \cap \mathfrak{s})} |W(L'_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_L^L(\eta, Z, \phi_L^{M_Q}(f_{\overline{Q}}^{\eta})) e_d(Z) dZ \\ =& d(\phi_L^{M_Q}(f_{\overline{Q}}^{\eta})) = \hat{I}_M^L(\eta, X, \phi_L^{M_Q}(f_{\overline{Q}}^{\eta})).\end{aligned}$$

Then by (8.1.1), we get

$$\begin{aligned}&\sum_{L \in \mathscr{L}^{M_Q}(M)} \hat{I}_M^L(\eta, X, \phi_L^{M_Q}(f_{\overline{Q}}^{\eta})) = \sum_{L \in \mathscr{L}^{M_Q}(M)} \hat{I}_M^{L, M_Q, w}(\eta, X, f_{\overline{Q}}^{\eta}) \\ =& \hat{J}_M^{M_Q}(\eta, X, f_{\overline{Q}}^{\eta}) = J_M^{M_Q}(\eta, X, \hat{f}_{\overline{Q}}^{\eta}) = J_M^{\overline{Q}}(\eta, X, \hat{f}).\end{aligned}$$

Hence,

$$\begin{aligned}\varphi_2(y) &= \sum_{Q \in \mathscr{F}^G(M)} v'_Q(y) J_M^{\overline{Q}}(\eta, X, \hat{f}) \\ &= |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{A_M(F) \backslash H(F)} \hat{f}(\mathrm{Ad}(x^{-1})(X)) \eta(\mathrm{Nrd}(x)) \sum_{Q \in \mathscr{F}^G(M)} v'_Q(y) v_M^{\overline{Q}}(x) dx.\end{aligned}$$

But

$$\sum_{Q \in \mathscr{F}^G(M)} v'_Q(y) v_M^{\overline{Q}}(x) = v_M(x, y),$$

which implies that $\varphi_1(y) = \varphi_2(y)$. □

Proof of Theorem 9.1. We use induction on the dimension of G. Suppose that the equality is true for $L \in \mathscr{L}^{G,\omega}(M_0), L \neq G$, which is actually a product form in lower dimensions. Now we would like to prove the equality for G. The argument below is also valid for the case $\mathscr{L}^{G,\omega}(M_0) = \{G\}$, i.e., $n = 1$.

First of all, suppose that $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}_{\mathrm{rs}}(F))$. Applying Lemma 9.2 to the definition (5.1.2) of $J^G(\eta, \hat{f}, f')$, we obtain

$$\begin{aligned}
J^G(\eta, \hat{f}, f') = & \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} \\
& J_M^G(\eta, X, \hat{f}, f')dX \\
= & \sum_{M \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{M_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_M/A_G)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} \\
& \sum_{L \in \mathscr{L}^G(M)} \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) \hat{I}_M^L(\eta, X, \phi_L^{L_1}(f_{\overline{Q}_1}^\eta)) I_L^L(\eta, X, \phi_L^{L_2}({f'}_{Q_2}^\eta))dX \\
= & \sum_{L \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{L_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_L/A_G)} B_L(\eta, f, f'),
\end{aligned}$$

where

$$\begin{aligned}
B_L(\eta, f, f') := & \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) \sum_{\{M \in \mathscr{L}^{G,\omega}(M_0) : M \subseteq L\}} |W_0^{M_n}||W_0^{L_n}|^{-1}(-1)^{\dim(A_M/A_L)} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m}\cap\mathfrak{s})} \\
& |W(M_H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} \hat{I}_M^L(\eta, X, \phi_L^{L_1}(f_{\overline{Q}_1}^\eta)) I_L^L(\eta, X, \phi_L^{L_2}({f'}_{Q_2}^\eta))dX \\
= & \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) I^L(\eta, \phi_L^{L_1}(f_{\overline{Q}_1}^\eta), \phi_L^{L_2}({f'}_{Q_2}^\eta)).
\end{aligned}$$

Here we have used the absolute convergence of the expressions above to exchange the order of sums, and $I^L(\eta, \phi_L^{L_1}(f_{\overline{Q}_1}^\eta), \phi_L^{L_2}({f'}_{Q_2}^\eta))$ is defined by a product form of (9.1.1).

By the noninvariant trace formula (Theorem 5.3) and Remark 5.2, we have the equality $J^G(\eta, \hat{f}, f') = J^G(\eta, \hat{f}', f)$. Therefore,

$$\sum_{L \in \mathscr{L}^{G,\omega}(M_0)} |W_0^{L_n}||W_0^{GL_n}|^{-1}(-1)^{\dim(A_L/A_G)}(B_L(\eta, f, f') - B_L(\eta, f', f)) = 0. \tag{9.1.6}$$

Let $L \in \mathscr{L}^{G,\omega}(M_0), L \neq G$. Applying the induction hypothesis, we have

$$B_L(\eta, f, f') = \sum_{L_1, L_2 \in \mathscr{L}^G(L)} d_L^G(L_1, L_2) I^L(\eta, \phi_L^{L_2}({f'}_{Q_2}^\eta), \phi_L^{L_1}(f_{\overline{Q}_1}^\eta)).$$

By exchanging L_1 and L_2 and by using (2) and (5) in Section 2.7, we obtain $B_L(\eta, f, f') = B_L(\eta, f', f)$. We deduce from (9.1.6) that $B_G(\eta, f, f') = B_G(\eta, f', f)$. But

$$B_G(\eta, f, f') = I^G(\eta, f, f'),$$

which implies $I^G(\eta, f, f') = I^G(\eta, f', f)$.

Now consider $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ in general. Let $\{\Omega_i\}_{i \geq 1}$ be a sequence of increasing open compact subsets of $\mathfrak{s}_{\mathrm{rs}}(F)$ such that $\bigcup_{i \geq 1}^{\infty} \Omega_i = \mathfrak{s}_{\mathrm{rs}}(F)$. Such a consequence exists. For example, one may take $\Omega_i := \{X \in \mathfrak{s}_{\mathrm{rs}}(F) : \|X\| \leq i\}$ for all $i \geq 1$, where $\|\cdot\|$ denotes the abstract norm on $\mathfrak{s}_{\mathrm{rs}}(F)$ defined by [35, (18.2.1) in §18.2]. From [35, Proposition 18.1.(3)], since $\|\cdot\|$ is continuous, we deduce that Ω_i is compact for all $i \geq 1$. It is obvious that Ω_i is open for all $i \geq 1$ and that $\bigcup_{i \geq 1}^{\infty} \Omega_i = \mathfrak{s}_{\mathrm{rs}}(F)$. For all $i \geq 1$, denote by 1_{Ω_i} the characteristic function of Ω_i. Let $f_i := f 1_{\Omega_i}$ and $f_i' := f' 1_{\Omega_i}$.

Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $\mathfrak{c} \in \mathscr{L}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})$. For all $X \in \mathfrak{c}_{\mathrm{reg}}(F)$, by Lebesgue's dominated convergence theorem, we have $\lim_{i \to \infty} I_G^G(\eta, X, f_i') = I_G^G(\eta, X, f')$. For $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, because $\hat{I}_M^G(\eta, X, \cdot) \in \mathscr{D}^{\mathfrak{s}}$ (see Proposition 8.1), again by Lebesgue's dominated convergence theorem, we have $\lim_{i \to \infty} \hat{I}_M^G(\eta, X, f_i) = \hat{I}_M^G(\eta, X, f)$. Because of Lemma 3.3 applied to $\mathrm{Supp}(f')$, there exists a compact subset $r \subseteq \mathfrak{c}(F)$ such that for all $X \in \mathfrak{c}_{\mathrm{reg}}(F) - r, I_G^G(\eta, X, f_i') = 0$ for all $i \geq 1$. By Lemma 8.3.4) applied to r and $\mathrm{Supp}(f')$,

there exists $c > 0$ and $N \in \mathbb{N}$ such that

$$|\hat{I}_M^G(\eta, X, f_i)| = \int_{\mathfrak{s}(F)} |f_i(Y)\hat{i}_M^G(\eta, X, Y)||D^{\mathfrak{s}}(Y)|_F^{-1/2} dY$$
$$\leq c \sup\{1, -\log|D^{\mathfrak{s}}(X)|_F\}^N \int_{\mathfrak{s}(F)} |f(Y)| \sup\{1, -\log|D^{\mathfrak{s}}(Y)|_F\}^N |D^{\mathfrak{s}}(Y)|_F^{-1/2} dY$$

for all $i \geq 1$ and $X \in r \cap \mathfrak{c}_{\mathrm{reg}}$. For all $X \in \mathfrak{s}_{\mathrm{rs}}(F)$, we also have $|I_G^G(\eta, X, f_i')| \leq I_X(|f'|)$, where I_X is defined by (3.1.3). Combining Corollary 3.15, Corollary 3.6 and Proposition 3.13, we deduce that $\{\hat{I}_M^G(\eta, X, f_i) I_G^G(\eta, X, f_i')\}_{i \geq 1}$ is bounded by an integrable function on $\mathfrak{c}_{\mathrm{reg}}(F)$. Using Lebesgue's dominated convergence theorem once again, we obtain

$$\int_{\mathfrak{c}_{\mathrm{reg}}(F)} \hat{I}_M^G(\eta, X, f) I_G^G(\eta, X, f') dX = \lim_{i \to \infty} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} \hat{I}_M^G(\eta, X, f_i) I_G^G(\eta, X, f_i') dX.$$

Therefore,

$$I^G(\eta, f, f') = \lim_{i \to \infty} I^G(\eta, f_i, f_i').$$

By exchanging f and f' and using the regular semi-simple support case that we have proved, we draw the conclusion. □

Corollary 9.3. *Let $M, L \in \mathscr{L}^{G,\omega}(M_0)$, $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ and $Y \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. Then we have the equality*

$$(-1)^{\dim(A_M/A_G)} \hat{i}_M^G(\eta, X, Y) = (-1)^{\dim(A_L/A_G)} \hat{i}_L^G(\eta, Y, X).$$

Proof. By Lemma 8.3.2), up to $M_H(F)$-conjugation on X and $L_H(F)$-conjugation on Y, we may and shall suppose that there exists $\mathfrak{c}_1 \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{m} \cap \mathfrak{s})$ and $\mathfrak{c}_2 \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{l} \cap \mathfrak{s})$ such that $X \in \mathfrak{c}_{1,\mathrm{reg}}(F)$ and $Y \in \mathfrak{c}_{2,\mathrm{reg}}(F)$. As in the proof of Proposition 7.2, we can choose an open compact neighbourhood V_1 of X in $\mathfrak{c}_{1,\mathrm{reg}}$ (resp. V_2 of Y in $\mathfrak{c}_{2,\mathrm{reg}}$) such that if two elements in V_1 (resp. V_2) are $H(F)$-conjugate, then they are the same. Let $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ with $\mathrm{Supp}(f) \subseteq \mathrm{Ad}(H(F))(V_2)$ and $\mathrm{Supp}(f') \subseteq \mathrm{Ad}(H(F))(V_1)$. By an analogous calculation to that of $J^G(\eta, \hat{f}, f')$ in the proof of Proposition 7.2, with the help of Lemma 8.3.2) and Proposition 4.1.3), we show the equalities

$$I^G(\eta, f, f') = (-1)^{\dim(A_M/A_G)} \int_{V_1} \hat{I}_M^G(\eta, X_1, f) I_G^G(\eta, X_1, f') dX_1$$

and

$$\hat{I}_M^G(\eta, X_1, f) = \int_{V_2} \hat{i}_M^G(\eta, X_1, Y_2) I_G^G(\eta, Y_2, f) dY_2$$

for all $X_1 \in V_1$ by (7.1.2). Then

$$I^G(\eta, f, f') = (-1)^{\dim(A_M/A_G)} \int_{V_1 \times V_2} \hat{i}_M^G(\eta, X_1, Y_2) I_G^G(\eta, Y_2, f) I_G^G(\eta, X_1, f') dY_2 dX_1.$$

Similarly, we have

$$I^G(\eta, f', f) = (-1)^{\dim(A_L/A_G)} \int_{V_2 \times V_1} \hat{i}_L^G(\eta, Y_2, X_1) I_G^G(\eta, X_1, f') I_G^G(\eta, Y_2, f) dX_1 dY_2.$$

By Harish-Chandra's submersion principle (Lemma 3.8), when f' varies, the function $X_1 \mapsto I_{X_1}(f') = \kappa(X_1) I_G^G(\eta, X_1, \kappa f')$ on V_1 runs over all $\mathcal{C}_c^\infty(V_1)$, so the function $I_G^G(\eta, \cdot, f')$ on V_1 also runs over all $\mathcal{C}_c^\infty(V_1)$. Similarly, when f varies, the function $I_G^G(\eta, \cdot, f)$ on V_2 runs over all $\mathcal{C}_c^\infty(V_2)$. Then from the invariant trace formula (Theorem 9.1), we deduce that

$$(-1)^{\dim(A_M/A_G)} \hat{i}_M^G(\eta, X_1, Y_2) = (-1)^{\dim(A_L/A_G)} \hat{i}_L^G(\eta, Y_2, X_1)$$

for all $(X_1, Y_2) \in V_1 \times V_2$. We conclude by $(X, Y) \in V_1 \times V_2$. □

9.2. The case of (G', H'). For $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define

$$I^{H'}(f, f') := \sum_{M' \in \mathscr{L}^{H'}(M_0')} |W_0^{H'}||W_0^{M'}|^{-1}(-1)^{\dim(A_{M'}/A_{H'})} \sum_{\mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')} |W(M', \mathfrak{c}')|^{-1} \int_{\mathfrak{c}'_{\mathrm{reg}}(F)} \hat{I}_{M'}^{H'}(Y, f) I_{H'}^{H'}(Y, f') dY. \tag{9.2.1}$$

From Proposition 4.4.2), for any $\mathfrak{c}' \in \mathscr{T}_{\mathrm{ell}}(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')$, $I_{H'}^{H'}(\cdot, f')$ vanishes outside a compact subset of $\mathfrak{c}'(F)$, so one may apply Lemma 8.8.5) to show that this expression is absolutely convergent with the help of Proposition 3.20 and Corollary 3.6. One may extend in the obious way the definition (9.2.1) to the symmetric pair $(\widetilde{M'}, M', \mathrm{Ad}(\alpha))$, where $M' \in \mathscr{L}^{H'}(M_0')$.

THEOREM 9.4 (Invariant trace formula). *For all $f, f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we have the equality*

$$I^{H'}(f, f') = I^{H'}(f', f).$$

PROOF. We may apply the argument of Theorem 9.1 with obvious modifications. It is deduced from the noninvariant trace formula (Theorem 5.12) and other results that we have prepared in previous sections. □

COROLLARY 9.5. *Let $M', L' \in \mathscr{L}^{H'}(M_0')$, $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$ and $X \in (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$. Then we have the equality*

$$(-1)^{\dim(A_{M'}/A_{H'})} \hat{i}_{M'}^{H'}(Y, X) = (-1)^{\dim(A_{L'}/A_{H'})} \hat{i}_{L'}^{H'}(Y, X).$$

PROOF. We may apply the argument of Corollary 9.3 by using the invariant trace formula (Theorem 9.4) and consulting the proof of Proposition 7.10. □

10. A vanishing property at infinity

10.1. The case of (G, H). The following proposition is an analogue of [14, Proposition 2.2].

PROPOSITION 10.1. *Let $M \in \mathscr{L}^{G,\omega}(M_0)$, $M \neq G$. Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in \mathfrak{s}_{\mathrm{rs}}(F)$. Then there exists $N \in \mathbb{N}$ such that if $\lambda \in F^\times$ satisfies $v_F(\lambda) < -N$, we have*

$$\hat{i}_M^G(\eta, \lambda X, Y) = 0.$$

REMARK 10.2. A limit formula at infinity for $\hat{i}_G^G(\eta, \lambda X, Y)$ in the spirit of Laplace transform is given in [58, Proposition 7.1] (see also [57, Proposition 6.4]), which is an analogue of [51, Proposition VIII.1].

PROOF OF PROPOSITION 10.1. We shall imitate the proof of [14, Proposition 2.2].

By Lemma 8.3.2), up to $H(F)$-conjugation on Y, we may and shall suppose that there exists $L \in \mathscr{L}^{G,\omega}(M_0)$ and an L-elliptic Cartan subspace $\mathfrak{c} \subseteq \mathfrak{l} \cap \mathfrak{s}$ such that $Y \in \mathfrak{c}_{\mathrm{reg}}(F)$. By Lemma 8.2, we have the equality

$$\hat{i}_M^G(\eta, \lambda X, Y) = \hat{j}_M^G(\eta, \lambda X, Y).$$

Thus it suffices to prove that there exists $N \in \mathbb{N}$ such that if $\lambda \in F^\times$ satisfies $v_F(\lambda) < -N$, we have

$$\hat{j}_M^G(\eta, \lambda X, Y) = 0.$$

Fix an $\mathcal{O}_F$-lattice $k_{\mathfrak{h}}$ (resp. $k_{\mathfrak{s}}$) of $\mathfrak{h}(F)$ (resp. $\mathfrak{s}(F)$). Denote by $\widetilde{k_{\mathfrak{s}}}$ the dual $\mathcal{O}_F$-lattice of $k_{\mathfrak{s}}$ in $\mathfrak{s}(F)$, i.e.,

$$\widetilde{k_{\mathfrak{s}}} := \{Z \in \mathfrak{s}(F) : \forall Z' \in k_{\mathfrak{s}}, \Psi(\langle Z, Z' \rangle) = 1\}.$$

Set

$$\mathfrak{c}(X) := \{X' \in \mathfrak{c}(F) : \exists x \in H(F), \mathrm{Ad}(x)(X') = X\},$$

which is a finite (perhaps empty) set. For $\lambda \in F^\times$, choose $h_\lambda \in \mathbb{N}$ such that both of the functions $\hat{j}_M^G(\eta, \lambda X, \cdot)$ and $|D^{\mathfrak{s}}(\cdot)|_F$ are constant on $Y + \varpi^{h_\lambda} k_{\mathfrak{s}}$.

Let f (resp. f') $\in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ be the characteristic function of $Y + \varpi^{h_\lambda} k_{\mathfrak{s}}$ (resp. $\varpi^{-h_\lambda} \widetilde{k_{\mathfrak{s}}}$). Then for $Z \in \mathfrak{s}(F)$, we see that

$$\begin{aligned}\hat{f}(Z) =& c_\Psi(\mathfrak{s}(F)) \int_{Y + \varpi^{h_\lambda} k_{\mathfrak{s}}} \Psi(\langle Z, Z' \rangle) dZ' = c_\Psi(\mathfrak{s}(F)) \Psi(\langle Z, Y \rangle) \int_{\varpi^{h_\lambda} k_{\mathfrak{s}}} \Psi(\langle Z, Z' \rangle) dZ' \\ =& c_\Psi(\mathfrak{s}(F)) \mathrm{vol}(\varpi^{h_\lambda} k_{\mathfrak{s}}) \Psi(\langle Z, Y \rangle) f'(Z).\end{aligned}$$

Now there are two expressions for $J_M^G(\eta, \lambda X, \hat{f})$. One the one hand,
(10.1.1)
$$J_M^G(\eta, \lambda X, \hat{f}) = \int_{\mathfrak{s}(F)} f(Z)\hat{j}_M^G(\eta, \lambda X, Z)|D^{\mathfrak{s}}(Z)|_F^{-1/2}dZ = \mathrm{vol}(\varpi^{h_\lambda}k_{\mathfrak{s}})\hat{j}_M^G(\eta, \lambda X, Y)|D^{\mathfrak{s}}(Y)|_F^{-1/2}.$$
On the other hand,
(10.1.2)
$$J_M^G(\eta, \lambda X, \hat{f}) = |D^{\mathfrak{s}}(\lambda X)|_F^{1/2}\int_{H_{\lambda X}(F)\backslash H(F)} \hat{f}(\mathrm{Ad}(x^{-1})(\lambda X))\eta(\mathrm{Nrd}(x))v_M^G(x)dx$$
$$=c_\Psi(\mathfrak{s}(F))\mathrm{vol}(\varpi^{h_\lambda}k_{\mathfrak{s}})|D^{\mathfrak{s}}(\lambda X)|_F^{1/2}\int_{H_X(F)\backslash H(F)} \Psi(\langle \mathrm{Ad}(x^{-1})(\lambda X), Y\rangle)f'(\mathrm{Ad}(x^{-1})(\lambda X))\eta(\mathrm{Nrd}(x))v_M^G(x)dx.$$

Fix an open neighbourhood $\mathscr{V}_{\mathfrak{h}}$ of 0 in $\mathfrak{h}(F)$ which is invariant by the adjoint action of $H(F)$ such that a homeomorphic exponential map is defined on $\mathscr{V}_{\mathfrak{h}}$. Choose $a \in \mathbb{N}$ verifying the following conditions:

(1) $\varpi^a k_{\mathfrak{h}} \subseteq \mathscr{V}_{\mathfrak{h}}$;
(2) $K_a := \exp(\varpi^a k_{\mathfrak{h}})$ is a subgroup of K_H;
(3) $\eta(\mathrm{Nrd}(K_a)) = 1$;
(4) the adjoint action of K_a stabilises $k_{\mathfrak{s}}$ (and thus $\widetilde{k_{\mathfrak{s}}}$).

Fix a set Γ of representatives in $H(F)$ of double cosets $H_X(F)\backslash H(F)/K_a$. We may and shall suppose that if $x \in \Gamma$ and $y \in H_X(F)xK_a$ verify $\mathrm{Ad}(y^{-1})(X) \in \mathfrak{c}(F)$, then $\mathrm{Ad}(x^{-1})(X) \in \mathfrak{c}(F)$.

The integral in (10.1.2) can be decomposed as
$$\sum_{x\in\Gamma}\int_{H_X(F)\backslash H_X(F)xK_a} \Psi(\langle \mathrm{Ad}(y^{-1})(\lambda X), Y\rangle)f'(\mathrm{Ad}(y^{-1})(\lambda X))\eta(\mathrm{Nrd}(y))v_M^G(y)dy.$$
By the conditions (2), (3) and (4) on a respectively, the factors v_M^G, η and f' can be extracted from the integral. By comparing (10.1.1) and (10.1.2), since $\langle\cdot,\cdot\rangle$ is invariant by the adjoint action of K_a, we obtain
(10.1.3)
$$\begin{aligned}\hat{j}_M^G(\eta, \lambda X, Y) =& c_\Psi(\mathfrak{s}(F))|D^{\mathfrak{s}}(\lambda X)D^{\mathfrak{s}}(Y)|_F^{1/2}\sum_{x\in\Gamma} f'(\mathrm{Ad}(x^{-1})(\lambda X))\eta(\mathrm{Nrd}(x))v_M^G(x)\int_{H_X(F)\backslash H_X(F)xK_a}\\ &\Psi(\langle \mathrm{Ad}(y^{-1})(\lambda X), Y\rangle)dy\\ =&c_\Psi(\mathfrak{s}(F))|D^{\mathfrak{s}}(\lambda X)D^{\mathfrak{s}}(Y)|_F^{1/2}\mathrm{vol}(K_a)^{-1}\sum_{x\in\Gamma}\mathrm{vol}(H_X(F)\backslash H_X(F)xK_a)f'(\mathrm{Ad}(x^{-1})(\lambda X))\\ &\eta(\mathrm{Nrd}(x))v_M^G(x)i(x),\end{aligned}$$
where
$$i(x) := \int_{K_a}\Psi(\langle \mathrm{Ad}(x^{-1})(\lambda X), \mathrm{Ad}(y)(Y)\rangle)dy.$$

For $x \in \Gamma$, consider the map $K_a \to F$ defined by
$$\forall y \in K_a, y \mapsto \langle \mathrm{Ad}(x^{-1})(X), \mathrm{Ad}(y)(Y)\rangle. \tag{10.1.4}$$
Its differential at the point $y_0 \in K_a$ is the map $\mathfrak{h}(F) \to F$ defined by
$$\forall Z \in \mathfrak{h}(F), Z \mapsto \langle \mathrm{Ad}(x^{-1})(X), \mathrm{Ad}(y_0)([Z, Y])\rangle. \tag{10.1.5}$$
Since $\langle\cdot,\cdot\rangle$ is invariant by the adjoint action of $G(F)$, we see that
$$\langle \mathrm{Ad}(x^{-1})(X), \mathrm{Ad}(y_0)([Z, Y])\rangle = \langle [Y, \mathrm{Ad}(xy_0)^{-1}(X)], Z\rangle.$$
Because the restriction of $\langle\cdot,\cdot\rangle$ to $\mathfrak{h}(F)$ is non-degenerate, the map (10.1.5) is not surjective if and only if
$$[Y, \mathrm{Ad}(xy_0)^{-1}(X)] = 0.$$
Since $Y \in \mathfrak{c}_{\mathrm{reg}}(F)$, this condition is equivalent to
$$\mathrm{Ad}(xy_0)^{-1}(X) \in \mathfrak{c}(F).$$
From our choice of Γ, as $y_0 \in K_a$, it implies that
$$\mathrm{Ad}(x^{-1})(X) \in \mathfrak{c}(F).$$

Let
$$\Gamma' := \{x \in \Gamma : \mathrm{Ad}(x^{-1})(X) \in \mathfrak{c}(F)\},$$

which is a finite (perhaps empty) subset of Γ. Then for $x \in \Gamma - \Gamma'$, the map (10.1.4) is a submersion. Define

$$\Omega := \bigcup_{x \in \Gamma - \Gamma'} H_X(F) x K_a,$$

which is an open and closed subset of $H(F)$. Fix a basis of the F-linear space $\mathfrak{s}(F)$. For $Z \in \mathfrak{s}(F)$, define its norm $\|Z\| \in \mathbb{R}_{\geq 0}$ as the maximum of normalised absolute values of coefficients of Z with respect to the fixed basis. For $Z \in \mathfrak{s}(F) - \{0\}$, define $\nu(Z) \in \mathbb{Z}$ by $\|Z\| = |\varpi^{\nu(Z)}|_F$. Let S_X be the closure of

$$S_X^0 := \{\varpi^{-\nu(\mathrm{Ad}(y^{-1})(X))} \mathrm{Ad}(y^{-1})(X) : y \in \Omega\}$$

in the unit sphere $S_{\mathfrak{s}} := \{Z \in \mathfrak{s}(F) : \|Z\| = 1\}$. Then S_X is compact. Recall that we denote by $\mathcal{N}^{\mathfrak{s}}$ the set of nilpotent elements in $\mathfrak{s}(F)$.

