



H infinity approximation, analytic interpolation and convex optimisation : application to broadband impedance matching

Gibin Bose

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THÈSE DE DOCTORAT

Approximation H^∞ , Interpolation Analytique et Optimisation Convexe :
Application à l'Adaptation d'Impédance Large Bande

Présentée en vue de l'obtention du grade de docteur en
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Doctoral School Sciences et Technologies de l'Information et de la Communication
Research Center Inria Sophia Antipolis - Méditerranée

PhD Thesis

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of the Université Côte d'Azur

Specialized in: Electronics, Microelectronics, Microwaves

by

Gibin Bose

H^∞ Approximation, Analytic Interpolation and Convex Optimisation : Application to Broadband Impedance Matching

Defended on January 8th 2021 in front of the jury composed by:

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To my parents
VASANTHA and BOSE

To my partner
ASWANI

and my entire family

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Gibin Bose
Nice, October 2020

Notations

\mathbb{N}	the set of natural numbers
\mathbb{R}	field of real numbers
\mathbb{C}	field of complex numbers
\mathbb{D}	open unit disc
Π^+	open right half plane
\mathbb{B}	the set of analytic functions from Π^+ into $\overline{\mathbb{D}}$
$H^\infty(\Pi^+)$	the space of bounded holomorphic functions in the right half plane
$H^2(\Pi^+)$	the Hardy space of exponent 2 of the right half plane
$L^\infty(j\mathbb{R})$	the space of essentially bounded functions on the imaginary axis
$L^2(j\mathbb{R})$	the space of square integrable functions on the imaginary axis
$\delta(a, b)$	pseudo-hyperbolic distance between a and b , where $a, b \in \mathbb{D}$
\underline{P}	the orthogonal projection from L^2 onto $\bar{H}^2 \stackrel{\text{def}}{=} L^2 \ominus H^2$
H_Φ	the Hankel operator with symbol Φ
\mathbb{P}_N	polynomials of degree at most N
\mathbb{P}_{2N}^+	non-negative polynomials of degree at most $2N$
\mathbb{SB}_N	polynomials of degree at most N , having no zeros inside Π^+
\mathbb{I}	finite union of disjoint compact intervals over the frequencies
\mathbb{S}_m	the set of $(m \times m)$ Hermitian matrices
\mathbb{S}_m^+	the set of $(m \times m)$ positive semi-definite Hermitian matrices
\mathbb{S}_m^-	the set of $(m \times m)$ negative semi-definite Hermitian matrices
$\text{Re}(z)$	real part of complex number z
$F \circ L$	chaining of two-port scattering matrices F and L
$F^*(s)$	para-hermitian conjugate of matrix valued function $F(s)$
$\det(M)$	determinant of matrix M
$\text{tr}(M)$	trace of matrix M
$\deg(p)$	degree of polynomial p
S	scattering matrix in belevitch form associated with a 2×2 system
$\mathbf{Tz}(S)$	the set of transmission zeros associated with 2×2 system S
$\Delta(P)$	the Pick matrix associated with the sequence of interpolation data $(\xi_k, \frac{L_{22}(\xi_k)}{U_P(\xi_k)})$
\mathbf{I}_k	identity matrix of order k
$\mathbb{B}_{p \times q}(\Pi^+)$	the set of $p \times q$ matrix valued functions $F(s)$ which are analytic in Π^+ and contractive, $(\overline{F(j\omega)})^t F(j\omega) \preceq I_q, \omega \in \mathbb{R}$
$H_k^p(\Pi^+)$	the space of $k \times 1$ matrix valued functions, each entry of which belongs to $H^p(\Pi^+)$

Abstract

The thesis makes an in-depth study of one of the classical problems in RF circuit design, the problem of impedance matching. Matching problem addresses the issue of transmitting the maximum available power from a source to a load within a frequency band. Antennas are one of the classical devices in which impedance matching plays an important role. The design of a matching circuit for a given load primarily amounts to find a lossless scattering matrix which when chained to the load minimize the reflection of power in the total system.

In this work, both the theoretical aspects of the broadband matching problem and the practical applicability of the developed approaches are given due importance. Part I of the thesis covers two different yet closely related approaches to the matching problem. These are based on the classical approaches developed by Helton and Fano-Youla to study the broadband matching problems. The framework established in the first approach entails in finding the best H^∞ approximation to an L^∞ function, Φ via Nehari's theory. This amounts to reduce the problem to a generalized eigen value problem based on an operator defined on H^2 , the Hankel operator, H_Φ . The realizability of a given gain is provided by the constraint, operator norm of H_Φ less than or equal to one. The second approach formulates the matching problem as a convex optimisation problem where in further flexibility is provided to the gain profiles compared to the previous approach. It is based on two rich theories, namely Fano-Youla matching theory and analytic interpolation. The realizability of a given gain is based on the Fano-Youla de-embedding conditions which reduces to the positivity of a classical matrix in analytic interpolation theory, the Pick matrix. The concavity of the concerned Pick matrix allows finding the solution to the problem by means of implementing a non-linear semi-definite programming problem. Most importantly, we estimate sharp lower bounds to the matching criterion for finite degree matching circuits and furnish circuits attaining those bounds.

Part II of the thesis aims at realizing the matching circuits as ladder networks consisting of inductors and capacitors and discusses some important realizability constraints as well. Matching circuits are designed for several mismatched antennas, testing the robustness of the developed approach. The theory developed in the first part of the thesis provides an efficient way of comparing the matching criterion obtained to the theoretical limits.

Keywords: Broadband matching, Scattering parameters, Antennas, LC filters, Belevitch representation, Nehari theorem, Hankel Operator, Analytic interpolation, Convex optimisation, Positive polynomials, Schur functions, Nevanlinna-Pick.

Résumé

La thèse étudie en profondeur l'un des problèmes classiques de la conception de circuits RF, le problème de l'adaptation d'impédance. L'adaptation d'impédance consiste à maximiser le transfert de puissance d'une source à une charge dans une bande de fréquences. Les antennes sont l'un des dispositifs classiques dans lesquels l'adaptation d'impédance joue un rôle important. La conception d'un circuit d'adaptation pour une charge donnée revient principalement à trouver une matrice de diffusion sans perte qui, lorsqu'elle est enchaînée à la charge, minimise la réflexion de la puissance dans l'ensemble du système.

Dans ce travail, les aspects théoriques du problème de l'adaptation et l'applicabilité pratique des approches développées sont dûment pris en compte. La partie I de la thèse couvre deux approches différentes mais étroitement liées du problème de l'adaptation large bande. Le cadre développé dans la première approche consiste à trouver la meilleure approximation H^∞ d'une fonction L^∞ , Φ via la théorie de Nehari. Cela revient à réduire le problème à un problème généralisé de valeurs propres basé sur un opérateur défini sur H^2 , l'opérateur de Hankel, H_Φ . La réalisabilité d'un gain donné est fournie par la contrainte, opérateur norme de H_Φ inférieure ou égale à un. La seconde approche formule le problème de l'adaptation comme un problème d'optimisation convexe où une plus grande flexibilité est fournie aux profils de gain par rapport à l'approche précédente. Il est basé sur deux théories riches, à savoir la théorie de l'adaptation de Fano-Youla et l'interpolation analytique. La réalisabilité d'un gain donné est basée sur les conditions de dé-chaînage de Fano-Youla qui se réduisent à la positivité d'une matrice classique en théorie d'interpolation analytique, la matrice de Pick. La concavité de la matrice de Pick concernée permet de trouver la solution au problème au moyen de l'implémentation d'un problème de programmation semi-défini non linéaire. Ainsi, nous estimons des limites inférieures nettes au niveau d'adaptation pour les circuits d'adaptation de degré fini et fournissons des circuits atteignant ces limites.

La partie II de la thèse vise à réaliser les circuits d'adaptation sous forme de réseaux en échelle constitués d'inductances et de condensateurs et aborde également certaines contraintes importantes de réalisabilité. Les circuits d'adaptation sont conçus pour plusieurs antennes non-adaptées, testant la robustesse de l'approche développée. La théorie développée dans la première partie de la thèse offre un moyen efficace de comparer le niveau d'adaptation atteint aux limites théoriques.

Mots clés: Adaptation large bande, Paramètres de diffusion, Antennes, Filtres LC, Représentation Belevitch, Théorème de Nehari, Opérateur de Hankel, Interpolation analytique, Optimisation convexe, Polynômes positifs, Fonctions Schur, Nevanlinna-Pick.

Contribution of the Thesis

In this chapter, we will give a brief overview of the important contributions and the main results of the work presented in the thesis. The thesis has made a substantial contribution to the problem of impedance matching - both the theoretical aspects and the practical applicability of the developed approaches.

The main contribution of chapter 2 constitute an approach to furnish lower bounds for matching criterion in the case of finite degree matching networks. It is based on the original idea by Helton which utilizes Nehari theory and non-Euclidean geometry to find optimal matching criterion in the case of no degree constraint on the matching circuit. In this chapter, the set of possible reflection coefficients of the total system (matching network together with the load) are restricted to certain subclass of the class of Schur functions (named reference functions) helping to gain control over the degree of the matching network to be synthesized. A numerical implementation scheme to find the optimal finite degree matching network with respect to the family of rational reference functions is provided. The degree bound of the matching network obtained using the developed approach is estimated and the computation of the best finite dimensional matching filter response obtained using this approach on some concrete antenna examples and analytic antenna examples are presented as well. A comparative study of the results obtained on analytic antenna examples with the matching bound obtained by the classical matching theory of Fano is also made.

The main contribution of chapter 3 is to provide a general formulation of the finite degree matching problem by allowing further flexibility to the gain functions compared to chapter 2. The possible set of responses of the global system is generalized to a set of “realizable” rational Schur functions of fixed maximal degree and having fixed transmission polynomial. In this approach, we are able to estimate sharp lower bounds to the matching criterion for finite degree matching circuits and furnish circuits attaining those bounds. The realizability constraint is imposed using the Fano-Youla de-embedding conditions which reduces to a constraint on the positivity of Pick matrix. The chapter presents the formulation of the matching problem as a convex optimisation problem over a subset of positive polynomials of fixed maximal degree and discusses various theoretical results concerning the problem like existence and uniqueness results, convexity of the problem, necessary optimality conditions and critical point equation. The concavity of the concerned pick matrix over the subset of positive polynomials involved in the problem is also proved. This allows the numerical implementation of the problem as a non-linear semi-definite programming problem. The details of numerical implementation is not provided in the thesis since an elaborate description is present in the PhD thesis of David Martínez Martínez [64]. Finally some illustrations

of the numerical implementation of the developed approach on some antenna examples are presented. A comparison of the results obtained with the results from chapter 2 and also the classical matching bounds derived by Fano is made. At the end of this chapter, a section is also devoted to illustrate the connection between two approaches discussed in part I of the thesis, serving to provide an in depth understanding of the core of these two approaches in solving the matching problem.

Chapter 4 contributes to the development of matching techniques by considering various practical constraints regarding the realization of the matching circuits in the case of PCB antennas. In general, in this chapter the transmission zeros of the matching circuit to be synthesized are fixed at either zero or infinity (or both) but a series of different problems are solved in order to better describe the realizable circuits. The realization of the matching circuits as ladder networks consisting of inductors and capacitors is presented. The main contribution of the chapter constitute of describing well formulated optimisation problems to take into consideration different practical constraints in matching circuit synthesis and numerical implementation schemes to solve these problems as well. This includes a matching circuit synthesis scheme allowing transformerless synthesis, including microstrip lines in between the lumped elements and bounding the element values of lumped inductors and capacitors to any desirable range as required by the user.

Chapter 5 provides a contribution of various illustrations of the results obtained by solving different problems in the thesis and their comparisons by considering different antenna prototypes. The theoretical bounds obtained in part I of the thesis helps in providing an idea of how far the matching criterion of the realized matching circuits is from the theoretical limits. The illustrations of various PCB antennas designed together with the matching circuits obtained from the developed approach are also made, contributing to the validation of developed schemes in the thesis.

Chapter 6 provides a practical contribution by indicating some of the important possible directions in which the work presented in the thesis can be carried on in the future.

Structure of the Manuscript

In this chapter we will give a brief overview of the organization of the manuscript and a short summary of all the chapters presented. The manuscript is based on the template available at <https://bu.univ-cotedazur.fr/fr/utiliser-nos-services/deposer-sa-these-ou-son-memoire/deposer-sa-these-de-doctorat>. The LaTeX editor Texmaker [15] is used for writing and formatting the document. The numerical implementation of approaches discussed in the thesis have been performed in MATLAB 2018b [65] and the plots displayed in the document have been converted to tikz format using MATLAB2TikZ toolbox [43]. Figures containing electric circuits and block diagrams are generated using the LaTeX package circuitikz [35].

The main contents of the thesis are divided into two parts where each part consists of three chapters. Part one covers the introductory chapter to the thesis along with two chapters of a detailed study of the broadband impedance matching problem. In part two of the thesis, two chapters discussing the realization of the matching circuits as ladder networks and a study of some of the important practical constraints in the realization are presented. Along with these two chapters, a chapter concluding the thesis and providing future perspectives is also provided in part two.

- In chapter 1, a brief introduction to the thesis is made, providing necessary details about the concepts and tools that would be required during the development of the manuscript. In particular, the necessary mathematical preliminaries in the field of functional analysis are provided along with a general introduction to the description of RF circuits using power waves and scattering matrix. Finally some important concepts related to the rational form of the scattering matrix of any finite lossless two port and an introduction to the broadband impedance matching problem are provided as well.
- Chapter 2 begins with the first formal formulation of the matching problem in the thesis and discusses an operator theoretic approach developed by Helton to solve it. A brief description of the Helton's approach based on Nehari theory is provided. Later on in the chapter, a formulation of the finite degree matching problem and finding solution to it using Helton's approach is presented. The realizability of gain functions which are fixed to be specific type is characterized by the operator norm of the Hankel operator of specific symbol being less than or equal to one. Finding the optimal matching circuit by solving an eigen value problem based on the Hankel operator and the degree bounds of the matching circuit obtained are discussed. The chapter is concluded by providing some illustrations of the numerical implementation of the developed approach on some concrete antenna examples.

- The idea of chapter 3 is to provide a more general formulation of the finite degree matching problem by allowing further flexibility to the gain functions. The possible set of responses of the global system is generalized to a set of “realizable” rational Schur functions of fixed maximal degree and having fixed transmission polynomial. The realizability constraint is imposed using the Fano-Youla de-embedding conditions which reduces to a constraint on the positivity of Pick matrix. This allows the formulation of the matching problem as a convex optimisation problem over a subset of positive polynomials of fixed maximal degree. The chapter presents various theoretical results concerning the problem like existence and uniqueness results, convexity of the problem, necessary optimality conditions and critical point equation. The concavity of the concerned pick matrix over the subset of positive polynomials involved in the problem is also proved. This allows the numerical implementation of the problem as a non-linear semi-definite programming problem. In this approach, we are able to estimate sharp lower bounds to the matching criterion for finite degree matching circuits and furnish circuits attaining those bounds. Finally some illustrations of the numerical implementation of the developed approach on some antenna examples are presented. At the end of this chapter, a section is devoted to illustrate the connection between approaches discussed in part I of the thesis to solve the matching problem as well.
- In chapter 4, various practical constraints regarding the realization of the matching circuits in the case of PCB antennas are taken into consideration. In general, in this chapter the transmission zeros of the matching circuit to be synthesized are fixed at either zero or infinity (or both) but a series of different problems are solved in order to better describe the realizable circuits. The realization of the matching circuits as ladder networks consisting of inductors and capacitors is presented. A matching circuit synthesis scheme allowing transformerless synthesis, including microstrip lines in between the lumped elements and bounding the element values of lumped inductors and capacitors to any desirable range as required by the user is also presented.
- In chapter 5, various illustrations of the results obtained by solving different problems in the thesis and their comparisons are made by considering different antenna prototypes. The theoretical bounds obtained in part I of the thesis helps in providing an idea of how far the matching criterion of the realized matching circuits is from the theoretical limits. The illustrations of some PCB antennas designed together with the matching circuits obtained from the developed approach are also made.
- Chapter 6 provides a conclusion to the manuscript by providing a summary of all the work that has been carried out as a part of the thesis. It also provides an overview of some of the possible directions in which the work presented in the thesis can be carried on in the future.

PART I

Theory of Broadband Matching

CHAPTER 1

Introduction

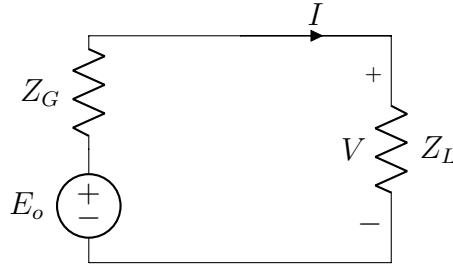
One of the important components in the design of an RF circuit is an impedance matching network which performs the task of minimising reflection of the power that is supposed to be transmitted from a source to a load within a frequency band. The design of impedance matching circuit plays a crucial role in many applications like antennas, multiplexers, amplifiers and other RF devices. The techniques for matching the impedance of a microwave device over a broadband has been evolving starting from around 1940's. The development of scattering techniques in network theory has played a crucial role in the design of broadband impedance matching networks.

A linear electrical network having n accessible terminal pairs (an n -port) may be analyzed in a variety of ways, for instance using impedance, admittance or transmission matrix depending upon the type of problem dealt with. The scattering parameters form a matrix of transformations between variables which are linear combinations of the voltages and currents in a network. The close association of scattering parameters with the power transfer properties of a network make them useful in problems involving insertion loss and matching networks. Moreover, at microwave frequencies, since the direct measurements usually involve the magnitude and phase of a wave travelling in a given direction or of a standing wave, scattering matrix is a representation in accordance with the direct measurements and the ideas of incident, reflected and transmitted waves.

1.1 Power Waves

In 1960's, K.Kurokawa introduced a new concept of waves known as incident and reflected power waves, a_i and b_i respectively for the i th port of any electrical network [55]. The main motivation behind the introduction of power waves was to replace the independent variables, current and voltage in a circuit by these variables in order to simplify the power calculations in circuits.

Let us consider the equivalent circuit of a generator connected to a load as shown in figure 1.1. In figure 1.1, E_o is the open circuit voltage of the generator and Z_G its internal


 Figure 1.1: A generator connected to a load impedance Z_L

impedance. From basic circuit theory, the voltage V across the load and the current I flowing into the load are

$$V = E_o \frac{Z_L}{Z_L + Z_G}, \quad I = \frac{E_o}{Z_L + Z_G}. \quad (1.1)$$

The power transferred from the generator to the load is (Section 4.3, [77]),

$$P_L = \frac{1}{2} \text{Re}\{V\bar{I}\}, \quad (1.2)$$

where ‘Re’ represent the real part and \bar{I} is the complex conjugate of I . Substituting for V and I in equation (1.2) from equations in (1.1), we have

$$P_L = \frac{|E_o|^2}{2} \frac{\text{Re}(Z_L)}{|Z_L + Z_G|^2} = \frac{|E_o|^2}{2} \frac{R_L}{(R_L + R_G)^2 + (X_L + X_G)^2} \quad (1.3)$$

$$= \frac{|E_o|^2}{2} \frac{1}{4R_G + \frac{(R_L - R_G)^2}{R_L} + \frac{(X_L + X_G)^2}{R_L}}, \quad (1.4)$$

where $Z_L = R_L + jX_L$ and $Z_G = R_G + jX_G$. For a given source voltage E_o and internal impedance Z_G with $R_G > 0$, it can be easily verified from equation (1.4) that the power P_L attains the maximum value when

$$R_L = R_G \quad \text{and} \quad X_L = -X_G, \quad (1.5)$$

that is $Z_L = \overline{Z_G}$. It is known as the maximum power transfer theorem (Section 1.4.1, [75]). Thus, we have the following expression for the maximal power that can be delivered to the load,

$$P_{max} = \frac{|E_o|^2}{8R_G} = \frac{|E_o|^2}{8R_L}. \quad (1.6)$$

Now, in order to describe the power waves, it should be noted that in the analysis of electric circuits, even though voltage and current at the terminals are generally chosen as the independent variables, any linear transformation of them can be chosen as well, as long as the inverse transformation exists. The incident and reflected power wave amplitudes, a and b respectively, are defined as the following linear transformations of voltage and current (Section 4.3, [77]),

$$a = \frac{V + Z_0 I}{2\sqrt{R_0}}, \quad b = \frac{V - \overline{Z_0} I}{2\sqrt{R_0}}, \quad (1.7)$$

where $Z_0 = R_0 + jX_0$ is known as the *reference impedance*. In practice, the reference impedance is usually the characteristic impedance of transmission line used between network analyzer and network under test and for each port, it can differ (say for the n -th port it is denoted by Z_{0n}). By convention, reference impedance is chosen to be 50 ohm for most of the commercial test equipments. That being said, we keep the possibility of the reference impedance being complex in the discussion.

It is also common in literature to find the definition of power waves in (1.7) made with reference impedance Z_0 equal to the generator impedance Z_G (for example, Kurokawa's definition of power waves, section 1.4, [56]). In this case, referring to figure 1.1, it can be noticed that since $V + Z_G I = E_o$, we have

$$\frac{1}{2}|a|^2 = \frac{|E_o|^2}{8R_G} = P_{max}. \quad (1.8)$$

So, $\frac{1}{2}|a|^2$ can be identified as the maximum power that the generator can supply to the load in this definition of power waves. We will keep the definition of power waves using reference impedance Z_0 since this can be more useful in general (will be reasoned at the end of the section). Inverting the transformation in (1.7), the voltage and current in terms of the power waves follows,

$$V = \frac{a\overline{Z_0} + bZ_0}{\sqrt{R_0}}, \quad I = \frac{a - b}{\sqrt{R_0}}. \quad (1.9)$$

Then the power transferred from the generator to the load can be calculated using this as

$$P_L = \frac{1}{2}\text{Re}\{V\bar{I}\} = \frac{1}{2R_0}\text{Re}\{|a|^2\overline{Z_0} - a\overline{bZ_0} + \overline{a}bZ_0 - |b|^2\overline{Z_0}\} \quad (1.10)$$

$$= \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2. \quad (1.11)$$

The last equality follows since the quantity $\overline{a}bZ_0 - a\overline{bZ_0}$ is pure imaginary. So, we have the power transferred to the load to be equal to the difference between the powers of incident and reflected power waves. The reflection coefficient Γ of the power waves a and b is defined to be the ratio,

$$\Gamma = \frac{b}{a}. \quad (1.12)$$

Making use of (1.7) and the fact that $V = Z_L I$, we have,

$$\Gamma = \frac{V - \overline{Z_0}I}{V + Z_0I} = \frac{Z_L - \overline{Z_0}}{Z_L + Z_0}. \quad (1.13)$$

This implies, when the reference impedance Z_0 is equal to the conjugate of the load impedance,

$$Z_0 = \overline{Z_L}, \quad (1.14)$$

we have the reflected power wave amplitude, b equal to zero. So, under the condition (1.14), using the equations of V and I mentioned in (1.1), the power wave amplitudes can be found

as

$$a = \frac{V + Z_0 I}{2\sqrt{R_0}} = \frac{E_o}{2\sqrt{R_L}} \left(\frac{Z_L}{Z_L + Z_G} + \frac{\overline{Z_L}}{Z_L + Z_G} \right) = E_o \frac{\sqrt{R_L}}{Z_L + Z_G} \quad (1.15)$$

$$b = \frac{V - \overline{Z_0} I}{2\sqrt{R_0}} \frac{E_o}{2\sqrt{R_L}} \left(\frac{Z_L}{Z_L + Z_G} - \frac{\overline{Z_L}}{Z_L + Z_G} \right) = 0. \quad (1.16)$$

It follows from equation (1.11) that

$$P_L = \frac{1}{2} |a|^2 = \frac{|E_o|^2}{2} \frac{R_L}{|Z_L + Z_G|^2}, \quad (1.17)$$

which agrees with equation (1.3). The equations (1.11) and (1.17) leads us to the following interpretation and the physical meaning of power waves : the generator sends the power $\frac{|a|^2}{2}$ towards a load irrespective of the load impedance and if the load impedance doesn't satisfy equation (1.14), a part of the incident power is reflected back. This reflected power is given by $\frac{|b|^2}{2}$ and so the net power delivered to the load is $\frac{|a|^2}{2} - \frac{|b|^2}{2}$.

It should be noted that the condition, $Z_0 = \overline{Z_L}$ doesn't necessarily mean that the maximal power is delivered to the load, which happens only if the conjugate matching condition, $Z_L = \overline{Z_G}$ is satisfied (Maximum power transfer theorem). Referring back to the definition of power waves using $Z_0 = Z_G$, we can see that the condition equivalent to (1.14) will be $Z_G = \overline{Z_L}$ and so, in this case, not only the reflected wave amplitude is zero but maximal power is delivered to the load as well. Following the definition 1.7, the condition $Z_0 = \overline{Z_L}$ results in a zero reflected wave even when the conjugate matching condition between generator and load impedance is not satisfied and hence can be more useful in general.

1.2 Scattering Parameters and Chain Parameters

1.2.1 Scattering Matrix (S)

The scattering parameters for a two-port network is one of the most commonly studied and mainly serves to quantify how energy propagates through microwave devices.

In order to describe the scattering parameters, let us consider a two-port network connected to a generator and a load as shown in figure 1.2. In figure 1.2, E_o is the open circuit voltage of the generator and Z_G its internal impedance, V_1 and V_2 , the voltages across ports 1 and 2 respectively, I_1 and I_2 , the currents entering port 1 and port 2 respectively. For this two-port network, the incident and reflected power waves, a_1, b_1 at port 1 and a_2, b_2 at port 2 respectively, follows the definition as described in the previous section,

$$a_1 = \frac{V_1 + Z_{01} I_1}{2\sqrt{R_{01}}}, \quad b_1 = \frac{V_1 - \overline{Z_{01}} I_1}{2\sqrt{R_{01}}}, \quad (1.18)$$

$$a_2 = \frac{V_2 + Z_{02} I_2}{2\sqrt{R_{02}}}, \quad b_2 = \frac{V_2 - \overline{Z_{02}} I_2}{2\sqrt{R_{02}}}, \quad (1.19)$$

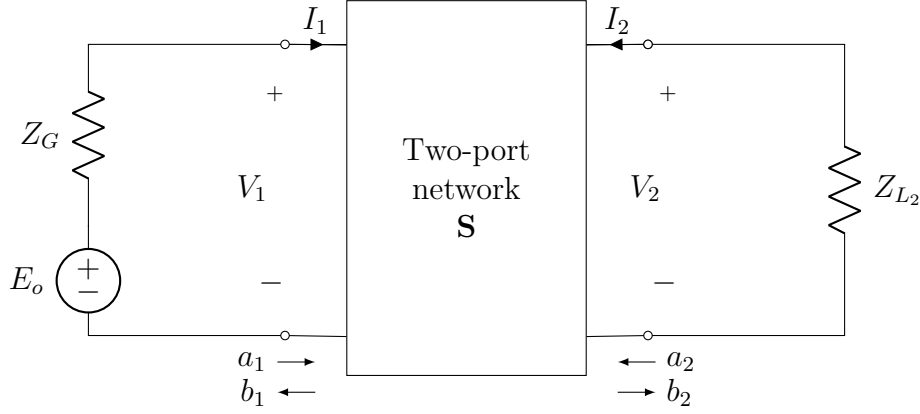


Figure 1.2: A two-port network connected to generator and load

where $Z_{01} = R_{01} + jX_{01}$ and $Z_{02} = R_{02} + jX_{02}$ are the reference impedances for port 1 and port 2 respectively.

The scattering matrix relates the reflected power waves b_1, b_2 to the incident power waves a_1, a_2 on the two port,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \quad (1.20)$$

The matrix elements $S_{11}, S_{12}, S_{21}, S_{22}$ are known as the scattering parameters or the S -parameters.

The measurements of scattering parameters, S_{11} and S_{21} are made by connecting the port one to a generator whereas port two is terminated in a load that satisfies $Z_{L_2} = \overline{Z_{02}}$. As mentioned in (1.14), it is the condition to ensure that there will be no reflected wave from the load, i.e $a_2 = 0$. So, (1.20) gives,

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1, \quad (1.21)$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1. \quad (1.22)$$

This provides the definition of input reflection coefficient, S_{11} and forward transmission coefficient, S_{21} as follows,

$$S_{11} = \left. \frac{b_1}{a_1} \right|_{Z_{L_2} = \overline{Z_{02}}}, \quad S_{21} = \left. \frac{b_2}{a_1} \right|_{Z_{L_2} = \overline{Z_{02}}}. \quad (1.23)$$

Similarly, for S_{12} and S_{22} , reversing the roles of generator and load and denoting the load impedance as Z_{L_1} , under the condition $Z_{L_1} = \overline{Z_{01}}$, there will be no reflected wave from the load, i.e $a_1 = 0$. This provides the definition of reverse transmission coefficient, S_{12} and output reflection coefficient, S_{22} as follows,

$$S_{12} = \left. \frac{b_1}{a_2} \right|_{Z_{L_1} = \overline{Z_{01}}}, \quad S_{22} = \left. \frac{b_2}{a_2} \right|_{Z_{L_1} = \overline{Z_{01}}}. \quad (1.24)$$

The scattering matrix (S-matrix) for the two-port network \mathbf{S} consisting of these scattering parameters,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad (1.25)$$

completely describes the response of network to incident signals on all its ports.

1.2.2 Chain Matrix (T)

The two-port can be represented by means of chain parameters (also known as scattering transfer parameters or T -parameters) as well, which are closely related to the corresponding S -parameters. Following figure 1.2, chain parameters (T -parameters) relates the incident and reflected power waves at each of the two ports by,

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}. \quad (1.26)$$

The matrix T , consisting of these chain parameters,

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad (1.27)$$

is known as the Chain matrix (also known as T -parameter matrix).

Sometimes it is convenient to use the chain matrix over scattering matrix since it provides the utility of calculating the effect of cascading two or more 2-port networks just by means of matrix multiplication of associated individual T -parameter matrices. If the T -parameter matrices of N two-ports are represented by T_1, T_2, \dots, T_N , the chain matrix, T of the cascade of these N two-ports (from left to right), can be easily calculated using the matrix multiplication (Section 1.5, [7]),

$$T = T_1 T_2 T_3 \dots T_N. \quad (1.28)$$

Even though the direct measurement of T -parameters is difficult physically, there exists classic conversion formulas between S -matrix and T -matrix (Section 2.2.4, [34]),

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \frac{1}{S_{21}} \begin{bmatrix} -\det(S) & S_{11} \\ -S_{22} & 1 \end{bmatrix}, \quad (1.29)$$

where, $\det(S) = S_{11}S_{22} - S_{12}S_{21}$ indicates the determinant of matrix S . For the conversion from T -matrix to S -matrix, we have,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \frac{1}{T_{22}} \begin{bmatrix} T_{12} & \det(T) \\ 1 & -T_{21} \end{bmatrix}, \quad (1.30)$$

where, $\det(T) = T_{11}T_{22} - T_{12}T_{21}$ indicates the determinant of matrix T .

It should be noted that the scattering parameters and chain parameters are complex-valued functions of frequency.

1.2.3 Mathematical Preliminaries

In this section, we introduce some of the functional analysis preliminaries which are relevant to the thesis. Throughout the thesis, the imaginary axis, $j\mathbb{R}$ will be the frequency axis, unless specified differently and as usual in electronics or control theory, the frequency variable is $j\omega$, where $\omega \in \mathbb{R}$. In order to describe the stability of systems under consideration, it is necessary to define the domain of analyticity in the complex plane. We use,

$$\Pi^+ = \{x + j\omega \in \mathbb{C} : x > 0\}, \quad (1.31)$$

as the domain of analyticity, unless specified otherwise. An analytic function defined on any open set, function which can be locally expressed as a convergent power series around any point in the domain, corresponds to a stable function in the electronics terminology. It is a classical result in complex analysis that a function is analytic in an open set U iff it is holomorphic in U (i.e complex differentiable at every point in U) (Chapter II, Theorem 5.1 and Chapter III, Theorem 7.2, [57]). In this framework, a stable filter is one whose scattering parameters belong to the following space, $H^\infty(\Pi^+)$ (Definition 1.2.4, [70]),

Definition 1.2.1. (*Hardy space, $H^\infty(\Pi^+)$*). The space of functions, f , such that f is holomorphic in Π^+ and such that,

$$\|f\|_{H^\infty} = \sup_{z \in \Pi^+} |f(z)| < \infty.$$

Generally, for $1 \leq p < \infty$, we have the definition,

Definition 1.2.2. (*Hardy space, $H^p(\Pi^+)$*). The space of functions, f , such that f is holomorphic in Π^+ and such that,

$$\|f\|_{H^p} = \left(\sup_{x>0} \int_{-\infty}^{\infty} |f(x + j\omega)|^p d\omega \right)^{\frac{1}{p}} < \infty.$$

In particular, the closed unit ball of $H^\infty(\Pi^+)$, Schur functions, will be of great interest in the thesis as well.

Definition 1.2.3. (*Schur function*). The holomorphic function, f in Π^+ , satisfying,

$$\sup_{z \in \Pi^+} |f(z)| \leq 1.$$

From here on, we use, \mathbb{B} , to denote the set of Schur functions on Π^+ .

It can be shown that for $p > 0$, functions in $H^p(\Pi^+)$ have nontangential limits at almost every point of $j\mathbb{R}$ (Theorem 2.2, [32]). This allows to speak of their boundary values $\tilde{f}(j\omega) = \lim_{x \rightarrow 0^+} f(x + j\omega)$ a.e and the boundary function, \tilde{f} lies in the space $L^p(j\mathbb{R})$. This leads us to the following definition (Definition 3.6, [83]),

Definition 1.2.4. (*L^∞ space*). The Banach space of essentially bounded (bounded on the complement of a set of measure zero) Lebesgue measurable complex valued functions on $j\mathbb{R}$ equipped with essential supremum norm,

$$\|f\|_{L^\infty} = \text{ess} \sup_{\omega \in (-\infty, \infty)} |f(j\omega)|.$$

Generally, for $1 \leq p < \infty$, we have the L^p norm of f to be defined as,

$$\|f\|_{L^p} = \left(\int_{-\infty}^{\infty} |f(j\omega)|^p d\omega \right)^{\frac{1}{p}}$$

and $L^p(j\mathbb{R})$ consists of all f for which $\|f\|_{L^p} < \infty$. For $f \in H^p(\Pi^+)$, we have its H^p norm to be equal to that of the L^p norm of its boundary function, i.e $\|f\|_{H^p} = \|\tilde{f}\|_{L^p}$ (Theorem 17.11, [83]). We can identify f and \tilde{f} and thus $H^p(\Pi^+)$ can be naturally regarded as a closed subspace of $L^p(j\mathbb{R})$ (chapter 8, [49] and chapter II, section 3, [42]). Another important property to note is that, knowing the boundary values on an arbitrary subset of positive measure of $j\mathbb{R}$ determines the function, $f(z) \in H^p(\Pi^+)$ uniquely (Ch. II, Corollary 4.2, [42]).

The basic structure of H^∞ functions is also going to play an important role later on in the thesis. It is a classical topic in functional analysis with a wide literature available ([70], [32], [83], [49]). We introduce the two important types of functions to which every function in $H^\infty(\Pi^+)$ can be factored into.

Definition 1.2.5. (*Inner Function*). An inner function is any function $g(z)$ holomorphic in Π^+ and satisfying,

$$|g(z)| \leq 1, \quad z \in \Pi^+ \quad \text{and} \quad |g(j\omega)| = 1 \text{ a.e. } \omega \in \mathbb{R}.$$

Definition 1.2.6. (*Outer Function*). An outer function for the class $H^\infty(\Pi^+)$ is a function of the form,

$$h(x + j\omega) = \alpha \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} P_x(\omega - t) \log(\psi(t)) dt \right),$$

where $|\alpha| = 1$, $\psi(t) \in L^\infty$ satisfies $\psi(t) \geq 0$, $\log(\psi(t)) \in L^1$ and $P_x(\omega) = \frac{x}{x^2 + \omega^2}$ is the Poisson Kernel for right half plane.

It should be noted in particular that outer functions do not vanish in Π^+ . Now, we state below the factorisation theorem (Theorem 2.5, [32]) by F.Riesz in particular for the H^∞ function.

Theorem 1.2.7. (*Factorisation Theorem*). Every function $f(z) \not\equiv 0$ of the class $H^\infty(\Pi^+)$ can be factored into the form,

$$f(z) = g(z)h(z),$$

where g is an inner function and h is an outer function in $H^\infty(\Pi^+)$.

Finally, it is useful to recollect the *Maximum Modulus Principle* (Chapter 4, Theorem 12, [5]) and Cauchy's integral formula (Chapter 4, Theorem 6, [5]) in complex analysis,

Theorem 1.2.8. (*Maximum Modulus Principle*). Suppose f is a holomorphic function in an open set \mathbb{U} of \mathbb{C} . If f is not a constant function, then $|f|$ does not attain a maximum on \mathbb{U} .

Theorem 1.2.9. (*Cauchy's Integral Formula*). Suppose that f is analytic in an open disk Δ and γ be a closed curve in Δ oriented counterclockwise. For any point, “ a ” not on γ ,

$$\eta(\gamma, a)f(a) = \frac{1}{2\pi j} \int_{\gamma} \frac{f(z)}{z-a} dz,$$

where $\eta(\gamma, a)$ is the index (winding number) of “ a ” with respect to γ .

Having discussed the necessary mathematical prerequisites we are in a position to continue the discussion about two-port.

1.2.4 Properties of Two-port

A detailed construction of a rigorous theory of the physical realizability postulates of an n -port can be found in [94]. In the work presented in [94], the connection between the physical realizability of n -ports and its scattering matrix being *Bounded Real* was made and it laid the foundations of linear passive network theory.

Definition 1.2.10. (*Bounded Real*). An $n \times n$ scattering matrix, $S = S(s)$ is *Bounded Real* if it satisfies:

- (*Stability*). S is analytic in Π^+ .
- (*Passivity*). $\mathbf{I}_n - \overline{S(j\omega)}^t S(j\omega) \succeq 0$, where \mathbf{I}_n represent the identity matrix of size n .
- (*Reality*). Each element of $S(s)$ is real when s is real.

The reader can refer to chapters 1 and 4 of [20] for a detailed literature about the linear circuits and systems and physical properties of n -ports. In particular, theorem 4.5.1 in [20] provides a detailed proof of the fact that the necessary and sufficient condition for an n -port to be physically realizable is that its scattering matrix S is bounded real.

The utility of power waves and scattering matrix in calculating the quantities like power flow into and out of the two-port and in defining the important characteristics of two-port becomes evident with the following calculations. For any point on the frequency axis, say $j\omega$, denoting the vector of input and output power waves in figure (1.2) as $a(j\omega) = \begin{bmatrix} a_1(j\omega) \\ a_2(j\omega) \end{bmatrix}$

and $b(j\omega) = \begin{bmatrix} b_1(j\omega) \\ b_2(j\omega) \end{bmatrix}$, we have,

$$b(j\omega) = S(j\omega)a(j\omega). \quad (1.32)$$

Fixing the reference impedances at port 1 and port 2 as $Z_0 = R_0 + jX_0$ and inverting the linear transformations in (1.18) and (1.19), we get the voltage and current in terms of power waves and reference impedance as follows,

$$V_1(j\omega) = \frac{a_1(j\omega)\overline{Z_0} + b_1(j\omega)Z_0}{\sqrt{R_0}}, \quad I_1(j\omega) = \frac{a_2(j\omega) - b_2(j\omega)}{\sqrt{R_0}}, \quad (1.33)$$

$$V_2(j\omega) = \frac{a_2(j\omega)\overline{Z_0} + b_2(j\omega)Z_0}{\sqrt{R_0}}, \quad I_2(j\omega) = \frac{a_2(j\omega) - b_2(j\omega)}{\sqrt{R_0}}. \quad (1.34)$$

Then the power entering the two-port, $P_1(j\omega)$ and power leaving the two-port, $P_2(j\omega)$ can be calculated as done in (1.11),

$$P_1(j\omega) = \frac{1}{2} \text{Re}\{V_1(\omega) \overline{I_1(j\omega)}\} = \frac{1}{2}(|a_1(j\omega)|^2 - |b_1(j\omega)|^2) \quad (1.35)$$

$$P_2(j\omega) = \frac{1}{2} \text{Re}\{V_2(j\omega) (-\overline{I_2(j\omega)})\} = \frac{1}{2}(|b_2(j\omega)|^2 - |a_2(j\omega)|^2). \quad (1.36)$$

So, we have the difference in power entering and leaving the two-port,

$$\delta P(j\omega) = P_1(j\omega) - P_2(j\omega) = \frac{1}{2}(|a_1(j\omega)|^2 + |a_2(j\omega)|^2) - \frac{1}{2}(|b_1(j\omega)|^2 + |b_2(j\omega)|^2) \quad (1.37)$$

$$= \frac{1}{2}(\overline{a(j\omega)})^t a(j\omega) - \frac{1}{2}(\overline{b(j\omega)})^t b(j\omega), \quad (1.38)$$

where superscript “ t ” represents the transpose. Since we have $b(j\omega) = S(j\omega)a(j\omega)$, as mentioned in (1.32), it follows that,

$$\delta P(j\omega) = \frac{1}{2}(\overline{a(j\omega)})^t a(j\omega) - \frac{1}{2}(\overline{a(j\omega)})^t (\overline{S(j\omega)})^t S(j\omega) a(j\omega) \quad (1.39)$$

$$= \frac{1}{2}(\overline{a(j\omega)})^t \{\mathbf{I}_2 - (\overline{S(j\omega)})^t S(j\omega)\} a(j\omega), \quad (1.40)$$

where \mathbf{I}_2 represent the identity matrix of size two. If the two port is passive, $\delta P(j\omega)$ has to be positive, which implies that the matrix $(\mathbf{I}_2 - (\overline{S(j\omega)})^t S(j\omega))$ must be positive semi-definite. Hence we have the following definition,

Definition 1.2.11. (*Passivity*). A (2×2) scattering matrix S is said to be passive, if,

$$(\overline{S(j\omega)})^t S(j\omega) \preceq \mathbf{I}_2, \quad \omega \in \mathbb{R}, \quad (1.41)$$

i.e, the matrix $\mathbf{I}_2 - (\overline{S(j\omega)})^t S(j\omega)$ must be positive semi-definite.

Hence, one of the important characteristics of scattering parameters in the case of passive two-ports is the fact that,

$$|S_{i,j}(j\omega)| \leq 1, \quad 1 \leq i, j \leq 2, \quad \omega \in \mathbb{R}. \quad (1.42)$$

It easily follows from *Maximum Modulus Principle* mentioned in theorem 1.2.8 that,

$$|S_{i,j}(s)| \leq 1, \quad 1 \leq i, j \leq 2, \quad s \in \Pi^+. \quad (1.43)$$

This together with the fact that $S_{ij} \in H^\infty(\Pi^+)$ implies that scattering parameters of any passive two-port are Schur functions. So we have the scattering matrix S of a passive two-port to be a *Schur matrix*, i.e the entries of matrix are Schur functions. We will be mostly dealing with passive systems in the thesis, unless specified otherwise.

Going further ahead, if the two port is lossless, we have, $\delta P(j\omega) = 0$ and it follows from (1.40) that the S-matrix should satisfy,

$$(\overline{S(j\omega)})^t S(j\omega) = \mathbf{I}_2 \quad \text{and} \quad S(j\omega) (\overline{S(j\omega)})^t = \mathbf{I}_2 \quad \omega \in \mathbb{R}. \quad (1.44)$$

This leads us to the definition,

Definition 1.2.12. (*Losslessness*). A (2×2) scattering matrix S is said to be lossless if it satisfies,

$$(\overline{S(j\omega)})^t S(j\omega) = S(j\omega) (\overline{S(j\omega)})^t = \mathbf{I}_2, \quad \omega \in \mathbb{R}. \quad (1.45)$$

Remark 1.2.13. It should be noted that a lossless matrix is a Schur matrix (passive). Also, definition 1.2.12 of losslessness is the matrix case of definition 1.2.5 of inner function in the scalar case. So, equivalently, the scattering matrix satisfying equation (1.45) can be called inner in $H^\infty(\Pi^+)$ as well.

Expanding the first matrix equality in (1.44), we get the following equations,

$$S_{11}(j\omega) \overline{S_{11}(j\omega)} + S_{21}(j\omega) \overline{S_{21}(j\omega)} = 1, \quad (1.46)$$

$$S_{22}(j\omega) \overline{S_{22}(j\omega)} + S_{12}(j\omega) \overline{S_{12}(j\omega)} = 1, \quad (1.47)$$

$$S_{11}(j\omega) \overline{S_{12}(j\omega)} + S_{21}(j\omega) \overline{S_{22}(j\omega)} = 0. \quad (1.48)$$

Similarly, expanding the second matrix equality in (1.44), we get the following equations,

$$S_{11}(j\omega) \overline{S_{11}(j\omega)} + S_{12}(j\omega) \overline{S_{12}(j\omega)} = 1, \quad (1.49)$$

$$S_{22}(j\omega) \overline{S_{22}(j\omega)} + S_{21}(j\omega) \overline{S_{21}(j\omega)} = 1, \quad (1.50)$$

$$S_{11}(j\omega) \overline{S_{21}(j\omega)} + S_{12}(j\omega) \overline{S_{22}(j\omega)} = 0. \quad (1.51)$$

Equation (1.48) gives, $|S_{11}(j\omega)|^2 |S_{12}(j\omega)|^2 = |S_{21}(j\omega)|^2 |S_{22}(j\omega)|^2$ and this together with (1.46) and (1.47) gives,

$$|S_{11}(j\omega)|^2 (1 - |S_{22}(j\omega)|^2) = (1 - |S_{11}(j\omega)|^2) |S_{22}(j\omega)|^2, \quad \omega \in \mathbb{R}.$$

This yields,

$$|S_{11}(j\omega)| = |S_{22}(j\omega)|, \quad \omega \in \mathbb{R}. \quad (1.52)$$

So, for a lossless two-port network, power reflection coefficients at one port is equal to that at the other port. Inserting (1.52) into (1.46) and comparing with (1.47), we also have,

$$|S_{12}(j\omega)| = |S_{21}(j\omega)|, \quad \omega \in \mathbb{R} \quad (1.53)$$

a kind of power reciprocal theorem, which is satisfied when the two-port is lossless.

Finally, we can define the notion of reciprocity, i.e the transmission of signal between any two ports does not depend upon the direction of propagation.

Definition 1.2.14. (*Reciprocity*). A (2×2) scattering matrix S is said to be reciprocal if it satisfies,

$$S_{12}(j\omega) = S_{21}(j\omega), \quad \omega \in \mathbb{R}. \quad (1.54)$$

Most of the passive electrical components like resistors, inductors and capacitors are reciprocal even though exceptions exist such as devices based on ferrites, plasmas etc. This completes the description of properties of two-ports. Throughout the thesis, we assume losslessness for the two-port networks which ensures that we deal with lossless scattering matrices. Even though it is not entirely true in practice, it is a reasonable approximation to systems where losses are not very high.

1.3 Rational Form of Scattering Matrices

In this section, we discuss about the rational form of the scattering matrix of a finite lossless two port. It was around the 1960's that V. Belevitch described the scattering matrix of any finite lossless two port as a rational (2×2) -matrix function of the complex variable $s = j\omega$ (Chapter 9, [12]). Before describing the Belevitch representation, it is important to introduce some further notions about rational matrix functions.

Definition 1.3.1. (*McMillan Degree*). The McMillan degree of an $(m \times n)$ rational matrix function, $S(s)$, is the smallest non-negative integer N for which one can write,

$$S(s) = D + C(s\mathbf{I}_N - A)^{-1}B, \quad s \in \mathbb{C}, \quad (1.55)$$

where A, B, C and D are complex matrices of size $(N \times N)$, $(N \times n)$, $(m \times N)$ and $(m \times n)$ and \mathbf{I}_N is the identity matrix of size $(N \times N)$. The representation in (1.55) is called the realisation of $S(s)$.

Remark 1.3.2. It should be noted that every rational $(n \times n)$ matrix function $S(s)$ which is finite at infinity (proper rational matrix function) admits a realisation as mentioned in equation 1.55 (Theorem 4.1.1, [8]). More detailed literature about the linear systems, in particular about the realisation theory can be found in [53].

Definition 1.3.3. (*Para-Conjugate*). For a matrix valued function $S(s)$, the para-conjugate, denoted by $S^*(s)$, is defined as

$$S^*(s) = (\overline{S(-\bar{s})})^t, \quad s \in \mathbb{C}. \quad (1.56)$$

Similarly, for a polynomial, $p(s)$, the para-conjugate is defined as,

$$p^*(s) = \overline{p(-\bar{s})}, \quad s \in \mathbb{C}. \quad (1.57)$$

It should be noted that on the imaginary axis, S^* agrees with the conjugate transpose of S and for polynomials, we have,

$$p^*(j\omega)p(j\omega) = p(j\omega)p^*(j\omega) = |p(j\omega)|^2, \quad \omega \in \mathbb{R}. \quad (1.58)$$

Definition 1.3.4. (*Stable Polynomials or Strictly Hurwitz Polynomials*). A polynomial is called “stable” (or strictly Hurwitz) if it has no roots in $\overline{\Pi^+}$, the closure of analyticity domain and it will be called “stable in the broad sense” (or Hurwitz) if it has no roots in Π^+ , the analyticity domain. We use \mathbb{SB}_N to denote the polynomials of degree at most N which are “stable in the broad sense”.

Definition 1.3.5. (*Blaschke Product*). A function of the form $\epsilon \frac{d^*}{d}$, where d is a stable polynomial of degree N and ϵ is a uni-modular constant is called a Blaschke product of degree N .

Definition 1.3.6. (*Degree of rational function*). The degree of a rational function, $f = \frac{p}{q}$, denoted by $\deg(f)$, where polynomials p and q are co-prime (no common factors) is,

$$\deg(f) = \max(\deg(p), \deg(q)).$$

1.3.1 Belevitch Representation

Lemma 1.3.7. (*Determinant of Rational Lossless Matrix*). *The determinant of a rational lossless matrix with McMillan degree N is a Blaschke product of degree N .*

Proof. Let us denote the rational lossless matrix, S with McMillan degree N as,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad (1.59)$$

where S_{ij} , $1 \leq i, j \leq 2$ are rational Schur functions. We have the expression for determinant of S at any frequency point as,

$$\det(S)(j\omega) = S_{11}(j\omega)S_{22}(j\omega) - S_{12}(j\omega)S_{21}(j\omega), \quad \omega \in \mathbb{R}. \quad (1.60)$$

Since the scattering matrix S is lossless, substituting for $S_{12}(j\omega)$ in the above expression from equation (1.48), we have,

$$\begin{aligned} \det(S)(j\omega) &= S_{11}(j\omega)S_{22}(j\omega) + \frac{\overline{S_{21}(j\omega)}S_{22}(j\omega)}{\overline{S_{11}(j\omega)}}S_{21}(j\omega), \quad \omega \in \mathbb{R} \\ &= \frac{S_{22}(j\omega)}{\overline{S_{11}(j\omega)}} \left(\overline{S_{11}(j\omega)}S_{11}(j\omega) + \overline{S_{21}(j\omega)}S_{21}(j\omega) \right), \quad \omega \in \mathbb{R} \\ &= \frac{S_{22}(j\omega)}{\overline{S_{11}(j\omega)}}. \end{aligned} \quad (1.61)$$

The last equality follows using equation (1.46). From equation (1.52), we have, $|S_{11}(j\omega)| = |S_{22}(j\omega)|$ and this yields the fact that $\det(S)$ is uni-modular on the imaginary axis. From equation (1.60), we also have $\det(S)$ is rational and $\det(S) \in H^\infty(\Pi^+)$ and thus $\det(S)$ is a Blaschke product. The degree of $\det(S)$ in the lowest form (numerator and denominator co-prime) being equal to the McMillan degree can be referred to Chapter 1, Theorem 10-3, [41]. \square

Remark 1.3.8. *A similar calculation for determinant of rational lossless scattering matrix, S can be done by starting from equation 1.60. Substituting for $S_{12}(j\omega)$ from equation (1.51) and later using equation (1.50) will yield,*

$$\det(S)(j\omega) = \frac{S_{11}(j\omega)}{\overline{S_{22}(j\omega)}}, \quad \omega \in \mathbb{R}. \quad (1.62)$$

So, we have the following expression for $\det(S)$ at any frequency point,

$$\det(S)(j\omega) = \frac{S_{22}(j\omega)}{\overline{S_{11}(j\omega)}} = \frac{S_{11}(j\omega)}{\overline{S_{22}(j\omega)}}, \quad \omega \in \mathbb{R}. \quad (1.63)$$

Remark 1.3.9. *The fact that $\det(S)$ is uni-modular on the imaginary axis when S is a rational lossless matrix can be proved more directly as well but the above calculation is useful*

to express $\det(S)$ just in terms of the reflection coefficients, S_{11} and S_{22} . In the direct way, from $(\overline{S(j\omega)})^t S(j\omega) = \mathbf{I}_2$, we have,

$$\begin{aligned} 1 &= \det((\overline{S(j\omega)})^t S(j\omega)), \quad \omega \in \mathbb{R} \\ &= \det(\overline{S(j\omega)}) \det(S(j\omega)), \quad \omega \in \mathbb{R} \\ &= |\det(S(j\omega))|^2, \quad \omega \in \mathbb{R}. \end{aligned}$$

This yields the result.

