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# Renormalized solutions to a class of quasilinear elliptic problems with weak data: existence, uniqueness and homogenization

Rheadel Fulgencio

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Normandie Université

## THÈSE

Pour obtenir le diplôme de doctorat

Spécialité : Mathématiques

Préparée au sein de l'Université de Rouen Normandie

**Solutions renormalisées d'une classe de problèmes elliptiques quasi-linéaires avec saut : existence, unicité et homogénéisation**

Présentée et soutenue par  
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*“You must forge your own path for it to mean anything.”*

-Rick Riordan, *The Lost Hero*



# Abstract

In this thesis, we study a class of quasilinear elliptic equations posed in a two-component domain with an  $L^1$  data and its asymptotic analysis. More precisely, we consider a two-component domain, denoted by  $\Omega$ , which can be written as the disjoint union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , where the open sets  $\Omega_1$  and  $\Omega_2$  are the two components of  $\Omega$ , and  $\Gamma$  is the interface between these components. We study the following quasilinear elliptic problem posed in  $\Omega$ :

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\nu_1$  is the unit outward normal to  $\Omega_1$ ,  $f$  is an  $L^1$  function, and  $B$  is a coercive matrix field which has a restricted growth assumption ( $B(x, r)$  is bounded on any compact set of  $\mathbb{R}$ ).

The first part of this thesis is dedicated to existence and uniqueness results for this problem in the framework of renormalized solutions, which was introduced by R.J. DiPerna and P.L. Lions.

In the second part, we study the corresponding homogenization problem for a two-component domain with a (disconnected) periodic second component by combining the notion of renormalized solutions and the periodic unfolding method, introduced D. Cioranescu, A. Dalmaoian and G. Griso. It has been successively adapted to two-component domains by P. Donato, K.H. Le Nguyen, and R. Tardieu.

In order to obtain a uniqueness result for the homogenized problem, we study the properties of the corresponding cell problem. In particular, we show that if the matrix field in the cell problem, denoted  $A(y, t)$ , is local Lipschitz-continuous with respect to  $t$ , then the resulting homogenized matrix  $A^0$  keeps this property. This uniqueness result ensures that the convergences obtained in the homogenization process hold for the whole sequence of the periodicity parameter (and not only a subsequence).



# Résumé

Dans cette thèse, nous étudions une classe de problèmes elliptiques quasi-linéaires posés dans un domaine à deux composantes avec une donnée  $L^1$  et son analyse asymptotique. Plus précisément, on considère un domaine  $\Omega$ , que l'on écrit comme une réunion disjointe  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , où les ensembles ouverts  $\Omega_1$  et  $\Omega_2$  sont les deux composantes de  $\Omega$ , et  $\Gamma$  est l'interface entre les composantes. Nous étudions le problème elliptique quasi-linéaire suivant posé dans  $\Omega$  :

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{dans } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{dans } \Omega_2, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{sur } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{sur } \Gamma, \\ u_1 = 0 & \text{sur } \partial\Omega, \end{cases}$$

où  $\nu_1$  est le vecteur normal unitaire extérieur à  $\Omega_1$ ,  $f \in L^1(\Omega)$  et  $B$  est une matrice coercitive qui vérifie une hypothèse assez générale ( $B(x, r)$  n'est pas uniformément borné mais borné sur tout ensemble compact de  $\mathbb{R}$ ).

La première partie de cette thèse est donc dédiée à des résultats d'existence et d'unicité de ce problème dans le cadre des solutions renormalisées, qui a été introduit par R.J. DiPerna et P.L. Lions.

Dans la deuxième partie, nous étudions l'homogénéisation d'un problème du même type, posé dans un domaine à deux composantes dont la deuxième est une réunion périodique d'ensembles déconnectés, en mélangeant le notion des solution renormalisées et la méthode de l'éclatement périodique. Cette méthode a été introduite par D. Cioranescu, A. Dalmlamian and G. Griso et adaptée aux domaines à deux composantes par P. Donato, K.H. Le Nguyen, et R. Tardieu.

Pour obtenir un résultat d'unicité pour le problème homogénéisé qui puisse assurer que les convergences obtenues sont valables pour toute la suite du paramètre de périodicité (et non pas à une sous-suite près), nous étudions les propriétés du problème périodique correspondant, posé dans la cellule de référence. En particulier, nous démontrons que si la matrice  $A(y, t)$  du problème dans la cellule de référence est localement lipschitzienne par rapport à  $t$ , alors la matrice homogénéisée résultante  $A^0(t)$  garde cette propriété.





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# Résumé de la thèse

Dans cette thèse, nous étudions une classe de problèmes elliptiques quasi-linéaires posés dans un domaine à deux composantes avec une donnée  $L^1$  et nous en faisons l'analyse asymptotique dans un domaine périodique à deux composantes. Plus précisément, on considère un domaine  $\Omega$ , que l'on écrit comme une réunion disjointe

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma,$$

où les ensembles ouverts  $\Omega_1$  et  $\Omega_2$  sont les deux composantes de  $\Omega$ , et  $\Gamma$  est l'interface entre les composantes. Nous étudions le problème elliptique quasi-linéaire suivant posé dans  $\Omega$  :

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{dans } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{dans } \Omega_2, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{sur } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{sur } \Gamma, \\ u_1 = 0 & \text{sur } \partial\Omega, \end{cases} \quad (1)$$

où  $\nu_1$  est le vecteur normal unitaire extérieur à  $\Omega_1$ ,  $f \in L^1(\Omega)$  et  $B$  est une matrice coercitive qui vérifie une hypothèse assez générale ( $B(x, r)$  n'est pas uniformément borné mais borné sur tout ensemble compact de  $\mathbb{R}$ ).

Observons que sous les hypothèses précédentes sur  $f$  et  $B$  nous ne pouvons pas, en général, obtenir l'existence d'une solution faible. Même si  $f \in L^2(\Omega)$  sans hypothèse de bornitude sur  $B(x, r)$  par rapport à  $r$ , on ne sait pas démontrer en général l'existence d'une solution faible (de même si  $B$  est borné et  $f \in L^1(\Omega)$ ).

Rappelons que le problème

$$-\operatorname{div}(A(x, u)\nabla u) = f$$

avec des conditions de Dirichlet sur le bord, si  $A(x, r)$  est bornée, elliptique et  $f$  appartient à  $L^1(\Omega)$  (ou même est une mesure bornée de Radon), il existe

d'après Boccardo-Gallouët [18] une solution au sens des distributions (et ces résultats sont valable pour une classe plus général d'opérateurs non linéaires à croissance  $p$ ). Les auteurs démontrent que  $u \in W_0^{1,q}(\Omega)$ ,  $\forall 1 < q < \frac{N}{N-1}$  et verifie

$$\int_{\Omega} A(x) \nabla u \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Cependant, cette solution au sens des distributions ne peut pas avoir, en général, une énergie finie, au sens où  $u \notin H_0^1(\Omega)$ . De plus même dans le cas linéaire, c'est-à-dire  $A(x, r) = A(x)$ , la solution au sens des distributions n'est pas unique en général d'après le contre exemple de Serrin [78] (voir aussi [76]).

Pour pallier cet inconvénient, plusieurs notions de solutions ont été développées : solutions entropiques (voir [10]), SOLA (solutions obtenues comme limite d'approximation, voir [37]) et solutions renormalisées.

Pour mener à bien notre étude sur le problème (1) (existence, unicité, analyse asymptotique), nous avons besoin d'une notion de solution qui permet des résultats d'unicité et de stabilité. Nous utiliserons dans cette thèse la notion de solution renormalisée.

La notion de solution renormalisée a été introduite dans [39] par R.J. DiPerna et P.L. Lions pour des équations du premier ordre. Elle a été développée ensuite par F. Murat dans [71], par P.L. Lions et F. Murat dans [63] pour des équations elliptiques avec conditions de Dirichlet et données  $L^1$ , puis par G. Dal Maso et al. dans [36] pour des équations elliptiques avec données mesures. La plupart des travaux concernant le développement des solutions renormalisées traite de problèmes elliptiques (ou paraboliques) à données  $L^1$  et avec des conditions de Dirichlet, mais plus rarement le cas d'autres conditions sur le bord (citons [13] pour des conditions de Neumann, [59] pour un domaine perforé). L'existence et l'unicité d'une solution renormalisée ont été étudiées pour des domaines perforés dans [59], avec une condition de Fourier sur le bord des trous, mais, à notre connaissance les équations de type (1) avec donnée  $L^1$  et saut à l'interface n'ont pas été abordées dans la littérature.

La première partie de cette thèse est donc dédiée à des résultats d'existence et d'unicité pour la solution du problème (1). Nous avons donné d'abord une définition appropriée de solution renormalisée du problème. Cette définition, ainsi que le résultat d'existence, sont présentés dans le chapitre 2. L'unicité de cette solution est démontrée dans le chapitre 3, où une hypothèse supplémentaire de lipschitzianité locale pour la matrice  $B$  est nécessaire.

Dans la deuxième partie, nous étudions l'homogénéisation d'un problème du même type, posé dans un domaine à deux composantes dont la deuxième

est une réunion périodique d'ensembles déconnectés, qui est présentée dans le chapitre 5. Dans ce chapitre, nous identifions d'abord, en utilisant les estimations a priori obtenues dans la première partie, le problème éclaté (théorème 12). Nous obtenons ensuite le problème homogénéisé dans  $\Omega$  (théorème 13).

Pour y parvenir, nous utilisons la méthode de l'éclatement périodique, qui a été introduite dans [31] pour des domaines fixes et dans [29] pour des domaines perforés. Elle a été étendue successivement au cas de domaines à deux composantes dans [46] et [45] (pour une présentation générale nous renvoyons au livre récent [32]).

Pour obtenir un résultat d'unicité pour le problème homogénéisé qui puisse assurer que les convergences obtenues sont valables pour toute la suite du paramètre de périodicité (et non pas à une sous-suite près), nous étudions dans le chapitre 4 les propriétés du problème périodique correspondant, posé dans la cellule de référence (voir (10)). En particulier, nous démontrons que si la matrice  $A(y, t)$  du problème dans la cellule de référence est localement lipschitzienne par rapport à  $t$ , alors la matrice homogénéisée résultante  $A^0(t)$  (voir (12)) garde cette propriété. Les résultats obtenus dans cette thèse sont présentés ci-dessous en détails.

Le chapitre 1 est dédié à l'introduction de la thèse en anglais.

## Partie I

Dans cette partie, nous étudions l'existence et l'unicité de la solution renormalisée de (1). On définit d'abord le domaine à deux composantes  $\Omega$ , qui est un ensemble ouvert borné connexe de  $\mathbb{R}^N$  de frontière  $\partial\Omega$ . Nous décomposons le domaine comme la réunion disjointe  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , où  $\Omega_2$  est un ensemble ouvert tel que  $\overline{\Omega_2} \subset \Omega$  de frontière lipschitzienne  $\Gamma$ , et  $\Omega_1 = \Omega \setminus \overline{\Omega_2}$  (voir figure 1).

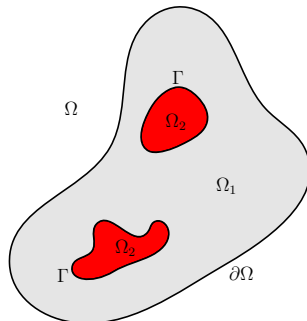


Figure 1: Le domaine à deux composantes  $\Omega$



Pour définir une solution renormalisée du (1), nous allons introduire d'abord les espaces fonctionnels nécessaires.

On définit l'espace  $V$  par

$$V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ et } v_2 \in H^1(\Omega_2)\},$$

muni de la norme

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2,$$

où  $V_1$  est l'espace défini par

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ sur } \partial\Omega\} \quad \text{avec} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

On introduit aussi, pour  $k > 0$ , la fonction de troncature  $T_k$  définie par (voir figure 2)

$$T_k(t) = \max(-k, \min(k, t)).$$

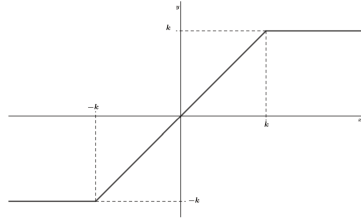


Figure 2: La fonction de troncature  $T_k$

## Chapitre 2 : Existence d'une solution renormalisée

Dans ce chapitre, nous introduisons une notion de solution renormalisée de (1) pour laquelle nous démontrons l'existence.

La définition s'inspire des travaux [36] avec l'ajout du terme sur la frontière et de la condition de saut. Plus précisément :

**Définition 1.** Soit  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  une fonction mesurable. Alors,  $u$  est une solution renormalisée de (1) si

$$T_k(u) \in V, \quad \forall k > 0; \tag{2a}$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \tag{2b}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u \, dx = 0; \tag{3a}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) d\sigma = 0; \quad (3b)$$

et pour tout  $S_1, S_2 \in C^1(\mathbb{R})$  (ou  $S_1, S_2 \in W^{1,\infty}(\mathbb{R})$ ) à support compact,  $u$  satisfait

$$\begin{aligned} & \int_{\Omega_1} S_1(u_1)B(x, u_1)\nabla u_1 \cdot \nabla v_1 dx + \int_{\Omega_1} S_1'(u_1)B(x, u_1)\nabla u_1 \cdot \nabla u_1 v_1 dx \\ & + \int_{\Omega_2} S_2(u_2)B(x, u_2)\nabla u_2 \cdot \nabla v_2 dx + \int_{\Omega_2} S_2'(u_2)B(x, u_2)\nabla u_2 \cdot \nabla u_2 v_2 dx \\ & + \int_{\Gamma} h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2)) d\sigma \\ & = \int_{\Omega_1} f v_1 S_1(u_1) dx + \int_{\Omega_2} f v_2 S_2(u_2) dx, \end{aligned} \quad (4)$$

pour tout  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ .

Notons que, dans le cadre des solutions renormalisées, une solution  $u$  peut ne pas avoir assez de régularité pour avoir un gradient et une trace dans le sens classique des espaces de Sobolev. Nous devons donc d'abord donner une définition appropriée du gradient et de la trace d'une solution renormalisée. Dans ce but, nous démontrons la proposition suivante (qui est une généralisation de [10, Lemma 2.1] et [59, Proposition 2.3]) :

**Proposition 2.** *Soit  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  une fonction mesurable telle que  $T_k(u) \in V$  pour tout  $k > 0$ .*

1. *Pour  $i = 1, 2$ , il existe une fonction mesurable unique  $G_i : \Omega_i \rightarrow \mathbb{R}^N$  telle que pour tout  $k > 0$ ,*

$$\nabla T_k(u_i) = G_i \chi_{\{|u_i| < k\}} \quad \text{p.p. dans } \Omega_i,$$

où  $\chi_{\{|u_i| < k\}}$  dénote la fonction caractéristique de l'ensemble

$$\{x \in \Omega_i : |u_i(x)| < k\}.$$

On définit  $G_i$  comme le gradient de  $u_i$  et on écrit  $G_i = \nabla u_i$ .

2. *Si*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty,$$

alors il existe une fonction mesurable unique

$$w_i : \Gamma \rightarrow \mathbb{R}, \quad \text{for } i = 1, 2,$$

telle que pour tout  $k > 0$ ,

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad p.p. \text{ sur } \Gamma,$$

où  $\gamma_i : H^1(\Omega_i) \longrightarrow L^2(\Gamma)$  est l'opérateur de trace. On définit la fonction  $w_i$  comme la trace de  $u_i$  sur  $\Gamma$  et on écrit

$$\gamma_i(u_i) = w_i, \quad i = 1, 2.$$

L'originalité de cette définition réside dans la régularité (2b), la décroissance d'une énergie sur le bord (3b) ainsi que la présence du terme sur le bord

$$\int_{\Gamma} h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2)) d\sigma.$$

La régularité (2a), la décroissance de l'énergie (3a) sont classiques et permettent via notamment 1 de la proposition 2 de donner un sens à tous les termes de (4) excepté le terme sur le bord.

En effet, soit  $S_i \in C^1(\mathbb{R})$ ,  $i = 1, 2$  à support compact. Pour tout  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ , si  $\text{supp } S_i \subset [-k, k]$  ( $i = 1, 2$ ), alors pour  $i = 1, 2$ , on a

$$S_i(u_i)B(x, u_i)\nabla u_i \cdot \nabla v_i = S_i(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla v_i \in L^1(\Omega_i),$$

$$S'_i(u_i)B(x, u_i)\nabla u_i \cdot \nabla u_i v_i = S'_i(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla T_k(u_i) v_i \in L^1(\Omega_i),$$

et

$$f v_i S_i(u_i) \in L^1(\Omega_i).$$

Concernant le terme sur le bord  $\Gamma$ , il est important de remarquer que (2a) et (3a) ne suffisent pas à donner un sens à  $h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2))$ . En général, il n'y a aucune raison d'avoir

$$\begin{aligned} h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2)) \\ = h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2))\chi_{\{|u_1| \leq n\}}\chi_{\{|u_2| \leq n\}} \end{aligned}$$

pour  $n$  grand.

Pour traiter l'intégrale sur  $\Gamma$ , nous allons utiliser (2b). Pour tout  $n \in \mathbb{N}$ , on définit  $\theta_n : \mathbb{R} \longrightarrow \mathbb{R}$  (voir figure 3) par

$$\theta_n(s) = \begin{cases} 0, & \text{si } s \leq -2n, \\ \frac{s}{n} + 2, & \text{si } -2n \leq s \leq -n, \\ 1, & \text{si } -n \leq s \leq n, \\ -\frac{s}{n} + 2, & \text{si } n \leq s \leq 2n, \\ 0, & \text{si } s \geq 2n. \end{cases}$$

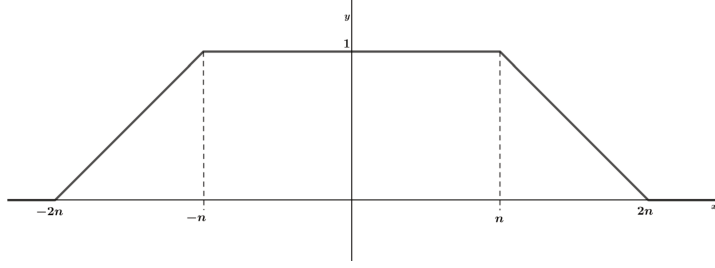


Figure 3: La fonction  $\theta_n$

Comme  $S_1$  a un support compact, on a pour  $n$  assez grand,

$$h(u_1 - u_2)v_1S_1(u_1) = hv_1(u_1 - u_2)(S_1(u_1) - S_1(u_2))\theta_n(u_1) \\ + hv_1(u_1 - u_2)S_1(u_2)\theta_n(u_1).$$

Puisque  $\theta_n$  est aussi à support compact, alors  $hv_1(u_1 - u_2)S_1(u_2)\theta_n(u_1)$  est bornée, et donc appartient à  $L^1(\Gamma)$ . De plus, puisque

$$S_1(u_1) - S_1(u_2) = S_1(T_{2n}(u_1)) - S_1(T_{2n}(u_2))$$

et  $S_1$  est lipschitzienne, on a

$$|hv_1(u_1 - u_2)(S_1(u_1) - S_1(u_2))\theta_n(u_1)| \\ \leq \|hv_1\|_{L^\infty(\Gamma)} \|S_1'\|_{L^\infty(\mathbb{R})} |u_1 - u_2| |T_{2n}(u_1) - T_{2n}(u_2)|,$$

p.p. sur  $\Gamma$ . Donc, en appliquant (2b),  $h(u_1 - u_2)v_1S_1(u_1) \in L^1(\Gamma)$ . De même,  $h(u_1 - u_2)v_2S_2(u_2) \in L^1(\Gamma)$ . On en conclut que la formulation renormalisée (4) est bien définie.

Le résultat principal de ce chapitre est le théorème d'existence ci-dessous.

**Théorème 3.** *On suppose les hypothèses suivantes :*

(A1) *La donnée  $f$  appartient à  $L^1(\Omega)$ .*

(A2) *La fonction  $h$  vérifie*

$$h \in L^\infty(\Gamma) \quad \text{et} \quad 0 < h_0 < h(y) \text{ p.p. sur } \Gamma, \quad (5)$$

*avec  $h_0 \in \mathbb{R}^+$ .*

(A3) *La matrice  $B$  est une fonction Carathéodory, i.e.,*

*(a) la fonction  $r \mapsto B(x, r)$  est continue p.p.  $x \in \Omega$ ;*

(b) la fonction  $x \mapsto B(x, r)$  est mesurable p.p.  $r \in \mathbb{R}$ ,

et vérifie les hypothèses suivantes :

$$(A3.1) \quad B(x, r)\xi \cdot \xi \geq \alpha|\xi|^2, \text{ avec } \alpha > 0,$$

pour p.p.  $x \in \Omega$ ,  $\forall r \in \mathbb{R}$ ,  $\forall \xi \in \mathbb{R}^N$ ;

$$(A3.2) \quad \text{pour tout } k > 0, B(x, r) \in L^\infty(\Omega \times (-k, k))^{N \times N}.$$

Alors, il existe une solution renormalisée de (1) dans le sens de définition 1.

La preuve du théorème 3 se fait par passage à la limite dans un problème approché. La première étape, si  $\{f^\varepsilon\} \subset L^2(\Omega)$  telle que

$$f^\varepsilon \longrightarrow f \text{ fortement dans } L^1(\Omega),$$

et  $B_\varepsilon(x, t) = B(x, T_{1/\varepsilon}(t))$ , nous considérons une solution  $u^\varepsilon \in V$  vérifiant

$$\begin{cases} -\operatorname{div}(B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{dans } \Omega_1, \\ -\operatorname{div}(B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{dans } \Omega_2, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{sur } \Gamma, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{sur } \Gamma, \\ u_1^\varepsilon = 0 & \text{sur } \partial\Omega. \end{cases}$$

La deuxième étape consiste à établir des estimations a priori, puis de construire (à l'aide de résultats de compacité de Rellich-Kondrachov)  $u$  telle que, à une sous-suite près,

$$\begin{cases} u_i^\varepsilon \longrightarrow u_i & \text{p.p. dans } \Omega, \\ T_k(u_i^\varepsilon) \rightharpoonup T_k(u_i) & \text{faiblement dans } H^1(\Omega_i), \\ \gamma_i(u_i^\varepsilon) \longrightarrow \gamma_i(u_i) & \text{p.p. dans } \Gamma, \\ \gamma_i(T_k(u_i^\varepsilon)) \longrightarrow \gamma_i(T_k(u_i)) & \text{fortement dans } L^2(\Gamma), \text{ p.p. sur } \Gamma. \end{cases}$$

La nouveauté ici vient une fois encore des termes sur le bord et d'exploiter efficacement, par exemple, dans la troisième étape, l'estimation a priori

$$\forall k > 0, \quad (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \text{ borné dans } L^1(\Gamma),$$

et la limite

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left( \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) dx + \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) = 0.$$

Dans la quatrième étape, on passe à la limite avec un choix judicieux de fonction test et on démontre que  $u$  est une solution renormalisée de (1).

**Remarque 4.** *Pour définir le terme sur le bord, il est possible de remplacer (2b) et (3b) par une condition de régularité, donnée par*

$$u_1 - u_2 \in W^{1-\frac{1}{q},q}(\Gamma), \quad \text{avec } q > 1. \quad (6)$$

*Ce type de régularité découle des estimations du type Boccardo-Gallouët, mais dépend fortement des constantes de Sobolev. Comme ces constantes de Sobolev peuvent exploser dans l'analyse asymptotique, il n'est pas possible d'utiliser (6). Ajoutons que (6) sera mise en défaut pour des problèmes non linéaires plus généraux. C'est pour ces raisons que nous avons choisi (2b) et (3b) dans la définition de solution renormalisée.*

### Chapitre 3 : L'unicité de la solution renormalisée

Dans ce chapitre, nous démontrons l'unicité d'une solution renormalisée de (1). Pour cela, nous rajoutons aux hypothèses (A1)-(A3) du théorème 3 la condition de lipschitzianité locale suivante sur la matrice  $B$  :

(A4)  $B(x, r)$  est localement lipschitzienne par rapport à  $r$ , c'est-à-dire, pour tout compact  $K$  de  $\mathbb{R}$  il existe  $M_K > 0$  tel que

$$|B(x, r) - B(x, s)| \leq M_K |r - s|, \quad \forall r, s \in K, \quad \text{p.p. } x \in \Omega.$$

Pour démontrer le résultat d'unicité, nous appliquons la méthode développée dans [16, 38]. Cette méthode utilise l'existence d'une fonction auxiliaire  $\varphi \in C^1(\mathbb{R})$  qui vérifie des propriétés intéressantes. Plus précisément, nous utilisons la proposition suivante de [38] :

**Proposition 5** ([38]). *On suppose (A4). Alors, il existe une fonction  $\varphi \in C^1(\mathbb{R})$  telle que*

$$\varphi(0) = 0 \quad \text{et} \quad \varphi' \geq 1.$$

*De plus, il existe des constantes  $\delta > 1/2$ ,  $0 < k_0 < 1$ , et  $L > 0$  telles que*

$$\frac{\varphi'}{(1 + |\varphi|)^{2\delta}} \in L^\infty(\mathbb{R}).$$

*En outre, pour tout  $r, s \in \mathbb{R}$  tels que*

$$|\varphi(r) - \varphi(s)| \leq k, \quad \text{pour } 0 < k < k_0,$$

*on a*

$$\left| \frac{B(x, r)}{\varphi'(r)} - \frac{B(x, s)}{\varphi'(s)} \right| \leq \frac{1}{\varphi'(s)} \frac{Lk}{(1 + |\varphi(r)| + |\varphi(s)|)^\delta}$$

et

$$\frac{1}{L} \leq \frac{\varphi'(s)}{\varphi'(r)} \leq L.$$

Une des difficultés vient de l'intégrale sur le bord  $\Gamma$  qui est liée au saut de la solution. La proposition suivante est un outil important pour la démonstration du théorème d'unicité, liée aux estimations de Boccardo-Gallouët :

**Proposition 6.** *Pour  $i = 1, 2$ , soit  $\gamma_i$  l'opérateur de trace défini sur  $H^1(\Omega_i)$ . Sous les hypothèses de théorème 3, si  $u$  est une solution renormalisée de (1), alors  $\gamma_i(u_i) \in L^1(\Gamma)$ ,  $i = 1, 2$ .*

La méthode développée dans [16, 38] consiste formellement à utiliser  $T_k(\varphi(u) - \varphi(v))$  comme fonction test. La justification est très technique et se fait par passage à la limite à l'aide de la fonction  $\theta_n(u)T_k(\varphi(u) - \varphi(v))$  autorisée dans la définition de solution renormalisée.

Cependant, la non linéarité du terme  $T_k(\varphi(u) - \varphi(v))$  n'est pas compatible avec le terme du bord dans le sens où le terme obtenu

$$\int_{\Gamma} h(x)((u_1 - v_1) - (u_2 - v_2))(T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))) d\sigma$$

n'est pas nécessairement de signe positif.

Pour surmonter cette difficulté, nous démontrons une propriété de signe sur  $\Gamma$  :

**Lemme 7.** *On suppose (A1)-(A4). Si  $u$  et  $v$  sont deux solutions renormalisées de (1), alors  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  p.p. sur  $\Gamma$ .*

Ce lemme est essentiel pour la démonstration du théorème d'unicité suivant, qui est le résultat principal de ce chapitre :

**Théorème 8.** *Sous les hypothèses (A1)-(A4), la solution renormalisée de (1) est unique.*

Une des étapes principales de la démonstration du théorème 8 est de montrer que

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{\Omega_i} \left( \frac{1}{\varphi'(u_i)} + \frac{1}{\varphi'(v_i)} \right) |\nabla T_k(\varphi(u_i) - \varphi(v_i))|^2 dx = 0, \quad i = 1, 2, \quad (7)$$

où  $u = (u_1, u_2)$  et  $v = (v_1, v_2)$  sont deux solutions renormalisées de (1) et  $\varphi$  est la fonction introduite dans proposition 5.

Pour le démontrer, nous obtenons d'abord, après de long calculs, l'inégalité suivante :

$$\begin{aligned} \limsup_{k \rightarrow 0} & \left( \frac{1}{k^2} \int_{U_1^k} \left( \frac{1}{\varphi'(u_1)} + \frac{1}{\varphi'(v_1)} \right) |\nabla \varphi(u_1) - \nabla \varphi(v_1)|^2 dx \right. \\ & \left. + \frac{1}{k^2} \int_{U_2^k} \left( \frac{1}{\varphi'(u_2)} + \frac{1}{\varphi'(v_2)} \right) |\nabla \varphi(u_2) - \nabla \varphi(v_2)|^2 dx + \frac{1}{k^2} C^k \right) \leq 0, \end{aligned}$$

où

$$U_i^k = \{x \in \Omega_i : 0 < |\varphi(u_i) - \varphi(v_i)| < k\}, \quad i = 1, 2,$$

et

$$C^k = \int_{\Gamma} h(x) [(u_1 - v_1) - (u_2 - v_2)] [T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))] d\sigma.$$

Alors, pour montrer (7), il suffit de montrer que

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} C^k \geq 0. \quad (8)$$

Comme écrit précédemment à cause de la présence de la fonction  $\varphi$ , nous ne connaissons pas en général le signe de  $C^k$ . Le contrôle de  $\frac{1}{k^2} C^k$  quand  $k \rightarrow 0$  est l'une des difficultés principale. Pour ce faire, on utilise la propriété de signe sur  $\Gamma$  (lemme 7) et on divise l'ensemble  $\{x \in \Gamma ; u_1(x) - v_1(x) > 0\}$  (défini à un ensemble de mesure nulle près) en 4 sous-ensembles disjoints,

$$\{u_1 - v_1 > 0\} = P_1 \cup P_2 \cup P_3 \cup P_4,$$

où

$$\begin{aligned} P_1 & := \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}, \\ P_2 & := \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\ P_3 & = \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\ P_4 & := \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}. \end{aligned}$$

Grâce au lemme 7 et à ce découpage, on démontre (8).

Une fois que (7) est démontrée, on s'inspire de [16, 38] pour obtenir dans un premier temps que  $u_1 = v_1$  dans  $\Omega_1$  (car  $u_1$  et  $v_1$  ont une trace nulle sur  $\partial\Omega$ , ce qui permet d'utiliser l'inégalité de Poincaré). Ainsi  $u_1 = v_1$  sur  $\Gamma$ , ce qui, combiné au lemme 7 et (7) donnent  $u_2 = v_2$  dans  $\Omega_2$ .



## Partie II

Dans cette partie, nous étudions l'homogénéisation du même problème dans un domaine à deux composantes avec une deuxième composante périodique.

Pour définir le domaine, on introduit d'abord la cellule de référence  $Y$  (qui est généralement définie comme un pavé de  $\mathbb{R}^N$ ,  $N \geq 2$ ). La cellule  $Y$  a deux composantes,  $Y_1$  et  $Y_2$ , avec  $\Gamma$  comme l'interface entre les deux. L'ensemble ouvert  $Y_2$ , de frontière lipschitzienne  $\Gamma$ , est tel que  $\overline{Y_2} \subset Y$  et l'on a  $Y_1 = Y \setminus \overline{Y_2}$ .

Le domaine à deux composantes correspondant, noté  $\Omega$ , est un ouvert connexe et borné de  $\mathbb{R}^N$  avec une frontière lipschitzienne  $\partial\Omega$ . Soit  $\varepsilon$  un paramètre qui prends ses valeurs dans une suite positive qui tend vers zéro. On peut écrire  $\Omega$  comme la réunion disjointe

$$\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon,$$

où  $\Omega_1^\varepsilon$  et  $\Omega_2^\varepsilon$  sont les deux composantes de  $\Omega$  et  $\Gamma^\varepsilon$  est l'interface entre les composantes. Ici,  $\Omega_2^\varepsilon$  est une réunion disjointe des translatées  $\varepsilon$ -périodiques de l'ensemble  $\varepsilon Y_2$  telles que  $\overline{\Omega_2^\varepsilon} \subset \Omega$  et  $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ , et  $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$  (voir figure 4).

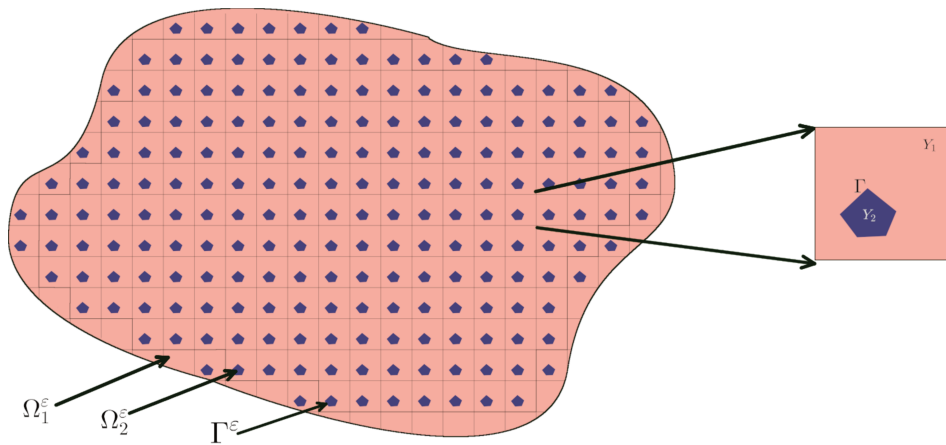


Figure 4: Le domaine à deux composantes  $\Omega$  et la cellule de référence  $Y$ .

Nous étudions le comportement asymptotique du problème elliptique

quasi-linéaire suivant :

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_1^\varepsilon\right)\nabla u_1^\varepsilon\right) = f & \text{dans } \Omega_1^\varepsilon, \\ -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_2^\varepsilon\right)\nabla u_2^\varepsilon\right) = f & \text{dans } \Omega_2^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_1^\varepsilon\right)\nabla u_1^\varepsilon\right)\nu_1^\varepsilon = \left(A\left(\frac{x}{\varepsilon}, u_2^\varepsilon\right)\nabla u_2^\varepsilon\right)\nu_1^\varepsilon & \text{sur } \Gamma^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_1^\varepsilon\right)\nabla u_1^\varepsilon\right)\nu_1^\varepsilon = -\varepsilon^{-1}h\left(\frac{x}{\varepsilon}\right)(u_1^\varepsilon - u_2^\varepsilon) & \text{sur } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{sur } \partial\Omega, \end{cases} \quad (9)$$

sous les mêmes hypothèses considérées dans la première partie, pour  $\varepsilon$  fixé. On suppose de plus ici la condition de périodicité usuelle pour  $A$  et  $h$ .

Concernant l'étude de l'homogénéisation de (9) avec  $f \in L^2(\Omega)$  (voir par exemple [46, 45]), l'hypothèse de proportionnalité du saut de la solution et du flux sur  $\Gamma^\varepsilon$  dépend de  $\varepsilon^\gamma$  (au lieu de  $\varepsilon^{-1}$ ), où  $\gamma \leq 1$  est un paramètre. L'homogénéisation est étudiée alors dans les trois cas :  $\gamma \in (-1, 1]$ ,  $\gamma = -1$ , et  $\gamma \in (-\infty, -1)$ . La différence principale entre ces cas réside dans le problème périodique posé dans la cellule de référence, qui permet de décrire la matrice homogénéisée. Nous nous bornons ici au cas  $\gamma = -1$ , dont la particularité est la présence dans le problème elliptique dans la cellule de référence, du saut de la solution sur l'interface de référence  $\Gamma$ .

## Chapitre 4 : Propriétés du problème dans la cellule de référence

Dans ce chapitre, nous étudions le problème elliptique suivant, posé dans la cellule de référence qui est lié au problème homogénéisé de (9) :

$$\begin{cases} -\operatorname{div}(A\nabla\chi_1^\lambda) = G_1^\lambda & \text{dans } Y_1, \\ -\operatorname{div}(A\nabla\chi_2^\lambda) = G_2^\lambda & \text{dans } Y_2, \\ A\nabla\chi_1^\lambda \cdot n_1 = -A\nabla\chi_2^\lambda \cdot n_2 & \text{sur } \Gamma, \\ A\nabla\chi_1^\lambda \cdot n_1 = -h(y)(\chi_1^\lambda - \chi_2^\lambda) & \text{sur } \Gamma, \\ \chi_1^\lambda \text{ } Y\text{-périodique,} \\ \mathcal{M}_\Gamma(\chi_1^\lambda) = 0, \end{cases} \quad (10)$$

où  $\lambda \in \mathbb{R}^N$  et  $G_i^\lambda$  est défini par

$$\langle G_i^\lambda, v \rangle = \int_{Y_i} A\lambda\nabla v \, dy, \quad \forall v \in H^1(Y_i), \quad i = 1, 2, \quad (11)$$

qui appartient à  $(H^1(Y_i))'$ .

Nous nous sommes intéressés aux propriétés de la solution de (10), qui n'étaient pas étudiées dans la littérature. Ceci est motivé par le fait que la matrice homogénéisée  $A^0$  qui correspond à (9) est définie en fonction de la solution  $\chi^\lambda$  de (10), et pas seulement de  $A$ . Plus précisément, la matrice  $A^0$  est définie par

$$A^0(t) = A_1^0(t) + A_2^0(t), \quad (12)$$

où

$$A_i^0(t)\lambda = \frac{1}{|Y_i|} \int_{Y_i} A(y, t) \nabla_y w_i^\lambda(y, t) dy, \quad i = 1, 2, \quad \forall \lambda \in \mathbb{R}^N,$$

avec

$$w_i^\lambda(y, t) = \lambda \cdot y - \chi_i^\lambda(y, t),$$

et  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$  solution de (10).

Les propriétés démontrées dans ce chapitre sont, à notre avis, intéressantes en elles-mêmes.

Nous démontrons, en particulier, que si la matrice  $A$  est lipschitzienne par rapport à la deuxième variable, alors  $A^0$  garde cette propriété. Grâce à cette propriété, nous pouvons obtenir un résultat d'unicité pour le problème homogénéisé posé dans  $\Omega$  (voir théorème 13) correspondant à (9).

Rappelons que dans [23] les auteurs ont démontré un résultat similaire pour l'homogénéisation des problèmes elliptiques dans un domaine perforé, en utilisant une estimation de type Meyers bien connue.

Dans notre cas, où la solution de (10) présente un saut sur l'interface  $\Gamma$ , nous avons dû d'abord démontrer le résultat suivant, qui établit une estimation du type Meyers, adaptée à notre problème périodique.

**Théorème 9.** *Soit  $\lambda \in \mathbb{R}^N$  et soit  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda) \in \mathcal{H}$  la solution de (10). Alors, pour tout  $\lambda \in \mathbb{R}^N$ , il existe  $p_i > 2$ ,  $i = 1, 2$ , tel que*

$$\chi_i^\lambda \in W^{1, p_i}(Y_i).$$

*De plus, pour tout  $q_i$  tel que  $2 \leq q_i \leq p_i$ ,  $i = 1, 2$ , il existe une constante positive  $c_i$ , qui dépend de  $\alpha$ ,  $\beta$ ,  $q_i$  et  $Y_i$ , telle que*

$$\|\nabla \chi_i^\lambda\|_{L^{q_i}(Y_i)} \leq c_i |\lambda|.$$

Nous montrons ce théorème en utilisant les estimations prouvées par T. Gallouët et A. Monier dans [55] pour des équations elliptiques avec des conditions de Neumann non homogènes.

Ceci nous a permis de démontrer le résultat principal de ce chapitre, énoncé ci-dessous :

**Théorème 10.** Soit  $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) \in \mathbb{R}^{N \times N}$  une matrice réelle qui vérifie :

(P1)  $A(\cdot, t)$  appartient à  $M(\alpha, \beta, Y)$  pour tout  $t \in \mathbb{R}$ ;

(P2)  $A(\cdot, t) = \{a_{ij}\}_{i,j=1,\dots,N}$  est  $Y$ -périodique pour tout  $t$ ;

(P3)  $A(y, t)$  est localement lipschitzienne par rapport à la deuxième variable, i.e., pour tout  $r > 0$ , il existe une constante positive  $M_r$  telle que

$$|A(y, s) - A(y, t)| \leq M_r |s - t| \quad \forall s, t \in (-r, r), \quad p.p. \ y \in Y.$$

Alors, la matrice homogénéisée  $A^0$  (voir (12)) est aussi localement lipschitzienne, i.e., pour tout  $r > 0$ , il existe une constante positive  $C_r$  telle que

$$|A^0(s) - A^0(t)| \leq C_r |s - t| \quad \forall s, t \in (-r, r).$$

Ce théorème est ce dont nous avons besoin pour démontrer un résultat d'unicité pour le problème homogénéisé, qui est présenté dans la section suivante.

## Chapitre 5 : Résultats d'homogénéisation

Dans ce chapitre, nous étudions le comportement asymptotique du problème (9). Nous utilisons une adaptation de la méthode de l'éclatement périodique aux domaines à deux composantes introduite dans [46]. Cette méthode utilise l'opérateur d'éclatement périodique  $\widetilde{\mathcal{V}}_i^\varepsilon$ ,  $i = 1, 2$ , agissant pour toute fonction mesurable  $u_i$ , définie dans  $\Omega_i^\varepsilon$ . Son intérêt principal est qu'il transforme les intégrales sur les ensembles variables  $\Omega_i^\varepsilon$  en des intégrales sur les ensembles  $\Omega \times Y_i$  qui sont indépendants de  $\varepsilon$ .

À notre connaissance, la première étude qui combine le cadre des solutions renormalisées et la méthode de l'éclatement périodique a été faite dans [43], pour des problèmes elliptiques dans des domaines périodiquement perforés avec des conditions de Robin sur le bord des trous. Nous avons adopté ici une approche similaire.

L'homogénéisation dans le cadre des solutions renormalisées est encore plus difficile que celle dans le cas de données  $L^2$ . Ceci est dû au fait que la fonction  $u_i^\varepsilon$ , qui est la restriction de la solution  $u^\varepsilon$  de (9) à  $\Omega_i^\varepsilon$ , n'appartient pas nécessairement à  $H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$ .

On rappelle que quand la donnée  $f$  dans (9) appartient à  $L^2(\Omega)$ , on peut obtenir des estimations a priori sur  $u_i^\varepsilon$  dans  $H^1(\Omega_i^\varepsilon)$  (voir [7]). Donc,

en utilisant les résultats montrés dans [46], on en déduit les convergences suivantes :

$$\begin{cases} \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_1 & \text{fortement dans } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2 \\ \mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_i & \text{faible dans } L^2(\Omega \times Y_i), \quad i = 1, 2, \end{cases} \quad (13)$$

avec  $u_1 \in H_0^1(\Omega)$  et  $\widehat{u}_i \in L^2(\Omega, H^1(Y_i))$ ,  $i = 1, 2$ . Ainsi, grâce à ces convergences, on obtient le problème homogénéisé dans  $\Omega$ , qui est satisfait par la fonction  $u_1$ .

Par contre, dans notre cas,  $u_i^\varepsilon$  n'appartient pas à  $H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$  et par conséquent nous ne pouvons pas procéder de cette façon. Nous considérons alors les troncatures de  $u_i^\varepsilon$  (i.e.  $T_k(u_i^\varepsilon)$ ), puisque dans le cadre des solutions renormalisées,  $T_k(u_i^\varepsilon) \in H^1(\Omega_i^\varepsilon)$  (voir définition 1),  $i = 1, 2$ , pour tout  $k > 0$ . Donc, au lieu à (13), en mélangeant les techniques des solutions renormalisées et celles de l'éclatement périodique (en particulier les résultats de compacité), nous démontrons qu'il existe  $u_1$  et une suite  $\{\widehat{u}_i^n\}_{n \in \mathbb{N}} \subset L^2(\Omega, H^1(Y_i))$ ,  $i = 1, 2$ , vérifiant pour tout  $n \in \mathbb{N}$ ,  $i = 1, 2$ ,

$$\begin{cases} T_n(u_1) \in H_0^1(\Omega), \\ \mathcal{T}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow T_n(u_1) & \text{fortement dans } L^2(\Omega, H^1(Y_i)), \\ \mathcal{T}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \nabla T_n(u_1) + \nabla_y \widehat{u}_i^n & \text{faiblement dans } L^2(\Omega \times Y_i). \end{cases} \quad (14)$$

Même s'il y a des similitudes entre l'homogénéisation de (9) et celle étudiée dans [43], il y a des difficultés supplémentaires dans notre cas, à cause de la présence du saut de la solution à l'interface.

La première différence peut être vue dans la définition d'une solution renormalisée de (9) (voir définition 1), qui contient des hypothèses supplémentaires, comme discutées dans le chapitre 2. De plus, la démonstration du théorème suivant, qui est la construction de la partie oscillante  $\widehat{u}_i$ ,  $i = 1, 2$ , à partir de la suite de fonctions  $\{\widehat{u}_i^n\}_{n \in \mathbb{N}}$  et d'un résultat d'identification, est encore plus délicate que celle du théorème analogue dans [43] :

**Théorème 11.** *Soit  $\widehat{u}_1^n \in L^2(\Omega, H_{per}^1(Y_1))$  et  $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$ ,  $n \in \mathbb{N}$ , les fonctions introduites dans (14), avec  $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$ . Alors, il existe une unique fonction*

$$\widehat{u}_i : \Omega \times Y_i \longrightarrow \mathbb{R}, \quad i = 1, 2,$$

telle que pour tout  $\mathcal{R} \in C^1(\mathbb{R})$  à support compact vérifiant  $\text{supp } \mathcal{R} \subset [-m, m]$ , pour un  $m \in \mathbb{N}$ , on a

$$\mathcal{R}(u_1) \widehat{u}_i^n = \mathcal{R}(u_1) \widehat{u}_i \quad \text{p.p. dans } \Omega \times Y_i,$$

pour tout  $n \geq m$ , où  $u_1$  est la fonction donnée par (14).

De plus,

$$\widehat{u}_i(x, \cdot) \in H^1(Y_i), \quad i = 1, 2, \text{ avec } \mathcal{M}_\Gamma(\widehat{u}_1) = 0, \quad p.p. \ x \in \Omega.$$

La partie la plus originale de la démonstration est liée au fait que la moyenne de  $\widehat{u}_2^n$ , pour  $n \in \mathbb{N}$ , n'est pas nécessairement zéro.

Grâce à ce théorème et aux convergences dans (14), on peut alors démontrer le théorème suivant, qui décrit le problème homogénéisé éclaté satisfait par  $(u_1, \widehat{u}_1, \widehat{u}_2)$  :

**Théorème 12** (Le problème homogénéisé éclaté). *Soit  $u_1, \widehat{u}_1$  et  $\widehat{u}_2$  les fonctions définies par (14). Soient  $\psi_1, \psi_2$  des fonctions appartenant à  $C^1(\mathbb{R})$  (ou  $\psi_1, \psi_2 \in W^{1,\infty}(\mathbb{R})$ ) à support compact. Alors,  $(u_1, \widehat{u}_1, \widehat{u}_2)$  satisfait*

$$\left\{ \begin{array}{l} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) (\nabla(\psi_1(u_1)\varphi) + \psi_2(u_1)\nabla_y \Phi_i) \, dx \, dy \\ + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)(\Phi_1 - \Phi_2) \, dx \, d\sigma_y = \int_{\Omega} f(x)\psi_1(u_1)\varphi(x) \, dx \\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \Phi_i \in L^2(\Omega, H_{per}^1(Y_i)), \quad i = 1, 2. \end{array} \right.$$

De plus, pour  $k > 0$ , on a les limites suivantes :

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\} \times Y_i} A(y, u_1) (\nabla T_k(u_1) + \nabla_y \widehat{u}_i) (\nabla T_k(u_1) + \nabla_y \widehat{u}_i) \, dx \, dy = 0,$$

pour  $i = 1, 2$ , et

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 \, dx \, d\sigma_y = 0.$$

Enfin, nous obtenons le problème homogénéisé dans  $\Omega$ , ce qui complète le chapitre :

**Théorème 13** (Le problème homogénéisé dans  $\Omega$ ). *Soit  $u_1$  une valeur d'adhérence de la suite  $\{\mathcal{G}_i^\varepsilon(u_i^\varepsilon)\}$ ,  $i = 1, 2$ . Alors  $u_1$  est une solution renormalisée du problème*

$$\left\{ \begin{array}{ll} -\operatorname{div}(A^0(u_1)\nabla u_1) = f & \text{dans } \Omega \\ u_1 = 0 & \text{sur } \partial\Omega, \end{array} \right. \quad (15)$$

i.e.,

$$T_k(u_1) \in H_0^1(\Omega), \quad \text{pour tout } k > 0, \quad (16)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\}} A^0(u_1)\nabla u_1 \nabla u_1 \, dx = 0, \quad (17)$$

et pour tout  $\psi \in C^1(\mathbb{R})$  (ou  $\psi \in W^{1,\infty}(\mathbb{R})$ ) à support compact,

$$\begin{aligned} \int_{\Omega} \psi(u_1) A^0(u_1) \nabla u_1 \nabla \varphi \, dx \\ + \int_{\Omega} \psi'(u_1) A^0(u_1) \nabla u_1 \nabla u_1 \varphi \, dx = \int_{\Omega} f \psi(u_1) \varphi \, dx, \end{aligned} \quad (18)$$

pour tout  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , où  $A^0$  est la matrice homogénéisée définie au dessus (voir (12)).

Si de plus (A4) est vérifiée, alors  $u_1$  est l'unique solution renormalisée de (9), les fonctions  $\widehat{u}_1, \widehat{u}_2$  sont définies de manière unique et toutes les suites dans (14) convergent (et pas seulement les sous-suites).

La démonstration de la dernière affirmation de ce théorème utilise le théorème 10 du chapitre précédent. Soulignons ici que la preuve de la condition de décroissance de l'énergie (condition (17)) n'est pas classique.

En conclusion, soulignons que, comme on peut voir tout au long de cette thèse, gérer l'intégrale sur le bord qui provient du saut de la solution sur l'interface est délicat. Cette difficulté ne se limite pas aux résultats d'homogénéisation. Elle peut aussi être observée dans l'étude de l'existence et de l'unicité de la solution renormalisée de (1), ainsi que dans l'étude des propriétés de la solution du problème dans la cellule de référence (10).

# Chapter 1

## Introduction

The aim of this thesis is to study a class of elliptic partial differential equations (PDE) with weak data. More precisely, we study a quasilinear elliptic problem posed in a domain with an imperfect interface, where the data is an  $L^1$  function and the matrix field of the quasilinear term has a restricted growth assumption (it is only bounded with respect to the solution on the compact sets of  $\mathbb{R}$ ) and we consider the corresponding asymptotic analysis in a periodic two-component domain.

Let us point out that since we have these weak assumptions, a weak solution may not exist (even in the presence of just one of them). Hence, the convenient framework of renormalized solution needs to be introduced for our problem.

This notion is related to partial differential equations (PDE) with data less summable than  $L^2$ , e.g. data in  $L^1$ , or measure data (more on renormalized solutions is presented in the next section).

The existence and uniqueness results in the framework of renormalized solution for a domain with imperfect interface has previously not been studied in the literature. The first part of this study is dedicated to obtaining such results.

The second part of this thesis is concerned with the corresponding homogenization for a two-component domain with a (disconnected) periodic second component. We use the periodic unfolding method, originally introduced in [30] (see also the recent book [32]), and adapted to a domain with imperfect interface in [46] (more on the periodic unfolding method in Section 1.2.4).

In the study of the homogenization, the framework usually considered is the variational setting, where the data are  $L^2$  (or  $H^{-1}$ ) functions. The framework considered in this thesis presents non-trivial additional difficulties, which need to be treated specifically. However, combining homogenization with the notion of renormalized solution is not completely new, one may refer



to the pioneer paper [70], see also [8, 22] for some cases. More recently, in [56], the authors studied the homogenization of a linear elliptic problem with Neumann boundary conditions, highly oscillating boundary and  $L^1$  data.

In addition, P. Donato, O. Guibé and A. Oropeza studied in [43] the homogenization of a quasilinear elliptic problem with nonlinear boundary conditions and  $L^1$  data. In their study, as far as we know, the notion of renormalized solution is combined for the first time with the period unfolding method. We follow a similar approach for this thesis.

In this chapter, we present in details the framework of renormalized solutions and the homogenization theory. In the next section, dedicated to the notion of renormalized solution, we present the existence and uniqueness results for a model case. In addition, we give a summary of the results obtained in the first part of this thesis.

In Section 1.2, devoted to the homogenization theory, we discuss several methods that were developed for periodic homogenization. We also present there a summary of the second part of this thesis.

## 1.1 Renormalized solutions

In this section, we discuss the framework of renormalized solutions.

When considering a weak data (e.g.,  $L^1$  data and measure data), we cannot, in general, show the existence of a weak solution. Recall that the problem

$$-\operatorname{div}(A(x, u)\nabla u) = f$$

with Dirichlet boundary conditions, if the matrix field  $A(x, r)$  is bounded and coercive, and  $f$  belongs to  $L^1(\Omega)$  (similarly, when  $f$  is a bounded Radon measure), from Boccardo-Gallouët [18], there exists a solution in the sense of distributions (this result is also true for a class of more general nonlinear operators with increasing  $p$ ). The authors show that  $u \in W_0^{1,q}(\Omega)$ ,  $\forall 1 < q < \frac{N}{N-1}$  and verifies

$$\int_{\Omega} A(x)\nabla u\nabla\varphi dx = \int_{\Omega} f\varphi dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

However, this solution in the sense of distributions can not have, in general, a finite energy, in the sense that  $u \notin H_0^1(\Omega)$ . Moreover, even in the linear case, that is,  $A(x, r) = A(x)$ , the solution in the sense of distributions is not unique in general, one can see the counter example by Serrin in [78] (see also [76]).

To overcome this inconvenience, a number of notions of solutions were developed: entropy solutions (see [10]), SOLA (solutions obtained as limit approximations, see [37]) and renormalized solutions.

The notion of renormalized solutions was originally introduced in [39] by R.J. Di Perna and P.L. Lions for first order equations. It was then further developed by F. Murat in [71], by P.L. Lions and F. Murat in [63] for elliptic equations with Dirichlet boundary conditions and  $L^1$  data, and by G. Dal Maso et al. in [36] for elliptic equations with general measure data. Most of the works concerning the development of renormalized solutions consider elliptic (or parabolic) problems with  $L^1$  data and Dirichlet boundary conditions, but rarely other boundary conditions (see for example [13] for Neumann conditions and [59] for perforated domains).

There are some physical motivations in considering a weaker data (e.g.  $L^1$  or measure data). As an example, in [66] the authors considered a reaction-diffusion system with  $L^1$  data, which is then applied to image processing. In addition, some engineering problems can require the source to be a mass concentration in a point, which is represented by a measure.

To have more understanding of the notion of renormalized solutions, we present in the sequel the existence and uniqueness results of a model case (see (1.1)).

### 1.1.1 A model case

Let  $\Omega$  be an open bounded set in  $\mathbb{R}^N$  with Lipschitz continuous boundary  $\partial\Omega$ , where  $N \geq 2$ .

We are interested to study the following quasilinear Dirichlet problem:

$$\begin{cases} -\operatorname{div}(A(x, u)\nabla u) + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\lambda \geq 0$  and the matrix  $A : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$  is a Carathéodory function, that is,

1. the map  $r \mapsto A(x, r)$  is continuous for a.e.  $x \in \Omega$ ; and
2. the map  $x \mapsto A(x, r)$  is measurable for a.e.  $r \in \mathbb{R}$ ,

which satisfies the following properties for some  $\alpha, \beta \in \mathbb{R}$  with  $0 < \alpha < \beta$ ,

$$\begin{cases} A(x, r)\xi \cdot \xi \geq \alpha|\xi|^2, & \forall \xi \in \mathbb{R}^N, \quad \forall r \in \mathbb{R}, \\ |A(x, r)| \leq \beta, & \forall r \in \mathbb{R}. \end{cases} \quad (1.2)$$

Before we consider the case when the function  $f$  belongs to  $L^1(\Omega)$ , let us first look at the variational case, where  $f \in L^2(\Omega)$ . Under this assumption, the variational formulation of problem (1.1) is

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \int_{\Omega} A(x, u) \nabla u \nabla v \, dx + \lambda \int_{\Omega} uv \, dx = \int_{\Omega} fv \, dx, \\ \text{for any } v \in H_0^1(\Omega). \end{cases} \quad (1.3)$$

Note that all three of the integrals in this formulation are well-defined. Moreover, using Lax-Milgram Theorem and Schauder's Fixed Point Theorem, we can show the existence of a solution to (1.3).

As for showing the uniqueness of said solution, we need to separate the cases for  $\lambda > 0$  and  $\lambda = 0$ . When  $\lambda > 0$ , an additional assumption that  $A(x, r)$  is locally Lipschitz continuous with respect to the second variable  $r$  must be made to show uniqueness. The case  $\lambda = 0$  is even more difficult and in order to show the uniqueness of solution of (1.3), the additional condition of global Lipschitz continuity with respect to  $r$  of  $A(x, r)$ , has to be done (see e.g. [5, 18]).

Now, let us consider the case when the function  $f$  belongs to  $L^1(\Omega)$ . As mentioned before, the weak formulation can not be used, since the integrals involved may not make sense. We then consider the notion of renormalized solution. We present here the definition of a renormalized solution of (1.1).

For simplicity, we only prove in this section the existence and uniqueness results for the linear case (that is,  $A(x, r) = A(x)$ ) with  $\lambda > 0$ . The proof of existence and uniqueness for the case where  $\lambda = 0$  requires more arguments (see Remark 1.6).

Before giving the definition of renormalized solution, we need to make sure that the integrals that will be in the renormalized formulation make sense. In particular, it is not clear if a solution  $u$  of (1.1) (with  $f \in L^1(\Omega)$ ) belongs to any Sobolev space. This means that a solution  $u$  may not have enough regularity to have a gradient in the usual sense of Sobolev spaces. Hence, one must first give a proper definition for the gradient of any measurable function using its truncate, where the truncation operator is defined by  $T_k(t) = \min\{k, \max\{t, -k\}\}$  (see Figure 1.1).

**Proposition 1.1** ([10, Lemma 2.1]). *Let  $u$  be a measurable function defined from  $\Omega$  to  $\mathbb{R}$ . If  $T_k(u) \in H_0^1(\Omega)$  for any  $k > 0$ , then there exists a unique measurable function  $v$  defined from  $\Omega$  to  $\mathbb{R}^N$  such that*

$$\nabla T_k(u) = v \chi_{\{|u| < k\}}, \quad \forall k > 0,$$

where  $\chi_{\{|u|<k\}}$  is the characteristic function of  $\{x \in \Omega : |u(x)| < k\}$ . We define  $v$  as the gradient of  $u$  and write  $v = \nabla u$ .

With this proposition, we can now define a renormalized solution of (1.1).

**Definition 1.2.** Let  $u$  be a measurable function from  $\Omega$  to  $\mathbb{R}$  with  $u$  finite almost everywhere in  $\Omega$ . Then  $u$  is a renormalized function of (1.1) if

$$T_k(u) \in H_0^1(\Omega) \quad \forall k > 0, \quad (1.4)$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_{\{|u|<n\}} A(x, u) \nabla u \cdot \nabla u \, dx = 0, \quad (1.5)$$

and for any  $S \in C^1(\mathbb{R})$  (or equivalently, for any  $S \in W^{1,\infty}(\mathbb{R})$ ) with compact support,  $u$  satisfies

$$\begin{aligned} \int_{\Omega} S(u) A(x, u) \nabla u \cdot \nabla v \, dx + \int_{\Omega} S'(u) A(x, u) \nabla u \cdot \nabla u \, v \, dx \\ + \lambda \int_{\Omega} S(u) u v \, dx = \int_{\Omega} f S(u) v \, dx, \end{aligned} \quad (1.6)$$

for any  $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ .

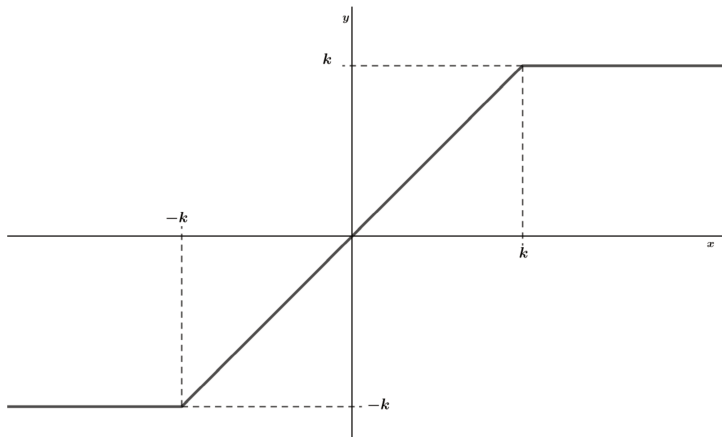


Figure 1.1: The truncation function  $T_k$

**Remark 1.3.** Observe that (1.6) can be obtained by formally choosing  $S(u)v$  as test function in (1.3). However, this does not justify how the integrals in (1.6) make sense. In fact, we need (1.4) to show that the integrals in (1.6) are well-defined. Indeed, note that if  $u$  is a renormalized solution of (1.1), then  $\nabla u$  is well-defined by Proposition 1.1, with

$$\nabla T_k(u) = \nabla u \chi_{\{|u|<k\}}, \quad \forall k > 0.$$

Let  $S \in C^1(\mathbb{R})$  with  $\text{supp } S \subset [-k, k]$ , for some  $k > 0$ . Then, we have

$$\begin{aligned} S(u)u &= S(u)T_k(u) \\ S'(u)\nabla u\nabla u &= S'(u)\nabla u\nabla u\chi_{|u|<k} = S'(u)\nabla T_k(u)\nabla T_k(u). \end{aligned}$$

Let  $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ . The functions  $S(u)$ ,  $T_k(u)$ , and  $v$  are bounded in  $\Omega$ , which implies that  $S(u)u$  and  $S(u)v$  both belong to  $L^\infty(\Omega)$ . It follows that the third and fourth integral of (1.6) are well-defined. Furthermore, since  $T_k(u) \in H_0^1(\Omega)$ , we have by (1.29),

$$\begin{aligned} S(u)A(x, u)\nabla u &\in (L^2(\Omega))^N \\ S'(u)A(x, u)\nabla u\nabla u &= S'(u)A(x, u)\nabla T_k(u)\nabla T_k(u) \in L^1(\Omega). \end{aligned}$$

Hence, the first two integrals of (1.6) make sense.