Lemma 10.3. *We have*

$$S_X - S_X^0 \subseteq \mathcal{N}^{\mathfrak{s}} - \{0\}.$$

Proof of Lemma 10.3. Since $S_X \subseteq S_{\mathfrak{s}}$, it is obvious that $\{0\} \notin S_X$. Let $Z \in S_X$. There exists a sequence $\{y_i\}$ in Ω such that when $i \to \infty$,

$$\varpi^{-\nu(\mathrm{Ad}(y_i^{-1})(X))} \mathrm{Ad}(y_i^{-1})(X) \to Z.$$

We distinguish two cases.

i) Suppose that the sequence $\{\|\mathrm{Ad}(y_i^{-1})(X)\|\}$ remains bounded. By Harish-Chandra's compactness lemma for symmetric spaces (Lemma 3.2), the projection of the sequence $\{y_i\}$ to $H_X(F)\backslash H(F)$ is contained in a compact subset. By taking a subsequence, since the projection of Ω to $H_X(F)\backslash H(F)$ is closed, we may assume that when $i \to \infty$, $\mathrm{Ad}(y_i^{-1})(X) \to \mathrm{Ad}(y^{-1})(X)$ with $y \in \Omega$. Thus $Z \in S_X^0$ in this case.

ii) Suppose that the sequence $\{\|\mathrm{Ad}(y_i^{-1})(X)\|\}$ is unbounded. By taking a subsequence, we may assume that when $i \to \infty$, $\|\mathrm{Ad}(y_i^{-1})(X)\| \to +\infty$. The eigenvalues of $\mathrm{ad}(\mathrm{Ad}(y_i^{-1})(X))$ are the same as those of $\mathrm{ad}(X)$; here $\mathrm{ad}(\mathrm{Ad}(y_i^{-1})(X))$ and $\mathrm{ad}(X)$ are viewed as linear endomorphisms of $\mathfrak{g}$. Thus the eigenvalues of $\mathrm{ad}(\varpi^{-\nu(\mathrm{Ad}(y_i^{-1})(X))} \mathrm{Ad}(y_i^{-1})(X))$ tend to zero when $i \to \infty$. Hence $\mathrm{ad}(Z)$ is nilpotent. We shall prove that $Z \in \mathcal{N}^{\mathfrak{s}}$ in this case.

Since $\mathfrak{g}$ is reductive, one has $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_{\mathrm{der}}$, where $\mathfrak{z}$ denotes the centre of $\mathfrak{g}$ and $\mathfrak{g}_{\mathrm{der}}$ denotes the derived algebra of $\mathfrak{g}$, and $\mathfrak{g}_{\mathrm{der}}$ is semisimple. Let $Z = Z_1 + Z_2$ with $Z_1 \in \mathfrak{z}(F)$ and $Z_2 \in \mathfrak{g}_{\mathrm{der}}(F)$. Since $\mathrm{ad}(Z)$ is nilpotent as a linear endomorphism of $\mathfrak{g}$, we deduce that $\mathrm{ad}(Z_2)$ is nilpotent as a linear endomorphism of $\mathfrak{g}_{\mathrm{der}}$. As $\mathfrak{g}_{\mathrm{der}}$ is semisimple, we obtain that Z_2 is a nilpotent element in $\mathfrak{g}$. Let $X = X_1 + X_2$ with $X_1 \in \mathfrak{z}(F)$ and $X_2 \in \mathfrak{g}_{\mathrm{der}}(F)$. The projection of $\varpi^{-\nu(\mathrm{Ad}(y_i^{-1})(X))} \mathrm{Ad}(y_i^{-1})(X)$ to $\mathfrak{z}(F)$ is equal to $\varpi^{-\nu(\mathrm{Ad}(y_i^{-1})(X))} X_1$, which tends to zero when $i \to \infty$. Thus $Z_1 = 0$, and $Z = Z_2$ is a nilpotent element in $\mathfrak{g}$. Hence $Z \in \mathcal{N}^{\mathfrak{s}}$. □

For $U \in \mathcal{N}^{\mathfrak{s}} - \{0\}$, consider the map $K_a \to F$ defined by

$$\forall y \in K_a, y \mapsto \langle U, \mathrm{Ad}(y)(Y) \rangle. \tag{10.1.6}$$

Its differential at the point $y_0 \in K_a$ is the map $\mathfrak{h}(F) \to F$ defined by

$$\forall Z \in \mathfrak{h}(F), Z \mapsto \langle U, \mathrm{Ad}(y_0)([Z, Y]) \rangle = \langle [Y, \mathrm{Ad}(y_0^{-1})(U)], Z \rangle$$

by the $G(F)$-invariance of $\langle \cdot, \cdot \rangle$. Since $Y \in \mathfrak{c}_{\mathrm{reg}}(F)$ and $\mathrm{Ad}(y_0^{-1})(U) \in \mathcal{N}^{\mathfrak{s}} - \{0\}$, we have

$$[Y, \mathrm{Ad}(y_0^{-1})(U)] \neq 0.$$

Then the map (10.1.6) is a submersion by the non-degeneration of $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}(F)$.

Using Lemma 10.3 and combining our discussion on the maps (10.1.4) and (10.1.6), we deduce that there exists an open compact neighbourhood $\widetilde{S_X}$ of S_X in $S_{\mathfrak{s}}$ such that the map $\varphi : K_a \times \widetilde{S_X} \to F \times \widetilde{S_X}$ defined by

$$\forall (y, Z) \in K_a \times \widetilde{S_X}, (y, Z) \mapsto (\langle Z, \mathrm{Ad}(y)(Y) \rangle, Z)$$

is a submersion. Since any submersion is open, the image of φ (denoted by $\mathrm{Im}(\varphi)$) is an open compact subset of $F \times \widetilde{S_X}$. Then the map φ induces a surjective submersion $\varphi' : K_a \times \widetilde{S_X} \to \mathrm{Im}(\varphi)$. Applying Harish-Chandra's submersion principle [26, Theorem 11] to φ', there exists a function $\phi \in \mathcal{C}_c^\infty(\mathrm{Im}(\varphi))$ such that for all $\Phi' \in \mathcal{C}_c^\infty(\mathrm{Im}(\varphi))$,

$$\int_{K_a \times \widetilde{S_X}} \Phi'(\langle Z, \mathrm{Ad}(y)(Y) \rangle, Z) dZ dy = \int_{\mathrm{Im}(\varphi)} \phi(t, Z) \Phi'(t, Z) dZ dt.$$

Fix such a ϕ. Denote by $\mathcal{C}^\infty(F\times\widetilde{S_X})$ the space of locally constant, complexed-valued functions on $F\times\widetilde{S_X}$. For $\Phi\in\mathcal{C}^\infty(F\times\widetilde{S_X})$, the restriction of Φ to $\mathrm{Im}(\varphi)$ belongs to $\mathcal{C}_c^\infty(\mathrm{Im}(\varphi))$, so we obtain

$$\int_{K_a\times\widetilde{S_X}}\Phi(\langle Z,\mathrm{Ad}(y)(Y)\rangle,Z)dZdy=\int_{F\times\widetilde{S_X}}\phi(t,Z)\Phi(t,Z)dZdt.$$

By taking $\Phi(t,Z):=\Psi(\mu t)\beta(Z)$ with $\mu\in F$ and $\beta\in\mathcal{C}_c^\infty(\widetilde{S_X})$, we deduce that for all $Z\in\widetilde{S_X}$,

$$\int_{K_a}\Psi(\mu\langle Z,\mathrm{Ad}(y)(Y)\rangle)dy=\int_F\phi(t,Z)\Psi(\mu t)dt.$$

Since $\mathrm{Im}(\phi)$ is an open compact subset of $F\times\widetilde{S_X}$, we see that $\phi\in\mathcal{C}_c^\infty(\mathrm{Im}(\varphi))\subseteq\mathcal{C}_c^\infty(F\times\widetilde{S_X})=\mathcal{C}_c^\infty(F)\otimes\mathcal{C}_c^\infty(\widetilde{S_X})$. Suppose that $\phi=\sum_{1\leq j\leq m}c_j\cdot\xi_j\otimes\chi_j$ with $c_j\in\mathbb{C},\xi_j\in\mathcal{C}_c^\infty(F)$ and $\chi_j\in\mathcal{C}_c^\infty(\widetilde{S_X})$. Then

$$\int_F\phi(t,Z)\Psi(\mu t)dt=\sum_{1\leq j\leq m}c_j\hat{\xi}_j(\mu)\chi_j(Z),$$

where $\hat{\xi}_j\in\mathcal{C}_c^\infty(F)$ is the Fourier transform of ξ_j. We see that there exists $N_0\in\mathbb{N}$ such that for all $\mu\in F^\times$ satisfying $v_F(\mu)<-N_0$ and all $Z\in S_X$, we have

$$\int_{K_a}\Psi(\mu\langle Z,\mathrm{Ad}(y)(Y)\rangle)dy=0.$$

Fix such an N_0.

Recall that for $x\in\Gamma$,

$$i(x)=\int_{K_a}\Psi(\mu\langle Z,\mathrm{Ad}(y)(Y)\rangle)dy,$$

where $\mu:=\lambda\varpi^{\nu(\mathrm{Ad}(x^{-1})(X))}$ and $Z:=\varpi^{-\nu(\mathrm{Ad}(x^{-1})(X))}\mathrm{Ad}(x^{-1})(X)$. For $x\in\Gamma-\Gamma'$, we have $Z\in S_X$, so $i(x)=0$ if

$$v_F(\lambda)+\nu(\mathrm{Ad}(x^{-1})(X))<-N_0.$$

Set

$$\nu_0:=\sup_{x\in\Gamma}\nu(\mathrm{Ad}(x^{-1})(X)),$$

which is finite thanks to Harish-Chandra's compactness lemma for symmetric spaces (Lemma 3.2). Now let

$$N:=N_0+\nu_0.$$

Suppose that $v_F(\lambda)<-N$. From (10.1.3), to show $\hat{j}_M^G(\eta,\lambda X,Y)=0$, it suffices to prove $v_M^G(x)=0$ for all $x\in\Gamma'$.

For $x\in\Gamma'$, we have $\mathrm{Ad}(x^{-1})(X)\in\mathfrak{c}_{\mathrm{reg}}(F)$. Then $\mathrm{Ad}(x^{-1})(H_X)=T_{\mathfrak{c}}$. Since $X\in(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$, we see that $\mathrm{Ad}(x^{-1})(A_M)$ is an F-split torus in $T_{\mathfrak{c}}$. As $\mathfrak{c}\subseteq\mathfrak{l}\cap\mathfrak{s}$ is L-elliptic, A_L is the maximal F-split torus in $T_{\mathfrak{c}}$. Thus $\mathrm{Ad}(x^{-1})(A_M)\subseteq A_L$. Then $A_M\subseteq\mathrm{Ad}(x)(A_L)\subseteq\mathrm{Ad}(x)(A_{M_0})$. We deduce that $\mathrm{Ad}(x)(A_{M_0})$ is a maximal F-split torus in M_H, so it is $M_H(F)$-conjugate to A_{M_0}. Therefore, $x\in M_H(F)\mathrm{Norm}_{H(F)}(M_0)\subseteq M_H(F)K_H$. Consequently, we have $v_M^G(x)=0$ and conclude. □

10.2. The case of (G',H').

Proposition 10.4. *Let $M'\in\mathscr{L}^{H'}(M_0'),M'\neq H'$. Let $Y\in(\widetilde{\mathfrak{m}'}\cap\mathfrak{s}'_{\mathrm{rs}})(F)$ and $X\in\mathfrak{s}'_{\mathrm{rs}}(F)$. Then there exists $N\in\mathbb{N}$ such that if $\lambda\in F^\times$ satisfies $v_F(\lambda)<-N$, we have*

$$\hat{i}_{M'}^{H'}(\lambda Y,X)=0.$$

Proof. It is almost the same as the proof of Proposition 10.1, except that one needs to use Lemma 8.8.2) and Lemma 8.7. □

CHAPTER 5

On certain identities between Fourier transforms of weighted orbital integrals on infinitesimal symmetric spaces of Guo-Jacquet

In an infinitesimal variant of Guo-Jacquet trace formulae, the regular semi-simple terms are expressed as noninvariant weighted orbital integrals on two global infinitesimal symmetric spaces. We prove some relations between the Fourier transforms of invariant weighted orbital integrals on the corresponding local infinitesimal symmetric spaces. These relations should be useful in the noninvariant comparison of the infinitesimal variant of Guo-Jacquet trace formulae.

1. Introduction

Inspired by Jacquet's new proof [29] of Waldspurger's well-known result [50] on the central values of automorphic L-functions for GL_2, Guo-Jacquet have suggested comparison of two relative trace formulae in [23] in order to generalise this theorem to higher ranks. This approach has also been followed by Feigon-Martin-Whitehouse [21] via a simple trace formula. However, if one wants to remove the restrictive conditions in [21], some additional terms in the Guo-Jacquet trace formula other than relative orbital integrals can not be neglected.

Our starting point is an infinitesimal analogue of Guo-Jacquet trace formulae and their comparison. It means that we first work on the tangent space of a symmetric space (called an infinitesimal symmetric space). A reason for this is that at the infinitesimal level, the spectral side of the relative trace formula is replaced by the Fourier transform of the geometric side where the harmonic analysis is simpler. Another reason is that the comparison of trace formulae for infinitesimal symmetric spaces should imply the comparison of the original relative trace formulae for symmetric spaces. For example, one may consult Zhang's proof of the transfer of relative local orbital integrals [58].

We have established an infinitesimal variant of Guo-Jacquet trace formulae in Chapter 2 and Chapter 3, where the main (namely regular semisimple) terms are explicit weighted orbital integrals. These distributions should be the first ones to be studied and compared after orbital integrals. However, some new difficulties arise since these distributions are noninvariant. Instead of making the trace formula invariant as Arthur did (see [5] and others), we would like to follow Labesse's proposal [37] of noninvariant comparison which seems more direct. For example, we have established the weighted fundamental lemma for infinitesimal Guo-Jacquet trace formulae in Chapter 3 as a noninvariant and infinitesimal avatar of Guo's fundamental lemma [23]. The strategy of noninvariant comparison has been also adopted in [14] and [15] on the stable base change. These works provide some indications to our work.

Let us recall some basic objects in the local setting. Let E/F be a quadratic extension of local fields of characteristic zero. Let η be the quadratic character of $F/NE^\times$ attached to E/F, where $NE^\times$ denotes the norm of $E^\times$. The first symmetric pair is $(G,H)=(GL_{2n},GL_n\times GL_n)$. Let $\mathfrak{s}\simeq\mathfrak{gl}_n\oplus\mathfrak{gl}_n$ be the corresponding infinitesimal symmetric space. Denote by $\mathfrak{s}_{\mathrm{rs}}$ the set of regular semi-simple elements in $\mathfrak{s}$ (see Section 2.2). Let M be an ω-stable (see Section 2.3) Levi subgroup of G, and $X\in(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$. Let f be a locally constant and compactly supported function on $\mathfrak{s}(F)$. We define the weighted orbital integral $J_M^G(\eta,X,f)$ by (2.3.2). We have proved in Chapter 4 that its Fourier transform is represented by a locally constant function $\hat{j}_M^G(\eta,X,\cdot)$ on $\mathfrak{s}_{\mathrm{rs}}(F)$. We have also defined the (H,η)-invariant weighted orbital integrals $I_M^G(\eta,X,f)$ in *loc. cit.* by Arthur's standard method, whose Fourier transform is represented by a locally constant function $\hat{\imath}_M^G(\eta,X,\cdot)$ on $\mathfrak{s}_{\mathrm{rs}}(F)$. The second symmetric pair is (G',H'), where G' is the group of invertible elements in a central simple algebra over F containing E, and H' is the centraliser of $E^\times$ in G'. It is inspired by the related local conjecture of Prasad and Takloo-Bighash [44] and more general than Guo-Jacquet's original setting. Denote by $\mathfrak{s}'$ the corresponding infinitesimal

symmetric space. For a Levi subgroup M' of H' and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ (see Section 2.4), we similarly obtain local constant functions $\hat{j}^{H'}_{M'}(Y,\cdot)$ and $\hat{i}^{H'}_{M'}(Y,\cdot)$ on $\mathfrak{s}_{\mathrm{rs}}(F)$.

The functions $\hat{j}^{G}_{M}(\eta,X,\cdot)$ is decomposed as their invariant analogues $\hat{i}^{G}_{M}(\eta,X,\cdot)$ and weight functions v^{G}_{M}. The decomposition for the functions $\hat{j}^{H'}_{M'}(Y,\cdot)$ is similar. In order to obtain relations between $\hat{j}^{G}_{M}(\eta,X,\cdot)$ and $\hat{j}^{H'}_{M'}(Y,\cdot)$, which is part of the noninvariant comparison of the infinitesimal variant of Guo-Jacquet trace formulae, we shall focus on the relations between $\hat{i}^{G}_{M}(\eta,X,\cdot)$ and $\hat{i}^{H'}_{M'}(Y,\cdot)$ in this paper.

There is an injection $M' \mapsto M$ from the set of Levi subgroups of H' into the set of ω-stable Levi subgroups of G (see Section 4). We fix such a matching pair of Levi subgroups. We define the notion of matching orbits between $\mathfrak{s}_{\mathrm{rs}}(F)$ and $\mathfrak{s}'_{\mathrm{rs}}(F)$ by Definition 4.1. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, we denote $\eta(X) := \eta(\det(AB))$. Our main result is as follows.

THEOREM 1.1 (see Corollary 5.6 and Proposition 5.9). *1) Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits. Let $U \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $V \in \mathfrak{s}'_{\mathrm{rs}}(F)$ have matching orbits. Then we have the equality*

$$\gamma_\psi(\mathfrak{h}(F))^{-1}\kappa(X)\kappa(U)\hat{i}^{G}_{M}(\eta,X,U) = \gamma_\psi(\mathfrak{h}'(F))^{-1}\hat{i}^{H'}_{M'}(Y,V),$$

where $\gamma_\psi(\mathfrak{h}(F))$ and $\gamma_\psi(\mathfrak{h}'(F))$ are Weil constants (see Section 2.2).

2) Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in \mathfrak{s}_{\mathrm{rs}}(F)$. If $\eta(X) \neq \eta(U)$, then

$$\hat{i}^{G}_{M}(\eta,X,U) = 0.$$

This theorem generalises some of the main results in [58] to the weighted context. As in *loc. cit.*, we use Waldspurger's global method on the endoscopic transfer [52] to show 1) and a local method to show 2). To show 1), we define a notion of matching weighted orbital integrals (see Definition 5.2) and prove that this property commutes with Fourier transform under some restriction (see Theorem 5.3). Then we may extract the relations between $\hat{i}^{G}_{M}(\eta,X,\cdot)$ and $\hat{i}^{H'}_{M'}(Y,\cdot)$ with the help of Labesse's lemma [37, Lemma 1.7.1]. These steps are close to those in [14]. However, there is an important distinction. While the weighted fundamental lemma for inner forms is tautological in *loc. cit.*, the vanishing condition of Lemma 8.1 here is more subtle. It makes the comparison of global trace formulae by Waldspurger's method, which is a simple case of the noninvariant comparison, even trickier. We translate our definition of matching orbits into the language of cohomology (see Sections 4.3 and 4.4) and use abelian Galois cohomology (see [38]) to go through some technical difficulties.

This paper is organised as follows. We introduce some notations and recall some preliminaries in Section 2. Then we recall Labesse's lemma in Section 3. We define the notion of matching orbits and give a cohomological criterion in Section 4. Our main results are stated in Section 5. The rest of the paper is devoted to the proof Proposition 5.5 by Waldspurger's global method. We recall limit formulae of $\hat{i}^{G}_{M}(\eta,X,\cdot)$ and $\hat{i}^{H'}_{M'}(Y,\cdot)$, the weighted fundamental lemma and an infinitesimal variant of Guo-Jacquet trace formulae in Sections 6, 8 and 10 respectively. We explain the construction of test functions and the globalisation of local data in Sections 7 and 9. These results are prepared for our final proof in Section 11.

2. Notation and preliminaries

2.1. Groups. Let F be a local field of characteristic zero or a number field. Denote by $\mathcal{O}_F$ the ring of integers of F. Let E be a quadratic extension of F. If F is a local (resp. global) field, denote by η the quadratic character of $F^\times/NE^\times$ (resp. $\mathbb{A}^\times/F^\times$) attached to E/F, where $NE^\times = N_{E/F}E^\times$ denotes the norm of $E^\times$ in $F^\times$ (resp. $\mathbb{A} = \mathbb{A}_F$ denotes the ring of adèles of F).

Let G be a reductive group over F. Denote by $\mathrm{rk}_F(G)$ the F-rank of G. Let $\widehat{G}$ be the Langlands dual of G which is a complex reductive group. All algebraic groups (except $\widehat{G}$) and varieties are assumed to be defined over F until further notice. Denote by G_{ad} the adjoint group of G, by G_{der} the derived subgroup of G and by G_{sc} the simple connected cover of G_{der}. Denote by Z_G the centre of G and by $C_G := G/G_{\mathrm{der}}$ the cocentre of G. Fix an algebraic closure $\overline{F}$ of F. Let $\Gamma := \mathrm{Gal}(\overline{F}/F)$. For an F-variety V, we sometimes abuse notation and also write V for $V(\overline{F})$ when there is no confusion.

We use a minuscule Fraktur letter to denote the Lie algebra of its corresponding algebraic group. For example, we write $\mathfrak{g} := \mathrm{Lie}(G)$. Denote by Ad the adjoint action of G on itself or $\mathfrak{g}$. If G acts on an F-variety V and $X \in V(F)$, denote by G_X the centraliser of X in G. If $\mathfrak{v}$ is an F-subvariety of $\mathfrak{g}$, denote

by $\mathfrak{v}_X$ the centraliser of $X \in \mathfrak{g}(F)$ in $\mathfrak{v}$. If θ is an automorphism on G, denote by G^θ the subgroup of fix points of G under θ.

Fix a Levi F-factor M_0 of a minimal parabolic F-subgroup of G. By a Levi subgroup of G, we mean a Levi F-factor of some parabolic F-subgroup of G. For a semi-standard (namely containing M_0) Levi subgroup M of G, denote by $\mathscr{F}^G(M)$, $\mathscr{P}^G(M)$ and $\mathscr{L}^G(M)$ the sets of parabolic F-subgroups of G containing M, parabolic F-subgroups of G with Levi factor M and Levi subgroups of G containing M respectively. For $P \in \mathscr{F}^G(M_0)$, denote by M_P the unique Levi factor containing M_0 and by N_P the unipotent radical. Let $\overline{P}$ be the parabolic subgroup opposite to P.

For $M \in \mathscr{L}^G(M_0)$, define the Weyl group of (G, M) by

$$W^G(M) := \mathrm{Norm}_{G(F)}(M)/M(F).$$

In particular, we also write $W_0^G := W^G(M_0)$. For $M, L \in \mathscr{L}^G(M_0)$, denote

$$\mathrm{Tran}_G(M, L) := \{w \in W_0^L \backslash W_0^G : \mathrm{Ad}(w)(M) \subseteq L\}.$$

Denote by A_G the maximal F-split central torus of G. Let $X(G)_F$ be the group of F rational characters of G. Define the $\mathbb{R}$-linear space

$$\mathfrak{a}_G := \mathrm{Hom}_{\mathbb{Z}}(X(G)_F, \mathbb{R}),$$

whose dual space is denoted by $\mathfrak{a}_G^*$. Fix a scalar product on $\mathfrak{a}_{M_0}$ which is invariant under the action of W_0^G, from which we deduce Haar measures on all subspaces of $\mathfrak{a}_{M_0}$. Denote by $\mathfrak{a}_M^G$ the orthogonal complement of $\mathfrak{a}_G$ in $\mathfrak{a}_M$.

