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Thèse de doctorat  
de l'université de Paris

ÉCOLE DOCTORALE DE SCIENCES MATHÉMATIQUES DE  
PARIS CENTRE (ED 386)  
Spécialité: Informatique

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# Intermittent Lévy Walks

## and applications to biological searches

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par **Brieuc GUINARD**

*conduite à l'Institut de Recherche en Informatique (IRIF)*

*sous la direction d'Amos KORMAN*

Soutenue le 4 novembre 2020, devant le jury composé de

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# Résumé

Ces deux dernières décennies, la recherche en éthologie, et plus spécifiquement celle de l'investigation des comportements des animaux lorsqu'ils traquent de la nourriture, ont exhibé qu'une famille de motifs de trajectoires, connue sous le nom de Motifs de Lévy, est prévalente à travers le règne animal et même au-delà. Dans ces motifs auto-similaires, la longueur d'un pas en ligne droite suit une distribution de puissance, d'exposant  $\mu \in (1, 3)$ . Ces découvertes ont posé notamment deux questions: la première est de savoir si ces motifs émergent spontanément, par un mécanisme interne à l'organisme biologique, ou si ils sont la résultante d'interactions avec l'environnement (de la même manière que le mouvement Brownien s'explique par la collision de particules). La seconde est de savoir quelles informations pertinentes sur le plan biologique ces motifs, et notamment l'exposant  $\mu$ , nous révèlent.

À travers cette thèse, j'essaie d'apporter des éléments de réponse à ces questions complexes en étudiant le modèle des marches de Lévy, des marches aléatoires dont la longueur des pas est donnée par une loi de puissance, et qui génère, naturellement, des motifs de Lévy. Plus particulièrement, je l'étudie dans le contexte où la détection d'une cible ne peut être faite que de manière intermittente. Dans le premier chapitre, je parle plus en détail desdites recherches en éthologie, et je donne les bases mathématiques des modèles probabilistes de cette thèse (chaîne de Markov, marches aléatoires dans les espaces euclidiens et, dans une mesure moins importante, dans des graphes).

Au second chapitre, je discute des propriétés générales des marches aléatoires en espace euclidiens: comment obtenir des bornes sur les temps de recherche d'une marche aléatoire lorsque l'on en connaît la distribution du marcheur; des bornes sur la distance parcourue par un marcheur après un certain temps; ainsi qu'une propriété utile de monotonie. En introduction aux preuves plus complexes des chapitres suivants, j'étudie un modèle de recherche intermittente sur un graphe.

Au troisième chapitre, je montre comment les performances des marches de Lévy, dans le modèle intermittent de détection, dépend de manière cruciale de la taille des cibles, et je montre que ces considérations sont opérantes à un niveau biologiquement pertinent. Ce chapitre est basé sur un travail commun avec Amos Korman, à paraître (Guinard and Korman, 2020a).

L'ultime chapitre est consacré à la question suivante: quelles sont les performances d'un agent incapable d'exécuter une marche de Lévy, mais qui peut en réaliser une approximation en utilisant  $k$  différentes longueurs fixées? De tels modèles ont été suggérés en biologie avec  $k = 2, 3$ , et je montre notamment qu'utiliser seulement trois modes est efficace pour un espace d'une taille biologiquement pertinente. Ce chapitre est basé sur (Boczkowski et al., 2018a) et (Guinard and Korman, 2020b).

# Summary

Throughout the last two decades, a type of trajectories has been found to be almost ubiquitous in biological searches: the Lévy Patterns. Such patterns are fractal, with searches happening in self-similar clusters. Their hallmark is that their step-lengths are distributed in a power-law with some exponent  $\mu \in (1, 3)$ . This discovery led to two intriguing questions: first, do these patterns emerge from an internal mechanism of the searcher, or from the interaction with the environment? Second, and independently of the previous question: what do these searchers have in common? When can we expect to see a Lévy Pattern of exponent  $\mu$ ? And how much does the knowledge of  $\mu$  inform on the biological situation?

This dissertation is an attempt at shedding some light on the topic, especially when the searcher can only detect targets intermittently, by studying the Lévy Walk model, a random walk model in which the lengths of the steps are drawn according to a power-law of exponent  $\mu$ . In the first chapter, I will provide more background in the foraging literature, especially in the Lévy Foraging literature. I will also provide the definitions of the probability models – Markov Chains, random walks on Euclidean spaces and, to a minor extent, on graphs – we will need in the theoretical analyses.

In Chapter 2, I will present general facts about random walks on Euclidean spaces: how to analyse their search performances based on pointwise probability bounds, what is the distance achieved by a random walk with a general step-length distribution, and a useful monotonicity property. I will also study, as both a preliminary to the more involved proofs of later chapter, and for its own sake, a model of intermittent search on general graphs.

Chapter 3 returns to the Lévy Walks, and contains an analysis of their efficiency when detection is intermittent, and targets appear in various sizes. In particular, I show that the much-debated inverse-square Lévy Walk is uniquely efficient in this setting. This is based on a joint work with Amos Korman (Guinard and Korman, 2020a), to be published.

The question of how animals can perform Lévy Patterns has been much debated. Among possible solutions, it has been suggested that animals could approximate a Lévy distribution by having  $k$  different modes of movement, where  $k = 2, 3$ . Chapter 4, which condenses (Boczkowski et al., 2018a) and its refinement, (Guinard and Korman, 2020b), proves tight bounds for the performances of such an algorithm, and shows, in accordance with the literature, that having  $k = 3$  modes may be sufficiently efficient in biological scenarios.

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# Chapter 1

## Foundations

### 1.1 Motivations

There is one problem faced by all biological organisms: searching. Animals foraging, cells of the immune system looking for pathogens, people looking for each other in a crowded market, are basic examples of search. Searches can also occur in the context of information: searching within a book, or on the web, for a piece of information, even trying to recall a memory is a kind of search, albeit one whose space is not visible. Perception itself involves a kind of search: for instance, visual perception in humans is done by diverse eye movements, that depend, among other parameters, on the salience of the elements of the scene.

Of course search, understood as a process aiming at acquiring information, can be found everywhere. A more specific case is that of an agent moving through a space, aiming at detecting some target. Since moving costs time and energy, the agent has a strong incentive to find its aim quickly. In Computer Science, the task of *exploring* a graph has attracted a lot of attention. The ability of the agents are crucial in determining its performances: the agent may be able to map the space while discovering it (map-building), or, on the contrary, they may have no memory at all, moving at random through space.

It is quite obvious that having a map contributes to fast searches – think of the popularity of GPS. If you have to find a building that you know is located at most at 5 minutes of walking, and it takes you only one minute to search the precise location on a map, you will likely do so (unless you want to get to know the surroundings). The advantage of advanced navigational abilities (self-localization, map-building, path-planning), perceptual abilities (target detecting) and possibly environmental clues for searching is quite obvious. However, these complex capacities are not always fully developed or even could lack entirely in some organisms. What strategies could such organisms, with minimal search abilities, adopt to perform relatively efficient searches?

This question is relevant also for more complex organisms. Indeed, it seems likely that the search strategies of simple ancient organisms would be the founding blocks of their more complex descendants. Understanding the strategies of simple organisms would then be a first, perhaps necessary, step to understand those of more sophisticated nature. Indeed, as we shall see in the next section, there is one model that seems to be relevant across many species: the Lévy Walks.



## 1.2 Lévy Walks in Biology

**Lévy Patterns are everywhere.** Considering the diversity of environmental contexts and navigational abilities, it should be the case that a wide diversity is observed in the searching strategies of biological organisms, in their corresponding performances, as well in the emerging patterns of movement. While this is true to an extent, it is remarkable that one family of patterns has consistently been found across the biological realm, the Lévy Patterns.

*Lévy Walks* are random walks that, at each step, choose a direction uniformly at random, and a length according to a power-law  $\mathbf{p}_\mu(l) = \Theta(l^{-\mu})$ , for a predetermined  $\mu \in (1, 3]$ . Over time, due to the generalized central limit theorem proved by the mathematician Paul Lévy, the position of such a walker will follow a Lévy stable distribution (hence their name), and the trajectories will form patterns I will call *Lévy Patterns*. Remarkably, the distribution  $\mathbf{p}_\mu$  has an infinite variance. As a consequence, it is often said that Lévy Walks are *scale-free*, i.e., they have no characteristic scale, or, in other words, all scales appear in these walks.

The Lévy Walk with exponent  $\mu = 2$ , that I will call *Cauchy Walk* (because they eventually are distributed according to a Cauchy distribution, which is a particular Lévy distribution) received special attention, as it is able to balance the local and global scales of exploration. In fact, that this process quickly escapes regions without food was the first argument invoked to surmise that Cauchy Walks might be used by foraging animals (Shlesinger and Klafter, 1986).

A decade after, in 1995, Blaine Cole studied the activity of drosophila (where inactivity is evaluated as the fly being immobile for at least some threshold duration), and found a power-law distribution for the duration of both activity and pauses, dependent on the threshold defining inactivity (Cole, 1995). Assuming the fly goes in a ballistic line between pauses, Cole surmised Lévy walks would emerge from such patterns of activity, and argued by simulations this would be a more efficient way of searching for food than Brownian motion.

Perhaps the most influential paper in the domain of Lévy Foraging, though, is (Viswanathan et al., 1999). Therein the authors argued *via* an approximate mathematical argument, that Cauchy Walks should be “optimal” for sparse and patchy resources.<sup>1</sup>

The combination of the idea that Cauchy Walks are theoretically optimal, and the central notion of Optimal Foraging Theory, which is that biological organisms should have been selected to optimize foraging, naturally led to the *Lévy Flight Foraging Hypothesis*, namely the hypothesis that evolution should have selected for Lévy Walks (or, more precisely, Cauchy Walks) in biological organisms.

Inspired by this attractive idea, there have been multiple works showing evidence of Lévy Patterns (not necessarily of parameter  $\mu = 2$ ) in a myriad of biological systems. Immune cells (Harris et al., 2012), swarming bacteria (Ariel et al., 2015), snails (Reynolds et al., 2017), bees (Reynolds et al., 2007; Wolf et al., 2016), deer

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<sup>1</sup>It is to be noted that, in this domain, authors rarely argue for optimality in absolute ; rather, it is often the case that optimality is conceived only with respect to a limited family of distribution. Thus, authors have argued that Lévy Walks are “optimal” because there is an exponent (e.g.  $\mu = 2$  in (Viswanathan et al., 1999)) for which Lévy Walks perform better than other Lévy Walks, or Brownian Motion, without making explicit that no other processes, submitted to the same constraints, would be able to perform better. I will try to reserve the word “optimal” to refer to a process that matches tightly, or almost tightly, a proved lower bound.

(Focardi et al., 2009), marine predators (Humphries et al., 2010; Sims et al., 2008), albatrosses (Humphries et al., 2012), primates (Boyer et al., 2006; Raichlen et al., 2014; Rhee et al., 2011; Murakami et al., 2019), eye movements (Brockmann and Geisel, 2000; Stephen et al., 2009), all have been reported to show some Lévy Patterns.

### **Lévy Walks in foraging: trajectory description or behavioural model?**

This body of work did attract its criticism (Pyke, 2015). One is that such random walk models assume uncorrelated directions of distinct steps. However, animals' paths are generally sinuous, and, in particular, on the scale at which animals take decisions, this sinuosity cannot be ignored. Furthermore, the Lévy Walks models apparently make no use of environmental clues. Hence, Lévy Walks would be a crude model that cannot account for complex animals' behaviour.

This criticism is not unjustified: indeed the distinction I have proposed here between Lévy Walks and Lévy Patterns has not always been explicitly used, especially in earlier studies (Viswanathan et al., 1999), which resulted in some level of confusion between the two. Indeed, it is not because, on the global scale, Lévy Patterns are observed that, on the local scale, animals perform Lévy Walks (i.e. choose a length according to  $\mathbf{p}_\mu$ ) (Benhamou, 2007). It is entirely possible that these patterns emerge due to interactions with the environment, even if the behaviour of the agents is deterministic. This alludes to two distinct parts of (foraging) research:

- the phenomenological description of trajectories. Reporting a Lévy Pattern of stability parameter  $\mu = 2$ , for instance, is a simple, abridged, description of an otherwise complex empirical event. This description would be successful if, for instance, Lévy Patterns of common stability patterns would arise under common circumstances, or yield comparable search performances.
- the understanding of the local decision rules obeyed by a searching organism.

Both must be compatible. For instance, if a Lévy Pattern of parameter  $\mu = 2$  is observed, then the local model of the agent (and its environment) should account for that. In fact numerous studies have reported on the possibility for Lévy Patterns to be emergent phenomena (see (Reynolds, 2015b)).

Let us note that Brownian Motion itself falls into the first category. Indeed, as suggested by (Einstein, 1911), the erratic behaviour of particles suspended in fluid is well-explained by their collisions with other particles of high velocity. The movement of the particles itself is ballistic.<sup>2</sup> Hence, Brownian Motion is an empirical description, and its models (simple random walk, Wiener processes) are not meant to show the local rules followed by particles. Instead, they are good models to understand the trajectory of particles in the long-run. In biology, we can surmise, then, that even if Lévy Walks are not an accurate behavioural description of animals, they could be employed to understand processes evolving at suitable temporal and spatial scales, such as foraging and, possibly, the migration of animal population (Dannemann et al., 2018).

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<sup>2</sup>Similarly, Brownian Patterns in animals can be explained by the interruption of long ballistic steps by the encounter of targets (de Jager et al., 2014).

**Do the simplest organisms perform Lévy Walks?** Of course, one of the simplest model that can account for Lévy Patterns is Lévy Walks, and this model might be suitable in some circumstances. Since the Lévy Walk model does not incorporate interaction with the environment, nor memory, the ideal candidates for it are organisms of low memory, and having relatively few perceptual and cognitive skills. This is quite hard to find: even Lévy Patterns found in cells could be the product of interaction with the environment (Huda et al., 2018). Nevertheless, a team of biologists managed to show Cauchy Patterns in brain-blocked *Drosophila* larvæ, thereby blocking their sensory functions (Sims et al., 2019). In contrast, non brain-blocked larvæ exhibited Lévy, but not necessarily Cauchy, Patterns of movement. This is in line with the hypothesis that Cauchy Walks could be a default behaviour modified by sensory inputs (adaptive Lévy Walks).

**Intermittent search.** In the very influential (Viswanathan et al., 1999), the agent was supposed to be able to detect while moving. Unsurprisingly, then, when targets were sparse and consumed upon encounter, the optimal process was simply a ballistic one, i.e. just always moving in the same direction. This particular assumption, however, is not necessarily verified, and in fact is unrealistic in many cases. Indeed, perception usually degrades with motion (Bell, 2012; O’Brien et al., 1990; Kramer and McLaughlin, 2001). This prompted physicists (Bénichou et al., 2011) to formulate the *intermittent* model of search. In this setting, the detection of targets is only possible intermittently, i.e. agents cannot detect targets at all times and must alternate between exploration phases (typically, a ballistic phase aiming to bring the searcher far away) and exploitation phases (for instance, detecting while immobile, or while moving diffusively). Markovian models of intermittent search also assume that the animal is memoryless, in the sense that after switching its state to a detection mode, it “forgets” the previous direction, and for its next step it must choose a direction uniformly at random. While this assumption, used to make the models analytically tractable, may seem too unrealistic, let us note that it was verified in desert locusts that, the longer the duration of a pause, the less correlated the direction of their next step will be with the direction they previously came from (Bazazi et al., 2012).

Because other terms have been used to refer to intermittent search in the ethological literature (especially in foraging studies), I feel compelled to provide the reader with some notes of vocabulary.

*Intermittent locomotion*, sometimes referred to as *stop-and-go* or *pause-travel*, is the alternation of pauses and moves during an activity, without presuming of the reason behind the pauses. The phenomenon is observed in a many animals, such as lizards, birds, flies, medusae, squirrels (Kramer and McLaughlin, 2015). In this survey, the authors found that the duration of these pauses would fall between 6% and 94% of the locomotion time of these animals. The reasons for pausing may be to increase endurance by partial recovery, to avoid being detected by a predator, or to stabilize the perceptual field. When this alternation of pauses and moves is absent, the behaviour is sometimes referred to as *cruising*.

*Saltatory* foragers are stop-and-go foragers that can detect targets only when immobile (O’Brien et al., 1989). As such, the terms saltatory and intermittent are equivalent, although “saltatory” generally applies to empirical evidence (behaviour), while “intermittent” generally applies to models.

*Sit-and-wait, or ambush*, predators, such as many spiders (Uetz, 1992), typically move rarely, and instead wait for prey to come to their neighbourhood (even setting traps for them in the case of spiders). While this behaviour might be seen as an extreme intermittent search, the energy cost associated with detection (i.e. assessing the quality of a spot) is the most prevalent factor and, as such, an intermittent model of search is probably not the best model for this predatory behaviour. In other words, the intermittent search model is appropriate when detection is neither very easy or fast (in this case, a non-intermittent model can be used) nor too difficult or time-consuming.

*Truncated Lévy Walks* may mean either:

- a Lévy Walk whose step-length distribution is truncated, i.e.  $\mathbf{p}_{\mu, \ell_{max}} = \Theta(l^{-\mu})$  for  $l = O(\ell_{max})$ , and  $p_{\mu, \ell_{max}} = 0$  for  $l = \Omega(\ell_{max})$ .
- a Lévy Walk, in the non-intermittent model of detection, that can stop in the middle of a jump, upon encountering a target. The name “truncated” for this second sense comes from the analysis of (Viswanathan et al., 1999), in which it was approximated that no step of length more than the mean distance between targets would appear, thus the distribution of the step-length effectively became a truncated power-law.

This latest sense is quite common, and has also been referred to as *prey-targeting* in (Humphries and Sims, 2014). In contrary, the non-intermittent model wherein the food is consumed when passing over it, but the step is not halted, has received little attention, to my knowledge. In dimension 2 or more, prey-targeting can be an inconvenient – when targets are sparse and non-patchy, it is better not to take the risk to return to a previously visited area – or an advantage – when targets are sparse and patchy, it is better if the walker is able to explore diffusively the area by halting frequently its steps.

To avoid confusion, I will prefer the term “prey-targeting” to “truncated”. Note that, although possible, there is *a priori* little interest to talk about prey-targeting in the intermittent detection model, as such models typically assume the detection to happen while the agent is immobile or moving diffusively. Note also that prey-targeting is a minimal form of interaction with the environment, of which (memoryless) intermittent searchers are devoid. As such, memoryless intermittent models represent a model for organisms of minimal abilities, and could represent a good approximation of the behaviour of ancient animals (Sims et al., 2014).

The acute reader will have remarked that Lévy Walks with truncated step-length distribution  $\mathbf{p}_{\mu, \ell_{max}}$ , because they have finite variance, fall under the central limit theorem and will eventually converge to a normal distribution. However, this convergence is slow (Mantegna and Stanley, 1994) and, as we will see *via* the short-time pointwise probability bounds in Chapter 3, can only happen after a time proportional to  $\ell_{max}$ .

Finally, while sometimes both terms are used equivalently, let us highlight the distinction between *Lévy Walks* and *Lévy Flights*. “Lévy Flights” refer to a process with infinite speed, while “Lévy Walks” are of finite velocity. (Zaburdaev et al., 2015) Biological processes, of course, are of the second nature, and the term “Lévy Flights”, when contrasted to “Lévy Walks” in this context, may simply indicate that the agent is not able to detect targets while moving (because it flies over targets),

i.e., the search is non-intermittent. I will use the term “Flight” only to denote a process of infinite speed.

I will return in Chapters 3 and 4 to this notion of intermittent detection, as the models will be framed within this setting. Specifically, in Chapter 3, I will provide an optimality result for intermittent Cauchy walk in dimension 2, and in Chapter 4 I will show, in dimension 1, an optimality result for a discretization of the Lévy Walk.

**On the applications of Lévy Walks.** The practical-minded reader might be interested to know potential applications of this Lévy Walks (and Patterns) theory.

In biology, Lévy Patterns were reported for metastatic cancer cells (Huda et al., 2018). In contrast, simple diffusive patterns were reported for non-metastatic cancerous cells. Furthermore, the authors were able to reprogram the trajectories of the cells from Lévy-like to diffusive by using chemical inhibitors targeting actin-binding proteins. At best, this may eventually be useful to prevent cancer cells to seed metastatic cancers, or slow down the process.

Pollen dispersion by honeybees was usually modelled by Brownian motion. However, if Lévy Walks are a more representative model of honeybees patterns, the model of Brownian motion leads to underestimate the isolation distance needed to prevent genetically modified pollens from outcrossing with conventional crops (Val-laeyes et al., 2017).

The understanding that humans trajectories are, at times, well-described by Lévy Patterns is relevant in modelling the spread of epidemics, although it was argued in (Fofana and Hurford, 2017) that the choice of a particular model of movement might not be so crucial to the threshold condition for a disease outbreak.

Inspired by the prevalence of Lévy Patterns in biology, computer scientists introduced Lévy Walks to robotic searches (Krivonosov et al., 2016). Interestingly, since robots are often equipped with sensors, the implemented algorithms are not pure Lévy Walks, but involve interaction with the environment. Hence, this field can in turn inform on biological processes, as it allows to tests search algorithms in a realistic environment.

## 1.3 Search Problems in Computer Science

The problem of an agent exploring a space has been extensively studied in Computer Science and a detailed survey is out of the scope of this dissertation. Instead, I mention only works which I think can inform the reader on the more general context of this research.

**Deterministic searching and memory requirements.** One of the simplest possible setting is the deterministic searching in the line and in the plane, which was studied in (Baezayates et al., 1993). An agent searches for a target located at an unknown distance  $D$ . They proved that, on the line, the Linear Spiral Search, that takes jumps of length  $2^i$  alternating right and left, finds the target in time no more than  $9D$ , which is optimal up to lower order terms. On a two-dimensional lattice, they showed that a lower bound  $2D^2 + 4D + 1$  holds for any algorithm, nearly matched by a spiral search.

The previous scenario assumed that the topology of the space is known to the searcher. This might not be the case, as in graph exploration problems. Such problems include the case of an agent wishing to cover a graph, i.e., visit every node (and, possibly, to go through every edge) of the graph. Since (Rollik, 1980), that proved that the problem was untractable, on general graphs, for a single agent (or a constant number of agents) with sublogarithmic (in the size of the graph) memory, attention was devoted to finding an algorithm with minimal memory requirements, until (Reingold, 2008) furnished one with a memory only logarithmic. The problem of which graphs an agent is able to explore with only  $k$  bits of memory was also investigated (Fraigniaud et al., 2008).

The problem can be facilitated with advice or if the agent is able to interact with the graph. For instance, in (Disser et al., 2019), it was shown that, if the agent has only constant memory, it can still explore any graph if it is allowed to drop (and retrieve)  $\Theta(\log \log n)$  pebbles on nodes, and this number of pebbles is necessary to explore any  $n$ -node graph. Perhaps surprisingly, multiple agents do not perform significantly better than a single agent with pebbles, as a number  $\Theta(\log \log n)$  of agents is also necessary and sufficient to solve the exploration problem. In some settings, however, it is more interesting to pre-process the graph. Indeed, it was shown that adding only 3 labels in a pre-processing stage can enable a searcher with constant memory to explore any graph (Cohen et al., 2008).

**Searching with randomness.** If the agent can make use of random moves, the problem is altogether different. Indeed, it is known that any simple random walk (that moves, at each step, to a neighbour of the current node, selected uniformly at random among all neighbours), on a simple connected graph with  $n$  nodes and  $m$  edges, will visit every vertex in time at most  $2m(n-1)$  (Aleliunas et al., 1979), and, in the worst case, the cover time is  $4/27n^3 + O(n^{5/2})$  (Feige, 1995). The hitting and cover times of simple random walks on graphs and graph families have, in fact, been studied extensively, including in Erdos-Renyi random graphs (Barlow et al., 2010) and in random regular graphs (Cooper et al., 2009).

**Extensions of the random walk.** The simple random walks on graphs uses only local information, however it is possible to improve appreciably the time performances of the algorithm by allowing the agent to know the degrees of the neighbours of the current vertex. Specifically, (Ikeda et al., 2009) proved that a random walk on a graph selecting node  $x$  with a probability proportional to  $\deg(x)^{-1/2}$ , covers any graph in time at most  $O(n^2)$ , which is an  $\Omega(n)$  improvement on the worst case of the simple random walk. See also (Nonaka et al., 2010) for a random walk achieving better cover time.

**Multiple agents and space-time tradeoffs.** Another way to extend the simple random walks is to consider multiple agents, which is a relevant scenario for robotics and biology. See (Elsässer and Sauerwald, 2009) for multiple simple random walks, and (Clementi et al., 2020) for multiple Lévy walks in the plane. In (Feinerman and Korman, 2017), multiple agents search for a treasure in the plane. Starting from their nests, they perform ballistic steps of length given by a power-law distribution, explore the region by a (deterministic) spiral search and, in case of failure, return to their nests to begin anew. Feinerman and Korman also showed lower bounds

on the memory of agents, as well as the estimation by the agents of their numbers, that achieve certain time performances. In general graphs, families of algorithms achieving some space-time tradeoff were also obtained by simulating multiple walkers (Broder et al., 1994). An improvement using the random walks of Nonaka et al. was built by (Kosowski, 2013), proposing randomized global algorithms achieving a space-time tradeoff of roughly  $n^2$  at worst.

**Random walk models for collective navigation by ants.** Finally, let us remark that models based on random walks were used to study, with success, collective animals' navigation. Specifically, the collective transport of large food items was studied in crazy longhorn (*Paratrechina longicornis*) ants (Fonio et al., 2016) by a team of physicists and computer scientists. The authors showed that ants that bear a large food item are guided by the pheromone marks left by surrounding ants. They demonstrated this by verifying that this transport was well modelled by a random walk with unreliable advice. However, the scent trail can be at times wrong, as the larger food items can be blocked by obstacles single ants can go through. This explains why the motion of the load could deviate from the scent trail. This work was followed by (Gelblum et al., 2020), which established that these ants, in a labyrinth, achieve performances that outperform known lower bounds on the time performance of bias random walkers evolving in a percolated grid. Gelblum et al. showed theoretically and empirically that this speed-up was due to ants using their numbers to collectively extend their sensing range.

## 1.4 Mathematical background

### 1.4.1 Notations

**Asymptotic notations.** For two real-valued functions  $f$  and  $g$ , defined on  $\mathbb{N}$  or  $\mathbb{R}^+$ , we write:

- $f = O(g)$  if there are a constant  $c > 0$  and  $x_0$  such that, for all  $x \geq x_0$ , we have  $f(x) \leq cg(x)$ ,
- $f = \Omega(g)$  if  $g = O(f)$ ,
- $f = \Theta(g)$  if  $f = O(g)$  and  $g = O(f)$ ,
- $f = o(g)$  if for any  $\varepsilon > 0$ , there is a  $x_0$  such that, for all  $x \geq x_0$ , we have  $f(x) \leq \varepsilon g(x)$ .

The symbols  $\tilde{O}$ ,  $\tilde{\Theta}$  mean that we omit polylogarithmic factors, in other words  $f = \tilde{O}(g)$  if there is some constant  $c > 0$  such that  $f(x) \log(x)^c = O(g(x))$ .

**Spaces.** A graph  $G = (V, E)$  is given by a set of nodes  $V$ , and a set of edges  $E$ . Unless mentioned otherwise, our graphs will be simple (i.e., without multiple edges), without loops, nor weights.

Given a positive integer  $\Delta$ , we denote by  $\mathbb{T}_n^\Delta$  the (continuous)  $\Delta$ -dimensional torus of volume  $n$ . When needed, we will identify  $\mathbb{T}_n^\Delta$  and the  $\Delta$ -dimensional cube  $[-n^{1/\Delta}/2, n^{1/\Delta}/2]^\Delta \subset \mathbb{R}^\Delta$ . We will denote by  $C_n$  the discrete cycle of length  $n$ .

**Probability notations.** For a real-valued random variable  $X$ , we denote by  $\mathbb{E}(X)$  its expectation and  $\sigma(X)$  its standard deviation. For  $x$  in a probability space, we will denote by  $\delta_x$  the Dirac distribution, given, for any measurable set  $A$ , by  $\delta_x(A) = 1$  if  $x \in A$ , otherwise  $\delta_x(A) = 0$ . We will denote by  $\mathbf{1}_A$  the indicator function of  $A$ , i.e.,  $\mathbf{1}_A(y) = 1$  if  $y \in A$  and  $\mathbf{1}_A(y) = 0$  otherwise.

### 1.4.2 Some reminders about probability theory

I will assume some basic familiarity with discrete probability theory. The reader can refer to (Mitzenmacher and Upfal, 2005) for a simple introduction to discrete probability theory, with applications to computer science. I will also use some concepts from measure-theoretical probability Theory in Chapter 3, although I will keep the use of these notions to a minimum. In fact, I will almost only use the following definition.

**Definition 1.1.** A *probability density function* (p.d.f.) for a random variable  $X$  on  $\Omega \in \{\mathbb{R}^\Delta, \mathbb{T}_n^\Delta\}$  is an integrable function  $f$  such that, for any measurable set  $A$  in  $\Omega$ , we have

$$\mathbb{P}(X \in A) = \int_A f(x)dx. \quad (1.1)$$

For a random variable  $X$ , we will denote by  $\mathbf{p}^X$  the p.d.f. of  $X$ , if it admits one.

The reader who ignores what is a measurable set is not to feel afraid; all of our sets will be measurable, as they will be balls, or their unions. An integrable function  $f$  is one that is measurable (which is the case if  $f$  is piecewise continuous) and that verifies  $\int_\Omega |f| < \infty$ . Likewise, this integrability condition will be obvious for the power-law functions we will study.

I recall also some basic inequalities of probability theory.

**Theorem 1.2** (Markov's inequality). *Let  $X$  be a real-valued, positive random variable. Then, for any  $a > 0$ ,*

$$\mathbb{P}(X \geq a\mathbb{E}(X)) \leq \frac{1}{a}. \quad (1.2)$$

**Theorem 1.3** (Chebyshev's inequality). *Let  $X$  be a random variable with finite mean and finite variance. Then for any  $a > 0$ ,*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a\sigma(X)) \leq \frac{1}{a^2} \quad (1.3)$$

**Theorem 1.4** (Kolmogorov's inequality). *Let  $(X_i)_{1 \leq i \leq n}$  be real-valued independent random variables, of common zero mean, and all admitting a finite variance. Then, for any  $a > 0$ ,*

$$\mathbb{P}(\max_{1 \leq i \leq n} |X_1 + \cdots + X_n| \geq a) \leq \frac{1}{a^2} \sum_{i=1}^n \mathbb{E}(X_i^2). \quad (1.4)$$

### 1.4.3 Memoryless processes: Markov Chains, Random Walks

In the remaining of this chapter, I recall basic definitions of probability theory, Markov Chains and random walks. I begin by a brief account on the origins of random walks.



## Historical roots of random walks

Random walks, in general, can be said to date back to more than a century ago. Perhaps the first mathematical example of a random walk we have, is the study of the error arising from a combination of multiple, uncorrelated, error terms. One of the first treatise on modern probability, Pierre-Simon de Laplace's *Théorie Analytique des Probabilités* (1812), includes a discussion on this matter, and a derivation of what we know today as the central limit theorem.

One century later, and in the two-dimensional setting, the bio-statistician Karl Pearson inquired, in a 1905 letter to *Nature*, about a walker in the plane, going to a pre-determined, fixed, distance at each step, a process he termed *random walk*. In the same year, the mathematician Lord Rayleigh answered to Pearson's letter by referring to a 1880 work of his in the domain of acoustics, that solves the problem when the number of steps is large. The 1905 letter also prompted the mathematician J.C. Kluyver to provide an exact, albeit impractical, form for the distribution of the walk after any number of steps. In 1906, Pearson applied these ideas in an article entitled *A Mathematical Theory of Random Migration*. He was motivated by the problem of the distribution of a species composed of  $N$  individuals, after  $n$  number of steps, each individual pursuing their own random walk. In the same work, Karl Pearson also endeavoured to apply his theory to the study of the infiltration of mosquitoes into cleared areas.

While Pearson's problem was in a discrete-time setting, the continuous-time, one-dimensional, equivalent was also studied by Louis Bachelier in his 1900s thesis *Théorie de la Spéculation*. Bachelier investigated the variations of stock prices in the financial market. Five years later, in 1905, Albert Einstein, aiming to prove the existence of atoms, published two articles on the theory of brownian motion, i.e., the random motion of particles immersed in fluid, described, among others, by the botanist Robert Brown in 1827. As often, it would take many more years before mathematicians develop a fully rigorous account of these continuous-time models of Brownian motion, the first of which was formulated by Norbert Wiener, in 1923. At this time, the random walk model was already a common object of study, and a particular example of the chains introduced by Andrei Markov in 1906.

## Markov Chains

Going in the reverse direction of history, I will first present the more general model of (discrete-time) Markov Chains, and then define random walks on Euclidean spaces and graphs.

**Definition 1.5.** A *Markov Chain*  $(X_n)_{n \in \mathbb{N}}$  on a probability space  $\Omega$  (called the *state space*) is a sequence of random variables on  $\Omega$  such that, for any  $(a_0, \dots, a_{n-1}) \in \Omega^n$ , the three following variables admit the same law:

- $X_n$  conditioning on the past values  $X_{n-1} = a_{n-1}, \dots, X_0 = a_0$ , whenever  $\mathbb{P}(X_{n-1} = a_{n-1}, \dots, X_0 = a_0) > 0$ ,
- $X_n$  conditioning on the single past value  $X_{n-1} = a_{n-1}$ , whenever  $\mathbb{P}(X_{n-1} = a_{n-1}) > 0$ ,
- $X_m$  conditioning on the single past value  $X_{m-1} = a_{n-1}$ , for every  $m \geq 1$  for which  $\mathbb{P}(X_{m-1} = a_{n-1}) > 0$ .

Sometimes authors call the equivalence between the first two items as *Markov property* (or *memoryless property*), and the additional information brought by the second equivalence as the *time-homogeneous property*. Since I will only study time-homogeneous processes, I will refer to the combination of these equivalences as the Markov property (which is not uncommon, see, e.g., (Levin et al., 2008)).

*Remark 1.6.* Whenever the space  $\Omega$  is finite, these equivalences can be formulated as follows:  $(X_n)_{n \in \mathbb{N}}$  is a Markov Chain on  $\Omega$  if and only if there is a probability matrix  $\mathbf{P} \in [0, 1]^{\Omega^2}$  such that, for any  $(a_0, \dots, a_n) \in \Omega^n$  for which  $\mathbb{P}(X_n = a_n, X_{n-1} = a_{n-1}, \dots, X_0 = a_0) > 0$ , we have:

$$\begin{aligned} \mathbb{P}(X_n = a_n \mid X_{n-1} = a_{n-1}, \dots, X_0 = a_0) &= \mathbb{P}(X_n = a_n \mid X_{n-1} = a_{n-1}) \\ &= \mathbf{P}_{a_{n-1}, a_n}. \end{aligned} \quad (1.5)$$

$\mathbf{P}$  is called the transition matrix.

When  $X$  is a Markov Chain on  $\Omega$  and  $x \in \Omega$ , we will denote by  $X^x$  the Markov Chain with same evolution law as  $X$ , starting at  $x$ , i.e.,  $X^x(0) = x$ .

An easy example of Markov Chain is called *Coupon Collecting*. A collector wishes to collect all  $n$  types of coupons found in cereal boxes. Each box contains only one coupon and all coupons are equally likely to be found in a box. If  $X(m)$  is the number of distinct coupons obtained by the collector after buying  $m$  boxes (with  $X(0) = 0$ ), then the sequence  $(X(m))_m$  is a Markov Chain. Indeed, if for  $m \geq 0$  and  $0 \leq k \leq n$ ,  $X(m) = k$ , then the law of  $X(m+1)$  is determined. Specifically, as all coupons are equally likely, and there are  $n - k$  new possible coupons to obtain, among  $n$ , the law of  $X(m+1) \mid X(m) = k$  is given by:

$$\mathbb{P}(X(m+1) = k+1 \mid X(m) = k) = \frac{n-k}{n} \text{ and } \mathbb{P}(X(m+1) = k \mid X(m) = k) = \frac{k}{n},$$

which define, respectively,  $\mathbf{P}_{k, k+1}$  and  $\mathbf{P}_{k, k}$ .