Theorem 1.3.10. (Belevitch representation). The rational scattering matrix, S , with McMillan degree N , of a lossless two port, can be represented as

$$S = \frac{1}{q} \begin{bmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{bmatrix}, \quad (1.64)$$

where, $p, r \in \mathbb{P}_N$, polynomials of degree at most N , q is stable polynomial of degree N , satisfying the Feldtkeller equation,

$$qq^* = pp^* + rr^*, \quad (1.65)$$

and ϵ is a uni-modular constant. We refer equation (1.65) as spectral equation and q as the stable spectral factor of $pp^* + rr^*$.

Proof. Let us denote the rational matrix, S , of the lossless two port as

$$S = \begin{bmatrix} \frac{p_{11}}{q_{11}} & \frac{p_{12}}{q_{12}} \\ \frac{p_{21}}{q_{21}} & \frac{p_{22}}{q_{22}} \end{bmatrix}, \quad (1.66)$$

where p_{ij} and q_{ij} , $1 \leq i, j \leq 2$, are polynomials and q_{ij} 's in particular are stable (strictly Hurwitz) polynomials. Let us denote the least common denominator of the polynomials q_{ij} as q_c , which will be stable as well. So let us put the above rational matrix over q_c , where the numerators are again denoted by p_{ij} for the sake of simplicity,

$$S = \frac{1}{q_c} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}. \quad (1.67)$$

Since the two-port is lossless, from definition 1.2.12 of losslessness, we have,

$$(\overline{S(j\omega)})^t S(j\omega) = S(j\omega) (\overline{S(j\omega)})^t = \mathbf{I}_2, \quad \omega \in \mathbb{R}.$$

By definition 1.3.3 of para-conjugate, $S^*(j\omega)$ and $(\overline{S(j\omega)})^t$ are equal and hence on the imaginary axis, we have,

$$S^* S = \mathbf{I}_2 \quad \text{and} \quad S S^* = \mathbf{I}_2. \quad (1.68)$$

Expanding the first matrix equality, we get the following equations,

$$p_{11}^* p_{11} + p_{21}^* p_{21} = q_c^* q_c, \quad (1.69)$$

$$p_{11}^* p_{12} + p_{21}^* p_{22} = 0, \quad (1.70)$$

$$p_{12}^* p_{12} + p_{22}^* p_{22} = q_c^* q_c. \quad (1.71)$$

Similarly, expanding the second matrix equality, we get,

$$p_{11}p_{11}^* + p_{12}p_{12}^* = q_c q_c^*, \quad (1.72)$$

$$p_{11}p_{21}^* + p_{12}p_{22}^* = 0, \quad (1.73)$$

$$p_{22}p_{22}^* + p_{21}p_{21}^* = q_c q_c^*. \quad (1.74)$$

It follows from (1.69) and (1.74) that,

$$p_{11}^* p_{11} = p_{22} p_{22}^*, \quad (1.75)$$

and similarly from (1.69) and (1.72), we have,

$$p_{21}^* p_{21} = p_{12} p_{12}^*, \quad (1.76)$$

So, (1.75) and (1.76) implies that $\frac{p_{11}}{p_{22}^*}$ and $\frac{p_{12}}{p_{21}^*}$ are uni-modular on the imaginary axis. It follows from (1.73),

$$\frac{p_{11}}{p_{22}^*} = -\frac{p_{12}}{p_{21}^*} \quad (1.77)$$

on the imaginary axis. It should be noted that $\frac{p_{11}}{p_{22}^*}$ is equal to $\frac{S_{11}}{S_{22}^*}$ and hence it represents $\det(S)$ which is a Blaschke product using lemma (1.3.7). This means that we have the following representation,

$$\frac{p_{11}}{p_{22}^*} = \epsilon \frac{\theta^*}{\theta}, \quad \frac{p_{12}}{p_{21}^*} = -\epsilon \frac{\theta^*}{\theta}, \quad (1.78)$$

where θ is a stable polynomial and ϵ is a uni-modular constant. If we use h_0 to represent the polynomials consisting of the common factors of p_{11} and p_{22}^* and similarly f_0 to represent the polynomials consisting of the common factors of p_{12} and p_{21}^* , we have,

$$\begin{aligned} p_{11} &= \epsilon h_0 \theta^* & p_{12} &= -\epsilon f_0 \theta^* \\ p_{21} &= f_0^* \theta^* & p_{22} &= h_0^* \theta^* \end{aligned}$$

It is evident from equation (1.69) that $q_c q_c^*$ must contain the factor $\theta^* \theta$ and hence it is possible to represent the stable polynomial q_c as,

$$q_c = q_0 \theta,$$

where q_0 is again a stable polynomial. This means that we have the following representation for the scattering matrix S ,

$$S = \frac{1}{q_0 \theta} \begin{bmatrix} \epsilon h_0 \theta^* & -\epsilon f_0 \theta^* \\ f_0^* \theta^* & h_0^* \theta^* \end{bmatrix}, \quad (1.79)$$

where, from equation (1.69), we have,

$$h_0 h_0^* + f_0 f_0^* = q_0 q_0^*. \quad (1.80)$$

For simplicity of representation, if we multiply all entries of matrix S by $\frac{\theta}{\theta}$, and use the following notations,

$$p = h_0^* \theta^* \theta, \quad r = f_0^* \theta^*, \quad q = q_0 \theta^2,$$

we have the following representation for S ,

$$S = \frac{1}{q} \begin{bmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{bmatrix}, \quad (1.81)$$

where q is stable polynomial and again from equation (1.69), we have,

$$qq^* = pp^* + rr^*. \quad (1.82)$$

Even though the above representation is in a reducible form (including cancellations in equation (1.79)), it is important to note that, we have,

$$\det(S) = \frac{\epsilon pp^* + \epsilon rr^*}{q^2} = \frac{\epsilon qq^*}{q^2} = \epsilon \frac{q^*}{q}. \quad (1.83)$$

Thus, from lemma (1.3.7) it follows that q is a stable polynomial of degree N after possible cancellations in the above representation of S and then using spectral equation, we also have $p, r \in \mathbb{P}_N$. This completes the proof. \square

Another important notion associated with two-port that needs to be defined is the transmission zero. It represents the frequencies at which no signal is transmitted through the system. We define it as follows,

Definition 1.3.11. (*Transmission Zeros*). The set of transmission zeros (possibly at ∞) associated to a lossless two port with scattering matrix S is defined as follows:

$$Tz(S) = \{s \in \overline{\Pi}^+ : S_{12}(s)S_{21}(s) = 0\}, \quad (1.84)$$

where the zeros on the imaginary axis are counted for half their multiplicity.

Remark 1.3.12. It should be noted that in Belevitch form, the zeros of the expression $S_{12}S_{21}$ in the closed right half plane are the same as the zeros of the expression, $\frac{rr^*}{q^2}$ in the closed right half plane. We will refer to the non-negative polynomial, $R = rr^*$ as the transmission polynomial of scattering matrix S . We will use \mathbb{P}_n^+ to denote non-negative polynomials of degree at most n on the imaginary axis and we have $R \in \mathbb{P}_{2N}^+$.

1.3.2 Darlington's Theorem

Now, we introduce another classical result related to two-ports which will be useful in the thesis. It was around the late 1930's that Darlington developed a synthesis procedure based on the observation that any given rational reflection coefficient $S_{11} \in \mathbb{B}$ can be realised as a lossless two-port having as input reflection coefficient S_{11} and terminated on port two by a constant impedance [27]. Darlington's synthesis is a widely studied topic with lots of literature available ([12], [20]).

To begin with, we will present the result by Darlington which states that any reflection coefficient $S_{11} = \frac{p}{q} \in \mathbb{B}$ can be extended to a lossless two port having as input reflection coefficient S_{11} . We do not suppose that p and q are co-prime. The set of finite transmission zeros of this pair (p, q) can be defined as

$$\{s \in \overline{\Pi}^+ : R(s) = 0, R \stackrel{\text{def}}{=} qq^* - pp^*\}. \quad (1.85)$$

If $R = rr^*$ denote the spectral factorisation of R with r^* having roots only in $\overline{\Pi}^+$, the finite transmission zeros of the pair (p, q) are the roots of r^* , which is called the transmission zero polynomial. The multiplicity of the transmission zeros are defined as the multiplicity of the zeros of r^* and if $\deg(q) = n$ and $\deg(p) = k$, the pair (p, q) by definition is said to have $n - k$ transmission zeros at infinity. A Darlington extension S of such a pair (p, q) is a lossless scattering matrix satisfying $S_{11} = \frac{p}{q}$ and such that $Tz(S)$ defined in (1.84) is equal to the set of transmission zeros of the pair (p, q) . The result by Darlington states that, if $R \neq 0$, any such extension of McMillan degree at most n takes the form

$$S = \frac{1}{q} \begin{bmatrix} p & -\epsilon u^* \\ u & \epsilon p^* \end{bmatrix} \quad (1.86)$$

where ϵ is a uni-modular constant and u is any polynomial satisfying $uu^* = R$. This can be seen to be following immediately from Belevitch theorem : without loss of generality, let us assume q is monic and Belevitch theorem states that there exist polynomials u, v and w in \mathbb{P}_n such that any extension sought for the pair (p, q) can be expressed as

$$S = \frac{1}{v} \begin{bmatrix} w & -\epsilon u^* \\ u & \epsilon w^* \end{bmatrix}$$

where v is the unique monic polynomial satisfying $vv^* = ww^* + uu^*$. By hypothesis, $\frac{p}{q} = \frac{w}{v}$ and because of the degree constraint, we have $uu^* = \alpha R$ for some $\alpha > 0$. It follows from the spectral equation that

$$\frac{ww^*}{vv^*} + \frac{uu^*}{vv^*} = 1$$

which implies that

$$\frac{pp^*}{qq^*} + \frac{\alpha R}{vv^*} = 1.$$

This leads to the equality

$$qq^* - pp^* = R = \alpha R \frac{qq^*}{vv^*}.$$

Since q and v are monic and stable and $R \neq 0$, we have $v = q$ and so $w = p$ and $\alpha = 1$. Thus we have the (2×2) extension of the pair (p, q) to be in the form mentioned in (1.86). It should be noted that if $R \equiv 0$, then this extension still holds true if p and q are co-prime.

An important point to be noted is that the extension in (1.86) is not necessarily reciprocal. To obtain reciprocity in (1.86), we must have $u = -\epsilon u^*$ that is $\epsilon uu^* = -u^2$, that is the zeros of uu^* must be of even order and paired. In general this may not be the case, but we can always obtain so as follows: decompose uu^* as

$$uu^* = u_1 u_1^* \phi \phi^* \quad (1.87)$$

where $u_1 u_1^*$ includes all $j\omega$ -axis zeros and are of even order, ϕ^* and ϕ includes the right half plane and left half plane zeros respectively. After multiplying (1.87) by $\phi \phi^*$, we have the modified uu^* to be

$$\tilde{u} \tilde{u}^* = u_1 u_1^* \phi^2 \phi^{*2}. \quad (1.88)$$

This modification implies that we have to multiply $\phi\phi^*$ to both qq^* and pp^* because of the spectral equation $qq^* - pp^* = uu^*$. Now, it can be seen that that reciprocal scattering matrix can be accomplished as

$$S = \frac{1}{\phi q} \begin{bmatrix} \phi p & u_1 \phi \phi^* \\ u_1 \phi \phi^* & \mp \phi^* p^* \end{bmatrix} \quad (1.89)$$

where the sign in S_{22} is minus if $u_1 = u_1^*$ and plus if $u_1 = -u_1^*$. It should be noted that because of the surplus factor, there is an augmentation in the McMillan degree of the scattering matrix compared to (1.86) unless we are in the case where all the transmission zeros are on the imaginary axis. The reader can refer to theorem 5.7.1 in [20] for a more detailed description.

Now, we are in a position to state the main result of Darlington which states that it is always possible to realise the Darlington (2×2) extension by a cascade of lossless circuital sections terminated on a unit resistor (Theorem 5.7.1, [20]).

Theorem 1.3.13. (*Darlington's Theorem*). *Any rational reflection coefficient $S_{11} \in \mathbb{B}$ can be seen as the input reflection coefficient of a lossless two-port, terminated at port two by a constant impedance.*

The lossless two-port in theorem 1.3.13 can be always chosen to be reciprocal as well. A detailed survey of the Darlington's procedure of synthesis including the necessary details of realising different types of transmission zeros using the appropriate elementary sections (Type A and B section, Brune's section, Darlington's C and D section) is available in chapter 5 in [20]. One of the main utility of Darlington's theorem in the thesis would be to extend the measured reflection coefficient, $S_{11} \in \mathbb{B}$ of any given antenna to a lossless scattering matrix S . A rational approximation can be done for the given scattering parameter measurements as $\frac{p}{q}$, where $p, q \in \mathbb{P}_N$, N is any fixed positive integer and q is a stable polynomial. Darlington's theorem and Belevitch representation can be utilized to extend the measured reflection coefficient of the given antenna to a lossless scattering matrix S . It will have as S_{11} , the rational approximation of measurement data,

$$S = \begin{bmatrix} \frac{p}{q} & S_{12} \\ S_{21} & \epsilon \frac{p^*}{q} \end{bmatrix}. \quad (1.90)$$

The rational Schur functions S_{12} and S_{21} in the above equation satisfy the following equality on imaginary axis,

$$S_{12}S_{21} = -\epsilon \frac{(qq^* - pp^*)}{q^2}, \quad (1.91)$$

with ϵ arbitrarily chosen uni-modular constant. Most part of the thesis where the theory of broadband matching is developed, we do not necessarily consider a reciprocal Darlington extension as mentioned in (1.89) but a general Darlington extension mentioned in (1.86). The reader should note that it is always possible to consider the reciprocal Darlington extension as discussed in this section as well at the cost of a possible augmentation in the McMillan degree of the scattering matrix of the load.

Remark 1.3.14. *It should be noted that throughout the thesis wherever we mention the degree of a lossless two-port (say degree of the load or degree of the matching network), we imply the McMillan degree of the scattering matrix of the lossless two-port.*

1.4 Impedance Matching Problem

Finally, in the last section of this chapter, we introduce the impedance matching problem in circuit design which is the main topic of interest in the thesis. The problem of impedance matching in communication systems is to minimize the power reflection that is to be transmitted by a generator to a given load within a frequency band. Consider figure 1.3 in

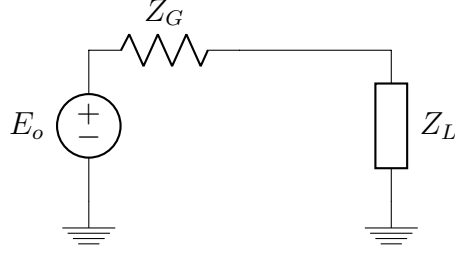


Figure 1.3: A generator connected to a load

which Z_G represents the internal impedance of generator and $Z_L(j\omega)$, the impedance of load, which is frequency dependent. It is a simple transmitting circuit intended to deliver maximum power available from the generator to the load. It follows from the maximum power transfer theorem stated in (1.5) that for a given source impedance, maximum power transfer to the load happens when the load impedance is equal to the complex conjugate of the source impedance in the frequency band of interest (say \mathbb{I}), i.e,

$$\forall j\omega \in \mathbb{I}, \quad Z_L(j\omega) = \overline{Z_G}. \quad (1.92)$$

Equation (1.92) is called the conjugate matching condition. The practical way of achieving this is to introduce a matching network between load and generator so that the new load impedance (load together with matching network) is as close as possible to the conjugate of generator impedance in the frequency band of interest. This is depicted in figure 1.4, where we say the load is “matched”.

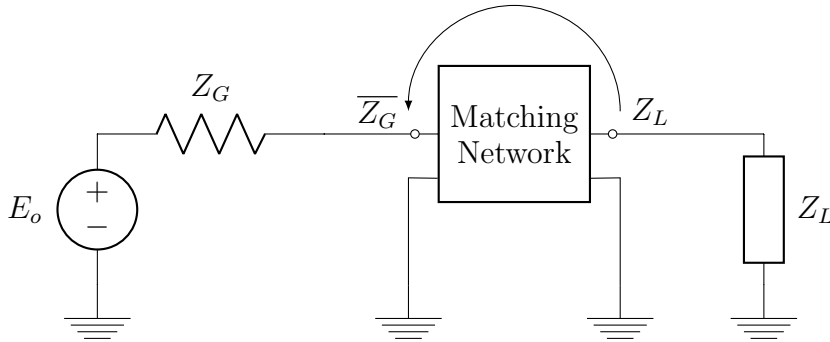


Figure 1.4: A generator connected to a matched load

Antennas are one of the classical devices in which matching plays a crucial role. When feeding the antenna with a signal, it is not desirable that power is reflected back to the input because it results in a loss of usable power. In addition, the unwanted reflected power

affecting devices like generators and amplifiers can also happen. In this problem, the main difficulty arises due the fact that antenna impedance being frequency dependent, we need to design a matching network such that $Z_L(j\omega)$ is matched in the entire frequency band of interest (one or more disjoint bands). The classical matching techniques where matching is done at the central frequency of the band might be good enough for some applications if the targeted band is narrow and the specifications are not so hard. This approach won't be sufficient most of the time when the interest is in broadband or with stringent specifications. The applications of impedance matching are not limited to antennas but include various other RF components such as multiplexers, filters and amplifiers as well.

The wide range of applications together with the in-depth theory that needs to be mastered in different domains have attracted many researchers to broadband matching starting from around the 1940's. The foundational work in matching theory goes back to the fifties where Fano and Youla developed a synthesis procedure for matching networks ([36], [96]). It is based on the use of Darlington's two-port equivalent networks and yields the necessary and sufficient conditions to be satisfied by the functions representing the reflection coefficient of the total system. These interpolation conditions, however, compromise the convexity of the associated optimization problem and led to practical applications only for loads of limited complexity and for reflection coefficients belonging to restricted classes, namely Chebyshev or Butterworth. This method was therefore progressively replaced by a non-convex optimization method called real frequency technique introduced by Carlin [19]. Even though this method yields reasonable results in practice, no results are known about the global optimality of the obtained matching network. More recently, in the eighties, J.W Helton proposed a more general approach using non-Euclidean functional analysis ([48], [47]). In the latter, the broadband impedance matching problem is formulated as a minimization problem of a pseudo hyperbolic distance in the supremum norm over H^∞ . The optimal point, if it exists, is obtained thanks to Nehari's theory which computes the supremum norm distance of an L^∞ function to H^∞ [97]. This H^∞ approach guarantees the global optimality of the obtained response but at the cost of the absence of a degree constraint on the circuital response. The relative mathematical complexity of this approach together with the impossibility to realize in practice an infinite degree matching network limited its impact in electronics.

The main contribution of this thesis is to further enhance the theory of broadband impedance matching and at the same time take into account the practical applicability of the developed techniques as well. We have made use of some of the classical approaches existing in the topic like Helton's operator theoretic approach and Fano-Youla's global system approach to study the broadband impedance matching problem. Together with the theoretical study of different formulations of the matching problem developed during the thesis, practical realization of the optimal matching networks have been given due importance as well. This has led to the division of the thesis into two main parts, where in the first part a detailed study of the broadband impedance matching problem is done to yield many interesting theoretical results regarding the formulated problems, including existence and uniqueness results, optimality conditions and critical point theory. The practical implementation of developed approaches and several examples of matching filter synthesis are illustrated as well. Furnishing lower bounds for matching criterion related to the computation of finite degree matching networks is one of the important results obtained. In

the second part of the thesis, different practical constraints for the implementation of the matching network as per the requirements of the application are taken into consideration as well. This has lead to the development of an approach which is able to take care of these requirements and at the same time provide an estimate of how far the obtained matching criterion in reality is to the theoretical lower bounds obtained in part I.

1.5 Conclusion

In this chapter, we have made a brief introduction to the thesis, giving details about broadband impedance matching and most important tools that would be required during the development of the manuscript. The other necessary references and results required during the course of the thesis will be discussed as and when required. Now, we are in a position to dive into chapter two, where we start with the first formal formulation of impedance matching problem and then discuss an operator theoretic approach to solve it. This approach was first introduced by J.W Helton and provides hard bounds to the matching criterion for any given load and a specified pass band. We will make use of this approach to solve the matching problem in the case of the finite degree matching networks.

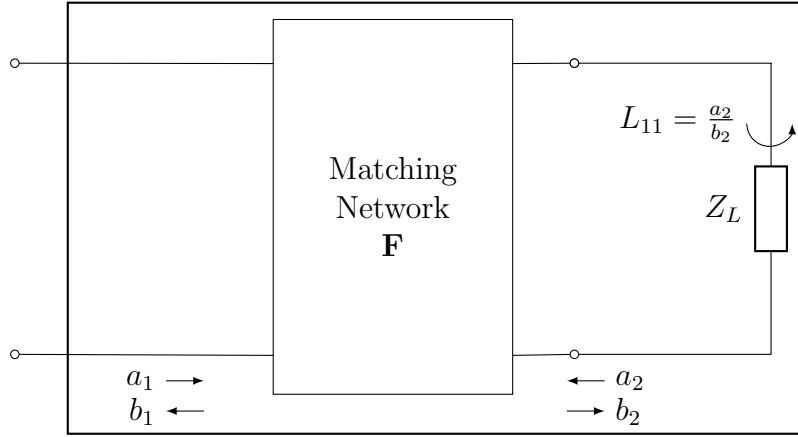
CHAPTER 2

Matching Problem : An Operator Theoretic Approach

2.1 Introduction

As mentioned in the previous chapter, it was J.W Helton who introduced an H^∞ approach to solve the impedance matching problem around the 1980's. The matching problem is casted as a quasi-convex optimization problem involving the minimization of a pseudo-hyperbolic distance. The absence of any degree constraint on the circuitual response is here traded for the guaranteed global optimality of the obtained response. The method developed by Helton is mainly based on a clever application of Nehari's theorem (Theorem 15.16, [97]) : finding best H^∞ approximation to any given L^∞ function in the L^∞ -norm. The implementation of Nehari's theorem depends largely on an operator defined on H^2 , namely the Hankel operator and Helton's H^∞ method reduces the broadband impedance matching problem to finding singular values and singular vectors of the Hankel matrix. In this chapter, we will give an overview of the optimisation problem solved by Helton and subsequently illustrate the reason why its impact in the electronics world was limited, despite its undeniable elegance. In brief terms, the relative mathematical complexity of this procedure coupled to the impossibility to realize in practice an infinite dimensional H^∞ response have severely contributed to its limited impact in electronics.

The main contribution of this chapter would be to describe how Helton's approach can be utilized to furnish lower bounds for matching criterion in the case of finite degree matching networks. The idea is to limit the set of possible reflection coefficients of the total system (matching network together with the load) to certain subclass of the class of Schur functions. This will help to gain control over the degree of the matching network to be synthesized. Then the original idea of Helton's approach based on the resolution of a bounded extremal problem in H^∞ will be used for the practical implementation. The degree bound of the optimal matching network obtained using this approach will be estimated. The computation and behaviour of the best finite dimensional matching filter response obtained using this approach on some concrete antenna examples will be presented at the end of the chapter as well.


 Figure 2.1: Scalar chaining of F and L_{11}

2.2 Matching Problem

The problem of interest is the synthesis of a matching network for a given frequency varying load. Given the measured reflection coefficient of the load under consideration in the frequency band of interest, say \mathbb{I} (a finite union of disjoint compact intervals), the goal is to design a matching network so as to minimize, when plugged on the load, the power reflected by the load in the frequency band \mathbb{I} . We suppose that we possess sufficiently enough values of the load's reflection parameter L_{11} in the frequency band of interest \mathbb{I} . This is usually obtained in practice directly from the data taken from a vector network analyzer while measuring the scattering parameter of the load. Before stating the problem formally, it is necessary to detail the chaining operation, so as to understand the effect of chaining a two-port to a rational schur f , in our case, to the reflection coefficient of the load, denoted by L_{11} . It should be noted that from here on, whenever we speak of scattering parameters, they will be the ones normalized to generator internal impedance. In most of the practical scenarios, this would be $50\ \Omega$ and hence the given reflection coefficient of the load would be the one normalised to $50\ \Omega$. We assume the same reference impedance for the scattering matrix of the matching circuit to be synthesized. As mentioned in equation (1.13), the reflection coefficient of the load normalised to the reference impedance is obtained from the frequency varying impedance of the load as follows,

$$L_{11}(j\omega) = \frac{Z_L(j\omega) - \overline{Z_0}}{Z_L(j\omega) + Z_0}, \quad \omega \in \mathbb{R}, \quad (2.1)$$

where Z_0 is the fixed reference impedance.

In order to describe the chaining operation, let us consider figure 2.1 in which a two-port with scattering matrix

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

is connected to a one-port load with reflection coefficient L_{11} . We suppose that all the scattering parameters are given with respect to the same reference impedance. The scalar

chaining of F with L_{11} denoted by $F \circ L_{11}$ produces the overall reflection coefficient $S_{11} = \frac{b_1}{a_1}$. The overall reflection coefficient S_{11} with respect to the same reference impedance can be easily expressed in terms of the scattering parameters of F and the reflection coefficient L_{11} with the help of power waves. From the definition of the scattering matrix of F , we have

$$b_1 = F_{11}a_1 + F_{12}a_2 \quad (2.2)$$

$$b_2 = F_{21}a_1 + F_{22}a_2. \quad (2.3)$$

It should be noted from the figure (2.1) that the input power wave and output power wave at the second port of the matching network is equal to the output power wave and input power wave respectively of the load Z_L . This implies that $a_2 = b_2L_{11}$. Substituting this for a_2 in (2.3) and calculating b_2 , we obtain

$$b_2 = \frac{F_{21}a_1}{1 - F_{22}L_{11}}. \quad (2.4)$$

Now, substituting $a_2 = b_2L_{11} = \frac{F_{21}L_{11}a_1}{1 - F_{22}L_{11}}$ in (2.2), we get

$$b_1 = F_{11}a_1 + \frac{F_{12}F_{21}L_{11}a_1}{1 - F_{22}L_{11}}.$$

This yields the input reflection coefficient of the overall system $S_{11} = \frac{b_1}{a_1}$ as

$$S_{11} = F_{11} + \frac{F_{12}F_{21}L_{11}}{1 - F_{22}L_{11}}.$$

The reader can also refer to section 2.6 in [45] for a proof of the chaining equation based on signal flow graphs.

Definition 2.2.1. (*Scalar Chaining*). Let us consider a two-port scattering matrix, say,

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

and a rational Schur function $L_{11} \in \mathbb{B}$ representing a one-port reflection coefficient, where all scattering parameters are given with respect to the same reference impedance. The scalar chaining of F with L_{11} represents the one-port reflection coefficient $S_{11} = F \circ L_{11}$ with respect to the same reference impedance, satisfying for any given frequency,

$$S_{11}(j\omega) = F_{11}(j\omega) + \frac{F_{12}(j\omega)F_{21}(j\omega)L_{11}(j\omega)}{1 - F_{22}(j\omega)L_{11}(j\omega)}. \quad (2.5)$$

Now, in order to describe the crux of the matching problem, let us consider figure 2.2 in which the system consisting of the matching network together with the load is depicted. It should be duly noted that throughout the thesis, the matrix,

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad (2.6)$$

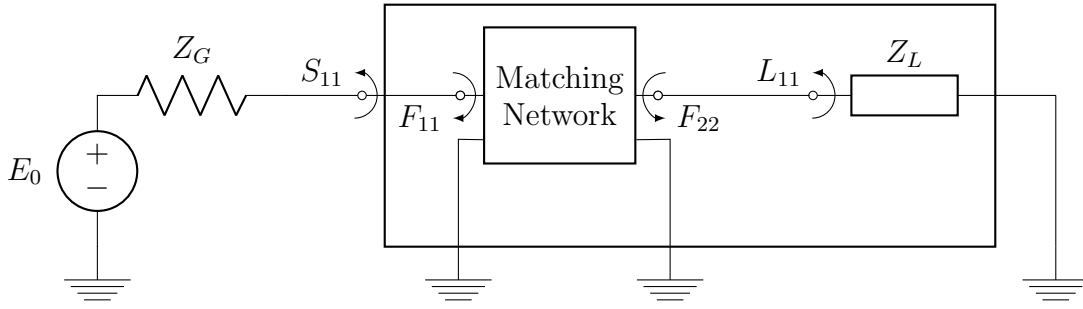


Figure 2.2: Matching Network Connected to One-Port Load

will represent the lossless scattering matrix (definition 1.2.12 in chapter 1) of the matching network that needs to be synthesized, L_{11} the reflection coefficient of the given load and S_{11} , the reflection coefficient of the overall system. The lossless character of F allows us to obtain another expression for S_{11} starting from the expression in (2.5). This expression described below will play the central role in this chapter. As mentioned in equation (2.5), we have the reflection coefficient of the system produced as a result of chaining F with L_{11} to be,

$$S_{11}(j\omega) = F_{11}(j\omega) + \frac{F_{12}(j\omega)F_{21}(j\omega)L_{11}(j\omega)}{1 - F_{22}(j\omega)L_{11}(j\omega)} \quad (2.7)$$

$$= \frac{F_{11}(j\omega) - L_{11}(j\omega)\det(F(j\omega))}{1 - F_{22}(j\omega)L_{11}(j\omega)} \quad (2.8)$$

$$= \det(F(j\omega)) \frac{\overline{F_{22}(j\omega)} - L_{11}(j\omega)}{1 - F_{22}(j\omega)L_{11}(j\omega)} \quad (2.9)$$

The last equality follows, since from equation (1.62), we have $\det(F(j\omega)) = \frac{F_{11}(j\omega)}{F_{22}(j\omega)}$, when F is a lossless scattering matrix. Now, again using the same fact, we have $|\det(F(j\omega))| = 1$, for any $\omega \in \mathbb{R}$ and hence it follows that,

$$|S_{11}(j\omega)| = \left| \frac{\overline{F_{22}(j\omega)} - L_{11}(j\omega)}{1 - F_{22}(j\omega)L_{11}(j\omega)} \right|, \quad \omega \in \mathbb{R}. \quad (2.10)$$

So, we have obtained an expression for the modulus of the reflection coefficient of the system obtained by chaining the lossless scattering matrix F to the load with reflection coefficient L_{11} . It is the quantity of interest which we would like to make lower or equal to a prescribed value in the entire passband of interest as required by the application specifications.

2.2.1 Pseudo-Hyperbolic Distance

The expression for $|S_{11}|$ on the imaginary axis as mentioned in equation (2.10) is having a form which is peculiar to non-Euclidean geometry, namely the pseudo-hyperbolic distance.

Definition 2.2.2. (*Pseudo-Hyperbolic Distance*). The pseudo-hyperbolic distance is defined on the unit disk, $\mathbb{D} = \{s \in \mathbb{C} : |s| < 1\}$, by,

$$\delta(a, b) = \left| \frac{a - b}{1 - \bar{a}b} \right|, \quad a, b \in \mathbb{D}. \quad (2.11)$$

It should be noted that being a distance, it satisfies all the properties of a metric on \mathbb{D} :

- Non-negativity : $\delta(a, b) \geq 0$, Moreover $\delta(a, b) = 0 \iff a = b$.
- Symmetry : $\delta(a, b) = \delta(b, a)$.
- Sub-additivity or Triangle inequality : $\delta(a, b) \leq \delta(a, c) + \delta(c, b)$.

Moreover, an additional point which follows from the definition of the pseudo-hyperbolic distance is

$$0 \leq \delta(a, b) < 1. \quad (2.12)$$

Next, we state a result which will be of importance in this chapter as we go ahead : any pseudo-hyperbolic disk is a Euclidean disk contained in the unit Euclidean disk, \mathbb{D} . To be precise,

Theorem 2.2.3. *Any pseudo-hyperbolic disk, denoted by,*

$$D_H(c, r) = \{z \in \mathbb{D} : \delta(z, c) < r\}, \quad (2.13)$$

is an Euclidean disk,

$$D_E(C, R) = \{z \in \mathbb{C} : |z - C| < R\}, \quad (2.14)$$

where,

$$C = \frac{1 - r^2}{1 - r^2|c|^2}c, \quad R = \frac{1 - |c|^2}{1 - r^2|c|^2}r. \quad (2.15)$$

A proof to theorem 2.2.3 can be found in Chapter I, [42]. Now, we are in a position to introduce the matching problem that was solved by J.W Helton using an operator theoretic approach.

2.2.2 Helton's Optimization Problem

We will continue to use the notation F , as mentioned in equation 2.6, to denote the lossless scattering matrix of the matching circuit to be synthesized. In the infinite dimensional setting, where the output reflection coefficient of matching network, F_{22} , is sought for in the set of Schur functions, \mathbb{B} (i.e $F_{22} \in H^\infty(\Pi^+)$ and $\forall \omega \in \mathbb{R}$, $|F_{22}(j\omega)| \leq 1$), the matching problem can be formulated as,

Problem. \mathcal{P}_1 : *Given a passband, \mathbb{I} and the reflection coefficient, $L_{11} \in \mathbb{B}$ of the load,*

$$\text{Find : } l_{inf} = \min_{F_{22} \in \mathbb{B}} \max_{j\omega \in \mathbb{I}} \delta(F_{22}(j\omega), \overline{L_{11}(j\omega)}),$$

where \mathbb{I} is a finite union of disjoint compact intervals on the imaginary axis.

Theorem 2.2.4. *Problem \mathcal{P}_1 is quasi-convex.*

Proof. The set of Schur functions, \mathbb{B} , is clearly a convex set. For a given passband \mathbb{I} and the reflection coefficient $L_{11} \in \mathbb{B}$ of the load, let us denote the objective function by,

$$\begin{aligned} \Psi : \mathbb{B} &\rightarrow \mathbb{R} \\ \Psi(f) &= \max_{j\omega \in \mathbb{I}} \delta(f(j\omega), \overline{L_{11}(j\omega)}) \end{aligned} \quad (2.16)$$

The aim is to prove that Ψ defined on the convex set \mathbb{B} is quasi-convex, that is, to prove the set $C_r(\Psi)$ defined by,

$$C_r(\Psi) = \{f \in \mathbb{B} : \Psi(f) \leq r\},$$

is a convex set for any fixed $r \in \mathbb{R}$. Let f_1, f_2 denote two functions in $C_r(\Psi)$. From the definition of $C_r(\Psi)$, we have,

$$\begin{aligned} \forall j\omega \in \mathbb{I}, \quad \delta(f_1(j\omega), \overline{L_{11}(j\omega)}) &\leq r \\ \forall j\omega \in \mathbb{I}, \quad \delta(f_2(j\omega), \overline{L_{11}(j\omega)}) &\leq r, \end{aligned}$$

that is,

$$\forall j\omega \in \mathbb{I}, \quad f_1(j\omega) \in \text{cl}(D_H(\overline{L_{11}(j\omega)}, r)) \quad (2.17)$$

$$\forall j\omega \in \mathbb{I}, \quad f_2(j\omega) \in \text{cl}(D_H(\overline{L_{11}(j\omega)}, r)), \quad (2.18)$$

where ‘cl’ represents the closure. Making use of theorem 2.2.3, we have, for any $j\omega \in \mathbb{I}$,

$$\begin{aligned} f \in \text{cl}(D_H(\overline{L_{11}(j\omega)}, r)) &\iff f \in \text{cl}(D_E(C(j\omega), R(j\omega))), \quad \text{where,} \\ C(j\omega) &= \frac{1-r^2}{1-r^2|L_{11}(j\omega)|^2} \overline{L_{11}(j\omega)} \quad \text{and} \quad R(j\omega) = \frac{1-|L_{11}(j\omega)|^2}{1-r^2|L_{11}(j\omega)|^2} r. \end{aligned}$$

This implies, from (2.17) and (2.18) we have,

$$\begin{aligned} \forall j\omega \in \mathbb{I}, \quad f_1(j\omega) &\in \text{cl}(D_E(C(j\omega), R(j\omega))) \quad \text{and} \quad f_2(j\omega) \in \text{cl}(D_E(C(j\omega), R(j\omega))), \\ \text{i.e. } \forall j\omega \in \mathbb{I}, \quad |f_1(j\omega) - C(j\omega)| &\leq R(j\omega) \quad \text{and} \quad |f_2(j\omega) - C(j\omega)| \leq R(j\omega). \end{aligned} \quad (2.19)$$

So, for any $j\omega \in \mathbb{I}$, we have for $\alpha \in (0, 1)$,

$$\begin{aligned} |\alpha f_1(j\omega) + (1-\alpha)f_2(j\omega) - C(j\omega)| &\leq \alpha |f_1(j\omega) - C(j\omega)| + (1-\alpha) |f_2(j\omega) - C(j\omega)| \\ &\leq \alpha R(j\omega) + (1-\alpha) R(j\omega). \\ &= R(j\omega) \end{aligned}$$

This implies, for any $j\omega \in \mathbb{I}$, we have, $\alpha f_1(j\omega) + (1-\alpha)f_2(j\omega) \in \text{cl}(D_E(C(j\omega), R(j\omega)))$, equivalently, $\alpha f_1(j\omega) + (1-\alpha)f_2(j\omega) \in \text{cl}(D_H(\overline{L_{11}(j\omega)}, r))$. This yields us the fact that for any fixed $\alpha \in (0, 1)$, the function $\alpha f_1 + (1-\alpha)f_2 \in \mathbb{B}$ satisfies,

$$\max_{j\omega \in \mathbb{I}} \delta(\alpha f_1(j\omega) + (1-\alpha)f_2(j\omega), \overline{L_{11}(j\omega)}) = \Psi(\alpha f_1 + (1-\alpha)f_2) \leq r.$$

This shows the convexity of set $C_r(\Psi)$ and hence the proof. \square

Helton's approach to solve broadband matching problem is mainly based on a theorem by Nehari to calculate the sup norm distance to H^∞ . A detailed literature about a non-Euclidean functional analysis approach by Helton for solving different problems in electronics like broadband impedance matching problem and optimizing the gain of amplifiers can be found in [48]. In order to tackle the problem of broadband matching introduced in problem \mathcal{P}_1 , Helton developed a powerful method of gain equalization. We will give below a brief outline of Helton's approach to broadband matching and the associated techniques to find theoretical bounds for the gain in the passband. The transducer power gain, G is the ratio of the power delivered to the load to the power available from the source. Given a desired gain profile $G(j\omega)$, Helton's approach find the largest multiple αG of it which is realizable. Throughout the original work, Helton consider the case where the reflection coefficient of the load, $L_{11}(j\omega)$ are known within a tolerance for $j\omega$ in some band \mathbb{I}^+ which contains the passband \mathbb{I} or the function L_{11} is given exactly as a rational function. Also, for the simplicity of the exposition and the fact that it is the common industrial case, it is assumed that $|L_{11}|$ does not take the value 1 on \mathbb{I}^+ . The problem of gain equalization can be stated as follows.

Problem. \mathcal{P}_{GE} . (*Gain Equalization Problem*). Given $L_{11} \in \mathbb{B}$ (reflection coefficient of the load) satisfying $|L_{11}(j\omega)| \leq r < 1$ on the frequency band \mathbb{I} and given a function $G(j\omega)$ (the transducer gain) on \mathbb{I} satisfying $0 < G(j\omega) < 1$, does there exist a lossless scattering matrix $F(s)$ such that

$$G(j\omega) = 1 - (\delta(F_{22}(j\omega), \overline{L_{11}(j\omega)}))^2, \quad \omega \in \mathbb{I} ? \quad (2.20)$$

If yes, find the $F(s)$ which realizes this gain.

The method proposed by Helton provides a test based on Nehari theory and Hankel operator to check the realizability of a given gain G . The idea then is to use this test to find the biggest realizable gain αG where G is given and $\alpha > 0$ is the constant to be found. The key starting point of Helton approach is to consider the realizability of a given gain $G(j\omega)$ on the frequency band \mathbb{I} for a given $L_{11} \in \mathbb{B}$ by replacing the equality in equation (2.20) with an inequality \leq . The gain is completed on the imaginary axis by defining

$$g(j\omega) = \begin{cases} G(j\omega) & j\omega \in \mathbb{I} \\ 0 & j\omega \notin \mathbb{I}. \end{cases} \quad (2.21)$$

So the problem can be stated as does there exist a lossless scattering matrix $F(s)$ such that

$$(\delta(F_{22}(j\omega), \overline{L_{11}(j\omega)}))^2 \leq 1 - g(j\omega), \quad \omega \in \mathbb{R}. \quad (2.22)$$

From theorem 2.2.3, the problem translates to finding $F_{22} \in H^\infty(\Pi^+)$ such that,

$$|F_{22}(j\omega) - C(j\omega)|^2 \leq R(j\omega)^2, \quad \omega \in \mathbb{R}, \quad (2.23)$$

where we have,

$$C(j\omega) = \frac{g(j\omega)\overline{L_{11}(j\omega)}}{1 - |L_{11}(j\omega)|^2(1 - g(j\omega))}, \quad (2.24)$$

$$R(j\omega)^2 = \left(\frac{1 - |L_{11}(j\omega)|^2}{1 - |L_{11}(j\omega)|^2(1 - g(j\omega))} \right)^2 (1 - g(j\omega)). \quad (2.25)$$

It should be noted that $C \equiv 0$ and $R \equiv 1$ outside the passband \mathbb{I} . Dividing equation (2.23) by $|V(j\omega)|^2$, where V is the outer function in $H^\infty(\Pi^+)$ satisfying, $VV^* = R^2$ on the imaginary axis, we are interested in finding $F_{22} \in H^\infty(\Pi^+)$ such that,

$$|F_{22}(j\omega)V^{-1}(j\omega) - C(j\omega)V^{-1}(j\omega)| \leq 1, \quad \omega \in \mathbb{R}, \quad (2.26)$$

if it exists. Let us define $\Phi = CV^{-1}$ and $\Psi = F_{22}V^{-1}$. It should be noted that the modulus of the given load, $|L_{11}(j\omega)|$ and the gain function $G(j\omega)$ fixed by the user are assumed to be strictly less than one. This implies that the outer function V is invertible in $H^\infty(\Pi^+)$ and we have $\Phi \in L^\infty(j\mathbb{R})$. Now, we are interested in solving the following classical problem of approximation by analytic functions:

Problem. (Nehari). Given $\Phi \in L^\infty(j\mathbb{R})$,

$$\min_{\Psi \in H^\infty} \|\Psi - \Phi\|_{L^\infty} \quad (2.27)$$

and find a Ψ at which the infimum is attained if the minimum is less than or equal to one.

A detailed literature about these approximation problems can be found in Chapter 15, [97]. The solution to (2.27) and a $\Psi \in H^\infty(\Pi^+)$ at which the infimum in (2.27) is attained can be obtained with the help of Hankel operators.

Definition 2.2.5. (Hankel Operator). Let $\phi \in L^\infty(j\mathbb{R})$ and $\bar{H}^2 \stackrel{\text{def}}{=} L^2 \ominus H^2$, the orthogonal complement of H^2 in L^2 . The Hankel operator with symbol ϕ , denoted by $H_\phi : H^2(\Pi^+) \rightarrow \bar{H}^2(\Pi^+)$ is the operator defined by

$$H_\phi(f) = \underline{P}(\phi f),$$

where \underline{P} represents the orthogonal projection from $L^2(j\mathbb{R})$ to $\bar{H}^2(\Pi^+)$.

We have the following solution to Nehari's problem (Theorem 15.16, [97]).

Theorem 2.2.6. Let $\phi \in L^\infty(j\mathbb{R})$ and suppose that the Hankel operator, $H_\phi : H^2(\Pi^+) \rightarrow \bar{H}^2(\Pi^+)$ has a maximizing vector f (vector at which H_ϕ attains its norm). There is a unique best approximation of $g \in H^\infty(\Pi^+)$ to ϕ in the L^∞ -norm. It is given by,

$$g = \phi - \frac{H_\phi(f)}{f}. \quad (2.28)$$

Furthermore, we have, $\min_{g \in H^\infty} \|g - \phi\|_{L^\infty}$ is equal to $\|H_\phi\|$, the operator norm of Hankel operator with symbol ϕ .

So, using theorem 2.2.6, we have a solution to problem stated in (2.22) if and only if $\|H_\Phi\|$ is less than or equal to one and in this case, we have the minimiser,

$$\Psi = \Phi - \frac{H_\Phi(W)}{W}, \quad (2.29)$$

where H_Φ is the Hankel operator with symbol Φ and W one of its maximizing vectors. We remind the reader that $\Phi = CV^{-1}$ is known by construction from the gain g and reflection

coefficient of the load, L_{11} . It should be noted from theorem 1.1.4 in [71] that as long as Φ is the sum of an H^∞ function and a continuous function on the imaginary axis, H_Φ is compact and have maximizing vectors. The reader can refer to section II in [47] for a methodology proposed to find a maximizing vector from a truncation of the matrix of H_Φ that is close to the maximizing vector of infinite dimensional matrix of H_Φ . The idea presented is based on the notion of *approximate realizability* of a gain G for a given load L_{11} , which means that there is a sequence of F_n of lossless matching circuits such that the resulting gains G_n converge uniformly to G on each closed set in the interior of \mathbb{I} . One smooths the functions C and R at the end points of the passband and then after truncating the matrix of H_Φ , the approximate realizability of the gain G can be tested using the test $\|H_\Phi\| \leq 1$. If it holds, there exists a lossless matching network F achieving the gain G for the load.

The solution to (2.22), namely $F_{22} \in H^\infty(\Pi^+)$ can be obtained by multiplying the outer function V to Ψ , that is $F_{22} = V\Psi$. We remind the reader that the outer function V is known by construction from the Euclidean radius R defined in (2.25). So, the problem of finding the biggest realizable gain of the form αG for a given gain G can be solved by performing the above test of checking $\|H_\Phi\| \leq 1$ on a grid of α 's where G is replaced by αG . Beginning with a small value of $\alpha > 0$, we can iterate on the gain $g(j\omega)$ mentioned in equation (2.21) by increasing the value of α until the operator norm of Hankel operator is equal to one (say it is achieved at $\hat{\alpha}$). This provides us \hat{F}_{22} which achieves the biggest realizable gain $\hat{\alpha}G$ in \mathbb{I} , that is

$$\hat{\alpha}G(j\omega) = 1 - (\delta(\hat{F}_{22}(j\omega), \overline{L_{11}(j\omega)}))^2, \quad \omega \in \mathbb{I}. \quad (2.30)$$

The idea proposed in [47] concerning the smoothing of the functions C and R in (2.24), (2.25) and then using the approximate realizability test can be interpreted more practically by starting off the procedure with a smooth gain function $G(j\omega)$, say it vanishes at the end points of the passband \mathbb{I} and its derivative with respect to the frequency variable, denoted by G' is Lipschitz continuous in \mathbb{I} . This implies that the function $g(j\omega)$ defined on the imaginary axis in (2.21) is in \mathbf{C}^1 . This together with the fact that $|L_{11}| \leq r < 1$ implies that the Euclidean center, $C(j\omega)$ and radius, $R(j\omega)$ defined in (2.24) is \mathbf{C}^1 -smooth. In addition, it should be noted that R' is in fact Lipschitz continuous in this setting. It also follows from (2.25) that R is a strictly positive function and so V is non-vanishing on the imaginary axis. Theorem 1.3 in chapter III, [42] implies that for such a smooth R as described here, its outer factor V build so using the Hilbert transform is \mathbf{C}^1 -smooth. This together with the fact that the derivative of Hilbert transform is the Hilbert transform of the derivative (chapter 3, theorem 3, [68]) yields the symbol Φ build on the imaginary axis as CV^{-1} to be \mathbf{C}^1 -smooth as well. Now, it follows from theorem 2.1, chap IV in [42] that the approximant function $\Psi \in H^\infty$ of such a Φ is continuous on the imaginary axis and hence belongs to the half-plane algebra, $A(\Pi^+)$. In this case, theorem 1.3.9 in [69] ensures that the harmonic extension of the approximant $\Psi(j\omega)$ converges to Ψ in the sup-norm. Thus, Runge's theorem (theorem 13.6, [83]) guarantees that a rational approximation of the optimal \hat{F}_{22} can be obtained and finally Darlington's construction (Theorem 1.3.13) can be used to form the lossless scattering matrix $\hat{F}(s)$ realizing the biggest gain $\hat{\alpha}G$. It should be noted that in general there is no control in the final degree of the matching circuit obtained by this scheme. With this, we conclude this short description of Helton's approach to broadband matching and theoretical gain-bandwidth limitations. For a given load, if we

consider the uniform gain profile in \mathbb{I} , say

$$G(j\omega) = l, \quad 0 < l < 1, \quad (2.31)$$

Helton's method provides hard upper bound of $\hat{\alpha}l$ for the realizable gain. Correspondingly it provides a hard lower bound, $\sqrt{1 - \hat{\alpha}l}$ on the power mismatch for the same load. The first implementation of this wide-band impedance matching technique can be found in [88]. Thus, we have the solution $\hat{F}_{22} \in \mathbb{B}$ to problem \mathcal{P}_1 which achieves the optimal matching criterion $l_{inf} = \sqrt{1 - \hat{\alpha}l}$. It should be noted that the solution \hat{F}_{22} to Helton's approach which realizes the biggest gain $\hat{\alpha}G$ will be infinite dimensional. This limits the practical usefulness of Helton's method in providing optimal finite degree matching networks. It is with this purpose in mind, we move to the next section, where we formulate the matching problem in such a way that if there exists a solution to the problem, it will be guaranteed to be realisable by finite degree matching networks.

2.3 Finite Degree Matching Problem

In this section, Helton's approach of solving gain equalization problem will be used to solve finite degree broadband matching problem. If there exists a solution to this problem, it will be ensured to be a finite degree rational Schur function. Throughout the section, we suppose that we possess a passive rational model of antenna's reflection parameter L_{11} obtained via rational approximation techniques at hand of scattering measurements [1]. In comparison to the setting described in the last section where a rational approximation of the reflection parameter of the load is not done at the beginning but an approximation of the matching filter response is done at the end, the current setting will be practically more useful. In most of the practical scenarios, the measured reflection parameter of the load can be approached very closely by a rational approximation in the frequency band of interest. For example, a single resonance antenna can be approximated by a degree one rational function in most of the cases. If a more accurate model is required, it is always possible to increase the degree of the approximation and then perform the optimisation scheme to be discussed in this section. In the practical world, degree bound on the matching circuit is one of the major constraint concerning its realization and in the previous setting, once the optimisation procedure is completed, it is highly likely that a response close to the optimal response cannot be achieved with the prescribed degree bound. We will be able to overcome this problem in the setting described in this section since the optimisation scheme ensures the desired degree bound on the matching circuits. Before moving to the problem formulation, it is necessary to describe rational "reference functions" which will play an important role in the problem to guarantee finite degree for the solution, if it exists. They will form a special class of Schur functions which will help to parameterise the modulus of the total system formed by the chaining of matching network and the load. It is important to note that if the McMillan degree of the load is M and that of the matching network is N_{mc} , then the degree of the overall system is at most $N_{mc} + M$ and so it is natural to consider the reference functions (whose construction is explained in the next subsection) to be of degree $N = N_{mc} + M$.

2.3.1 Reference Functions

As already mentioned, the idea of introducing reference functions is to gain control over the degree of the matching network to be synthesized. It is done at the cost of limiting the set of possible reflection coefficients of the total system (obtained by chaining the scattering matrix of the matching network and the reflection coefficient of the load) to certain subclasses of the class of Schur functions. We will form a family of rational outer reference functions $\{k_\alpha\}$ parametrized by $\alpha \in \mathbb{R}^+$, the modulus of which mimic an ideal step function k_{opt} :

$$k_{opt}(j\omega) = \begin{cases} l & j\omega \in \mathbb{I} \\ 1 & j\omega \notin \mathbb{I} \end{cases} \quad 0 < l < 1 \quad (2.32)$$

There are multiple ways to approach rationally a step function. We choose here to follow the classical Darlington insertion loss synthesis for filters. Considering the Belevitch form of a general loss-less rational scattering matrix (Theorem 1.3.10), modulus square of the reflection coefficient of the total system can be expressed on the imaginary axis as :

$$|S_{11}|^2 = \frac{pp^*}{qq^*} = \frac{pp^*}{pp^* + rr^*} = \frac{1}{1 + \frac{rr^*}{pp^*}} \quad (2.33)$$

As mentioned at the beginning of the section, we suppose that we possess a passive rational model of the loads reflection parameter L_{11} denoted by $\frac{p_L}{q_L} \in \mathbb{B}$. Using Darlington's theorem (Theorem 1.3.13), we can represent the lossless scattering matrix of McMillan degree M of the given load as

$$L = \frac{1}{q_L} \begin{bmatrix} p_L & r_L \\ -\epsilon r_L^* & \epsilon p_L^* \end{bmatrix}, \quad (2.34)$$

where, $p_L, r_L \in \mathbb{P}_M$ and q_L is a stable polynomial of degree M satisfying the Feldtkeller equation, $q_L q_L^* = p_L p_L^* + r_L r_L^*$. Throughout this chapter, we will assume that all the transmission zeros of the load are strictly inside the right half plane and in the spectral factorisation $r_L r_L^*$ of the transmission polynomial of the load, we choose r_L to be Hurwitz. Now, we build the reference functions by considering two cases, (i). $r = r_F$ and (ii). $r = r_L r_F$, where r_F represent the transmission polynomial associated to the scattering matrix of the matching circuit to be synthesized. For simplicity, in both cases we will consider r_F is a non-zero constant and denote $\alpha = r_F r_F^* > 0$. This corresponds to a classical low pass network topology for the matching circuit to be synthesized (equation 9.22, [92]).