On the other hand, condition (1.5) is important in showing the stability and uniqueness of the renormalized solution.

We now consider, for simplicity, the linear case and present the proof of the following theorem from [71]:

**Theorem 1.4** ([71]). *Suppose that  $A(x, r) = A(x)$  for a.e.  $(x, r) \in \Omega \times \mathbb{R}$  (that is, the problem is linear) and  $\lambda > 0$ . Then there exists a unique renormalized solution to (1.1) in the sense of Definition 1.2.*

*Proof.* The proof is divided into 2 steps. The first step is dedicated to show the existence of a renormalized solution. This will be done by approximating  $f$  by a sequence  $\{f_\varepsilon\}$  in  $L^2(\Omega)$ , and considering a sequence of approximate solutions  $\{u_\varepsilon\}$  in  $H_0^1(\Omega)$ . The limit of  $\{u_\varepsilon\}$  will be a candidate for a renormalized solution, and we show that this is the case. In the second step, we show the uniqueness of the obtained solution.

*Step 1. Existence of a renormalized solution.*

Let  $\{f_\varepsilon\}$  be a sequence in  $L^2(\Omega)$  such that as  $\varepsilon$  tends to 0,

$$f_\varepsilon \longrightarrow f \quad \text{strongly in } L^1(\Omega). \quad (1.7)$$

For a fixed  $\varepsilon > 0$ , we consider the following variational formulation:

$$\int_{\Omega} A(x)\nabla u_\varepsilon\nabla v \, dx + \lambda \int_{\Omega} u_\varepsilon v \, dx = \int_{\Omega} f_\varepsilon v \, dx, \quad \forall v \in H_0^1(\Omega). \quad (1.8)$$

Since  $\lambda > 0$  and  $A$  satisfies (1.29), the Lax-Milgram Theorem gives the existence and uniqueness of the solution  $u_\varepsilon \in H_0^1(\Omega)$  of (1.8).

Now, we want to show that  $u_\varepsilon$  is Cauchy in  $L^1(\Omega)$ . For  $\varepsilon, \varepsilon' > 0$ , by linearity, we obtain that  $u_\varepsilon - u_{\varepsilon'}$  satisfies

$$\int_{\Omega} A(x) \nabla(u_\varepsilon - u_{\varepsilon'}) \nabla v \, dx + \lambda \int_{\Omega} (u_\varepsilon - u_{\varepsilon'}) v \, dx = \int_{\Omega} (f_\varepsilon - f_{\varepsilon'}) v \, dx, \quad \forall v \in H_0^1(\Omega).$$

Note that  $T_k(u_\varepsilon - u_{\varepsilon'}) \in H_0^1(\Omega)$ , and so, we can use it as a test function in this last formulation. We then obtain

$$\begin{aligned} \int_{\Omega} A(x) \nabla(u_\varepsilon - u_{\varepsilon'}) \nabla T_k(u_\varepsilon - u_{\varepsilon'}) \, dx + \lambda \int_{\Omega} (u_\varepsilon - u_{\varepsilon'}) T_k(u_\varepsilon - u_{\varepsilon'}) \, dx \\ = \int_{\Omega} (f_\varepsilon - f_{\varepsilon'}) T_k(u_\varepsilon - u_{\varepsilon'}) \, dx. \end{aligned} \quad (1.9)$$

Note that from Proposition 1.1, we have

$$\begin{aligned} A(x) \nabla(u_\varepsilon - u_{\varepsilon'}) \nabla T_k(u_\varepsilon - u_{\varepsilon'}) \\ = \chi_{\{|u_\varepsilon - u_{\varepsilon'}| < k\}} A(x) \nabla(u_\varepsilon - u_{\varepsilon'}) \nabla(u_\varepsilon - u_{\varepsilon'}) \geq 0, \end{aligned}$$

a.e. in  $\Omega$ . Moreover, by Hölder's inequality and the fact that  $|T_k(t)| \leq k$  for any  $t \in \mathbb{R}$ , we have

$$\int_{\Omega} (f_\varepsilon - f_{\varepsilon'}) T_k(u_\varepsilon - u_{\varepsilon'}) \, dx \leq k \|f_\varepsilon - f_{\varepsilon'}\|_{L^1(\Omega)}.$$

It then follows from (1.9) that

$$\frac{1}{k} \int_{\Omega} (u_\varepsilon - u_{\varepsilon'}) T_k(u_\varepsilon - u_{\varepsilon'}) \, dx \leq \frac{1}{\lambda} \|f_\varepsilon - f_{\varepsilon'}\|_{L^1(\Omega)}.$$

Taking the limit of both sides of this inequality as  $k$  tends to 0, and noting that

$$\left| (u_\varepsilon - u_{\varepsilon'}) \frac{T_k(u_\varepsilon - u_{\varepsilon'})}{k} \right| \leq |u_\varepsilon - u_{\varepsilon'}| \in L^1(\Omega),$$

and

$$\frac{T_k(u_\varepsilon - u_{\varepsilon'})}{k} \longrightarrow \operatorname{sgn}(u_\varepsilon - u_{\varepsilon'}) \quad \text{as } k \rightarrow 0 \text{ a.e. in } \Omega,$$

where  $\operatorname{sgn}$  is the usual sign function (i.e.,  $\operatorname{sgn}(r) = r/|r|$  if  $r \neq 0$  and  $\operatorname{sgn}(0) = 0$ ), we obtain by Lebesgue dominated convergence theorem,

$$\|u_\varepsilon - u_{\varepsilon'}\|_{L^1(\Omega)} \leq \frac{1}{\lambda} \|f_\varepsilon - f_{\varepsilon'}\|. \quad (1.10)$$

We then conclude from (1.7) that  $\{u_\varepsilon\}$  is a Cauchy sequence in  $L^1(\Omega)$ . Hence, there exists  $u \in L^1(\Omega)$  (and so  $u$  is finite a.e. in  $\Omega$ ) such that (up to a subsequence)

$$\begin{cases} u_\varepsilon \longrightarrow u & \text{strongly in } L^1(\Omega), \\ u_\varepsilon \longrightarrow u & \text{a.e. in } \Omega. \end{cases} \quad (1.11)$$

We now claim that  $u$  is a renormalized solution of (1.1). To show this, we need to show that  $u$  satisfies (1.4), (1.5), and (1.6).

We first obtain some estimates for  $T_k(u_\varepsilon)$ , for any  $k > 0$ , then show some convergences for the sequence  $\{T_k(u_\varepsilon)\}$  as  $\varepsilon$  approaches 0. We start by using  $T_k(u_\varepsilon) \in H_0^1(\Omega)$  as a test function in (1.8), which gives

$$\int_{\Omega} A(x) \nabla u_\varepsilon \nabla T_k(u_\varepsilon) dx + \lambda \int_{\Omega} u_\varepsilon T_k(u_\varepsilon) dx = \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) dx. \quad (1.12)$$

Since  $T_k$  is an increasing function, we know that  $u_\varepsilon T_k(u_\varepsilon) \geq 0$  a.e. in  $\Omega$ . Then, we deduce from (1.12) and Hölder's inequality that

$$\int_{\Omega} A(x) \nabla T_k(u_\varepsilon) \nabla T_k(u_\varepsilon) dx \leq \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) dx \leq k \|f_\varepsilon\|_{L^1(\Omega)}. \quad (1.13)$$

By coercivity of  $A$  and (1.7), we have

$$\|\nabla T_k(u_\varepsilon)\|_{(L^2(\Omega))^N} \leq kM,$$

for any  $k > 0$ , for any  $\varepsilon > 0$ , and for some  $M > 0$  independent of  $k$  and  $\varepsilon$ . This implies that for any  $k > 0$ , the sequence  $\{T_k(u_\varepsilon)\}_\varepsilon$  is uniformly bounded in  $H_0^1(\Omega)$ .

By a diagonal process and Rellich Theorem, we can extract a subsequence (which will still be denoted by  $\varepsilon$ ) such that for any  $k > 0$  (taken from a countable set), there exists  $v_k \in H_0^1(\Omega)$  such that

$$\begin{cases} T_k(u_\varepsilon) \rightharpoonup v_k & \text{weakly in } H_0^1(\Omega), \\ T_k(u_\varepsilon) \rightarrow v_k & \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega. \end{cases}$$

From (1.11), the continuity of  $T_k$ , and the uniqueness of the limit, we deduce that for any  $k > 0$ ,  $T_k(u) = v_k \in H_0^1(\Omega)$  (this shows that  $u$  satisfies (1.4)), and

$$\begin{cases} T_k(u_\varepsilon) \rightharpoonup T_k(u) & \text{weakly in } H_0^1(\Omega), \\ T_k(u_\varepsilon) \rightarrow T_k(u) & \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \\ \nabla T_k(u_\varepsilon) \rightharpoonup \nabla T_k(u) & \text{weakly in } (L^2(\Omega))^N. \end{cases} \quad (1.14)$$

We now show that  $u$  satisfies (1.5). By (1.13), (1.14), and (1.7), for all  $k > 0$ , we have

$$\begin{aligned} \frac{1}{k} \int_{\Omega} A(x) \nabla T_k(u) \nabla T_k(u) dx &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{k} \int_{\Omega} A(x) \nabla T_k(u_\varepsilon) \nabla T_k(u_\varepsilon) dx \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{k} \int_{\Omega} f_\varepsilon T_k(u_\varepsilon) dx \\ &= \frac{1}{k} \int_{\Omega} f T_k(u) dx. \end{aligned}$$

Since  $u$  is finite a.e. in  $\Omega$ , we have

$$\left| \frac{T_k(u)}{k} \right| \rightarrow 0 \quad \text{a.e. in } \Omega \quad \text{as } k \rightarrow \infty.$$

Moreover, from the definition of  $T_k$ , we have for any  $k > 0$ ,

$$\left| f \frac{T_k(u)}{k} \right| \leq |f| \in L^1(\Omega).$$

Then the Lebesgue dominated convergence theorem implies

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\Omega} A(x) \nabla T_k(u) \nabla T_k(u) dx = 0$$

(which gives that  $u$  satisfies (1.5)), and

$$\lim_{k \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{k} \int_{\Omega} A(x) \nabla T_k(u_\varepsilon) \nabla T_k(u_\varepsilon) dx = 0. \quad (1.15)$$

To end this step, we show that  $u$  satisfies (1.6). For  $n > 0$ , let us define the function  $h_n$  as (see Figure 1.2)

$$h_n(s) = \begin{cases} 0, & \text{if } s \leq -2n \\ \frac{s}{n} + 2, & \text{if } -2n \leq s \leq -n \\ 1, & \text{if } -n \leq s \leq n \\ -\frac{s}{n} + 2, & \text{if } n \leq s \leq 2n \\ 0, & \text{if } s \geq 2n. \end{cases}$$



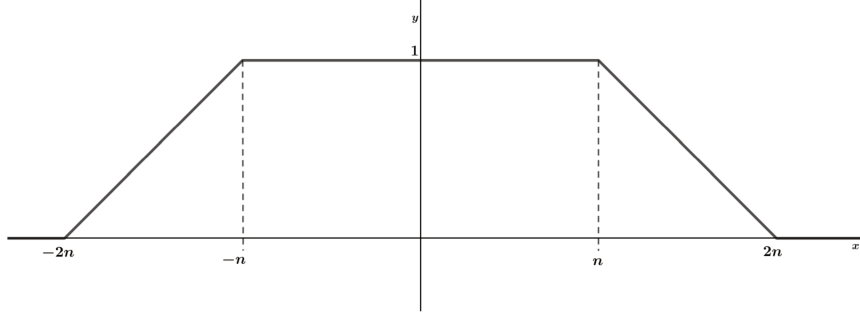


Figure 1.2: The function  $h_n$

Let  $S \in C^1(\mathbb{R})$  with compact support and  $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ . Note that  $S(u)h_n(u_\varepsilon)v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , and thus, it can be considered as a test function in (1.8). This gives for  $n > 0$ ,

$$\begin{aligned} & \int_{\Omega} A(x) \nabla u_\varepsilon \nabla v S(u) h_n(u_\varepsilon) dx + \int_{\Omega} S'(u) A(x) \nabla u_\varepsilon \nabla u h_n(u_\varepsilon) v dx \\ & + \int_{\Omega} h'_n(u_\varepsilon) A(x) \nabla u_\varepsilon \nabla u_\varepsilon S(u) v dx + \lambda \int_{\Omega} u_\varepsilon S(u) h_n(u_\varepsilon) v dx \quad (1.16) \\ & = \int_{\Omega} f_\varepsilon S(u) h_n(u_\varepsilon) v dx. \end{aligned}$$

Our goal now is to pass to the limit first as  $\varepsilon$  tends to 0 and then as  $n$  approaches  $+\infty$ .

We first observe the last two integrals of (1.16). The Lebesgue dominated convergence theorem, together with the continuity of  $h_n$ , (1.11), and (1.7), gives

$$\lim_{\varepsilon \rightarrow 0} \lambda \int_{\Omega} u_\varepsilon S(u) h_n(u_\varepsilon) v dx = \lambda \int_{\Omega} u S(u) h_n(u) v dx, \quad (1.17)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f_\varepsilon S(u) h_n(u_\varepsilon) v dx = \int_{\Omega} f S(u) h_n(u) v dx. \quad (1.18)$$

Note that from the definition of  $h_n$ , it can be deduced that  $\text{supp } h_n \subset [-2n, 2n]$  and  $|h'_n| \leq \frac{1}{n}$ . It follows that

$$\begin{aligned} & \left| \int_{\Omega} h'_n(u_\varepsilon) A(x) \nabla u_\varepsilon \nabla u_\varepsilon S(u) v \right| \\ & \leq \|S\|_{L^\infty(\mathbb{R})} \|v\|_{L^\infty(\Omega)} \frac{1}{n} \int_{\Omega} A(x) \nabla T_{2n}(u_\varepsilon) \nabla T_{2n}(u_\varepsilon) dx. \end{aligned}$$

By (1.15), we obtain

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} h'_n(u) A(x) \nabla u_\varepsilon \nabla u_\varepsilon S(u) v dx = 0. \quad (1.19)$$

Moreover, we have

$$h_n(u_\varepsilon)\nabla u_\varepsilon = h_n(u_\varepsilon)\nabla T_{2n}(u_\varepsilon),$$

which implies that

$$\int_{\Omega} A(x)\nabla u_\varepsilon\nabla v S(u)h_n(u_\varepsilon) dx = \int_{\Omega} A(x)\nabla T_{2n}(u_\varepsilon)\nabla v S(u)h_n(u_\varepsilon) dx,$$

and

$$\int_{\Omega} S'(u)A(x)\nabla u_\varepsilon\nabla u h_n(u_\varepsilon)v dx = \int_{\Omega} S'(u)A(x)\nabla T_{2n}(u_\varepsilon)\nabla u h_n(u_\varepsilon)v dx.$$

Since  $h_n$  is continuous, we know from (1.11) that  $h_n(u_\varepsilon) \rightarrow h_n(u)$  a.e. in  $\Omega$  as  $\varepsilon$  tends to 0. We also have  $|h_n| \leq 1$  from the definition of  $h_n$ , which implies that

$$h_n(u_\varepsilon) \longrightarrow h_n(u) \quad \text{in } L^\infty(\Omega) \text{ weak-}^* \quad \text{as } \varepsilon \rightarrow 0.$$

Combining this with (1.14), we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} A(x)\nabla u_\varepsilon\nabla v S(u)h_n(u_\varepsilon) dx = \int_{\Omega} A(x)\nabla u\nabla v S(u)h_n(u) dx, \quad (1.20)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} S'(u)A(x)\nabla u_\varepsilon\nabla u h_n(u_\varepsilon)v dx = \int_{\Omega} S'(u)A(x)\nabla u\nabla u h_n(u)v dx. \quad (1.21)$$

If we choose  $n$  sufficiently large such that  $\text{supp } S \subset [-n, n]$ , then we have

$$S(u)h_n(u) = S(u) \quad \text{a.e. in } \Omega.$$

Applying this identity to (1.17), (1.18), (1.20), and (1.21), we can pass to the limit as  $n$  tends to infinity. Together with (1.19) and (1.16), we have the desired result.

*Step 2. Uniqueness of the renormalized solution.*

From the previous step, we know that there exists at least one solution. Suppose that  $u$  and  $v$  are two renormalized solutions of (1.1). Let  $n \in \mathbb{N}$  and  $k > 0$ . From the definitions of  $T_k$  and  $h_n$ , it follows that

$$h_n(u)h_n(v)T_k(T_{2n}(u) - T_{2n}(v)) \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Hence, this can be used as a test function for the renormalized formulation (1.6) for  $u$  (with  $S = h_n$  and  $v = h_n(v)T_k(T_{2n}(u) - T_{2n}(v))$ ) and also for  $v$

(with  $S = h_n$  and  $v = h_n(u)T_k(T_{2n}(u) - T_{2n}(v))$ ). Subtracting the resulting equations and the fact that

$$h_n(u)h_n(v)T_k(T_{2n}(u) - T_{2n}(v)) = h_n(u)h_n(v)T_k(u - v) \quad \text{a.e. in } \Omega,$$

yield

$$I_1 + I_2 + I_3 - I_4 - I_5 + I_6 = 0,$$

where

$$I_1 = \int_{\Omega} A(x)(\nabla u - \nabla v)(\nabla u - \nabla v)\chi_{\{|u-v|<k\}}h_n(u)h_n(v) dx$$

$$I_2 = \int_{\Omega} h'_n(u)A(x)\nabla u\nabla u h_n(v)T_k(u - v) dx$$

$$I_3 = \int_{\Omega} h'_n(v)A(x)\nabla u\nabla v h_n(u)T_k(u - v) dx$$

$$I_4 = \int_{\Omega} h'_n(u)A(x)\nabla v\nabla u h_n(v)T_k(u - v) dx$$

$$I_5 = \int_{\Omega} h'_n(v)A(x)\nabla v\nabla v h_n(u)T_k(u - v) dx$$

$$I_6 = \lambda \int_{\Omega} (u - v)h_n(u)h_n(v)T_k(u - v) dx.$$

We want to evaluate the limit of each term as  $n$  approaches  $+\infty$ .

By the decay of the truncate energy (1.5), the properties of the matrix  $A$  in (1.29), the fact that  $\text{supp } h_n \subset [-2n, 2n]$ , and Young's inequality (for  $I_3$  and  $I_4$ ), it follows that

$$\lim_{n \rightarrow \infty} I_2 = \lim_{n \rightarrow \infty} I_3 = \lim_{n \rightarrow \infty} I_4 = \lim_{n \rightarrow \infty} I_5 = 0.$$

Note that  $I_1 \geq 0$ , and thus, from Fatou's Lemma

$$\int_{\Omega} (u - v)T_k(u - v) dx \leq \liminf_{n \rightarrow \infty} I_6 \leq 0.$$

Since  $T_k$  is an increasing function, it follows that  $u - v = 0$  a.e. in  $\Omega$ . This concludes this step and this proof.  $\square$

We can also have the following stability result. This can be proved by using the same method used in the proof above.

**Remark 1.5** (Stability). *Let  $\{f_\varepsilon\}$  be a sequence that strongly converges to a function  $f$  in  $L^1(\Omega)$ . Suppose  $u_\varepsilon$  is the renormalized solution corresponding to the data  $f_\varepsilon$ . Then  $\{u_\varepsilon\}$  converges to  $u$  in  $L^1(\Omega)$ , where  $u$  is the renormalized solution corresponding to the function  $f$ . In addition,  $T_k(u_\varepsilon)$  converges strongly to  $T_k(u)$  in  $H_0^1(\Omega)$ , for any  $k > 0$ .*

**Remark 1.6** (The case  $\lambda = 0$ ). *Note that if we consider the case  $\lambda = 0$ , we clearly do not have the estimate (1.10) from the proof of the existence result of Theorem 1.4. To show existence for this case, we argue first by showing that the approximate solution  $u_\varepsilon$  of (1.8) is Cauchy in measure then use the Rellich-Kondrachov compactness result to obtain convergences (1.14).*

*For the uniqueness result, if  $u$  and  $v$  are two renormalized solutions, notice that from the proof of the uniqueness part of Theorem 1.4, the argument to show that  $u = v$  involves mainly integral  $I_6$ , which is the integral related to the term with  $\lambda$ . As  $\lambda = 0$ , we do not have this term when showing uniqueness for this case. The proof of uniqueness of the renormalized solution for this case requires more delicate arguments.*

*These delicate arguments can be observed in the first part of this thesis, as we consider there the case  $\lambda = 0$ .*

### 1.1.2 Our renormalized solution results

In the first part of this thesis, we study the existence and uniqueness of renormalized solution of a class of quasilinear elliptic equations posed in a two-component domain with an  $L^1$  data. More precisely, the domain  $\Omega$  is a connected bounded open set in  $\mathbb{R}^N$  with boundary  $\partial\Omega$ . We write  $\Omega$  as the disjoint union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , where  $\Omega_2$  is an open set such that  $\overline{\Omega_2} \subset \Omega$  with a Lipschitz boundary  $\Gamma$ , and  $\Omega_1 = \Omega \setminus \overline{\Omega_2}$  (see Figure 1.3). We study the following quasilinear elliptic problem posed in  $\Omega$ :

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.22)$$

where  $\nu_1$  is the unit outward normal to  $\Omega_1$ ,  $f$  is an  $L^1$  function, and  $B$  is a coercive matrix field which has a restricted growth assumption ( $B(x, r)$  is bounded on any compact set of  $\mathbb{R}$ ).

To properly define a renormalized solution of (1.22), we first introduce the space that we will be working with, that is well adapted to our problem.

We define the space  $V$  as

$$V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\},$$

equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2,$$

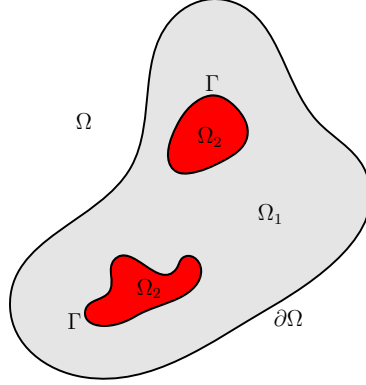


Figure 1.3: The two-component domain  $\Omega$

where  $V_1$  is the space defined by

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

We follow a similar definition from [36] with the additional term on the interface  $\Gamma$  and condition for the jump. More precisely:

**Definition 1.7.** *Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  be a measurable function. Then  $u$  is a renormalized solution of (1) if*

$$T_k(u) \in V, \quad \forall k > 0; \quad (1.23a)$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \quad (1.23b)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u \, dx = 0; \quad (1.24a)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \quad (1.24b)$$

and for any  $S_1, S_2 \in C^1(\mathbb{R})$  (or equivalently for any  $S_1, S_2 \in W^{1,\infty}(\mathbb{R})$ ) with compact support,  $u$  satisfies

$$\begin{aligned} & \int_{\Omega_1} S_1(u_1) B(x, u_1) \nabla u_1 \cdot \nabla v_1 \, dx + \int_{\Omega_1} S_1'(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 v_1 \, dx \\ & + \int_{\Omega_2} S_2(u_2) B(x, u_2) \nabla u_2 \cdot \nabla v_2 \, dx + \int_{\Omega_2} S_2'(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 v_2 \, dx \\ & + \int_{\Gamma} h(x) (u_1 - u_2) (v_1 S_1(u_1) - v_2 S_2(u_2)) \, d\sigma \\ & = \int_{\Omega_1} f v_1 S_1(u_1) \, dx + \int_{\Omega_2} f v_2 S_2(u_2) \, dx, \end{aligned} \quad (1.25)$$

for all  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ .

Note that since we are in the renormalized framework, a solution  $u$  of (1) may not have enough regularity to have a gradient and trace in the usual sense of Sobolev spaces. Hence, we first have to make sure that the gradient and the trace of a solution are properly defined. To this aim, we prove the following proposition (which is a generalization of [10, Lemma 2.1] and [59, Proposition 2.3]):

**Proposition 1.8.** *Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$  be a measurable function such that  $T_k(u) \in V$  for every  $k > 0$ .*

1. *For  $i = 1, 2$ , there exists a unique measurable function  $G_i : \Omega_i \longrightarrow \mathbb{R}^N$  such that for all  $k > 0$ ,*

$$\nabla T_k(u_i) = G_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i,$$

where  $\chi_{\{|u_i| < k\}}$  denotes the characteristic function of

$$\{x \in \Omega_i : |u_i(x)| < k\}.$$

We define  $G_i$  as the gradient of  $u_i$  and write  $G_i = \nabla u_i$ .

2. *If*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty,$$

then there exists a unique measurable function

$$w_i : \Gamma \longrightarrow \mathbb{R}, \quad \text{for } i = 1, 2,$$

such that for all  $k > 0$ ,

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma,$$

where  $\gamma_i : H^1(\Omega_i) \longrightarrow L^2(\Gamma)$  is the trace operator. We define the function  $w_i$  as the trace of  $u_i$  on  $\Gamma$  and set

$$\gamma_i(u_i) = w_i, \quad i = 1, 2.$$

The originality of this definition is found in the regularity (1.23b) and the decay of the energy on the interface (1.24b), together with the presence of the term on  $\Gamma$ ,

$$\int_{\Gamma} h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2)) \, d\sigma.$$

The regularity (1.23a) and the decay of the truncated energy (1.24a) are classical and it allows us (with Proposition 1.8) to give a sense to all the integrals in (1.25) except for the boundary integral on  $\Gamma$ .

Indeed, let  $S_i \in C^1(\mathbb{R})$ ,  $i = 1, 2$ , with compact support. Then for all  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ , if  $\text{supp } S_i \subset [-k, k]$  ( $i = 1, 2$ ), then for  $i = 1, 2$ , we have

$$\begin{aligned} S_i(u_i)B(x, u_i)\nabla u_i \cdot \nabla v_i &= S_i(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla v_i \in L^1(\Omega_i), \\ S'_i(u_i)B(x, u_i)\nabla u_i \cdot \nabla u_i v_i &= S'_i(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla T_k(u_i) v_i \in L^1(\Omega_i), \\ f v_i S_i(u_i) &\in L^1(\Omega_i). \end{aligned}$$

Concerning the boundary integral on  $\Gamma$ , note that (1.23a) and (1.24a) are not enough to give a sense to  $h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2))$ . In general, there is no reason to have

$$\begin{aligned} h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2)) \\ = h(x)(u_1 - u_2)(v_1 S_1(u_1) - v_2 S_2(u_2))\chi_{\{|u_1| \leq n\}}\chi_{\{|u_2| \leq n\}} \end{aligned}$$

for large  $n$ .

For the boundary term, for any  $n \in \mathbb{N}$ , let us define  $\theta_n : \mathbb{R} \rightarrow \mathbb{R}$  by (see Figure 1.4)

$$\theta_n(s) = \begin{cases} 0, & \text{if } s \leq -2n, \\ \frac{s}{n} + 2, & \text{if } -2n \leq s \leq -n, \\ 1, & \text{if } -n \leq s \leq n, \\ -\frac{s}{n} + 2, & \text{if } n \leq s \leq 2n, \\ 0, & \text{if } s \geq 2n. \end{cases}$$

Then since  $S_1$  has a compact support, we have for some large enough  $n$ ,

$$\begin{aligned} h(u_1 - u_2)v_1 S_1(u_1) &= h v_1(u_1 - u_2)(S_1(u_1) - S_1(u_2))\theta_n(u_1) \\ &\quad + h v_1(u_1 - u_2)S_1(u_2)\theta_n(u_1). \end{aligned}$$

Since  $\theta_n$  also has a compact support, then  $h v_1(u_1 - u_2)S_1(u_2)\theta_n(u_1)$  is bounded, and is therefore in  $L^1(\Gamma)$ . Moreover, since

$$S_1(u_1) - S_1(u_2) = S_1(T_{2n}(u_1)) - S_1(T_{2n}(u_2))$$

and  $S_1$  are Lipschitz-continuous, we have

$$\begin{aligned} |h v_1(u_1 - u_2)(S_1(u_1) - S_1(u_2))\theta_n(u_1)| \\ \leq \|h v_1\|_{L^\infty(\Gamma)} \|S'_1\|_{L^\infty(\mathbb{R})} |u_1 - u_2| |T_{2n}(u_1) - T_{2n}(u_2)|, \end{aligned}$$

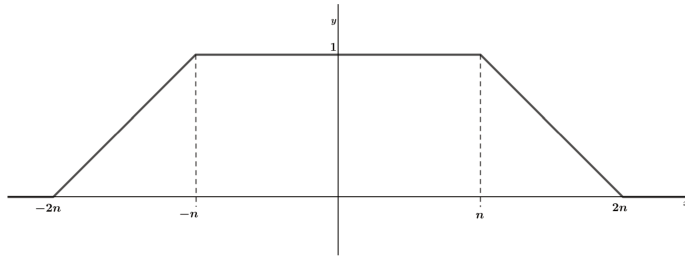


Figure 1.4: The function  $\theta_n$

a.e. in  $\Gamma$ . Thus, in view of (1.23b),  $h(u_1 - u_2)v_1S_1(u_1) \in L^1(\Gamma)$ . Similarly,  $h(u_1 - u_2)v_2S_2(u_2) \in L^1(\Gamma)$ . Hence, the renormalized formulation (1.25) is well-defined.

The main result for this chapter is the existence theorem below.

**Theorem 1.9.** *Suppose the following assumptions hold:*

(A1) *The data  $f$  belongs to  $L^1(\Omega)$ .*

(A2) *The function  $h$  satisfies*

$$h \in L^\infty(\Gamma) \quad \text{and} \quad 0 < h_0 < h(y) \text{ a.e. on } \Gamma,$$

*for some  $h_0 \in \mathbb{R}^+$ .*

(A3) *The matrix field  $B$  is a Carathéodory function, that is,*

(a) *the map  $r \mapsto B(x, r)$  is continuous for a.e.  $x \in \Omega$ ;*

(b) *the map  $x \mapsto B(x, r)$  is measurable for a.e.  $r \in \mathbb{R}$ ,*

*and it has the following properties:*

(A3.1)  *$B(x, r)\xi \cdot \xi \geq \alpha|\xi|^2$ , for some  $\alpha > 0$ , for a.e.  $x \in \Omega$ ,  $\forall r \in \mathbb{R}$ ,  $\forall \xi \in \mathbb{R}^N$ ;*

(A3.2) *for any  $k > 0$ ,  $B(x, r) \in L^\infty(\Omega \times (-k, k))^{N \times N}$ .*

*Then there exists a renormalized solution to (1) in the sense of Definition 1.7.*

The proof of this existence theorem is done by passing to the limit in an approximate problem. The first step consists of considering a sequence  $\{f^\varepsilon\} \subset L^2(\Omega)$  such that

$$f^\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega),$$



and we let  $B_\varepsilon(x, t) = B(x, T_{1/\varepsilon}(t))$ . We consider a solution  $u^\varepsilon \in V$  of

$$\begin{cases} -\operatorname{div}(B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{in } \Omega_1, \\ -\operatorname{div}(B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{in } \Omega_2, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The second step is to obtain a priori estimates, then construct (with the help of Rellich-Kondrachov compactness results) a function  $u$  such that (up to a subsequence)

$$\begin{cases} u_i^\varepsilon \rightarrow u_i & \text{a.e. in } \Omega, \\ T_k(u_i^\varepsilon) \rightharpoonup T_k(u_i) & \text{weakly in } H^1(\Omega_i), \\ \gamma_i(u_i^\varepsilon) \rightarrow \gamma_i(u_i) & \text{a.e. in } \Gamma, \\ \gamma_i(T_k(u_i^\varepsilon)) \rightarrow \gamma_i(T_k(u_i)) & \text{strongly in } L^2(\Gamma), \text{ a.e. on } \Gamma. \end{cases}$$

The originality here comes again from the boundary integral on  $\Gamma$ , and using, for example, in the third step, the a priori estimate

$$\forall k > 0, \quad (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \text{ bounded in } L^1(\Gamma),$$

and the limit

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left( \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) dx + \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) = 0.$$

In the final step of the proof, we are able then to pass to the limit with an appropriate choice of test function and show that  $u$  is a renormalized solution of (1).

**Remark 1.10.** *To give a sense to the integral on  $\Gamma$ , it is possible to replace (1.23b) and (1.24b) by a regularity condition given by*

$$u_1 - u_2 \in W^{1-\frac{1}{q}, q}(\Gamma), \quad \text{for some } q > 1. \quad (1.26)$$

*This type of regularity comes from the Boccardo-Gallouët type estimates, but depends heavily on the Sobolev embedding constants. Since these constants may blow up with asymptotic analysis, we are not able to use (6). In addition, (1.26) will not work when considering a more general nonlinear problem. Thus, we chose (1.23b) and (1.24b) in the definition of renormalized solution.*

To prove the uniqueness result, we adopt the method developed in [16, 38]. This method makes use of an auxiliary function  $\varphi \in C^1(\mathbb{R})$  with interesting properties. To be more precise, we use the following proposition from [38]:

**Proposition 1.11** ([38]). *Suppose that (A4) holds. Then there exists a function  $\varphi \in C^1(\mathbb{R})$  that satisfies the following properties:*

$$\varphi(0) = 0 \quad \text{and} \quad \varphi' \geq 1.$$

In addition, there are constants  $\delta > 1/2$ ,  $0 < k_0 < 1$ , and  $L > 0$  such that

$$\frac{\varphi'}{(1 + |\varphi|)^{2\delta}} \in L^\infty(\mathbb{R}).$$

Moreover, for any  $r, s \in \mathbb{R}$  satisfying  $|\varphi(r) - \varphi(s)| \leq k$ , for  $0 < k < k_0$ ,

$$\left| \frac{B(x, r)}{\varphi'(r)} - \frac{B(x, s)}{\varphi'(s)} \right| \leq \frac{1}{\varphi'(s)} \frac{Lk}{(1 + |\varphi(r)| + |\varphi(s)|)^\delta}$$

and

$$\frac{1}{L} \leq \frac{\varphi'(s)}{\varphi'(r)} \leq L.$$

One of the difficulties in the proof of uniqueness comes from the boundary integral on  $\Gamma$ , which is related to the jump of the solution. The following proposition is an important tool in the proof of the uniqueness theorem, which is related to the Boccardo-Gallouët estimates:

**Proposition 1.12.** *For  $i = 1, 2$ , let  $\gamma_i$  be the trace function defined on  $H^1(\Omega_i)$ . Under the assumptions of Theorem 1.9, if  $u$  is a renormalized solution of (1), then  $\gamma_i(u_i) \in L^1(\Gamma)$ ,  $i = 1, 2$ .*

The method developed in [16, 38] consists of using  $T_k(\varphi(u) - \varphi(v))$  as test function. The proof is very technical and is done by passing to the limit with the help of the function  $\theta_n(u)T_k(\varphi(u) - \varphi(v))$  in the renormalized formulation in the definition of a renormalized solution.

However, the nonlinearity of the term  $T_k(\varphi(u) - \varphi(v))$  is not compatible with the boundary integral on  $\Gamma$ , since the term obtained

$$\int_{\Gamma} h(x)((u_1 - v_1) - (u_2 - v_2))(T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))) d\sigma$$

is not necessarily positive.

To overcome this difficulty, we show a sign property on  $\Gamma$ :

**Lemma 1.13.** *Suppose assumptions (A1)–(A4) hold. If  $u$  and  $v$  are two renormalized solutions of (1), then  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  a.e. on  $\Gamma$ .*

This lemma is essential in the proof of the following uniqueness theorem, which is the main result of this chapter:

**Theorem 1.14.** *If assumptions (A1)–(A4) hold, then the renormalized solution of (1) is unique.*

One of the major steps of the proof of Theorem 1.14 consists of showing that

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{\Omega_i} \left( \frac{1}{\varphi'(u_i)} + \frac{1}{\varphi'(v_i)} \right) |\nabla T_k(\varphi(u_i) - \varphi(v_i))|^2 dx = 0, \quad i = 1, 2, \quad (1.27)$$

where  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  are two renormalized solutions of (1), and  $\varphi$  is the function in Proposition 1.11.

To prove this, after some long computations, we obtain

$$\begin{aligned} & \limsup_{k \rightarrow 0} \left( \frac{1}{k^2} \int_{U_1^k} \left( \frac{1}{\varphi'(u_1)} + \frac{1}{\varphi'(v_1)} \right) |\nabla \varphi(u_1) - \nabla \varphi(v_1)|^2 dx \right. \\ & \quad \left. + \frac{1}{k^2} \int_{U_2^k} \left( \frac{1}{\varphi'(u_2)} + \frac{1}{\varphi'(v_2)} \right) |\nabla \varphi(u_2) - \nabla \varphi(v_2)|^2 dx + \frac{1}{k^2} C^k \right) \leq 0, \end{aligned}$$

where

$$U_i^k = \{x \in \Omega_i : 0 < |\varphi(u_i) - \varphi(v_i)| < k\}, \quad i = 1, 2,$$

and

$$C^k = \int_{\Gamma} h(x) [(u_1 - v_1) - (u_2 - v_2)] [T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))] d\sigma.$$

Hence, to show (1.27), it is enough to show that

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} C^k \geq 0.$$

As already mentioned above, due to the presence of the function  $\varphi$ , we don't know in general the sign of  $C^k$ . The limit of  $\frac{1}{k^2} C^k$  as  $k \rightarrow 0$  is one of the main difficulties. To proceed, we use the sign property on  $\Gamma$  (Lemma 1.13) and we divide the set  $\{x \in \Gamma; u_1(x) - v_1(x) > 0\}$  (up to a zero measure subset) into 4 disjoint subsets,

$$\{u_1 - v_1 > 0\} = P_1 \cup P_2 \cup P_3 \cup P_4,$$

where

$$\begin{aligned}
P_1 &:= \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}, \\
P_2 &:= \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\
P_3 &:= \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\
P_4 &:= \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}.
\end{aligned}$$

This division is possible due to Lemma 1.13. Then we are able to compute the limit of the boundary integral over each subset.

Once (1.27) is shown, we follow the same approach as in [16, 38] to show first that  $u_1 = v_1$  in  $\Omega_1$  (since  $u_1$  and  $v_1$  have zero trace on  $\partial\Omega$ , which allows us to use the Poincaré inequality). As a consequence, we have  $u_1 = v_1$  on  $\Gamma$ . This, combined with Lemma 1.13 and (1.27), gives  $u_2 = v_2$  in  $\Omega_2$ .

## 1.2 Homogenization theory

Homogenization theory is motivated from the study of the macroscopic behaviour of microscopic composite materials. Composite materials are composed of 2 or more finely mixed components and their main physical characteristics (e.g. thermal or electric conductivity) can be modelled by PDEs with oscillating coefficients, describing the heterogeneities at the micro-scale. Then, the mathematical homogenization theory allows to give a macroscopic description of these materials, considered as homogeneous, at the macro-scale.

There are different types of composite materials that can be studied by homogenization, some examples are plywood (a layered material) and concrete (a periodic material). Also, additional oscillations can come, in some case, from the domain. Let us mention the ones with oscillating boundaries, such as heat radiators and engines, or perforated materials like sponge (a porous medium), or trusses.

In this thesis, we are concerned in the homogenization of finely periodically mixed materials, that is, periodic homogenization. Indeed, when the components of these materials are finely mixed, a reasonable assumption is that the distribution of the heterogeneities is periodic.

To explain intuitively how periodic homogenization works, imagine a composite material with two components, Material 1 and Material 2. We suppose that an  $\varepsilon$ -sized Material 1 is periodically scattered throughout Material 2, with  $\varepsilon$ -periodicity in each axis-direction. This is obtained by a change of scale from a fixed unit cell (see Figure 1.2).

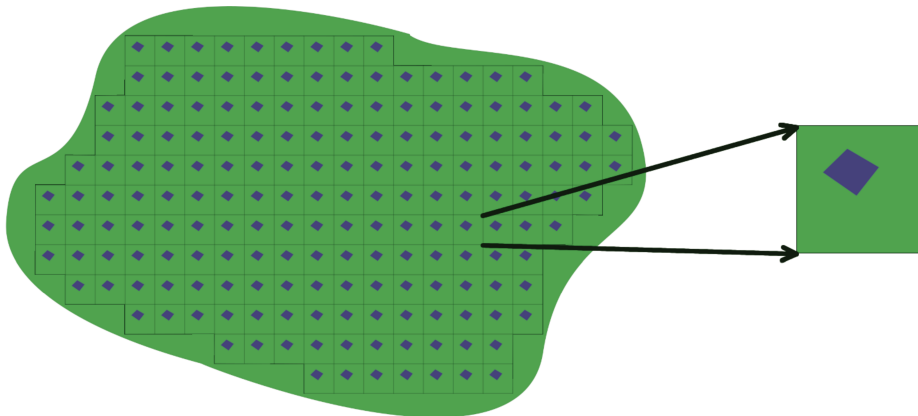


Figure 1.5: The composite material with Material 1 (purple) and Material 2 (green), and the unit cell.

Here  $\varepsilon$  is a very small positive number and to study the behaviour of this composite material, one should take the value of  $\varepsilon$  as small as possible, since the components are finely mixed.

In the mathematical theory of homogenization, the characteristics of the material that we are looking for, are modelled as the coefficient of a PDE depending on a positive parameter  $\varepsilon$  taking its values in a positive sequence that goes to zero, and representing the periodicity. Then, the aim is to study the asymptotic behaviour of the PDE as  $\varepsilon$  go to zero. This gives a PDE with constant coefficients, the so called effective matrix field, describing a homogeneous material, with no more heterogeneities. The behaviour of this “fictitious” homogenized material now approximates the behaviour of the original composite material. Moreover, it is easier to study, both theoretically and numerically, since the oscillations are removed, due to the absence of the parameter  $\varepsilon$  in the problem.

Mathematically, the main difficulty in periodic homogenization (or in any type of homogenization) when passing to the limit as  $\varepsilon$  goes to zero is that one needs to pass to the limit in products of two weakly convergent sequences (as will be seen in the sequel).

Now, we explain the homogenization process theoretically and we state the results for the model case of a linear Dirichlet problem in a fixed domain. Their proofs are presented in the subsequent subsections using different methods for the periodic homogenization. We also explain the main points where the difficulties arise.

First, let  $\Omega$  be an open connected bounded set in  $\mathbb{R}^N$ ,  $N \geq 2$ , with boundary  $\partial\Omega$ . This  $\Omega$  represents the domain occupied by our composite material.

Let  $\varepsilon$  be a positive parameter taken from a positive sequence that goes to 0. This parameter characterizes the periodicity of the domain.

We now introduce the reference cell  $Y$ , defined by

$$Y = \prod_{j=1}^N [0, l_j),$$

for some  $l_j > 0$ ,  $j = 1, \dots, N$ .

The heat conductivity in  $\Omega$ , can be modelled by the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.28)$$

where  $f \in L^2(\Omega)$  is the heat source,  $u^\varepsilon$  is the temperature distribution, and  $A^\varepsilon$  represents the conductivity of the material. We assume that

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right),$$

where  $A(y)$  is a  $Y$ -periodic measurable matrix field, that satisfies the following properties, for some fixed  $\alpha, \beta \in \mathbb{R}^+$ ,  $0 < \alpha < \beta$ :

$$\begin{cases} A(y)\xi \cdot \xi \geq \alpha|\xi|^2, \\ |A(y)\xi| \leq \beta|\xi|, \end{cases} \quad (1.29)$$

for all  $\xi \in \mathbb{R}^N$ , and for all  $y \in Y$ .

Note that the conductivity  $A^\varepsilon$  is, in general, discontinuous due to the nature of the domain  $\Omega$ . This means that the heat flux  $A^\varepsilon \nabla u^\varepsilon$  may not be differentiable in the classical sense. Thus, we need to consider derivatives in the weak sense, and we look for a weak solution  $u^\varepsilon$  in an appropriate Sobolev space.

In order to find a weak solution, we consider the variational formulation of problem (1.28), which is the following:

$$\begin{cases} \text{Find } u^\varepsilon \in H_0^1(\Omega) \text{ such that} \\ \int_{\Omega} A^\varepsilon \nabla u^\varepsilon \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega). \end{cases} \quad (1.30)$$

Since  $f \in L^2(\Omega)$ , for a fixed  $\varepsilon$ , it is easily established by the Lax-Milgram Theorem that this problem has a unique solution with the following a priori estimate:

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha} \|f\|_{L^2(\Omega)}. \quad (1.31)$$

Our goal is to pass to the limit in the integrals in (1.30) as  $\varepsilon$  goes to 0. Actually, we are only concerned with the integral

$$\int_{\Omega} A^\varepsilon \nabla u^\varepsilon \nabla v \, dx, \quad (1.32)$$

since the other integral in (1.30) is independent of  $\varepsilon$ . Let us examine the sequences  $\{A^\varepsilon\}$  and  $\{u^\varepsilon\}$ .

From (1.31), we know that the sequence  $\{u^\varepsilon\}$  is uniformly bounded in  $H_0^1(\Omega)$ . Thus, we can extract a subsequence from  $\{u^\varepsilon\}$  such that

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega), \quad (1.33)$$

which then implies that

$$\nabla u^\varepsilon \rightharpoonup \nabla u^0 \quad \text{weakly in } L^2(\Omega)^N.$$

Now, to easily pass to the limit of (1.32), we need a strong convergence for  $A^\varepsilon$ . However, this is not the case. In fact, from classical results (see Theorem 2.6 of [33]), we only have a weak convergence for functions with the form similar to  $A^\varepsilon$ .

In general, we cannot pass to the limit of an integral of a product of two weakly convergent sequences. This difficulty justifies the interest and development of the homogenization theory. The various methods for periodic homogenization were developed specifically to solve this problem.

It is worth noting that the weak limit  $u^0$  of  $u^\varepsilon$  describes the thermal distribution in the homogenized material. However, except for this  $u^0$ , we have no information about this homogenized material. By passing to the limit of (1.30), we will obtain an elliptic problem satisfied by this  $u^0$ . This homogenized problem gives the conductivity matrix  $A^0$  (see (1.35)) for the homogenized material.

We now discuss the different methods for periodic homogenization by proving the following homogenization result for problem (1.28):

**Theorem 1.15.** *Let  $A^\varepsilon$  be the matrix  $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$ , where  $A$  is  $Y$ -periodic and satisfies (1.29), and  $f$  be a function in  $L^2(\Omega)$ . If  $u^\varepsilon$  is the weak solution of (1.30), then there exists  $u^0 \in H_0^1(\Omega)$  such that*

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega), \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0 & \text{weakly in } (L^2(\Omega))^N, \end{cases} \quad (1.34)$$

where  $A^0$  is the constant matrix defined by (see [11])

$$A^0 \lambda = \frac{1}{|Y|} \int_Y A(y, t) \nabla_y \widehat{w}_\lambda(y) \, dy, \quad \forall \lambda \in \mathbb{R}^N, \quad (1.35)$$

or equivalently,

$${}^tA^0\lambda = \frac{1}{|Y|} \int_Y {}^tA(y, t) \nabla_y w_\lambda(y) dy, \quad \forall \lambda \in \mathbb{R}^N. \quad (1.36)$$

The functions  $\widehat{w}_\lambda$  and  $w_\lambda$  are respectively the unique solutions of the following cell problems:

$$\begin{cases} -\operatorname{div}(A(y)\nabla\widehat{w}_\lambda) = 0 & \text{in } Y, \\ \widehat{w}_\lambda - \lambda \cdot y & Y\text{-periodic} \\ \mathcal{M}_Y(\widehat{w}_\lambda - \lambda \cdot y) = 0, \end{cases} \quad (1.37)$$

and

$$\begin{cases} -\operatorname{div}({}^tA(y)\nabla w_\lambda) = 0 & \text{in } Y, \\ w_\lambda - \lambda \cdot y & Y\text{-periodic} \\ \mathcal{M}_Y(w_\lambda - \lambda \cdot y) = 0, \end{cases} \quad (1.38)$$

for every  $\lambda \in \mathbb{R}^N$ .

The function  $u^0$  is the unique solution in  $H_0^1(\Omega)$  of the homogenized problem

$$\begin{cases} -\operatorname{div}(A^0\nabla u^0) = f & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.39)$$

**Remark 1.16.** *The uniqueness of the function  $u^0$  implies that the convergences in (1.34) applies to the whole sequence and not just for a subsequence. This is very important in homogenization since this means that the function  $u^0$  approximates the thermal distribution in  $\Omega$  for whatever value of  $\varepsilon$  we choose.*

In the sequel, we discuss briefly some methods, originally developed to prove this last theorem. For more in-depth discussion of these methods, one can see [33].

### 1.2.1 Multiple-scales method

The multiple-scales method, which is also known as asymptotic expansion method, is a non-rigorous method of homogenization introduced in [11, 12] (see also [77]). It features the formal asymptotic expansion of the solution  $u^\varepsilon$  of (1.30) (see (1.40)). It must be mentioned first that the problem can be characterized in two scales: the macroscopic scale, which gives the position of a point on the domain  $\Omega$ , and the microscopic scale  $y = \frac{x}{\varepsilon}$ , which locates a point in the cell  $Y$  and describes the micro-oscillations.



This leads to the consideration of the following formal expansion (which is also called ansatz) for  $u^\varepsilon$ :

$$u^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots = \sum_{i=0}^{\infty} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right), \quad (1.40)$$

where the functions  $u_i = u_i(x, y)$  are functions from  $\Omega \times Y$  to  $\mathbb{R}$  and are  $Y$ -periodic with respect to the second variable  $y$ . It is clear that  $u_0$  is the limit of  $u^\varepsilon$  as  $\varepsilon$  tends to 0. The idea of this method is to substitute this formal expansion of  $u^\varepsilon$  in (1.28) and then solve for the functions  $u_j$ ,  $j \in \mathbb{N} \cup \{0\}$ . We need a few preliminaries before we can further discuss this method.

Let  $\Psi = \Psi(x, y)$  be a function defined on  $\mathbb{R}^N \times \mathbb{R}^N$ . Let  $\Psi^\varepsilon$  be defined by

$$\Psi^\varepsilon(x) = \Psi\left(x, \frac{x}{\varepsilon}\right).$$

Define the operator  $\mathcal{A}^\varepsilon$  by

$$\mathcal{A}^\varepsilon = -\operatorname{div}(A^\varepsilon \nabla) = -\sum_{1 \leq i, j \leq N} \frac{\partial}{\partial x_i} \left( a_{ij}^\varepsilon \frac{\partial}{\partial x_j} \right).$$

Note that

$$\frac{\partial \Psi^\varepsilon}{\partial x_i}(x) = \frac{1}{\varepsilon} \frac{\partial \Psi}{\partial y_i}\left(x, \frac{x}{\varepsilon}\right) + \frac{\partial \Psi}{\partial x_i}\left(x, \frac{x}{\varepsilon}\right).$$

Then, we can write

$$\mathcal{A}^\varepsilon \Psi^\varepsilon(x) = [(\varepsilon^{-2} \mathcal{A}_0 + \varepsilon^{-1} \mathcal{A}_1 + \mathcal{A}_2) \Psi]\left(x, \frac{x}{\varepsilon}\right),$$

where

$$\begin{aligned} \mathcal{A}_0 &= -\sum_{1 \leq i, j \leq N} \frac{\partial}{\partial y_i} \left( a_{ij}(y) \frac{\partial}{\partial y_j} \right), \\ \mathcal{A}_1 &= -\sum_{1 \leq i, j \leq N} \frac{\partial}{\partial x_i} \left( a_{ij}(y) \frac{\partial}{\partial y_j} \right) - \sum_{1 \leq i, j \leq N} \frac{\partial}{\partial y_i} \left( a_{ij}(y) \frac{\partial}{\partial x_j} \right), \\ \mathcal{A}_2 &= -\sum_{1 \leq i, j \leq N} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right). \end{aligned}$$

It then follows, using (1.40), that we have the following infinite systems of equations:

$$\begin{cases} \mathcal{A}_0 u_0 = 0 & \text{in } Y \\ u_0 & Y\text{-periodic in } y, \end{cases}$$

$$\begin{cases} \mathcal{A}_0 u_1 = -\mathcal{A}_1 u_0 & \text{in } Y \\ u_1 & Y\text{-periodic in } y, \end{cases}$$

$$\begin{cases} \mathcal{A}_0 u_2 = f - \mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 & \text{in } Y \\ u_2 & Y\text{-periodic in } y, \end{cases}$$

and

$$\begin{cases} \mathcal{A}_0 u_{s+2} = -\mathcal{A}_1 u_{s+1} - \mathcal{A}_2 u_s & \text{in } Y \\ u_{s+2} & Y\text{-periodic in } y, \end{cases}$$

for  $s \geq 1$ . The following theorem is the result of solving these systems. One can refer to [33] for details.

**Theorem 1.17.** *Let  $f \in L^2(\Omega)$  and  $u^\varepsilon$  be the unique solution of (1.28). Then  $u^\varepsilon$  can be written as*

$$u^\varepsilon = u^0 - \varepsilon \sum_{k=1}^N \widehat{\chi}_k \left( \frac{x}{\varepsilon} \right) \frac{\partial u^0}{\partial x_k} + \varepsilon^2 \sum_{k,l=1}^N \widehat{\theta}^{kl} \left( \frac{x}{\varepsilon} \right) \frac{\partial^2 u^0}{\partial x_k \partial x_l} + \dots,$$

where  $u^0$  is the solution to (1.39),  $\widehat{\chi}_k = y_k - \widehat{w}_{e_k}$ , with  $\widehat{w}_{e_k}$  the unique solution of the cell problem (1.37),  $\{e_k\}$  is the canonical basis of  $\mathbb{R}^N$ , and  $\widehat{\theta}^{kl}$  is the solution of

$$\begin{cases} -\operatorname{div}(A(y)\nabla \widehat{\theta}^{kl}) = -a_{kl} - \sum_{i,j=1}^N \frac{\partial(a_{ij}\delta_{ki}\widehat{\chi}_l)}{\partial y_i} - \sum_{j=1}^N a_{kj} \frac{\partial(\widehat{\chi}_l - y_l)}{\partial y_j} & \text{in } Y, \\ \widehat{\theta}^{kl} & Y\text{-periodic, } \mathcal{M}_Y(\widehat{\theta}^{kl}) = 0. \end{cases}$$

As mentioned earlier, this is a non-rigorous method. In particular, there is no justification as to why we can write  $u^\varepsilon$  as the formal expansion (1.40). However, the results of this method is justified by the subsequent methods.

## 1.2.2 Tartar's method of oscillating test functions

Tartar's method of oscillating test functions is introduced by L. Tartar in [82, 83]. As the name suggests, this method makes use of a suitable oscillating test function to remove the problem of passing to the limit of an integral with a product of two weakly convergent sequences. This is a more mathematically rigorous method than the multiple-scales method and it also confirms the results obtained in the previous method.

We now present a sketch of the proof of Theorem 1.15 using this method (the details of which can be found in Section 8 of [33]).

We first recall that we have convergence (1.33), and it follows that we have for a subsequence

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega) \\ u^\varepsilon \rightarrow u^0 & \text{strongly in } L^2(\Omega) \\ \nabla u^\varepsilon \rightharpoonup \nabla u^0 & \text{weakly in } (L^2(\Omega))^N. \end{cases} \quad (1.41)$$

From (1.29) and (1.31), we deduce that

$$\|A^\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega)} \leq \frac{\beta}{\alpha} \|f\|_{L^2(\Omega)}.$$

This implies that there exists  $\xi^0 \in (L^2(\Omega))^N$  such that

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup \xi^0 \quad \text{weakly in } (L^2(\Omega))^N. \quad (1.42)$$

This allows us to pass to the limit of the integral in (1.30) as  $\varepsilon$  tends to 0, which gives

$$\int_{\Omega} \xi^0 \nabla v \, dx = \int_{\Omega} f v \, dx,$$

for any  $v \in H_0^1(\Omega)$ . Hence, to prove Theorem 1.15, it is enough to prove that

$$\xi^0 = A^0 \nabla u^0,$$

where  $A^0$  is defined in (1.35).

Define

$$w_\lambda^\varepsilon(x) = \varepsilon w_\lambda \left( \frac{x}{\varepsilon} \right) = \lambda \cdot x - \varepsilon \chi_\lambda \left( \frac{x}{\varepsilon} \right),$$

where  $w_\lambda$  is the solution of (1.38) and  $\chi_\lambda$  is the solution of

$$\begin{cases} \text{Find } \chi_\lambda \in H_{per}^1(Y) \text{ with } \mathcal{M}_Y(\chi_\lambda) = 0 \text{ such that} \\ \int_Y {}^t A(y) \nabla \chi_\lambda \nabla v \, dy = \int_Y {}^t A(y) \lambda \nabla v \, dy, \quad \forall v \in H_{per}^1(Y). \end{cases} \quad (1.43)$$

Functions taking the form similar to  $w_\lambda^\varepsilon$  are the main feature of this method.

By classical results, it can be shown that

$$\nabla_x w_\lambda^\varepsilon \rightharpoonup \lambda \quad \text{weakly in } L^2(\Omega).$$

By compact embedding of  $H^1(\Omega)$  in  $L^2(\Omega)$ , we have

$$\begin{cases} w_\lambda^\varepsilon \rightharpoonup \lambda \cdot x & \text{weakly in } H^1(\Omega) \\ w_\lambda^\varepsilon \rightarrow \lambda \cdot x & \text{strongly in } L^2(\Omega). \end{cases} \quad (1.44)$$

Now, let  $\eta_\lambda^\varepsilon$  be the vector function

$$\eta_\lambda^\varepsilon = {}^t A^\varepsilon \nabla w_\lambda^\varepsilon = ({}^t A \nabla_y w_\lambda) \left( \frac{x}{\varepsilon} \right).$$

Note that  ${}^t A \nabla_y w_\lambda$  is  $Y$ -periodic, and it follows from classical results on functions of the form  $\eta_\lambda^\varepsilon$  that

$$\eta_\lambda^\varepsilon \rightharpoonup \mathcal{M}_Y({}^t A \nabla w_\lambda) = {}^t A^0 \lambda \quad \text{weakly in } (L^2(\Omega))^N. \quad (1.45)$$

Using (1.43), it can be shown that

$$\int_\Omega \eta_\lambda^\varepsilon \nabla v \, dx = 0, \quad \forall v \in H_0^1(\Omega). \quad (1.46)$$

Note also that

$$A^\varepsilon \nabla u^\varepsilon \nabla w_\lambda^\varepsilon = {}^t A^\varepsilon \nabla w_\lambda^\varepsilon \nabla u^\varepsilon = \eta_\lambda^\varepsilon \nabla u^\varepsilon.$$

Let  $\varphi \in \mathcal{D}(\Omega)$ . Choose  $\varphi w_\lambda^\varepsilon$  as test function in (1.38) and  $\varphi u^\varepsilon$  in (1.46). We then obtain, by subtracting the resulting equations,

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \nabla \varphi w_\lambda^\varepsilon \, dx - \int_\Omega \eta_\lambda^\varepsilon \nabla \varphi u^\varepsilon \, dx = \int_\Omega f \varphi w_\lambda^\varepsilon \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Note that since the integrals with a product of two weakly convergent sequences cancel, we don't have any problem passing to the limit of this equation. Using (1.41), (1.42), (1.44), and (1.45), and doing some calculations, we have

$$\int_\Omega \xi^0 \lambda \varphi \, dx = \int_\Omega {}^t A^0 \lambda \nabla u^0 \varphi \, dx, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

By the arbitrary nature of  $\lambda$ , we have the desired result, that is

$$\xi^0 = A^0 \nabla u^0.$$

### 1.2.3 Two-scale convergence method

The two-scale convergence method was introduced by Nguetseng in [72], and further developed by Allaire in [1, 2]. This method introduced a new type of convergence which is called the two-scale convergence. This kind of convergence is concerned with the limit of the integrals of the form

$$\int_\Omega v^\varepsilon(x) \psi \left( x, \frac{x}{\varepsilon} \right) \, dx,$$

where  $\{v^\varepsilon\}$  is a sequence in  $L^2(\Omega)$  and  $\psi(x, y)$  is a smooth  $Y$ -periodic function.

Before we present the definition of this convergence, we need to define the appropriate space for the function  $\psi$ . The space  $\mathcal{D}(\Omega; C_{per}^\infty(Y))$  is the space of measurable functions defined from  $\Omega \times \mathbb{R}^N$  to  $\mathbb{R}$  such that for any fixed  $x \in \Omega$ ,  $u(x, \cdot) \in C_{per}^\infty(Y)$ , and the map  $x \in \Omega \mapsto u(x, \cdot) \in C_{per}^\infty(Y)$  is indefinitely differentiable with a compact support inside  $\Omega$ .

We now have the following definition of the two-scale convergence:

**Definition 1.18.** *Let  $\{v^\varepsilon\}$  be a sequence of functions in  $L^2(\Omega)$ . The sequence  $\{v^\varepsilon\}$  is said to be two-scale convergent to a function  $v_0 = v_0(x, y) \in L^2(\Omega \times Y)$  if*

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} v^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \frac{1}{|Y|} \int_{\Omega} \int_Y v_0(x, y) \psi(x, y) dy dx, \quad (1.47)$$

for any  $\psi = \psi(x, y) \in \mathcal{D}(\Omega; C_{per}^\infty(Y))$ .