Let D a central division algebra over F. Denote by $\deg(D)$ the degree of D, i.e., $\dim_F(D) = \deg(D)^2$. Denote by $GL_{n,D}$ the reductive group over F whose F-points are $GL_n(D)$. For $x \in \mathfrak{gl}_n(D)$, we write $\mathrm{Nrd}(x), \mathrm{Trd}(x)$ and Prd_x for its reduced norm, reduced trace and reduced characteristic polynomial respectively. If $D = F$, we also write them as $\det(x)$, $\mathrm{Tr}(x)$ and χ_x respectively.

Now suppose that F is a local field of characteristic zero. Denote by $|\cdot|_F$ the normalised absolute value on F. Define a homomorphism $H_G : G(F) \to \mathfrak{a}_G$ by

$$\langle H_G(x), \chi\rangle = \log(|\chi(x)|_F)$$

for all $x \in G(F)$ and $\chi \in X(G)_F$. Fix a maximal compact subgroup K of $G(F)$ which is admissible relative to M_0 in the sense of [5, p. 9]. In this paper, we choose the standard maximal compact subgroup when $G(F) = GL_n(D)$, where D is a central division algebra over a finite field extension of F. That is to say, if F is non-archimedean, $K = GL_n(\mathcal{O}_D)$ with $\mathcal{O}_D$ being the ring of integers of D (see [54, p. 191]), while if F is archimedean, K is the unitary group with respect to some hermitian form (see [54, p. 199]). We may extend the function H_M to a map $H_P : G(F) \to \mathfrak{a}_{M_P}$ using the decomposition $G(F) = M_P(F)N_P(F)K$.

Fix the Haar measure on K such that $\mathrm{vol}(K) = 1$. For $P \in \mathscr{F}^G(M_0)$, fix a Haar measure on $N_P(F)$ such that

$$\int_{N_P(F)} \exp(2\rho_{\overline{P}}(H_{\overline{P}}(n)))dn = 1,$$

where $\rho_{\overline{P}}$ is the half of the sum of roots (with multiplicity) associated to $\overline{P}$. For $M \in \mathscr{L}^G(M_0)$, there are compatible Haar measures on $G(F)$ and $M(F)$ in the sense of [8, (1.1), p. 12] such that for all $P \in \mathscr{P}^G(M)$ and all continuous and compactly supported function f on $G(F)$, we have the equality

$$\int_{G(F)} f(x)dx = \int_{M(F)\times N_P(F)\times K} f(mnk)dkdndm.$$

We shall choose such measures.

Let V be an F-linear space of finite dimension. If F is non-archimedean, denote by $\mathcal{C}_c^\infty(V) = \mathcal{S}(V)$ the space of locally constant, compactly supported and complex-valued functions on V. If F is archimedean, denote by $\mathcal{S}(V)$ the space of Schwartz functions on V. For $f \in \mathcal{S}(V)$, denote by $\mathrm{Supp}(f)$ its support.

Fix a continuous and nontrivial unitary character $\psi : F \to \mathbb{C}^\times$. Let $\langle\cdot,\cdot\rangle$ be a non-degenerate symmetric bilinear form on $\mathfrak{g}(F)$ which is invariant under conjugation. Let V be an F-linear subspace of $\mathfrak{g}(F)$, on which the restriction of $\langle\cdot,\cdot\rangle$ is non-degenerate. It is equipped with the unique self-dual Haar measure with respect to $\psi(\langle\cdot,\cdot\rangle)$. For $f \in \mathcal{S}(V)$, define its Fourier transform $\hat{f} \in \mathcal{S}(V)$ by

$$\forall X \in V, \hat{f}(X) := \int_V f(Y)\psi(\langle X, Y\rangle)dY.$$

Let $M \in \mathscr{L}^G(M_0)$ and $Q \in \mathscr{F}^G(M)$. By [5, p. 40-41], for $x \in G(F)$,

$$v_P(\lambda, x) := e^{-\lambda(H_P(x))}, \forall \lambda \in i\mathfrak{a}_M^*, P \in \mathscr{P}^G(M),$$

is a (G, M)-family in the sense of [5, p. 36]. Define the weight function

$$v_M^Q(x) := \lim_{\lambda \to 0} \sum_{\{P \in \mathscr{P}^G(M) : P \subseteq Q\}} v_P(\lambda, x) \theta_P^Q(\lambda)^{-1}, \forall x \in G(F). \tag{2.1.1}$$

Let V be an F-linear space of finite dimension equipped with a non-degenerate symmetric bilinear form $q(\cdot, \cdot)$ and a Haar measure. Denote by $\gamma_\psi(q)$ the Weil constant given in [53, Théorème 2].

2.2. Symmetric pairs. Let F be a local field of characteristic zero or a number field. A symmetric pair in the sense of [1, Definition 7.1.1] is a triple (G, H, θ) where $H \subseteq G$ are a pair of reductive groups, and θ is an involution of G such that $H = G^\theta$. Let $\mathfrak{s}$ be the tangent space at the neutral element of the symmetric space $S := G/H$. We shall always view $\mathfrak{s}$ as a subspace of $\mathfrak{g}$. Thus

$$\mathfrak{s} = \{X \in \mathfrak{g} : (d\theta)(X) = -X\},$$

on which H acts by the restriction of Ad. By [1, Lemma 7.1.9], there exists a G-invariant θ-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$.

An element $X \in \mathfrak{s}$ is said to be semi-simple if $\mathrm{Ad}(H)(X)$ is Zariski closed in $\mathfrak{s}$. If F is a local field of characteristic zero, $X \in \mathfrak{s}(F)$ is semi-simple if and only if $\mathrm{Ad}(H(F))(X)$ is closed in $\mathfrak{s}(F)$ in the analytic topology by [45, Fact A, p. 108-109]. We say an element $X \in \mathfrak{s}$ is regular if H_X has minimal dimension. Denote by $\mathfrak{s}_{\mathrm{rs}}$ the subset of $\mathfrak{s}$ consisting of regular semi-simple elements in $\mathfrak{s}$.

Now suppose that F is a local field of characteristic zero. A Cartan subspace of $\mathfrak{s}$ is defined as a maximal abelian subspace $\mathfrak{c} \subseteq \mathfrak{s}$ defined over F consisting of semi-simple elements. Denote by $\mathscr{T}^{\mathfrak{s}}$ the set of Cartan subspaces of $\mathfrak{s}$. Fix a (finite) set of representatives $\mathscr{T}_0^{\mathfrak{s}}$ for $H(F)$-conjugacy classes in $\mathscr{T}^{\mathfrak{s}}$. Let $\mathfrak{c} \in \mathscr{T}^{\mathfrak{s}}$. Denote by $T_{\mathfrak{c}}$ the centraliser of $\mathfrak{c}$ in H, which is a torus. Define the Weyl group

$$W(H, \mathfrak{c}) := \mathrm{Norm}_{H(F)}(\mathfrak{c})/T_{\mathfrak{c}}(F).$$

For $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathscr{T}^{\mathfrak{s}}$, denote by $W(H, \mathfrak{c}_1, \mathfrak{c}_2)$ the set of isomorphisms from $\mathfrak{c}_1$ onto $\mathfrak{c}_2$ induced by $\mathrm{Ad}(x)$ for some $x \in H(F)$. If $\mathfrak{c}_1 = \mathfrak{c}_2$, we see that $W(H, \mathfrak{c}_1, \mathfrak{c}_1)$ is nothing but $W(H, \mathfrak{c}_1)$ (viewed as a set).

For $\mathfrak{c} \in \mathscr{T}^{\mathfrak{s}}$ and $X \in (\mathfrak{c} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, define the Weyl discriminant factor

$$|D^{\mathfrak{s}}(X)|_F := |\det(\mathrm{ad}(X)|_{\mathfrak{h}/\mathfrak{t}_{\mathfrak{c}} \oplus \mathfrak{s}/\mathfrak{c}})|_F^{1/2}.$$

Let $\langle \cdot, \cdot \rangle$ be a G-invariant θ-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. For any F-linear subspace $\mathfrak{v}$ of $\mathfrak{g}(F)$ such that the restriction of $\langle \cdot, \cdot \rangle$ on $\mathfrak{v}$ is non-degenerate, denote by $\gamma_\psi(\mathfrak{v})$ the Weil constant associated to $\mathfrak{v}$. Let $\mathfrak{c} \in \mathscr{T}^{\mathfrak{s}}$. For $X, Y \in (\mathfrak{c} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, define a bilinear form $q_{X,Y}$ on $\mathfrak{h}(F)/\mathfrak{t}_{\mathfrak{c}}(F)$ by

$$q_{X,Y}(Z, Z') := \langle [Z, X], [Y, Z'] \rangle.$$

It is non-degenerate and symmetric and we have $q_{X,Y} = q_{Y,X}$. Write

$$\gamma_\psi(X, Y) := \gamma_\psi(q_{X,Y}). \tag{2.2.1}$$

2.3. The case of (G, H). Let F be a local field of characteristic zero or a number field. Let $G := GL_{2n}$ and denote by $H := GL_n \times GL_n$ its subgroup via diagonal embedding. In fact, H is the subgroup of fixed points of the involution $\mathrm{Ad}(\epsilon)$ on G, where $\epsilon := \begin{pmatrix} 1_n & \\ & -1_n \end{pmatrix}$. We shall embed G into $\mathfrak{g}$ in the standard way. For an F-subvariety $\mathfrak{v}$ of $\mathfrak{g}$, we write $\mathfrak{v}^\times := \mathfrak{v} \cap G$. Recall that $\mathfrak{s}_{\mathrm{rs}} \subseteq \mathfrak{s}^\times$ in our case. Let $\langle \cdot, \cdot \rangle$ be the non-degenerate symmetric bilinear form on $\mathfrak{g}(F)$ defined by

$$\langle X, Y \rangle := \mathrm{Tr}(XY), \forall X, Y \in \mathfrak{g}(F), \tag{2.3.1}$$

which is invariant by the adjoint action of $G(F)$ and $\mathrm{Ad}(\epsilon)$.

Let M_0 be the group of diagonal matrices in G. Set $\omega := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in G(F)$. For $P \in \mathscr{F}^G(M_0)$, we say that P is "ω-stable" if $\omega \in P$. Denote by $\mathscr{F}^{G,\omega}(M_0)$ the subset in $\mathscr{F}^G(M_0)$ consisting of ω-stable parabolic subgroups. For $M \in \mathscr{L}^G(M_0)$, we say that M is "ω-stable" if $M = M_P$ for some $P \in \mathscr{F}^{G,\omega}(M_0)$. Denote by $\mathscr{L}^{G,\omega}(M_0)$ the subset in $\mathscr{L}^G(M_0)$ consisting of ω-stable Levi subgroups. Let A_n be the group of diagonal matrices in GL_n. Recall that there is a bijection between $\mathscr{L}^{GL_n}(A_n)$ and $\mathscr{L}^{G,\omega}(M_0)$ induced by $M_n \mapsto M = \begin{pmatrix} \mathfrak{m}_n & \mathfrak{m}_n \\ \mathfrak{m}_n & \mathfrak{m}_n \end{pmatrix}^\times$. We shall always write M_n for the preimage of M

under this bijection. Notice that if $M \in \mathscr{L}^{G,\omega}(M_0)$ and $Q \in \mathscr{F}^G(M)$, then $Q \in \mathscr{F}^{G,\omega}(M_0)$. There is also a bijection between $\mathscr{F}^{GL_n}(A_n)$ and $\mathscr{F}^{G,\omega}(M_0)$ induced by $P_n \mapsto P = \begin{pmatrix} \mathfrak{p}_n & \mathfrak{p}_n \\ \mathfrak{p}_n & \mathfrak{p}_n \end{pmatrix}^\times$. We shall always write P_n for the preimage of P under this bijection. For $\mathfrak{c} \in \mathscr{T}^{\mathfrak{s}}$ and $L \in \mathscr{L}^{G,\omega}(M_0)$, define

$$W(H, \mathfrak{c}, \mathfrak{l} \cap \mathfrak{s}) := \bigsqcup_{\mathfrak{c}_2 \in \mathscr{T}_0^{\mathfrak{l} \cap \mathfrak{s}}} W(H, \mathfrak{c}, \mathfrak{c}_2).$$

Let $M \in \mathscr{L}^{G,\omega}(M_0)$. We say an element $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ is elliptic if A_M is the maximal F-split torus in H_X. Denote by $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ the set of elliptic elements in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Write $M_H := M \cap H$. Denote by $\Gamma_{\mathrm{ell}}((\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F))$ the set of $M_H(F)$-conjugacy classes in $(\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$. For $X \in \mathfrak{s}_{\mathrm{rs}}(F)$, $X \in \mathfrak{s}_{\mathrm{rs}}(F)_{\mathrm{ell}}$ if and only if $\chi_X(\lambda) = p(\lambda^2)$ for some irreducible polynomial $p(\lambda) \in F[\lambda]$ of degree n.

Now suppose that F is a local field of characteristic zero. Let $P \in \mathscr{F}^{G,\omega}(M_0)$. Then $\mathfrak{m}_P = \begin{pmatrix} \mathfrak{m}_n & \mathfrak{m}_n \\ \mathfrak{m}_n & \mathfrak{m}_n \end{pmatrix}$ and $\mathfrak{n}_P = \begin{pmatrix} \mathfrak{n}_n & \mathfrak{n}_n \\ \mathfrak{n}_n & \mathfrak{n}_n \end{pmatrix}$, where we denote $M_n := M_{P_n}$ and $N_n := N_{P_n}$. We shall choose the same Haar measure for any of the four copies in $\mathfrak{m}_P(F)$ or $\mathfrak{n}_P(F)$ under these identifications. For $f \in \mathcal{S}(\mathfrak{s}(F))$, we define a function $f_P^\eta \in \mathcal{S}((\mathfrak{m}_P \cap \mathfrak{s})(F))$ by

$$f_P^\eta(Z) := \int_{K_H \times (\mathfrak{n}_P \cap \mathfrak{s})(F)} f(\mathrm{Ad}(k^{-1})(Z+U))\eta(\det(k))dUdk$$

for all $Z \in (\mathfrak{m}_P \cap \mathfrak{s})(F)$. Recall that $(\hat{f})_P^\eta = (f_P^\eta)\hat{}$, and we shall denote it by $\hat{f}_P^\eta$ without confusion.

Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $Q \in \mathscr{F}^G(M)$. For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, define the weighted orbital integral

$$J_M^Q(\eta, X, f) := |D^{\mathfrak{s}}(X)|_F^{1/2} \int_{H_X(F)\backslash H(F)} f(\mathrm{Ad}(x^{-1})(X))\eta(\det(x))v_M^Q(x)dx. \tag{2.3.2}$$

For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, define a transfer factor $\kappa(X) := \eta(\det(A))$ (see [58, Definition 5.7]). Then $\kappa(\mathrm{Ad}(x^{-1})(X)) = \eta(\det(x))\kappa(X)$, and the function $\kappa(\cdot)J_M^Q(\eta, \cdot, f)$ is constant on $\mathrm{Ad}(M_H(F))(X)$.

Now suppose additionally that F is non-archimedean. Let $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. In §8.1 in Chapter 4, we deduce from $J_M^G(\eta, X, \cdot)$ an (H, η)-invariant distribution $I_M^G(\eta, X, \cdot)$ on $\mathfrak{s}(F)$. By Propositions 7.2 and 8.1 in Chapter 4, there are unique locally constant functions $\hat{j}_M^G(\eta, X, \cdot)$ and $\hat{i}_M^G(\eta, X, \cdot)$ on $\mathfrak{s}_{\mathrm{rs}}(F)$ representing the distributions $\hat{J}_M^G(\eta, X, \cdot)$ and $\hat{I}_M^G(\eta, X, \cdot)$ respectively. That is to say, for all $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we have

$$\hat{J}_M^G(\eta, X, f) := J_M^G(\eta, X, \hat{f}) = \int_{\mathfrak{s}(F)} f(U)\hat{j}_M^G(\eta, X, U)|D^{\mathfrak{s}}(U)|_F^{-1/2}dU$$

and

$$\hat{I}_M^G(\eta, X, f) := I_M^G(\eta, X, \hat{f}) = \int_{\mathfrak{s}(F)} f(U)\hat{i}_M^G(\eta, X, U)|D^{\mathfrak{s}}(U)|_F^{-1/2}dU.$$

2.4. The case of (G', H'). Let F be a local field of characteristic zero or a number field. Let E be a quadratic extension of F. Let $\mathfrak{g}'$ be a central simple algebra over F with a fixed embedding $E \hookrightarrow \mathfrak{g}'(F)$ as F-algebras. Let $\mathfrak{h}' := \mathrm{Cent}_{\mathfrak{g}'}(E)$ be the centraliser of E in $\mathfrak{g}'$. Then $\mathfrak{h}'(F)$ is a central simple algebra over E by the double centraliser theorem. Denote by $G' := \mathfrak{g}'^\times$ (resp. $H' := \mathfrak{h}'^\times$) the group of invertible elements in $\mathfrak{g}'$ (resp. $\mathfrak{h}'$). Let $\alpha \in E \backslash F$ be such that $\alpha^2 \in F$, so $E = F(\alpha)$. In fact, H' is the subgroup of fixed points of the involution $\mathrm{Ad}(\alpha)$ on G'. Denote by $\mathfrak{s}'$ the corresponding tangent space of G'/H' at the neutral element. For a linear subspace $\mathfrak{v}' \subseteq \mathfrak{g}'$, we write $\mathfrak{v}'^\times := \mathfrak{v}' \cap G'$. Then $\mathfrak{s}'_{\mathrm{rs}} \subseteq \mathfrak{s}'^\times$ in our case. Let $\langle \cdot, \cdot \rangle$ be the non-degenerate symmetric bilinear form on $\mathfrak{g}'(F)$ defined by

$$\langle X, Y \rangle := \mathrm{Trd}(XY), \forall X, Y \in \mathfrak{g}'(F), \tag{2.4.1}$$

which is invariant by the adjoint action of $G'(F)$ and $\mathrm{Ad}(\alpha)$.

By the Wedderburn-Artin theorem, G' is isomorphic to $GL_{r,D}$ for some integer $r \geq 1$ and some central division algebra D over F such that $r\deg(D)$ is even. By the Noether-Skolem theorem, up to conjugation by $G'(F)$, the emdedding $H' \hookrightarrow G'$ is isomorphic to one of the two cases below (see [18, §2.1 and §3.1] and §3.4 in Chapter 3).

Case I: if there is an embedding $E \to D$ as F-algebras, then $(G', H') \simeq (GL_{r,D}, \mathrm{Res}_{E/F}GL_{r,D'})$, where $D' := \mathrm{Cent}_D(E)$ is a central division algebra over E of degree $\frac{\deg(D)}{2}$. Let $M_0' \simeq (\mathrm{Res}_{E/F}\mathbb{G}_{m,D'})^r$

(resp. $M'_{\widetilde{0}} \simeq (\mathbb{G}_{m,D})^r$) be the subgroup of diagonal elements in H' (resp. G'). Recall that there is a bijection $M' \mapsto \widetilde{M'}$ between $\mathscr{L}^{H'}(M'_0)$ and $\mathscr{L}^{G'}(M'_{\widetilde{0}})$. We shall always denote by $\widetilde{M'}$ the image of M' under this bijection. Notice that $M' = \widetilde{M'} \cap H'$ and that we can identify $A_{M'}$ with $A_{\widetilde{M'}}$.

Case II: if there is no embedding $E \to D$ as F-algebras, then $(G', H') \simeq (GL_{r,D}, \mathrm{Res}_{E/F} GL_{\frac{r}{2}, D\otimes_F E})$, where $D \otimes_F E$ is a central division algebra over E of degree $\deg(D)$. Let $M'_0 \simeq (\mathrm{Res}_{E/F}\mathbb{G}_{m,D\otimes_F E})^{\frac{r}{2}}$ (resp. $M'_{\widetilde{0}} \simeq (\mathbb{G}_{m,D})^r$) be the subgroup of diagonal elements in H' (resp. G'). Denote by $\mathscr{L}^{G'}(M'_{\widetilde{0}}, M'_0)$ the subset of elements in $\mathscr{L}^{G'}(M'_{\widetilde{0}})$ containing M'_0. Recall that there is a bijection $M' \mapsto \widetilde{M'}$ between $\mathscr{L}^{H'}(M'_0)$ and $\mathscr{L}^{G'}(M'_{\widetilde{0}}, M'_0)$. We shall always denote by $\widetilde{M'}$ the image of M' under this bijection. Notice that $M' = \widetilde{M'} \cap H'$ and that we can identify $A_{M'}$ with $A_{\widetilde{M'}}$.

If $\mathrm{rk}_F(G') = r$, we also write $G'_r := G'$, $H'_r := H'$ and $\mathfrak{s}'_r := \mathfrak{s}'$. Notice that $\mathrm{rk}_F(H'_r) = r$ in **Case I** (resp. $= \frac{r}{2}$) in **Case II**. There is also a bijection $P' \mapsto \widetilde{P'}$ between $\mathscr{F}^{H'}(M'_0)$ and $\mathscr{F}^{G'}(\widetilde{M'_0})$ in both of **Case I** and **Case II**. We shall always denote by $\widetilde{P'}$ the image of P' under this bijection. Let $\tau \in D^\times$ in **Case I** (resp. $\tau \in GL_2(D)$ in **Case II**) be an element such that $\mathrm{Ad}(\alpha)(\tau) = -\tau$. Let $P' \in \mathscr{F}^{H'}(M'_0)$. By Proposition 3.12 in Chapter 3, we have $\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}' = \mathfrak{m}_{P'}\tau = \tau\mathfrak{m}_{P'}$ and $\mathfrak{n}_{\widetilde{P'}} \cap \mathfrak{s}' = \mathfrak{n}_{P'}\tau = \tau\mathfrak{n}_{P'}$. For $\mathfrak{c}' \in \mathscr{T}^{\mathfrak{s}'}$ and $L' \in \mathscr{L}^{H'}(M'_0)$, define

$$W(H', \mathfrak{c}', \widetilde{\mathfrak{l}'} \cap \mathfrak{s}') := \bigsqcup_{\mathfrak{c}'_2 \in \mathscr{T}_0^{\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'}} W(H', \mathfrak{c}', \mathfrak{c}'_2).$$

Let $M' \in \mathscr{L}^{H'}(M'_0)$. We say an element $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ is elliptic if $A_{M'}$ is the maximal F-split torus in H'_Y. Denote by $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$ the set of elliptic elements in $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Denote by $\Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F))$ the set of $M'(F)$-conjugacy classes in $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)_{\mathrm{ell}}$. For $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$, $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)_{\mathrm{ell}}$ if and only if $\mathrm{Prd}_Y(\lambda) = p(\lambda^2)$ for some irreducible polynomial $p(\lambda) \in F[\lambda]$ of degree $\frac{r\deg(D)}{2}$.

Now suppose that F is a local field of characteristic zero. Let $P' \in \mathscr{F}^{H'}(M'_0)$. We shall choose the same Haar measures on $(\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}')(F)$ and $(\mathfrak{n}_{\widetilde{P'}} \cap \mathfrak{s}')(F)$ using above identifications induced by τ. Such Haar measures are independent of the choice of τ. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we define a function $f'_{P'} \in \mathcal{C}_c^\infty((\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}')(F))$ by

$$f'_{P'}(Z) := \int_{K_{H'} \times (\mathfrak{n}_{\widetilde{P'}} \cap \mathfrak{s}')(F)} f'(\mathrm{Ad}(k^{-1})(Z + U))dUdk$$

for all $Z \in (\mathfrak{m}_{\widetilde{P'}} \cap \mathfrak{s}')(F)$. Recall that $(\hat{f}')_{P'} = (f'_{P'})\hat{}$, and we shall denote it by $\hat{f}'_{P'}$ without confusion.

Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $Q' \in \mathscr{F}^{H'}(M')$. For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, define the weighted orbital integral

$$J^{Q'}_{M'}(Y, f') := |D^{\mathfrak{s}'}(Y)|_F^{1/2} \int_{H'_Y(F)\backslash H'(F)} f'(\mathrm{Ad}(x^{-1})(Y))v^{Q'}_{M'}(x)dx.$$

Now suppose additionally that F is non-archimedean. Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. In §8.2 in Chapter 4, we deduce from $J^{H'}_{M'}(Y, \cdot)$ an H'-invariant distribution $I^{H'}_{M'}(Y, \cdot)$ on $\mathfrak{s}'(F)$. By Propositions 7.10 and 8.6 in Chapter 4, there are unique locally constant functions $\hat{j}^{H'}_{M'}(Y, \cdot)$ and $\hat{i}^{H'}_{M'}(Y, \cdot)$ on $\mathfrak{s}'_{\mathrm{rs}}(F)$ representing the distributions $\hat{J}^{H'}_{M'}(Y, \cdot)$ and $\hat{I}^{H'}_{M'}(Y, \cdot)$ respectively. That is to say, for all $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we have

$$\hat{J}^{H'}_{M'}(Y, f') := J^{H'}_{M'}(Y, \hat{f}') = \int_{\mathfrak{s}'(F)} f'(V)\hat{j}^{H'}_{M'}(Y, V)|D^{\mathfrak{s}'}(V)|_F^{-1/2}dV$$

and

$$\hat{I}^{H'}_{M'}(Y, f') := I^{H'}_{M'}(Y, \hat{f}') = \int_{\mathfrak{s}'(F)} f'(V)\hat{i}^{H'}_{M'}(Y, V)|D^{\mathfrak{s}'}(V)|_F^{-1/2}dV.$$

3. Labesse's lemma

Let F be a non-archimedean local field of characteristic zero and E be a quadratic extension of F. Let η the quadratic character of $F^\times/NE^\times$ attached to E/F.