**Random walks.** Another example of Markov Chains, and a very important one, is the random walk. I will provide two definitions, depending on the state space being a graph or an Euclidean space. Intuitively, an agent walking on a graph is simply an agent that, at each step, goes from one node to an adjacent node. The walk is random if the agent chooses uniformly at random the edge it will be going through. For technical reasons, I introduce also a parameter  $p$ , that governs whether the agent decides to move or not.

**Definition 1.7.** A random walk  $X$  on a graph  $G$  is a Markov Chain on  $G$  with transition matrix  $\mathbf{P}$  given by

$$\mathbf{P}(x, y) = \begin{cases} 0 & \text{if } (x, y) \notin E \\ p & \text{if } x = y \\ \frac{1-p}{\deg(x)} & \text{if } (x, y) \in E \end{cases}. \quad (1.6)$$

The parameter  $p$  is called the laziness of  $X$ .  $X$  is said to be non-lazy if  $p = 0$ .

In a vector space, a random walk is simply an additive process of increments given by some common distribution  $\mathbf{P}$ . In other words, at each step, the agent goes to a direction, and in a certain length, according to  $\mathbf{P}$ .

**Definition 1.8.** A random walk  $X$  on  $\Omega \in \{\mathbb{R}^\Delta, \mathbb{T}_n^\Delta\}$  is a random variable  $(X(m))_{m \in \mathbb{N}}$  verifying, for every  $m \in \mathbb{N}$ :

$$X(m+1) = X(m) + V(m+1), \quad (1.7)$$

where  $(V(m+1))_{m \in \mathbb{N}}$  is a sequence of i.i.d. variables chosen according to some distribution  $\mathbf{P}$ .  $X$  is called *unbiased* if the directions of the incremental vectors  $(V(m))_m$  are chosen uniformly at random. Otherwise,  $X$  is *biased*.

Unless explicitly specified, subsequently, random walks will be unbiased. Note that  $X$  is unbiased if, and only if,  $\mathbf{P}$  is radial, i.e., for any measurable set  $A$  and any rotation  $rot_\theta$  of angle  $\theta \in [0, 2\pi]$ ,  $\mathbf{P}(A) = \mathbf{P}(rot_\theta(A))$ . When  $\mathbf{P}$  admits a p.d.f.  $F$ , this means that  $F$  is radial, i.e., there is a function  $f$  on  $\mathbb{R}$  such that, for any  $x \in \Omega$ ,  $F(x) = f(\|x\|)$ .

When  $X$  is unbiased, we will denote by  $\mathbf{p}$  the distribution of the lengths of the step.

Examples of unbiased random walks include:

- The simple random walk on  $\mathbb{Z}^\Delta$ , where the step-length distribution is  $\delta_1$ , i.e., all steps have unit length.
- Its continuous-space equivalent, where the step-length distribution is a gaussian distribution with mean zero and unit variance.
- Lévy Walks, for which  $\mathbf{p}_\mu(x) = \Theta(x^{-\mu})$ , for  $\mu \in (1, 3]$ .
- Truncated Lévy Walks, for which  $\mathbf{p}_{\mu, \ell_{max}}(x) = \Theta(x^{-\mu})$  for  $\mu \in (1, 3]$  and  $x = O(\ell_{max})$ , and  $\mathbf{p}_{\mu, \ell_{max}}(x) = 0$  for  $x = \Omega(\ell_{max})$ .
- $k$ -Scales random walks, that can use only  $k$  distinct lengths (i.e., the support of  $\mathbf{p}$  has cardinal  $k$ ). The Weierstrassian Walks are a particular example, with step-length distribution  $\sum_{i=0}^k p_i \delta_{b^i}$ , with  $p_i$  proportional to  $a^{-i}$  for some  $a > 1$  and  $b > 1$ . In words, a step of length  $b^i$  is chosen with probability  $p_i$ .

**Convergence.** What does a Markov Chain look like in the long run? Since a Markov Chain evolves according to some distribution  $\mathbf{P}$ , it is expected that, if the Markov Chain converges, it will converge to a distribution invariant under  $\mathbf{P}$ . This motivates the following definition.

**Definition 1.9.** The stationary distribution is the distribution  $\pi$  such that, if  $X(0)$  is distributed according to  $\pi$ , then  $X(1)$  (and thus, all  $X(m)$  for  $m \geq 0$ ) is distributed according to  $\pi$ .

The unicity of the stationary distribution is not necessarily assured. For instance, imagine a random walk  $X$  on the graph composed of just two nodes  $x, y$ , with no edge. The distributions  $\delta_x$  and  $\delta_y$  are both stationary. This motivates the following definition:

**Definition 1.10.** An irreducible Markov Chain  $X$  on a countable space  $\Omega$  is one such that any state  $y$  is accessible starting from any initial state  $X(0) = x$ , i.e.  $\mathbb{P}(X^x(m) = y) > 0$  for some  $m > 0$ .

*Remark 1.11.* All random walks evolving on a simple connected graph are irreducible.

*Remark 1.12.* If  $X$  is irreducible, then its stationary distribution, if it exists, is unique.

*Remark 1.13.* When  $G$  is a finite connected graph, the stationary distribution (of a random walk on  $G$ ) exists and is given by

$$\pi : x \in G \mapsto \frac{\deg(x)}{2|E|} \in [0, 1], \quad (1.8)$$

where  $\deg(x)$  is the degree of the node  $x$ .

In many cases, such as random walk on graphs, the stationary distribution is rather easy to compute. The true unknown, then, is how much time is needed before the Markov Chain is mixed. This question is often found in practice; think, for instance, of how many times you must shuffle a deck so as that it will be well mixed.

The mixing time is the (expected) time before a chain is approximately distributed according to the stationary distribution. There are several possible definitions according to how we understand this approximation; we will use the following one.

**Definition 1.14.** For an irreducible Markov Chain, the (total variation) *mixing time* ((Levin et al., 2008, Section 4.5)) is defined as:

$$m_{mix} = \min \left\{ m \geq 1 : \max_x \sum_y |\mathbb{P}(X^x(m) = y) - \pi(y)| \leq \frac{1}{2} \right\},$$

where  $\pi$  is the stationary distribution.

**Hitting time.** Let us return to the Coupon Collector example. The collector might be interested in a specific coupon, and would like to know how much boxes they will have to buy before finding it. This is an example of the *hitting time* of a Markov Chain.

**Definition 1.15.** For a (measurable) set  $A \subset \Omega$ , we say that  $X$  *visits*  $A$  at step  $m$  if and only if  $X(m) \in A$ .

**Definition 1.16.** Let  $\Omega$  be a probability space,  $S \subseteq \Omega$  be measurable, and  $X$  be a Markov chain on  $\Omega$ . The (random) hitting time  $m_{hit}^X(S)$  of  $S$  is equal to the number of steps before  $S$  is visited, i.e.

$$m_{hit}^X(S) = \min\{m \text{ s.t. } X(m) \in S\},$$

where we consider that  $\min \emptyset = +\infty$ , i.e., if the set  $S$  is never reached, then the random hitting time of  $S$  is infinite.

When  $S = \{x\}$ , we write  $m_{hit}^X(x)$ .

The *hitting time*  $m_{hit}^X$  of  $X$  is the maximal expected time between two states of  $\Omega$ , starting from any node  $x$  to reach any node  $y$ , i.e. it is equal to  $\max_{x,y} \mathbb{E}(m_{hit}^{X^x}(y))$ .

In the case of the coupon collector, all coupons are equally likely, therefore the obtention of a specific coupon is governed by a Bernoulli process, of chance of success  $p = \frac{1}{n}$ . Let us recall briefly why the expectation before getting a success is  $p^{-1} = n$ . For this, remark that the probability that the coupon is *not* obtained in the steps  $1, 2, \dots, m$  is equal to the probability that any other coupon was chosen at each of these steps, i.e., is equal to  $(1 - p)^m$ . Therefore,

$$\mathbb{E}(m_{hit}^X(x)) = \sum_{m=0}^{\infty} \mathbb{P}(m_{hit}^X(x) \geq m) = \sum_{m=0}^{\infty} (1 - p)^m = \frac{1}{1 - 1 - p} = \frac{1}{p}. \quad (1.9)$$

**Cover time.** A slightly more complicated question, for the Coupon Collector, is to determine how many boxes they will have to buy before they complete their collection. This is known as the *cover time* of the coupon collector process, and is defined generally as follows.

**Definition 1.17.** The random cover time  $m_{cov}(X^x)$  of a Markov Chain  $X$ , on a finite probability space  $\Omega$ , starting from  $x \in \Omega$ , is a random variable equal to the time before every state of  $\Omega$  has been visited. The *cover time* of  $X$  is the maximal expected cover time, where the maximum is taken with respect to the initial location  $x \in \Omega$ , i.e.,  $\max_{x \in \Omega} \mathbb{E}(m_{cov}(X^x))$ .

The cover time of a random walk on a disconnected graph is infinite. Indeed, a random walk on a graph stays within the connected component of its initial state.

In the case of the Coupon Collector, the cover time is  $nH_n$ , where  $H_n = \sum_{k=1}^n k^{-1} = \log n + \Theta(1)$ . The derivation of this result is rather easy. When there are  $k \in [0, n-1]$  coupons obtained by the collector, the probability at each step to get a new coupon is  $\frac{n-k}{n}$ , as explained above. Therefore, the expected number of steps  $m_k$  between the  $k$ -th and the  $k+1$ -th coupon is  $\frac{n}{n-k}$ . By summation, the cover time is  $\sum_{k=0}^{n-1} m_k = nH_n$ .

**Matthew's upper bound.** The cover time is always at least the hitting time of any node. The following bound indicates that the reverse is true, up to a logarithmic factor.

**Theorem 1.18** (Matthew's upper bound). *For any Markov Chain on a state space  $\Omega$  of size  $n$ , we have*

$$\mathbb{E}(m_{cov}) \leq (1 + \log n) \cdot m_{hit}^X,$$

The proof of Matthew's upper bound is simple and elegant. I provide it as an introduction to the more involved proofs that will come later.

*Proof.* We will proceed in a similar manner to that of the Coupon Collector problem. Let  $x_0$  be the initial state, and denote by  $x_1, x_2, \dots, x_{n-1}$  the other states of  $\Omega$ . Now, choose an enumeration  $x_{\sigma(1)}, \dots, x_{\sigma(n-1)}$  uniformly at random, and independently from the Markov Chain, among all possible enumerations. (This is equivalent to choosing a random permutation  $\sigma$ .)

For  $k \in [1, n-1]$ , let  $\mathcal{A}_k$  be the event that the states  $x_{\sigma(1)}, \dots, x_{\sigma(k)}$  have been visited. Let  $m_k$  be the random number of steps between the achievement of  $\mathcal{A}_{k-1}$  and that of  $\mathcal{A}_k$ . We have  $m_{cov} = \sum_{k=1}^{n-1} m_k$ . Note that  $m_k > 0$  only if  $x_{\sigma(k)}$  is visited after the  $k-1$  states  $x_{\sigma(1)}, \dots, x_{\sigma(k-1)}$ . Remarkably, because we have chosen

a random ordering of the states, this happens with probability exactly  $\frac{1}{k}$ . Hence, we may write

$$\mathbb{E}(m_k) = 0 + \mathbb{P}(\mathcal{A}_k)\mathbb{E}(m_k \mid \mathcal{A}_k).$$

Next, bound  $\mathbb{E}(m_k \mid \mathcal{A}_k)$  by the hitting time  $m_{hit} = \max_{x_0, x} \mathbb{E}(m_{hit}^{X^{x_0}}(x))$ , and use the linearity of expectation to obtain:

$$\mathbb{E}(m_{cov}(X^{x_0})) \leq m_{hit}^X \sum_{k=1}^{n-1} \frac{1}{k},$$

and use that  $\sum_{k=1}^{n-1} \frac{1}{k} = H_{n-1} \leq 1 + \log n$ . To conclude, observe that since this bound holds for any initial point  $x_0$ , it holds in particular for the worst-case starting point.  $\square$

# Chapter 2

## Properties of general random walks

In this chapter, I provide some properties of Markov Chains, and random walks on  $\mathbb{R}^\Delta$  or  $\mathbb{T}_n^\Delta$  with a general distribution  $\mathbf{p}$ . After providing some supplementary definitions, including a definition of the time of the process (with constant speed), I show in Sections 2.2 and 2.3, how bounds on the hitting times of a Markov Chain can be obtained by studying its pointwise probability. Next, I show in Section 2.4 a monotonicity property. In Section 2.5, I provide asymptotics on the distance achieved by a random walk of step-length distribution  $\mathbf{p}$ . Finally, I introduce in Section 2.6 the Walk or Probe problem, as a model of intermittent search defined on general graphs, and provide an analysis of the performances of a simple algorithm.

### 2.1 Definitions

Recall from Chapter 1 that an unbiased random walk  $X$  on  $\Omega \in \{\mathbb{R}^\Delta, \mathbb{T}_n^\Delta\}$  is a random variable  $(X_m)_{m \in \mathbb{N}}$  verifying, for every  $m \in \mathbb{N}$ :

$$X(m+1) = X(m) + V(m+1),$$

where  $(V_m)_{m \geq 1}$  are i.i.d., the direction of the vector  $Y_m$  is chosen uniformly at random, and  $\|V_m\|$  follows a probability distribution  $\mathbf{p}$ . Denote by  $\tau$  and  $\sigma$  the mean and standard deviation of  $\mathbf{p}$  (when they are defined).

**Definition 2.1.** To a random walk  $X$  defined on the torus  $\mathbb{T}_n^\Delta$  by Eq. (1.7), we associate an *extension*  $Z$  to  $\mathbb{R}^\Delta$ , defined by:

$$Z(0) = X(0), \text{ and } Z(m+1) = Z(m) + V(m+1) \quad (2.1)$$

In this case,  $X$  is the projection of  $Z$  on  $\mathbb{T}_n^\Delta$ .

**Definition 2.2.** For an integer  $m$ , the (*random*) *time*,  $T(m)$ , taken by the walk  $X$  (or  $Z$ ) up to step  $m$  is

$$T(m) = \sum_{s=1}^m \|V(s)\|. \quad (2.2)$$

*Remark 2.3.* The usual terminology of Markov Chains, presented in Chapter 1, also use the word “time”, e.g. “hitting time” or “stopping time”. In order to avoid

confusion, I will use the word *step* when talking about the Markov Chain  $(X(m))_m$ . Hence, if  $m^*$  is a stopping step for  $(X(m))_m$ , the corresponding stopping time is  $t^* = T(m^*)$ .

Let us denote by  $m_{hit}(S)$  the random hitting step of  $S$ , and  $t_{hit}(S) = \mathbb{E}(T(m_{hit}(S)))$  the expected hitting time of  $S$ . As we shall see in the next section, there is a simple link between the two.

## 2.2 From hitting time to hitting step

Indeed, we establish that the hitting time is simply obtained by multiplying the hitting step by the expected time of one step.

**Claim 2.4.** Let  $X$  be a (possibly biased) random walk on  $\mathbb{T}_n^\Delta$  that does not always stay in place (i.e.  $\mathbf{p} \neq \delta_0$ ). Let  $S$  be a subset of  $\mathbb{T}_n^\Delta$ , such that, with positive probability,  $X(0) \notin S$ . We have

$$t_{hit}^X(S) = \mathbb{E}(m_{hit}^X(S)) \cdot \tau,$$

where  $\tau$  is the expected time of a step.

Claim 2.4 reminds of Wald's identity with respect to the lengths  $(\|V(s)\|)_s$ . However, Wald's identity cannot be applied directly because  $m_{hit}^X(S)$  is not a stopping step for the sequence  $(\|V(s)\|)_s$ . Instead, we prove the claim by the Martingale Stopping Theorem (that can also be used to prove Wald's identity).

*Proof.* To prove the claim, note that we can suppose that  $\tau < \infty$  and  $\mathbb{E}(m_{hit}^X(S)) < \infty$ . Indeed, if  $\tau = \infty$ , then even one step takes an infinite expected time. With some positive probability  $X(0) \notin S$ , hence at least one step must be done to visit  $S$ , and thus  $t_{hit}^X(S) = \infty$ . If  $\mathbb{E}(m_{hit}^X(S)) = \infty$ , then, since  $\mathbf{p}(0) < 1$ , there is at each step a positive probability  $\delta$  to do a step of length at least  $\varepsilon$ . Therefore, after  $m$  steps, where  $m$  is large, there are roughly  $\delta m$  steps of length at least  $\varepsilon$ . Hence, if there is an infinite number of steps, then with probability 1 there is an infinite number of steps of length (and hence, time) at least  $\varepsilon$ . In this case, we have again  $t_{hit}^X(S) = \infty$ , and the equality is verified. In what follows we therefore assume that both  $\tau < \infty$  and  $\mathbb{E}(m_{hit}^X(S)) < \infty$ .

We start the proof by defining:

$$W(m) := \sum_{s \leq m} (\|V(s)\| - \tau).$$

The claim is proven by showing first that  $(W(m))_m$  is a martingale with respect to  $(X(m))_m$ . Then, as  $m_{hit}^X(S)$  is a stopping step for  $(X(m))_m$  (i.e., the event  $\{m_{hit}^X(S) = m\}$  depends only on  $X(s)$ , for  $s \leq m$ ), we can apply the Martingale Stopping Theorem which gives  $\sum_{s \leq m_{hit}^X(S)} (\|V_s\| - \tau) = 0$ . In more details, recall, e.g., from (Mitzenmacher and Upfal, 2005)[Definition 12.1], that a sequence of random variables  $(W(m))_m$  is a *martingale* with respect to the sequence  $(X(m))_m$  if, for all  $m \geq 0$ , the following conditions hold:

- $W(m)$  is a function of  $X(0), X(1), \dots, X(m)$ ,



- $\mathbb{E}(|W(m)|) < \infty$ ,
- $\mathbb{E}(W(m+1) \mid X(0), \dots, X(m)) = W(m)$ .

I first claim that  $W(m)$  is a martingale with respect to  $X(0), X(1), \dots$ . Indeed, since  $V(s) = X(s) - X(s-1)$ , the first condition holds. Since  $\mathbb{E}(|W(m)|) \leq \sum_{s \leq m} \mathbb{E}(|V_s - \tau|) \leq 2\tau m < \infty$ , the second condition holds. Finally, since  $W(m+1) = W(m) + \|V(m+1)\| - \tau$ , we have  $\mathbb{E}(W(m+1) \mid X(0), \dots, X(m)) = W(m) + \mathbb{E}(\|V(m+1)\|) - \tau = W(m)$ , and hence the third condition holds as well.

Next, recall the Martingale Stopping Theorem (e.g., (Mitzenmacher and Upfal, 2005)[Theorem 12.2]) which implies that  $\mathbb{E}(W(M)) = \mathbb{E}(W(0))$ , whenever the following three conditions hold:

- $W(0), W(1), \dots$  is a martingale with respect to  $X(0), X(1), \dots$ ,
- $M$  is a stopping step for  $X(0), X(1), \dots$  such that  $\mathbb{E}(M) < \infty$ , and
- there is a constant  $c$  such that  $\mathbb{E}(|W(m+1) - W(m)| \mid X(0), \dots, X(m)) < c$ .

Let us prove that the conditions of the Martingale Stopping theorem hold. We have already seen that the first condition holds. Secondly, we have  $\mathbb{E}(m_{hit}^X(S)) < \infty$  by hypothesis. Finally, we need to prove that  $\mathbb{E}(|W(m+1) - W(m)| \mid X(0), \dots, X(m)) < c$  for some  $c$  independent of  $m$ . Since  $W(m+1) - W(m) = \|V(m+1)\| - \tau$ , we have  $\mathbb{E}(|W(m+1) - W(m)| \mid X(0), \dots, X(m)) = \mathbb{E}(\|V(m+1)\| - \tau) \leq 2\tau$ . Therefore, the conditions hold and the theorem gives:

$$\mathbb{E}(W(m_{hit}(S))) = \mathbb{E}(W(0)) = 0.$$

Hence,

$$\begin{aligned} 0 &= \mathbb{E}(W(m_{hit}^X(S))) = \mathbb{E}\left(-m_{hit}^X(S)\tau + \sum_{s \leq m_{hit}^X(S)} \|V(s)\|\right) \\ &= -\mathbb{E}(m_{hit}^X(S))\tau + \mathbb{E}\left(\sum_{s \leq m_{hit}^X(S)} \|V_s\|\right) \\ &= -\mathbb{E}(m_{hit}^X(S))\tau + t_{hit}^X(S), \end{aligned}$$

which establishes Claim 2.4. □

## 2.3 Linking hitting step to pointwise probabilities

With the next two claims, we provide an explicit way of computing the expected hitting step of a random walk, by means of the study of the pointwise probability of the process  $Z$ , i.e.  $\mathbb{P}(Z(m) = x)$  for  $m$  and  $x$ . First, we prove that, once we know that, with at least some probability,  $S$  is visited before step  $m$ , then we can obtain an upper bound on the hitting step of  $S$ .

**Claim 2.5.** Let  $Z$  be a Markov Chain on a probability space  $G$  and  $S$  be a (measurable) subset of  $G$ . Denote by  $Z^x$  the process  $Z$  starting at  $Z(0) = x$ . If, for any  $x \in G$ , we have  $\mathbb{P}(m_{hit}^{Z^x}(S) \leq m) \geq q$  then  $\mathbb{E}(m_{hit}^{Z^x}(S)) \leq mq^{-1}$ .

*Proof of Claim 2.5.* Define a Bernoulli variable  $\chi$  as follows. Consider  $m$  steps of the process and define  $\chi$  to be “success” if and only if the process hits  $S$  within these  $m$  steps. Note that  $\chi$  has probability at least  $q$  to be “success” regardless of where the process starts, by hypothesis. Hence, the expected number of trials until  $\chi$  succeeds is at most  $1/q$ . This translates to  $\mathbb{E}(m_{hit}^X(S)) \leq mq^{-1}$ , and establishes Claim 2.5.  $\square$

Next, we prove a link between the pointwise probability of a Markov Chain and its hitting time. It involves the number  $N_S = \sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) \in S)$  of visits to  $S$  between  $m_0$  and  $2m_0$  and the oversample  $O_S = \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in S)$ , starting from the worst point  $z_0$ , of  $Z$  with respect to  $S$ . We establish that the probability that  $S$  is visited before time  $2m_0$  is at least  $V_S/O_S$ .

**Claim 2.6.** Consider a Markov Chain  $Z$  on a domain  $G$ . Let  $S$  be a measurable subset of  $G$  (resp. a subset of  $G$ ), with  $Z(0) \notin S$ . For any  $m_0 > 0$ , we have

$$\mathbb{P}(m_{hit}^Z(S) \leq 2m_0) \geq \frac{\sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) \in S)}{\sup_{z_0 \in S} \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in S)}. \quad (2.3)$$

*Proof.* In order to prove Eq. (2.3), we rely on the following identity (see also (Adler et al., 2003)[Proposition 2] and (Kanade et al., 2016)). If  $N$  is a non-negative random variable then:

$$\mathbb{P}(N \geq 1) = \frac{\mathbb{E}(N)}{\mathbb{E}(N \mid N \geq 1)}. \quad (2.4)$$

We employ this identity for the random variable  $N_S(m_0, 2m_0)$  which is the number of times  $Z$  visits  $S$  between steps  $m_0$  and  $2m_0$  included. Note that this quantity is positive if and only if  $S$  is visited during this interval by  $Z$ , therefore

$$\mathbb{P}(m_{hit}^Z(S) \leq 2m_0) \geq \mathbb{P}(N_S(m_0, 2m_0) \geq 1). \quad (2.5)$$

Note that  $N_S(m_0, 2m_0) = \sum_{m=m_0}^{2m_0} \mathbf{1}_{Z(m) \in S}$ , so that

$$\mathbb{E}(N_S(m_0, 2m_0)) = \sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) \in S). \quad (2.6)$$

Note also that the denominator in Eq. (2.4) applied to  $N_S(m_0, 2m_0)$  verifies

$$\begin{aligned} \mathbb{E}(N_S(m_0, 2m_0) \mid N_S(m_0, 2m_0) \geq 1) &= \mathbb{E}(N_S(m_0, 2m_0) \mid Z(m) \in S \text{ for some } m \in [m_0, 2m_0]) \\ &\leq \mathbb{E}(N_S(m_0, 2m_0) \mid Z(m_0) \in S) \\ &\leq \sup_{z_0 \in S} \mathbb{E}(N_S(0, m_0) \mid Z(0) = z_0), \end{aligned}$$

where the first inequality comes from the fact that visiting  $S$  earlier (i.e., for  $m = m_0$  instead of  $m > m_0$ ) can only increase the number of returns to  $S$ , and the second inequality is a consequence of the Markov property. Finally, write, as above,

$$\mathbb{E}(N_S(0, m_0) \mid Z(0) = z_0) = \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in S). \quad (2.7)$$

Therefore, when applied to  $N_S(m_0, 2m_0)$ , Eq. (2.4), combined with Eqs. (2.5), (2.6) and (2.7), implies that

$$\mathbb{P}_0(m_{hit}^X(S) \leq 2m_0) \geq \frac{\sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) \in S)}{\sup_{z_0 \in S} \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in S)}.$$

This establishes Eq. (2.3), and thus completes the proof of Claim 2.6.  $\square$

Combining the two previous claims, we are able to formulate the following lemma:

**Lemma 2.7.** *Consider a Markov Chain  $Z$  on a probability space  $\Omega$ . Let  $m_0 \geq 1$  and  $S$  be a measurable subset of  $G$ . Define*

$$q := \frac{\inf_{x \in \Omega} \sum_{m=m_0}^{2m_0} \mathbb{P}(Z^x(m) \in S)}{\sup_{z_0 \in S} \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in S)}. \quad (2.8)$$

Then we have

$$\mathbb{E}(m_{hit}^Z(S)) = O(m_0 \cdot q^{-1}). \quad (2.9)$$

*Remark 2.8.* If  $m_0$  is such that there is a constant  $c > 0$  for any  $m \geq m_0$ , we have  $\mathbb{P}(Z(m) \in S) \geq c\pi(S)$ , then

$$\mathbb{E}(m_{hit}^Z(S)) = O\left(\pi(S)^{-1} \sum_{m=0}^{m_0} \sup_{z_0 \in S} \mathbb{P}(Z^{z_0}(m) \in S)\right).$$

This happens, in particular, in finite Markov Chains, when  $m_0 \geq 4m_{mix}$ , with  $c = \frac{1}{2}$  (Aldous and Fill, 2002)[Lemma 4.11], and this  $m_0$  is known as the separation time.

*Remark 2.9.* If  $Z$  is a random walk defined on  $\mathbb{R}^\Delta$  (resp.  $\mathbb{Z}$ ) and  $X$  is its projection on the torus  $\mathbb{T}_n^\Delta$  (resp. the discrete cycle  $C_n$ ), and  $S$  is a measurable subset of  $T_n^\Delta$  (resp.  $C_n$ ), then, if  $q$  is defined by Eq. (2.8) (with the process  $Z$ ), we have

$$t_{hit}^Z(S) = O(m_0 \cdot q^{-1} \cdot \tau). \quad (2.10)$$

Indeed, if  $Z \in S$ , then  $X \in S$ , therefore  $m_{hit}^X(S) \leq m_{hit}^Z(S)$ , and by Claim 2.4 the bound is obtained by multiplying by the average time of a step  $\tau$ .

## 2.4 Monotonicity

A function  $f$  on  $\mathbb{R}^\Delta$  is called *radial* if there is a function  $\tilde{f}$  on  $\mathbb{R}^+$  such that for any  $x \in \mathbb{R}^\Delta$ ,  $f(x) = \tilde{f}(\|x\|)$ . In this case we say that  $f$  is non-increasing if  $\tilde{f}$  is. The goal of this section is to prove the following.

**Claim 2.10.** Let  $X$  and  $Y$  be two independent random variables with values in  $\mathbb{R}^\Delta$ , admitting probability density functions respectively  $f$  and  $g$ . Let  $h$  be the probability density functions of  $X+Y$ . If  $f$  and  $g$  are both radial and non-increasing functions then so is  $h$ .

I shall soon show the claim, but first, let me give a corollary, assuming the claim is true.

*Corollary 2.11* (Monotonicity). Let  $Z$  be a random walk process on  $\mathbb{R}^\Delta$ , starting at  $Z(0) = 0$ , with admits a p.d.f.  $f$ . If  $f$  is non-increasing, then for any  $m \geq 1$  the p.d.f.  $\mathbf{p}^{Z(m)}$  of the distribution of  $Z(m)$  is radial and non-increasing. In particular, for any  $x, x'$  points in  $\mathbb{R}^2$  with  $\|x\| \leq \|x'\|$ , we have  $\mathbf{p}^{Z(m)}(x') \leq \mathbf{p}^{Z(m)}(x)$ . Furthermore, for any  $x \in \mathbb{R}^2$  and any  $m \geq 1$ ,  $\mathbf{p}^{Z(m)}(x) \leq \frac{1}{c_\Delta \|x\|^2}$ , where  $c_\Delta$  is the measure of the unit ball in  $\mathbb{R}^\Delta$ .

*Proof of Corollary 2.11.* The fact that  $\mathbf{p}^{Z(m)}$  is radial and non-increasing follows from Claim 2.10 by induction. Indeed, the step-length vectors  $V(1), V(2), \dots$  are independent and, by hypothesis, admit a radial, non-increasing p.d.f. Hence so does  $Z(m) = V(1) + V(2) + \dots + V(m)$ . The upper bound on  $\mathbf{p}^{Z(m)}(x)$  follows easily. Indeed, for  $x \in \mathbb{R}^2 \setminus \{(0,0)\}$ , consider the ball  $B$  of radius  $\|x\|$  and centred at 0. We have  $\int_B \mathbf{p}_m^Z(y) dy \leq 1$ , and by the monotonicity,  $\int_B \mathbf{p}_m^Z(y) dy \geq \mathbf{p}_m^Z(x) |B| = \mathbf{p}_m^Z(x) \cdot c_\Delta \|x\|^2$ .  $\square$

*Proof of Claim 2.10.* Let  $\theta \in [0, 2\pi)$ . For  $x \in \mathbb{R}^2$ , denote by  $\text{rot}_\theta(x)$  the point obtained by rotating  $x$  around the center 0 with an angle of  $\theta$ . Then, by a change of variable, we have:

$$\begin{aligned} h(\text{rot}_\theta(x)) &= \int_{y \in \mathbb{R}^2} f(\text{rot}_\theta(x) - y) g(y) dy \\ &= \int_{y \in \mathbb{R}^2} f(\text{rot}_\theta(x) - \text{rot}_\theta(y)) g(\text{rot}_\theta(y)) dy \\ &= \int_{y \in \mathbb{R}^2} f(x - y) g(y) dy = h(x), \end{aligned}$$

where we used in the last equality the radially of  $f$  and  $g$ . This establishes the fact that  $h$  is radial. Next, we prove, in a manner inspired by (Adler et al., 2003), that  $h(x)$  is non-increasing with  $\|x\|$ . Since  $h$  is radial, we can restrict the study to points of the non-negative  $y$ -axis. Let us fix  $x = (0, x_2) \in \mathbb{R} \times \mathbb{R}^{\geq 0}$ , and  $x' = (0, x'_2) \in \mathbb{R} \times \mathbb{R}^{\geq 0}$  with  $x'_2 \geq x_2$ . Our goal is to show that  $h(x) \geq h(x')$ .

Let  $\gamma = \frac{x'_2 - x_2}{2}$ . Note that  $f(0, x_2 + y) \geq f(0, x'_2 - y)$  for every  $y \in (-\infty, \gamma]$ . Define, for  $y = (y_1, y_2) \in \mathbb{R}^2$ , the function  $H_{x, y_1}(y_2) = f(x - y) g(y)$ . When  $y_1$  is clear from the context, we shall write  $H_x(y_2)$  instead of  $H_{x, y_1}(y_2)$  for simplicity of notation. Now write, beginning with the change of variable  $y_2 \mapsto -y_2$ ,

$$\begin{aligned} h(x) &= \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} H_x(-y_2) dy_1 dy_2 = \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} H_x(-y_2 - \gamma) dy_1 dy_2 \\ &= \int_{y_1 \in \mathbb{R}} \left( \int_{y_2 \geq 0} H_x(-y_2 - \gamma) dy_2 + \int_{y_2 \leq 0} H_x(-y_2 - \gamma) dy_2 \right) dy_1 \\ &= \int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} H_x(-y_2 - \gamma) + H_x(y_2 - \gamma) dy_2 dy_1, \end{aligned}$$

and

$$\begin{aligned}
 h(x') &= \int_{y_1 \in \mathbb{R}} \int_{y_2 \in \mathbb{R}} H_{x'}(y_2) dy_1 dy_2 \\
 &= \int_{y_1 \in \mathbb{R}} \int_{y_2 \geq \gamma} H_{x'}(y_2) + \int_{y_2 \leq \gamma} H_{x'}(y_2) dy_1 dy_2 \\
 &= \int_{y_1 \in \mathbb{R}} \left( \int_{y_2 \geq 0} H_{x'}(y_2 + \gamma) dy_2 + \int_{y_2 \leq 0} H_{x'}(y_2 + \gamma) dy_2 \right) dy_1 \\
 &= \int_{y_1 \in \mathbb{R}} \left( \int_{y_2 \geq 0} H_{x'}(y_2 + \gamma) + H_{x'}(-y_2 + \gamma) dy_2 \right) dy_1
 \end{aligned}$$

Hence, we have that  $h(x) - h(x')$  is equal to

$$\begin{aligned}
 &\int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} f(-y_1, x_2 + y_2 + \gamma) g(y_1, -y_2 - \gamma) + f(-y_1, x_2 - y_2 + \gamma) g(y_1, y_2 - \gamma) \\
 &\quad - f(-y_1, x'_2 - y_2 - \gamma) g(y_1, y_2 + \gamma) - f(-y_1, x'_2 + y_2 - \gamma) g(y_1, \gamma - y_2) dy_1 dy_2
 \end{aligned}$$

Since  $g$  is radial, we have  $g(y_1, -y_2 - \gamma) = g(y_1, y_2 + \gamma)$  and  $g(y_1, \gamma - y_2) = g(y_1, y_2 - \gamma)$ . Furthermore, using that  $x_2 + \gamma = x'_2 - \gamma$ , we obtain that  $h(x) - h(x')$  is equal to:

$$\begin{aligned}
 &\int_{y_1 \in \mathbb{R}} \int_{y_2 \geq 0} (f(-y_1, x_2 + y_2 + \gamma) - f(-y_1, x_2 - y_2 + \gamma)) \\
 &\quad \cdot (g(y_1, y_2 + \gamma) - g(y_1, y_2 - \gamma)) dy_1 dy_2
 \end{aligned}$$

In this summation, since  $x_2 \geq 0$ ,  $\gamma \geq 0$  and  $y_2 \geq 0$ , we have  $|x_2 + y_2 + \gamma| \geq |x_2 - y_2 + \gamma|$  and  $|y_2 + \gamma| \geq |y_2 - \gamma|$ . Since  $f$  and  $g$  are radial and non-increasing, both factors of the integrand are non-negative, hence the integrand is non-negative and  $h(x) - h(x') \geq 0$ .  $\square$

## 2.5 Propagation of random walks

In this section I show the following claim that gives a lower bound on the number of steps needed for a random walk to achieve a certain distance. See also (Comtet and Majumdar, 2005) for more precise asymptotics (in one-dimension).

**Claim 2.12.** Consider an unbiased random walk  $X$  on  $\Omega \in \{\mathbb{T}_n^\Delta, \mathbb{R}^\Delta\}$ . Let  $\sigma'$  be the standard deviation of a projected step-length of  $X$  over one coordinate. We have:

- The expected maximal distance achieved in  $m$  steps is  $O(\sqrt{m}\sigma')$ .
- The expected number of steps needed to reach a distance  $d$  is  $\Omega(\frac{d}{\sigma'^2})$ .
- The time needed to reach a distance  $d$  is  $\Omega(\frac{d^2}{\sigma'^2})$ . In particular, if  $L$  is the maximal length in the support of  $\mathbf{p}$ , then the expected time needed to go to a distance  $d$  is  $\Omega(\frac{d^2}{L})$ .

This claim may appear crude in some situations, for instance in Lévy Walks, for which the variance is infinite. However, as we shall see in the next chapter, simply conditioning by the event that all steps are sufficiently small (e.g., no steps have length more than  $d$ ) allows to yield satisfying results.

*Proof of Claim 2.12.* Denote by  $m_d$  the (random) number of steps needed to reach a distance at least  $d$ . We are interested in  $\mathbb{E}(m_d)$ .