We remark that the two-scale convergence justifies the formal asymptotic expansion that was introduced in the multiple-scale method (see Remark 9.6 of [33] for more details).

The following theorems are two of the main results of this method. These compactness results allow us to pass to the limit when proving the homogenization results.

**Theorem 1.19.** *Let  $\{v^\varepsilon\}$  be a bounded sequence in  $L^2(\Omega)$ . Then, we can extract a subsequence  $\{v^{\varepsilon'}\}$  such that  $\{v^{\varepsilon'}\}$  two-scale converges to a function  $v_0 \in L^2(\Omega \times Y)$ .*

**Theorem 1.20.** *Let  $\{v^\varepsilon\}$  be a sequence of functions in  $H^1(\Omega)$  such that*

$$v^\varepsilon \rightharpoonup v_0 \quad \text{weakly in } H^1(\Omega).$$

*Then  $\{v^\varepsilon\}$  two-scale converges to  $v_0$ , and there exist a subsequence  $\varepsilon'$  and  $v_1 = v_1(x, y) \in L^2(\Omega \times Y)$  such that*

$$\nabla v^{\varepsilon'} \quad \text{two-scale converges to} \quad \nabla_x v_0 + \nabla_y v_1.$$

We now give the idea of the proof of Theorem 1.15 using the two-scale convergence method.

Note that we still have the convergences (1.41), specifically the first two. Then, from Theorem 1.20,

$$u^\varepsilon \quad \text{two-scale converges to} \quad u^0.$$

In addition, we can find  $u_1 = u_1(x, y) \in L^2(\Omega \times Y)$  such that

$$\nabla u^\varepsilon \quad \text{two-scale converges to} \quad \nabla_x u^0 + \nabla_y u_1.$$

We choose  $v_0(x) + \varepsilon v\left(x, \frac{x}{\varepsilon}\right)$  as test function in (1.30), where  $v_0 \in \mathcal{D}(\Omega)$  and  $v_1 = v_1(x, y) \in \mathcal{D}(\Omega; C_{per}^\infty(Y))$ . We then have

$$\begin{aligned} & \int_{\Omega} A^\varepsilon \nabla u^\varepsilon \left[ \nabla v_0(x) + \varepsilon \nabla_x v_1\left(x, \frac{x}{\varepsilon}\right) + \nabla_y v_1\left(x, \frac{x}{\varepsilon}\right) \right] dx \\ &= \int_{\Omega} f\left(v_0(x) + v_1\left(x, \frac{x}{\varepsilon}\right)\right) dx. \end{aligned}$$

By using the two-scale convergence of  $\nabla u^\varepsilon$  and other classical results, we can pass to the limit of both of these integrals to obtain

$$\frac{1}{|Y|} \int_{\Omega} \int_Y A(y) (\nabla u^0 + \nabla_y u_1(x, y)) (\nabla v_0(x) + \nabla_y v_1(x, y)) dx dy = \int_{\Omega} f v_0 dx, \quad (1.48)$$

for any  $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega \times Y)$ . This variational formulation can be shown to have a unique solution  $(u^0, u_1) \in H_0^1 \times L^2(\Omega \times Y)$  by the Lax-Milgram Theorem.

Choosing first  $(0, v_1)$  and then  $(v_0, 0)$  as test functions in (1.48), it can be shown that (1.48) is equivalent to the problem

$$\begin{cases} -\operatorname{div}_y(A(y)\nabla_y u_1(x, y)) = \operatorname{div}_y(A(y)\nabla u^0) & \text{in } \Omega \times Y \\ -\operatorname{div}_x \left[ \int_Y A(y) (\nabla u^0(x) + \nabla_y u_1(x, y)) dy \right] = |Y|f & \text{in } \Omega \\ u^0 = 0 & \text{on } \partial\Omega \\ u_1(x, \cdot) & Y\text{-periodic.} \end{cases}$$

By performing some calculations, it follows that  $u_1$  can be written as

$$u_1(x, y) = - \sum_{j=1}^N \widehat{\chi}_j(y) \frac{\partial u^0}{\partial x_j}, \quad (1.49)$$

where, for  $j = 1, \dots, n$ ,  $\widehat{\chi}_j$  satisfies

$$\begin{cases} \text{Find } \widehat{\chi}_j \in H_{per}^1(Y) \text{ with } \mathcal{M}_Y(\widehat{\chi}_j) = 0 \text{ such that} \\ \int_Y A(y) \nabla \widehat{\chi}_j \nabla v dy = \int_Y A(y) e_j \nabla v dy, \quad \forall v \in H_{per}^1(Y), \end{cases}$$

with  $\{e_j\}_{j=1}^N$  the canonical basis of  $\mathbb{R}^N$ .

Replacing  $u_1$  in (1.48) with its rewritten form in (1.49), and performing some calculations, we can deduce that  $u^0$  satisfies (1.39).

## 1.2.4 The periodic unfolding method

The periodic unfolding method was introduced by D. Cioranescu, A. Damlamian and G. Griso in [30] and further developed in [31] for the study of periodic homogenization in classical domains (that is, with no holes or interfacial resistance). It was then later developed for perforated domains in [29] and for two-component domains in [46].

This method relies heavily on an operator which is called the unfolding operator and usually denoted by  $\mathcal{T}_\varepsilon$ . This operator, which sends an  $L^p(\Omega)$  function to an  $L^p(\Omega \times Y)$  function, has a lot of interesting properties to help in passing to the limit. One of which is it transforms an integral over  $\Omega$  to an integral over  $\Omega \times Y$ . The resulting integral over  $\Omega \times Y$  is then relatively easier to manage than the original integral. One can refer to [32] for a very detailed discussion on the periodic unfolding method.

In the sequel, we will discuss the proof of Theorem 1.15 using the periodic unfolding method on classical domains. As we will be using this method for this thesis, we will give more details in the proof using this method. In addition, since this thesis focuses on two-component domains, we will also say some few words about the Periodic Unfolding Method on two-component domains.

### The periodic unfolding method for classical domains

Let  $B = \{b_1, b_2, \dots, b_N\}$  be a basis for  $\mathbb{R}^N$ . In this method, one can define a more general reference cell  $Y$  as

$$Y = \left\{ \ell \in \mathbb{R}^N : \ell = \sum_{i=1}^N \ell_i b_i, (\ell_1, \dots, \ell_N) \in (0, 1)^N \right\}. \quad (1.50)$$

We now denote for almost every  $z \in \mathbb{R}^N$ ,

$$[z]_Y = \sum_{j=1}^N l_j b_j, \quad l_j \in \mathbb{Z}, \quad j = 1, \dots, N,$$

the unique integer combination such that  $z - [z]_Y \in Y$ . We set  $\{z\}_Y = z - [z]_Y \in Y$ , for a.e.  $z \in \mathbb{R}^N$ . Then we can write

$$z = \{z\}_Y + [z]_Y, \quad \text{for a.e. } z \in \mathbb{R}^N,$$

that is, we can decompose  $z$  as the sum of  $\{z\}_Y$ , which is its local position in the cell  $Y$ , and  $[z]_Y$ , which gives the location of the cell where  $z$  is. Then, writing this decomposition for  $z = \frac{x}{\varepsilon}$ , for any  $x \in \mathbb{R}^N$ , we have

$$\frac{x}{\varepsilon} = \left\{ \frac{x}{\varepsilon} \right\}_Y + \left[ \frac{x}{\varepsilon} \right]_Y.$$

Now, we define the following sets:

- $G = \left\{ \xi \in \mathbb{R}^N : \xi = \sum_{i=1}^N k_i b_i, (k_1, \dots, k_N) \in \mathbb{Z}^N \right\},$
- $\Xi_\varepsilon = \{\xi \in G : \varepsilon(\xi + Y) \subset \Omega\},$
- $\widehat{\Omega}_\varepsilon = \text{interior} \left\{ \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + \overline{Y}) \subset \Omega \right\},$
- $\Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon.$

Clearly, from the definition of  $\Xi_\varepsilon$ , the set  $\widehat{\Omega}_\varepsilon$  is the interior of the largest union of cells  $\varepsilon(\xi + \overline{Y})$  such that the cells  $\varepsilon(\xi + Y)$  are entirely inside  $\Omega$ . We can also see that its complement  $\Lambda_\varepsilon$  is the subset of  $\Omega$  that contains the parts of the cells  $\varepsilon(\xi + \overline{Y})$  that intersects the boundary  $\partial\Omega$ .

We now present the following definition of the unfolding operator  $\mathcal{T}_\varepsilon$  for classical domains.

**Definition 1.21.** *Let  $\varphi$  be a measurable function. The unfolding operator  $\mathcal{T}_\varepsilon$  is given by*

$$\mathcal{T}_\varepsilon(\varphi)(x, y) = \begin{cases} \varphi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases} \quad (1.51)$$

It must be noted that the unfolding operator  $\mathcal{T}_\varepsilon$  doubles the dimension of the space, and all the oscillations are in the second variable  $y$ .

We also need the following definition for the mean value operator.

**Definition 1.22.** *For  $p \in [1, \infty]$ , the mean value operator  $\mathcal{M}_Y$  is defined from  $L^p(\Omega \times Y)$  to  $L^p(\Omega)$ , and is given by*

$$\mathcal{M}_Y(\varphi)(x) = \frac{1}{|Y|} \int_Y \varphi(x, y) dy \quad \text{for a.e. } x \in \Omega. \quad (1.52)$$

We now present the following proposition which states the interesting properties of the unfolding operator  $\mathcal{T}_\varepsilon$ . These properties are very helpful in passing to the limit during the homogenization process. The proof of this proposition can be found in [31, 32].

**Proposition 1.23** ([31, 32]). *Let  $p \in [1, \infty)$ .*

1.  $\mathcal{T}_\varepsilon$  is a linear and continuous operator from  $L^p(\Omega)$  to  $L^p(\Omega \times Y)$ .



2. For any measurable functions  $\varphi$  and  $\psi$ , we have

$$\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi).$$

3. For any  $\varphi \in L^p(\Omega)$ ,

$$\mathcal{T}_\varepsilon(\varphi) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y).$$

4. For any  $\varphi \in L^1(\Omega)$ , we have

$$\begin{aligned} \int_{\widehat{\Omega}_\varepsilon} \varphi(x) dx &= \int_{\Omega} \varphi(x) dx - \int_{\Lambda_\varepsilon} \varphi(x) dx \\ &= \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\varphi)(x, y) dx dy. \end{aligned}$$

5. For any  $\varphi \in L^p(\Omega)$ ,

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^p(\Omega \times Y)} \leq |Y|^{1/p} \|\varphi\|_{L^p(\Omega)}.$$

For the case  $p = \infty$ ,  $\|\mathcal{T}_\varepsilon(\varphi)\|_{L^\infty(\Omega \times Y)} \leq \|\varphi\|_{L^\infty(\Omega)}$ .

6. Let  $\{w_\varepsilon\}$  be a bounded sequence in  $L^p(\Omega)$ . It follows from the previous item that the sequence  $\{\mathcal{T}_\varepsilon(w_\varepsilon)\}$  is bounded in  $L^p(\Omega \times Y)$  (which also holds when  $p = +\infty$ ). Suppose further that

$$\mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y),$$

then

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w}) \quad \text{weakly in } L^p(\Omega).$$

If  $p = +\infty$  and  $\mathcal{T}_\varepsilon(w_\varepsilon)$  converges weakly-\* to  $\widehat{w}$  in  $L^\infty(\Omega \times Y)$ , then

$$w_\varepsilon \rightharpoonup \mathcal{M}_Y(\widehat{w}) \quad \text{weakly-* in } L^\infty(\Omega).$$

7. Let  $\{w_\varepsilon\}$  be a sequence in  $L^p(\Omega)$  such that for some  $w \in L^p(\Omega)$ ,

$$w_\varepsilon \longrightarrow w \quad \text{strongly in } L^p(\Omega).$$

Then

$$\mathcal{T}_\varepsilon(w_\varepsilon) \longrightarrow w \quad \text{strongly in } L^p(\Omega \times Y).$$

The following proposition gives an interesting result on the action of the unfolding operator to a highly oscillating function. This is very helpful in periodic homogenization as we deal with highly oscillating coefficients.

**Proposition 1.24.** *Let  $f$  be a function defined on  $Y$ , then extended by  $Y$ -periodicity to the whole of  $\mathbb{R}^N$ . Define the sequence of functions  $\{f^\varepsilon\}$  by*

$$f^\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Then

$$\mathcal{T}_\varepsilon(f^\varepsilon|_\Omega)(x, y) = \begin{cases} f(y) & \text{for a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{for a.e. } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Let  $p \in [1, \infty)$ . If  $f \in L^p(Y)$ , then

$$\mathcal{T}_\varepsilon(f^\varepsilon|_\Omega) \longrightarrow f \quad \text{strongly in } L^p(\Omega \times Y).$$

It is worth noting that there is a relationship between the two-scale convergence of a sequence and the weak convergence of the unfolded sequence. This is stated in the following proposition, which is proved in [31].

**Proposition 1.25** ([31]). *Let  $\{w_\varepsilon\}$  be a bounded sequence in  $L^p(\Omega)$ , with  $p \in (1, \infty)$ , and  $w \in L^p(\Omega)$ . The following statements are equivalent:*

1.  $\{\mathcal{T}_\varepsilon(w_\varepsilon)\}$  converges weakly to  $w$  in  $L^p(\Omega)$ .
2.  $\{w_\varepsilon\}$  two-scale converges to  $w$ .

Now, we present a proposition that gives the convergence results for unfolded sequence of functions and unfolded sequence of gradients of these functions. These convergences are essential to the homogenization process when using this method.

**Proposition 1.26** ([31]). *Let  $p \in (1, +\infty)$ . Let  $\{w_\varepsilon\}$  be a bounded sequence in  $W^{1,p}(\Omega)$ . Then there exist a subsequence (which will still be denoted by  $\varepsilon$ ),  $w \in W^{1,p}(\Omega)$ , and  $\widehat{w} \in L^p(\Omega; W_{per}^{1,p}(Y))$  such that*

$$\begin{cases} \mathcal{T}_\varepsilon(w_\varepsilon) \rightharpoonup w & \text{weakly in } L^p(\Omega; W^{1,p}(Y)) \\ \mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla w + \nabla_y \widehat{w} & \text{weakly in } L^p(\Omega \times Y). \end{cases}$$

Moreover, if  $p = +\infty$ , then we have the same convergences in the weak-\* topology.

We are now ready to prove Theorem 1.15 using the periodic unfolding method.

Note that we still have estimate (1.31) and convergence (1.33). Then from Proposition 1.23 and Proposition 1.26, we can extract a subsequence

(which we will still denote by  $\varepsilon$ ), such that for some  $u^0 \in H_0^1(\Omega)$ , and  $u_1 \in L^2(\Omega; H_{per}^1(Y))$ , we have the following convergences:

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 & \text{weakly in } H_0^1(\Omega), \\ \mathcal{T}_\varepsilon(u^\varepsilon) \rightharpoonup u^0 & \text{weakly in } L^2(\Omega; H^1(Y)), \\ \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \rightharpoonup \nabla u^0 + \nabla_y u_1 & \text{weakly in } L^2(\Omega \times Y). \end{cases} \quad (1.53)$$

Now, we choose a test function  $\varphi \in H_0^1(\Omega)$  in the variational formulation (1.30) and we obtain

$$\int_{\Omega} A^\varepsilon(x) \nabla u^\varepsilon \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx.$$

Using the properties of the unfolding operator in Proposition 1.23 and the convergences in (1.53), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \nabla \varphi \, dx &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon\left(A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \nabla \varphi\right) \, dx \, dy \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} A(y) \mathcal{T}_\varepsilon(\nabla u^\varepsilon) \mathcal{T}_\varepsilon(\nabla \varphi) \, dx \, dy \\ &= \frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u^0 + \nabla_y u_1) \nabla \varphi \, dx \, dy. \end{aligned}$$

Moreover, by Proposition 1.23 and the fact that both functions  $f$  and  $\varphi$  are independent of  $y$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} f \varphi \, dx &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(f \varphi) \, dx \, dy = \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(f) \mathcal{T}_\varepsilon(\varphi) \, dx \, dy \\ &= \frac{1}{|Y|} \int_{\Omega \times Y} f \varphi \, dx \, dy = \int_{\Omega} f \varphi \, dx. \end{aligned}$$

It follows that

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u^0 + \nabla_y u_1) \nabla \varphi \, dx \, dy = \int_{\Omega} f \varphi \, dx, \quad (1.54)$$

for all  $\varphi \in H_0^1(\Omega)$ .

We now consider the function  $v_\varepsilon(x) = \varepsilon \varphi(x) \psi(\frac{x}{\varepsilon})$ , where  $\varphi \in \mathcal{D}(\Omega)$  and  $\psi \in H_{per}^1(Y)$ . By the properties of the unfolding operator in Proposition 1.23, we have

$$\mathcal{T}_\varepsilon(v_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\varphi) \psi(y), \quad \mathcal{T}_\varepsilon(\nabla v_\varepsilon) = \varepsilon \mathcal{T}_\varepsilon(\nabla \varphi) \psi + \mathcal{T}_\varepsilon(\varphi) \nabla_y \psi.$$

Then, as  $\varepsilon$  tends to 0, we obtain

$$\begin{cases} \mathcal{T}_\varepsilon(v_\varepsilon) \longrightarrow 0 & \text{strongly in } L^2(\Omega \times Y), \\ \mathcal{T}_\varepsilon(\nabla v_\varepsilon) \longrightarrow \varphi \nabla_y \psi & \text{strongly in } L^2(\Omega \times Y). \end{cases} \quad (1.55)$$

Choosing  $v_\varepsilon$  as a test function in (1.30) gives

$$\int_{\Omega} A^\varepsilon(x) \nabla u^\varepsilon \nabla v_\varepsilon \, dx = \int_{\Omega} f v_\varepsilon \, dx.$$

Similar to what was done above, we again use Proposition 1.23 to pass to the limit of the left-hand side of this equation. Together with the convergences in (1.55), we will obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} A\left(\frac{x}{\varepsilon}\right) \nabla u^\varepsilon \nabla v_\varepsilon \, dx = \frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u^0 + \nabla_y u_1) \varphi \nabla_y \psi \, dx \, dy$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f v_\varepsilon \, dx = 0.$$

This gives

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u^0 + \nabla_y u_1) \varphi \nabla_y \psi \, dx \, dy = 0.$$

Adding this last equation to (1.54), and using the density of the tensor product  $\mathcal{D}(\Omega) \otimes H_{per}^1(Y)$  in  $L^2(\Omega; H_{per}^1(Y))$ , we obtain

$$\frac{1}{|Y|} \int_{\Omega \times Y} A(y) (\nabla u^0 + \nabla_y u_1) (\nabla \varphi + \nabla_y \Phi) \, dx \, dy = \int_{\Omega} f \varphi \, dx,$$

for any  $\varphi \in H_0^1(\Omega)$  and for all  $\Phi \in L^2(\Omega; H_{per}^1(Y))$ . This is equivalent to the formulation (1.48) obtained in the previous section by the two-scale convergence method. To obtain the homogenized problem (1.39) that is satisfied by  $u^0$ , we argue as in the proof above. This concludes the proof.

### The periodic unfolding method for an imperfect interface

We briefly discuss here the method for a two-component domain, which is the domain considered in this thesis. The main difference of a classical domain and a two-component domain is the jump of the solution on the interface of the two materials. This jump may be caused by an imperfect contact in the interface.

The method will be presented in more detail in Chapter 5, which is dedicated to the homogenization. One can also see [45, 46] for a full discussion of this method.

As the name suggests, a two-component domain is composed of two components (or two different materials). One component can be considered as a perforated domain and the other component consists of disconnected  $\varepsilon$ -periodic domains. Because of this, there are two unfolding operators defined in this method, the unfolding operator for perforated domains (one can see [29, 32] for a detailed presentation of this operator) and the unfolding operator for classical domains (which was briefly discussed above).

### 1.2.5 Our homogenization results

In this part, we study the homogenization corresponding to the problem presented in the first part.

To properly define the two-component domain  $\Omega$ , we first have to define the reference cell  $Y$  (which is usually taken as a parallelepiped in  $\mathbb{R}^N$ ,  $N \geq 2$ ). The cell  $Y$  also has two components,  $Y_1$  and  $Y_2$ , with the interface  $\Gamma$  between them. The open set  $Y_2$  is such that  $\overline{Y_2} \subset Y$  with a Lipschitz-continuous boundary  $\Gamma$ , and  $Y_1 = Y \setminus \overline{Y_2}$ .

Now, the two-component domain, denoted by  $\Omega$ , is an open bounded connected set in  $\mathbb{R}^N$  with a Lipschitz-continuous boundary  $\partial\Omega$ . Let  $\varepsilon$  be a parameter from a positive sequence that goes to 0. Then we can write  $\Omega$  as the disjoint union

$$\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon,$$

where  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$  are the two components of  $\Omega$  and  $\Gamma^\varepsilon$  is the interface between them. In particular,  $\Omega_2^\varepsilon$  is a disjoint union of  $\varepsilon$ -periodic translated sets  $\varepsilon Y_2$  with  $\overline{\Omega_2^\varepsilon} \subset \Omega$  and  $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ , and  $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$  (see figure below).

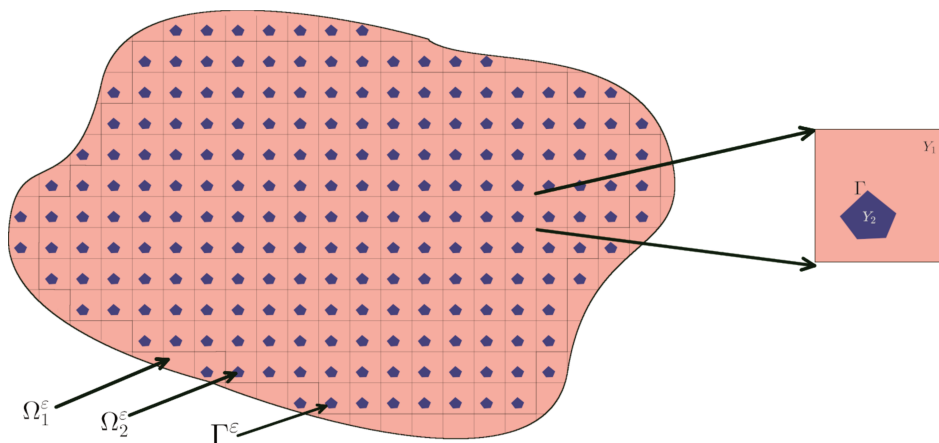


Figure 1.6: The two-component domain  $\Omega$  and the reference cell  $Y$ .

We study the asymptotic behavior of the following quasilinear elliptic problem in  $\Omega$ :

$$\left\{ \begin{array}{ll} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_1^\varepsilon\right)\nabla u_1^\varepsilon\right) = f & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u_2^\varepsilon\right)\nabla u_2^\varepsilon\right) = f & \text{in } \Omega_2^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_1^\varepsilon\right)\nabla u_1^\varepsilon\right)\nu_1^\varepsilon = \left(A\left(\frac{x}{\varepsilon}, u_2^\varepsilon\right)\nabla u_2^\varepsilon\right)\nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u_1^\varepsilon\right)\nabla u_1^\varepsilon\right)\nu_1^\varepsilon = -\varepsilon^{-1}h\left(\frac{x}{\varepsilon}\right)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{array} \right. \quad (1.56)$$

where  $\nu_1^\varepsilon$  is the unit outward normal to  $\Omega_1^\varepsilon$ ,  $f \in L^1(\Omega)$ ,  $h$  is a  $Y$ -periodic  $L^\infty(\Gamma)$  function and  $A(y, t)$  is a coercive matrix field with restricted growth assumption which is also  $Y$ -periodic.

In the literature, in the study of homogenization of (1.56) for  $f \in L^2(\Omega)$ , the proportionality assumption on  $\Gamma^\varepsilon$  depends on  $\varepsilon^\gamma$  (instead of  $\varepsilon^{-1}$ ), where  $\gamma \leq 1$ . The homogenization then is done in three cases:  $\gamma \in (-1, 1]$ ,  $\gamma = -1$ , and  $\gamma \in (-\infty, -1)$ . The major difference between these cases is the corresponding cell problem. We only consider here the case  $\gamma = -1$ , which has the particularity that also the cell problem presents a jump on the reference interface.

Before we obtain homogenization results, we first study the following periodic elliptic cell problem, related to the homogenized problem of (1.56):

$$\left\{ \begin{array}{ll} -\operatorname{div}(A\nabla\chi_1^\lambda) = G_1^\lambda & \text{in } Y_1, \\ -\operatorname{div}(A\nabla\chi_2^\lambda) = G_2^\lambda & \text{in } Y_2, \\ A\nabla\chi_1^\lambda \cdot n_1 = -A\nabla\chi_2^\lambda \cdot n_2 & \text{on } \Gamma, \\ A\nabla\chi_1^\lambda \cdot n_1 = -h(y)(\chi_1^\lambda - \chi_2^\lambda) & \text{on } \Gamma, \\ \chi_1^\lambda \text{ } Y\text{-periodic,} \\ \mathcal{M}_\Gamma(\chi_1^\lambda) = 0, \end{array} \right. \quad (1.57)$$

where  $\lambda \in \mathbb{R}^N$  and  $G_i^\lambda$  is defined by

$$\langle G_i^\lambda, v \rangle = \int_{Y_i} A\lambda\nabla v \, dy, \quad \forall v \in H^1(Y_i), \quad i = 1, 2, \quad (1.58)$$

which belongs to  $(H^1(Y_i))'$ .

We are interested in the properties of the solution of (1.57), which were not studied in the literature. This is motivated by the fact that the corresponding homogenized matrix  $A^0$  corresponding to (1.56) is defined with

through the solution  $\chi^\lambda$  of (1.57). More precisely, the definition of  $A^0$  is as follows:

$$A^0(t) = A_1^0(t) + A_2^0(t), \quad (1.59)$$

where

$$A_i^0(t)\lambda = \frac{1}{|Y_i|} \int_{Y_i} A(y, t) \nabla_y w_i^\lambda(y, t) dy, \quad i = 1, 2, \quad \forall \lambda \in \mathbb{R}^N,$$

with

$$w_i^\lambda(y, t) = \lambda \cdot y - \chi_i^\lambda(y, t),$$

and  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$  the solution of (1.57).

The properties that we obtained in this thesis are interesting in itself. In particular, we prove that if the matrix field  $A$  is Lipschitz-continuous with respect to the second variable, then also  $A^0$  retains this property. This allows us to obtain a uniqueness result for the homogenized problem in  $\Omega$  (see Theorem 1.31) corresponding to (1.56).

Let us recall, in [23], the authors proved a similar result for the homogenization of elliptic problems in a perforated domains, which involves the use of a Meyers-type estimate.

We then proceed by proving first the following theorem, which states that a suitable Meyers-type estimate holds for the periodic solution of (1.57):

**Theorem 1.27.** *Let  $\lambda \in \mathbb{R}^N$  and let  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda) \in \mathcal{H}$  be the solution of (1.57). Then for any  $\lambda \in \mathbb{R}^N$ , there exists  $p_i > 2$ ,  $i = 1, 2$ , such that*

$$\chi_i^\lambda \in W^{1, p_i}(Y_i).$$

Moreover, for  $i = 1, 2$ , for every  $2 \leq q_i \leq p_i$ , there exists a positive constant  $c_i$ , dependent on  $\alpha$ ,  $\beta$ ,  $q_i$ , and  $Y_i$ , such that

$$\|\nabla \chi_i^\lambda\|_{L^{q_i}(Y_i)} \leq c_i |\lambda|.$$

This theorem is proved using the estimates given by Gallouët and Monier in [55] for elliptic equations with nonhomogeneous Neumann boundary conditions.

We are then able to prove the following main result:

**Theorem 1.28.** *Let  $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) \in \mathbb{R}^{N \times N}$  be a real matrix field with the following properties:*

(P1)  $A(\cdot, t)$  belongs to  $M(\alpha, \beta, Y)$  for all  $t \in \mathbb{R}$ ;

(P2)  $A(\cdot, t) = \{a_{ij}\}_{i,j=1,\dots,N}$  is  $Y$ -periodic for every  $t$ ;

(P3)  $A(y, t)$  is locally Lipschitz with respect to the second variable, that is, for every  $r > 0$ , there exists a positive constant  $M_r$  such that

$$|A(y, s) - A(y, t)| \leq M_r |s - t| \quad \forall s, t \in (-r, r).$$

Then the corresponding homogenized matrix  $A^0$  (see (1.59)) is also locally Lipschitz, that is, for every  $r > 0$ , there exists a positive constant  $C_r$  such that

$$|A^0(s) - A^0(t)| \leq C_r |s - t| \quad \forall s, t \in (-r, r).$$

This theorem is used to show uniqueness results for the homogenized problem in  $\Omega$  presented below (see Theorem 1.31).

We now proceed with the homogenization of (1.56). We use the periodic unfolding method adapted to the domain with imperfect interface introduced in [46]. This method makes use of the periodic unfolding operators  $\mathcal{T}_i^\varepsilon$ ,  $i = 1, 2$ , which is defined for any measurable function  $u_i$  defined in  $\Omega_i^\varepsilon$ . One of the main interesting properties of this operator is that it transforms the integrals over the varying domain  $\Omega_i^\varepsilon$  to integrals over the set  $\Omega \times Y_i$ ,  $i = 1, 2$ , which is independent of  $\varepsilon$ .

As far as we know, a study that combines the framework of renormalized solution and the periodic unfolding method has been first done in [43], and we adopt a similar approach in this thesis.

The homogenization in the framework of renormalized solution is not as straightforward as in the case with  $L^2$  data. This is because the restrictions to  $\Omega_i^\varepsilon$  of the solution  $u^\varepsilon$  of (9) do not necessarily belong to  $H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$ .

In particular, when  $f$  in (9) belongs to  $L^2(\Omega)$ , one can obtain some a priori estimates on  $u_i^\varepsilon$  in  $H^1(\Omega_i^\varepsilon)$  (see [7]). Then, using the results in [46], these estimates lead to the following convergences:

$$\begin{cases} \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_1 & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2 \\ \mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_i & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2, \end{cases} \quad (1.60)$$

for some  $u_1 \in H_0^1(\Omega)$  and  $\widehat{u}_i \in L^2(\Omega, H^1(Y_i))$ ,  $i = 1, 2$ . Then using these convergences, one obtains the homogenized problem in  $\Omega$  satisfied by  $u_1$ .

However, in our case,  $u_i^\varepsilon$  does not belong to  $H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$  and hence, this is not how we will proceed. We instead work on the truncates of  $u_i^\varepsilon$  (i.e.,  $T_k(u_i^\varepsilon)$ ), since in the framework of renormalized solutions,  $T_k(u_i^\varepsilon) \in H^1(\Omega_i^\varepsilon)$  (see Definition 1.7),  $i = 1, 2$ , for all  $k > 0$ . Thus, in place of (1.60), combining the techniques of the framework of renormalized solutions and that of the periodic unfolding method (in particular, the compactness results), we show



that there exists  $u_1$  and a sequence  $\{\widehat{u}_i^n\}_{n \in \mathbb{N}} \subset L^2(\Omega, H^1(Y_i))$ ,  $i = 1, 2$ , such that for any  $n \in \mathbb{N}$ ,  $i = 1, 2$ ,

$$\begin{cases} T_n(u_1) \in H_0^1(\Omega) \\ \overline{\mathcal{V}}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow T_n(u_1) & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ \overline{\mathcal{V}}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \nabla T_n(u_1) + \nabla_y \widehat{u}_i^n & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2. \end{cases} \quad (1.61)$$

Even if there are similarities between the homogenization of (9) and that studied in [43], there are additional difficulties that arise here, due to the presence of the jump of the solution on the interface  $\Gamma$ .

The first main difference can be observed in the definition of a renormalized solution of (9) (see Definition 1.7), containing additional conditions, as presented in Chapter 2. In addition, the proof of the following theorem, which is the construction of the oscillating part  $\widehat{u}_i$ ,  $i = 1, 2$ , from the sequence of functions  $\{\widehat{u}_i^n\}_{n \in \mathbb{N}}$  and an identification result, is also more delicate than the corresponding theorem in [43]:

**Theorem 1.29.** *Let  $\widehat{u}_1^n \in L^2(\Omega, H_{per}^1(Y_1))$  and  $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$ ,  $n \in \mathbb{N}$ , be the functions given in (1.61) with  $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$ . Then there exists a unique measurable function*

$$\widehat{u}_i : \Omega \times Y_i \longrightarrow \mathbb{R}, \quad i = 1, 2,$$

such that for every  $\mathcal{R} \in C^1(\mathbb{R})$  with compact support, that is,  $\text{supp } \mathcal{R} \subset [-m, m]$ , for some  $m \in \mathbb{N}$ , we have

$$\mathcal{R}(u_1) \widehat{u}_i^n = \mathcal{R}(u_1) \widehat{u}_i \quad \text{a.e. in } \Omega \times Y_i,$$

for all  $n \geq m$ , where  $u_1$  is the function given above.

Moreover, we have

$$\widehat{u}_i(x, \cdot) \in H^1(Y_i), \quad i = 1, 2, \quad \text{with } \mathcal{M}_\Gamma(\widehat{u}_1) = 0, \quad \text{for a.e. } x \in \Omega.$$

The delicate part of the proof comes from the fact that the average value on  $\Gamma$  of  $\widehat{u}_2^n$ ,  $n \in \mathbb{N}$ , is not necessarily zero.

Using this theorem and the convergences in (1.61), we are able to show the following theorem, describing the unfolded homogenized problem satisfied by the triple  $(u_1, \widehat{u}_1, \widehat{u}_2)$ :

**Theorem 1.30** (The unfolded homogenized problem). *Let  $u_1$ ,  $\widehat{u}_1$ , and  $\widehat{u}_2$  be functions introduced in (1.61). Let  $\psi_1, \psi_2$  be functions in  $C^1(\mathbb{R})$  (or equivalently,  $\psi_1, \psi_2 \in W^{1, \infty}(\mathbb{R})$ ) with compact supports.*

Then the triple  $(u_1, \widehat{u}_1, \widehat{u}_2)$  satisfies

$$\begin{cases} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) (\nabla(\psi_1(u_1)\varphi) + \psi_2(u_1)\nabla_y \Phi_i) dx dy \\ + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)(\Phi_1 - \Phi_2) dx d\sigma_y = \int_{\Omega} f(x)\psi_1(u_1)\varphi(x) dx \\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \Phi_i \in L^2(\Omega, H_{per}^1(Y_i)), \quad i = 1, 2. \end{cases}$$

In addition, for  $k > 0$ , the following limits hold:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\} \times Y_i} A(y, u_1) (\nabla T_k(u_1) + \nabla_y \widehat{u}_i) (\nabla T_k(u_1) + \nabla_y \widehat{u}_i) dx dy = 0,$$

for  $i = 1, 2$ , and

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 dx d\sigma_y = 0.$$

From this unfolded problem, we obtain the homogenized problem in  $\Omega$ , which completes the chapter.

**Theorem 1.31** (The homogenized problem in  $\Omega$ ). *Let  $u_1$  be a cluster point of the sequence  $\{\mathcal{U}_i^\varepsilon(u_i^\varepsilon)\}$ ,  $i = 1, 2$ . Then  $u_1$  is a renormalized solution of*

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla u_1) = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.62)$$

that is,

$$T_k(u_1) \in H_0^1(\Omega), \quad \text{for any } k > 0, \quad (1.63)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\}} A^0(u_1)\nabla u_1 \nabla u_1 dx = 0, \quad (1.64)$$

and for every  $\psi \in C^1(\mathbb{R})$  (or equivalently,  $\psi \in W^{1,\infty}(\mathbb{R})$ ) with compact support,

$$\begin{aligned} & \int_{\Omega} \psi(u_1) A^0(u_1) \nabla u_1 \nabla \varphi dx \\ & + \int_{\Omega} \psi'(u_1) A^0(u_1) \nabla u_1 \nabla u_1 \varphi dx = \int_{\Omega} f \psi(u_1) \varphi dx, \end{aligned} \quad (1.65)$$

for every  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , where  $A^0$  is the homogenized matrix as defined above.

If in addition,  $(A_4)$  holds, then  $u_1$  is the unique renormalized solution of (9) and all of the sequences in (1.61) converge (not just a subsequence).

The proof of the last assertion of this theorem makes use of Theorem 10 of the previous chapter. It is worth noting that proving of the decay of the “truncated” energy (condition (1.64)) is not standard and also delicate.

Finally, let us point out that, as observed along this thesis, managing the boundary integral that arise from the jump of the solution on the interface is not an easy task. This difficulty is not limited to the homogenization result. This can also be seen in the existence and uniqueness of the renormalized solution of (1) and also in studying the properties of the solution of the cell problem (10).

# Part I

## Existence and uniqueness results



## Chapter 2

# Quasilinear elliptic problems in a two-component domain with $L^1$ data

### 2.1 Introduction

In the present chapter, we study the existence of a solution  $u := (u_1, u_2)$  of the following class of quasi-linear equations:

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2 \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases} \quad (2.1)$$

Here,  $\Omega$  is our two-component domain and  $\partial\Omega$  is its boundary. The open sets  $\Omega_1$  and  $\Omega_2$  are the two disjoint components of  $\Omega$ ,  $\Gamma$  is the interface between them (see Figure 1), and the vector  $\nu_i$  is the unit outward normal to  $\Omega_i$ . The matrix field  $B(x, r)$  is coercive and is not restricted by any growth condition with respect to  $r$  ( $B(x, r)$  is bounded on any compact set of  $\mathbb{R}$ ), and the data  $f$  is an  $L^1$ -function. On the boundary  $\partial\Omega$ , we have a Dirichlet boundary condition, while on the interface  $\Gamma$ , we have a continuous flux and the jump of the solution is proportional to the flux. We refer to [26] for a justification of the model in the case of the conduction of heats in solids.

The existence and uniqueness of solution of (2.1) when  $f \in L^2(\Omega)$  was studied in [7, 45, 46]. In [45, 46] the equations are linear, that is, the matrix field  $B$  does not depend on the solution  $u$ , while in [7], the equations are

quasilinear, which is also the case in this study. The above mentioned papers are all motivated by homogenization, which is also our main goal (see [42]).

Since we consider in this chapter an  $L^1$  data, we need an appropriate notion of solution. Let us recall that, for the elliptic equation

$$-\operatorname{div}(A(x, u)\nabla u) = f$$

with Dirichlet boundary condition, if the matrix  $A$  is bounded, a solution in the sense of distribution exists (see [18]) but it is not unique in general (see the counterexamples in [76, 78]). If the matrix field is not bounded, then we cannot expect to have a solution in the sense of distribution since there is no reason to have  $A(x, u) \in L^1_{loc}$ . In this chapter, we use the notion of renormalized solution, which was first discussed in [39] by R.J. DiPerna and P.L. Lions for first order equations. This notion was then further developed by F. Murat in [71], by P.L. Lions and F. Murat in [63] for elliptic equations with Dirichlet boundary conditions and  $L^1$  data, and by G. Dal Maso et al. in [36] for elliptic equations with general measure data. There is a wide range of literature for elliptic equations with Dirichlet boundary condition and  $L^1$  data, among them are [10, 18, 36, 37, 63, 71]. Considering elliptic equations with Neumann or Robin boundary conditions and  $L^1$  data, which are connected to our problem, gives, in general, additional difficulties due the lack of Poincaré inequality or the low regularity of the solution (definition of the trace for e.g.). In the case of one-component domain,  $L^1$  data and Neumann or Robin boundary conditions, let us mention [3, 4, 73] using the framework of entropy solutions, [49] using a duality method and [13, 59] using the framework of renormalized solutions.

The main originality here is the jump of the solution which produces in the formulation a term in the interface  $\Gamma$ . Recalling that the regularity of the renormalized solution is given through the truncate, the first difficulty is to give a sense on the interface for functions  $(u_1, u_2)$  whose truncates belong to  $H^1$  only in each component. Following the ideas of [3, 59] (but in the case of one-component domain), we define an appropriate notion of trace (see Proposition 2.2). The second difficulty is the regularity of  $\gamma_1(u_1) - \gamma_2(u_2)$  (where  $\gamma_1$  is the trace function for  $H^1(\Omega_1)$ -functions and  $\gamma_2$  is the trace function for  $H^1(\Omega_2)$ -functions), since we have to deal, in the renormalized formulation, with terms on the boundary like  $(\gamma_1(u_1) - \gamma_2(u_2))S(\gamma(u_1))$ , where  $S$  is a  $C^1$  function with compact support. To have  $(\gamma_1(u_1) - \gamma_2(u_2))S(\gamma(u_1))$  belonging to  $L^1(\Gamma)$  is then equivalent to have  $\gamma_2(u_2)S(\gamma(u_1)) \in L^1(\Gamma)$ , which is unusual and is in some sense a coupled regularity on the boundary. It is worth noting that it is not a direct consequence of  $T_k(u_1) \in H^1(\Omega_1)$  and  $T_k(u_2) \in H^1(\Omega_2)$  ( $T_k$  is the usual truncation function at height  $\pm k$ , see

(2.4)). Using the structure of the equation, we impose an extra regularity (see (2.12b)), namely

$$(\gamma_1(u_1) - \gamma_2(u_2))[T_k(\gamma(u_1)) - T_k(\gamma(u_2))] \in L^1(\Gamma), \quad \text{for any } k > 0,$$

which allows one to prove that  $\gamma_2(u_2)S(\gamma(u_1)) \in L^1(\Gamma)$  and then  $(\gamma_1(u_1) - \gamma_2(u_2))S(\gamma(u_1)) \in L^1(\Gamma)$  (see Remark 2.5). We also impose a decay of the energy of the trace (see (2.13b)) in addition to the usual decay of the energy which are crucial to obtain stability results (see Remark 2.10). Consequently, we are able to give a definition of renormalized solution for problem (2.1) for which we prove the existence (see Theorem 2.8).

This chapter is organized as follows. The next section discusses the assumptions on our problem and some definitions, including the definition of a renormalized solution of (2.1) (see Definition 2.4). Section 2.3 is devoted to the proof of the existence of a renormalized solution for (2.1). We also remark here that by using the Boccardo-Gallouët estimates, we can actually replace conditions (2.12b) and (2.13b) of Definition 2.4 by another regularity condition on the interface. However, we prefer not to use these estimates because we have the homogenization process in mind (see Remark 2.9).

## 2.2 Assumptions and Definitions

In this section, we present the assumptions and definitions necessary for our problem. We begin by introducing the two-component domain  $\Omega$ . The domain  $\Omega$  is a connected bounded open set in  $\mathbb{R}^N$  with boundary  $\partial\Omega$ . We can write  $\Omega$  as the disjoint union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , where  $\Omega_2$  is an open set such that  $\overline{\Omega_2} \subset \Omega$  with a Lipschitz boundary  $\Gamma$ , and  $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ . We denote by  $\nu_i$  the unit outward normal to  $\Omega_i$ .

If we have a function  $u$  defined on  $\Omega \setminus \Gamma$ , then we denote  $u_i = u|_{\Omega_i}$  the restriction of  $u$  in  $\Omega_i$ . Furthermore, we have the following assumptions:

(A1) The data  $f$  belongs to  $L^1(\Omega)$ .

(A2) The function  $h$  satisfies

$$h \in L^\infty(\Gamma) \quad \text{and} \quad 0 < h_0 < h(y) \text{ a.e. on } \Gamma, \quad (2.2)$$

for some  $h_0 \in \mathbb{R}^+$ .

(A3) The matrix field  $B$  is a Carathéodory function, that is,

(a) the map  $r \mapsto B(x, r)$  is continuous for a.e.  $x \in \Omega$ ;



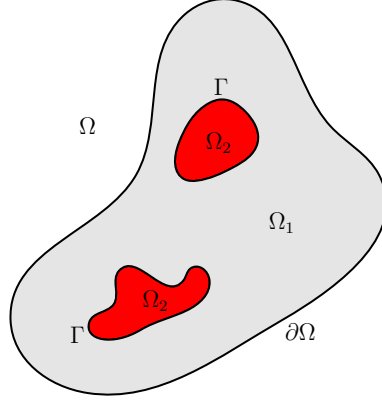


Figure 2.1: The two-component domain  $\Omega$

(b) the map  $x \mapsto B(x, r)$  is measurable for a.e.  $r \in \mathbb{R}$ ,

and it has the following properties:

$$(A3.1) \quad B(x, r)\xi \cdot \xi \geq \alpha|\xi|^2, \text{ for some } \alpha > 0, \text{ for a.e. } x \in \Omega, \forall r \in \mathbb{R}, \\ \forall \xi \in \mathbb{R}^N;$$

$$(A3.2) \quad \text{for any } k > 0, B(x, r) \in L^\infty(\Omega \times (-k, k))^{N \times N}.$$

The space for this class of equations is not a usual  $L^p$ -space or a Sobolev space due to the jump on the interface. We need the normed space  $V$  defined as follows. Let  $V_1$  be the space defined by

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

Define  $V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\}$ , equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2. \quad (2.3)$$

Identifying  $\nabla v := \widetilde{\nabla} v_1 + \widetilde{\nabla} v_2$ , we have that  $\|v\|_V^2 = \|\nabla v\|_{L^2(\Omega \setminus \Gamma)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2$ .

**Proposition 2.1** ([68]). *The norm given in (2.3) is equivalent to the norm of  $V_1 \times H^1(\Omega_2)$ , that is, there exist two positive constants  $c_1, c_2$  such that*

$$c_1\|v\|_V \leq \|v\|_{V_1 \times H^1(\Omega_2)} \leq c_2\|v\|_V, \quad \forall v \in V.$$

We now define the function  $T_k$ , which is known as the truncation function at height  $\pm k$ . The function  $T_k : \mathbb{R} \rightarrow \mathbb{R}$  is given by

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq -k, \\ t, & \text{if } -k \leq t \leq k, \\ k, & \text{if } t \geq k. \end{cases} \quad (2.4)$$

This function will be crucial in the definition of a renormalized solution of (2.1).

In the case of  $L^1$  data, we cannot expect to have the solution  $u$  belonging to  $V$ . In general, in the framework of renormalized solution, the regularity of the solution is given through the regularity of any truncate. So it is necessary in our case to define the gradient and the trace of the solution  $u$ . For the gradient, we follow the definition given in [10]. For the trace, we have to precise the trace of  $u_1$  on  $\Gamma$  and the one of  $u_2$  on  $\Gamma$ . With respect to [3, 59], we have the additional difficulty for  $u_2$  since we do not have the Poincaré inequality.

**Proposition 2.2.** *Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  be a measurable function such that  $T_k(u) \in V$  for every  $k > 0$ .*

1. *For  $i = 1, 2$ , there exists a unique measurable function  $G_i : \Omega_i \rightarrow \mathbb{R}^N$  such that for all  $k > 0$ ,*

$$\nabla T_k(u_i) = G_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i, \quad (2.5)$$

where  $\chi_{\{|u_i| < k\}}$  denotes the characteristic function of

$$\{x \in \Omega_i : |u_i(x)| < k\}.$$

We define  $G_i$  as the gradient of  $u_i$  and write  $G_i = \nabla u_i$ .

2. *If*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty, \quad (2.6)$$

then there exists a unique measurable function

$$w_i : \Gamma \rightarrow \mathbb{R}, \quad \text{for } i = 1, 2,$$

such that for all  $k > 0$ ,

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma, \quad (2.7)$$

where  $\gamma_i : H^1(\Omega_i) \rightarrow L^2(\Gamma)$  is the trace operator. We define the function  $w_i$  as the trace of  $u_i$  on  $\Gamma$  and set

$$\gamma_i(u_i) = w_i, \quad i = 1, 2.$$

*Proof.*

1. This is proved in [10] (see Lemma 2.1).
2. The case  $i = 1$ , or more generally the truncates have a zero trace on a part of the boundary (which allows one to use Poincaré-kind inequality) is presented in [59]. We just have to prove the result for  $i = 2$ .

The uniqueness is in the almost everywhere sense. Note that if we find two functions that satisfies (2.7), then the uniqueness of  $w_2$  is assured by the monotonicity of  $T_k$  and the fact that  $w_2$  is finite a.e. in  $\Gamma$ .

By Proposition 2.1, we know that

$$\|T_k(u_2)\|_{H^1(\Omega_2)} \leq c_1 \|T_k(u)\|_V,$$

for some positive constant  $c_1$ , independent of  $k$ . It follows from (2.6) that

$$\|T_k(u_2)\|_{H^1(\Omega_2)}^2 \leq Mk, \quad (2.8)$$

with  $M \in \mathbb{R}^+$  independent of  $k$ . Due to the regularity of  $\Gamma$ ,  $\gamma_2(T_n(u_2))$  is well-defined and

$$\begin{aligned} k^2 \text{meas}_\Gamma \{|\gamma_2(T_k(u_2))| \geq k\} &= \int_{\Gamma \cap \{|T_k(u_2)| \geq k\}} (\gamma_2(T_k(u_2)))^2 d\sigma \\ &\leq \|\gamma_2(T_k(u_2))\|_{L^2(\Gamma)}^2. \end{aligned}$$

Hence, by Trace Theorem and (2.8), we have

$$\begin{aligned} k^2 \text{meas}_\Gamma \{|\gamma_2(T_k(u_2))| \geq k\} &\leq \|\gamma_2(T_k(u_2))\|_{L^2(\Gamma)}^2 \\ &\leq \|T_k(u_2)\|_{L^2(\Omega_2)}^2 + \|\nabla T_k(u_2)\|_{L^2(\Omega_2)}^2 \\ &\leq Mk. \end{aligned}$$

As a result,

$$\text{meas}_\Gamma \{|\gamma_2(T_k(u_2))| \geq k\} \longrightarrow 0 \text{ as } k \longrightarrow 0. \quad (2.9)$$

Define  $\Gamma_n = \{x \in \Gamma : |\gamma_2(T_n(u_2))| < n\}$  for  $n \in \mathbb{N}$ .

From (2.9), it follows that

$$\Gamma = \bigcup_{n \geq 1} \Gamma_n \cup A, \quad (2.10)$$

where  $A$  is a subset of  $\Gamma$  with zero measure.

Note that for  $k < n$ , we have  $T_k(T_n(u_2)) = T_k(u_2)$ . Fix  $k > 0$ . Then for every  $n \in \mathbb{N}$  such that  $n > k$ , we have the following equality

$$T_k(\gamma_2(T_n(u_2))) = \gamma_2(T_k(T_n(u_2))) = \gamma_2(T_k(u_2)) \quad \text{a.e. on } \Gamma,$$

and then

$$\gamma_2(T_k(u_2)) = \gamma_2(T_n(u_2)) \quad \text{a.e. on } \Gamma_k. \quad (2.11)$$

Since for every  $n_1 \leq n$ , we have  $\Gamma_{n_1} \subseteq \Gamma_n$ , in view of (2.10) and (2.11), we can define  $w_2$  in the following way:

$$w_2 = \gamma_2(T_n(u_2)) \quad \text{on } \Gamma_n,$$

and noting that  $\Gamma = \bigcup_{n \geq 1} \Gamma_n$  (up to measure zero set), we have for any  $k > 0$

$$\gamma_2(T_k(u_2)) = T_k(w_2) \quad \text{a.e. on } \Gamma.$$

This concludes the proof. □

**Remark 2.3.** *In the following, we give an example of a measurable function  $u$  where  $T_k(u) \in V$  but  $u_2$  is not defined on a part of the interface. We consider  $\Omega = (-1, 2)$  with  $\Omega_1 = (-1, 0) \cup (1, 2)$  and  $\Omega_2 = (0, 1)$  (so  $\Gamma = \{0, 1\}$ ), and  $u = (u_1, u_2)$  is defined as*

$$u(x) = \begin{cases} u_1(x) = (x+1)(x-2) & \text{if } x \in \Omega_1 \\ u_2(x) = x^{-2} & \text{if } x \in \Omega_2. \end{cases}$$

We have for some positive constants  $C_1, C_2$ ,

$$\|\nabla T_k(u_1)\|_{L^2(\Omega_1)}^2 = \int_{\{|u_1| < k\}} (2x-1)^2 dx \leq \int_{\Omega_1} (2x-1)^2 dx \leq C_1,$$

and

$$\|\nabla T_k(u_2)\|_{L^2(\Omega_2)}^2 = \int_{k^{1/2}}^1 (-2x^{-3})^2 dx = 4 \left[ -\frac{x^7}{7} \right]_{x=k^{1/2}}^1 = \frac{4}{7}(k^{7/2} - 1).$$

Thus, we can see that

$$\frac{k^{7/2}}{C} \leq \|T_k(u)\|_V^2 \leq Ck^{7/2},$$

for some  $C > 0$  but clearly  $u_2$  does not have a trace on  $\{0\} \subset \Gamma$ .

We are now in a position to give the definition of renormalized solution.

**Definition 2.4.** Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \longrightarrow \mathbb{R}$  be a measurable function. Then  $u$  is a renormalized solution of (2.1) if

$$T_k(u) \in V, \quad \forall k > 0; \quad (2.12a)$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \quad (2.12b)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u \, dx = 0; \quad (2.13a)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \quad (2.13b)$$

and for any  $S_1, S_2 \in C^1(\mathbb{R})$  (or equivalently for any  $S_1, S_2 \in W^{1,\infty}(\mathbb{R})$ ) with compact support,  $u$  satisfies

$$\begin{aligned} & \int_{\Omega_1} S_1(u_1) B(x, u_1) \nabla u_1 \cdot \nabla v_1 \, dx + \int_{\Omega_1} S_1'(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 v_1 \, dx \\ & + \int_{\Omega_2} S_2(u_2) B(x, u_2) \nabla u_2 \cdot \nabla v_2 \, dx + \int_{\Omega_2} S_2'(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 v_2 \, dx \\ & + \int_{\Gamma} h(x) (u_1 - u_2) (v_1 S_1(u_1) - v_2 S_2(u_2)) \, d\sigma \\ & = \int_{\Omega_1} f v_1 S_1(u_1) \, dx + \int_{\Omega_2} f v_2 S_2(u_2) \, dx, \end{aligned} \quad (2.14)$$

for all  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ .

**Remark 2.5.** Conditions (2.12a) (the regularity of the truncate) and (2.13a) (the decay of the "truncated energy") are standard in the framework of renormalized solutions. As mentioned in the introduction, the main originality here is the presence of the traces in conditions (2.12b) and (2.13b).

In view of Proposition 2.2,  $\gamma(u_1)$  and  $\gamma(u_2)$  are well-defined. Condition (2.12b) is an extra regularity of  $(u_1 - u_2)(T_k(u_1) - T_k(u_2))$ .

Indeed,  $(u_1 - u_2)(T_k(u_1) - T_k(u_2))$  cannot be written as

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \chi_{\{|u_1| < n\}} \chi_{\{|u_2| < n\}},$$

for any  $n \in \mathbb{N}$ , so that having  $(u_1 - u_2)(T_k(u_1) - T_k(u_2))$  belonging to  $L^1(\Gamma)$  is not a consequence of (2.12a).

Conditions (2.12a) and (2.12b) allow one to give a sense of all the terms in (2.14). Let  $S_i \in C^1(\mathbb{R})$ ,  $i = 1, 2$ , with compact support.

Then for all  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ , we have if  $\text{supp } S_i \subset [-k, k]$  ( $i = 1, 2$ ), then for  $i = 1, 2$ ,

$$\begin{aligned} S_i(u_i)B(x, u_i)\nabla u_i \cdot \nabla v_i &= S_i(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla v_i \in L^1(\Omega_i), \\ S'_i(u_i)B(x, u_i)\nabla u_i \cdot \nabla v_i &= S'_i(u_i)B(x, T_k(u_i))\nabla T_k(u_i) \cdot \nabla T_k(u_i) v_i \in L^1(\Omega_i), \\ f v_i S_i(u_i) &\in L^1(\Omega_i). \end{aligned}$$

For the boundary term, for any  $n \in \mathbb{N}$ , let us define  $\theta_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\theta_n(s) = \begin{cases} 0, & \text{if } s \leq -2n, \\ \frac{s}{n} + 2, & \text{if } -2n \leq s \leq -n, \\ 1, & \text{if } -n \leq s \leq n, \\ -\frac{s}{n} + 2, & \text{if } n \leq s \leq 2n, \\ 0, & \text{if } s \geq 2n. \end{cases} \quad (2.15)$$

Then since  $S_1$  has a compact support, for some large enough  $n$ , we have

$$\begin{aligned} h(u_1 - u_2)v_1 S_1(u_1) &= h v_1(u_1 - u_2)(S_1(u_1) - S_1(u_2))\theta_n(u_1) \\ &\quad + h v_1(u_1 - u_2)S_1(u_2)\theta_n(u_1). \end{aligned}$$

Since both  $S_1$  and  $\theta_n$  have compact support, we have that  $h v_1(u_1 - u_2)S_1(u_2)\theta_n(u_1)$  is bounded and is therefore in  $L^1(\Gamma)$ . Moreover, since

$$S_1(u_1) - S_1(u_2) = S_1(T_{2n}(u_1)) - S_1(T_{2n}(u_2))$$

and  $S_1$  is Lipschitz, we have

$$\begin{aligned} |h v_1(u_1 - u_2)(S_1(u_1) - S_1(u_2))\theta_n(u_1)| &\leq \|h v_1\|_{L^\infty(\Gamma)} \|S'_1\|_{L^\infty(\mathbb{R})} \\ &\quad \times |u_1 - u_2| |T_{2n}(u_1) - T_{2n}(u_2)|, \end{aligned}$$

a.e. in  $\Gamma$ . Thus, in view of (2.12b),  $h(u_1 - u_2)v_1 S_1(u_1) \in L^1(\Gamma)$ . Similarly,  $h(u_1 - u_2)v_2 S_2(u_2) \in L^1(\Gamma)$ .

It is worth noting that condition (2.12b) is equivalent to have

$$u_2 \chi_{\{|u_1| < k\}} \in L^1(\Gamma) \quad \text{and} \quad u_1 \chi_{\{|u_2| < k\}} \in L^1(\Gamma), \quad (2.16)$$

for any  $k > 0$ . Indeed,

$$\begin{aligned} u_2 \chi_{\{|u_1| < k\}} &= (u_2 - u_1) \chi_{\{|u_1| < k\}} (\theta_n(u_1) - \theta_n(u_2)) + u_2 \theta_n(u_2) \chi_{\{|u_1| < k\}} \\ &\quad + u_1 \theta_n(u_1) \chi_{\{|u_1| < k\}} - u_1 \theta_n(u_2) \chi_{\{|u_1| < k\}}, \end{aligned}$$

and by condition (2.12b), the first term on the right-hand side belongs to  $L^1(\Gamma)$  while the next 3 terms are bounded and thus also belong to  $L^1(\Gamma)$ .

Finally, let us comment that conditions (2.13a) and (2.13b) are crucial to recover that formally, for any  $k > 0$ ,  $T_k(u)$  is an admissible function in (2.1), that is,

$$\begin{aligned} \int_{\Omega_1} B(x, u_1) \nabla u_1 \nabla T_k(u_1) dx + \int_{\Omega_2} B(x, u_2) \nabla u_2 \nabla T_k(u_2) dx \\ + \int_{\Gamma} h(x)(u_1 - u_2)(T_k(u_1) - T_k(u_2)) d\sigma = \int_{\Omega} f T_k(u_1) dx. \end{aligned}$$

To prove this, fix  $k > 0$ . For  $n \in \mathbb{N}$ , using  $S_1 = S_2 = \theta_n$  and  $v = T_k(u)$  as a test function in (2.14), we have

$$\begin{aligned} \int_{\Omega_1} \theta_n(u_1) B(x, u_1) \nabla u_1 \cdot \nabla T_k(u_1) dx \\ + \int_{\Omega_1} \theta'_n(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 T_k(u_1) dx \\ + \int_{\Omega_2} \theta_n(u_2) B(x, u_2) \nabla u_2 \cdot \nabla T_k(u_2) dx \\ + \int_{\Omega_2} \theta'_n(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 T_k(u_2) dx \\ + \int_{\Gamma} h(x)(u_1 - u_2)(\theta_n(u_1) T_k(u_1) - \theta_n(u_2) T_k(u_2)) d\sigma \\ = \int_{\Omega} f T_k(u) \theta_n(u) dx. \quad (2.17) \end{aligned}$$

Condition (2.13a) allows one to pass to the limit of the second and fourth integral in (2.17) while condition (2.13b) is useful for passing to the limit of the integral on the boundary in (2.17).

**Remark 2.6.** As observed in the previous remark, the main purpose of introducing condition (2.12b) is to allow us to make sense of the integral on the interface. We can avoid introducing this extra regularity condition on  $\Gamma$  by using the Boccardo-Gallouët estimates presented in [18]. However, these estimates are heavily dependent on the Sobolev constants. With the final aim of doing the homogenization process, we try as much as possible to refrain from using these estimates (see Remark 2.9).

**Remark 2.7.** *In the variational case (i.e.  $B$  is a bounded matrix field and  $f \in L^q$  with  $q \geq (N + 2)/(2N)$ ), if  $B(x, r)$  is global Lipschitz continuous with respect to  $r$  or if its modulus of continuity is strongly controlled, the (variational) solution is unique (see [48]). In the  $L^1$  case, the uniqueness question is addressed in [54]: under assumptions (A1)–(A3) and a local Lipschitz condition on  $B(x, r)$  with respect to  $r$ , we prove that the renormalized solution is unique.*

## 2.3 Existence Results

In this section, we present the proof for the existence of a renormalized solution of (2.1).