For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, we say that the weighted orbital integrals of f vanish for nontrivial weights if for all $M \in \mathscr{L}^{G,\omega}(M_0), Q \in \mathscr{F}^G(M) - \mathscr{P}^G(M)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we have

$$J^Q_M(\eta, X, f) = 0.$$

Suppose that f satisfies this condition. By definition (8.1.1) in Chapter 4, for all $M \in \mathscr{L}^{G,\omega}(M_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we have the equality

$$J_M^G(\eta, X, \hat{f}) = I_M^G(\eta, X, \hat{f}).$$

For $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$, we say that the weighted orbital integrals of f' vanish for nontrivial weights if for all $M' \in \mathscr{L}^{H'}(M'_0), Q' \in \mathscr{F}^{H'}(M') - \mathscr{P}^{H'}(M')$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, we have

$$J_{M'}^{Q'}(Y, f') = 0.$$

Suppose that f' satisfies this condition. By definition (8.2.1) in Chapter 4, for all $M' \in \mathscr{L}^{H'}(M'_0)$ and $Y \in (\widetilde{\mathfrak{m}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, we have the equality

$$J_{M'}^{H'}(Y, \hat{f}') = I_{M'}^{H'}(Y, \hat{f}').$$

Lemma 3.1 (see [37, Lemma I.7.1]). *1) Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ be such that $\mathrm{Supp}(f) \subseteq \mathfrak{s}_{\mathrm{rs}}(F)$. Then there exists $\phi \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ such that*

(a) $\mathrm{Supp}(\phi) \subseteq \mathrm{Ad}(H(F))(\mathrm{Supp}(f))$;
(b) for all $X \in \mathfrak{s}_{\mathrm{rs}}(F)$, $J_G^G(\eta, X, \phi) = J_G^G(\eta, X, f)$;
(c) the weighted orbital integrals of f vanish for nontrivial weights.

2) Let $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ be such that $\mathrm{Supp}(f') \subseteq \mathfrak{s}'_{\mathrm{rs}}(F)$. Then there exists $\phi' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ such that

(a) $\mathrm{Supp}(\phi') \subseteq \mathrm{Ad}(H'(F))(\mathrm{Supp}(f'))$;
(b) for all $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$, $J_{H'}^{H'}(Y, \phi') = J_{H'}^{H'}(Y, f')$;
(c) the weighted orbital integrals of f' vanish for nontrivial weights.

4. Matching of orbits

Let F be a local field of characteristic zero or a number field. Let E be a quadratic extension of F. Assume that $\dim_{\overline{F}}(G) = \dim_{\overline{F}}(G')$, i.e., $2n = r\deg(D)$.

4.1. Definition by invariants. There is an injection $M' \mapsto M$ from $\mathscr{L}^{H'}(M'_0)$ into $\mathscr{L}^{G,\omega}(M_0)$ induced by the injection from the set of partitions of r in **Case I** (resp. $\frac{r}{2}$ in **Case II**) into the set of partitions of n. We shall always denote by M the image of M' under this injection.

Denote by $\mathbf{A}^n$ the affine space over F of dimension n. By Proposition 3.3 in Chapter 2, the map $\mathfrak{s} \to \mathbf{A}^n, X \mapsto \chi_X$ defines a categorical quotient $\mathfrak{s}//H$ over F, where χ_X denotes the characteristic polynomial of $X \in \mathfrak{g}$. By Proposition 3.5 in Chapter 3, the map $\mathfrak{s}' \to \mathbf{A}^n, Y \mapsto \mathrm{Prd}_Y$ defines a categorical quotient $\mathfrak{s}'//H'$ over F, where Prd_Y denotes the reduced characteristic polynomial of $Y \in \mathfrak{g}'$. Therefore, we can identify $\mathfrak{s}//H \simeq \mathbf{A}^n \simeq \mathfrak{s}'//H'$. By Proposition 3.3 in Chapter 3, it induces an injection from the set of $H'(F)$-orbits in $\mathfrak{s}'_{\mathrm{rs}}(F)$ into the set of $H(F)$-orbits in $\mathfrak{s}(F)$.

Definition 4.1. *Let $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$. If $\chi_X = \mathrm{Prd}_Y$, we say that X and Y have matching orbits and write $X \leftrightarrow Y$. For $X \in \mathfrak{s}_{\mathrm{rs}}(F)$, if there is an element $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$ such that $X \leftrightarrow Y$, we also say that X comes from $\mathfrak{s}'_{\mathrm{rs}}(F)$.*

Remark 4.2. Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Then X comes from $\mathfrak{s}'_{\mathrm{rs}}(F)$ if and only if there is an element $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ such that $X \leftrightarrow Y$, in which case we also say that X comes from $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.

Now suppose that F is a local field of characteristic zero. For $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$, we denote $\eta(X) := \eta(\det(AB))$.

Lemma 4.3. *Let $X \in \mathfrak{s}_{\mathrm{rs}}(F)_{\mathrm{ell}}$. Then the following conditions are equivalent:*

(1) X comes from $\mathfrak{s}'_{\mathrm{rs}}(F)$;
(2) $\eta(X) = (-1)^r$, where $r := \mathrm{rk}_F(G')$;
(3) $\eta(X) = e(G')$, where $e(G')$ denotes the Kottwitz sign of G' in the sense of [33].

Proof. See the proof of [55, Lemma 2.7] for (1)⇔(2). Now we show (2)⇔(3). Let $\frac{i}{2n} = \frac{i_0}{\deg(D)} \in \mathbb{Q} \cap [0, 1)$ be the invariant of $\mathfrak{g}'(F)$, where i_0 and $\deg(D)$ are coprime. Since $2n = r\deg(D)$ is even, by [33, Corollary (7)], $e(G') = -1$ if and only if i is odd. We have $i = \frac{2ni_0}{\deg(D)} = ri_0$. If i is odd, then r is odd. Conversely, if r is odd, then $\deg(D)$ is even, so i_0 is odd, which implies that i is odd. We have shown that $e(G') = (-1)^r$ when $\deg(\mathfrak{g}'(F))$ is even and thus proved (2)⇔(3). □

Let $M' \in \mathscr{L}^{H'}(M'_0)$. Assume that

$$(M, M_H) \simeq (GL_{2n_1}, GL_{n_1} \times GL_{n_1}) \times \cdots \times (GL_{2n_\ell}, GL_{n_\ell} \times GL_{n_\ell})$$

and that

$$(\widetilde{M'}, M') \simeq (G'_{r_1}, H'_{r_1}) \times \cdots \times (G'_{r_\ell}, H'_{r_\ell}).$$

Definition 4.4. *Let $M' \in \mathscr{L}^{H'}(M'_0)$, $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. If each pair of factors of X and Y have matching orbits, we say that X and Y have M-matching orbits.*

Let M' be as above and $X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, where

$$A := \mathrm{diag}(A_1, \cdots, A_\ell), B := \mathrm{diag}(B_1, \cdots, B_\ell) \in \mathfrak{gl}_{n_1}(F) \oplus \cdots \oplus \mathfrak{gl}_{n_\ell}(F).$$

Denote $\eta_M(X) := (\eta(\det(A_i B_i)))_{1 \le i \le \ell} \in \{\pm 1\}^\ell$. Denote $e_{M'} := ((-1)^{r_i})_{1 \le i \le \ell} \in \{\pm 1\}^\ell$.

Definition 4.5. *Let $M' \in \mathscr{L}^{H'}(M'_0)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. If $\eta_M(X) = e_{M'}$, we say that X comes potentially from $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.*

Corollary 4.6. *Let $M' \in \mathscr{L}^{H'}(M'_0)$. Then we have*

(1) if $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ comes from $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, then X comes potentially from $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$;
(2) if $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ comes potentially from $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, then X comes from $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.

4.2. Centralisers. Recall that $E = F(\alpha)$ with $\alpha^2 \in F$. Set $\alpha_0 := \begin{pmatrix} 0 & \alpha^2 1_n \\ 1_n & 0 \end{pmatrix} \in G(F)$. Then $E \simeq F(\alpha_0) \subseteq \mathfrak{g}(F)$. Denote $H_0 := \mathrm{Cent}_G(\alpha) \simeq \mathrm{Res}_{E/F} GL_{n,E}$. Then

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} A & \alpha^2 C \\ C & A \end{pmatrix} : A, C \in \mathfrak{gl}_n \right\}.$$

Denote by $\mathfrak{s}_0$ the corresponding tangent space of G/H_0 at the neutral element. Then

$$\mathfrak{s}_0 = \left\{ \begin{pmatrix} A & -\alpha^2 C \\ C & -A \end{pmatrix} : A, C \in \mathfrak{gl}_n \right\}.$$

Set $\omega_0 := \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \in \mathfrak{s}_0(F)$. Then $\omega_0^2 = 1_{2n}$ and $\mathfrak{s}_0 = \mathfrak{h}_0 \omega_0 = \omega_0 \mathfrak{h}_0$. The action $\mathrm{Ad}(\omega_0)$ induces an involution on H_0 and $\mathfrak{h}_0$.

Recall that $\mathfrak{g}'$ is a central simple algebra over F with a fixed embedding $E \hookrightarrow \mathfrak{g}'(F)$ as F-algebras. Let $\varphi : \mathfrak{g}' \to \mathfrak{g}$ be an isomorphism over $\overline{F}$ such that for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have

$$\varphi \circ \sigma \circ \varphi^{-1} \circ \sigma^{-1} = \mathrm{Ad}(u_\sigma),$$

where u_σ is a Galois 1-cocycle with values in G_{ad}. Let α' be the image of α in $\mathfrak{g}'(F)$. Then $\mathrm{Prd}_{\alpha'}$ is defined over F and ${\alpha'}^2 - \alpha^2 = 0$. We deduce that $\mathrm{Prd}_{\alpha'}(\lambda) = (\lambda^2 - \alpha^2)^n = \chi_{\alpha_0}(\lambda) \in F[\lambda]$. Because both of α_0 and α' are semi-simple in the classical sense, there exists $x \in G$ such that $\mathrm{Ad}(x) \circ \varphi(\alpha') = \alpha_0$. Since $\alpha' \in \mathfrak{g}'(F)$, we have $\varphi(\alpha') = \phi \circ \sigma(\alpha') = \mathrm{Ad}(u_\sigma) \circ \sigma \circ \varphi(\alpha')$. As $\alpha_0 \in G(F)$, we obtain $\mathrm{Ad}(x^{-1})(\alpha_0) = \mathrm{Ad}(u_\sigma) \circ \sigma \circ \mathrm{Ad}(x^{-1})(\alpha_0) = \mathrm{Ad}(u_\sigma) \circ \mathrm{Ad}(\sigma(x)^{-1})(\alpha_0)$. It implies that $x u_\sigma \sigma(x)^{-1} \in H_0$. It turns out that by changing u_σ in its class in $H^1(F, G_{\mathrm{ad}})$, we may and shall suppose that $\varphi(\alpha') = \alpha$ and that u_σ is a Galois 1-cocycle with values in H_0/Z_G.

Set $\alpha_1 := \begin{pmatrix} \alpha 1_n & \\ & -\alpha 1_n \end{pmatrix} \in G(E)$. Then $H = \mathrm{Cent}_G(\alpha_1)$. Recall that $\omega = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. Thus $\mathfrak{s} = \mathfrak{h}\omega = \omega\mathfrak{h}$. The action $\mathrm{Ad}(\omega)$ induces an involution on H and $\mathfrak{h}$.

Let $y \in G$ such that

$$\mathrm{Ad}(y) \circ \varphi(\alpha') = \alpha_1, \tag{4.2.1}$$

i.e., $\mathrm{Ad}(y)(\alpha_0) = \alpha_1$. Then the morphism $\mathrm{Ad}(y) \circ \varphi$ induces an isomorphism over $\overline{F}$ from $\mathfrak{s}'$ to $\mathfrak{s}$. For all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have $\mathrm{Ad}(\sigma(y)) \circ \sigma \circ \varphi(\alpha') = \sigma(\alpha_1) = \varepsilon_\sigma \alpha_1$, where ε_σ denotes the quadratic character of $\mathrm{Gal}(\overline{F}/F)$ associated to E/F. Since $\sigma \circ \varphi(\alpha') = \mathrm{Ad}(u_\sigma^{-1}) \circ \varphi \circ \sigma(\alpha') = \mathrm{Ad}(u_\sigma^{-1}) \circ \varphi(\alpha') = \mathrm{Ad}(u_\sigma^{-1}) \circ \mathrm{Ad}(y^{-1})(\alpha_1)$, we obtain

$$\mathrm{Ad}(y u_\sigma \sigma(y)^{-1})(\alpha_1) = \varepsilon_\sigma \alpha_1.$$

For all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, define $w_\sigma := 1_{2n}$ if $\sigma \in \mathrm{Gal}(\overline{F}/E)$ and $w_\sigma := \omega$ otherwise. Then $\mathrm{Ad}(w_\sigma)(\alpha_1) = \varepsilon_\sigma \alpha_1$. We deduce that

$$yu_\sigma \sigma(y)^{-1} w_\sigma \in H. \tag{4.2.2}$$

The elements y verifying (4.2.1) form an H-torsor of the form Hv where

$$v := \begin{pmatrix} 1_n & \alpha 1_n \\ 1_n & -\alpha 1_n \end{pmatrix} \in G(E). \tag{4.2.3}$$

We easily check that $\sigma(v) = w_\sigma v$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ and that $\omega v = v\omega_0$.

Let $Y \in \mathfrak{s}'(F)$ be a semi-simple element. There exists $y \in Hv$ and a semi-simple element $X \in \mathfrak{s}(F)$ such that

$$\mathrm{Ad}(y) \circ \varphi(Y) = X.$$

Then $\mathrm{Ad}(y) \circ \varphi$ induces an isomorphism over $\overline{F}$ from H'_Y to H_X. For all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have $\mathrm{Ad}(\sigma(y)) \circ \sigma \circ \varphi(Y) = \sigma(X) = X$. Since $\sigma \circ \varphi(Y) = \mathrm{Ad}(u_\sigma^{-1}) \circ \varphi \circ \sigma(Y) = \mathrm{Ad}(u_\sigma^{-1}) \circ \varphi(Y) = \mathrm{Ad}(u_\sigma^{-1}) \circ \mathrm{Ad}(y^{-1})(X)$, we obtain

$$yu_\sigma \sigma(y)^{-1} \in G_X.$$

Combined with (4.2.2), we have $yu_\sigma\sigma(y)^{-1} \in G_X \cap Hw_\sigma$. Since w_σ normalises H (resp. $\mathfrak{s}$), we see that $yu_\sigma\sigma(y)^{-1}$ normalises H_X (resp. $\mathfrak{s}_X$).

Lemma 4.7. *Let $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $Y \in \mathfrak{s}'_{\mathrm{rs}}(F)$ be such that $X \leftrightarrow Y$. There exists $y \in Hv$ such that $\mathrm{Ad}(y) \circ \varphi(Y) = X$ and that $\mathrm{Ad}(y) \circ \varphi$ induces isomorphisms $H'_Y \to H_X$ and $\mathfrak{s}'_Y \to \mathfrak{s}_X$ over F.*

Proof. This is a generalisation of [58, Lemma 7.4]. For all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we see that $(\mathrm{Ad}(y)\circ\varphi)\circ\sigma\circ(\mathrm{Ad}(y)\circ\varphi)^{-1}\circ\sigma^{-1} = \mathrm{Ad}(y)\circ(\varphi\circ\sigma\circ\varphi^{-1}\circ\sigma^{-1})\circ(\sigma\circ\mathrm{Ad}(y^{-1})\circ\sigma^{-1}) = \mathrm{Ad}(yu_\sigma\sigma(y)^{-1})$. For $X \in \mathfrak{s}_{\mathrm{rs}}$, we know that X is also regular semi-simple in G in the classical sense. Since $yu_\sigma\sigma(y)^{-1} \in G_X$, the action $\mathrm{Ad}(yu_\sigma\sigma(y)^{-1})$ on H_X (resp. $\mathfrak{s}_X$) is trivial, which implies that $\mathrm{Ad}(y)\circ\varphi$ induces an isomorphism $H'_Y \to H_X$ (resp. $\mathfrak{s}'_Y \to \mathfrak{s}_X$) over F. □

4.3. Cohomological criterion. Let $X = \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$. Then $GL_{n,A}$ is a maximal F-torus in GL_n. Denote $X_0 := \mathrm{Ad}(v^{-1})(X) \in \mathfrak{s}_0(E)$, where v is defined by (4.2.3). Notice that $H_{0,X_0} = \mathrm{Ad}(v^{-1})(H_X) = H_X \simeq GL_{n,A}$. Let $H_A := \mathrm{Cent}_H(\mathrm{diag}(A,A)) = GL_{n,A} \times GL_{n,A}$, which is a maximal F-torus in H. Denote $T_X := \mathrm{Cent}_{H_0}(\mathrm{diag}(A,A)) = \mathrm{Ad}(v^{-1})(H_A)$. For all closed subvariety V of H defined over F and stable by ω, we easily check that $\mathrm{Ad}(v^{-1})(V)$ is a closed subvariety of H_0 defined over F and stable by ω_0. Thus T_X is a maximal F-torus in H_0. Notice that $H_X = H_A^\omega$ and that $H_{0,X_0} = T_X^{\omega_0}$. We see that $T_X \simeq \mathrm{Res}_{E/F}(GL_{n,A})_E$ and that the inclusion $H_{0,X_0} \subseteq T_X$ is isomorphic to the inclusion $GL_{n,A} \subseteq \mathrm{Res}_{E/F}(GL_{n,A})_E$. For simplicity, we also write $T := T_X$ and $R := H_{0,X_0}$.

Lemma 4.8. *There exists a unique Galois 1-cocycle t_σ with values in T/R such that for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have*

$$\mathrm{Ad}(t_\sigma) \circ \sigma(X_0) = X_0.$$

Proof. The uniqueness is obvious by definition. It suffices to consider the existence. The cocycle condition is also automatic. We only need to check the equality. If $\sigma \in \mathrm{Gal}(\overline{F}/E)$, then $\sigma(X_0) = X_0$, so it suffices to take $t_\sigma = 1$. Now suppose that $\sigma \notin \mathrm{Gal}(\overline{F}/E)$. Since $\sigma(X_0) = \mathrm{Ad}(\sigma(v)^{-1}) \circ \sigma(X) = \mathrm{Ad}(v^{-1}w_\sigma^{-1})(X)$, it suffices to find an element $t_\sigma \in T$ such that $vt_\sigma v^{-1} w_\sigma^{-1} \in G_X$. In fact, we can take $t_\sigma = \mathrm{Ad}(v^{-1}) \begin{pmatrix} 1 & \\ & A \end{pmatrix}$. □

By the inflation-restriction exact sequence and Hilbert's Theorem 90, the cohomology group $H^1(F, T/R)$ is identified with $H^1(\mathrm{Gal}(E/F), T(E)/R(E))$. For all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ and $h' = \mathrm{Ad}(v^{-1})(h) \in T$ where $h \in H_A$, we see that $\sigma(h') = \mathrm{Ad}(\sigma(v)^{-1}) \circ \sigma(h) = \mathrm{Ad}(v^{-1}) \circ \mathrm{Ad}(w_\sigma) \circ \sigma(h)$. Therefore, we can regard T as the subgroup H_A of G equipped with the Galois action $\mathrm{Ad}(w_\sigma) \circ \sigma$. Then the inclusion $R(E) \subseteq T(E)$ is isomorphic to the inclusion $H_X(E) \subseteq H_A(E)$. Let σ be the nontrivial element in $\mathrm{Gal}(E/F)$. We see that $u_\sigma = (B, C) \in H_A(E)/H_X(E)$ is a 1-cocycle if and only if $u_\sigma \mathrm{Ad}(w_\sigma) \circ \sigma(u_\sigma) \in H_X(E)$, i.e., $BC^\sigma = CB^\sigma$. We also see that $u_\sigma \in H_A(E)/H_X(E)$ is a 1-coboundary if and only if $u_\sigma = (B,C)^{-1}\mathrm{Ad}(w_\sigma) \circ \sigma(B,C)$ for some $(B,C) \in H_A(E)/H_X(E)$, i.e.,

$u_\sigma = (C^\sigma B^{-1}, B^\sigma C^{-1}) \in H_A(E)/H_X(E)$. Then $H^1(\mathrm{Gal}(E/F), T(E)/R(E))$ is the quotient of

$$\begin{aligned}&\{(B,C) \in H_A(E)/H_X(E) : BC^\sigma = CB^\sigma\} = \{(1, CB^{-1}) \in H_A(E)/H_X(E) : (CB^{-1})^\sigma = CB^{-1}\}\\ =&\{(1,B) \in H_A(E)/H_X(E) : B^\sigma = B\} = \{(1,B) \in H_A(E)/H_X(E) : B \in GL_{n,A}(F)\}\end{aligned}$$

by

$$\begin{aligned}&\{(C^\sigma B^{-1}, B^\sigma C^{-1}) \in H_A(E)/H_X(E) : B, C \in GL_{n,A}(E)\}\\ =&\{(B^{-\sigma}, B) \in H_A(E)/H_X(E) : B \in GL_{n,A}(E)\} = \{(1, BB^\sigma) \in H_A(E)/H_X(E) : B \in GL_{n,A}(E)\}.\end{aligned}$$

That is to say, it is the quotient of $G_A(F)$ by the group of norms of $GL_{n,A}(E)$. Under the identification of T and H_A with the twisted Galois action, the Galois 1-cocycle t_σ in Lemma 4.8 corresponds exactly to the class of A.

Since A is regular semi-simple in $GL_n(F)$ in the classical sense, its characteristic polynomial χ_A is separable. Let $\chi_A = \prod_{i\in I} \chi_i$ be the factorisation of χ_A into a product of monic irreducible polynomials over F. We see that

$$GL_{n,A} \simeq \prod_{i\in I} \mathrm{Res}_{F_i/F}\mathbb{G}_{m,F_i},$$

where $F_i = F[\lambda]/(\chi_i(\lambda))$. Denote $E_i := F_i \otimes_F E$. Let I_0 be the subset of I consisting of i such that E_i is a field. We have

$$H^1(F, T/R) = \prod_{i\in I} F_i^\times / N_{E_i/F_i}(E_i^\times) = \prod_{i\in I_0} F_i^\times / N_{E_i/F_i}(E_i^\times).$$

If F is a local field of characteristic zero, then $H^1(F, T/R) = (\mathbb{Z}/2\mathbb{Z})^{I_0}$.

Lemma 4.9. *Let $t \in H^1(F, T/R)$ be the class of the Galois 1-cocycle t_σ in Lemma 4.8. Let $u \in H^1(F, H_0/Z_G)$ be the class of the Galois 1-cocycle u_σ associated to $\mathfrak{g}'$. There exists $Y \in \mathfrak{s}'(F)$ and $h \in H_0$ such that $\mathrm{Ad}(h) \circ \varphi(Y) = X_0$ if and only if there exists an element of $H^1(F, T/Z_G)$ which has images $t \in H^1(F, T/R)$ and $u \in H^1(F, H_0/Z_G)$ under the natural maps:*

$$\begin{tikzcd} H^1(F, T/Z_G) \arrow[r] \arrow[d] & H^1(F, T/R) \\ H^1(F, H_0/Z_G) & \end{tikzcd}$$

Remark 4.10. The condition in the above lemma says exactly that X comes from $\mathfrak{s}'_{\mathrm{rs}}(F)$.