- Case (i). $r = r_F$: In this case, we have the modulus of the input reflection coefficient of the total system satisfying

$$|S_{11}|^2 = \frac{1}{1 + \alpha |\frac{1}{p}|^2} \quad (2.35)$$

on the imaginary axis. So, for any fixed p , the modulus of S_{11} can be varied monotonously using the parameter α . Thus it is possible to form a family of rational Schur functions $\{\frac{p}{q}\}$ where q is a solution of spectral factorisation and the modulus of the Schur function can be parametrised using α . We will make use of Chebyshev polynomials, one of the classical tools in Filter design in order to build these family of reference functions.

Definition 2.3.1. (*Chebyshev Polynomials*). The Chebyshev polynomial of degree N , denoted by T_N is defined on $[-1j, 1j]$ by the recurrence relation,

$$T_N(s) = -2jsT_{N-1}(s) - T_{N-2}(s), \quad N \geq 2,$$

where $T_0(s) = 1$ and $T_1(s) = -js$.

It should be noted that in contrast to the usual setting of Chebyshev polynomials on $[-1, 1]$, we have transformed it to $[-1j, 1j]$ since in our problem, the passband \mathbb{I} is normalized to $[-1j, 1j]$. For a fixed positive constant c , we fix p_k as the Hurwitz polynomial satisfying

$$p_k p_k^* = T_N T_N^* + c \quad (2.36)$$

on the imaginary axis and q_k the strict Hurwitz polynomial satisfying

$$q_k q_k^* = p_k p_k^* + \alpha. \quad (2.37)$$

Thus, we have a family $\{k_\alpha = \frac{p_k}{q_k}\}$ of rational outer functions of degree N satisfying,

$$|k_\alpha|^2 = \frac{|p_k|^2}{|q_k|^2} = \frac{1}{1 + \frac{\alpha}{|p_k|^2}} = \frac{1}{1 + \frac{\alpha}{|T_N|^2 + c}} \quad (2.38)$$

on the imaginary axis. It should be noted that $\{k_\alpha\}$ form a family of equi-oscillating rational reference functions and $|k_\alpha|^2$ equi-oscillate on \mathbb{I} between the values $1/(1 + \alpha/c)$ and $1/(1 + \alpha/(c + 1))$. The reader should not confuse the subscript k in polynomials p_k and q_k to be associated with degree but it is named so to mark its association with reference function named k_α . They are in fact degree N polynomials.

- Case (ii). $r = r_L r_F$: In this case, we have the modulus of the input reflection coefficient of the total system satisfying

$$|S_{11}|^2 = \frac{1}{1 + \alpha \left| \frac{r_L}{p} \right|^2} \quad (2.39)$$

on the imaginary axis. Similar to the previous case, we will make use of Chebyshev approximation, but this time a weighted approximation because of the factor r_L in equation 2.39. We are interested in solving the problem

$$\text{Find : } t_0 = \min_{p \in \mathbb{RMP}_N} \max_{j\omega \in \mathbb{I}} \left| \frac{p(j\omega)}{\tilde{r}_L(j\omega)} \right|^2, \quad (2.40)$$

where \mathbb{RMP}_N represent the monic polynomials of degree N which are real-valued on the imaginary axis and the polynomial r_L associated with the given load is normalized to be monic, denoted by \tilde{r}_L . This problem can be solved using the classical Remez algorithm for the weighted Chebyshev approximation [60]. It can also be solved by defining an extra variable $\Gamma \in \mathbb{R}$ satisfying for all $j\omega \in \mathbb{I}$,

$$\Gamma \geq \left| \frac{p(j\omega)}{\tilde{r}_L(j\omega)} \right|^2$$

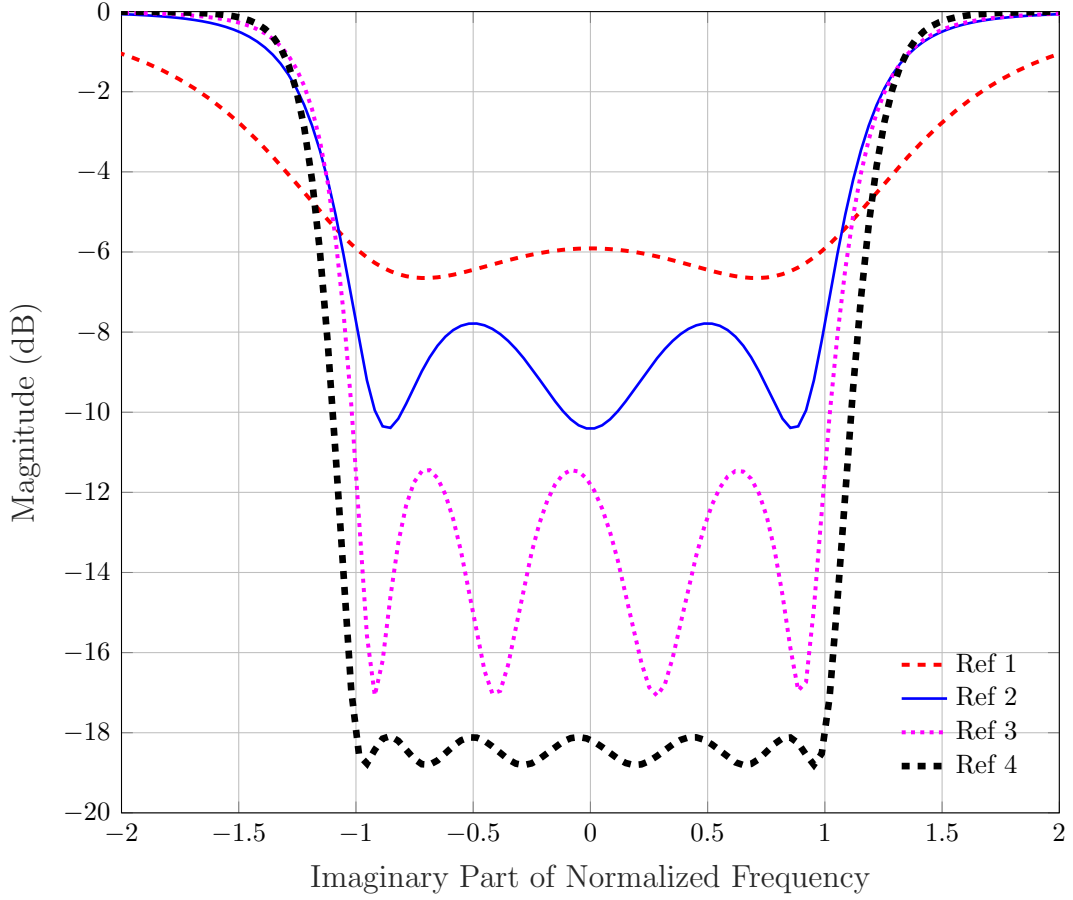


Figure 2.3: Reference Functions

and then using matlab solvers like `fmincon` to solve

$$\begin{aligned} & \min_{(p, \Gamma) \in \mathbb{RMP}_N \times \mathbb{R}} \Gamma \\ & \text{sub : } \left| \frac{p(j\omega)}{\tilde{r}_L(j\omega)} \right|^2 - \Gamma \leq 0, \quad j\omega \in \mathbb{I}. \end{aligned}$$

The solution to problem 2.40 provides the weighted Chebyshev polynomial of degree N . After de-normalizing the leading coefficient of the solution to be the leading coefficient of r_L , we denote it by T_N^L . Now, in order to build the family of reference functions, for a fixed positive constant c , we define p_k as the Hurwitz polynomial satisfying

$$p_k p_k^* = T_N^L T_N^{L*} + c r_L r_L^* \quad (2.41)$$

and q_k the strict Hurwitz polynomial satisfying

$$q_k q_k^* = p_k p_k^* + \alpha r_L r_L^*. \quad (2.42)$$

Thus, we have the family $\{k_\alpha = \frac{p_k}{q_k}\}$ of rational outer functions of degree N satisfying,

$$|k_\alpha|^2 = \frac{|p_k|^2}{|q_k|^2} = \frac{1}{1 + \alpha \left| \frac{r_L}{p_k} \right|^2} = \frac{1}{1 + \frac{\alpha}{\left| \frac{T_N^L}{r_L} \right|^2 + c}} \quad (2.43)$$

on the imaginary axis. Similar to case (i), $\{k_\alpha\}$ form a family of equi-oscillating rational reference functions and $|k_\alpha|^2$ equi-oscillate on \mathbb{I} between the values $1/(1+\alpha/c)$ and $1/(1+\alpha/(c+t_0))$. Fig. 2.3 shows a representation of some of the reference functions in the described family $\{k_\alpha\}$. Ref 1 and Ref 2 represent two reference functions of degree N equal to 2 and 3 respectively build by following case (i). The value of α and c were set to $\alpha = 8, c = 2$ and $\alpha = 10, c = 1$ respectively in Ref 1 and Ref 2. Similarly, Ref 3 and Ref 4 represent two reference functions of degree N equal to 4 and 6 respectively build by following case (ii) where the polynomial r_L was fixed, $r_L(s) = -1.2s^2 - (5.1 + 1.2j)s - (3.1 + 2j)$. The value of α and c were set to $\alpha = 0.02, c = 0.02$ and $\alpha = 0.03, c = 0.02$ respectively in Ref 3 and Ref 4.

In summary, we have considered two cases of reference functions representing the modulus of the input reflection coefficient of the total system, one in which transmission zeros of the load are not included in the reference function (case (i)) and other one in which they are included (case (ii)). At the end of this section (subsection 2.3.4), we will illustrate the main difference in the results obtained using these two types of reference functions in the optimisation scheme that will be described for solving the matching problem in the next subsection. It will become clear that there is an increase in the degree of the optimal matching circuit that we will have to pay if we do not include the transmission zeros of the load in the reference functions (case (i)). The reader should also note that choosing $r_F r_F^* = \alpha$ in the construction of reference functions was done just for the purpose of simplicity of explanation, instead we can consider r_F to be any fixed transmission polynomial of the matching circuit to be synthesized with no zeros inside passband \mathbb{I} . It can be handled the same way as case (ii) in which we can consider α to be a positive multiplicative factor for r and replace r_L with $r_L r_F$ in equation (2.39). Then the same construction scheme of reference functions discussed in case (ii) can be utilized for handling this case.

2.3.2 Bound for the Reflection Level Using H^∞ Approach

Having described the family of reference functions, we move to the important part of this chapter, that is to describe the finite degree matching problem and use Helton's approach to solve it. We will continue to use the notation F to denote the lossless scattering matrix of matching circuit to be synthesized and its parameterisation using the output reflection coefficient, $F_{22} \in H^\infty(\Pi^+)$. Following the Helton's approach of gain equalization problem, we are interested in finding the smallest power mismatch of the form $|k_\alpha|$. In the finite dimensional setting, the problem can be formulated as,

Problem. \mathcal{P}_2 : *Given a passband \mathbb{I} , the reflection coefficient $L_{11} \in \mathbb{B}$ of the load and a reference function k_α of degree N as described in subsection 2.3.1, does there exist a lossless scattering matrix $F(s)$ such that,*

$$\forall \omega \in \mathbb{R}, \delta(F_{22}(j\omega), \overline{L_{11}(j\omega)}) = |k_\alpha(j\omega)| ? \quad (2.44)$$

If yes, find the $F(s)$ which achieves this.

We will make use of Nehari theory to provide the test based on Hankel operator to answer problem \mathcal{P}_2 . It easily follows from equations (2.38) and (2.43) that $|k_\alpha|$ is a decreasing

function of α on the imaginary axis for any fixed polynomial p in both the cases of reference functions. So, the idea is to use this test to find the smallest possible power mismatch of the form $|k_\alpha|$. This is done by iterating on $|k_\alpha|$ by increasing the value of α . The first step of solving problem \mathcal{P}_2 is to relax the equality in equation (2.44) with an inequality \leq . So the problem reduces to finding a lossless scattering matrix $F(s)$ such that

$$\delta(F_{22}(j\omega), \overline{L_{11}(j\omega)}) \leq |k_\alpha(j\omega)|, \quad \omega \in \mathbb{R}. \quad (2.45)$$

The pseudo-hyperbolic disk, $\delta(F_{22}, \overline{L_{11}}) \leq |k_\alpha|$ with centre $\overline{L_{11}}$ and radius $|k_\alpha|$ translates to the following Euclidean disk using theorem 2.2.3 :

$$\left| F_{22} - \frac{(1 - |k_\alpha|^2)}{1 - |k_\alpha|^2 |L_{11}|^2} \overline{L_{11}} \right| \leq \frac{(1 - |L_{11}|^2)}{1 - |k_\alpha|^2 |L_{11}|^2} |k_\alpha| \quad (2.46)$$

Now, following the same approach of solving gain equalization problem described in subsection 2.2.2, the problem can be casted as the following

Problem.

$$\min_{\Psi \in H^\infty} \|\Psi - \Phi\|_{L^\infty} \quad (2.47)$$

and find a Ψ at which the infimum is attained if the minimum is less than or equal to one, where $\Phi = \frac{1}{V} \left(\frac{1 - |k_\alpha|^2}{1 - |k_\alpha|^2 |L_{11}|^2} \right) \overline{L_{11}}$ and V is the outer function satisfying $|V| = \left| \frac{(1 - |L_{11}|^2)}{1 - |k_\alpha|^2 |L_{11}|^2} \right| |k_\alpha|$ on the imaginary axis.

It should be noted that since we assume the load has no transmission zeros on the imaginary axis and the function k_α by construction has no zeros on the imaginary axis, the outer function V is invertible in $H^\infty(\Pi^+)$. So, we have $\Phi \in L^\infty(j\mathbb{R})$ and we are interested in finding an H^∞ function in problem 2.47 whose L^∞ -distance to Φ is less than or equal to one. So, using theorem 2.2.6, we have a solution to the problem described in (2.45) if and only if $\|H_\Phi\| \leq 1$.

Nehari Test I : The problem in (2.45) is solvable if and only if $\|H_\Phi\| \leq 1$ where Φ is described in the Nehari problem in (2.47).

In this case, we have the minimiser,

$$\Psi = \Phi - \frac{H_\Phi(W)}{W}, \quad (2.48)$$

where H_Φ is the Hankel operator with symbol Φ and W one of its maximizing vectors. The solution to problem in (2.45), namely $F_{22} \in H^\infty(\Pi^+)$ can be obtained by multiplying the outer function V to Ψ , that is $F_{22} = V\Psi$.

Before answering the problem of finding the smallest possible power mismatch of the form k_α , we will show that the symbol Φ in the Nehari problem (2.47) is in a peculiar rational form $\bar{b}g$, where b is a finite Blaschke product and g is a rational H^∞ function. The importance of this form lies in the fact that it provides direct information about the kernel and range of the Hankel operator with this symbol. Theorem 2.4 in [71] states the following.

Theorem 2.3.2. *Let $g \in H^\infty$ and b be an inner function such that b and g are coprime (no common nonconstant inner divisors). Then*

$$\text{Ker } H_{bg} = bH^2 \quad \text{and} \quad \text{clos Range } H_{bg} = \bar{H}^2 \ominus \bar{b}\bar{H}^2,$$

where $\bar{H}^2 \stackrel{\text{def}}{=} L^2 \ominus H^2$, the orthogonal complement of H^2 in L^2

We had two cases of reference functions, $k_\alpha = \frac{p_k}{q_k}$, described in subsection 2.3.1,

- Case (i).

$$|k_\alpha|^2 = \frac{|p_k|^2}{|p_k|^2 + \alpha} \text{ on the imaginary axis.}$$

In this case, first of all, we do some elementary calculations to provide a rational expression for the outer function V and the symbol Φ mentioned in problem 2.47.

V is the outer function satisfying $|V| = \left| \frac{(1-|L_{11}|^2)}{1-|k_\alpha|^2|L_{11}|^2} \right| |k_\alpha|$ on the imaginary axis and

$$\Phi = \frac{1}{V} \left(\frac{1-|k_\alpha|^2}{1-|k_\alpha|^2|L_{11}|^2} \right) \overline{L_{11}}. \text{ We have,}$$

$$\begin{aligned} L_{11} &= \frac{p_L}{q_L}, \text{ where } q_L q_L^* - p_L p_L^* = r_L r_L^*, \\ k_\alpha &= \frac{p_k}{q_k}, \text{ where } q_k q_k^* - p_k p_k^* = \alpha. \end{aligned} \tag{2.49}$$

Now, we will express the terms, $1 - |L_{11}|^2$, $1 - |k_\alpha|^2$ and $1 - |k_\alpha|^2 |L_{11}|^2$ present in the expressions of $|V|$ and Φ in a rational form using (2.49). We have

$$\begin{aligned} 1 - |L_{11}|^2 &= 1 - \frac{p_L p_L^*}{q_L q_L^*} = \frac{r_L r_L^*}{q_L q_L^*}, \\ 1 - |k_\alpha|^2 &= 1 - \frac{p_k p_k^*}{q_k q_k^*} = \frac{\alpha}{q_k q_k^*}, \\ 1 - |k_\alpha|^2 |L_{11}|^2 &= 1 - \frac{p_k p_k^* p_L p_L^*}{q_k q_k^* q_L q_L^*} = \frac{xx^*}{q_k q_k^* q_L q_L^*}, \end{aligned} \tag{2.50}$$

where x is used to denote the Hurwitz polynomial of degree $N + M$ satisfying

$$xx^* = q_k q_k^* q_L q_L^* - p_k p_k^* p_L p_L^*. \tag{2.51}$$

Substituting the required expressions from (2.50) in the expression of $|V|$, we have,

$$|V| = \frac{r_L r_L^*}{q_L q_L^*} \frac{q_k q_k^* q_L q_L^*}{xx^*} \left| \frac{p_k}{q_k} \right|.$$

This provides the rational expression of the outer function V , $V = \frac{r_L^2 p_k q_k}{x^2}$. Now, using this rational expression of V and the required expressions from (2.50), we have

$$\Phi = \frac{1}{V} \left(\frac{1 - |k_\alpha|^2}{1 - |k_\alpha|^2 |L_{11}|^2} \right) \overline{L_{11}} = \frac{x^2}{r_L^2 p_k q_k} \frac{\alpha}{q_k q_k^*} \frac{q_k q_k^* q_L q_L^*}{xx^*} \frac{p_L^*}{q_L^*} = \alpha \frac{x}{x^*} \frac{p_L^* q_L}{r_L^2 p_k q_k}.$$

So, we have,

$$V = \frac{r_L^2 p_k q_k}{x^2} \quad \text{and} \quad \Phi = \alpha \frac{x}{x^*} \frac{p_L^* q_L}{r_L^2 p_k q_k}. \quad (2.52)$$

It should be noted that the unstable poles of Φ are the roots of x^* since r_L, p_k and q_k are Hurwitz polynomials. If we denote the roots of x^* by $\beta_1, \beta_2, \dots, \beta_{N+M}$, each $\beta_i \in \Pi^+$, we have $\Phi = \bar{b}g$ where

$$g = \frac{\alpha p_L^* q_L}{r_L^2 p_k q_k} \quad \text{and} \quad b(s) = \frac{x^*(s)}{x(s)} = \prod_{i=1}^{N+M} \frac{s - \beta_i}{s + \bar{\beta}_i}. \quad (2.53)$$

- Case (ii).

$$|k_\alpha|^2 = \frac{|p_k|^2}{|p_k|^2 + \alpha |r_L|^2} \quad \text{on the imaginary axis.} \quad (2.54)$$

Similar to the previous case, we can derive the following expressions in this case of reference functions,

$$V = \frac{p_k q_k}{x_0^2}, \quad \Phi = \alpha \frac{x_0 p_L^* q_L}{x_0^* p_k q_k}, \quad (2.55)$$

where x_0 is the Hurwitz polynomial of degree N satisfying

$$x_0 x_0^* = \alpha q_L q_L^* + p_k p_k^*. \quad (2.56)$$

The unstable poles of Φ are the roots of x_0^* since p_k and q_k are Hurwitz polynomials. If we denote the roots of x_0^* by $\beta_1, \beta_2, \dots, \beta_N$, each $\beta_i \in \Pi^+$, we have $\Phi = \bar{b}g$ where

$$g = \frac{\alpha p_L^* q_L}{p_k q_k} \quad \text{and} \quad b(s) = \frac{x_0^*(s)}{x_0(s)} = \prod_{i=1}^N \frac{s - \beta_i}{s + \bar{\beta}_i}. \quad (2.57)$$

So, the rational symbol $\Phi \in L^\infty$ in the Nehari problem (2.47) has the following expression in both cases of reference functions,

$$\Phi = \bar{b}g,$$

where b and g follow the equations mentioned in (2.53) and (2.57) in cases (i) and (ii) respectively. So, it follows from theorem 2.3.2 that

$$\text{Ker } H_\Phi = bH^2 \quad \text{and} \quad \text{Range } H_\Phi = \bar{H}^2 \ominus \bar{b}\bar{H}^2.$$

So, the spaces, orthogonal complement of kernel of Hankel operator H_Φ in H^2 and the image of Hankel operator are finite dimensional and are explicitly

$$(\text{Ker } H_\Phi)^\perp = H^2 \ominus bH^2 \stackrel{\text{def}}{=} \mathcal{H}(b) = \text{span} \left\langle g_i = \frac{1}{s + \bar{\beta}_i} \right\rangle, \quad (2.58)$$

$$\text{Range } H_\Phi = \bar{H}^2 \ominus \bar{b}\bar{H}^2 \stackrel{\text{def}}{=} \mathcal{H}(\bar{b}) = \text{span} \left\langle h_i = \frac{1}{s - \beta_i} \right\rangle. \quad (2.59)$$

A detailed literature about the collection of analytic functions that are in the image of an operator can be found in [38] and [39]. In particular, corollary 14.8 in [38] provides the result, $\text{span}\langle g_i \rangle = \mathcal{H}(b)$. The bases $\{g_i\}$ and $\{h_i\}$ of $(\text{Ker } H_\Phi)^\perp$ and $\text{Range } H_\Phi$ respectively will be useful in the numerical implementation of the Nehari problem 2.47 as described in the next subsection.

So, the problem of finding the smallest possible power mismatch of the form k_α can be solved by performing the Nehari test I of $\|H_\Phi\| \leq 1$ by iterating $|k_\alpha|$. This can be done by performing the test on a grid of α 's and finding the α at which $\|H_\Phi\| = 1$ (say it is achieved at $\hat{\alpha}$). It should be noted that when L_{11} is a non-constant rational Schur function, the existence of such an $\hat{\alpha}$ which is unique is guaranteed. This can be seen by the fact that perfect matching on interval \mathbb{I} would imply $F_{22}(j\omega) = \overline{L_{11}(j\omega)}$ for all $\omega \in \mathbb{I}$ and this is not possible since $\overline{L_{11}(s)}$ corresponds to the evaluation of $L_{11}^*(s)$ on the imaginary axis, which is an anti-analytic function. So, for any $\alpha > \hat{\alpha}$, the answer to the problem in (2.45) is that there doesn't exist $F_{22} \in \mathbb{B}$ satisfying (2.45). The optimal $\hat{\alpha}$ provides us \hat{F}_{22} which achieves the smallest possible power mismatch $|k_{\hat{\alpha}}|$, that is

$$\delta(\hat{F}_{22}(j\omega), \overline{L_{11}(j\omega)}) = |k_{\hat{\alpha}}(j\omega)|, \quad \omega \in \mathbb{R}. \quad (2.60)$$

Finally Belevitch theorem (Theorem 1.3.10) can be used to form the lossless scattering matrix $\hat{F}(s)$ realizing the smallest power mismatch $|k_{\hat{\alpha}}|$. In the next subsection, we will briefly explain the important steps in the numerical implementation of the above mentioned approach in finding \hat{F}_{22} , the solution to problem \mathcal{P}_2 which realizes the smallest possible power mismatch of the form $|k_\alpha|$.

2.3.3 Numerical Implementation : Generalized EVP

Even though it is more common to find Nehari theory described in the framework of Hardy spaces of the unit disc \mathbb{D} , $H^\infty(\mathbb{D})$, we choose to continue in the setting of right half plane, $H^\infty(\Pi^+)$. It should be noted that the implementation described below of the Nehari's solution to the extremal problem (2.47) can be easily shifted to the setting of $H^\infty(\mathbb{D})$ using the map, $s \rightarrow (s - 1)/(s + 1)$ sending the right half plane to the unit disk. It should be evident from the last subsection that solving problem \mathcal{P}_2 for a reference function k_α essentially reduces to solving Nehari problem for $\Phi = \bar{b}g \in L^\infty$ in (2.47). For a rational function Φ of this form, in order to find the maximizing vector of the Hankel operator, $H_\Phi : H_2 \rightarrow \bar{H}_2$, we follow the steps below:

- (i) Let $\{\beta_i\}_{i=1}^d$ be the poles of Φ inside the right half plane. When the reference functions are build as in case (i), we have $d = N + M$ and in case (ii), we have $d = N$.
- (ii) For $i = 1, 2, \dots, d$, $\{g_i\} = \{\frac{1}{s+\beta_i}\}$ form a basis of $(\text{Ker}(H_\Phi))^\perp$ and $\{h_i\} = \{\frac{1}{s-\beta_i}\}$ form a basis of the image of H_Φ as mentioned in equations (2.58) and (2.59).

(iii) Let the Gram matrix of the $\{g_i\}'s$ be denoted by G_1 . We have

$$\begin{aligned} G_1 = [a_{m,n}] = \langle g_m, g_n \rangle &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{(s - \beta_m)} \frac{1}{(s + \bar{\beta}_n)} d\omega \\ &= \frac{1}{2\pi j} \int_{-\infty}^{+\infty} \frac{h(s)}{s - \beta_m} ds, \end{aligned}$$

where $h(s) = \frac{1}{s + \bar{\beta}_n}$. Since $\beta_n \in \Pi^+$, h is analytic in Π^+ and hence Cauchy's integral formula (Theorem 1.2.9) applies to give

$$G_1 = h(\beta_m) = \left[\frac{1}{\beta_m + \bar{\beta}_n} \right]_{m,n=1}^d. \quad (2.61)$$

Similarly, the Gram matrix of the $\{h_i\}'s$ can be calculated as,

$$\begin{aligned} G_2 = [b_{m,n}] = \langle h_m, h_n \rangle &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{(s + \bar{\beta}_m)} \frac{1}{(s - \beta_n)} d\omega \\ &= \left[\frac{1}{\bar{\beta}_m + \beta_n} \right]_{m,n=1}^d. \end{aligned} \quad (2.62)$$

(iv) Denote the matrix of the Hankel operator, H_Φ by A in the chosen bases $\{g_i\}$ and $\{h_i\}$ for $(\text{Ker}(H_\Phi))^\perp$ and the image of H_Φ in \bar{H}^2 respectively. If we represent the rational function Φ by :

$$\Phi(s) = \frac{\mathcal{N}(s)}{(s - \beta_1)(s - \beta_2) \dots (s - \beta_d)(s - \gamma_1)(s - \gamma_2) \dots (s - \gamma_t)}, \quad (2.63)$$

where β_i 's are in Π^+ , γ_i 's are in Π^- and $\mathcal{N}(s)$ a polynomial, the matrix A of dimension $(d \times d)$ can be calculated as follows. We can use $f(s) \in H^\infty(\Pi^+)$ to denote

$$f(s) = \frac{\mathcal{N}(s)}{\prod_{i=1}^t (s - \gamma_i)}. \quad (2.64)$$

Now, we can compute the application of the Hankel operator H_Φ on some element $u = (u_1, u_2, \dots, u_d)$ in $(\text{Ker } H_\Phi)^\perp$. From equation 2.58, we have

$$u(s) = \sum_{i=1}^d u_i g_i(s),$$

where $g_i(s) = \frac{1}{s+\bar{\beta}_i}$. The application of Hankel operator H_Φ on g_n for $1 \leq n \leq d$ produces the following:

$$H_\Phi(g_n) = \underline{P}(\Phi g_n) = \underline{P}\left(\frac{f(s)}{\prod_{i=1}^d (s - \beta_i)} \frac{1}{s + \bar{\beta}_n}\right) \quad (2.65)$$

$$= \sum_{m=1}^d \frac{f(\beta_m)}{(\beta_m + \bar{\beta}_n) \prod_{\substack{i=1 \\ i \neq m}}^d (\beta_m - \beta_i)} \frac{1}{s - \beta_m} \quad (2.66)$$

$$= \sum_{m=1}^d \frac{f(\beta_m)}{(\beta_m + \bar{\beta}_n) \prod_{\substack{i=1 \\ i \neq m}}^d (\beta_m - \beta_i)} h_m, \quad (2.67)$$

where $h_m = \frac{1}{s - \beta_m}$. It should be noted that (2.66) is obtained from (2.65) after doing a partial fraction decomposition of the function Φg_n . Now, (2.67) implies that we have the $(m, n)^{th}$ element ($1 \leq m, n \leq d$) of the matrix of Hankel operator,

$$A_{mn} = \frac{f(\beta_m)}{(\beta_m + \bar{\beta}_n) \prod_{\substack{i=1 \\ i \neq m}}^d (\beta_m - \beta_i)} = \frac{\mathcal{N}(\beta_m)}{(\beta_m + \bar{\beta}_n) \prod_{\substack{i=1 \\ i \neq m}}^d (\beta_m - \beta_i) \prod_{j=1}^t (\beta_m - \gamma_j)}.$$

The last equality follows from equation (2.64). Thus, we have the matrix A of the Hankel operator H_Φ ,

$$A = \begin{bmatrix} \frac{\mathcal{N}(\beta_1)}{(\beta_1 + \bar{\beta}_1) \prod_{\substack{i=1 \\ i \neq 1}}^d (\beta_1 - \beta_i) \prod_{j=1}^t (\beta_1 - \gamma_j)} & \frac{\mathcal{N}(\beta_1)}{(\beta_1 + \bar{\beta}_2) \prod_{\substack{i=1 \\ i \neq 1}}^d (\beta_1 - \beta_i) \prod_{j=1}^t (\beta_1 - \gamma_j)} & \cdots & \frac{\mathcal{N}(\beta_1)}{(\beta_1 + \bar{\beta}_d) \prod_{\substack{i=1 \\ i \neq 1}}^d (\beta_1 - \beta_i) \prod_{j=1}^t (\beta_1 - \gamma_j)} \\ \frac{\mathcal{N}(\beta_2)}{(\beta_2 + \bar{\beta}_1) \prod_{\substack{i=1 \\ i \neq 2}}^d (\beta_2 - \beta_i) \prod_{j=1}^t (\beta_2 - \gamma_j)} & \frac{\mathcal{N}(\beta_2)}{(\beta_2 + \bar{\beta}_2) \prod_{\substack{i=1 \\ i \neq 2}}^d (\beta_2 - \beta_i) \prod_{j=1}^t (\beta_2 - \gamma_j)} & \cdots & \frac{\mathcal{N}(\beta_2)}{(\beta_2 + \bar{\beta}_d) \prod_{\substack{i=1 \\ i \neq 2}}^d (\beta_2 - \beta_i) \prod_{j=1}^t (\beta_2 - \gamma_j)} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\mathcal{N}(\beta_d)}{(\beta_d + \bar{\beta}_1) \prod_{\substack{i=1 \\ i \neq d}}^d (\beta_d - \beta_i) \prod_{j=1}^t (\beta_d - \gamma_j)} & \frac{\mathcal{N}(\beta_d)}{(\beta_d + \bar{\beta}_2) \prod_{\substack{i=1 \\ i \neq d}}^d (\beta_d - \beta_i) \prod_{j=1}^t (\beta_d - \gamma_j)} & \cdots & \frac{\mathcal{N}(\beta_d)}{(\beta_d + \bar{\beta}_d) \prod_{\substack{i=1 \\ i \neq d}}^d (\beta_d - \beta_i) \prod_{j=1}^t (\beta_d - \gamma_j)} \end{bmatrix}.$$

So, in order to find the operator norm of Hankel operator and the maximizing vector, we are interested in solving the following problem,

$$\text{Find : } \tau^2 = \max_{u \in (\text{Ker } H_\Phi)^\perp} \frac{\|Au\|_2}{\|u\|_2}. \quad (2.68)$$

This gives $\|H_\Phi\| = \tau$. We have $\|Au\|_2 = u^* A^* G_2 A u$ and $\|u\|_2 = u^* G_1 u$, where G_1 and G_2 are Gram matrices as described in equations (2.61) and (2.62) respectively. Thus, we solve the generalized eigenvalue problem: $A^* G_2 A u = \lambda G_1 u$.

The eigenvector corresponding to the largest eigenvalue will provide the maximizing vector, W of the Hankel operator and the square root of largest eigenvalue will provide the value of the minimum in (2.47). This directly provides us the solution, $\Psi \in H^\infty(\Pi^+)$ to Nehari problem in (2.47),

$$\Psi = \Phi - \frac{H_\Phi(W)}{W}.$$

The solution F_{22} to problem 2.45 is obtained by multiplying back the outer factor V , that is $F_{22} = V\Psi$. The process is continued by iterating over α until we obtain $\|H_\Phi\| = 1$ (say it is achieved at $\hat{\alpha}$). This provides us the solution \hat{F}_{22} to problem \mathcal{P}_2 realizing the smallest possible power mismatch of the form $|k_\alpha|$.

2.3.4 Degree of the Optimal Matching Circuit

In this subsection, we will provide an important result which states a bound for the degree of rational Schur function F_{22} which is obtained as a solution of problem (2.45). This provides information about the degree of optimal matching circuit obtained as a result of solving problem \mathcal{P}_2 . It should be noted that we had two cases of building the rational reference functions k_α described in subsection 2.3.1, where, in one case the transmission zeros of the load was included in the reference and one in which it was not done. The proposition stated below provides the estimate of degree of the matching circuit obtained in both cases.

Proposition 2.3.3. *The scattering matrix of the matching circuit obtained by solving problems (2.45) and \mathcal{P}_2 satisfies the following degree bounds on its McMillan degree :*

(1). *In the case of reference function, $k_\alpha = \frac{p_k}{q_k}$ satisfying $r_k r_k^* = q_k q_k^* - p_k p_k^* = \alpha$ (case (i) in subsection 2.3.1), if there exists a solution $F(s)$ to problem 2.45, the degree of the rational Schur function $F_{22}(s)$ is bounded by $2(N + M) - 1$, where N is the degree of $k_\alpha(s)$ and M is the McMillan degree of the scattering matrix of the load. In this case, the degree of the solution $F_{22}(s)$ to problem \mathcal{P}_2 is bounded by $N + M - 1$.*

(2). *If the reference function, $k_\alpha = \frac{p_k}{q_k}$ satisfy $r_k r_k^* = q_k q_k^* - p_k p_k^* = \alpha r_L r_L^*$ (case (ii) in subsection 2.3.1), the degree of the solution $F_{22}(s)$ to problem 2.45 is bounded by $2N - 1$, where N is the degree of $k_\alpha(s)$. In this case, the degree of the solution $F_{22}(s)$ to problem \mathcal{P}_2 is bounded by $N - 1$.*

Proof. We will do the proof separately for both cases of reference functions.

(1). In the first case, reference function, $k_\alpha = \frac{p_k}{q_k}$ satisfy

$$r_k r_k^* = q_k q_k^* - p_k p_k^* = \alpha.$$

In this case, as per equations in 2.52, we have the following expressions for the symbol Φ of the Hankel operator and the outer function V ,

$$\Phi = \alpha \frac{x}{x^*} \frac{p_L^* q_L}{r_L^2 p_k q_k}, \quad V = \frac{r_L^2 p_k q_k}{x^2}.$$

As mentioned in equation (2.48), we have a solution to the problem 2.45 if and only if $\|H_\Phi\| \leq 1$ and in this case, we have the minimiser,

$$\Psi = \Phi - \frac{H_\Phi(W)}{W}, \quad (2.69)$$

where H_Φ is the Hankel operator with symbol Φ and W one of its maximizing vectors. The solution to problem 2.45, $F_{22} \in H^\infty(\Pi^+)$ is obtained by multiplying the outer function V to Ψ , that is $F_{22} = V\Psi$. We have $W \in \mathcal{H}(b)$ and $H_\Phi(W) \in \mathcal{H}(\bar{b})$, where $b(s)$ is the following Blaschke product of degree $(N + M)$ as per equation in (2.53)

$$b(s) = \frac{x^*(s)}{x(s)} = \prod_{i=1}^{N+M} \frac{s - \beta_i}{s + \bar{\beta}_i}.$$

It should be noted that from equations (2.58) and (2.59), we have

$$\mathcal{H}(b) = \frac{\mathbb{P}_{N+M-1}}{x}, \quad \mathcal{H}(\bar{b}) = \frac{\mathbb{P}_{N+M-1}}{x^*}. \quad (2.70)$$

This implies that the maximizing vector W and the Hankel operator acted upon W can be expressed as

$$W = \frac{a_0}{x}, \quad H_\Phi(W) = \frac{b_0}{x^*}, \quad (2.71)$$

where a_0 and b_0 are polynomials of at most degree $N + M - 1$. In addition, in this case, from theorem 1.4 in [71] it follows that the error function, $\Phi - \Psi = \frac{H_\Phi(W)}{W}$ has constant modulus on the imaginary axis and we have

$$b_0 = \tau a_0^*, \quad \tau \in \mathbb{C} \text{ satisfying } |\tau| = \|H_\Phi\| \quad (2.72)$$

and a_0 is Hurwitz polynomial. So, we have

$$\Psi = \Phi - \frac{H_\Phi(W)}{W} = \frac{x}{x^*} \left(\frac{\alpha p_L^* q_L}{r_L^2 p_k q_k} - \frac{b_0}{a_0} \right) = \frac{x}{x^*} \left(\frac{\alpha p_L^* q_L a_0 - b_0 r_L^2 p_k q_k}{r_L^2 p_k q_k a_0} \right). \quad (2.73)$$

It should be noted that since $\Psi \in H^\infty(\Pi^+)$, x^* should divide the polynomial $(\alpha p_L^* q_L a_0 - b_0 r_L^2 p_k q_k)$ which is of maximal degree $3(N + M) - 1$. This follows from equation 2.71, $H_\Phi(\frac{a_0}{x}) = \underline{P}(\Phi \frac{a_0}{x}) = \frac{b_0}{x^*}$, i.e, we have

$$\Phi \frac{a_0}{x} = \frac{\alpha p_L^* q_L a_0}{x^* r_L^2 p_k q_k} = \frac{b_0}{x^*} + \frac{c_0}{r_L^2 p_k q_k},$$

where $c_0 \in \mathbb{P}_{2(N+M)-1}$. This implies that we have

$$\alpha p_L^* q_L a_0 - b_0 r_L^2 p_k q_k = c_0 x^*. \quad (2.74)$$

Substituting this in equation 2.73, we have,

$$\Psi = \frac{x c_0}{r_L^2 p_k q_k a_0}.$$

So, when $\|H_\Phi\| \leq 1$, we have the solution $F_{22} \in H^\infty(\Pi^+)$ taking the following rational expression,

$$F_{22} = V\Psi = \frac{c_0}{xa_0}. \quad (2.75)$$

Thus, we have the degree of the solution $F_{22}(s)$ to problem (2.45) bounded by $2(N+M)-1$.

The solution to problem \mathcal{P}_2 is obtained when $\|H_\Phi\| = 1$ and when this happen, we claim that in the expression for F_{22} , the polynomial x divides c_0 and hence giving a degree bound of $N+M-1$ for F_{22} . In order to prove this claim, following equation 2.74, we will prove that the polynomial $(\alpha p_L^* q_L a_0 - b_0 r_L^2 p_k q_k)$ which vanish at the roots of x^* vanish at the roots of x as well when $\|H_\Phi\| = 1$. From equation 2.74, for $1 \leq i \leq N+M$, we have

$$\alpha[p_L^* q_L a_0](\beta_i) = [b_0 r_L^2 p_k q_k](\beta_i), \quad (2.76)$$

where $\beta_i \in \Pi^+$ are the roots of x^* . Conjugating equation 2.76, we have

$$\alpha[p_L q_L^* a_0^*](\bar{\beta}_i) = [b_0^* (r_L^*)^2 p_k^* q_k^*](\bar{\beta}_i),$$

Multiplying this equation by $[r_L p_L^* p_k](\bar{\beta}_i)$, we get

$$\alpha[p_L q_L^* r_L p_L^* p_k a_0^*](\bar{\beta}_i) = [b_0^* (r_L^*)^2 r_L p_L^* p_k p_k^* q_k^*](\bar{\beta}_i). \quad (2.77)$$

Recall from equation 2.51 that the Hurwitz polynomial x satisfy

$$xx^* = q_k q_k^* q_L q_L^* - p_k p_k^* p_L p_L^*. \quad (2.78)$$

It follows from equation (2.78) that

$$[q_k q_k^* q_L q_L^*](\bar{\beta}_i) = [p_k p_k^* p_L p_L^*](\bar{\beta}_i).$$

This equation along with the spectral equations, $q_k q_k^* = p_k p_k^* + \alpha$ and $q_L q_L^* = p_L p_L^* + r_L r_L^*$ provides

$$\alpha[p_L p_L^*](\bar{\beta}_i) = -[r_L r_L^* q_k q_k^*](\bar{\beta}_i), \quad [r_L r_L^* p_k p_k^*](\bar{\beta}_i) = -\alpha[q_L q_L^*](\bar{\beta}_i). \quad (2.79)$$

Substituting these values of $\alpha[p_L p_L^*](\bar{\beta}_i)$ and $[r_L r_L^* p_k p_k^*](\bar{\beta}_i)$ in equation 2.77, we get

$$[a_0^* r_L^2 q_L^* r_L^* q_k q_k^* p_k](\bar{\beta}_i) = \alpha[p_L^* r_L^* q_L q_L^* q_k^* b_0^*](\bar{\beta}_i).$$

Simplifying the factor $[r_L^* q_L^* q_k^*](\bar{\beta}_i)$ (note that it does not vanish since r_L, q_L and q_k are Hurwitz polynomials), we get

$$[a_0^* r_L^2 p_k q_k](\bar{\beta}_i) = \alpha[p_L^* q_L b_0^*](\bar{\beta}_i).$$

When $\|H_\Phi\| = 1$, we have $a_0^* = \tau^* b_0$ and $b_0^* = \tau^* a_0$ from equation 2.72 and thus we have,

$$[b_0 r_L^2 p_k q_k](\bar{\beta}_i) = \alpha[p_L^* q_L a_0](\bar{\beta}_i). \quad (2.80)$$

It should be noted that if the multiplier $[r_L p_L^* p_k](\bar{\beta}_i)$ in equation 2.77 is zero, then we have the above result trivially since $[r_L p_k](\bar{\beta}_i) = 0$ implies $q_L(\bar{\beta}_i) = 0$ and when $p_L^*(\bar{\beta}_i) = 0$,

$[r_L q_k](-\bar{\beta}_i) = 0$ from equations in (2.79). So, equation 2.80 proves our claim that the polynomial $(\alpha p_L^* q_L a_0 - b_0 r_L^2 p_k q_k)$ vanish at the roots of x as well when $\|H_\Phi\| = 1$ and hence the maximal degree of solution F_{22} to problem \mathcal{P}_2 is $N + M - 1$. This completes the proof in the first case.

(2). In the second case of reference function, $k_\alpha = \frac{p_k}{q_k}$, we have

$$r_k r_k^* = q_k q_k^* - p_k p_k^* = \alpha r_L r_L^*.$$

In this case, as per equations in (2.55), we have the following expressions for the symbol Φ of the Hankel operator and the outer function V ,

$$\Phi = \alpha \frac{x_0 p_L^* q_L}{x_0^* p_k q_k}, \quad V = \frac{p_k q_k}{x_0^2}. \quad (2.81)$$

We will follow the same approach of previous case in order to calculate the degree bounds of the solution in this case and hence will only present the important steps. We have a solution to the problem 2.45 if and only if $\|H_\Phi\| \leq 1$ and in this case, we have the minimiser,

$$\Psi = \Phi - \frac{H_\Phi(W)}{W}, \quad (2.82)$$

We have $W \in \mathcal{H}(b)$ and $H_\Phi(W) \in \mathcal{H}(\bar{b})$, where $b(s)$ is the following Blaschke product of degree N as per equation in (2.57),

$$b(s) = \frac{x_0^*(s)}{x_0(s)} = \prod_{i=1}^N \frac{s - \beta_i}{s + \bar{\beta}_i}.$$

The maximizing vector W and the Hankel operator acted upon W can be expressed as

$$W = \frac{a_0}{x_0}, \quad H_\Phi(W) = \frac{b_0}{x_0^*}. \quad (2.83)$$

where a_0 and b_0 are polynomials of at most degree $N - 1$. In addition, we have

$$b_0 = \tau a_0^*, \quad \tau \in \mathbb{C} \text{ satisfying } |\tau| = \|H_\Phi\| \quad (2.84)$$

and a_0 is Hurwitz polynomial. So, we have

$$\Psi = \Phi - \frac{H_\Phi(W)}{W} = \frac{x_0}{x_0^*} \left(\frac{\alpha p_L^* q_L}{p_k q_k} - \frac{b_0}{a_0} \right) = \frac{x_0}{x_0^*} \left(\frac{\alpha p_L^* q_L a_0 - b_0 p_k q_k}{p_k q_k a_0} \right). \quad (2.85)$$

Since $\Psi \in H^\infty(\Pi^+)$, x_0^* should divide the polynomial $(\alpha p_L^* q_L a_0 - b_0 p_k q_k)$ which is of maximal degree $3N - 1$. This implies that we have

$$\alpha p_L^* q_L a_0 - b_0 p_k q_k = c_0 x_0^*, \quad (2.86)$$

where $c_0 \in \mathbb{P}_{2N-1}$. Substituting this in equation 2.85, we have,

$$\Psi = \frac{x_0 c_0}{p_k q_k a_0}.$$

2.3. FINITE DEGREE MATCHING PROBLEM

So, when $\|H_\Phi\| \leq 1$, we have the solution $F_{22} \in H^\infty(\Pi^+)$ taking the following rational expression,

$$F_{22} = V\Psi = \frac{c_0}{x_0 a_0}. \quad (2.87)$$

Thus, we have the degree of the solution $F_{22}(s)$ to problem (2.45) bounded by $2N - 1$.

The solution to problem \mathcal{P}_2 is obtained when $\|H_\Phi\| = 1$ and when this happen, we claim that the polynomial $(\alpha p_L^* q_L a_0 - b_0 p_k q_k)$ which vanish at the roots of x_0^* vanish at the roots of x_0 as well. From equation 2.86, for $1 \leq i \leq N$, we have

$$\alpha[p_L^* q_L a_0](\beta_i) = [b_0 p_k q_k](\beta_i), \quad (2.88)$$

where $\beta_i \in \Pi^+$ are the roots of x_0^* . Conjugating equation 2.88, we have

$$\alpha[p_L q_L^* a_0^*](\bar{\beta}_i) = [b_0^* p_k^* q_k^*](\bar{\beta}_i),$$

Multiplying this equation by $[p_L^* p_k](\bar{\beta}_i)$, we get

$$\alpha[p_L q_L^* p_L^* p_k a_0^*](\bar{\beta}_i) = [b_0^* p_L^* p_k p_k^* q_k^*](\bar{\beta}_i). \quad (2.89)$$

Recall from equation (2.56) that the Hurwitz polynomial x_0 satisfy

$$x_0 x_0^* = \alpha q_L q_L^* + p_k p_k^*. \quad (2.90)$$

Substituting $p_k p_k^* = q_k q_k^* - \alpha r_L r_L^*$, we also have,

$$x_0 x_0^* = q_k q_k^* + \alpha p_L p_L^*. \quad (2.91)$$

Equations (2.90) and (2.91) provides

$$\alpha[p_L p_L^*](\bar{\beta}_i) = -[q_k q_k^*](\bar{\beta}_i), \quad [p_k p_k^*](\bar{\beta}_i) = -\alpha[q_L q_L^*](\bar{\beta}_i). \quad (2.92)$$

Substituting these values of $\alpha[p_L p_L^*](\bar{\beta}_i)$ and $[p_k p_k^*](\bar{\beta}_i)$ in equation 2.89, we get

$$[q_L^* p_k q_k q_k^* a_0^*](\bar{\beta}_i) = \alpha[b_0^* q_L q_L^* p_L^* q_k^*](\bar{\beta}_i) \quad (2.93)$$

Simplifying the factor $[q_L^* q_k^*](\bar{\beta}_i)$ (note that it does not vanish since q_L and q_k are Hurwitz polynomials), we get

$$[a_0^* p_k q_k](\bar{\beta}_i) = \alpha[p_L^* q_L b_0^*](\bar{\beta}_i).$$

When $\|H_\Phi\| = 1$, we have $a_0^* = \tau^* b_0$ and $b_0^* = \tau^* a_0$ from equation 2.84 and thus we have,

$$[b_0 p_k q_k](\bar{\beta}_i) = \alpha[p_L^* q_L a_0](\bar{\beta}_i). \quad (2.94)$$

If the multiplier $[p_L^* p_k](\bar{\beta}_i)$ in equation 2.89 is zero, then we have the above result trivially since $p_k(\bar{\beta}_i) = 0$ implies $q_L(\bar{\beta}_i) = 0$ and when $p_L^*(\bar{\beta}_i) = 0$, $q_k(\bar{\beta}_i) = 0$ from equations in (2.92). So, equation (2.94) proves our claim that the polynomial $(\alpha p_L^* q_L a_0 - b_0 p_k q_k)$ vanish at the roots of x_0 as well when $\|H_\Phi\| = 1$ and hence the maximal degree of solution F_{22} to problem \mathcal{P}_2 is $N - 1$. This completes the proof. \square

Proposition 2.3.3 provides some interesting insights into the maximal degree of optimal matching circuit obtained by the approach used in this chapter. If the reference function is considered to have the degree N equal to the sum of the desired degree of the matching circuit and the load ($N = N_{mc} + M$), then in case (i), the resulting optimal matching circuit can have an extra degree of $2M - 1$ and in case (ii) it can have an extra degree of $M - 1$ compared to the desired degree N_{mc} in the proposed scheme. However, the knowledge of degree bounds of the optimal matching circuit from proposition 2.3.3 allows us to perform the following : if the targeted McMillan degree of the scattering matrix of the matching circuit to be synthesized is constrained to be less than or equal to N_{mc} as per the application requirement,

- (i). in the case of reference functions $k_\alpha = \frac{p_k}{q_k}$ build such that $r_k r_k^* = \alpha$, the maximal degree of k_α should be constrained in the optimisation to be $N_{mc} + 1 - M$,
- (ii). in the case of reference functions $k_\alpha = \frac{p_k}{q_k}$ build such that $r_k r_k^* = \alpha r_L r_L^*$, the maximal degree of k_α should be constrained in the optimisation to be $N_{mc} + 1$.

The extra freedom of degree (which is equal to the degree of the load) in case (ii) compared to that in case (i) provides an advantage to the reference functions in which transmission zeros of the load are included in them ($r_k r_k^* = \alpha r_L r_L^*$) in providing a better optimal criterion compared to the ones in which they are not included. It is also more natural to include the transmission zeros of the load in the total system to be synthesized from the start and so it is always better to consider the reference functions described in case (ii) in the optimisation scheme. In the next section, we provide some illustrations of the results obtained via the numerical implementation of the optimisation scheme described in this chapter.

2.4 Results

For a given reflection coefficient, $L_{11} \in \mathbb{B}$ of the load, the solution to Problem \mathcal{P}_2 , when one exists, provides the output reflection coefficient F_{22} of the optimal matching circuit for reference functions of the form k_α . As we have discussed, there exists a solution to Problem \mathcal{P}_2 if and only if the operator norm of H_Φ is equal to one (say it is achieved at $\hat{\alpha}$). We call the so obtained reflection coefficient \hat{F}_{22} of matching circuit an approximate solution to Problem \mathcal{P}_1 with respect to the family of rational references $\{k_\alpha\}$. A remarkable property of the latter is that its degree is comparable to that of the reference $k_{\hat{\alpha}}$ as described in proposition 2.3.3. For $\alpha = \hat{\alpha}$, the modulus of the total system's reflection parameter equals that of $k_{\hat{\alpha}}$. We will present in this section some examples of the matching results obtained by the implementation of problem \mathcal{P}_2 using reference functions build as in case (i) and case (ii) of subsection 2.3.1.

2.4.1 Example : Superdirective Antenna

In this example, we consider matching a superdirective antenna presented in [52] with scattering matrix of McMillan degree, $M = 2$. The targeted passband for matching is 870 MHz - 900 MHz. The reflection coefficient of the this antenna, L_{11} is depicted in figure 2.4. The shaded rectangle in the figure represent the targeted passband.