**Theorem 2.8.** *Suppose assumptions (A1)–(A3) hold. Then there exists a renormalized solution to (2.1) in the sense of Definition 2.4.*

*Proof.* The proof is divided into 4 steps. In Step 1, we consider an approximate problem (see (2.18) below) in which  $B$  is approximated by a bounded function and  $f^\varepsilon$  is an  $L^2$ -data. Using Schauder's fixed point theorem, the existence of at least a variational solution of (2.18) can be shown. Step 2 is devoted to prove some a priori estimates and then extracting a convergent subsequence. In Step 3, we prove that conditions (2.12a), (2.12b), (2.13a) and (2.13b) are satisfied by the limit. Finally, in Step 4, we pass to the limit and we show that the constructed function is a renormalized solution.

From this point until the end of the proof, we let  $i \in \{1, 2\}$ .

**Step 1:** *Introducing the approximate problem and showing the existence of solution of the approximate problem*

Let  $\varepsilon > 0$ . Suppose  $\{f^\varepsilon\} \subset L^2(\Omega)$  such that

$$f^\varepsilon \longrightarrow f \text{ strongly in } L^1(\Omega),$$

as  $\varepsilon \rightarrow 0$ . Define  $B_\varepsilon(x, t) = B(x, T_{1/\varepsilon}(t))$ . We now consider the following approximate problem

$$\begin{cases} -\operatorname{div}(B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{in } \Omega_1, \\ -\operatorname{div}(B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{in } \Omega_2, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma. \end{cases} \quad (2.18)$$



The variational formulation of problem (2.18) is the following

$$\left\{ \begin{array}{l} \text{Find } u^\varepsilon \in V \text{ such that } \forall \varphi \in V \\ \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \cdot \nabla \varphi_1 dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \cdot \nabla \varphi_2 dx \\ \quad + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(\varphi_1 - \varphi_2) d\sigma = \int_{\Omega} f^\varepsilon \varphi dx. \end{array} \right. \quad (2.19)$$

Using Proposition 2.1 and Schauder's Fixed Point Theorem, the proof of the existence of solution for (2.19) is quite standard (see e.g. [7]).

**Step 2:** *Extracting subsequences and examining convergences*

Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a solution to the approximate problem (2.18). By Stampacchia's theorem, for  $k > 0$ ,  $T_k(u^\varepsilon) \in V$  since  $u^\varepsilon \in V$ .

Using  $T_k(u^\varepsilon)$  as a test function in the variational formulation (2.19), we have

$$\begin{aligned} & \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \nabla T_k(u_1^\varepsilon) dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \nabla T_k(u_2^\varepsilon) dx \\ & \quad + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma = \int_{\Omega} f^\varepsilon T_k(u^\varepsilon) dx. \end{aligned} \quad (2.20)$$

By the definition of  $T_k$ , the coercivity of  $B$ , and the assumption on  $h$ , we have

$$\begin{aligned} & \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \nabla T_k(u_1^\varepsilon) dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \nabla T_k(u_2^\varepsilon) dx \\ & \quad + \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\ & \geq \alpha \|\nabla T_k(u_1^\varepsilon)\|_{L^2(\Omega_1)}^2 + \alpha \|\nabla T_k(u_2^\varepsilon)\|_{L^2(\Omega_2)}^2 + h_0 \|T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)\|_{L^2(\Gamma)}^2 \\ & \geq C_1 \|T_k(u^\varepsilon)\|_V^2, \end{aligned}$$

for some positive constant  $C_1$ . On the other hand, by Hölder inequality,

$$\begin{aligned} \left| \int_{\Omega} f^\varepsilon T_k(u^\varepsilon) dx \right| &= \left| \int_{\Omega_1} f^\varepsilon T_k(u_1^\varepsilon) dx + \int_{\Omega_2} f^\varepsilon T_k(u_2^\varepsilon) dx \right| \\ &\leq \|f^\varepsilon\|_{L^1(\Omega)} k \leq Mk, \end{aligned}$$

for some positive constant  $M$ , which is independent of  $\varepsilon$ .

Thus,

$$\|T_k(u^\varepsilon)\|_V^2 \leq \frac{Mk}{C_1}, \quad (2.21)$$

that is, the sequence  $\{T_k(u^\varepsilon)\}$  is bounded in  $V$  for every  $k > 0$ .

By the Rellich theorem, the inclusions  $V \hookrightarrow L^2(\Omega_1) \times L^2(\Omega_2)$  and  $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$  are compact. Consequently, since  $\{T_k(u^\varepsilon)\}$  is bounded in  $V$  for every  $k > 0$  (countable), by a diagonal process, we can extract a subsequence of  $\{T_k(u^\varepsilon)\}$  such that for any  $k > 0$  ( $k$  being a rational number), there is a  $v_k \in V$  such that

$$\begin{cases} T_k(u_i^{\varepsilon'}) \longrightarrow v_{k,i} & \text{strongly in } L^2(\Omega_i), \text{ a.e. in } \Omega_i, \\ T_k(u_i^{\varepsilon'}) \rightharpoonup v_{k,i} & \text{weakly in } V, \\ \gamma_i(T_k(u_i^{\varepsilon'})) \longrightarrow \gamma_i(v_{k,i}) & \text{strongly in } L^2(\Gamma), \text{ a.e. on } \Gamma. \end{cases} \quad (2.22)$$

Now, we show that  $\{u_i^{\varepsilon'}\}$  and  $\{\gamma_i(u_i^{\varepsilon'})\}$  are Cauchy sequences in measure. For  $u_i^{\varepsilon'}$ , we follow the arguments developed in [10]. For  $\gamma_i(u_i^{\varepsilon'})$ , we have additional difficulties which are overcome by using Proposition 2.1.

Note that we have

$$\begin{aligned} \|T_k(u_i^{\varepsilon'})\|_{L^2(\Omega_i)}^2 &= \int_{\{|u_i^{\varepsilon'}| \geq k\}} |T_k(u_i^{\varepsilon'})|^2 dx + \int_{\{|u_i^{\varepsilon'}| < k\}} |T_k(u_i^{\varepsilon'})|^2 dx \\ &= \int_{\{|u_i^{\varepsilon'}| \geq k\}} k^2 dx + \int_{\{|u_i^{\varepsilon'}| < k\}} |u_i^{\varepsilon'}|^2 dx. \end{aligned}$$

It follows by Poincaré inequality, Proposition 2.1, and (2.21), we have

$$\begin{aligned} k^2 \text{meas}\{|u^{\varepsilon'}| \geq k\} &= \int_{\{|u_1^{\varepsilon'}| \geq k\}} k^2 dx + \int_{\{|u_2^{\varepsilon'}| \geq k\}} k^2 dx \\ &\leq \|T_k(u_1^{\varepsilon'})\|_{L^2(\Omega_1)}^2 + \|T_k(u_2^{\varepsilon'})\|_{L^2(\Omega_2)}^2 \\ &\leq C_3 \|\nabla T_k(u_1^{\varepsilon'})\|_{L^2(\Omega_1)}^2 + \|T_k(u_2^{\varepsilon'})\|_{H^1(\Omega_2)}^2 \\ &\leq C_4 \|T_k(u^{\varepsilon'})\|_V^2 \leq \frac{C_4 M k}{C_1}. \end{aligned}$$

for some  $C_3, C_4 \in \mathbb{R}^+$ . Thus, we can find a positive constant  $C$  independent of  $\varepsilon$  such that

$$\text{meas}\{|u_i^{\varepsilon'}| \geq k\} \leq \frac{C}{k}. \quad (2.23)$$

For  $\gamma_1(u_1^{\varepsilon'})$ , observe that by Poincaré inequality and (2.21),

$$\begin{aligned} k^2 \text{meas}_\Gamma\{|\gamma_1(u_1^{\varepsilon'})| \geq k\} &= \int_{\{|\gamma_1(u_1^{\varepsilon'})| \geq k\}} k^2 d\sigma \\ &= \int_{\{|\gamma_1(u_1^{\varepsilon'})| \geq k\}} \gamma_1(T_k(u_1^{\varepsilon'}))^2 d\sigma \\ &\leq \|\gamma_1(T_k(u_1^{\varepsilon'}))\|_{L^2(\Gamma)}^2 \\ &\leq C_5 \|\nabla T_k(u_1^{\varepsilon'})\|_{L^2(\Omega_2)}^2 \leq C_6 k. \end{aligned}$$

Consequently,

$$\text{meas}_\Gamma\{|\gamma_1(u_1^{\varepsilon'})| \geq k\} \leq \frac{C_6}{k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (2.24)$$

For  $\gamma_2(u_2^{\varepsilon'})$ , by the Trace Theorem, Proposition 2.1, and (2.21), we have

$$\begin{aligned} k^2 \text{meas}_\Gamma\{|\gamma_2(u_2^{\varepsilon'})| \geq k\} &= \int_{\{|\gamma_2(u_2^{\varepsilon'})| \geq k\}} k^2 d\sigma \\ &= \int_{\{|\gamma_2(u_2^{\varepsilon'})| \geq k\}} \gamma_2(T_k(u_2^{\varepsilon'}))^2 d\sigma \\ &\leq \|\gamma_2(T_k(u_2^{\varepsilon'}))\|_{L^2(\Gamma)}^2 \\ &\leq C_7 \|T_k(u_2^{\varepsilon'})\|_{H^1(\Omega_2)}^2 \leq C_8 k. \end{aligned}$$

It follows that

$$\text{meas}_\Gamma\{|\gamma_2(u_2^{\varepsilon'})| \geq k\} \leq \frac{C_8}{k} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \quad (2.25)$$

By (2.24) and (2.25), for every  $\eta > 0$ , there exists  $k_0$  such that for every  $k \geq k_0$ ,

$$\text{meas}_\Gamma\{x \in \Gamma; |\gamma_i(u_i^{\varepsilon'})| \geq k\} < \eta. \quad (2.26)$$

Let  $\omega, \eta > 0$ . By (2.23) and (2.26), we can find  $k$  large enough such that

$$\text{meas}\{|u_i^{\varepsilon'}| \geq k\} \leq \frac{\eta}{3}, \quad (2.27)$$

$$\text{meas}_\Gamma\{x \in \Gamma; |\gamma_i(u_i^{\varepsilon'})| \geq k\} \leq \frac{\eta}{3}, \quad (2.28)$$

for every  $\varepsilon' > 0$ . Note that from (2.22), we can deduce that the sequences  $\{T_k(u_i^{\varepsilon'})\}$ ,  $\{\gamma_i(T_k(u_i^{\varepsilon'}))\}$  are Cauchy in measure. Hence, there exists  $\varepsilon_0 > 0$  such that

$$\text{meas}\{|T_k(u_i^{\varepsilon'}) - T_k(u_i^{\varepsilon''})| \geq \omega\} < \frac{\eta}{3}, \quad (2.29)$$

$$\text{meas}_\Gamma\{|\gamma_i(T_k(u_i^{\varepsilon'})) - \gamma_i(T_k(u_i^{\varepsilon''}))| \geq \omega\} < \frac{\eta}{3}, \quad (2.30)$$

for every  $0 < \varepsilon', \varepsilon'' < \varepsilon_0$ .

Observe that

$$\{|u_i^{\varepsilon'} - u_i^{\varepsilon''}| \geq \omega\} \subset \{|u_i^{\varepsilon'}| \geq k\} \cup \{|u_i^{\varepsilon''}| \geq k\} \cup \{|T_k(u_i^{\varepsilon'}) - T_k(u_i^{\varepsilon''})| \geq \omega\},$$

and thus,

$$\begin{aligned} \text{meas}\{|u_i^{\varepsilon'} - u_i^{\varepsilon''}| \geq \omega\} &\leq \text{meas}\{|u_i^{\varepsilon'}| \geq k\} + \text{meas}\{|u_i^{\varepsilon''}| \geq k\} \\ &\quad + \text{meas}\{|T_k(u_i^{\varepsilon'}) - T_k(u_i^{\varepsilon''})| \geq \omega\}. \end{aligned}$$

It follows from (2.27) and (2.29) that

$$\text{meas}\{|u_i^{\varepsilon'} - u_i^{\varepsilon''}| \geq \omega\} < \eta,$$

that is,  $\{u_i^{\varepsilon'}\}$  is actually Cauchy in measure. Using the inequalities (2.28) and (2.30), and similar arguments, it can be shown that  $\{\gamma_i(u_i^{\varepsilon'})\}$  is Cauchy in measure.

Consequently, there is a subsequence of  $\{u_i^{\varepsilon'}\}$  that is convergent a.e. to some measurable function  $u_i : \Omega_i \rightarrow \overline{\mathbb{R}}$ , that is

$$u_i^{\varepsilon'} \rightarrow u_i \quad \text{a.e. in } \Omega_i. \quad (2.31)$$

It follows from (2.23) that  $u_i$  is finite a.e. in  $\Omega_i$ . This  $u := (u_1, u_2)$  is our candidate for a renormalized solution for problem (2.1).

We now prove that  $u$  satisfies the conditions (2.12). Indeed, by the continuity of  $T_k$ , we have

$$T_k(u^{\varepsilon'}) \rightarrow T_k(u) = v_k \in V \quad \text{a.e. in } \Omega \setminus \Gamma. \quad (2.32)$$

Moreover, we can deduce that  $\{\gamma_i(u_i^{\varepsilon'})\}$  is convergent a.e. on  $\Gamma$  up to a subsequence. Hence, there exists  $\omega_i : \Gamma \rightarrow \overline{\mathbb{R}}$  such that

$$\gamma_i(u_i^{\varepsilon'}) \rightarrow \omega_i \quad \text{a.e. on } \Gamma, \quad (2.33)$$

with  $\omega_i$  finite a.e. on  $\Gamma$  by (2.26). We now identify  $w_i$  and  $\gamma_i(u_i)$ . Using (2.21) and (2.22), we obtain

$$\frac{1}{k} \|T_k(u)\|_V^2 \leq \frac{M}{C_1},$$

for any  $k > 0$ .

By Proposition 2.2,  $\gamma_i(u_i)$  (the trace in the truncate sense) is well defined. From (2.22), (2.32) and (2.33), we obtain that for any  $k > 0$ ,

$$T_k(\omega_i) = \gamma_i(v_{k,i}) = \gamma_i(T_k(u_i)) = T_k(\gamma_i(u_i)) \quad \text{a.e. on } \Gamma.$$

Then we have  $\omega_i = \gamma_i(u_i)$  a.e. on  $\Gamma$ . By Fatou's Lemma,  $T_k$  being non-decreasing, we have for all  $k > 0$ ,

$$\begin{aligned} \int_{\Gamma} (u_1 - u_2)(T_k(u_1) - T_k(u_2)) \, d\sigma &\leq \liminf_{\varepsilon' \rightarrow 0} \int_{\Gamma} (u_1^{\varepsilon'} - u_2^{\varepsilon'})(T_k(u_1^{\varepsilon'}) - T_k(u_2^{\varepsilon'})) \, d\sigma \\ &\leq kM, \end{aligned}$$

which is (2.12b).

From this point, we just denote our sequence by  $\varepsilon$ . Rewriting all the results we got in terms of  $\varepsilon$ , we have the following: for all  $k > 0$ ,

$$\begin{cases} u_i^\varepsilon \longrightarrow u_i & \text{a.e. in } \Omega, \\ T_k(u_i^\varepsilon) \longrightarrow T_k(u_i) & \text{strongly in } L^2(\Omega_i), \text{ a.e. in } \Omega_i, \\ \gamma_i(u_i^\varepsilon) \longrightarrow \gamma_i(u_i) & \text{a.e. on } \Gamma, \\ \gamma_i(T_k(u_i^\varepsilon)) \longrightarrow \gamma_i(T_k(u_i)) & \text{strongly in } L^2(\Gamma), \text{ a.e. in } \Gamma. \end{cases} \quad (2.34)$$

In addition, we have

$$\nabla T_k(u_i^\varepsilon) \rightharpoonup \nabla T_k(u_i) \quad \text{weakly in } (L^2(\Omega_i))^N. \quad (2.35)$$

**Step 3:** *Showing conditions (2.13) of Definition 2.4.*

From the continuity of  $B$  and (2.34), we have that for any fixed  $n > 0$ ,

$$B(x, T_n(u^\varepsilon)) \longrightarrow B(x, T_n(u)) \quad \text{a.e. in } \Omega \text{ and weakly* in } L^\infty(\Omega). \quad (2.36)$$

Due to assumption (A3.1) and the lower semi-continuity of the weak convergence,

$$\begin{aligned} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u \, dx &= \frac{1}{n} \int_{\Omega \setminus \Gamma} B(x, T_n(u)) \nabla T_n(u) \cdot \nabla T_n(u) \, dx \\ &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) \, dx, \end{aligned}$$

and by Fatou's Lemma,

$$\frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma \leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma.$$

Since

$$\int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) \, dx \quad \text{and} \quad \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma$$

are nonnegative, it is sufficient to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left( \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) \, dx \right. \\ \left. + \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma \right) = 0. \end{aligned} \quad (2.37)$$

We use  $\frac{1}{n}T_n(u^\varepsilon)$  as a test function in (2.19) to obtain

$$\begin{aligned} & \frac{1}{n} \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon) \nabla u_1^\varepsilon \nabla T_n(u_1^\varepsilon) dx + \frac{1}{n} \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon) \nabla u_2^\varepsilon \nabla T_n(u_2^\varepsilon) dx \\ & + \frac{1}{n} \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma = \frac{1}{n} \int_{\Omega} f^\varepsilon T_n(u^\varepsilon) dx. \end{aligned}$$

Consequently, for  $\varepsilon$  small enough, we have

$$\begin{aligned} & \frac{1}{n} \int_{\Omega_1} B(x, T_n(u_1^\varepsilon)) \nabla u_1^\varepsilon \nabla T_n(u_1^\varepsilon) dx + \frac{1}{n} \int_{\Omega_2} B(x, T_n(u_2^\varepsilon)) \nabla u_2^\varepsilon \nabla T_n(u_2^\varepsilon) dx \\ & + \frac{1}{n} \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma = \frac{1}{n} \int_{\Omega} f^\varepsilon T_n(u^\varepsilon) dx. \end{aligned}$$

Furthermore, since  $T_n(u^\varepsilon)$  converges to  $T_n(u)$  weakly\* in  $L^\infty(\Omega)$  and  $f^\varepsilon$  converges to  $f$  in  $L^1(\Omega)$ , we have

$$\frac{1}{n} \int_{\Omega} f^\varepsilon T_n(u^\varepsilon) dx \longrightarrow \frac{1}{n} \int_{\Omega} f T_n(u) dx \quad \text{as } \varepsilon \longrightarrow 0.$$

It follows that

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \left( \int_{\Omega \setminus \Gamma} B(x, T_n(u^\varepsilon)) \nabla T_n(u^\varepsilon) \nabla T_n(u^\varepsilon) dx \right. \\ \left. + \int_{\Gamma} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) = \frac{1}{n} \int_{\Omega} f T_n(u) dx. \end{aligned}$$

Observe that since  $u$  is finite a.e.,

$$\frac{1}{n} T_n(u) \longrightarrow 0 \quad \text{a.e. in } \Omega.$$

In addition, for any  $n > 0$ ,  $|T_n(u)| \leq n$  a.e. and thus,

$$\left| \frac{1}{n} f T_n(u) \right| \leq |f| \in L^1(\Omega).$$

By the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} f T_n(u) dx = 0,$$

which gives (2.37).

**Step 4.** Show that  $u$  satisfies (2.14) of Definition 2.4.

Let  $S_1, S_2 \in C^1(\mathbb{R})$  with compact support and let  $k > 0$  such that

$$\text{supp } S_i \subset [-k, k]. \quad (2.38)$$

We need to show that for any  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ ,  $u$  satisfies (2.14).

We use the function  $\theta_n$  defined in (2.15). Note that

$$\theta_n(u_i^\varepsilon) = \theta_n(T_{2n}(u_i^\varepsilon)) \in H^1(\Omega_i) \cap L^\infty(\Omega_i),$$

and thus, if we define

$$\psi_i = v_i S_i(u_i) \theta_n(u_i^\varepsilon),$$

for  $v \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ , we have that

$$\psi = (\psi_1, \psi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2)).$$

Using  $\psi$  as a test function in (2.19), we have

$$I_{11} + I_{12} + I_{21} + I_{22} + I_{31} + I_{32} + I_4 = I_{51} + I_{52}, \quad (2.39)$$

where

$$\begin{aligned} I_{1i} &= \int_{\Omega_i} B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla v_i S_i(u_i) \theta_n(u_i^\varepsilon) dx \\ I_{2i} &= \int_{\Omega_i} B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla u_i S_i'(u_i) \theta_n(u_i^\varepsilon) dx \\ I_{3i} &= \int_{\Omega_i} B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon S_i(u_i) S_n'(u_i^\varepsilon) dx \\ I_4 &= \int_\Gamma h(x) (u_1^\varepsilon - u_2^\varepsilon) (v_1 S_1(u_1) \theta_n(u_1^\varepsilon) - v_2 S_2(u_2) \theta_n(u_2^\varepsilon)) d\sigma \\ I_{5i} &= \int_{\Omega_i} f v_i S_i(u_i) \theta_n(u_i^\varepsilon) dx. \end{aligned}$$

We look at the behavior of each integral. In particular, we will pass to the limit as  $\varepsilon \rightarrow 0$  and then as  $n \rightarrow \infty$ .

Note that for  $n \geq k$ , since  $\text{supp } S_i \subset [-k, k]$ , we have

$$\theta_n(s) S_i(s) = S_i(s) \quad \text{and} \quad \theta_n(s) S_i'(s) = S_i'(s), \quad \text{for a.e. } s \in \mathbb{R}. \quad (2.40)$$

We first look at  $I_{1i}$ . Observe that if  $\varepsilon$  is small enough, we have

$$B_\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla v_i S_i(u_i) \theta_n(u_i^\varepsilon) = B(x, T_{1/\varepsilon}(u_i^\varepsilon)) \nabla T_{2n}(u_i^\varepsilon) S_i(u_i) \theta_n(u_i^\varepsilon).$$

Choosing  $\varepsilon$  small enough, we have

$$B(x, T_{1/\varepsilon}(u_i^\varepsilon))\theta_n(u_i^\varepsilon) = B(x, T_{2n}(u_i^\varepsilon))\theta_n(u_i^\varepsilon) \longrightarrow \theta_n(u_i)B(x, T_n(u_i)),$$

a.e. in  $\Omega_i$ . Moreover, by the assumptions on  $B$ , we have

$$|B(x, T_{1/\varepsilon}(u_i^\varepsilon))\theta_n(u_i^\varepsilon)| \leq \sup_{\Omega_i \times [-2n, 2n]} |B(x, r)|.$$

It follows from the Lebesgue Dominated Convergence Theorem that

$$B(x, T_{1/\varepsilon}(u_i^\varepsilon))\theta_n(u_i^\varepsilon) = B(x, T_{2n}(u_i^\varepsilon))\theta_n(u_i^\varepsilon) \longrightarrow \theta_n(u_i)B(x, T_{2n}(u_i)).$$

a.e. in  $\Omega_i$  and in  $L^\infty(\Omega_i)$  weak-\*. This and (2.35) imply as  $\varepsilon \longrightarrow 0$ ,

$$\begin{aligned} I_{1i} &\longrightarrow \int_{\Omega_i} B(x, T_{2n}(u_i))\nabla T_{2n}(u_i)\nabla v_i S_i(u_i)\theta_n(u_i) dx \\ &= \int_{\Omega_i} B(x, u_i)\nabla u_i\nabla v_i S_i(u_i)\theta_n(u_i) dx. \end{aligned}$$

By (2.40) we have,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{1i} = \int_{\Omega_i} B(x, u_i)\nabla u_i\nabla v_i S_i(u_i) dx. \quad (2.41)$$

We now observe the behavior of  $I_{2i}$ . For small enough  $\varepsilon$ , we have

$$B_\varepsilon(x, u_i^\varepsilon)\nabla u_i^\varepsilon\nabla v_i S_i'(u_i)\theta_n(u_i^\varepsilon) = B(x, T_{2n}(u_i^\varepsilon))\nabla T_{2n}(u_i^\varepsilon)\nabla u_i v_i S_i'(u_i)\theta_n(u_i^\varepsilon),$$

a.e. in  $\Omega_i$ .

Since  $\nabla u_i v_i S_i'(u_i) = \nabla T_{2n}(u_i)v S_i'(u_i) \in (L^2(\Omega_i))^N$ , by (2.35) we obtain as  $\varepsilon \longrightarrow 0$ ,

$$\begin{aligned} I_{2i} &\longrightarrow \int_{\Omega_i} B(x, T_{2n}(u_i))\nabla T_{2n}(u_i)\nabla u_i v_i S_i'(u_i)\theta_n(u_i) dx \\ &= \int_{\Omega_i} B(x, u_i)\nabla u_i\nabla u_i v_i S_i'(u_i)\theta_n(u_i) dx. \end{aligned}$$

By (2.40),

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{2i} = \int_{\Omega_i} B(x, u_i)\nabla u_i \cdot \nabla u_i v_i S_i'(u_i) dx. \quad (2.42)$$

For the behavior of  $I_{3i}$ , we observe that

$$|\theta_n'(s)| \leq \frac{1}{n}, \quad \text{for } |s| \leq 2n.$$



Consequently,

$$|I_{3i}| \leq \frac{\|v_i\|_{L^\infty(\Omega_i)} \|S_i\|_{L^\infty(\mathbb{R})}}{n} \int_{\{|u_i^\varepsilon| < 2n\}} B(x, u_i^\varepsilon) \nabla T_{2n}(u_i^\varepsilon) \nabla T_{2n}(u_i^\varepsilon) dx.$$

By (2.37), we have

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{3i} = 0. \quad (2.43)$$

For  $I_4$ , we note that

$$h(x)(u_1^\varepsilon - u_2^\varepsilon)v_i S_i(u_i)\theta_n(u_i^\varepsilon) = h(x)(u_1^\varepsilon - u_2^\varepsilon)v_i S_i(u_i)\theta_n(u_i^\varepsilon)\theta_{2n}(u_i^\varepsilon).$$

Then we can write  $I_4$  as

$$I_4 = I_{41} + I_{42} + I_{43} - I_{44},$$

where

$$\begin{aligned} I_{41} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_1 S_1(u_1)\theta_{2n}(u_1^\varepsilon)(\theta_n(u_1^\varepsilon) - \theta_n(u_2^\varepsilon)) d\sigma, \\ I_{42} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_1 S_1(u_1)\theta_n(u_2^\varepsilon)\theta_{2n}(u_1^\varepsilon) d\sigma, \\ I_{43} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_2 S_2(u_2)\theta_{2n}(u_2^\varepsilon)(\theta_n(u_1^\varepsilon) - \theta_n(u_2^\varepsilon)) d\sigma, \\ I_{44} &= \int_{\Gamma} h(x)(u_1^\varepsilon - u_2^\varepsilon)v_2 S_2(u_2)\theta_n(u_1^\varepsilon)\theta_{2n}(u_2^\varepsilon) d\sigma. \end{aligned}$$

Observe that  $\theta_n$  is Lipschitz and  $\theta_n(u_i^\varepsilon) = \theta_n(T_{2n}(u_i^\varepsilon))$ . This gives

$$\begin{aligned} |\theta_n(u_1^\varepsilon) - \theta_n(u_2^\varepsilon)| &= |\theta_n(T_{2n}(u_1^\varepsilon)) - \theta_n(T_{2n}(u_2^\varepsilon))| \\ &\leq \frac{1}{n} |T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)|. \end{aligned}$$

Consequently,

$$|I_{41}| \leq \frac{\|h\|_{L^\infty(\Gamma)} \|v_1\|_{L^\infty(\Gamma)} \|S_1\|_{L^\infty(\mathbb{R})} \|\theta_n\|_{L^\infty(\mathbb{R})}}{n} \int_{\Gamma} |u_1^\varepsilon - u_2^\varepsilon| |T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)| d\sigma,$$

and then by (2.37) we get

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{41} = 0. \quad (2.44)$$

By similar arguments, it can be shown that

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} I_{43} = 0. \quad (2.45)$$

For  $I_{42}$ , we observe that

$$|h(u_1^\varepsilon - u_2^\varepsilon)v_1S_1(u_1)\theta_{2n}(u_1)\theta_n(u_2^\varepsilon)| \leq M,$$

where the constant  $M$  depends only on the  $L^\infty$ -norms of  $h, S_1, \theta_n$  and  $\theta_{2n}$ , and  $n$ . Also,

$$h(x)(u_1^\varepsilon - u_2^\varepsilon)v_1S_1(u_1)\theta_{2n}(u_1^\varepsilon)\theta_n(u_2^\varepsilon) \longrightarrow h(x)(u_1 - u_2)v_1S_1(u_1)\theta_{2n}(u_1)\theta_n(u_2),$$

a.e. on  $\Gamma$  as  $\varepsilon \longrightarrow 0$ . By Lebesgue Dominated Convergence Theorem, as  $\varepsilon \longrightarrow 0$ ,

$$I_{42} \longrightarrow \int_{\Gamma} h(x)(u_1 - u_2)v_1S_1(u_1)\theta_{2n}(u_1)\theta_n(u_2) d\sigma,$$

and similarly,

$$I_{44} \longrightarrow \int_{\Gamma} h(x)(u_1 - u_2)v_2S_2(u_2)\theta_{2n}(u_2)\theta_n(u_1) d\sigma.$$

For large enough  $n$ , for  $j = 1, 2$ ,  $i \neq j$ , we have  $S_i(u_i)\theta_{2n}(u_i)\theta_n(u_j) = S_i(u_i)\theta_n(u_j)$ . In view of (2.16) in Remark 2.5,  $(u_1 - u_2)S_i(u_i) \in L^1(\Gamma)$ , for  $i = 1, 2$ , so that by Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{42} = \int_{\Gamma} h(u_1 - u_2)v_1S_1(u_1) d\sigma, \quad (2.46)$$

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{44} = \int_{\Gamma} h(u_1 - u_2)v_2S_2(u_2) d\sigma. \quad (2.47)$$

Combining (2.44)-(2.47), we conclude that

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_4 = \int_{\Gamma} h(u_1 - u_2)(v_1S_1(u_1) - v_2S_2(u_2)) d\sigma. \quad (2.48)$$

Finally for  $I_5$ , observing that  $\theta_n(u_i^\varepsilon)$  weakly converges to  $\theta_n(u_i)$  in  $L^\infty(\Omega_i)$  weakly\* and a.e. in  $\Omega_i$ ,  $f^\varepsilon$  converges strongly to  $f$  in  $L^1(\Omega)$ , we have

$$I_{5i} = \int_{\Omega_i} f^\varepsilon v_i S_i(u_i) \theta_n(u_i^\varepsilon) dx \longrightarrow \int_{\Omega_i} f v_i S_i(u_i) \theta_n(u_i) dx.$$

From (2.40), we have

$$\lim_{n \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} I_{5i} = \int_{\Omega_i} f v_i S_i(u_i) dx. \quad (2.49)$$

Passing through the limit of (2.39) and using (2.41), (2.42), (2.43), (2.48), and (2.49), we have the desired conclusion.

This concludes the proof for the existence of a renormalized solution.  $\square$

**Remark 2.9.** As explained in the introduction, it is possible to use the Boccardo-Gallouët estimates, that is, to show that  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  is bounded in  $W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ , and that leads to  $u = (u_1, u_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ , for all  $q < \frac{N}{N-1}$ . Such a result may simplify the proof since it implies that  $\gamma_1(u_1^\varepsilon), \gamma_2(u_2^\varepsilon) \in W^{1-\frac{1}{q},q}(\Gamma)$  and in particular,  $\gamma_1(u_1^\varepsilon), \gamma_2(u_2^\varepsilon)$  are bounded in  $L^{1+\eta}(\Gamma)$ , for some small enough  $\eta$ . It follows that we can give another definition of renormalized solution including  $u_1 - u_2 \in W^{1-\frac{1}{q},q}(\Gamma)$  instead of (2.12b) and then (2.13b) is not necessary since it is a direct consequence of the regularity  $u_1 - u_2 \in W^{1-\frac{1}{q},q}(\Gamma)$ .

However, since we plan to deal with the periodic homogenization of this problem (see [42]), we cannot use the Boccardo-Gallouët estimates since they are strongly related to the Sobolev constant which may blow up in a varying domain. Moreover, our techniques allow us to consider more general equations (with a nonlinear boundary terms) for which the Boccardo-Gallouët estimates are not useful.

**Remark 2.10** (Stability). By adapting the proof of Theorem 2.8, it is possible to derive a stability result. More precisely, let us consider  $u^\varepsilon$ , a renormalized solution of

$$\begin{cases} -\operatorname{div}(B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f^\varepsilon & \text{in } \Omega_1, \\ -\operatorname{div}(B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f^\varepsilon & \text{in } \Omega_2, \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma, \\ (B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = -h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma, \end{cases} \quad (2.50)$$

where

1.  $f^\varepsilon \in L^1(\Omega)$ ;
2.  $B_\varepsilon(x, t)$  is a Carathéodory matrix verifying
  - (a)  $B_\varepsilon(x, r)\xi \cdot \xi \geq \alpha|\xi|^2$ , a.e.  $x \in \Omega$ , for all  $r \in \mathbb{R}$ , for any  $\xi \in \mathbb{R}^N$ , and
  - (b) for any  $k > 0$ ,  $B_\varepsilon(x, r) \in L^\infty(\Omega \times (-k, k))^{N \times N}$ ;
3.  $h^\varepsilon \in L^\infty(\Gamma)$  with  $0 < h_0 < h^\varepsilon(y)$  a.e. on  $\Gamma$  and  $h^\varepsilon(y) < M$  (uniform), for some  $M > 0$ .

Let  $f \in L^1(\Omega)$ ,  $B : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$  is a Carathéodory function, and  $h : \Gamma \rightarrow \mathbb{R}$  with  $h \geq 0$ . If

$$f^\varepsilon \rightarrow f \quad \text{strongly in } L^1(\Omega);$$

$$\begin{cases} B_\varepsilon(x, r_\varepsilon) \longrightarrow B(x, r) \\ \text{for every sequence } r_\varepsilon \in \mathbb{R} \text{ such that} \\ r_\varepsilon \longrightarrow r \text{ a.e. on } \mathbb{R}; \\ h^\varepsilon \longrightarrow h \text{ a.e. in } \Gamma, \end{cases}$$

then  $u^\varepsilon$  converges to  $u$  a.e. where  $u$  is a renormalized solution of

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2 \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases}$$

The main point is to obtain the a priori estimates of Step 2. In view of (2.17) in Remark 2.5,  $T_k(u^\varepsilon)$  is an "admissible" test function, so that

$$\begin{aligned} & \int_{\Omega_1} B_\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon \cdot \nabla T_k(u_1^\varepsilon) dx + \int_{\Omega_2} B_\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon \cdot \nabla T_k(u_2^\varepsilon) dx \\ & + \int_{\Gamma} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma = \int_{\Omega} f^\varepsilon T_k(u^\varepsilon) dx, \end{aligned}$$

which gives all the necessary estimates. Then we can extract subsequences so that (2.34) hold true. In view of conditions on  $f^\varepsilon$ ,  $B_\varepsilon$ , and  $h^\varepsilon$ , we can perform Step 3 of the proof of Theorem 2.8.



# Chapter 3

## Uniqueness for quasilinear elliptic problems in a two-component domain with $L^1$ data

### 3.1 Introduction

In this chapter, we study the uniqueness of the renormalized solution of the following class of quasilinear elliptic problems:

$$\begin{cases} -\operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1, \\ -\operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2, \\ u_1 = 0 & \text{on } \partial\Omega, \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma, \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma. \end{cases} \quad (3.1)$$

The domain  $\Omega$  can be written as the disjoint union of  $\Omega_1$ ,  $\Omega_2$  and  $\Gamma$ , where  $\Omega_1$  and  $\Omega_2$  are the two open components of  $\Omega$ , and  $\Gamma$  is the interface between them. The matrix field  $B$  is a Carathéodory function that is uniformly elliptic (see assumption (A3)). The function  $h$  is in  $L^\infty(\Gamma)$  while  $f$  belongs to  $L^1(\Omega)$ .

When  $f$  belongs to  $L^2(\Omega)$  and the domain is composed of only one component, that is,  $-\operatorname{div}(B(x, u)\nabla u) = f$  in  $\Omega$  with Dirichlet boundary conditions, the uniqueness of the solution was obtained in [5] and [25] under a global Lipschitz-kind condition on  $B$  with respect to the second variable. Some generalizations to nonlinear elliptic problems were addressed in [19], [27] and [28].

As far as the two-component domain is concerned, additional difficulties arise due to the jump at the interface. When  $f \in L^2(\Omega)$ , the uniqueness of

the variational solution was proved in [7] by adapting the method of Chipot introduced in [27] (see also [48] for (3.1) with a singular term).

Elliptic problems in the usual one component domain and  $L^1$  data, that is,  $-\operatorname{div}(B(x, u)\nabla u) = f$  in  $\Omega$  with Dirichlet boundary conditions, are not in the standard variational setting. Furthermore, in the sense of distribution, we do not have uniqueness of the solution (see [78]). Thus, we need a convenient framework to prove the uniqueness of the solution.

Uniqueness results were proved by using the notion of entropy solutions (see [75]) or by using the (equivalent) notion of renormalized solutions (see [16], [38], and [52]).

Since we consider  $L^1$  data, we choose the appropriate framework of renormalized solutions (see [36, 71]). The existence of a renormalized solution (which is motivated by homogenization, see [42]) has been obtained in [54] (see Definition 3.2).

The main novelty of this chapter is the uniqueness of the renormalized solution under a fairly used assumption on the matrix field  $B(x, s)$  in  $s$  (similar to [38], see assumption (A4)). With respect to the already mentioned references, let us point out that mixing technical test functions developed in [16] for  $L^1$  problem and the jump give additional difficulties.

In particular, we cannot expect to control the sign of the contribution of the interface terms. To overcome this, we first prove in Lemma 3.6 that if  $u$  and  $v$  are two renormalized solutions of (3.1), then  $u_1 - v_1$  and  $u_2 - v_2$  have the same sign on the interface  $\Gamma$ . This sign property is crucial to prove the uniqueness result, Theorem 3.8, which we accomplish by adapting the method of [38].

This chapter is organized as follows. The next section is devoted to the assumptions and the definitions that are necessary to achieve our aim. Here, we present the definition of a renormalized solution of (3.1) (see Definition 3.2). Section 3.3 is devoted to prove some properties of the renormalized solution to (3.1), in particular, the sign property (see Lemma 3.6) mentioned above. Our uniqueness result (see Theorem 3.8) is proved in Section 3.4.

## 3.2 Assumptions and Definitions

We now present the assumptions and some definitions for our problem. The domain  $\Omega$  is a connected bounded open set in  $\mathbb{R}^N$  with its boundary  $\partial\Omega$ . We can write  $\Omega$  as the disjoint union  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , where  $\Omega_2$  is an open set such that  $\overline{\Omega_2} \subset \Omega$  with a Lipschitz continuous boundary  $\Gamma$  and  $\Omega_1 = \Omega \setminus \overline{\Omega_2}$  (see figure below).

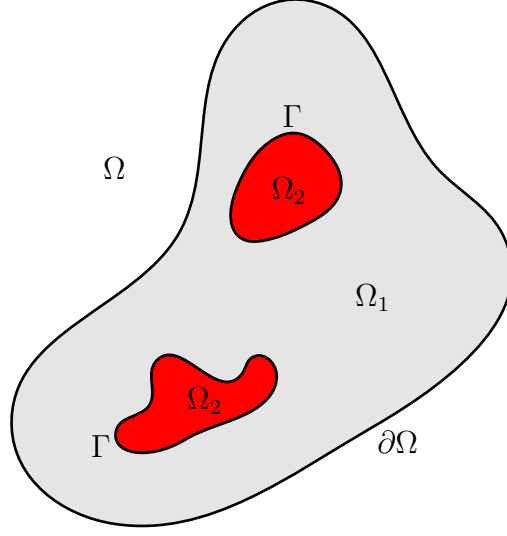


Figure 3.1: The two-component domain  $\Omega$

We denote by  $u_i = u|_{\Omega_i}$  the restriction of  $u$  in  $\Omega_i$ , where  $u$  is any measurable function defined on  $\Omega \setminus \Gamma$ .

We prescribe the following assumptions on  $f$ ,  $h$  and  $B$ :

(A1) The function  $f$  is in  $L^1(\Omega)$ .

(A2) The function  $h$  belongs to  $L^\infty(\Gamma)$  and for some  $h_0 > 0$ ,

$$h(y) \geq h_0 \quad \text{for a.e. } y \in \Gamma. \quad (3.2)$$

(A3) The matrix field  $B$  is a Carathéodory function, that is,

- (a) the map  $r \mapsto B(x, r)$  is continuous for a.e.  $x \in \Omega$ ;
- (b) the map  $x \mapsto B(x, r)$  is measurable for a.e.  $r \in \mathbb{R}$ ,

and it satisfies the following properties:

- (a)  $B(x, r)\xi \cdot \xi \geq \alpha|\xi|^2$ , for some  $\alpha > 0$  a.e.  $x \in \Omega$ ,  $\forall r \in \mathbb{R}$ ,  $\forall \xi \in \mathbb{R}^N$ ;
- (b) for any  $k > 0$ ,  $B(x, r) \in L^\infty(\Omega \times (-k, k))^{N \times N}$ ;

(A4)  $B(x, r)$  is Locally Lipschitz with respect to  $r$ , that is, for any compact subset  $K$  of  $\mathbb{R}$ , there exists  $M_K > 0$  such that

$$|B(x, r) - B(x, s)| \leq M_K|r - s|, \quad \forall r, s \in K. \quad (3.3)$$



Due to the jump of a solution on the interface  $\Gamma$ , the usual Sobolev spaces are not suitable to work with for our problem. Hence, we need to define a special normed space  $V$ .

Let  $V_1$  be the normed space defined as

$$V_1 = \{v \in H^1(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|\nabla v\|_{L^2(\Omega_1)}.$$

The space  $V$  is defined as

$$V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in H^1(\Omega_2)\},$$

equipped with the norm

$$\|v\|_V^2 := \|\nabla v_1\|_{L^2(\Omega_1)}^2 + \|\nabla v_2\|_{L^2(\Omega_2)}^2 + \|v_1 - v_2\|_{L^2(\Gamma)}^2. \quad (3.4)$$

As presented in [54], since the data  $f$  is in  $L^1(\Omega)$ , we do not expect a solution  $u$  of (3.1) to be in any  $L^p$ -space. Moreover, it is also not expected to have the regularity required to have a gradient and trace in the usual sense. The following proposition was proved in [54] to give a definition for the gradient and trace of any measurable function. This proposition made use of the truncation function  $T_k : \mathbb{R} \rightarrow \mathbb{R}$ , given by

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq -k, \\ t, & \text{if } -k \leq t \leq k, \\ k, & \text{if } t \geq k. \end{cases} \quad (3.5)$$

**Proposition 3.1** ([54]). *Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  be a measurable function such that  $T_k(u) \in V$  for every  $k > 0$ .*

1. *For  $i = 1, 2$ , there exists a unique measurable function  $G_i : \Omega_i \rightarrow \mathbb{R}^N$  such that for all  $k > 0$ ,*

$$\nabla T_k(u_i) = G_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i, \quad (3.6)$$

where  $\chi_{\{|u_i| < k\}}$  denotes the characteristic function of

$$\{x \in \Omega_i : |u_i(x)| < k\}.$$

We define  $G_i$  as the gradient of  $u_i$  and write  $G_i = \nabla u_i$ .

2. *If*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_V^2 < \infty, \quad (3.7)$$

then there exists a unique measurable function  $w_i : \Gamma \rightarrow \mathbb{R}$ , for  $i = 1, 2$ , such that for all  $k > 0$

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. in } \Gamma, \quad (3.8)$$

where  $\gamma_i : H^1(\Omega_i) \rightarrow L^2(\Gamma)$  is the trace operator. We define the function  $w_i$  as the trace of  $u_i$  on  $\Gamma$  and set

$$\gamma_i(u_i) = w_i.$$

With this proposition, we can now present the definition of a renormalized solution of (3.1) given in [54].

**Definition 3.2.** Let  $u = (u_1, u_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R}$  be a measurable function. Then  $u$  is a renormalized solution of (3.1) if

$$T_k(u) \in V, \quad \forall k > 0; \quad (3.9a)$$

$$(u_1 - u_2)(T_k(u_1) - T_k(u_2)) \in L^1(\Gamma), \quad \forall k > 0; \quad (3.9b)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u| < n\}} B(x, u) \nabla u \cdot \nabla u \, dx = 0; \quad (3.10a)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma} (u_1 - u_2)(T_n(u_1) - T_n(u_2)) \, d\sigma = 0; \quad (3.10b)$$

and for any  $S_1, S_2 \in C^1(\mathbb{R})$  (or equivalently for any  $S_1, S_2 \in W^{1,\infty}(\mathbb{R})$ ) with compact support,  $u$  satisfies

$$\begin{aligned} & \int_{\Omega_1} S_1(u_1) B(x, u_1) \nabla u_1 \cdot \nabla \psi_1 \, dx + \int_{\Omega_1} S_1'(u_1) B(x, u_1) \nabla u_1 \cdot \nabla u_1 \psi_1 \, dx \\ & + \int_{\Omega_2} S_2(u_2) B(x, u_2) \nabla u_2 \cdot \nabla \psi_2 \, dx + \int_{\Omega_2} S_2'(u_2) B(x, u_2) \nabla u_2 \cdot \nabla u_2 \psi_2 \, dx \\ & + \int_{\Gamma} h(x) (u_1 - u_2) (\psi_1 S_1(u_1) - \psi_2 S_2(u_2)) \, d\sigma \\ & = \int_{\Omega_1} f \psi_1 S_1(u_1) \, dx + \int_{\Omega_2} f \psi_2 S_2(u_2) \, dx, \end{aligned} \quad (3.11)$$

for all  $\psi \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ .

**Remark 3.3.** In the previous definition, conditions (3.9a) and (3.10a) are standard in the definition of renormalized solution. However, due to the presence of the boundary integral on  $\Gamma$ , (3.9b) and (3.10b) have to be added. In particular, since  $\gamma_i(u_i) \in L^1(\Gamma)$  is not an assumption in Definition 3.2, we need (2.12b) to give sense to the integral on  $\Gamma$  in (3.11) (see [54, Remark 2]).

As presented in [54], assumptions (A1)-(A3) are enough to show the existence of a renormalized solution of (3.1) in the sense of this previous definition. However, to have uniqueness of the solution, an additional assumption on matrix  $B$  must be added (see (A4)), as will be seen in the next section.

### 3.3 Preliminary Results

In this section, we prove some properties on renormalized solutions of (3.1) (see Lemma 3.4 and Proposition 3.5), which are standard in the  $L^1$  framework. Moreover, we prove Lemma 3.6, which states that if  $u$  and  $v$  are two renormalized solutions of (3.1) for the same data  $f$ , then we have the sign condition on the interface  $\Gamma$ , that is  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  on  $\Gamma$ . This result is crucial for the proof of our uniqueness result (see Theorem 3.8).

**Lemma 3.4.** *Let  $u$  be a renormalized solution of (3.1). If  $\varphi$  is a bounded and increasing function belonging in  $C^1(\mathbb{R})$  such that  $\varphi(0) = 0$ , then*

$$\varphi'(u_i)B(x, u_i)\nabla u_i \cdot \nabla u_i \in L^1(\Omega_i), \quad i = 1, 2, \quad (3.12)$$

$$(u_1 - u_2)(\varphi(u_1) - \varphi(u_2)) \in L^1(\Gamma). \quad (3.13)$$

*Proof.* Let  $\varphi$  be a bounded increasing function that belongs in  $C^1(\mathbb{R})$  such that  $\varphi(0) = 0$ . Let  $n > 0$ . Define the function  $\theta_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$\theta_n(s) = \begin{cases} 0, & \text{if } s \leq -2n \\ \frac{s}{n} + 2, & \text{if } -2n \leq s \leq -n \\ 1, & \text{if } -n \leq s \leq n \\ -\frac{s}{n} + 2, & \text{if } n \leq s \leq 2n \\ 0, & \text{if } s \geq 2n. \end{cases} \quad (3.14)$$

We can clearly see from the definition of  $\theta_n$  that it is a continuous Lipschitz function verifying

$$|\theta_n(r)| \leq 1 \quad \text{and} \quad |\theta'_n(r)| \leq \frac{1}{n}, \quad \text{a.e. in } \mathbb{R}. \quad (3.15)$$

Substituting  $S_1 = S_2 = \theta_n$  and  $\psi = \varphi(T_{2n}(u))$  in (3.11) of Definition 3.2,

we have

$$\begin{aligned}
& \int_{\Omega_1} \theta_n(u_1) \varphi'(u_1) B(x, u_1) \nabla u_1 \nabla u_1 dx + \int_{\Omega_2} \theta_n(u_2) \varphi'(u_2) B(x, u_2) \nabla u_2 \nabla u_2 dx \\
& + \int_{\Omega_1} \theta'_n(u_1) \varphi(u_1) B(x, u_1) \nabla u_1 \nabla u_1 dx + \int_{\Omega_2} \theta'_n(u_2) \varphi(u_2) B(x, u_2) \nabla u_2 \nabla u_2 dx \\
& + \int_{\Gamma} h(x) (u_1 - u_2) (\theta_n(u_1) \varphi(u_1) - \theta_n(u_2) \varphi(u_2)) d\sigma = \int_{\Omega} f \theta_n(u) \varphi(u) dx.
\end{aligned} \tag{3.16}$$

We now study the terms in (3.16) to pass to the limit as  $n$  goes to infinity. Regarding the third and fourth terms we have, for  $i = 1, 2$ ,

$$\begin{aligned}
& \left| \int_{\Omega_i} \theta'_n(u_i) \varphi(u_i) B(x, u_i) \nabla u_i \nabla u_i dx \right| \\
& \leq \frac{\|\varphi\|_{L^\infty(\mathbb{R})}}{n} \int_{\{n < |u_i| < 2n\}} B(x, u_i) \nabla u_i \nabla u_i dx,
\end{aligned}$$

so that the decay of the energy of the truncates (3.10a) implies that

$$\lim_{n \rightarrow +\infty} \left| \int_{\Omega_i} \theta'_n(u_i) \varphi(u_i) B(x, u_i) \nabla u_i \nabla u_i dx \right| = 0, \quad \text{for } i = 1, 2. \tag{3.17}$$

As far as the fifth term of (3.16) is concerned, we have in view of the definition of  $\theta_n$ ,

$$\begin{aligned}
& h(x) (u_1 - u_2) (\theta_n(u_1) \varphi(u_1) - \theta_n(u_2) \varphi(u_2)) \\
& = h(x) (u_1 - u_2) \theta_n(u_1) (\theta_{2n}(u_1) - \theta_{2n}(u_2)) \varphi(u_1) \\
& \quad + h(x) (u_1 - u_2) \theta_n(u_1) \theta_{2n}(u_2) (\varphi(u_1) - \varphi(u_2)) \\
& \quad - h(x) (u_1 - u_2) \theta_{2n}(u_2) (\theta_n(u_1) - \theta_n(u_2)) \varphi(u_2).
\end{aligned} \tag{3.18}$$

Since the functions  $\theta_n$  and  $\theta_{2n}$  are Lipschitz continuous and recalling that  $\varphi$  is bounded, we deduce that

$$\begin{aligned}
& \left| h(x) (u_1 - u_2) \theta_n(u_1) (\theta_{2n}(u_1) - \theta_{2n}(u_2)) \varphi(u_1) \right| \\
& \leq \frac{\|h\|_{L^\infty(\Gamma)}}{2n} (u_1 - u_2) (T_{4n}(u_1) - T_{4n}(u_2)) \|\varphi\|_{L^\infty(\mathbb{R})}, \\
& \left| h(x) (u_1 - u_2) \theta_{2n}(u_2) (\theta_n(u_1) - \theta_n(u_2)) \varphi(u_2) \right| \\
& \leq \frac{\|h\|_{L^\infty(\Gamma)}}{2n} (u_1 - u_2) (T_{2n}(u_1) - T_{2n}(u_2)) \|\varphi\|_{L^\infty(\mathbb{R})},
\end{aligned}$$

so that condition (3.10b) leads to

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \left| h(x)(u_1 - u_2)\theta_n(u_1)(\theta_{2n}(u_1) - \theta_{2n}(u_2))\varphi(u_1) \right| dx = 0, \quad (3.19)$$

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \left| h(x)(u_1 - u_2)\theta_{2n}(u_2)(\theta_n(u_1) - \theta_n(u_2))\varphi(u_2) \right| dx = 0. \quad (3.20)$$

At last, the integral on the right-hand side of (3.16), by (3.15), is bounded by

$$\left| \int_{\Omega} f\theta_n(u)\varphi(u) dx \right| \leq \|f\|_{L^1(\Omega)}\|\varphi\|_{L^\infty(\mathbb{R})}. \quad (3.21)$$

Combining (3.16)-(3.21), we obtain

$$\begin{aligned} & \int_{\Omega_1} \theta_n(u_1)\varphi'(u_1)B(x, u_1)\nabla u_1\nabla u_1 dx + \int_{\Omega_2} \theta_n(u_2)\varphi'(u_2)B(x, u_2)\nabla u_2\nabla u_2 dx \\ & \quad + \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(u_1)\theta_{2n}(u_2)(\varphi(u_1) - \varphi(u_2)) d\sigma \\ & \leq \omega(n) + \|f\|_{L^1(\Omega)}\|\varphi\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

where  $w(n) \rightarrow 0$  as  $n$  goes to infinity.

Since  $u_1$  (resp.  $u_2$ ) is finite almost everywhere in  $\Omega_1$  (resp.  $\Omega_2$ ), the definition of  $\theta_n$  and Fatou's Lemma allow one to deduce that

$$\begin{aligned} & \int_{\Omega_1} B(x, u_1)\varphi'(u_1)\nabla u_1\nabla u_1 dx + \int_{\Omega_2} B(x, u_2)\varphi'(u_2)\nabla u_2\nabla u_2 dx \\ & \quad + \int_{\Gamma} h(x)(u_1 - u_2)(\varphi(u_1) - \varphi(u_2)) d\sigma \leq \|f\|_{L^1(\Omega)}\|\varphi\|_{L^\infty(\mathbb{R})}. \quad (3.22) \end{aligned}$$

This gives (3.12) and (3.13).  $\square$

As mentioned in Remark 3.3, we do not impose in Definition 3.2 that  $\gamma_i(u_i)$  ( $i = 1, 2$ ) belongs to  $L^1(\Gamma)$ . However, having no regularity on  $\gamma_i(u_i)$ , for  $i = 1, 2$ , seems to be an obstacle to prove Theorem 3.8. By adapting the estimates of Boccardo-Gallouët (see [18]) to our two-component domain, we are able to prove in Proposition 3.5 that  $\gamma_i(u_i)$  belongs to  $L^1(\Gamma)$ , for  $i = 1, 2$ .

**Proposition 3.5.** *For  $i = 1, 2$ , let  $\gamma_i$  be the trace function defined on  $H^1(\Omega_i)$ . If  $u$  is a renormalized solution of (3.1), then  $\gamma_i(u_i) \in L^1(\Gamma)$ ,  $i = 1, 2$ .*

*Proof.* By taking  $\varphi = T_k$  in Lemma 3.4, and by observing the precise estimate (3.22) at the end of the proof of Lemma 3.4, we have

$$\|T_k(u)\|_V^2 \leq k\|f\|_{L^1(\Omega)}, \quad \forall k > 0.$$

Since on  $V$ , the norm  $\|\cdot\|_V$  is equivalent to the norm of  $H^1(\Omega_1) \times H^1(\Omega_2)$ , the Boccardo-Gallouët estimates hold true, so that  $u_i \in W^{1,p}(\Omega_i)$ ,  $i = 1, 2$ , for any  $p < \frac{N}{N-1}$ . We are then able to conclude that  $\gamma_i(u_i) \in L^1(\Gamma)$ ,  $i = 1, 2$ .  $\square$

In proving the uniqueness result, one of the main difficulties we encountered is managing the integral on the interface  $\Gamma$  with test functions which are nonlinear with respect to the unknown. The very first step to overcome this difficulty is the following lemma which establishes a sign property of the difference of any two renormalized solutions of (3.1) on the interface. We will denote by  $\text{sgn}$  the usual sign function ( $\text{sgn}(r) = r/|r|$  if  $r \neq 0$  and  $\text{sgn}(0) = 0$ ).

**Lemma 3.6.** *Suppose (A1)–(A4) hold. If  $u$  and  $v$  are two renormalized solutions of (3.1), then  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  a.e. on  $\Gamma$ .*

*Proof.* Let  $u$  and  $v$  be renormalized solutions of (3.1). Writing (3.11) of Definition 3.2 for  $S_1 = S_2 = \theta_n$  and  $\psi = \frac{1}{k}T_k(u - v)$ , where  $0 < k < 1$ , for  $u$  and  $v$ , and subtracting the resulting equations, we have

$$I_1^{k,n} + I_2^{k,n} + J_1^{k,n} + J_2^{k,n} + L^{k,n} = M^{k,n}, \quad (3.23)$$

where for  $i = 1, 2$ ,

$$\begin{aligned} I_i^{k,n} &= \frac{1}{k} \int_{\Omega_i} (\theta_n(u_i)B(x, u_i)\nabla u_i - \theta_n(v_i)B(x, v_i)\nabla v_i)\nabla T_k(u_i - v_i) dx, \\ J_i^{k,n} &= \frac{1}{k} \int_{\Omega_i} (\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i - \theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i)T_k(u_i - v_i) dx, \\ L^{k,n} &= \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)(\theta_n(u_1)T_k(u_1 - v_1) - \theta_n(u_2)T_k(u_2 - v_2)) d\sigma \\ &\quad - \frac{1}{k} \int_{\Gamma} h(x)(v_1 - v_2)(\theta_n(v_1)T_k(u_1 - v_1) - \theta_n(v_2)T_k(u_2 - v_2)) d\sigma, \\ M^{k,n} &= \frac{1}{k} \int_{\Omega} f(\theta_n(u) - \theta_n(v))T_k(u - v) dx. \end{aligned}$$

We study the behavior of each term first as  $k \rightarrow 0$  and then as  $n \rightarrow \infty$ .