If  $X$  is defined on  $\mathbb{T}_n^\Delta$ , let  $Z$  be its extension to  $\mathbb{R}^\Delta$ . Since the distance between  $Z(m)$  and  $Z(0)$ , in  $\mathbb{R}^2$ , is always at least that of  $X(m)$  and  $X(0)$ , in  $\mathbb{T}_n$ , the number of steps needed to go to distance  $d$  in  $\mathbb{T}_n$  is at least as high as in  $\mathbb{R}^2$ . Hence, we may analyse the process  $Z$  instead of  $X$ . (If  $X$  is defined on  $\mathbb{R}^\Delta$  then take  $Z = X$ .)

Define  $d_{max}(m)$  as the maximal distance (from the initial point) that the process reached from step 0 up to step  $m$ , i.e.,

$$d_{max}(m) = \max_{s \leq m} \|Z(0) - Z(s)\|.$$

Now write  $Z = (Z_1, Z_2, \dots, Z_\Delta)$ , let  $\mathbf{p}'$  be the distribution of the projected step-lengths (i.e. the distribution of the step-lengths of  $Z_1$ ), and let  $\tau'$  and  $\sigma'$  be respectively its mean and standard deviation.

Next, let  $d_{i,max}(m)$  be the maximal distance reached by the projection on coordinate  $i \in \{1, \dots, \Delta\}$ . Since steps are independent, the standard deviation of  $Z_i(s)$ , for  $s \leq m$ , is  $\sqrt{s}\sigma' \leq \sqrt{m}\sigma'$ .

By Kolmogorov's inequality (Theorem 1.4), we have for any  $\lambda > 0$ ,  $\mathbb{P}(d_{i,max}(m) \geq \lambda\sqrt{m}\sigma') \leq \frac{1}{\lambda^2}$ . Furthermore, since  $d_{max}(m) \leq \sqrt{\Delta} \max_i \{d_{i,max}(m)\}$ , we have by a union bound argument, for any  $\lambda > 0$ ,

$$\mathbb{P}(d_{max}(m) \geq \lambda\sqrt{m}\sigma') \leq \Delta \mathbb{P}\left(d_{1,max}(m) \geq \frac{\lambda}{\sqrt{\Delta}}\sqrt{m}\sigma'\right) \leq \frac{\Delta^2}{\lambda^2}.$$

We thus obtain the first part of the claim:

$$\begin{aligned} \mathbb{E}(d_{max}(m)) &= \int_{s=0}^{\infty} \mathbb{P}(d_{max}(m) \geq s) ds \leq \sum_{\lambda=0}^{\infty} \int_{\lambda'=0}^{\sqrt{m}\sigma'} \mathbb{P}(d_{max}(m) \geq \lambda\sqrt{m}\sigma' + \lambda') d\lambda' \\ &\leq \sqrt{m}\sigma' \left( \sum_{\lambda \geq 0} \mathbb{P}(d_{max}(m) \geq \lambda\sqrt{m}\sigma') \right) = O(\sqrt{m}\sigma'). \end{aligned} \quad (2.11)$$

For the second part, note that for  $m \geq m_d$ , we have  $d_{max}(m) \geq d_{max}(m_d) \geq d$ . Therefore, by Markov's inequality (Theorem 1.2),

$$\mathbb{E}(d_{max}(2\mathbb{E}(m_d))) \geq \mathbb{E}(d_{max}(2\mathbb{E}(m_d)) \mid m_d < 2\mathbb{E}(m_d)) \cdot \mathbb{P}(m_d < 2\mathbb{E}(m_d)) \geq d \cdot \frac{1}{2}. \quad (2.12)$$

Now using Eq. (2.11) with  $m = 2\mathbb{E}(m_d)$ , we have  $\mathbb{E}(d_{max}(2\mathbb{E}(m_d))) = O(\sqrt{\mathbb{E}(m_d)}\sigma')$  and hence, by Eq. (2.12),

$$\mathbb{E}(m_d) = \Omega\left(\frac{d^2}{\sigma'^2}\right),$$

which is as stated. Finally, to compute the expected time  $T_d = \mathbb{E}(T(m_d))$  needed to reach distance  $d$ , let  $S_d$  be the set containing all points at distance more than  $d$ . We now use Claim 2.4 with respect to  $S_d$  to obtain:

$$T_d = \mathbb{E}(m_d)\tau = \Omega\left(\frac{d^2}{\sigma'^2} \cdot \tau\right).$$

To conclude, simply remark that

$$\sigma'^2 = \int_0^L p'(\ell)\ell^2 d\ell \leq \int_0^L p'(\ell)\ell \cdot L d\ell = L\tau' \leq L\tau, \quad (2.13)$$

where the last inequality is justified by the fact that the projection reduces distances.  $\square$

## 2.6 Walk or Probe

In this section, I use the previous results in order to study a model of intermittent random walks defined on graphs. This model, and its study, was presented in (Boczkowski et al., 2018a).

Consider a simple random walker that walks on a connected graph  $G$  and aims to *probe* all nodes in  $G$  as quickly as possible. The walker at a node is unable to detect whether it has previously probed it. At this point it needs to decide whether to continue the walk or probe it and then continue the walk. Crucially, probing a node is time consuming, and can potentially be very slow with respect to the time required to move between neighbours. Specifically, let us assume that each edge traversal costs 1 unit of time, while probing a node costs  $C \geq 0$  time units, where  $C$  can be a function of several parameters of  $G$  (e.g., the number of nodes, edges, or maximal degree).

The Walk or Probe problem aims to find a strategy that balances the time spent in walking vs. probing so as to minimize the *probing cover time*, that is, the expected time until all nodes are probed.

### 2.6.1 Results

Let  $G = (V, E)$  be a connected graph with  $n$  nodes. Denote by  $m_{cov}$  (resp.  $m_{mix}$ ) the cover time (resp. mixing time) of a random walk on  $G$ . The most naive strategy is to probe after each step, in which case we get a probing cover time of:

$$(C + 1) \cdot m_{cov}. \tag{2.14}$$

I show that there exists another simple strategy whose running time depends on  $m_{mix}$ , which can be much more efficient in some cases. Let us call *t-strategy* the tactic that consists in probing only once every  $t$  steps. Let  $G = (V, E)$  be an  $n$ -node connected graph.

**Theorem 2.13.** *The probing cover time of the  $\sqrt{m_{mix}}$ -strategy is*

$$O((C + \sqrt{m_{mix}}) \cdot |E| \log n). \tag{2.15}$$

*And if  $G$  is regular, the probing cover time of the  $\sqrt{m_{mix}}$ -strategy is*

$$O((C + \sqrt{m_{mix}}) \cdot n \log n). \tag{2.16}$$

Note that, by taking  $C = 0$ , we can recover from Theorem 2.13 the following bounds. The first one appears also in (Kanade et al., 2016, Theorem 1.4).

*Corollary 2.14.* For any  $n$ -node connected graph  $G = (V, E)$ ,

$$m_{cov} = \begin{cases} O(\sqrt{m_{mix}} n \log n) & \text{if } G \text{ is regular} \\ O(\sqrt{m_{mix}} |E| \log n) & \text{otherwise.} \end{cases} \tag{2.17}$$

In comparison to the upper bound, since the search should cover every node and probe every node of  $G$ , we have the following trivial lower bound for the probing cover time:

$$m_{cov} + C \cdot n. \quad (2.18)$$

Let us compare this lower bound to the upper bounds on the probing cover time of the 1- and  $\sqrt{m_{mix}}$ -strategies, given by Eq. (2.14) and (2.16), in the case of regular graphs.

One family of instances consists of regular graphs for which Eq. (2.17) is tight (up to poly-logarithmic factors). In this case Eq. (2.16) becomes, up to poly-logarithmic factors,  $Cn + m_{cov}$ , and so the  $\sqrt{m_{mix}}$ -strategy is near optimal. This happens, for example, if the mixing time is poly-logarithmic (e.g. cliques and all expander graphs, such as random  $r$ -regular graphs (Friedman, 2004)) and in such cases, by Eq. (2.14), the 1-strategy is also near optimal since the cover time is almost linear as Eq. (2.17) shows. On the other hand, Eq. (2.17) is also tight for the path and the cycle, on which the mixing time and cover time are  $\Theta(n^2)$ . Hence, in this case the  $\sqrt{m_{mix}}$ -strategy significantly outperforms the trivial 1-strategy when  $C$  is large.

If Eq. (2.17) is not tight then the  $\sqrt{m_{mix}}$ -strategy may not be optimal. An example where this happens is the torus of dimension  $d \geq 2$ : Indeed, the mixing time of such a torus is  $\Theta(n^{2/d})$ , while the cover time is almost linear (Levin et al., 2008, Sections 5.3.2 and 10.4). In this case the 1-strategy has probing cover time near optimal by Eq. (2.14), but the  $\sqrt{m_{mix}}$ -strategy, which needs at least  $\sqrt{m_{mix}}n$  steps (because it must probe at least  $n$  times), is not optimal.

Finally, regarding the general bound in Eq. 2.15 (for not-necessarily regular graphs), let us look at Erdos-Renyi random graphs  $\mathcal{G}(n, p)$  with  $p = c/n$  for a constant  $c > 1$ . In this case the mixing time is a.a.s.  $\Theta(\log^2 n)$  (Benjamini et al., 2014, Theorem 1.1) and the cover time is a.a.s.  $\Theta(n \log^2 n)$  (Cooper and Frieze, 2008, Theorem 2.(a)), and the number of vertices is a.a.s. linear in  $n$ . Therefore, by Eq. (2.15), the  $\sqrt{m_{mix}}$ -strategy has probing cover time  $O(Cn \log n + n \log^2 n)$ , while the 1-strategy, by Eq. (2.14), takes time  $(C + 1)n \log^2 n$ , and hence the former strategy gains a logarithmic factor in the number of probes.

## 2.6.2 Proof of Theorem 2.13

Since the probing cover time can only be increased by adding laziness, we will suppose that  $X$  is lazy with laziness  $\frac{1}{2}$ . Proving Theorem 2.13, then, is equivalent to proving a bound on the cover time of the Markov chain  $(Y(m))_{m \geq 0}$ , where  $Y(m) = X_{m\sqrt{m_{mix}}}$ . Indeed, if the agent probes only at steps  $0, t, 2t, \dots$ , then, each  $t$  steps cost  $C + t$ . Furthermore, the agent has probed the whole graph whenever the chain  $Y$  has visited every vertex. Hence, the probing cover time of the  $t$ -strategy is exactly

$$(C + t)m_{cov}^Y. \quad (2.19)$$

Next, consider  $m_0 = \sqrt{m_{mix}}$  and define  $N = \min_{y_0 \in G} \sum_{m=m_0}^{2m_0} \mathbb{P}(Y^{y_0}(m) = x)$  and  $O = \max_{x \in G} \sum_{m=0}^{m_0} \mathbb{P}(Y^x(m) = x)$ . By Lemma 2.5, we have that the cover time of  $Y$  is  $m_{cov}^Y = O(m_0 \cdot O/N)$ .



**Lower bound on  $N$ .** I first recall some basic results about mixing time, defined in Chapter 1 as:

$$m_{mix} = \min \left\{ m \geq 1 : \max_{y_0} \sum_{x \in G} |\mathbb{P}(X^{y_0}(m) = x) - \pi(x)| \leq \frac{1}{2} \right\},$$

where  $\pi$  is the stationary distribution. Lemmas 4.5 and 4.7 in (Aldous and Fill, 2002) imply that, for  $m \geq 4m_{mix}$ ,  $\mathbb{P}(X^{y_0}(m) = x) \geq \frac{\pi(x)}{2}$ . Since  $\pi(x) = \deg(x)/2|E|$ , where  $\deg(x)$  is the degree of node  $x$ , for  $m \geq 4m_{mix}$ , we have:

$$\mathbb{P}(X^{y_0}(m) = x) \geq \frac{\deg(x)}{4|E|}. \quad (2.20)$$

By Eq. (2.20) we thus have:

$$\sum_{m=0}^{c\sqrt{m_{mix}}} \mathbb{P}(X_{4m_{mix}+m\sqrt{m_{mix}}}^{y_0} = x) \geq c\sqrt{m_{mix}} \frac{\deg(x)}{4|E|},$$

and since this lower bound holds for any starting point  $y_0$ , it holds also for  $N$ .

**Upper bound on  $O$ .** We next use the following bounds on the probability of returns. For any  $x \in G$ ,

$$\mathbb{P}(X^x(m) = x) \leq 5/\sqrt{m} \quad \text{if } m \leq 5n^2 \text{ and } G \text{ is regular,} \quad (2.21)$$

$$\leq \deg(x)/\sqrt{m} \quad \text{if } m \leq |E|^2 - 1. \quad (2.22)$$

The bound for regular graphs is taken from Proposition 6.18 in (Aldous and Fill, 2002), while the general bound follows from the more elaborate bound in Lemma 3.4 in (Lyons, 2005). Since  $X$  is lazy with parameter  $\frac{1}{2}$ , we have that  $\mathbb{P}(X^x(m) = x)$  is non-increasing with  $m$ , hence we can write these bounds as:

$$\mathbb{P}(X(m) = x) \leq \frac{\beta_x}{\sqrt{t}}$$

with  $\beta_x = O(1)$  if  $G$  is regular, and  $\beta_x = O(\deg(x))$  otherwise, valid for  $m = O(n^2)$  and, respectively,  $m = O(|E|^2)$ .

Note that we can use these bounds for  $m \leq m_{mix}$ , since  $m_{mix} \leq 3m_{cov}$  ((Levin et al., 2008, Eq. (10.24))) and, in connected graphs  $m_{cov} \leq 2|E|(n-1) \leq 2|E|^2$  ((Aleliunas et al., 1979, Theorem)), while in regular graphs we have  $m_{cov} \leq 2n^2$  ((Feige, 1994, Corollary 6)). Thus, using that  $\sum_{k=1}^t \frac{1}{\sqrt{k}} \leq 2\sqrt{t}$ , we have:

$$\sum_{m=0}^{\sqrt{m_{mix}}} \mathbb{P}(X_{m\sqrt{m_{mix}}}^x = x) \leq 1 + \frac{\beta_x}{m_{mix}^{\frac{1}{4}}} \sum_{m=1}^{\sqrt{m_{mix}}} \frac{1}{\sqrt{k}} \leq 1 + 2\beta_x,$$

and since this upper bound stands for any  $x \in G$ , it stands also for  $O$ .

**Conclusion.** Hence, we have

$$m_{hit}^Y(x) = O(m_0 \cdot O/N) = O\left(\sqrt{m_{mix}} \frac{4|E|}{\deg(x)} \frac{1+2\beta_x}{\sqrt{m_{mix}}}\right) = O(\gamma),$$

where  $\gamma = n$  if  $G$  is regular and  $\gamma = |E|$  otherwise. By Matthew's bound (Theorem 1.18), we have  $m_{cov}^Y = O(m_{hit}^Y \log n)$  and the probing cover time is thus given by Eq. (2.19)  $O((C + \sqrt{m_{mix}})\gamma \log n)$ , as announced by Theorem 2.13.  $\square$

### 2.6.3 Discussion

The bounds obtained show that there is room for improvement, especially in general graphs. Simulations seem to indicate that the cover time of  $(X(mt))_{m \geq 0}$  on a connected graph  $G$  follows the law  $a + b/t$ , where  $a + b = m_{cov}(G)$ , obtained for  $t = 1$ , and  $a = t_{cc}(G)$ , obtained for  $t = \infty$ , where  $t_{cc}(G)$  is a generalized coupon collector time, i.e. the time to get all coupons when they are given according to the stationary distribution  $\pi$  (Anceaume et al., 2015). It would be of interest to prove (or disprove) this conjecture.

The strategy probes once every  $t$  steps exactly. Another strategy, related more directly to Lévy Walks, would be to select  $t$  according to a power-law, truncated at  $t_{max} = m_{mix}^\alpha$  where  $\alpha$  might depend on the chosen power-law.

# Chapter 3

## Lévy Walks perform differently according to target sizes

This chapter is based on an incoming article (Guinard and Korman, 2020a) which is a joint work with Amos Korman. I discuss the optimality of the Cauchy Walk in the non-intermittent detection model, and show that the Cauchy Walk performs quasi-optimally when targets are sparse and have varied sizes.

### 3.1 Lévy Walks: In search of lost optimality

As was discussed in Chapter 1 (to which I refer the reader for the terminology), Lévy Patterns are ubiquitous (but not universal) in biological searches scenario. Since evolution should have favoured efficient foragers, a natural question thus arises: under which circumstances is it advantageous for a forager to adopt a behaviour that will result in a Lévy Pattern of exponent  $\mu$ ?

**Performances of Lévy Walks in the prey-targeting setting** The first theoretical argument for the efficiency of Lévy Walks appeared in the influential paper (Viswanathan et al., 1999). In a non-intermittent model, Viswanathan et al. argued that when food resources are scarce and revisitable (i.e., targets are not depleted once found), Cauchy Walk should perform much better than other Lévy Walks. While the result was analytically proved in one-dimensional topologies (Buldyrev et al., 2001), it was shown in (Levernier et al., 2020) to be an exaggeration in higher dimensions. Indeed, Levernier et al. proved, in the same setting, and in dimension  $d \geq 2$  that there is only a constant difference of mean capture rate between Lévy Walks.<sup>1</sup> Not only this, in fact, their simulations seem to indicate that the constant dependency on  $\alpha$  is not very high, and, in any cases, the Cauchy Walk is not the optimal algorithm in all settings, depending on the distribution of targets.

Other authors, however, have used extensive simulations to show that Cauchy Walks in the non-intermittent setting are, nonetheless, appreciably more efficient than Brownian Motion or other Lévy Walks (Humphries and Sims, 2014). It is, in fact, in the prey-targeting scenarios that the difference in performances between algorithms is the most obvious (essentially because the ballistic and exponential

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<sup>1</sup>This result is perhaps not that surprising, in fact, since the hitting time of the simple random walk on a torus (of dimension 2) is  $\Theta(n \log n)$  where  $n$  is the volume of the torus (Levin and Peres, 2017)[Eq (11.11)], which stands in contrast to the  $\Theta(n^2)$  performance on a cycle of length  $n$ .

walks perform very badly in these cases). We can note also that Cauchy Walks seem to perform relatively well in all scenarios, even when they are not the optimal algorithm (a feature also present in the simulations of (Levernier et al., 2020)). This stands in contrast with, e.g., ballistic walks or other Lévy walks which performances are more affected by the setting.

The 2D and 3D topologies were studied in (Reynolds, 2015a). In this work, Reynolds argued, on the basis of an approximate theoretical heuristic, supported by simulations, that (non-intermittent) Lévy Walks are an efficient strategy when targets are large compared to the perceptual range of the searcher. However, when targets are small, Reynolds argued that ballistic movement is better. Cauchy Walks, the author argued, are the most efficient, comparatively to other Lévy Walks, when targets are large and sparse.

Finally, in dimension 2, and independently from the work presented here, computer scientists studied the search efficiency of multiple agents performing Lévy Walks. (Clementi et al., 2020). The common goal of the agents is to find in the minimal time a unique target, located at an unknown distance. The agents are able to detect the target while performing their ballistic steps (i.e., the detection is not intermittent). Interestingly, they proved that performances of Lévy Walks with exponent  $\mu \in (1, 2]$  are comparable. In contrast, the search time, as well as the optimal number of agents, of the Lévy Walks with exponent  $\mu \in [2, 3]$  depend polynomially on  $\mu$ .

**Performances of Lévy Walks in the intermittent setting** The theoretical evidence of the efficiency of the Cauchy Walk is very limited in the intermittent scenario.

In (Reynolds, 2006), the author argued, on the basis of simulations, that Cauchy Walks could be optimal in the intermittent scenario. This was confirmed analytically in (Lomholt et al., 2008). However, both studies are limited to one-dimensional topologies.

Many biological searches, of course, happen on 2D and 3D topologies. The lack of result in this setting is inconvenient because it represents searchers of minimal abilities: without memory, interaction with the environment, nor the ability to detect while moving. As discussed in Chapter 1, this may apply to ancient organisms and, as such, may be the founding blocks of the strategies of more complex animals.

**Performance of Lévy Flights as a prey strategy** Adler et al. proved in (Adler et al., 2003) that Lévy Flights are an efficient strategy for a rabbit aiming to escape a hunter on the discrete cycle of length  $n$ , whatever the strategy used by the hunter. The more precise setting can be described as the following turn-by-turn game. The goal of the hunter is to find the rabbit (i.e., be at some step at the same node as the rabbit) in the minimal (expected) time, while the rabbit wants to be caught in the maximal (expected) time. The hunter can only move through edges of the graph, while the rabbit is able to teleport (i.e., can go from any node to any other node in one step only). It is then proved that the rabbit, when choosing to move according to a Cauchy Flight, is able to escape the hunter for at least  $\Omega(n \log n)$  steps, whatever the strategy employed by the hunter. This is optimal, as the hunter has a strategy that allows them to capture the rabbit in  $O(n \log n)$  time.

Note that (Adler et al., 2003) differs from previously mentioned studies as the

aim of the agent is to escape a predator (which seems to be opposite to a goal of search efficiency). However, this result is valid for a Lévy Flight (i.e., the prey must be able to go much faster than the predator), and the search operates in a bounded one-dimensional domain. While the first assumption seem to be necessary, it would be interesting to study the process in 2D and 3D topologies. This is especially relevant as few studies on the effect of predation on Lévy Walks' efficiency exist and they show no conclusive evidence for optimality or sub-optimality of Brownian Motion nor Lévy Walks. (Reynolds, 2010; Abe and Shimada, 2015)

**Target sizes in foraging animals** Like (Reynolds, 2015a), I will investigate the influences of target sizes on the efficiency of Lévy Walks, albeit in the intermittent mode of detection. This is motivated by the fact that searching for targets that significantly vary in size prevails in multiple contexts, including ones for which Lévy patterns have been reported. To name a few examples, this occurs when marine predators search for patches of fish (Sims et al., 2008), bees search for assemblages of flowers (Wolf et al., 2016), swarming bacteria search for food concentrations (Ariel et al., 2015), immune cells search for bacterial infections (Harris et al., 2012), and even when the eye scans the visual field (Brockmann and Geisel, 2000). Among saltatory foragers, but that have not been shown to have Lévy Patterns of movement, the influence of target sizes has been studied in plovers (Pienkowski, 1983) and white crappies (O'Brien et al., 1989).

In principle, as larger targets often entail higher rewards, an animal could benefit from optimizing the search with respect to such targets (O'Brien et al., 1989). However, tuning the intermittent search for larger targets may potentially cause inefficiency with respect to finding smaller, possibly more abundant, targets. For example, in the non-intermittent setting, the ballistic strategy seems to find (small) targets the fastest (Viswanathan et al., 1999; Reynolds, 2015a). However, as I shall show, in the intermittent setting, while this strategy is efficient at finding very large targets it is highly inefficient at finding small targets. Conversely, an intermittent walk with very small steps is close to a Brownian motion, and is hence very slow at going far away, and thus, as I shall show, inefficient at finding large targets. Overall, when targets appear in unpredictable sizes and detection is intermittent, it is unclear which strategy is best to employ. I will prove that, in two-dimensions, the Cauchy Walks represent an optimal memoryless strategy, in the sense that it matches (up to a logarithmic factor) an unconditional lower bound.

## 3.2 Model

Consider a searcher that aims to quickly find a single target in a finite two-dimensional terrain. This is modelled as a continuous two-dimensional square torus  $\mathbb{T}_n$  whose area is parametrized by  $n$ , identified as  $[-\sqrt{n}/2, \sqrt{n}/2]^2 \subset \mathbb{R}^2$ . For  $x = (x_1, x_2)$  in  $\mathbb{T}_n$  or  $\mathbb{R}^2$ , I define  $\|x\| = \sqrt{x_1^2 + x_2^2}$ .

The searcher starts the search at a point of the torus, denoted  $X(0)$ , which is either fixed, or chosen uniformly at random (u.a.r), and then moves according to some strategy.

A Lévy walk  $Z^\mu$  on  $\mathbb{R}^2$ , or  $X^\mu$  on  $\mathbb{T}_n$ , for a given  $\mu \in (1, 3]$  and maximal step-length  $\ell_{max} > 1$  (possibly  $\ell_{max} = \infty$ ), is the random walk process whose step-lengths

are distributed according to

$$\mathbf{P}_{\mu, \ell_{max}}(\ell) = \begin{cases} a_{\mu, \ell_{max}} & \text{if } \ell \leq 1 \\ a_{\mu, \ell_{max}} \ell^{-\mu} & \text{if } \ell \in (1, \ell_{max}) \\ 0 & \text{if } \ell \geq \ell_{max} \end{cases}, \quad (3.1)$$

where  $a_{\mu, \ell_{max}} = (1 + \int_1^{\ell_{max}} \ell^{-\mu} d\ell)^{-1} > 0$  is the normalization factor. We denote by  $X^{\text{cauchy}} = X^2$  the Cauchy Walk.

For all processes, speed is assumed to be constant. Specifically, doing a step of length  $\ell$  necessitates  $\ell$  time units.

A *target*  $S$  is a connected subset of the torus. An *intermittent* searcher can detect  $S$  only in-between steps. That is,  $S$  is *detected* if, at the end of a ballistic step, it is located within distance 1 — the sensing range — from the current location of the searcher. See Figure 3.1(a). Detecting  $S$  is therefore equivalent to visiting a point of the “extended set”  $B(S)$ , containing all points at distance at most 1 from  $S$ . (Note that  $B(S)$  is measurable.)

The *detection time* of a process  $X$  with respect to  $S$ , denoted  $t_{detect}^X(n, S)$ , is the expected time until  $X$  detects  $S$  for the first time.

As I shall show, it turns out that the important parameter governing the detection time is not the area of  $S$ , but rather its diameter  $D$ , namely, the maximal distance between any two points of  $S$ . Since the detection radius is 1, finding targets of diameter less than 1 takes roughly the same time, hence, in what follows we assume that  $D \geq 1$ .

**Competitive analysis.** To evaluate the efficiency of  $X$  with respect to a certain target  $S$ , I compare  $t_{detect}^X(n, S)$  to  $\text{opt}(n, S)$ , namely, the best achievable detection time of  $S$ . Importantly, when computing this optimal value, there is no restriction on the search strategy, allowing it to detect while moving (i.e., be non-intermittent), use infinite memory, and, furthermore, be tuned to the shape and the diameter of the target. The following tight bound holds for every connected target  $S$  whose diameter is  $D \in [1, \sqrt{n}/2]$ :

$$\text{opt}(n, S) = \Theta(n/D). \quad (3.2)$$

The proof of Eq. (3.2) appears in Section 3.6.1. A sketch of the lower bound is given in Fig 3.1(b).

I define the *competitiveness* of  $X$  with respect to a target  $S$  of diameter  $1 \leq D \leq \sqrt{n}/2$ , as an indicator of how well  $X$  performs in comparison to the optimal algorithm:

$$\text{Comp}^X(S) = \frac{t_{detect}^X(S)}{\text{opt}(n, S)} = \Theta\left(t_{detect}^X(S) \cdot \frac{D}{n}\right).$$

The *competitiveness* of  $X$  with respect to a given diameter  $1 \leq D \leq \sqrt{n}/2$  is then defined as the worst competitiveness, taken over all connected targets of diameter  $D$ , that is,

$$\text{Comp}^X(n, D) = \sup\{\text{Comp}^X(n, S) \mid S \text{ is of diameter } D\}. \quad (3.3)$$

To demonstrate the definition of competitiveness, let us consider the intermittent process  $X$  in which all step lengths are some pre-determined integer  $\ell$ . Note that

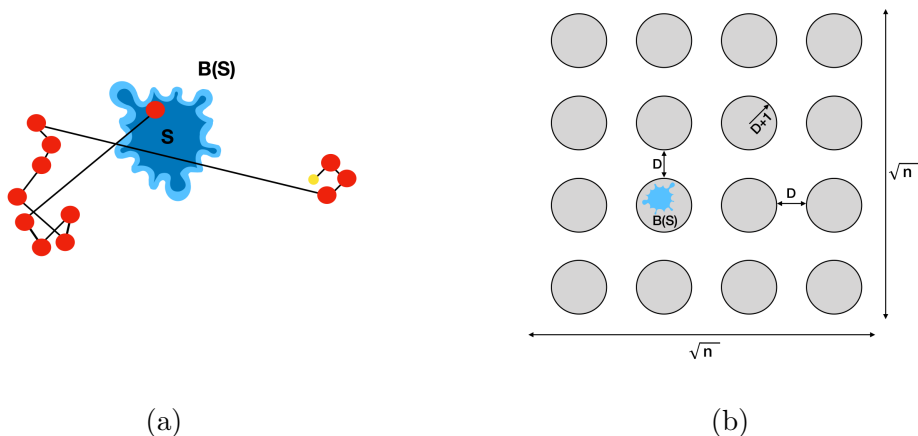


Figure 3.1: (a) Intermittent Lévy walk. The target  $S$  is marked in dark blue. The extended target  $B(S) \supset S$  is marked in both light and dark blue. The Lévy searcher starts at the yellow point, and moves in discrete steps. A red circle signifies the area inspected at the end of a step - the ball  $B(x)$  of radius 1 around the location  $x$ . The target  $S$  is detected on the 12th step of the process. (b) Illustration of the lower bound proof of Eq. (3.2). Consider a target  $S$  (colored blue) of diameter  $D$  (of any given shape). Color roughly  $n/(3D + 2)^2$  discs gray, so that each has radius  $D + 1$  and is located at a distance of  $D$  from its neighboring discs. Furthermore, align this structure so that the extended target  $B(S)$  is located in one of the discs. Since the initial location of the searcher is uniform in the torus, with probability  $\frac{1}{2}$ , at least half of the discs need to be visited before detecting  $S$ . The time required to visit a new disc is at least the smallest distance between two discs, i.e.,  $D$ . The detection time is therefore at least roughly  $D \cdot n/(3D + 2)^2 \approx n/D$ .

the case  $\ell = 1$  corresponds to the simple random walk, and that taking  $\ell = \Theta(\sqrt{n})$  may be viewed as a ballistic strategy.

**Claim 3.1.** If  $X$  is a random walk on  $\mathbb{T}_n$  with step-length distribution  $\mathbf{p} = \delta_\ell$ , for some fixed  $\ell$ , and targets have diameter  $D \in [1, D_{max}]$ , then the competitiveness of  $X$  with respect to either  $D = 1$  or  $D = D_{max}$  is  $\Omega(\sqrt{D_{max}})$ .

*Proof.* Consider a disc target of diameter  $D < \sqrt{n}/2$ . Since the searcher starts at a random point, with constant probability, the target is located at a distance of at least  $\sqrt{n}/4$  from the initial location of the searcher. In this case, merely traversing this distance by the random walk process requires expected  $\Omega(n/\ell^2)$  steps and hence consumes  $\Omega(n/\ell)$  time on expectation. This implies that  $\text{Comp}^X(n, D) = \Omega(D/\ell)$ . Furthermore, as illustrated in Figure 3.1, there are  $\Omega(n/D^2)$  possible locations of the target. Since the agent must, in average, visit half of those, it will overall need  $\Omega(n\ell/D^2)$  time to find the target, since each step takes  $\ell$  time. Thus, we also have  $\text{Comp}^X(n, D) = \Omega(\ell/D)$ .

Altogether, these arguments imply that  $\text{Comp}^X(n, D) = \Omega(\max\{\ell/D, D/\ell\})$ . While  $\ell$  can be tuned to optimize the competitiveness w.r.t. a specific value of  $D$ , if we know only an upper bound  $D_{max}$  on the value of  $D$  then the competitiveness would be large w.r.t. either  $D = 1$  or  $D = D_{max}$ . Specifically, for  $D = 1$  we have  $\text{Comp}^X(n, 1) = \Omega(\ell)$ , while for  $D = D_{max}$ , we have  $\text{Comp}^X(n, D_{max}) = \Omega(D_{max}/\ell)$ . Hence, for at least one value of  $D$  among the two, we obtain  $\text{Comp}^X(n, D) = \Omega(\max\{\sqrt{D_{max}}\})$ .  $\square$

## 3.3 Results

### 3.3.1 Analytical results

**The Cauchy Walk at step  $m$  is (roughly) uniform in the ball of radius  $m$ .** Let  $Z$  be the (truncated) Cauchy Flight on  $\mathbb{R}^2$  that starts at the origin. I establish lower and upper bounds on its distribution at step  $m$ . Roughly speaking, I show that the location of this process at step  $m$  is more or less uniform in the ball of radius  $m$  around the origin.<sup>2</sup> Since the expected duration of a Cauchy Walk step is  $\Theta(\log n)$ , this is also true for the Cauchy Walk. I suggest this explains why the Cauchy Walk is so efficient (indeed, the proof for the competitiveness of the Cauchy Walk uses only these properties, and would therefore hold for any Markovian process for which they are true). Indeed, for a Markovian searcher, sampling uniformly at each step leads to a hitting time  $\Theta(n/|B(S)|)$  which is optimal. When agents have a constant speed, however, going at each time at a position chosen uniformly at random would be very costly for them, as each step needs  $\Theta(\sqrt{n})$  time to be executed. The Cauchy Walk, that imitates at time  $m$  the behaviour of the uniform distribution on a ball of radius  $\tilde{\Theta}(m)$ , while keeping the expected time of a jump to  $\Theta(\log n)$ , balances well the tradeoff between local exploitation and global exploration. (Hills et al., 2015)

Specifically, I provide first the following lower bound on the probability density function of the Cauchy Flight  $Z(m)$ .

<sup>2</sup>I believe a similar statement would hold for the Lévy Flight of exponent  $\mu \in (1, 3)$ :  $Z^\mu(m)$  is more or less uniform in the ball of radius  $m^{\frac{1}{\mu-1}}$ . See also (Comtet and Majumdar, 2005).



**Lemma 3.2.** *For any constant  $\alpha > 0$ , there exists a constant  $c > 0$  such that for any integer  $m \in [1, \alpha \ell_{max}]$ , and any  $x \in \mathbb{R}^2$ , with  $\|x\| \leq m$ , we have  $\mathbf{p}^{Z(m)}(x) \geq \frac{c}{m^2}$ .*

I then complement this lower bound with an upper bound on the probability that  $Z(m)$  detects  $x$ .

**Lemma 3.3.** *For any constant  $\alpha > 0$ , there exists a constant  $c' > 0$  such that, for any integer  $m \in [2, \alpha \ell_{max}]$  and any  $x \in \mathbb{R}^2$ , we have  $\mathbb{P}(\|Z(m) - x\| \leq 1) \leq \frac{c' \log^2 m}{m^2}$ .*

**The Cauchy walk is polylogarithmic-competitive w.r.t. any diameter.** We next study the intermittent Cauchy process  $X^{\text{cauchy}}$  on the torus  $\mathbb{T}_n$ , and use the lower and upper bounds in Lemmas 3.2 and 3.3 to establish an upper bound on its detection time.

**Theorem 3.4.** *Consider the Cauchy walk  $X^{\text{cauchy}}$  on the torus  $\mathbb{T}_n$  starting at 0, with cut-off  $\ell_{max} = \sqrt{n}/2$ . The detection time of  $X^{\text{cauchy}}$  with respect to a target  $S$  of diameter  $D$  is  $t_{\text{detect}}^{X^{\text{cauchy}}}(n, S) = O\left(\frac{n \log^3 n}{D}\right)$ . Consequently, for every  $1 \leq D \leq \frac{\sqrt{n}}{2}$ , the competitiveness of  $X^{\text{cauchy}}$  is*

$$\text{Comp}^X(n, D) = O(\log^3 n).$$

**Lower bounds.** I prove that for  $1 < \mu < 2$ , the competitiveness of the intermittent Lévy Walk  $X^\mu$  is large with respect to small diameter targets, and that for  $2 < \mu < 3$ , the competitiveness is large with respect to large diameter targets.