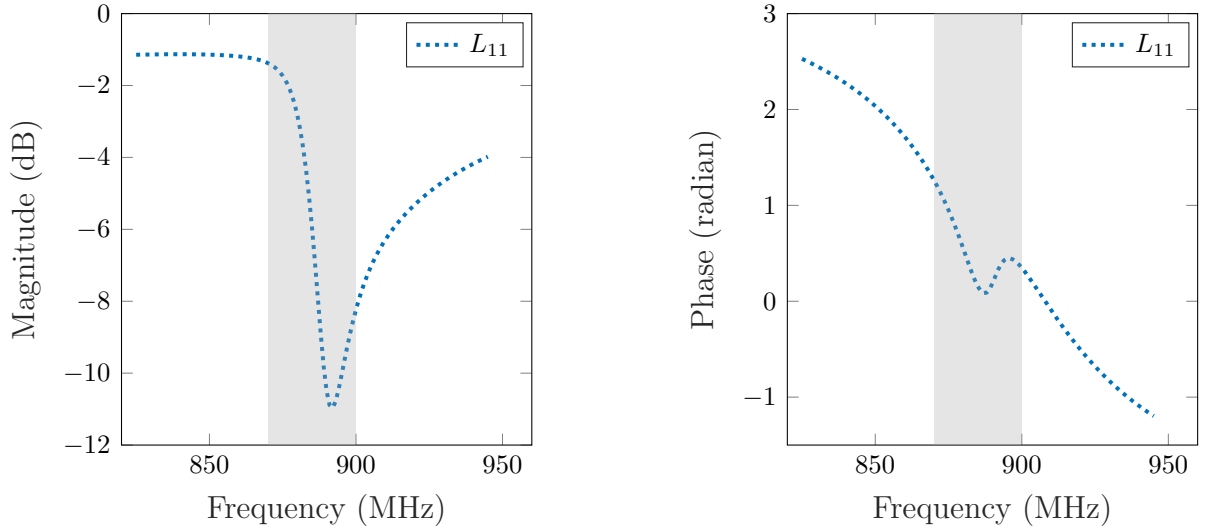


Figure 2.4: Scattering parameter of load with McMillan Degree 2.

For this antenna, we will be presenting the results obtained with a matching circuit of degree, $N_{MC} = 5$ in both cases of the reference functions as described in subsection 2.3.1.

Case (i) : $r_k r_k^* = \alpha$

In this case, we did the implementation of problem \mathcal{P}_2 with degree of the reference function

$$N = N_{MC} + 1 - M = 4$$

and the fixed positive constant $c = 2$ in equation 2.36. The iterations on α provided $\hat{\alpha} = 26.1$ at which $\|H_\Phi\| = 1$. The result of solving the Nehari problem mentioned in 2.47 is shown in figure 2.5. It shows the function $\Phi_{\hat{\alpha}}$ to be approximated in the problem and the optimal Hankel approximant $\Psi_{\hat{\alpha}}$ that is obtained when $\|H_\Phi\| = 1$. The error function in the Nehari approximation, $\Phi_{\hat{\alpha}} - \Psi_{\hat{\alpha}}$ is plotted in the normalized passband $[-1j, 1j]$ in figure 2.6. It can be noticed that the modulus of the error function is constant and equal to one. The scattering parameter of the matching circuit of McMillan degree 5 obtained by solving \mathcal{P}_2 is depicted in figure 2.7. The result of chaining this matching circuit to the load is shown in figure 2.8. It can be noticed in figure 2.8 that $|k_{\hat{\alpha}}|$ and $|S_{11}|$ are equal and so we have the solution to the gain equalization problem in the finite setting, problem \mathcal{P}_2 . The optimisation scheme produced a significant reduction in the maximal mismatch value of the load in the targeted passband from -1.37 dB (L_{11} at 870 MHz) to -7.94 dB.

Case (ii) : $r_k r_k^* = \alpha r_L r_L^*$

In this case, we did the implementation of problem \mathcal{P}_2 with degree of the reference function

$$N = N_{MC} + 1 = 6$$

and the fixed positive constant $c = 0.02$ in equation 2.41. The iterations on α provided $\hat{\alpha} = 0.00282$ at which $\|H_\Phi\| = 1$. The scattering parameter of the matching circuit of

McMillan degree 5 obtained by solving \mathcal{P}_2 is depicted in figure 2.9. The result of chaining this matching circuit to the load is shown in figure 2.10. In this case, the optimisation scheme produced a reduction in the maximal mismatch value of the load in the targeted passband from -1.37 dB (L_{11} at 870 MHz) to -8.42 dB.

It should be noted that fixed value of positive constant c in problem \mathcal{P}_2 can have an effect on the optimal criterion obtained. The best possible matching criterion (lowest value of maximal mismatch in the passband) with the described family of reference functions can be always obtained by solving the problem \mathcal{P}_2 on a grid of c 's and choosing the best c . The possible variation in the optimal matching criterion obtained for different values of fixed c in the example provided in case (i) is shown in figure 2.11. In the examples depicted, the c providing the best matching criterion for the given load using the described family of reference functions is chosen. As mentioned at the end of the last section, it can be observed that the reference function in which transmission zeros of the load are included in them (case (ii)) provides a better criterion compared to the family in which they are not included.

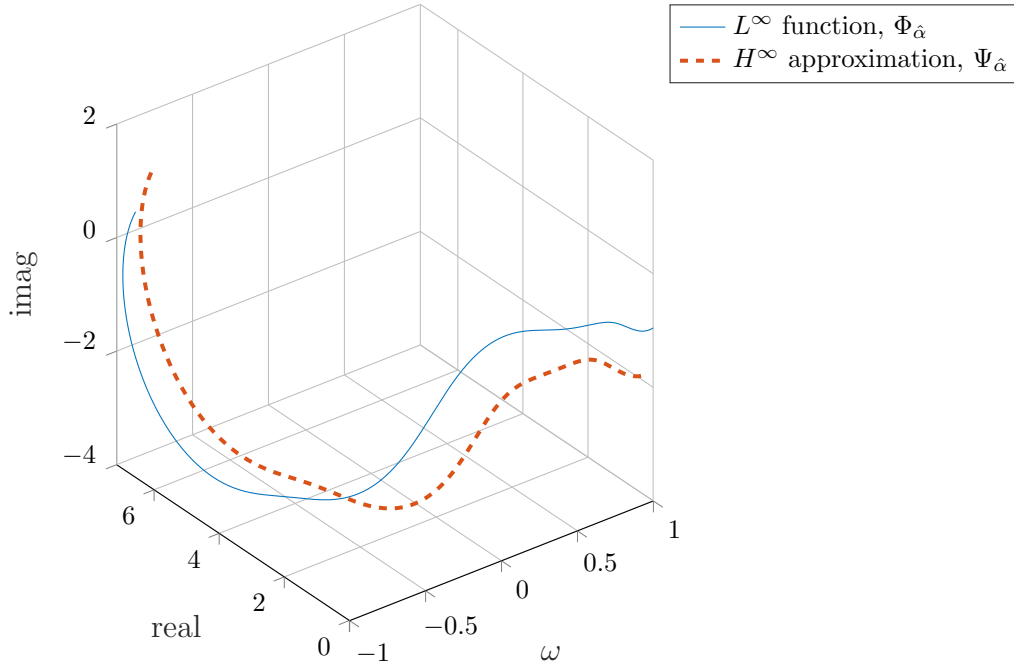


Figure 2.5: H^∞ approximation result.

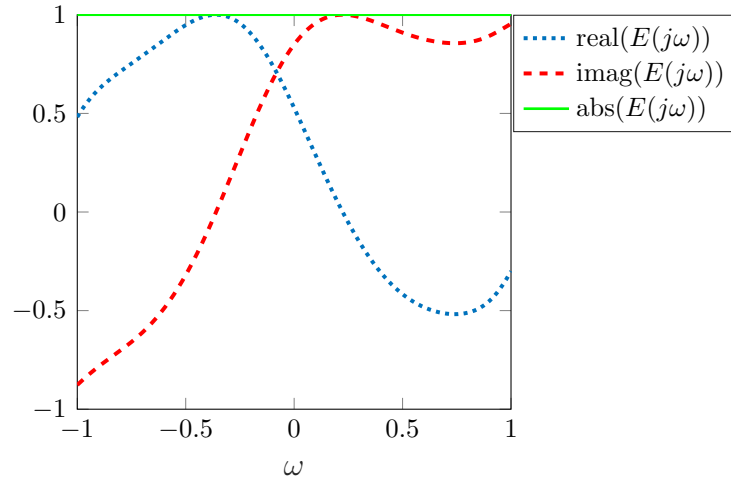


Figure 2.6: H^∞ approximation error, $E(j\omega) = |\Phi_{\hat{\alpha}}(j\omega) - \Psi_{\hat{\alpha}}(j\omega)|$.

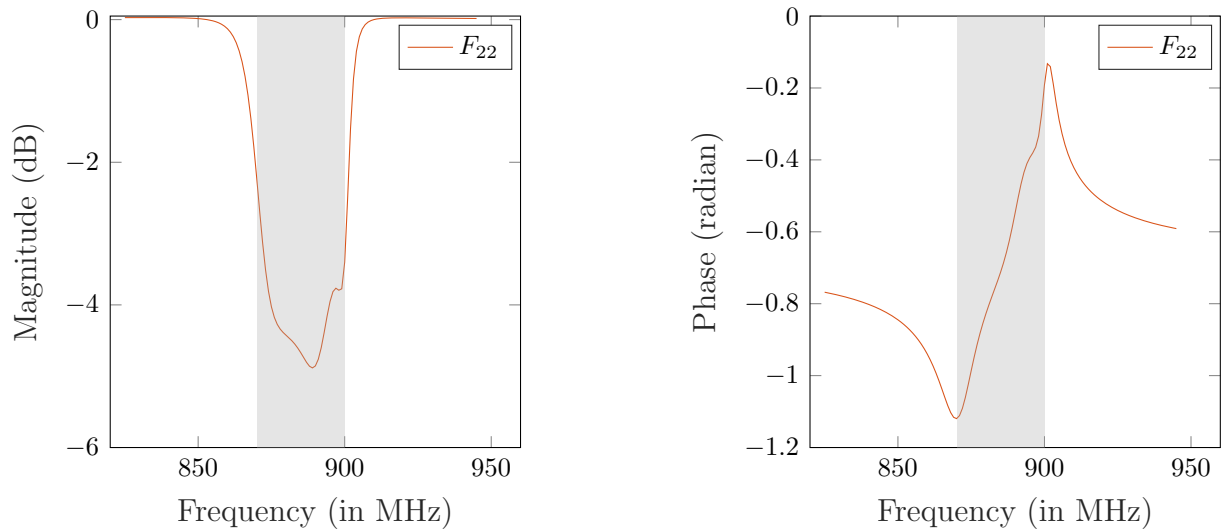


Figure 2.7: Scattering parameter of the optimal matching circuit in Case (i) ($N_{MC} = 5$).

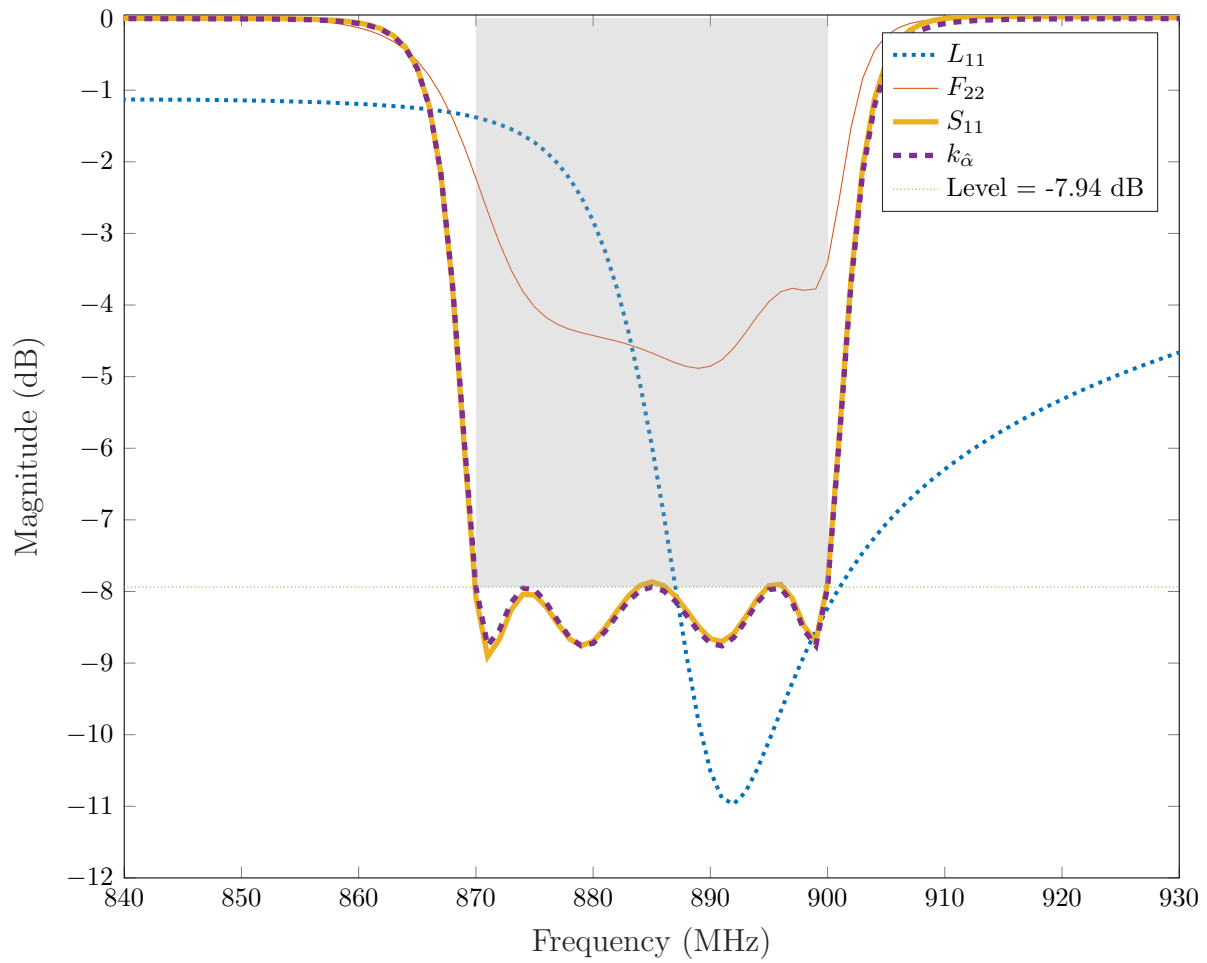


Figure 2.8: Optimal System reflection S_{11} obtained by solving \mathcal{P}_2 in Case (i) ($N = 4$).

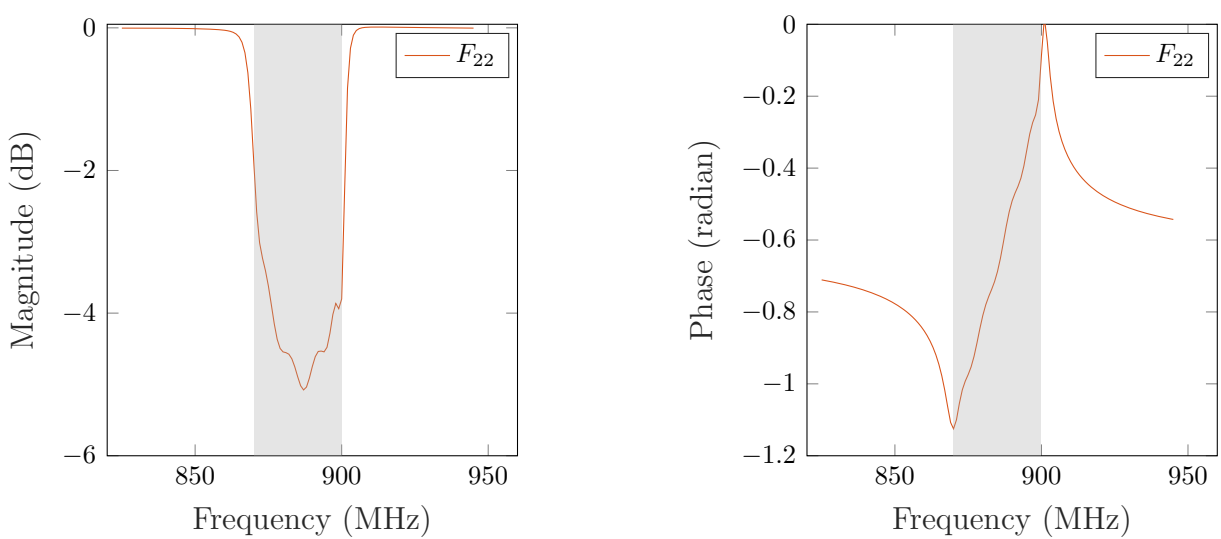


Figure 2.9: Scattering parameter of the optimal matching circuit in Case (ii) ($N_{MC} = 5$).

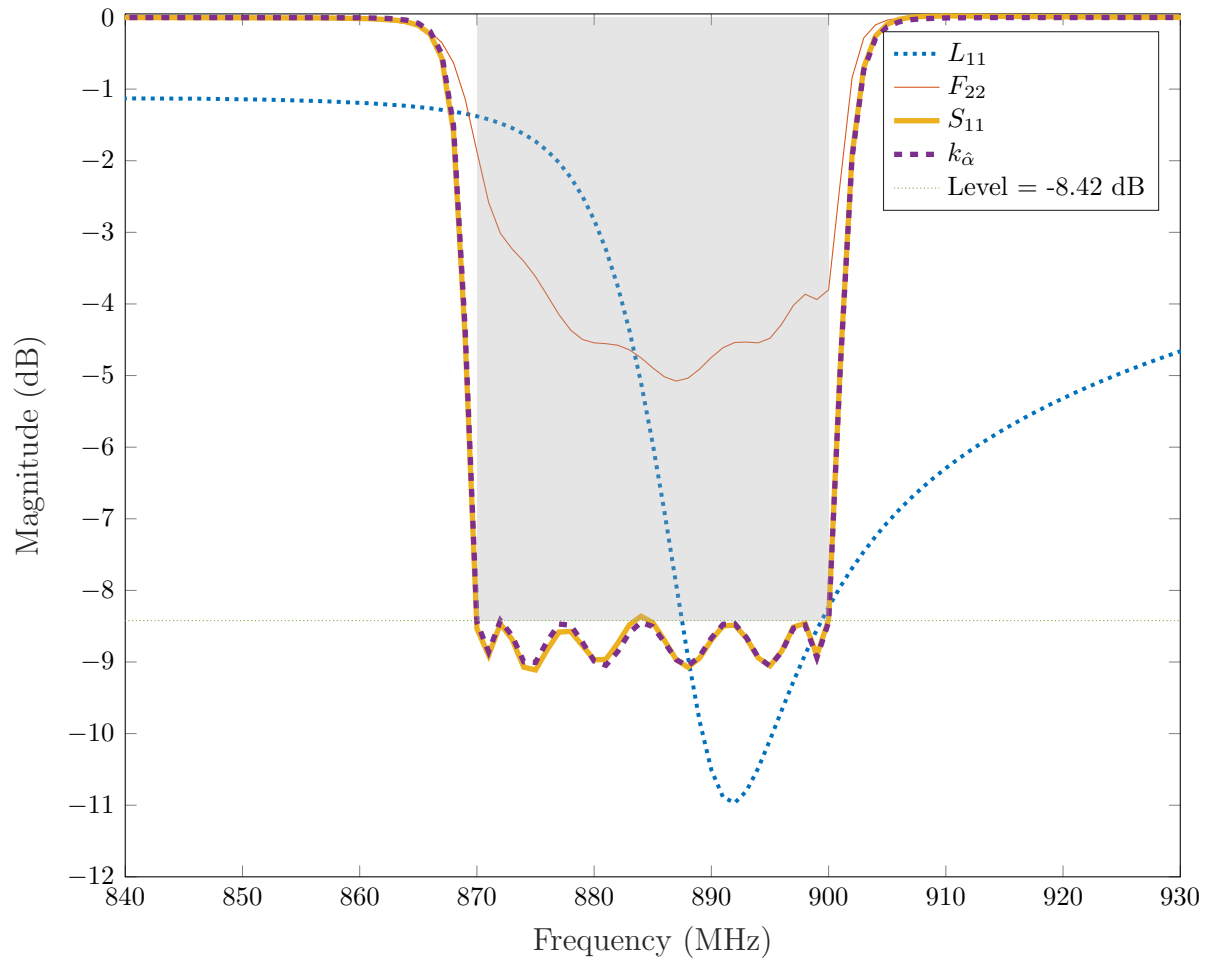


Figure 2.10: Optimal System reflection S_{11} obtained by solving \mathcal{P}_2 in Case (ii) ($N = 6$).

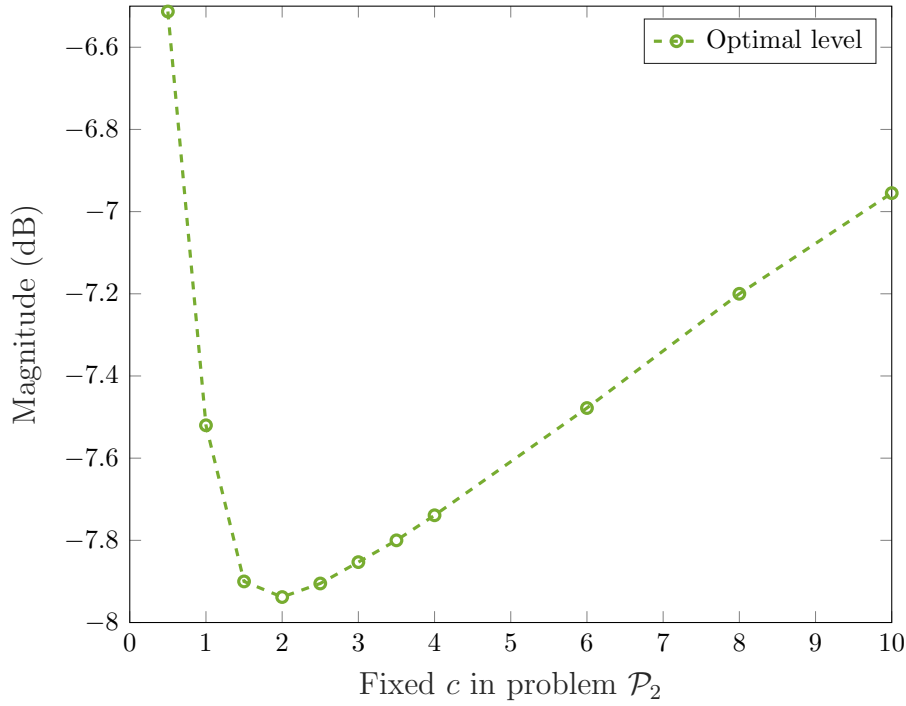


Figure 2.11: Variation of optimal matching criterion with value of fixed c in Case (i).

2.4.2 Analytic Examples

In this subsection, we will consider two analytic examples of antenna, meaning the polynomials representing the Belevitch representation of the scattering matrix of the antenna will be explicitly provided. In these cases, we will discuss the results obtained by the implementation of problem \mathcal{P}_2 for different degrees of matching circuits and also its comparison to the classical matching criterion bound obtained by Fano in [36].

(i). Degree 1 Antenna

In this example, we will consider matching an antenna with scattering matrix of McMillan degree, $M = 1$. The polynomials p_L, q_L and r_L and uni-modular constant ϵ in the Belevitch representation of the scattering matrix of the load in (2.34) were fixed to be

$$p_L(s) = s, \quad r_L(s) = -1, \quad q_L(s) = s + 1 \text{ and } \epsilon = -1.$$

So, we have a load of degree 1 with $L_{11}(s) = L_{22}(s) = \frac{s}{s+1}$ and a transmission zero at infinity. The targeted passband for matching is $[-1j, 1j]$. The reflection coefficient of the this antenna, L_{11} is depicted in figure 2.12. The shaded rectangle in the figure represent the targeted passband and $\text{Im}(\cdot)$ is used to denote the imaginary part.

Before presenting the results obtained by solving problem \mathcal{P}_2 , we estimate the Fano bound for this antenna. For this purpose, we initially provide the general expression of angular derivative, $\text{ang}[L_{22}](\infty)$ for a given reflection coefficient,

$$L_{22}(s) = \frac{p_N s^N + p_{N-1} s^{N-1} + \dots + p_1 s + p_0}{q_N s^N + q_{N-1} s^{N-1} + \dots + q_1 s + q_0}, \quad (2.95)$$

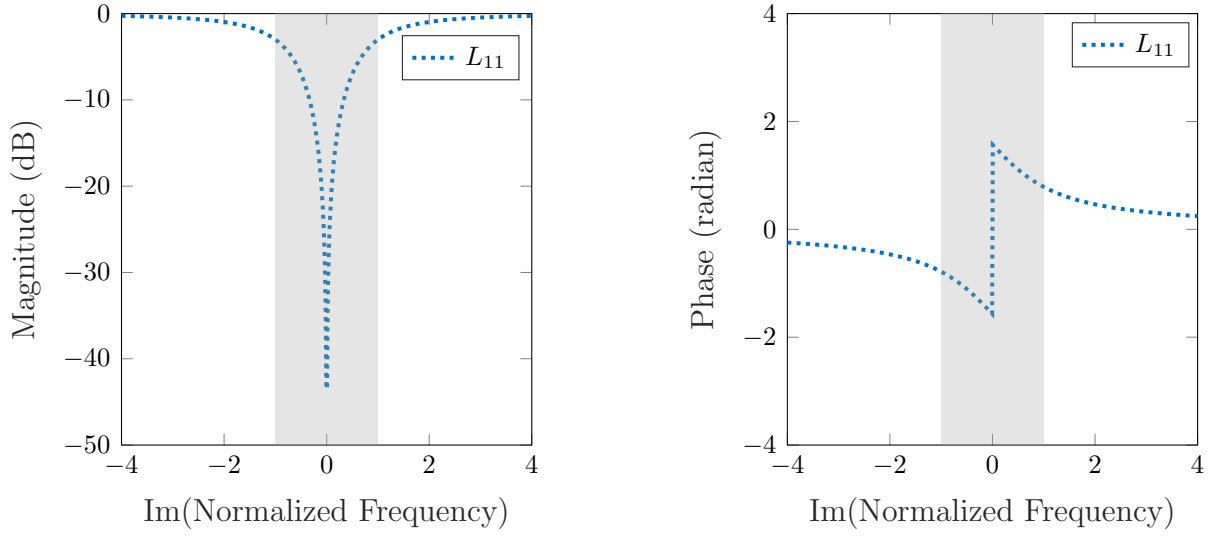


Figure 2.12: Scattering parameter of load with McMillan Degree 1.

satisfying $|L_{22}(\infty)| = 1$. We have the angular derivative of L_{22} at infinity as follows (detailed in the derivation of (4.71)):

$$\text{ang}[L_{22}](\infty) = \frac{1}{j} \left(\frac{d}{d\omega} \ln \left(L_{22} \left(\frac{1}{j\omega} \right) \right) \right) \Big|_{\omega=0} = \frac{1}{j} \frac{\frac{d}{d\omega} (L_{22}(\frac{1}{j\omega}))}{L_{22}(\frac{1}{j\omega})} \Big|_{\omega=0} = \frac{p_{N-1}}{p_N} - \frac{q_{N-1}}{q_N}. \quad (2.96)$$

A detailed summary of angular derivatives is provided later in subsection 4.1.1 in chapter 4. Now, we will provide a brief outline of the optimal matching criterion bound that was provided by Fano [36] for an antenna consisting of at least a transmission zero at infinity. This will serve the purpose of comparison of results obtained on the analytic antenna examples with the Fano bound. Denoting the output reflection coefficient of the overall system (matching circuit together with load) by S_{22} , we have the angular derivative of S_{22} greater than or equal to the angular derivative of L_{22} at infinity (proposition B.1.1), that is

$$\text{ang}[S_{22}](\infty) \geq \text{ang}[L_{22}](\infty).$$

By denoting the overall reflection coefficient $S_{22} \in \mathbb{B}$ by $S_{22} = S_{22}^O b$, where S_{22}^O is outer and b is Blaschke product (theorem 1.2.7), it follows from the fact that the angular derivative of the product of two functions is the sum of angular derivative of these two functions (4.3),

$$\text{ang}[S_{22}^O](\infty) + \text{ang}[b](\infty) \geq \text{ang}[L_{22}](\infty).$$

That is

$$\text{ang}[S_{22}^O](\infty) \geq \text{ang}[L_{22}](\infty) - \text{ang}[b](\infty) \geq \text{ang}[L_{22}](\infty) \quad (2.97)$$

since the angular derivative of b is non-positive (proposition 4.1.3). It follows from Riesz-Herglotz representation [22] that the angular derivative of an outer function has an integral

expression as described below. For $z \in \Pi^+$, from Riesz-Herglotz representation, the evaluation of logarithm of outer function S_{22}^O is given in terms of its real part on the boundary (equation 5, [9] and page 133, [49]),

$$\ln(S_{22}^O(z)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln(|S_{22}^O(j\omega)|) \left(\frac{1}{z - j\omega} - \frac{j\omega}{1 + \omega^2} \right) d\omega.$$

The representation is unique upto a purely imaginary constant. By a change of variable from $z \rightarrow \frac{1}{z}$,

$$\ln \left(S_{22}^O\left(\frac{1}{z}\right) \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln(|S_{22}^O(j\omega)|) \left(\frac{z}{1 - j\omega z} - \frac{j\omega}{1 + \omega^2} \right) d\omega$$

and then by differentiating with respect to z , we have

$$\frac{d}{dz} \left(\ln \left(S_{22}^O\left(\frac{1}{z}\right) \right) \right) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\ln(|S_{22}^O(j\omega)|)}{(1 - j\omega z)^2} d\omega.$$

Taking the limit $z \rightarrow 0$, we obtain

$$\text{ang}[S_{22}^O](\infty) = \frac{1}{\pi} \int_{-\infty}^{\infty} \ln |S_{22}^O(j\omega)| d\omega.$$

This together with (2.97) provides

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \ln |S_{22}^O(j\omega)| d\omega \geq \text{ang}[L_{22}](\infty). \quad (2.98)$$

For the optimal S_{22} which minimize $|S_{22}(j\omega)|$ in an interval $\mathbb{I} = [-1j, 1j]$, Fano considered S_{22}^{opt} whose modulus is constant (denoted by FB) within the passband \mathbb{I} and zero outside,

$$|S_{22}^{opt}(j\omega)| = \begin{cases} \text{FB} & j\omega \in \mathbb{I} \\ 0 & j\omega \notin \mathbb{I}. \end{cases} \quad (2.99)$$

Thus it follows from (2.98) that

$$\frac{2 \ln(\text{FB})}{\pi} \geq \text{ang}[L_{22}](\infty)$$

and it provides the following expression of Fano Bound in dB

$$\text{FB}(\text{in dB}) = 20 \log(\text{FB}) = \frac{10\pi \text{ang}[L_{22}](\infty)}{\ln 10}. \quad (2.100)$$

2.4. RESULTS

In this example, from (2.96), we have the angular derivative of $L_{22}(s) = \frac{\epsilon p_L^*(s)}{q_L(s)} = \frac{s}{s+1}$ of the given antenna evaluated at the transmission zero, infinity,

$$\text{ang}[L_{22}](\infty) = 0 - 1 = -1$$

and so the Fano bound (FB) can be evaluated from the expression,

$$\text{FB(in dB)} = \frac{10\pi \text{ang}[L_{22}](\infty)}{\ln 10} = -13.64 \text{ dB}. \quad (2.101)$$

Now, we present the result obtained by solving problem \mathcal{P}_2 . We did the implementation with degree of reference function, $N = 5$. The optimisation scheme provided the optimal $c = 0.09$ and $\hat{\alpha} = 0.166$ at which $||H_\Phi|| = 1$. The result obtained is depicted in figure 2.13 together with the Fano bound. The green shaded rectangle inside the passband is used to denote the difference in the optimal criterion obtained and the Fano bound. The reference

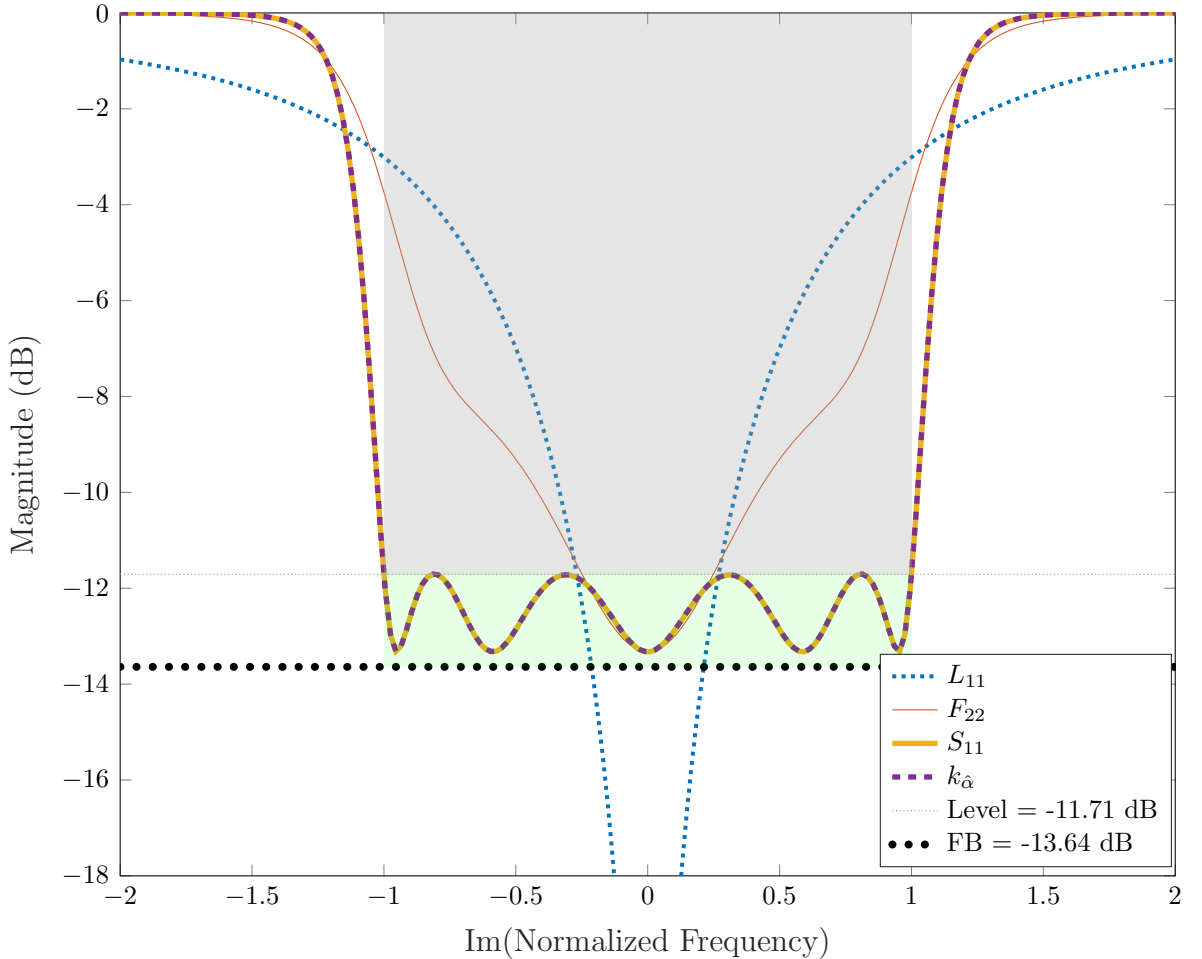


Figure 2.13: Optimal System reflection S_{11} obtained by solving \mathcal{P}_2 in Case (ii) ($N = 5$).

function which provided $\|H_\Phi\| = 1$ was

$$k_{\hat{\alpha}}(s) = \frac{s^5 + 0.76s^4 + 1.5s^3 + 0.75s^2 + 0.48s + 0.09}{s^5 + 1.8s^4 + 2.8s^3 + 2.4s^2 + 1.5s + 0.42}.$$

The optimal reflection coefficient of the overall system was obtained to be

$$S_{11}(s) = \frac{-s^5 + 0.76s^4 - 1.5s^3 + 0.75s^2 - 0.48s + 0.09}{s^5 + 1.8s^4 + 2.8s^3 + 2.4s^2 + 1.5s + 0.42}$$

and the reflection coefficient of the optimal matching filter of degree 4,

$$F_{22}(s) = \frac{-s^4 - 0.28s^3 - 0.44s^2 + 0.31s + 0.18}{s^4 + 2.2s^3 + 2.9s^2 + 2.3s + 0.83}.$$

It can be noted that the optimal level attained using degree 4 matching filter, -11.71 dB is only 1.91 dB higher than the Fano bound for this antenna, -13.64 dB. Since the antenna under consideration in this example is of degree 1 which has a transmission zero at infinity, the Fano bound in this case coincides with the Helton's bound for infinite degree matching network (solution to problem \mathcal{P}_1). So the difference of 1.91 dB we have from the Fano bound essentially comes from the finite degree constraint of the matching circuit and the fixed equi-oscillating shape of reference function in problem \mathcal{P}_2 . In figure 2.14, we present the optimal level obtained by solving problem \mathcal{P}_2 using different degrees of reference functions (N) in a tabular form. It can be noticed that as the degree of the matching circuit (M.C) increases, we are getting closer to the Fano bound, -13.64 dB.

Degree of ref fn (N)	Degree of M.C ($N-1$)	Optimal level
2	1	-7.75 dB
3	2	-9.90 dB
4	3	-11.15 dB
5	4	-11.71 dB
6	5	-12.13 dB
7	6	-12.27 dB
8	7	-12.40 dB
9	8	-12.52 dB
10	9	-12.75 dB

Figure 2.14: Optimal level obtained by solving \mathcal{P}_2 for different N .

(ii). Degree 3 Antenna

In this example, we will consider matching an antenna with scattering matrix of McMillan degree, $M = 3$. The polynomials p_L, q_L and r_L and uni-modular constant ϵ in the Belevitch representation of the scattering matrix of the load in (2.34) were fixed to be

$$\begin{aligned} p_L(s) &= -s^3 + (0.1 + 0.2j)s^2 + (0.09 + 0.35j)s + (0.046 + 0.2j), \\ r_L(s) &= s^2 + (0.8 + 0.6j)s + (0.42 + 0.12j), \\ q_L(s) &= s^3 + (2.022 - 0.2j)s^2 + (1.449 + 0.25j)s + (0.483 - 0.023j) \text{ and} \\ \epsilon &= 1. \end{aligned} \tag{2.102}$$

So, we have a load of degree 3 with transmission zeros at $(0.5 - 0.9j)$, $(0.3 + 0.3j)$, ∞ . The targeted passband for matching is $[-1j, 1j]$. The reflection coefficient of the this antenna, L_{11} is depicted in figure 2.15. The shaded rectangle in the figure represent the targeted passband and $\text{Im}(\cdot)$ is used to denote the imaginary part.

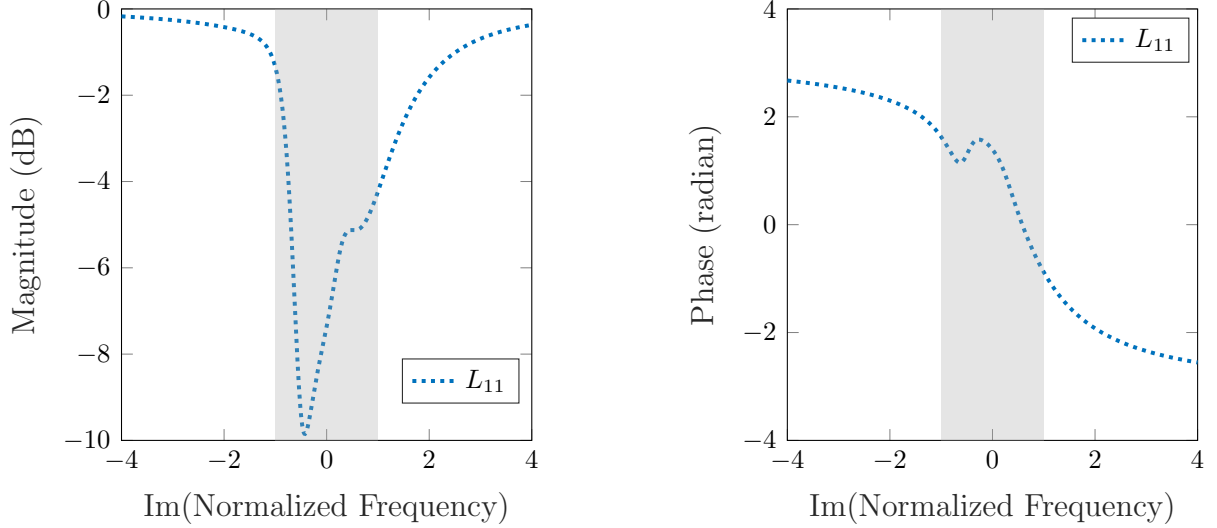


Figure 2.15: Scattering parameter of load with McMillan Degree 3.

Similar to the previous example, initially, we estimate the Fano bound for this antenna. We have

$$L_{22}(s) = \frac{\epsilon p_L^*(s)}{q_L(s)} = \frac{s^3 + (0.1 - 0.2j)s^2 - (0.09 - 0.35j)s + (0.046 - 0.2j)}{s^3 + (2.022 - 0.2j)s^2 + (1.449 + 0.25j)s + (0.483 - 0.023j)}$$

and from (2.96), we have the angular derivative of L_{22} evaluated at infinity of this antenna,

$$\text{ang}[L_{22}](\infty) = (0.1 - 0.2j) - (2.022 - 0.2j) = -1.922.$$

This provides the Fano bound for this antenna,

$$\text{FB}(\text{in dB}) = \frac{10\pi \text{ang}[L_{22}](\infty)}{\ln 10} = -26.22 \text{ dB}. \quad (2.103)$$

Now, we present the result obtained by solving problem \mathcal{P}_2 . We did the implementation with degree of reference function, $N = 5$. The optimisation scheme provided the optimal $c = 0.19$ and $\hat{\alpha} = 0.285$ at which $\|H_\Phi\| = 1$. The result obtained is depicted in figure 2.16 together with the Fano bound. The green shaded rectangle inside the passband is used to denote the difference in the optimal criterion obtained and the Fano bound. The reference function which provided $\|H_\Phi\| = 1$ was

$$k_{\hat{\alpha}}(s) = \frac{s^5 + (0.97 + 0.16j)s^4 + (1.6 + 0.17j)s^3 + (0.89 + 0.21j)s^2 + (0.45 + 0.088j)s + (0.083 + 0.0048j)}{s^5 + (1.7 + 0.16j)s^4 + (2.6 + 0.31j)s^3 + (2.1 + 0.43j)s^2 + (1 + 0.27j)s + (0.25 + 0.017j)}.$$

The optimal reflection coefficient of the overall system was obtained to be

$$S_{11}(s) = \frac{(1.8 + 0.64j)s^7 - (0.92 - 0.34j)s^6 + (2.1 + 0.54j)s^5 - (0.31 - 0.45j)s^4 + (0.15 + 0.22j)s^3 + (0.14 - 0.084j)s^2 + (0.0033 + 0.082j)s - (0.011 + 0.026j)}{(1.8 - 0.64j)s^7 + (4.3 - 0.89j)s^6 + (7.2 - 0.89j)s^5 + (7.5 - 0.044j)s^4 + (5.2 + 0.73j)s^3 + (2.2 + 0.61j)s^2 + (0.54 + 0.27j)s + (0.071 + 0.048j)}$$

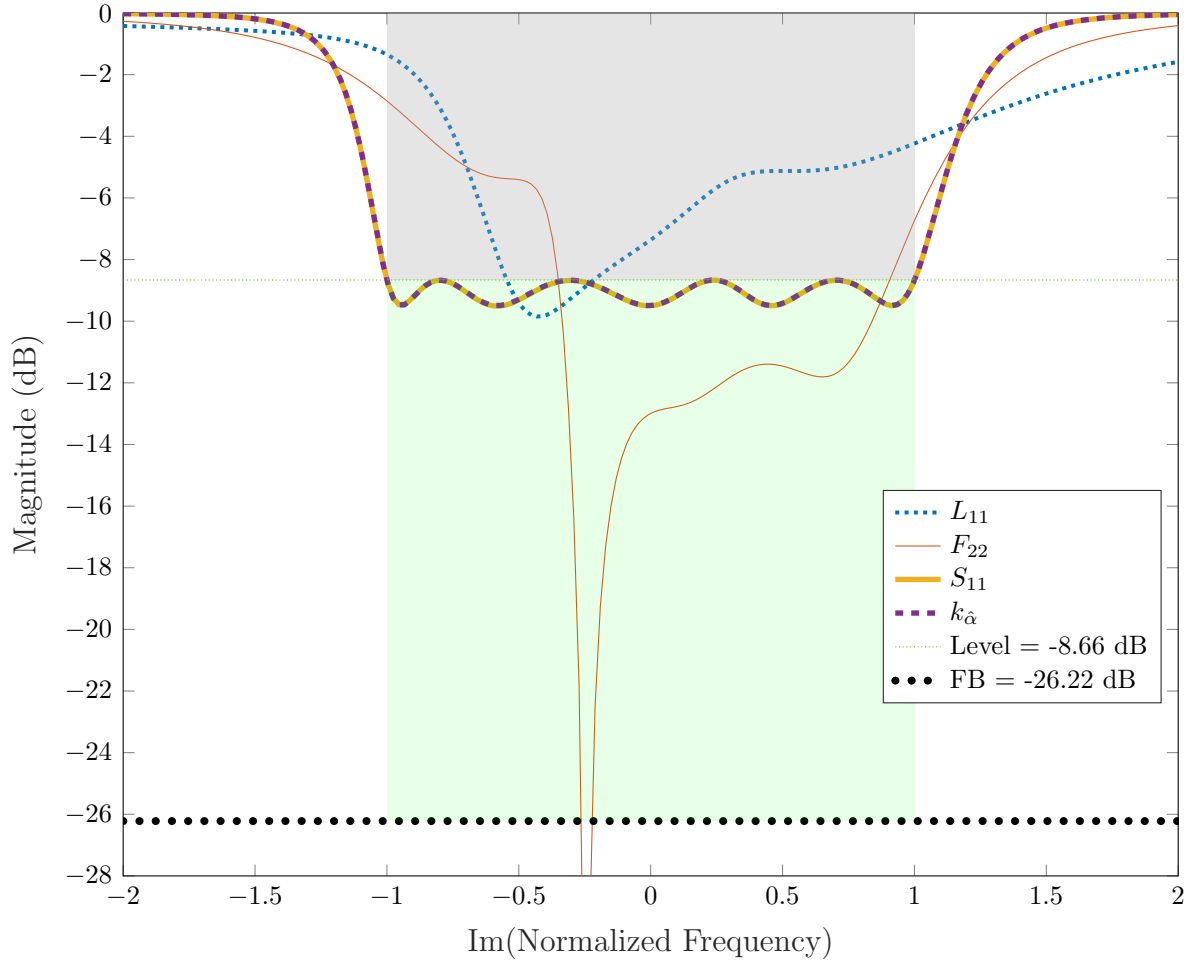


Figure 2.16: Optimal System reflection S_{11} obtained by solving \mathcal{P}_2 in Case (ii) ($N = 5$).

and the reflection coefficient of the optimal matching filter of degree 4,

$$F_{22}(s) = \frac{(0.77 - 0.64j)s^4 + (0.7 - 0.59j)s^3 + (0.57 - 0.54j)s^2 + (0.18 - 0.08j)s + (0.042 + 0.0045j)}{s^4 + (1.8 - 0.0088j)s^3 + (2 + 0.073j)s^2 + (0.81 + 0.24j)s + (0.14 + 0.12j)}.$$

In contrast to the previous analytic example in which the optimal level attained was relatively close to the Fano bound, it can be observed in this example that the optimal level attained using degree 4 matching circuit is 17.56 dB higher than the Fano bound. The constraint of finite degree for matching circuit and fixed equi-oscillating shape of the overall response is still present in this example but the dominant factor contributing to the big difference in the optimal level from the Fano bound is essentially arising due to the presence of two transmission zeros of the antenna which are at finite complex frequencies, $(0.5 - 0.9j)$ and $(0.3 + 0.3j)$. It can be noted that in the estimation of Fano bound in (2.100), the restriction on the bound of $|S_{22}|$ in the passband is dependent of only the transmission zero at infinity of the load, not the ones at finite frequencies. This leads to the fact that Fano bound in (2.100) is not very tight in the kind of antenna depicted in this example which

have transmission zeros at finite frequencies. In the first analytic example, the antenna had only one transmission zero and it was located at infinity. In figure 2.17, we present the optimal level obtained by solving problem \mathcal{P}_2 using different degrees of reference functions (N) in a tabular form. It can be observed that even with high degrees of matching circuit, we are still far from the Fano bound of -26.22 dB. So, essentially by solving problem \mathcal{P}_2 , we are able to provide a matching circuit synthesis scheme for finite degree circuits irrespective of the location of transmission zeros of the load and in general, having optimal levels for matching criterion which are more realistic than the Fano bound.

Degree of ref fn (N)	Degree of M.C ($N - 1$)	Optimal level
4	3	-7.57 dB
5	4	-8.66 dB
6	5	-9.42 dB
7	6	-10.13 dB
8	7	-10.36 dB
9	8	-10.49 dB
10	9	-10.66 dB

Figure 2.17: Optimal level obtained by solving \mathcal{P}_2 for different N for degree 3 antenna.

Flip Zeroes of p_L in Π^- to Π^+ in degree 3 example

The limitation of Fano bound in estimating realistic matching criterion bounds becomes more clear by making a flip of the zeros of $p_L(s)$ which are present in the left half plane to right half plane in the degree 3 antenna example we have just discussed. The zeros of the polynomial $p_L(s)$ in (2.102) are $(0.7 + 0.5j)$, $(-0.2 - 0.5j)$, $(-0.4 + 0.2j)$. We flip the two zeros, $(-0.2 - 0.5j)$ and $(-0.4 + 0.2j)$ to $(0.2 - 0.5j)$ and $(0.4 + 0.2j)$ respectively, and hence have the new polynomial p_L ,

$$p_L(s) = -s^3 + (1.3 + 0.2j)s^2 - (0.75 - 0.07j)s + (0.21 - 0.022j),$$

replacing p_L in (2.102). It should be noted that we have polynomials r_L , q_L and ϵ same as in (2.102) and the modulus of L_{11} remains same as in (2.102), only the phase changes. The reflection coefficient of the this antenna, L_{11} is depicted in figure 2.18. We have

$$L_{22}(s) = \frac{s^3 + (1.3 - 0.2j)s^2 + (0.75 + 0.07j)s + (0.21 + 0.022j)}{s^3 + (2.022 - 0.2j)s^2 + (1.449 + 0.25j)s + (0.483 - 0.023j)}$$

and from (2.96), we have the angular derivative of L_{22} evaluated at infinity of this antenna,

$$\text{ang}[L_{22}](\infty) = (1.3 - 0.2j) - (2.022 - 0.2j) = -0.722.$$

It should be noted that zeros of p_L^* , $(-0.2 - 0.5j)$, $(-0.7 + 0.5j)$, $(-0.4 + 0.2j)$ are now in Π^- making $L_{22} = \frac{\epsilon p_L^*}{q_L}$ outer. The angular derivative of L_{22} is now different from the case when L_{22} was not outer and we have the new Fano bound,

$$\text{FB} = \frac{10\pi \text{ang}[L_{22}](\infty)}{\ln 10} = -9.85 \text{ dB}.$$

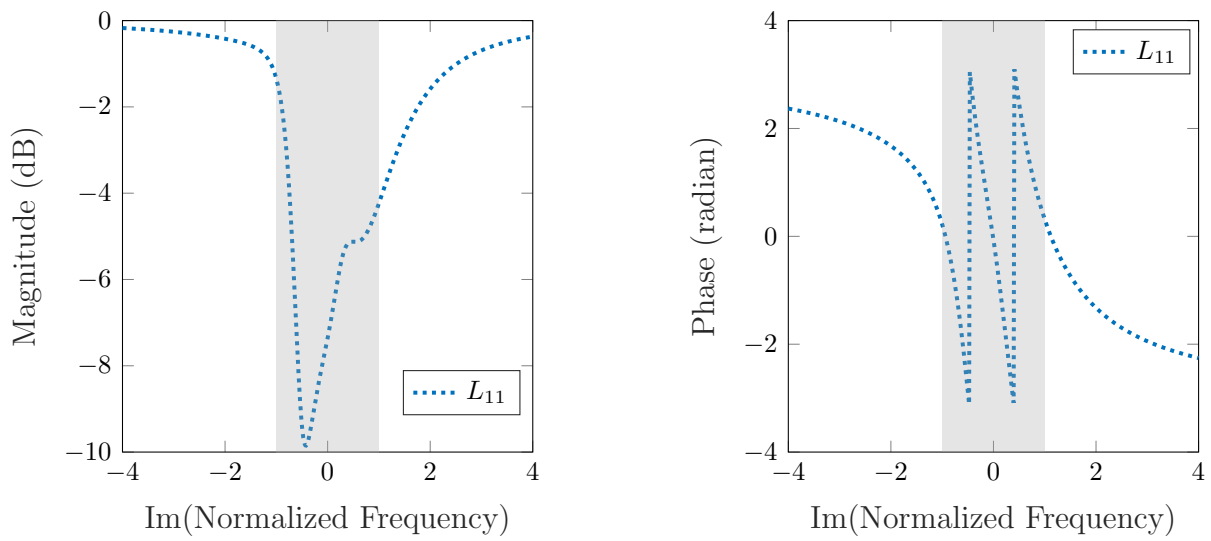
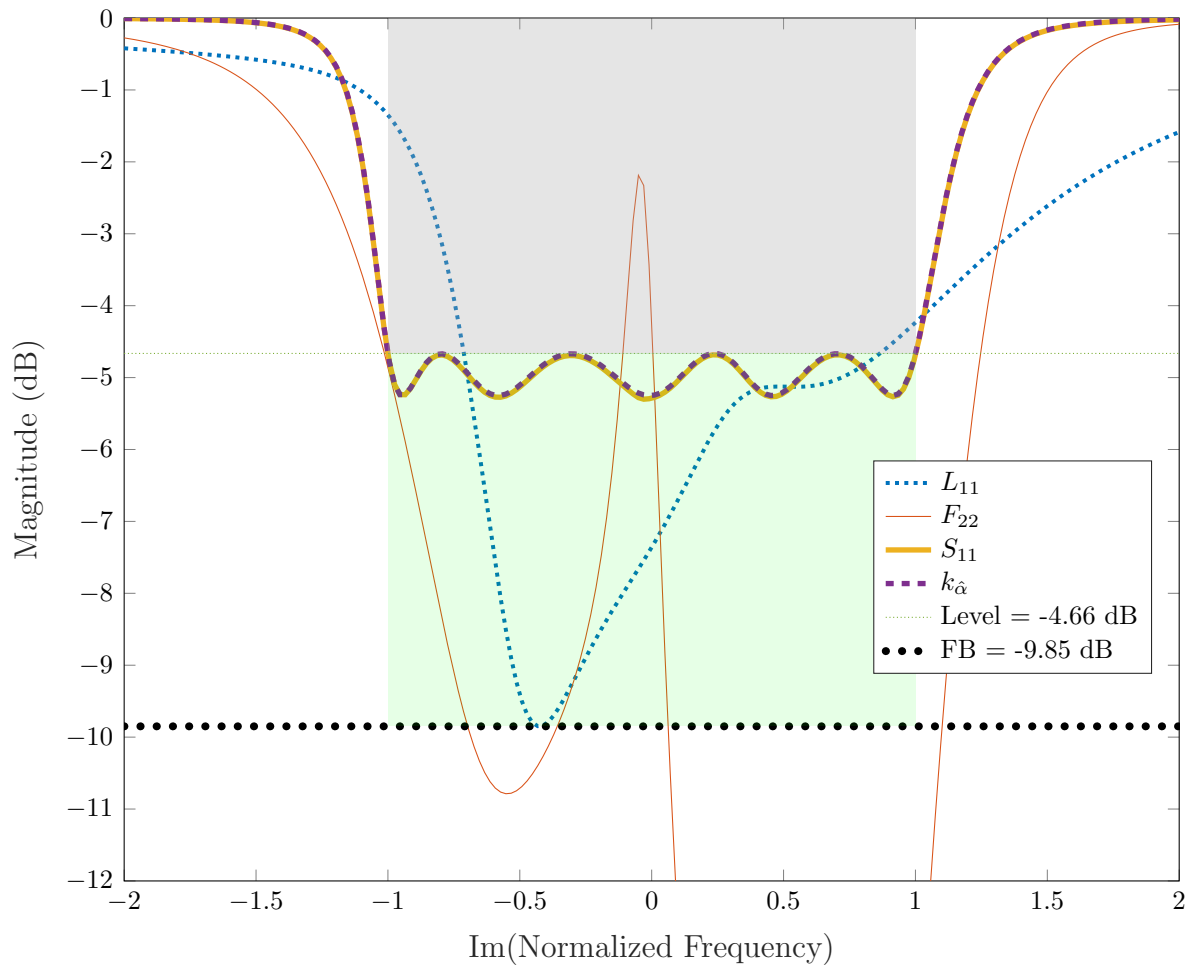


Figure 2.18: Scattering parameter of load with McMillan Degree 3.