Proof of Lemma 4.9. For all $h \in H_0$, let $Y \in \mathfrak{s}'$ be the unique element such that $\mathrm{Ad}(h) \circ \varphi(Y) = X_0$. For all $\sigma \in \mathrm{Gal}(\overline{F}/F)$, we have $\mathrm{Ad}(\sigma(h)) \circ \sigma \circ \varphi(Y) = \sigma(X_0) = \mathrm{Ad}(t_\sigma^{-1})(X_0)$ and $\sigma \circ \varphi(Y) = \mathrm{Ad}(u_\sigma^{-1}) \circ \varphi \circ \sigma(Y)$. But $Y \in \mathfrak{s}'(F)$ if and only if $\varphi \circ \sigma(Y) = \varphi(Y) = \mathrm{Ad}(h^{-1})(X_0)$. We have shown that there exists $Y \in \mathfrak{s}'(F)$ and $h \in H_0$ such that $\mathrm{Ad}(h) \circ \varphi(Y) = X_0$ if and only if there exists $h \in H_0$ such that

$$\mathrm{Ad}(\sigma(h) u_\sigma^{-1} h^{-1})(X_0) = \mathrm{Ad}(t_\sigma^{-1})(X_0)$$

for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$. If this equality is satisfied, then $hu_\sigma\sigma(h)^{-1} \in Rt_\sigma \subseteq T$ defines a Galois 1-cocycle with values in T/Z_G which has desired images. Conversely, any Galois 1-cocycle with values in T/Z_G having image u is of the form $hu_\sigma\sigma(h)^{-1}$ where $h \in H_0$. If it also has image t, it means that by replacing h with $t'h$ where $t' \in T$, we may suppose that $hu_\sigma\sigma(h)^{-1} \in Rt_\sigma$. Then the above equality is satisfied for such an h. □

Recall that $H_0^{\omega_0} = H^\omega \simeq GL_n$. We shall abuse notation and denote by R_{der} (resp. T_{der}) the preimage of R (resp. T) in $H_{0,\mathrm{der}}^{\omega_0} = H_{0,\mathrm{sc}}^{\omega_0}$ (resp. $H_{0,\mathrm{der}} = H_{0,\mathrm{sc}}$). We shall the index "ab" to denote the abelianised cohomology defined in [38, §1.6 and 1.8]. Recall that the abelianisation maps $H^1(F, H_0^{\omega_0}/Z_G) \to H^1_{\mathrm{ab}}(F, H_0^{\omega_0}/Z_G)$ and $H^1(F, H_0/Z_G) \to H^1_{\mathrm{ab}}(F, H_0/Z_G)$ are surjective (see [38, Proposition 1.6.7]). If F is a non-archimedean local field of characteristic zero, they are also injective by Kneser's theorem (see *loc. cit.*). We have the following commutative diagram with exact columns and rows.

$$\begin{tikzcd}
H^1(F,R/Z_G) \arrow[r] \arrow[d] & H^1(F,T/Z_G) \arrow[r] \arrow[d] & H^1(F,T/R) \arrow[r] \arrow[d] & H^2(F,R/Z_G) \arrow[d] \\
H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G) \arrow[r] \arrow[d] & H^1_{\mathrm{ab}}(F,H_0/Z_G) \arrow[r] \arrow[d] & H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G) \arrow[r] \arrow[d] & H^2_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G) \\
H^2(F,R_{\mathrm{der}}) \arrow[r] & H^2(F,T_{\mathrm{der}}) \arrow[r] & H^2(F,T_{\mathrm{der}}/R_{\mathrm{der}}) &
\end{tikzcd}$$

PROPOSITION 4.11. *There is a canonical identification*

$$H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G) \simeq H^1(F,C_{H_0}/C_{H_0^{\omega_0}}).$$

PROOF. The group on the left is isomorphic to $H^1(F,T_{\mathrm{der}}/R_{\mathrm{der}} \to T/R)$. Since $[T_{\mathrm{der}}/R_{\mathrm{der}} \to T/R]$ is injective, it is quasi-isomorphic to $[1 \to (T/R)/(T_{\mathrm{der}}/R_{\mathrm{der}})]$. But $(T/R)/(T_{\mathrm{der}}/R_{\mathrm{der}}) \simeq (T/T_{\mathrm{der}})/(R/R_{\mathrm{der}})$, where $T/T_{\mathrm{der}} = C_{H_0}$ and $R/R_{\mathrm{der}} = C_{H_0^{\omega_0}}$. □

COROLLARY 4.12. *Suppose that F is a local field of characteristic zero. Then*

$$H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G) = \mathbb{Z}/2\mathbb{Z}.$$

The morphism

$$H^1(F,T/R) = (\mathbb{Z}/2\mathbb{Z})^{I_0} \to H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G) = \mathbb{Z}/2\mathbb{Z}$$

is the sum of components.

Denote by ε_X the image of t (defined in Lemma 4.8) under the morphism

$$H^1(F,T/R) \to H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G).$$

COROLLARY 4.13. *Suppose that F is a local field of characteristic zero. The map*

$$\mathfrak{s}_{\mathrm{rs}}(F) \to H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G) = \mathbb{Z}/2\mathbb{Z} : X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \mapsto \varepsilon_{X'}$$

where $X' := \begin{pmatrix} 0 & 1_n \\ AB & 0 \end{pmatrix} \in \mathfrak{s}_{\mathrm{rs}}(F)$ *is understood as the map*

$$\mathfrak{s}_{\mathrm{rs}}(F) \to \{\pm 1\} : X \mapsto \eta(X).$$

PROOF. We may reduce ourselves to the case where $X \in \mathfrak{s}_{\mathrm{rs}}(F)_{\mathrm{ell}}$. In the elliptic case, it results from [10, Lemmas 1.3 and 1.4] □

Denote by ε' the image of u (associated to $\mathfrak{g}'$) under the morphism

$$H^1(F,H_0/Z_G) \simeq H^1_{\mathrm{ab}}(F,H_0/Z_G) \to H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G).$$

LEMMA 4.14. *If X comes from $\mathfrak{s}'_{\mathrm{rs}}(F)$, then $\varepsilon_X = \varepsilon'$.*

PROOF. It results from Lemma 4.9 and the above commutative diagram. □

PROPOSITION 4.15. *Suppose that F is a non-archimedean local field of characteristic zero and that A is elliptic in $GL_n(F)$. Then X comes from $\mathfrak{s}'_{\mathrm{rs}}(F)$ if and only if $\varepsilon_X = \varepsilon'$.*

PROOF. By Lemma 4.14, it suffices to prove the reverse direction. Since F is non-archimedean, we have $H^2_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G) = 0$ by [38, Lemme 1.5.1]. Since F is local and the tori $R/Z_G, R_{\mathrm{der}}$ and T_{der} are F-anisotropic, the groups $H^2(F,R/Z_G), H^2(F,R_{\mathrm{der}}), H^2(F,T_{\mathrm{der}})$ and $H^2(F,T_{\mathrm{der}}/R_{\mathrm{der}})$ vanish (see *loc. cit.*). The above commutative diagram is simplified as follows.

$$\begin{tikzcd}
H^1(F,R/Z_G) \arrow[r, "f"] \arrow[d] & H^1(F,T/Z_G) \arrow[r] \arrow[d] & H^1(F,T/R) \arrow[r] \arrow[d] & 0 \\
H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G) \arrow[r, "g"] \arrow[d] & H^1_{\mathrm{ab}}(F,H_0/Z_G) \arrow[r] \arrow[d] & H^1_{\mathrm{ab}}(F,H_0^{\omega_0}/Z_G \to H_0/Z_G) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Choose an arbitary preimage $t' \in H^1(F, T/Z_G)$ of t. The image of t' in $H^1_{\mathrm{ab}}(F, H_0/Z_G)$ is of the form $ug(u_1)$ where $u_1 \in H^1_{\mathrm{ab}}(F, H_0^{\omega_0}/Z_G)$ because both of u and this image map to $\varepsilon' = \varepsilon_X \in H^1_{\mathrm{ab}}(F, H_0^{\omega_0}/Z_G \to H_0/Z_G)$ by our assumption. Let $t_1 \in H^1(F, R/Z_G)$ be a preimage of u_1. Then $t'f(t_1)^{-1} \in H^1(F, T/Z_G)$ has images t and u. We may conclude by the bijectivity of the abelianisation map $H^1(F, H_0/Z_G) \to H^1_{\mathrm{ab}}(F, H_0/Z_G)$ and Lemma 4.9. □

Corollary 4.16. *Suppose that F is a non-archimedean local field of characteristic zero. The map*

$$H^1(F, H_0/Z_G) \simeq H^1_{\mathrm{ab}}(F, H_0/Z_G) \to H^1_{\mathrm{ab}}(F, H_0^{\omega_0}/Z_G \to H_0/Z_G) = \mathbb{Z}/2\mathbb{Z} : u \mapsto \varepsilon'$$

is understood as the Kottwitz sign

$$H^1(F, H_0/Z_G) \to H^1(F, G_{\mathrm{ad}}) \to \{\pm 1\}.$$

Proof. It can be computed directly. However, it is also a consequence of Lemma 4.3, Corollary 4.13 and Proposition 4.15. □

4.4. Levi subgroups. Let $M \in \mathscr{L}^{G,\omega}(M_0)$. Denote $M_{H_0} := \mathrm{Ad}(v^{-1})(M) \cap H_0$, where v is defined by (4.2.3). Then M_{H_0} is a Levi subgroup of H_0 defined over F. Let $X = \begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix} \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Then $T \subseteq M_{H_0}$ and $R \subseteq M_{H_0}^{\omega_0}$. We have an obvious generalisation to the product form of some results in the previous section.

Proposition 4.17 (cf. Proposition 4.11). *There is a canonical identification*

$$H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M) \simeq H^1(F, C_{M_{H_0}}/C_{M_{H_0}^{\omega_0}}).$$

Corollary 4.18 (cf. Corollary 4.12). *Suppose that F is a local field of characteristic zero. Then*

$$H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M) = (\mathbb{Z}/2\mathbb{Z})^{\ell},$$

where ℓ is the number of blocks of M. The morphism

$$H^1(F, T/R) = (\mathbb{Z}/2\mathbb{Z})^{I_0} \to H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M) = (\mathbb{Z}/2\mathbb{Z})^{\ell}$$

is the sum of components in each block of M. If $L \in \mathscr{L}^G(M)$, then the morphism

$$H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M) = (\mathbb{Z}/2\mathbb{Z})^{\ell} \to H^1_{\mathrm{ab}}(F, L_{H_0}^{\omega_0}/Z_L \to L_{H_0}/Z_L) = (\mathbb{Z}/2\mathbb{Z})^{\ell'}$$

is the sum of components in each block of L, where ℓ' is the number of blocks of L.

Proposition 4.19. *Suppose that F is a non-archimedean local field of characteristic zero. The group $H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M)$ is canonically isomorphic to the Pontryagin dual of the finite group $Z_{\widehat{M}}[2]$ of elements $z \in Z_{\widehat{M}}$ such that $z^2 = 1$. If $L \in \mathscr{L}^G(M)$, then the morphism*

$$H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M) \to H^1_{\mathrm{ab}}(F, L_{H_0}^{\omega_0}/Z_L \to L_{H_0}/Z_L)$$

is the dual of the canonical embedding $Z_{\widehat{L}}[2] \hookrightarrow Z_{\widehat{M}}[2]$.

Proof. By [34, Proposition 6.4] and Proposition 4.17, the group $H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M)$ is canonically isomorphic to the Pontryagin dual of $\pi_0((Z_{\widehat{M_{H_0}}}/Z_{\widehat{M_{H_0}^{\omega_0}}})^{\Gamma})$. Notice that $Z_{\widehat{M_{H_0}}} = Z_{\widehat{M_{H_0}^{\omega_0}}} \times Z_{\widehat{M_{H_0}^{\omega_0}}}$, on which the nontrivial element $\sigma \in \mathrm{Gal}(E/F)$ acts by exchanging two components. Via the morphism $(z, z') \mapsto zz'^{-1}$ and the diagonal embedding, we obtain

$$Z_{\widehat{M_{H_0}}}/Z_{\widehat{M_{H_0}^{\omega_0}}} \simeq Z_{\widehat{M_{H_0}^{\omega_0}}} \simeq Z_{\widehat{M}}.$$

These isomorphisms are Γ-equivariant if we define the action of Γ on $Z_{\widehat{M_{H_0}^{\omega_0}}}$ and $Z_{\widehat{M}}$ by the lift of the action of $\mathrm{Gal}(E/F)$ with $\sigma(z) = z^{-1}$. Then

$$\pi_0((Z_{\widehat{M_{H_0}}}/Z_{\widehat{M_{H_0}^{\omega_0}}})^{\Gamma}) \simeq Z_{\widehat{M}}[2].$$

□

Denote by ε_X^M the image of t (defined in Lemma 4.8) under the morphism

$$H^1(F, T/R) \to H^1_{\mathrm{ab}}(F, M_{H_0}^{\omega_0}/Z_M \to M_{H_0}/Z_M).$$

Proposition 4.20 (cf. Corollary 4.13). *Suppose that F is a local field of characteristic zero. The map*

$$(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)\to H^1_{\mathrm{ab}}(F,M_{H_0}^{\omega_0}/Z_M\to M_{H_0}/Z_M)=(\mathbb{Z}/2\mathbb{Z})^\ell : X=\begin{pmatrix}0 & A\\ B & 0\end{pmatrix}\mapsto \varepsilon^M_{X'}$$

where $X':=\begin{pmatrix}0 & 1_n\\ AB & 0\end{pmatrix}\in(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$ *is understood as the map*

$$(\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)\to\{\pm1\}^\ell : X\mapsto\eta_M(X).$$

For our purpose in this paper, we shall fix $M'\in\mathscr{L}^{H'}(M_0')$. It means that we may and shall start from some $M\in\mathscr{L}^{G,\omega}(M_0)$ such that u (associated to $\mathfrak{g}'$) belongs to the image of the injective map $H^1(F,M_{H_0}/Z_G)\to H^1(F,H_0/Z_G)$. Denote by $u_M\in H^1(F,M_{H_0}/Z_G)$ the preimage of u. Denote by ${\varepsilon'}^M$ the image of u_M under the morphism

$$H^1(F,M_{H_0}/Z_G)\to H^1(F,M_{H_0}/Z_M)\to H^1_{\mathrm{ab}}(F,M_{H_0}^{\omega_0}/Z_M\to M_{H_0}/Z_M).$$

Proposition 4.21 (cf. Corollary 4.16). *Suppose that F is a non-archimedean local field of characteristic zero. The map*

$$H^1(F,M_{H_0}/Z_G)\to H^1(F,M_{H_0}/Z_M)\to H^1_{\mathrm{ab}}(F,M_{H_0}^{\omega_0}/Z_M\to M_{H_0}/Z_M)=(\mathbb{Z}/2\mathbb{Z})^\ell : u_M\mapsto {\varepsilon'}^M$$

is understood as the map

$$H^1(F,M_{H_0}/Z_G)\to H^1(F,M_{H_0}/Z_M)\to H^1(F,M_{\mathrm{ad}})\to\{\pm1\}^\ell : u_M\mapsto e_{M'}.$$

5. Statement of results

Let F be a non-archimedean local field of characteristic zero and E be a quadratic extension of F. Fix $M'\in\mathscr{L}^{H'}(M_0')$. Recall that its image in $\mathscr{L}^{G,\omega}(M_0)$ is denoted by M.

Definition 5.1. *Let $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f'\in\mathcal{C}_c^\infty(\mathfrak{s}'(F))$. We say that f and f' are partially M-associated if they satisfy the following condition: for all $L\in\mathscr{L}^G(M)$ and all $Q\in\mathscr{F}^G(L)$, if $X\in(\mathfrak{l}\cap\mathfrak{s}_{\mathrm{rs}})(F)$ and $Y\in(\widetilde{\mathfrak{l}'}\cap\mathfrak{s}'_{\mathrm{rs}})(F)$ have L-matching orbits, then*

$$\kappa(X)J_L^Q(\eta,X,f)=J_{L'}^{Q'}(Y,f').$$

Definition 5.2. *Let $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f'\in\mathcal{C}_c^\infty(\mathfrak{s}'(F))$. We say that f and f' are M-associated if they are partially M-associated and satisfy the additional condition: for all $L\in\mathscr{L}^G(M)$, $Q\in\mathscr{F}^G(L)$ and $X\in(\mathfrak{l}\cap\mathfrak{s}_{\mathrm{rs}})(F)$, we have*

$$J_L^Q(\eta,X,f)=0$$

unless X comes potentially from $(\mathfrak{m}_{\widetilde{Q'}}\cap\mathfrak{s}'_{\mathrm{rs}})(F)$ (see Definition 4.5).

Theorem 5.3. *Let $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f'\in\mathcal{C}_c^\infty(\mathfrak{s}'(F))$ be partially M-associated and satisfy the following conditions.*

(a) The weighted orbital integrals of f and f' vanish for nontrivial weights.
(b) If $X\in\mathfrak{s}_{\mathrm{rs}}(F)$ does not come from $\mathfrak{s}'_{\mathrm{rs}}(F)$, then

$$J_G^G(\eta,X,f)=0.$$

Then $\gamma_\psi(\mathfrak{h}(F))^{-1}\hat{f}$ and $\gamma_\psi(\mathfrak{h}'(F))^{-1}\hat{f}'$ are M-associated.

Proof. Combine Proposition 5.5 and Corollary 5.11 below. □

Remark 5.4. The two conditions in the above theorem imply the additional condition in Definition 5.2. They may be weakened, but they are enough for our purpose.

Recall that we denote by $\gamma_\psi(\mathfrak{h}(F))$ (resp. $\gamma_\psi(\mathfrak{h}'(F))$) the Weil constants associated to $\mathfrak{h}(F)$ (resp. $\mathfrak{h}'(F)$).

Proposition 5.5. *Let $f\in\mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f'\in\mathcal{C}_c^\infty(\mathfrak{s}'(F))$ be partially M-associated and satisfy the two conditions in Theorem 5.3. Then $\gamma_\psi(\mathfrak{h}(F))^{-1}\hat{f}$ and $\gamma_\psi(\mathfrak{h}'(F))^{-1}\hat{f}'$ are also partially M-associated.*

The rest of this paper will be denoted to the proof of Proposition 5.5. Its corollary below may be more useful for applications.

Corollary 5.6. *Let $X \in (\mathfrak{m}\cap\mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'}\cap\mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits. Let $U \in \mathfrak{s}_{\mathrm{rs}}(F)$ and $V \in \mathfrak{s}'_{\mathrm{rs}}(F)$ be such that $U \leftrightarrow V$. Then we have the equality*

$$\gamma_\psi(\mathfrak{h}(F))^{-1}\kappa(X)\kappa(U)\hat{i}_M^G(\eta, X, U) = \gamma_\psi(\mathfrak{h}'(F))^{-1}\hat{i}_{M'}^{H'}(Y, V).$$

Proof. By Lemma 4.7, we may and shall fix an isomorphism $\varphi : \mathfrak{s}'_V(F) \to \mathfrak{s}_U(F)$ such that $\varphi(V) = U$. Recall that $W(H, \mathfrak{s}_U)$ (resp. $W(H', \mathfrak{s}'_V)$) denotes the Weyl group associated to $\mathfrak{s}_U \in \mathscr{T}^{\mathfrak{s}}$ (resp. $\mathfrak{s}'_V \in \mathscr{T}^{\mathfrak{s}'}$). Choose open compact neighbourhoods ω of U in $(\mathfrak{s}_U \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and ω' of V in $(\mathfrak{s}'_V \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ which are small enough such that

(i) the sets $i(\omega)$ where $i \in W(H, \mathfrak{s}_U)$ are mutually disjoint;
(ii) the sets $i'(\omega')$ where $i' \in W(H', \mathfrak{s}'_V)$ are mutually disjoint;
(iii) $\varphi(\omega') = \omega$;
(iv) $\kappa(\cdot)\hat{i}_M^G(\eta, X, \cdot)$ is constant on ω;
(v) $\hat{i}_{M'}^{H'}(Y, \cdot)$ is constant on ω'.

Notice that the conditions (iv) and (v) are assured by Lemmas 8.3.1) and 8.8.1) in Chapter 4.

By the condition (i) on ω and Lemma 3.1.1), we can construct a function $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ such that

(i) $\mathrm{Supp}(f) \subseteq \mathrm{Ad}(H(F))(\omega)$;
(ii) for all $Z \in \omega$, $\kappa(Z)J_G^G(\eta, Z, f) = 1$;
(iii) the weighted orbital integrals of f vanish for nontrivial weights.

By the condition (ii) on ω' and Lemma 3.1.2), we can construct a function $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ such that

(i) $\mathrm{Supp}(f') \subseteq \mathrm{Ad}(H'(F))(\omega')$;
(ii) for all $Z' \in \omega'$, $J_{H'}^{H'}(Z', f') = 1$;
(iii) the weighted orbital integrals of f' vanish for nontrivial weights.

We see that f and f' are partially M-associated and satisfy the two conditions in Theorem 5.3. By Proposition 5.5, we have the equality

$$\gamma_\psi(\mathfrak{h}(F))^{-1}\kappa(X)J_M^G(\eta, X, \hat{f}) = \gamma_\psi(\mathfrak{h}'(F))^{-1}J_{M'}^{H'}(Y, \hat{f}').$$

By the condition (iii) on f and the Weyl integration formula (7.1.2) in Chapter 4, we have

$$J_M^G(\eta, X, \hat{f}) = I_M^G(\eta, X, \hat{f}) = \sum_{\mathfrak{c}\in\mathscr{T}_0^{\mathfrak{s}}} |W(H, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_G^G(\eta, Z, f)\hat{i}_M^G(\eta, X, Z)dZ.$$

By the conditions (i) and (ii) on f and the condition (iv) on ω, the last expression equals

$$\int_\omega J_G^G(\eta, Z, f)\hat{i}_M^G(\eta, X, Z)dZ = \mathrm{vol}(\omega)\kappa(U)\hat{i}_M^G(\eta, X, U).$$

Similarly, with the help of the conditions on f', (7.2.2) in Chapter 4 and the condition (iv) on ω, we obtain

$$J_M^G(\eta, X, \hat{f}) = I_M^G(\eta, X, \hat{f}) = \mathrm{vol}(\omega')\hat{i}_{M'}^{H'}(Y, V).$$

Since $\mathrm{vol}(\omega) = \mathrm{vol}(\omega')$, we deduce the equality in the corollary. □

Recall that $\omega = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. For $X \in \mathfrak{s}(F)$, denote $X^\omega := \mathrm{Ad}(\omega)(X)$. For $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$, define $f^\omega(X) := f(X^\omega)$ for all $X \in \mathfrak{s}(F)$.

Lemma 5.7. *Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Then we have*

(1) $(\hat{f})^\omega = (f^\omega)\hat{}$;
(2) $J_M^G(\eta, X, f^\omega) = \eta(X)J_M^G(\eta, X, f)$.

Proof. Similar properties are used in the proof of [58, Lemma 8.3] though our involutions are slightly different. It suffices to notice additionally that $v_M^G(\mathrm{Ad}(w)(x)) = v_M^G(x)$ for $x \in H(F)$. □

Lemma 5.8. *Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in \mathfrak{s}_{\mathrm{rs}}(F)$. Then we have the equality*

$$\hat{i}_M^G(\eta, X, U^\omega) = \eta(X)\hat{i}_M^G(\eta, X, U).$$

Proof. From Lemma 5.7, we deduce that

$$\hat{j}_M^G(\eta, X, U^\omega) = \eta(X)\hat{j}_M^G(\eta, X, U). \tag{5.0.1}$$

There exists $x \in H(F)$, $L \in \mathscr{L}^{G,\omega}(M_0)$ and $Z \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$ such that $U = \mathrm{Ad}(x)(Z)$. By Lemmas 8.3.2) and 8.2 in Chapter 4, we have

$$\eta(X)\hat{i}_M^G(\eta, X, U) = \eta(\det(x))\eta(X)\hat{i}_M^G(\eta, X, Z) = \eta(\det(x))\eta(X)\hat{j}_M^G(\eta, X, Z).$$

Applying (5.0.1) to X and Z, we have

$$\eta(X)\hat{j}_M^G(\eta, X, Z) = \hat{j}_M^G(\eta, X, Z^\omega).$$

Since $Z^\omega \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)_{\mathrm{ell}}$, by Lemma 8.2 in Chapter 4 again, we obtain

$$\hat{j}_M^G(\eta, X, Z^\omega) = \hat{i}_M^G(\eta, X, Z^\omega).$$

Thus

$$\eta(X)\hat{i}_M^G(\eta, X, U) = \eta(\det(x))\hat{i}_M^G(\eta, X, Z^\omega).$$

We see that $U^\omega = \mathrm{Ad}(\omega x \omega^{-1})(Z^\omega)$, where $\omega x \omega^{-1} \in H(F)$. By Lemma 8.3.2) in Chapter 4 again, we have

$$\hat{i}_M^G(\eta, X, U^\omega) = \eta(\det(\omega x \omega^{-1}))\hat{i}_M^G(\eta, X, Z^\omega) = \eta(\det(x))\hat{i}_M^G(\eta, X, Z^\omega).$$

Then the lemma follows. □

Proposition 5.9. *Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in \mathfrak{s}_{\mathrm{rs}}(F)$. If $\eta(X) \neq \eta(U)$, then*

$$\hat{i}_M^G(\eta, X, U) = 0.$$

Proof. We see that $U^\omega = \mathrm{Ad}(\omega U)(U)$, where $\omega U \in H(F)$. By Lemma 8.3.2) in Chapter 4, we have

$$\hat{i}_M^G(\eta, X, U^\omega) = \eta(\det(\omega U))\hat{i}_M^G(\eta, X, U) = \eta(U)\hat{i}_M^G(\eta, X, U).$$

One may conclude by comparing this equality with Lemma 5.8. □

Remark 5.10. By the same argument, we can generalise the above proposition to the following form. Let $L \in \mathscr{L}^G(M)$. Let $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in (\mathfrak{l} \cap \mathfrak{s})_{\mathrm{rs}}(F)$. If $\eta_L(X) \neq \eta_L(U)$, then

$$\hat{i}_M^L(\eta, X, U) = 0.$$

Corollary 5.11. *Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ satisfies the additional condition in Definition 5.2. Then $\hat{f}$ also satisfies this condition.*

Proof. By induction, it suffices to show that for all $Q \in \mathscr{F}^G(M)$ and $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we have

$$J_M^Q(\eta, X, \hat{f}) = 0$$

unless X comes potentially from $(\mathfrak{m}_{\widetilde{Q'}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.