**Theorem 3.5.** *Let  $\mu \in (1, 2)$  and  $D \in [1, \sqrt{n}/2]$ . Write  $\mu = 2 - \varepsilon$ . The competitiveness of the Lévy Walk  $X^\mu$  w.r.t.  $D$  is:*

$$\text{Comp}(X^\mu, D) = \Omega(n^{\varepsilon/2}/D). \quad (3.4)$$

In a few words, in these cases, the average step length is already polynomial in  $n$ , implying that the process is slow at finding small targets. Note that, in particular, with respect to constant diameters, the competitiveness is  $n^{\varepsilon/2}$ . The proof of this theorem, and of the following one, are given in Section 3.6.2

**Theorem 3.6.** *Let  $\mu \in [2, 3]$  and  $D \in [2, \sqrt{n}/6 - 1]$ . Write  $\mu = 2 + \varepsilon$  where  $0 \leq \varepsilon \leq 1$ . The competitiveness of  $X^\mu$  with respect to  $D$  is:*

$$\text{Comp}(X^\mu, D) = \begin{cases} \Omega(\log D) & \text{if } \mu = 2, \\ \Omega(D^\varepsilon) & \text{if } \mu = 2 + \varepsilon, \text{ where } 0 < \varepsilon < 1, \\ \Omega\left(\frac{D}{\log D}\right) & \text{if } \mu = 3. \end{cases}$$

These lower bounds stem from the fact that such processes take a long time to reach faraway locations. Hence, in comparison to the optimal strategy, these strategies are slow at finding large faraway targets. Note that taking  $D = n^\delta$  for some  $0 < \delta < 1/2$  would give polynomial competitiveness for Lévy walks with  $2 < \mu \leq 3$ .

Altogether, the intermittent Cauchy walk is  $O(\log^3 n)$ -competitiveness w.r.t any diameter, while for any Lévy walk with  $\mu \neq 2$ , there is a large range of diameters for which the competitiveness is polynomial in  $n$ .

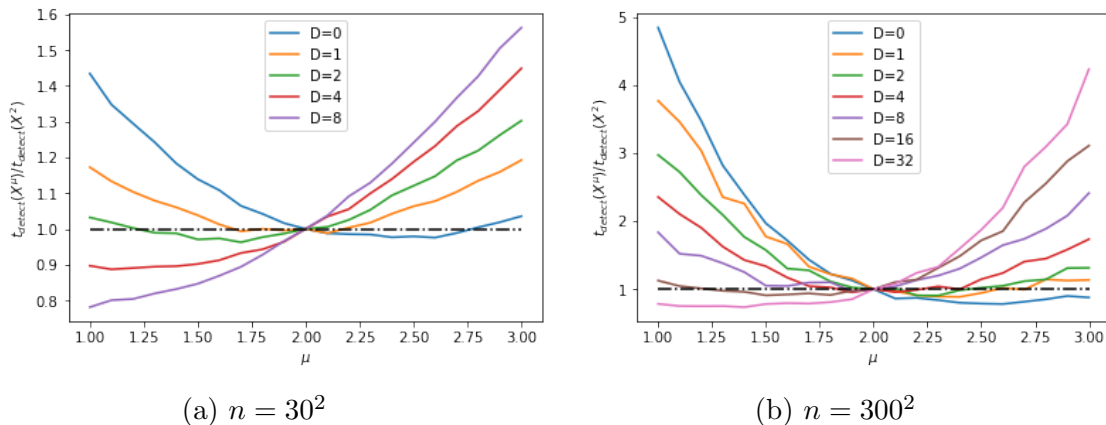


Figure 3.2: Detection time of the Lévy Walk  $X^\mu$  (cut-off  $\ell_{max} = \sqrt{n}/2$ ) on  $\mathbb{T}_n$ , as a function of  $\mu$ , for various ball targets of diameter  $D$ , with detection range equal to 1. To facilitate comparisons, the detection time of the Cauchy Walk  $X^2$  is scaled to 1.

### 3.3.2 Simulations

I used computer simulations to verify, first, if the results could be seen even for small values of  $n$  and  $D$ , and second to measure the influence of considering that the agent has a probability  $p > 0$  to detect while moving. As in the theoretical scenario, the search happens in  $\mathbb{T}_n$  and the initial location of the searcher is chosen uniformly at random in the torus. The searcher performs a Lévy walk with step-lengths from a discrete Pareto distribution of exponent  $\mu$ , truncated at  $\ell_{max} = \sqrt{n}/2$ . The detection mode depends on the scenario.

**Do the results stand for small  $n$  values?** This setting is identical to that of the theoretical results: the detection can only happen between steps, and the target is detected if it is at distance at most 1 from the searcher. In Fig. 3.2, we can see that even for a relatively small  $n$ , this intermittent setting advantages a Cauchy Walker. When  $n$  is multiplied by 100 (hence,  $\sqrt{n}/2$  is multiplied by 5), the difference is even more pronounced (note that the lower values attained in both graphs are around 0.8, while the upper values are, respectively, 1.6 and 5).

Biologically speaking, it is doubtful whether very large values of  $D$  would be relevant. However, the simulations indicate that even for plausible target diameters (e.g.,  $D \in [1, 4]$ ), the Cauchy Walk is favoured. This tends to indicate that, if targets have very similar sizes, a Brownian motion could be slightly more advantageous than a Cauchy Walk. However, as target sizes become more diverse, a Cauchy Walk becomes increasingly better.

**On the impact of weak detection: intermittent vs. non-intermittent.** The optimal lower bound  $\text{opt}(n, D) = \Theta(n/D)$  holds in the non-intermittent setting, and since the intermittent setting is the more restrictive scenario of the two, the upper bound  $O(\log^3 n)$  I gave for the Cauchy Walk also holds in the non-intermittent scenario. Furthermore, the lower bounds given for  $2 \leq \mu \leq 3$  also hold in the non-intermittent setting, as they depend only on the time needed to go far. For  $1 < \mu < 2$ , however, these bounds do not hold, as now steps can be halted mid-way.

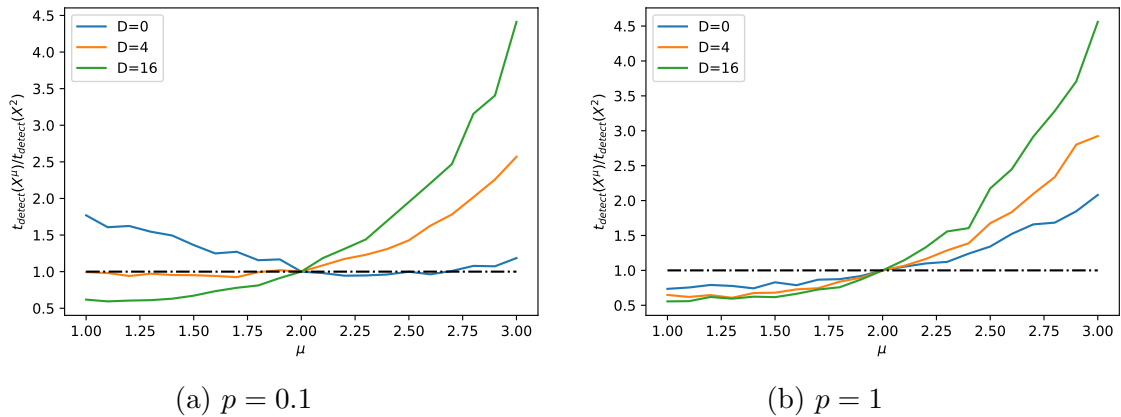


Figure 3.3: On the torus  $\mathbb{T}_n$  with  $n = 300^2$ , we study the influence of  $p$  on detection time. The walker has probability  $p$  to detect  $S$ , a ball of diameter  $D$ , for each unit of time spent within a distance 1 of the target. In-between steps, the detection probability is 1.

To study the influence of detection while moving on the results, I simulated the search with the following mode of detection. While moving, for each unit of time that the searcher spends at distance at most 1 from the target, it detects it with probability  $p$ . In-between steps, the target is detected (w.p. 1) if it is at distance 1 from the searcher.

I compared the competitiveness of the Cauchy walk to those of other Lévy walks in the semi-intermittent setting (Fig. 3.3(a)) and the fully non-intermittent setting (Fig. 3.3(b)). As expected, we see that with respect to  $\mu > 2$  and large targets, the gap remains very large.<sup>3</sup> Also as expected, in the fully non-intermittent detection mode, the best strategy consists in doing ballistic steps (this is in line with (Viswanathan et al., 1999), since searching for a single target in a large torus can be seen as searching for a sparse target destroyed upon encounter). Note, however, that the performances of all Lévy Walks with  $\mu \in (1, 2)$  are comparable, a feature also encountered in (Clementi et al., 2020) in the context of multiple Lévy Walkers searching in the plane, in the non-intermittent regime of detection.

In the semi-intermittent setting, the performances of the Lévy Walks with exponent  $\mu < 2$  also become significantly better than in the intermittent regime. The value of  $p$ , however, was not taken to represent a specific biological process. Thus, it would be of interest to simulate a more realistic model, where detection (range and, possibly, probability) would depend on the speed of the searcher.

### 3.3.3 Implications

As implied by Eq. (3.2), all connected targets of a given diameter share a common lower bound for their detection time. More surprisingly, Theorem 3.4 implies that such targets are found by roughly this time by the Cauchy strategy. These results

<sup>3</sup>Note, however, that all these Lévy Walks do not have the same expected step-length, which is an advantage to the Lévy Walks with  $\mu \leq 2$  in the non-intermittent setting. A way of improving the Lévy Walks with parameter  $\mu > 2$ , then, would be to allow them to do steps only bigger than some  $\ell_{min}$ .

suggest that, at least asymptotically, the right parameter to consider is indeed the diameter of the target and not, e.g., its area. This seems rather surprising, as, in contrast to the non-intermittent searcher, crossing the target’s boundary by an intermittent searcher does not suffice for detection. Hence, for example, a ball-shaped target appears to be, at least at a first glance, significantly more susceptible for detection than its one dimensional perimeter. A consequence of this insight is that a large prey aiming to hide from an efficient searcher would benefit by organizing itself in a bulging shape that minimizes its diameter.

Perhaps the most impressive feature of the Cauchy walk, though, is that it manages to perform almost optimally for all target shapes and sizes, without the need for any *a priori* information about the target. Furthermore, contrarily to the simple deterministic algorithm I have built in the lower bound (see Section 3.6), the Cauchy walk will find efficiently a target even if it appears after some time, or if it moves.

Animals might have more navigational abilities than the minimal ones assumed here, as they may have memory, or may adapt their search to their environment, and, as a consequence, may behave in a more complicated manner. However, the optimality of the Cauchy Walk shows that the gain incurred by such abilities is relatively modest – i.e., not polynomial. Of course, even constant improvements may be critical for biological organisms, but this improvement comes with the energy cost associated with higher perceptual and cognitive abilities.

Early studies on white crappies (O’Brien et al., 1989) and plovers (Pienkowski, 1983), who are saltatory foragers, have shown that they predate first on larger preys. While the interpretations of the authors is that these animals ignore at first small, but more common, preys, and progressively become more acceptive of smaller preys, it seems also possible that smaller targets are simply not detected as quickly as larger targets. Hence, it is not clear whether or not there really is a behavioural mode switch when no large targets are found, especially as the Cauchy Walk (which, in this case, may model the eye path of plovers when they are paused and try to detect prey) performs quasi optimally.

In conclusion, the theorems presented here suggest to experimentally study the correlation between the distribution of target sizes (Sims et al., 2008) and the detection ability of animals presenting Lévy Patterns. Specifically, I expect that when detection abilities are low (as, e.g., in some insects (Reynolds, 2015a)) and target vary in sizes, Cauchy Patterns of movement will be encountered. Furthermore, if detection abilities are improved, then Lévy Patterns could be observed, of exponent  $\mu \in (1, 2)$ , with exponent going increasingly closer to 1 as detection becomes more efficient.

### 3.4 Semi-formal proof of the upper bound

In this section, I provide the outline of the proof of Theorem 3.4, i.e., we prove that the detection time of  $X = X^{\text{cauchy}}$  with respect to a target  $S$  of diameter  $D$  is  $t_{\text{detect}}^{X^{\text{cauchy}}}(n, S) = O\left(\frac{n \log^3 n}{D}\right)$ , for any diameter  $D \leq \sqrt{n}/2$ . More precisely, the formal proof of Theorem assuming Lemmas 3.2 and 3.3 is given, and before that, a sketch of the proof of these lemmas is drawn.

Theorem 3.4 concerns the Cauchy walk on the two-dimensional torus. As the one-

dimensional Cauchy walk is relatively well understood (Adler et al., 2003; Lomholt et al., 2008; Zaburdaev et al., 2015), it is tempting to analyze the two-dimensional walk by projecting it on the two axes and using the projected walks. However, this approach needs to somehow handle the fact that these projections are not independent of each other. As A. Korman and I could not find a way to overcome this dependency issue, let us prove Theorem 3.4 by extending the one-dimensional arguments of (Adler et al., 2003) to the dimension 2.

Let  $Z$  be the extension of  $X$  to  $\mathbb{R}^2$ . Given a set  $S$ , recall that  $B(S)$  is the set of points at distance at most 1 from  $S$ , and that  $Z(m)$  detects  $S$  if and only if  $Z(m) \in B(S)$ .

We have seen in Chapter 2 the general technique for linking the hitting time of  $B(S)$  to the pointwise probabilities of the process. If, for some  $m_0$ ,

$$N = \inf_{x \in \mathbb{T}_n} \sum_{m=m_0}^{2m_0} \mathbb{P}(Z^x(m) \in B(S))$$

and

$$O = \sup_{z_0 \in B(S)} \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in B(S)),$$

then, since the average time of a jump is  $\tau = \Theta(\log n)$ , we have, by Remark 2.9,

$$t_{detect}(S) = t_{hit}(B(S)) = O(m_0 \cdot ON^{-1} \cdot \log n). \quad (3.5)$$

We then use Lemmas 3.2 and 3.3, which are formally proved in Section 3.7, to obtain bounds on  $N$  and  $O$ . Let us give here a sketch of the proofs of both lemmas.

### 3.4.1 Sketch proof of Lemma 3.2

Recall that in Chapter 2, we proved that, since the step-length distribution of our process is non-increasing,  $Z(m)$  is non-increasing, and its p.d.f. verifies:

$$\mathbf{p}^{Z(m)}(x) \leq \frac{1}{\pi \|x\|^2}. \quad (3.6)$$

Using the monotonicity property, the lower bound stated in Lemma 3.2 follows once we prove that with at least some constant probability, the process at step  $m$  belongs to the ring  $\{x \mid \|x\| \in [m, cm]\}$  for some constant  $c > 1$ . This is because the area of this ring is roughly  $m^2$ , and each point in it is further from 0 than  $x$ , and hence, by monotonicity, less likely to be visited at step  $m$ . In order to establish the lower bound on the probability to be in the ring at step  $m$ , we first prove that with some constant probability, at some step before  $m$ , the walk goes to a distance at least  $2m$ .

Next, conditioning on that event, we prove that with constant probability, the walk does not get much further away, i.e., it stays at a distance of at least  $m$ . To prove the latter claim, we use Claim 2.12 that implies that the distance travelled in  $m$  steps is governed by  $\sqrt{m}\sigma'$  where  $\sigma'$  is the standard deviation of the projected step-lengths. Here the standard deviation is too large (roughly  $\ell_{max} = \sqrt{n}$ ), however, we can reduce it by conditioning on the event that none of the  $m$  step-lengths are significantly larger than  $m$ , which occurs with constant probability. Finally, we prove

that by taking a sufficiently large constant  $c$ , it can be guaranteed that with a large constant probability, the walk at step  $m$  is at most at distance  $cm$ . Making sure that all these constant probability events happen simultaneously, we then establish the desired constant lower bound on the probability to be in the aforementioned ring at step  $m$ .

### 3.4.2 Sketch proof of Lemma 3.3

For the proof of the upper bound in Lemma 3.3, we first remark that because of the monotonicity property, it is sufficient to prove that the probability to detect 0 at step  $m$  is small, i.e., that

$$\mathbb{P}(\|Z(m)\| \leq 1) = O\left(\frac{\log^2 m}{m^2}\right).$$

Intuitively, to establish this, we first argue that with high probability in  $m$ , at some step before step  $m$ , the process has gone to a distance  $d = \Omega\left(\frac{m}{\log m}\right)$ . By Eq. (3.6), the probability density function at any point in  $B(0)$  would then be at most  $O\left(\frac{1}{d^2}\right)$ , which is the desired bound.

### 3.4.3 Formal proof of the theorem, assuming the lemmas

**Replacing  $S$  by a “cylinder”  $S'$ .** Given the connected set  $S$  of diameter  $D \geq 1$ , we first construct a subset  $S'$ , containing  $\Theta(D)$  isolated points of  $S$  that stretch over distance of roughly  $D$ , as follows. Take two points  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  in  $S$  that are at distance  $D$  from each other, so that  $\max\{|u_1 - v_1|, |u_2 - v_2|\} \geq D/2$ . Let us assume, without loss of generality, that  $v_1 - u_1 \geq D/2$ . Since  $S$  is connected, for every  $z \in [u_1, v_1]$ , there exists  $\phi(z)$  such that  $(z, \phi(z)) \in S$ . Let  $d = \lceil v_1 - u_1 \rceil = \Theta(D)$ . For integer  $i \in \{0, 1, \dots, \lfloor d \rfloor\}$ , define

$$s(i) = (u_1 + i, \phi(u_1 + i)),$$

and let  $S' = \{s(i) \mid i = 0, 1, \dots, \lfloor d \rfloor\}$ . Note that  $|S'| = \Theta(D)$ . Since  $S' \subseteq S$ , an upper bound on the detecting time of  $S'$  is an upper bound on the detecting time of  $S$ . It is therefore sufficient to restrict attention to  $S'$  and upper bound its detecting time. For that purpose we need to bound the time until visiting a point in  $B(S')$ , the set of points of distance at most 1 from  $S'$ . Note that the area of  $B(S')$  is  $|B(S')| = \Omega(D)$ . We also remark, that although  $B(S')$  may not be connected, it may help the reader to imagine  $B(S')$  as a horizontal cylinder of length  $\Theta(D)$  and radius 1, i.e., to consider that  $\phi(u_1 + i)$  does not depend on  $i$ . Indeed, we will not require any condition on the  $y$ -coordinates of the  $s(i)$ 's.

Now, let us take  $m_0 = \sqrt{n}$ . Note that  $2m_0 \leq \alpha \ell_{max}$  for  $\alpha = 4$ . Hence, we will be able to use Lemmas 3.2 and 3.3 (with  $\ell_{max} = \sqrt{n}/2$ ). Define

$$N' = \inf_{x \in \mathbb{T}_n} \sum_{m=m_0}^{2m_0} \mathbb{P}(Z^x(m) \in B(S')),$$

and

$$O' = \sup_{z_0 \in B(S)} \sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in B(S')).$$

By Remark 2.9, we have:

$$t_{\text{detect}}(S) \leq t_{\text{detect}}(S') = O(m_0 \cdot O' N'^{-1} \cdot \log n). \quad (3.7)$$

**Lower bound on  $N'$ .** Let us begin with the lower bound. With this setting of  $m_0$ , any  $x \in B(S') \subseteq B(\mathbb{T}_n) \subseteq [-\sqrt{n}/2 - 1, \sqrt{n}/2 + 1]^2$  trivially satisfies  $\|x\| \leq m$ , for any  $m \geq m_0 + 1$ , and we can apply Lemma 3.2 to get:

$$\sum_{m=m_0+1}^{2m_0} \mathbb{P}(Z(m) \in B(S')) = \sum_{m=m_0+1}^{2m_0} \int_{x \in B(S')} p_m^Z(x) dx \geq \sum_{m=m_0+1}^{2m_0} \frac{c}{m^2} |B(S')| = \Omega\left(\frac{D}{\sqrt{n}}\right).$$

Note that, since this is true for any connected set  $S$  of diameter  $D$  starting from 0, it is then also true starting from a node  $x \in \mathbb{T}_n$  (by translation of the location of the target). Hence, we have  $N' = \Omega\left(\frac{D}{\sqrt{n}}\right)$ .

**Upper bound on  $O'$ .** Let  $z_0 \in B(S')$ . We have:

$$\sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in B(S')) \leq 2 + \sum_{m=2}^{m_0} \mathbb{P}(Z^{z_0}(m) \in B(S')). \quad (3.8)$$

Clearly, the probability density function  $\mathbf{p}^{Z^{z_0}(m)}$  of  $Z^{z_0}(m)$  is obtained by a translation from  $\mathbf{p}^{Z(m)}$ . Thus, by the monotonicity property Eq. (3.6), we have for any  $y \in \mathbb{R}^2$ ,  $\mathbf{p}^{Z^{z_0}(m)}(y) \leq \frac{1}{\|y-z\|^2}$ . In particular, for  $y$  such that  $\|y - z_0\| \geq 2$ ,

$$\mathbb{P}(Z^{z_0}(m) \in B(y)) \leq \frac{1}{(\|y - z_0\| - 1)^2}, \quad (3.9)$$

since every  $w \in B(y)$  satisfies  $\|w - z_0\| \geq \|y - z_0\| - 1 \geq 0$ .

Next, as  $z_0 \in B(S')$ , consider an index  $i_z \in \{0, \dots, d-1\}$  for which  $z_0 \in B(s(i_z))$ . Let

$$r_m = \frac{m}{\sqrt{c} \log m}$$

with  $c$  being the constant  $c'$  mentioned in Lemma 3.3. To exploit Eq. (3.9), we define

$$I = \{i \in \{0, \dots, d-1\} \mid |s(i)_1 - s(i_z)_1| = |i - i_z| \leq r_m + 2\},$$

and  $I^c = \{0, \dots, d-1\} \setminus I$ . We proceed with the following decomposition:

$$\mathbb{P}(Z^{z_0}(m) \in B(S')) \leq \sum_{i \in I} \mathbb{P}(Z^{z_0}(m) \in B(s(i))) + \sum_{i \in I^c} \mathbb{P}(Z^{z_0}(m) \in B(s(i))). \quad (3.10)$$

By construction,  $|I| \leq 2(r_m + 2) + 1$ . Hence, using Lemma 3.3, the first sum in the r.h.s of Eq. (3.10) is at most:

$$\sum_{i \in I} \mathbb{P}(Z^{z_0}(m) \in B(s(i))) \leq \frac{|I|}{r_m^2} = O\left(\frac{1}{r_m}\right).$$

Next, we aim to upper bound the sum on  $I^c$ . By the triangle inequality, for any  $i \in I^c$ , we have  $\|s(i) - z_0\| \geq \|s(i) - s(i_z)\| - 1 \geq |i - i_z| - 1 > 1$ . Hence, by Eq. (3.9), we get:

$$\begin{aligned} \sum_{i \in I^c} \mathbb{P}(Z^{z_0}(m) \in B(s(i))) &\leq \sum_{i \in I^c} \frac{1}{(\|s(i) - z_0\| - 1)^2} \\ &\leq \sum_{i \in I^c} \frac{1}{(|i - i_z| - 2)^2} \\ &\leq \sum_{k \in \mathbb{Z}, |k| \geq \lceil r_m \rceil} \frac{1}{k^2} = O\left(\frac{1}{r_m}\right), \end{aligned}$$

where we used in the last line that  $i \in I^c \subset \{i_z + k \mid k \in \mathbb{Z} \text{ and } |k| > r_m + 2\}$ . Thus, by Eq. (3.10):

$$\mathbb{P}(Z^{z_0}(m) \in B(S')) = O\left(\frac{1}{r_m}\right),$$

which stands for any  $z_0 \in B(S')$ . Plugging this in Eq. (3.8), together with the definition  $r_m = \frac{m}{\sqrt{c \log m}}$ , and the fact that  $m_0 = \sqrt{n}$ , we get:

$$\sum_{m=0}^{m_0} \mathbb{P}(Z^{z_0}(m) \in B(S')) = 2 + O\left(\sum_{m=2}^{m_0} \frac{\log m}{m}\right) = O(\log^2 n).$$

Since this is true for any starting point  $z_0 \in B(S')$ , we have  $O' = O(\log^2 n)$ .

**Conclusion.** By the previous bounds on  $N'$  and  $O'$ , Eq. (3.7) becomes:

$$t_{detect}(S) = O\left(\sqrt{n} \cdot \log^2 n \frac{\sqrt{n}}{D} \cdot \log n\right) = O\left(\frac{n \log^3 n}{D}\right).$$

This concludes the proof of Theorem 3.4 assuming Lemmas 3.2 and 3.3.  $\square$

## 3.5 Preliminaries to lower and upper bounds proofs

### 3.5.1 Expectations and variances of step-lengths

**Claim 3.7.** Consider the Lévy walk  $X^\mu$  with maximal step length  $\ell_{max}$ . The average length of a step (and hence the average time to take a step) is

$$\tau = \begin{cases} \Theta(\ell_{max}^{2-\mu}) & \text{if } \mu \in (1, 2) \\ \Theta(\log \ell_{max}) & \text{if } \mu = 2 \\ \Theta(1) & \text{if } \mu \in (2, 3] \end{cases}, \quad (3.11)$$

where the asymptotics are with respect to  $\ell_{max}$ , and the variance  $\sigma^2$  and second moment  $M$  of a step-length are

$$\sigma^2 = \Theta(M) = \begin{cases} \Theta(\ell_{max}^{3-\mu}) & \text{if } \mu \in (1, 3) \\ \Theta(\log \ell_{max}) & \text{if } \mu = 3 \end{cases}. \quad (3.12)$$



*Proof.* Given the definition of  $\mathbf{p}_{\mu, \ell_{max}}$ , the expected step-length is

$$\tau = \int_0^1 a_{\mu, \ell_{max}} \ell d\ell + \int_1^{\ell_{max}} a_{\mu, \ell_{max}} \ell^{1-\mu} d\ell.$$

The first term is  $\frac{a_{\mu, \ell_{max}}}{2}$ , a constant with respect to  $\ell_{max}$ , as  $a_{\mu, \ell_{max}} \in [\frac{\mu-1}{\mu}, 1]$ . The second term is  $\Theta(\ell_{max}^{2-\mu})$  if  $\mu \neq 2$ , and  $\Theta(\log \ell_{max})$  if  $\mu = 2$ . The second moment  $M$  is computed likewise:

$$M = \int_0^{\ell_{max}} \ell^2 p^\mu(\ell) d\ell = \int_0^1 a_{\mu, \ell_{max}} \ell^2 d\ell + \int_1^{\ell_{max}} a_{\mu, \ell_{max}} \ell^{2-\mu} d\ell.$$

We have  $\int_0^1 a_{\mu, \ell_{max}} \ell^2 d\ell = \frac{a}{3}$  for the first term, and for the second term

$$\int_1^{\ell_{max}} \ell^{2-\mu} d\ell = \begin{cases} \Theta(\ell_{max}^{3-\mu}) & \text{if } \mu < 3 \\ \Theta(\log(\ell_{max})) & \text{if } \mu = 3 \end{cases}.$$

Now remark that  $\tau^2 = o(M)$ , so that  $\sigma^2 = \Theta(M)$ .  $\square$

### 3.5.2 Projections of 2-dimensional Lévy walks are also Lévy

Consider a Lévy walk  $Z^\mu$  with parameter  $\mu$  on  $\mathbb{R}^2$ , that has maximal step length  $\ell_{max}$  (including the case  $\ell_{max} = \infty$ ). The goal of this section is to prove that its projection on each of the axes is also a Lévy walk with parameter  $\mu$ . The conservation of the power-law distribution under projection was also established in (Sims et al., 2008). I nevertheless provide a proof here, for completeness purposes, and also because (Sims et al., 2008) does not examine the case  $\ell_{max} < \infty$ .

Because the walk is unbiased, we may, without loss of generality, consider only the projection  $Z_1^\mu$  on the  $x$ -axis. Hence, we aim to prove the following.

**Theorem 3.8.** *The projection  $Z_1^\mu$  of  $Z^\mu$  is a Lévy walk on  $\mathbb{R}$  with parameter  $\mu$ , in the sense that the p.d.f. of the step-lengths of  $X_1^\mu$  is  $p(\ell) \sim 1/\ell^\mu$ , for  $\ell \in [1, \frac{\ell_{max}}{2}]$ . Furthermore, the variance of  $X_1^\mu$  is*

$$\sigma'^2 = \begin{cases} \Theta(\ell_{max}^{3-\mu}) & \text{if } \mu \in (1, 3) \\ \Theta(\log \ell_{max}) & \text{if } \mu = 3 \end{cases}.$$

*Proof.* It is clear that  $Z_1^\mu$  is also a random walk that moves incrementally, with the increments between  $Z_1^\mu(m)$  and  $Z_1^\mu(m+1)$  being the projection  $Z_1(m+1)$  of the chosen 2-dimensional vector  $V(m+1) = Z^\mu(m+1) - Z^\mu(m)$ . These projections are i.i.d. variables as the vectors  $(V(m))_m$  are i.i.d. variables, and their signs are  $\pm$  with equal probability. Hence, all that needs to be verified is that  $l_1 := |V_1(1)|$  has a Lévy distribution with parameter  $\mu$ .

Let  $V$  be one step-length drawn according to a Lévy distribution  $p^\mu$ . Recall that

$$p^\mu(\ell) = \begin{cases} a_\mu & \text{if } \ell \leq 1 \\ a_\mu \ell^{-\mu} & \text{if } \ell \in [1, \ell_{max}] \\ 0 & \text{if } \ell \geq \ell_{max} \end{cases},$$

where  $a_\mu$  is the normalization factor, with  $a_\mu = \frac{1}{1 + \int_1^{\ell_{max}} \ell^{-\mu} d\ell} = \frac{1}{1 + \frac{1 - \ell_{max}^{1-\mu}}{\mu-1}} \in [1 - \frac{1}{\mu}]$ .

Hence the distribution of  $V = (V_1, V_2) \in \mathbb{R}^2$  is

$$p^V(x) = \frac{1}{2\pi} \frac{1}{\|x\|} p^\mu(\|x\|) = \begin{cases} \frac{a_\mu}{2\pi} \|x\|^{-1} & \text{if } \|x\| \leq 1 \\ \frac{a_\mu}{2\pi} \|x\|^{-\mu-1} & \text{if } \|x\| \in [1, \ell_{max}) \\ 0 & \text{if } \|x\| \geq \ell_{max} \end{cases} \quad (3.13)$$

For  $x_1 \in (0, \ell_{max})$ , we have

$$\begin{aligned} p^{h^1}(x_1) &= 2 \int_0^{\sqrt{\ell_{max}^2 - x_1^2}} p^V(x_1, x_2) dx_2 \\ &= \frac{2a_\mu}{2\pi} \int_0^{\sqrt{\ell_{max}^2 - x_1^2}} \mathbf{1}_{\|x\| < 1} \frac{1}{\|x\|} + \mathbf{1}_{\|x\| \geq 1} \frac{1}{\|x\|^{1+\mu}} dx_2, \end{aligned}$$

where  $x = (x_1, x_2)$ . If  $|x_1| \geq 1$ , then  $\|x\| \geq 1$  for any  $x_2 \in \mathbb{R}$ , so that

$$\begin{aligned} p^{h^1}(x_1) &= \frac{a_\mu}{\pi} \int_0^{\sqrt{\ell_{max}^2 - x_1^2}} \frac{1}{(x_1^2 + x_2^2)^{\frac{1+\mu}{2}}} dx_2 \\ &= \frac{a_\mu}{\pi} \frac{1}{x_1^\mu} I(x_1), \end{aligned}$$

where

$$I(x_1) := \int_0^{\sqrt{\frac{\ell_{max}^2}{x_1^2} - 1}} \frac{1}{(1 + y^2)^{\frac{1+\mu}{2}}} dy.$$

For any  $x_1 \in (1, \ell_{max})$ , we have  $I(x_1) \leq \int_0^\infty \frac{1}{(1+y^2)^{\frac{1+\mu}{2}}} dy = O(1)$  since  $\frac{1}{(1+y^2)^{\frac{1+\mu}{2}}} = \Theta(y^{-\mu})$ , for large  $y$ , and this function of  $y$  is integrable as  $\mu > 1$ . Furthermore, if  $|x_1| \leq \ell_{max}/2$ , we have  $I(x_1) \geq \int_0^1 \frac{1}{(1+y^2)^{\frac{1+\mu}{2}}} dy$  which is a positive constant. Hence, if  $|x_1| \in (1, \ell_{max}/2)$ , we have

$$p^{h^1}(x_1) = \Theta\left(\frac{1}{x_1^\mu}\right), \quad (3.14)$$

and for  $\ell_{max}/2 \leq x_1 \leq \ell_{max}$ , we have

$$p^{h^1}(x_1) = O\left(\frac{1}{x_1^\mu}\right). \quad (3.15)$$

Hence, the projection of the Lévy walk on the axes are Lévy-like, in the sense that their step-lengths distributions generally follow a power-law of same exponent  $\mu$ . The expected length, second moment and variance of one projected step are computed as in Claim 3.7. Indeed write, for  $i \in \{1, 2\}$ ,

$$\int_0^{\ell_{max}} x_1^i p^{h^1}(x_1) dx_1 = \Theta\left(\int_0^1 x_1^i p^{h^1}(x_1) dx_1 + \int_1^{\ell_{max}/2} x_1^{i-\mu} dx_1 + \int_{\ell_{max}/2}^{\ell_{max}} x_1^i p^{h^1}(x_1) dx_1\right).$$

We have  $\int_0^1 x_1^i p^{h^1}(x_1) dx_1 \leq 1$ . Also, it is easy to verify from Eq. (3.14) and (3.15) that the third term is dominated by the second term, which in turn, is  $\Theta(\int_1^{\ell_{max}} x_1^{i-\mu} dx_1)$ . Hence, the expected length, second moment and variance of one projected step are of the same order as those of the non-projected steps given by Claim 3.7, which concludes the proof of Theorem 3.8.  $\square$

## 3.6 Proofs of the lower bounds

### 3.6.1 A general lower bound

Let us begin with a general proposition that holds for any search process  $X$  on the torus whose speed is constant (i.e., it takes  $O(\ell)$  units of time to do a ballistic step of length  $\ell$ ). We may assume without loss of generality that the speed is normalized to 1.

I next define a quantity, termed  $T_d$ , which will be used to lower bound the time needed to detect an extended target  $B(S)$  at distance  $d$  or more. Formally, we distinguish between two cases, according to the given process  $X$ .

- If  $X$  is an intermittent random walk, we let  $T_d$  be the expected time needed before the end point of a step is at distance at least  $d$  from the initial location.
- Otherwise, we simply define  $T_d = d$ .

Note that Claim 2.12 from Chapter 2 gives a lower bound on  $T_d$  for an intermittent random walk.

**Claim 3.9.** Let  $X$  be any search process on the torus. Consider any target  $S$  of diameter  $D < \sqrt{n}/6 - 1$ . The expected time to detect  $S$  is  $\Omega(n \frac{T_D}{D^2})$ .

The proof of Claim 3.9 is illustrated in Figure 3.1

*Proof.* Consider a target  $S$  of diameter  $D$  and of an arbitrary shape. Instead of considering that  $S$  is fixed and that the initial location  $X(0)$  is chosen u.a.r, we may assume without loss of generality that  $X(0)$  is fixed, say at the origin, and that the center of mass  $u^*$  of  $S$  is chosen uniformly at random in the torus.

Let us first construct an  $s \times s$  grid, where  $s = \lfloor \sqrt{n}/(3D + 2) \rfloor$ . Note that since  $D < \sqrt{n}/6 - 1$ , we have  $s \geq 2$ . To make the grid symmetric, we let the distance between two neighboring points be precisely  $\sqrt{n}/s$ . We next align the grid so that  $u^*$  is a point of the grid, and construct a disc of radius  $D + 1$  around each node. Note that the number of discs is  $M = s^2 = \Omega(n/D^2)$ , and that the distance between any two discs is at least  $D$ . See Figure 1(b). Furthermore, note that the disc  $U^*$  corresponding to  $u^*$  fully contains the extended target  $B(S)$ . Let us therefore lower bound the time until visiting  $U^*$  for the first time. This will serve as the desired lower bound for detecting  $S$ .

Assume that the information about the collection of discs is given to the searcher. We may assume this, since it can only decrease the best detection time. Because the location of  $S$  is chosen u.a.r in the torus, from the perspective of the searcher, each of the discs has an equal probability to be  $U^*$ . It follows that with probability  $1/2$ , at least half of the discs are visited, before the searcher visits  $U^*$ . Since the discs are separated by distance of at least  $D$ , we immediately get that the expected time until visiting  $U^*$  is  $\Omega(MD) = \Omega(n/D)$ , which is the desired claim when  $X$  is not an intermittent random walk (and hence  $T_D = D$ ).