 Figure 2.19: Optimal System reflection S_{11} obtained by solving \mathcal{P}_2 in Case (ii) ($N = 5$).

The flipping of zeros inside Π^- of p_L has increased the Fano bound significantly from -26.22 dB to -9.85 dB. The reader can notice that after flipping the zeros of p_L , L_{22} is outer and it is only when L_{22} is outer itself than one can “compare” the surface above the curve of the $\ln(|L_{22}|)$ of the load with the surface above the $\ln(|S_{22}|)$ following from (2.98). When L_{22} has zeros in Π^+ its angular derivatives decreases (because of the angular derivative of the Blaschke) which can be observed in degree 3 example by comparing the cases : before flipping zeros ($\text{ang}[L_{22}](\infty) = -1.922$) and after flipping zeros of p_L ($\text{ang}[L_{22}](\infty) = -0.722$). This explains why different Fano bounds are obtained in each case and the reason why Fano bound is comparatively more optimistic in the case when L_{22} has zeros inside Π^+ . In terms of the reflection coefficient of antenna measured in practical scenarios, which is the input reflection coefficient, L_{11} , it means Fano bound relative to minimum phase (outer) L_{11} are “more optimistic” than other cases. One can say that minimum phase load L_{11} ’s are usually easier to match than others. After flipping the zeros in our example, L_{11} has all zeros inside the analyticity domain Π^+ and it is more difficult to match. The Fano bound becomes more restrictive and corresponds in this case to the surface above the curve of $\ln(|L_{22}|)$.

Now, we present the result obtained by solving problem \mathcal{P}_2 in this case. We did the implementation with degree of reference function, $N = 5$. The optimisation scheme provided the optimal $c = 0.2$ and $\hat{\alpha} = 0.0945$ at which $\|H_\Phi\| = 1$. The result obtained is depicted in figure 2.19 together with the Fano bound. The green shaded rectangle inside the passband is used to denote the difference in the optimal criterion obtained and the Fano bound. The reference function which provided $\|H_\Phi\| = 1$ was

$$k_{\hat{\alpha}}(s) = \frac{s^5 + (1 + 0.16j)s^4 + (1.6 + 0.17j)s^3 + (0.93 + 0.22j)s^2 + (0.46 + 0.093j)s + (0.087 + 0.005j)}{s^5 + (1.4 + 0.16j)s^4 + (2.1 + 0.25j)s^3 + (1.5 + 0.31j)s^2 + (0.73 + 0.17j)s + (0.16 + 0.0098j)}.$$

The optimal reflection coefficient of the overall system was obtained to be

$$S_{11}(s) = \frac{-(1.2 - 0.98j)s^7 + (3.7 + 1.6j)s^6 - (8.1 - 0.15j)s^5 + (8.8 + 1.7j)s^4 - (8.7 - 0.26j)s^3 + (4.4 - 0.4j)s^2 - (1.7 - 0.39j)s + (0.28 - 0.076j)}{(1.2 + 0.98j)s^7 + (4.4 - 0.93j)s^6 + (10 + 0.22j)s^5 + (13 - 0.96j)s^4 + (12 + 0.81j)s^3 + (7.3 + 1.2j)s^2 + (2.7 + 0.85j)s + (0.5 + 0.18j)}$$

and the reflection coefficient of the optimal matching filter of degree 4,

$$F_{22}(s) = \frac{(0.18 + 0.98j)s^4 + (0.73 + 0.29j)s^3 + (0.6 + 0.33j)s^2 + (0.3 - 0.13j)s + (0.036 - 0.05j)}{s^4 + (2 - 0.67j)s^3 + (2.3 - 1.1j)s^2 + (1.3 - 0.93j)s + (0.099 - 0.015j)}.$$

In figure 2.20, we present the optimal level obtained by solving problem \mathcal{P}_2 using different degrees of reference functions (N) in a tabular form.

Degree of ref fn (N)	Degree of M.C ($N - 1$)	Optimal level
4	3	-4.14 dB
5	4	-4.66 dB
6	5	-4.80 dB
7	6	-4.92 dB
8	7	-5.12 dB
9	8	-5.21 dB
10	9	-5.29 dB

Figure 2.20: Optimal level obtained by solving \mathcal{P}_2 for different N for degree 3 antenna.

We will conclude this section here. In the final section, we will provide a short summary of the chapter along with the main objectives of next chapter.

2.5 Conclusion

In this chapter, we have showed that Helton's H^∞ approach to matching theory, supposed to yield a guaranteed optimal response at the cost of an infinite degree matching network can be successfully adapted to yield finite degree matching networks by using families of rational reference functions as described in the chapter. The McMillan degree of the scattering matrix of the optimal matching circuit obtained using this approach is comparable to the degree of the reference function as explained in proposition 2.3.3. This is an important property with respect to the practical applications and the procedure can therefore be put into practice to derive matching networks for mismatched antennas. In practice the constraints that need to be imposed on the scattering matrix of the matching network can be more complicated than just the McMillan degree but the degree constraint is the first requirement. As mentioned in the structure of the manuscript, we will be addressing more constraints regarding the practical realization of the matching circuit in the second part of the thesis.

Another important point to note regarding the finite degree degree matching problem described in this chapter in comparison to Helton's original problem is the variation of gain with respect to the parameter α . In case (i) and case (ii) of reference functions k_α , we have the following gains $G_1(j\omega)$ and $G_2(j\omega)$ respectively,

$$G_1(j\omega) = \frac{\alpha}{|p_k(j\omega)|^2 + \alpha}, \quad G_2(j\omega) = \frac{\alpha|r_L(j\omega)|^2}{|p_k(j\omega)|^2 + \alpha|r_L(j\omega)|^2}$$

which do not vary linearly with the parameter α contrary to the original Helton approach where in for a fixed gain G , αG is used to perform the test on a grid of α 's. In this approach we have limited the possible set of responses of the matching circuit chained to the load by fixing the polynomial p_k of the reference function with the help of Chebyshev approximation. This fixes the shape of the system response and allows the parameterisation of its modulus using α .

The optimal matching criterion possible with a finite degree matching circuit for a given load can be limited when we have parametrised the modulus of the system response using the one dimensional parameter α as discussed in the chapter. The natural question to address is allowing further flexibility in the choice of reference functions. It is with this purpose in mind we move to the next chapter. In this chapter we will generalize the set of responses of the system to a set of "realizable" rational Schur functions of fixed maximal degree and having a fixed transmission polynomial. The realizability constraint will be included in the optimisation scheme using Fano and Youla's matching theory rather than the one proposed by Helton. This will allow the formulation of the matching problem as a convex optimisation approach based on the theory of analytic interpolation. A detailed description of the theory of the developed approach along with some practical examples of matching circuit synthesis will be presented.

At the end of the next chapter, we will also provide a comparison of the approach presented in this chapter to solve the finite degree matching problem and the one that will

be discussed in the next chapter. Furthermore, the connection between these approaches will be presented as well. An additional point to be noted in this chapter : it follows from the formulas in (2.24) that when we move to the Euclidean setting, the gain appears non-linearly in the test provided. So, in [47], Helton propose a similar kind of second test with the help of a linear fractional coordinate change applied to the pseudo-hyperbolic disk in (2.22). In this test, he gets the gain to appear linearly. Even though we have not described it in this chapter, we will describe a direct way of arriving at a similar test starting from the chaining equation at the end of the next chapter and discuss the details.

Matching Problem : A Convex Optimisation Approach

3.1 Introduction

In this chapter, the impedance matching problem in communication systems is formulated as a convex optimisation problem using two rich theories, namely Youla's matching theory and analytic interpolation. In comparison to the previous chapter, this chapter will provide further flexibility to the set of responses of the system formed by chaining the matching circuit and the load. Some close comparisons and the connection of the approaches developed in both chapters will be done at the end of this chapter. In this chapter, the set of possible responses of the global system will be defined to be a set of "realizable" rational Schur functions of fixed maximal degree and having a fixed transmission polynomial. The de-embedding theory of Fano-Youla's approach to the matching problems will be used to include the constraints on the realizability of the system response. These de-embedding conditions are based on the characterisation of the global system containing the Darlington equivalent of the load. It consists of the reflection coefficient of the global system satisfying certain conditions at the transmission zeros of the Darlington equivalent of the load.

The matching problem formulation in the chapter will provide accurate lower hard bounds for the best feasible matching level, as well as the practical synthesis of matching filters approaching those bounds. The much celebrated Nevanlinna-Pick theorem is utilized to characterize the convex set of constraints of the problem and thereby essential properties of the problem, including feasibility, existence and uniqueness of solution are studied. An interesting result about a necessary condition for the optimality of the solution and critical point equation of the problem under consideration is studied as well. The concavity of the Pick matrix as a matrix-valued function with respect to the set of positive polynomials plays a crucial role in solving the problem as a non-linear semi-definite programming problem. Finally, the chapter will be concluded by providing some practical examples of matching filter synthesis using the presented approach.

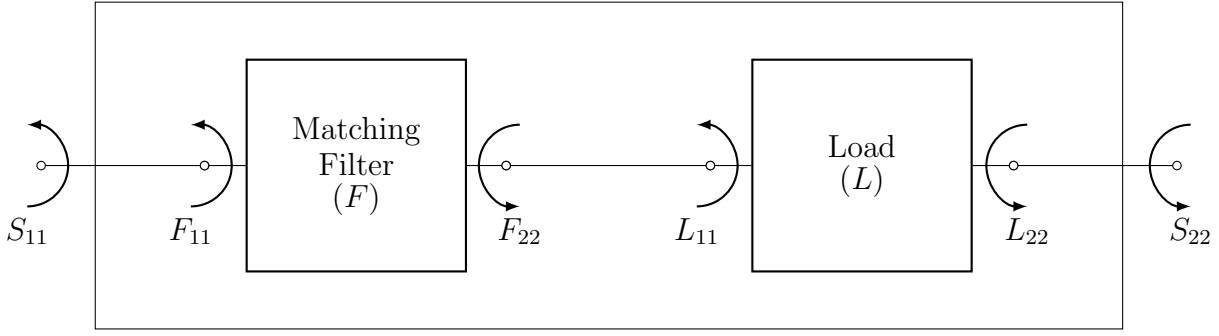


Figure 3.1: Global system (Matching Filter + Load) and Scattering Parameters

3.2 Formulation of the Problem

The matching problem is formulated as a convex optimization problem based on Youla's matching theory and the existence of finite degree matching networks that can achieve the optimal reflection level in the desired frequency band when plugged on to the given load is discussed. Given measured frequency data of the reflection coefficient of the load on the frequency band of interest \mathbb{I} , the first step is to obtain a rational approximation. Darlington's representation is utilized to construct an equivalent lossless two port, with the same input reflection. Thus, we have a load characterized by its (2×2) scattering matrix, L , while the matching filter to be synthesized is determined by its output reflection coefficient F_{22} . Following the Fano-Youla matching theory, physical realizability of the global system (S), composed of the matching filter (F) chained to the Darlington equivalent of load (L), is first considered. The global system is characterised here by its reflection coefficient at port 2, denoted S_{22} . Fig. 3.1 represents the cascade operation between F and L to form $S = F \circ L$. The problem of interest, namely the minimization, over the passband, of the reflection coefficient associated with the global system, is cast to a convex optimization problem involving Nevanlinna-Pick interpolation theory. Finally, the matching filter providing the optimal reflection level is afterwards obtained by de-embedding the load from the "optimal" global system.

We continue to use the setting from the last chapters, where, $\Pi^+ = \{x + j\omega \in \mathbb{C} : x > 0\}$ is considered as the domain of analyticity and \mathbb{I} denote the normalized passband of interest on the imaginary axis. Recall from definition 1.3.3 that for a rational matrix valued function $S(s)$,

$$S^*(s) = \overline{(S(-\bar{s}))^t}, \quad s \in \mathbb{C}, \quad (3.1)$$

represent the para-conjugate. The load (L) is supposed to be a system which does not completely reflect all frequencies, i.e $|L_{11}|$ is not uniformly equal to one for all frequencies. Before getting to the de-embedding approach, it is necessary to introduce the chaining of two-ports, a generalisation of chaining of one-port introduced in Chapter 2.

Definition 3.2.1. (*Chaining of two-ports*). The chaining of a two-port scattering matrix, say,

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

to a two-port scattering matrix,

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix},$$

which represents the load (where $(1, 1)$ th element is the given reflection coefficient measurement L_{11} , which follows by Darlington extension, Theorem 1.3.13), the global scattering matrix, say,

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

produced as the result of chaining F with L , denoted by $F \circ L$ is

$$S = \begin{bmatrix} F_{11} + \frac{F_{12}F_{21}L_{11}}{1-F_{22}L_{11}} & \frac{F_{12}L_{12}}{1-F_{22}L_{11}} \\ \frac{F_{21}L_{21}}{1-F_{22}L_{11}} & L_{22} + \frac{L_{12}L_{21}F_{22}}{1-F_{22}L_{11}} \end{bmatrix}. \quad (3.2)$$

In particular, we should note that in the above definition, the output reflection coefficient S_{22} of the global system at each frequency is

$$S_{22} = L_{22} + \frac{L_{12}L_{21}F_{22}}{1-F_{22}L_{11}} = \det(L) \left(\frac{L_{11}^* - F_{22}}{1 - F_{22}L_{11}} \right). \quad (3.3)$$

3.2.1 De-embedding Approach

In this section, we will introduce the concept of de-chaining and state and prove a theorem describing the necessary and sufficient condition for a lossless two-port two be de-chainable from a given reflection coefficient. This will play a crucial role in the formulation of the matching problem in the next subsection.

Definition 3.2.2. (*De-chaining*). A lossless two-port, L is said to be de-chainable from a function, $S_{22} \in \mathbb{B}$ iff there exists $F_{22} \in \mathbb{B}$, such that $S_{22} = F_{22} \circ L$. Expression for F_{22} is obtained by inverting (3.3),

$$F_{22} = \frac{S_{22} - L_{22}}{S_{22}L_{11} - \det(L)}. \quad (3.4)$$

Before stating the de-embedding theorem, it would be useful to recall the Rouches's theorem from complex analysis (Theorem 10.43, [83]), which will be used in the proof of the same.

Theorem 3.2.3. (*Rouche's Theorem*). Let f and g be two complex valued holomorphic functions defined on an open connected subset G of \mathbb{C} with simple closed contour ∂G . If for $z \in \partial G$,

$$|g(z)| < |f(z)|,$$

then f and $f + g$ have the same number of zeros inside G , counting multiplicity.

For simplicity, throughout this chapter unless specified otherwise we will consider the case of rational loads having only $M \geq 1$ simple transmission zeros inside the analyticity domain (Π^+) , say $\{\xi_1, \xi_2, \dots, \xi_M\}$. So, we will be dealing with the case of strictly contractive load ($|L_{11}(j\omega)| < 1$, $\omega \in \mathbb{R}$) and the trivial case of the input reflection coefficient of the load, L_{11} being a constant is avoided as well. It should be noted that under the given conditions on the input reflection coefficient L_{11} of the load, the transmission polynomial of the load, $R_L \neq 0$ and the L_{22} build in the Darlington extension of the load (1.86 in chapter 1) cannot be zero at all the transmission zeros, ξ_k , $1 \leq k \leq M$. Now, we are in a position to introduce the de-embedding theorem which will help to characterize the set of all reflection coefficients from which the matrix L can be de-chained.

Theorem 3.2.4. (*De-embedding Theorem*). *Let L represent the (2×2) rational lossless scattering matrix of McMillan degree M of a strictly contractive non-constant load and $\xi_1, \xi_2, \dots, \xi_M \in \Pi^+$ represent the M simple transmission zeros of L . The matrix L is de-chainable of any given $S_{22} \in \mathbb{B}$ iff at each transmission zero ξ_k , the following conditions are satisfied:*

$$S_{22}(\xi_k) = L_{22}(\xi_k), \quad 1 \leq k \leq M. \quad (3.5)$$

Proof. Let us suppose that L is de-chainable from S_{22} . Using definition 3.2.2, there exists $F_{22} \in \mathbb{B}$, such that,

$$S_{22} = F_{22} \circ L = L_{22} + \frac{L_{12}L_{21}F_{22}}{1 - F_{22}L_{11}}.$$

Since, at each transmission zero of L , ξ_k , $1 \leq k \leq M$, $L_{12}(\xi_k)L_{21}(\xi_k) = 0$ and the load L being strictly contractive, we have,

$$S_{22}(\xi_k) = L_{22}(\xi_k), \quad 1 \leq k \leq M.$$

Now, in order to prove sufficiency, let us assume that at each transmission zero of L , ξ_k , $1 \leq k \leq M$ we have,

$$S_{22}(\xi_k) = L_{22}(\xi_k), \quad 1 \leq k \leq M. \quad (3.6)$$

Then consider,

$$F_{22} = \frac{S_{22} - L_{22}}{S_{22}L_{11} - \det(L)} \quad (3.7)$$

$$= \frac{1}{\det(L)} \left(\frac{S_{22} - L_{22}}{S_{22}L_{22}^* - 1} \right). \quad (3.8)$$

Equation (3.8) indicates that $\forall \omega \in \mathbb{R}$, $|F_{22}(j\omega)| \leq 1$, since on the imaginary axis, $|\det(L)| = 1$ and $\delta(S_{22}, L_{22}^*)$ is bounded by one. Now, we can prove that F_{22} is analytic in Π^+ . At each ξ_k , $1 \leq k \leq M$, both the numerator and denominator of (3.7) vanishes,

$$S_{22}(\xi_k)L_{11}(\xi_k) - \det(L)(\xi_k) = S_{22}(\xi_k)L_{11}(\xi_k) - L_{11}(\xi_k)L_{22}(\xi_k) = L_{11}(\xi_k)(S_{22}(\xi_k) - L_{22}(\xi_k))$$

Thus, M zeros in the denominator of expression (3.7) gets cancelled with the M zeros of numerator. Now, we will prove the fact that the denominator has no other zeros in Π^+ . Since $S_{22} \in \mathbb{B}$ and $|L_{11}(j\omega)| < 1$, $\omega \in \mathbb{R}$ (all transmission zeros being in Π^+), we have,

$$|S_{22}(j\omega)L_{11}(j\omega)| < 1, \quad |\det(L)(j\omega)| = 1, \quad \omega \in \mathbb{R} \quad (3.9)$$

3.2. FORMULATION OF THE PROBLEM

It follows from Rouché's theorem (Theorem 3.2.3) that $S_{22}L_{11} - \det(L)$ has the same number of zeros in Π^+ as that of $\det(L)$, which is M . This shows that F_{22} is analytic in Π^+ . Thus, we have a rational Schur function F_{22} , such that $S_{22} = F_{22} \circ L$ and hence proving the de-chainability of L from S_{22} . This completes the proof. \square

If we use \mathbb{F} to denote the set of all functions, $S_{22} \in \mathbb{B}$ such that the matrix L is de-chainable from S_{22} , we have the following characterisation of \mathbb{F} from theorem 3.2.4,

$$\mathbb{F} = \{S_{22} \in \mathbb{B} : S_{22}(\xi_k) = L_{22}(\xi_k), 1 \leq k \leq M\}. \quad (3.10)$$

We are now in a position to introduce the classical Nevanlinna-Pick theorem which will play a crucial role in this chapter. For a given load, it will provide a necessary and sufficient condition for the existence of schur function S_{22} in \mathbb{F} . Nevanlinna-Pick theorem (Section 18.1, [8] and Theorem 2.2, Chap. I, [42]) can be stated as the following.

Theorem 3.2.5. (*Nevanlinna-Pick Theorem*). *Given a collection of points $s_1, s_2, \dots, s_d \in \Pi^+$ and $\gamma_1, \gamma_2, \dots, \gamma_d \in \mathbb{C}$, there exists, $b \in \mathbb{B}$, satisfying the interpolation,*

$$b(s_i) = \gamma_i, \quad i = 1, 2, \dots, d \quad (3.11)$$

if and only if the Pick matrix, $\Delta(s_i, \gamma_i) = \left[\frac{1 - \overline{\gamma_i} \gamma_j}{s_i + s_j} \right]_{1 \leq i, j \leq d}$ is positive semi-definite.

Also, when $\Delta(s_i, \gamma_i) \succeq 0$, there is a Blaschke product of degree at most d which solves the interpolation problem. About the uniqueness of solution, if $\Delta(s_i, \gamma_i) \succeq 0$, then the problem has a unique solution iff $\det(\Delta(s_i, \gamma_i)) = 0$. If $\det(\Delta(s_i, \gamma_i)) = 0$ and $d_0 < d$ is the rank of $\Delta(s_i, \gamma_i)$, then the interpolating function is a Blaschke product of degree d_0 . A brief review of the Nevanlinna-Pick theorem is provided in section A.2 in Appendix A.

3.2.2 The Global Matching Problem

As mentioned in Theorem 1.3.10 in Chapter 1, any general lossless rational scattering matrix can be expressed in Belevitch form as follows :

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{1}{q} \begin{pmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{pmatrix} \quad (3.12)$$

where ϵ is a unimodular complex constant and p, q, r are complex polynomials of degree at most N (hereinafter denoted by \mathbb{P}_N) satisfying the Feldtkeller equation $qq^* = pp^* + rr^*$. In particular, the scattering matrix of the global system, S , can be put in Belevitch form. Its transmission polynomial r is $r = r_L r_F$, the product of the transmission polynomials associated to the load, $r_L \neq 0$ and the matching filter, r_F . We suppose that $r_F \neq 0$ is fixed by the user and let $N \geq \deg(r^F r^L)$ be the target degree of the global system G . This provides the characterisation of the global system by its output reflection coefficient, $S_{22} = \frac{p}{q}$ with the associated transmission polynomial $R \neq 0$ such that $R = rr^* = qq^* - pp^*$. Note that R is a fixed non-negative polynomial of degree at most $2N$, i.e $R \in \mathbb{P}_{2N}^+$. Following the Fano-Youla theory of de-embedding, we define the following set of functions S_{22} of

degree atmost N associated to a fix transmission polynomial $R \in \mathbb{P}_{2N}^+$, with the necessary constraint that L is de-chainable from S_{22} ,

$$\mathbb{F}_R^N = \left\{ S_{22} = \frac{p}{q} \in \mathbb{F} \mid p, q \in \mathbb{P}_N, qq^* - pp^* = R \right\} \quad (3.13)$$

As in previous chapters, let \mathbb{I} be the desired passband, a finite union of disjoint compact intervals over the frequencies. We assume that $R^F = r^F r^{F*}$ is fixed and has no zeros inside \mathbb{I} . It should be noted that if R_F has zeros inside \mathbb{I} , this means that R vanishes at these points as well and yields $|S_{22}(s)|$ to be 1 at these points. That is, in this case, we will have points inside the passband where there is no transmission through the global system. Now, we state the problem as,

Problem. \mathfrak{P} : Find $l_{opt} = \min_{S_{22} \in \mathbb{F}_R^N} \max_{j\omega \in \mathbb{I}} |S_{22}(j\omega)|$.

Problem \mathfrak{P} is clearly not convex since \mathbb{F}_R^N is not a convex set. Nevertheless, note that the modulus of S_{22} can be expressed in function of the polynomials p and r only. With the change of variable $P = pp^*$ and $R = rr^*$, we have on the imaginary axis:

$$|S_{22}|^2 = \frac{pp^*}{qq^*} = \frac{pp^*}{pp^* + rr^*} = \frac{P}{P + R}. \quad (3.14)$$

It should be noted that now we will be moving towards a formulation of the problem \mathfrak{P} in terms of the positive polynomial $P \in \mathbb{P}_{2N}^+$ and the fixed transmission polynomial $R \in \mathbb{P}_{2N}^+$ of the global system.

3.2.3 Convex Formulation

The main aim of this section is to derive a convex relaxation of problem \mathfrak{P} . Let us denote by \mathbb{X} , the finite set of at most $2N$ points on the imaginary axis where the fixed transmission polynomial $R \not\equiv 0$ of the global system vanishes,

$$\mathbb{X} = \{j\omega \in j\mathbb{R} : R(j\omega) = 0\}. \quad (3.15)$$

Now, for the given load L , we define the following subset of \mathbb{P}_{2N}^+ ,

Definition 3.2.6.

$$\mathbb{H}_R^N = \left\{ P \in \mathbb{P}_{2N}^+ : \exists \rho \in \mathbb{F} : |\rho(j\omega)| \leq \sqrt{\frac{P(j\omega)}{P(j\omega) + R(j\omega)}}, j\omega \in j\mathbb{R} \setminus \mathbb{X} \right\}. \quad (3.16)$$

The reader can verify that under the given assumptions on the load for this chapter, that is the load is strictly contractive and non-constant, $P \equiv 0$ is not in \mathbb{H}_R^N . This follows from the fact that if $0 \in \mathbb{H}_R^N$, then it means $\rho \equiv 0$ is in \mathbb{F} which is not possible since $L_{22}(\xi_k)$, $1 \leq k \leq M$ cannot be all zero.

It should be noted that given the non-zero polynomials $P, R \in \mathbb{P}_{2N}^+$, an outer function U_P in $H^\infty(\Pi^+)$ with prescribed modulus on the imaginary axis except the finite number of elements in the set \mathbb{X} can be build,

$$|U_P(j\omega)|^2 = \frac{P(j\omega)}{P(j\omega) + R(j\omega)}, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}. \quad (3.17)$$

3.2. FORMULATION OF THE PROBLEM

To be more precise about the formation of this outer function, consider the following proposition, whose proof can be found in proposition 2, [10].

Proposition 3.2.7. *Let \mathbb{SB}_N represent the set of polynomials of degree at most N which are stable in the broad sense (no zeros inside Π^+). To any non-zero $P \in \mathbb{P}_{2N}^+$, one can associate $q \in \mathbb{SB}_N$, such that,*

$$P(j\omega) = |q(j\omega)|^2 = q(j\omega)q^*(j\omega), \quad \omega \in \mathbb{R}. \quad (3.18)$$

The polynomial q is unique up to a multiplicative unimodular constant and if P has exact degree $2N$ then q has exact degree N . For fixed $z_0 \in \Pi^+$, define the map $\Phi_{z_0} : \mathbb{P}_{2N}^+ \rightarrow \mathbb{SB}_N$, with $\Phi_{z_0}(P)$ the unique solution to (3.18), satisfying $q(z_0) > 0$. The map Φ_{z_0} is continuous.

Now, for any non-zero $P \in \mathbb{P}_{2N}^+$, the outer function U_P in $H^\infty(\Pi^+)$ can be defined as

$$U_P = \frac{\Phi_{z_0}(P)}{\Phi_{z_0}(P + R)}, \quad (3.19)$$

where the function Φ_{z_0} is defined in proposition 3.2.7. The normalization at the point z_0 inside the right half plane allows to avoid the ambiguity in determining the phase of the outer function formed from a given modulus on the imaginary axis. It should be noted that U_P is a continuous function in P since Φ_{z_0} is continuous in P (Proposition 3.2.7) and the denominator does not vanish inside Π^+ . Moreover U_P is a rational outer function of degree at most N satisfying $|U_P|^2 = \frac{P}{P+R}$ on the imaginary axis except the finite set \mathbb{X} as mentioned in (3.17). The dependence of U_P on R is not indicated in the notation since in our scheme, the polynomial R which represent the transmission polynomial of the global system is fixed.

The reader can refer to chapter 8, [49] for a general exposition of H^p spaces in half plane. In general, given the non-zero polynomials $P, R \in \mathbb{P}_{2N}^+$, the modulus of any outer function U in $H^\infty(\Pi^+)$ inside the right half plane satisfying

$$|U(j\omega)| = \sqrt{\frac{P(j\omega)}{P(j\omega) + R(j\omega)}}, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}$$

can be calculated irrespective of a normalization on its phase. For $x > 0$,

$$|U(x + j\omega)| = \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} P_x(\omega - t) \log \sqrt{\frac{P(jt)}{P(jt) + R(jt)}} dt \right), \quad (3.20)$$

where $P_x(\omega) = \frac{x}{x^2 + \omega^2}$ represents the Poisson Kernel for the right half plane.

For a polynomial $P \in \mathbb{H}_R^N$, by definition there exists $\rho \in \mathbb{B}$ satisfying the de-embeddding conditions for the given load and satisfying

$$|\rho(j\omega)| \leq |U_P(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}.$$

Next, we claim that for such a polynomial, it is always possible to find a corresponding $\rho \in \mathbb{F}$ which is rational with degree at most $N + M$.

Proposition 3.2.8. (*Bound on the degree of ρ*). Consider $P \in \mathbb{H}_R^N$. Then there exists $\rho \in \mathbb{F}$ that is rational with degree at most $N + M$ satisfying,

$$|\rho(j\omega)| \leq \sqrt{\frac{P(j\omega)}{P(j\omega) + R(j\omega)}}, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}.$$

Proof. Let $P \in \mathbb{H}_R^N$ and U_P be the outer function in $H^\infty(\Pi^+)$ of at most degree N defined in (3.19) with prescribed modulus on $j\mathbb{R} \setminus \mathbb{X}$, $|U_P| = \sqrt{\frac{P}{P+R}}$. Since $P \in \mathbb{H}_R^N$, there exists $\rho \in \mathbb{B}$ satisfying

$$\begin{aligned} \rho(\xi_k) &= L_{22}(\xi_k), \quad 1 \leq k \leq M \\ |\rho(j\omega)| &\leq |U_P(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}. \end{aligned}$$

It should be noted that the function $b = \frac{\rho}{U_P}$ is holomorphic in Π^+ since $\rho, U_P \in H^\infty(\Pi^+)$ and U_P has no zeroes inside Π^+ . Since $\rho \in \mathbb{H}^\infty(\Pi^+)$, it follows from result (ii), page 133, [49] that if $P_x(\omega) = \frac{x}{x^2 + \omega^2}$ represents the Poisson Kernel for right half plane, we have, for $x > 0$,

$$\begin{aligned} |\rho(x + j\omega)| &\leq \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} P_x(\omega - t) \log |\rho(jt)| dt \right) \\ &\leq \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} P_x(\omega - t) \log |U_P(jt)| dt \right) \\ &= |U_P(x + j\omega)|. \end{aligned}$$

The second inequality follows because $|\rho(jt)| \leq \sqrt{\frac{P(jt)}{P(jt) + R(jt)}}$, $jt \in j\mathbb{R} \setminus \mathbb{X}$ for $P \in \mathbb{H}_R^N$ (Definition 3.2.6), log and exp are monotonically increasing functions and $P_x(\omega)$ is a positive kernel. The last equality follows from (3.20). This yields, $\|\frac{\rho}{U_P}\|_\infty = \sup_{x>0} |\frac{\rho}{U_P}(x + i\omega)| \leq 1$,

implying $b = \frac{\rho}{U_P} \in \mathbb{B}$. It should be noted that $b(\xi_k) = \frac{L_{22}(\xi_k)}{U_P(\xi_k)}$ follows from the fact that $\rho(\xi_k) = L_{22}(\xi_k)$ and $U_P(\xi_k) \neq 0$. This implies that using Nevanlinna-Pick theorem, the Pick matrix $\Delta(\xi_k, \frac{L_{22}(\xi_k)}{U_P(\xi_k)}) \succeq 0$ and hence there exists a Blaschke product, b_0 (depending on P) of at most degree M solving the interpolation problem, $f \in \mathbb{B}$, $f(\xi_k) = \frac{L_{22}(\xi_k)}{U_P(\xi_k)}$. So, we have $\rho = b_0 \cdot U_P \in \mathbb{F}$ which is rational of at most degree $N + M$. \square

It is evident from proposition 3.2.8 that for $\rho \in \mathbb{F}$ corresponding to $P \in \mathbb{H}_R^N$, the constraint, degree of ρ to be less than or equal to N (as in \mathbb{F}_R^N) has been relaxed, but we do gain the important property, convexity of \mathbb{H}_R^N as proved below along with other properties.

Theorem 3.2.9. \mathbb{H}_R^N is a non-empty, closed, convex subset of \mathbb{P}_{2N}^+ .

Proof. By definition, \mathbb{H}_R^N is a subset of \mathbb{P}_{2N}^+ . The proof of each property follows.

- \mathbb{H}_R^N is non-empty.

The fact that \mathbb{H}_R^N is nonempty can be proved by showing the existence of a $P_0 \in \mathbb{P}_{2N}^+$ such that there exist a $\rho_0 \in \mathbb{F}$ which satisfies for all $j\omega \in j\mathbb{R} \setminus \mathbb{X}$,

$$|\rho_0(j\omega)|^2 \leq \frac{P_0(j\omega)}{P_0(j\omega) + R(j\omega)}. \quad (3.21)$$

Let us represent the lossless scattering matrix of the given load (McMillan degree M) by

$$L = \frac{1}{q_L} \begin{bmatrix} \epsilon p_L^* & -\epsilon r_L^* \\ r_L & p_L \end{bmatrix}, \quad (3.22)$$

where $Q_L \stackrel{\text{def}}{=} q_L q_L^* = p_L p_L^* + r_L r_L^*$. In the inequality (3.21), we have $R = R_F R_L$, where $R_F = r_F r_F^* \in \mathbb{P}_{2N-2M}^+$ and $R_L = r_L r_L^* \in \mathbb{P}_{2M}^+$ as mentioned at the beginning of the subsection 3.2.2. If we denote $P_L = p_L p_L^* \in \mathbb{P}_{2M}^+$, it can be easily verified that $P_0 = R_F P_L \in \mathbb{P}_{2N}^+$ is in \mathbb{H}_R^N since we have $\rho_0 = L_{22} = \frac{p_L}{q_L} \in \mathbb{F}$ satisfying for all $\omega \in \mathbb{R}$,

$$|\rho_0(j\omega)|^2 = \frac{P_L(j\omega)}{Q_L(j\omega)} = \frac{P_L(j\omega)}{P_L(j\omega) + R_L(j\omega)} = \frac{P_0(j\omega)}{P_0(j\omega) + R(j\omega)}.$$

- \mathbb{H}_R^N is closed.

In order to prove that the set \mathbb{H}_R^N is closed, we consider any sequence (P_n) in \mathbb{H}_R^N converging to a polynomial \tilde{P} (convergence in any of the equivalent norms defined on the finite dimensional vector space of polynomials, \mathbb{P}_{2N}) and prove that $\tilde{P} \in \mathbb{H}_R^N$. If we consider such a sequence, using the definition of \mathbb{H}_R^N , we have $\tilde{P} \in \mathbb{P}_{2N}^+$, since \mathbb{P}_{2N}^+ is a closed set. We have a sequence of rational functions, $(\frac{P_n}{P_n + R})$ and since $(P_n) \in \mathbb{H}_R^N$, corresponding to each P_n , there exist $\rho_n \in \mathbb{F}$ satisfying the inequality in the definition of \mathbb{H}_R^N . The sequence $(\rho_n) \in \mathbb{F}$ having uniform degree bound of $(M + N)$ (proposition 3.2.8), satisfy,

$$|\rho_n(j\omega)|^2 \leq \frac{P_n(j\omega)}{P_n(j\omega) + R(j\omega)}, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}. \quad (3.23)$$

The sequence, $(\frac{P_n}{P_n + R})$ converges pointwise to $\frac{\tilde{P}}{\tilde{P} + R}$ everywhere on the imaginary axis except the finite set of zeros of $(\tilde{P} + R)$ on the imaginary axis (denote this set by \mathbb{X}_1). Proposition A.1.2 in Appendix guarantees that a subsequence $(\rho_{\phi(n)})$ of (ρ_n) can be extracted such that it converges (in sup norm) to a rational Schur function $\tilde{\rho}$ (in lowest form) on every compact set inside the right half plane and the pointwise convergence holds everywhere on the imaginary axis except the finite set of limit points of poles of $(\rho_{\phi(n)})$, say set \mathbb{X}_2 . Since $\rho_{\phi(n)}(\xi_k) = L_{22}(\xi_k)$, where ξ_k 's are inside the right half plane, pointwise convergence hold at any point in the right half plane, implying we have the equality holding at the limit, i.e $\tilde{\rho}(\xi_k) = L_{22}(\xi_k)$, hence $\tilde{\rho} \in \mathbb{F}$. The inequality, $|\rho_{\phi(n)}(s)|^2 \leq \frac{P_{\phi(n)}(j\omega)}{P_{\phi(n)}(j\omega) + R(j\omega)}$ on the imaginary axis and the fact that $(\frac{P_{\phi(n)}}{P_{\phi(n)} + R})$ and $(\rho_{\phi(n)})$ converges pointwise to $\frac{\tilde{P}}{\tilde{P} + R}$ and $\tilde{\rho}$ on the imaginary axis except finite sets \mathbb{X}_1 and \mathbb{X}_2 respectively yields $|\tilde{\rho}(j\omega)|^2 \leq \frac{\tilde{P}(j\omega)}{\tilde{P}(j\omega) + R(j\omega)}$ for all $j\omega \in j\mathbb{R} \setminus (\mathbb{X}_1 \cup \mathbb{X}_2)$. After cancelling the common zeros of $(\tilde{P} + R)$ and \tilde{P} on the imaginary axis (elements of finite set, \mathbb{X}_1 , if it is non-empty), the continuity of $|\tilde{\rho}|^2$ and $\frac{\tilde{P}}{\tilde{P} + R}$ on the imaginary axis and the fact that $(\mathbb{X}_1 \cup \mathbb{X}_2)$ is a finite set ensures $|\tilde{\rho}(j\omega)|^2 \leq \frac{\tilde{P}(j\omega)}{\tilde{P}(j\omega) + R(j\omega)}$ on the whole imaginary axis. So, we have $\tilde{\rho} \in \mathbb{F}$, satisfying, $|\tilde{\rho}(j\omega)| \leq \sqrt{\frac{\tilde{P}(j\omega)}{\tilde{P}(j\omega) + R(j\omega)}}$, $\omega \in \mathbb{R}$ (since square root function is continuous in $[0, \infty)$), implying $\tilde{P} \in \mathbb{H}_R^N$, proving that the set \mathbb{H}_R^N is closed.

- \mathbb{H}_R^N is convex.

Let P_1, P_2 be distinct polynomials in \mathbb{H}_R^N . By (3.16), there exists Schur functions $\rho_1, \rho_2 \in \mathbb{F}$, satisfying at any $j\omega \in j\mathbb{R} \setminus \mathbb{X}$, $|\rho_1(j\omega)| \leq \sqrt{\frac{P_1(j\omega)}{P_1(j\omega)+R(j\omega)}}$ and $|\rho_2(j\omega)| \leq \sqrt{\frac{P_2(j\omega)}{P_2(j\omega)+R(j\omega)}}$. We need to verify, for any $\gamma \in (0, 1)$, $P_3 \stackrel{\text{def}}{=} \gamma P_1 + (1 - \gamma)P_2$ is in \mathbb{H}_R^N . To begin with, we show that at every $j\omega \in j\mathbb{R} \setminus \mathbb{X}$, $\sqrt{\frac{P(j\omega)}{P(j\omega)+R(j\omega)}}$ is a concave function of P . For $j\omega \in j\mathbb{R} \setminus \mathbb{X}$, note $x = P(j\omega) \geq 0$, $\alpha = R(j\omega) > 0$ and define,

$$\sqrt{\frac{P(j\omega)}{P(j\omega) + R(j\omega)}} = \tilde{h}(x) \stackrel{\text{def}}{=} \sqrt{\frac{x}{x + \alpha}}. \quad (3.24)$$

The first and second derivative of \tilde{h} with respect to the non-negative real variable x are,

$$\begin{aligned} \frac{d}{dx}(\tilde{h}(x)) &= \frac{\alpha}{2} \frac{x^{-\frac{1}{2}}}{(x + \alpha)^{\frac{3}{2}}} \\ \frac{d^2}{dx^2}(\tilde{h}(x)) &= \frac{-\alpha}{4} \left[\frac{(x + \alpha)^{\frac{3}{2}} x^{-\frac{3}{2}} + 3x^{-\frac{1}{2}}(x + \alpha)^{\frac{1}{2}}}{(x + \alpha)^3} \right] < 0, \end{aligned} \quad (3.25)$$

showing that \tilde{h} is a concave function in $x \geq 0$. It follows that at any point on the imaginary axis except \mathbb{X} ,

$$\begin{aligned} \sqrt{\frac{P_3(j\omega)}{P_3(j\omega) + R(j\omega)}} &\geq \gamma \sqrt{\frac{P_1(j\omega)}{P_1(j\omega) + R(j\omega)}} + (1 - \gamma) \sqrt{\frac{P_2(j\omega)}{P_2(j\omega) + R(j\omega)}} \\ &\geq \gamma |\rho_1(j\omega)| + (1 - \gamma) |\rho_2(j\omega)| \\ &\geq |\gamma \rho_1(j\omega) + (1 - \gamma) \rho_2(j\omega)|. \end{aligned} \quad (3.26)$$

It should be noted that at the finite set of at most $2N$ points where the evaluation of P_1 and P_2 are equal, we have equality in (3.26). So, we have, $\rho_3 \stackrel{\text{def}}{=} \gamma \rho_1 + (1 - \gamma) \rho_2$, satisfying, $|\rho_3(j\omega)| \leq \sqrt{\frac{P_3(j\omega)}{P_3(j\omega)+R(j\omega)}}$, $j\omega \in j\mathbb{R} \setminus \mathbb{X}$. Additionally, since ρ_3 is a linear combination of ρ_1 and ρ_2 , it satisfies the de-embedding conditions in Theorem 3.2.4 as well i.e $\rho_3 \in \mathbb{F}$. Thus, $P_3 \in \mathbb{H}_R^N$. □

Remark 3.2.10. For every $j\omega \in j\mathbb{R} \setminus \mathbb{X}$, we proved in the last part of the above theorem that the function $\sqrt{\frac{P(j\omega)}{P(j\omega)+R(j\omega)}}$ is concave in P . It means that we have the function $|U_P|$ concave in P on the imaginary axis except \mathbb{X} . It should also be noted that we in fact have $|U_P|^2$ concave in P on the imaginary axis except \mathbb{X} as well.

Now, we are in position to formulate the problem which will be central to this chapter. In comparison to problem \mathfrak{P} which was casted in the setting of Schur functions, we will now move to the setting of positive polynomials in this problem.

Problem. $\mathfrak{P}_C : \text{Find} : L_{opt} = \min_{P \in \mathbb{H}_R^N} \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)}.$

The constraint P in \mathbb{H}_R^N in problem \mathfrak{P}_C guarantees that there exists $S_{22} \in \mathbb{B}$ which satisfy the de-embedding conditions for the given load (i.e $S_{22} \in \mathbb{F}$) and satisfying $|S_{22}| \leq |U_P|$ on the imaginary axis. However, in comparison to problem \mathfrak{P} where $S_{22} \in \mathbb{F}$ was restricted to be of degree N , it should be noted that the degree bound on $S_{22} \in \mathbb{F}$ in problem \mathfrak{P}_C is relaxed to $N + M$ (proposition 3.2.8). Nevertheless, in this new setting we have a convex feasible set \mathbb{H}_R^N (theorem 3.2.9) in problem \mathfrak{P}_C , obtained at the cost of a higher degree bound on the global system. In addition, in problem \mathfrak{P}_C , the objective function to be minimized, $\Psi : \mathbb{P}_{2N}^+ \rightarrow \mathbb{R}$, $\Psi(P) = \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)}$ is convex and so problem \mathfrak{P}_C is convex. Also, with respect to problem \mathfrak{P} , we have

$$\forall S_{22} \in \mathbb{F}_R^N, \quad \max_{j\omega \in \mathbb{I}} |S_{22}(j\omega)| \geq \sqrt{\frac{L_{opt}}{L_{opt} + 1}}$$

which implies that $l_{opt} \geq \sqrt{\frac{L_{opt}}{L_{opt} + 1}}$. Hence, problem \mathfrak{P}_C provides a hard lower bound for the optimal matching criterion in problem \mathfrak{P} and it is this convex relaxation of problem \mathfrak{P} which will be studied in the remaining part of the chapter.

3.3 Analysis and Resolution of the Problem

In this section, we will do a detailed analysis of the convex optimisation problem \mathfrak{P}_C .

Theorem 3.3.1. (Solution to \mathfrak{P}_C). *There exist $P_{opt} \in \mathbb{H}_R^N$ which solves problem \mathfrak{P}_C .*

Proof. Denoting the cost function as $\Psi(P) = \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)}$, we have the problem \mathfrak{P}_C , find : $L_{opt} = \min_{P \in \mathbb{H}_R^N} \Psi(P)$. It can be easily verified that the cost function $\Psi(P)$ is a continuous function in P in the compact interval \mathbb{I} . Since \mathbb{H}_R^N is a non-empty set as proved in Theorem 3.2.9, we have, $\{\Psi(P) : P \in \mathbb{H}_R^N\}$ to be a non-empty set of real numbers that is bounded below by zero and hence it has an infimum, which is a finite real number. Let us denote, $m_0 = \inf_{P \in \mathbb{H}_R^N} \Psi(P)$. Now, \mathbb{H}_R^N being non-empty, we can always consider a sequence, $(P_n) \in \mathbb{H}_R^N$ such that, $(\Psi(P_n))$ form a decreasing sequence of real numbers, satisfying, $\lim_{n \rightarrow \infty} (\Psi(P_n)) = m_0$, say, $\forall n \in \mathbb{N}$, $\Psi(P_n) \leq m_0 + \frac{1}{n}$. Now, we can prove that such a sequence of positive polynomials, (P_n) , itself is a bounded sequence. For any $j\omega \in \mathbb{I}$, we have the following for all $n \in \mathbb{N}$,

$$\left| \frac{P_n(j\omega)}{R(j\omega)} \right| = \frac{P_n(j\omega)}{R(j\omega)} \leq \Psi(P_n) \leq m_0 + 1. \quad (3.27)$$

The first equality follows since $P_n, R \in \mathbb{P}_{2N}^+$. If we denote $k_0 = \max_{j\omega \in \mathbb{I}} R(j\omega)$, from equation (3.27), for any $j\omega \in \mathbb{I}$ and for all $n \in \mathbb{N}$,

$$P_n(j\omega) \leq k_0(m_0 + 1). \quad (3.28)$$

Using Definition A.1.1 in Appendix for the norm of polynomials of degree at most N , we have,

$$\begin{aligned} \|P_n\| &= \left(\sum_{j=0}^{2N} |P_n(a_j)|^2 \right)^{\frac{1}{2}}, \quad a_j \in \mathbb{I} \\ &\leq \left(\sum_{j=0}^{2N} k_0^2(m_0 + 1)^2 \right)^{\frac{1}{2}} = k_0(m_0 + 1)\sqrt{2N + 1}. \end{aligned}$$

The inequality followed from equation (3.28). So, the sequence of positive polynomials (P_n) is bounded. Since the set \mathbb{H}_R^N is closed (Theorem 3.2.9), we have a subsequence of (P_n) in \mathbb{H}_R^N , converging to an element \hat{P} in \mathbb{H}_R^N and Ψ being a continuous function in P yields $\Psi(\hat{P}) = m_0$. Since m_0 is the infimum of $\{\max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)} : P \in \mathbb{H}_R^N\}$, we have $m_0 = L_{opt}$ and so, $\Psi(\hat{P}) = L_{opt}$. This verifies the existence of a $P_{opt} \in \mathbb{H}_R^N$ at which L_{opt} is attained. \square

Note that we have not mentioned anything about the uniqueness of P_{opt} which solves problem \mathfrak{P}_C , but it will be proved later to be unique. In the next section, we move to a characterisation of the constraint set, \mathbb{H}_R^N of problem \mathfrak{P}_C .

3.3.1 Characterisation of \mathbb{H}_R^N

In this subsection, we will describe how a characterisation of the set \mathbb{H}_R^N is obtained using Nevanlinna-Pick interpolation theory ([8], [42]). Recall from equation (3.19) that U_P is a rational outer function in $H^\infty(\Pi^+)$ of degree at most N with the prescribed modulus

$$|U_P(j\omega)|^2 = \frac{P(j\omega)}{P(j\omega) + R(j\omega)}, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X},$$

where $\mathbb{X} = \{j\omega \in j\mathbb{R} : R(j\omega) = 0\}$. A practical way of the characterisation of the set

$$\mathbb{H}_R^N = \left\{ P \in \mathbb{P}_{2N}^+ : \exists \rho \in \mathbb{F} : |\rho(j\omega)| \leq |U_P(j\omega)|, j\omega \in j\mathbb{R} \setminus \mathbb{X} \right\}$$

is provided in the following proposition.

Proposition 3.3.2. *The set \mathbb{H}_R^N defined in (3.16) is characterised by*

$$\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\}, \quad (3.29)$$

where the Pick matrix $\Delta(P) = [\Delta_{ij}]_{1 \leq i, j \leq M}$ is defined by,

$$\Delta_{ij} = \frac{1 - \left(\frac{\overline{L_{22}(\xi_i)}}{U_P(\xi_i)} \right) \left(\frac{L_{22}(\xi_j)}{U_P(\xi_j)} \right)}{\xi_i + \xi_j}. \quad (3.30)$$

Proof. The set \mathbb{H}_R^N consists of positive polynomials $P \in \mathbb{P}_{2N}^+$ for which there exists a function $\rho \in \mathbb{B}$, satisfying,

$$\forall k = \{1, 2, \dots, M\}, \quad \rho(\xi_k) = L_{22}(\xi_k), \quad (3.31)$$

$$\forall j\omega \in j\mathbb{R} \setminus \mathbb{X}, \quad |\rho(j\omega)| \leq |U_P(j\omega)|. \quad (3.32)$$

As explained in the proof of proposition 3.2.8, this implies that for $P \in \mathbb{H}_R^N$, there exists $b = \frac{\rho}{U_P} \in \mathbb{B}$ satisfying,

$$\forall k = \{1, 2, \dots, M\}, \quad b(\xi_k) = \frac{L_{22}(\xi_k)}{U_P(\xi_k)}. \quad (3.33)$$

Using classical Nevanlinna-Pick theorem, there exist a Schur function, $b : \Pi^+ \rightarrow \overline{\mathbb{D}}$ satisfying (3.33) if and only if the Pick matrix, $\Delta(P) = [\Delta_{ij}]_{1 \leq i, j \leq M}$ defined by,

$$\Delta_{ij} = \frac{1 - \left(\frac{\overline{L_{22}(\xi_i)}}{U_P(\xi_i)} \right) \left(\frac{L_{22}(\xi_j)}{U_P(\xi_j)} \right)}{\overline{\xi_i} + \xi_j},$$

is positive semidefinite. Thus $\mathbb{H}_R^N \subseteq \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\} \stackrel{\text{def}}{=} \tilde{\mathbb{H}}_R^N$. To prove the converse inclusion, $\tilde{\mathbb{H}}_R^N \subseteq \mathbb{H}_R^N$, consider a $P \in \tilde{\mathbb{H}}_R^N$. We have, $\Delta(P) \succeq 0$ and hence there exists a Schur function $b : \Pi^+ \rightarrow \overline{\mathbb{D}}$ such that :

$$\forall k = \{1, 2, \dots, M\}, \quad b(\xi_k) = \frac{L_{22}(\xi_k)}{U_P(\xi_k)} \quad (3.34)$$

If we define, $\rho = U_P \cdot b$, we have, $|\rho| \leq |U_P|$ on the imaginary axis except \mathbb{X} . Also, $\rho \in \mathbb{F}$, since U_P and b are Schur functions and using equation (3.34),

$$\rho(\xi_k) = L_{22}(\xi_k), \quad 1 \leq k \leq M. \quad (3.35)$$

This implies that $P \in \mathbb{H}_R^N$, implying $\tilde{\mathbb{H}}_R^N \subseteq \mathbb{H}_R^N$ and hence $\mathbb{H}_R^N = \tilde{\mathbb{H}}_R^N$. \square

Remark 3.3.3. The above mentioned characterisation of $\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\}$,

where $\Delta_{ij} = \frac{1 - \left(\frac{\overline{L_{22}(\xi_i)}}{U_P(\xi_i)} \right) \left(\frac{L_{22}(\xi_j)}{U_P(\xi_j)} \right)}{\overline{\xi_i} + \xi_j}$, $1 \leq i, j \leq M$ directly provides the fact that \mathbb{H}_R^N is closed. We have the outer function U_P to be continuous in P (Proposition 3.2.7) and since the pointwise evaluation is continuous as well, we have $\Delta(P)$ to be continuous in P and hence the set \mathbb{H}_R^N is closed.