We can write  $I_i^{k,n}$ ,  $i = 1, 2$ , as

$$I_i^{k,n} = I_{i,1}^{k,n} + I_{i,2}^{k,n} + I_{i,3}^{k,n},$$

where

$$\begin{aligned}
I_{i,1}^{k,n} &= \frac{1}{k} \int_{\Omega_i} \theta_n(u_i) B(x, u_i) \nabla T_k(u_i - v_i) \nabla T_k(u_i - v_i) dx, \quad i = 1, 2, \\
I_{i,2}^{k,n} &= \frac{1}{k} \int_{\Omega_i} \theta_n(u_i) (B(x, u_i) - B(x, v_i)) \nabla v_i \nabla T_k(u_i - v_i) dx, \quad i = 1, 2, \\
I_{i,3}^{k,n} &= \frac{1}{k} \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) B(x, v_i) \nabla v_i \nabla T_k(u_i - v_i) dx, \quad i = 1, 2.
\end{aligned}$$

Clearly,  $I_{i,1}^{k,n} \geq 0$ ,  $i = 1, 2$ . For  $I_{i,2}^{k,n}$ ,  $i = 1, 2$ , we use (3.3) and (3.15) to obtain

$$\begin{aligned}
|I_{i,2}^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} \theta_n(u_i) (B(x, u_i) - B(x, v_i)) \nabla v_i \nabla T_k(u_i - v_i) dx \right| \\
&\leq \frac{1}{k} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| \leq 2n\} \\ \cap \{|v_i| \leq 2n+1\}}} |\theta_n(u_i)| |B(x, u_i) - B(x, v_i)| |\nabla v_i| |\nabla T_k(u_i - v_i)| dx \\
&\leq \frac{1}{k} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| \leq 2n\} \\ \cap \{|v_i| \leq 2n+1\}}} C |u_i - v_i| |\nabla v_i \nabla T_k(u_i - v_i)| dx \\
&\leq C \int_{\{0 < |u_i - v_i| < k\}} |\nabla T_{2n+1}(v_i) \nabla T_k(u_i - v_i)| dx.
\end{aligned}$$

From (3.9a) of Definition 3.2, we know that for any  $0 < k < 1$ ,  $i = 1, 2$ ,

$$|\nabla T_{2n+1}(v_i) \nabla T_k(u_i - v_i) \chi_{\{0 < |u_i - v_i| < k\}}| \leq |\nabla T_{2n+1}(v_i) \nabla T_1(u_i - v_i)| \in L^1(\Omega_i).$$

In addition, we have,

$$\nabla T_k(u_i - v_i) \chi_{\{0 < |u_i - v_i| < k\}} \longrightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2.$$

By Lebesgue Dominated Convergence Theorem, we conclude that

$$C \int_{\{0 < |u_i - v_i| < k\}} |\nabla T_{2n+1}(v_i) \nabla T_k(u_i - v_i)| dx \longrightarrow 0 \quad \text{as } k \longrightarrow 0, \quad i = 1, 2,$$

which gives

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} I_{i,2}^{k,n} = 0, \quad i = 1, 2. \quad (3.24)$$

For  $I_{i,3}^{k,n}$ , noting that  $\theta_n$  is Lipschitz continuous with (3.15), we have for

$i = 1, 2$ ,

$$\begin{aligned}
|I_{i,3}^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) B(x, v_i) \nabla v_i \nabla T_k(u_i - v_i) dx \right| \\
&\leq \frac{1}{k} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |\theta_n(u_i) - \theta_n(v_i)| |B(x, u_i) \nabla v_i \nabla(u_i - v_i)| dx \\
&\leq \frac{1}{n} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |B(x, u_i) \nabla v_i \nabla(u_i - v_i)| dx \\
&\leq \frac{1}{n} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |B(x, v_i) \nabla u_i \nabla v_i| dx \\
&\quad + \frac{1}{n} \int_{\substack{\{0 < |u_i - v_i| < k\} \\ \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, v_i) \nabla v_i \nabla v_i dx \\
&\leq \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} |B(x, v_i) \nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| dx \\
&\quad + \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} B(x, v_i) \nabla T_{2n+1}(v_i) \nabla T_{2n+1}(v_i) dx.
\end{aligned}$$

Again, from (3.9a) of Definition 3.2, we deduce that for  $i = 1, 2$ ,

$$|\nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| \chi_{\{0 < |u_i - v_i| < k\}} \leq |\nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| \in L^1(\Omega_i)$$

and

$$|\nabla T_{2n+1}(v_i)|^2 \chi_{\{0 < |u_i - v_i| < k\}} \leq |\nabla T_{2n+1}(v_i)|^2 \in L^1(\Omega_i).$$

Furthermore,

$$\nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i) \chi_{\{0 < |u_i - v_i| < k\}} \longrightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2,$$

and

$$|\nabla T_{2n+1}(v_i)|^2 \chi_{\{0 < |u_i - v_i| < k\}} \longrightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2.$$

Using the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}
&\frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} |B(x, v_i) \nabla T_{2n+1}(u_i) \nabla T_{2n+1}(v_i)| dx \\
&\quad + \frac{1}{n} \int_{\{0 < |u_i - v_i| < k\}} B(x, v_i) |\nabla T_{2n+1}(v_i)|^2 dx \longrightarrow 0 \quad \text{as } k \rightarrow 0.
\end{aligned}$$



Hence,

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} I_{i,3}^{k,n} = 0, \quad i = 1, 2. \quad (3.25)$$

For  $J_i^{k,n}$ ,  $i = 1, 2$ , since we have (3.15) and

$$|T_k(r)| \leq k, \quad \forall r \in \mathbb{R}, \quad \forall k > 0, \quad (3.26)$$

we obtain for  $i = 1, 2$ ,

$$\begin{aligned} |J_i^{k,n}| &= \left| \frac{1}{k} \int_{\Omega_i} (\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i - \theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i)T_k(u_i - v_i) dx \right| \\ &\leq \int_{\Omega_i} |\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i| dx + \int_{\Omega_i} |\theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i| dx \\ &\leq \frac{1}{n} \int_{\{|u_i| < 2n\}} B(x, u_i)\nabla u_i\nabla u_i dx + \frac{1}{n} \int_{\{|v_i| < 2n\}} B(x, v_i)\nabla v_i\nabla v_i dx. \end{aligned}$$

By (3.10a) of Definition 3.2, it follows that

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} J_i^{k,n} = 0, \quad i = 1, 2. \quad (3.27)$$

For the integral on the boundary, we use Proposition 3.5 to pass to the limit. Note that by (3.15) and (3.26), we have

$$\left| \frac{1}{k} h(x)(u_1 - u_2)\theta_n(u_1)T_k(u_1 - v_1) \right| \leq \|h\|_{L^\infty(\Gamma)} |u_1 - u_2| \in L^1(\Gamma).$$

Furthermore,

$$\frac{1}{k} h(x)(u_1 - u_2)\theta_n(u_1)T_k(u_1 - v_1) \xrightarrow[k \rightarrow 0]{} h(x)(u_1 - u_2)\theta_n(u_1) \operatorname{sgn}(u_1 - v_1)$$

a.e. on  $\Gamma$ , and

$$h(x)(u_1 - u_2)\theta_n(u_1) \operatorname{sgn}(u_1 - v_1) \xrightarrow[n \rightarrow \infty]{} h(x)(u_1 - u_2) \operatorname{sgn}(u_1 - v_1),$$

a.e. on  $\Gamma$ . The Lebesgue Dominated Convergence Theorem implies

$$\begin{aligned} \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(u_1)T_k(u_1 - v_1) d\sigma \\ = \int_{\Gamma} h(x)(u_1 - u_2) \operatorname{sgn}(u_1 - v_1) d\sigma, \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(u_2)T_k(u_2 - v_2) d\sigma \\
& \qquad \qquad \qquad = \int_{\Gamma} h(x)(u_1 - u_2) \operatorname{sgn}(u_2 - v_2) d\sigma, \\
& \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Gamma} h(x)(u_1 - u_2)\theta_n(v_i)T_k(u_2 - v_2) d\sigma \\
& \qquad \qquad \qquad = \int_{\Gamma} h(x)(u_1 - u_2) \operatorname{sgn}(u_2 - v_2) d\sigma, \quad i = 1, 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} L^{k,n} \\
& \qquad \qquad \qquad = \int_{\Gamma} h(x)[(u_1 - v_1) - (u_2 - v_2)](\operatorname{sgn}(u_1 - v_1) - \operatorname{sgn}(u_2 - v_2)) d\sigma.
\end{aligned} \tag{3.28}$$

For the integral on the right-hand side of (3.23),

$$\begin{aligned}
|M^{k,n}| &= \left| \frac{1}{k} \int_{\Omega} f(\theta_n(u) - \theta_n(v))T_k(u - v) dx \right| \\
&\leq \int_{\Omega} |f||\theta_n(u) - \theta_n(v)| dx.
\end{aligned}$$

Note that  $|f||\theta_n(u) - \theta_n(v)| \leq 2|f| \in L^1(\Omega)$  with

$$\theta_n(u) - \theta_n(v) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad \text{a.e. in } \Omega.$$

Thus, by the Lebesgue Dominated Convergence Theorem,

$$\int_{\Omega} |f||\theta_n(u) - \theta_n(v)| dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which gives

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} M^{k,n} = 0. \tag{3.29}$$

From (3.23) and the fact that  $I_{i,1}^{k,n} \geq 0$ , we obtain

$$I_{i,2}^{k,n} + I_{i,3}^{k,n} + X^{k,n} + L^{k,n} \leq M^{k,n}.$$

Taking the limit of both sides of the last inequality first as  $k \longrightarrow 0$  then as  $n \longrightarrow \infty$ , we get

$$\int_{\Gamma} h(x)((u_1 - v_1) - (u_2 - v_2))(\operatorname{sgn}(u_1 - v_1) - \operatorname{sgn}(u_2 - v_2)) d\sigma = 0.$$

That is,  $\operatorname{sgn}(u_1 - v_1) = \operatorname{sgn}(u_2 - v_2)$  on  $\Gamma$ .

□

### 3.4 Main Result

This section is devoted to our main result, Theorem 3.8, namely the uniqueness of the renormalized solution under assumptions (A1)-(A4). The proof of this uniqueness result makes use of the results of the previous section and the method developed in [16, 38]. The following proposition, proved in [38], states that assuming a very local Lipschitz control of  $B(x, s)$  with respect to  $s$ , we have the existence of a function  $\varphi$  which controls the Lipschitz continuous character of the matrix field  $B$  through very technical conditions.

**Proposition 3.7** ([38]). *Suppose that (3.3) holds. Then there exists a function  $\varphi \in C^1(\mathbb{R})$  that satisfies the following properties:*

$$\varphi(0) = 0 \quad \text{and} \quad \varphi' \geq 1. \quad (3.30)$$

*In addition, there are constants  $\delta > 1/2$ ,  $0 < k_0 < 1$ , and  $L > 0$  such that*

$$\frac{\varphi'}{(1 + |\varphi|)^{2\delta}} \in L^\infty(\mathbb{R}), \quad (3.31)$$

*and for any  $r, s \in \mathbb{R}$  satisfying  $|\varphi(r) - \varphi(s)| \leq k$ , for  $0 < k < k_0$ ,*

$$\left| \frac{B(x, r)}{\varphi'(r)} - \frac{B(x, s)}{\varphi'(s)} \right| \leq \frac{1}{\varphi'(s)} \frac{Lk}{(1 + |\varphi(r)| + |\varphi(s)|)^\delta} \quad (3.32)$$

*and*

$$\frac{1}{L} \leq \frac{\varphi'(s)}{\varphi'(r)} \leq L. \quad (3.33)$$

We now state and prove the main theorem.

**Theorem 3.8.** *If assumptions (A1)-(A4) hold, then the renormalized solution of (3.1) is unique.*

*Proof.* In view of Theorem 1 in [54], assumptions (A1)–(A3) are sufficient to give the existence of at least one solution to (3.1). Let  $u$  and  $v$  be two renormalized solutions of (3.1).

Since (3.3) holds, by Proposition 3.7, we can find a function  $\varphi \in C^1(\mathbb{R})$ , such that for some constants  $\delta > 1/2$ ,  $0 < k_0 < 1$ , and  $L > 0$ ,  $\varphi$  satisfies (3.30)–(3.33).

The proof is then decomposed into two steps. Step 1 is devoted to show the very technical result (3.34). Roughly speaking, (3.34) is an extension of the method developed by Artola in [5] (see also [19]), and allows one to consider very general dependency of  $B(x, s)$  with respect to  $s$  and  $L^1$  data.

Limit (3.34) was also derived in [16] (see also [38]) for elliptic equations with Dirichlet boundary condition. Since we have to deal with the boundary term, we give here a complete proof of (3.34). In Step 2, we are able to conclude that  $u = v$  a.e. in  $\Omega$ .

*Step 1.* In this step we prove that

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{\Omega_i} \left( \frac{1}{\varphi'(u_i)} + \frac{1}{\varphi'(v_i)} \right) |\nabla T_k(\varphi(u_i) - \varphi(v_i))|^2 dx = 0, \quad i = 1, 2. \quad (3.34)$$

Writing (3.11) of Definition 3.2 for  $u$  and  $v$ , with  $S_1 = S_2 = \theta_n$  and  $\psi = W_k := T_k(\varphi(T_{3n}(u)) - \varphi(T_{3n}(v)))$ , where  $n \in \mathbb{N}$ ,  $n \geq 1$  and  $0 < k < 1$ , and subtracting the resulting equations, we have

$$A_1^{k,n} + A_2^{k,n} + B_1^{k,n} + B_2^{k,n} + C^{k,n} = D^{k,n}, \quad (3.35)$$

where

$$\begin{aligned} A_i^{k,n} &= \int_{\Omega_i} (\theta_n(u_i)B(x, u_i)\nabla u_i - \theta_n(v_i)B(x, v_i)\nabla v_i)\nabla W_k dx, \quad i = 1, 2, \\ B_i^{k,n} &= \int_{\Omega_i} (\theta'_n(u_i)B(x, u_i)\nabla u_i\nabla u_i - \theta'_n(v_i)B(x, v_i)\nabla v_i\nabla v_i)W_k dx, \quad i = 1, 2, \\ C^{k,n} &= \int_{\Gamma} h(x)(u_1 - u_2)(W_{k,1}\theta_n(u_1) - W_{k,2}\theta_n(u_2)) d\sigma \\ &\quad - \int_{\Gamma} h(x)(v_1 - v_2)(W_{k,1}\theta_n(v_1) - W_{k,2}\theta_n(v_2)) d\sigma, \\ D^{k,n} &= \int_{\Omega} fW_k(\theta_n(u) - \theta_n(v)) dx. \end{aligned}$$

We will first look at the limit of each term as  $n \rightarrow \infty$ .

We can write  $A_i^{k,n}$  as

$$\begin{aligned} A_i^{k,n} &= \int_{\Omega_i} \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} (\nabla \varphi(u_i) - \nabla \varphi(v_i)) \nabla W_k dx \\ &\quad + \int_{\Omega_i} \theta_n(u_i) \left( \frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right) \varphi'(v_i) \nabla v_i \nabla W_k dx \\ &\quad + \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) B(x, v_i) \nabla v_i \nabla W_k dx, \quad i = 1, 2. \end{aligned}$$

Using the symmetry with respect to  $v_i$ , we obtain

$$A_i^{k,n} = A_{i,1}^{k,n} + A_{i,2}^{k,n} + A_{i,3}^{k,n}, \quad i = 1, 2,$$

where

$$\begin{aligned}
A_{i,1}^{k,n} &= \frac{1}{2} \int_{\Omega_i} \left( \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} + \theta_n(v_i) \frac{B(x, v_i)}{\varphi'(v_i)} \right) (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx, \\
A_{i,2}^{k,n} &= \frac{1}{2} \int_{\Omega_i} \left( \frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right) \\
&\quad \times (\theta_n(u_i) \varphi'(v_i) \nabla v_i + \theta_n(v_i) \varphi'(u_i) \nabla u_i) \nabla W_k \, dx, \\
A_{i,3}^{k,n} &= \frac{1}{2} \int_{\Omega_i} (\theta_n(u_i) - \theta_n(v_i)) (B(x, u_i) \nabla u_i + B(x, v_i) \nabla v_i) \nabla W_k \, dx.
\end{aligned}$$

For  $i = 1, 2$ , let us define  $U_i^k = \{x \in \Omega_i : 0 < |\varphi(u_i) - \varphi(v_i)| < k\}$ . For any  $k > 0$  small enough, since  $\text{supp } \theta_n = [-2n, 2n]$ , we have a.e. in  $U_i^k$ ,

$$\begin{aligned}
\theta_n(u_i) \nabla T_k(\varphi(u_i) - \varphi(v_i)) &= \theta_n(u_i) \nabla T_k(\varphi(T_{3n}(u_i)) - \varphi(T_{3n}(v_i))) \\
&= \theta_n(u_i) \nabla W_k, \quad i = 1, 2.
\end{aligned} \tag{3.36}$$

As a consequence of (3.36), for any  $k > 0$  small enough, we get for  $i = 1, 2$ ,

$$\begin{aligned}
A_{i,1}^{k,n} &= \frac{1}{2} \int_{\Omega_i} \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx \\
&\quad + \frac{1}{2} \int_{\Omega_i} \theta_n(v_i) \frac{B(x, v_i)}{\varphi'(v_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \nabla W_k \, dx \\
&= \frac{1}{2} \int_{U_i^k} \theta_n(u_i) \frac{B(x, u_i)}{\varphi'(u_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \, dx \\
&\quad + \frac{1}{2} \int_{U_i^k} \theta_n(v_i) \frac{B(x, v_i)}{\varphi'(v_i)} (\nabla\varphi(u_i) - \nabla\varphi(v_i)) (\nabla\varphi(u_i) - \nabla\varphi(v_i)) \, dx.
\end{aligned}$$

Using the coercivity of  $B$  we obtain that

$$\frac{\alpha}{2} \int_{U_i^k} \left( \frac{\theta_n(u_i)}{\varphi'(u_i)} + \frac{\theta_n(v_i)}{\varphi'(v_i)} \right) |\nabla(\varphi(u_i) - \varphi(v_i))|^2 \, dx \leq A_{i,1}^{k,n}, \quad i = 1, 2. \tag{3.37}$$

As far as  $A_{i,2}^{k,n}$  are concerned, by (3.32), we have for  $i = 1, 2$ , and any  $k > 0$  small enough

$$\begin{aligned}
|A_{i,2}^{k,n}| &\leq \frac{1}{2} \int_{U_i^k} \left| \frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right| |\theta_n(u_i) \nabla \varphi(v_i) + \theta_n(v_i) \nabla \varphi(u_i)| \\
&\quad \times |\nabla(\varphi(u_i) - \varphi(v_i))| dx \\
&\leq \frac{1}{2} \int_{U_i^k} \frac{1}{\varphi'(u_i)} \frac{Lk}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \\
&\quad \times |\theta_n(u_i) \nabla \varphi(v_i) + \theta_n(v_i) \nabla \varphi(u_i)| |\nabla(\varphi(u_i) - \varphi(v_i))| dx \\
&\leq \frac{1}{2} \int_{U_i^k} \frac{1}{\varphi'(u_i)} \frac{Lk}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \\
&\quad \times \theta_n(u_i) |\nabla \varphi(v_i)| |\nabla(\varphi(u_i) - \varphi(v_i))| dx \\
&\quad + \frac{1}{2} \int_{U_i^k} \frac{1}{\varphi'(u_i)} \frac{Lk}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \\
&\quad \times \theta_n(v_i) |\nabla \varphi(u_i)| |\nabla(\varphi(u_i) - \varphi(v_i))| dx.
\end{aligned}$$

For  $\varepsilon > 0$  (which will be chosen later), Young's inequality leads to, for  $i = 1, 2$ ,

$$\begin{aligned}
|A_{i,2}^{k,n}| &\leq \int_{U_i^k} \frac{\theta_n(u_i) \varphi'(v_i)}{2} \left[ \frac{1}{\varepsilon} \left( \frac{Lk}{\varphi'(u_i)} \frac{|\nabla \varphi(v_i)|}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \right)^2 \right. \\
&\quad \left. + \varepsilon \left( \frac{1}{\varphi'(v_i)} |\nabla \varphi(u_i) - \nabla \varphi(v_i)| \right)^2 \right] dx \\
&\quad + \int_{U_i^k} \frac{\theta_n(v_i) \varphi'(u_i)}{2} \left[ \frac{1}{\varepsilon} \left( \frac{Lk}{\varphi'(u_i)} \frac{|\nabla \varphi(u_i)|}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \right)^2 \right. \\
&\quad \left. + \varepsilon \left( \frac{1}{\varphi'(u_i)} |\nabla \varphi(u_i) - \nabla \varphi(v_i)| \right)^2 \right] dx \\
&\leq C_1 k^2 \int_{U_i^k} \frac{(\theta_n(u_i) \varphi'(v_i) |\nabla \varphi(v_i)|^2 + \theta_n(v_i) \varphi'(u_i) |\nabla \varphi(u_i)|^2)}{(\varphi'(u_i))^2 (1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
&\quad + C_2 \varepsilon \int_{U_i^k} \left( \frac{\theta_n(u_i)}{\varphi'(u_i)} + \frac{\theta_n(v_i)}{\varphi'(v_i)} \right) |\nabla \varphi(u_i) - \nabla \varphi(v_i)|^2 dx.
\end{aligned}$$

In view of assumption (3.33), we deduce that, for  $i = 1, 2$ ,

$$\begin{aligned}
& C_1 k^2 \int_{U_i^k} \frac{[\theta_n(u_i)\varphi'(v_i)|\nabla\varphi(v_i)|^2 + \theta_n(v_i)\varphi'(u_i)|\nabla\varphi(u_i)|^2]}{(\varphi'(u_i))^2(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
& \leq C_1 k^2 \int_{U_i^k} \frac{(L^2\theta_n(u_i)\varphi'(v_i)|\nabla v_i|^2 + \theta_n(v_i)\varphi'(u_i)|\nabla u_i|^2)}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
& \leq C_3 k^2 \int_{U_i^k} (\theta_n(u_i) + \theta_n(v_i)) \frac{\varphi'(u_i)|\nabla u_i|^2 + \varphi'(v_i)|\nabla v_i|^2}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx.
\end{aligned}$$

It follows that

$$\begin{aligned}
|A_{i,2}^{k,n}| & \leq C_3 k^2 \int_{U_i^k} (\theta_n(u_i) + \theta_n(v_i)) \frac{\varphi'(u_i)|\nabla u_i|^2 + \varphi'(v_i)|\nabla v_i|^2}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^{2\delta}} dx \\
& + C_2 \varepsilon \int_{U_i^k} \left( \frac{\theta_n(u_i)}{\varphi'(u_i)} + \frac{\theta_n(v_i)}{\varphi'(v_i)} \right) |\nabla\varphi(u_i) - \nabla\varphi(v_i)|^2 dx, \quad i = 1, 2,
\end{aligned} \tag{3.38}$$

where  $C_2$  and  $C_3$  are positive constants independent of  $k$  and  $n$  (with  $C_2$  also independent of  $\varepsilon$ ).

We now turn to the term  $A_{i,3}^{k,n}$ . By (3.30) and (3.33), we have

$$|u_i - v_i| \leq \frac{L}{\varphi'(u_i)} |\varphi(u_i) - \varphi(v_i)| \quad \text{a.e. in } U_i^k, \quad i = 1, 2, \tag{3.39}$$

and since  $\theta_n$  is a Lipschitz continuous function verifying  $|\theta'_n(r)| = \frac{1}{n}\chi_{\{n < |r| < 2n\}}$  a.e. in  $\mathbb{R}$ , we obtain

$$|\theta_n(u_i) - \theta_n(v_i)| \leq \frac{1}{n} |u_i - v_i| \leq \frac{Lk}{n\varphi'(u_i)} \quad \text{a.e. in } U_i^k, \quad i = 1, 2. \tag{3.40}$$

Observe that this inequality still holds if the roles of  $u_i$  and  $v_i$  are interchanged.

Therefore using (3.40), we obtain, for  $i = 1, 2$ ,

$$\begin{aligned}
|A_{i,3}^{k,n}| &\leq \frac{1}{2} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} |\theta_n(u_i) - \theta_n(v_i)| \left| B(x, u_i) \varphi'(u_i) \nabla u_i \nabla u_i \right. \\
&\quad \left. + B(x, u_i) \varphi'(u_i) \nabla u_i \nabla v_i - B(x, v_i) \varphi'(v_i) \nabla u_i \nabla v_i \right. \\
&\quad \left. - B(x, v_i) \varphi'(v_i) \nabla v_i \nabla v_i \right| dx \\
&\leq \frac{Lk}{2n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) \nabla u_i \nabla u_i + B(x, v_i) \nabla v_i \nabla v_i) dx \\
&\quad + \frac{Lk}{2n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) |\nabla u_i \nabla v_i| + B(x, v_i) |\nabla u_i \nabla v_i|) dx
\end{aligned}$$

Applying Young's Inequality on the second term of the previous inequality, we get, for  $i = 1, 2$ ,

$$\begin{aligned}
&\frac{Lk}{2n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) |\nabla u_i \nabla v_i| + B(x, v_i) |\nabla u_i \nabla v_i|) dx \\
&\leq \frac{Lk}{4n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} \left[ B(x, u_i) (\nabla u_i \nabla u_i + \nabla v_i \nabla v_i) \right. \\
&\quad \left. + [B(x, v_i) (\nabla u_i \nabla u_i + \nabla v_i \nabla v_i)] \right] dx.
\end{aligned}$$

It follows that, for  $i = 1, 2$ ,

$$\begin{aligned}
|A_{i,3}^{k,n}| &\leq \frac{C_4 k}{n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) \nabla u_i \nabla u_i + B(x, v_i) \nabla v_i \nabla v_i) dx \\
&\quad + \frac{C_5 k}{n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} (B(x, u_i) \nabla v_i \nabla v_i + B(x, v_i) \nabla u_i \nabla u_i) dx.
\end{aligned} \tag{3.41}$$

By (3.10a) of Definition 3.2, the first term of the right-hand side of (3.41) goes to zero as  $n$  goes to infinity. It is worth noting that the second term of the right-hand side of (3.41) contains non symmetric terms in  $u_i$  and  $v_i$ , so that without any bound on  $B$ , the behavior of this term is not a direct consequence of the decay of the truncate energy (3.10a). Using (3.10a) and condition (3.32), we claim that the second term also goes to zero as  $n$  goes to infinity.



Indeed, writing for  $i = 1, 2$ ,

$$\begin{aligned}
& \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, u_i) \nabla v_i \nabla v_i \, dx \\
&= \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} \left( \frac{B(x, u_i)}{\varphi'(u_i)} - \frac{B(x, v_i)}{\varphi'(v_i)} \right) \varphi'(u_i) \nabla v_i \nabla v_i \, dx \\
&\quad + \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} \frac{B(x, v_i)}{\varphi'(v_i)} \varphi'(u_i) \nabla v_i \nabla v_i \, dx,
\end{aligned}$$

and using (3.32) and (3.33), we have, for  $i = 1, 2$ ,

$$\begin{aligned}
& \frac{C_5 k}{n} \int_{\substack{U_i^k \cap \\ \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, u_i) \nabla v_i \nabla v_i \, dx \\
&\leq \frac{C_5 L k^2}{n} \int_{\{|v_i| < 2n+1\}} \frac{|\nabla v_i|^2}{(1 + |\varphi(u_i)| + |\varphi(v_i)|)^\delta} \, dx \\
&\quad + \frac{C_5 L k}{n} \int_{\{|v_i| < 2n+1\}} B(x, v_i) \nabla v_i \nabla v_i \, dx.
\end{aligned}$$

It follows that by (3.10a) of Definition 3.2, we have

$$\lim_{n \rightarrow \infty} \frac{C}{n} \int_{\substack{U_i^k \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, u_i) \nabla v_i \nabla v_i \, dx = 0, \quad i = 1, 2.$$

By similar computations, it can be shown that

$$\lim_{n \rightarrow \infty} \frac{C}{n} \int_{\substack{U_i^k \cap \{|u_i| < 2n+1\} \\ \cap \{|v_i| < 2n+1\}}} B(x, v_i) \nabla u_i \nabla u_i \, dx = 0, \quad i = 1, 2.$$

Consequently,

$$\lim_{n \rightarrow \infty} A_{i,3}^{k,n} = 0, \quad i = 1, 2. \tag{3.42}$$

Regarding the term  $B_i^{k,n}$ ,  $i = 1, 2$ , we have

$$\begin{aligned}
|B_i^{k,n}| &= \left| \int_{\Omega_i} (\theta'_n(u_i) B(x, u_i) \nabla u_i \nabla u_i - \theta'_n(v_i) B(x, v_i) \nabla v_i \nabla v_i) W_k \, dx \right| \\
&\leq \int_{\Omega_i} |\theta'_n(u_i)| B(x, u_i) \nabla u_i \nabla u_i |W_k| \, dx + \int_{\Omega_i} |\theta'_n(v_i)| B(x, v_i) \nabla v_i \nabla v_i |W_k| \, dx \\
&\leq \frac{k}{n} \int_{\{|u_i| < 2n\}} B(x, u_i) \nabla u_i \nabla u_i \, dx + \frac{k}{n} \int_{\{|v_i| < 2n\}} B(x, v_i) \nabla v_i \nabla v_i \, dx.
\end{aligned}$$

These last two integrals go to zero as  $n$  goes to infinity by (3.10a) of Definition 3.2. Thus,

$$\lim_{n \rightarrow \infty} B_i^{k,n} = 0, \quad i = 1, 2. \quad (3.43)$$

To pass to the limit of  $C^{k,n}$  as  $n$  goes to  $\infty$ , we use Proposition 3.5. Note that

$$|h(x)(u_1 - u_2)W_{k,1}\theta_n(u_1)| \leq \|h\|_{L^\infty(\Gamma)}|u_1 - u_2|k \in L^1(\Gamma),$$

and

$$h(x)(u_1 - u_2)W_{k,1}\theta_n(u_1) \xrightarrow[n \rightarrow \infty]{} h(x)(u_1 - u_2)T_k(\varphi(u_1) - \varphi(v_1)) \quad \text{a.e. on } \Gamma.$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\Gamma} h(x)(u_1 - u_2)W_{k,1}\theta_n(u_1) d\sigma = \int_{\Gamma} h(x)(u_1 - u_2)T_k(\varphi(u_1) - \varphi(v_1)) d\sigma.$$

Using similar arguments, we obtain that

$$\lim_{n \rightarrow \infty} \int_{\Gamma} h(x)(u_1 - u_2)W_{k,2}\theta_n(u_2) d\sigma = \int_{\Gamma} h(x)(u_1 - u_2)T_k(\varphi(u_2) - \varphi(v_2)) d\sigma$$

and for  $i = 1, 2$ ,

$$\lim_{n \rightarrow \infty} \int_{\Gamma} h(x)(v_1 - v_2)W_{k,i}\theta_n(v_i) d\sigma = \int_{\Gamma} h(x)(v_1 - v_2)T_k(\varphi(u_i) - \varphi(v_i)) d\sigma.$$

Therefore, we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} C^{k,n} &= \int_{\Gamma} h(x)[(u_1 - u_2) - (v_1 - v_2)] \\ &\quad \times [T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))] d\sigma. \end{aligned} \quad (3.44)$$

Finally, concerning  $D^{k,n}$ , since

$$|fW_k(\theta_n(u) - \theta_n(v))| \leq 2k|f| \in L^1(\Omega),$$

while

$$\theta_n(u) - \theta_n(v) \longrightarrow 0 \quad \text{a.e. in } \Omega \text{ as } n \longrightarrow \infty,$$

the Lebesgue Dominated Convergence Theorem leads to

$$\lim_{n \rightarrow \infty} D^{k,n} = 0. \quad (3.45)$$

Combining (3.37), (3.38), (3.42), (3.43), and (3.45), and choosing  $\varepsilon$  small enough, we obtain

$$\begin{aligned}
& \int_{U_1^k} \left( \frac{\theta_n(u_1)}{\varphi'(u_1)} + \frac{\theta_n(v_1)}{\varphi'(v_1)} \right) |\nabla\varphi(u_1) - \nabla\varphi(v_1)|^2 dx \\
& + \int_{U_2^k} \left( \frac{\theta_n(u_2)}{\varphi'(u_2)} + \frac{\theta_n(v_2)}{\varphi'(v_2)} \right) |\nabla\varphi(u_2) - \nabla\varphi(v_2)|^2 dx + C^{k,n} \\
& \leq Ck^2 \int_{U_1^k} (\theta_n(u_1) + \theta_n(v_1)) \frac{\varphi'(u_1)|\nabla u_1|^2 + \varphi'(v_1)|\nabla v_1|^2}{(1 + |\varphi(u_1)| + |\varphi(v_1)|)^{2\delta}} dx \\
& + Ck^2 \int_{U_2^k} (\theta_n(u_2) + \theta_n(v_2)) \frac{\varphi'(u_2)|\nabla u_2|^2 + \varphi'(v_2)|\nabla v_2|^2}{(1 + |\varphi(u_2)| + |\varphi(v_2)|)^{2\delta}} dx + \rho(n),
\end{aligned} \tag{3.46}$$

where  $C$  is a positive constant independent of  $k$  and  $n$ , and where  $\rho(n)$  goes to zero as  $n$  goes to  $\infty$ .

Let  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\tau(r) = \int_0^r \frac{\varphi'(t)}{(1 + |\varphi(t)|)^{2\delta}} dt.$$

Clearly,  $\tau$  is an increasing  $C^1(\mathbb{R})$ -function and since  $2\delta > 1$ ,  $\tau$  is bounded. Then, by Lemma 3.4 and (3.30) of Proposition 3.7, we deduce that

$$\frac{\varphi'(u_i)|\nabla u_i|^2}{(1 + |\varphi(u_i)|)^{2\delta}} \quad \text{and} \quad \frac{\varphi'(v_i)|\nabla v_i|^2}{(1 + |\varphi(v_i)|)^{2\delta}} \in L^1(\Omega_i). \tag{3.47}$$

Hence, we can pass to the limit on the right-hand side of (3.46). Furthermore,

$$\theta_n(u_i) \rightarrow 1 \quad \text{as } n \rightarrow \infty, \quad \text{a.e. in } \Omega_i, \quad i = 1, 2.$$

By Fatou's Lemma and (3.44), we have

$$\begin{aligned}
& \int_{U_1^k} \left( \frac{1}{\varphi'(u_1)} + \frac{1}{\varphi'(v_1)} \right) |\nabla\varphi(u_1) - \nabla\varphi(v_1)|^2 dx \\
& + \int_{U_2^k} \left( \frac{1}{\varphi'(u_2)} + \frac{1}{\varphi'(v_2)} \right) |\nabla\varphi(u_2) - \nabla\varphi(v_2)|^2 dx + C^k \\
& \leq Ck^2 \int_{U_1^k} \frac{\varphi'(u_1)|\nabla u_1|^2 + \varphi'(v_1)|\nabla v_1|^2}{(1 + |\varphi(u_1)| + |\varphi(v_1)|)^{2\delta}} dx \\
& + Ck^2 \int_{U_2^k} \frac{\varphi'(u_2)|\nabla u_2|^2 + \varphi'(v_2)|\nabla v_2|^2}{(1 + |\varphi(u_2)| + |\varphi(v_2)|)^{2\delta}} dx,
\end{aligned} \tag{3.48}$$

where

$$C^k = \int_{\Gamma} h(x)[(u_1 - v_1) - (u_2 - v_2)][T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))] d\sigma.$$

Dividing both sides of (3.48) by  $k^2$  and noting that  $\chi_{U_i^k} \rightarrow 0$  a.e. in  $\Omega_i$  as  $k \rightarrow 0$ , (3.47) and the Lebesgue Dominated Convergence Theorem allow one to conclude that

$$\begin{aligned} & \limsup_{k \rightarrow 0} \left( \frac{1}{k^2} \int_{U_1^k} \left( \frac{1}{\varphi'(u_1)} + \frac{1}{\varphi'(v_1)} \right) |\nabla \varphi(u_1) - \nabla \varphi(v_1)|^2 dx \right. \\ & \quad \left. + \frac{1}{k^2} \int_{U_2^k} \left( \frac{1}{\varphi'(u_2)} + \frac{1}{\varphi'(v_2)} \right) |\nabla \varphi(u_2) - \nabla \varphi(v_2)|^2 dx + \frac{1}{k^2} C^k \right) \leq 0. \end{aligned} \quad (3.49)$$

As a consequence, proving (3.34) is equivalent to showing that

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} C^k \geq 0. \quad (3.50)$$

We now study the behavior of  $C^k/k^2$  as  $k$  goes to zero. To shorten the notation, we will denote by  $g_k$  the function given by

$$g_k = h[(u_1 - v_1) - (u_2 - v_2)] \times [T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2))].$$

The main difficulty in managing this term is its non-linearity. Indeed, even if  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  a.e. on  $\Gamma$ , there is no reason to have  $g_k \geq 0$  nor to give a bound of  $g_k/k^2$ . In order to study the behavior of  $C^k/k^2$ , we decompose the integral on  $\Gamma$  into the integral on different subsets. Since from Lemma 3.6,  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  a.e. on  $\Gamma$ , in view of the symmetry of  $g_k$  with respect to  $u_i$  and  $v_i$  ( $i = 1, 2$ ), proving (3.50) is equivalent to prove

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} \int_{\{u_1 - v_1 > 0\}} g_k d\sigma \geq 0.$$

We now split the set  $\{x \in \Gamma; u_1(x) - v_1(x) > 0\}$  (up to a zero measure subset) into 4 subsets,

$$\{u_1 - v_1 > 0\} = P_1 \cup P_2 \cup P_3 \cup P_4,$$

where

$$\begin{aligned} P_1 &:= \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}, \\ P_2 &:= \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\ P_3 &:= \{\varphi(u_1) - \varphi(v_1) \geq k\} \cap \{0 < \varphi(u_2) - \varphi(v_2) < k\}, \\ P_4 &:= \{0 < \varphi(u_1) - \varphi(v_1) < k\} \cap \{\varphi(u_2) - \varphi(v_2) \geq k\}. \end{aligned}$$

Since we have

$$T_k(\varphi(u_1) - \varphi(v_1)) - T_k(\varphi(u_2) - \varphi(v_2)) = 0 \text{ a.e. on } P_1,$$

we obtain that

$$\frac{1}{k^2} \int_{P_1} g_k d\sigma = 0. \quad (3.51)$$

As far as  $\int_{P_2} g_k d\sigma$  is concerned, recalling that  $\varphi \in C^1(\mathbb{R})$  with  $\varphi'(t) \geq 1$  for any  $t \in \mathbb{R}$  gives

$$0 < u_1 - v_1 \leq \varphi(u_1) - \varphi(v_1) < k \text{ a.e. on } P_2$$

and

$$0 < u_2 - v_2 \leq \varphi(u_2) - \varphi(v_2) < k \text{ a.e. on } P_2.$$

As a consequence, we deduce that

$$\begin{aligned} \frac{1}{k^2} |g_k| &= \frac{1}{k^2} |h| |(u_1 - v_1) - (u_2 - v_2)| |(\varphi(u_1) - \varphi(v_1)) - (\varphi(u_2) - \varphi(v_2))| \\ &\leq \|h\|_{L^\infty(\Gamma)} \text{ a.e. on } P_2. \end{aligned}$$

Since

$$\chi_{P_2} \rightarrow 0 \quad \text{as } k \rightarrow 0 \quad \text{a.e. on } \Gamma,$$

the Lebesgue Dominated Convergence Theorem leads to

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{P_2} g_k d\sigma = 0. \quad (3.52)$$

We now study  $\int_{P_3} g_k d\sigma$  by splitting  $P_3$  into  $P_3 \cap \{u_1 - v_1 \geq k\}$  and  $P_3 \cap \{u_1 - v_1 < k\}$ . With already used arguments, we have

$$0 < u_2 - v_2 < k \text{ a.e. on } P_3.$$

It follows that

$$[(u_1 - v_1) - (u_2 - v_2)] \geq 0 \text{ a.e. on } P_3 \cap \{u_1 - v_1 \geq k\},$$

so that

$$g_k \geq 0 \text{ a.e. on } P_3 \cap \{u_1 - v_1 \geq k\}.$$

On the other hand, we have

$$\begin{aligned} \frac{1}{k^2} |g_k| &= \frac{1}{k^2} |h| |(u_1 - v_1) - (u_2 - v_2)| (k - (\varphi(u_2) - \varphi(v_2))) \\ &\leq \|h\|_{L^\infty(\Gamma)} \text{ a.e. on } P_3 \cap \{0 < u_1 - v_1 < k\}. \end{aligned}$$

Since

$$\chi_{P_3} \rightarrow 0 \quad \text{as } k \rightarrow 0, \quad \text{a.e. on } \Gamma,$$

the Lebesgue Dominated Convergence Theorem leads to

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{P_3 \cap \{0 < u_1 - v_1 < k\}} g_k d\sigma = 0.$$

Noting that we can write

$$\int_{P_3} g_k d\sigma = \int_{P_3 \cap \{u_1 - v_1 \geq k\}} g_k d\sigma + \int_{P_3 \cap \{0 < u_1 - v_1 < k\}} g_k d\sigma,$$

we deduce that

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} \int_{P_3} g_k d\sigma \geq 0. \quad (3.53)$$

At last, by writing

$$\int_{P_4} g_k d\sigma = \int_{P_4 \cap \{u_2 - v_2 \geq k\}} g_k d\sigma + \int_{P_4 \cap \{0 < u_2 - v_2 < k\}} g_k d\sigma,$$

and by proving with similar arguments that

$$\int_{P_4 \cap \{u_2 - v_2 \geq k\}} g_k d\sigma \geq 0$$

and

$$\lim_{k \rightarrow 0} \frac{1}{k^2} \int_{P_4 \cap \{0 < u_2 - v_2 < k\}} g_k d\sigma = 0,$$

yield

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} \int_{P_4} g_k d\sigma \geq 0. \quad (3.54)$$

The results (3.51)–(3.54) give

$$\limsup_{k \rightarrow 0} \frac{1}{k^2} C^k \geq 0. \quad (3.55)$$

Therefore, (3.34) holds true.

*Step 2.* In this step, we prove that  $u_1 = v_1$  a.e. in  $\Omega_1$  and  $u_2 = v_2$  a.e. in  $\Omega_2$ .

We first show that  $u_1 = v_1$  a.e. in  $\Omega_1$ . To do this, we consider the function

$$\theta_n(u_1) T_k(\varphi(u_1) - \varphi(v_1)) \in L^\infty(\Omega_1) \cap V_1.$$

Since  $u_1 = v_1 = 0$  on  $\partial\Omega$ , we can apply Poincaré inequality which leads to

$$\begin{aligned} & \int_{\Omega_1} (\theta_n(u_1))^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx \\ & \leq C \int_{\Omega_1} (\theta_n(u_1))^2 \left( \frac{\nabla T_k(\varphi(u_1) - \varphi(v_1))}{k} \right)^2 dx \\ & \quad + C \int_{\Omega_1} (\theta'_n(u_1))^2 |\nabla u_1|^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx, \end{aligned} \quad (3.56)$$

where  $C > 0$  does not depend on  $k$  and  $n$ . The second integral on the right-hand side of (3.56) can be bounded by

$$\int_{\Omega_1} (\theta'_n(u_1))^2 |\nabla u_1|^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx \leq \frac{1}{n^2} \int_{\{|u_1| < 2n\}} |\nabla u_1|^2 dx.$$

The integral on the right-hand side goes to zero as  $n$  goes to  $\infty$  by (3.10a) of Definition 3.2. This implies

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \int_{\Omega_1} (\theta'_n(u_1))^2 |\nabla u_1|^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx = 0. \quad (3.57)$$

For the first integral on the right-hand side of (3.56), we have

$$\begin{aligned} & \int_{\Omega_1} (\theta_n(u_1))^2 \left( \frac{\nabla T_k(\varphi(u_1) - \varphi(v_1))}{k} \right)^2 dx \\ & \leq \frac{\max_{s \in [-2n, 2n]} \varphi'(s)}{k^2} \int_{U_k} \frac{1}{\varphi'(u_1)} |\nabla \varphi(u_1) - \nabla \varphi(u_2)|^2 dx. \end{aligned}$$

The integral on the right-hand side of this inequality goes to 0 as  $k$  goes to 0 by (3.34). Thus,

$$\begin{aligned} & \int_{\Omega_1} \chi_{\{\varphi(u_1) \neq \varphi(v_1)\}} dx \\ & = \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \int_{\Omega_1} (\theta_n(u_1))^2 \left| \frac{T_k(\varphi(u_1) - \varphi(v_1))}{k} \right|^2 dx = 0, \end{aligned} \quad (3.58)$$

that is,  $\varphi(u_1) = \varphi(v_1)$  a.e. in  $\Omega_1$ . Since  $\varphi' \geq 1$ , we have  $u_1 = v_1$  a.e. in  $\Omega_1$ . As a consequence,  $\gamma_1(u_1) = \gamma_1(v_1)$ , that is,  $u_1 = v_1$  a.e. on  $\Gamma$ .

From Lemma 3.6,  $\text{sgn}(u_1 - v_1) = \text{sgn}(u_2 - v_2)$  on  $\Gamma$ . Since  $u_1 - v_1 = 0$  a.e. on  $\Gamma$ , we also have  $u_2 - v_2 = 0$  a.e. on  $\Gamma$ . Thus,  $u_2 = v_2$  a.e. on  $\Gamma$ .

It only remains to prove that  $u_2 = v_2$  in  $\Omega_2$ . Consider the function

$$\theta_n(u_2)T_k(\varphi(u_2) - \varphi(v_2)) \in L^\infty(\Omega_2) \cap H^1(\Omega_2),$$

which has a zero value a.e. on  $\Gamma$  since  $u_2 = v_2$  a.e. on  $\Gamma$ . We can then apply Poincaré inequality which implies that

$$\begin{aligned} & \int_{\Omega_2} (\theta_n(u_2))^2 \left| \frac{T_k(\varphi(u_2) - \varphi(v_2))}{k} \right|^2 dx \\ & \leq C \int_{\Omega_2} (\theta_n(u_2))^2 \left( \frac{\nabla T_k(\varphi(u_2) - \varphi(v_2))}{k} \right)^2 dx \\ & \quad + C \int_{\Omega_2} (\theta'_n(u_2))^2 |\nabla u_2|^2 \left| \frac{T_k(\varphi(u_2) - \varphi(v_2))}{k} \right|^2 dx. \end{aligned}$$

Using the same arguments to show (3.58), we conclude that

$$\int_{\Omega_2} \chi_{\{\varphi(u_2) \neq \varphi(v_2)\}} dx = \lim_{n \rightarrow \infty} \lim_{k \rightarrow 0} \int_{\Omega_2} (\theta_n(u_2))^2 \left| \frac{T_k(\varphi(u_2) - \varphi(v_2))}{k} \right|^2 dx = 0.$$

This implies  $\varphi(u_2) = \varphi(v_2)$  a.e. in  $\Omega_2$ . Therefore, since  $\varphi' \geq 1$ ,  $u_2 = v_2$  a.e. in  $\Omega_2$ .

This concludes the proof of the uniqueness of the renormalized solution of (3.1).  $\square$





## Part II

# Homogenization results



# Chapter 4

## Some properties of an elliptic periodic problem with an interfacial resistance

### 4.1 Introduction

In several models of partial differential equations, an imperfect contact on the interface between two materials gives rise to an interfacial resistance, often modelled by a jump of the solution on the interface, which is proportional to the normal derivative. For a heat diffusion model, this means that the jump of the temperature is proportional to the flux (see [26] for a justification of the model).

When the coefficients are periodically oscillating, and the interface is  $\varepsilon$ -periodic, the asymptotic behaviour of the problem as  $\varepsilon \rightarrow 0$  has been widely studied in literature. In this case, the proportionality factor is of order  $\varepsilon^\gamma$ , and the homogenized problem depends on the value of  $\gamma$ . Let us briefly describe the domain, more precisely defined in Section 4, and the problem we are concerned about in this chapter (see Figure 4.1).

The two-component domain is an open bounded set  $\Omega$  of  $\mathbb{R}^N$  (with  $N \geq 2$ ), which is the union of two open subsets  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$ , and of their common boundary (the interface) denoted  $\Gamma^\varepsilon$ . The component  $\Omega_2^\varepsilon$  is the union of the  $\varepsilon$ -periodic translated sets of  $\varepsilon Y_2$ ,  $\overline{Y_2}$  being contained in the reference periodicity cell  $Y$ . The other component  $\Omega_1^\varepsilon$  is connected and defined by  $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$ , while the interface is given by  $\Gamma^\varepsilon = \partial\Omega_2^\varepsilon$ .

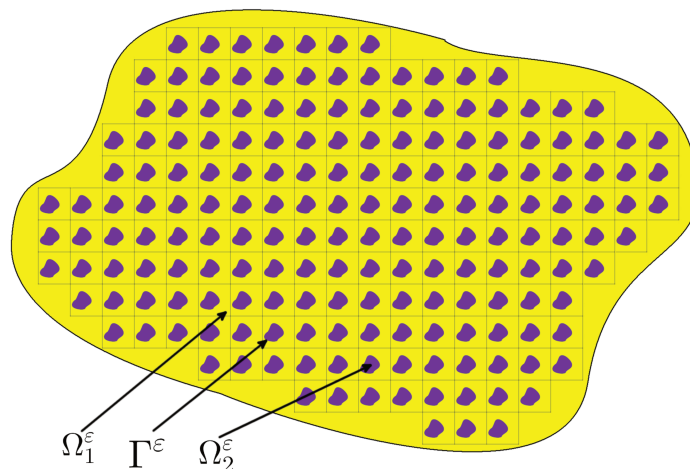


Figure 4.1: The two-component domain  $\Omega$  and its periodic interface.

The corresponding elliptic homogenization problem with jump is

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon, \\ [A^\varepsilon \nabla u^\varepsilon] \cdot \nu^\varepsilon = 0 & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon \nabla u^\varepsilon) \nu^\varepsilon = -\varepsilon^\gamma h^\varepsilon(x) [u^\varepsilon] & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

where  $\nu^\varepsilon$  is the unit outward normal to  $\Omega_1^\varepsilon$ ,  $[\cdot]$  denotes the jump through  $\Gamma^\varepsilon$ ,  $\gamma$  is a real parameter, and  $f \in L^2(\Omega)$ .

Also,

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right), \quad h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right),$$

where  $A$  is a real uniformly elliptic and bounded  $Y$ -periodic matrix field, and  $h$  is a bounded  $Y$ -periodic function in  $L^\infty(\Gamma)$ , positively bounded from below.

The pioneer paper on this subject is due to J.L. Auriault and H.I. Ene [6], via asymptotic expansions for some particular value of  $\gamma$  (see also [50], [51] for the case of two-connected component). As proved in [60], in order to have uniform a priori estimates, one has to suppose  $\gamma \leq 1$ .

Homogenization results for the elliptic case when one component is a disconnected union of periodic domains have been studied using the Tartar's method of oscillating test function by S. Monsurrò in [68, 69] for  $\gamma \leq -1$ , and by P. Donato, S. Monsurrò in [47] for  $\gamma > -1$  (see also [40] for a unified presentation). Successively, these results were recovered and completed by using the periodic unfolding method in [46] (see also [45]).

Let us recall the results for the case  $\gamma = -1$  (see [47], [46]), which motivate our study. For  $i = 1, 2$ , set  $\theta_i = \frac{|Y_i|}{|Y|}$ , with  $Y_1 = Y \setminus \overline{Y_2}$ .

Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be the solution of problem (4.1) for  $\gamma = -1$ . Then, there exists  $u_1 \in H_0^1(\Omega)$  such that

$$\begin{cases} \widetilde{u}_i^\varepsilon \rightharpoonup \theta_i u_1 & \text{weakly in } L^2(\Omega), i = 1, 2, \\ A^\varepsilon \widetilde{\nabla} u_1^\varepsilon \rightharpoonup A_1^0 \nabla u_1 & \text{weakly in } (L^2(\Omega))^N, \\ A^\varepsilon \widetilde{\nabla} u_2^\varepsilon \rightharpoonup A_2^0 \nabla u_1 & \text{weakly in } (L^2(\Omega))^N, \end{cases} \quad (4.2)$$

where  $\widetilde{\cdot}$  denotes the zero extension to the whole of  $\Omega$ , and  $u_1$  is the unique solution of the problem

$$\begin{cases} -\operatorname{div}(A^0 \nabla u_1) = f & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.3)$$

The homogenized matrix  $A^0$  is defined as

$$A^0 = A_1^0 + A_2^0, \quad A_i^0 \lambda = \frac{1}{|Y_i|} \int_{Y_i} A(y)(\lambda - \nabla \chi_i^\lambda(y)) dy, \quad i = 1, 2, \forall \lambda \in \mathbb{R}^N,$$

where  $\chi_i^\lambda = \chi^\lambda|_{Y_i}$ ,  $\chi^\lambda$  is the solution of

$$\begin{cases} -\operatorname{div}(A \nabla \chi^\lambda) = -\operatorname{div}(A \lambda) & \text{in } Y_1 \cup Y_2, \\ [A \nabla \chi^\lambda \cdot n_1] = 0 & \text{on } \Gamma, \\ A \nabla \chi_1^\lambda \cdot n_1 = -h(y)[\chi^\lambda] & \text{on } \Gamma, \\ \chi_1^\lambda & Y\text{-periodic}, \\ \mathcal{M}_\Gamma(\chi_1^\lambda) = 0, \end{cases} \quad (4.4)$$

and  $[\cdot]$  denotes the jump through  $\Gamma$ , the term  $-\operatorname{div}(A \lambda)$  being intended as in (4.13)-(4.14) below.

For similar elliptic homogenization problems we refer to [24], [50], [51], [64], [65] and [74]. More recently, a similar boundary condition has been considered in [67], where the authors study the homogenization of linearized elasticity in a two-component medium with slip displacement conditions.

We also refer to [23] for the homogenization, by unfolding, of a quasilinear elliptic problem in perforated domains with Robin conditions on the interface, which motivates this study.

Indeed, in the homogenization of quasilinear elliptic problems, it is crucial for the homogenized matrix to have some kind of Lipschitz continuity to prove

the uniqueness of the solution of the limit problem. For example, in [23, Theorem 3.3], it is proved that if a sequence of periodic matrices satisfies the assumption introduced by Chipot in [27] (see Remark 4.12), then this assumption still holds true for the homogenized matrix field. To prove this result, a Meyers type estimate for the solution of the periodic cell problem is needed.

In this chapter, we present some results similar to Theorem 3.3 of [23], but in the case where the solution of the cell problem describing the homogenized matrix is the solution of problem (4.4), which present a jump, which corresponds to the homogenization of problem (4.1) in the case  $\gamma = -1$ .

To do that, we need first to prove a Meyers type estimate for the solution of the cell problem in this case, which makes use of a Meyers type result proved by T. Gallouët and A. Monier in [55] for elliptic equations with nonhomogeneous Neumann boundary conditions. We also prove, for completeness, a boundedness result for that solution.

The main difficulty here comes from the fact that a weak solution of this problem is not an  $H^1(Y)$ -function, since only the restriction of the solution to each component  $Y_i$  belongs to  $H^1(Y_i)$ . Hence, we have to deal with a jump on the interface  $\Gamma$ , which gives rise to a boundary term in the variational formulation (see (4.15)). This implies that one cannot merely repeat the arguments of the classical case without jump in the proofs, but most of the steps are more delicate and need suitable arguments.

The present chapter is organised as follows:

In the next section, we introduce some definitions and we state the periodic problem in a two-component cell.

Section 4.3 is devoted to the Meyers type result.

In Section 4.4, we prove some results for a rescaled version of (4.15), which will be needed for the main result of the last section.

Section 4.5 contains some additional properties adapted to the quasilinear case, where  $A = B(y, t)$  is a Carathéodory matrix field depending also on a real parameter  $t$ . More precisely, we prove an estimate of the difference of two homogenized matrices in terms of the difference of the oscillating ones. As a consequence, in Corollary 4.11 (see also Remark 4.12) we deduce a generalization of Theorem 3.3 of [23] to the case with jump.

Finally, in Section 4.6, we prove the boundedness of the solution to (4.15).

We believe that these results can be very helpful not only to extend the homogenization results proved for problem (4.1) with  $\gamma = -1$  to the quasilinear case, but also in the homogenization of other elliptic problems in progress (e.g. quasilinear problems with singularity or with  $L^1$  data in two-component domains).

## 4.2 Statement of the problem

We present in this section the assumptions for our problem and some definitions.

Let  $Y = \prod_{j=1}^N [0, l_j)$ , where  $l_j > 0$  for  $j = 1, 2, \dots, N$  (see Figure 4.2).

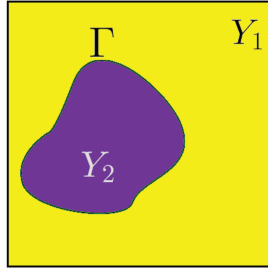


Figure 4.2: The two-component cell  $Y$ .

The open sets  $Y_1$  and  $Y_2$  are subsets of  $Y$  such that  $Y_1 = Y \setminus \overline{Y_2}$ , and  $Y_2$  satisfies the following:

$$\left\{ \begin{array}{l} (1) \quad \overline{Y_2} \subset Y; \\ (2) \quad \partial Y_2 = \Gamma \text{ is Lipschitz continuous; and} \\ (3) \quad Y_2 \text{ has a finite number } m \text{ of open connected components} \\ \quad \quad Y_2^j, j = 1, 2, \dots, m, \text{ with } \overline{Y_2^j} \cap \overline{Y_2^k} = \emptyset \text{ if } j \neq k. \end{array} \right. \quad (4.5)$$

For  $i = 1, 2$ , we denote by  $n_i$  the unit exterior normal to  $Y_i$ .

We define  $M(\alpha, \beta, Y)$  as the set of  $N \times N$  matrix-valued functions  $A$  in  $(L^\infty(Y))^{N \times N}$  such that for any  $\lambda \in \mathbb{R}^N$ ,

- $A(y)\lambda \cdot \lambda \geq \alpha|\lambda|^2$  for a.e.  $y \in Y$ ;
- $|A(y)\lambda| \leq \beta|\lambda|$  for a.e.  $y \in Y$ .

For any integrable function  $v$ , we define the mean value of  $v$  on  $\Gamma$  as

$$\mathcal{M}_\Gamma(v) = \frac{1}{|\Gamma|} \int_\Gamma v \, dy. \quad (4.6)$$

Let

$$W_{per}(Y_1) = \{u \in H_{per}^1(Y_1) \mid \mathcal{M}_\Gamma(u) = 0\} \quad (4.7)$$

equipped with the norm

$$\|u\|_{W_{per}(Y_1)} = \|\nabla u\|_{L^2(Y_1)}. \quad (4.8)$$



Recall that (4.8) defines a norm since a Poincaré-Wirtinger inequality holds on  $W_{per}(Y_1)$ .

From now on, for any measurable function  $u$  defined on  $Y$ , for  $i = 1, 2$ , we denote by  $u_i = u|_{Y_i}$ , the restriction of  $u$  on  $Y_i$  and by  $\tilde{\cdot}$  the zero extension to the whole of  $Y$ .

Let us introduce the space

$$\mathcal{H} = \{u \in L^2(Y) \mid u_1 \in W_{per}(Y_1), u_2 \in H^1(Y_2)\}$$

equipped with the norm

$$\|u\|_{\mathcal{H}}^2 = \|\nabla u_1\|_{L^2(Y_1)}^2 + \|\nabla u_2\|_{L^2(Y_2)}^2 + \|u_1 - u_2\|_{L^2(\Gamma)}^2. \quad (4.9)$$

**N.B.** In the sequel, we identify the gradient  $\nabla u$  of a function  $u$  in  $\mathcal{H}$  with its absolutely continuous part  $\widetilde{\nabla u_1} + \widetilde{\nabla u_2}$ , and we set

$$\|\nabla u\|_{L^2(Y \setminus \Gamma)}^2 = \|\nabla u_1\|_{L^2(Y_1)}^2 + \|\nabla u_2\|_{L^2(Y_2)}^2, \quad (4.10)$$

so that, (4.9) reads

$$\|u\|_{\mathcal{H}}^2 = \|\nabla u\|_{L^2(Y \setminus \Gamma)}^2 + \|u_1 - u_2\|_{L^2(\Gamma)}^2.$$

**Remark 4.1.** The norm given in (4.9) is equivalent to the norm

$$\|u\|_{W_{per}(Y_1) \times H^1(Y_2)}^2 = \|u_1\|_{W_{per}(Y_1)}^2 + \|u_2\|_{H^1(Y_2)}^2, \quad (4.11)$$

that is, there exist 2 positive constants  $c_1, c_2$ , depending on  $Y_1, Y_2$ , such that

$$c_1 \|u\|_{W_{per}(Y_1) \times H^1(Y_2)} \leq \|u\|_{\mathcal{H}} \leq c_2 \|u\|_{W_{per}(Y_1) \times H^1(Y_2)}. \quad (4.12)$$

The proof of the equivalence is similar to that of Proposition 2.6 of [68], where the result is presented for the space  $V_1 \times H^1(Y_2)$ , with

$$V_1 = \{v \in H^1(Y_1) : v = 0 \text{ on } \partial Y\} \quad \text{with } \|v\|_{V_1} = \|\nabla v\|_{L^2(Y_1)}.$$

In [68], the estimates for  $V_1 \times H^1(Y_2)$  is obtained by using the Poincaré inequality in  $V_1$ . The only difference here is that one has to use the Poincaré-Wirtinger inequality on  $W_{per}(Y_1)$  instead.

Our aim is to study some properties of the solution of the following problem:

$$\begin{cases} -\operatorname{div}(A\nabla\chi_1^\lambda) = G_1^\lambda & \text{in } Y_1, \\ -\operatorname{div}(A\nabla\chi_2^\lambda) = G_2^\lambda & \text{in } Y_2, \\ A\nabla\chi_1^\lambda \cdot n_1 = -A\nabla\chi_2^\lambda \cdot n_2 & \text{on } \Gamma, \\ A\nabla\chi_1^\lambda \cdot n_1 = -h(y)(\chi_1^\lambda - \chi_2^\lambda) & \text{on } \Gamma, \\ \chi_1^\lambda \text{ } Y\text{-periodic,} \\ \mathcal{M}_\Gamma(\chi_1^\lambda) = 0, \end{cases} \quad (4.13)$$

where  $\lambda \in \mathbb{R}^N$  and  $G_i^\lambda$  is defined by

$$\langle G_i^\lambda, v \rangle = \int_{Y_i} A \lambda \nabla v \, dy, \quad \forall v \in H^1(Y_i), \quad i = 1, 2, \quad (4.14)$$

which belongs to  $(H^1(Y_i))'$ .

**Remark 4.2.** *Observe that a Poincaré-Wirtinger inequality still holds in  $W_{per}(Y_1)$  if in (4.7) we replace the condition  $\mathcal{M}_\Gamma(u) = 0$  by  $\mathcal{M}_{Y_1}(u) = 0$ , where*

$$\mathcal{M}_{Y_1}(u) = \frac{1}{|Y_1|} \int_{Y_1} u \, dy.$$

Consequently, if we set

$$\widehat{\chi}_i^\lambda = \chi_i^\lambda - \mathcal{M}_{Y_1}(\chi_1^\lambda), \quad i = 1, 2, \quad \forall \lambda \in \mathbb{R}^N,$$

then  $\widehat{\chi}^\lambda = (\widehat{\chi}_1^\lambda, \widehat{\chi}_2^\lambda)$  is the unique solution of

$$\begin{cases} -\operatorname{div}(A \nabla \widehat{\chi}_1^\lambda) = G_1^\lambda & \text{in } Y_1, \\ -\operatorname{div}(A \nabla \widehat{\chi}_2^\lambda) = G_2^\lambda & \text{in } Y_2, \\ A \nabla \widehat{\chi}_1^\lambda \cdot n_1 = -A \nabla \widehat{\chi}_2^\lambda \cdot n_2 & \text{on } \Gamma, \\ A \nabla \widehat{\chi}_1^\lambda \cdot n_1 = -h(y)(\widehat{\chi}_1^\lambda - \widehat{\chi}_2^\lambda) & \text{on } \Gamma, \\ \widehat{\chi}_1^\lambda \text{ } Y\text{-periodic,} \\ \mathcal{M}_{Y_1}(\widehat{\chi}_1^\lambda) = 0, \end{cases}$$

which is a problem of the same type as (4.13).

Hence, all the results stated here for  $\chi^\lambda$  are still true for  $\widehat{\chi}^\lambda$ . We present here the problem under the form (4.13), since it is the problem which is used for describing the homogenization of elliptic problems with a jump when  $\gamma = -1$  (see [46]-[47]).

We make the following assumptions on  $A$  and  $h$ :

- (A1) The matrix field  $A$  belongs to  $M(\alpha, \beta, Y)$ , where  $0 < \alpha < \beta$ .
- (A2) The function  $h$  is in  $L^\infty(\Gamma)$  with  $0 < h_0 < h(y)$  for a.e.  $y \in \Gamma$ , for some  $h_0 \in \mathbb{R}$ .