Let us next consider the case that  $X$  is an intermittent random walk. The arguments are similar, yet slightly more subtle. We aim to lower bound the time until visiting  $U^*$  for the first time, where by visiting a disc, I mean that the end of a ballistic step of  $X$  is in that disc. For this purpose, we may assume that the process terminates when it visits  $U^*$ . Let  $U_1, U_2, \dots$  denote the newly visited discs, in order

of visitation, with all the  $U_i$  distinct. Let  $A_i$  be the event that  $U^* \notin \{U_1, \dots, U_i\}$ . Note that  $\mathbb{P}(A_i) = 1 - \frac{i}{M}$ . Let  $t_i$  denote the time from visiting  $U_i$  (for the first time) until visiting  $U_{i+1}$  (for the first time), in the event that  $A_i$  occurs. If the event  $A_i$  does not occur, let us say that  $t_i = 0$ . The time before visiting  $U^*$  can therefore be written as  $\sum_{i=1}^{M-1} t_i$ . Furthermore, we have  $\mathbb{E}(t_i) = \mathbb{E}(t_i | A_i)\mathbb{P}(A_i)$ . Hence, the expected time before visiting  $U^*$  is:

$$\sum_{i=1}^{M-1} \mathbb{E}(t_i | A_i)\mathbb{P}(A_i).$$

Now recall that  $X$  is an intermittent Markovian process, and that  $A_i$  corresponds to an event that is relevant up to (and including) the detection of  $U_i$ . Hence,  $\mathbb{E}(t_i | A_i)$  is lower bounded by the minimal expected time that the intermittent random walk  $X$ , starting at some point  $u \in U_i$ , visits another disc, where the minimization is taken w.r.t  $u \in U_i$ . Since discs are separated by distance of at least  $D$ , the process starting at any such  $u$  needs to visit a disc at distance at least  $D$ . It therefore follows that  $\mathbb{E}(t_i | A_i) \geq T_D$ . Altogether, the expected time to detect  $S$  is at least:

$$\sum_{i=1}^{M-1} T_D \mathbb{P}(A_i) = \sum_{i=1}^{M-1} T_D (1 - i/M) = \Omega(T_D M) = \Omega\left(n \frac{T_D}{D^2}\right),$$

as desired.  $\square$

*Corollary 3.10.* For every  $1 \leq D \leq \sqrt{n}/2$ , the best possible detection time is  $\Theta(n/D)$ , when we allow the strategy to be non-intermittent, unrestricted in terms of its internal computational power and navigation abilities, and fully tuned to the diameter. In other words,  $\text{opt}(n, D) = \Theta(n/D)$ .

*Proof.* The fact that  $\text{opt}(n, D) = \Omega(n/D)$  for every  $D < \sqrt{n}/6 - 1$  follows immediately from Claim 3.9 and the fact that  $T_D \geq D$ . For  $\sqrt{n}/6 - 1 < D \leq \sqrt{n}/2$  the bound  $\Omega(n/D) = \Omega(\sqrt{n})$  follows simply because with constant probability, the target is at distance  $\Omega(\sqrt{n})$  from the initial location of the searcher.

In order to see why  $\text{opt}(n, D) = O(n/D)$ , let us tile the torus with horizontal and vertical lines partitioning the torus into squares of size  $D/2 \times D/2$  each. In the case that  $\sqrt{n}$  is not a multiple of  $D/2$ , we might have few of these squares smaller than  $D/2 \times D/2$ . It is clear that this can be constructed while maintaining that the number of horizontal and vertical lines is  $O(\sqrt{n}/D)$ . For any connected target  $S$  of diameter  $D$ , the set  $B(S)$  must intersect at least one of these lines. Now consider a deterministic strategy that repeatedly walks over this tiling exhaustively, without doing much repetition in each exhaustive search. E.g., by first walking on the horizontal lines exhaustively (with occasional steps to move between horizontal lines) and then walking on the vertical lines exhaustively. It is easy to see that such a strategy exists and requires at most  $O(\sqrt{n}/D \cdot \sqrt{n}) = O(n/D)$  time to pass over all the lines, and hence to detect the target. This establishes the required upper bound.  $\square$

Claim 3.9, applied with  $D = 1$ , also yields the following corollary, by remarking that for intermittent random walk processes,  $T_D$ , namely, the expected time until the end point of a step is at a distance of at least  $D$  is at least the expected time for one step  $\tau$ , i.e.,  $T_D \geq \tau$ .

*Corollary 3.11.* Consider an intermittent random walk strategy  $X$  on the torus  $\mathbb{T}_n$ . The detection time of any target of diameter  $D$  is  $\Omega(n\tau/D^2)$ .

### 3.6.2 Lower bounds for Lévy walks

Let  $X^\mu$  be the intermittent Lévy walk on the torus  $\mathbb{T}_n$ . The goal of this section is to prove lower bounds on the competitiveness and scale-sensitivities of Lévy walks other than Cauchy. We start by analysing the detection times of small targets by Lévy walk corresponding to  $1 < \mu < 2$ .

**Theorem 3.5.** *Let  $\mu \in (1, 2)$  and  $D \in [1, \sqrt{n}/2]$ . Write  $\mu = 2 - \varepsilon$ . The competitiveness of the Lévy Walk  $X^\mu$  w.r.t.  $D$  is:*

$$\text{Comp}(X^\mu, D) = \Omega(n^{\varepsilon/2}/D). \quad (3.4)$$

*Proof.* By Corollary 3.11, the detection time of a target  $S$  with diameter  $D$  is  $\Omega(n\tau/D^2)$  where  $\tau$  is the expected step length. Using that  $\ell_{max} = \Theta(\sqrt{n})$ , Claim 3.7 implies that this expected step length is:

$$\tau = \Theta(n^{1-\frac{\mu}{2}}) = \Theta(n^{\varepsilon/2}),$$

writing  $\mu = 2 - \varepsilon$  for  $0 < \varepsilon < 1$ . Hence, the detection time  $X^\mu$  for a target of diameter  $D$  is  $\Omega(n^{1+\varepsilon/2}/D^2)$ . Dividing this by the unconditional optimal time  $\Theta(n/D)$ , we get the desired lower bound.  $\square$

Theorem 3.5 implies that the competitiveness of a Lévy walk  $X^\mu$  for  $1 < \mu < 2$  is very large with respect to small targets, i.e, when  $D \ll n^{\varepsilon/2}$ . We next aim to prove that the competitiveness of a Lévy walk  $X^\mu$  for  $2 < \mu \leq 3$  is very large with respect to large targets. Towards proving this, we first establish the following.

**Claim 3.12.** Let  $X^\mu$  be an intermittent Lévy walk process on the torus  $\mathbb{T}_n$ , for  $\mu \in [2, 3]$ , with  $\ell_{max} = \sqrt{n}/2$ . The expected time required to reach a distance of  $d \geq 1$  from the starting point is:

$$T_d = \begin{cases} \Omega(d \log d) & \text{if } \mu = 2 \\ \Omega(d^{\mu-1}) & \text{if } \mu \in (2, 3) \\ \Omega\left(\frac{d^2}{\log d}\right) & \text{if } \mu = 3 \end{cases} .$$

*Proof.* We may suppose that  $d \in [1, \sqrt{n}/4]$ . Denote by  $m_d$  the random number of steps before the process reaches a distance of at least  $d$ . Let us define  $m_0 = \lceil d^{\mu-1} \rceil$ , and say that a step is *small* if it has length at most  $d$ . Define the event  $\mathcal{A}$  that all the steps  $1, 2, \dots, m_0$  are small. Note that since  $d \leq \ell_{max}/2$ , the probability for any given step not to be small is  $q = \int_d^{\ell_{max}} \frac{a}{\ell^\mu} d\ell \geq \frac{c}{d^{\mu-1}}$  for some constant  $c \in (0, 1)$ . Hence, the probability for a step to be small is  $1 - q$ , and since the steps are independent, we have:

$$\mathbb{P}(\mathcal{A}) = (1 - q)^{m_0} = \exp(m_0 \log(1 - q)) \geq \exp(d^{\mu-1} \log(1 - cd^{1-\mu})).$$

We have:

$$\exp(d^{\mu-1} \log(1 - cd^{1-\mu})) = \exp(d^{\mu-1}(-cd^{1-\mu} + o(d^{1-\mu}))) = \exp(-c + o(1)),$$

which is a positive constant. Since this is a continuous, strictly positive, function of  $d \in [1, \infty)$ , we have  $\mathbb{P}(\mathcal{A}) \geq c'$  for some constant  $c' > 0$  independent of  $d$ .

Next, note that

$$\mathbb{E}(T(m_d)) \geq \mathbb{P}(\mathcal{A}) \cdot \mathbb{E}(m_d \mid \mathcal{A}) = c' \cdot \mathbb{E}(T(m_d) \mid \mathcal{A}).$$

Hence, for the purposes of obtaining a lower bound, it is sufficient to examine the process when conditioned on  $\mathcal{A}$ . This is a Lévy process of parameter  $\mu$ , with cut-off  $\ell_{max} = d$ . The expected length  $\tau$  of a jump is given by Claim 3.7:

$$\tau = \begin{cases} \Theta(\log d) & \text{if } \mu = 2 \\ \Theta(1) & \text{if } \mu \in (2, 3] \end{cases}, \quad (3.16)$$

and the variance  $\sigma'^2$  of the step-length of a jump projected onto one of the axes is given by Theorem 3.8:

$$\sigma'^2 = \begin{cases} \Theta(d^{3-\mu}) & \text{if } \mu \in (1, 3) \\ \Theta(\log d) & \text{if } \mu = 3 \end{cases}.$$

To conclude, combine these values with Claim. (2.12):

$$T_d = \Omega\left(\frac{d^2}{\sigma'^2} \cdot \tau\right) = \begin{cases} \Omega(d \log d) & \text{if } \mu = 2 \\ \Omega(d^{\mu-1}) & \text{if } \mu \in (2, 3) \\ \Omega\left(\frac{d^2}{\log d}\right) & \text{if } \mu = 3 \end{cases}.$$

This concludes the proof of Claim 3.12.  $\square$

Combining Claim 3.12 with the fact that the expected time to detect a target of diameter  $D$  is  $\Omega(n \frac{T_D}{D^2})$ , as established by Claim 3.9, and comparing to the unconditional optimal detection time  $\Theta(n/D)$  for targets of diameter  $D$ , we obtain Theorem 3.6.

## 3.7 Proof of the pointwise probability bounds of $Z(m)$

I provide the proofs of Lemma 3.2 in Section 3.7.2, and of Lemma 3.3 in Section 3.7.3. Before presenting these proofs, let us first establish lower and upper bounds on the distance travelled by the walk at step  $m$ .

### 3.7.1 Superdiffusive properties of the Cauchy walk on $\mathbb{R}^2$

We first remark that the probability to choose a length in a given interval is easily computed from Eq. (3.1):

*Observation 3.13.* The probability to do a step of length at most  $\ell > 0$  is  $a_{2, \ell_{max}} \ell$  if  $\ell \leq 1$  and  $a_{2, \ell_{max}}(2 - \frac{1}{\ell})$  if  $\ell \in (1, \ell_{max})$ . For integers  $\ell_{max} \geq \ell_2 \geq \ell_1 \geq 1$ , the probability to choose a length in  $[\ell_1, \ell_2]$  is  $a_{2, \ell_{max}}(\frac{1}{\ell_1} - \frac{1}{\ell_2})$ .

The next claim quantifies the probability that the Cauchy process goes to a distance of at least  $d$  after  $m$  steps. In particular, it shows that in step  $m$ , the process is at a distance of  $\Omega(m)$  with constant probability, and that it is at a distance of  $\Omega(m/\log m)$  with high probability in  $m$ .

**Claim 3.14.** For any integer  $m \geq 2$  and any real  $d \in [1, \frac{\ell_{max}}{3}]$  we have,

$$\mathbb{P}(\exists s \leq m \text{ s.t. } \|Z(s)\| \geq d) \geq 1 - e^{-cm/d},$$

for some constant  $c > 0$ . In particular this lower bound is at least

- $1 - O(m^{-2})$  if  $d = c' \frac{m}{\log m}$  with  $c' > 0$  a small enough constant,
- $\Omega(1)$  if  $d = c'm$  for any constant  $c' > 0$  with  $c'm \leq \ell_{max}/3$ .

*Proof.* By Observation 3.13, the probability that a given step has a length at least  $2d$  is  $a_{2, \ell_{max}}(\frac{1}{2d} - \frac{1}{\ell_{max}}) \geq \frac{a_{2, \ell_{max}}}{6d}$ . Since the steps are independent, the probability of the event  $\mathcal{A}$  that at least one of the steps  $1, \dots, m$  has a length at least  $2d$  is

$$\mathbb{P}(\mathcal{A}) \geq 1 - \left(1 - \frac{a_{2, \ell_{max}}}{6d}\right)^m.$$

Writing  $(1 - a_{2, \ell_{max}}/6d)^m = e^{m \log(1 - \frac{a_{2, \ell_{max}}}{6d})} \leq e^{-cm/d}$ , for some constant  $c > 0$ , we get

$$\mathbb{P}(\mathcal{A}) \geq 1 - e^{-cm/d}.$$

To conclude, it suffices to show that  $\mathcal{A}$  implies that there exists a step  $s \leq m$  for which  $\|Z(s)\| \geq d$ . Indeed, suppose that  $\mathcal{A}$  occurs and let  $s \leq m$  be the first step of length  $2d$  or more. Then,

- Either  $\|Z(s-1)\| \geq d$ , in which case we are done.
- Or  $\|Z(s-1)\| < d$ . In this case, as  $Z(s) = Z(s-1) + V(s)$ , we have  $\|Z(s)\| \geq \|V(s)\| - \|Z(s-1)\| > 2d - d = d$ .

This concludes the proof of Claim 3.14.  $\square$

Claim 3.14 asserts that, with some probability, the walk goes far from 0. Conversely, the next claim says that with some constant probability, the walk does not get too far.

**Claim 3.15.** • For any constant  $c > 0$ , there exists a constant  $\delta > 0$  such that, for any two integers  $1 \leq s \leq m$ , we have  $\mathbb{P}(\|Z(s)\| \leq cm) \geq \delta$ .

- For any constant  $0 < \delta < 1$ , there exists a (large enough) constant  $c > 0$  such that, for any two integers  $1 \leq s \leq m$ , we have  $\mathbb{P}(\|Z(s)\| \leq cm) \geq \delta$ .

*Proof.* Fix an integer  $m \geq 1$  and let  $c''$  be a constant, to be chosen later. Let  $\mathcal{A}$  denote the event that each of the first  $m$  steps has length at most  $\ell = c''m$ . We have, for any integer  $s \leq m$ , and any constant  $c > 0$ ,

$$\mathbb{P}(\|Z(s)\| \leq cm) \geq \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\|Z(s)\| \leq cm \mid \mathcal{A}). \quad (3.17)$$

We shall study separately each term in the r.h.s of Eq. (3.17), and establish the following:

- For the first item of Claim 3.15, we shall take  $c'' > 0$  so that both factors are constants (hence their multiplication is at least some constant  $\delta$ ),
- For the second item of Claim 3.15, where the bound  $\delta$  is given, we will show that both terms can be made at least  $\sqrt{\delta}$  by choosing  $c$  and  $c''$  appropriately.

Proceeding with the first term in the r.h.s of Eq. (3.17), by Observation 3.13, we have:

$$\mathbb{P}(\mathcal{A}) = \begin{cases} (a_{2,\ell_{max}} c'' m)^m & \text{if } c'' m \leq 1 \\ (2a_{2,\ell_{max}})^m (1 - \frac{1}{2c'' m})^m & \text{if } c'' m \in [1, \ell_{max}] \\ 1 & \text{if } c'' m \geq \ell_{max} \end{cases} .$$

For  $1 \leq m \leq \frac{1}{c''}$ , we have  $(a_{2,\ell_{max}} c'' m)^m \geq (a_{2,\ell_{max}} c'' m)^{\frac{1}{c''}}$  as  $a_{2,\ell_{max}} c'' m \leq c'' m \leq 1$ , and  $(a_{2,\ell_{max}} c'' m)^{\frac{1}{c''}} \geq (a_{2,\ell_{max}} c'')^{\frac{1}{c''}}$  as  $m \geq 1$ . For the second item, note that the function  $(1 - \frac{\alpha}{x})^x = e^{x \log(1 - \frac{\alpha}{x})}$  is increasing in  $x \geq \alpha$  and thus, for  $x \geq 2\alpha$ , we have  $(1 - \frac{\alpha}{x})^x \geq 2^{-2\alpha}$ . Applying this with  $\alpha = \frac{1}{2c''}$ , we have,  $(1 - \frac{1}{2c'' m})^m \geq 2^{-\frac{1}{c''}}$ , for  $m \geq \frac{1}{c''}$ . Overall, using  $2a_{2,\ell_{max}} \geq 1$ , we get

$$\mathbb{P}(\mathcal{A}) \geq \begin{cases} (\frac{c''}{2})^{\frac{1}{c''}} & \text{if } c'' m \leq 1 \\ 2^{-\frac{1}{c''}} & \text{if } c'' m \in [1, \ell_{max}] \\ 1 & \text{if } c'' m \geq \ell_{max} \end{cases} .$$

Hence,

- $\mathbb{P}(\mathcal{A}) = \Omega(1)$  for any given  $c'' > 0$ .
- Furthermore, with respect to the second item of Claim 3.15 where  $0 < \delta < 1$  is given, we can choose  $c''$  large enough (in particular, we take  $c'' \geq 1$  so that  $c'' m \geq 1$ ), to ensure that  $\mathbb{P}(\mathcal{A}) \geq 2^{-\frac{1}{c''}} \geq \sqrt{\delta}$ .

We are now ready to lower bound the second factor in Eq. (3.17), namely,  $\mathbb{P}(\|Z(s)\| \leq cm \mid \mathcal{A})$ . We begin with a notation: if  $X$  is a random variable, let us write  $X^{\mathcal{A}}$  for the random variable  $X$  conditioned on the occurrence of  $\mathcal{A}$ . Our first goal is to prove that

$$\mathbb{P}(\|Z^{\mathcal{A}}(s)\| \leq cm) \geq 1 - \frac{8s\mathbb{E}(\|V^{\mathcal{B}}\|^2)}{c^2 m^2}, \quad (3.18)$$

where  $V^{\mathcal{B}} = (V_1^{\mathcal{B}}, V_2^{\mathcal{B}})$  is one step-vector of the walk on  $\mathbb{R}^2$ , conditioned on the event  $\mathcal{B}$  that it is at most  $c'' m$ . Eq. (3.18) will be established by applying Chebyshev's inequality on each of the projections on the axes and using a union bound argument. Specifically, decomposing the walk  $Z$  on the two axes, by writing  $Z = (Z_1, Z_2)$ , we first use a union bound to obtain:

$$\begin{aligned} \mathbb{P}(\|Z^{\mathcal{A}}(s)\| > cm) &\leq \mathbb{P}(\exists i = 1, 2 \text{ s.t. } |Z_i^{\mathcal{A}}(s)| > cm/2) \\ &\leq \mathbb{P}(|Z_1^{\mathcal{A}}(s)| > cm/2) + \mathbb{P}(|Z_2^{\mathcal{A}}(s)| > cm/2) \\ &\leq 2\mathbb{P}(|Z_1^{\mathcal{A}}(s)| > cm/2), \end{aligned}$$

where we used the symmetry to deduce that  $Z_1$  and  $Z_2$  share the same distribution. Hence,

$$\mathbb{P}(\|Z^{\mathcal{A}}(s)\| \leq cm) \geq 1 - 2\mathbb{P}(|Z_1^{\mathcal{A}}(s)| > cm/2).$$



Next, we aim to lower bound the r.h.s. Relying on the fact that the expectation of  $Z_1^A(s)$  is 0 for any  $s$ , by Chebyshev's inequality, we have:

$$\mathbb{P}(|Z_1^A(s)| > cm/2) \leq \frac{4\text{Var}(Z_1^A(s))}{c^2m^2}.$$

Since  $Z_1^A(s)$  is the sum of  $s$  independent steps that follow the same law as  $V_1^B$ , we have:

$$\text{Var}(Z_1^A(s)) = s\text{Var}(V_1^B).$$

As the expectation of  $V_1^B$  is zero, we have  $\text{Var}(V_1^B) = \mathbb{E}((V_1^B)^2)$ . Furthermore, since  $|V_1^B| \leq \|V^B\|$ , we obtain:

$$\text{Var}(Z_1^A(s)) \leq s\mathbb{E}(\|V^B\|^2),$$

which concludes the proof of Eq. (3.18). Next, let us estimate  $\mathbb{E}(\|V^B\|^2)$ . If, on the one hand,  $c''m \leq 1$ , then, when conditioning on  $\mathcal{A}$ , the length of a step is chosen uniformly at random in  $[0, c''m]$ . Thus, its second moment is

$$\mathbb{E}(\|V^B\|^2) = \int_0^{c''m} \ell^2 \frac{d\ell}{c''m} = \frac{(c''m)^2}{3}. \quad (3.19)$$

On the other hand, if  $c''m \geq 1$ , then  $V^B$  is a Cauchy walk with cut off  $\ell_{max} = c''m$ . Hence, its second moment is

$$\begin{aligned} \mathbb{E}(\|V^B\|^2) &= a' \int_0^1 \ell^2 d\ell + a' \int_1^{c''m} \ell^2 \ell^{-2} d\ell \\ &\leq a' \int_0^{c''m} 1 d\ell = a'c''m \leq c''m. \end{aligned} \quad (3.20)$$

Overall, by Eqs. (3.18), (3.19) and (3.20) we find that, for  $s \leq m$ ,

$$\begin{aligned} \mathbb{P}(\|Z^A(s)\| \leq cm) &\geq \begin{cases} 1 - \frac{8sc''^2}{3c^2} & \text{if } c''m \leq 1 \\ 1 - \frac{8sc''}{c^2m} & \text{if } c''m \geq 1 \end{cases} \\ &\geq \begin{cases} 1 - \frac{8c''}{3c^2} & \text{if } c''m \leq 1 \\ 1 - \frac{8c''}{c^2} & \text{if } c''m \geq 1 \end{cases}. \end{aligned}$$

We then conclude the proof of Claim 3.15 by observing the following.

- For the first item of Claim 3.15, we have proved that  $\mathbb{P}(\mathcal{A}) = \Omega(1)$  for any constant  $c'' > 0$ . Hence, we may now choose  $c''$  small enough so that  $\mathbb{P}(\|Z^A(s)\| \leq cm) = \Omega(1)$ .
- For the second item of Claim 3.15, we have already chosen  $c''$  to be large (in order to have  $\mathbb{P}(\mathcal{A}) \geq \sqrt{\delta}$ , but we are free to choose  $c$  large enough so that  $\mathbb{P}(\|Z^A(s)\| \leq cm) \geq \sqrt{\delta}$ .

□

### 3.7.2 Proof of Lemma 3.2 (lower bound)

In this section we prove the following:

**Lemma 3.2.** *For any constant  $\alpha > 0$ , there exists a constant  $c > 0$  such that for any integer  $m \in [1, \alpha \ell_{max}]$ , and any  $x \in \mathbb{R}^2$ , with  $\|x\| \leq m$ , we have  $\mathbf{p}^{Z(m)}(x) \geq \frac{c}{m^2}$ .*

*Proof.* First note that for  $m = 1$ , the lemma holds by the definition of the Lévy process. Let us therefore consider an integer  $m \geq 2$ .

By the monotonicity property (Corollary 2.11), it is enough to prove that there is some constant  $c' > 1$  such that,

$$\mathbb{P}(m \leq \|Z(m)\| \leq c'm) = \Omega(1). \quad (3.21)$$

Indeed, if this holds, then, since the area of the ring  $\{y \in \mathbb{R}^2 \text{ s.t. } m \leq \|y\| \leq c'm\}$  is  $\Theta(m^2)$ , then we would have that for at least one point  $u$  in this ring,  $p^{Z(m)}(u) = \Omega(m^{-2})$ . Then, by monotonicity, for  $x \in \mathbb{R}^2$  such that  $\|x\| \leq m$ , we would have  $p^{Z(m)}(x) \geq p^{Z(m)}(u) = \Omega(m^{-2})$  which is the desired lower bound.

We thus proceed to prove Eq. (3.21). For this, let us define, for a given  $m \in [2, \alpha \ell_{max}]$ , the event

$$\mathcal{A}_{far} = \exists s \leq m \text{ s.t. } \|Z(s)\| \geq 2m.$$

We next prove the following claim.

**Claim 3.16.**  $\mathbb{P}(\mathcal{A}_{far}) = \Omega(1)$ , where the constant in lower bound does not depend on  $m$ .

*Proof of Claim 3.16.* By Claim 3.14, we immediately get that the claim holds for any  $m \in [2, \ell_{max}/6]$ . I next show that the claim holds also for  $m \in [\ell_{max}/6, \alpha \ell_{max}]$ . Intuitively, we prove this using a constant number of iterations. Each iteration consists of at most  $m' = \alpha' \ell_{max}$  steps, with  $\alpha'$  a small constant, during which we are guaranteed to go a distance of  $\ell_{max}/3$  with constant probability. Because the direction is chosen uniformly at random, at the cost of reducing this probability by a constant factor, we can further impose that the  $x$ -coordinate increases by a factor of, say,  $\ell_{max}/5$ . As these iterations are independent, and since  $\alpha$  is a constant, we can guarantee that up to step  $m = \alpha \ell_{max}$ , the process goes away to a distance of at least  $2\alpha \ell_{max}$  with constant probability.

Formally, first notice that we can take  $\alpha > 1$  without loss of generality. Note now that since  $m \in [\ell_{max}/6, \alpha \ell_{max}]$ , the second item in Claim 3.14 implies that:

$$\mathbb{P}\left(\exists s \leq \frac{m}{10\alpha} \text{ s.t. } \|Z(s)\| \geq \frac{\ell_{max}}{3}\right) \geq c'_\alpha,$$

for some constant  $c'_\alpha > 0$ . As a consequence, since the direction of  $Z(s)$  is distributed uniformly at random, we have:

$$\mathbb{P}\left(\exists s \leq \frac{m}{10\alpha}, Z_1(s) \geq \frac{\ell_{max}}{4}\right) \geq c_\alpha, \quad (3.22)$$

for some constant  $c_\alpha > 0$ . When this occurs, let  $s_1 \leq \frac{m}{10\alpha}$  be such that  $Z_1(s_1) \geq \frac{\ell_{max}}{4}$ . By the Markov property, starting from step  $s_1$ , we can then apply again (3.22) to show that with probability  $c_\alpha$ , there is a  $s_2 \leq s_1 + \frac{m}{10\alpha} \leq 2\frac{m}{10\alpha}$  such that  $Z_1(s_2) \geq$

$Z_1(s_1) + \frac{\ell_{max}}{4} \geq 2\frac{\ell_{max}}{4}$ . Overall, this happens with probability  $c_\alpha^2$ . Repeating this  $\lceil 9\alpha \rceil$  times, we finally get:

$$\mathbb{P}\left(\exists s \leq \lceil 9\alpha \rceil \frac{m}{10\alpha}, Z_1(s) \geq \lceil 9\alpha \rceil \frac{\ell_{max}}{4}\right) \geq c_\alpha^{\lceil 9\alpha \rceil},$$

which is a positive constant. Because  $\alpha > 1$ , this implies  $\mathbb{P}(\exists s \leq m, Z_1(s) \geq 2\alpha\ell_{max}) = \Omega(1)$ . As  $2\alpha\ell_{max} \geq 2m$  and  $\|Z\|(s) \geq |Z_1(s)|$ , this, in turn, implies  $\mathbb{P}(\mathcal{A}_{far}) = \Omega(1)$ , completing the proof of Claim 3.16.  $\square$

Next, conditioning on  $\mathcal{A}_{far}$ , we write:

$$\mathbb{P}(\|Z(m)\| \geq m \mid \mathcal{A}_{far}) \geq \min_{s \leq m} \mathbb{P}(\|Z(m)\| \geq m \mid \|Z(s)\| \geq 2m) \quad (3.23)$$

$$\geq \min_{s \leq m} \mathbb{P}(\|Z(m-s)\| \leq m), \quad (3.24)$$

where we used the Markov property, and the spatial homogeneity of the process, in the latter inequality. In words, in the r.h.s. of Inequality (3.23), we examine the probability to be at a high distance (i.e.,  $m$ ), knowing that the process was even further (at some point  $x$  at distance at least  $2m$ ). In Inequality (3.24) we bound this by the probability of staying within distance  $m$ .

By the first item of Claim 3.15, the r.h.s of Inequality (3.24) is at least some positive constant (again, independent of  $m$ ). Overall, for any  $m \geq 2$ , we have:

$$\mathbb{P}(\|Z(m)\| \geq m) \geq \mathbb{P}(\|Z(m)\| \geq m \mid \mathcal{A}_{far}) \cdot \mathbb{P}(\mathcal{A}_{far}) \geq \gamma,$$

for some constant  $\gamma > 0$  (independent of  $m$ ). Next, using the second item of Claim 3.15, with  $\delta = 1 - \frac{\gamma}{2}$ , we get that there exists a large enough constant  $c' > 0$  (again, independent of  $m$ ), such that:

$$\mathbb{P}(\|Z(m)\| \leq c'm) \geq \delta. \quad (3.25)$$

Hence, using a union bound argument, we have:

$$\begin{aligned} \mathbb{P}(m \leq \|Z(m)\| \leq c'm) &\geq \mathbb{P}(\|Z(m)\| \geq m) + \mathbb{P}(\|Z(m)\| \leq c'm) - 1 \\ &\geq \gamma + \delta - 1 = \frac{\gamma}{2} > 0. \end{aligned}$$

This establishes Eq. (3.21) and thus concludes the proof of Lemma 3.2.  $\square$

### 3.7.3 Proof of Lemma 3.3 (upper bound)

Recall the statement of the lemma:

**Lemma 3.3.** *For any constant  $\alpha > 0$ , there exists a constant  $c' > 0$  such that, for any integer  $m \in [2, \alpha\ell_{max}]$  and any  $x \in \mathbb{R}^2$ , we have  $\mathbb{P}(\|Z(m) - x\| \leq 1) \leq \frac{c' \log^2 m}{m^2}$ .*

*Proof.* Let  $\alpha > 0$  and  $m \in [2, \alpha\ell_{max}]$ . Due to the monotonicity property stated in Corollary 2.11, it is sufficient to prove this result for  $x = 0$ . Indeed, for any  $x \in \mathbb{R}^2$ , the sets  $B(0) \setminus B(x)$  and  $B(x) \setminus B(0)$  have the same area  $A$ , and

$$\begin{aligned} \mathbb{P}(Z(m) \in B(x) \setminus B(0)) &\leq A \max_{y \in B(x) \setminus B(0)} \{p^{\|Z(m)\|(y)}\} \\ &\leq A \min_{y \in B(0) \setminus B(x)} \{p^{\|Z(m)\|(y)}\} \\ &\leq \mathbb{P}(Z(m) \in B(0) \setminus B(x)), \end{aligned}$$

where the second inequality is due to the monotonicity property and the fact that any point in  $B(x) \setminus B(0)$  is at distance more than 1 from the origin, and hence, further from 0 than any point in  $B(0) \setminus B(x)$ . This shows that  $\mathbb{P}(Z(m) \in B(x)) \leq \mathbb{P}(Z(m) \in B(0))$ , hence it is sufficient to prove the required upper bound for  $x = 0$ .

Intuitively, to establish this, we say that with high probability, there is some step  $s \leq m$  for which  $Z(s)$  is “distant” (at least  $cm/\log m$ ). Conditioning on this, the probability to be located in  $B(0)$  at step  $m$  is found out to be small, due to the monotonicity of the process (Corollary 2.11). Formally, consider a (small) positive constant  $c$ , and let  $\mathcal{A}$  be the event that there is some  $s \leq m$  for which  $\|Z(s)\| \geq cm/\log m$ .

Consider  $B(0)$  the ball of radius 1 with center 0. Write

$$\begin{aligned} \mathbb{P}(Z(m) \in B(0)) &= \mathbb{P}(Z(m) \in B(0) \cap \mathcal{A}) + \mathbb{P}(Z(m) \in B(0) \cap \neg\mathcal{A}) \\ &\leq \mathbb{P}(Z(m) \in B(0) \mid \mathcal{A}) + \mathbb{P}(\neg\mathcal{A}), \end{aligned} \quad (3.26)$$

By the first item of Claim 3.14, taking  $c$  to be sufficiently small, we have

$$\mathbb{P}(\neg\mathcal{A}) = O(m^{-2}).$$

In order to express the remaining term of Eq. (3.26), we will denote in the following equation  $Z^x$  the Cauchy process on  $\mathbb{R}^2$  with cut off  $\ell_{max}$  starting with  $Z(0) = x$ . Since our process was defined to start at 0, we have  $Z = Z^0$ . Remark that the law of  $Z^x$  is obtained by a translation of that of  $Z^0$ . With this notation in mind, we have, using the Markov property for the second inequality:

$$\begin{aligned} \mathbb{P}(Z^0(m) \in B(0) \mid \mathcal{A}) &\leq \max_{s \leq m} \mathbb{P}(Z^0(m) \in B(0) \mid \|Z^0(s)\| \geq cm/\log m) \\ &\leq \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \mathbb{P}(Z^x(m-s) \in B(0)) \\ &= \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \mathbb{P}(Z^x(s) \in B(0)) \\ &= \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \mathbb{P}(Z^0(s) \in B(-x)) \\ &= \max_{s \leq m} \sup_{\|x\| \geq cm/\log m} \mathbb{P}(Z(s) \in B(x)) \end{aligned}$$

Use now Corollary 2.11 that gives  $p^{Z(m)}(x) \leq \frac{1}{\pi\|x\|^2}$ . Hence, for any  $x \in \mathbb{R}^2$  with  $\|x\| > 1$ , we have

$$\mathbb{P}(Z(m) \in B(x)) = \int_{B(x)} p^{Z(m)}(y) dy \leq \int_{B(x)} \frac{1}{\pi(\|x\| - 1)^2} dy = \frac{1}{(\|x\| - 1)^2}.$$

Let  $m(c)$  be the largest integer  $m > 0$  such that  $cm/\log m \leq 2$ . For  $m > m(c)$ , we have

$$\mathbb{P}(Z(s) \in B(x)) \leq \max_{s \leq m} \frac{1}{(cm \log m - 1)^2} = \frac{1}{(cm \log m - 1)^2}$$

Overall, we find that, for  $m > m(c)$

$$\mathbb{P}(Z(m) \in B(0)) \leq \frac{1}{(cm/\log m - 1)^2} + \frac{c'}{m^2},$$

which we can bound by  $\frac{c_2 \log^2 m}{m^2}$  for some constant  $c_2 > 0$ . Since  $m(c)$  is a constant, there is some other constant  $c_3 > 0$  for which, for any  $m \in [2, m(c)]$ , we have  $\mathbb{P}(Z(m) \in B(0)) \leq \frac{c_3 \log^2 m}{m^2}$ . We then obtain, for any  $m \geq 2$ ,

$$\mathbb{P}(Z(m) \in B(0)) \leq \frac{\max\{c_2, c_3\} \log^2 m}{m^2},$$

which concludes the proof of Lemma 3.3. □

# Chapter 4

## On the efficiency of discrete approximations of the Cauchy Walk

In this chapter, based on (Boczkowski et al., 2018a) and (Guinard and Korman, 2020b), I discuss the efficiency of random walks when the agent is capable of using  $k$  different modes of search, each characterized by a typical length scale. I will review the significance of this model in biology, and prove formally that one model, known as the (truncated) Weierstrassian Walk, performs optimally, among models with  $k$  modes, in one-dimension.

### 4.1 Introduction

**Searchers alternating modes of movement.** In many biological settings, it was observed that organisms appear to execute different modes of search. This may happen as a response to environment (*cue-driven search*), or simply be an internal mechanism of the forager (*internal search*). For instance, elks were found to execute long steps when resources are sparse, and shorter sinuous moves when resources are abundant (Fryxell et al., 2008), although this may also be explained if steps are halted by the encounter of food. In order to study the internal search, i.e., performed in the absence of external stimuli, the movement of desert locusts (*Schistocerca gregaria*) was studied when they are placed in isolation in a homogeneous experimental arena, and they were shown to alternate between local search behaviour and relocation behaviour. (Bazazi et al., 2012) In the microscopic word, restriction enzymes targeting a DNA site were shown to have the possibility of either diffusing along the DNA (a one-dimensional search), or detaching, doing an excursion in the ambient fluid and relocating at another DNA site (Coppey et al., 2004).