By Proposition 4.1.4) and (8.1.1) in Chapter 4, we obtain

$$J_M^Q(\eta, X, \hat{f}) = J_M^{M_Q}(\eta, X, \hat{f}_Q^\eta) = \sum_{L \in \mathscr{L}^{M_Q}(M)} \hat{I}_M^{L, M_Q, w}(\eta, X, f_Q^\eta),$$

where

$$\hat{I}_M^{L, M_Q, w}(\eta, X, f_Q^\eta) = \sum_{\{R \in \mathscr{L}^{G,\omega}(M_0) : R \subseteq L\}} |W_0^{R_n}||W_0^{L_n}|^{-1} \sum_{\mathfrak{c} \in \mathscr{T}_{\mathrm{ell}}(\mathfrak{r} \cap \mathfrak{s})} |W(R_H, \mathfrak{c})|^{-1}$$
$$\int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_L^{M_Q}(\eta, Z, f_Q^\eta)\hat{i}_M^L(\eta, X, Z)dZ.$$

By Proposition 4.1.4) in Chapter 4 again, we have

$$J_L^{M_Q}(\eta, Z, f_Q^\eta) = J_L^Q(\eta, Z, f).$$

If $J_M^Q(\eta, X, \hat{f}) \neq 0$, then $J_L^Q(\eta, Z, f)\hat{i}_M^L(\eta, X, Z) \neq 0$ for some Z. Since $J_L^Q(\eta, Z, f) \neq 0$, by our assumption on f, we see that Z comes potentially from $(\mathfrak{m}_{\widetilde{Q'}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Since $\hat{i}_M^L(\eta, X, Z) \neq 0$, by Remark 5.10, we have $\eta_L(X) = \eta_L(Z)$. As $L \subseteq M_Q$, it implies that $\eta_{M_Q}(X) = \eta_{M_Q}(Z)$. Thus X also comes potentially from $(\mathfrak{m}_{\widetilde{Q'}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. □

6. Limit formulae

Let F be a non-archimedean local field of characteristic zero and E be a quadratic extension of F. Recall that for $\mathfrak{c} \in \mathfrak{s}$ and $X, U \in (\mathfrak{c} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we define $\gamma_\psi(X, U)$ by (2.2.1). A similar notation is used in the case of (G', H').

Let $M \in \mathscr{L}^{G,\omega}(M_0)$. For all $L \in \mathscr{L}^G(M), X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we define

$$\hat{a}_M^M(\eta, X, U) := \sum_{x \in (M_H)_U(F) \backslash M_H(F), \mathrm{Ad}(x)(X) \in \mathfrak{s}_U(F)} \eta(\mathrm{Nrd}(x)) \gamma_\psi(\mathrm{Ad}(x)(X), U) \psi(\langle \mathrm{Ad}(x)(X), U \rangle)$$

if $L = M$ and

$$\hat{a}_M^L(\eta, X, U) := 0$$

if $L \neq M$.

Let $M' \in \mathscr{L}^{H'}(M'_0)$. For all $L' \in \mathscr{L}^{H'}(M'), Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ and $V \in (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$, we define

$$\hat{a}_{M'}^{M'}(Y, V) := \sum_{x \in M'_V(F) \backslash M'(F), \mathrm{Ad}(x)(Y) \in \mathfrak{s}'_V(F)} \gamma_\psi(\mathrm{Ad}(x)(Y), V) \psi(\langle \mathrm{Ad}(x)(Y), V \rangle)$$

if $L' = M'$ and

$$\hat{a}_{M'}^{L'}(Y, V) := 0$$

if $L' \neq M'$.

Proposition 6.1. *1) Let $M \in \mathscr{L}^{G,\omega}(M_0)$, $L \in \mathscr{L}^G(M), X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $U \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(F)$. Then there exists $N \in \mathbb{N}$ such that if $\mu \in F^\times$ satisfies $v_F(\mu) < -N$, we have the equality*

$$\hat{i}_M^L(\eta, \mu X, U) = \hat{a}_M^L(\eta, \mu X, U).$$

2) Let $M' \in \mathscr{L}^{H'}(M'_0)$, $L' \in \mathscr{L}^{H'}(M'), Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ and $V \in (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$. Then there exists $N \in \mathbb{N}$ such that if $\mu \in F^\times$ satisfies $v_F(\mu) < -N$, we have the equality

$$\hat{i}_{M'}^{L'}(\mu Y, V) = \hat{a}_{M'}^{L'}(\mu Y, V).$$

Proof. This is a generalisation of [58, Proposition 7.1] and Propositions 10.1 and 10.4 in Chapter 4. □

7. Construction of test functions

Let F be a non-archimedean local field of characteristic zero and E be a quadratic extension of F.

Lemma 7.1. *Let X, Y, y be as in Lemma 4.7. Let $V \in (\mathfrak{s}'_Y \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ and $U := \mathrm{Ad}(y) \circ \varphi(V)$. Then we have*

(1) $\langle X, U \rangle = \langle Y, V \rangle$;
(2) $\gamma_\psi(\mathfrak{h}(F))^{-1} \gamma_\psi(X, U) = \gamma_\psi(\mathfrak{h}'(F))^{-1} \gamma_\psi(Y, V)$.

Proof. This is a generalisation of [58, Lemma 7.5]. □

Fix $M' \in \mathscr{L}^{H'}(M'_0)$. Recall that its image in $\mathscr{L}^{G,\omega}(M_0)$ is denoted by M.

Proposition 7.2. *Let $X_0 \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y_0 \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ be such that $X_0 \leftrightarrow Y_0$. Then there exists $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ satisfying the following conditions.*

(a) If $X \in \mathrm{Supp}(f)$, there exists $Y \in (\mathfrak{s}'_{Y_0} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ such that $X \leftrightarrow Y$.
(b) If $Y \in \mathrm{Supp}(f')$, then Y is $H'(F)$-conjugate to an element in $(\mathfrak{s}'_{Y_0} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.
(c) The weighted orbital integrals of f and f' vanish for nontrivial weights.
(d) The functions f and f' are partially G-associated and satisfy the condition: if $X \in \mathfrak{s}_{\mathrm{rs}}(F)$ does not come from $\mathfrak{s}'_{\mathrm{rs}}(F)$, then

$$J_G^G(\eta, X, f) = 0.$$

(e) For $Q \in \mathscr{F}^G(M) - \mathscr{P}^G(M)$,

$$J_M^Q(\eta, X_0, \hat{f}) = J_{M'}^{Q'}(Y_0, \hat{f}') = 0.$$

(f) We have the equality

$$\gamma_\psi(\mathfrak{h}(F))^{-1} \kappa(X_0) J_G^G(\eta, X_0, \hat{f}) = \gamma_\psi(\mathfrak{h}'(F))^{-1} J_{H'}^{H'}(Y_0, \hat{f}') \neq 0.$$

Proof. This is a generalisation of [58, Proposition 7.6] whose additional ingredients are Lemma 3.1 and Proposition 6.1 (cf. [14, §6]).

Fix an isomorphism $\varphi : \mathfrak{s}'_{Y_0}(F) \to \mathfrak{s}_{X_0}(F)$ such that $\varphi(Y_0) = X_0$ as in Lemma 4.7. Choose $V_0 \in (\mathfrak{s}'_{Y_0} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ and denote $U_0 := \varphi(V_0) \in (\mathfrak{s}_{X_0} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ such that

(i) for all $i \in W(H, \mathfrak{s}_{X_0}), i \neq 1$, we have $\langle i(X_0) - X_0, U_0\rangle \neq 0$;
(ii) for all $i' \in W(H', \mathfrak{s}'_{Y_0}), i' \neq 1$, we have $\langle i'(Y_0) - Y_0, V_0\rangle \neq 0$;
(iii) $\kappa(U_0) = \kappa(X_0)$.

Fix $r \in \mathbb{N}^*$ such that

(i) $1 + \varpi^r \mathcal{O}_F \subseteq F^{\times 2}$;
(ii) the sets $i((1 + \varpi^r \mathcal{O}_F)U_0)$ where $i \in W(H, \mathfrak{s}_{X_0})$ are mutually disjoint;
(iii) the sets $i'((1 + \varpi^r \mathcal{O}_F)V_0)$ where $i' \in W(H', \mathfrak{s}'_{Y_0})$ are mutually disjoint.

By Propositions 6.1, there exists $N \in \mathbb{N}$ such that if $\mu \in F^\times$ satisfies $v_F(\mu) < -N$, then

(i) for $L_1, L_2 \in \mathscr{L}^G(M)$ with $L_1 \subseteq L_2$, $i_1 \in W(L_{1,H}, \mathfrak{s}_{X_0}, \mathfrak{l}_1 \cap \mathfrak{s})$ and $i_2 \in W(H, \mathfrak{s}_{X_0}, \mathfrak{l}_2 \cap \mathfrak{s})$, we have
$$\hat{i}^{L_2}_{L_1}(\eta, i_1(X_0), i_2(\mu U_0)) = \hat{a}^{L_2}_{L_1}(\eta, i_1(X_0), i_2(\mu U_0));$$
(ii) for $L'_1, L'_2 \in \mathscr{L}^{H'}(M')$ with $L'_1 \subseteq L'_2$, $i'_1 \in W(L'_1, \mathfrak{s}'_{Y_0}, \widetilde{\mathfrak{l}}'_1 \cap \mathfrak{s}')$ and $i'_2 \in W(H', \mathfrak{s}'_{Y_0}, \widetilde{\mathfrak{l}}'_2 \cap \mathfrak{s}')$, we have
$$\hat{i}^{L'_2}_{L'_1}(i'_1(Y_0), i'_2(\mu V_0)) = \hat{a}^{L'_2}_{L'_1}(i'_1(Y_0), i'_2(\mu V_0)).$$

Fix such an integer N.

Fix $\mu \in F^\times$ with $v_F(\mu) < -N$ such that

(i) $\eta(\mu) = 1$;
(ii) for all $i \in W(H, \mathfrak{s}_{X_0}), i \neq 1$, the character $\lambda \mapsto \psi(\varpi^r \mu\lambda\langle i(X_0) - X_0, U_0\rangle)$ is nontrivial on $\mathcal{O}_F$;
(iii) for all $i' \in W(H', \mathfrak{s}'_{Y_0}), i' \neq 1$, the character $\lambda \mapsto \psi(\varpi^r \mu\lambda\langle i'(Y_0) - Y_0, V_0\rangle)$ is nontrivial on $\mathcal{O}_F$.

Notice that the conditions (ii) and (iii) are possible because of the conditions (i) and (ii) on U_0 and V_0.

Set $\omega'_0 := \mu(1 + \varpi^r \mathcal{O}_F)V_0$. Denote by $\mathfrak{d}'$ the F-vector space generated by V_0. Fix a complement $\mathfrak{e}'$ of $\mathfrak{d}'$ in $\mathfrak{s}'_{Y_0}(F)$. For $V \in \mathfrak{s}'_{Y_0}(F)$, denote by $V_{\mathfrak{d}'}$ its projection to $\mathfrak{d}'$ with respect to the decomposition $\mathfrak{s}'_{Y_0}(F) = \mathfrak{d}' \oplus \mathfrak{e}'$.

Set $\omega_0 := \mu(1 + \varpi^r \mathcal{O}_F)U_0$. Denote by $\mathfrak{d} := \varphi(\mathfrak{d}')$ the F-vector space generated by U_0. Let $\mathfrak{e} := \varphi(\mathfrak{e}')$ be the complement of $\mathfrak{d}$ in $\mathfrak{s}_{X_0}(F)$. For $U \in \mathfrak{s}_{X_0}(F)$, denote by $U_{\mathfrak{d}}$ its projection to $\mathfrak{d}$ with respect to the decomposition $\mathfrak{s}_{X_0}(F) = \mathfrak{d} \oplus \mathfrak{e}$.

Choose open compact neighbourhoods $\omega_{\mathfrak{e}}$ of 0 in $\mathfrak{e}$ and $\omega_{\mathfrak{e}'}$ of 0 in $\mathfrak{e}'$ which are small enough such that $\omega := \omega_0 \oplus \omega_{\mathfrak{e}}$ and $\omega' := \omega'_0 \oplus \omega'_{\mathfrak{e}'}$ satisfy

(i) the sets $i(\omega)$ where $i \in W(H, \mathfrak{s}_{X_0})$ are mutually disjoint;
(ii) the sets $i'(\omega')$ where $i' \in W(H', \mathfrak{s}'_{Y_0})$ are mutually disjoint;
(iii) $\omega \subseteq (\mathfrak{s}_{X_0} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, $\omega' \subseteq (\mathfrak{s}'_{Y_0} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ and $\varphi(\omega') = \omega$;
(iv) for $L_1, L_2 \in \mathscr{L}^G(M)$ with $L_1 \subseteq L_2$, $i_1 \in W(L_{1,H}, \mathfrak{s}_{X_0}, \mathfrak{l}_1 \cap \mathfrak{s})$, $i_2 \in W(H, \mathfrak{s}_{X_0}, \mathfrak{l}_2 \cap \mathfrak{s})$ and $U \in \omega$, we have
$$\hat{i}^{L_2}_{L_1}(\eta, i_1(X_0), i_2(U)) = \hat{a}^{L_2}_{L_1}(\eta, i_1(X_0), i_2(U)) = \hat{a}^{L_2}_{L_1}(\eta, i_1(X_0), i_2(U_{\mathfrak{d}}));$$
(v) for $L'_1, L'_2 \in \mathscr{L}^{H'}(M')$ with $L'_1 \subseteq L'_2$, $i'_1 \in W(L'_1, \mathfrak{s}'_{Y_0}, \widetilde{\mathfrak{l}}'_1 \cap \mathfrak{s}')$, $i'_2 \in W(H', \mathfrak{s}'_{Y_0}, \widetilde{\mathfrak{l}}'_2 \cap \mathfrak{s}')$ and $V \in \omega'$, we have
$$\hat{i}^{L'_2}_{L'_1}(i'_1(Y_0), i'_2(V)) = \hat{a}^{L'_2}_{L'_1}(i'_1(Y_0), i'_2(V)) = \hat{a}^{L'_2}_{L'_1}(i'_1(Y_0), i'_2(V_{\mathfrak{d}'}));$$
(vi) the function κ is constant on ω.

Notice that the conditions (i) and (ii) follow from the conditions (ii) and (iii) on r. Besides, the conditions (iv) and (v) are assured by $v_F(\mu) < -N$ and $r \geq 1$. Morover, the condition (vi) results from the condition (i) on μ and the condition (i) on r. Combined with the condition (iii) on U_0, the condition (vi) says that the restriction of κ to ω equals $\kappa(X_0)$.

Define a function f_ω on ω by
$$f_\omega(U) := \psi(-\langle X_0, U_{\mathfrak{d}}\rangle)$$
for all $U \in \omega$. By the condition (i) on ω and Lemma 3.1.1), we can construct a function $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ such that

(i) $\mathrm{Supp}(f) \subseteq \mathrm{Ad}(H(F))(\omega)$;
(ii) for all $U \in \omega$, $\kappa(U) J^G_G(\eta, U, f) = f_\omega(U)$;
(iii) the weighted orbital integrals of f vanish for nontrivial weights.

Define a function $f'_{\omega'}$ on ω' by

$$f'_{\omega'}(V) := \psi(-\langle Y_0, V_{\mathfrak{d}'}\rangle)$$

for all $V \in \omega'$. By the condition (ii) on ω' and Lemma 3.1.2), we can construct a function $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ such that

(i) $\mathrm{Supp}(f') \subseteq \mathrm{Ad}(H'(F))(\omega')$;
(ii) for all $V \in \omega'$, $J_{H'}^{H'}(V, f') = f'_{\omega'}(V)$;
(iii) the weighted orbital integrals of f' vanish for nontrivial weights.

We shall check that f and f' satisfy the conditions in the proposition. The conditions (a) and (b) result from the condition (iii) on ω and ω', the condition (i) on f and the condition (i) on f'. The condition (c) is exactly the condition (iii) on f and the condition (iii) on f'. The condition (d) is deduced from the condition (iii) on ω and ω', the conditions (i) and (ii) on f and the conditions (i) and (ii) on f'.

We now verify the condition (e). By Proposition 4.1.4) in Chapter 4, we write

$$J_M^Q(\eta, X_0, \hat{f}) = J_M^{M_Q}(\eta, X_0, \hat{f}_Q^\eta).$$

For the same reason and the condition (iii) on f, the weighted orbital integrals of f_Q^η vanish for nontrivial weights. Then we have

$$J_M^{M_Q}(\eta, X_0, \hat{f}_Q^\eta) = I_M^{M_Q}(\eta, X_0, \hat{f}_Q^\eta).$$

From the Weyl integration formula (7.1.2) in Chapter 4, we deduce that

$$I_M^{M_Q}(\eta, X_0, \hat{f}_Q^\eta) = \sum_{\mathfrak{c} \in \mathscr{T}_0^{\mathfrak{m}_Q \cap \mathfrak{s}}} |W(M_{Q_H}, \mathfrak{c})|^{-1} \int_{\mathfrak{c}_{\mathrm{reg}}(F)} J_{M_Q}^{M_Q}(\eta, Z, f_Q^\eta) \hat{\imath}_M^{M_Q}(\eta, X_0, Z) dZ.$$

Using Proposition 4.1.4) Chapter 4 again, we obtain

$$J_{M_Q}^{M_Q}(\eta, Z, f_Q^\eta) = J_{M_Q}^Q(\eta, Z, f) = J_G^G(\eta, Z, f).$$

Suppose that $J_G^G(\eta, Z, f) \neq 0$. By the condition (i) on f, there exists $x \in H(F)$ such that $\mathrm{Ad}(x^{-1})(Z) \in \omega$. For such an x, we have $\mathrm{Ad}(x)(\mathfrak{s}_{X_0}) = \mathfrak{c}$. By the condition (iv) on ω, if $M_Q \neq M$, we get $\hat{\imath}_M^{M_Q}(\eta, X_0, Z) = 0$ and thus

$$J_M^Q(\eta, X_0, \hat{f}) = 0.$$

Similarly, if $M_{Q'} \neq M'$, we prove the vanishing of $J_{M'}^{Q'}(Y_0, \hat{f}')$ with the help of Proposition 4.4.4) and (7.2.2) in Chapter 4, the condition (i) on f' and the condition (v) on ω'.

The condition (f) is shown as in the last paragraph of the proof of [58, Proposition 7.6] with the aide of Lemma 7.1. □

8. The weighted fundamental lemma

Let F be a non-archimedean local field of characteristic zero and E be a quadratic extension of F. Assume that F has odd residue characteristic and that E/F is unramified. Assume that $(G', H') \simeq (GL_{2n}, \mathrm{Res}_{E/F} GL_{n,E})$. Let

$$\mathfrak{s}(\mathcal{O}_F) := \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A, B \in \mathfrak{gl}_n(\mathcal{O}_F) \right\}.$$

We identify $\mathfrak{s}'(F) \simeq \mathfrak{h}'(F)$ and let

$$\mathfrak{s}'(\mathcal{O}_F) := \mathfrak{gl}_n(\mathcal{O}_E).$$

Denote by $f_0 \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ (resp. $f'_0 \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$) the characteristic function of $\mathfrak{s}(\mathcal{O}_F)$ (resp. of $\mathfrak{s}'(\mathcal{O}_F)$).

Lemma 8.1 (see Theorem 10.9 in Chapter 3). *For all $M \in \mathscr{L}^{G,\omega}(M_0)$ and all $Q \in \mathscr{F}^G(M)$, we have*

(a) if $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ have M-matching orbits, then

$$\kappa(X) J_M^Q(\eta, X, f_0) = J_{M'}^{Q'}(Y, f'_0);$$

(b) for $X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$, we have

$$J_M^Q(\eta, X, f_0) = 0$$

unless X comes potentially from $(\mathfrak{m}_{\widetilde{Q'}} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$.

9. Approximation of local data by global data

Let k'/k be a quadratic extension of number fields and $\mathbb{D}$ be a central division algebra over k. For a place v of k, denote $k'_v := k' \otimes_k k_v$ and $\mathbb{D}_v := \mathbb{D} \otimes_k k_v$.

PROPOSITION 9.1. *Let E/F be a quadratic extension of non-archimedean local fields of characteristic 0.*

I) Let D be a central division algebra over F containing E. Then there exists a quadratic extension of number fields k'/k, a central division algebra $\mathbb{D}$ over k containing k', and a finite set S of finite places of k satisfying the following conditions.
   (a) The number field k is totally imaginary.
   (b) $|S| \geq 2$.
   (c) For all $v \in S$, we have $k_v \simeq F$, $k'_v \simeq E$ and $\mathbb{D}_v \simeq D$.
   (d) For all $v \notin S$, k_v splits $\mathbb{D}_v$.

II) Let D be a central division algebra over F such that $D \otimes_F E$ is a central division algebra over E. Then there exists a quadratic extension of number fields k'/k, a central division algebra $\mathbb{D}$ over k such that $\mathbb{D} \otimes_k k'$ is a central division algebra over k', and a finite set S of finite places of k satisfying the following conditions.
   (a) The number field k is totally imaginary.
   (b) $|S| \geq 2$.
   (c) For all $v \in S$, we have $k_v \simeq F$, $k'_v \simeq E$ and $\mathbb{D}_v \simeq D$.
   (d) For all $v \notin S$, k_v splits $\mathbb{D}_v$.

PROOF. By [14, Proposition 9.1], there exists a number field k, a central simple algebra $\mathbb{D}$ over k and a finite set S of finite places of k such that

(i) the number field k is totally imaginary;
(ii) $|S| \geq 2$;
(iii) for all $v \in S$, we have $k_v \simeq F$ and $\mathbb{D}_v \simeq D$;
(iv) for all $v \notin S$, k_v splits $\mathbb{D}_v$.

From the condition (iii), we know that $\mathbb{D}$ is a central division algebra over k. By [20, Theorem 3.1], there exists a quadratic extension k' of k such that $k'_v \simeq E$ for all $v \in S$.

I) We shall use [48, Theorem 1.2] to show that there exists an k-embedding of k' into $\mathbb{D}$. It is clear that there exists an k_v-embedding of k'_v into $\mathbb{D}_v$ for all place v of k. Let v' be a place of k' and v be the place of k below v'. Denote by c_v (resp. d_v) the capacity (resp. index) of $\mathbb{D}_v$. Since D is a central division algebra over F of even degree, we deduce that $\mathbb{D}$ is a central division algebra over k of even degree. Then $c_v d_v$ is even. Define $x_{v'} := \frac{c_v \cdot \gcd([k'_{v'}:k_v], d_v)}{2} \in \mathbb{Q}_{>0}$. If v is archimedean or $v \notin S$, i.e., k_v splits $\mathbb{D}_v$, then $d_v = 1$ and c_v is even, so $x_{v'} \in \mathbb{Z}$. If $v \in S$, then $[k'_{v'} : k_v] = [E : F] = 2$, which implies that $x_{v'} \in \mathbb{Z}$ since $c_v d_v$ is even. We may use [48, Theorem 1.2] to conclude.

II) Let $v \in S$ and v' be the unique place of k' over v. Since $(\mathbb{D} \otimes_k k')_{v'} \simeq \mathbb{D} \otimes_k k_v \otimes_{k_v} k'_{v'} \simeq D \otimes_F E$ is a central division algebra over E, we know that $\mathbb{D} \otimes_k k'$ is a central division algebra over k'.

□

10. An infinitesimal variant of Guo-Jacquet trace formulae

Let k'/k be a quadratic extension of number fields and η the quadratic character of $\mathbb{A}^\times/k^\times$ attached to it, where $\mathbb{A}$ denotes the ring of adèles of k.

Let $\mathbf{G}$ be a reductive group over k. Fix a maximal compact subgroup K of $\mathbf{G}(\mathbb{A})$ which is admissible relative to $\mathbf{M}_0$ in the sense of [5, p. 9]. In this paper, we choose the standard maximal compact subgroup when $\mathbf{G}(k) = \mathbf{GL}_n(\mathbb{D})$, where $\mathbb{D}$ is a central division algebra over a finite field extension of k. That is to say, $K := \prod_v K_v$ where each K_v is the standard maximal compact subgroup of $\mathbf{G}(k_v)$. For $\mathbf{M} \in \mathscr{L}^{\mathbf{G}}(\mathbf{M}_0)$, we define the weight function $v^{\mathbf{G}}_{\mathbf{M}}$ with respect to K as in the local case (2.1.1). We fix the Haar measure on K such that $\mathrm{vol}(K) = 1$. We fix a Haar measure on $\mathbf{G}(\mathbb{A})$ which is compatible with the Iwasawa decomposition.