3.3.2 Optimal Characteristics and Uniqueness of P_{opt}

Proposition 3.3.4. Let $\Delta(P)$ represent the pick matrix as defined in (3.30) and P_{opt} the optimal solution to problem \mathfrak{P}_C . Then, $\Delta(P_{opt})$ is singular.

Proof. Let us suppose that at $P_{opt} \in \mathbb{H}_R^N$ at which L_{opt} is attained, the pick matrix is strictly positive definite, $\Delta(P_{opt}) \succ 0$. It should be noted that $|U_{\alpha P_{opt}}(s)|$ is a strictly monotonous function in $\alpha > 0$ and for $0 < \alpha < 1$, $|U_{\alpha P_{opt}}| < |U_{P_{opt}}|$ on the imaginary axis. We also

have $\Delta(P)$ continuous in P (remark 3.3.3) and therefore all its principal minors. Under the assumption that $\Delta(P_{opt})$ is strictly positive definite, there exists $0 < \tilde{\alpha} < 1$ such that all the principal minors of $\Delta(\tilde{\alpha}P_{opt})$ including $\det(\Delta(\tilde{\alpha}P_{opt}))$ are non-negative and by Sylvester's criterion (Theorem 7.2.5, [50]), we have $\Delta(\tilde{\alpha}P_{opt}) \succeq 0$. This implies that we are able to find an $\tilde{\alpha}P_{opt}$ in \mathbb{H}_R^N such that $\max_{j\omega \in \mathbb{I}} \frac{\tilde{\alpha}P_{opt}(j\omega)}{R(j\omega)}$ is strictly lower than $\max_{j\omega \in \mathbb{I}} \frac{P_{opt}(j\omega)}{R(j\omega)}$, contradicting the optimality of P_{opt} . \square

Proposition 3.3.5. *If $\rho \in H^\infty(\Pi^+)$ satisfy the interpolation conditions, $\rho(\xi_k) = L_{22}(\xi_k)$ and for $j\omega \in j\mathbb{R} \setminus \mathbb{X}$, $|\rho(j\omega)| \leq |U_{P_{opt}}(j\omega)|$, then,*

$$|\rho(j\omega)| = |U_{P_{opt}}(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}. \quad (3.36)$$

Proof. Suppose there exists a set of non-zero measure such that, $|\rho(j\omega)| < |U_{P_{opt}}(j\omega)|$ on $j\mathbb{R} \setminus \mathbb{X}$. Since, $\rho \in H^\infty(\Pi^+)$ satisfy the interpolation conditions, $\rho(\xi_k) = L_{22}(\xi_k)$, there exists a Schur function, $b_{opt} = \frac{\rho}{U_{P_{opt}}}$, satisfying, $b_{opt}(\xi_k) = \frac{L_{22}(\xi_k)}{U_P(\xi_k)}$ and $|b_{opt}| < 1$ on the imaginary axis. Using Nevanlinna-Pick theorem, this implies $\Delta(P_{opt}) \succ 0$. This contradicts Proposition 3.3.4 that $\Delta(P_{opt})$ is singular. Hence, $|\rho| = |U_{P_{opt}}|$ on $j\mathbb{R} \setminus \mathbb{X}$. \square

Theorem 3.3.6. *The $P_{opt} \in \mathbb{H}_R^N$ at which L_{opt} is attained for Problem \mathfrak{P}_C is unique.*

Proof. For any $P \in \mathbb{H}_R^N$, there exist $\rho \in \mathbb{F}$ such that, for $j\omega \in j\mathbb{R} \setminus \mathbb{X}$, $|\rho(j\omega)| \leq |U_P(j\omega)|$. At the optimum, using Proposition 3.3.5, we have,

$$|\rho_{opt}(j\omega)| = |U_{P_{opt}}(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}. \quad (3.37)$$

Let us suppose that there are two distinct polynomials, P_1 and P_2 in \mathbb{H}_R^N at which L_{opt} is attained. We have,

$$L_{opt} = \max_{j\omega \in \mathbb{I}} \frac{P_1(j\omega)}{R(j\omega)} = \max_{j\omega \in \mathbb{I}} \frac{P_2(j\omega)}{R(j\omega)}. \quad (3.38)$$

Let \mathbb{Y} denote the finite set of at most $2N$ points on the imaginary axis where the evaluation of P_1 and P_2 are equal. It should be noted from (3.26) that $|U_P|$ is strictly concave on the imaginary axis except the finite set $\mathbb{X} \cup \mathbb{Y}$. That is, for any $\gamma \in (0, 1)$, if $P_3 = \gamma P_1 + (1 - \gamma)P_2$, we have

$$\gamma|U_{P_1}(j\omega)| + (1 - \gamma)|U_{P_2}(j\omega)| < |U_{P_3}(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus (\mathbb{X} \cup \mathbb{Y}). \quad (3.39)$$

Using (3.37), we have $\rho_1, \rho_2 \in \mathbb{F}$, corresponding to P_1 and P_2 respectively, satisfying,

$$|\rho_1(j\omega)| = |U_{P_1}(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}, \quad (3.40)$$

$$|\rho_2(j\omega)| = |U_{P_2}(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus \mathbb{X}. \quad (3.41)$$

Now, if we consider the convex combination of ρ_1 and ρ_2 , we have, for all $j\omega \in j\mathbb{R} \setminus (\mathbb{X} \cup \mathbb{Y})$,

$$\begin{aligned} |\gamma\rho_1(j\omega) + (1 - \gamma)\rho_2(j\omega)| &\leq \gamma|\rho_1(j\omega)| + (1 - \gamma)|\rho_2(j\omega)| \\ &= \gamma|U_{P_1}| + (1 - \gamma)|U_{P_2}| \\ &< |U_{P_3}(j\omega)|. \end{aligned}$$

The first inequality follows from triangle inequality, the second equality from (3.40) and (3.41) and the last strict inequality from (3.39). So, we have,

$$|\gamma\rho_1(j\omega) + (1 - \gamma)\rho_2(j\omega)| < |U_{P_3}(j\omega)|, \quad j\omega \in j\mathbb{R} \setminus (\mathbb{X} \cup \mathbb{Y}). \quad (3.42)$$

This contradicts (3.37) since if P_1 and P_2 are optimal, we have, for $P_3 = \gamma P_1 + (1 - \gamma)P_2$,

$$\max_{j\omega \in \mathbb{I}} \frac{P_3(j\omega)}{R(j\omega)} = L_{opt}.$$

□

3.3.3 Necessary Condition for Optimality

In this subsection, a useful result regarding the minimal number of times $\frac{P_{opt}}{R}$ attains L_{opt} in the passband \mathbb{I} in problem \mathfrak{P}_C will be discussed, equivalently the minimal number of zeroes of the polynomial, $P_{opt} - L_{opt}R$ in \mathbb{I} . Before getting to the details of the result, it should be duly noted that the counting of zeros for the polynomial will be defined in a specific manner as follows:

Definition 3.3.7. (*Counting function*). Let $P \in \mathbb{P}_{2N}$ and $j\omega_k$'s, $k = 1, 2, \dots, \tilde{N}$, where $\tilde{N} \leq 2N$ be its' zeroes in the passband \mathbb{I} with corresponding multiplicities m_k for each $j\omega_k$. Define the count n_k for each $j\omega_k$ as follows,

$$n_k = \left\lceil \frac{m_k}{2} \right\rceil = \begin{cases} \frac{m_k}{2} & , \quad m_k \text{ even} \\ \frac{m_k+1}{2} & , \quad m_k \text{ odd} \end{cases}, \quad k = 1, 2, \dots, \tilde{N} \quad (3.43)$$

where, $\lceil \cdot \rceil$ represents the ceiling function. Now, we define the counting function, $\mathcal{N}(P)$ for $P \in \mathbb{P}_{2N}$ as follows,

$$\mathcal{N}(P) = \sum_{k=1}^{\tilde{N}} n_k. \quad (3.44)$$

Theorem 3.3.8. If \mathcal{N} represent the counting function defined in (3.44), in problem \mathfrak{P}_C , we have,

$$\mathcal{N}(P_{opt} - L_{opt}R) \geq N + 1,$$

that is we have, $\frac{P_{opt}}{R}$ equal to the value L_{opt} at least $(N + 1)$ times in \mathbb{I} if counted as per the definition 3.3.7.

Proof. Let us denote the zeros of $P_{opt} - L_{opt}R$ in \mathbb{I} as $j\omega_k$'s, $k = 1, 2, \dots, \tilde{N}$, where $\tilde{N} \leq 2N$ and m_k their corresponding multiplicity. We follow proof by contradiction, i.e by assuming, $\mathcal{N}(P_{opt} - L_{opt}R) < N + 1$, we show that there exists $\hat{P} \neq P_{opt}$ in \mathbb{P}_{2N}^+ , satisfying,

$$\forall \omega \in \mathbb{R}, \quad \hat{P}(j\omega) \geq P_{opt}(j\omega), \quad (3.45)$$

$$\forall j\omega \in \mathbb{I}, \quad \hat{P}(j\omega) - L_{opt}R(j\omega) \leq 0. \quad (3.46)$$

Once we show this, we have $\hat{P} \in \mathbb{H}_R^N$ as well, since, from equation (3.45) and proposition 3.3.5, we have almost everywhere on the imaginary axis,

$$|U_{\hat{P}}(j\omega)| \geq |U_{P_{opt}}(j\omega)| = |\rho_{opt}(j\omega)|. \quad (3.47)$$

This together with (3.46) contradicts the uniqueness of $P_{opt} \in \mathbb{H}_R^N$ for problem \mathfrak{P}_C (Theorem 3.3.6) and hence the result follows.

So, in order to prove (3.45) and (3.46), assume the optimal polynomial for problem \mathfrak{P}_C , namely, $P_{opt} \in \mathbb{P}_{2N}^+$, satisfy,

$$\mathcal{N}(P_{opt} - L_{opt}R) < N + 1. \quad (3.48)$$

Consider the polynomial,

$$P_\epsilon(j\omega) = P_{opt}(j\omega) + \epsilon\Phi(j\omega), \quad (3.49)$$

where ϵ is a positive constant and $\Phi(j\omega) = \prod_{k=1}^{\tilde{N}} (-1)^{n_k} (j\omega - j\omega_k)^{2n_k} = \prod_{k=1}^{\tilde{N}} (\omega - \omega_k)^{2n_k}$.

It should be noted that $\Phi(j\omega)$ has $2n_k$ roots at each point $j\omega_k$ and so, using (3.48), $\deg(\Phi(j\omega)) \leq 2N$. Thus we have, $\Phi(j\omega) \in \mathbb{P}_{2N}^+$. This implies $P_\epsilon \in \mathbb{P}_{2N}^+$ and we also have,

$$\forall \omega \in \mathbb{R}, P_\epsilon(j\omega) \geq P_{opt}(j\omega). \quad (3.50)$$

Now, we are interested in finding a strictly positive ϵ such that for $j\omega \in \mathbb{I}$, $P_\epsilon(j\omega) \leq L_{opt}R(j\omega)$, i.e

$$\forall j\omega \in \mathbb{I}, (P_{opt} - L_{opt}R)(j\omega) + \epsilon\Phi(j\omega) \leq 0. \quad (3.51)$$

Defining $Q = (P_{opt} - L_{opt}R)$, we have for $j\omega \in \mathbb{I}$, $Q(j\omega) \leq 0$ and we need,

$$\forall j\omega \in \mathbb{I}, Q(j\omega) + \epsilon\Phi(j\omega) \leq 0. \quad (3.52)$$

Let t be any positive natural number. For $P(j\omega) \in \mathbb{P}_{2N}$, let $P^{(t)}(j\omega_k)$ denote the value of t -th derivative of P with respect to $j\omega$ at $j\omega_k$ and $P_\omega^{(t)}(j\omega_k)$ denote the value of t -th derivative of P with respect to ω at $j\omega_k$. Using chain rule it can be verified that,

$$\Phi_\omega^{(t)}(j\omega_k) = j^t \Phi^{(t)}(j\omega_k), \quad (3.53)$$

$$Q_\omega^{(t)}(j\omega_k) = j^t Q^{(t)}(j\omega_k). \quad (3.54)$$

Let $j\omega \in \mathbb{I}$ be any point in the open neighbourhood of given point $j\omega_k$. Consider the Taylor development of Φ and Q at $j\omega$. Since $\Phi(j\omega)$ has $2n_k$ zeros at $j\omega_k$, we have the first $(2n_k - 1)$ derivatives with respect to ω vanishing at $j\omega_k$, and so,

$$\Phi(j\omega) = \frac{\Phi_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!} (\omega - \omega_k)^{2n_k} + o(|\omega - \omega_k|^{2n_k}). \quad (3.55)$$

If m_k is even, we have $m_k = 2n_k$ and since $Q(j\omega)$ has $2n_k$ zeros at $j\omega_k$, we have,

$$Q(j\omega) = \frac{Q_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!} (\omega - \omega_k)^{2n_k} + o(|\omega - \omega_k|^{2n_k}). \quad (3.56)$$

If m_k is odd, we have $m_k = 2n_k - 1$ and since $Q(j\omega)$ has $2n_k - 1$ zeros at $j\omega_k$, we have,

$$Q(j\omega) = \frac{Q_\omega^{(2n_k-1)}(j\omega_k)}{(2n_k-1)!}(\omega - \omega_k)^{2n_k-1} + \frac{Q_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!}(\omega - \omega_k)^{2n_k} + o(|\omega - \omega_k|^{2n_k}). \quad (3.57)$$

Since we have $\Phi(j\omega) \geq 0$ for $\omega \in \mathbb{R}$, $(\omega - \omega_k)^{2n_k}$ being non-negative yields $\Phi_\omega^{(2n_k)}(j\omega_k) > 0$. We have $Q(j\omega) \leq 0$ for $j\omega \in \mathbb{I}$. If m_k is even, $(\omega - \omega_k)^{2n_k}$ being non-negative yields $Q_\omega^{(2n_k)}(j\omega_k) < 0$. If m_k is odd, we have two cases : when $(\omega - \omega_k) \geq 0$, we have, $Q_\omega^{(2n_k-1)}(j\omega_k) < 0$ and when $(\omega - \omega_k) < 0$, we have, $Q_\omega^{(2n_k-1)}(j\omega_k) > 0$.

Now, we can evaluate $Q(j\omega) + \epsilon\Phi(j\omega)$ using the above equations. If m_k is even,

$$Q(j\omega) + \epsilon\Phi(j\omega) = \left(\frac{Q_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!} + \epsilon \frac{\Phi_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!} \right) (\omega - \omega_k)^{2n_k} + o(|\omega - \omega_k|^{2n_k}) \quad (3.58)$$

Denote, $|\frac{Q_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!}| = q_k^e$. Choose $\epsilon_k \in (0, 1)$, such that,

$$\epsilon_k \frac{\Phi_\omega^{(2n_k)}(j\omega_k)}{(2n_k)!} \leq \frac{q_k^e}{4} \text{ (we have } q_k^e > 0, \Phi_\omega^{(2n_k)}(j\omega_k) > 0). \quad (3.59)$$

Then fix an open neighbourhood in \mathbb{I} around $j\omega_k$, name it $\mathbb{V}_{\delta_k^e}$, in which $|o(|\omega - \omega_k|^{2n_k})| \leq \frac{q_k^e}{4}$ is satisfied. This will ensure that if $\epsilon \leq \epsilon_k$,

$$\forall j\omega \in \mathbb{V}_{\delta_k^e}, \quad Q(j\omega) + \epsilon\Phi(j\omega) \leq 0. \quad (3.60)$$

If m_k is odd, we have the following,

$$Q(j\omega) + \epsilon\Phi(j\omega) = \frac{Q_\omega^{(2n_k-1)}(j\omega_k)}{(2n_k-1)!}(\omega - \omega_k)^{2n_k-1} + o(|\omega - \omega_k|^{2n_k-1}). \quad (3.61)$$

Denote, $|\frac{Q_\omega^{(2n_k-1)}(j\omega_k)}{(2n_k-1)!}| = q_k^o$. Fix an open neighbourhood in \mathbb{I} around $j\omega_k$, name it $\mathbb{V}_{\delta_k^o}$, in which $|o(|\omega - \omega_k|^{2n_k-1})| \leq \frac{q_k^o}{2}$ is satisfied. This will ensure that, if $\epsilon \leq \epsilon_k$,

$$\forall j\omega \in \mathbb{V}_{\delta_k^o}, \quad Q(j\omega) + \epsilon\Phi(j\omega) \leq 0. \quad (3.62)$$

Defining, $\cup_{k=1}^{\tilde{N}} \mathbb{V}_{\delta_k^f} = \mathbb{V}$, where f is used to denote e or o depending upon the multiplicity m_k is even or odd respectively and by fixing $\epsilon_v = \min_{k \in \{1, 2, \dots, \tilde{N}\}} \epsilon_k$ (clearly $\epsilon_v \in (0, 1)$), we have, if $\epsilon \leq \epsilon_v$,

$$\forall j\omega \in \mathbb{V}, \quad Q(j\omega) + \epsilon\Phi(j\omega) \leq 0 \quad (3.63)$$

$$\text{i.e., } \forall j\omega \in \mathbb{V}, \quad P_{opt}(j\omega) - L_{opt}R(j\omega) + \epsilon\Phi(j\omega) \leq 0. \quad (3.64)$$

Now, let us consider the compact interval $\mathbb{I} \setminus \mathbb{V}$. We have for all $j\omega \in \mathbb{I} \setminus \mathbb{V}$, $P_{opt}(j\omega) < L_{opt}R(j\omega)$ and $\Phi(j\omega) > 0$. Let us denote $k_1 = \min_{j\omega \in \mathbb{I} \setminus \mathbb{V}} (L_{opt}R(j\omega) - P(j\omega)) > 0$ and $k_2 = \max_{j\omega \in \mathbb{I} \setminus \mathbb{V}} \Phi(j\omega) > 0$. By choosing $\epsilon_c = \frac{k_1}{k_2}$, it can be ensured that if $\epsilon \leq \epsilon_c$,

$$\forall j\omega \in \mathbb{I} \setminus \mathbb{V}, \quad P_{opt}(j\omega) - L_{opt}R(j\omega) + \epsilon\Phi(j\omega) \leq 0. \quad (3.65)$$

Finally, by choosing, $\hat{\epsilon} = \min(\epsilon_v, \epsilon_c)$ (clearly $\hat{\epsilon} \in (0, 1)$), it can be ensured that (3.52) is satisfied for all $j\omega$ in \mathbb{V} and $\mathbb{I} \setminus \mathbb{V}$ as well, i.e, we have,

$$\forall j\omega \in \mathbb{I}, P_{opt}(j\omega) - L_{opt}R(j\omega) + \hat{\epsilon}\Phi(j\omega) \leq 0. \quad (3.66)$$

So, by defining, $\hat{P}(j\omega) = P_{opt}(j\omega) + \hat{\epsilon}\Phi(j\omega)$, we have $\hat{P} \in \mathbb{P}_{2N}^+$ and,

$$\forall j\omega \in \mathbb{I}, \hat{P}(j\omega) \leq L_{opt}R(j\omega). \quad (3.67)$$

It satisfies, $\hat{P} \neq P$ since $\hat{\epsilon} \neq 0$. Also, since $\Phi(j\omega) \in \mathbb{P}_{2N}^+$ and $\hat{\epsilon} > 0$, we have,

$$\forall \omega \in \mathbb{R}, \hat{P}(j\omega) \geq P_{opt}(j\omega). \quad (3.68)$$

This completes the proof of the theorem. \square

3.3.4 Concavity of Pick Matrix and Non-Linear SDP

In this subsection, the implementation of problem \mathfrak{P}_C as a non-linear semi-definite programming problem (SDP) is discussed. The concavity of Pick matrix which appears in the characterisation of constraint set \mathbb{H}_R^N of problem \mathfrak{P}_C will be proved first. It plays an important role in solving problem \mathfrak{P}_C using non-linear semi-definite programming techniques. Let us recall problem \mathfrak{P}_C ,

$$\text{Find: } L_{opt} = \min_{P \in \mathbb{H}_R^N} \left(\max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)} \right).$$

According to proposition 3.3.2, the set \mathbb{H}_R^N is characterised as,

$$\mathbb{H}_R^N = \{P \in \mathbb{P}_{2N}^+ : \Delta(P) \succeq 0\}.$$

Theorem 3.3.9. *Let \mathbb{S}_M and \mathbb{S}_M^- denote the set of $(M \times M)$, Hermitian matrices and negative semi-definite Hermitian matrices respectively. The matrix valued function, $\mathcal{A} : \mathbb{P}_{2N}^+ \rightarrow \mathbb{S}_M$, defined as,*

$$\mathcal{A}(P) := -\Delta(P), \quad (3.69)$$

maps an element in \mathbb{H}_R^N to an element in \mathbb{S}_M^- , and is convex, i.e , $\forall \alpha \in (0, 1)$,

$$\alpha\mathcal{A}(P_1) + (1 - \alpha)\mathcal{A}(P_2) - \mathcal{A}(\alpha P_1 + (1 - \alpha)P_2) \succeq 0, \quad P_1, P_2 \in \mathbb{P}_{2N}^+. \quad (3.70)$$

Proof. The matrix valued function \mathcal{A} maps an element in the convex set \mathbb{H}_R^N to an element in the set \mathbb{S}_M^- since $\forall P \in \mathbb{H}_R^N$, we have $\Delta(P) \succeq 0$. In order to prove that the function \mathcal{A} is convex, let us consider the element (i, j) of $\Delta(P)$, where $1 \leq i, j \leq M$,

$$\Delta_{ij} = \frac{1 - \left(\frac{L_{22}(\xi_i)}{U_P(\xi_i)} \right) \left(\frac{L_{22}(\xi_j)}{U_P(\xi_j)} \right)}{\bar{\xi}_i + \xi_j} \quad (3.71)$$

Consider $\alpha \in (0, 1)$ and $P_1, P_2 \in \mathbb{P}_{2N}^+$. Let us denote $P = \alpha P_1 + (1 - \alpha)P_2$ and $L_{22}(\xi_i) = \gamma_i$. We need to prove that the matrix, $T^O \stackrel{\text{def}}{=} \Delta(P) - \alpha\Delta(P_1) - (1 - \alpha)\Delta(P_2)$ is positive semi-definite. Evaluating,

$$T_{ij}^O = \Delta_{ij}(P) - \alpha\Delta_{ij}(P_1) - (1 - \alpha)\Delta_{ij}(P_2),$$

using (3.71) yields,

$$T_{ij}^O = \frac{\overline{\gamma_i}\gamma_j}{(\xi_i + \xi_j)} \left(\frac{\alpha}{\overline{U_{P_1}(\xi_i)}U_{P_1}(\xi_j)} + \frac{(1-\alpha)}{\overline{U_{P_2}(\xi_i)}U_{P_2}(\xi_j)} - \frac{1}{\overline{U_P(\xi_i)}U_P(\xi_j)} \right)$$

Applying a change of basis to T^O by using an $(M \times M)$ diagonal transition matrix, C with $C_{ii} = U_P(\xi_i)$, $1 \leq i \leq M$, we obtain the base changed matrix, T^{OB} , where,

$$T_{ij}^{OB} = \frac{\overline{\gamma_i}\gamma_j}{(\xi_i + \xi_j)} \left(\frac{\alpha \overline{U_P(\xi_i)}U_P(\xi_j)}{\overline{U_{P_1}(\xi_i)}U_{P_1}(\xi_j)} + \frac{(1-\alpha) \overline{U_P(\xi_i)}U_P(\xi_j)}{\overline{U_{P_2}(\xi_i)}U_{P_2}(\xi_j)} - 1 \right). \quad (3.72)$$

Let us denote by $F \in H_2^\infty(\Pi^+)$, the following vector,

$$F = \left[\sqrt{\alpha} \frac{U_{P_1}}{U_P}, \sqrt{1-\alpha} \frac{U_{P_2}}{U_P} \right]^t. \quad (3.73)$$

Evaluating $F^*(s)F(s)$, on the imaginary axis, we have,

$$F^*F = \frac{\alpha|U_{P_1}|^2 + (1-\alpha)|U_{P_2}|^2}{|U_P|^2} \leq 1.$$

The inequality follows from the concavity of $|U_P|^2$ (Remark 3.2.10). So, by maximum modulus theorem we have $F \in \mathbb{B}_{2 \times 1}(\Pi^+)$. If we denote $x_i = \left[\sqrt{\alpha} \frac{\gamma_i U_P(\xi_i)}{U_{P_1}(\xi_i)}, \sqrt{1-\alpha} \frac{\gamma_i U_P(\xi_i)}{U_{P_2}(\xi_i)} \right]$ and $y_i = \gamma_i$ for all $i = 1, 2, \dots, M$, $F \in \mathbb{B}_{2 \times 1}$ introduced in (3.73) satisfies the following left-interpolation conditions,

$$x_i F(\xi_i) = y_i. \quad (3.74)$$

It follows from the Nevanlinna-Pick theorem for left interpolation problem (Theorem A.2.2 in Appendix) that $\Lambda(F) \succeq 0$, where,

$$\begin{aligned} \Lambda_{ij}(F) &= \frac{x_i \bar{x}_j^t - y_i \bar{y}_j}{\xi_i + \bar{\xi}_j} \\ &= \frac{\gamma_i \overline{\gamma_j}}{(\xi_i + \bar{\xi}_j)} \left(\frac{\alpha \overline{U_P(\xi_i)}U_P(\xi_j)}{\overline{U_{P_1}(\xi_i)}U_{P_1}(\xi_j)} + \frac{(1-\alpha) \overline{U_P(\xi_i)}U_P(\xi_j)}{\overline{U_{P_2}(\xi_i)}U_{P_2}(\xi_j)} - 1 \right) \\ &= T_{ji}^{OB}. \end{aligned} \quad (3.75)$$

The last equality follows from equation (3.72). So, we have $(T^{OB})^t \succeq 0$, implying $T^{OB} \succeq 0$, and hence $T^O \succeq 0$, that is,

$$\Delta(P) - \alpha \Delta(P_1) - (1-\alpha) \Delta(P_2) \succeq 0,$$

showing that $\Delta(P)$ is concave in P . This proves the convexity of \mathcal{A} ,

$$\alpha \mathcal{A}(P_1) + (1-\alpha) \mathcal{A}(P_2) - \mathcal{A}(P) \succeq 0.$$

□

Now, the fact that $\mathcal{A} = -\Delta$ is convex can be utilized to cast problem \mathfrak{P}_C in the form of a semi-definite programming problem. Let us denote the cost function as $\Psi : \mathbb{P}_{2N}^+ \rightarrow \mathbb{R}$, where, $\Psi(P) = \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)}$. Then, we have the following minimisation problem,

Problem. *SDP*

$$\begin{aligned} \text{Find: } & \min_{P \in \mathbb{P}_{2N}^+} \Psi(P) \\ \text{subject to: } & \mathcal{A}(P) \preceq 0. \end{aligned}$$

We have put problem \mathfrak{P}_C in the standard form of a semi-definite programming problem, problem *SDP* for which algorithms exist to solve the problem. Non-linear semidefinite programming techniques can be used to solve the problem *SDP*. Polynomials in \mathbb{P}_{2N}^+ can be characterized by means of $(N+1) \times (N+1)$ positive semi-definite matrices, denoted by \mathbb{S}_{N+1}^+ (Theorem 2.5, [30]), i.e. the polynomial $P \in \mathbb{P}_{2N}$ is positive iff there exist a Gram matrix Θ_P parametrising P that is positive semi-definite.

$$P \in \mathbb{P}_{2N}^+ \iff \exists \Theta_P \in \mathbb{S}_{N+1}^+ \mid v(s)\Theta_P v^t(s) = P(s), \quad s \in \mathbb{C}, \quad (3.76)$$

where $v(s) = [s^N, s^{N-1}, \dots, 1]$ is the vector of standard basis of set of polynomials \mathbb{P}_N . The admissibility of $P \in \mathbb{P}_{2N}^+$ is ensured by the non-linear constraint $\mathcal{A}(P) \preceq 0$. Positivity and admissibility constraints are implemented using the standard logarithmic barrier and a special penalty function used in section 8.2, [89] respectively. The reader can refer to chapter 7 and 8, [64] for a detailed description of the numerical implementation of the *SDP* problem and the final product MATLAB toolbox named PUMA in [62].

3.3.5 Critical Point Equation

In this section, we will discuss about the dual problem associated to problem \mathfrak{P}_C and thereby the critical point equation for problem \mathfrak{P}_C . From Theorem 3.3.9 in the previous section, we have, $\Delta : \mathbb{P}_{2N}^+ \rightarrow \mathbb{S}_M$ to be concave in P . We have the following problem for which we are interested to write down the critical point equation,

Problem. \mathfrak{P}_C

$$\begin{aligned} \text{Find: } & l_{opt} = \min_{P \in \mathbb{P}_{2N}} \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)} \\ \text{subject to: } & P(j\omega) \geq 0, \quad \omega \in \mathbb{R} \\ & \Delta(P) \succeq 0. \end{aligned}$$

We remind the reader that in problem \mathfrak{P}_C , the polynomial $R \in \mathbb{P}_{2N}^+$ is fixed, it doesn't vanish in \mathbb{I} and $\Delta(P)$ represent the Pick matrix defined in (3.30). $\Delta(P)$ as proved in the previous subsections is a continuous concave function in P . The first step is to restate problem \mathfrak{P}_C by introducing an extra variable, $\Gamma \in \mathbb{R}$, which becomes the criterium to be minimized. It is defined by the inequality $\Gamma \geq \frac{P(j\omega)}{R(j\omega)}, \quad \forall j\omega \in \mathbb{I}$. Now, we have

Problem. \mathfrak{P}_C

$$\begin{aligned} \text{Find: } l_{opt} &= \min_{(P, \Gamma) \in \mathbb{P}_{2N} \times \mathbb{R}} \Gamma \\ \text{subject to: } \Gamma - \frac{P(j\omega)}{R(j\omega)} &\geq 0, \quad j\omega \in \mathbb{I} \\ P(j\omega) &\geq 0, \quad \omega \in \mathbb{R} \\ \Delta(P) &\succeq 0. \end{aligned}$$

It should be noted that for $P \in \mathbb{P}_{2N}$, we can always estimate $\Gamma \in \mathbb{R}$ as $\Gamma = \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)}$. The main result of this subsection is the following theorem which states that the optimal solution to problem \mathfrak{P}_C , (P_{opt}, l_{opt}) can be obtained as the solution to an unconstrained optimization problem as well.

Theorem 3.3.10. *There exists non-negative real numbers $\tilde{\nu}_1, \tilde{\nu}_2, \dots, \tilde{\nu}_{k_1}, \tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{k_2}$, $k_1 \leq 2N, k_2 \leq 2N$ where $\sum_{i=1}^{k_1} \tilde{\nu}_i = 1$, matrix $\tilde{U} \in \mathbb{S}_M^+$, points $j\omega_1, j\omega_2, \dots, j\omega_{k_1}$ in \mathbb{I} and points $j\gamma_1, j\gamma_2, \dots, j\gamma_{k_2}$ in $j\mathbb{R}$ such that the solution to problem \mathfrak{P}_C , $P_{opt} \in \mathbb{P}_{2N}^+$ satisfies*

$$\frac{P_{opt}(j\omega_i)}{R(j\omega_i)} = l_{opt}, \quad i = 1, 2, \dots, k_1$$

and it coincides with the solution to the following unconstrained optimization problem

$$\min_{P \in \mathbb{P}_{2N}} \left(\sum_{i=1}^{k_1} \frac{P(j\omega_i)}{R(j\omega_i)} \tilde{\nu}_i - \sum_{i=1}^{k_2} P(j\gamma_i) \tilde{\lambda}_i + \langle \mathcal{A}(P), \tilde{U} \rangle \right), \quad (3.77)$$

where

$$\langle \mathcal{A}(P), \tilde{U} \rangle = \text{tr}(\overline{\mathcal{A}(P)}^t \tilde{U}), \quad \mathcal{A}(P) = \begin{cases} -\Delta(P), & P \in \mathbb{P}_{2N}^+, \\ +\infty I_M, & \text{otherwise,} \end{cases}$$

“tr” represent the trace, I_M denote the identity matrix of size M . Moreover, if there are no common zeros for P_{opt} and R on the imaginary axis and if the coefficients of P_{opt} are denoted by $[p_{2N}, p_{2N-1}, \dots, p_1, p_0]$ in the standard basis $\{s^{2N}, s^{2N-1}, \dots, s, 1\}$, then for $m = 0, 1, \dots, 2N$,

$$\sum_{i=1}^{k_1} \frac{1}{R(j\omega_i)} \frac{\partial P_{opt}(j\omega_i)}{\partial p_m} \tilde{\nu}_i - \sum_{i=1}^{k_2} \frac{\partial P_{opt}(j\gamma_i)}{\partial p_m} \tilde{\lambda}_i + \left\langle \frac{\partial \mathcal{A}(P_{opt})}{\partial p_m}, \tilde{U} \right\rangle = 0. \quad (3.78)$$

The case when P_{opt} and R have some common zeros on the imaginary axis, the Pick matrix $\Delta(P_{opt})$ is non-differentiable and this special case is discussed separately at the end of this subsection. A proof of the above theorem is provided in the following main steps : (i) writing down the Lagrange dual problem of problem \mathfrak{P}_C and verifying weak duality, (ii) proving strong duality under a constraint qualification and hence verifying complementary slackness conditions, (iii) complementary slackness conditions providing the optimality condition. So, the proof of theorem 3.3.10 is divided into three subsections

and a brief introduction and prerequisite to each section is provided at the beginning of corresponding sections.

Theorem 3.3.10 is essentially stating a typical *Karush-Kuhn-Tucker* (KKT) condition for problem \mathfrak{P}_C . The main idea of the proof of theorem 3.3.10 is to show that strong duality holds in the convex optimisation problem \mathfrak{P}_C (i.e the primal and dual optimal are equal, leading to an optimal duality gap equal to zero). It will help to prove that the solution to problem \mathfrak{P}_C coincides with the solution to an unconstrained optimization problem and thus provide the KKT conditions associated with problem \mathfrak{P}_C . This is precisely presented in the remaining part of this subsection.

(i). Dual Problem and Weak Duality

In order to introduce the Lagrange dual problem of problem \mathfrak{P}_C , let us introduce positive linear functionals, Riesz representation theorems for bounded linear functionals in the space of continuous functions and trace operator for \mathbb{S}_M , the set of $(M \times M)$ Hermitian matrices.

Definition 3.3.11. (*Positive Linear Functional*). A positive linear functional on a real vector space V is a mapping Φ of V to \mathbb{R} such that

$$\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g), \quad f, g \in V, \quad \alpha, \beta \in \mathbb{R}$$

and also satisfying

$$\Phi(f) \geq 0 \text{ whenever } f \geq 0.$$

Definition 3.3.12. (*Bounded Linear Functional*). Let $(X, ||\cdot||)$ be a normed vector space. A bounded linear functional on X is a linear functional Φ such that

$$||\Phi|| \stackrel{\text{def}}{=} \sup\{|\Phi(x)| : x \in X, ||x|| \leq 1\} < \infty.$$

We remind the reader that a linear functional is bounded if and only if it is continuous on X (Theorem 5.4, [83]). The reader can refer to chapter 2 and 6 in [83] or chapter 14, [6] for a detailed literature about the Riesz representation theorems, an integral representation of bounded linear functionals.

Theorem 3.3.13. Let X be a compact Hausdorff space and $C(X)$ the vector space of continuous real-valued functions on X . Then to each bounded linear functional $\Phi : C(X) \rightarrow \mathbb{R}$, there corresponds a unique finite signed Baire measure ν on X such that

$$\forall f \in C(X), \quad \Phi(f) = \int_X f d\nu. \quad (3.79)$$

Theorem 3.3.14. Let X be a locally compact Hausdorff space, $C_0(X)$ the space of continuous real-valued functions on X vanishing at infinity and $\Phi : C_0(X) \rightarrow \mathbb{R}$ a bounded linear functional. Then there exists a unique bounded regular signed Borel measure λ which represents Φ in the sense that

$$\forall f \in C_0(X), \quad \Phi(f) = \int_X f d\lambda. \quad (3.80)$$

A general version of theorem 3.3.13 and 3.3.14 and its proof can be found in theorem 2.14 and theorem 6.19 respectively in [83]. For theorem 3.3.13, see chapter 13, theorem 25, [82]. In the case of Hermitian matrices, say for $A, B \in \mathbb{S}_M$, we define the inner product,

$$\langle A, B \rangle = \text{tr}(\bar{A}^t B), \quad (3.81)$$

where “tr” represent the trace. Let us put the primal problem \mathfrak{P}_C in the standard form,

Problem. \mathfrak{P}_C

$$\begin{aligned} \text{Find: } l_{opt} &= \min_{(P, \Gamma) \in \mathbb{P}_{2N} \times \mathbb{R}} \Gamma \\ \text{subject to: } & \frac{P(j\omega)}{R(j\omega)} - \Gamma \leq 0, \quad j\omega \in \mathbb{I} \\ & -P(j\omega) \leq 0, \quad \omega \in \mathbb{R} \\ & -\Delta(P) \preceq 0, \end{aligned}$$

where, $-\Delta(P)$ is continuous and convex in P . It should be noted that the fixed polynomial R doesn't vanish in \mathbb{I} and since polynomials are continuous functions and for polynomial p of degree less than or equal to n , $\frac{p(s)}{1+|s|^{n+1}}$ is a continuous function vanishing at infinity, theorem 3.3.13 and theorem 3.3.14 implies that there exist integral representation of linear forms associated to the first two constraint functions in problem \mathfrak{P}_C . Also, the inner product defined in (3.81) can be used to define the linear form associated to the matrix constraint function. Now, for measures ν on \mathbb{I} , λ on $j\mathbb{R}$ and matrix $U \in \mathbb{S}_M$, we define the Lagrangian dual function,

$$\inf_{(P, \Gamma) \in \mathbb{P}_{2N} \times \mathbb{R}} \left(\Gamma + \int_{\mathbb{I}} \left(\frac{P}{R} - \Gamma \right) d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle \right), \quad (3.82)$$

where

$$\mathcal{A}(P) \stackrel{\text{def}}{=} \begin{cases} -\Delta(P), & P \in \mathbb{P}_{2N}^+, \\ +\infty I_M, & \text{otherwise,} \end{cases} \quad (3.83)$$

I_M denote the identity matrix of size M . The definition of $\mathcal{A}(P)$ ensures that in the dual function in (3.82) the subset of polynomials in \mathbb{P}_{2N} which are not positive is rendered not feasible. Since the pick matrix, $\Delta(P)$ defined for $P \in \mathbb{P}_{2N}^+$ is concave in P (theorem 3.3.9), we have the matrix $\mathcal{A}(P)$ defined in (3.83) convex in $P \in \mathbb{P}_{2N}$, that is $\forall \alpha \in (0, 1)$,

$$\alpha \mathcal{A}(P_1) + (1 - \alpha) \mathcal{A}(P_2) - \mathcal{A}(\alpha P_1 + (1 - \alpha) P_2) \succeq 0, \quad P_1, P_2 \in \mathbb{P}_{2N}. \quad (3.84)$$

If P_1 and P_2 are both in \mathbb{P}_{2N}^+ , (3.84) follows from the concavity of pick matrix $\Delta(P)$ and in the case when atleast one of the polynomials is not positive, it follows trivially as well since we have atleast one of $\mathcal{A}(P_1)$ or $\mathcal{A}(P_2)$ equal to $+\infty I_M$. It should be noted that the objective function in (3.82) has the equivalent form

$$\Gamma \left(1 - \int_{\mathbb{I}} d\nu \right) + \int_{\mathbb{I}} \frac{P}{R} d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle$$

and if $\int_{\mathbb{I}} d\nu \neq 1$, the objective function is not bounded from below. So, we will only consider ν satisfying, $\nu(\mathbb{I}) = \int_{\mathbb{I}} d\nu = 1$ and we have the Lagrangian associated with problem \mathfrak{P}_C as

$$L(P, \nu, \lambda, U) = \int_{\mathbb{I}} \frac{P}{R} d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle. \quad (3.85)$$

So, we have the Lagrangian dual function,

$$g(\nu, \lambda, U) = \min_{P \in \mathbb{P}_{2N}} L(P, \nu, \lambda, U) \quad (3.86)$$

$$= \min_{P \in \mathbb{P}_{2N}} \left(\int_{\mathbb{I}} \frac{P}{R} d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle \right). \quad (3.87)$$

This leads to the following optimization problem, which is the Lagrange dual problem associated with the problem \mathfrak{P}_C ,

Problem. \mathfrak{P}_d

$$\begin{aligned} \text{Find: } d_{opt} &= \max g(\nu, \lambda, U) \\ \text{subject to: } \nu(\mathbb{I}) &= 1, \nu \geq 0, \lambda \geq 0, U \succeq 0 \end{aligned}$$

Let us suppose that $\nu(\mathbb{I}) = 1, \nu \geq 0, \lambda \geq 0, U \succeq 0$. Then, clearly, for $(P, \Gamma) \in \mathbb{P}_{2N} \times \mathbb{R}$, if $\frac{P}{R}|_{\mathbb{I}} - \Gamma \leq 0, -P|_{j\mathbb{R}} \leq 0$ and $\mathcal{A}(P) \preceq 0$ we have,

$$\begin{aligned} \Gamma &\geq \Gamma + \int_{\mathbb{I}} \left(\frac{P}{R} - \Gamma \right) d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle \\ &= \int_{\mathbb{I}} \frac{P}{R} d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle. \end{aligned}$$

This follows from the facts that $(\Gamma - \frac{P}{R})$ and P being non-negative on \mathbb{I} and $j\mathbb{R}$ respectively, using Riesz theorem, the linear forms represented using positive measures ν and λ as $\int_{\mathbb{I}} (\Gamma - \frac{P}{R}) d\nu$ and $\int_{j\mathbb{R}} (\frac{P}{1 + |\omega|^{2N+1}}) d\lambda$ are non-negative and also for matrices $A, B \in \mathbb{S}_M^+$, we have, $\langle A, B \rangle = \text{tr}(A^t B) \geq 0$. So, we have,

$$\begin{aligned} l_{opt} &= \min \left\{ \Gamma \mid \mathcal{A}(P) \preceq 0, -P|_{j\mathbb{R}} \leq 0, \frac{P}{R}|_{\mathbb{I}} - \Gamma \leq 0 \right\} \\ &\geq \min \left\{ \int_{\mathbb{I}} \frac{P}{R} d\nu + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\lambda + \langle \mathcal{A}(P), U \rangle \mid \frac{P}{R}|_{\mathbb{I}} - \Gamma \leq 0, \right. \\ &\quad \left. -P|_{j\mathbb{R}} \leq 0, \mathcal{A}(P) \preceq 0 \right\} \\ &= g(\nu, \lambda, U). \end{aligned}$$

This implies that we have weak duality, that is, $d_{opt} \leq l_{opt}$. It follows since $d_{opt} = g(\tilde{\nu}, \tilde{\lambda}, \tilde{U})$ for some $\tilde{\nu}(\mathbb{I}) = 1, \tilde{\nu} \geq 0, \tilde{\lambda} \geq 0, \tilde{U} \succeq 0$ which is less than or equal to l_{opt} as shown above.

(ii). Strong Duality and Complementary Slackness Condition

Now, in order to prove strong duality, we will follow an epigraph variation approach similar to the one presented in section 5.3, [14]. This method is based on the idea of defining two non-empty disjoint convex sets and making use of Hahn Banach separation theorem to find Lagrange multipliers which will be proved to be non-negative. Slater's constraint qualification will play the important role of ensuring strong duality. We suppose that we have the Slater condition,

$$\exists (P_0, \Gamma_0) \in \mathbb{P}_{2N} \times \mathbb{R}, \text{ satisfying, } \frac{P_0}{R}|_{\mathbb{I}} - \Gamma_0 < 0, \quad -P_0|_{j\mathbb{R}} < 0, \quad \mathcal{A}(P_0) \prec 0. \quad (3.88)$$

It should be noted that in problem \mathfrak{P}_C , the existence of such a (P_0, Γ_0) is guaranteed, since for example, if we fix $P_0 \in \mathbb{P}_{2N}^+$, $P_0 = P_L + 1$, where $P_L = p_L p_L^*$ is the reflection polynomial associated with the given load and Γ_0 satisfying $\Gamma_0 > \max_{j\omega \in \mathbb{I}} \frac{P_0(j\omega)}{R(j\omega)}$, we have the conditions in (3.88) satisfied. This follows, since for this P_0 , we have,

$$|U_{P_0}|^2 = \frac{P_0}{P_0 + R} = \frac{P_L + 1}{P_L + 1 + R} = \frac{1}{1 + \frac{R}{P_L + 1}} \text{ on } j\omega$$

and so, $|L_{22}|^2 = \frac{P_L}{P_L + R} = \frac{1}{1 + \frac{R}{P_L}} < |U_{P_0}|^2$ on $j\omega$. This implies that we have P_0 in the interior of set \mathbb{H}_R^N (the same idea as in we prove the set \mathbb{H}_R^N is non-empty in theorem 3.2.9). So we have the pick matrix $\Delta(P_0) \succ 0$ implying $\mathcal{A}(P_0) \prec 0$. Before getting to the details of this section, we introduce the separating hyperplane theorem by Eidelheit (Theorem 1.13, [98]) which will be used in the proof,

Theorem 3.3.15. (*Separating Hyperplane Theorem*). *Let A and B be two non-empty convex subsets of a topological vector space X over \mathbb{R} and X^* denote the topological dual of X . If $\text{int}(A) \neq \emptyset$ and $\text{int}(A) \cap B = \emptyset$, where 'int' represents the interior, then there exist $\Phi \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that*

$$\forall a \in A, \forall b \in B : \Phi(a) \leq \alpha \leq \Phi(b).$$

To begin the main proof of this section, we define the set \mathcal{B}_1 which can be geometrically interpreted as a sort of epigraph of the set of values taken on by the objective and constraint functions of problem \mathfrak{P}_C ,

$$\mathcal{B}_1 = \left\{ (l, f, g, V) \in \mathbb{R} \times C(\mathbb{I}) \times C_0(j\mathbb{R}) \times \mathbb{S}_M \mid \exists (P, \Gamma) \in \mathbb{P}_{2N} \times \mathbb{R} : \right. \\ \left. \Gamma \leq l, \frac{P}{R} - \Gamma \leq f, \frac{-P}{1 + |\omega|^{2N+1}} \leq g, \mathcal{A}(P) \preceq V \right\}.$$

It can be easily verified that \mathcal{B}_1 is a convex set since the underlying problem \mathfrak{P}_C is convex. It should also be noted that the interior of \mathcal{B}_1 is non-empty since for example, for any $l > \Gamma_0$, $(l, 0, 0, 0)$ is in the interior of \mathcal{B}_1 . We define the second non-empty convex set \mathcal{B}_2 , as follows,

$$\mathcal{B}_2 = \{(l, 0, 0, 0) \in \mathbb{R} \times C(\mathbb{I}) \times C_0(j\mathbb{R}) \times \mathbb{S}_M \mid l < l_{opt}\}. \quad (3.89)$$

We claim $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Let us suppose that the claim is not true, say $(l, f, g, V) \in \mathcal{B}_1 \cap \mathcal{B}_2$. Since $(l, f, g, V) \in \mathcal{B}_2$, we have,

$$l < l_{opt}, \quad f = 0 \in C(\mathbb{I}), \quad g = 0 \in C_0(j\mathbb{R}), \quad V = 0 \in \mathbb{S}_M. \quad (3.90)$$

Now, since $(l, 0, 0, 0) \in \mathcal{B}_1$, using definition of \mathcal{B}_1 , $\exists(P, \Gamma) \in \mathbb{P}_{2N} \times \mathbb{R}$, satisfying,

$$\Gamma \leq l < l_{opt}, \quad -P|_{j\mathbb{R}} \leq 0, \quad \frac{P}{R}|_{\mathbb{I}} - \Gamma \leq 0, \quad \mathcal{A}(P) \preceq 0. \quad (3.91)$$

This contradicts the optimality of P_{opt} at which l_{opt} is attained. So, we have $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ as claimed. Now, using the separating hyperplane theorem for disjoint convex sets \mathcal{B}_1 and \mathcal{B}_2 (Theorem 3.3.15), there exist $\hat{\mu} \in \mathbb{R}$, measures $\hat{\nu}, \hat{\lambda}$ and matrix $\hat{U} \in \mathbb{S}_M$ satisfying $(\hat{\mu}, \hat{\nu}, \hat{\lambda}, \hat{U}) \neq (0, 0, 0, 0)$ (all cannot be zero simultaneously) and $\alpha \in \mathbb{R}$, such that,

$$(l, f, g, V) \in \mathcal{B}_1 \implies \hat{\mu}l + \int_{\mathbb{I}} f d\hat{\nu} + \int_{j\mathbb{R}} g d\hat{\lambda} + \langle V, \hat{U} \rangle \geq \alpha, \quad (3.92)$$

$$(l, f, g, V) \in \mathcal{B}_2 \implies \hat{\mu}l + \int_{\mathbb{I}} f d\hat{\nu} + \int_{j\mathbb{R}} g d\hat{\lambda} + \langle V, \hat{U} \rangle \leq \alpha. \quad (3.93)$$

(3.93) implies that $\hat{\mu}l \leq \alpha$ for all $l < l_{opt}$. This together with the fact that $l_{opt} < \infty$ in problem \mathfrak{P}_C provides

$$\hat{\mu}l_{opt} \leq \alpha. \quad (3.94)$$

Now, from (3.92) and (3.94), it follows that

$$\forall (l, f, g, V) \in \mathcal{B}_1 : \hat{\mu}l + \int_{\mathbb{I}} f d\hat{\nu} + \int_{j\mathbb{R}} g d\hat{\lambda} + \langle V, \hat{U} \rangle \geq \hat{\mu}l_{opt}. \quad (3.95)$$

Now, we make certain claims regarding $\hat{\mu}, \hat{\nu}, \hat{\lambda}$ and \hat{U} .

- Claim 1 : $\hat{\mu} \in \mathbb{R}$ satisfies, $\hat{\mu} \geq 0$. Let us suppose $\hat{\mu} < 0$. Fix any $l \in \mathbb{R}$, satisfying, $l > l_{opt}$. It should be noted that $(l, 0, 0, 0) \in \mathcal{B}_1$. So, from (3.95), we have, $\hat{\mu}l \geq \hat{\mu}l_{opt}$, which implies, $l \leq l_{opt}$ (since we assume $\hat{\mu} < 0$). This contradicts $l > l_{opt}$ and hence, we have, $\hat{\mu} \geq 0$ as claimed.
- Claim 2 : Measure $\hat{\nu}$ satisfies, $\hat{\nu}|_{\mathbb{I}} \geq 0$. Using Riesz theorem, in order to prove the claim it is sufficient to prove that for all $f \in C(\mathbb{I})$ satisfying $f \geq 0$, we have, $\int_{\mathbb{I}} f d\hat{\nu} \geq 0$. If, $f \geq 0$, we have, $(l_{opt}, f, 0, 0) \in \mathcal{B}_1$, and so using (3.95), we have, $\int_{\mathbb{I}} f d\hat{\nu} \geq 0$. Thus, $\hat{\nu}|_{\mathbb{I}} \geq 0$.
- Claim 3 : Measure $\hat{\lambda}$ satisfies, $\hat{\lambda}|_{j\mathbb{R}} \geq 0$. It follows similarly as Claim 2.
- Claim 4 : Matrix $\hat{U} \in \mathbb{S}_M$, satisfies, $\hat{U} \succeq 0$. Using the fact that the positive semi-definite cone \mathbb{S}_M^+ is self-dual (Example 2.24, [14]), in order to prove the claim, it is sufficient to prove, for all $V \in \mathbb{S}_M^+$, we have $\langle V, \hat{U} \rangle \geq 0$. If, $V \succeq 0$, we have, $(l_{opt}, 0, 0, V) \in \mathcal{B}_1$, and so using (3.95), we have, $\langle V, \hat{U} \rangle \geq 0$. Thus, $\hat{U} \succeq 0$.