**Models of searches alternating modes.** One of the most common model in the foraging literature is motivated by such examples. It is the Composite (Random) Walk, or Composite Brownian Walk, in which the searcher alternates ballistic and diffusive steps. The switch can happen either due to interaction with the environment, or happens with some distribution dependent on the mode of search. The first one was investigated by simulations, in (Benhamou and Collet, 2015), in

a semi-intermittent detection regime, i.e., when there is a low, but non-zero, probability to detect while moving. In comparison to the Lévy Walk model, which just halts its step and then begins a new one, they found better performances for their Composite Walk model, which, they argued, uses information from patch-discovery more rationally.

The internal model of search, where the switch between modes happens randomly, was also studied in detail in the (fully) intermittent detection regime, where one mode is devoted to moving, and another is used to detect targets, see the review (Bénichou et al., 2011). Both phases have durations that follow an exponential distribution with some rate dependent on the phase. These models follow a setting minimal in terms of agents' abilities: the detection is intermittent, there is no memory (setting aside the question of how the agent stops a ballistic step), and there is no tactical change according to external or internal stimuli. In one-dimension, (Lomholt et al., 2008) proved that such a model with two modes would find a target in a cycle of length  $n$  in time roughly  $n^{4/3}$ , which represents a considerable improvement over the quadratic time  $n^2$  of the simple random walk. However, when the lengths of the ballistic step follow a Cauchy distribution, Lomholt et al. found that the detection time fell down to quasi-linear. This lead my co-authors and me, in (Boczkowski et al., 2018a; Guinard and Korman, 2020b) to enquire how the time performances depended on the number of (exponentially distributed) modes of movement.

**Weierstrassian Walks: a discretization of the Lévy Walk.** Let us consider in more detail a specific model with  $k$  modes, the Weierstrassian Walk model, first introduced in (Hughes et al., 1981) as a discrete approximation to Lévy Walks. This random walk has distribution of step-lengths

$$\mathbf{p}_W = \sum_{i=0}^k p_i \delta_{b^i},$$

where  $p_i = c_a a^{-i}$ , for some  $a, b > 1$  and  $c_a = \frac{1-a^{-1}}{1-a^{-(k+1)}}$ . Note that, if we discretize a Lévy Walk of exponent  $\mu$  by assigning to every  $\ell \in [b^i, b^{i+1})$  the value  $b^i$ , the probability of  $b^i$  is

$$\int_{b^i}^{b^{i+1}} \ell^{-\mu} d\ell = \frac{1}{\mu-1} (b^i)^{1-\mu} - (b^{i+1})^{1-\mu} = \Theta(a^{-i}),$$

for  $a = b^{\mu-1}$ . And inversely, when  $a$  and  $b$  go to 1, and  $k = \infty$ , this distribution converges to a Lévy Walk with parameter  $\mu = 1 + \log(a)/\log(b)$  (Hughes et al., 1981). This explains in what sense Weierstrassian Walks are discrete equivalents of the Lévy Walks. In particular, the Weierstrassian Walk with  $a = b$  is the discrete equivalent of the Cauchy Walk. In Figure 4.1, I illustrated the trajectories of Weierstrassian and Lévy Walks. They look very similar. Since both distributions are truncated, after some time they begin to look like a Brownian Motion (Fig. 4.1(b)).

**Weierstrassian Patterns in animals.** In biology, tri-modal Weierstrassian Walks were used to model the movement patterns of mussels (Reynolds, 2014), mud snails (Reynolds et al., 2017), the Australian desert ant *Melophorus bagoti* (Reynolds et al.,

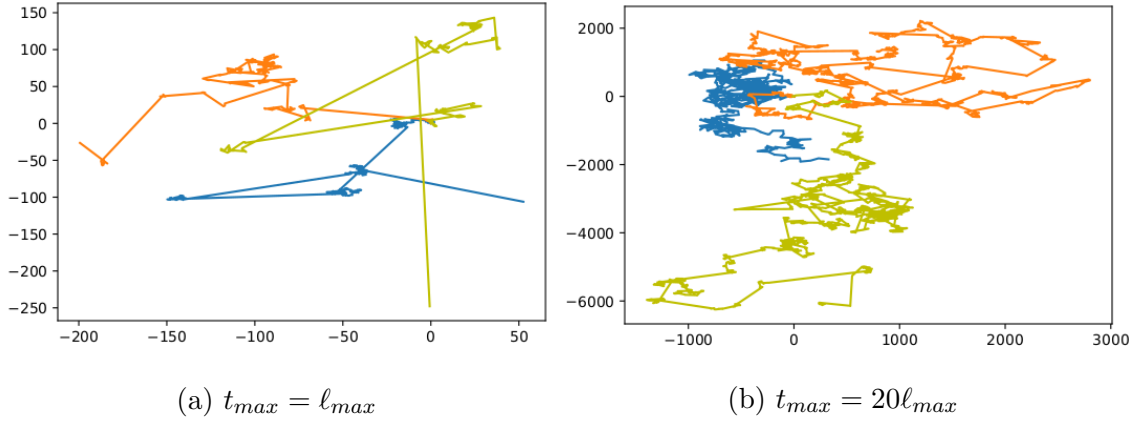


Figure 4.1: Patterns of movement. Cauchy Walk (yellow), compared with Weierstrassian Walks (blue: discrete steps of length exactly 1 and  $b$ ; orange: Hyperexponential steps) with  $k = 2$  modes, of lengths 1 and  $b = 100$ , and probability to do a step of length  $b$   $p_1 = \Theta(b^{-1})$ . All walks are truncated with  $\ell_{max} = 1000$  and were run until the first step for which the time exceeded  $t_{max}$ .

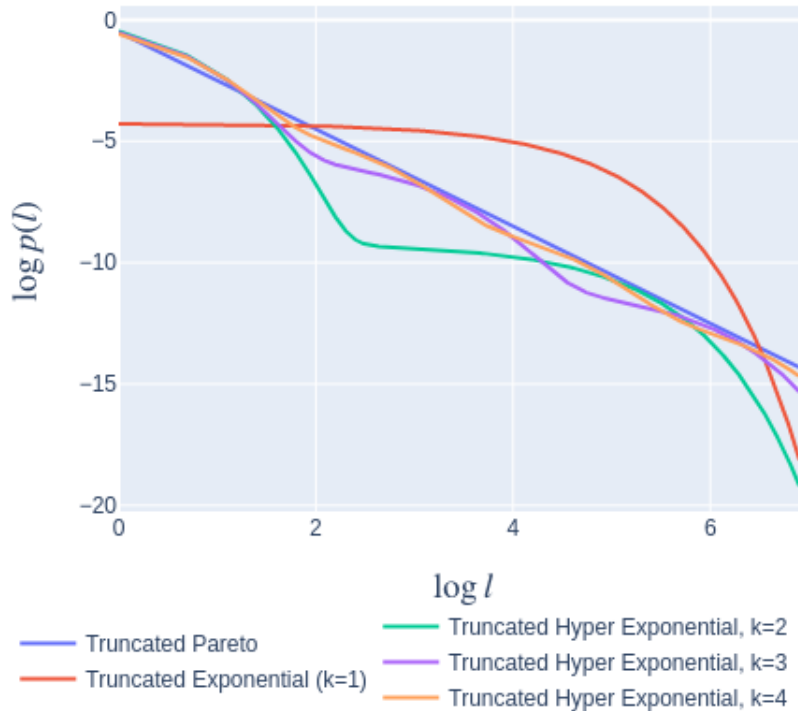


Figure 4.2: Probability density functions of the Truncated Pareto distribution (of exponent 2) and the Truncated Hyperexponential Weierstrassian distribution, with  $\ell_{max} = 1000$ . The exponential distribution has mean  $L_0 = 72$ , chosen to minimize the multiplicative ratio between it and the Pareto distribution. The hyperexponential distributions are Weierstrassian, with  $b = n^{2/(2k-1)}$ , as suggested by Theorem 4.3.



2014), and even extinct 50 My-old sea urchins (Sims et al., 2014). More precisely, these model follow an hyperexponential distribution

$$\mathbf{p}_{hW} = \sum_{i=0}^k p_i \mathbf{e}_{b^i},$$

where  $\mathbf{e}_{b^i}$  is the exponential distribution with mean  $b^i$ , and  $p_i = c_i b^{-i}$ . Since the mean step-lengths are a geometric series, as in the (discrete) Weierstrassian distribution, I will refer to  $\mathbf{p}_{hW}$  as a hyperexponential Weierstrassian distribution. In the aforementioned papers, Reynolds argue that using several exponential modes allow animals to approach a (truncated) Lévy Walk, and in this sense, Composite Brownian Walk models are not as opposed to Lévy Walk models as was previously believed (Lévy Walks model had earlier been described as “strange kinetics” (Benhamou and Collet, 2015)). Figure 4.2 illustrates how such hyperexponential weierstrassian distribution can approximate a (truncated) Pareto distribution. Note how close the 4-modal hyperexponential distribution is to the Pareto distribution.

If the argument that Weierstrassian Walks approximate Lévy Walks is certainly true for a large number of modes, it was not clear what was the connection between the number of modes  $k$  and the search performances, especially when  $k$  is small. Neither was it proved that, among all possible walks with  $k$  modes, the Weierstrassian Walk represented an optimal. In this chapter, I will provide answers to this question by showing that, in a cycle of length  $n$ , a Weierstrassian Walk tuned to the size  $n$  finds a target in time roughly  $n^{1+\frac{1}{2k-1}}$ , which is optimal.

**One-dimensional search in biology.** The one-dimensional cycle was chosen mainly because, analytically speaking, it is simpler than higher-dimensional spaces. This is due to the fact that the simple random walk on a cycle of length  $n$  find a target in quadratic time, while the Lévy Walk finds a target in quasi-linear time; hence, it was to be expected that a random walk that uses  $k$  lengths would perform in time  $n^{f(k)}$ . In contrast, on 2 and 3-dimensional tori, the random walk already finds a target in almost linear space, and it is typically harder to get the constants and logarithmic factors associated to a quasi-linear time performance, than it is to get a bound of the form  $\Theta(n^{f(k)})$  for some function  $f(k)$ . However, a one-dimensional search is not completely irrelevant to biology. Indeed, it can serve as a model to long and narrow topologies. As noted above, it is also relevant for modelling the search of DNA binding proteins (Berg et al., 1981; Coppey et al., 2004; Bénichou et al., 2011).

## 4.2 Theorem statements

### 4.2.1 Definitions

Let us model the one-dimension space as a discrete,  $n$ -node, cycle, termed  $C_n$ . For an integer  $k$ , define the random walks process with  $k$  step lengths as follows.

**Definition 4.1** ( $k$ -scales search). A random walk process  $X$  is called a  $k$ -scales search on  $C_n$  if the support of its step-length distribution  $\mathbf{p}$  has cardinal  $k$ . In this case, we will denote by  $L_0, L_1, \dots, L_{k-1}$  this support, and  $p_i = \mathbf{p}(L_i)$  the probability

of length  $L_i$ . Weierstrassian Walks are the  $k$ -scales random walk with  $L_i = b^i$  for some integer  $b \geq 2$ , with  $p_i = c_b b^{-i}$ .

The goal of this chapter is to show upper and lower bounds on the *cover time* of a  $k$ -scales search, that is, the expected time to visit every node of the ambient graph  $C_n$ . The parameters  $n$  and  $k$  are omitted when clear from the context.

### 4.2.2 The Lower Bound

I begin with the statement of the lower bound. The formal proof is given in Section 4.4.

**Theorem 4.2.** *Let  $k$  and  $n$  be positive integers. The cover time of any  $k$ -scales search  $X$  on  $C_n$  is*

$$\Omega(n^{1+\frac{1}{2k-1}} \cdot k^{-1}).$$

The proof of the lower bound implies also that, in order to be efficient, a  $k$ -scales search should have ratios  $L_i/L_{i-1}$  small. In fact, in the proof, the best situation is when they are all equal – which corresponds to a Weierstrassian Walk.

### 4.2.3 The Upper Bound

The following theorem implies that up to lower order terms, the cover time of the Weierstrassian random walk matches the lower bound of the cover time of any  $k$ -scales search, as given by Theorem 4.2, for  $2 \leq k \leq \log n$ , i.e., for all potential scales.

**Theorem 4.3.** *Let  $k$  be an integer such that  $2 \leq k \leq \log_2 n$ . The Weierstrassian random walk with parameter  $b = \lfloor n^{\frac{2}{2k-1}} \rfloor$  is a  $k$ -scales search that achieves a cover time of:*

$$n^{1+\frac{1}{2k-1}} \cdot O(k^2 \log^2 n).$$

Observe that combining Theorems 4.2 and 4.3 we obtain the best cover time  $Cov_{k,n}$  achievable by a  $k$ -scales search on  $C_n$ , which is  $\tilde{\Theta}\left(n^{1+\frac{1}{2k-1}}\right)$  for any  $2 \leq k \leq \log n$ . For particular values of  $k$ , we thus have:

$k$	1	2	3	4	5	...	$\log n$
$Cov_{k,n}$	$\Theta(n^2)$	$\tilde{\Theta}(n^{\frac{4}{3}})$	$\tilde{\Theta}(n^{\frac{6}{5}})$	$\tilde{\Theta}(n^{\frac{8}{7}})$	$\tilde{\Theta}(n^{\frac{10}{9}})$	...	$O(n \log^3 n)$

In practice, when  $n = 1000$ , then  $\log_2(n) = 10$ . The ratios  $\frac{n^{\frac{2k}{2k-1}}}{n} = n^{\frac{1}{2k-1}}$ , which indicate roughly the time performance of a Weierstrassian Walk in comparison to a Lévy Walk, are, for  $k = 1, 2, 3, 4, 5, 6$ , respectively, 1000, 10, 3.98, 2.7, 1.87, 1.70. We can observe a sharp decrease in time performances from the Brownian motion ( $k = 1$ ) to a composite walk with two or three modes.

Theorem 4.3 follows immediately from the following more general theorem, by taking  $b = n^{\frac{2}{2k-1}}$ .

**Theorem 4.4.** *Let  $b, k, n$  be integers such that  $b^{k-1} < n \leq b^k$ . The cover time of the Weierstrassian random walk on  $C_n$  with parameter  $b$  is*

$$O\left(n \max\left\{\frac{b^k}{n}, \frac{n}{b^{k-1}}\right\} \cdot k^2 \cdot \log b \cdot \log n\right) = \tilde{O}\left(\max\left\{b^k, \frac{n^2}{b^{k-1}}\right\}\right).$$

The formal proof of Theorem 4.4 is deferred to Section 4.6. In Section 4.5 I provide a sketch of the proof.

As mentioned, Theorem 4.4 using the particular value  $b = n^{\frac{2}{2k-1}}$  gives a tight upper bound for  $k$ -scales search. As a side note, the lower bounds from Section 4.4, when applied to the Weierstrassian random walk on  $C_n$ , show that its cover time is at least  $\tilde{\Omega}\left(\max\left\{n\sqrt{b}, \frac{n^2}{b^{k-1}}\right\}\right)$ . This is quite close to the bound  $\tilde{O}\left(\max\left\{b^k, \frac{n^2}{b^{k-1}}\right\}\right)$  of the theorem. Indeed if  $n \geq b^{k-\frac{1}{2}}$ , both bounds match, up to logarithmic terms. If  $n \leq b^{k-\frac{1}{2}}$ , the ratio of the bounds is  $\frac{b^{k-\frac{1}{2}}}{n}$ .

### 4.3 Discussion

The idea that lengths appear in all scales has been recognized as fundamental to the efficiency of Lévy Walks. Nevertheless, it had never been proved exactly that, if one scale is missing, then the search efficiency is deteriorated, and, more importantly, by how much the efficiency would be decreased. With lower and upper bounds combined, I suggest that using a Weierstrassian Walk with two modes drastically improves the efficiency, compared to a Brownian Motion or, in a more moderate manner, compared to using only the two modes. This may suggest that indeed, for animals, using only three modes may be efficient enough (note that hyperexponential Weierstrassian Walks can be expected to perform better than the discrete Weierstrassian Walks studied here). This is compatible with (Sims et al., 2014) that found that a three-modal hyperexponential walk modelled well the patterns of movement found in fossil trails.

When combined with appropriate empirical measurements, the lower bound can potentially be used to indirectly show that a given one-dimensional intermittent process uses strictly more than a certain number of step lengths. For example, if the process is empirically shown as a random walk, of constant speed, whose cover time is almost linear, then Theorem 4.2 implies that it must use roughly logarithmic number of step lengths. From a methodological perspective, such a result would be of particular appeal as demonstrating lower bounds in biology through mathematical arguments is extremely rare (Boczkowski et al., 2018b; Feinerman and Korman, 2013).

### 4.4 Proof of the lower bound

The goal of this section is to establish the lower bound in Theorem 4.2. For this purpose, consider a  $k$ -scales search  $X$  on the cycle  $C_n$  and denote  $(L_i)_{i=0}^{k-1}$  its step lengths with  $L_i < L_{i+1}$  for all  $i \in [k-2]$ . For convenience of writing let us also set  $L_k = n$ , but it should be clear that it is actually not a step length of the walk. Let  $p_i$  denote the probability of taking the step length  $L_i$ .

The theorem will follow from the combination of the following lemma, with our Claim 2.12 on the maximal distance achieved by a random walk. Specifically, this claim implies that the expected maximal distance achieved after  $m$  steps by a random walk with maximal length  $L$  is  $O(\sqrt{m}L)$ , and the time needed to go to distance  $d$  is at least  $\Omega(d^2L^{-1})$ . Lemma 4.5 stems from the analysis of the number of nodes that can be visited during  $L_{i+1}$  time steps. It forces  $L_0L_1$  as well as the

ratios  $L_{i+1}/L_i$  for all  $1 \leq i \leq k-1$  to be small enough in order to have a small cover time. Claim 2.12 forces  $L_{k-1}$  to be big enough to have a small cover time.

**Lemma 4.5.** *The cover time of  $X$  is at least*

- $\mathbb{E}(t_{cov}) = \Omega(n\sqrt{L_0L_1})$ .
- $\mathbb{E}(t_{cov}) = \Omega\left(\frac{n}{k}\sqrt{\frac{L_{i+1}}{L_i}}\right)$  for any  $1 \leq i \leq k-1$ .

*Proof of Lemma 4.5. Semi-formal proof.* I sketch here the ideas behind the proof of the first part, namely, that the cover time is at least of order  $n\sqrt{L_0L_1}$  (the second part is similar). Essentially, we count the expected number of nodes  $N$  that can be visited in a time duration of  $L_1$ , which we call a *phase*. A jump of length  $L_i \geq L_1$  will not contribute to visiting a new node during this time duration. Thus, we may suppose that there are only jumps of length  $L_0$ . Since  $L_1 \leq n$ , the process does not do a turn of the cycle and, therefore, it can be viewed as a walk on  $\mathbb{Z}$ . Furthermore, since every jump has length  $L_0$ , we can couple this walk by a corresponding simple random walk, that does steps of length 1, during a time duration of  $L_1/L_0$ . The expected number of nodes visited during a phase is thus of order  $\sqrt{L_1/L_0}$ . It follows that we need at least  $n/(\sqrt{L_1/L_0})$  such phases before covering the cycle. Since a phase lasts for  $L_1$  time, the cover time is at least of order  $n\sqrt{L_0L_1}$ .

**Formal proof.** In what follows, note that we will count the *time* and not the number of moves. Fix an index  $0 \leq i < k$ . We divide time into consecutive *i-phases*, each of time-duration precisely  $L_{i+1}$  (the last one may be shorter). We next prove the following.

**Claim 4.6.** The expected number of nodes visited during the  $\ell$ 'th *i-phase* is

- For  $i = 0$ ,  $\mathbb{E}(N_\ell) = O\left(\sqrt{\frac{L_1}{L_0}}\right)$ .
- For  $0 < i < k$ ,  $\mathbb{E}(N_\ell) = O\left(k\sqrt{L_i \cdot L_{i+1}}\right)$ .

*Proof of Claim 4.6.* Fix an index  $i$  and consider the *i-phases*. As the last *i-phase* may be shorter and intermediate *i-phases* may start when the process is executing a jump, the value of  $\mathbb{E}(N_\ell)$  is at most  $\mathbb{E}(N_1)$ , namely, the expected number of nodes that are visited during the first *i-phase*. Let us therefore concentrate on upper bounding  $\mathbb{E}(N_1)$ . The first *i-phases* lasts during the time period  $[0, L_{i+1})$ . Since only endpoints of jumps are visited, if during the *i-phase* the process starts any jump of length at least  $L_{i+1}$ , then the number of nodes does not increase. Thus, to get an upper bound on  $\mathbb{E}(N_1)$ , we may consider only trajectories that do not use such large jumps, i.e., we may restrict the process to jumps of length  $L_j$ , for  $j \leq i$ .

Denote by  $D$  the maximal distance achieved by the process in the time interval  $[0, L_{i+1})$ . We have  $N_1 \leq 2D + 1$ . In this phase of duration  $L_{i+1}$ , there are at most  $\frac{L_{i+1}}{L_j}$  steps of length  $L_j$  that can be made, for  $j \leq i$ , because a jump of length  $L_j$  takes  $L_j$  time. Let  $D_j$  be the maximal distance travelled by the jumps of length  $L_j$  (when ignoring jumps of length different than  $L_j$ ). We have  $D \leq \sum_{j \leq i} D_j$ . By Claim 2.12, we have:

$$\mathbb{E}(D_j) = O\left(\sqrt{\frac{L_{i+1}}{L_j}} \cdot L_j\right).$$

Thus,

$$\mathbb{E}(N_1) \leq 2\mathbb{E}(D) + 1 = O\left(\sum_{j \leq i} \sqrt{L_{i+1}L_j}\right) = O\left(k\sqrt{L_iL_{i+1}}\right).$$

This establishes the second item in the claim.

Bounding the number of visited nodes  $N_1$  by the distance  $D$ , as was done above, is not very precise, since there may be non-visited points between jumps. In order to establish the first item in the claim, i.e., the case where  $i = 0$ , let us be more precise. In this case, we may replace the equation  $N_1 \leq 2D + 1$  by the more precise inequality

$$N_1 \leq 2\frac{D}{L_0} + 1.$$

Indeed, since there are only jumps of length  $L_0$ , and there is no time to do a full turn of the cycle in the duration  $L_1$ , we visit only multiples of  $L_0$ . Thus, when  $i = 0$ , we have:

$$\mathbb{E}(N_1) = O\left(\frac{\sqrt{L_1L_0}}{L_0}\right) = O\left(\sqrt{\frac{L_1}{L_0}}\right),$$

as desired. This completes the proof of Claim 4.6.  $\square$

Let us end the proof of Lemma 4.5. By Claim 4.6, the number of nodes visited during the  $s$  first  $i$ -phases is

$$\mathbb{E}\left(\sum_{\ell=1}^s N_\ell\right) \leq s \cdot O(E_i).$$

where  $E_0 = \sqrt{\frac{L_1}{L_0}}$  and  $E_i = \sqrt{L_iL_{i+1}}$  for  $1 \leq i \leq k-1$ . Next, let us set  $s_1 := n \cdot \frac{c}{E_i}$  for a sufficiently small constant  $c$ , such that the previous bound becomes less than  $n/2$ . Using Markov's inequality, we get

$$\mathbb{P}\left(\sum_{\ell=1}^{s_1} N_\ell \geq n\right) < \frac{1}{2}.$$

Therefore, with probability at least  $1/2$ , the process needs at least  $s_1$  phases before visiting all nodes. Since the duration of a phase is  $L_{i+1}$ , the cover time is at least

$$s_1 \cdot L_{i+1} = \Omega\left(n \cdot \frac{L_{i+1}}{E_i}\right),$$

which is  $\Omega(n \cdot \sqrt{L_1L_0})$  if  $i = 0$  and  $\Omega(n \cdot \sqrt{\frac{L_{i+1}}{L_i}})$  otherwise. This completes the proof of Lemma 4.5.  $\square$

Next, it remains to show how Theorem 4.2 follows by combining Claim 2.12 and Lemma 4.5. First, as the process needs to go to distance at least  $n/3$  in order to cover the cycle, by Claim 2.12, the cover time is at least  $\Omega(n^2/L_{k-1})$ . If  $L_{k-1} \leq n^{1-\frac{1}{2k-1}}$  then the bound in Theorem 4.2 immediately follows. Let us therefore assume that  $L_{k-1} > n^{1-\frac{1}{2k-1}}$ .

Define  $\alpha_0 = L_0L_1$  and  $\alpha_i = \frac{L_{i+1}}{L_i}$  for  $i \in \{1, 2, \dots, k-2\}$ . As

$$\prod_{i=0}^{k-2} \alpha_i = L_0 L_{k-1},$$

there must exist an index  $0 \leq i \leq k-2$  such that  $\alpha_i \geq (L_0 L_{k-1})^{\frac{1}{k-1}}$ . Thus, by Lemma 4.5, the cover time is at least

$$\Omega\left(\frac{n}{k} (L_0 L_{k-1})^{\frac{1}{2(k-1)}}\right).$$

Since  $L_{k-1} > n^{1-\frac{1}{2k-1}} = n^{\frac{2k-2}{2k-1}}$  and  $L_0 \geq 1$ , we conclude that the cover time is at least

$$\mathbb{E}(t_{cov}) = \Omega\left(\frac{n}{k} \cdot n^{\frac{1}{2k-1}}\right),$$

as desired. This completes the proof of Theorem 4.2.  $\square$

## 4.5 Proof of the upper bound

In this section, I give the key ideas of the proof of Theorem 4.4. The detailed, formal, proof is the object of the next section.

In more details, let us consider the Weierstrassian walk on  $C_n$ , termed  $X$ , starting at  $X(0) = 0$ . We can assume that  $X$  is lazy with parameter  $\frac{1}{2}$ , as this only multiplies the cover time by a factor 2. The average length of a step of  $X$  is  $\tau = c_b k/2 = \Theta(k)$ . For some  $m_0$  to be fixed later, and any  $x \in C_n$ , consider  $N_x = \sum_{m=m_0}^{2m_0} \mathbb{P}(X(m) = x)$  and  $O = \sum_{m=0}^{m_0} \mathbb{P}(X(m) = 0)$ . By combining Lemma 2.5, and Matthew's bound (Theorem 1.18), we have that the cover time is at most

$$O(m_0 \cdot O \max_x \{N_x^{-1}\}) \cdot k \cdot \log n. \quad (4.1)$$

In order to simplify the presentation, assume first that  $n = b^k$ . Let us view the  $k$ -lengths Weierstrassian random walks as  $k$  (dependent) random walks, by grouping together the jumps of the same length (see Figure 4.3). Define  $S_i(m)$  as the algebraic count of the jumps of lengths  $b^i$ . E.g., if, by step  $m$ , there are exactly four positive jumps of length  $b^i$ , and one negative, then  $S_i(m) = 3$ . We have:

$$X(m) = \sum_{i=0}^{k-1} S_i(m) b^i.$$

Define also the following decomposition of  $C_n$ .

**Definition 4.7** (Base  $b$  decomposition). For any  $x \in C_n$ , we may decompose  $x$  in base  $b$  as

$$x = \sum_{i=0}^{k-1} x_i b^i,$$

with  $0 \leq x_i < b$ . We call  $x_i$  the  $i$ -th coordinate of  $x$  (in base  $b$ ).

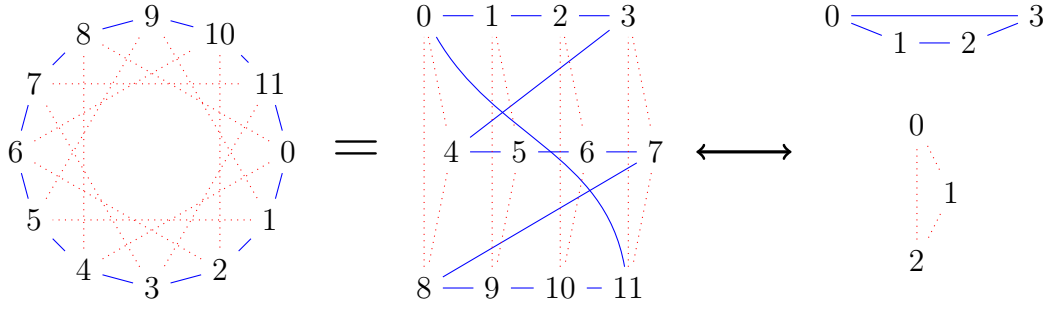


Figure 4.3: The first two graphs represent, in different node disposition, the Weierstrassian walk on  $C_{12}$  with parameter  $b = 4$ . There are  $k = 2$  jump lengths,  $L_0 = 1$  (blue edges) and  $L_1 = b = 4$  (red, dotted edges). To the right, I show the decomposition of  $C_{12}$  as  $C_4 \times C_3$ . For instance the node  $x = 7 \in C_{12}$  will be represented by  $x_0 = 3 \in C_4$  and  $x_1 = 1 \in C_3$ .

It follows from Euclidean division, and the fact that  $n = b^k$ , that the base  $b$  decomposition is well-defined and unique for every  $x \in C_n$ . This decomposition is illustrated in Figure 4.3 (where we have taken  $n = \hat{n}b^{k-1}$  to anticipate the more general case to follow).

Note that  $X(m) = x$  in  $C_n$  if and only if

$$\sum_i (S_i(m) - x_i)b^i = 0 \pmod{n}. \quad (4.2)$$

By taking Eq. (4.2) modulo  $b^i$ , for  $i \leq k - 1$ , it is easy to show that Eq. (4.2) is equivalent to

$$S_i(m) = y_i \pmod{b},$$

for  $y_i := x_i - b^{-i} \sum_{j < i} (S_j(m) - x_j)b^j \pmod{b}$ .

Thus,  $X(m) = x$  is equivalent to  $R_i(m) = y_i$  for all  $i$ , where  $R_i = S_i \pmod{b}$  is a random walk on  $C_b$  that moves with probability  $\frac{p_i}{2}$ . This process is illustrated in Figure 4.3, where  $X(m) = 7$  is equivalent to  $R_0(m) = 3$  and  $R_1(m) = 2$ .

Unfortunately, the  $R_i$ 's and the  $y_i$ 's are not independent, due to the fact that only one of the  $R_i$  can change between steps  $m$  and  $m + 1$ , however, let us overlook this issue in this informal outline. We then have:

$$\mathbb{P}(X(m) = x) \approx \prod_{i=0}^{k-1} \mathbb{P}(R_i(m) = y_i). \quad (4.3)$$

Recall that  $R_i$  is a random walk over  $C_b$  that moves with probability  $p_i/2$ . The following is a well-known property of the random walk a cycle (see, e.g., Example 5.7 and Proposition 6.18 in (Aldous and Fill, 2002)):

**Claim 4.8.** For a simple random walk  $R$  on  $C_b$  that moves with probability  $\frac{1}{2}$ , and any  $y \in C_b$ ,

$$\mathbb{P}(R(m) = y) = \begin{cases} O(1/\sqrt{m}) & \text{if } m < b^2 \\ b^{-1}(1 \pm \varepsilon_m) & \text{if } m \geq b^2, \end{cases} \quad (4.4)$$

with  $\varepsilon_m = O(e^{-cmb^{-2}})$  where  $c > 0$ .

Considering that  $R_i$  moves with probability  $\frac{p_i}{2} = \Theta(b^{-i})$ , we can expect that, at step  $m$ ,  $R_i(m)$  has the same distribution as the lazy random walk with  $mp_i$  steps that moves with probability  $\frac{1}{2}$ . This is proved formally in the next section. Hence, by substituting  $m$  with  $mp_i$  in Claim 4.8, we obtain:

$$\mathbb{P}(R_i(m) = y_i) = \begin{cases} O(1/\sqrt{mp_i}) & \text{if } m < b^{i+2} \\ b^{-1}(1 \pm \varepsilon_{mp_i}) & \text{if } m \geq b^{i+2}. \end{cases} \quad (4.5)$$

Theorem 4.4 then follows from Eq. (4.1), Eq. (4.3) and Eq. (4.5). Essentially, to cover  $C_n$ , we need that each  $R_i(m)$  is mixed, i.e., has some significant probability to visit any node  $y_i$  in  $C_b$ , which happens, as shown by Eq. (4.5), for  $m > b^{k-1+2} = b^{k+1}$ . Let us now define:

$$m_0 := b^{k+1}.$$

We first establish a lower bound on  $V = \sum_{m=m_0}^{2m_0} \mathbb{P}(X(m) = x)$ . By Eq. (4.3) and Eq. (4.5), we have, for  $m > m_0$ ,

$$\mathbb{P}(X(m) = x) \approx \prod_{0 \leq i \leq k-1} b^{-1}(1 - \varepsilon_{mp_i}) = \Theta(b^{-k}),$$

where the last equality is justified in the full proof. Thus,

$$\sum_{m=m_0}^{2m_0} \mathbb{P}(X(m) = x) = \Omega(m_0 b^{-k}) = \Omega(b).$$

We need also to upper bound the oversample  $O = \sum_{m=0}^{m_0} \mathbb{P}(X(m) = 0)$ . To do this, we shall use the short-time bounds of Eq. (4.5).

Let us decompose the aforementioned sum as follows.

$$\sum_{m=0}^{m_0} \mathbb{P}(X(m) = 0) = 1 + \frac{1}{2} + \sum_{j=0}^{k-1} \sum_{m=1+b^j}^{b^{j+1}} \mathbb{P}(X(m) = 0) + \sum_{m=1+b^k}^{m_0} \mathbb{P}(X(m) = 0). \quad (4.6)$$

Fix  $j$ , such that  $1 \leq j \leq k-1$  and let  $m \in (b^j, b^{j+1}]$ . By Eq. (4.3), in order to upper bound  $\mathbb{P}(X(m) = 0)$  it is enough to bound  $\mathbb{P}(R_i(m) = y_i)$  for every  $i \leq k-1$ . For  $i > j$ , we bound  $\mathbb{P}(R_i(m) = y_i)$  by 1. For  $i \leq j-2$ , we use Eq. (4.5) to upper bound  $\mathbb{P}(R_i(m) = y_i)$  by  $b^{-1}(1 + \varepsilon_{mp_j})$ . For  $i = j-1$  and  $i = j$ , we bound  $\mathbb{P}(R_i(m) = y_i)$  by  $O(1/\sqrt{mp_{j-1}})$  and  $O(1/\sqrt{mp_j})$ , respectively. We thus obtain, by Eq. (4.3),

$$\begin{aligned} \mathbb{P}(X(m) = x) &= O\left(\frac{1}{\sqrt{mp_{j-1}}} \cdot \frac{1}{\sqrt{mp_j}} \cdot \prod_{0 \leq i \leq j-2} b^{-1}(1 + \varepsilon_{mp_j})\right) \\ &= O\left(b^{-(j-1)} \cdot \frac{\sqrt{b}b^{j-1}}{m}\right) = O\left(\frac{\sqrt{b}}{m}\right), \end{aligned}$$

where we justify in the full proof that  $\prod_{0 \leq i \leq j-2} (1 + \varepsilon_{mp_j}) = O(1)$ . Hence, we get:

$$\sum_{m=1+b^j}^{b^{j+1}} \mathbb{P}(X(m) = 0) = O(\sqrt{b} \log b), \quad (4.7)$$



by using that  $\sum_{m=1+b^j}^{b^{j+1}} m^{-1} = \Theta\left(\int_{m=b^j}^{b^{j+1}} u^{-1} du\right) = \Theta(\log b)$ . For the case  $j = 0$ , we bound  $\mathbb{P}(R_i(m) = y_i)$  by 1 for  $i > 1$  and  $\mathbb{P}(R_0(m) = y_0)$  by  $O(m^{-\frac{1}{2}})$ , so that, by Eq. (4.3),  $\mathbb{P}(X(m) = 0) = O(\frac{1}{\sqrt{m}})$ . Hence, we get:

$$\sum_{m=2}^b \mathbb{P}(X(m) = 0) = O(\sqrt{b}). \quad (4.8)$$

Similarly, for  $m \in (b^k, b^{k+1}]$ ,  $\mathbb{P}(R_i(m) = y_i)$  is bounded by  $b^{-1}(1 + \varepsilon_{mp_i})$  for  $i \leq k-2$ , and by  $\frac{1}{\sqrt{mp_{k-1}}}$  for  $i = k-1$ . Thus, for  $m \in (b^k, b^{k+1}]$ ,

$$\mathbb{P}(X(m) = 0) = O\left(\frac{1}{\sqrt{m}\sqrt{b^{k-1}}}\right)$$

and, since  $\sum_{m=1+b^k}^{b^{k+1}} \frac{1}{\sqrt{m}} = O\left(\int_{b^k}^{b^{k+1}} \frac{1}{\sqrt{u}} du\right) = O(\sqrt{b^{k+1}})$ , we get:

$$\sum_{m=1+b^k}^{b^{k+1}} \mathbb{P}(X(m) = x) = O\left(\frac{\sqrt{b^{k+1}}}{\sqrt{b^{k-1}}}\right) = O(b). \quad (4.9)$$

In total, by Eq. (4.6), combining Eqs. (4.7), (4.8) and (4.9), we find that the expected number of returns to the origin up to step  $b^{k+1}$  is

$$O = \sum_{m=0}^{m_0} \mathbb{P}(X(m) = 0) = O(k\sqrt{b} \log b + b) = O(kb \log b).$$

Hence, by Eq. (4.1), we have:

$$t_{cov} = O(m_0 \cdot k \log b \cdot k \log n) = O(b^{k+1} k^2 \log b \log n) = O(nbk^2 \log b \log n), \quad (4.10)$$

as claimed by Theorem 4.4, for the case where  $n = b^k$ .