Let $(\mathbf{G}, \mathbf{H}, \theta)$ be a symmetric pair defined over k. Denote by $\mathcal{S}(\mathfrak{s}(\mathbb{A}))$ the Bruhat-Schwartz space of $\mathfrak{s}(\mathbb{A})$. Let $\langle \cdot, \cdot \rangle$ be a G-invariant θ-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. Fix a continuous and nontrivial unitary character $\Psi : \mathbb{A}/k \to \mathbb{C}^\times$. For $f \in \mathfrak{s}(\mathbb{A})$, we define its Fourier transform $\hat{f} \in \mathfrak{s}(\mathbb{A})$

by

$$\forall \widehat{X} \in \mathfrak{s}(\mathbb{A}), \hat{f}(\widehat{X}) := \int_{\mathfrak{s}(\mathbb{A})} f(X)\Psi(\langle X, \widehat{X}\rangle)dX.$$

We fix the self-dual on $\mathfrak{s}(\mathbb{A})$ with respect to the $\Psi(\langle\cdot,\cdot\rangle)$.

We shall write $(\mathbf{G}, \mathbf{H})$ and $(\mathbf{G}', \mathbf{H}')$ for the global symmetric pairs over k with respect to k' defined in Sections 2.3 and 2.4 respectively. We still use a minuscule Fraktur letter to denote a global Lie algebra and write $\mathfrak{s}$ and $\mathfrak{s}'$ for the corresponding global symmetric spaces. We also fix minimal Levi k-subgroups of these four groups as in the local case and denote them by bold letters.

For $\phi \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$, we define

$$J^{\mathbf{G}}(\eta, \phi) = \sum_{\mathfrak{o} \in \mathcal{O}^{\mathfrak{s}}_{\mathrm{rs}}} J^{\mathbf{G}}_{\mathfrak{o}}(\eta, \phi), \tag{10.0.1}$$

where $\mathcal{O}^{\mathfrak{s}}_{\mathrm{rs}}$ denotes the set of $\mathbf{H}(k)$-orbits in $\mathfrak{s}_{\mathrm{rs}}(k)$, and $J^{\mathbf{G}}_{\mathfrak{o}}(\eta, \phi)$ is the constant term of (5.0.1) with $s = 0$ in Chapter 2. From Theorem 4.11 and Corollary 5.6 in Chapter 2, we know that the right hand side of (10.0.1) is absolutely convergent. For $\mathfrak{o} \in \mathcal{O}^{\mathfrak{s}}_{\mathrm{rs}}$, let $\mathbf{L} \in \mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0)$ and $X \in \mathfrak{o} \cap (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(k)_{\mathrm{ell}}$. By Theorem 9.2 in Chapter 2, we have

$$J^{\mathbf{G}}_{\mathfrak{o}}(\eta, \phi) = \mathrm{vol}(A^{\infty}_{\mathbf{L}}\mathbf{H}_X(k)\backslash\mathbf{H}_X(\mathbb{A})) \int_{\mathbf{H}_X(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \phi(\mathrm{Ad}(x^{-1})(X))\eta(\mathrm{Nrd}(x))v^{\mathbf{G}}_{\mathbf{L}}(x)dx.$$

We denote

$$\tau(\mathbf{H}_X) := \mathrm{vol}(A^{\infty}_{\mathbf{L}}\mathbf{H}_X(k)\backslash\mathbf{H}_X(\mathbb{A}))$$

for $X \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(k)_{\mathrm{ell}}$ and define

$$J^{\mathbf{G}}_{\mathbf{L}}(\eta, X, \phi) := \int_{\mathbf{H}_X(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \phi(\mathrm{Ad}(x^{-1})(X))\eta(\mathrm{Nrd}(x))v^{\mathbf{G}}_{\mathbf{L}}(x)dx$$

for all $X \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(k)$. From Lemma 3.10 in Chapter 4, we obtain

$$J^{\mathbf{G}}(\eta, \phi) = \sum_{\mathbf{L} \in \mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0)} |W_0^{\mathbf{L}_n}||W_0^{\mathbf{G}\mathbf{L}_n}|^{-1} \sum_{X \in \Gamma_{\mathrm{ell}}((\mathfrak{l}\cap\mathfrak{s}_{\mathrm{rs}})(k))} \tau(\mathbf{H}_X)J^{\mathbf{G}}_{\mathbf{L}}(\eta, X, \phi). \tag{10.0.2}$$

Denote by $\mathcal{S}(\mathfrak{s}'(\mathbb{A}))$ the Bruhat-Schwartz space of $\mathfrak{s}'(\mathbb{A})$. For $\phi' \in \mathcal{S}(\mathfrak{s}'(\mathbb{A}))$, we define

$$J^{\mathbf{H}'}(\phi') = \sum_{\mathfrak{o} \in \mathcal{O}^{\mathfrak{s}'}_{\mathrm{rs}}} J^{\mathbf{H}'}_{\mathfrak{o}}(\phi'), \tag{10.0.3}$$

where $\mathcal{O}^{\mathfrak{s}'}_{\mathrm{rs}}$ denotes the set of $\mathbf{H}'(k)$-orbits in $\mathfrak{s}'_{\mathrm{rs}}(k)$, and $J^{\mathbf{H}'}_{\mathfrak{o}}(\phi')$ is the constant term of (5.0.1) in Chapter 3. From Theorem 4.2 and Corollary 5.3 in Chapter 3, we know that the right hand side of (10.0.3) is absolutely convergent. For $\mathfrak{o} \in \mathcal{O}^{\mathfrak{s}'}_{\mathrm{rs}}$, let $\mathbf{L}' \in \mathscr{L}^{\mathbf{H}'}(\mathbf{M}'_0)$ and $Y \in \mathfrak{o} \cap (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k)_{\mathrm{ell}}$. By Theorem 9.2 in Chapter 3, we have

$$J^{\mathbf{H}'}_{\mathfrak{o}}(\phi') = \mathrm{vol}(A^{\infty}_{\mathbf{L}'}\mathbf{H}'_Y(k)\backslash\mathbf{H}'_Y(\mathbb{A})) \int_{\mathbf{H}'_Y(\mathbb{A})\backslash\mathbf{H}'(\mathbb{A})} \phi'(\mathrm{Ad}(x^{-1})(Y))v^{\mathbf{H}'}_{\mathbf{L}'}(x)dx.$$

We denote

$$\tau(\mathbf{H}'_Y) := \mathrm{vol}(A^{\infty}_{\mathbf{L}'}\mathbf{H}'_Y(k)\backslash\mathbf{H}'_Y(\mathbb{A}))$$

for $Y \in (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k)_{\mathrm{ell}}$ and define

$$J^{\mathbf{H}'}_{\mathbf{L}'}(Y, \phi') := \int_{\mathbf{H}'_Y(\mathbb{A})\backslash\mathbf{H}'(\mathbb{A})} \phi'(\mathrm{Ad}(x^{-1})(Y))v^{\mathbf{H}'}_{\mathbf{L}'}(x)dx$$

for all $Y \in (\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k)$. From Lemma 3.16 in Chapter 4, we obtain

$$J^{\mathbf{H}'}(\phi') = \sum_{\mathbf{L}' \in \mathscr{L}^{\mathbf{H}'}(\mathbf{M}'_0)} |W_0^{\mathbf{L}'}||W_0^{\mathbf{H}'}|^{-1} \sum_{Y \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{l}'}\cap\mathfrak{s}'_{\mathrm{rs}})(k))} \tau(\mathbf{H}'_Y)J^{\mathbf{H}'}_{\mathbf{L}'}(Y, \phi'). \tag{10.0.4}$$

Proposition 10.1 (see Theorem 7.1 in Chapter 2 and Theorem 7.1 in Chapter 3). *1) Let $\phi \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ be such that $\mathrm{Supp}(\phi) \subseteq \mathfrak{s}_{\mathrm{rs}}(k_{v_1})$ and $\mathrm{Supp}(\hat{\phi}) \subseteq \mathfrak{s}_{\mathrm{rs}}(k_{v_2})$ at some places v_1, v_2 of k. Then we have the equality*

$$J^{\mathbf{G}}(\eta, \phi) = J^{\mathbf{G}}(\eta, \hat{\phi}).$$

2) Let $\phi' \in \mathcal{S}(\mathfrak{s}'(\mathbb{A}))$ be such that $\mathrm{Supp}(\phi') \subseteq \mathfrak{s}'_{\mathrm{rs}}(k_{v_1})$ *and* $\mathrm{Supp}(\hat{\phi}') \subseteq \mathfrak{s}'_{\mathrm{rs}}(k_{v_2})$ *at some places v_1, v_2 of k. Then we have the equality*

$$J^{\mathbf{H}'}(\phi') = J^{\mathbf{H}'}(\hat{\phi}').$$

11. Proof of Proposition 5.5

Let F be a non-archimedean local field of characteristic zero and E be a quadratic extension of F. Fix $M' \in \mathscr{L}^{H'}(M'_0)$. Recall that its image in $\mathscr{L}^{G,\omega}(M_0)$ is denoted by M. Let $f \in \mathcal{C}_c^\infty(\mathfrak{s}(F))$ and $f' \in \mathcal{C}_c^\infty(\mathfrak{s}'(F))$ be partially M-associated and satisfy the two conditions in Theorem 5.3. Fix $X_0 \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(F)$ and $Y_0 \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(F)$ with M-matching orbits. It is obvious from Definition 5.1 that a pair of partially M-associated functions are also partially L-associated for all $L \in \mathscr{L}^G(M)$. Then by induction, to prove Proposition 5.5, it suffices to show that for all $Q \in \mathscr{F}^G(M)$, we have the equality

$$\kappa(X_0) J_M^Q(\eta, X_0, \hat{f}) = J_{M'}^{Q'}(Y_0, \hat{f}'). \tag{11.0.1}$$

11.1. Global data. Fix a quadratic extension of number fields k'/k, a central division algebra $\mathbb{D}$ over k, and a finite set S of finite places of k satisfying the conditions in Proposition 9.1. Fix $w \in S$. Denote by V (resp. V_∞, V_f) the set of places (resp. archimedean places, finite places) of k.

Define the global symmetric pairs $(\mathbf{G}, \mathbf{H})$ and $(\mathbf{G}', \mathbf{H}')$ over k with respect to k' and $\mathbb{D}$ as in the local case. There is a bijection $L \mapsto \mathbf{L}$ from $\mathscr{L}^G(M_0)$ to $\mathscr{L}^{\mathbf{G}}(\mathbf{M}_0)$ such that $\mathbf{L}_w \simeq L$ and we denote by $\mathbf{M}$ the image of M under this bijection. Similarly, there is a bijection $L' \mapsto \mathbf{L}'$ from $\mathscr{L}^{H'}(M'_0)$ to $\mathscr{L}^{\mathbf{H}'}(\mathbf{M}'_0)$ such that $\mathbf{L}'_w \simeq L'$ and we denote by $\mathbf{M}'$ the image of M' under this bijection.

11.2. Places. Fix a continuous and nontrivial unitary character $\Psi : \mathbb{A}/k \to \mathbb{C}^\times$ whose local component at w is ψ. Let $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ (resp. $\langle \cdot, \cdot \rangle_{\mathfrak{s}'}$) be the $\mathbf{H}$-invariant (resp. $\mathbf{H}'$-invariant) non-degenerate symmetric bilinear form on $\mathfrak{s}$ (resp. on $\mathfrak{s}'$) defined by (2.3.1) (resp. by (2.4.1)). Then we deduce local data Ψ_v, $\langle \cdot, \cdot \rangle_{\mathfrak{s}_v}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{s}'_v}$ for all $v \in V$. Fix a finite set $S_1 \subseteq V$ and for $v \in V - S_1$ lattices $\mathfrak{k}_v \subseteq \mathfrak{s}(k_v)$ and $\mathfrak{k}'_v \subseteq \mathfrak{s}'(k_v)$ such that

(i) $V_\infty \sqcup S \subseteq S_1$;
(ii) if $v \in V - S_1$, then
- k_v has odd residue characteristic and v is unramified in k';
- $\mathfrak{k}_v = \left\{ \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} : A, B \in \mathfrak{gl}_n(\mathcal{O}_{k_v}) \right\}$;
- $\mathfrak{k}'_v = \mathfrak{k}_v$ if v splits in k', while $\mathfrak{k}'_v = \mathfrak{gl}_n(\mathcal{O}_{k'_v})$ if v is inert in k';
- $\mathfrak{k}_v$ (resp. $\mathfrak{k}'_v$) is self-dual respect to $\Psi_v(\langle \cdot, \cdot \rangle_{\mathfrak{s}_v})$ (resp. $\Psi_v(\langle \cdot, \cdot \rangle_{\mathfrak{s}'_v})$).

For $v \in V - S_1$, notice that $\widehat{1_{\mathfrak{k}_v}} = 1_{\mathfrak{k}_v}$ (resp. $\widehat{1_{\mathfrak{k}'_v}} = 1_{\mathfrak{k}'_v}$), where $1_{\mathfrak{k}_v}$ (resp. $1_{\mathfrak{k}'_v}$) denotes the characteristic function of $\mathfrak{k}_v$ (resp. $\mathfrak{k}'_v$).

11.3. Orbits. For each $v \in S_1 - V_\infty$, we fix an open compact non-empty subset $\Omega_v \subseteq (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k_v)$ such that

(i) if $v = w$, then $Y_0 \in \Omega_w \subseteq \mathrm{Ad}(\mathbf{M}'(k_w))((\mathfrak{s}'_{Y_0} \cap \mathfrak{s}'_{\mathrm{rs}})(k_w))$, $J_{M'}^{Q'}(\cdot, \hat{f}')$ is constant on Ω_w and $\kappa(\cdot) J_M^Q(\eta, \cdot, \hat{f})$ is constant on $\{X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(k_w) : \exists Y \in \Omega_w, X \text{ and } Y \text{ have } M\text{-matching orbits}\}$;
(ii) if $v \in S - \{w\}$, then $\Omega_v \subseteq (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k_v)_{\mathrm{ell}}$.

Notice that the condition (i) is achievable because of Lemma 4.7, Propositions 4.4.2) and 4.1.2) in Chapter 4 and the constancy of $J_{M'}^{Q'}(\cdot, \hat{f}')$ (resp. $\kappa(\cdot) J_M^Q(\eta, \cdot, \hat{f})$) on $\mathbf{M}'(k_w)$ (resp. $\mathbf{M}_{\mathbf{H}}(k_w)$)-orbits. Besides, the set $(\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k_v)_{\mathrm{ell}}$ in the condition (ii) is not empty (see [55, Lemma 2.7 and p. 14]). Since $X_0 \leftrightarrow Y_0$, the condition (i) implies that the restriction of $\kappa(\cdot) J_M^Q(\eta, \cdot, \hat{f})$ to $\{X \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(k_w) : \exists Y \in \Omega_w, X \leftrightarrow Y\}$ equals $\kappa(X_0) J_M^Q(\eta, X_0, \hat{f})$.

By the strong approximation theorem, there exists $Y^0 \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}')(k)$ such that

(i) for $v \in S_1 - V_\infty$, $Y^0 \in \Omega_v$;
(ii) for $v \in V - S_1$, $Y^0 \in \widetilde{\mathfrak{m}'}(k_v) \cap \mathfrak{k}'_v$.

Combined with the condition (ii) on Ω_v, the condition (i) implies that $Y^0 \in (\widetilde{\mathfrak{m}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k)_{\mathrm{ell}}$. Choose an element $X^0 \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(k)_{\mathrm{ell}}$ such that X^0 and Y^0 have $\mathbf{M}$-matching orbits.

11.4. Choice of functions. For each $v \in V$, we fix functions $\phi_v \in \mathcal{S}(\mathfrak{s}(k_v))$ and $\phi'_v \in \mathcal{S}(\mathfrak{s}'(k_v))$ as follows.

(i) If $v = w$, let $\phi_w = f$ and $\phi'_w = f'$.
(ii) If $v \in S_1 - V_\infty - \{w\}$, then ϕ_v and ϕ'_v verify the conditions in Proposition 7.2 with respect to $\mathbf{M}'_v$, X^0 and Y^0.
(iii) If $v \in V - S_1$, let $\phi_v = 1_{\mathfrak{k}_v}$ and $\phi'_v = 1_{\mathfrak{k}'_v}$.
(iv) If $v \in V_\infty$, we identify $(\mathbf{H}_v, \mathfrak{s}_v)$ and $(\mathbf{H}'_v, \mathfrak{s}'_v)$ by the condition (a) in Proposition 9.1, and choose $\phi_v = \phi'_v$ such that
- $J^{\mathbf{G}_v}_{\mathbf{G}_v}(\eta, X^0, \hat{\phi}_v) = J^{\mathbf{H}'_v}_{\mathbf{H}'_v}(Y^0, \hat{\phi}'_v) \neq 0$;
- if $X \in \mathfrak{s}(k)$ is $\mathbf{H}(k_v)$-conjugate to an element in $\mathrm{Supp}(\hat{\phi}_v)$ for all $v \in V$, then X is $\mathbf{H}(k)$-conjugate to X^0;
- if $Y \in \mathfrak{s}'(k)$ is $\mathbf{H}'(k_v)$-conjugate to an element in $\mathrm{Supp}(\hat{\phi}'_v)$ for all $v \in V$, then Y is $\mathbf{H}'(k)$-conjugate to Y^0.

Notice that the condition (ii) is easier to be satisfied if v splits in k'. In fact, we can identify $(\mathbf{H}_v, \mathfrak{s}_v)$ and $(\mathbf{H}'_v, \mathfrak{s}'_v)$ by the condition (d) in Proposition 9.1, and choose $\phi_v = \phi'_v$. Additionally, the condition (iv) is possible (see [58, p. 1874]).

In the rest of the proof of Proposition 5.5, we consider the global functions $\phi := \prod_{v \in V} \phi_v \in \mathcal{S}(\mathfrak{s}(\mathbb{A}))$ and $\phi' := \prod_{v \in V} \phi'_v \in \mathcal{S}(\mathfrak{s}'(\mathbb{A}))$.

11.5. Comparison of $J^{\mathbf{G}}(\eta, \phi)$ and $J^{\mathbf{H}'}(\phi')$.

Lemma 11.1 (cf. [14, Lemme 10.1]). *For our choice of ϕ and ϕ', we have*

$$J^{\mathbf{G}}(\eta, \phi) = \sum_{\mathbf{L} \in \mathscr{L}^{\mathbf{G}}(\mathbf{M})} |\mathrm{Tran}_{\mathbf{GL}_n}(\mathbf{M}_n, \mathbf{L}_n)|^{-1} \sum_{X \in \Gamma_{\mathrm{ell}}((\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(k))} \tau(\mathbf{H}_X) J^{\mathbf{G}}_{\mathbf{L}}(\eta, X, \phi)$$

and

$$J^{\mathbf{H}'}(\phi') = \sum_{\mathbf{L}' \in \mathscr{L}^{\mathbf{H}'}(\mathbf{M}')} |\mathrm{Tran}_{\mathbf{H}'}(\mathbf{M}', \mathbf{L}')|^{-1} \sum_{Y \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k))} \tau(\mathbf{H}'_Y) J^{\mathbf{H}'}_{\mathbf{L}'}(Y, \phi').$$

Proof. For the first formula, we start with (10.0.2). Fix $\mathbf{L} \in \mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0)$ and $X \in \Gamma_{\mathrm{ell}}((\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(k))$ such that $J^{\mathbf{G}}_{\mathbf{L}}(\eta, X, \phi) \neq 0$. By the condition (b) in Proposition 9.1, there exists $u \in S - \{w\}$. From the condition imposed on ϕ_u (see (a) in Proposition 7.2) and Lemma 4.7, we see that X is $\mathbf{H}(k_u)$-conjugate to an element in $(\mathfrak{s}_{X^0} \cap \mathfrak{s}_{\mathrm{rs}})(k_u)$. Choose $\mathbf{R} \in \mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0)$ such that $\mathbf{R} \subseteq \mathbf{L}$ and that X is $\mathbf{L}_{\mathbf{H}}(k_u)$-conjugate to an element in $(\mathfrak{r} \cap \mathfrak{s}_{\mathrm{rs}})(k_u)_{\mathrm{ell}}$. Then there exists $x \in \mathbf{H}(k_u)$ and $Z \in (\mathfrak{r} \cap \mathfrak{s}_{\mathrm{rs}})(k_u)_{\mathrm{ell}}$ such that $Z = \mathrm{Ad}(x)(X^0)$. But by the condition (ii) on Ω_u, we know that $X^0 \in (\mathfrak{m} \cap \mathfrak{s}_{\mathrm{rs}})(k_u)_{\mathrm{ell}}$. Thus by Lemma 3.10.1) in Chapter 4, there exists $w \in \left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{\mathbf{GL}_{n,u}} \right\}$ such that $\mathrm{Ad}(w)(\mathbf{M}_u) = \mathbf{R}_u$. Since we may identify $W_0^{\mathbf{GL}_{n,u}}$ with $W_0^{\mathbf{GL}_n}$, we deduce that $\mathrm{Tran}_{\mathbf{GL}_n}(\mathbf{M}_n, \mathbf{L}_n) \neq \emptyset$. That is to say, we may restrict the sum on $\mathbf{L} \in \mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0)$ in (10.0.2) to those conjugate to an element in $\mathscr{L}^{\mathbf{G}}(\mathbf{M})$ under $\left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{\mathbf{GL}_n} \right\}$.

We reindex the sum on $\mathbf{L} \in \mathscr{L}^{\mathbf{G}}(\mathbf{M})$. On the one hand, the number of elements in $\mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0)$ conjugate to $\mathbf{L}$ under $\left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{\mathbf{GL}_n} \right\}$ is

$$|\mathrm{Norm}_{W_0^{\mathbf{GL}_n}}(\mathbf{L}_n) \backslash W_0^{\mathbf{GL}_n}|.$$

On the other hand, the number of elements in $\mathscr{L}^{\mathbf{G}}(\mathbf{M})$ conjugate to $\mathbf{L}$ under $\left\{ \begin{pmatrix} \omega_n & \\ & \omega_n \end{pmatrix} : \omega_n \in W_0^{\mathbf{GL}_n} \right\}$ is

$$|W^{\mathbf{GL}_n}(\mathbf{L}_n)|^{-1} |\mathrm{Tran}_{\mathbf{GL}_n}(\mathbf{M}_n, \mathbf{L}_n)|.$$

Since $W^{\mathbf{GL}_n}(\mathbf{L}_n) = \mathrm{Norm}_{W_0^{\mathbf{GL}_n}}(\mathbf{L}_n)/W_0^{\mathbf{L}_n}$ (see [35, (7.12.2)]), we obtain

$$|W_0^{\mathbf{L}_n}||W_0^{\mathbf{GL}_n}|^{-1} \cdot |\mathrm{Norm}_{W_0^{\mathbf{GL}_n}}(\mathbf{L}_n) \backslash W_0^{\mathbf{GL}_n}| \cdot |W^{\mathbf{GL}_n}(\mathbf{L}_n)||\mathrm{Tran}_{\mathbf{GL}_n}(\mathbf{M}_n, \mathbf{L}_n)|^{-1} = |\mathrm{Tran}_{\mathbf{GL}_n}(\mathbf{M}_n, \mathbf{L}_n)|^{-1}.$$

Then the first formula of the lemma follows.

The second formula can be proved in a similar way with the help of (10.0.4), the condition (b) in Proposition 7.2 of ϕ'_u and Lemma 3.16.1) in Chapter 4. The only additional ingredient is the condition (c) in Proposition 9.1, by which we may identify $W_0^{\mathbf{H}'_u}$ with $W_0^{\mathbf{H}'}$. $\square$

Recall that we have bijections $\mathscr{L}^{\mathbf{H}'}(\mathbf{M}') \to \mathscr{L}^{\mathbf{G}}(\mathbf{M})$ and $\mathscr{F}^{\mathbf{H}'}(\mathbf{M}') \to \mathscr{F}^{\mathbf{G}}(\mathbf{M})$. For all $\mathbf{L} \in \mathscr{L}^{\mathbf{G}}(\mathbf{M})$, we also have $\mathbb{R}$-linear isomorphisms between $\mathfrak{a}_{\mathbf{L}'}$ and $\mathfrak{a}_{\mathbf{L}}$ and, for all $v \in V$, between $\mathfrak{a}_{\mathbf{L}'_v}$ and $\mathfrak{a}_{\mathbf{L}_v}$. By modifying the scalar products on these Euclidean spaces, we may and shall suppose that these bijections are isometries. Then for all $(\mathbf{R}_v)_{v\in V} \in \prod_{v\in V} \mathscr{L}^{\mathbf{G}_v}(\mathbf{L}_v)$, we have the equality

$$d_{\mathbf{L}'}^{\mathbf{H}'}((\mathbf{R}'_v)_{v\in V}) = d_{\mathbf{L}}^{\mathbf{G}}((\mathbf{R}_v)_{v\in V}),$$

where both sides are Arthur's coefficients defined in [6, p. 356]. We shall also choose compatible sections

$$(\mathbf{R}'_v)_{v\in V} \mapsto (\mathbf{Q}_{\mathbf{R}'_v})_{v\in V} \in \prod_{v\in V} \mathscr{P}^{\mathbf{H}'_v}(\mathbf{R}'_v)$$

and

$$(\mathbf{R}_v)_{v\in V} \mapsto (\mathbf{Q}_{\mathbf{R}_v})_{v\in V} \in \prod_{v\in V} \mathscr{P}^{\mathbf{G}_v}(\mathbf{R}_v)$$

in the sense that for all $v \in V$,

$$\mathbf{Q}_{\mathbf{R}'_v} = (\mathbf{Q}_{\mathbf{R}_v})'.$$

Proposition 11.2. *For our choice of ϕ and ϕ', we have the equality*

$$J^{\mathbf{G}}(\eta, \phi) = J^{\mathbf{H}'}(\phi').$$

The rest of this section is devoted to the proof of Proposition 11.2.