Using (4.14), the variational formulation of (4.13) is

$$\begin{cases} \text{Find } \chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda) \in \mathcal{H} \text{ such that} \\ \int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla v \, dy + \int_\Gamma h(y)(\chi_1^\lambda - \chi_2^\lambda)(v_1 - v_2) \, d\sigma = \int_{Y \setminus \Gamma} A \lambda \nabla v \, dy, \\ \text{for any } v = (v_1, v_2) \in \mathcal{H}. \end{cases} \quad (4.15)$$

The existence and uniqueness of the solution of (4.15) can be easily established by the Lax-Milgram theorem. In the next sections, we investigate some properties of the solution of (4.15).

### 4.3 A Meyers type estimate

We prove here that the gradient of the solution, which is in  $L^2(Y \setminus \Gamma)$ , has a better summability.

**Theorem 4.3.** *Let  $\lambda \in \mathbb{R}^N$  and let  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda) \in \mathcal{H}$  be the solution of (4.15). Then for any  $\lambda \in \mathbb{R}^N$ , there exists  $p_i > 2$ ,  $i = 1, 2$ , such that*

$$\chi_i^\lambda \in W^{1,p_i}(Y_i). \quad (4.16)$$

Moreover, for  $i = 1, 2$ , for every  $2 \leq q_i \leq p_i$ , there exists a positive constant  $c_i$ , dependent on  $\alpha$ ,  $\beta$ ,  $q_i$ , and  $Y_i$ , such that

$$\|\nabla \chi_i^\lambda\|_{L^{q_i}(Y_i)} \leq c_i |\lambda|. \quad (4.17)$$

To prove this theorem, we apply the following Meyers type result, which has been proved by Gallouët and Monier in [55].

**Theorem 4.4** ([55]). *Let  $N \geq 2$  and  $\Omega$  be a bounded connected open subset of  $\mathbb{R}^N$  with a Lipschitz continuous boundary  $\partial\Omega$ . Suppose  $u$  is the unique solution of*

$$\begin{cases} -\operatorname{div}(A\nabla u) = F & \text{in } \Omega \\ A\nabla u \cdot n + hu = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.18)$$

where  $A \in M(\alpha, \beta, \Omega)$ , for some  $0 < \alpha < \beta$ , and  $h \in L^\infty(\partial\Omega)$  with  $0 < h_0 \leq h(y)$ , for a.e.  $y \in \partial\Omega$ , for some real number  $h_0$ .

Define

$$2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N \geq 3 \\ \text{any number in } (2, +\infty), & \text{if } N = 2. \end{cases} \quad (4.19)$$

Then there exists  $p_0 > 2$  (with  $p_0 < 2^*$  for  $N \geq 3$ ) such that for every  $2 < q < p_0$ , if  $F \in (W^{1,q'}(\Omega))'$  then  $u \in W^{1,q}(\Omega)$ , and for some  $C > 0$ ,

$$\|u\|_{W^{1,q}(\Omega)} \leq C \|F\|_{(W^{1,q'}(\Omega))'}, \quad (4.20)$$

where  $C$  depends on  $\alpha$ ,  $\beta$ ,  $\Omega$ , and  $q$ .

**Remark 4.5.** 1. It is easy to check that Theorem 4.4 still holds if  $\Omega = Y_1$  and we prescribe the boundary condition only on  $\Gamma$ , and the  $Y$ -periodicity of  $u$ .

2. Theorem 4.4 also holds if  $\Omega = Y_2$  as defined in (4.5) since we can apply it to every connected component  $Y_2^j$  of  $Y_2$ .

We establish the following lemma which is needed to prove Theorem 4.3.

**Lemma 4.6.** Let  $\lambda \in \mathbb{R}^N$  and let  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$  be the solution to (4.15). For  $i = 1, 2$ , set

$$t = \begin{cases} \frac{2(N-1)}{N-2} & \text{if } N \geq 3, \\ \text{any number in } (2, +\infty) & \text{if } N = 2, \end{cases} \quad (4.21)$$

and let  $\mu_1^\lambda$  and  $\mu_2^\lambda$  be the linear functionals defined by

$$\mu_1^\lambda(v_1) = \int_{\Gamma} h \chi_2^\lambda v_1 d\sigma, \quad \forall v_1 \in W^{1,t}(Y_1), \quad (4.22)$$

and

$$\mu_2^\lambda(v_2) = \int_{\Gamma} h \chi_1^\lambda v_2 d\sigma, \quad \forall v_2 \in W^{1,t}(Y_2), \quad (4.23)$$

respectively. Then, for  $i = 1, 2$ , for every  $1 \leq r_i \leq t$  we have

$$\mu_i^\lambda \in (W^{1,r_i}(Y_i))', \quad (4.24)$$

and for some  $b_i > 0$ ,

$$\|\mu_i^\lambda\|_{(W^{1,r_i}(Y_i))'} \leq b_i |\lambda|, \quad \forall \lambda \in \mathbb{R}^N. \quad (4.25)$$

*Proof.* We first observe that

$$\|\chi^\lambda\|_{W_{per}(Y_1) \times H^1(Y_2)} \leq c_1^{-1} \|\chi^\lambda\|_{\mathcal{H}} \leq c |\lambda|, \quad (4.26)$$

for some positive constant  $c$  that depends on  $\alpha, \beta, h_0, Y_1$ , and  $Y_2$ . Indeed, the first inequality is a consequence of the equivalence of the norms in (4.12). To prove the second inequality, take  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$  as a test function in (4.15), which gives

$$\int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla \chi^\lambda dy + \int_{\Gamma} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\chi_1^\lambda - \chi_2^\lambda) d\sigma = \int_{Y \setminus \Gamma} A \lambda \nabla \chi^\lambda dy.$$

Then, by the ellipticity of  $A$ , the assumptions on  $h$ , and Hölder's inequality, we obtain

$$\begin{aligned} \min\{\alpha, h_0\} \|\chi^\lambda\|_{\mathcal{H}}^2 &\leq \alpha \|\nabla \chi^\lambda\|_{L^2(Y \setminus \Gamma)}^2 + h_0 \|\chi_1^\lambda - \chi_2^\lambda\|_{L^2(\Gamma)}^2 \\ &\leq \int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla \chi^\lambda dy + \int_{\Gamma} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\chi_1^\lambda - \chi_2^\lambda) d\sigma \\ &= \int_{Y \setminus \Gamma} A \lambda \nabla \chi^\lambda dy \leq \beta |\lambda| \|\nabla \chi^\lambda\|_{L^2(Y \setminus \Gamma)} \leq \beta |\lambda| \|\chi^\lambda\|_{\mathcal{H}}. \end{aligned}$$

Hence, (4.26) holds true.

Note that for  $i = 1, 2$ ,  $r_i \leq t$ , we have  $r'_i \geq t'$  and hence  $v_i \in W^{1, r'_i}(Y_i) \subseteq W^{1, t'}(Y_i)$ . Moreover, we have the following continuous embeddings,

$$\begin{aligned} H^1(Y_1) &\subset L^t(\Gamma) \subseteq L^{r_2}(\Gamma), \\ H^1(Y_2) &\subset L^t(\Gamma) \subseteq L^{r_1}(\Gamma). \end{aligned}$$

Then, since  $\chi_i^\lambda \in H^1(Y_i)$  for  $i = 1, 2$ , it follows that for some positive constants  $a_1$  and  $a_2$ ,

$$\chi_1^\lambda \in L^{r_2}(\Gamma) \quad \text{with} \quad \|\chi_1^\lambda\|_{L^{r_2}(\Gamma)} \leq a_1 \|\chi_1^\lambda\|_{H^1(Y_1)}, \quad (4.27)$$

$$\chi_2^\lambda \in L^{r_1}(\Gamma) \quad \text{with} \quad \|\chi_2^\lambda\|_{L^{r_1}(\Gamma)} \leq a_2 \|\chi_2^\lambda\|_{H^1(Y_2)}. \quad (4.28)$$

Note that for  $i = 1, 2$ ,  $\mu_i^\lambda$  is linear and using the Trace Theorem, Hölder's inequality, (4.27), (4.28), and (4.26), we have

$$\begin{aligned} |\langle \mu_1^\lambda, v_1 \rangle| &\leq \|h\|_{L^\infty(\Gamma)} \|\chi_2^\lambda\|_{L^{r_1}(\Gamma)} \|v_1\|_{L^{r'_1}(\Gamma)} \\ &\leq a_2 \|h\|_{L^\infty(\Gamma)} \|\chi_2^\lambda\|_{H^1(Y_2)} \|v_1\|_{W^{1, r'_1}(Y_1)} \\ &\leq b_1 |\lambda| \|v_1\|_{W^{1, r'_1}(Y_1)}, \end{aligned}$$

and

$$\begin{aligned} |\langle \mu_2^\lambda, v_2 \rangle| &\leq \|h\|_{L^\infty(\Gamma)} \|\chi_1^\lambda\|_{L^{r_2}(\Gamma)} \|v_2\|_{L^{r'_2}(\Gamma)} \\ &\leq a_1 \|h\|_{L^\infty(\Gamma)} \|\chi_1^\lambda\|_{H^1(Y_1)} \|v_2\|_{W^{1, r'_2}(Y_2)} \\ &\leq b_2 |\lambda| \|v_2\|_{W^{1, r'_2}(Y_2)}, \end{aligned}$$

for some positive constants  $b_1, b_2$  which depend on  $\alpha, \beta, h, Y_1, Y_2, r_1$ , and  $r_2$ . These imply (4.24) and (4.25).  $\square$

**Proof of Theorem 4.3:** Fix  $\lambda \in \mathbb{R}^N$ . From (4.13)-(4.14), we can write the following system for  $\chi_1^\lambda$ :

$$\begin{cases} -\operatorname{div}(A\nabla\chi_1^\lambda) = G_1^\lambda & \text{in } Y_1, \\ A\nabla\chi_1^\lambda \cdot n_1 + h\chi_1^\lambda = h\chi_2^\lambda & \text{on } \Gamma, \\ \chi_1^\lambda \text{ } Y\text{-periodic,} \\ \mathcal{M}_\Gamma(\chi_1^\lambda) = 0, \end{cases} \quad (4.29)$$

while for  $\chi_2^\lambda$  we have

$$\begin{cases} -\operatorname{div}(A\nabla\chi_2^\lambda) = G_2^\lambda & \text{in } Y_2, \\ A\nabla\chi_2^\lambda \cdot n_2 + h\chi_2^\lambda = h\chi_1^\lambda & \text{on } \Gamma. \end{cases} \quad (4.30)$$

System (4.29) can be rewritten as

$$\begin{cases} -\operatorname{div}(A\nabla\chi_1^\lambda) = G_1^\lambda + \mu_1^\lambda & \text{in } Y_1, \\ A\nabla\chi_1^\lambda \cdot n_1 + h\chi_1^\lambda = 0 & \text{on } \Gamma, \\ \chi_1^\lambda \text{ } Y\text{-periodic,} \\ \mathcal{M}_\Gamma(\chi_1^\lambda) = 0, \end{cases} \quad (4.31)$$

and (4.30) as

$$\begin{cases} -\operatorname{div}(A\nabla\chi_2^\lambda) = G_2^\lambda + \mu_2^\lambda & \text{in } Y_2, \\ A\nabla\chi_2^\lambda \cdot n_2 + h\chi_2^\lambda = 0 & \text{on } \Gamma, \end{cases} \quad (4.32)$$

where  $\mu_1^\lambda$  and  $\mu_2^\lambda$  are as defined in (4.22) and (4.23), respectively.

For  $i = 1, 2$ , set

$$F_i = G_i^\lambda + \mu_i^\lambda. \quad (4.33)$$

Let us apply Theorem 4.4 (taking also into account Remark 4.5) first to (4.31) with  $F = F_1$  and then to (4.32) with  $F = F_2$ . Then for  $i = 1, 2$ , we can find  $2 < p_0^i < 2^*$  such that

$$\forall q_i \in (2, p_0^i), \text{ if } F_i \in (W^{1,q_i'}(Y_i))', \text{ then the solution } \chi_i^\lambda \in W^{1,q_i}(Y_i), \quad (4.34)$$

and due to (4.20), for some  $c_i > 0$ ,

$$\forall q_i \in (2, p_0^i), \quad \|\chi_i^\lambda\|_{W^{1,q_i}(Y_i)} \leq c_i \|F_i\|_{(W^{1,q_i'}(Y_i))'}. \quad (4.35)$$

Let now  $t$  be as in (4.21). Set

$$s_i = \min \{t, p_0^i\}, \quad i = 1, 2.$$

From Lemma 4.6 (written for  $t = s_i$  and  $r_i = p_i$ ), we deduce that

$$\mu_i^\lambda \in (W^{1,p'_i}(Y_i))', \quad \forall p_i \in (2, s_i), \quad i = 1, 2. \quad (4.36)$$

Moreover, for  $i = 1, 2$ , since  $A\lambda \in L^\infty(Y_i)$ , for every  $v_i \in W^{1,p'_i}(Y_i)$ , we have by Hölder's inequality,

$$\begin{aligned} |\langle G_i^\lambda, v_i \rangle| &= \left| \int_{Y_i} A\lambda \nabla v_i \, dy \right| \\ &\leq \|A\lambda\|_{L^{p_i}(Y_i)} \|\nabla v_i\|_{L^{p'_i}(Y_i)} \\ &\leq d_i |\lambda| \|v_i\|_{W^{1,p'_i}(Y_i)}, \end{aligned}$$

where  $d_i$  depends on  $\beta$  and  $Y_i$ . This implies that for  $i = 1, 2$ , for every  $p_i < s_i$ ,

$$G_i^\lambda \in (W^{1,p'_i}(Y_i))'$$

with

$$\|G_i^\lambda\|_{(W^{1,p'_i}(Y_i))'} \leq d_i |\lambda|. \quad (4.37)$$

This, together with (4.33) and (4.36), implies

$$F_i \in (W^{1,p'_i}(Y_i))', \quad \forall p_i \in (2, s_i),$$

which, due to (4.34), proves the first statement of the theorem for every  $p_i$  in  $(2, s_i)$ , since  $s_i < p_0^i$ .

Fix now, for  $i = 1, 2$ , such a  $p_i$  and let us prove the second statement. For every  $q_i \in (2, p_i)$ ,  $i = 1, 2$ , from (4.35),

$$\|\nabla \chi_i^\lambda\|_{L^{q_i}(Y_i)} \leq c_i \|F_i\|_{(W^{1,q'_i}(Y_i))'}.$$

Hence, to prove (4.17), it suffices to show that for some positive constant  $c_i$ ,

$$\|F_i\|_{(W^{1,q'_i}(Y_i))'} \leq c_i |\lambda|, \quad \forall \lambda \in \mathbb{R}^N, \quad i = 1, 2. \quad (4.38)$$

Since  $q_i < p_i < s_i$ , from (4.33), (4.25) of Lemma 4.6 (written for  $r_i = q_i$ ), and (4.37) (written for  $p_i = q_i$ ), we have

$$\|F_i\|_{(W^{1,q'_i}(Y_i))'} \leq \|G_i^\lambda\|_{(W^{1,q'_i}(Y_i))'} + \|\mu_i^\lambda\|_{(W^{1,q'_i}(Y_i))'} \leq c_i |\lambda|.$$

This shows (4.38) and ends the proof.  $\square$

## 4.4 The rescaled problem

In this section, we rescale the function  $\chi^\lambda$  and prove that it satisfies a rescaled version of (4.15) in  $\mathbb{R}^N$ .

Let  $\Omega$  be a connected open set of  $\mathbb{R}^N$  with a Lipschitz continuous boundary  $\partial\Omega$ . Under the notations of Section 2, for  $k \in \mathbb{Z}^N$ , let  $k_l = (k_1 l_1, k_2 l_2, \dots, k_N l_N)$  and define

$$Y^k = k_l + Y \quad \text{and} \quad Y_i^k = k_l + Y_i, \quad i = 1, 2,$$

and

$$Y_2^\# = \bigcup_{k \in \mathbb{Z}^N} Y_2^k, \quad \Gamma^\# = \partial Y_2^\#, \quad Y_1^\# = \mathbb{R}^N \setminus \overline{Y_2^\#}. \quad (4.39)$$

Let  $\{\varepsilon\}$  be a sequence of positive real numbers that converges to zero. Set  $G_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon \overline{Y_2^k} \subset \Omega\}$ .

We define the sets  $\Omega_1^\varepsilon$ ,  $\Omega_2^\varepsilon$ , and  $\Gamma^\varepsilon$  as

$$\Omega_2^\varepsilon = \bigcup_{k \in G_\varepsilon} \varepsilon Y_2^k \quad \text{with} \quad \Gamma^\varepsilon = \partial \Omega_2^\varepsilon, \quad \text{and} \quad \Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}. \quad (4.40)$$

In what follows, for any measurable function  $u$  defined in  $\Omega$ , we set

$$u_i = u|_{\Omega_i^\varepsilon}, \quad \text{the restriction of } u \text{ in } \Omega_i^\varepsilon, \quad i = 1, 2,$$

and we denote by  $\tilde{u}_i$ , the zero extension to  $\Omega$  of any function  $u_i$  defined in  $\Omega_i^\varepsilon$ .

**N.B.** *In the sequel, we identify a function in  $H^1(Y_2)$  with its extension by periodicity to  $Y_2^\#$ . Moreover as in (4.10), for any function  $u$  such that  $u_i \in H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$  we identify the gradient with its absolutely continuous part, and we set*

$$\|\nabla u\|_{L^2(\Omega \setminus \Gamma^\varepsilon)}^2 = \|\nabla u_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla u_2\|_{L^2(\Omega_2^\varepsilon)}^2.$$

Let now  $\chi^\lambda$  the solution of (4.15) and set

$$w^\lambda(y) = -\chi^\lambda(y) + \lambda y \quad \text{for a.e. } y \in Y. \quad (4.41)$$

For a.e.  $x \in \Omega$ , define

$$\chi_\varepsilon^\lambda(x) = \varepsilon \chi^\lambda\left(\frac{x}{\varepsilon}\right), \quad w_\varepsilon^\lambda(x) = \varepsilon w^\lambda\left(\frac{x}{\varepsilon}\right) = -\chi_\varepsilon^\lambda(x) + \lambda x. \quad (4.42)$$



Let us first prove the following result:

**Proposition 4.7.** *The solution  $\chi^\lambda$  of (4.15) satisfies*

$$\int_{\mathbb{R}^N \setminus \Gamma^\#} A \nabla \chi^\lambda \nabla \varphi \, dy + \int_{\Gamma^\#} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\varphi_1 - \varphi_2) \, d\sigma = \int_{\mathbb{R}^N \setminus \Gamma^\#} A \lambda \nabla \varphi \, dy, \quad (4.43)$$

for any  $\varphi \in L^2(\mathbb{R}^N)$  with  $\text{supp } \varphi$  compact and the restriction of  $\varphi$  in  $Y_i^\#$ , denoted by  $\varphi_i$ , belongs to  $H^1(Y_i^\#)$ .

*Proof.* We adapt here the proof of Theorem 4.28 of [33] to the case with a jump on  $\Gamma$ . First, for  $i = 1, 2$ , let  $\psi_i \in C_{per}^\infty(Y_i)$ , and set  $v_i = \psi_i - \mathcal{M}_\Gamma(\psi_i)$ . Then  $v = (v_1, v_2) \in \mathcal{H}$  and choosing  $v$  as test function in (4.15), we have

$$\int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla \psi \, dy + \int_{\Gamma} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\psi_1 - \psi_2) \, d\sigma = \int_{Y \setminus \Gamma} A \lambda \nabla \psi \, dy. \quad (4.44)$$

Let  $\varphi$  be a function in  $L^2(\mathbb{R}^N)$  with a compact support  $K$  and  $\varphi_i \in H^1(Y_i)$ ,  $i = 1, 2$ . Let  $\{Y^j\}_{j=1}^m$  be a finite set of translated cells of  $Y$  such that

$$K \subset \bigcup_{j=1}^m Y^j.$$

Let  $\Gamma^j \subset Y^j$  be the corresponding translated sets of  $\Gamma$ ,  $j = 1, \dots, m$ .

Let  $\{\theta^j\}_{j=0}^m$  be a partition of unity associated to this covering, i.e., a family of functions such that,

$$\begin{cases} \theta^j \in \mathcal{D}(\mathbb{R}^N), & 0 \leq \theta^j \leq 1, \quad \forall j \in \{0, 1, \dots, m\}, & \sum_{j=0}^m \theta^j = 1, \\ \text{supp } \theta^j \subset Y^j, & \forall j \in \{1, \dots, m\}, & \text{supp } \theta^0 \subset \mathbb{R}^N \setminus K. \end{cases} \quad (4.45)$$

Since  $\varphi = 0$  in  $\text{supp } \theta^0$ , we have

$$\varphi = \varphi \sum_{j=1}^m \theta^j = \sum_{j=1}^m (\varphi \theta^j) \quad \text{in } \mathbb{R}^N. \quad (4.46)$$

We denote the extension by periodicity of  $\varphi \theta^j$  by  $(\varphi \theta^j)^\#$ ,  $j = 1, \dots, m$ . By (4.45),  $\theta^j = 0$  in a neighborhood of  $\partial Y^j$ ,  $j = 1, \dots, m$ , and hence, for  $i = 1, 2$ ,  $(\varphi_i \theta^j)^\# \in C_{per}^\infty(Y_i)$ .

Writing (4.44) for  $\psi_i = (\varphi_i \theta^j)^\#$ ,  $i = 1, 2$ , and then adding the resulting equations for  $j = 1, \dots, m$ , we obtain,

$$\begin{aligned} \sum_{j=1}^m \int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla (\varphi \theta^j)^\# dy + \sum_{j=1}^m \int_{\Gamma} h(y) (\chi_1^\lambda - \chi_2^\lambda) ((\varphi_1 \theta^j)^\# - (\varphi_2 \theta^j)^\#) d\sigma \\ = \sum_{j=1}^m \int_{Y \setminus \Gamma} A \lambda \nabla (\varphi \theta^j)^\# dy. \end{aligned} \quad (4.47)$$

Note that by (4.45), and (4.46), we have by periodicity

$$\begin{aligned} \sum_{j=1}^m \int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla (\varphi \theta^j)^\# dy &= \sum_{j=1}^m \int_{Y^j \setminus \Gamma^j} A \nabla \chi^\lambda \nabla (\varphi \theta^j) dy \\ &= \int_{\mathbb{R}^N \setminus \Gamma^\#} A \nabla \chi^\lambda \nabla \varphi dy. \end{aligned} \quad (4.48)$$

Similarly,

$$\sum_{j=1}^m \int_{Y \setminus \Gamma} A \lambda \nabla (\varphi \theta^j)^\# dy = \int_{\mathbb{R}^N \setminus \Gamma^\#} A \lambda \nabla \varphi dy. \quad (4.49)$$

Also, from (4.45), and (4.46), we have by periodicity, for the integral on the boundary,

$$\begin{aligned} \sum_{j=1}^m \int_{\Gamma} h(y) (\chi_1^\lambda - \chi_2^\lambda) ((\varphi_1 \theta^j)^\# - (\varphi_2 \theta^j)^\#) d\sigma \\ = \sum_{j=1}^m \int_{\Gamma^j} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\varphi_1 \theta^j - \varphi_2 \theta^j) d\sigma \\ = \sum_{j=1}^m \int_{\Gamma^\#} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\varphi_1 \theta^j - \varphi_2 \theta^j) d\sigma \\ = \int_{\Gamma^\#} h(y) (\chi_1^\lambda - \chi_2^\lambda) (\varphi_1 - \varphi_2) d\sigma \end{aligned} \quad (4.50)$$

Finally, combining (4.47) with (4.48)-(4.50), we get the desired result.  $\square$

The main result of this section is the following one:

**Theorem 4.8.** *For  $\psi = (\psi_1, \psi_2) \in H^1(Y_1) \times H^1(Y_2)$ , set  $\psi^\varepsilon(x) = \varepsilon \psi \left( \frac{x}{\varepsilon} \right)$  for a.e.  $x \in \Omega$ . Then, for every  $\varphi \in \mathcal{D}(\Omega)$  and for  $\varepsilon$  sufficiently small, the function  $w_\varepsilon^\lambda$  defined by (4.42) satisfies*

$$\int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon \nabla w_\varepsilon^\lambda \nabla (\varphi \psi^\varepsilon) dx = -\varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (w_{1,\varepsilon}^\lambda - w_{2,\varepsilon}^\lambda) (\psi_1^\varepsilon - \psi_2^\varepsilon) \varphi d\sigma, \quad (4.51)$$

where

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right). \quad (4.52)$$

*Proof.* Let  $\varphi \in \mathcal{D}(\Omega)$  and  $\varepsilon$  sufficiently small such that  $\text{supp } \varphi \cap \varepsilon\Gamma^\# \subset \Gamma^\varepsilon$ . Set  $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$  and

$$v_i(y) = \varphi_\varepsilon(y)\psi_i(y), \quad i = 1, 2.$$

Using  $v = (v_1, v_2)$  as a test function in (4.43), we have

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \Gamma^\#} A \nabla \chi^\lambda \nabla(\varphi_\varepsilon \psi) dy + \int_{\Gamma^\#} h(y)(\chi_1^\lambda - \chi_2^\lambda)(\psi_1 - \psi_2)\varphi_\varepsilon d\sigma \\ = \int_{\mathbb{R}^N \setminus \Gamma^\#} A \lambda \nabla(\varphi_\varepsilon \psi) dy. \end{aligned}$$

Using Proposition 4.7 and making the change of scale  $x = \varepsilon y$ , by construction we have

$$\begin{aligned} \varepsilon^{-N} \int_{\Omega \setminus \Gamma^\varepsilon} A\left(\frac{x}{\varepsilon}\right) \nabla_y \chi^\lambda\left(\frac{x}{\varepsilon}\right) \nabla_y \left(\varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) \psi\left(\frac{x}{\varepsilon}\right)\right) dx \\ + \varepsilon^{-N+1} \int_{\Gamma^\varepsilon} h\left(\frac{x}{\varepsilon}\right) \left(\chi_1^\lambda\left(\frac{x}{\varepsilon}\right) - \chi_2^\lambda\left(\frac{x}{\varepsilon}\right)\right) \left(\psi_1\left(\frac{x}{\varepsilon}\right) - \psi_2\left(\frac{x}{\varepsilon}\right)\right) \varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) d\sigma \\ = \varepsilon^{-N} \int_{\Omega \setminus \Gamma^\varepsilon} A\left(\frac{x}{\varepsilon}\right) \lambda \nabla_y \left(\varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) \psi\left(\frac{x}{\varepsilon}\right)\right) dx. \end{aligned} \quad (4.53)$$

Observe now that

$$\nabla_y \chi^\lambda\left(\frac{x}{\varepsilon}\right) = \nabla_x \chi_\varepsilon^\lambda(x), \quad \nabla_y \psi\left(\frac{x}{\varepsilon}\right) = \nabla_x \psi^\varepsilon(x), \quad \nabla_y \varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) = \varepsilon \nabla_x \varphi(x).$$

It follows that

$$\begin{aligned} \nabla_y \left(\varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) \psi\left(\frac{x}{\varepsilon}\right)\right) &= \varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) \nabla_y \psi\left(\frac{x}{\varepsilon}\right) + \psi\left(\frac{x}{\varepsilon}\right) \nabla_y \varphi_\varepsilon\left(\frac{x}{\varepsilon}\right) \\ &= \varphi(x) \nabla_x \psi^\varepsilon(x) + \psi^\varepsilon(x) \nabla_x \varphi(x) \\ &= \nabla_x(\varphi(x)\psi^\varepsilon(x)). \end{aligned}$$

Then, we can rewrite (4.53) as

$$\begin{aligned} \varepsilon^{-N} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon \nabla \chi_\varepsilon^\lambda \nabla(\varphi \psi^\varepsilon) dx + \varepsilon^{-N-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(\chi_{1,\varepsilon}^\lambda - \chi_{2,\varepsilon}^\lambda)(\psi_1^\varepsilon - \psi_2^\varepsilon)\varphi d\sigma \\ = \varepsilon^{-N} \int_{\Omega \setminus \Gamma^\varepsilon} A^\varepsilon \lambda \nabla(\varphi \psi^\varepsilon) dx. \end{aligned}$$

Multiplying both sides of this equation by  $\varepsilon^N$  and using (4.42) we obtain (4.51).  $\square$

## 4.5 Further properties for a quasilinear case

In this section, we want to prove that some estimates providing the uniqueness for a class of quasilinear problems in a periodic two-component domain, remain valid after the homogenization process.

More precisely, we show that if a sequence of periodic matrices satisfies the assumption introduced by Chipot in [27] (see (4.62)), then this assumption still holds true for the homogenized matrix field. To do that, we combine the Meyer-like estimate obtained in Theorem 4.3 with some ideas from [23].

**Lemma 4.9.** *There exists  $p > 2$  such that for  $i = 1, 2$ ,  $w_i^\lambda \in W^{1,p}(Y_i)$ . Furthermore, for all  $q \in [2, p]$ , there exists  $c > 0$ , depending on  $\alpha, \beta, Y_1, Y_2$ , and  $q$  such that*

$$\|\nabla w^\lambda\|_{L^q(Y \setminus \Gamma)} \leq c|\lambda|, \quad \text{for every } \lambda \in \mathbb{R}^N. \quad (4.54)$$

In addition,

$$\lim_{\varepsilon \rightarrow 0} \|\nabla w_\varepsilon^\lambda\|_{L^q(\Omega \setminus \Gamma^\varepsilon)} \leq c \frac{|\Omega|^{\frac{1}{q}}}{|Y|^{\frac{1}{q}}} |\lambda| \quad \text{for every } \lambda \in \mathbb{R}^N. \quad (4.55)$$

*Proof.* From Theorem 4.3, for  $i = 1, 2$ , we can find  $p_i$ ,  $2 < p_i < 2^*$  (where  $2^*$  is defined in (4.19)), such that  $\chi_i^\lambda \in W^{1,p_i}(Y_i)$ . It follows that for  $p = \min\{p_1, p_2\}$ ,  $w_i^\lambda \in W^{1,p}(Y_i)$ ,  $i = 1, 2$ . Furthermore, for every  $2 \leq q \leq p$ , there exists a positive constant  $b$  such that

$$\|\nabla \chi^\lambda\|_{L^q(Y \setminus \Gamma)} \leq b|\lambda|.$$

This, together with (4.41), gives (4.54) with  $c = b + 1$ .

Moreover, since the function  $|\widetilde{\nabla_y w_i^\lambda}|^{q_i}$  is a  $Y$ -periodic function in  $L^1(Y)$ , from classical results on functions of the form  $f(x/\varepsilon)$  (see for instance [33, Chapter 2]), we get

$$\lim_{\varepsilon \rightarrow 0} \|\widetilde{\nabla w_{i,\varepsilon}^\lambda}\|_{L^{q_i}(\Omega)}^{q_i} = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left| \widetilde{\nabla_y w_i^\lambda} \left( \frac{x}{\varepsilon} \right) \right|^{q_i} dx = \frac{|\Omega|}{|Y|} \int_{Y_i} |\nabla_y w_i^\lambda(y)|^{q_i} dy. \quad (4.56)$$

This gives, using (4.54),

$$\lim_{\varepsilon \rightarrow 0} \|\nabla w_\varepsilon^\lambda\|_{L^q(\Omega \setminus \Gamma^\varepsilon)} = \frac{|\Omega|^{\frac{1}{q}}}{|Y|^{\frac{1}{q}}} \|\nabla w^\lambda\|_{L^q(Y \setminus \Gamma)} \leq c \frac{|\Omega|^{\frac{1}{q}}}{|Y|^{\frac{1}{q}}} |\lambda|,$$

which completes the proof.  $\square$

For  $A \in M(\alpha, \beta, Y)$ , let us introduce now the homogenized matrix  $A^0$  (see [47]), defined as

$$A^0 = A_1^0 + A_2^0, \quad A_i^0 \lambda = \frac{1}{|Y_i|} \int_{Y_i} A(y) \nabla w_i^\lambda(y) dy, \quad i = 1, 2, \quad \forall \lambda \in \mathbb{R}^N, \quad (4.57)$$

where  $w^\lambda$  is defined by (4.41).

From classical results (see for instance [33, Chapter 6]) we know that the following convergences hold, for  $i = 1, 2$ :

$$\begin{cases} \widetilde{\chi_{i,\varepsilon}^\lambda} \longrightarrow 0 & \text{strongly in } L^2(\Omega), \\ A^\varepsilon \widetilde{\nabla w_{i,\varepsilon}^\lambda} \rightharpoonup A_i^0 \lambda & \text{weakly in } [L^2(\Omega)]^N, \end{cases} \quad (4.58)$$

where  $A^\varepsilon$  is defined in (4.52).

The following result extends to the periodic case with a jump on  $\Gamma$  a similar result for a fixed domain, proved in the general framework the G-convergence in Colombini-Spagnolo [35] and for the H-convergence in [20]. A simpler proof for the case of a periodic matrix field and a periodically perforated domain, has been given in [23, Proposition 3.2]. We adapt here to the case with jump some arguments used therein.

**Theorem 4.10.** *Let  $B$  and  $D$  be in  $M(\alpha, \beta, Y)$ , with the corresponding homogenized matrices  $B^0$  and  $D^0$ , given by (4.57). Then there exists  $c_0 > 0$  dependent on  $\alpha, \beta, Y_1$  and  $Y_2$  such that*

$$|B^0 - D^0| \leq c_0 \|B - D\|_{L^\infty(Y)}. \quad (4.59)$$

*Proof.* Let  ${}^tD$  denote the transposed matrix of  $D$ . Let  $\chi_{i,\varepsilon}^{\lambda,B}$ ,  $w_{i,\varepsilon}^{\lambda,B}$ ,  $\chi_{i,\varepsilon}^{\mu,{}^tD}$  and  $w_{i,\varepsilon}^{\mu,{}^tD}$ ,  $i = 1, 2$ , be the functions defined in (4.41) and (4.42) for  $A = B$  and  $A = {}^tD$ , respectively.

Let us show that for every  $\lambda, \mu \in \mathbb{R}^N$ , we have

$$(B^\varepsilon - D^\varepsilon) (\widetilde{\nabla w_{1,\varepsilon}^{\lambda,B} \nabla w_{1,\varepsilon}^{\mu,{}^tD}} + \widetilde{\nabla w_{2,\varepsilon}^{\lambda,B} \nabla w_{2,\varepsilon}^{\mu,{}^tD}}) \rightharpoonup (B^0 - D^0) \lambda \mu \quad (4.60)$$

weakly in  $L^1(\Omega)$ . To do this, we first prove that for any  $\varphi \in \mathcal{D}(\Omega)$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} (B^\varepsilon - D^\varepsilon) \nabla w_\varepsilon^{\lambda,B} \nabla w_\varepsilon^{\mu,{}^tD} \varphi dx = \int_{\Omega} (B^0 - D^0) \lambda \mu \varphi dx. \quad (4.61)$$

Let  $\varphi \in \mathcal{D}(\Omega)$ . We have

$$\begin{aligned} I_\varepsilon &= \int_{\Omega \setminus \Gamma^\varepsilon} (B^\varepsilon - D^\varepsilon) \nabla w_\varepsilon^{\lambda,B} \nabla w_\varepsilon^{\mu,{}^tD} \varphi dx \\ &= \int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda,B} \nabla w_\varepsilon^{\mu,{}^tD} \varphi dx - \int_{\Omega \setminus \Gamma^\varepsilon} {}^tD^\varepsilon \nabla w_\varepsilon^{\mu,{}^tD} \nabla w_\varepsilon^{\lambda,B} \varphi dx. \end{aligned}$$

From Theorem 4.8, written for  $\psi = w^{\mu, tD}$ , we obtain

$$\begin{aligned}
\int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda, B} \nabla w_\varepsilon^{\mu, tD} \varphi \, dx &= \int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda, B} \nabla (w_\varepsilon^{\mu, tD} \varphi) \, dx \\
&\quad - \int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda, B} w_\varepsilon^{\mu, tD} \nabla \varphi \, dx \\
&= -\varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (w_{1, \varepsilon}^{\lambda, B} - w_{2, \varepsilon}^{\lambda, B}) (w_{1, \varepsilon}^{\mu, tD} - w_{2, \varepsilon}^{\mu, tD}) \varphi \, d\sigma \\
&\quad - \int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda, B} w_\varepsilon^{\mu, tD} \nabla \varphi \, dx.
\end{aligned}$$

Similarly, from Theorem 4.8, written for  $\psi = w^{\lambda, B}$ , we have

$$\begin{aligned}
\int_{\Omega \setminus \Gamma^\varepsilon} {}^tD^\varepsilon \nabla w_\varepsilon^{\mu, tD} \nabla w_\varepsilon^{\lambda, B} \varphi \, dx &= - \int_{\Omega \setminus \Gamma^\varepsilon} {}^tD^\varepsilon \nabla w_\varepsilon^{\mu, tD} w_\varepsilon^{\lambda, B} \nabla \varphi \, dx \\
&\quad - \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (w_{1, \varepsilon}^{\mu, tD} - w_{2, \varepsilon}^{\mu, tD}) (w_{1, \varepsilon}^{\lambda, B} - w_{2, \varepsilon}^{\lambda, B}) \varphi \, d\sigma.
\end{aligned}$$

Then

$$I_\varepsilon = - \int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda, B} w_\varepsilon^{\mu, tD} \nabla \varphi \, dx + \int_{\Omega \setminus \Gamma^\varepsilon} {}^tD^\varepsilon \nabla w_\varepsilon^{\mu, tD} w_\varepsilon^{\lambda, B} \nabla \varphi \, dx.$$

From (4.57), (4.42), and (4.58) (written for  $A = B$ ), we have

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} B^\varepsilon \nabla w_\varepsilon^{\lambda, B} w_\varepsilon^{\mu, tD} \nabla \varphi \, dx &= \lim_{\varepsilon \rightarrow 0} \sum_{i=1}^2 \int_{\Omega} B^\varepsilon \widetilde{\nabla w_{i, \varepsilon}^{\lambda, B}} (\mu \cdot x - \widetilde{\chi_{i, \varepsilon}^{\mu, tD}}) \nabla \varphi \, dx \\
&= \sum_{i=1}^2 \int_{\Omega} B_i^0 \lambda(\mu \cdot x) \nabla \varphi \, dx \\
&= - \int_{\Omega} B^0 \lambda \mu \varphi \, dx.
\end{aligned}$$

Similarly,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus \Gamma^\varepsilon} {}^tD^\varepsilon \nabla w_\varepsilon^{\mu, tD} w_\varepsilon^{\lambda, B} \nabla \varphi \, dx = - \int_{\Omega} D^0 \lambda \mu \varphi \, dx.$$

It follows that

$$\lim_{\varepsilon \rightarrow 0} I_\varepsilon = \int_{\Omega} (B^0 - D^0) \lambda \mu \varphi \, dx.$$

This proves (4.61).

By Lemma 4.9, there exists  $\eta > 0$  such that  $(B^\varepsilon - D^\varepsilon) \widetilde{\nabla w_\varepsilon^{\lambda, B}} \widetilde{\nabla w_\varepsilon^{\mu, tD}}$  is bounded in  $L^{1+\eta}(\Omega)$ . This, together with (4.61), gives us (4.60).

Noting that  $B^0$  and  $D^0$  are constants, and  $\|B^\varepsilon - D^\varepsilon\|_{L^\infty(\Omega)} = \|B - D\|_{L^\infty(Y)}$ , using the Cauchy-Schwarz inequality, from (4.55) written for  $q = 2$  and (4.60) we have,

$$\begin{aligned} |\Omega| |(B^0 - D^0)\lambda\mu| &= \|(B^0 - D^0)\lambda\mu\|_{L^1(\Omega)} \\ &\leq \liminf_{\varepsilon \rightarrow 0} \left\| (B^\varepsilon - D^\varepsilon) (\widetilde{\nabla w_{1,\varepsilon}^{\lambda, B}} \widetilde{\nabla w_{1,\varepsilon}^{\mu, tD}} + \widetilde{\nabla w_{2,\varepsilon}^{\lambda, B}} \widetilde{\nabla w_{2,\varepsilon}^{\mu, tD}}) \right\|_{L^1(\Omega)} \\ &\leq \|B - D\|_{L^\infty(Y)} \lim_{\varepsilon \rightarrow 0} \left( \|\nabla w_\varepsilon^{\lambda, B}\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} \|\nabla w_\varepsilon^{\mu, tD}\|_{L^2(\Omega \setminus \Gamma^\varepsilon)} \right) \\ &\leq c^2 \frac{|\Omega|}{|Y|} \|B - D\|_{L^\infty(Y)} |\lambda| |\mu|. \end{aligned}$$

It follows that for some  $c_0$  depending on  $\alpha, \beta, Y, Y_1$ , and  $Y_2$ ,

$$|(B^0 - D^0)\lambda\mu| \leq c_0 \|B - D\|_{L^\infty(Y)} |\lambda| |\mu|.$$

Writing this inequality with  $\lambda = e_j$  and  $\mu = e_k$ , where  $\{e_j\}_{j=1}^N$  is the canonical basis, gives the desired result.  $\square$

As a consequence, we have the following corollaries:

**Corollary 4.11.** *Let  $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) \in \mathbb{R}^{N \times N}$  be a real matrix field with the following properties:*

- (P1)  $A(\cdot, t)$  belongs to  $M(\alpha, \beta, Y)$  for all  $t \in \mathbb{R}$ ;
- (P2)  $A(\cdot, t) = \{a_{ij}\}_{i,j=1,\dots,N}$  is  $Y$ -periodic for every  $t$ ;
- (P3) there exists a function  $\omega : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$\begin{cases} (a) \ \omega \text{ is continuous, non-decreasing and } \omega(t) > 0, \forall t > 0; \\ (b) \ |A(y, s) - A(y, t)| \leq \omega(|s - t|) \text{ for a.e. } y \in Y, \forall s, t \in \mathbb{R}. \end{cases} \quad (4.62)$$

Let  $A^0$  be the corresponding homogenized matrix to  $A$  given by (4.57). Then

1.  $A^0$  is continuous on  $\mathbb{R}$  and  $A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right)$  for all  $t \in \mathbb{R}$ ;
2. there exists  $C$  depending on  $\alpha, \beta, Y_1$ , and  $Y_2$  such that

$$|A^0(s) - A^0(t)| \leq C\omega(|s - t|),$$

for all  $s, t \in \mathbb{R}$ .

*Proof.* The result  $A^0(t) \in M\left(\alpha, \frac{\beta^2}{\alpha}, \Omega\right)$  is a classical result. The others are direct consequences of Theorem 4.10 written for  $B = A(\cdot, s)$  and  $D = A(\cdot, t)$ .  $\square$

**Remark 4.12.** *To show the interest of Corollary 4.11, let us mention that M. Chipot introduced in [27] the class of matrix field satisfying the assumptions of Corollary 4.11, and the following additional condition:*

$$\text{For any } r > 0, \quad \lim_{s \rightarrow 0^+} \int_s^r \frac{dt}{\omega(t)} = +\infty. \quad (4.63)$$

*He proved that under these assumptions, which are weaker than the Lipschitz-continuity, the corresponding quasilinear problem with Dirichlet boundary conditions admits a unique solution. Consequently, from Corollary 4.11 we can derive that under these assumptions on the periodic matrix field, also the homogenized matrix has the same property, so that the homogenized problem has a unique solution. This is crucial in homogenization, since it implies that the whole sequences under consideration converge (and not only subsequences). A similar result has been proved in [23, Theorem 3.3] for the homogenization of a quasilinear problem in perforated domains with some nonlinear Robin conditions.*

*As mentioned in the introduction, we believe that this can be applied to many other interesting homogenization problems.*

**Corollary 4.13.** *Suppose that the matrix field  $A(y, t)$  satisfies (P1) and (P2) from Corollary 4.11 and*

*(P4)  $A(y, t)$  is locally Lipschitz with respect to the second variable, that is, for every  $r > 0$ , there exists a positive constant  $M_r$  such that*

$$|A(y, s) - A(y, t)| \leq M_r |s - t| \quad \forall s, t \in (-r, r).$$

*Then the homogenized matrix  $A^0$  is also locally Lipschitz, that is, for every  $r > 0$ , there exists a positive constant  $C_r$  such that*

$$|A^0(s) - A^0(t)| \leq C_r |s - t| \quad \forall s, t \in (-r, r).$$

*Proof.* It follows again from Theorem 4.10 written for  $B = A(\cdot, s)$  and  $D = A(\cdot, t)$ .  $\square$



## 4.6 Boundedness of the solution

To complete the study done in the previous sections, adapting classical arguments from G. Stampacchia [80], we prove here that for every  $\lambda$ , the solution  $\chi^\lambda$  of (4.15) belongs to  $L^\infty(Y)$ .

**Theorem 4.14.** *For any  $\lambda \in \mathbb{R}^N$ , if  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda) \in \mathcal{H}$  is the solution of (4.15), then  $\chi_i^\lambda \in L^\infty(Y_i)$ ,  $i = 1, 2$ .*

*Proof.* Fix  $\lambda \in \mathbb{R}^N$ . Define for any  $h \geq 1$ ,

$$A_h = \{y \in Y \setminus \Gamma \mid \chi^\lambda(y) \geq h\}. \quad (4.64)$$

Let  $g_h(x) = (x - h)^+$ , where

$$(x - h)^+ = \begin{cases} x - h, & \text{if } x - h > 0 \\ 0, & \text{if } x - h \leq 0. \end{cases}$$

Using  $g_h(\chi^\lambda)$  as a test function in (4.15), we have

$$\begin{aligned} \int_{Y \setminus \Gamma} A \nabla \chi^\lambda \nabla g_h(\chi^\lambda) dy + \int_{\Gamma} h(y) (\chi_1^\lambda - \chi_2^\lambda) (g_h(\chi_1^\lambda) - g_h(\chi_2^\lambda)) d\sigma \\ = \int_{Y \setminus \Gamma} A \lambda \nabla g_h(\chi^\lambda) dy. \end{aligned} \quad (4.65)$$

Let us first show that if  $w$  and  $z$  are two functions defined on  $\Gamma$ , we have

$$(w - z)[g_h(w) - g_h(z)] \geq [g_h(w) - g_h(z)]^2. \quad (4.66)$$

If  $w \geq h$  and  $z \geq h$ , then  $g_h(w) = w - h$  and  $g_h(z) = z - h$ . Thus

$$(w - z)[g_h(w) - g_h(z)] = [(w - h) - (z - h)][g_h(w) - g_h(z)] = [g_h(w) - g_h(z)]^2.$$

When  $w \leq h$  and  $z \leq h$ , we have  $g_h(z) = 0 = g_h(w)$ , and it follows that

$$(w - z)[g_h(w) - g_h(z)] = 0 = [g_h(w) - g_h(z)]^2.$$

If  $w \leq h$  and  $z \geq h$ , then  $g_h(w) = 0$  and  $g_h(z) = z - h$ . Hence,

$$\begin{aligned} (w - z)[g_h(w) - g_h(z)] &= (w - z)[-g_h(z)] = (z - w)g_h(z) \\ &\geq (z - h)g_h(z) = [g_h(z)]^2 = [g_h(w) - g_h(z)]^2. \end{aligned}$$

Finally, by symmetry, we also have the result when  $w \geq h$  and  $z \leq h$ . This proves (4.66).

By the ellipticity of  $A$ , the assumption on  $h$ , (4.66) (written for  $w = \chi_1^\lambda$  and  $z = \chi_2^\lambda$ ), and Hölder's inequality, we obtain from (4.65),

$$\begin{aligned}
\min\{\alpha, h_0\} \|(\chi^\lambda - h)^+\|_{\mathcal{H}}^2 &\leq \alpha \|\nabla(\chi^\lambda - h)^+\|_{L^2(Y \setminus \Gamma)}^2 \\
&\quad + h_0 \|(\chi_1^\lambda - h)^+ - (\chi_2^\lambda - h)^+\|_{L^2(\Gamma)}^2 \\
&\leq \int_{Y \setminus \Gamma} A \lambda \nabla(\chi_1^\lambda - h)^+ dy \\
&= \int_{A_h} A \lambda \nabla(\chi^\lambda - h)^+ dy \\
&\leq \beta |\lambda| |A_h|^{1/2} \|\nabla(\chi^\lambda - h)^+\|_{L^2(Y \setminus \Gamma)} \\
&\leq \beta |\lambda| |A_h|^{1/2} \|(\chi^\lambda - h)^+\|_{\mathcal{H}}.
\end{aligned}$$

Then from the equivalence of the two norms in (4.12), for some positive constant  $C$ ,

$$\|(\chi^\lambda - h)^+\|_{W_{per}(Y_1) \times H^1(Y_2)}^2 \leq c_1^{-2} \|(\chi^\lambda - h)^+\|_{\mathcal{H}}^2 \leq C |A_h|. \quad (4.67)$$

By the Sobolev embeddings, we have from (4.67),

$$\left( \int_{Y_1} [(\chi_1^\lambda - h)^+]^{2^*} dy \right)^{2/2^*} + \left( \int_{Y_2} [(\chi_2^\lambda - h)^+]^{2^*} dy \right)^{2/2^*} \leq C |A_h|, \quad (4.68)$$

where  $2^*$  is defined in (4.19).

For  $r \geq h$ , we have from (4.64) that  $A_r \subseteq A_h$ . It follows that  $(\chi_i^\lambda - h)^+ = \chi_i^\lambda - h \geq r - h$  in  $A_r$ . Thus,

$$\begin{aligned}
(r - h)^2 |A_r|^{2/2^*} &\leq \left( \int_{A_r} [(\chi_1^\lambda - h)^+]^{2^*} dy \right)^{2/2^*} + \left( \int_{A_r} [(\chi_2^\lambda - h)^+]^{2^*} dy \right)^{2/2^*} \\
&\leq C |A_h|.
\end{aligned}$$

Fix  $k_0 > 1$  and for  $h \geq k_0$ , define  $\varphi(h) = |A_h|$ . Then  $\varphi$  is decreasing and for all  $r > h$ ,

$$\varphi(r) \leq \frac{C}{(r - h)^{2^*}} (\varphi(h))^{2^*/2}.$$

Since  $\frac{2^*}{2} > 1$ , by the classical Stampacchia's lemma (see Lemma 4.1 in [80]), there exists  $d > 0$  such that

$$\varphi(k_0 + d) = |A_{k_0+d}| = 0,$$

where  $d$  depends on  $C$  and  $k_0$ . This gives us the desired result, with

$$\|\chi_i^\lambda\|_{L^\infty(Y_i)} \leq k_0 + d. \quad \square$$



# Chapter 5

## Homogenization results for quasilinear elliptic problems in a two-component domain with $L^1$ data

### 5.1 Introduction

The main goal of this chapter is to study the homogenization of a class of quasilinear elliptic problems in a two-component domain with  $L^1$  data and a matrix field not necessarily bounded with respect to the solution.

To describe the domain, we first introduce the reference cell  $Y$ , which has two components  $Y_1$  and  $Y_2$ , with  $\overline{Y_2} \subset Y$  and  $Y_1 = Y \setminus \overline{Y_2}$ , and the interface between them denoted by  $\Gamma$ . The two-component domain, denoted  $\Omega$ , can be written as the union  $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$ , where  $\Omega_2^\varepsilon$  is a disjoint union of  $\varepsilon$ -periodic translated sets  $\varepsilon Y_2$  with  $\overline{\Omega_2^\varepsilon} \subset \Omega$ ,  $\Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}$ , and  $\Gamma^\varepsilon$  is the interface between  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$ .

We prescribe a Dirichlet boundary condition on  $\partial\Omega$ , while on  $\Gamma^\varepsilon$  we assume that the flux is continuous and proportional to the jump of the solution, with a proportionality function of order  $\varepsilon^{-1}$ . This models an imperfect contact between the two components  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$  (see [26] for the justification of this model).

To be more precise, the elliptic problem has the following form:

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\nabla u^\varepsilon\right) = f & \text{in } \Omega_1^\varepsilon \cup \Omega_2^\varepsilon, \\ \left[A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\nabla u^\varepsilon\right] \cdot \nu^\varepsilon = 0 & \text{on } \Gamma^\varepsilon, \\ \left(A\left(\frac{x}{\varepsilon}, u^\varepsilon\right)\nabla u^\varepsilon\right) \nu^\varepsilon = -\varepsilon^{-1}h^\varepsilon(x)[u^\varepsilon] & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where  $\nu^\varepsilon$  is the unit outward normal to  $\Omega_1^\varepsilon$ ,  $[\cdot]$  denotes the jump on  $\Gamma^\varepsilon$ , and  $f \in L^1(\Omega)$ . Moreover, we do not prescribe a restricted growth assumption on the matrix field  $A(y, t)$  with respect to  $t$  (see assumption (A3.2) in the next section).

In general, in the study of homogenization of (5.1), the proportionality assumption on  $\Gamma^\varepsilon$  depends on  $\varepsilon^\gamma$  (instead of  $\varepsilon^{-1}$ ), where  $\gamma \leq 1$ . The homogenization then has 3 cases:  $\gamma \in (-1, 1]$ ,  $\gamma = -1$ , and  $\gamma \in (-\infty, -1)$ . The major difference between these cases is the corresponding cell problem. In this chapter, we consider the case  $\gamma = -1$ , which has the particularity that also the cell problem presents a jump on the reference interface. The remaining cases will be studied in a forthcoming paper.

We consider a quasilinear term since for several composite materials, the thermal conductivity depends, nonlinearly, on the temperature itself. Some of the dependence are monotone, for example, glass or wood, the conductivity of which nonlinearly increases with the temperature, while for ceramics, the dependence is decreasing. There are also cases where the dependence is not monotone, for instance, aluminium or semi-conductors (see [9, 15] for more details).

On the other hand, taking an  $L^1$  data is motivated by thermo-dynamics. A thermoviscoelasticity model that considers the nonlinear mechanical dissipation (which depends on the square of the gradient of the displacement), by the methods in [57, 81] gives a balance energy equation, where the right-hand side is expected to be in  $L^1$ .

Since we are considering an  $L^1$  data, we cannot work in a variational setting. Hence, we choose the appropriate framework of renormalized solution for our problem, which provides existence, stability and, with some additional assumptions, uniqueness results.

The notion of renormalized solution was first introduced by R. J. DiPerna and P. L. Lions in [39] for first order equations. It was then further developed for elliptic equations with  $L^1$  data by F. Murat in [71] and by P. L. Lions and F. Murat in [63], and for elliptic equations with measure data by G. Dal Maso et al. in [36].

The existence of a renormalized solution to (5.1) for fixed  $\varepsilon$  is proved in [54], while the uniqueness has been studied in [53], where an additional local Lipschitz continuity assumption must be prescribed on the matrix field  $A$ .

As mentioned above, we study in this chapter the homogenization of (5.1). Homogenization theory deals with the study of the macroscopic behaviour of composite materials. Composite materials are composed of two or more finely mixed components and their main physical characteristics (e.g. thermal or electric conductivity) can be modelled by problems with oscillating coefficients, describing the heterogeneities at the micro-scale. Then, the mathematical homogenization theory allows to give a macroscopic description of these materials, considered as homogeneous, at the macro-scale.

There are various literatures concerning the homogenization theory, as there is a wide interest on the subject. One can refer to the following references: [11, 33, 34, 83]. It is worth mentioning that the first results on homogenization was by Spagnolo in [79].

There are several methods developed for homogenization. Some of which are the multiple-scale method introduced by Bensoussan et al. in [11] (one can also see [77]), the oscillating test function method developed by Tartar in [83], and the two-scale convergence method first studied by Nguetseng in [72] which is further developed by Allaire in [1].

The most recently developed method, which is what we use in this chapter, is the periodic unfolding method. This method was introduced by D. Cioranescu et al. in [30] and further developed in [31] for the study of periodic homogenization in classical domains (that is, with no holes or interfacial resistance). It was then later developed for perforated domains in [29] and for two-component domains in [46]. For a detailed and comprehensive presentation of the method with several applications we refer to the recent book [32].

The periodic unfolding method for two-component domains makes use of the unfolding operator  $\mathcal{T}_i^\varepsilon$ ,  $i = 1, 2$ , which is defined on any measurable functions in  $\Omega_i^\varepsilon$  (see Definition 5.8 for more details). This operator has some interesting properties that can be used in homogenization, one of which is that it allows us to transform an integral in  $\Omega_i^\varepsilon$  to an integral on the fixed domain  $\Omega \times Y_i$ .

Moreover, when the data  $f$  in (5.1) belongs to  $L^2(\Omega)$  and  $A(y, t)$  is bounded, we can obtain some a priori estimates on the restriction of the solution  $u^\varepsilon$  to each component  $\Omega_i^\varepsilon$ , which we denote by  $u_i^\varepsilon = u^\varepsilon|_{\Omega_i^\varepsilon}$ , in  $H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$  (see [7]).

Using the results in [46], these estimates lead to the following convergences:

$$\begin{cases} \mathcal{T}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_1 & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2 \\ \mathcal{T}_i^\varepsilon(\nabla u_i^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \hat{u}_i & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2, \end{cases} \quad (5.2)$$

for some  $u_1 \in H_0^1(\Omega)$  and  $\hat{u}_i \in L^2(\Omega, H^1(Y_i))$ ,  $i = 1, 2$ .

These convergences allow us to identify first the unfolded limit problem (Theorem 5.22), satisfied by  $(u_1, \hat{u}_1, \hat{u}_2)$ . This is the most delicate proof, which is long and quite labourous. Then we obtain, by standard computations, the homogenized problem in  $\Omega$ , satisfied by  $u_1$  (Theorem 5.25). Nevertheless, the proof of the decay of the “truncated” energy (see (5.111)), which proves that  $u_1$  is a renormalized solution, is not standard and also delicate.

In literature, the homogenization results for (5.1) (with  $f \in L^2(\Omega)$ ) have been studied by different methods. The linear case was studied in [68, 69] using the oscillating test function method, and in [46, 45] using the periodic unfolding method. Some results in the quasilinear case are contained in [21], with the use the periodic unfolding method.

So far, we have only referenced literatures that dealt with homogenization in the variational setting, that is, when the data belongs to  $L^2(\Omega)$  and the matrix field  $A(y, t)$  is globally bounded. There are various works in homogenization in the framework of renormalized solution, one can see for instance [8, 22, 70]. More recent works are [56, 43]. The authors in [56] worked on the homogenization of a linear elliptic problem with Neumann boundary conditions, highly oscillating boundary, and  $L^1$  data.

On the other hand, with the use of periodic unfolding method, the authors in [43] studied the asymptotic behaviour of a quasilinear elliptic problem in perforated domains with nonlinear Robin boundary condition and  $L^1$  data. To our knowledge, [43] is the first study to mix the periodic unfolding method and the notion of renormalized solution, we adopt a similar approach in this chapter.

Let us recall that for a fixed  $\varepsilon$ , the solutions  $u_i^\varepsilon$ ,  $i = 1, 2$ , does not necessarily belong to  $H^1(\Omega_i^\varepsilon)$ , due to the fact that the data  $f$  is an  $L^1$  function. This means that we cannot have the usual convergences as in (5.2). We instead consider the truncates of  $u_i^\varepsilon$ ,  $i = 1, 2$ , where the truncation function is defined as  $T_k(r) = \min(k, \max(-k, r))$ , for any  $k > 0$ . This is possible since, in the framework of renormalized solution, the truncated of the solution for any  $k > 0$  is an  $H^1$  function. Thus, in place of (5.2), combining the techniques of the framework of renormalized solutions and that of the periodic unfolding method (in particular, the compactness results), we show that there exist a

function  $u_1 : \Omega \rightarrow \mathbb{R}$  and a sequence  $\{\widehat{u}_i^n\}_{n \in \mathbb{N}} \subset L^2(\Omega, H^1(Y_i))$ ,  $i = 1, 2$ , such that for any  $n \in \mathbb{N}$ ,  $i = 1, 2$ ,

$$\begin{cases} T_n(u_1) \in H_0^1(\Omega) \\ \mathcal{T}_i^\varepsilon(T_n(u_i^\varepsilon)) \rightarrow T_n(u_1) & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ \mathcal{T}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \nabla T_n(u_1) + \nabla_y \widehat{u}_i^n & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2. \end{cases}$$

Even if we are using the process developed in [43], important additional differences and difficulties arise here, due to the presence of the jump. We use the definition of a renormalized solution adapted to our problem, introduced in [53], and which contains an additional condition on the interface integral. In particular, Theorem 5.18 has a more delicate proof in our case compared to the corresponding theorem in [43], since the second component  $\widehat{u}_2^n$ , for  $n \in \mathbb{N}$ , does not necessarily have a zero average on  $\Gamma$ . Furthermore, throughout this study, it can be observed that managing the boundary integral resulting from the jump is not a straightforward task and must be handled with care.

This chapter is organized as follows. The next section is dedicated to all the preliminary definitions and assumptions that we need. In particular, we define there the renormalized solution of (5.1) (Definition 5.6) and the periodic unfolding operator  $\mathcal{T}_i^\varepsilon$ ,  $i = 1, 2$  (Definition 5.8). We also present the properties of  $\mathcal{T}_i^\varepsilon$ ,  $i = 1, 2$  (see Proposition 5.10).

In Section 5.3, we obtain some a priori estimates for the renormalized solution of (5.1). We also prove some pointwise convergence of  $\mathcal{T}_i^\varepsilon(u_i^\varepsilon)$  on  $\Omega \times Y_i$ ,  $i = 1, 2$  and on  $\Omega \times \Gamma$  (see Theorem 5.17).