Consider now a more general case, in which  $n$  is a multiple of  $b^{k-1}$ . Here, we can write  $n = \hat{n}b^{k-1}$ , where  $\hat{n} \in (0, b]$  is an integer. What changes in this case is that the last coordinate,  $R_{k-1}$ , is now a random walk over  $C_{\hat{n}}$  instead of over  $C_b$ , as depicted in Figure 4.3.  $R_{k-1}$  is thus mixed after number of steps:

$$\hat{n}^2 p_{k-1}^{-1} = \Theta(b^{k-1} \hat{n}^2) = \Theta(n^2 / b^{k-1}).$$

On the other hand, after  $\Theta(b^{k-2+2}) = \Theta(b^k)$  steps, the other coordinates are mixed. Thus, the number of steps needed before every coordinate  $R_i$  is mixed is:

$$m_0 = \Theta(\max\{b^k, n^2 / b^{k-1}\}), \quad (4.11)$$

which is again the order of magnitude of the cover time of  $X$ , up to polylogarithmic factors. Note that when  $n = b^k$ , Eq. (4.11) recovers the cover time of order  $\tilde{\Theta}(b^{k+1})$ . Furthermore, the ratio of the cover time for  $n = b^k$  and  $n = \hat{n}b^{k-1}$  is of order  $\frac{b^{k+1}}{\max\{b^k, b^{k-1} \hat{n}^2\}} = \min\{b, \frac{b^2}{\hat{n}^2}\}$ . When  $b$  is large (which corresponds to  $k$  being small), this can be significant. Hence, naively bounding  $\hat{n}$  from above by  $b$  would not suffice to yield an optimal bound.

The general case, when  $n$  is not necessarily a multiple of  $b^{k-1}$ , needs to be treated with more care. What changes in this case is that we can no longer decompose  $X$  as  $k$  dependent random walks on  $C_b \times \cdots \times C_b \times C_{\frac{n}{b^{k-1}}}$ , since  $\frac{n}{b^{k-1}}$  is not an integer. Instead, we define  $Z$  as the process that does the same jumps as  $X$ , but on the infinite line  $\mathbb{Z}$ , and we also define

$$\hat{n} := \lfloor n/b^{k-1} \rfloor.$$

Then, we use almost the same decomposition, where  $Z$  is viewed as  $k$  dependent random walks over  $C_b \times \cdots \times C_b \times \mathbb{Z}$ . The process corresponding to the last coordinate,  $R_{k-1}$ , is now a random walk on  $\mathbb{Z}$ , and we are interested especially on the probability of the event  $R_{k-1}(m) = x_{k-1}$  for  $x_{k-1} \in [0, \hat{n}]$ . As the coordinate  $R_{k-1}$  is not restricted to  $[0, \hat{n}]$ , we need to pay attention that the walk does not go too far.

## 4.6 Proof of the upper bound (formal)

In this section, I prove the following theorem.

**Theorem 4.4.** *Let  $b, k, n$  be integers such that  $b^{k-1} < n \leq b^k$ . The cover time of the Weierstrassian random walk on  $C_n$  with parameter  $b$  is*

$$O\left(n \max\left\{\frac{b^k}{n}, \frac{n}{b^{k-1}}\right\} \cdot k^2 \cdot \log b \cdot \log n\right) = \tilde{O}\left(\max\left\{b^k, \frac{n^2}{b^{k-1}}\right\}\right).$$

### 4.6.1 Preliminaries

We may assume that  $X$  is lazy with laziness  $\frac{1}{2}$ . We define:

$$Z(m) = \sum_{s=1}^m \xi_s \cdot V_s, \quad X(m) = Z(m) \pmod n, \quad (4.12)$$

where  $V_s$  and  $\xi_s$  are, respectively, the length and the sign of the  $s$ -th jump, i.e.,  $V_s$  is a random variable taking value  $b^i$  with probability  $p_i = c_b b^{-i}$  for every  $i \leq k-1$ , and  $\xi_s$  takes value 0, 1 or  $-1$ , with probabilities  $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}$ , and the variables  $(V_s)_{s \in \mathbb{N}}$  and  $(\xi_s)_{s \in \mathbb{N}}$  are independent.

The time it takes to accomplish the first  $m$  moves, denoted  $T(m)$ , is then:

$$T(m) = \sum_{s=1}^m |\xi_s| \cdot V_s. \quad (4.13)$$

For some  $m_0$  to be fixed later, and any  $x \in \{0, \dots, n-1\}$ , consider  $N_x = \sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) = x)$  and  $O = \sum_{m=0}^{m_0} \mathbb{P}(Z(m) = 0)$ . Remark that the average step length of  $X$  is

$$\sum_{i=0}^{k-1} b^i p_i / 2 = \sum_{i=0}^{k-1} c_b / 2 = c_b k / 2 = \Theta(k),$$

since  $c_b = \frac{1-b^k}{1-b} = \Theta(1)$ , as  $b \geq 2$ . By combining Remark 2.9 with Matthew's bound (Theorem 1.18), we have that the cover time is at most

$$O(m_0 \cdot O \max_{x \in \{0, \dots, n-1\}} \{N_x^{-1}\} \cdot k \cdot \log n). \quad (4.14)$$

### 4.6.2 From $k$ -scales search on $\mathbb{Z}$ to $k$ (dependent) random walks on $C_b \times \cdots \times C_b \times \mathbb{Z}$

This section is the conceptual core of the proof. I show how the Weierstrassian walk with  $k$  scales  $Z$  can be studied as  $k$  dependent random walks on the space  $C_b \times \cdots \times C_b \times \mathbb{Z}$ . For this we first define in Section 4.6.2 the  $k$  random walks  $Z_0, \dots, Z_{k-1}$  on  $C_b \times \cdots \times C_b \times \mathbb{Z}$ . Then, in Section 4.6.2, we establish how the pointwise probabilities of  $Z$  can be obtained by the pointwise probabilities of the  $Z_i$ .

#### Definitions and Notations

**Definitions.** Define, for any  $i \in [k-1]$ :

$$S_i(m) := \sum_{s=1}^m \xi_s \cdot \mathbf{1}_{V_s=b^i},$$

the simple (unitary) random walk on the line corresponding to the steps of length  $b^i$ , and

$$J_i(m) := b^i S_i(m),$$

the sum of the steps of length  $b^i$ . Note that

$$Z(m) = \sum_{i \leq k-1} J_i(m) = \sum_{i \leq k-1} S_i(m) b^i. \quad (4.15)$$

Define also:

$$J'_i(m) := \sum_{j=0}^{i-1} J_j,$$

the sum of the steps of length at most  $b^{i-1}$ .

**Base  $b$  decomposition.** Define, for any  $x \in \mathbb{Z}$ , the (truncated) base  $b$  decomposition of  $x$  as:

$$x = \sum_{i=0}^{k-1} x_i b^i,$$

with  $x_i \in [0, b-1]$  for any  $i \in [0, k-2]$  and  $x_{k-1} \in \mathbb{Z}$ . This decomposition exists for any  $x \in \mathbb{Z}$  and is unique.

*Remark 4.9.* For any  $x \in \mathbb{Z}$ , and any  $i \in [0, k-2]$ , we have  $x_i = \lfloor x b^{-i} \rfloor \bmod b$ . We have also  $x_{k-1} = \lfloor x b^{-(k-1)} \rfloor$ . To see why, note that for any  $i \in [0, k-2]$ ,  $x b^{-i} = \sum_{j \leq k-1} x_j b^{j-i} = \sum_{j \leq i-1} x_j b^{j-i} + x_i + \sum_{j \in [i+1, k-1]} x_j b^{j-i}$ , so that  $\lfloor x b^{-i} \rfloor \bmod b = x_i + \lfloor \sum_{j \leq i-1} x_j b^{j-i} \rfloor \bmod b$ . Since  $0 \leq x_j \leq b-1$ , we have  $0 \leq \sum_{j \leq i-1} x_j b^j \leq b^i - 1$ , hence  $\lfloor x b^{-i} \rfloor \bmod b = x_i$ . For  $i = k-1$ , the proof is similar, except we do not need to take modulo  $b$  (as  $x_{k-1} \in \mathbb{Z}$ ).

**Decomposition of  $Z$  in the base  $b$ .** In this base, let us denote by  $Z_i$  the  $i$ -th coordinate of  $Z$ , so that:

$$Z(m) = \sum_{i=0}^{k-1} Z_i(m)b^i.$$

By Remark 4.9 and Eq. (4.15), we have, for  $i \leq k-2$ ,

$$\begin{aligned} Z_i(m) &= Z(m)b^{-i} \pmod{b} = \sum_{j \leq k-1} S_j(m)b^{j-i} \pmod{b} = \sum_{j \leq i} S_j(m)b^{j-i} \pmod{b} \\ &= R_i(m) + N_i(m) \pmod{b} \end{aligned}$$

where we define, for  $i \leq k-2$ ,

$$R_i(m) := S_i(m) \pmod{b},$$

and

$$N_i(m) := \left\lfloor \left( \sum_{j \leq i-1} S_j(m)b^j \right) b^{-i} \right\rfloor \pmod{b} = \lfloor J'_i(m)b^{-i} \rfloor \pmod{b}.$$

Similarly, we decompose  $Z_{k-1}(m)$  as the sum of  $R_{k-1}(m) = S_{k-1}(m)$  and

$$N_{k-1}(m) = \lfloor J'_{k-1}(m)b^{-(k-1)} \rfloor. \quad (4.16)$$

$R_i$  corresponds to the steps of length  $b^i$  and is a lazy random walk on  $C_b$  that moves with probability  $\frac{p_i}{2}$ .  $N_i$  can be thought of as the noise from smaller coordinates. For instance, if  $Z_{i-1}(m)$  is  $b-1$  and a step of length  $b^{i-1}$  is done, then  $N_i(m)$  will be incremented of one. Note that  $N_0(m) = 0$  always, and that the  $k-1$ 'st coordinate is defined on  $\mathbb{Z}$ , thus  $R_{k-1}$  and  $N_{k-1}$  are not defined modulo  $b$ .

So far, we have decomposed  $Z$  as a linear combination of  $k-1$  simple, dependent, random walks. Now we will define additional variables that will allow to control the dependencies between the  $Z_i$ .

**Number of steps of length  $b^i$ .** Denote by  $M_i(m)$  the number of steps of length  $b^i$  done up to move  $m$ , i.e.,

$$M_i(m) = \sum_{s \leq m} \mathbf{1}_{V_s = b^i}.$$

This random variable follows a binomial distribution with parameter  $p_i = c_b b^{-i}$ , and is thus concentrated around its mean:

$$\mu_i := mp_i = c_b mb^{-i} = \Theta(mb^{-i}).$$

Precisely, I will show that  $M_i(m)$  is likely to belong to the interval:

$$Q_i := \begin{cases} [\frac{1}{2}\mu_i, \frac{3}{2}\mu_i], & \text{for } i > 0, \text{ and} \\ [\frac{1}{4}m, m] & \text{for } i = 0. \end{cases} \quad (4.17)$$

### From $Z$ to the $Z_i$

**Fixing the number of steps  $M_i(m)$ .** Here, we look at what happens when we fix the number of steps of length  $b^i$ ,  $M_i(m)$ , to be  $q_i$ . I start with the following important remark.

*Remark 4.10.* In general, the variables  $R_i(m)$  and  $N_i(m)$  are dependent. For example if  $M_1(1) = 1$ , then  $M_0(1) = 0$ , since we choose only one step-length between 0 and 1. However, once we condition on  $M_i(m) = q_i$ ,  $R_i(m)$  and  $N_i(m)$  become independent and  $R_i(m)$  has then the law of a lazy (with parameter  $\frac{1}{2}$ ) random walk after  $q_i$  steps. I.e., we have, for any  $y \in C_b$ , or  $y \in \mathbb{Z}$  if  $i = k - 1$ , and any  $q_i \leq m$ ,

$$\mathbb{P}(R_i(m) = y \mid M_i(m) = q_i) = \mathbf{p}_{q_i}^{G_i}(y) \quad (4.18)$$

where

$$G_i = \begin{cases} C_b & \text{if } i \in [0, k - 2] \\ \mathbb{Z} & \text{if } i = k - 1, \end{cases}$$

and  $\mathbf{p}_{q_i}^G(y)$  is the law of a lazy (with parameter  $\frac{1}{2}$ ) random walk on  $G \in \{\mathbb{Z}, C_b\}$ , that starts at 0, to visit the node  $y$  at step  $q_i$ .

Considering this remark, we write, with  $m \geq 0$  and  $x = \sum_{j=0}^{k-1} x_j b^j \in \mathbb{Z}$ ,

$$\mathbb{P}(Z(m) = x) = \sum_{q_0 + \dots + q_{k-1} = m} \mathcal{P}_{x, \mathbf{q}} \cdot \mathcal{M}_{\mathbf{q}}, \quad (4.19)$$

where  $\mathbf{q} = (q_0, \dots, q_{k-1})$ ,

$$\mathcal{P}_{x, \mathbf{q}} = \mathbb{P}(Z(m) = x \mid \forall t \leq k - 1, M_t(m) = q_t),$$

and

$$\mathcal{M}_{\mathbf{q}} = \mathbb{P}(\forall t \leq k - 1, M_t(m) = q_t).$$

Since the base  $b$  decomposition is unique, we have  $Z(m) = x$  if and only if  $Z_s(m) = x_s$  for all  $s \leq k - 1$ . Hence,

$$\mathcal{P}_{x, \mathbf{q}} = \prod_{s=0}^{k-1} \mathbb{P}(Z_s(m) = x_s \mid \mathcal{A}_{s, x, \mathbf{q}}),$$

where  $\mathcal{A}_{s, x, \mathbf{q}}$  denotes the event  $(\forall j < s, Z_j(m) = x_j) \cap (\forall t \leq k - 1, M_t(m) = q_t)$ .

Since  $Z_i(m) = R_i(m) + N_i(m) = x_i$  if and only if  $R_i(m) = x_i - y$  and  $N_i(m) = y$  for some  $y \in C_b$  ( $\mathbb{Z}$  if  $i = k - 1$ ), using Remark 4.10, we have:

$$\mathcal{P}_{x, \mathbf{q}} = \prod_{s=0}^{k-1} \sum_y \mathbb{P}(R_s(m) = x_s - y \mid \mathcal{A}_{s, x, \mathbf{q}}) \cdot \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s, x, \mathbf{q}}). \quad (4.20)$$

Using Eq. (4.18), we have:

$$\mathbb{P}(R_s(m) = x_s - y \mid \mathcal{A}_{s, x, \mathbf{q}}) = \mathbf{p}_{q_s}^{G_s}(x_s - y).$$

Inserting this in Eq. (4.20), we obtain that

$$\mathcal{P}_{x, \mathbf{q}} = \prod_{s=0}^{k-1} \sum_y \mathbf{p}_{q_s}^{G_s}(x_s - y) \cdot \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s, x, \mathbf{q}}). \quad (4.21)$$

Hence, in Eq. (4.19), we have:

$$\mathbb{P}(Z(m) = x) = \sum_{q_0 + \dots + q_{k-1} = m} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \sum_y \mathbf{p}_{q_s}^{G_s}(x_s - y) \cdot \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}). \quad (4.22)$$

**Bounds on pointwise probabilities of  $Z$  taking dependencies between the coordinates into account.** Recall, with Eq. (4.14), that we need only bounds on  $\mathbb{P}(Z(m) = 0)$  and  $\mathbb{P}(Z(m) = x)$  for  $x \in [n]$  to estimate the cover time of the Weierstrassian random walk on  $C_n$ . In the following two lemmas, I show how such bounds can be obtained by the independent study of:

- the distributions of  $M_i(m)$ , studied in Section 4.6.3
- the probability  $\mathbf{p}_q^{G_i}(y)$ . It is given in Section 4.6.3, and
- the noise in the last coordinate,  $N_{k-1}(m)$ , studied in Section 4.6.3.

Note that, when neglecting the dependencies between the coordinates, and assuming that  $M_i(m)$  is exactly its expected value  $mp_i$ , we have, as detailed in the main text,

$$\mathbb{P}(Z(m) = x) = \prod_{s=0}^{k-1} \mathbb{P}(R_s(m) = x_s) = \prod_{s=0}^{k-1} \mathbf{p}_{mp_s}^{G_s}(x_s).$$

Note also that we have  $\prod_{s=0}^{k-1} \mathbf{p}_{mp_s}^{G_s}(x_s) \leq \prod_{s=0}^i \mathbf{p}_{mp_s}^{G_s}(x_s)$  for any  $i \leq k-1$ . This is useful in particular when  $mp_{i+1} \leq 1 \leq mp_i$ , i.e. when  $m \in [c_b b^i, c_b b^{i+1}]$ . The following two lemmas provide the additional components that appear when taking into account the noise and the fact that the number of steps  $M_i(m)$  does not always equal its expected mean  $mp_i$ . We shall first prove the following upper bound.

**Lemma 4.11.** *For any  $m \geq 0$  and any  $i \leq k-1$ ,*

$$\mathbb{P}(Z(m) = 0) \leq \prod_{s=0}^i \max_{y, q_s \in Q_s} \mathbf{p}_q^{G_s}(y) + \sum_{j=0}^i \left( \mathbb{P}(M_j(m) \notin Q_j) \prod_{s=0}^{j-1} \max_{y, q_s \in Q_s} \mathbf{p}_q^{G_s}(y) \right) \quad (4.23)$$

We will prove in Section 4.6.4, that the dominating term of this upper bound is

$$\prod_{s=0}^i \max_{y, q_s \in Q_s} \mathbf{p}_q^{G_s}(y),$$

as is hinted by the intuition. We will also prove the following lower bound. It uses the event  $\mathcal{A}_{k-1,x,\mathbf{q}}$  that we recall as  $(\forall j \leq k-2, Z_j(m) = x_j) \cap (\forall t \leq k-1, M_t(m) = q_t)$ .

**Lemma 4.12.** *For any  $m \geq 0$ , any  $x \in \mathbb{Z}$ , and any  $I$  interval of  $\mathbb{Z}$ ,*

$$\begin{aligned} \mathbb{P}(Z(m) = x) &\geq \min_{\forall i, q_i \in Q_i} \mathbb{P}(N_{k-1}(m) \in I \mid \mathcal{A}_{k-1,x,\mathbf{q}}) \\ &\cdot \min_{y \in I, q \in Q_{k-1}} \mathbf{p}_q^{\mathbb{Z}}(x_{k-1} - y) \cdot \prod_{s=0}^{k-2} \min_{y \in C_b, q \in Q_s} \mathbf{p}_q^{C_b}(y) \\ &\cdot \mathbb{P}(\forall j \leq k-1, M_j(m) \in Q_j). \end{aligned} \quad (4.24)$$

We will prove in Section 4.6.4, that the dominating term of this lower bound, when  $m \geq b^k$ , is the second one, namely,  $\min_{y \in I, q \in Q_{k-1}} \mathbf{p}_q^Z(x_{k-1} - y) \cdot \prod_{s=0}^{k-2} \min_{y \in C_b, q \in Q_s} \mathbf{p}_q^{G_s}(y)$ . Indeed, with  $I$  well-chosen, and for  $m \geq b^k$ , I will prove that the first and last factors are  $\Omega(1)$ , in Sections 4.6.3 and 4.6.3, respectively.

*Proof of Lemma 4.11.* We start with Eq. (4.22)

$$\begin{aligned}
 \mathbb{P}(Z(m) = x) &= \sum_{q_0 + \dots + q_{k-1} = m} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \sum_y \mathbf{p}_{q_s}^{G_s}(x_s - y) \cdot \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}) \\
 &\leq \sum_{q_0 + \dots + q_{k-1} = m} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \max_y \mathbf{p}_{q_s}^{G_s}(y) \sum_y \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}) \\
 &\leq \sum_{q_0 + \dots + q_{k-1} = m} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \max_y \mathbf{p}_{q_s}^{G_s}(y), \tag{4.25}
 \end{aligned}$$

where we used in the last inequality that  $\sum_y \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}) = 1$ . As the number of steps of length  $b^j$ ,  $M_j(m)$ , is likely to belong to  $Q_j$  (defined by Eq. (4.17)), we make the following decomposition of the sum in Eq. (4.25), for any  $i \leq k-1$ :

$$\sum_{q_0 + \dots + q_{k-1} = m} = \sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_i \in Q_i}} + \sum_{j=0}^i \sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{j-1} \in Q_{j-1}, q_j \notin Q_j}} \tag{4.26}$$

The intuition behind this decomposition is that when  $q_0, \dots, q_i \in Q_0 \times \dots \times Q_i$ , we may obtain a good bound on the pointwise probability of the coordinates 0 to  $i$ , giving an upper bound on  $\prod_{s=0}^{k-1} \max_y \mathbf{p}_{q_s}^{G_s}(y)$  (bounding the factors for  $s > i$  by 1). When for some  $j \leq i$ ,  $q_0 \in Q_0, \dots, q_{j-1} \in Q_{j-1}, q_j \notin Q_j$ , we have such a bound for the coordinates 0 to  $j-1$ , yielding a (weaker) bound on  $\prod_{s=0}^{k-1} \max_y \mathbf{p}_{q_s}^{G_s}(y)$ . To compensate for this weaker bound, we use that the event  $M_j(m) \notin Q_j$  is unlikely, to get a bound on  $\mathcal{M}_{\mathbf{q}}$ .

Let us first consider the inner sum in the second sum of Eq. (4.26). We have:

$$\begin{aligned}
 &\sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{j-1} \in Q_{j-1}, q_j \notin Q_j}} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \max_y \mathbf{p}_{q_s}^{G_s}(y) \\
 &\leq \left( \prod_{s=0}^{j-1} \max_{y, q_j \in Q_s} \mathbf{p}_{q_s}^{G_s}(y) \right) \cdot \sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{j-1} \in Q_{j-1}, q_j \notin Q_j}} \mathcal{M}_{\mathbf{q}} \\
 &\leq \left( \prod_{s=0}^{j-1} \max_{y, q_s \in Q_s} \mathbf{p}_{q_s}^{G_s}(y) \right) \cdot \mathbb{P}(M_0(m) \in Q_0, \dots, M_{j-1}(m) \in Q_{j-1}, M_j(m) \notin Q_j) \\
 &\leq \left( \prod_{s=0}^{j-1} \max_{y, q_s \in Q_s} \mathbf{p}_{q_s}^{G_s}(y) \right) \cdot \mathbb{P}(M_j(m) \notin Q_j).
 \end{aligned}$$

By similar computations, we bound the first sum:

$$\sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_i \in Q_i}} \mathcal{P}_{x,\mathbf{q}} \cdot \mathcal{M}_{\mathbf{q}} \leq \prod_{s=0}^i \max_{y, q_s \in Q_s} \mathbf{p}_{q_s}^{G_s}(y).$$

Inserting into Eq. (4.25), we get:

$$\mathbb{P}(Z(m) = x) \leq \prod_{s=0}^i \max_{y, q_s \in Q_s} \mathbf{p}_q^{G_s}(y) + \sum_{j=0}^i \mathbb{P}(M_j(m) \notin Q_j) \prod_{s=0}^{j-1} \max_{y, q_s \in Q_s} \mathbf{p}_{q_s}^{G_s}(y),$$

as desired.  $\square$

*Proof of Lemma 4.12.* Let us recall Eq. (4.22):

$$\begin{aligned} \mathbb{P}(Z(m) = x) &= \sum_{q_0 + \dots + q_{k-1} = m} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \sum_y \mathbf{p}_{q_s}^{G_s}(x_s - y) \cdot \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}) \\ &\geq \sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{k-1} \in Q_{k-1}}} \mathcal{M}_{\mathbf{q}} \cdot \prod_{s=0}^{k-1} \sum_{y \in I_s} \mathbf{p}_{q_s}^{G_s}(x_s - y) \cdot \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}), \end{aligned}$$

where  $I_s = C_b$  for  $s \leq k-2$  and  $I_{k-1} = I$  is any interval of  $\mathbb{Z}$ . We then lower bound  $\mathbf{p}_{q_s}^{G_s}(x_s - y)$  by  $\min_{y \in I_s} \mathbf{p}_{q_s}^{G_s}(x_s - y)$ , and use that  $\sum_{y \in I_s} \mathbb{P}(N_s(m) = y \mid \mathcal{A}_{s,x,\mathbf{q}}) = \mathbb{P}(N_s(m) \in I_s \mid \mathcal{A}_{s,x,\mathbf{q}})$ , which is 1 for  $s \leq k-2$ , and  $\mathbb{P}(N_{k-1}(m) \in I \mid \mathcal{A}_{k-1,x,\mathbf{q}})$  for  $s = k-1$ , to get:

$$\begin{aligned} \mathbb{P}(Z(m) = x) &\geq \sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{k-1} \in Q_{k-1}}} \mathcal{M}_{\mathbf{q}} \cdot \mathbb{P}(N_{k-1}(m) \in I \mid \mathcal{A}_{k-1,x,\mathbf{q}}) \cdot \prod_{s=0}^{k-1} \min_{y \in I_s} \mathbf{p}_{q_s}^{G_s}(x_s - y) \\ &\geq \min_{\forall i, q_i \in Q_i} \left\{ \mathbb{P}(N_{k-1}(m) \in I \mid \mathcal{A}_{k-1,x,\mathbf{q}}) \cdot \prod_{s=0}^{k-1} \min_{y \in I_s} \mathbf{p}_{q_s}^{G_s}(x_s - y) \right\} \cdot \sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{k-1} \in Q_{k-1}}} \mathcal{M}_{\mathbf{q}} \end{aligned}$$

To conclude, we use the definition of  $\mathcal{M}_{\mathbf{q}}$  to see that

$$\sum_{\substack{q_0 + \dots + q_{k-1} = m \\ q_0 \in Q_0, \dots, q_{k-1} \in Q_{k-1}}} \mathcal{M}_{\mathbf{q}} = \mathbb{P}(M_0(m) \in Q_0, \dots, M_{k-1}(m) \in Q_{k-1}).$$

$\square$

### 4.6.3 Estimating the terms in Lemmas 4.11 and 4.12

In order to estimate the terms in Lemmas 4.11 and 4.12, we need to understand

- the distribution of  $M_i(m)$ ,
- the distribution  $\mathbf{p}_q^{G_s}$ .
- the distribution of the noise  $N_{k-1}(m)$ ,

They will be studied in Sections 4.6.3, 4.6.3, and 4.6.3, respectively. But first, let us start with a very technical claim.



### Preliminary technical computations

In what follows, we will use several times the following technical claim.

**Claim 4.13.** For any  $i \geq 0$ , and any constants  $c \in (0, 1)$  and  $c' > 0$ , we have

$$\prod_{s=0}^i \left(1 - ce^{-c'b^{i-s}}\right) = \Theta(1), \quad \text{and} \quad \prod_{s=0}^i \left(1 + ce^{-c'b^{i-s}}\right) = \Theta(1).$$

*Proof.* Let us consider the first product. Remark that it is upper bounded by 1. For the lower bound, as  $c < 1$  all terms are positive and we can take its logarithm,

$$\sum_{s=0}^i \log \left(1 - ce^{-c'b^{i-s}}\right),$$

which is negative as  $c > 0$ . To lower bound it, we upper bound its absolute value. For this, we use that  $e^{-c'b^{i-s}} \leq e^{-c'} < 1$  and  $-\log(1 - t) = O(t)$  for  $t \in (0, e^{-c'})$  to get:

$$-\sum_{s=0}^i \log \left(1 - ce^{-c'b^{i-s}}\right) = O\left(\sum_{s=0}^i e^{-c'b^{i-s}}\right).$$

Then, use that  $e^{-c't} = O(t^{-1})$  for any  $t > 0$  to get:

$$-\sum_{s=0}^i \log \left(1 - ce^{-c'b^{i-s}}\right) = O\left(\sum_{s=0}^i b^{s-i}\right) = O\left(\sum_{s=0}^i b^{-s}\right) = O(1).$$

Taking the opposite of this, and then the exponential, proves the first part of Claim 4.13. The second part is done similarly.  $\square$

### Concentration of $M_i(m)$ around its mean $mp_i$

For any  $i \leq k - 1$ , and any  $m \geq 1$ ,  $M_i(m)$  follows a binomial distribution with parameter  $p_i$  and is thus concentrated around its mean  $mp_i$ . Since  $Q_i = [\frac{1}{2}mp_i, \frac{3}{2}mp_i]$  for  $i > 0$  and  $Q_0 = [\frac{1}{4}m, m]$ , we can use Chernoff's bound (Theorems 4.4 and 4.5 in (Mitzenmacher and Upfal, 2005)) to obtain:

$$\mathbb{P}(M_i(m) \notin Q_i) \leq e^{-cmp_i} = e^{-cc_b mb^{-i}}, \quad (4.27)$$

for some constant  $c > 0$ . This is the basis for the following lemma, which will essentially ensure that, for  $m \geq b^k$ , we can suppose that, for all  $i \leq k - 1$ ,  $M_i(m) \in Q_i$

**Lemma 4.14.** *There are positive constants  $c'$  and  $c''$  such that for  $m \geq c'b^k$ ,*

$$\mathbb{P}(\forall i \leq k - 1, M_i(m) \in Q_i) > c''.$$

*Proof of Lemma 4.14.* Using the union bound and Eq. (4.27), we get:

$$\begin{aligned}
 \mathbb{P}(\exists i \leq k-1, M_i(m) \notin Q_i) &\leq \sum_{i \leq k-1} \mathbb{P}(M_i(m) \notin Q_i) \\
 &\leq \sum_{i \leq k-1} e^{-cc_b m b^{-i}} \leq \sum_{i \leq k-1} e^{-cc_b c' b^{k-i}} \\
 &\leq \frac{1}{ecc'c_b} \sum_{i \leq k-1} \frac{1}{b^{k-i}} \\
 &\leq \frac{1}{ecc'c_b} \frac{1-b^{-k}}{b-1} \leq \frac{2}{ecc'},
 \end{aligned}$$

where we used that  $m \geq c'b^k$  and  $e^{-t} = O(\frac{1}{t})$  for  $t > 0$ . For  $c'$  well-chosen, this is less than  $1 - c''$  with  $c'' > 0$ . Hence, we have:

$$\mathbb{P}(\forall i \leq k-1, M_i(m) \in Q_i) = 1 - \mathbb{P}(\exists i \leq k-1, M_i(m) \notin Q_i) \geq c'',$$

as claimed by Lemma 4.14.  $\square$

### Random walks distributions

I need to recall estimations for the distribution of a random walk over the infinite line, and over the cycle  $C_b$ . Since the random walk over the cycle is obtained by projecting the random walk on  $\mathbb{Z}$  modulo  $b$ , let us first state the results on  $\mathbb{Z}$ .

**Claim 4.15.** For a  $\frac{1}{2}$ -lazy random walk on  $\mathbb{Z}$  that begins at 0, we have, for any  $q \geq 1$ , and any  $y \in \mathbb{Z}$ , the probability to visit  $y$  at step  $q$  is:

$$\mathbf{p}_q^{\mathbb{Z}}(y) \leq cq^{-\frac{1}{2}},$$

with  $c > 0$  some constant. Furthermore, for any constant  $c'' > 0$ , there is a constant  $c' > 0$  such that for any  $y \in [-c'\sqrt{q}, c'\sqrt{q}]$ , we have

$$\mathbf{p}_q^{\mathbb{Z}}(y) \geq c'q^{-\frac{1}{2}}.$$

*Proof.* It is easy to prove that, due to the laziness of parameter  $\frac{1}{2}$ , we have  $\mathbf{p}_q^{\mathbb{Z}}(y) \geq \mathbf{p}_q^{\mathbb{Z}}(y+1)$  for any  $y \geq 0$ . Hence we can restrict what follows to  $y = O(\sqrt{q})$ . In this case, the bounds in (Lawler and Limic, 2010)[Proposition 2.5.3] show that the distribution of a non-lazy random walk on  $\mathbb{Z}$  is of order  $\Theta(q^{-\frac{1}{2}})$ . Going from there to a lazy random walk that moves with probability  $\frac{1}{2}$ , we just need to apply again a concentration argument for a Bernoulli variable. This allows to link the behaviour of the lazy random walk with  $m$  steps with that of the non-lazy random walk with  $\Theta(m)$  steps.  $\square$

**Claim 4.16.** For a  $\frac{1}{2}$ -lazy random walk on  $C_b$  that begins at 0, we have, for any  $q \geq 1$ , and any  $y \in C_b$ :

$$\mathbf{p}_q^{C_b}(y) \leq \begin{cases} cq^{-\frac{1}{2}}, & q \leq b^2 \\ b^{-1}(1 + ce^{-c'qb^{-2}}), & q \geq b^2 \end{cases}$$

where  $c$  and  $c'$  are positive constants. Furthermore there are constants  $c'' \in (0, 1)$  and  $c''' > 0$  such that for any  $q \geq b^2$ ,

$$\mathbf{p}_q^{C_b}(y) \geq b^{-1}(1 - c''e^{-c'''qb^{-2}}).$$

Note that  $c'' < 1$  ensures that this lower bound (which holds for all  $q \geq b^2$ ) is at least  $\Omega(\frac{1}{b})$ . *Proof.* First, the upper bound simply follows as a particular case of the distribution of a random walk in regular graphs (Aldous and Fill, 2002)[Prop 6.18].

The lower bound requires more explanation. Informally, it stems from the mixing properties of the cycle. Recall that the mixing time of the cycle is  $\Theta(b^2)$ , which means that after this time, the nodes have probability roughly  $\frac{1}{b}$  to be visited. In what follows, we make this statement more precise.

Define the *separation distance* as:

$$s(q) = \min_{y \in C_b} \{1 - b \cdot \mathbf{p}_q^{C_b}(y)\} = \inf\{s : \mathbf{p}_q^{C_b}(y) \geq \frac{1-s}{b}, \forall y \in C_b\},$$

and the *total variation distance* as:

$$d(q) = \frac{1}{2} \sum_{y \in C_b} |\mathbf{p}_q^{C_b}(y) - \frac{1}{b}|.$$

We have, as a consequence of the mixing time of the cycle being less than  $b^2$ , that  $d(q) \leq \varepsilon$  for  $q \geq b^2 \log(\varepsilon^{-1})$  (see (Levin et al., 2008)[5.3.1 and Eq. (4.36)]). Furthermore, by (Levin et al., 2008)[Lemma 19.3 and Eq. (4.24)], we have  $s(2q) \leq 1 - (1 - 2d(q))^2$  for any  $q \geq 1$ . Hence, for  $q \geq 2b^2 \log(\varepsilon^{-1})$ , we have  $s(q) \leq 1 - (1 - 2\varepsilon)^2 = 4\varepsilon - 4\varepsilon^2 < 4\varepsilon$ . That is, when  $q \geq 2b^2 \log(\varepsilon^{-1})$ , we have, for any  $y \in C_b$ :

$$\mathbf{p}_q^{C_b}(y) \geq \frac{1}{b}(1 - 4\varepsilon).$$

With the change of variable  $\varepsilon = \exp(-\frac{q}{2b^2})$ , we have

$$\mathbf{p}_q^{C_b}(y) \geq \frac{1}{b}(1 - 4 \exp(-\frac{q}{2b^2})),$$

which is not meaningful (as the bound is negative) when  $q \leq 2b^2$ . In fact, we will use this bound only for  $q \geq Cb^2$ , with  $C = 2 \log(8) > 1$ . This ensures that  $1 - 4 \exp(-\frac{q}{2b^2}) \geq \frac{1}{2}$  which makes for a more useful lower bound.