Let $\mathbf{L} \in \mathscr{L}^{\mathbf{G}}(\mathbf{M})$. There is a canonical bijection between $\mathrm{Tran}_{\mathbf{GL}_n}(\mathbf{M}_n, \mathbf{L}_n)$ and $\mathrm{Tran}_{\mathbf{H}'}(\mathbf{M}', \mathbf{L}')$ since both of them are understood as permutations. If $X \in \Gamma_{\mathrm{ell}}((\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})(k))$ and $Y \in \Gamma_{\mathrm{ell}}((\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k))$ have matching orbits, by Lemma 4.7, there is an isomorphism $\mathbf{H}_X \simeq \mathbf{H}'_Y$ over F. By choosing compatible Haar measures on them, we have $\tau(\mathbf{H}_X) = \tau(\mathbf{H}'_Y)$. Since X and Y have matching orbits at each $v \in V$, we see that $J_{\mathbf{L}}^{\mathbf{G}}(\eta, X, \phi) = J_{\mathbf{L}'}^{\mathbf{H}'}(Y, \phi')$ by our choice of ϕ and ϕ'.

Now assume that X does not come from $(\widetilde{\mathfrak{l}'} \cap \mathfrak{s}'_{\mathrm{rs}})(k)$ (or equivalently $\mathfrak{s}'_{\mathrm{rs}}(k)$). By Lemma 11.1, to show Proposition 11.2, it suffices to show that $J_{\mathbf{L}}^{\mathbf{G}}(\eta, X, \phi) = 0$.

By the splitting formula of $(\mathbf{G}, \mathbf{M})$-families applied to the weight function (see [6, Corollary 7.4]), we write

$$J_{\mathbf{L}}^{\mathbf{G}}(\eta, X, \phi) = \sum_{(\mathbf{R}_v)_{v\in V} \in \prod_{v\in V} \mathscr{L}^{\mathbf{G}_v}(\mathbf{L}_v)} d_{\mathbf{L}}^{\mathbf{G}}((\mathbf{R}_v)_{v\in V}) \prod_{v\in V} \kappa_v(X) J_{\mathbf{L}_v}^{\mathbf{Q}_{\mathbf{R}_v}}(\eta, X, \phi_v).$$

If $J_{\mathbf{L}}^{\mathbf{G}}(\eta, X, \phi) \neq 0$, we fix $(\mathbf{R}_v)_{v\in V} \in \prod_{v\in V} \mathscr{L}^{\mathbf{G}_v}(\mathbf{L}_v)$ such that

$$d_{\mathbf{L}}^{\mathbf{G}}((\mathbf{R}_v)_{v\in V}) \prod_{v\in V} \kappa_v(X) J_{\mathbf{L}_v}^{\mathbf{Q}_{\mathbf{R}_v}}(\eta, X, \phi_v) \neq 0.$$

We choose a representative of X (still denoted by X) in the form of $\begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix} \in (\mathfrak{l} \cap \mathfrak{s}_{\mathrm{rs}})_{\mathrm{ell}}(k)$.

Lemma 11.3. *For all $v \in V$, we have $\eta_{\mathbf{R}_v}(X) = e_{\mathbf{R}'_v}$.*

Proof. It is trivial if $v \in V$ splits in k' (in particular, if $v \in V_\infty$). For $v \in V - S_1$ which does not split in k', it results from Lemma 8.1.(b). For $v \in S_1 - V_\infty - \{w\}$ which does not split in k', it is deduced from Proposition 7.2.(a)(c) and Corollary 4.6.(1). For $v = w$, it is because f satisfies the additional condition in Definition 5.2. $\square$

Lemma 11.4. *For all $v \in V$, we have $\eta_{\mathbf{L}_v}(X) = e_{\mathbf{L}'_v}$.*

Proof. Let S_2 be the subset of V_f consisting of v which does not split in k'. The assertion is trivial for $v \in V - S_2$. Consider $v \in S_2$. By Propositions 4.20 and 4.21, $\eta_{\mathbf{L}_v}(X)$ and $e_{\mathbf{L}'_v}$ are understood as $\varepsilon_X^{\mathbf{L}_v}$ and $\varepsilon'^{\mathbf{L}_v}$ respectively. By Proposition 4.19, we view $\varepsilon_X^{\mathbf{L}_v}$ and $\varepsilon'^{\mathbf{L}_v}$ as complex characters of $Z_{\widehat{\mathbf{L}}}[2]$. Denote by ζ_v the character obtained by their quotient.

For all $v \in S_2$, by Lemma 11.3, we know that the character ζ_v is trivial on $Z_{\widehat{\mathbf{R}_v}}[2]$. By the product formula of Kottwitz signs [33, the last proposition] and the fundamental exact sequence of global class

field theory for k'/k (see [42, Example 4.4.(a) in Chapter VIII] for example), we also have a global condition: the product $\prod_{v\in S_2}\zeta_v$ is trivial on $Z_{\widehat{\mathbf{L}}}[2]$.

Let $v\in S_2$. We need to show that ζ_v is trivial on $Z_{\widehat{\mathbf{L}}}[2]$. We first deduce from $d^{\mathbf{G}}_{\mathbf{L}}((\mathbf{R}_v)_{v\in V})\neq 0$ that $Z_{\widehat{\mathbf{L}}}=A_vA^v$, where $A_v:=Z_{\widehat{\mathbf{R}_v}}$ and $A^v:=\bigcap_{v'\in S_2-\{v\}}Z_{\widehat{\mathbf{R}_{v'}}}$. By [15, Lemme 4.5], we also have $A_v\cap A^v=Z_{\widehat{\mathbf{G}}}$ since $d^{\mathbf{G}}_{\mathbf{L}}((\mathbf{R}_v)_{v\in V})\neq 0$.

We claim that $Z_{\widehat{\mathbf{L}}}[2]=A_v[2]A^v[2]$. To see this, let $s=a_va^v\in Z_{\widehat{\mathbf{L}}}[2]$, where $a_v\in A_v$ and $a^v\in A^v$. Then $a_v^2=(a^v)^{-2}\in Z_{\widehat{\mathbf{G}}}$. Since $z\mapsto z^2$ is a surjective endomorphism of $Z_{\widehat{\mathbf{G}}}$, there exists $y\in Z_{\widehat{\mathbf{G}}}$ such that $a_v^2=(a^v)^{-2}=y^2$. Thus $s=y^{-1}a_v\cdot(ya^v)$ with $y^{-1}a_v\in A_v[2]$ and $ya^v\in A^v[2]$. We have shown our claim.

Now, let $s=a_va^v\in Z_{\widehat{\mathbf{L}}}[2]$, where $a_v\in A_v[2]$ and $a^v\in A^v[2]$. Since the character ζ_v is trivial on $Z_{\widehat{\mathbf{R}_v}}[2]$, we have $\zeta_v(s)=\zeta_v(a_v)\zeta_v(a^v)=\zeta_v(a^v)$. By the global condition above, we have $\zeta_v(a_v)=\prod_{v'\in S_2-\{v\}}\zeta_{v'}^{-1}(a^v)$. But for $v'\in S_2-\{v\}$, the character $\zeta_{v'}$ is trivial on $A^v[2]$. We have proved $\zeta_v(s)=1$ for all $s\in Z_{\widehat{\mathbf{L}}}[2]$ and thus the lemma. □

Lemma 11.5. *For all $v\in V$, X comes from $\mathfrak{s}'_{\mathrm{rs}}(k_v)$.*

Proof. It suffices to consider $v\in V-S_1$ which does not split in k'. Notice that $e_{\mathbf{L}'_v}=(1,\cdots,1)$. We are in a similar situation as the base change for GL_n. There exists $\mathbf{L}_0\in\mathscr{L}^{\mathbf{G},\omega}(\mathbf{M}_0),\mathbf{L}_0\subseteq\mathbf{L}$ such that X is $(\mathbf{L}_v\cap\mathbf{H}_v)(k_v)$ conjugate to an element $Z\in(\mathfrak{l}_0\cap\mathfrak{s}_{\mathrm{rs}})(k_v)_{\mathrm{ell}}$. If X does not come from $\mathfrak{s}'_{\mathrm{rs}}(k_v)$, by [10, Lemma 1.4 in Chapter 1], there exists $\eta_{\mathbf{L}_{0,v}}(Z)\neq e_{\mathbf{L}'_{0,v}}=(1,\cdots,1)$. By Lemma 8.1.(b) and the argument of [37, Lemma III.3.4], we deduce that $J^{\mathbf{Q}_{\mathbf{R}_v}}_{\mathbf{L}_v}(\eta,X,\phi_v)=0$, which contradicts our assumption. □

Proposition 11.6. *The element X comes from $\mathfrak{s}'_{\mathrm{rs}}(k)$.*

Proof. We start with two lemmas.

Lemma 11.7. *Let F_0 be a field. Let $Z_C\subseteq A\subseteq B\subseteq C$ be reductive groups defined over F_0, where Z_C denotes the centre of C. Suppose that $H^1(F_0,A_\beta)$ is a singleton for all inner form A_β of A. Then the natural map*

$$H^1(F_0,A/Z_C)\to H^1(F_0,B/Z_C)$$

is injective.

Proof of Lemma 11.7. We begin with the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & Z_C \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & A/Z_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_C \arrow[r] & B \arrow[r] & B/Z_C \arrow[r] & 0
\end{tikzcd}$$

Then we obtain the following commutative diagram of pointed sets.

$$\begin{tikzcd}
H^1(F_0,A/Z_C) \arrow[r] \arrow[d, "g"] & H^2(F_0,Z_C) \arrow[d, equal] \\
H^1(F_0,B/Z_C) \arrow[r] & H^2(F_0,Z_C)
\end{tikzcd}$$

Since $H^1(F_0,A_\beta)$ is a singleton for all inner form A_β of A, the map

$$H^1(F_0,A/Z_C)\to H^2(F_0,Z_C)$$

is injective by [32, Corollary (28.13)]. As the above diagram is commutative, we deduce that g is injective. □

Lemma 11.8. *Let F_0 be a global field. Let $Z_B\subseteq A\subseteq B$ be reductive groups defined over F_0, where Z_B denotes the centre of B. Suppose that $H^1(F_0,A_\beta)$ is a singleton for all inner form A_β of A. Then the map*

$$H^1(F_0,A/Z_B)\to H^1(\mathbb{A}_{F_0},A/Z_B)$$

is injective.

Proof of Lemma 11.8. We use the commutative diagram of pointed sets:

$$\begin{tikzcd} H^1(F_0, A/Z_B) \arrow[r, "i"] \arrow[d, "g"] & H^1(\mathbb{A}_{F_0}, A/Z_B) \arrow[d] \\ H^1(F_0, B_{\mathrm{ad}}) \arrow[r, "h"] & H^1(\mathbb{A}_{F_0}, B_{\mathrm{ad}}) \end{tikzcd}$$

By Lemma 11.7, the map g is injective. By the Hasse principle for B_{ad}, the map h is injective. Since the diagram is commutative, we deduce that i is injective. □

Return to the proof of Proposition 11.6. We shall use the notation in Section 4.3 but bold letters for reductive groups defined over k. Let $t \in H^1(k, \mathbf{T}/\mathbf{R})$ be the class associated to X in Lemma 4.8. Let $u \in H^1(k, \mathbf{H}_0/Z_{\mathbf{G}})$ be the class associated to $\mathfrak{g}'$. Let $t' \in H^1(\mathbb{A}, \mathbf{T}/\mathbf{R})$ (resp. $u' \in H^1(\mathbb{A}, \mathbf{H}_0/Z_{\mathbf{G}})$) be the image of t (resp. u). By Lemmas 11.5 and 4.9, there exists $v' \in H^1(k, \mathbf{T}/Z_{\mathbf{G}})$ with images t' and u'.

By [38, Proposition 1.6.12], we have the following commutative diagram of pointed sets with exact rows.

$$\begin{tikzcd} H^1(k, \mathbf{T}/Z_{\mathbf{G}}) \arrow[r] \arrow[d] & H^1(\mathbb{A}, \mathbf{T}/Z_{\mathbf{G}}) \arrow[r] \arrow[d] & H^1(\mathbb{A}/k, \mathbf{T}/Z_{\mathbf{G}}) \arrow[d, "g"] \\ H^1(k, \mathbf{H}_0/Z_{\mathbf{G}}) \arrow[r, "i"] & H^1(\mathbb{A}, \mathbf{H}_0/Z_{\mathbf{G}}) \arrow[r] & H^1_{\mathrm{ab}}(\mathbb{A}/k, \mathbf{H}_0/Z_{\mathbf{G}}) \end{tikzcd}$$

Let $v'' \in H^1(\mathbb{A}, \mathbf{T}/\mathbf{R})$ (resp. $u'' \in H^1(\mathbb{A}, \mathbf{H}_0/Z_{\mathbf{G}})$) be the image of v' (resp. u'). Then $u'' = 0$. By a variant of Lemma 11.7 for $H^1_{\mathrm{ab}}(\mathbb{A}/k, \cdot)$, the map g is injective. Thus $v'' = 0$ and there exists $v \in H^1(k, \mathbf{T}/Z_{\mathbf{G}})$ with image v'. By Lemma 11.8, the map i is injective. Since the square on the left above is commutative, the class v maps to u.

We also have the following commutative diagram.

$$\begin{tikzcd} H^1(k, \mathbf{T}/Z_{\mathbf{G}}) \arrow[r] \arrow[d] & H^1(\mathbb{A}, \mathbf{T}/Z_{\mathbf{G}}) \arrow[d] \\ H^1(k, \mathbf{T}/\mathbf{R}) \arrow[r, "j"] & H^1(\mathbb{A}, \mathbf{T}/\mathbf{R}) \end{tikzcd}$$

By our discussion in Section 4.3, the map j is given by

$$\prod_{i \in I_0} k_i^\times / N_{k_i'/k_i}(k_i'^\times) \to \prod_{i \in I_0} \mathbb{A}_{k_i}^\times / N_{k_i'/k_i}(\mathbb{A}_{k_i'}^\times),$$

where $k_i = k[\lambda]/(\chi_i(\lambda))$ with χ_i being an irreducible polynomial over k, and $k_i' = k_i \otimes_k k'$. It is known to be injective. Since the diagram is commutative, the class v maps to t.

By Lemma 4.9, we draw our conclusion. □

Proof of Proposition 11.2. The statement of Proposition 11.6 contradicts our assumption, so we have finished the proof. □

11.6. End of the proof.

Lemma 11.9. *For our choice of ϕ and ϕ', we have*

$$J^{\mathbf{G}}(\eta, \hat{\phi}) = \tau(\mathbf{H}_{X^0}) J^{\mathbf{G}}_{\mathbf{M}}(\eta, X^0, \hat{\phi})$$

and

$$J^{\mathbf{H}'}(\hat{\phi}') = \tau(\mathbf{H}'_{Y^0}) J^{\mathbf{H}'}_{\mathbf{M}'}(Y^0, \hat{\phi}').$$

Proof. The first formula results from (10.0.1) for $\hat{\phi}$ and the condition (iv) in Section 11.4, while the second formula results from (10.0.3) for $\hat{\phi}'$ and the condition (iv) in Section 11.4. □

Proof of Proposition 5.5. Combining Propositions 10.1 and 11.2 and Lemma 11.9, for our choice of ϕ and ϕ', we have the equality

$$J^{\mathbf{G}}_{\mathbf{M}}(\eta, X^0, \hat{\phi}) = J^{\mathbf{H}'}_{\mathbf{M}'}(Y^0, \hat{\phi}').$$

But by the splitting formula and the conditions (iii) and (iv) on ϕ_v and ϕ'_v, the difference of two sides can be written as

$$\begin{aligned}(11.6.1)\qquad &\sum_{(\mathbf{L}_v)_{v\in V}\in\prod_{v\in V}\mathscr{L}^{\mathbf{G}_v}(\mathbf{M}_v)} d^{\mathbf{G}}_{\mathbf{M}}((\mathbf{L}_v)_{v\in V})\prod_{v\in(V-S_1)\cup V_\infty}\kappa_v(X^0)J^{\mathbf{Q}_{\mathbf{L}_v}}_{\mathbf{M}_v}(\eta,X^0,\hat{\phi}_v)\\ &\times\left[\prod_{v\in S_1-V_\infty}\kappa_v(X^0)J^{\mathbf{Q}_{\mathbf{L}_v}}_{\mathbf{M}_v}(\eta,X^0,\hat{\phi}_v)-\prod_{v\in S_1-V_\infty}J^{\mathbf{Q}'_{\mathbf{L}_v}}_{\mathbf{M}'_v}(Y^0,\hat{\phi}'_v)\right].\end{aligned}$$

By the condition (e) in Proposition 7.2, we may suppose that $\mathbf{L}_v=\mathbf{M}_v$ for all $v\in S_1-V_\infty-\{w\}$. Moreover, by the condition (f) in Proposition 7.2, we have

$$\kappa_v(X^0)J^{\mathbf{Q}_{\mathbf{M}_v}}_{\mathbf{M}_v}(\eta,X^0,\hat{\phi}_v)=\gamma_{\Psi_v}(\mathfrak{h}(k_v))\gamma_{\Psi_v}(\mathfrak{h}'(k_v))^{-1}J^{\mathbf{Q}'_{\mathbf{M}_v}}_{\mathbf{M}'_v}(Y^0,\hat{\phi}'_v)\neq 0$$

for all $v\in S_1-V_\infty-\{w\}$.

Recall the product formula

$$\prod_{v\in V}\gamma_{\Psi_v}(\mathfrak{h}(k_v))=\prod_{v\in V}\gamma_{\Psi_v}(\mathfrak{h}'(k_v))=1.$$

For $v\in(V-S_1)\cup V_\infty$, we have

$$\gamma_{\Psi_v}(\mathfrak{h}(k_v))=\gamma_{\Psi_v}(\mathfrak{h}'(k_v))=1.$$

Hence,

$$\prod_{v\in S_1-V_\infty-\{w\}}\gamma_{\Psi_v}(\mathfrak{h}(k_v))\gamma_{\Psi_v}(\mathfrak{h}'(k_v))^{-1}=\gamma_{\Psi_w}(\mathfrak{h}(k_w))^{-1}\gamma_{\Psi_w}(\mathfrak{h}'(k_w)).$$

Then the expression (11.6.1) equals

$$\begin{aligned}(11.6.2)\qquad &\sum_{(\mathbf{L}_v)_{v\in V}\in\prod_{v\in V}\mathscr{L}^{\mathbf{G}_v}(\mathbf{M}_v)} d^{\mathbf{G}}_{\mathbf{M}}((\mathbf{L}_v)_{v\in V})\prod_{v\in V-\{w\}}\kappa_v(X^0)J^{\mathbf{Q}_{\mathbf{L}_v}}_{\mathbf{M}_v}(\eta,X^0,\hat{\phi}_v)\\ &\times\left[\gamma_{\Psi_w}(\mathfrak{h}(k_w))^{-1}\gamma_{\Psi_w}(\mathfrak{h}'(k_w))\kappa_w(X^0)J^{\mathbf{Q}_{\mathbf{L}_w}}_{\mathbf{M}_w}(\eta,X^0,\hat{\phi}_w)-J^{\mathbf{Q}'_{\mathbf{L}_w}}_{\mathbf{M}'_w}(Y^0,\hat{\phi}'_w)\right].\end{aligned}$$

Since ϕ_w and ϕ'_w are partially $\mathbf{M}_w$-associated, by parabolic descent (see Propositions 4.1.4) and 4.4.4) in Chapter 4), we see that $\phi^{\eta}_{w,\mathbf{Q}_{\mathbf{L}_w}}\in\mathcal{C}^\infty_c((\mathfrak{l}_w\cap\mathfrak{s}_w)(k_w))$ and $\phi'_{w,\mathbf{Q}'_{\mathbf{L}_w}}\in\mathcal{C}^\infty_c((\widetilde{\mathfrak{l}'_w}\cap\mathfrak{s}'_w)(k_w))$ are partially $\mathbf{M}_w$-associated. We shall prove (11.0.1) by induction on the dimension of G. Then we may suppose that for $\mathbf{L}_w\neq\mathbf{G}_w$, we have the equality

$$\kappa_w(X^0)J^{\mathbf{L}_w}_{\mathbf{M}_w}(\eta,X^0,\hat{\phi}^{\eta}_{w,\mathbf{Q}_{\mathbf{L}_w}})=\gamma_{\Psi_w}((\mathfrak{l}_w\cap\mathfrak{h}_w)(k_w))\gamma_{\Psi_w}(\mathfrak{l}'_w(k_w))^{-1}J^{\mathbf{L}'_w}_{\mathbf{M}'_w}(Y^0,\hat{\phi'}_{w,\mathbf{Q}'_{\mathbf{L}_w}}).$$

Since the difference between the quadratic form on $\mathfrak{h}(k_w)$ (resp. $\mathfrak{h}'(k_w)$) and its restriction on $(\mathfrak{l}_w\cap\mathfrak{h}_w)(k_w)$ (resp. $\mathfrak{l}'_w(k_w)$) is a split quadratic form, we have $\gamma_{\Psi_w}((\mathfrak{l}_w\cap\mathfrak{h}_w)(k_w))=\gamma_{\Psi_w}(\mathfrak{h}(k_w))$ (resp. $\gamma_{\Psi_w}(\mathfrak{l}'_w(k_w))=\gamma_{\Psi_w}(\mathfrak{h}'(k_w))$). By parabolic descent again, we see that

$$\kappa_w(X^0)J^{\mathbf{Q}_{\mathbf{L}_w}}_{\mathbf{M}_w}(\eta,X^0,\hat{\phi}_w)=\gamma_{\Psi_w}(\mathfrak{h}(k_w))\gamma_{\Psi_w}(\mathfrak{h}'(k_w))^{-1}J^{\mathbf{Q}'_{\mathbf{L}_w}}_{\mathbf{M}'_w}(Y^0,\hat{\phi}'_w).$$

By the condition (i) on Ω_w, it implies (11.0.1) for all $\mathbf{Q}_{\mathbf{L}_w}\neq\mathbf{G}_w$ (actually for all $\mathbf{Q}_w\in\mathscr{L}^{\mathbf{G}_w}(\mathbf{M}_w)$, $\mathbf{Q}_w\neq\mathbf{G}_w$). Then we may suppose that $\mathbf{L}_w=\mathbf{G}_w$ in (11.6.2). But for all $\mathbf{L}\in\mathscr{L}^{\mathbf{G}}(\mathbf{M})$, we have $\mathfrak{a}^{\mathbf{G}}_{\mathbf{L}}\simeq\mathfrak{a}^{\mathbf{G}_w}_{\mathbf{L}_w}$. Thus $d^{\mathbf{G}}_{\mathbf{M}}((\mathbf{L}_v)_{v\in V})=0$ unless $\mathbf{L}_v=\mathbf{M}_v$ for all $v\in V-\{w\}$, in which case $d^{\mathbf{G}}_{\mathbf{M}}((\mathbf{L}_v)_{v\in V})=1$. That is to say, the sum in (11.6.2) is reduced to only one term. We obtain

$$\begin{aligned}&\prod_{v\in V-\{w\}}\kappa_v(X^0)J^{\mathbf{Q}_{\mathbf{M}_v}}_{\mathbf{M}_v}(\eta,X^0,\hat{\phi}_v)\\ &\times\left[\gamma_{\Psi_w}(\mathfrak{h}(k_w))^{-1}\gamma_{\Psi_w}(\mathfrak{h}'(k_w))\kappa_w(X^0)J^{\mathbf{G}_w}_{\mathbf{M}_w}(\eta,X^0,\hat{\phi}_w)-J^{\mathbf{G}'_w}_{\mathbf{M}'_w}(Y^0,\hat{\phi}'_w)\right]=0.\end{aligned}$$

By the condition (iv) on ϕ_v for $v\in V_\infty$, the condition (f) in Proposition 7.2, the condition (ii) on Y^0 and Lemma 8.1.(a), we know that the product on $v\in V-\{w\}$ does not vanish. Then we conclude by the condition (i) on Ω_w. □

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