It should be noted that for any $P \in \mathbb{P}_{2N}$, denoting $\Gamma = \max_{j\omega \in \mathbb{I}} \frac{P(j\omega)}{R(j\omega)}$, we have, $(\Gamma, \frac{P}{R} - \Gamma, \frac{-P}{1+|\omega|^{2N+1}}, \mathcal{A}(P)) \in \mathcal{B}_1$ and so (3.95) implies

$$\hat{\mu}\Gamma + \int_{\mathbb{I}} \left(\frac{P}{R} - \Gamma \right) d\hat{\nu} + \int_{j\mathbb{R}} \frac{-P}{1+|\omega|^{2N+1}} d\hat{\lambda} + \langle \mathcal{A}(P), \hat{U} \rangle \geq \hat{\mu}l_{opt}. \quad (3.96)$$

Now, let us suppose the non-negative real $\hat{\mu}$ is strictly positive, $\hat{\mu} > 0$. The positive measure $\tilde{\nu}$ on \mathbb{I} can be normalized to satisfy $\hat{\nu}(\mathbb{I}) = \hat{\mu}$ and for simplicity, we call this measure again by $\hat{\nu}$. So, it follows from (3.96),

$$\int_{\mathbb{I}} \frac{P}{R} d\hat{\nu} + \int_{j\mathbb{R}} \frac{-P}{1+|\omega|^{2N+1}} d\hat{\lambda} + \langle \mathcal{A}(P), \hat{U} \rangle \geq \hat{\mu}l_{opt}. \quad (3.97)$$

Dividing (3.97) by $\hat{\mu}$, we have the Lagrangian,

$$L\left(P, \frac{\hat{\nu}}{\hat{\mu}}, \frac{\hat{\lambda}}{\hat{\mu}}, \frac{\hat{U}}{\hat{\mu}}\right) \geq l_{opt}, \quad \forall P \in \mathbb{P}_{2N}.$$

Minimizing the Lagrangian over $P \in \mathbb{P}_{2N}$, we obtain,

$$g(\tilde{\nu}, \tilde{\lambda}, \tilde{U}) \geq l_{opt}, \quad \text{where } \tilde{\nu} = \frac{\hat{\nu}}{\hat{\mu}}, \quad \tilde{\lambda} = \frac{\hat{\lambda}}{\hat{\mu}}, \quad \tilde{U} = \frac{\hat{U}}{\hat{\mu}}. \quad (3.98)$$

The results of claims 2,3 and 4, the fact that $\hat{\nu}(\mathbb{I}) = \hat{\mu}$ and the assumption that $\hat{\mu} > 0$ implies $\tilde{\nu}(\mathbb{I}) = 1$, $\tilde{\nu} \geq 0$, $\tilde{\lambda} \geq 0$ and $\tilde{U} \succeq 0$. By weak duality, we also have $g(\tilde{\nu}, \tilde{\lambda}, \tilde{U}) \leq l_{opt}$. So, $g(\tilde{\nu}, \tilde{\lambda}, \tilde{U}) = l_{opt}$, when $\hat{\mu} > 0$.

Next, we claim : $\hat{\mu} \neq 0$. Let us suppose $\hat{\mu} = 0$. From (3.96),

$$\int_{\mathbb{I}} \left(\frac{P}{R} - \Gamma \right) d\hat{\nu} + \int_{j\mathbb{R}} \frac{-P}{1+|\omega|^{2N+1}} d\hat{\lambda} + \langle \mathcal{A}(P), \hat{U} \rangle \geq 0.$$

Applying this to (P_0, Γ_0) that satisfies the Slater condition, $\frac{P_0}{R}|_{\mathbb{I}} - \Gamma_0 < 0$, $-P_0|_{j\mathbb{R}} < 0$, $\mathcal{A}(P_0) \prec 0$, we have,

$$\int_{\mathbb{I}} \left(\frac{P_0}{R} - \Gamma_0 \right) d\hat{\nu} + \int_{j\mathbb{R}} \frac{-P_0}{1+|\omega|^{2N+1}} d\hat{\lambda} + \langle \mathcal{A}(P_0), \hat{U} \rangle \geq 0. \quad (3.99)$$

From the results of claims 2, 3 and 4, $\hat{\nu} \geq 0$, $\hat{\lambda} \geq 0$, $\hat{U} \succeq 0$ and so we also have,

$$\int_{\mathbb{I}} \left(\frac{P_0}{R} - \Gamma_0 \right) d\hat{\nu} \leq 0, \quad \int_{j\mathbb{R}} \frac{-P_0}{1+|\omega|^{2N+1}} d\hat{\lambda} \leq 0, \quad \langle \mathcal{A}(P_0), \hat{U} \rangle \leq 0. \quad (3.100)$$

The inequalities in (3.99) and (3.100) leads us to the conclusion that $\hat{\nu}$, $\hat{\lambda}$ and \hat{U} has to be zero. This contradicts the fact that $(\hat{\mu}, \hat{\nu}, \hat{\lambda}, \hat{U}) \neq (0, 0, 0, 0)$ which was announced by the hyperplane separating theorem. Thus, $\hat{\mu} \neq 0$ as claimed.

So, we have P_{opt} and $(\tilde{\nu}, \tilde{\lambda}, \tilde{U})$ to be the points where primal and dual optimal are attained with zero duality gap, i.e,

$$l_{opt} = d_{opt} = g(\tilde{\nu}, \tilde{\lambda}, \tilde{U}) \quad (3.101)$$

$$= \min_{P \in \mathbb{P}_{2N}} \left(\int_{\mathbb{I}} \frac{P}{R} d\tilde{\nu} + \int_{j\mathbb{R}} \left(\frac{-P}{1 + |\omega|^{2N+1}} \right) d\tilde{\lambda} + \langle \mathcal{A}(P), \tilde{U} \rangle \right) \quad (3.102)$$

$$\leq l_{opt} + \int_{j\mathbb{R}} \left(\frac{-P_{opt}}{1 + |\omega|^{2N+1}} \right) d\tilde{\lambda} + \langle \mathcal{A}(P_{opt}), \tilde{U} \rangle \quad (3.103)$$

$$\leq l_{opt}. \quad (3.104)$$

(3.103) follows since the minimum of Lagrangian over $P \in \mathbb{P}_{2N}$ is less than or equal to its value at P_{opt} and the facts that for all $j\omega \in \mathbb{I}$, $\frac{P_{opt}(j\omega)}{R(j\omega)} \leq l_{opt}$ and $\tilde{\nu}(\mathbb{I}) = 1$. (3.104) follows since $-P_{opt}|_{j\mathbb{R}} \leq 0$, $\tilde{\lambda} \geq 0$, $\mathcal{A}(P_{opt}) \preceq 0$, $\tilde{U} \succeq 0$. This implies that we have equality throughout and hence we have the complementary slackness condition, i.e there exists positive measures $\tilde{\nu}, \tilde{\lambda}$ on \mathbb{I} and $j\mathbb{R}$ respectively, $\tilde{U} \in \mathbb{S}_M^+$, satisfying,

$$\int_{\mathbb{I}} \frac{P_{opt}}{R} d\tilde{\nu} = l_{opt}, \quad (3.105)$$

$$\int_{j\mathbb{R}} \left(\frac{-P_{opt}}{1 + |\omega|^{2N+1}} \right) d\tilde{\lambda} = 0, \quad (3.106)$$

$$\langle \mathcal{A}(P_{opt}), \tilde{U} \rangle = 0. \quad (3.107)$$

(iii). Optimality Condition

The remaining section explains the fact that integrals in equation (3.105) and (3.106) can be replaced by finite sums. Let us denote by $j\omega_1, j\omega_2, \dots, j\omega_{k_1}$, the zeros of $P_{opt} - l_{opt}R$ in \mathbb{I} . Assuming that we are not at the special case when $P_{opt} \equiv l_{opt}R$, we have $k_1 \leq 2N$ (this special case is included in the discussion to follow in the next page : ‘Simplification at the optimum’). Then (3.105) clearly implies that the positive measure $\tilde{\nu}$ can have positive mass only at these points, $\tilde{\nu}(j\omega_i) = \tilde{\nu}_i$ and we have $\sum_{i=1}^{k_1} \tilde{\nu}_i = 1$. So, we have

$$\frac{P_{opt}(j\omega_i)}{R(j\omega_i)} = l_{opt}, \quad i = 1, 2, \dots, k_1$$

and equation (3.105) can be replaced as follows,

$$\sum_{i=1}^{k_1} \frac{P_{opt}(j\omega_i)}{R(j\omega_i)} \tilde{\nu}_i = l_{opt}, \quad k_1 \leq 2N. \quad (3.108)$$

If the optimal $P_{opt} > 0$ on $j\mathbb{R}$, i.e, the constraint, $P \geq 0$ is not active at the optimum, then (3.106) clearly implies the measure $\tilde{\lambda} = 0$. The case when P_{opt} is identically equal to zero cannot occur since $0 \notin \mathbb{H}_R^N$ as discussed immediately after definition 3.2.6 of \mathbb{H}_R^N . So, if we say P_{opt} has k_2 zeros on $j\mathbb{R}$ ($k_2 \leq 2N$), denote them as $j\gamma_1, j\gamma_2, \dots, j\gamma_{k_2}$, (3.106) clearly implies that the positive measure $\tilde{\lambda}$ can have positive mass only at these points.

After multiplying the positive weight $\frac{1}{1+|\omega|^{2N+1}}$ to the measure $\tilde{\lambda}$, we name it again as $\tilde{\lambda}$ and denote $\tilde{\lambda}(j\gamma_i) = \tilde{\lambda}_i$. So, equation (3.106) can be replaced as follows,

$$\sum_{i=1}^{k_2} -P_{opt}(j\gamma_i)\tilde{\lambda}_i = 0, \quad k_2 \leq 2N. \quad (3.109)$$

Thus, to sum up, it follows from (3.107), (3.108) and (3.109) that the solution to the following unconstrained optimization problem

$$\min_{P \in \mathbb{P}_{2N}} \left(\sum_{i=1}^{k_1} \frac{P(j\omega_i)}{R(j\omega_i)} \tilde{\nu}_i - \sum_{i=1}^{k_2} P(j\gamma_i) \tilde{\lambda}_i + \langle \mathcal{A}(P), \tilde{U} \rangle \right), \quad (3.110)$$

coincides with the solution to problem \mathfrak{P}_C . We have P_{opt} and $(\tilde{\nu}, \tilde{\lambda}, \tilde{U})$ to be the primal and dual optimal with zero duality gap. Since P_{opt} minimizes $L(P, \tilde{\nu}, \tilde{\lambda}, \tilde{U})$ over $P \in \mathbb{P}_{2N}$, if we assume that P_{opt} has no common zeros with R on the imaginary axis, the Lagrangian is differentiable and its gradient must vanish at P_{opt} , that is,

$$\sum_{i=1}^{k_1} \frac{1}{R(j\omega_i)} \frac{\partial P_{opt}(j\omega_i)}{\partial p_m} \tilde{\nu}_i - \sum_{i=1}^{k_2} \frac{\partial P_{opt}(j\gamma_i)}{\partial p_m} \tilde{\lambda}_i + \left\langle \frac{\partial \mathcal{A}(P_{opt})}{\partial p_m}, \tilde{U} \right\rangle = 0 \quad (3.111)$$

for $m = 0, 1, \dots, 2N$, where $[p_{2N}, p_{2N-1}, \dots, p_1, p_0]$ denote the coefficients of $P_{opt} \in \mathbb{P}_{2N}^+$ in the standard basis $\{s^{2N}, s^{2N-1}, \dots, s, 1\}$. In summary, the *Karush-Kuhn-Tucker* (KKT) conditions are satisfied and this completes the proof of theorem 3.3.10. The reader can note that the derivative of the matrix $\mathcal{A}(P_{opt}) = -\Delta(P_{opt})$ in (3.111) with respect to the coefficients of P_{opt} can be calculated once we know the derivative of the outer function $U_{P_{opt}}$ with respect to the coefficients of P_{opt} . This in fact can be calculated once we know the Jacobian of the maps providing the spectral factor q_{opt} of $P_{opt} + R$ and p_{opt} of P_{opt} . The calculation of this Jacobian is explained in the next chapter in (4.38).

Simplification at the optimum

In theorem 3.3.10, now let us consider the case when P_{opt} and R have some common zeros on the imaginary axis, denoted by $j\alpha_1, j\alpha_2, \dots, j\alpha_k$, where $k \leq 2N$. These common zeros can occur only outside the passband since the fixed transmission polynomial R of the global system is assumed not to have any zeros in the passband \mathbb{I} . It should be noted that in this case, the Pick matrix $\Delta(P_{opt})$ as defined in (3.30) is non-differentiable. The common zeros of P_{opt} and R , represented by $j\alpha_i$'s are not assumed to be distinct and it should be noted that since P_{opt} and R are positive polynomials, these zeros on the imaginary axis necessarily have even multiplicity. If we denote, $C(s) = (s - j\alpha_1)(s - j\alpha_2) \dots (s - j\alpha_k)$, then $C(s)$ is a positive polynomial of degree k . Let us denote, $\tilde{R}(s) = \frac{R(s)}{C(s)} \in \mathbb{P}_{2N-k}^+$. It should be noted that $\tilde{P}_{opt} = \frac{P_{opt}}{C} \in \mathbb{P}_{2N-k}^+$ is the optimal solution for lower order version of problem \mathfrak{P}_C with optimisation space \mathbb{P}_{2N} replaced by \mathbb{P}_{2N-k} and R replaced by \tilde{R} . This can be seen by the fact, if this is not the case, say if $\hat{P} \neq \frac{P_{opt}}{C}$ in \mathbb{P}_{2N-k}^+ solves the lower order problem with a better criterion, then $\hat{P}(s)C(s)$ in \mathbb{P}_{2N}^+ which is not identically equal to P_{opt} solves the primal problem \mathfrak{P}_C with a better criterion, contradicting the optimality of the

solution P_{opt} to \mathfrak{P}_C . So, it follows from the previous proof that there exists non-negative real numbers $\tilde{\nu}_1, \tilde{\nu}_2, \dots, \tilde{\nu}_{\tilde{k}_1}, \tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_{\tilde{k}_2}, \tilde{k}_1 \leq 2N - k, \tilde{k}_2 \leq 2N - k$, where $\sum_{i=1}^{\tilde{k}_1} \tilde{\nu}_i = 1$, matrix $\tilde{U} \in \mathbb{S}_M^+$, points $j\omega_1, j\omega_2, \dots, j\omega_{\tilde{k}_1}$ in \mathbb{I} and points $j\gamma_1, j\gamma_2, \dots, j\gamma_{\tilde{k}_2}$ in $j\mathbb{R}$, such that, $\tilde{P}_{opt} = \frac{P_{opt}}{C} \in \mathbb{P}_{2N-k}^+$ satisfies

$$\frac{\tilde{P}_{opt}(j\omega_i)}{\tilde{R}(j\omega_i)} = l_{opt}, \quad i = 1, 2, \dots, \tilde{k}_1$$

and it solves the following unconstrained optimization problem

$$\min_{P \in \mathbb{P}_{2N-k}} \left(\sum_{i=1}^{\tilde{k}_1} \frac{P(j\omega_i)}{\tilde{R}(j\omega_i)} \tilde{\nu}_i - \sum_{i=1}^{\tilde{k}_2} P(j\gamma_i) \tilde{\lambda}_i + \langle \tilde{\mathcal{A}}(P), \tilde{U} \rangle \right), \quad (3.112)$$

where

$$\tilde{\mathcal{A}}(P) = \begin{cases} -\tilde{\Delta}(P), & P \in \mathbb{P}_{2N-k}^+, \\ +\infty I_M, & \text{otherwise,} \end{cases}$$

$\tilde{\Delta}(P)$ represent the Pick matrix defined in (3.30) with R replaced by \tilde{R} in the definition of U_P in (3.19). Now, as in the previous case, we have the KKT condition satisfied,

$$\sum_{i=1}^{\tilde{k}_1} \frac{1}{\tilde{R}(j\omega_i)} \frac{\partial \tilde{P}_{opt}(j\omega_i)}{\partial p_m} \tilde{\nu}_i - \sum_{i=1}^{\tilde{k}_2} \frac{\partial \tilde{P}_{opt}(j\gamma_i)}{\partial p_m} \tilde{\lambda}_i + \left\langle \frac{\partial \tilde{\mathcal{A}}(\tilde{P}_{opt})}{\partial p_m}, \tilde{U} \right\rangle = 0 \quad (3.113)$$

for $m = 0, 1, \dots, 2N - k$, where $[p_{2N-k}, \dots, p_1, p_0]$ denote the coefficients of $\tilde{P}_{opt} \in \mathbb{P}_{2N-k}^+$. So, we have written down the KKT condition for problem \mathfrak{P}_C in the cases when P_{opt} and R have common zeros or not. In this subsection, ‘Simplifications at the optimum’, we have tackled the case, the Pick matrix $\Delta(P_{opt})$ as defined in (3.30) is non-differentiable by formulating the unconstrained dual problem in a lower dimension. We will conclude this section here and in the next section, we move to the illustration of some results of the numerical implementation of problem \mathfrak{P}_C on some antenna prototypes.

3.4 Results

In this section, we will present some numerical illustrations of the matching results obtained on some concrete antenna examples. Before this illustration, we will provide a brief description of the synthesis of the optimal global reflection for a given load of McMillan degree M starting from the optimal solution $P_{opt} \in \mathbb{H}_R^N$ to problem \mathfrak{P}_C and then the extraction of the matching filter from this global response as well.

As discussed in proposition 3.3.2 about the characterisation of \mathbb{H}_R^N , if $P \in \mathbb{H}_R^N$ i.e the Pick matrix $\Delta(P) \succeq 0$, there exist a Schur function $b_P(s)$ of at most degree M such that the global reflection coefficient $S_{22} = b_P U_P$ of degree at most $N + M$ satisfy the de-embedding conditions for the given load, $S_{22}(\xi_i) = L_{22}(\xi_i)$, $1 \leq i \leq M$. For any $P \in \mathbb{H}_R^N$,

the corresponding function $b_P(s)$ can be computed by the Schur recursion described in appendix C. Let us define

$$\mathbb{E}_M^P = \left\{ b \in \mathbb{B} : b(\xi_i) = \frac{L_{22}(\xi_i)}{U_P(\xi_i)}, 1 \leq i \leq M \right\}. \quad (3.114)$$

As stated in theorem C.1.1 in appendix C, the recursion procedure provides the interpolant function $b_P(s) \in \mathbb{E}_M^P$ of at most degree M . In the case of non-optimal feasible solution for problem \mathfrak{P}_C , i.e for $P \in \mathbb{P}_{2N}^+$ at which $\Delta(P) \succ 0$, \mathbb{E}_M^P contains more than one Schur function and is parametrised by theorem C.1.1 as

$$b_P(s) = \frac{A_M(s) + B_M(s)f_M(s)}{C_M(s) + D_M(s)f_M(s)}, \quad (3.115)$$

for some $f_M \in \mathbb{B}$ and the polynomials $A_M, B_M, C_M, D_M \in \mathbb{P}_M$ are computed by means of Schur recursion from the interpolation data. If we choose $f_M(s) = c$, a uni-modular constant, then the Schur interpolant takes the form

$$b_P(s) = \frac{A_M(s) + B_M(s)c}{C_M(s) + D_M(s)c}, \quad (3.116)$$

and as discussed in remark C.1.2 in appendix C, $b(s)$ is a blaschke product of degree M . This provides the global reflection $S_{22} = b_P U_P$ of degree $N + M$ which satisfy the de-embedding conditions for the given load. Furthermore, at the optimal solution P_{opt} of problem \mathbb{H}_R^N (Pick matrix $\Delta(P_{opt})$ is singular), $\mathbb{E}_M^{P_{opt}}$ contains only a blaschke product $b_{P_{opt}}$ of degree equal to the rank of the Pick matrix. Let us denote this degree by $d \leq M - 1$. So, we have the optimal global reflection $S_{22} = b_{P_{opt}} U_{P_{opt}}$ of degree $N + d$ which satisfy the de-embedding conditions for the given load.

3.4.1 Extraction of the Matching Filter

Once we obtain the optimal global reflection S_{22} , it is important in practice to compute the matching filter which provides this S_{22} . Following the Fano-Youla approach of the global system synthesis, throughout this chapter, we have discussed the optimisation scheme focused on the minimisation of the magnitude of the output reflection coefficient of the global system, $|S_{22}(j\omega)|$ within a given passband \mathbb{I} . It should be noted that this also provides the optimal input reflection coefficient for the global system since we are dealing with lossless systems (for all $\omega \in \mathbb{R}$, $|S_{11}(j\omega)| = |S_{22}(j\omega)|$). The classical matching problem is to minimise the input reflection coefficient of the total system (matching circuit + load) in the passband of interest.

The matching filter providing the optimal global reflection can be computed directly by de-embedding the load from the global system. Let us denote the optimal global reflection coefficient

$$S_{22} = \frac{p}{q} = b_{P_{opt}} U_{P_{opt}} = \frac{a^* p_U}{a q_U}, \quad (3.117)$$

where $b_{P_{opt}} = \frac{a^*}{a}$ is a blaschke product of degree $d \leq M - 1$ ($a \in \mathbb{P}_d$ is a stable polynomial) and $U_{P_{opt}} = \frac{p_U}{q_U}$ is an outer function which satisfies $p_U, q_U \in \mathbb{P}_N$ and $q_U q_U^* - p_U p_U^* = R$,

the fixed transmission polynomial stated at the beginning of subsection 3.2.2. It should be noted that, because of the additional degree introduced by the Blaschke, with respect to the constraint set of problem \mathfrak{P} defined in equation (3.13), the optimal global reflection S_{22} obtained by solving problem \mathfrak{P}_C only satisfies $S_{22} \in \mathbb{F}_{aa^*R}^{N+d}$ since $qq^* - pp^* = aa^*R$. This means that compared to the initial fixed transmission polynomial R for the global system, now it is aa^*R and d additional transmission zeros are introduced in the global system at the points where the polynomial a^* vanishes. In general, once we have the optimal global reflection, the output reflection coefficient of the matching filter F_{22} can be computed by equation (3.4) :

$$F_{22} = \frac{S_{22} - L_{22}}{S_{22}L_{11} - \det(L)}. \quad (3.118)$$

If we denote the Darlington extension of the load by

$$L = \frac{1}{q_L} \begin{pmatrix} p_L^* & -r_L^* \\ r_L & p_L \end{pmatrix},$$

such that $r_L r_L^* \stackrel{\text{def}}{=} R_L = q_L q_L^* - p_L p_L^*$, the de-chaining formula for F_{22} in (3.118) reduces to

$$F_{22} = \frac{pq_L - p_L q}{pp_L^* - qq_L^*}. \quad (3.119)$$

In the above rational expression of F_{22} , it can be noted that the numerator and the denominator are of degree $N + d + M$. Since we have $S_{22}(\xi_i) = L_{22}(\xi_i)$, $1 \leq i \leq M$ and all ξ_i 's in Π^+ belonging to the set of transmission zeros of the global system as well, it can be easily verified that in the above expression for F_{22} there are pole-zero cancellations at ξ_i and $-\bar{\xi}_i$, $1 \leq i \leq M$ (see section C.1 in appendix C) leading to $2M$ simplification. Thus, we have $F_{22} \stackrel{\text{def}}{=} \frac{p_F}{q_F}$, where $p_F, q_F \in \mathbb{P}_{N-M+d}$ and so, we have the following scattering matrix of McMillan degree $N - M + d$ for the optimal matching filter,

$$F = \frac{1}{q_F} \begin{pmatrix} p_F^* & -r_F^* \\ r_F & p_F \end{pmatrix},$$

where the polynomial $r_F \in \mathbb{P}_{N-M+d}$ satisfies $r_F r_F^* \stackrel{\text{def}}{=} R_F = q_F q_F^* - p_F p_F^*$ and $R_F R_L = R$. So there is an additional degree d to the desired McMillan degree $N - M$ for the matching filter. This increase in the degree of the matching filter is the price to pay for the convex relaxation of the problem. It should be noted that since $d \leq M - 1$, the maximal degree of the optimal matching filter is $N - 1$. Nevertheless, it should also be noted that if the load is of degree $M = 1$, there occurs no increase in the degree, S_{22} belongs to \mathbb{F}_R^N and it solves problem \mathfrak{P} as well. So, in practice for most of the antennas which feature single resonance, the approach provides sharp lower bounds for the matching criterion and a procedure to extract the matching circuit of desired degree which achieves this optimal global reflection. When the load is of higher degree, the approach provides optimal global reflections level that can be achieved using matching circuits of degree d more than the initial desired degree.

In addition, it should be noted that, in problem \mathfrak{P}_C , for any $P \in \mathbb{H}_R^N$ where $P \neq P_{opt}$, we are able to furnish a global reflection coefficient S_{22} of degree $N + M$ as well,

$$S_{22} = b_P U_P, \quad (3.120)$$

where b_P is a blaschke product of degree M and U_P is an outer function of degree N which satisfies $|U_P| > |U_{P_{opt}}|$ on the imaginary axis. So, following the same explanation above, we can furnish a matching filter of degree N which provides this global reflection. It means there is an increase of degree M compared to the desired McMillan degree $N - M$ for the matching filter. We conclude here this brief description about the extraction of the matching filter providing the synthesized global response. In the next subsection, we will provide a couple of examples to demonstrate the optimisation scheme discussed in this chapter.

3.4.2 Example 1 : Superdirective Antenna

In the first example, we present the same antenna example that was illustrated in subsection 2.4.1 in chapter 2. The scattering parameter of the load with McMillan degree $M = 2$ is depicted in figure 3.2. The passband \mathbb{I} of interest represented by the shaded rectangle extends from 870MHz to 900MHz.

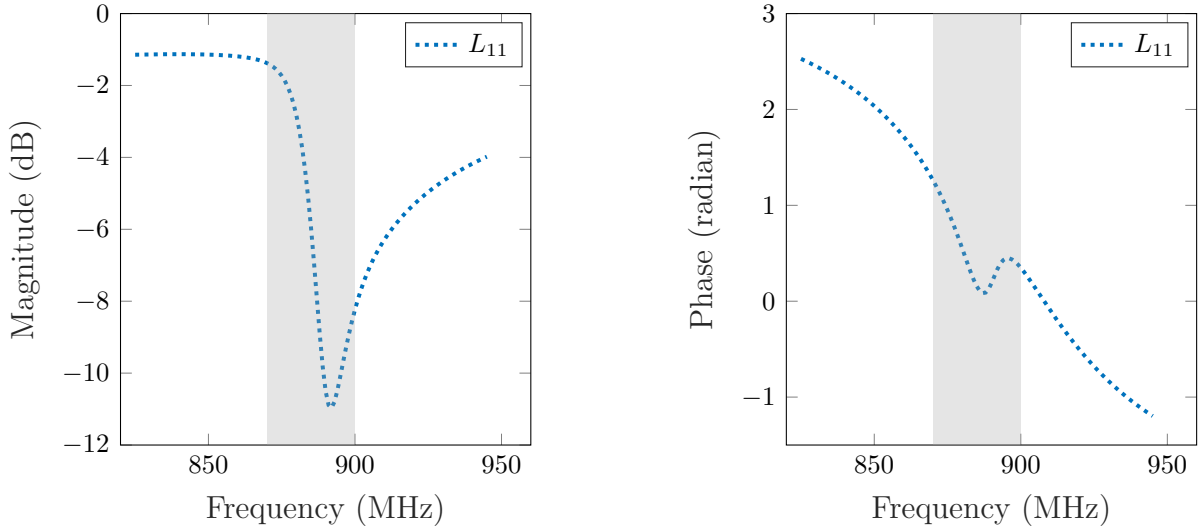


Figure 3.2: Scattering parameter of load with McMillan Degree 2.

The result of solving problem \mathfrak{P}_C with $N = 6$ is presented in figure 3.3. The transmission zeros of the matching circuit were all fixed at infinity ($R_F = 1$). The solution to problem \mathfrak{P}_C provided the optimal matching circuit of McMillan degree 5 which provided a reflection level $\hat{L}_{opt} = 20 \log \left(\sqrt{\frac{L_{opt}}{L_{opt}+1}} \right) = -8.53$ dB. It can be observed in figure 3.3 that the global reflection S_{11} attains \hat{L}_{opt} at $(N + 1) = 7$ times in the passband \mathbb{I} characterizing the optimality of the obtained response as mentioned in theorem 3.3.8. By using the matching circuit, it can be noted that the value of the maximal mismatch of the load in the passband has been improved from -1.37 dB (L_{11} at 870 MHz) to -8.53 dB. The scattering parameter of the matching circuit F_{22} that provides this optimal criterion is shown in figure 3.4.

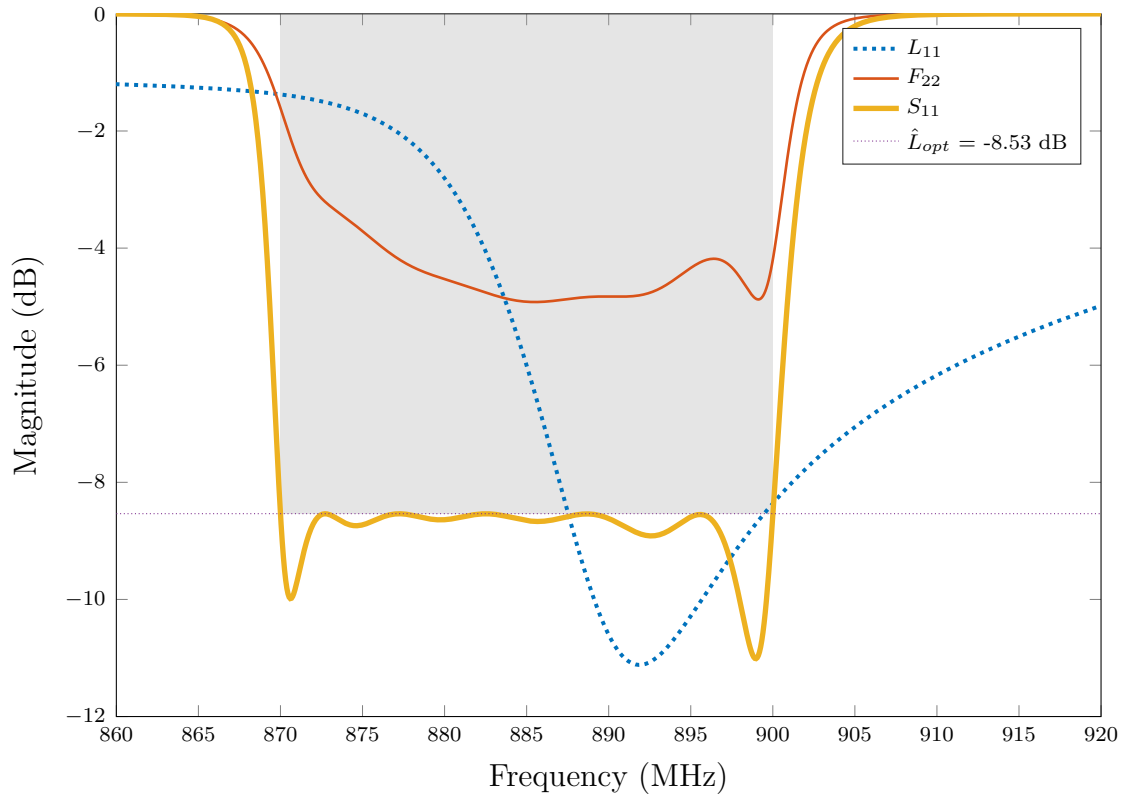


Figure 3.3: Result of problem \mathfrak{P}_C with $N = 6$.

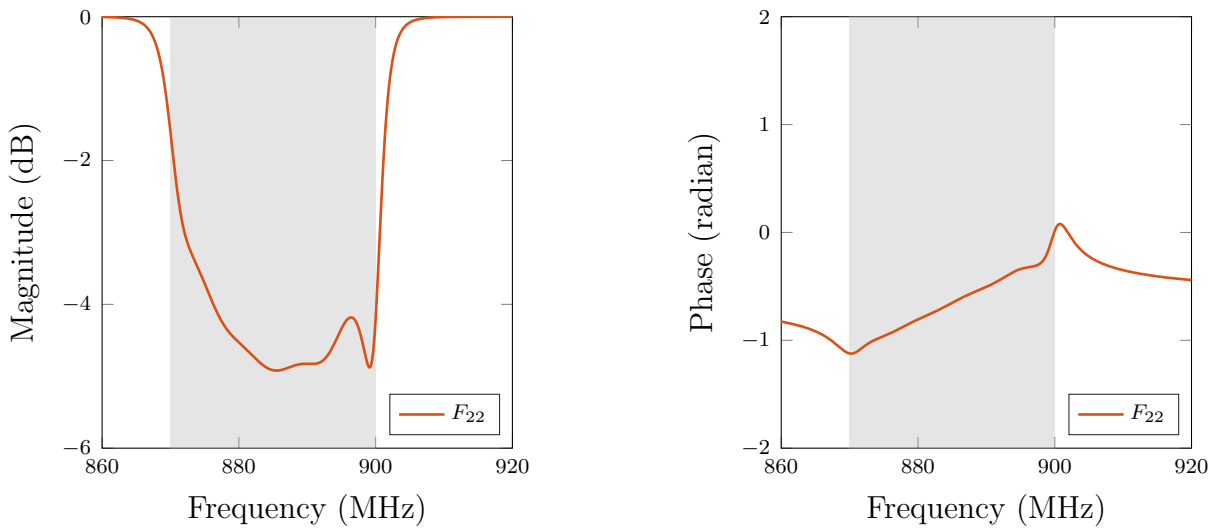


Figure 3.4: Scattering parameter of the optimal matching circuit with McMillan Degree 5.

3.4.3 Example 2 : Dual Band Antenna

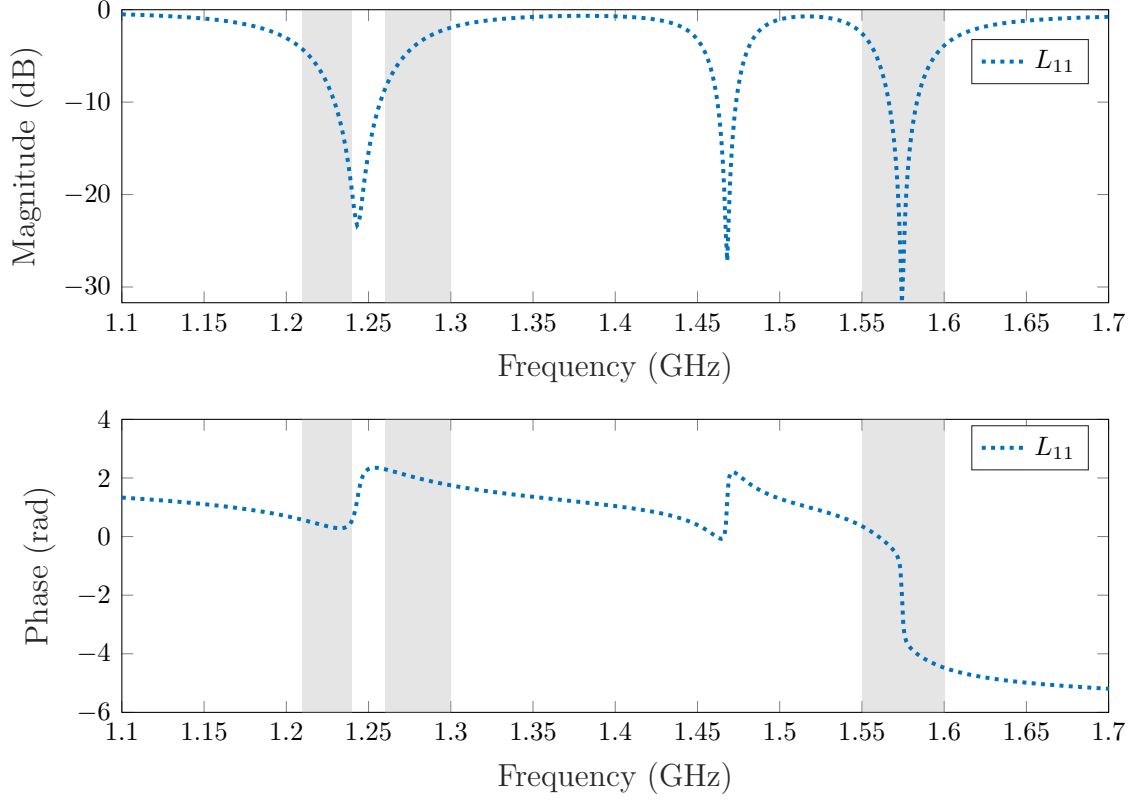
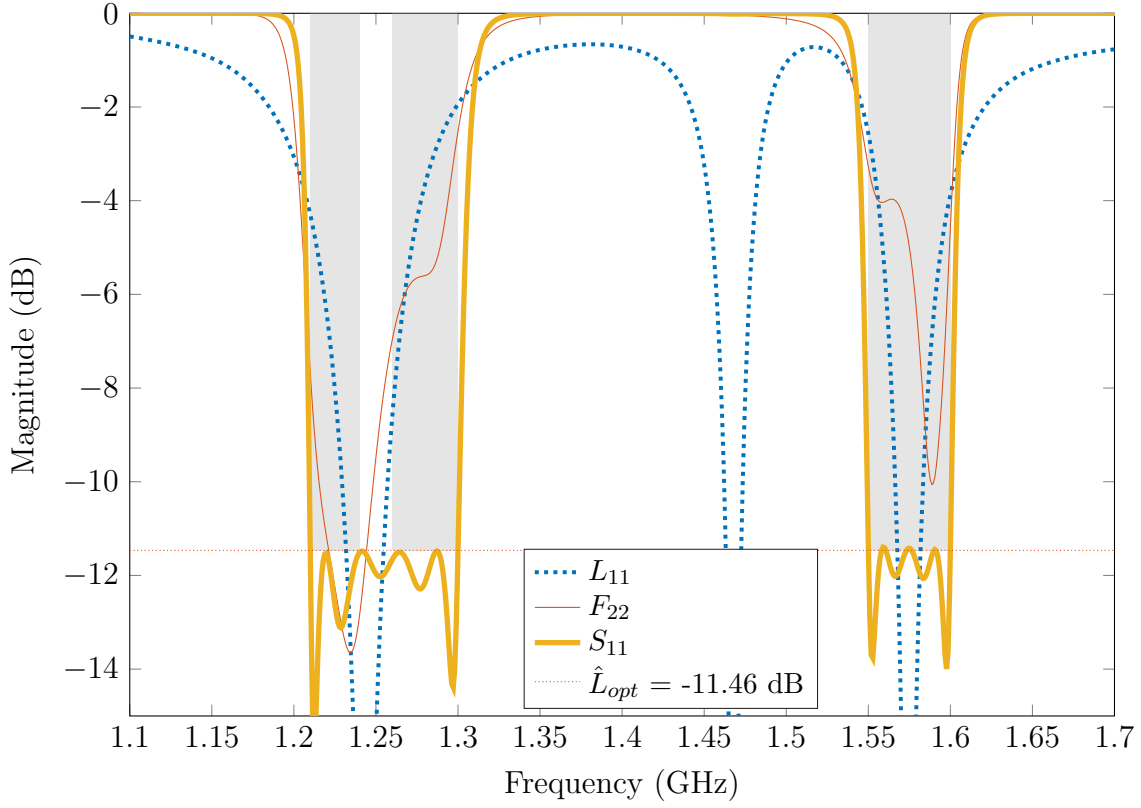


Figure 3.5: Scattering parameter of load with McMillan Degree 3.

In the second example, we present the matching results obtained for a dual band antenna presented in [37]. The scattering parameter of the load with McMillan degree $M = 3$ is depicted in figure 3.5. In this example, the targeted passband in the matching problem was fixed to be 1.21 GHz - 1.24 GHz (GPS L2 band), 1.26 GHz - 1.3 GHz (Galileo E6 band) and 1.55 GHz - 1.6 GHz (GPS L2 band). The result of solving problem \mathfrak{P}_C with $N = 10$ is presented in figure 3.6. The transmission zeros of the matching circuit were all fixed at infinity ($R_F = 1$). The solution to problem \mathfrak{P}_C provided the optimal matching circuit of McMillan degree 9 which provided a reflection level $\hat{L}_{opt} = 20 \log \left(\sqrt{\frac{L_{opt}}{L_{opt}+1}} \right) = -11.46$ dB. It can be observed in figure 3.6 that the global reflection S_{11} attains \hat{L}_{opt} at $(N + 1) = 11$ times in the passband \mathbb{I} characterizing the optimality of the obtained response as mentioned in theorem 3.3.8. By using the matching circuit, it can be noted that the value of the maximal mismatch of the load in the passband has improved significantly from -1.9 dB (L_{11} at 1.3 GHz) to -11.46 dB.


 Figure 3.6: Result of problem \mathfrak{P}_C with $N = 10$.

3.4.4 Analytic Examples

In this subsection, we will consider the two analytic examples of antenna that was presented in subsection 2.4.2 in chapter 2. In these cases, we will discuss the results obtained by the implementation of problem \mathfrak{P}_C for different degrees of matching circuits and also its comparison to the classical matching criterion bound obtained by Fano in [36].

(i). Degree 1 Antenna

In this example, the polynomials p_L , q_L and r_L and uni-modular constant ϵ in the Belevitch representation of the scattering matrix of the load in (2.34) were fixed to be

$$p_L(s) = s, \quad r_L(s) = -1, \quad q_L(s) = s + 1 \quad \text{and} \quad \epsilon = -1.$$

So, we have a load of degree 1 with $L_{11}(s) = L_{22}(s) = \frac{s}{s+1}$ and a transmission zero at infinity. The targeted passband for matching is $[-1j, 1j]$. The reflection coefficient of the this antenna, L_{11} is depicted in figure 3.7. The shaded rectangle in the figure represent the targeted passband and $\text{Im}(\cdot)$ is used to denote the imaginary part.

For this antenna, as detailed in (2.101) in chapter 2, we have the Fano bound,

$$\text{FB} = -13.64 \text{ dB}.$$

Now, we present the result obtained by solving problem \mathfrak{P}_C with $N = 5$ for this antenna. We remind the reader that throughout this chapter, in the discussion of problem \mathfrak{P}_C , for

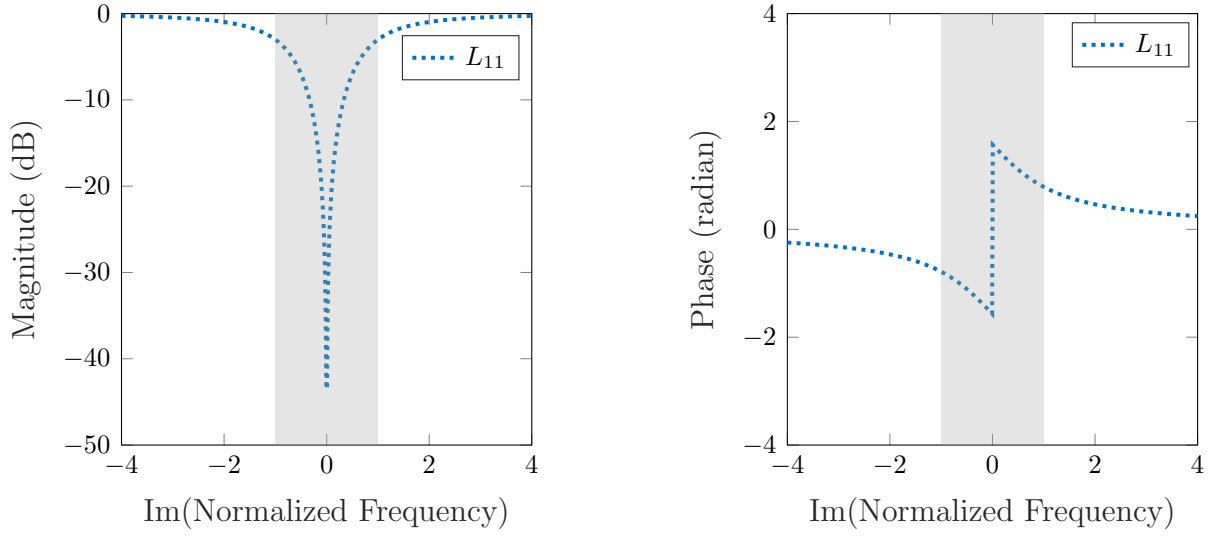


Figure 3.7: Scattering parameter of load with McMillan Degree 1.

simplicity we considered the transmission zeros of the given load, ξ_i , $1 \leq i \leq M$ in Π^+ . In this example, for the purpose of comparing the obtained results with the Fano bound, we consider the load with transmission zero at ∞ . To handle such cases of transmission zero on the boundary of the analyticity domain, $j\mathbb{R}$, it should be noted that we can compute the Pick matrix as the limit of the classical Pick matrix defined in (3.30) when the interpolation points tend non-tangentially to the boundary. The reader can find a detailed description of boundary Nevanlinna-Pick interpolation in chapter 21, [8]. In the analytic examples we consider in this section, the given load has one transmission zero at infinity with the value of L_{22} at infinity equal to 1. In order to handle this case, we can obtain the new Pick matrix as the limit of the Pick matrix $[\Delta]_{1 \leq i, j \leq M}$,

$$\Delta_{ij} = \frac{1 - \frac{\overline{L_{22}(\xi_i)} L_{22}(\xi_j)}{\overline{U_P(\xi_i)} U_P(\xi_j)}}{\overline{\xi_i} + \xi_j}, \quad (3.121)$$

defined in (3.30) when the interpolation point, say ξ_1 tends non-tangentially to infinity. In order to detail this, let us consider the interpolation problem : Let $\xi_1 = \infty$, ξ_m , $2 \leq m \leq M$ be distinct points in Π^+ and let γ_m , $2 \leq m \leq M$ be complex numbers such that $|\gamma_m| < 1$, $2 \leq m \leq M$ and $\gamma_1 = 1$; under which conditions, there exists a function $f \in H^\infty(\Pi^+)$ such that

$$f(\xi_m) = \gamma_m, \quad 1 \leq m \leq M, \quad \|f\|_{H^\infty} \leq 1. \quad (3.122)$$

For the problem with all the interpolation points in Π^+ , i.e ξ_m , $1 \leq m \leq M$ in Π^+ and γ_m : complex numbers such that $|\gamma_m| < 1$, $1 \leq m \leq M$, it follows from Nevanlinna-Pick theorem that the interpolation problem is solvable if and only if the matrix

$$\Delta = \left[\frac{1 - \overline{f(\xi_m)} f(\xi_n)}{\overline{\xi_m} + \xi_n} \right]_{1 \leq m, n \leq M} \quad (3.123)$$

is positive semi-definite. To deal the case when $\xi_1 = \infty$ and $f(\infty) = \gamma_1 = 1$, we will multiply the first line and column in matrix in (3.123) by $\overline{\xi_1}$ and ξ_1 respectively and then

take the limit of the Pick matrix $[\Delta]_{1 \leq i, j \leq M}$ when ξ_1 tends non-tangentially to infinity. Denoting $\delta_1 = \frac{1}{\epsilon_1}$, we obtain the first element of the modified Pick matrix, $\tilde{\Delta}$ as

$$\tilde{\Delta}_{11} = \lim_{\delta_1 \rightarrow 0} \frac{1}{|\delta_1|^2} \frac{1 - |f(\frac{1}{\delta_1})|^2}{(\frac{1}{\delta_1} + \frac{1}{\delta_1})} = \lim_{\delta_1 \rightarrow 0} \frac{1 - |f(\frac{1}{\delta_1})|^2}{(\delta_1 + \delta_1)}.$$

Applying L'Hôpital's rule and from the assumption that $f(\infty) = 1$, we have

$$\tilde{\Delta}_{11} = -f'(\infty) = -\text{ang}[f](\infty).$$

We can obtain the non-diagonal elements in the first row of $\tilde{\Delta}$ as, for $2 \leq n \leq M$,

$$\tilde{\Delta}_{1n} = \lim_{\delta_1 \rightarrow 0} \frac{1}{\delta_1} \frac{1 - \overline{f(\frac{1}{\delta_1})} f(\xi_n)}{\frac{1}{\delta_1} + \xi_n} = 1 - f(\xi_n).$$

Similarly, the non-diagonal elements in the first column of $\tilde{\Delta}$ can be obtained as, for $2 \leq m \leq M$,

$$\tilde{\Delta}_{m1} = \lim_{\delta_1 \rightarrow 0} \frac{1}{\delta_1} \frac{1 - \overline{f(\xi_m)} f(\frac{1}{\delta_1})}{\xi_m + \frac{1}{\delta_1}} = 1 - \overline{f(\xi_m)}.$$

Thus, for problem in (3.122), we have the $(M \times M)$ Pick matrix as follows :

$$\tilde{\Delta} = \begin{bmatrix} -\text{ang}[f](\infty) & 1 - f(\xi_2) & 1 - f(\xi_3) & \dots & 1 - f(\xi_M) \\ 1 - \overline{f(\xi_2)} & \Delta_{22} & \Delta_{23} & \dots & \Delta_{2M} \\ 1 - \overline{f(\xi_3)} & \Delta_{32} & \Delta_{33} & \dots & \Delta_{3M} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 - \overline{f(\xi_M)} & \Delta_{M2} & \Delta_{M3} & \dots & \Delta_{MM} \end{bmatrix}, \quad (3.124)$$

where Δ is the classical Pick matrix defined in (3.123). Now, in the analytic examples we consider, in problem in (3.122), we have $f = \frac{L_{22}}{U_P}$, where the given load satisfies, $L_{22}(\infty) = 1$ and in the construction of U_P in (3.19), we assume $U_P(\infty) = 1$. Thus, by the calculation described above, in problem \mathfrak{P}_C if one interpolation point is at infinity ($\xi_1 = \infty$) and $L_{22}(\infty) = 1$, the theory described in this chapter can be used with the modification in the Pick matrix as shown in (3.124), where we have,

$$-\text{ang}[f](\infty) = \text{ang}[U_P](\infty) - \text{ang}[L_{22}](\infty), \quad (3.125)$$

$$1 - f(\xi_i) = 1 - \frac{L_{22}(\xi_i)}{U_P(\xi_i)}, \quad 2 \leq i \leq M, \quad (3.126)$$

$$1 - \overline{f(\xi_i)} = 1 - \frac{\overline{L_{22}(\xi_i)}}{\overline{U_P(\xi_i)}}, \quad 2 \leq i \leq M, \quad (3.127)$$

$$\Delta_{ij} = \frac{1 - \frac{\overline{L_{22}(\xi_i)} L_{22}(\xi_j)}{\overline{U_P(\xi_i)} U_P(\xi_j)}}{\xi_i + \xi_j}, \quad 2 \leq i, j \leq M. \quad (3.128)$$

3.4. RESULTS

The positive semi definite character of the Pick matrix imposed the non negativity of its diagonal terms. As shown by equation (3.125) the condition at infinity translates as,

$$\text{ang}[U_P](\infty) \geq \text{ang}[L_{22}](\infty)$$

which is exactly the Fano condition applied to the exterior factor U_P . It is therefore readily seen than when the load is of degree one with a transmission zero at infinity the condition imposed by the Pick matrix positivity is equivalent to the restriction that yields the Fano bound. For more complex loads with additional transmission zeros, Fano's condition appears only as one of many scalar conditions weighing on U_P . A complete set of such conditions can be derived by Sylvester's criterion that asks for the non negativity of all principal minors of the Pick matrix.