Section 5.4 is devoted to the statement and proof of the main homogenization results. In particular, we identify the unfolded problem in Theorem 5.22 and the homogenized problem on  $\Omega$  in Theorem 5.25.

## 5.2 Preliminaries and Position of the Problem

In this section, we present some definitions and the assumptions relevant to our problem. We also discuss here the unfolding operator that will be used in the homogenization results, as well as its important properties.

We first define our two-component domain. Let  $\Omega$  be a connected open bounded subset of  $\mathbb{R}^N$  with a Lipschitz continuous boundary  $\partial\Omega$ . We then define the reference cell  $Y$  as

$$Y = \prod_{j=1}^N [0, l_j),$$



where  $l_j > 0$ ,  $j = 1, \dots, N$ . Let  $Y_1$  and  $Y_2$  be two disjoint connected open subsets of  $Y$  such that

$$\overline{Y_2} \subset Y \quad \text{and} \quad Y_1 = Y \setminus \overline{Y_2},$$

where the boundary  $\Gamma = \partial Y_2$  is Lipschitz continuous.

For any  $k \in \mathbb{Z}^N$ , let  $k_l = (k_1 l_1, \dots, k_N l_N)$  and define

$$Y^k = k_l + Y \quad \text{and} \quad Y_i^k = k_l + Y_i, \quad i = 1, 2.$$

Let  $\varepsilon$  be a positive real parameter taking its values in a sequence that goes to 0. Define

$$K_\varepsilon = \{k \in \mathbb{Z}^N \mid \varepsilon \overline{Y_2^k} \subset \Omega\},$$

and set

$$\Omega_2^\varepsilon = \bigcup_{k \in K_\varepsilon} \varepsilon Y_2^k, \quad \Omega_1^\varepsilon = \Omega \setminus \overline{\Omega_2^\varepsilon}, \quad \Gamma^\varepsilon = \partial \Omega_2^\varepsilon.$$

It follows that for every  $\varepsilon$ ,

$$\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$$

is the two-component domain (see figure below).

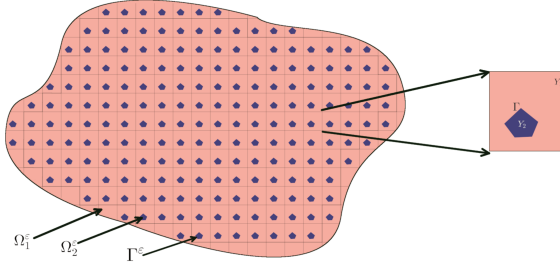


Figure 5.1: The two-component domain  $\Omega$  and the reference cell  $Y$ .

We also introduce here the following sets, needed in the sequel:

- $\widehat{K}_\varepsilon = \{k \in \mathbb{Z}^N : \varepsilon Y^k \subset \Omega\};$
- $\widehat{\Omega}_\varepsilon = \text{int} \left( \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon (k_l + \overline{Y}) \right), \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon;$
- $\widehat{\Omega}_i^\varepsilon = \bigcup_{k \in \widehat{K}_\varepsilon} \varepsilon Y_i^k, \quad \Lambda_i^\varepsilon = \Omega_i^\varepsilon \setminus \widehat{\Omega}_i^\varepsilon, \quad i = 1, 2, \quad \widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_2^\varepsilon.$

The main goal of this research is to study the homogenization of the following quasilinear problem in the two-component domain described above:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div}(A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f & \text{in } \Omega_2^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = (A^\varepsilon(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ (A^\varepsilon(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1^\varepsilon = -\varepsilon^{-1}h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \end{cases} \quad (5.3)$$

with a  $L^1$  data and a not globally bounded matrix field.

More precisely, we prescribe the following assumptions:

(A1)  $f \in L^1(\Omega)$ ;

(A2)  $h$  is a  $Y$ -periodic function in  $L^\infty(\Gamma)$  and there exists  $h_0 \in \mathbb{R}$  such that  $0 < h_0 < h(y)$  a.e. on  $\Gamma$ , and set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \quad \text{a.e. on } \Gamma^\varepsilon;$$

(A3)  $A : (y, t) \in Y \times \mathbb{R} \mapsto A(y, t) \in \mathbb{R}^{N \times N}$  is a real matrix field such that  $A(\cdot, t) = \{a_{ij}\}_{i,j=1,\dots,N}$  is  $Y$ -periodic for every  $t$ ,  $A$  is a Carathéodory function with the following properties:

(A3.1)  $A(y, t)\xi \cdot \xi \geq \alpha|\xi|^2$ , a.e.  $y \in Y$ ,  $\forall t \in \mathbb{R}$ ,  $\forall \xi \in \mathbb{R}^N$ ;

(A3.2)  $A(y, t) \in L^\infty(\Omega \times (-k, k))^{N \times N}$ ,  $\forall k > 0$ ,

and set

$$A^\varepsilon(x, t) = A\left(\frac{x}{\varepsilon}, t\right),$$

for every  $(x, t) \in \Omega \times \mathbb{R}$ .

We also introduce the following assumption, which is important in showing uniqueness results.

(A4) The matrix field  $A(y, t)$  is locally Lipschitz continuous with respect to the second variable, that is, for every  $r > 0$ , there exists a positive constant  $M_r$  such that

$$|A(y, s) - A(y, t)| < M_r|s - t|, \quad \forall s, t \in [-r, r], \quad \forall y \in Y.$$

In the sequel, we use the following notations:

- $\theta_i = \frac{|Y_i|}{|Y|}$ ,  $i = 1, 2$ ;
- $\tilde{u}_i$  is the zero extension to the whole  $\Omega$  of a function  $u_i$  defined on  $\Omega_i^\varepsilon$ , for  $i = 1, 2$ ;
- $u_i = u|_{\Omega_i^\varepsilon}$  is the restriction to  $\Omega_i^\varepsilon$ ,  $i = 1, 2$ , of a function  $u$  defined in  $\Omega$ ;
- $\nu_i^\varepsilon$  is the unit outward normal to  $\Omega_i^\varepsilon$  for  $i = 1, 2$ ;
- $\chi_\omega$  is the characteristic function of a set  $\omega$ ;
- $\mathcal{M}_\omega(f) := \frac{1}{|\omega|} \int_\omega f dx$ , for any open set  $\omega$  and for any  $f \in L^1(\omega)$ .

We now define the Sobolev space that we will be working with, adapted to the fact that we are dealing with a jump on the interface.

**Definition 5.1.** *The functional space  $H^\varepsilon$  is defined by*

$$H^\varepsilon := \{u = (u_1, u_2) : u_1 \in V^\varepsilon \text{ and } u_2 \in H^1(\Omega_2^\varepsilon)\}, \quad (5.4)$$

*equipped with the norm*

$$\|u\|_{H^\varepsilon}^2 := \|\nabla u_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \|\nabla u_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon^{-1} \|u_1 - u_2\|_{L^2(\Gamma^\varepsilon)}^2, \quad (5.5)$$

*where  $V^\varepsilon = \{u \in H^1(\Omega_1^\varepsilon) : u = 0 \text{ on } \partial\Omega\}$  is endowed with the norm*

$$\|u\|_{V^\varepsilon} := \|\nabla u\|_{L^2(\Omega_1^\varepsilon)}. \quad (5.6)$$

**Remark 5.2.** *The norm (5.6) is equivalent to that of  $H^1(\Omega_1^\varepsilon)$ , with constants independent of  $\varepsilon$  (see [68] for details).*

We have the following proposition on the equivalence of the norms of  $V^\varepsilon \times H^1(\Omega_2^\varepsilon)$  and  $H^\varepsilon$ , which is proved in [44].

**Proposition 5.3** ([44]). *The norm (5.5) is equivalent to that of  $V^\varepsilon \times H^1(\Omega_2^\varepsilon)$ , that is, there are positive constants  $c_1$  and  $c_2$  (independent of  $\varepsilon$ ) such that for any  $u = (u_1, u_2) \in H^\varepsilon$ ,*

$$c_1 \|u\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2 \leq \|u\|_{H^\varepsilon}^2 \leq c_2 (1 + \varepsilon^{-2}) \|u\|_{V^\varepsilon \times H^1(\Omega_2^\varepsilon)}^2. \quad (5.7)$$

We also have the following lemma from [68]:

**Lemma 5.4** ([68]). *There exists a constant  $C > 0$ , independent of  $\varepsilon$ , such that*

$$\|v_i\|_{L^2(\Gamma^\varepsilon)}^2 \leq C(\varepsilon^{-1}\|v_i\|_{L^2(\Omega_i^\varepsilon)}^2 + \varepsilon\|\nabla v_i\|_{L^2(\Omega_i^\varepsilon)}^2),$$

for every  $v_i \in H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$ .

Note that the above assumptions for problem (5.3) give rise to two main difficulties: the data  $f$  belongs to  $L^1(\Omega)$ , and assumption (A3.2) on  $A$  only supposes local boundedness with respect to  $t$ . This means that a solution of (5.3) in the weak sense may not exist (even in the presence of only one of the two difficulties). Hence, we need to introduce a framework that provides existence and uniqueness of a solution of (5.3). Remark that, if so, a solution  $u^\varepsilon$  may not have enough regularity in order to have a gradient and a trace in the usual sense of Sobolev spaces. In the following proposition, which is proved in [54], we give a definition of gradient and trace of any measurable function  $u$ , using the truncation function  $T_k$ , defined by

$$T_k(t) = \begin{cases} -k, & \text{if } t \leq -k \\ t, & \text{if } -k \leq t \leq k \\ k, & \text{if } t \geq k. \end{cases} \quad (5.8)$$

**Proposition 5.5** ([54]). *Let  $u = (u_1, u_2) : \Omega \setminus \Gamma^\varepsilon \rightarrow \mathbb{R}$  be a measurable function such that  $T_k(u) \in H^\varepsilon$  for every  $k > 0$ . Then*

1. *there exists a unique measurable function  $v_i : \Omega_i^\varepsilon \rightarrow \mathbb{R}^N$  such that for all  $k > 0$ ,*

$$\nabla T_k(u_i) = v_i \chi_{\{|u_i| < k\}} \quad \text{a.e. in } \Omega_i^\varepsilon, \quad i = 1, 2. \quad (5.9)$$

*We define  $v_i$  as the gradient of  $u_i$  and write  $v_i = \nabla u_i$ .*

2. *If*

$$\sup_{k \geq 1} \frac{1}{k} \|T_k(u)\|_{H^\varepsilon}^2 < \infty, \quad (5.10)$$

*there exists a unique measurable function  $w_i : \Gamma^\varepsilon \rightarrow \mathbb{R}$ , for  $i = 1, 2$ , such that for all  $k > 0$ ,*

$$\gamma_i(T_k(u_i)) = T_k(w_i) \quad \text{a.e. on } \Gamma^\varepsilon, \quad i = 1, 2, \quad (5.11)$$

*where  $\gamma_i : H^1(\Omega_i^\varepsilon) \rightarrow L^2(\Gamma^\varepsilon)$  is the trace operator. For  $i = 1, 2$ , we define the function  $w_i$  as the trace of  $u_i$  on  $\Gamma^\varepsilon$  and set*

$$\gamma_i(u_i) = w_i.$$

With this proposition, we are now in the position to define a renormalized solution of (5.3).

**Definition 5.6.** *The function  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  is a renormalized solution of (5.3) if*

$$T_k(u^\varepsilon) \in H^\varepsilon, \quad \forall k > 0; \quad (5.12a)$$

$$(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \in L^1(\Gamma^\varepsilon), \quad \forall k > 0; \quad (5.12b)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u^\varepsilon| < k\}} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx = 0; \quad (5.13a)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \, d\sigma = 0; \quad (5.13b)$$

and for any  $\psi \in C^1(\mathbb{R})$  (or equivalently for any  $\psi \in W^{1,\infty}(\mathbb{R})$ ) with compact support,  $u^\varepsilon$  satisfies

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega_i^\varepsilon} \psi(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla v_i \, dx + \sum_{i=1}^2 \int_{\Omega_i^\varepsilon} \psi'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \cdot \nabla u_i^\varepsilon v_i \, dx \\ & + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (v_1 \psi(u_1^\varepsilon) - v_2 \psi(u_2^\varepsilon)) \, d\sigma = \int_{\Omega} f v \psi(u^\varepsilon) \, dx, \end{aligned} \quad (5.14)$$

for all  $v = (v_1, v_2) \in H^\varepsilon \cap (L^\infty(\Omega_1^\varepsilon) \times L^\infty(\Omega_2^\varepsilon))$ .

**Remark 5.7.**

1. In the renormalized formulation presented in [54], convergences (5.13a)-(5.13b) are stated for  $k \in \mathbb{N}$ , which reads

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u^\varepsilon| < n\}} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx = 0; \quad (5.15a)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) \, d\sigma = 0. \quad (5.15b)$$

Actually, convergences (5.13a)-(5.13b) can be deduced from convergences (5.15a)-(5.15b). Indeed it is sufficient to observe that if  $n \leq k < n+1$  then the ellipticity condition (A3.1) gives

$$0 \leq \frac{1}{k} \int_{\{|u^\varepsilon| < k\}} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx \leq \frac{1}{n} \int_{\{|u^\varepsilon| < n+1\}} A^\varepsilon(x, u^\varepsilon) \nabla u^\varepsilon \cdot \nabla u^\varepsilon \, dx,$$

while properties of  $T_k$  imply

$$\begin{aligned} 0 &\leq \frac{1}{k} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\ &= \frac{1}{k} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_k(T_{n+1}(u_1^\varepsilon)) - T_k(T_{n+1}(u_2^\varepsilon))) d\sigma \\ &\leq \frac{1}{n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_{n+1}(u_1^\varepsilon) - T_{n+1}(u_2^\varepsilon)) d\sigma. \end{aligned}$$

2. Let us mention that the definition in [54] involves two different  $C^1(\mathbb{R})$  functions,  $\psi_1, \psi_2$ , for the two components  $\Omega_1^\varepsilon$  and  $\Omega_2^\varepsilon$  as opposed to the one function  $\psi$  for both as in (5.14). We use here the latter option for simplicity, as there is no difference in the proofs whichever renormalized formulation we use.
3. If we have assumptions (A1)-(A3), then the existence (as well as stability) of a renormalized solution of (5.3) in the sense of the previous definition is assured by [54]. If, in addition, (A4) holds, then the uniqueness of the renormalized solution is proved in [53].

For the homogenization process, we will use the periodic unfolding operator, which was introduced in [31] (for classical domains) and [29] (for perforated domains), and extended to the case of two-component domains in [46] and [45] (see also the recent book [32]).

For a.e.  $z \in \mathbb{R}^N$ , we denote by  $[z]_Y$  the integer part of  $z$ , i.e.,  $[z]_Y = (k_1 l_1, k_2 l_2, \dots, k_N l_N)$ , for some  $k_i \in \mathbb{Z}$ ,  $i = 1, \dots, N$  such that  $z - [z]_Y \in Y$ .

We now define the periodic unfolding operator adapted to the two-component domain.

**Definition 5.8.** For  $i = 1, 2$ , and for any function  $\varphi_i$  Lebesgue measurable on  $\Omega_i^\varepsilon$ , the periodic unfolding operator  $\mathfrak{T}_i^\varepsilon$  is defined by

$$\mathfrak{T}_i^\varepsilon(\varphi_i)(x, y) = \begin{cases} \varphi_i \left( \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon y \right) & \text{a.e. } (x, y) \in \widehat{\Omega}_\varepsilon \times Y_i \\ 0 & \text{a.e. } (x, y) \in \Lambda_\varepsilon \times Y_i. \end{cases}$$

**Remark 5.9.** In the sequel, if  $\varphi \in H^1(\Omega_i^\varepsilon)$ ,  $i = 1, 2$ , the trace of  $\mathfrak{T}_i^\varepsilon(\varphi)$  on  $\Omega \times \Gamma$  (sometimes denoted  $\mathfrak{T}_b^\varepsilon(\varphi)$  in the literature) is still denoted by  $\mathfrak{T}_i^\varepsilon(\varphi)$ , for  $i = 1, 2$ .

We now present the following result from [46], completed in [45], which states some very helpful properties of the unfolding operator for two-component domains.

**Proposition 5.10** ([45, 46]). For  $p \in [1, \infty)$ , the operators  $\mathcal{T}_i^\varepsilon$ ,  $i = 1, 2$  are linear and continuous from  $L^p(\Omega_i^\varepsilon)$  to  $L^p(\Omega \times Y)$ . Moreover, for  $i = 1, 2$ ,

(P1)  $\mathcal{T}_i^\varepsilon(\varphi\psi) = \mathcal{T}_i^\varepsilon(\varphi)\mathcal{T}_i^\varepsilon(\psi)$ , for every  $\varphi, \psi$  Lebesgue measurable on  $\Omega_i^\varepsilon$ .

(P2) For every  $\varphi \in L^1(\Omega_i^\varepsilon)$ , one has

$$\frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(\varphi)(x, y) dx dy = \int_{\hat{\Omega}_i^\varepsilon} \varphi(x) dx = \int_{\Omega_i^\varepsilon} \varphi(x) dx - \int_{\Lambda_i^\varepsilon} \varphi(x) dx.$$

(P3) For every  $\varphi \in L^p(\Omega_i^\varepsilon)$ ,

$$\|\mathcal{T}_i^\varepsilon(\varphi)\|_{L^p(\Omega \times Y_i)} \leq |Y|^{1/p} \|\varphi\|_{L^p(\Omega_i^\varepsilon)}.$$

(P4) For  $\varphi \in L^p(\Omega)$ ,

$$\mathcal{T}_i^\varepsilon(\varphi) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y_i).$$

(P5) Let  $\{\varphi_\varepsilon\}$  be a sequence in  $L^p(\Omega)$  such that  $\varphi_\varepsilon \longrightarrow \varphi$  strongly in  $L^p(\Omega)$ . Then

$$\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y_i).$$

(P6) Let  $\varphi \in L^p(Y_i)$  be a  $Y$ -periodic function and set  $\varphi^\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$ . Then

$$\mathcal{T}_i^\varepsilon(\varphi^\varepsilon) \longrightarrow \varphi \quad \text{strongly in } L^p(\Omega \times Y_i).$$

(P7) Let  $\varphi_\varepsilon \in L^p(\Omega_i^\varepsilon)$  satisfy  $\|\varphi_\varepsilon\|_{L^p(\Omega_i^\varepsilon)} \leq C$ . If  $\mathcal{T}_i^\varepsilon(\varphi_\varepsilon) \rightharpoonup \hat{\varphi}$  weakly in  $L^p(\Omega \times Y_i)$ , then

$$\tilde{\varphi}_\varepsilon \rightharpoonup \theta_i \mathcal{M}_{Y_i}(\hat{\varphi}) \quad \text{weakly in } L^p(\Omega).$$

(P8) For any  $\varphi \in W^{1,p}(\Omega_i^\varepsilon)$ , one has

$$\nabla_y[\mathcal{T}_i^\varepsilon(\varphi)] = \varepsilon \mathcal{T}_i^\varepsilon(\nabla \varphi) \quad \text{and} \quad \mathcal{T}_i^\varepsilon(\varphi) \in L^2(\Omega, W^{1,p}(Y_i)).$$

(P9) If  $\varphi \in L^p(\Gamma^\varepsilon)$ , then

$$\|\mathcal{T}_i^\varepsilon(\varphi)\|_{L^p(\Omega \times \Gamma)} \leq |Y|^{1/p} \varepsilon^{1/p} \|\varphi\|_{L^p(\Gamma^\varepsilon)}.$$

The following result is crucial for the sequel. It follows from [46] and [45], using the notations given in Remark 4.4 of [45] for  $\gamma = -1$ .

**Proposition 5.11** ([46, 45]). *Let  $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$  be a bounded sequence in  $H^\varepsilon$ . Then there exist a subsequence (still denoted by  $\varepsilon$ ), and three functions  $v_1 \in H_0^1(\Omega)$ ,  $\widehat{v}_1 \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{v}_1) = 0$  a.e. in  $\Omega$ ,  $\widehat{v}_2 \in L^2(\Omega, H^1(Y_2))$  such that as  $\varepsilon \rightarrow 0$ ,*

$$\begin{cases} \overline{\mathcal{T}}_i^\varepsilon(v_i^\varepsilon) \longrightarrow v_i & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ \overline{\mathcal{T}}_i^\varepsilon(\nabla v_i^\varepsilon) \rightharpoonup \nabla v_i + \nabla_y \widehat{v}_i & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2. \end{cases}$$

Furthermore,

$$\frac{\overline{\mathcal{T}}_1^\varepsilon(v_1^\varepsilon) - \overline{\mathcal{T}}_2^\varepsilon(v_2^\varepsilon)}{\varepsilon} \rightharpoonup \widehat{v}_1 - \widehat{v}_2 \quad \text{weakly in } L^2(\Omega \times \Gamma), \quad \text{as } \varepsilon \rightarrow 0.$$

### 5.3 A priori estimates

We now prove some results that we will use when proving the homogenization results.

**Proposition 5.12.** *Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a renormalized solution of (5.3). Then there exists a positive constant  $C$  such that for every  $k > 0$  and  $\varepsilon > 0$ ,*

$$\|T_k(u^\varepsilon)\|_{H^\varepsilon}^2 \leq Ck, \quad (5.16)$$

where  $C$  is independent of  $\varepsilon$  and  $k$ .

*Proof.* Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a renormalized solution of (5.3). Fix  $k > 0$ . For  $n > 0$ , define the function  $S_n : \mathbb{R} \rightarrow \mathbb{R}$  by

$$S_n(s) = \begin{cases} 0 & \text{if } s \leq -2n, \\ \frac{s}{n} + 2 & \text{if } -2n \leq s \leq -n, \\ 1 & \text{if } -n \leq s \leq n, \\ -\frac{s}{n} + 2 & \text{if } n \leq s \leq 2n, \\ 0, & \text{if } s \geq 2n. \end{cases} \quad (5.17)$$

Let  $n > 0$ . Choosing  $\psi = S_n$  and  $v = T_k(u^\varepsilon)$  as test function in (5.14) of



Definition 5.6, we obtain

$$\begin{aligned}
& \sum_{i=1}^2 \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla T_k(u_i^\varepsilon) dx \\
& + \sum_{i=1}^2 \int_{\Omega_i^\varepsilon} S_n'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon T_k(u_i^\varepsilon) dx \\
& + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (S_n(u_1^\varepsilon) T_k(u_1^\varepsilon) - S_n(u_2^\varepsilon) T_k(u_2^\varepsilon)) d\sigma \\
& = \int_{\Omega} f S_n(u^\varepsilon) T_k(u^\varepsilon) dx.
\end{aligned} \tag{5.18}$$

For  $i = 1, 2$ , note that

$$\begin{aligned}
& \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla T_k(u_i^\varepsilon) dx \\
& = \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) dx.
\end{aligned} \tag{5.19}$$

On the other hand, since

$$|S_n'| \leq \frac{1}{n}, \quad \text{for any } n > 0, \tag{5.20}$$

$$|T_k| \leq k, \quad \text{for any } k > 0, \tag{5.21}$$

by (5.13a) of Definition 5.6,

$$\begin{aligned}
& \left| \int_{\Omega_i^\varepsilon} S_n'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon T_k(u_i^\varepsilon) dx \right| \\
& \leq 2k \cdot \frac{1}{2n} \int_{\{|u_i^\varepsilon| \leq 2n\}} A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx = k \rho_i(n), \quad i = 1, 2,
\end{aligned} \tag{5.22}$$

where

$$\lim_{n \rightarrow \infty} \rho_i(n) = 0, \quad i = 1, 2. \tag{5.23}$$

Moreover, since we have (5.21) and

$$|S_n| \leq 1, \quad \text{for any } n > 0, \tag{5.24}$$

we obtain

$$\left| \int_{\Omega} f S_n(u^\varepsilon) T_k(u^\varepsilon) dx \right| \leq k \|f\|_{L^1(\Omega)}. \tag{5.25}$$

Combining (5.18), (5.19), (5.22), and (5.25), we deduce that

$$\begin{aligned}
& \sum_{i=1}^2 \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) dx \\
& + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (S_n(u_1^\varepsilon) T_k(u_1^\varepsilon) - S_n(u_2^\varepsilon) T_k(u_2^\varepsilon)) d\sigma \\
& \leq k(\rho_1(n) + \rho_2(n) + \|f\|_{L^1(\Omega)}).
\end{aligned} \tag{5.26}$$

We wish to pass to the limit as  $n$  approaches  $\infty$  on both sides of (5.26). The right-hand side goes to  $k\|f\|_{L^1(\Omega)}$ . For the first integral of (5.26), since for  $i = 1, 2$  and  $n > k$  we have  $S_n(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) = \nabla T_k(u_i^\varepsilon)$  stt a.e. in  $\Omega$ , we obtain

$$\begin{aligned}
& \int_{\Omega_i^\varepsilon} A^\varepsilon(x, u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) dx \\
& = \lim_{n \rightarrow \infty} \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) \nabla T_k(u_i^\varepsilon) dx, \quad i = 1, 2.
\end{aligned} \tag{5.27}$$

To pass to the limit of the integral on the boundary of (5.26), we write

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (S_n(u_1^\varepsilon) T_k(u_1^\varepsilon) - S_n(u_2^\varepsilon) T_k(u_2^\varepsilon)) d\sigma \\
& = \lim_{n \rightarrow \infty} \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) S_n(u_1^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\
& \quad + \lim_{n \rightarrow \infty} \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) T_k(u_2^\varepsilon) d\sigma.
\end{aligned} \tag{5.28}$$

From (5.24) and (5.12b) of Definition 5.6, we have

$$\begin{aligned}
& |h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) S_n(u_1^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))| \\
& \leq \|h\|_{L^\infty(\Gamma)} (u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) \in L^1(\Gamma^\varepsilon).
\end{aligned}$$

Moreover,

$$S_n(u_i^\varepsilon) \longrightarrow 1 \quad \text{a.e. on } \Gamma^\varepsilon \quad \text{as } n \longrightarrow \infty, \quad i = 1, 2. \tag{5.29}$$

Then, by Lebesgue Dominated Convergence Theorem, we obtain

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) S_n(u_1^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \\
& = \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma.
\end{aligned} \tag{5.30}$$

In addition, since  $S_n$  is Lipschitz continuous with (5.20), we have

$$|S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)| = |S_n(T_{2n}(u_1^\varepsilon)) - S_n(T_{2n}(u_2^\varepsilon))| \leq \frac{1}{n} |T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)|, \quad (5.31)$$

a.e. on  $\Gamma^\varepsilon$ . Consequently, by (5.21),

$$\begin{aligned} & \left| \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon))T_k(u_2^\varepsilon) d\sigma \right| \\ & \leq \frac{\|h\|_{L^\infty(\Gamma)} k \varepsilon^{-1}}{n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)) d\sigma, \end{aligned}$$

where the integral on the right-hand side goes to zero as  $n$  approaches  $\infty$  by (5.13b) of Definition 5.6. As a result,

$$\lim_{n \rightarrow \infty} \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon))T_k(u_2^\varepsilon) d\sigma = 0. \quad (5.32)$$

Combining (5.28), (5.30), and (5.32), we conclude that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(S_n(u_1^\varepsilon)T_k(u_1^\varepsilon) - S_n(u_2^\varepsilon)T_k(u_2^\varepsilon)) d\sigma \\ & = \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma. \end{aligned} \quad (5.33)$$

Consequently, using (5.23), (5.27), and (5.33), we can pass to the limit in (5.26). We obtain

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} A^\varepsilon(x, u_1^\varepsilon) \nabla T_k(u_1^\varepsilon) \nabla T_k(u_1^\varepsilon) dx + \int_{\Omega_2^\varepsilon} A^\varepsilon(x, u_2^\varepsilon) \nabla T_k(u_2^\varepsilon) \nabla T_k(u_2^\varepsilon) dx \\ & + \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)(T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon)) d\sigma \leq C_1 k, \end{aligned}$$

where  $C_1$  is independent of  $\varepsilon$  and  $k$ .

Now, observe that for any functions  $f, g$  defined on  $\Gamma^\varepsilon$ , we have

$$(f - g)(T_k(f) - T_k(g)) \geq (T_k(f) - T_k(g))^2, \quad \forall k > 0, \quad (5.34)$$

since  $T_k$  is an increasing Lipschitz continuous function with Lipschitz constant 1.

By the ellipticity of  $A$ , assumption on  $h$ , and (5.34) (written for  $f = u_1^\varepsilon$  and  $g = u_2^\varepsilon$ ), we have

$$\begin{aligned} C_2 \|T_k(u^\varepsilon)\|_{H^\varepsilon}^2 & \leq \alpha \int_{\Omega_1^\varepsilon} |\nabla T_k(u_1^\varepsilon)|^2 dx + \alpha \int_{\Omega_2^\varepsilon} |\nabla T_k(u_2^\varepsilon)|^2 dx \\ & + \varepsilon^{-1} h_0 \int_{\Gamma^\varepsilon} (T_k(u_1^\varepsilon) - T_k(u_2^\varepsilon))^2 d\sigma \leq C_1 k, \end{aligned}$$

for some positive constants  $C_1, C_2$  independent of  $\varepsilon$  and  $k$ , which gives the desired estimate.  $\square$

**Proposition 5.13.** *Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a renormalized solution of (5.3). Then, for any  $k > 0$ ,*

$$\|T_k(u_i^\varepsilon)\|_{L^2(\Omega_i^\varepsilon)}^2 \leq C_1 k, \quad i = 1, 2, \quad (5.35)$$

for some positive constant  $C_1$  independent of  $k$  and  $\varepsilon$ .

Moreover, for  $\varepsilon < 1$ , we have for any  $k > 0$

$$\|T_k(u_i^\varepsilon)\|_{L^2(\Gamma^\varepsilon)}^2 \leq C_2 k \varepsilon^{-1}, \quad i = 1, 2, \quad (5.36)$$

where  $C_2$  is a positive constant independent of  $k$  and  $\varepsilon$ .

*Proof.* To prove (5.35), we use the first inequality in (5.7), Remark 5.2, and Proposition 5.12. We then obtain for  $i = 1, 2$ ,

$$\|T_k(u_i^\varepsilon)\|_{L^2(\Omega_i^\varepsilon)}^2 \leq C_1 \|T_k(u_i^\varepsilon)\|_{H^1(\Omega_i^\varepsilon)}^2 \leq C_1 \|T_k(u_i^\varepsilon)\|_{H^\varepsilon}^2 \leq C_1 k,$$

where  $C_1$  is a positive constant independent of  $k$  and  $\varepsilon$ .

To prove (5.36), when  $\varepsilon < 1$  (and hence  $\varepsilon < \varepsilon^{-1}$ ), we have from Lemma 5.4 (taking  $v_i = T_k(u_i^\varepsilon)$ ,  $i = 1, 2$ ),

$$\|T_k(u_i)\|_{L^2(\Gamma^\varepsilon)}^2 \leq C \varepsilon^{-1} \|T_k(u_i)\|_{H^1(\Omega_i^\varepsilon)}^2, \quad i = 1, 2.$$

Arguing as in the proof of (2.21), we obtain (5.36).  $\square$

**Corollary 5.14.** *Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a renormalized solution of (5.3). We can find a subsequence (still denoted by  $\varepsilon$ ) such that for any  $n \in \mathbb{N}$ , there exist  $u_1^n \in H_0^1(\Omega)$ ,  $\widehat{u}_1^n \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$  a.e. in  $x \in \Omega$ , and  $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$  such that the following convergences hold:*

$$\begin{cases} \widetilde{\mathcal{T}}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow u_1^n & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ \widetilde{\mathcal{T}}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \nabla u_1^n + \nabla_y \widehat{u}_i^n & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2, \\ \widetilde{\mathcal{T}}_n^\varepsilon(T_n(u_i^\varepsilon)) \rightharpoonup \theta_i u_1^n & \text{weakly in } L^2(\Omega), \quad i = 1, 2, \end{cases} \quad (5.37)$$

as  $\varepsilon$  goes to 0.

Furthermore, for any  $n \in \mathbb{N}$ , as  $\varepsilon$  tends to 0,

$$\frac{\widetilde{\mathcal{T}}_1^\varepsilon(T_n(u_1^\varepsilon)) - \widetilde{\mathcal{T}}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \rightharpoonup \widehat{u}_1^n - \widehat{u}_2^n \quad \text{weakly in } L^2(\Omega \times \Gamma). \quad (5.38)$$

*Proof.* By Proposition 5.12, for any  $k > 0$ , the sequence

$$\{T_k(u^\varepsilon)\} = \{(T_k(u_1^\varepsilon), T_k(u_2^\varepsilon))\}$$

is bounded in  $H^\varepsilon$ . Choosing  $k = n$  with  $n \in \mathbb{N}$ , the first two convergences in (5.37) and convergence (5.38) follow from Proposition 5.11 and a standard diagonal process. To show the third one, we use the first one, estimate (2.21) of Proposition 5.13 and (P7) of Proposition 5.10. We obtain

$$\widetilde{T_n(u_i^\varepsilon)} \rightharpoonup \theta_i \mathcal{M}_{Y_i}(u_1^n) \quad \text{weakly in } L^2(\Omega), \quad i = 1, 2.$$

Since  $u_1^n$  is independent of  $y$ , we have the desired result.  $\square$

**Remark 5.15.** *The diagonal process used in the proof of Corollary 5.14 requires that  $k$  belongs to some countable set which tends to  $+\infty$ . For simplicity, we take  $k \in \mathbb{N}$ .*

**Remark 5.16.** *Let  $f$  be a function defined on  $\Omega$ . Since by definition,  $\widetilde{\mathcal{V}}_i^\varepsilon$  vanishes in  $\Lambda^\varepsilon \times Y_i$ , the identity*

$$T_i^\varepsilon(f(u_i^\varepsilon)) = f(\widetilde{\mathcal{V}}_i^\varepsilon(u_i^\varepsilon)), \quad i = 1, 2, \quad (5.39)$$

*does not hold in  $\Omega \times Y_i$ , unless  $f(0) = 0$  (see also Remark 3.3 of [43]). However, for any Lebesgue measurable function  $\varphi$ , we have*

$$\widetilde{\mathcal{V}}_i^\varepsilon(f(u_i^\varepsilon))\widetilde{\mathcal{V}}_i^\varepsilon(\varphi(x)) = f(\widetilde{\mathcal{V}}_i^\varepsilon(u_i^\varepsilon))\widetilde{\mathcal{V}}_i^\varepsilon(\varphi(x)), \quad i = 1, 2 \quad (5.40)$$

*in  $\Omega \times Y_i$  even if  $f(0) \neq 0$ , since  $\widetilde{\mathcal{V}}_i^\varepsilon(\varphi)(x, y) = 0$  in  $\Lambda^\varepsilon \times Y_i$ .*

The following result, stating a pointwise convergence of  $\widetilde{\mathcal{V}}_i^\varepsilon(u_i^\varepsilon)$ ,  $i = 1, 2$ , is an important tool for the identification of the homogenized problem in  $\Omega$ .

**Theorem 5.17.** *Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a renormalized solution of (5.3). Then there exists a measurable function  $u_1 : \Omega \rightarrow \mathbb{R}$ , finite almost everywhere, such that (up to a subsequence)*

$$\widetilde{\mathcal{V}}_i^\varepsilon(u_i^\varepsilon) \longrightarrow u_1 \quad \text{a.e. in } \Omega \times Y_i \text{ and on } \Omega \times \Gamma, \quad i = 1, 2, \quad (5.41)$$

*with*

$$T_n(u_1) = u_1^n, \quad \forall n \in \mathbb{N}, \quad (5.42)$$

*where  $u_1^n$  is given in Corollary 5.14, and*

$$\widetilde{\mathcal{V}}_i^\varepsilon(A^\varepsilon(x, T_n(u_i^\varepsilon))) \longrightarrow A(y, T_n(u_1)) \quad \text{a.e. in } \Omega \times Y_i, \quad i = 1, 2. \quad (5.43)$$

*Proof.* Let  $n \in \mathbb{N}$ . Note that (5.39) holds true, since  $T_n(0) = 0$ . Then, from Corollary 5.14, we have as  $\varepsilon$  tends to 0,

$$T_n(\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)) \longrightarrow u_1^n \quad \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2. \quad (5.44)$$

In addition, from the trace theorem,

$$\begin{aligned} \|\overline{\mathcal{G}}_i^\varepsilon(T_n(u_i^\varepsilon)) - u_1^n\|_{L^2(\Omega \times \Gamma)}^2 &= \int_{\Omega} \|\overline{\mathcal{G}}_i^\varepsilon(T_n(u_i^\varepsilon)) - u_1^n\|_{L^2(\Gamma)}^2 dx \\ &\leq \int_{\Omega} \|\overline{\mathcal{G}}_i^\varepsilon(T_n(u_i^\varepsilon)) - u_1^n\|_{H^1(Y_i)}^2 dx, \quad i = 1, 2. \end{aligned}$$

This last quantity goes to 0 in view of Corollary 5.14. It follows from (5.39) that

$$T_n(\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)) \longrightarrow u_1^n \quad \text{strongly in } L^2(\Omega \times \Gamma), \quad i = 1, 2. \quad (5.45)$$

We now claim that  $\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)$ ,  $i = 1, 2$ , is Cauchy in measure and hence, pointwise convergent (up to a subsequence).

Let  $n > 0$ . By (P3) of Proposition 5.10 and (2.21) of Proposition 5.13, we obtain for  $i = 1, 2$ ,

$$\begin{aligned} n^2 \text{meas}\{|\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)| \geq n\} &= \int_{\{|\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)| \geq n\}} (T_n(\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)))^2 dx dy \\ &\leq \int_{\Omega \times Y_i} (\overline{\mathcal{G}}_i^\varepsilon(T_n(u_i^\varepsilon)))^2 dx dy \\ &\leq |Y_i| \|T_n(u_i^\varepsilon)\|_{L^2(\Omega_i^\varepsilon)}^2 \\ &\leq |Y_i| Cn, \end{aligned}$$

where  $C$  is a positive constant independent of  $n$  and  $\varepsilon$ . Consequently,

$$\text{meas}\{|\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)| \geq n\} \leq \frac{C}{n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \quad i = 1, 2. \quad (5.46)$$

Since we have for  $\omega > 0$  and  $n \in \mathbb{N}$ ,

$$\begin{aligned} \text{meas}\{|\overline{\mathcal{G}}_i^{\varepsilon'}(u_i^{\varepsilon'}) - \overline{\mathcal{G}}_i^{\varepsilon''}(u_i^{\varepsilon''})| \geq \omega\} &\leq \text{meas}\{|\overline{\mathcal{G}}_i^{\varepsilon'}(u_i^{\varepsilon'})| \geq n\} \\ &\quad + \text{meas}\{|\overline{\mathcal{G}}_i^{\varepsilon''}(u_i^{\varepsilon''})| \geq n\} \\ &\quad + \text{meas}\{|T_k(\overline{\mathcal{G}}_i^{\varepsilon'}(u_i^{\varepsilon'})) - T_k(\overline{\mathcal{G}}_i^{\varepsilon''}(u_i^{\varepsilon''}))| \geq \omega\}, \end{aligned}$$

using similar arguments to the ones in [54], (5.37) and (5.46) allow one to show that  $\{\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)\}$ ,  $i = 1, 2$ , is Cauchy in measure. We can then find a subsequence of  $\{\overline{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)\}$ ,  $i = 1, 2$ , which converges almost everywhere.

Hence, there exists a subsequence (still denoted by  $\varepsilon$ ) and  $u_i : \Omega \times Y_i \rightarrow \mathbb{R}$ , such that

$$\mathcal{T}_i^\varepsilon(u_i^\varepsilon) \rightarrow u_i \quad \text{a.e. in } \Omega \times Y_i, \quad i = 1, 2. \quad (5.47)$$

Note that by (5.46),  $u_i$  is finite a.e. for  $i = 1, 2$ .

To show a pointwise convergence on  $\Omega \times \Gamma$ , let  $n > 0$ . Define

$$B_{i,\varepsilon}^n = \{(x, y) \in \Omega \times \Gamma \mid |\mathcal{T}_i^\varepsilon(u_i^\varepsilon)(x, y)| \geq n\}, \quad i = 1, 2.$$

Then by (5.39), (P9) of Proposition 5.10, and (5.36) of Proposition 5.13, we have for  $\varepsilon$  small enough, and  $i = 1, 2$ ,

$$\begin{aligned} n^2 \text{meas}(B_{i,\varepsilon}^n) &= \int_{B_{i,\varepsilon}^n} (T_n(\mathcal{T}_i^\varepsilon(u_i^\varepsilon)))^2 dx d\sigma \\ &\leq \int_{\Omega \times \Gamma} (\mathcal{T}_i^\varepsilon(T_n(u_i^\varepsilon)))^2 dx d\sigma \\ &\leq \varepsilon |Y| \|T_n(u_i^\varepsilon)\|_{L^2(\Gamma^\varepsilon)}^2 \\ &\leq \varepsilon |Y| (Cn\varepsilon^{-1}) \\ &\leq Cn|Y|, \end{aligned}$$

where  $C > 0$ , independent of  $n$  and  $\varepsilon$ .

Then, using similar arguments as above, we have by a trace argument (up to a subsequence)

$$\mathcal{T}_i^\varepsilon(u_i^\varepsilon) \rightarrow u_i \quad \text{a.e. on } \Omega \times \Gamma, \quad i = 1, 2.$$

Let us prove now (5.42). The continuity of  $T_n$ , (5.44), and (5.47) imply

$$T_n(u_i) = u_1^n, \quad \text{a.e. in } \Omega \times Y_i \quad \forall n \in \mathbb{N}, \quad i = 1, 2. \quad (5.48)$$

Since  $u_1^n$  is independent of  $y$  for any  $n > 0$ , it follows that for  $i = 1, 2$ ,  $u_i$  is independent of  $y$ . Moreover, (5.48) also implies that  $T_n(u_1) = T_n(u_2)$  for every  $n \in \mathbb{N}$ , and thus,  $u_1 = u_2$  a.e. in  $\Omega$ .

To end the proof, observe that convergence (5.43) can be proved as that given in Proposition 5.1 of [23] for perforated domains.  $\square$

The following theorem, essential for the homogenization procedure done in the next section, makes use of some ideas of Proposition 5.3 of [43]. Here, the situation is more delicate due to the presence of the second component and the jump on the boundary (see Remark 5.19).

**Theorem 5.18.** Let  $\widehat{u}_1^n \in L^2(\Omega, H_{per}^1(Y_1))$  and  $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$ ,  $n \in \mathbb{N}$ , be the functions given by Corollary 5.14 with  $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$ . Then there exists a unique measurable function

$$\widehat{u}_i : \Omega \times Y_i \longrightarrow \mathbb{R}, \quad i = 1, 2,$$

such that for every  $\mathcal{R} \in C^0(\mathbb{R})$  with compact support, verifying

$$\text{supp } \mathcal{R} \subset [-m, m], \quad \text{for some } m \in \mathbb{N}, \quad (5.49)$$

we have

$$\mathcal{R}(u_1)\widehat{u}_i^n = \mathcal{R}(u_1)\widehat{u}_i \quad \text{a.e. in } \Omega \times Y_i, \quad (5.50)$$

for all  $n \geq m$ , where  $u_1$  is the function given by Theorem 5.17.

Moreover, we have

$$\widehat{u}_i(x, \cdot) \in H^1(Y_i), \quad i = 1, 2, \quad \text{with } \mathcal{M}_\Gamma(\widehat{u}_1) = 0, \quad \text{for a.e. } x \in \Omega. \quad (5.51)$$

*Proof.* The uniqueness of  $\widehat{u}_i$ , for  $i = 1, 2$ , is in the almost everywhere sense.

We first prove (5.50). Let  $\mathcal{R} \in C^0(\mathbb{R})$  with compact support verifying (5.49) and  $n \geq m$ .

By (P1) of Proposition 5.10, Remark 5.16 Corollary 5.14, and using Theorem 5.17 together with the Lebesgue dominated convergence theorem, we obtain for  $i = 1, 2$ ,

$$\begin{aligned} \mathcal{T}_i^\varepsilon(\mathcal{R}(u_i^\varepsilon)\nabla T_n(u_i^\varepsilon)) &= \mathcal{R}(\mathcal{T}_i^\varepsilon(u_i^\varepsilon))\mathcal{T}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \mathcal{R}(u_1)(\nabla T_n(u_1) + \nabla_y \widehat{u}_i^n), \\ \mathcal{T}_i^\varepsilon(\mathcal{R}(u_i^\varepsilon)\nabla T_m(u_i^\varepsilon)) &= \mathcal{R}(\mathcal{T}_i^\varepsilon(u_i^\varepsilon))\mathcal{T}_i^\varepsilon(\nabla T_m(u_i^\varepsilon)) \rightharpoonup \mathcal{R}(u_1)(\nabla T_m(u_1) + \nabla_y \widehat{u}_i^m), \end{aligned}$$

both weakly in  $L^2(\Omega \times Y_i)$ . Since  $\text{supp } \mathcal{R} \subset [-m, m]$ , then for any  $n \geq m$ ,

$$\begin{aligned} \mathcal{R}(u_i^\varepsilon)\nabla T_n(u_i^\varepsilon) &= \mathcal{R}(u_i^\varepsilon)\nabla T_m(u_i^\varepsilon), \quad i = 1, 2, \\ \mathcal{R}(u_1)\nabla T_n(u_1) &= \mathcal{R}(u_1)\nabla T_m(u_1), \end{aligned}$$

and thus,

$$\mathcal{R}(u_1)\nabla_y \widehat{u}_i^n = \mathcal{R}(u_1)\nabla_y \widehat{u}_i^m, \quad \forall n \geq m, \quad i = 1, 2.$$

Since  $\mathcal{R}(u_1)$  is independent of  $y$ , for some  $\Phi_i^{m,n}(x) \in L^2(\Omega)$  (independent of  $y$ ),  $i = 1, 2$ , we have

$$\mathcal{R}(u_1)\widehat{u}_i^n - \mathcal{R}(u_1)\widehat{u}_i^m = \Phi_i^{m,n}(x), \quad i = 1, 2, \quad \forall n \geq m. \quad (5.52)$$



From Remark 5.16, convergence (5.38) of Corollary 5.14, and Theorem 5.17, using the boundedness of  $\mathcal{R}$ , we have

$$\begin{aligned} & \overline{\mathcal{V}}_1^\varepsilon(\mathcal{R}(u_1^\varepsilon))\overline{\mathcal{V}}_2^\varepsilon(\mathcal{R}(u_2^\varepsilon))\frac{\overline{\mathcal{V}}_1^\varepsilon(T_n(u_1^\varepsilon)) - \overline{\mathcal{V}}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \\ &= \mathcal{R}(\overline{\mathcal{V}}_1^\varepsilon(u_1^\varepsilon))\mathcal{R}(\overline{\mathcal{V}}_2^\varepsilon(u_2^\varepsilon))\frac{\overline{\mathcal{V}}_1^\varepsilon(T_n(u_1^\varepsilon)) - \overline{\mathcal{V}}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \\ &\rightharpoonup \mathcal{R}(u_1)\mathcal{R}(u_1)(\widehat{u}_1^n - \widehat{u}_2^n), \end{aligned} \quad (5.53)$$

weakly in  $L^2(\Omega \times \Gamma)$ . Similarly, we have

$$\begin{aligned} & \overline{\mathcal{V}}_1^\varepsilon(\mathcal{R}(u_1^\varepsilon))\overline{\mathcal{V}}_2^\varepsilon(\mathcal{R}(u_2^\varepsilon))\frac{\overline{\mathcal{V}}_1^\varepsilon(T_m(u_1^\varepsilon)) - \overline{\mathcal{V}}_2^\varepsilon(T_m(u_2^\varepsilon))}{\varepsilon} \\ &\rightharpoonup \mathcal{R}(u_1)\mathcal{R}(u_1)(\widehat{u}_1^m - \widehat{u}_2^m), \end{aligned} \quad (5.54)$$

weakly in  $L^2(\Omega \times \Gamma)$ .

On the other hand, using (5.49), we obtain

$$\mathcal{R}(u_i^\varepsilon)T_n(u_i^\varepsilon) = \mathcal{R}(u_i^\varepsilon)T_m(u_i^\varepsilon), \quad \forall n \geq m, \quad i = 1, 2,$$

which implies that

$$\begin{aligned} & \overline{\mathcal{V}}_1^\varepsilon(\mathcal{R}(u_1^\varepsilon))\overline{\mathcal{V}}_2^\varepsilon(\mathcal{R}(u_2^\varepsilon))\frac{\overline{\mathcal{V}}_1^\varepsilon(T_n(u_1^\varepsilon)) - \overline{\mathcal{V}}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \\ &= \overline{\mathcal{V}}_1^\varepsilon(\mathcal{R}(u_1^\varepsilon))\overline{\mathcal{V}}_2^\varepsilon(\mathcal{R}(u_2^\varepsilon))\frac{\overline{\mathcal{V}}_1^\varepsilon(T_m(u_1^\varepsilon)) - \overline{\mathcal{V}}_2^\varepsilon(T_m(u_2^\varepsilon))}{\varepsilon}. \end{aligned}$$

Hence, from convergences (5.53) and (5.54), we deduce that

$$[\mathcal{R}(u_1)]^2(\widehat{u}_1^n - \widehat{u}_2^n) = [\mathcal{R}(u_1)]^2(\widehat{u}_1^m - \widehat{u}_2^m) \quad \text{on } \Omega \times \Gamma, \quad \forall n \geq m,$$

or equivalently,

$$\mathcal{R}(u_1)(\widehat{u}_1^n - \widehat{u}_2^n) = \mathcal{R}(u_1)(\widehat{u}_1^m - \widehat{u}_2^m) \quad \text{on } \Omega \times \Gamma, \quad \forall n \geq m. \quad (5.55)$$

Recall now that from Corollary 5.14,

$$\mathcal{M}_\Gamma(\widehat{u}_1^m) = \mathcal{M}_\Gamma(\widehat{u}_1^n) = 0, \quad \forall m, n > 0. \quad (5.56)$$

Then, noting that  $\mathcal{R}(u_1)$  and  $\Phi_1^{m,n}$  are independent of  $y$ , taking the average on  $\Gamma$  in both sides of (5.52) (for  $i = 1$ ), we have

$$\Phi_1^{m,n} \equiv 0.$$

Similarly, from (5.56), taking the average on  $\Gamma$  in both sides of (5.55), we have

$$\mathcal{R}(u_1)\mathcal{M}_\Gamma(\widehat{u}_2^n) = \mathcal{R}(u_1)\mathcal{M}_\Gamma(\widehat{u}_2^m), \quad \forall n \geq m,$$

which, using again (5.52) (now for  $i = 2$ ), gives

$$\Phi_2^{m,n} \equiv 0.$$

As a consequence, (5.52) becomes

$$\mathcal{R}(u_1)\widehat{u}_i^n = \mathcal{R}(u_1)\widehat{u}_i^m \quad \text{a.e. in } \Omega \times Y_i, \quad \forall n \geq m, \quad i = 1, 2.$$

Define the set  $\Omega_n = \{x \in \Omega \mid |u_1| < n\}$ . The collection  $\{\Omega_n\}_{n \in \mathbb{N}}$  is an increasing sequence of subsets of  $\Omega$ . Let  $\{\alpha_n\}_{n \in \mathbb{N}}$  be a sequence of functions in  $C^0(\mathbb{R})$  with compact support such that  $\text{supp } \alpha_n \subset [-n, n]$  and  $\alpha_n > 0$  for all  $x \in (-n, n)$ .

Then, for any  $n_2 \geq n_1$ , we have

$$\alpha_{n_1}(u_1)\widehat{u}_i^{n_1} = \alpha_{n_1}(u_1)\widehat{u}_i^{n_2}, \quad i = 1, 2.$$

It follows that

$$\widehat{u}_i^{n_1} = \widehat{u}_i^{n_2} \quad \text{a.e. in } \Omega_{n_1} \times Y_i, \quad i = 1, 2, \quad \forall n_2 \geq n_1.$$

Then, we can define the function  $\widehat{u}_i : \Omega \times Y_i \rightarrow \mathbb{R}$  for  $i = 1, 2$ , as

$$\widehat{u}_i(x, y) = \widehat{u}_i^n(x, y), \quad \forall (x, y) \in \Omega_n \times Y_i, \quad n \in \mathbb{N}. \quad (5.57)$$

Since  $u_1$  is finite almost everywhere,

$$\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n \quad (\text{up to a zero measure set}).$$

This proves (5.50). Finally, to show (5.51), note that from (5.57), we can deduce that for any  $\mathcal{R} \in W^{1,\infty}(\mathbb{R})$  with compact support, the product  $\mathcal{R}(u_1)\widehat{u}_i$  belongs to  $L^2(\Omega, H^1(Y_i))$ , for  $i = 1, 2$ , with  $\mathcal{R}(u_1)\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ . Then, since  $u_1$  is finite a.e., we have (5.51).  $\square$

**Remark 5.19.** *Let us point out that in the proof of the previous theorem, the fact that  $\Phi_1^{m,n} \equiv 0$  in (5.52) follows from (5.56). However, in order to prove that  $\Phi_2^{m,n} \equiv 0$  in (5.52) (which is essential in the proof of Theorem 5.18), we need additional and more delicate arguments since the average on  $\Gamma$  of  $\widehat{u}_2^n$ , for  $n \in \mathbb{N}$ , is not necessarily zero.*

## 5.4 Homogenization Results

In this section, we will discuss our homogenization results. We begin by proving the following proposition.

**Proposition 5.20.** *Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be the subsequence of renormalized solutions of (5.3) given by Corollary 5.14 and  $n \in \mathbb{N}$ . Then*

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\{|u_i^\varepsilon| < n\}} A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx = 0, \quad i = 1, 2 \quad (5.58)$$

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-1}}{n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma = 0. \quad (5.59)$$

*Proof.* Let  $m, n \in \mathbb{N}$ . Writing (5.14) of Definition 5.6 for

$$\psi = S_m \quad \text{and} \quad v = \frac{1}{n} T_n(u^\varepsilon),$$

where  $S_m$  is defined by (5.17), we have

$$I_{11}^\varepsilon + I_{21}^\varepsilon + I_{12}^\varepsilon + I_{22}^\varepsilon + I_3^\varepsilon = I_4^\varepsilon, \quad (5.60)$$

where

$$I_{i1}^\varepsilon = \frac{1}{n} \int_{\Omega_i^\varepsilon} S_m(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla T_n(u_i^\varepsilon) dx, \quad i = 1, 2$$

$$I_{i2}^\varepsilon = \frac{1}{n} \int_{\Omega_i^\varepsilon} S'_m(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon T_n(u_i^\varepsilon) dx, \quad i = 1, 2$$

$$I_3^\varepsilon = \frac{\varepsilon^{-1}}{n} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (S_m(u_1^\varepsilon) T_n(u_1^\varepsilon) - S_m(u_2^\varepsilon) T_n(u_2^\varepsilon)) d\sigma$$

$$I_4^\varepsilon = \int_{\Omega} f S_m(u^\varepsilon) \frac{T_n(u^\varepsilon)}{n} dx.$$

We study the behavior of each integral first as  $m$  approaches  $\infty$ . As far as  $I_{i1}^\varepsilon$ ,  $I_{i2}^\varepsilon$  and  $I_3^\varepsilon$  are concerned, the proof of Proposition 5.12 (see in particular (5.22), (5.23), (5.27) and (5.30)) gives that

$$\lim_{m \rightarrow \infty} I_{i1}^\varepsilon = \frac{1}{n} \int_{\{|u_i^\varepsilon| < n\}} A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx, \quad i = 1, 2 \quad (5.61)$$

$$\lim_{m \rightarrow \infty} I_{i2}^\varepsilon = 0, \quad i = 1, 2 \quad (5.62)$$

$$\lim_{m \rightarrow \infty} I_3^\varepsilon = \frac{\varepsilon^{-1}}{n} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma. \quad (5.63)$$

For  $I_4^\varepsilon$ , since  $S_m(u^\varepsilon) \rightarrow 1$  a.e. in  $\Omega$ , as  $m \rightarrow +\infty$ , while  $|S_m(u^\varepsilon)| \leq 1$  a.e. in  $\Omega$  and since

$$\left| f S_m(u^\varepsilon) \frac{T_n(u^\varepsilon)}{n} \right| \leq |f| \in L^1(\Omega),$$

we can apply the Lebesgue Dominated Convergence Theorem, which gives

$$\lim_{m \rightarrow \infty} I_4^\varepsilon = \int_{\Omega} f \frac{T_n(u^\varepsilon)}{n} dx. \quad (5.64)$$

We now look at the limit of this as  $\varepsilon$  tends to 0. By convergence (5.37) of Corollary 5.14 and (5.42) of Theorem 5.17, we have

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega} f \frac{T_n(u^\varepsilon)}{n} dx &= \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} f \widetilde{T_n(u_1^\varepsilon)} dx + \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega} f \widetilde{T_n(u_2^\varepsilon)} dx \\ &= \frac{1}{n} \int_{\Omega} f \theta_1 u_1^n dx + \frac{1}{n} \int_{\Omega} f \theta_2 u_1^n dx \\ &= \int_{\Omega} f \frac{T_n(u_1)}{n} dx. \end{aligned}$$

Now, we evaluate the limit of this last integral as  $n$  goes to  $\infty$ . We again use the Lebesgue Dominated Convergence Theorem. Note that by (5.21),

$$\left| f \frac{T_n(u_1)}{n} \right| \leq |f| \in L^1(\Omega)$$

and since  $u_1$  is finite a.e.,

$$\frac{T_n(u_1)}{n} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \text{ a.e. in } \Omega.$$

It follows that

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \lim_{m \rightarrow \infty} I_4^\varepsilon = 0. \quad (5.65)$$

Combining (5.60)–(5.65), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \left( \sum_{i=1}^2 \frac{1}{n} \int_{\{|u_i^\varepsilon| < n\}} A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx \right. \\ \left. + \frac{\varepsilon^{-1}}{n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon) (T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma \right) = 0. \end{aligned}$$

Since all the integrals are nonnegative, this gives (5.58) and (5.59).  $\square$

The following lemma states the convergence results that we need to identify the limit problem.

**Lemma 5.21.** *Let  $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$  be a renormalized solution of (5.3). We can find a subsequence (still denoted by  $\varepsilon$ ) such that for any  $n \in \mathbb{N}$  the following convergences hold as  $\varepsilon$  tends to 0:*

$$\begin{cases} \widetilde{T_n(u_i^\varepsilon)} \rightharpoonup \theta_i T_n(u_1) & \text{weakly in } L^2(\Omega), \quad i = 1, 2, \\ \mathfrak{T}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow T_n(u_1) & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \end{cases} \quad (5.66)$$

and for any  $S \in C^0(\mathbb{R})$  with compact support and  $\text{supp } S \subset [-n, n]$ , for  $i = 1, 2$ ,

$$\begin{cases} \mathfrak{T}_i^\varepsilon(S(u_i^\varepsilon) \nabla T_n(u_i^\varepsilon)) \rightharpoonup S(u_1)(\nabla T_n(u_1) + \nabla_y \widehat{u}_i), & \text{weakly in } L^2(\Omega \times Y_i), \\ \mathfrak{T}_i^\varepsilon(S(u_i^\varepsilon)) \frac{\mathfrak{T}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathfrak{T}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \rightharpoonup S(u_1)(\widehat{u}_1 - \widehat{u}_2), & \text{weakly in } L^2(\Omega \times \Gamma), \end{cases} \quad (5.67)$$

where  $u_1 : \Omega \rightarrow \mathbb{R}$  is measurable and finite a.e., and  $\widehat{u}_i : \Omega \times Y_i \rightarrow \mathbb{R}$ , for  $i = 1, 2$  with  $\widehat{u}_i(x, \cdot) \in H^1(Y_i)$  for a.e.  $x \in \Omega$ , and  $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ .

*Proof.* From (5.37) of Corollary 5.14, there is a subsequence of  $\{\varepsilon\}$  (still denoted by  $\varepsilon$ ) such that for any  $n \in \mathbb{N}$ , we can find  $u_1^n \in H_0^1(\Omega)$ ,  $\widehat{u}_1^n \in L^2(\Omega, H_{per}^1(Y_1))$  with  $\mathcal{M}_\Gamma(\widehat{u}_1^n) = 0$  a.e. in  $x \in \Omega$ , and  $\widehat{u}_2^n \in L^2(\Omega, H^1(Y_2))$  such that the following convergences hold as  $\varepsilon$  tends to 0:

$$\begin{cases} s\widetilde{T_n(u_i^\varepsilon)} \rightharpoonup \theta_i u_1^n & \text{weakly in } L^2(\Omega), \quad i = 1, 2, \\ \mathfrak{T}_i^\varepsilon(T_n(u_i^\varepsilon)) \longrightarrow u_1^n & \text{strongly in } L^2(\Omega, H^1(Y_i)), \quad i = 1, 2, \\ \mathfrak{T}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup \nabla u_1^n + \nabla_y \widehat{u}_i^n & \text{weakly in } L^2(\Omega \times Y_i), \quad i = 1, 2, \\ \frac{\mathfrak{T}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathfrak{T}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \rightharpoonup \widehat{u}_1^n - \widehat{u}_2^n & \text{weakly in } L^2(\Omega \times \Gamma). \end{cases} \quad (5.68)$$

Then, convergences (5.66) follow from the first two convergences in (5.68) and identity (5.42) of Theorem 5.17.