Now, for  $b^2 \leq q \leq Cb^2$ , we can lower bound  $\mathbf{p}_q^{C_b}(y)$  by  $\mathbf{p}_q^{\mathbb{Z}}(y)$ , and use Claim 4.15, to show that  $\mathbf{p}_q^{C_b}(y) \geq C' \frac{1}{b}$  for some  $C' \in (0, 1)$ . Altogether, we have  $\mathbf{p}_q^{C_b}(y) \geq \frac{1}{b}F(q)$  for any  $q \geq b^2$ , where:

$$F(q) := \begin{cases} C' & \text{for } q \in [b^2, Cb^2] \\ 1 - 4 \exp(-\frac{q}{2b^2}) & \text{for } q \geq Cb^2. \end{cases}$$

To conclude, we need to verify that we can bound  $F(q)$  from below, for all  $q \geq b^2$ , by  $(1 - c''e^{-c'''qb^{-2}})$ , for a good choice of  $c'' \in (0, 1)$  and  $c''' > 0$ . This is equivalent to: establishing that:

$$\begin{cases} c''e^{-c'''qb^{-2}} \geq 1 - C' & \text{for } q \in [b^2, Cb^2] \\ c''e^{-c'''qb^{-2}} \geq 4 \exp(-\frac{q}{2b^2}) & \text{for } q \geq Cb^2, \end{cases}$$

which is in turn equivalent to:

$$\begin{cases} c'' e^{-c''' q b^{-2}} \geq 1 - C' \text{ for } q \in [b^2, C b^2] \\ c'' e^{q b^{-2}(\frac{1}{2} - c''')} \geq 4 \text{ for } q \geq C b^2. \end{cases}$$

Since we are looking for  $c'' < 1$ , for the second condition to be true, we need that  $c''' < \frac{1}{2}$  (otherwise, it is obvious that the condition will not hold for  $q \rightarrow \infty$ ). Given  $c''' < \frac{1}{2}$ , the left hand side of the second equation is increasing with  $q \geq C b^2$  and thus it is enough to verify the condition at  $q = C b^2$ . Similarly, the left hand side of the first equation is decreasing with  $q$  and thus it is enough to verify the condition at  $q = C b^2$ . The system is thus equivalent to:

$$\begin{cases} c'' e^{-c''' C} \geq 1 - C' \\ c'' e^{C(\frac{1}{2} - c''')} \geq 4 \end{cases},$$

which is in turn equivalent to the condition  $c'' e^{-c''' C} \geq M$  for  $M := \max\{1 - C', 4e^{-\frac{C}{2}}\}$ . Since  $C = 2 \log 8$ , we have  $M := \max\{1 - C', \frac{1}{2}\}$ . Since  $M < 1$ , we may take  $c'' = \frac{1+M}{2} < 1$ . Then it suffices to take  $c'''$  small enough, e.g.,  $c''' = \frac{1}{C} \log(\frac{1}{2} + \frac{1}{2M}) > 0$ . With these parameters, we have proved:

$$\mathbf{p}_q^{C_b}(y) \geq \frac{1}{b} F(q) \geq \frac{1}{b} (1 - c'' e^{-c''' q b^{-2}}),$$

for any  $q \geq b^2$ , and with  $c'' < 1$ . This concludes the proof of Claim 4.16.  $\square$

With Claims 4.15 and 4.16, we can obtain the following Lemma. Intuitively, Lemma 4.17 gives the distribution of  $(R_0, \dots, R_{k-1})$  when they are approximated as independent. As I will show, the bounds of Lemma 4.17 are good approximations of the distributions of  $\mathbb{P}(Z(m) = 0)$ .

**Lemma 4.17.** *We have, for any  $m \geq b^k$ , any  $x \in \mathbb{Z}$ ,*

$$\prod_{s=0}^{k-2} \min_{y \in C_b, q \in Q_s} \mathbf{p}_q^{G_s}(x_s) = \Omega(b^{-(k-1)}). \quad (4.28)$$

We have also, for any  $i \leq k-1$ ,  $m \in (b^i, b^{i+1}]$ ,

$$\prod_{s=0}^j \max_{y, q \in Q_s} \mathbf{p}_q^{G_s}(y) = \begin{cases} O(b^{-j-1}) & \text{if } j \leq i-2, \\ O\left(\frac{1}{\sqrt{m b^{i-1}}}\right) & \text{if } j = i-1, \\ O\left(\frac{\sqrt{b}}{m}\right) & \text{if } j = i, \end{cases} \quad (4.29)$$

and, for any  $m \geq b^k$ ,

$$\prod_{s=0}^{k-1} \max_{y, q \in Q_s} \mathbf{p}_q^{G_s}(y) = O\left(\frac{1}{\sqrt{m b^{k-1}}}\right). \quad (4.30)$$

*Proof.* Let us show first Eq. (4.28). For  $j \leq k-2$ ,  $q \in Q_j$  and  $m \geq b^k$ , we have  $q = \Theta(m p_i) = \Theta(m b^{-i}) = \Omega(b^2)$ . Applying the lower bound in Claim 4.16, we have,

for some constants  $c \in (0, 1)$  and  $c'$ ,

$$\begin{aligned} & \prod_{j=0}^{k-2} \min_{y \in C_b, q \in Q_j} \mathbf{p}_q^{G_j}(x_j) \geq \prod_{j=0}^{k-2} \min_{q \in Q_j} \left( b^{-1} (1 - ce^{-c'qb^{-2}}) \right) \\ & \geq b^{-(k-1)} \prod_{j=0}^{k-2} \left( (1 - ce^{-c'c_b mb^{i-2}}) \right) \geq b^{-(k-1)} \prod_{j=0}^{k-2} \left( (1 - ce^{-c'c_b b^{k-2-i}}) \right). \end{aligned}$$

We conclude by applying Lemma 4.13 to show that  $\prod_{j=0}^{k-2} (1 - ce^{-c'c_b b^{k-2-i}}) = \Omega(1)$ .

To prove Eq. (4.29), we proceed similarly. Let  $i \leq k-1$  and  $m \in (b^i, b^{i+1}]$ . Using this time the upper bound from Claim 4.16, we have, for  $j \leq i-2$ ,

$$\prod_{s=0}^j \max_{y \in C_b, q \in Q_s} \mathbf{p}_q^{G_s}(y) \leq \prod_{s=0}^j \max_{q \in Q_s} \left( b^{-1} (1 + c''e^{-c'''qb^{-2}}) \right) = O(b^{-j-1}) \quad (4.31)$$

where the last equality is justified as above. For the cases  $j = i-1$ , by the upper bound in Claim 4.16, for  $q \in Q_{i-1}$ , we have  $\max_y \mathbf{p}_q^{G_{i-1}}(y) = O(\sqrt{\frac{b^{i-1}}{m}})$ . Using Eq. (4.31), we then have

$$\prod_{s=0}^{i-1} \max_{y \in C_b, q \in Q_s} \mathbf{p}_q^{G_s}(y) = O \left( b^{-(i-1)} \sqrt{\frac{b^{i-1}}{m}} \right) = O \left( \frac{1}{\sqrt{mb^{i-1}}} \right). \quad (4.32)$$

For  $j = i$ , with the upper bound in Claim 4.16 (or Claim 4.15 if  $i = k-1$ ), we have  $\max_y p_q^{(i-1)}(y) = O(\sqrt{\frac{b^i}{m}})$ , which, gives, with Eq. (4.32):

$$\prod_{s=0}^i \max_{y \in C_b, q \in Q_s} \mathbf{p}_q^{G_s}(y) = O \left( \sqrt{\frac{b^i}{m}} \cdot \frac{1}{\sqrt{mb^{i-1}}} \right) = O \left( \sqrt{\frac{b}{m}} \right).$$

Finally, for  $m \geq b^k$ , we use again Claims 4.15 and 4.16 to show that

$$\max_{y, q \in Q_s} \mathbf{p}_q^{G_s}(y) \leq b^{-1} \max_{q \in Q_s} (1 + c''e^{-c'''qb^{-2}}),$$

for  $s \leq k-2$ , and

$$\max_{y, q \in Q_{k-1}} \mathbf{p}_q^Z(y) = O \left( \max_{q_{k-1} \in Q_{k-1}} \frac{1}{\sqrt{q_{k-1}}} \right) = O \left( \sqrt{\frac{b^{k-1}}{m}} \right).$$

Hence, we have with Lemma 4.13,

$$\prod_{s=0}^{k-1} \max_{y, q \in Q_s} \mathbf{p}_q^{G_s}(y) = O \left( b^{-(k-1)} \sqrt{\frac{b^{k-1}}{m}} \right) = O \left( \frac{1}{\sqrt{b^{k-1}m}} \right).$$

This concludes the proof of Lemma 4.17.  $\square$

### Noise in the last coordinate

Recall that the last coordinate  $Z_{k-1}$  of  $Z$  verifies  $Z_{k-1}(m) = R_{k-1}(m) + N_{k-1}(m)$ . Since  $R_{k-1}$  is a walk on  $\mathbb{Z}$  that moves with probability  $p_{k-1}/2 = \Theta(b^{k-1})$ , we can expect that  $|R_{k-1}(m)| \approx \sqrt{\frac{m}{b^{k-1}}}$ .

With the following Lemma, we show that, when considering that the variables  $M_i(m)$  are close to their mean, we have  $N_{k-1}(m) = O(\sqrt{\frac{m}{b^k}})$  with at least constant probability, and hence the noise  $N_{k-1}(m)$  is of lesser order than  $R_{k-1}(m)$ , at least with constant probability.

**Lemma 4.18.** *There is a constant  $c' > 0$  such that, for any  $m \geq b^k$ , with  $I = (-u, u)$  and  $u = c' \sqrt{\frac{m}{b^k}}$ ,*

$$\min_{\forall i, q_i \in Q_i} \mathbb{P}(N_{k-1}(m) \in I \mid \mathcal{A}_{k-1, x, \mathbf{q}}) = \Omega(1). \quad (4.33)$$

*Proof.* It is enough to prove that there is a constant  $c'' > 0$  such that, for any  $u > 0$ , and any  $\mathbf{q} = (q_0, \dots, q_{k-1}) \in Q_0 \times \dots \times Q_{k-1}$ ,

$$\mathcal{N}_{x, \mathbf{q}} := \mathbb{P}(|N_{k-1}(m)| < u \mid \mathcal{A}_{k-1, x, \mathbf{q}}) \geq 1 - c'' \frac{\sqrt{\frac{m}{b^k}}}{u-1}. \quad (4.34)$$

Since, by Eq. (4.16),  $N_{k-1}(m) = \lfloor J'_{k-1}(m)b^{-(k-1)} \rfloor$ , we have  $|N_{k-1}(m)| \leq 1 + |J'_{k-1}(m)|b^{-k+1}$ . Thus, defining  $u' = (u-1)b^{k-1}$ , we have:

$$\mathcal{N}_{x, \mathbf{q}} \geq \mathbb{P}(|J'_{k-1}(m)| < u' \mid \mathcal{A}_{k-1, x, \mathbf{q}})$$

By Markov's inequality, we have  $\mathbb{P}(|J'_{k-1}(m)| \geq u' \mid \mathcal{A}_{k-1, x, \mathbf{q}}) \leq \mathbb{E}(|J'_{k-1}(m)| \mid \mathcal{A}_{k-1, x, \mathbf{q}}) \cdot \frac{1}{u'}$  and hence:

$$\mathcal{N}_{x, \mathbf{q}} \geq 1 - \mathbb{E}(|J'_{k-1}(m)| \mid \mathcal{A}_{k-1, x, \mathbf{q}}) \frac{1}{u'}.$$

Since  $J'_{k-1}(m) = \sum_{i \leq k-2} b^i S_i(m)$ , we have  $|J'_{k-1}(m)| \leq \sum_{i \leq k-2} b^i |S_i(m)|$ , therefore:

$$\mathbb{E}(|J'_{k-1}(m)| \mid \mathcal{A}_{k-1, x, \mathbf{q}}) \leq \sum_{i \leq k-2} b^i \mathbb{E}(|S_i(m)| \mid \mathcal{A}_{k-1, x, \mathbf{q}}).$$

Hence,

$$\mathcal{N}_{x, \mathbf{q}} \geq 1 - \frac{\sum_{i \leq k-2} b^i \mathbb{E}(|S_i(m)| \mid \mathcal{A}_{k-1, x, \mathbf{q}})}{u'}. \quad (4.35)$$

Our next goal is to bound  $\mathbb{E}(|S_i(m)| \mid \mathcal{A}_{k-1, x, \mathbf{q}})$ . Recall that conditioning on  $\mathcal{A}_{k-1, x, \mathbf{q}}$ ,  $S_i(m)$  is a lazy (with laziness  $\frac{1}{2}$ ) random walk on  $\mathbb{Z}$  with  $q_i$  (possibly lazy) steps, and we have  $Z_i(m) = x_i$  for every  $i \leq k-2$ . Thus, for every  $i \leq k-2$ ,  $S_i(m) + N_i(m) \bmod b = Z_i(m) = x_i$ . Conditioning on the value  $y_i \in C_b$  taken by  $N_i(m)$ , we have  $S_i(m) = x_i - y_i \bmod b$  and are in the setting of the following claim.

**Claim 4.19.** Let  $S_q$  be a lazy (with parameter  $\frac{1}{2}$ ) random walk on  $\mathbb{Z}$  at step  $q \geq b^2$ , and  $x \in [b]$ . Then there is a constant  $c > 0$  such that:

$$\mathbb{E}(|S_q| \mid S_q = x \bmod b) \leq c\sqrt{q}.$$

The claim essentially says that the conditioning on  $S_q = x \bmod b$ , for any  $x \in [0, b-1]$ , does not change significantly the distance travelled by the walk up to

step  $q$ . Let us delay the proof of Claim 4.19 and assume it for now. Then, by Claim 4.19, for any  $y_i \in C_b$ ,

$$\mathbb{E}(|S_i(m)| \mid \mathcal{A}_{k-1,x,\mathbf{q}} \cap N_i(m) = y_i) \leq c\sqrt{q_i}.$$

Hence,  $\mathbb{E}(|S_i(m)| \mid \mathcal{A}_{k-1,x,\mathbf{q}}) \leq c\sqrt{q_i}$ , and thus, by Eq. (4.35):

$$\mathcal{N}_{x,\mathbf{q}} \geq 1 - \frac{c}{u'} \sum_{i \leq k-2} b^i \sqrt{q_i}.$$

Since  $q_i \in Q_i$ , we have  $q_i = \Theta(mb^{-i})$ . Hence

$$\sum_{i \leq k-2} b^i \sqrt{q_i} = \Theta(\sqrt{m} \sum_{i \leq k-2} \sqrt{b^i}) = \Theta(\sqrt{m} \sqrt{b^{k-2}}).$$

Thus, for some constant  $c'' > 0$ , we have

$$\mathcal{N}_{x,\mathbf{q}} \geq 1 - \frac{c''}{u'} \sqrt{m} \sqrt{b^{k-2}}.$$

Replacing  $u'$  yields Eq. (4.34) and thus establishes Lemma 4.18, assuming Claim 4.19.  $\square$

We next proceed to prove Claim 4.19. [ Proof of Claim 4.19] Let  $x \in \{0, \dots, b-1\}$ . By definition,

$$\begin{aligned} \mathbb{E}(|S_q| \mid S_q = x \pmod{b}) &= \frac{1}{\mathbb{P}(S_q = x \pmod{b})} \sum_{k \geq 1} k \mathbb{P}(|S_q| = k \cap S_q = x \pmod{b}) \\ &= \frac{1}{\mathbb{P}(S_q = x \pmod{b})} \sum_{k \geq 1} \sum_{l \in \mathbb{Z}} k \mathbb{P}(|S_q| = k \cap S_q = x + lb) \\ &= \frac{1}{\mathbb{P}(S_q = x \pmod{b})} \sum_{k \geq 1} \sum_{l \in \mathbb{Z}} k (\mathbb{P}(S_q = k = x + lb) + \mathbb{P}(S_q = -k = x + lb)) \\ &= \frac{1}{\mathbb{P}(S_q = x \pmod{b})} (\theta_x + \gamma_x). \end{aligned} \tag{4.36}$$

where  $\theta_x = \sum_{l \geq 0} (x + lb) \mathbb{P}(S_q = x + lb)$  and  $\gamma_x = \sum_{l \geq 1} (lb - x) \mathbb{P}(S_q = -lb + x)$ . We will prove that  $\gamma_x + \theta_x$  is of order  $\frac{\sqrt{q}}{b}$ . For this, note that

$$\sum_{y=0}^{b-1} \theta_y + \gamma_y = \mathbb{E}(|S_q|) = O(\sqrt{q}). \tag{4.37}$$

Next, let us prove that  $\theta_y + \gamma_y$  does not significantly depend on  $y \in [b]$ , for  $q \geq b^2$ . First, by symmetry of the process, for any  $y \in \{0, \dots, b-1\}$ , we have  $\gamma_y = \sum_{l \geq 1} (lb - y) \mathbb{P}(S_q = lb - y) = \sum_{l \geq 0} (lb + b - y) \mathbb{P}(S_q = lb + b - y) = \Theta_{b-y}$ . Thus,

$$\sum_y \theta_y + \gamma_y = 2 \sum_y \theta_y = O(\sqrt{q}) \tag{4.38}$$

Furthermore, as  $S$  is lazy with parameter  $\frac{1}{2}$ , we have  $\mathbb{P}(S_q = z) \geq \mathbb{P}(S_q = z + 1)$  for any  $q > 0$  and  $z \geq 0$ . Hence,

$$\theta_y \leq \sum_{l \geq 0} (y + lb) \mathbb{P}(S_q = lb) \leq \sum_{l \geq 0} (b + lb) \mathbb{P}(S_q = lb) \leq b \mathbb{P}(S_q = 0 \pmod{b}) + \theta_0. \quad (4.39)$$

Using the same monotony property of the process, we have

$$\begin{aligned} \theta_y &\geq \sum_{l \geq 0} lb \mathbb{P}(S_q = (l+1)b) = \sum_{l \geq 0} (l+1)b \mathbb{P}(S_q = (l+1)b) - b \sum_{l \geq 0} \mathbb{P}(S_q = (l+1)b) \\ &\geq \theta_0 - b \mathbb{P}(S_q = 0 \pmod{b}). \end{aligned}$$

By Claim 4.16, we have, for  $q \geq b^2$ ,  $\mathbb{P}(S_q = 0 \pmod{b}) = \Theta(\frac{1}{b})$ . Hence

$$\theta_y = \theta_0 \pm \Theta(1)$$

and, by summing, we have

$$\sum_y \theta_y = b\theta_0 \pm \Theta(b).$$

Since  $\sum_y \theta_y = O(\sqrt{q})$ , and  $q \geq b^2$ , this implies  $\theta_0 = O(\frac{\sqrt{q}}{b})$ , and hence,  $\theta_y = O(\frac{\sqrt{q}}{b})$ . Combined with Eq. (4.36), we have, for  $q \geq b^2$ ,

$$\mathbb{E}(|S_q| \mid S_q = x \pmod{b}) = O\left(\frac{1}{\mathbb{P}(S_q = x \pmod{b})} \frac{\sqrt{q}}{b}\right) = O\left(b \frac{\sqrt{q}}{b}\right) = O(\sqrt{q}),$$

where in the last equality we use again Claim 4.16. This proves Claim 4.19.  $\square$

#### 4.6.4 Estimating the number of visits to 0 and $x$

Recall, with Eq. (4.14), that we want to find  $p > 0$  and  $m_0$  such that, for any  $x \in \{0, \dots, n-1\}$ ,

$$\frac{N_x}{O} = \frac{\sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) = x)}{\sum_{m=0}^{m_0} \mathbb{P}(Z(m) = 0)} \geq p$$

with  $m_0 p^{-1}$  as small as possible, since the cover time is then  $\tilde{O}(m_0 p^{-1})$ , by Eq. (4.14).

Let us explain intuitively how we find the right  $m_0$ . We want any  $x \in [0, n-1]$  to have a reasonable chance to be visited by  $Z(m_0)$ . As  $x = x_0 + \dots + x_{k-1} b^{k-1} \leq n$ , with nonnegative  $x_i$ , we have  $x_{k-1} \leq \hat{n}$ , where we define:

$$\hat{n} := \lfloor \frac{n}{b^{k-1}} \rfloor.$$

Hence, we are interested in the behaviour of  $Z_0, \dots, Z_{k-2}, Z_{k-1}$  on  $C_b \times \dots \times C_b \times [0, \hat{n}]$ . To ensure that any  $x \in [0, n-1]$  has a reasonable chance to be visited, we require that every coordinate  $R_i$ , for  $i \leq k-2$ , should be mixed. As  $R_i$  is a random walk on  $C_b$  which moves with probability  $p_i/2$ , this happens after  $\Theta(p_i^{-1} b^2) = O(p_{k-2}^{-1} b^2)$  steps. We also require that the coordinate  $R_{k-1}$  has gone to distance at least  $\hat{n}$ , which needs about  $\hat{n}^2 p_{k-1}^{-1} = \Theta(\frac{n^2}{b^{k-1}})$  steps. This leads us to define:

$$m_0 := \max\{p_{k-2}^{-1} b^2, \hat{n}^2 p_{k-1}^{-1}\} = c_b^{-1} b^{k-1} \max\{b, \hat{n}^2\} = \Theta\left(\max\{b^k, \frac{n^2}{b^{k-1}}\}\right),$$

as the minimal number of steps such that both of these conditions are satisfied. Note that  $m_0 \in [c_b^{-1} b^k, c_b^{-1} b^{k+1}]$ , with  $c_b^{-1} = p_0^{-1} \in (1, 2)$  as is explicit in the definition of the Weierstrassian process.



### Estimating the expected number of visits to $x$

**Lemma 4.20.** *The expected number of visits to  $x$  in between steps  $m_0$  and  $2m_0$  is:*

$$N_x = \sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) = x) = \Omega \left( \sqrt{\frac{m_0}{b^{k-1}}} \right). \quad (4.40)$$

*Proof.* To lower bound  $\mathbb{P}(Z(m) = x)$ , we use Eq. (4.24) with  $m \in [m_0, 2m_0]$ , and  $I = (-u, u)$  with  $u = \frac{m}{b^{k-2}} > 1$ . Let us write Eq. (4.24) as the product of four terms:

$$\mathbb{P}(Z(m) = x) \geq T_1 T_2 T_3 T_4.$$

- The first term is:

$$T_1 := \min_{\forall i, q_i \in Q_i} \mathbb{P}(N_{k-1}(m) \in I \mid \forall i (Z_i(m) = x_i) \cap (M_i(m) = q_i)) = \Omega(1).$$

where the last inequality is by Lemma 4.18.

- The second term of Eq. (4.24) is

$$T_2 := \min_{y \in I, q \in Q_{k-1}} \mathbf{P}_q^{\mathbb{Z}}(x_{k-1} - y),$$

in which, as  $q \in Q_{k-1}$ , we have  $q = \Theta(mp_{k-1}) = \Theta(\frac{m}{b^{k-1}})$ . As  $|x_{k-1}| \leq \hat{n} = \lfloor \frac{n}{b^{k-1}} \rfloor$  and  $|y| < u = 1 + c' \frac{m}{b^k}$ , we have  $|x_{k-1} - y| = O(\frac{n}{b^{k-1}} + \frac{m}{b^k}) = O(\frac{n}{b^{k-1}})$  where we verify the last equality easily by using the fact that  $m \in [m_0, 2m_0]$ . Thus,  $|x_{k-1} - y| = O(\hat{n})$ . As in addition,  $q = \Theta(\frac{m}{b^{k-1}}) = \Omega(\hat{n}^2)$  and  $p_q^{\mathbb{Z}}$  is the distribution of a lazy random walk on the line, which is given by Claim 4.15, we have:

$$T_2 = \Omega \left( \frac{1}{\sqrt{q}} \right) = \Omega \left( \sqrt{\frac{b^{k-1}}{m_0}} \right).$$

- The third term of Eq. (4.24) verifies, by Lemma 4.17,

$$T_3 := \prod_{j=0}^{k-2} \min_{y \in C_b, q \in Q_j} p_q^j(y) = \Omega(b^{-(k-1)}).$$

- Finally, the fourth term of Eq. (4.24) verifies, by Lemma 4.14,

$$T_4 := \mathbb{P}(\forall j \leq k-1, M_j(m) \in Q_j) = \Theta(1).$$

Altogether, we obtain:

$$\mathbb{P}(Z(m) = x) = \Omega(T_1 T_2 T_3 T_4) = \Omega \left( \frac{1}{\sqrt{b^{k-1} m_0}} \right),$$

which implies that the total expected number of visits to  $x$  between steps  $m_0$  and  $2m_0$  is

$$\sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) = x) = \Omega \left( \sqrt{\frac{m_0}{b^{k-1}}} \right),$$

as claimed by Lemma 4.20.  $\square$

### Estimating the expected number of returns to the origin

We want to bound the expected number of returns to 0 up to step  $m_0$ . Ideally, we would like to match the upper bound, found in Lemma 4.20, on the expected number of visits to  $x$ , which is  $O(\sqrt{m_0 b^{-(k-1)}})$ . The following Lemma shows this is nearly the case, up to a factor of  $k \log b$ .

**Lemma 4.21.** *The expected number of returns to 0 up to step  $m_0$  is*

$$O = \sum_{m=0}^{m_0} \mathbb{P}(Z(m) = 0) = O\left(\sqrt{\frac{m_0}{b^{k-1}}} k \log b\right).$$

*Proof.* To estimate  $O$ , the strategy, as presented in the sketch of the proof, starts with the following decomposition:

$$O = \sum_{m=0}^{m_0} \mathbb{P}(Z(m) = 0) = 1 + \frac{1}{2} + \sum_{i=0}^{k-1} \sum_{m=1+b^i}^{b^{i+1}} \mathbb{P}(Z(m) = 0) + \sum_{m=1+b^k}^{m_0} \mathbb{P}(Z(m) = 0).$$

The main idea is to use that, for  $i \leq k-1$ , between the steps  $b^i$  and  $b^{i+1}$ , the coordinates 0 to  $i-2$  are mixed, and that we know short-time probability bounds for the coordinates  $i-1$  and  $i$ .

Precisely, let  $i \in [1, k-1]$  and  $m \in (b^i, b^{i+1}]$ . Recall that Lemma 4.11 states that:

$$\mathbb{P}(Z(m) = 0) \leq \sum_{j=0}^i \left( \mathbb{P}(M_j(m) \notin Q_j) \prod_{s=0}^{j-1} \max_{q_s \in Q_s} p_q^{G_s}(0) \right) + \prod_{s=0}^i \max_{q_s \in Q_s} p_q^{G_s}(0). \quad (4.41)$$

By Eq. (4.29) in Lemma 4.17 and Eq. (4.27), we have:

$$\mathbb{P}(Z(m) = 0) = O\left(\sum_{j=0}^{i-1} \left(e^{-cmb^{-j}} b^{-j}\right) + e^{-cmb^{-i}} \frac{1}{\sqrt{m} \cdot \sqrt{b^{i-1}}} + \frac{\sqrt{b}}{m}\right). \quad (4.42)$$

Using that, for any  $t > 0$ ,  $e^{-t} \leq 2t^{-2}$ , we have

$$\sum_{j=0}^{i-1} \left(e^{-cmb^{-j}} b^{-j}\right) \leq \sum_{j=0}^{i-1} \left(\frac{2}{c^2 m^2 b^{-2j}} b^{-j}\right) = O\left(\frac{1}{m^2} \sum_{j=0}^{i-1} b^j\right) = O\left(\frac{b^i}{m^2}\right) = O\left(\frac{1}{m}\right),$$

where we used in the last equality that  $m \geq b^i$ . For the middle term of Eq. (4.41), we use that  $e^{-t} \leq t^{-1}$  for any  $t > 0$ . Hence,  $e^{-cmb^{-i}} \frac{1}{\sqrt{m} \cdot \sqrt{b^{i-1}}} = O\left(\frac{\sqrt{b}}{m}\right)$ . Altogether, we have

$$\mathbb{P}(Z(m) = 0) = O\left(\frac{\sqrt{b}}{m}\right) \quad (4.43)$$

We now sum Eq. (4.43) for  $m$  between  $b^i$  and  $b^{i+1}$ :

$$\begin{aligned} \sum_{m=1+b^i}^{b^{i+1}} \mathbb{P}(Z(m) = 0) &= O\left(\sum_{m=1+b^i}^{b^{i+1}} \frac{\sqrt{b}}{m}\right) = O\left(\int_{b^i}^{b^{i+1}} \frac{\sqrt{b}}{u} du\right) \\ &= O\left(\sqrt{b} \log\left(\frac{b^{i+1}}{b^i}\right)\right) = O\left(\sqrt{b} \log b\right). \end{aligned} \quad (4.44)$$

Summing Eq. (4.44) for  $i = 1, \dots, k-1$ , we have:

$$\sum_{m=1+b}^{b^k} \mathbb{P}(Z(m) = 0) = O\left(k\sqrt{b} \log b\right). \quad (4.45)$$

For  $m \in [2, b]$ , by Lemma 4.11 and Eq. (4.27) applied with  $i = 0$ , and Claim 4.16, we have  $\mathbb{P}(Z(m) = 0) = O(e^{-cc_b m} + \frac{1}{\sqrt{m}}) = O(\frac{1}{\sqrt{m}})$ . Thus,

$$\sum_{m=2}^b \mathbb{P}(Z(m) = 0) = O\left(\sum_{m=2}^b m^{-\frac{1}{2}}\right) = O(\sqrt{b}). \quad (4.46)$$

Finally, let us bound the expected number of returns to the origin between steps  $b^k$  and  $m_0$ . We use Eq. (4.23) (with  $i = k-1$ ), Eq. (4.27) and Lemma 4.17 to obtain, for  $m \geq b^k$ ,

$$\mathbb{P}(Z(m) = 0) = O\left(\sum_{j=0}^{k-1} \left(e^{-cmb^{-j}} b^{-j}\right) + \frac{1}{\sqrt{b^{k-1}}\sqrt{m}}\right) = O\left(\frac{1}{\sqrt{b^{k-1}}\sqrt{m}}\right),$$

where in the last equality, we use again that  $e^{-t} \leq t^{-2}$ , and  $m \geq b^k$ . Summing this for  $m \in (b^k, m_0]$ , we use again a comparison to an integral:

$$\begin{aligned} \sum_{m=1+b^k}^{m_0} \mathbb{P}(Z(m) = 0) &= O\left(\sum_{m=1+b^k}^{m_0} \frac{1}{\sqrt{b^{k-1}}\sqrt{m}}\right) = O\left(\int_{b^k}^{m_0} \frac{1}{\sqrt{b^{k-1}}\sqrt{u}} du\right) \\ &= O\left(\sqrt{\frac{m_0}{b^{k-1}}}\right). \end{aligned} \quad (4.47)$$

Combining Eqs. (4.45), (4.46) and (4.47), we have:

$$\sum_{m=0}^{m_0} \mathbb{P}(Z(m) = 0) = O\left(k\sqrt{b} \log b + \sqrt{\frac{m_0}{b^{k-1}}}\right) = O\left(\sqrt{\frac{m_0}{b^{k-1}}} k \log b\right),$$

where we used in that last inequality that  $m_0 \geq b^k$  and hence  $\sqrt{\frac{m_0}{b^{k-1}}} \geq \sqrt{b}$ . This concludes the proof of Lemma 4.21.  $\square$

#### 4.6.5 Concluding the Proof of Theorem 4.4

Now we have by Lemmas 4.20 and 4.21:

$$\frac{\sum_{m=m_0}^{2m_0} \mathbb{P}(Z(m) = x)}{\sum_{m=0}^{m_0} \mathbb{P}(Z(m) = 0)} = \Omega\left(\sqrt{\frac{m_0}{b^{k-1}}} \cdot \frac{1}{\sqrt{\frac{m_0}{b^{k-1}}} k \log b}\right) = \Omega\left(\frac{1}{k \log b}\right),$$

and, by Eq. (4.14), the cover time of the Weierstrassian random walk with parameter  $b$  on  $C_n$  is:

$$O(m_0 \cdot k \log b \cdot k \log n) = O(m_0 k^2 \log b \log n).$$

Since we have defined

$$m_0 = \Theta\left(b^{k-1} \max\{b, \hat{n}^2\}\right) = \Theta\left(b^{k-1} \max\left\{b, \frac{n^2}{b^{2(k-1)}}\right\}\right) = \Theta\left(n \max\left\{\frac{b^k}{n}, \frac{n}{b^{k-1}}\right\}\right),$$

this concludes the proof of Theorem 4.4.  $\square$

# Chapter 5

## Conclusion

**Summary of the results.** Throughout this dissertation, I have analysed theoretical models of search when the navigational abilities of the agent are minimal, i.e., the agent has to pause to detect targets and, after such a pause, has lost its previous orientation; more generally, the agent cannot remember information on the environment, nor does it respond to external or internal stimuli. Such a setting is highly restrictive; however I have shown that, if the agent is at least able to do ballistic steps of length given by a probability distribution  $\mathbf{p}$ , then it can search a two-dimensional space highly effectively. More precisely, I have shown in Chapter 3 that when  $\mathbf{p}$  is a truncated power-law  $\mathbf{p}_{\mu, \ell_{max}}$  of exponent  $\mu$ , truncated at an appropriately chosen  $\ell_{max}$ , and targets appear in different sizes, then  $\mu$  close to 1 is not efficient for small targets, while  $\mu$  close to 3 is not efficient for large targets. In contrast, the middle value  $\mu = 2$ , which gives rise to Cauchy Walks, perform almost optimally (at most a multiplicative factor  $O(\log^3 n)$  on the unconditional lower bound  $\Theta(n/D)$ ) for all target sizes.

Furthermore, with Chapter 4, I have shown that if the agent is able only to approximately reproduce  $\mathbf{p}_{2, \ell_{max}}$ , then the search efficiency is still appreciably improved in comparison to a Brownian motion. Precisely, I considered a model in which an agent is able to use  $k$  distinct lengths. I have then shown that the discrete approximation, called Weierstrassian Walk, to a Cauchy Walk, performs optimally (in time roughly  $n^{\frac{2k}{2k-1}}$ ) among all random walks that utilize  $k$  lengths; furthermore, I have suggested, by a lower bound, that such an algorithm, to be efficient, must resemble a Weierstrassian Walk.

**Implications in the foraging literature.** As discussed in Chapter 1, there is some ongoing debate as to whether Lévy Walks are just descriptions of the trajectories, or if they can be, at least in some instances, appropriate behavioural models of searchers. Moreover, in both cases, it was not clear under which setting a Lévy Pattern (or Walk) of exponent  $\mu$  is expected. While the definite answers to these questions are out of the scope of this thesis, I do hope that the results presented here shed some light on the problem. Indeed, I expect simple organisms with minimal navigational abilities to adopt strategies leading to Cauchy Patterns – and there is some evidence that they do, since (Sims et al., 2014) showed in extinct sea urchins the presence of three-modal Weierstrassian Patterns, which, I have argued, are optimal in comparison to other walks utilizing three step lengths. This is relevant for the study of more complex animals since the algorithms of early species could be

the founding blocks of later species' search strategies.

For instance, more complex animals might be able to adapt the exponent  $\mu$  of their step-length distribution to their environment. I have suggested in Chapter 3 how the dependency on  $\mu$  could evolve according to the target sizes. When targets appear in various sizes, a Cauchy Pattern could emerge. Interestingly, a deviation from this result, biased to smaller  $\mu$ , could represent an indicator of how much animals prefer to find large targets rather than small targets.

**Open problems.** I have shown that Cauchy Walks are surprisingly good at exploring an Euclidean space. Although such spaces are natural, in many cases the space can be crowded, i.e., full of obstacles. Hence, the question arises whether these walks also perform efficiently in percolated grids. More generally, the Walk or Probe problem, introduced in Chapter 2 as the extension of the intermittent, memoryless, search problem on general graphs, has not been solved completely, and has natural ties with the theory of random walks on graphs. In particular, it would be of interest to see if a strategy where the waiting time between two probes follow a power-law distribution would perform reasonably well in general. If the answer is positive, it would yield a search algorithm in general graphs, attractive by its simplicity of implementation. If the answer is negative, it would be interesting to see whether the bad-case graphs could arise in a biological situation, and, if so, how animals respond when confronted to such topologies. Such a methodology has recently been fruitful (Gelblum et al., 2020), and it is my hope that it continues to be so.

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