Now, we show (5.67). Let  $S \in C^0(\mathbb{R})$  with  $\text{supp } S \subset [-n, n]$ . Then  $S$  is bounded, and by (5.41) of Theorem 5.17, we have for  $i = 1, 2$ ,

$$S(\mathfrak{T}_i^\varepsilon(u_i^\varepsilon)) \longrightarrow S(u_1) \quad \text{in } L^\infty(\Omega \times Y_i) \text{ weak-}^* \text{ and a.e. in } \Omega \times Y_i. \quad (5.69)$$

This convergence also holds when we replace  $\Omega \times Y_i$  by  $\Omega \times \Gamma$ . Then, using Remark 5.16, by the third convergence in (5.68), we deduce for  $i = 1, 2$

$$\mathfrak{T}_i^\varepsilon(S(u_i^\varepsilon) \nabla T_n(u_i^\varepsilon)) = S(\mathfrak{T}_i^\varepsilon(u_i^\varepsilon)) \mathfrak{T}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \rightharpoonup S(u_1)(\nabla T_n(u_1) + \nabla_y \widehat{u}_i^n),$$

weakly in  $L^2(\Omega \times Y_i)$ , while by the fourth convergence in (5.68), it follows that for  $i = 1, 2$ ,

$$\begin{aligned} \mathcal{T}_i^\varepsilon(S(u_i^\varepsilon)) \frac{\mathcal{T}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathcal{T}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} &= S(\mathcal{T}_i^\varepsilon(u_i^\varepsilon)) \frac{\mathcal{T}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathcal{T}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \\ &\rightharpoonup S(u_1)(\widehat{u}_1^n - \widehat{u}_2^n), \end{aligned}$$

weakly in  $L^2(\Omega \times \Gamma)$ . Then (5.67) follows from Theorem 5.18 written for  $\mathcal{R} = S$ .  $\square$

We now identify the unfolded problem satisfied by the triple  $(u_1, \widehat{u}_1, \widehat{u}_2)$ . This is the most difficult proof, which is long and quite labourous.

**Theorem 5.22** (The unfolded homogenized problem). *Let  $u_1, \widehat{u}_1$ , and  $\widehat{u}_2$  be functions as in Lemma 5.21. Let  $\psi_1, \psi_2$  be functions in  $C^1(\mathbb{R})$  (or equivalently,  $\psi_1, \psi_2 \in W^{1,\infty}(\mathbb{R})$ ) with compact supports.*

*Then the triple  $(u_1, \widehat{u}_1, \widehat{u}_2)$  satisfies*

$$\left\{ \begin{aligned} &\sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_i)(\nabla(\psi_1(u_1)\varphi) + \psi_2(u_1)\nabla_y \Phi_i) dx dy \\ &+ \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)(\Phi_1 - \Phi_2) dx d\sigma_y = \int_{\Omega} f(x)\psi_1(u_1)\varphi(x) dx \\ &\forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \quad \Phi_i \in L^2(\Omega, H_{per}^1(Y_i)), \quad i = 1, 2. \end{aligned} \right. \quad (5.70)$$

In addition, for  $k > 0$ , the following limits hold:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\} \times Y_i} A(y, u_1)(\nabla T_k(u_1) + \nabla_y \widehat{u}_i)(\nabla T_k(u_1) + \nabla_y \widehat{u}_i) dx dy = 0, \quad (5.71)$$

for  $i = 1, 2$ , and

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 dx d\sigma_y = 0. \quad (5.72)$$

*Proof.* We do the proof in two steps. The first step is to obtain the unfolded equation (5.70). The next step is proving the convergences (5.94) and (5.95).

*Step 1. Obtaining the unfolded equation (5.70).*

Let  $v \in \mathcal{D}(\Omega)$  and  $n \in \mathbb{N}$ . Choosing  $v_i = v$  and  $\psi = S_n$  (see (5.17)) as test functions in (5.14) of Definition 5.6, we have

$$J_{11}^\varepsilon + J_{21}^\varepsilon + J_{12}^\varepsilon + J_{22}^\varepsilon + J_3^\varepsilon = J_4^\varepsilon, \quad (5.73)$$

where

$$\begin{aligned}
J_{i1}^\varepsilon &= \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla v \, dx, \quad i = 1, 2 \\
J_{i2}^\varepsilon &= \int_{\Omega_i^\varepsilon} S'_n(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon v \, dx, \quad i = 1, 2 \\
J_3^\varepsilon &= \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) v \, d\sigma \\
J_4^\varepsilon &= \int_{\Omega} f S_n(u^\varepsilon) v \, dx.
\end{aligned}$$

We will evaluate the limit of each term first as  $\varepsilon$  goes to zero, and then as  $n$  approaches infinity. To do this, we use convergences (5.66) of Lemma 5.21, Proposition 5.20, and some properties of the unfolding operator in Proposition 5.10.

By (P1) and (P2) of Proposition 5.10, and the fact that  $v$  has compact support, we have for  $i = 1, 2$ ,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} J_{i1}^\varepsilon &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega_i^\varepsilon} S_n(u_i^\varepsilon) A^\varepsilon(x, T_{2n}(u_i^\varepsilon)) \nabla T_{2n}(u_i^\varepsilon) \nabla v \, dx \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathfrak{T}_i^\varepsilon(S_n(u_i^\varepsilon) A^\varepsilon(x, T_{2n}(u_i^\varepsilon)) \nabla T_{2n}(u_i^\varepsilon) \nabla v) \, dx \, dy \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathfrak{T}_i^\varepsilon(A^\varepsilon(x, T_{2n}(u_i^\varepsilon))) \mathfrak{T}_i^\varepsilon(S_n(u_i^\varepsilon) \nabla T_{2n}(u_i^\varepsilon)) \mathfrak{T}_i^\varepsilon(\nabla v) \, dx \, dy.
\end{aligned}$$

Note that  $\mathfrak{T}_i^\varepsilon(A^\varepsilon(x, T_{2n}(u_i^\varepsilon)))$  is bounded and by (5.43) of Theorem 5.42,

$$A^\varepsilon(y, T_{2n}(\mathfrak{T}_i^\varepsilon(u_i^\varepsilon))) \rightharpoonup A(y, T_{2n}(u_1)) \quad \text{in } L^\infty(\Omega \times Y_i) \text{ weak-}^* . \quad (5.74)$$

Hence, it can be shown by Lebesgue Dominated Convergence Theorem that

$$\mathfrak{T}_i^\varepsilon({}^t A^\varepsilon(x, T_{2n}(u_i^\varepsilon))) \mathfrak{T}_i^\varepsilon(\nabla v) \longrightarrow {}^t A(y, T_{2n}(u_1)) \nabla v,$$

strongly in  $(L^2(\Omega \times Y_i))^N$ . Consequently, by (5.67) of Lemma 5.21 (written for  $S = S_n$ ), for  $i = 1, 2$ ,

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} J_{i1}^\varepsilon &= \frac{1}{|Y|} \int_{\Omega \times Y_i} {}^t A(y, T_{2n}(u_1)) \nabla v S_n(u_1) (\nabla T_{2n}(u_1) + \nabla_y \widehat{u}_i) \, dx \, dy \\
&= \frac{1}{|Y|} \int_{\Omega \times Y_i} S_n(u_1) A(y, T_{2n}(u_1)) (\nabla T_{2n}(u_1) + \nabla_y \widehat{u}_i) \nabla v \, dx \, dy. \quad (5.75)
\end{aligned}$$

On the other hand, for  $J_{i2}^\varepsilon$ ,  $i = 1, 2$ , by (5.20), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |J_{i2}^\varepsilon| &\leq \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_i^\varepsilon} |S'_n(u_i^\varepsilon)| |A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon| |v| dx \\ &\leq 2 \cdot \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{\|v\|_{L^\infty(\Omega)}}{2n} \int_{\{|u_i^\varepsilon| < 2n\}} A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx. \end{aligned}$$

Denoting

$$\omega_1(n) = 2 \cdot \limsup_{\varepsilon \rightarrow 0} \frac{1}{2n} \int_{\{|u_i^\varepsilon| < 2n\}} A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx,$$

we have by (5.58) of Proposition 5.20,

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} J_{i2}^\varepsilon \leq \|v\|_{L^\infty(\Omega)} \lim_{n \rightarrow \infty} \omega_1(n) = 0, \quad i = 1, 2. \quad (5.76)$$

For  $J_3^\varepsilon$ , by (5.31), we obtain

$$\begin{aligned} &\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} |J_3^\varepsilon| \\ &\leq \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \varepsilon^{-1} \int_{\Gamma^\varepsilon} |h^\varepsilon(x)| |u_1^\varepsilon - u_2^\varepsilon| |S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)| |v| d\sigma \\ &\leq 2 \cdot \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-1} \|v\|_{L^\infty(\Omega)} \|h\|_{L^\infty(\Gamma)}}{2n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon) (T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)) d\sigma. \end{aligned}$$

If we set

$$\omega_2(n) = 2 \|h\|_{L^\infty(\Gamma)} \limsup_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-1}}{2n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon) (T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon)) d\sigma,$$

then by (5.59) of Proposition 5.20,

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} J_3^\varepsilon \leq \|v\|_{L^\infty(\Omega)} \lim_{n \rightarrow \infty} \omega_2(n) = 0. \quad (5.77)$$

For the integral on the right-hand side of (5.73), by (P1) and (P2) of Proposition 5.10, and Remark 5.16, and since  $v$  has a compact support, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} J_4^\varepsilon &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \sum_{i=1}^2 \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(f S_n(u_i^\varepsilon) v) dx dy \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \sum_{i=1}^2 \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(f) S_n(\mathcal{T}_i^\varepsilon(u_i^\varepsilon)) \mathcal{T}_i^\varepsilon(v) dx dy. \end{aligned}$$



Using (P4) of Proposition 5.10, (5.69) (with  $S = S_n$ ), and the fact that  $f$ ,  $S_n(u_1)$ , and  $v$  are independent of  $y$ , we obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} J_4^\varepsilon &= \frac{1}{|Y|} \int_{\Omega \times Y_1} f S_n(u_1) v \, dx \, dy + \frac{1}{|Y|} \int_{\Omega \times Y_2} f S_n(u_1) v \, dx \, dy \\ &= \theta_1 \int_{\Omega} f S_n(u_1) v \, dx + \theta_2 \int_{\Omega} f S_n(u_1) v \, dx \\ &= \int_{\Omega} f S_n(u_1) v \, dx. \end{aligned} \quad (5.78)$$

Combining (5.75)-(5.78), we have

$$\begin{aligned} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} S_n(u_1) A(y, u_1) (\nabla u_1 + \nabla_y \hat{u}_i) \nabla v \, dx \, dy \\ + \omega(v, n) = \int_{\Omega} f S_n(u_1) v \, dx, \end{aligned} \quad (5.79)$$

where  $\omega(v, n) \leq \|v\|_{L^\infty(\Omega)} \omega(n)$ , with  $\omega(n) \rightarrow 0$  as  $n \rightarrow \infty$ . By density, this last identity holds for all  $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ .

Let  $\psi_1, \psi_2$  be functions in  $C^1(\mathbb{R})$  with compact support, verifying

$$\text{supp } \psi_i \subset [-m, m], \quad i = 1, 2,$$

for some  $m > 0$ . Using  $v = \psi_1(u_1)\varphi$ , where  $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ , as the test function in (5.79), we get

$$\begin{aligned} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} S_n(u_1) A(y, u_1) (\nabla u_1 + \nabla \hat{u}_i) \psi_1'(u_1) \nabla u_1 \varphi \, dx \, dy \\ + \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} S_n(u_1) A(y, u_1) (\nabla u_1 + \nabla_y \hat{u}_i) \psi_1(u_1) \nabla \varphi \, dx \, dy \\ + \omega(v, n) = \int_{\Omega} f S_n(u_1) \psi_1(u_1) \varphi \, dx. \end{aligned} \quad (5.80)$$

For  $n \geq m$ , since  $\text{supp } \psi_1 \subset [-m, m]$ , we have

$$\psi_1(u_1) S_n(u_1) = \psi_1(u_1) \quad \text{and} \quad \psi_1'(u_1) S_n(u_1) = \psi_1'(u_1).$$

Consequently, if we pass to the limit on both sides of (5.80) as  $n \rightarrow \infty$ , we obtain

$$\begin{aligned} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla \hat{u}_i) \psi_1'(u_1) \nabla u_1 \varphi_1 \, dx \, dy \\ + \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \hat{u}_i) \psi_1(u_1) \nabla \varphi \, dx \, dy = \int_{\Omega} f \psi_1(u_1) \varphi \, dx, \end{aligned}$$

which can be rewritten as

$$\sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla (\psi_1(u_1) \varphi) dx dy = \int_{\Omega} f \psi_1(u_1) \varphi dx. \quad (5.81)$$

Now, let  $v_i^\varepsilon(x) = \varepsilon \omega(x) \xi_i\left(\frac{x}{\varepsilon}\right)$ ,  $i = 1, 2$ , where

$$\omega \in \mathcal{D}(\Omega) \quad \text{and} \quad \xi_i \in H_{per}^1(Y_i) \cap L^\infty(Y_i), \quad i = 1, 2.$$

Note that we have

$$\mathcal{T}_i^\varepsilon(v_i^\varepsilon) = \varepsilon \mathcal{T}_i^\varepsilon(\omega) \xi_i \longrightarrow 0 \quad \text{strongly in } L^2(\Omega \times Y_i), \quad i = 1, 2. \quad (5.82)$$

Moreover, since, for  $i = 1, 2$ ,  $\nabla v_i^\varepsilon(x) = \varepsilon \nabla \omega(x) \xi_i\left(\frac{x}{\varepsilon}\right) + \omega(x) \nabla_y \xi_i\left(\frac{x}{\varepsilon}\right)$ , we get

$$\mathcal{T}_i^\varepsilon(\nabla v_i^\varepsilon) = \varepsilon \xi_i \mathcal{T}_i^\varepsilon(\nabla \omega) + \nabla_y \xi_i \mathcal{T}_i^\varepsilon(\omega) \longrightarrow \omega \nabla_y \xi_i, \quad i = 1, 2, \quad (5.83)$$

strongly in  $L^2(\Omega \times Y_i)$ .

Taking  $\psi = \psi_2$  and  $v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon)$  as test function in (5.14) we have

$$L_{11}^\varepsilon + L_{21}^\varepsilon + L_{12}^\varepsilon + L_{22}^\varepsilon + L_3^\varepsilon = L_{14}^\varepsilon + L_{24}^\varepsilon, \quad (5.84)$$

where

$$\begin{aligned} L_{i1}^\varepsilon &= \int_{\Omega_i^\varepsilon} \psi_2(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_1^\varepsilon \nabla v_i^\varepsilon dx, \quad i = 1, 2 \\ L_{i2}^\varepsilon &= \int_{\Omega_i^\varepsilon} \psi_2'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon v_i^\varepsilon dx, \quad i = 1, 2 \\ L_3^\varepsilon &= \varepsilon^{-1} \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (v_1^\varepsilon \psi_2(u_1^\varepsilon) - v_2^\varepsilon \psi_2(u_2^\varepsilon)) d\sigma \\ L_{i4}^\varepsilon &= \int_{\Omega_i^\varepsilon} f \psi_2(u_i^\varepsilon) v_i^\varepsilon dx, \quad i = 1, 2. \end{aligned}$$

We study each term as  $\varepsilon$  tends to 0. For  $L_{i1}^\varepsilon$ , using similar arguments to show (5.75), and by (5.67) of Lemma 5.21 (written for  $S = \psi_2$ ), we obtain for  $i = 1, 2$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} L_{i1}^\varepsilon &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(A^\varepsilon(x, T_n(u_i^\varepsilon))) \mathcal{T}_i^\varepsilon(\psi_2(u_i^\varepsilon) \nabla T_n(u_i^\varepsilon)) \mathcal{T}_i^\varepsilon(\nabla v_i^\varepsilon) dx dy \\ &= \frac{1}{|Y|} \int_{\Omega \times Y_i} \psi_2(u_1) A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \omega \nabla_y \xi_i dx dy. \end{aligned} \quad (5.85)$$

For  $L_{i2}^\varepsilon$ , we have for  $i = 1, 2$ ,

$$\begin{aligned} |L_{i2}^\varepsilon| &= \left| \int_{\Omega_i^\varepsilon} \psi_2'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon \left( \varepsilon \omega(x) \xi_i \left( \frac{x}{\varepsilon} \right) \right) dx \right| \\ &\leq \varepsilon \|\omega\|_{L^\infty(\Omega)} \|\xi_i\|_{L^\infty(Y_i)} \int_{\Omega_i^\varepsilon} \psi_2'(u_i^\varepsilon) A^\varepsilon(x, u_i^\varepsilon) \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx. \end{aligned}$$

By (5.16) of Proposition 5.12 and the fact that  $A \in L^\infty(\Omega \times (-m, m))^{N \times N}$ , this last integral is uniformly bounded. As a consequence,

$$\lim_{\varepsilon \rightarrow 0} L_{i2}^\varepsilon = 0. \quad (5.86)$$

For  $L_{i4}^\varepsilon$ , by (P1) of Proposition 5.10 and Remark 5.16, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} L_{i4}^\varepsilon &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(f \psi_2(u_i^\varepsilon) v_i^\varepsilon) dx dy \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathcal{T}_i^\varepsilon(f) \psi_2(\mathcal{T}_i^\varepsilon(u_i^\varepsilon)) \mathcal{T}_i^\varepsilon(v_i^\varepsilon) dx dy. \end{aligned}$$

Then by (P4) of Proposition 5.10, (5.69) (written for  $S = \psi_2$ ), and (5.82), we obtain

$$\lim_{\varepsilon \rightarrow 0} L_{i4}^\varepsilon = 0. \quad (5.87)$$

Using the definition of  $v^\varepsilon$ , we can write  $L_3^\varepsilon$  as

$$L_3^\varepsilon = \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) (\psi_2(u_1^\varepsilon) \omega(x) \xi_1^\varepsilon(x) - \psi_2(u_2^\varepsilon) \omega(x) \xi_2^\varepsilon(x)) d\sigma.$$

To pass to the limit of this expression, we want to be able to express the term  $\psi_2(u_i^\varepsilon)(u_1^\varepsilon - u_2^\varepsilon)$  as  $\psi_2(u_i^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon))$ , for  $i = 1, 2$ . This will enable us to use the second convergence in (5.67) of Lemma 5.21. But this is not a straightforward task since

$$\psi_2(u_i^\varepsilon)(u_1^\varepsilon - u_2^\varepsilon) \neq \psi_2(u_i^\varepsilon)(u_1^\varepsilon - u_2^\varepsilon) \chi_{\{|u_1^\varepsilon| < n\}} \chi_{\{|u_2^\varepsilon| < n\}}.$$

To achieve our aim, we make use again of the functions  $S_n$ . We choose  $n$  large enough, such that

$$\psi_2(t) = S_n(t) \psi_2(t) \quad \forall t \in \mathbb{R}. \quad (5.88)$$

Then, we can write

$$\begin{aligned} &\int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) \psi_2(u_1^\varepsilon) \omega(x) \xi_1^\varepsilon(x) d\sigma \\ &= \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) \psi_2(u_1^\varepsilon) \omega(x) \xi_1^\varepsilon(x) (S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon)) d\sigma \\ &\quad + \int_{\Gamma^\varepsilon} h^\varepsilon(x) (u_1^\varepsilon - u_2^\varepsilon) \psi_2(u_1^\varepsilon) \omega(x) \xi_1^\varepsilon(x) S_n(u_2^\varepsilon) d\sigma. \end{aligned} \quad (5.89)$$

By (5.31), we obtain

$$\begin{aligned} & \left| \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)\omega(x)\xi_1\left(\frac{x}{\varepsilon}\right)\psi_2(u_1^\varepsilon)(S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon))d\sigma \right| \\ & \leq 2\|h\|_{L^\infty(\Gamma)}\|\omega\|_{L^\infty(\Omega)}\|\xi_1\|_{L^\infty(\Gamma)}\|\psi_2\|_{L^\infty(\mathbb{R})} \\ & \quad \times \frac{1}{2n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon))d\sigma, \end{aligned}$$

and (5.59) of Proposition 5.20 implies

$$\lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)\omega(x)\xi_1\left(\frac{x}{\varepsilon}\right)\psi_2(u_1^\varepsilon)(S_n(u_1^\varepsilon) - S_n(u_2^\varepsilon))d\sigma = 0. \quad (5.90)$$

Note that (5.88) implies  $\text{supp } \psi_2 \subset \text{supp } S_n = [-2n, 2n]$ . Hence, we can write

$$\begin{aligned} & \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)\psi_2(u_1^\varepsilon)\omega(x)\xi_1^\varepsilon(x)S_n(u_2^\varepsilon)d\sigma \\ & = \int_{\Gamma^\varepsilon} h^\varepsilon(x)(T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon))\psi_2(u_1^\varepsilon)\omega(x)\xi_1^\varepsilon(x)S_n(u_2^\varepsilon)d\sigma. \end{aligned}$$

By Lemma 2.16 of [46], (P1) of Proposition 5.10, and Remark 5.16, we obtain

$$\begin{aligned} & \int_{\Gamma^\varepsilon} h^\varepsilon(x)(T_{2n}(u_1^\varepsilon) - T_{2n}(u_2^\varepsilon))\psi_2(u_1^\varepsilon)\omega(x)\xi_1^\varepsilon(x)S_n(u_2^\varepsilon)d\sigma \\ & = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \Gamma} h(y)\mathfrak{V}_1^\varepsilon(\psi_2(u_1^\varepsilon))\left(\mathfrak{V}_1^\varepsilon(T_{2n}(u_1^\varepsilon)) - \mathfrak{V}_2^\varepsilon(T_{2n}(u_2^\varepsilon))\right) \\ & \quad \times \mathfrak{V}_1^\varepsilon(\omega)\xi_1(y)\mathfrak{V}_2^\varepsilon(S_n(u_2^\varepsilon))dx d\sigma_y \\ & = \frac{1}{\varepsilon|Y|} \int_{\Omega \times \Gamma} h(y)\mathfrak{V}_1^\varepsilon(\psi_2(u_1^\varepsilon))\left(\mathfrak{V}_1^\varepsilon(T_{2n}(u_1^\varepsilon)) - \mathfrak{V}_2^\varepsilon(T_{2n}(u_2^\varepsilon))\right) \\ & \quad \times \mathfrak{V}_1^\varepsilon(\omega)\xi_1(y)S_n(\mathfrak{V}_2^\varepsilon(u_2^\varepsilon))dx d\sigma_y. \end{aligned}$$

Note that, by (5.24),

$$\|S_n(\mathfrak{V}_2^\varepsilon(u_2^\varepsilon))\|_{L^\infty(\Omega \times \Gamma)} \leq 1,$$

and hence, (5.41) of Theorem 5.17 gives

$$S_n(\mathfrak{V}_2^\varepsilon(u_2^\varepsilon)) \longrightarrow S_n(u_1) \quad \text{weakly-}^* \text{ in } L^\infty(\Omega \times \Gamma) \text{ and a.e. on } \Omega \times \Gamma. \quad (5.91)$$

By (5.67) of Lemma 5.21 (written for  $S = \psi_2$ ), (5.91), (P4) of Proposition

5.10, and (5.88), we have

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)\psi_2(u_1^\varepsilon)\omega(x)\xi_1^\varepsilon(x)S_n(u_2^\varepsilon) d\sigma \\
&= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon|Y|} \int_{\Omega \times \Gamma} h(y)\mathfrak{T}_1^\varepsilon(\psi_2(u_1^\varepsilon))\left(\mathfrak{T}_1^\varepsilon(T_{2n}(u_1^\varepsilon)) - \mathfrak{T}_2^\varepsilon(T_{2n}(u_2^\varepsilon))\right) \\
&\quad \times \mathfrak{T}_1^\varepsilon(\omega)\xi_1(y)S_n(\mathfrak{T}_2^\varepsilon(u_2^\varepsilon)) dx d\sigma_y \\
&= \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)\omega(x)\xi_1(y)S_n(u_1) dx d\sigma_y \\
&= \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)\omega(x)\xi_1(y) dx d\sigma_y.
\end{aligned}$$

Thus, combining this with (5.89) and (5.90), gives

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)\psi_2(u_1^\varepsilon)\omega(x)\xi_1^\varepsilon(x) d\sigma \\
&= \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)\omega(x)\xi_1(y) dx d\sigma_y.
\end{aligned}$$

Using similar arguments to obtain the preceding identity, we deduce that

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \int_{\Gamma^\varepsilon} h^\varepsilon(x)(u_1^\varepsilon - u_2^\varepsilon)\psi_2(u_2^\varepsilon)\omega(x)\xi_2^\varepsilon(x) d\sigma \\
&= \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2)\omega(x)\xi_2(y) dx d\sigma_y.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon \rightarrow 0} L_3 = \frac{1}{|Y|} \int_{\Omega \times \Gamma} \psi_2(u_1)h(y)(\widehat{u}_1 - \widehat{u}_2)(\omega\xi_1 - \omega\xi_2) dx d\sigma_y. \quad (5.92)$$

Hence, passing to the limit of (5.84) as  $\varepsilon$  goes to 0, using the results (5.85)-(5.87) and (5.92), we conclude that

$$\begin{aligned}
& \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} \psi_2(u_1)A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_i)\omega \nabla_y \xi_i dx dy \\
&\quad + \frac{1}{|Y|} \int_{\Omega \times \Gamma} \psi_2(u_1)h(y)(\widehat{u}_1 - \widehat{u}_2)(\omega\xi_1 - \omega\xi_2) dx d\sigma_y = 0,
\end{aligned}$$

for any  $\omega \in \mathcal{D}(\Omega)$  and for any  $\xi_i \in H_{per}^1(Y_i) \cap L^\infty(Y_i)$ ,  $i = 1, 2$ .

Note that  $\psi_2$  has a compact support, then from (5.51) of Theorem 5.18, we deduce that

$$\psi_2(u_1)A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_i) \in (L^2(\Omega \times Y_i))^N, \quad i = 1, 2,$$

and

$$h(y)\psi_2(u_1)(\widehat{u}_1 - \widehat{u}_2) \in L^2(\Omega \times \Gamma).$$

Hence, we obtain by density,

$$\begin{aligned} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} \psi_2(u_1) A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla_y (\omega \xi_i) dx dy \\ + \frac{1}{|Y|} \int_{\Omega \times \Gamma} \psi_2(u_1) h(y) (\widehat{u}_1 - \widehat{u}_2) (\omega \xi_1 - \omega \xi_2) dx d\sigma_y = 0, \end{aligned}$$

for every  $\omega \in L^2(\Omega)$  and  $\xi_i \in H_{per}^1(Y_i)$ ,  $i = 1, 2$ .

Since for  $i = 1, 2$ ,  $\omega \xi_i \in L^2(\Omega) \otimes H_{per}^1(Y_i)$  and  $L^2(\Omega) \otimes H_{per}^1(Y_i)$  is dense in  $L^2(\Omega, H_{per}^1(Y_i))$ , we have

$$\begin{aligned} \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} \psi_2(u_1) A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla_y \Phi_i dx dy \\ + \frac{1}{|Y|} \int_{\Omega \times \Gamma} \psi_2(u_1) h(y) (\widehat{u}_1 - \widehat{u}_2) (\Phi_1 - \Phi_2) dx d\sigma_y = 0, \end{aligned} \quad (5.93)$$

for every  $\Phi_i \in L^2(\Omega, H_{per}^1(Y_i))$ , for  $i = 1, 2$ .

Adding (5.81) and (5.93) gives the limit problem (5.70).

*Step 2. Proof of the convergences (5.71) and (5.72).*

Let us first prove that for  $n \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u_1| < n\} \times Y_i} A(y, u_1) (\nabla T_n(u_1) + \nabla_y \widehat{u}_i) (\nabla T_n(u_1) + \nabla_y \widehat{u}_i) dx dy = 0, \quad (5.94)$$

for  $i = 1, 2$ , and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u_1| < n\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 dx d\sigma_y = 0. \quad (5.95)$$

To prove convergence (5.94), for any  $n \in \mathbb{N}$ , and  $i = 1, 2$  set

$$E_{i,n} = \frac{1}{|Y|} \int_{\{|u_1| < n\} \times Y_i} A(y, u_1) (\nabla T_n(u_1) + \nabla_y \widehat{u}_i) (\nabla T_n(u_1) + \nabla_y \widehat{u}_i) dx dy,$$

Note that we have the following convergences as  $\varepsilon$  tends to 0:

$$\begin{aligned} \widetilde{\mathcal{G}}_i^\varepsilon(A^\varepsilon(x, u_i^\varepsilon)) &\longrightarrow A(y, u_1) && \text{a.e. in } \Omega \times Y_i, \\ \widetilde{\mathcal{G}}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) &\rightharpoonup \nabla T_n(u_1) + \nabla_y \widehat{u}_i^n && \text{weakly in } L^2(\Omega \times Y_i), \quad \forall n \in \mathbb{N}, \end{aligned}$$

where  $\widehat{u}_i^n$  is the function defined in (5.68) for  $i = 1, 2$ .

Taking into account (5.57), by Lemma 4.9 of [29] (with  $\mathcal{D}_\varepsilon = A^\varepsilon(x, u_1^\varepsilon)$  and  $\zeta_\varepsilon = \nabla T_n(u_1^\varepsilon)$ ), and (P2) of Proposition 5.10, we have

$$\begin{aligned} E_{i,n} &\leq \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla T_n(u_1) + \nabla_y \widehat{u}_i^n) (\nabla T_n(u_1) + \nabla_y \widehat{u}_i^n) dx dy \\ &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathfrak{G}_i^\varepsilon(A^\varepsilon(x, u_i^\varepsilon)) \mathfrak{G}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \mathfrak{G}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) dx dy \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times Y_i} \mathfrak{G}_i^\varepsilon(A^\varepsilon(x, u_i^\varepsilon)) \mathfrak{G}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) \mathfrak{G}_i^\varepsilon(\nabla T_n(u_i^\varepsilon)) dx dy \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_i^\varepsilon} A^\varepsilon(x, u_i^\varepsilon) \nabla T_n(u_i^\varepsilon) \nabla T_n(u_i^\varepsilon) dx. \end{aligned}$$

By (5.58) of Proposition 5.20,

$$\lim_{n \rightarrow \infty} \frac{1}{n} E_{i,n} \leq \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{1}{n} \int_{\Omega_i^\varepsilon} A^\varepsilon(x, u_i^\varepsilon) \nabla T_n(u_i^\varepsilon) \nabla T_n(u_i^\varepsilon) dx = 0.$$

Observing that for any  $0 < n \leq k < n + 1$  the ellipticity condition (A3.1) (see Remark 5.7 for a similar argument) gives

$$\frac{1}{k} \int_{\{|u_1| < k\} \times Y_i} A(y, u_1) (\nabla T_k(u_1) + \nabla_y \widehat{u}_i) (\nabla T_k(u_1) + \nabla_y \widehat{u}_i) dx dt \leq \frac{1}{n} E_{i,n+1},$$

we obtain (5.71).

It remains to show (5.95). By (5.38) of Corollary 5.14, we have

$$\|\widehat{u}_1^n - \widehat{u}_2^n\|_{L^2(\Omega \times \Gamma)} \leq \liminf_{\varepsilon \rightarrow 0} \left\| \frac{\mathfrak{G}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathfrak{G}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \right\|_{L^2(\Omega \times \Gamma)}.$$

Consequently, by (5.57), (P9) of Proposition 5.10, and (5.34) (written for  $f = u_1^\varepsilon$  and  $g = u_2^\varepsilon$ ), we obtain

$$\begin{aligned} \int_{\{|u_1| < n\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 dx d\sigma_y &\leq \frac{1}{|Y|} \int_{\Omega \times \Gamma} (\widehat{u}_1^n - \widehat{u}_2^n)^2 dx d\sigma_y \\ &\leq \liminf_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times \Gamma} \left( \frac{\mathfrak{G}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathfrak{G}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \right)^2 dx d\sigma_y \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{|Y|} \int_{\Omega \times \Gamma} \left( \frac{\mathfrak{G}_1^\varepsilon(T_n(u_1^\varepsilon)) - \mathfrak{G}_2^\varepsilon(T_n(u_2^\varepsilon))}{\varepsilon} \right)^2 dx d\sigma_y \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon} (T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon))^2 d\sigma \\ &\leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon) (T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma. \end{aligned}$$

By (5.59) of Proposition 5.20,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u_1| < n\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 dx d\sigma_y \\ \leq \lim_{n \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-1}}{n} \int_{\Gamma^\varepsilon} (u_1^\varepsilon - u_2^\varepsilon)(T_n(u_1^\varepsilon) - T_n(u_2^\varepsilon)) d\sigma = 0. \end{aligned}$$

This proves (5.95) and then (5.72) by a monotony argument. This completes the proof.  $\square$

In order to state the next theorem, we introduce the space  $W_{per}(Y_1)$ , by

$$W_{per}(Y_1) = \{u \in H_{per}^1(Y_1) \mid \mathcal{M}_\Gamma(u) = 0\}$$

equipped with the norm

$$\|u\|_{W_{per}(Y_1)} = \|\nabla u\|_{L^2(Y_1)}. \quad (5.96)$$

Since a Poincaré-Wirtinger inequality holds in  $W_{per}(Y_1)$ , (5.96) defines a norm.

The following results give a characterization of  $\widehat{u}_i$ ,  $i = 1, 2$ , which allows us to obtain the homogenized problem in  $\Omega$ . This kind of result is a classical and a quite simple step in the literature. For completeness, we prove it here in detail since the corresponding theorem in [46] was proved without giving details. With respect to the case of perforated domains studied in [43], the situation here has the additional difficulties due to the fact that the second component  $\widehat{u}_2$  does not necessarily have a zero average on  $\Gamma$ .

**Theorem 5.23.** *Let  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$  be the unique solution of the following variational problem:*

$$\left\{ \begin{array}{l} \text{Find } \chi^\lambda(\cdot, t) = (\chi_1^\lambda(\cdot, t), \chi_2^\lambda(\cdot, t)) \in W_{per}(Y_1) \times H^1(Y_2) \text{ such that} \\ \sum_{i=1}^2 \int_{Y_i} A(y, t) \nabla_y \chi_i^\lambda(y, t) \nabla_y v_i dy + \int_\Gamma h(y) (\chi_1^\lambda(y, t) - \chi_2^\lambda(y, t)) (v_1 - v_2) d\sigma \\ = \sum_{i=1}^2 \int_{Y_i} A(y, t) \lambda \nabla_y v_i dy \\ \text{for any } v = (v_1, v_2) \in H_{per}^1(Y_1) \times H^1(Y_2), \end{array} \right. \quad (5.97)$$

for all  $t \in \mathbb{R}$  and  $\lambda \in \mathbb{R}^N$ .

Using the assumptions and notations of Theorem 5.22, the function  $\widehat{u}_i$ ,  $i = 1, 2$ , can be expressed as

$$\widehat{u}_i(x, y) = - \sum_{j=1}^N \chi_i^j(y, u_1(x)) \frac{\partial u_1}{\partial x_j}(x), \quad i = 1, 2, \quad (5.98)$$



where  $\chi^j = (\chi_1^j, \chi_2^j)$  is the unique solution of (5.97) written for  $\lambda = e_j$  with  $\{e_j\}_{j=1}^N$  being the canonical basis.

*Proof.* Let  $\varphi \equiv 0$  and  $\Phi_i(x, y) = \omega(x)v_i(y)$  in (5.70) of Theorem 5.22, where  $\omega \in \mathcal{D}(\Omega)$  and  $v_i \in H_{per}^1(Y_i)$ . Substituting these values, we obtain

$$\begin{aligned} & \sum_{i=1}^2 \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \psi_2(u_1) \omega \nabla_y v_i \, dx \, dy \\ & \quad + \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) \psi_2(u_1) (\widehat{u}_1 - \widehat{u}_2) (v_1 - v_2) \omega \, dx \, d\sigma_y = 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} & \int_{\Omega} \left( \sum_{i=1}^2 \int_{Y_i} A(y, u_1) \psi_2(u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla_y v_i \, dy \right. \\ & \quad \left. + \int_{\Gamma} h(y) \psi_2(u_1) (\widehat{u}_1 - \widehat{u}_2) (v_1 - v_2) \, d\sigma_y \right) \omega \, dx = 0. \end{aligned}$$

Since this holds for all  $\omega \in \mathcal{D}(\Omega)$  and  $u_1$  is independent of  $y$ , we have

$$\begin{aligned} & \psi_2(u_1) \left( \sum_{i=1}^2 \int_{Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla_y v_i \, dy \right. \\ & \quad \left. + \int_{\Gamma} h(y) (\widehat{u}_1 - \widehat{u}_2) (v_1 - v_2) \, d\sigma_y \right) = 0. \end{aligned}$$

Let  $k > 0$  and choose  $\psi_2$  such that  $\psi_2(t) = 1$  whenever  $|t| \leq k$ . Consequently,

$$\begin{aligned} & \sum_{i=1}^2 \int_{Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla_y v_i \, dy \\ & \quad + \int_{\Gamma} h(y) (\widehat{u}_1 - \widehat{u}_2) (v_1 - v_2) \, d\sigma_y = 0, \end{aligned} \tag{5.99}$$

for all  $v = (v_1, v_2) \in H_{per}^1(Y_1) \times H_{per}^1(Y_2)$ , and a.e. in  $\{x \in \Omega \mid |u_1(x)| \leq k\}$ , for any  $k > 0$ . Since  $u_1$  is finite a.e. in  $\Omega$ , (5.99) holds for a.e.  $x \in \Omega$ .

Note that by linearity with respect to  $\lambda$ , we have

$$\chi_i^\lambda(y, t) = \sum_{j=1}^N \lambda_j \chi_i^j(y, t), \quad i = 1, 2, \quad \forall \lambda \in \mathbb{R}^N, \quad \forall t \in \mathbb{R},$$

where  $\chi^j = (\chi_1^j, \chi_2^j)$  is the unique solution of (5.97) with  $\lambda = e_j$ . More particularly, if we take

$$\lambda = \nabla u_1 \quad \text{and} \quad t = u_1(x), \quad \text{for some } x \in \Omega, \quad (5.100)$$

and denote the corresponding solution by  $\widehat{\chi} = (\widehat{\chi}_1, \widehat{\chi}_2)$ , we have

$$\widehat{\chi}_i(y, u_1(x)) = \sum_{j=1}^N \frac{\partial u_1}{\partial x_j}(x) \chi_i^j(y, u_1(x)), \quad i = 1, 2. \quad (5.101)$$

Let  $x \in \Omega$ . Replacing (5.100) in (5.97), we obtain

$$\begin{aligned} \sum_{i=1}^2 \int_{Y_i} A(y, u_1) \nabla_y \widehat{\chi}_i \nabla_y v_i \, dy + \int_{\Gamma} h(y) (\widehat{\chi}_1 - \widehat{\chi}_2) (v_1 - v_2) \, d\sigma \\ = \sum_{i=1}^2 \int_{Y_i} A(y, u_1) \nabla u_1 \nabla_y v_i \, dy. \end{aligned} \quad (5.102)$$

Now, let  $\varphi_1 \in \mathcal{D}(Y_1)$ . Consider  $v = (\varphi_1, 0)$  as test function for both (5.99) and (5.102). Taking into account that  $\varphi_1$  has a compact support, we obtain, from (5.99),

$$\int_{Y_1} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_1) \nabla_y \varphi_1 \, dy = 0,$$

and, from (5.102),

$$\int_{Y_1} A(y, u_1) \nabla_y \widehat{\chi}_1 \nabla_y \varphi_1 \, dy = \int_{Y_1} A(y, u_1) \nabla u_1 \nabla_y \varphi_1 \, dy.$$

Subtracting these last two equations, we have

$$\int_{Y_1} A(y, u_1) (\nabla_y \widehat{u}_1 + \nabla_y \widehat{\chi}_1) \nabla_y \varphi_1 \, dy = 0, \quad \forall \varphi_1 \in \mathcal{D}(Y_1). \quad (5.103)$$

By density, this holds for every  $\varphi_1 \in H^1(Y_1)$ . Then, (5.103) written for  $\varphi_1 = \widehat{u}_1 + \widehat{\chi}_1$  (taking into account (5.51) of Theorem 5.18) gives

$$\int_{Y_1} A(y, u_1) (\nabla_y \widehat{u}_1 + \nabla_y \widehat{\chi}_1) (\nabla_y \widehat{u}_1 + \nabla_y \widehat{\chi}_1) \, dy = 0.$$

The ellipticity of  $A$  implies

$$\|\nabla_y (\widehat{u}_1 + \widehat{\chi}_1)\|_{L^2(Y_1)}^2 = \|\nabla_y \widehat{u}_1 + \nabla_y \widehat{\chi}_1\|_{L^2(Y_1)}^2 = 0.$$

It follows that for some  $\xi_1(x) \in L^2(\Omega)$  (independent of  $y$ ),

$$\widehat{u}_1(x, y) = -\widehat{\chi}_1(y, u_1(x)) + \xi_1(x), \quad \text{a.e. in } \Omega \times Y_1. \quad (5.104)$$

Similary, taking  $v = (0, \varphi_2)$ , where  $\varphi_2 \in \mathcal{D}(Y_2)$ , as test function for both (5.99) and (5.102), we conclude that

$$\widehat{u}_2(x, y) = -\widehat{\chi}_2(y, u_1(x)) + \xi_2(x), \quad \text{a.e. in } \Omega \times Y_2, \quad (5.105)$$

for some  $\xi_2(x) \in L^2(\Omega)$  (independent of  $y$ ).

Consequently, from (5.101),

$$\widehat{u}_i(x, y) = -\sum_{j=1}^N \frac{\partial u_1}{\partial x_j}(x) \chi_i^j(y, u_1(x)) + \xi_i(x), \quad \text{a.e. in } \Omega \times Y_i. \quad (5.106)$$

To show (5.98), we need to prove that  $\xi_i \equiv 0$ , for  $i = 1, 2$ .

*Case  $i = 1$ .*

Taking the average on  $\Gamma$  on both sides of (5.104) and using (5.51) of Theorem 5.18, we obtain

$$0 = \int_{\Gamma} \widehat{u}_1(x, y) dy = - \int_{\Gamma} \widehat{\chi}_1(y, u_1(x)) dy + \xi_1(x)|\Gamma|, \quad \text{for a.e. } x \in \Omega.$$

Since  $\mathcal{M}_{\Gamma}(\widehat{\chi}_1) = 0$  for a.e.  $x \in \Omega$ , it follows that

$$\xi_1 \equiv 0. \quad (5.107)$$

*Case  $i = 2$ .*

To show that  $\xi_2 \equiv 0$ , note that from (5.104), (5.105), and (5.107), we have

$$\widehat{\chi}_1 - \widehat{\chi}_2 = -\widehat{u}_1 + \widehat{u}_2 - \xi_2, \quad \text{on } \Omega \times \Gamma,$$

with

$$\nabla_y \widehat{\chi}_i = -\nabla_y \widehat{u}_i \quad \text{in } \Omega \times Y_i, \quad i = 1, 2.$$

Replacing these identities in (5.102) gives

$$\begin{aligned} & \sum_{i=1}^2 \int_{Y_i} A(y, u_1) (-\nabla_y \widehat{u}_i) \nabla_y v_i dy \\ & + \int_{\Gamma} h(y) (-\widehat{u}_1 + \widehat{u}_2 - \xi_2) (v_1 - v_2) d\sigma = \sum_{i=1}^2 \int_{Y_i} A(y, u_1) \nabla u_1 \nabla_y v_i dy. \end{aligned}$$

Rearranging the terms, we obtain

$$\begin{aligned} \sum_{i=1}^2 \int_{Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla_y v_i \, dy \\ + \int_{\Gamma} h(y) (\widehat{u}_1 - \widehat{u}_2 + \xi_2) (v_1 - v_2) \, d\sigma = 0. \end{aligned}$$

This, together with (5.99), leads to

$$\xi_2(x) \int_{\Gamma} h(y) (v_1 - v_2) \, d\sigma = 0, \quad \text{for a.e. } x \in \Omega.$$

Choosing  $(v_1, v_2) \in H_{per}^1(Y_1) \times H_{per}^1(Y_2)$  such that  $v_1 > v_2$  a.e. on  $\Gamma$ , we conclude that

$$\xi_2 \equiv 0,$$

since  $h > 0$  a.e. on  $\Gamma$ . This, together with (5.106) and (5.107), gives (5.98).  $\square$

We are now able to describe the homogenized problem in  $\Omega$ . Let us first define, the homogenized matrix  $A^0(t)$ , introduced in [68] for the linear case and given by

$$A^0(t) = A_1^0(t) + A_2^0(t), \quad \text{for every } t \in \mathbb{R},$$

where

$$A_i^0(t)\lambda = \frac{1}{|Y_i|} \int_{Y_i} A(y, t) \nabla_y w_i^\lambda(y, t) \, dy, \quad i = 1, 2, \quad \forall \lambda \in \mathbb{R}^N,$$

with

$$w_i^\lambda(y, t) = \lambda \cdot y - \chi_i^\lambda(y, t),$$

and  $\chi^\lambda = (\chi_1^\lambda, \chi_2^\lambda)$  the solution of (5.97). Standard arguments (see for instance [33] or [32]), provide the following uniform ellipticity:

$$A^0(t)\xi \cdot \xi \geq \alpha|\xi|^2, \quad \forall t \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N. \quad (5.108)$$

We recall the following recent result from [41], which is important for uniqueness results.

**Theorem 5.24** ([41]). *Suppose that the matrix field  $A(y, t)$  satisfies the assumptions (A3) and (A4). Then the homogenized matrix  $A^0$  is locally Lipschitz, that is, for every  $r > 0$ , there exists a positive constant  $C_r$  such that*

$$|A^0(s) - A^0(t)| \leq C_r |s - t| \quad \forall s, t \in (-r, r).$$

**Theorem 5.25** (The homogenized problem in  $\Omega$ ). *Let  $u_1$  be a cluster point of the sequence  $\{\widehat{\mathcal{G}}_i^\varepsilon(u_i^\varepsilon)\}$ ,  $i = 1, 2$ , as in Lemma 5.21. Then  $u_1$  is a renormalized solution of*

$$\begin{cases} -\operatorname{div}(A^0(u_1)\nabla u_1) = f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.109)$$

that is,

$$T_k(u_1) \in H_0^1(\Omega), \quad \text{for any } k > 0, \quad (5.110)$$

$$\lim_{k \rightarrow \infty} \frac{1}{k} \int_{\{|u_1| < k\}} A^0(u_1)\nabla u_1 \nabla u_1 \, dx = 0, \quad (5.111)$$

and for every  $\psi \in C^1(\mathbb{R})$  (or equivalently,  $\psi \in W^{1,\infty}(\mathbb{R})$ ) with compact support,

$$\begin{aligned} & \int_{\Omega} \psi(u_1) A^0(u_1)\nabla u_1 \nabla \varphi \, dx + \int_{\Omega} \psi'(u_1) A^0(u_1)\nabla u_1 \nabla u_1 \varphi \, dx \\ & = \int_{\Omega} f \psi(u_1) \varphi \, dx, \quad \text{for every } \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{aligned} \quad (5.112)$$

If in addition, (A4) holds, then  $u_1$  is the unique renormalized solution of (5.109) and all of the sequences in Lemma 5.21 converge (not just a subsequence).

*Proof.* From Corollary 5.14 and (5.42) of Theorem 5.17, we have (5.110). To obtain (5.112), we take  $\Phi_i \equiv 0$  for  $i = 1, 2$  in (5.70) of Theorem 5.22. Hence, we have

$$\begin{aligned} & \frac{1}{|Y|} \int_{\Omega \times Y_1} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_1) \nabla(\psi_1(u_1)\varphi) \, dx \, dy \\ & + \frac{1}{|Y|} \int_{\Omega \times Y_2} A(y, u_1)(\nabla u_1 + \nabla_y \widehat{u}_2) \nabla(\psi_1(u_1)\varphi) \, dx \, dy = \int_{\Omega} f \psi_1(u_1) \varphi \, dx. \end{aligned}$$

Substituting the expressions for  $\widehat{u}_i$ ,  $i = 1, 2$ , from (5.98) of Theorem 5.23 and denoting  $\psi_1$  as  $\psi$  to simplify the notation, we get

$$\begin{aligned} & \int_{\Omega} A_1^0(u_1)\nabla u_1 \nabla(\psi(u_1)\varphi) \, dx + \int_{\Omega} A_2^0(u_1)\nabla u_1 \nabla(\psi(u_1)\varphi) \, dx \\ & = \int_{\Omega} f \psi(u_1) \varphi \, dx, \end{aligned}$$

which is equivalent to (5.112).

To show (5.111), let us first take  $k = n \in \mathbb{N}$ . Choose  $\varphi = T_{2n}(u_1)$ ,  $\Phi_i = \widehat{u}_i \chi_{\{|u_1| < 2n\}}$ , and  $\psi_i = S_n$ ,  $i = 1, 2$ , for (5.70) of Theorem 5.22. Substituting these functions, we have

$$J_1 + K_1 + J_2 + K_2 + L + M = 0, \quad (5.113)$$

where

$$\begin{aligned} J_i &= \frac{1}{|Y|} \int_{\Omega \times Y_i} S_n(u_1) A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) (\nabla u_1 + \nabla_y \widehat{u}_i) dx dy, \quad i = 1, 2 \\ K_i &= \frac{1}{|Y|} \int_{\Omega \times Y_i} A(y, u_1) (\nabla u_1 + \nabla_y \widehat{u}_i) \nabla u_1 S'_n(u_1) T_{2n}(u_1) dx dy, \quad i = 1, 2 \\ L &= \frac{1}{|Y|} \int_{\Omega \times \Gamma} h(y) S_n(u_1) (\widehat{u}_1 - \widehat{u}_2) (\widehat{u}_1 - \widehat{u}_2) dx d\sigma_y \\ M &= - \int_{\Omega} f S_n(u_1) T_{2n}(u_1) dx. \end{aligned}$$

We study the behavior of each integral as  $n$  approaches  $\infty$ .

Note that, for  $i = 1, 2$ ,

$$J_i \leq \frac{1}{|Y|} \int_{\{|u_1| < 2n\} \times Y_i} A(y, u_1) (\nabla T_{2n}(u_1) + \nabla_y \widehat{u}_i) (\nabla T_{2n}(u_1) + \nabla_y \widehat{u}_i) dx dy.$$

By (5.94) of Theorem 5.22,

$$\lim_{n \rightarrow \infty} \frac{1}{n} J_i = 0, \quad i = 1, 2. \quad (5.114)$$

If we substitute (5.98) of Theorem 5.23 to  $K_i$ ,  $i = 1, 2$ , then we get

$$\begin{aligned} K_i &= \int_{\Omega} A_i^0(u_1) \nabla u_1 \nabla u_1 S'_n(u_1) T_{2n}(u_1) dx \\ &= \int_{\{n < |u_1| < 2n\}} A_i^0(u_1) \nabla u_1 \nabla u_1 S'_n(u_1) T_{2n}(u_1) dx \leq 0, \end{aligned}$$

since  $S'_n(u_1) T_{2n}(u_1) \leq 0$  in  $\{n < |u_1| < 2n\}$ . Note that

$$\begin{aligned} |S'_n(u_1) T_{2n}(u_1)| &= \left| \frac{\operatorname{sgn}(u_1)}{n} \chi_{\{n < |u_1| < 2n\}} T_{2n}(u_1) \right| \\ &= \left| \frac{\operatorname{sgn}(u_1)}{n} \chi_{\{n < |u_1| < 2n\}} u_1 \right| \\ &\geq \chi_{\{n < |u_1| < 2n\}}. \end{aligned}$$

Consequently, for  $i = 1, 2$ ,

$$\begin{aligned} -K_i &= \int_{\{n < |u_1| < 2n\}} A_i^0(u_1) \nabla u_1 \nabla u_1 (-S'_n(u_1) T_{2n}(u_1)) dx \\ &\geq \int_{\{n < |u_1| < 2n\}} A_i^0(u_1) \nabla u_1 \nabla u_1 dx. \end{aligned} \quad (5.115)$$

For  $L$ , using Theorem 5.18 (written for  $\mathcal{R} = S_n$ ) and (5.24), we obtain

$$\begin{aligned} |L| &= \left| \frac{1}{|Y|} \int_{\{|u_1| < 2n\} \times \Gamma} h(y) S_n(u_1) (\widehat{u}_1 - \widehat{u}_2) (\widehat{u}_1 - \widehat{u}_2) dx d\sigma_y \right| \\ &\leq \frac{\|h\|_{L^\infty(\Gamma)}}{|Y|} \int_{\{|u_1| < 2n\} \times \Gamma} (\widehat{u}_1 - \widehat{u}_2)^2 dx d\sigma_y. \end{aligned}$$

We can conclude from (5.95) of Theorem 5.22 that

$$\lim_{n \rightarrow \infty} \frac{1}{n} L = 0. \quad (5.116)$$

For  $M$ , since  $u_1$  is finite a.e. in  $\Omega$ , we have

$$\frac{T_{2n}(u_1)}{n} \longrightarrow 0 \quad \text{a.e. in } \Omega,$$

and, by (5.21) and (5.24),

$$\frac{1}{n} |f S_n(u_1) T_{2n}(u_1)| \leq 2|f| \in L^1(\Omega).$$

By Lebesgue Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \frac{1}{n} M = 0. \quad (5.117)$$

As a result, by (5.113)-(5.117),

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{n < |u_1| < 2n\}} A^0(u_1) \nabla u_1 \nabla u_1 dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^2 \int_{\{n < |u_1| < 2n\}} A_i^0(u_1) \nabla u_1 \nabla u_1 dx \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} (-K_1 - K_2) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} |J_1 + J_2 + L + M| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} (|J_1| + |J_2| + |L| + |M|) \\ &= 0, \end{aligned}$$

that is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{n < |u_1| < 2n\}} A^0(u_1) \nabla u_1 \nabla u_1 \, dx = 0,$$

which is equivalent by a Cesaro argument (see [58, Remark 6]) to

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\{|u_1| < n\}} A^0(u_1) \nabla u_1 \nabla u_1 \, dx = 0.$$

At last, in view of the ellipticity of the homogenized matrix  $A^0(t)$  stated in (5.108), we obtain (5.111) (see Remark 5.7).

Moreover, with the additional assumption (A4) on the matrix field  $A$ , the uniqueness of  $u_1$  follows from Theorem 5.24 and [16]. Furthermore, since  $u_1$  is unique, it follows that both  $\widehat{u}_1$  and  $\widehat{u}_2$  are uniquely determined by (5.98). As a consequence, the limit problem (5.70) has a unique solution and the convergences in (5.66) and (5.67) of Lemma 5.21 in fact hold for the whole sequence  $\{\varepsilon\}$ .  $\square$





# Perspectives

In this thesis, we showed some results concerning a quasilinear problem in a two-component domain with  $L^1$  data. In the first part, we obtained existence and uniqueness results in the framework of renormalized solutions. In the second part, we proved homogenization results and also studied some properties of the corresponding cell problem.

There are very interesting open problems related to this problem that we wish to study in the future.

As mentioned in Chapter 5, the homogenization of the following quasilinear problem with  $L^1$  data

$$\left\{ \begin{array}{ll} -\operatorname{div} \left( A \left( \frac{x}{\varepsilon}, u_1^\varepsilon \right) \nabla u_1^\varepsilon \right) = f & \text{in } \Omega_1^\varepsilon, \\ -\operatorname{div} \left( A \left( \frac{x}{\varepsilon}, u_2^\varepsilon \right) \nabla u_2^\varepsilon \right) = f & \text{in } \Omega_2^\varepsilon, \\ \left( A \left( \frac{x}{\varepsilon}, u_1^\varepsilon \right) \nabla u_1^\varepsilon \right) \nu_1^\varepsilon = \left( A \left( \frac{x}{\varepsilon}, u_2^\varepsilon \right) \nabla u_2^\varepsilon \right) \nu_1^\varepsilon & \text{on } \Gamma^\varepsilon, \\ \left( A \left( \frac{x}{\varepsilon}, u_1^\varepsilon \right) \nabla u_1^\varepsilon \right) \nu_1^\varepsilon = -\varepsilon^\gamma h \left( \frac{x}{\varepsilon} \right) (u_1^\varepsilon - u_2^\varepsilon) & \text{on } \Gamma^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{array} \right.$$

for the cases  $\gamma \in (-1, 1]$  and  $\gamma \in (-\infty, -1)$  is not yet studied. These cases appear to be relatively easier compared to the case  $\gamma = -1$  except probably for the case  $\gamma = 1$ . This case is still an open problem even for the variational case, i.e.,  $f \in L^2(\Omega)$ , which we also plan to look at in the future. In addition, we want to obtain some corrector results for all the cases of  $\gamma$ . We will first consider the corrector results for the case  $\gamma = -1$  as it was what we considered in this thesis. It is worth mentioning that corrector results in the framework of renormalized solutions are yet to be studied.

We can also study the existence, uniqueness and homogenization results for problem (2.1) with a nonlinear boundary condition. That is, we consider the following condition on  $\Gamma$

$$(B(x, u_1) \nabla u_1) \nu_1 = -h(x, u_1 - u_2),$$

where the function  $h(y, t) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following properties:

1.  $h$  is a Carathéodory function,
2.  $h$  is an increasing function in  $C^1(\mathbb{R})$  such that  $h(y, 0) = 0$  for all  $y \in \mathbb{R}^N$ ,
3. there exists a constant  $c_1 > 0$  and an exponent  $q$ , with

$$1 \leq q < 2 \text{ if } N = 2, 3 \quad \text{and} \quad 1 \leq q < \frac{N}{N-2} \text{ if } N > 3,$$

such that

$$\left| \frac{\partial h}{\partial t}(y, t) \right| \leq c_1(1 + |t|^{q-1}) \quad \text{for a.e. } y \in Y \text{ and for all } t \in \mathbb{R},$$

4. there exists a constant  $c_2 > 0$  such that

$$th(y, t) \geq c_2|t|^2 \quad \text{for a.e. } y \in Y \text{ and for all } t \in \mathbb{R}.$$

This kind of nonlinear boundary condition was considered by [45], where the authors studied the existence, uniqueness and homogenization results for a linear problem in the variational setting (that is, the data  $f$  belongs to  $L^2(\Omega)$ ).

Another aim is to consider the parabolic version of (2.1), i.e.,

$$\left\{ \begin{array}{ll} \frac{\partial u_1}{\partial t} - \operatorname{div}(B(x, u_1)\nabla u_1) = f & \text{in } \Omega_1 \times (0, T), \\ \frac{\partial u_2}{\partial t} - \operatorname{div}(B(x, u_2)\nabla u_2) = f & \text{in } \Omega_2 \times (0, T) \\ (B(x, u_1)\nabla u_1)\nu_1 = (B(x, u_2)\nabla u_2)\nu_1 & \text{on } \Gamma \times (0, T), \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma \times (0, T), \\ u_1 = 0 & \text{on } \partial\Omega \times (0, T). \\ u_1(x, 0) = U_1^0(x) & \text{in } \Omega_1 \\ u_2(x, 0) = U_2^0(x) & \text{in } \Omega_2, \end{array} \right. \quad (5.118)$$

where  $f \in L^1(\Omega)$  and  $T$  is a given time.

Moreover, we would like to give a definition of a renormalized solution of the parabolic problem (5.118), and obtain existence and uniqueness results.

Then we can also study the homogenization of the corresponding periodic problem, that is

$$\left\{ \begin{array}{ll} \frac{\partial u_1^\varepsilon}{\partial t} - \operatorname{div}(B(x, u_1^\varepsilon)\nabla u_1^\varepsilon) = f & \text{in } \Omega_1^\varepsilon \times (0, T), \\ \frac{\partial u_2^\varepsilon}{\partial t} - \operatorname{div}(B(x, u_2^\varepsilon)\nabla u_2^\varepsilon) = f & \text{in } \Omega_2 \times (0, T) \\ (B(x, u_1^\varepsilon)\nabla u_1^\varepsilon)\nu_1 = (B(x, u_2^\varepsilon)\nabla u_2^\varepsilon)\nu_1 & \text{on } \Gamma^\varepsilon \times (0, T), \\ (B(x, u_1)\nabla u_1)\nu_1 = -h(x)(u_1 - u_2) & \text{on } \Gamma \times (0, T), \\ u_1^\varepsilon = 0 & \text{on } \partial\Omega \times (0, T). \\ u_1^\varepsilon(x, 0) = U_1^{0,\varepsilon}(x) & \text{in } \Omega_1^\varepsilon \\ u_2(x, 0) = U_2^{0,\varepsilon}(x) & \text{in } \Omega_2^\varepsilon, \end{array} \right.$$

where  $f \in L^1(\Omega \times (0, T))$  and for  $i = 1, 2$ ,  $\widetilde{U_i^{0,\varepsilon}}$  converges weakly to  $\theta_i U_i^0$  in  $L^1(\Omega)$ , for some  $U_i^0 \in L^1(\Omega)$ .

Finally, another possible extension for problem (2.1) is considering a non-linear problem with a Leray-Lions operator with  $p$ -growth in the same kind of domain and similar boundary conditions. That is, an operator of the form

$$- \operatorname{div}(a(x, u, \nabla u)),$$

where

$$a : (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow a(x, s, \xi) \in \mathbb{R}^N$$

is a Carathéodory function such that for some  $p \in (1, N)$ ,

$$\begin{aligned} a(x, s, \xi)\xi &\geq \alpha|\xi|^p, \quad \alpha > 0, \\ |a(x, s, \xi)| &\leq c(|\xi|^{p-1} + |s|^{p-1} + a_0(x)), \quad a_0 \in L^{p'}(\Omega), \quad c > 0, \\ \langle a(x, s, \xi) - a(x, s, \eta), \xi - \eta \rangle &> 0, \quad \xi \neq \eta. \end{aligned}$$

This operator was introduced by J. Leray and J.L. Lions in [61] and by J.L. Lions in [62]. In the framework of renormalized solutions with this kind of operator, we refer to [63, 36] for the case of Dirichlet boundary condition and to [13, 14] for the case of Neumann boundary condition.



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