The numerical result obtained by considering this modification in the Pick matrix and solving problem \mathfrak{P}_C is depicted in figure 3.8 together with the Fano bound. The green shaded rectangle inside the passband is used to denote the difference in the optimal criterion obtained and the Fano bound. The optimal system reflection was obtained to be

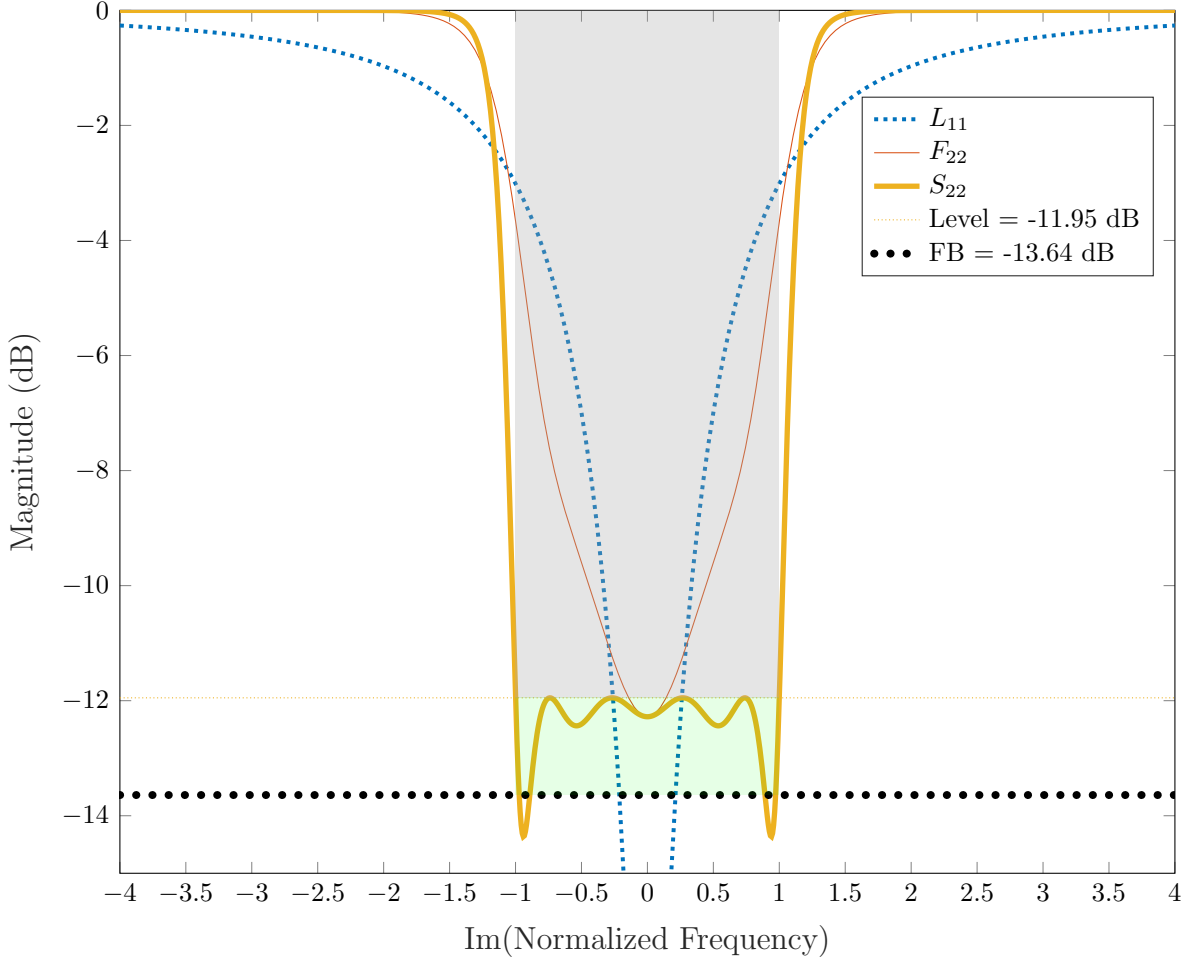


Figure 3.8: Optimal System reflection S_{22} obtained by solving \mathfrak{P}_C for $N = 5$.

$$S_{22}(s) = b_{P_{opt}}(s)U_{P_{opt}}(s), \quad (3.129)$$

where

$$b_{P_{opt}}(s) = 1, \quad U_{P_{opt}}(s) = \frac{s^5 + 1.1s^4 + 1.7s^3 + 1.1s^2 + 0.58s + 0.13}{s^5 + 2.1s^4 + 3.2s^3 + 3s^2 + 1.8s + 0.55}.$$

The output reflection coefficient of the degree 4 matching filter providing this optimal response was found be to

$$F_{22}(s) = \frac{-0.5s^4 - 0.27s^3 - 0.18s^2 + 0.16s + 0.13}{-0.5s^4 - 1.3s^3 - 1.8s^2 - 1.4s - 0.55}.$$

As expected, the optimal level attained using degree 4 matching circuit by solving convex problem \mathfrak{P}_C , -11.95 dB for this example is better (lower) than the one obtained using equi-oscillating response in chapter 2 (-11.71 dB). This is in agreement with the theory developed in chapter 2 and 3 since we haven't fixed the shape of global response in problem \mathfrak{P}_C compared to the global response in problem \mathcal{P}_2 . In figure 3.9, we present the optimal level obtained by solving problem \mathfrak{P}_C using different degrees of reference functions, U_P in a tabular form. Similar to the equi-oscillating optimal response in chapter 2, it can be noticed that the optimal level attained is close to the Fano bound (difference of 1.69 dB) and as the degree of the matching circuit (M.C) increases, the optimal level attained is getting closer to the Fano bound. Since this antenna is of degree one with the only transmission zero at infinity, the difference of the optimal level attained from the Fano bound comes only from the finite degree constraint of matching circuit we have in problem \mathfrak{P}_C . In problem \mathcal{P}_2 in chapter 2, we had the additional constraint of fixed equi-oscillating shape as well for the global response.

Degree of U_P (N)	Degree of M.C ($N - 1$)	Optimal level
2	1	-8.36 dB
3	2	-10.28 dB
4	3	-11.32 dB
5	4	-11.95 dB
6	5	-12.36 dB
7	6	-12.63 dB
8	7	-12.83 dB
9	8	-12.97 dB
10	9	-13.08 dB

Figure 3.9: Optimal level obtained by solving \mathfrak{P}_C for different N .

We also present a table of comparison of the optimal matching criterion obtained by solving problem \mathcal{P}_2 (chapter 2) and problem \mathfrak{P}_C (chapter 3) in figure 3.10. As expected, the optimal level attained in problem \mathfrak{P}_C is better (lower) than the one obtained in problem \mathcal{P}_2 for each degree of matching circuit.

Degree of M.C	Optimal level in \mathcal{P}_2	Optimal level in \mathfrak{P}_C
1	-7.76 dB	-8.36 dB
2	-9.90 dB	-10.28 dB
3	-11.15 dB	-11.32 dB
4	-11.71 dB	-11.95 dB
5	-12.13 dB	-12.36 dB
6	-12.27 dB	-12.63 dB
7	-12.40 dB	-12.83 dB
8	-12.52 dB	-12.97 dB
8	-12.75 dB	-13.08 dB

Figure 3.10: Comparison of optimal level obtained by solving \mathcal{P}_2 and \mathfrak{P}_C .**(ii). Degree 3 Antenna**

In this example, we consider the same degree 3 antenna that was presented in subsection 2.4.2 in chapter 2. The polynomials p_L, q_L and r_L and uni-modular constant ϵ in the Belevitch representation of the scattering matrix of the load in (2.34) were fixed to be

$$\begin{aligned}
 p_L(s) &= -s^3 + (0.1 + 0.2j)s^2 + (0.09 + 0.35j)s + (0.046 + 0.2j), \\
 r_L(s) &= s^2 + (0.8 + 0.6j)s + (0.42 + 0.12j), \\
 q_L(s) &= s^3 + (2.022 - 0.2j)s^2 + (1.449 + 0.25j)s + (0.483 - 0.023j) \text{ and} \\
 \epsilon &= 1
 \end{aligned}$$

as in (2.102). So, we have a load of degree 3 with transmission zeros at $(0.5 - 0.9j)$, $(0.3 + 0.3j)$, ∞ . The targeted passband for matching is $[-1j, 1j]$. The reflection coefficient of the this antenna, L_{11} is depicted in figure 3.11. The shaded rectangle in the figure represent the targeted passband and $\text{Im}(\cdot)$ is used to denote the imaginary part. For this antenna,

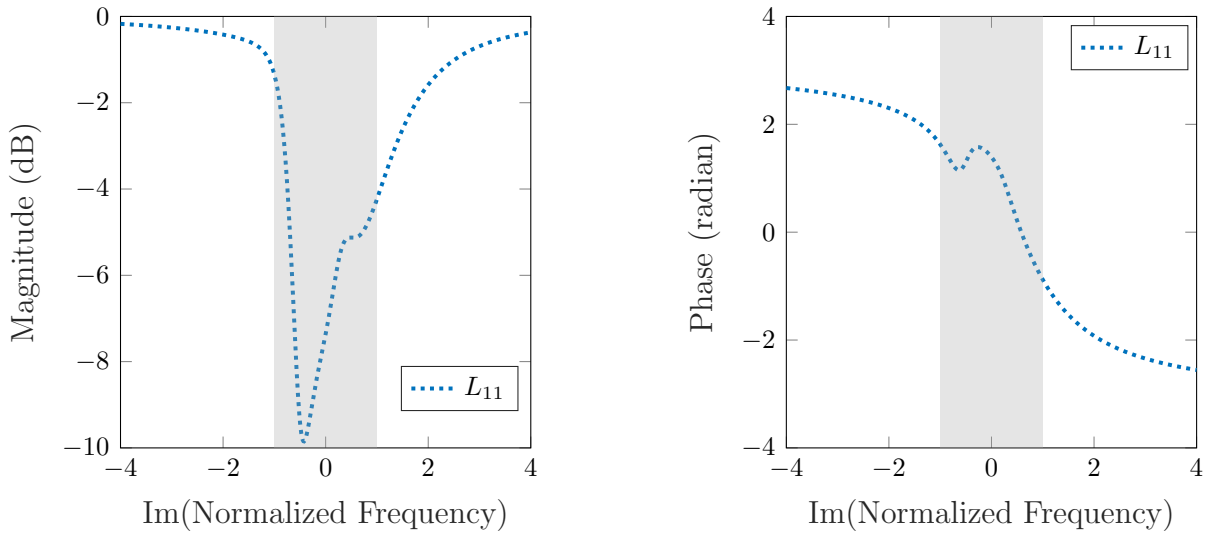


Figure 3.11: Scattering parameter of load with McMillan Degree 3.

as detailed in (2.103) in chapter 2, we have the Fano bound,

$$\text{FB} = \frac{10\pi(-1.922)}{\ln 10} = -26.22 \text{ dB}.$$

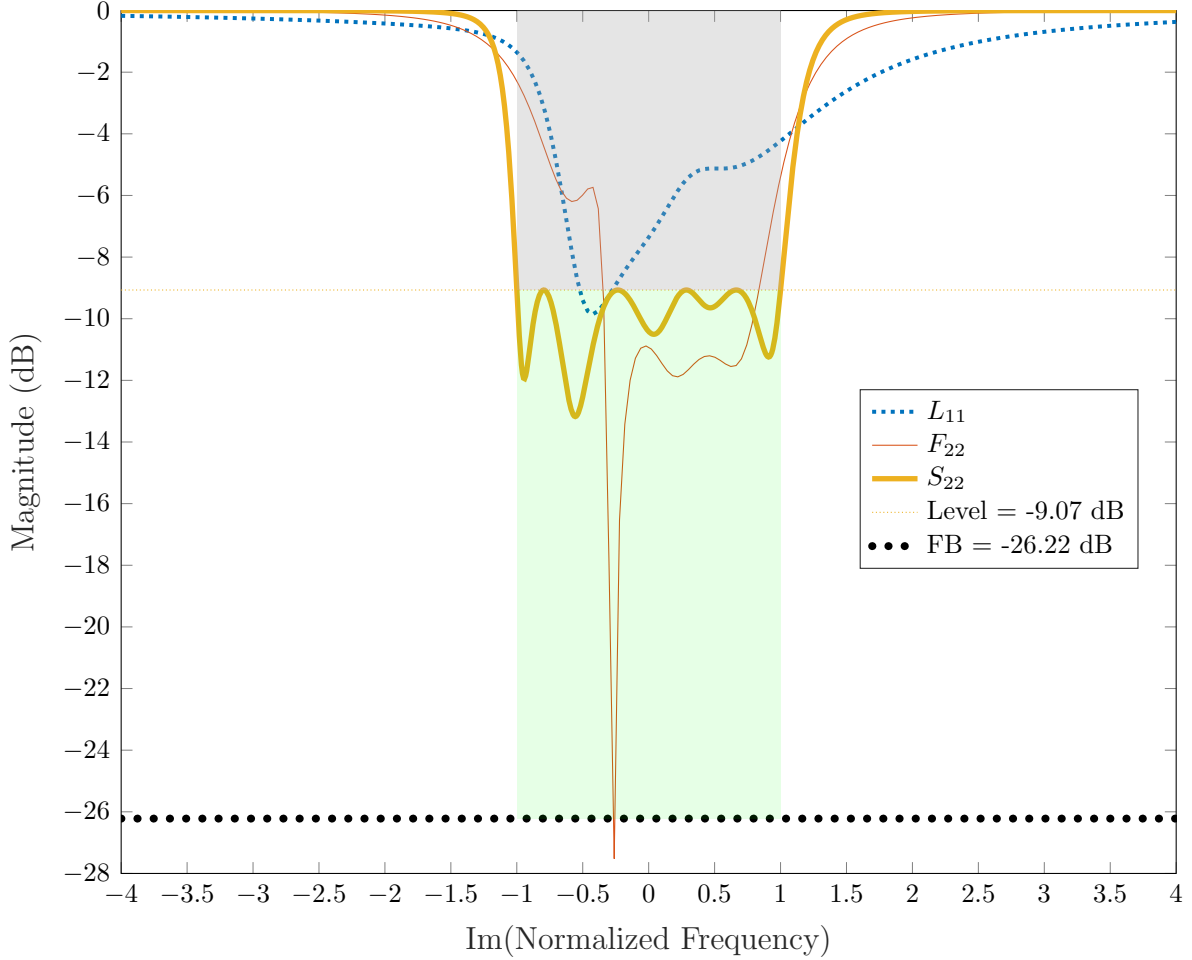


Figure 3.12: Optimal System reflection S_{22} obtained by solving \mathfrak{P}_C for $N = 5$.

Similar to the previous example, we present the result obtained by solving problem \mathfrak{P}_C using modified Pick matrix as detailed in last example with $N = 5$ for this antenna. The result obtained is depicted in figure 3.12 together with the Fano bound. The green shaded rectangle inside the passband is used to denote the difference in the optimal criterion obtained and the Fano bound. The optimal system reflection was obtained to be

$$S_{22}(s) = b_{P_{opt}}(s)U_{P_{opt}}(s), \quad (3.130)$$

where

$$b_{P_{opt}}(s) = \frac{s^2 - (0.55 - 0.17j)s + (0.1 - 0.15j)}{s^2 + (0.55 + 0.17j)s + (0.1 + 0.15j)},$$

$$U_{P_{opt}}(s) = \frac{s^5 + (0.71 + 0.079j)s^4 + (1.4 + 0.14j)s^3 + (0.6 + 0.1j)s^2 + (0.33 + 0.081j)s + (0.055 + 0.00091j)}{s^5 + (1.5 + 0.079j)s^4 + (2.3 + 0.21j)s^3 + (1.7 + 0.25j)s^2 + (0.81 + 0.17j)s + (0.18 + 0.0035j)}.$$

3.4. RESULTS

The output reflection coefficient of the degree 4 matching filter providing this optimal response was found to be

$$F_{22}(s) = \frac{-(0.5 - 0.17j)s^4 - (0.35 - 0.14j)s^3 - (0.33 - 0.11j)s^2 - (0.084 - 0.002j)s - (0.02 + 0.0083j)}{-(0.5 + 0.17j)s^4 - (0.76 + 0.24j)s^3 - (0.76 + 0.3j)s^2 - (0.26 + 0.2j)s - (0.033 + 0.068j)}.$$

In figure 3.13, we present the optimal level obtained by solving problem \mathfrak{P}_C using different degrees of reference functions, U_P in a tabular form.

Degree of U_P (N)	Degree of M.C ($N-1$)	Optimal level
4	3	-7.79 dB
5	4	-9.07 dB
6	5	-9.85 dB
7	6	-10.35 dB
8	7	-10.67 dB
9	8	-10.87 dB
10	9	-11.01 dB

Figure 3.13: Optimal level obtained by solving \mathfrak{P}_C for different N .

As expected, the optimal level attained using degree 4 matching circuit by solving convex problem \mathfrak{P}_C , -9.07 dB for this example is better (lower) than the one obtained using equi-oscillating response in chapter 2 (-8.66 dB). We also present a table of comparison of the optimal matching criterion obtained by solving problem \mathcal{P}_2 (chapter 2) and problem \mathfrak{P}_C (chapter 3) for different degrees of matching circuit (M.C) in figure 3.14.

Degree of M.C	Optimal level in \mathcal{P}_2	Optimal level in \mathfrak{P}_C
3	-7.57 dB	-7.79 dB
4	-8.66 dB	-9.07 dB
5	-9.42 dB	-9.85 dB
6	-10.13 dB	-10.35 dB
7	-10.36 dB	-10.67 dB
8	-10.49 dB	-10.87 dB
9	-10.66 dB	-11.01 dB

Figure 3.14: Comparison of optimal level obtained by solving \mathcal{P}_2 and \mathfrak{P}_C .

Similar to the equi-oscillating response in chapter 2, the optimal levels obtained by solving problem \mathfrak{P}_C for different degrees of matching circuit are relatively far from the Fano bound in this example (-26.22 dB). As explained in last chapter, this is due to the fact that the antenna under consideration has two transmission zeros at finite complex frequencies which are not taken into account in the derivation of Fano bound in (2.100).

Flip Zero of p_L in degree 3 example - making L_{22} outer

Now, we consider the same degree 3 antenna example with zeros of p_L in Π^- flipped to Π^+ as did in chapter 2. The polynomials p_L, q_L and r_L and uni-modular constant ϵ in the

Belevitch representation of the scattering matrix of the load in (2.34) were now fixed to be

$$\begin{aligned} p_L(s) &= -s^3 + (1.3 + 0.2j)s^2 - (0.75 - 0.07j)s + (0.21 - 0.022j), \\ r_L(s) &= s^2 + (0.8 + 0.6j)s + (0.42 + 0.12j), \\ q_L(s) &= s^3 + (2.022 - 0.2j)s^2 + (1.449 + 0.25j)s + (0.483 - 0.023j) \text{ and} \\ \epsilon &= 1. \end{aligned}$$

It should be noted that zeros of p_L^* , $(-0.2 - 0.5j)$, $(-0.7 + 0.5j)$, $(-0.4 + 0.2j)$ are now in Π^- making $L_{22} = \frac{ep_L^*}{q_L}$ outer. The angular derivative of L_{22} is now $\text{ang}[L_{22}](\infty) = -0.722$ which is different from the case when L_{22} was not outer and we have the new Fano bound,

$$\text{FB} = \frac{10\pi \text{ang}[L_{22}](\infty)}{\ln 10} = -9.85 \text{ dB}.$$

As we mentioned in chapter 2, the flipping of zeros inside Π^- of p_L has increased the Fano bound significantly from -26.22 dB to -9.85 dB. Now, we present the result obtained by solving problem \mathfrak{P}_C using modified Pick matrix using degree of reference function, $N = 5$ in this case. The optimal system reflection was obtained to be

$$S_{22}(s) = b_{P_{opt}}(s)U_{P_{opt}}(s), \quad (3.131)$$

where,

$$\begin{aligned} b_{P_{opt}}(s) &= \frac{s^2 - (0.14 + 3.2j)s + (0.19 + 0.075j)}{s^2 + (0.14 - 3.2j)s + (0.19 - 0.075j)}, \\ U_{P_{opt}}(s) &= \frac{s^5 + (0.84 + 0.2j)s^4 + (1.5 + 0.15j)s^3 + (0.72 + 0.21j)s^2 + (0.39 + 0.074j)s + (0.052 + 0.0051j)}{s^5 + (1.3 + 0.2j)s^4 + (1.9 + 0.25j)s^3 + (1.3 + 0.31j)s^2 + (0.62 + 0.15j)s + (0.12 + 0.011j)}. \end{aligned}$$

The output reflection coefficient of the degree 4 matching filter providing this optimal response was found to be

$$F_{22}(s) = \frac{-(0.5 + 0.46j)s^4 - (0.39 - 0.16j)s^3 - (0.4 - 0.018j)s^2 - (0.068 - 0.17j)s - (0.0022 - 0.016j)}{-(0.5 - 0.46j)s^4 - (0.67 - 1.1j)s^3 - (0.7 - 1.4j)s^2 - (0.27 - 0.86j)s - (0.052 - 0.019j)}.$$

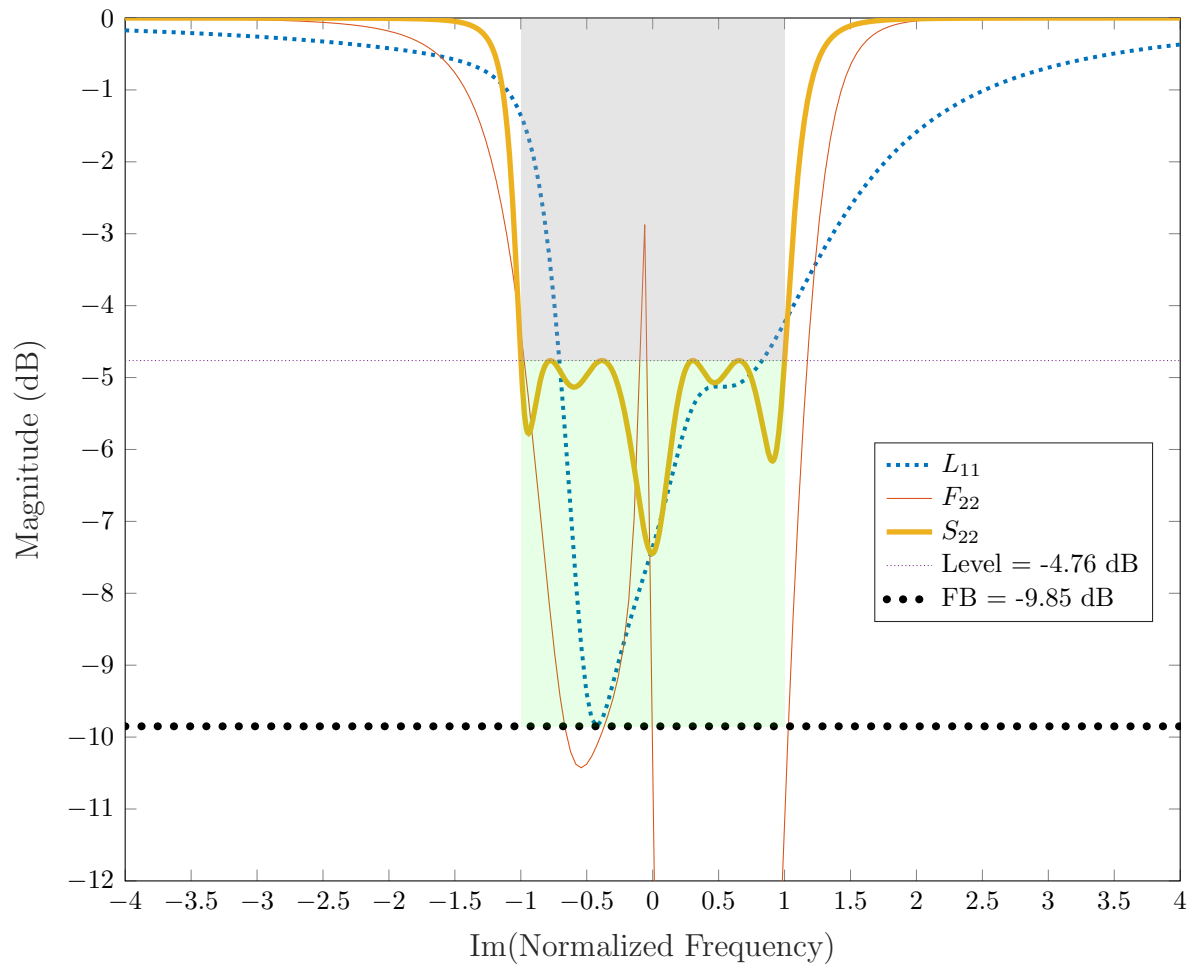


Figure 3.15: Optimal S_{22} obtained by solving \mathfrak{P}_C for $N = 5$ for outer L_{22} .

In figure 3.16, we present the optimal level obtained by solving problem \mathfrak{P}_C for this antenna using different degrees of reference functions, U_P in a tabular form.

Degree of U_P (N)	Degree of M.C ($N - 1$)	Optimal level
4	3	-4.25 dB
5	4	-4.76 dB
6	5	-5.05 dB
7	6	-5.26 dB
8	7	-5.40 dB
9	8	-5.51 dB
10	9	-5.59 dB

Figure 3.16: Optimal level obtained by solving \mathfrak{P}_C for different N for outer L_{22} .

We also present a table of comparison of the optimal matching criterion obtained by

solving problem \mathcal{P}_2 (chapter 2) and problem \mathfrak{P}_C (chapter 3) for different degrees of matching circuit (M.C) in figure 3.17 for this antenna with outer L_{22} .

Degree of M.C	Optimal level in \mathcal{P}_2	Optimal level in \mathfrak{P}_C
3	-4.14 dB	-4.25 dB
4	-4.66 dB	-4.76 dB
5	-4.80 dB	-5.05 dB
6	-4.92 dB	-5.26 dB
7	-5.12 dB	-5.40 dB
8	-5.21 dB	-5.51 dB
9	-5.29 dB	-5.59 dB

Figure 3.17: Comparison of optimal level obtained by solving \mathcal{P}_2 and \mathfrak{P}_C .

Before concluding the chapter, we will devote a section to make a comparison of the approaches discussed in chapter 2 and chapter 3 to solve the matching problem and present the connection between these approaches.

3.5 Nehari Theory and Nevanlinna-Pick Theory

In chapter 2, we have used Nehari theory and the realizability test based on the Hankel operator to solve the matching problem, whereas in this chapter, it is based on the interpolation theory and positivity of Nevanlinna-Pick matrix. We will make a brief comparison of these approaches and then present the connection between two approaches. This will be done by demonstrating an elegant way of arriving at the Pick matrix positivity constraint presented in this chapter starting from the norm constraint on the Hankel operator discussed in chapter 2.

In comparison to the approach presented in chapter 2, it should be noted that in this chapter we have provided further flexibility to the modulus of realizable global response by parameterizing it using the set of positive polynomials in \mathbb{P}_{2N}^+ . A brief summary of the important points in the presented approaches in both chapters is given below for the sake of comparison. In chapter 2, we had the following important steps in the presented approach.

- Relaxed Problem : Does there exist $F_{22} \in \mathbb{B}$ such that

$$|\delta(F_{22}(j\omega), \overline{L_{11}(j\omega)})| \leq |k_\alpha(j\omega)|, \quad \omega \in \mathbb{R} \quad (3.132)$$

- Optimisation over a class of reference functions k_α whose modulus on the imaginary axis is parametrised by the real parameter α ,

$$\text{Case (i). } |k_\alpha|^2 = \frac{|p_k|^2}{|p_k|^2 + \alpha|r_L|^2}, \quad \text{Case (ii). } |k_\alpha|^2 = \frac{|p_k|^2}{|p_k|^2 + \alpha}.$$

- k_α is realizable if and only if $\|H_\Phi\| \leq 1$, where the symbol Φ of the Hankel operator is $\Phi = \frac{1}{V} \left(\frac{1-|k_\alpha|^2}{1-|k_\alpha|^2|L_{11}|^2} \right) \overline{L_{11}}$ and V is the outer function satisfying $|V| = \left| \frac{(1-|L_{11}|^2)}{1-|k_\alpha|^2|L_{11}|^2} \right| |k_\alpha|$ on the imaginary axis.

- At the optimal $\hat{\alpha}$, we have $\|H_\Phi\| = 1$ and hence $|S_{11}(j\omega)| = |k_{\hat{\alpha}}(j\omega)|$.

In the current chapter, we have the following main steps in the presented approach.

- Relaxed Problem : Does there exist $F_{22} \in \mathbb{B}$ such that

$$|(F_{22} \circ L)(j\omega)| \leq |U_P(j\omega)|, \quad \omega \in \mathbb{R} ?$$

- Optimisation over a class of reference functions U_P whose modulus on the imaginary axis is parametrised by $P \in \mathbb{P}_{2N}^+$,

$$|U_P|^2 = \frac{P}{P + R}.$$

- U_P is realizable if and only if $\Delta(P) \succeq 0$, where $\Delta_{ij} = \frac{1 - \left(\frac{L_{22}(\xi_i)}{U_P(\xi_i)}\right) \left(\frac{L_{22}(\xi_j)}{U_P(\xi_j)}\right)}{\xi_i + \xi_j}$, $1 \leq i, j \leq M$.
- At the optimal P_{opt} , we have $\Delta(P_{opt})$ singular and hence $|S_{22}(j\omega)| = |U_{P_{opt}}(j\omega)|$.

In both chapters, we are able to furnish optimal matching circuits of maximal McMillan degree $N - 1$ by using reference functions of degree N .

3.5.1 Pick Matrix Positivity from Nehari Test

In this subsection we will give a brief overview of the classical connection between Hankel operators and Nevanlinna-Pick matrix to an analytic interpolation problem. Then, we will illustrate this connection in our setting with respect to the approaches discussed in chapter 2 and 3 for solving the matching problem.

Hankel Operator and Nevanlinna-Pick Matrix

The connection of Hankel operators and Nevanlinna-Pick matrix to analytic interpolation problems are quite classic (Chapter 1, Section 4, [71]). In order to briefly describe this connection, we consider the following problem: Let ξ_m , $1 \leq m \leq M$ be distinct points in Π^+ and let γ_m , $1 \leq m \leq M$ be complex numbers such that $|\gamma_m| < 1$, $1 \leq m \leq M$; under which conditions, there exists a function $f \in H^\infty(\Pi^+)$ such that

$$f(\xi_m) = \gamma_m, \quad 1 \leq m \leq M, \quad \|f\|_{H^\infty} \leq 1. \quad (3.133)$$

The problem (3.133) can be easily reduced to a problem about the Hankel operators. Let g be an arbitrary function in $H^\infty(\Pi^+)$ for which $g(\xi_m) = \gamma_m$, $1 \leq m \leq M$. For example, g can be constructed as the Lagrange interpolating polynomial that assumes at each value ξ_m , the corresponding value γ_m . If b is the Blaschke product with zeros at ξ_m ,

$$b(s) = \prod_{m=1}^M \frac{s - \xi_m}{s + \overline{\xi_m}},$$

then a function $f \in H^\infty(\Pi^+)$ interpolates the values γ_m at ξ_m if and only if f has the form $f = g - bh$, $h \in H^\infty$. So, it follows from Nehari theorem (Theorem 2.2.6) that the problem in (3.133) is solvable if and only if

$$\inf_{h \in H^\infty} \|g - bh\|_\infty = \inf_{h \in H^\infty} \|\bar{b}g - h\|_\infty = \|H_{\bar{b}g}\| \leq 1,$$

where $H_{\bar{b}g}$ is the Hankel operator with symbol $\bar{b}g$. The condition $\|H_{\bar{b}g}\| \leq 1$ can be easily reformulated in terms of the interpolation data as well (Chapter 1, Theorem 4.1, [71]):

Theorem 3.5.1. *The interpolation problem in (3.133) is solvable if and only if the matrix*

$$\left[\frac{1 - \bar{\gamma}_m \gamma_n}{\bar{\xi}_m + \xi_n} \right]_{1 \leq m, n \leq M}$$

is positive semi-definite.

The matrix in theorem 3.5.1 is the Nevanlinna-Pick matrix associated with the interpolation data in problem (3.133). Having described this classical connection between Hankel operators and Nevanlinna-Pick matrix to analytic interpolation problem in (3.133), we are in a position to connect the approaches discussed in chapter 2 and chapter 3.

Nehari Test and Pick Matrix Test for Matching Problem

In order to easily depict the connection between approaches discussed in chapter 2 and chapter 3, we consider the following setting : we are given a passband \mathbb{I} , reflection coefficient $L_{11} \in \mathbb{B}$ and a reference function $k_\alpha = \frac{p_k}{q_k}$ of degree N as described in case (ii) in subsection 2.3.1 in chapter 2 which satisfies $q_k q_k^* - p_k p_k^* = \alpha r_L r_L^*$. We will consider the Darlington extension L of the load with transmission zeros $\xi_1, \xi_2, \dots, \xi_M$ in Π^+ and consider the problem: does there exist $F_{22}(s) \in H^\infty(\Pi^+)$ such that

$$|(F_{22} \circ L)(j\omega)| \leq |k_\alpha(j\omega)|, \quad \omega \in \mathbb{R} ? \quad (3.134)$$

If yes, find the F_{22} which achieves this. It should be noted that this problem is equivalent to the problem (2.45) stated in chapter 2, but after considering the Darlington extension of the load and then using port 2 of the global system to state the problem. By the definition of chaining operation in (3.3), the inequality in (3.134) implies that on the imaginary axis,

$$\left| L_{22} + \frac{L_{12} L_{21} F_{22}}{1 - F_{22} L_{11}} \right| \leq |k_\alpha|. \quad (3.135)$$

We define the Blaschke product $b(s)$ with zeros ξ_m , $1 \leq m \leq M$,

$$b(s) = \frac{r_L^*}{r_L} = \prod_{m=1}^M \frac{s - \xi_m}{s + \bar{\xi}_m}. \quad (3.136)$$

So, problem (3.134) is equivalent to finding $F_{22} \in H^\infty(\Pi^+)$ such that

$$\left| \bar{b} L_{22} + \frac{\bar{b} L_{12} L_{21} F_{22}}{1 - F_{22} L_{11}} \right| \leq |k_\alpha| \quad (3.137)$$

on the imaginary axis. That is,

$$|\Phi(j\omega) - \Psi(j\omega)| \leq 1, \quad \omega \in \mathbb{R}, \quad (3.138)$$

where

$$\Phi = \frac{1}{k_\alpha} \bar{b} L_{22}, \quad \Psi = -\frac{1}{k_\alpha} \frac{\bar{b} L_{12} L_{21} F_{22}}{1 - F_{22} L_{11}}. \quad (3.139)$$

It should be noted that k_α , $\bar{b} L_{12} L_{21}$ and $1 - F_{22} L_{11}$ are invertible outer functions in $H^\infty(\Pi^+)$ and so if $F_{22} \in H^\infty(\Pi^+)$ satisfy (3.134), Ψ belongs to $H^\infty(\Pi^+)$ as well. Furthermore, we can also prove that, if $\Psi \in H^\infty(\Pi^+)$ satisfies (3.138), then F_{22} is in $H^\infty(\Pi^+)$ and satisfies $|F_{22}| \leq 1$ on the imaginary axis. This can be seen as follows, let us denote $V = -\frac{\bar{b} L_{12} L_{21}}{k_\alpha}$ which is an invertible outer function in $H^\infty(\Pi^+)$. From the equation for Ψ in (3.139), it follows that

$$F_{22} = \frac{\Psi}{V + \Psi L_{11}}. \quad (3.140)$$

We have $\Psi = \frac{V F_{22}}{1 - F_{22} L_{11}}$ and

$$V + \Psi L_{11} = V + \frac{V F_{22} L_{11}}{1 - F_{22} L_{11}} = \frac{V}{1 - F_{22} L_{11}}. \quad (3.141)$$

We have $\frac{V}{1 - F_{22} L_{11}}$ an invertible outer function in $H^\infty(\Pi^+)$ and hence equation (3.141) implies that $\frac{1}{V + \Psi L_{11}}$ is in $H^\infty(\Pi^+)$ as well. This implies that $F_{22} \in H^\infty(\Pi^+)$. Moreover, if $\Psi \in H^\infty(\Pi^+)$ satisfies (3.138), we have $F_{22} \in H^\infty(\Pi^+)$ of the form in (3.140) satisfying (3.135). This implies that $\delta(F_{22}(j\omega), \overline{L_{11}(j\omega)}) \leq |k_\alpha(j\omega)|$ for all $\omega \in \mathbb{R}$ and hence $|F_{22}| \leq 1$ on the imaginary axis. So, the problem reduces to the classical Nehari problem:

Problem. Given $\Phi \in L^\infty(j\mathbb{R})$,

$$\min_{\Psi \in H^\infty} \|\Phi - \Psi\|_{L^\infty} \quad (3.142)$$

and find a Ψ at which the infimum is attained if the minimum is less than or equal to one.

Let us denote $g = \frac{L_{22}}{k_\alpha} \in H^\infty$. So, similar to the Nehari Test I that was discussed in chapter 2 to solve the problem in (2.45), we are able to propose the following test using the Nehari theorem for solving the problem in (3.134).

Nehari Test II : The problem in (3.134) is solvable if and only if $\|H_{\bar{b}g}\| \leq 1$, where $H_{\bar{b}g}$ is the Hankel operator with symbol $\bar{b}g$.

It should be noted that unlike in chapter 2, we are able to arrive at this test without passing through the non-Euclidean setting. This condition can be equivalently expressed as the positivity of Nevanlinna Pick matrix as follows. Since b and g are coprime (no common nonconstant inner divisors), it follows from Theorem 2.4 in [71] that $\text{Ker } H_{\bar{b}g} = bH^2$ and we have the orthogonal complement of kernel of Hankel operator H_Φ in H^2 ,

$$(\text{Ker } H_\Phi)^\perp = H^2 \ominus bH^2 = \text{span} \left\langle f_m(s) \stackrel{\text{def}}{=} \frac{b(s)}{s - \xi_m} \right\rangle. \quad (3.143)$$

The condition $\|H_{\bar{b}g}\| \leq 1$ holds if and only if

$$H_{\bar{b}g}^* H_{\bar{b}g} \leq I_M, \quad (3.144)$$

where $H_{\bar{b}g}^*$ represents the adjoint of $H_{\bar{b}g}$ and I_M the identity matrix of size $M \times M$. So, we have for any complex numbers c_m , $1 \leq m \leq M$,

$$\sum_{m,n=1}^M c_m \bar{c}_n \langle H_{\bar{b}g} f_m, H_{\bar{b}g} f_n \rangle \leq \sum_{m,n=1}^M c_m \bar{c}_n \langle f_m, f_n \rangle. \quad (3.145)$$

We have,

$$H_{\bar{b}g} f_m = P \left(\frac{g}{s - \xi_m} \right) = P \left(\frac{g - g(\xi_m)}{s - \xi_m} + \frac{g(\xi_m)}{s - \xi_m} \right) = \frac{g(\xi_m)}{s - \xi_m}. \quad (3.146)$$

So,

$$\langle H_{\bar{b}g} f_m, H_{\bar{b}g} f_n \rangle = g(\xi_m) \overline{g(\xi_n)} \left\langle \frac{1}{s - \xi_m}, \frac{1}{s - \xi_n} \right\rangle = \overline{g(\xi_m)} g(\xi_n) \frac{1}{\bar{\xi}_m + \xi_n}. \quad (3.147)$$

The last equality follows using Cauchy's integral theorem. We also have,

$$\langle f_m, f_n \rangle = \left\langle \frac{b}{s - \xi_m}, \frac{b}{s - \xi_n} \right\rangle = \frac{1}{\bar{\xi}_m + \xi_n}. \quad (3.148)$$

Thus, it follows from (3.145) that

$$\sum_{m,n=1}^M c_m \bar{c}_n \frac{1 - \overline{g(\xi_m)} g(\xi_n)}{\bar{\xi}_m + \xi_n} \geq 0. \quad (3.149)$$

That is the problem in (3.142) is solvable if and only if the matrix

$$\left[\frac{1 - \frac{\overline{L_{22}(\xi_m)}}{k_\alpha(\xi_m)} \frac{L_{22}(\xi_n)}{k_\alpha(\xi_n)}}{\bar{\xi}_m + \xi_n} \right]_{1 \leq m, n \leq M}$$

is positive semi-definite. Thus, we have arrived at the Pick matrix positivity constraint as stated in proposition 3.3.2 with reference function U_P replaced by the reference function k_α .

Remark 3.5.2. *It should be noted that solving the Nehari problem in (3.142) with L^∞ function to be approximated, $\Phi = \frac{1}{k_\alpha} \bar{b} L_{22}$ is a possible alternative to the Nehari problem in (2.47) in chapter 2 which had $\Phi = \frac{1}{V} \left(\frac{1 - |k_\alpha|^2}{1 - |k_\alpha|^2 |L_{11}|^2} \right) \overline{L_{11}}$. The reference function k_α enters the formulas in (3.139) in the Nehari test linearly in this new setting (this was not the case in (2.47)).*

3.5.2 Nehari Test II for Finite Degree Matching Problem

Before concluding this section, we will discuss some more important details of solving the problem in (3.134) using the Nehari test II discussed in the previous subsection. As done in chapter 2, the idea is to find the smallest possible power mismatch of the form k_α . That is to solve the problem : does there exist $F_{22}(s) \in H^\infty(\Pi^+)$ such that

$$|(F_{22} \circ L)(j\omega)| = |k_\alpha(j\omega)|, \quad \omega \in \mathbb{R} ? \quad (3.150)$$

If yes, find the F_{22} which achieves this. This can be done by performing the Nehari test II of $\|H_\Phi\| \leq 1$ by iterating $|k_\alpha|$. The test can be performed on a grid of α 's and the solution to problem in (3.150) can be obtained once we find the $\hat{\alpha}$ at which $\|H_\Phi\| = 1$ (a possible update rule for α is suggested at the end of this subsection). The same numerical implementation scheme discussed in subsection 2.3.3 in chapter 2 can be used to solve the Nehari problem in (3.134).

Now, we will provide an important result about the estimate of the degree of the rational Schur function F_{22} and \hat{F}_{22} obtained as the solution of problems in (3.134) and (3.150) respectively. As discussed in the proof of the proposition (2.3.3) in chapter 2, if we denote the maximizing vector of the Hankel operator in Nehari problem in (3.142) by W , we have

$$W = \frac{a_0}{r_L}, \quad H_\Phi(W) = \frac{b_0}{r_L^*}, \quad (3.151)$$

where a_0 and b_0 are polynomials of at most degree $M - 1$. In addition, it from theorem 1.4 in [71] that

$$b_0 = \tau a_0^*, \quad \tau \in \mathbb{C} \text{ satisfying } |\tau| = \|H_\Phi\| \quad (3.152)$$

and a_0 is Hurwitz polynomial. So, the error function in Nehari approximation,

$$\Phi - \Psi = \frac{H_\Phi(W)}{W} = \tau \frac{a_0^* r_L}{a_0 r_L^*}. \quad (3.153)$$

It follows from the equations in (3.139) that

$$\Phi - \Psi = \frac{\bar{b}}{k_\alpha} (F_{22} \circ L) = \frac{S_{22} r_L}{k_\alpha r_L^*}, \quad (3.154)$$

where $S_{22} = F_{22} \circ L$ is the output reflection coefficient of the global system. Comparing equations in (3.153) and (3.154), we have the output reflection coefficient of the global system,

$$S_{22} \stackrel{\text{def}}{=} \frac{p}{q} = \tau \frac{a_0^*}{a_0} k_\alpha = \frac{\tau a_0^* p_k}{a_0 q_k}. \quad (3.155)$$

It should be noted that S_{22} is of at most degree $N + M - 1$ and we have,

$$qq^* - pp^* = a_0 a_0^* (q_k q_k^* - |\tau|^2 p_k p_k^*). \quad (3.156)$$

At the optimal α , namely $\hat{\alpha}$ at which $\|H_\Phi\| = 1$, we have $|\tau| = 1$ and hence we have,

$$|\hat{S}_{22}(j\omega)| = |k_{\hat{\alpha}}(j\omega)|, \quad \omega \in \mathbb{R},$$

the smallest possible power mismatch of the form k_α . Also, $\hat{S}_{22} = \frac{\hat{p}}{\hat{q}}$ satisfies,

$$\hat{q}\hat{q}^* - \hat{p}\hat{p}^* = a_0 a_0^* (q_k q_k^* - p_k p_k^*) = \hat{\alpha} a_0 a_0^* r_L r_L^* \quad (3.157)$$

which implies that the global system contains the transmission zeros of the load. The reflection coefficient of the matching filter providing the optimal global reflection can be obtained by de-chaining the load from \hat{S}_{22} ,

$$\hat{F}_{22} = \frac{\hat{S}_{22} - L_{22}}{\hat{S}_{22} L_{11} - \det(L)}. \quad (3.158)$$

As explained in subsection 3.4.1, since the optimal global system contains the transmission zeros of the load, the de-chaining gives $\hat{F}_{22} = \frac{p_{\hat{F}}}{q_{\hat{F}}}$, where $p_{\hat{F}}, q_{\hat{F}} \in \mathbb{P}_{N-1}$ and thus we have the optimal matching filter of McMillan degree $N - 1$. In the non-optimal feasible case, i.e when $\|H_\Phi\| < 1$, it follows from equation (3.156) that the global system synthesized doesn't contain the transmission zeros of the load. This implies that the matching filter obtained in this case by de-chaining the load from S_{22} will be of degree $N + M - 1$.

An important point to note, comparing the global reflection coefficient obtained in two approaches (3.117 and 3.155) is that among all the possible H^∞ functions, Nehari test II provides the minimal norm solution $S_{22} = F_{22} \circ L$ to the problem in 3.134. When it comes to the degree of the matching filter, considering the non-optimal feasible situations in the described approaches (pick matrix strictly positive definite and operator norm of Hankel operator strictly less than one), it follows from the explanation in this section and subsection 3.4.1 that the extraction of matching filter from S_{22} synthesized in (3.117 : Pick matrix approach) provides a better degree (lower) than the S_{22} synthesized in (3.155: Nehari approach). Before concluding this section, we will provide a couple of remarks concerning the Nehari test II and its implementation.

Remark 3.5.3. (*Monotonicity of Hankel Operator Norm*). Let us consider the problem in (3.134) using reference functions k_{α_1} and k_{α_2} where $\alpha_1 < \alpha_2 \leq \hat{\alpha}$. It should be noted that for any $\omega \in \mathbb{R}$, $|k_{\alpha_1}(j\omega)| > |k_{\alpha_2}(j\omega)|$. For the reference functions k_{α_1} and k_{α_2} , let us denote the Hankel symbol defined in (3.139) as Φ_{α_1} and Φ_{α_2} and the the global reflection coefficients obtained via Nehari test II as $S_{22}^{\alpha_1} = F_{22}^{\alpha_1} \circ L$ and $S_{22}^{\alpha_2} = F_{22}^{\alpha_2} \circ L$ respectively. We have,

$$|S_{22}^{\alpha_2}(j\omega)| \leq |k_{\alpha_2}(j\omega)| < |k_{\alpha_1}(j\omega)|, \quad \omega \in \mathbb{R} \quad (3.159)$$

and so $F_{22}^{\alpha_2}$ is a solution to the problem in (3.134) with reference function k_{α_1} as well. From Nehari theorem, we have

$$\|H_{\Phi_{\alpha_1}}\| = \left\| \frac{S_{22}^{\alpha_1}}{k_{\alpha_1}} \right\|_{L^\infty} \leq \left\| \frac{S_{22}^{\alpha_2}}{k_{\alpha_1}} \right\|_{L^\infty} < \left\| \frac{S_{22}^{\alpha_2}}{k_{\alpha_2}} \right\|_{L^\infty} = \|H_{\Phi_{\alpha_2}}\|.$$

The first inequality follows since $S_{22}^{\alpha_1}$ is build from the best H^∞ approximant Ψ_{α_1} to Φ_{α_1} in the L^∞ -norm and the second strict inequality follows since for any $\omega \in \mathbb{R}$, $|k_{\alpha_1}(j\omega)| > |k_{\alpha_2}(j\omega)|$.

Remark 3.5.4. (*Update Rule for α*). In order to solve the problem in (3.150) using Nehari test II, a possible update rule for α is described in this remark. Let α_i denote an α at which $\|H_{\Phi_{\alpha_i}}\| < 1$. From equation (3.155), on the imaginary axis we have

$$|S_{22}^{\alpha_i}|^2 = |\tau_i|^2 |k_{\alpha_i}|^2 = |\tau_i|^2 \frac{|p_k|^2}{|p_k|^2 + \alpha_i |r_L|^2},$$

where $|\tau_i| = \|H_{\Phi_{\alpha_i}}\|$. At the next iteration, imposing the following condition on the normalized passband $[-1j, 1j]$,

$$|S_{22}^{\alpha_i}(j\omega)|^2 \leq |k_{\alpha_{i+1}}(j\omega)|^2,$$

we have the following upper bound on α_{i+1} in the passband,

$$\alpha_{i+1} \leq \frac{(1 - |\tau_i|^2 |p_k|^2) + \alpha_i |r_L|^2}{|\tau_i|^2 |r_L|^2}.$$

Thus,

$$\alpha_{i+1} = \max_{j\omega \in [-1j, 1j]} \frac{(1 - |\tau_i|^2 |p_k(j\omega)|^2) + \alpha_i |r_L(j\omega)|^2}{|\tau_i|^2 |r_L(j\omega)|^2}$$

is a possible update for α_i . Starting from an α_0 at which $\|H_{\Phi_{\alpha_0}}\| < 1$, this update can be used to find the $\hat{\alpha}$ at which the operator norm of $\|H_{\Phi}\|$ become equal to one.

We conclude here this section about the brief description of the comparisons and the connection between approaches discussed in chapter 2 and chapter 3 to solve the matching problem.

3.6 Conclusion

In this chapter, we have presented a convex optimisation approach for broadband impedance matching problem mainly based on Youla's matching theory and theory of analytic interpolation. Most of the important theoretical aspects regarding the developed approach along with some illustrations of practical implementation have been done. As it is usual in classical filter synthesis, the transmission polynomial of the global system to be synthesized was supposed to be fixed. Then the positivity of the Pick matrix based on the de-embedding conditions was used to test the realizability criterion. As we have discussed, the solution to convex formulation of the global matching problem (problem \mathfrak{P}_C) gives us $P_{opt} \in \mathbb{H}_R^N$ and $b_{P_{opt}} U_{P_{opt}}$ provides the optimal global system of maximal McMillan degree $N + M - 1$. Hence, this provides an efficient method of finding optimal matching circuits of maximal McMillan degree $N - 1$ for a given load and a class of realizable Schur functions with fixed transmission polynomial. We have also provided a brief overview of the comparisons and the connection of approaches discussed in part I of the thesis to solve the matching problem. This study has made a clear link between the Nehari approach and Nevanlinna-Pick approach with reference to the broadband matching problem.

With this chapter, we conclude part I of the thesis. We have made a theoretical study of the broadband matching problem in this part along with the illustration of the methods

developed on some antenna examples. The second part of the thesis will move further ahead in the practical direction. In this part, the matching problems will be formulated in such a way as to easily take into account more practical considerations concerning the realization of the matching circuit. We will formulate the matching problems in the real setting and present results obtained by using the matching circuits obtained on some antenna prototypes. A direct formulation of the matching problem where it is casted over the set of reflection polynomials of the matching circuit (of finite degree) which we are seeking will be made. The practical constructability of matching circuits as ladder networks consisting of inductors and capacitors and methods to ensure transformerless synthesis of circuits, bounding element values of capacitors and inductors and including microstrip lines in between the elements will be presented as well.

PART II

Practical Applications and Design Implementation

Matching Problem in Real Setting and Ladder Network Synthesis

4.1 Introduction

In this chapter, we will do a detailed study of the different realizability constraints that need to be taken into account in the case of matching circuit synthesis using L, C components. The problem of impedance matching circuit synthesis has been evolving for around past 80 years and the reader can find a good amount of literature related to the topic (some of these are [73], [95], [58], [93], [90], [3], [51]). In comparison to the approaches developed in chapter 2 and 3 of the thesis, the problems studied in this chapter will impose additional constraints on the matching circuit to be synthesized in order to meet different specifications as required by the application. We will introduce the matching problem in the real setting where broadband matching problem is casted directly over the set of reflection polynomials of matching network (p in $\mathbb{P}_{\mathbb{R},N}$, polynomials of degree at most N with real coefficients). The importance of this problem lies in the fact that it is the simplest formulation of the matching problem in such a way that the optimal matching circuit can be realized using finite number of inductors and capacitors. As the chapter progress, it will also be clear that in this setting many practical constraints concerning the realization of the matching circuit can be easily incorporated and implemented as well. Some of the important theoretical results like existence of the solution for this optimisation problem and a result about the characterisation of the optimal solution will be done. Later on in the chapter, a series of different, yet closely related problems are introduced in order to tackle the problems of transformerless synthesis of the matching circuit, bounding the values of the lumped elements in the matching circuit to desired ranges as required by the user and presence of microstrip lines in between the lumped elements. A description about the numerical implementation of different problems considered in the chapter will be provided. An example of the numerical illustration of each problem introduced and the realization of the optimal matching circuit obtained using ladder networks will be done as well. For any given load, the matching criterion provided in this chapter will be inferior to the one in chapter 3 of the thesis for a fixed McMillan degree of the matching circuit but the optimal scattering matrix is ensured to meet the additional specifications of realizability as discussed.

Throughout the numerical implementations discussed in this chapter, the transmission zeros of the scattering matrix of the matching circuit to be synthesized will be fixed at either zero or infinity (or both) in order to ensure the realizability of the matching circuit using series or parallel inductors or capacitors. In the scattering matrix of the matching circuit to be synthesized, this corresponds to fixing $r = s^n$, $0 \leq n \leq N$ and it is a choice made for the practical ease of realization of the circuit. The angular derivative of the reflection coefficient at the transmission zero on the imaginary axis will play an important role in parameterising the S -matrix of these elementary L, C -sections. In the next subsection, we will provide a brief description of angular derivatives.

4.1.1 Angular Derivatives

Definition 4.1.1. (*Angular Derivative*). Let S represent the (2×2) scattering matrix of a lossless system having $\alpha_0 = j\omega_0$, $\omega_0 \in \mathbb{R}$ as a transmission zero on the imaginary axis (possibly $\alpha_0 = \infty$ as well). Then the angular derivative of S_{22} at α_0 is defined as the quantity,

$$\text{ang}[S_{22}](\alpha_0) = \overline{S_{22}(\alpha_0)} S'_{22}(\alpha_0),$$

where $S'_{22}(\alpha_0)$ represent the quantity $\frac{d}{ds}(S_{22}(s))\big|_{s=\alpha_0} = \frac{1}{j} \frac{d}{d\omega}(S_{22}(j\omega))\big|_{\omega=\omega_0}$.

It should be noted that in definition 4.1.1, since α_0 is a transmission zero of a lossless scattering matrix S on the imaginary axis, we have,

$$|S_{22}(\alpha_0)| = |S_{11}(\alpha_0)| = 1. \quad (4.1)$$

Remark 4.1.2. The angular derivative of S_{22} is the derivative of phase of S_{22} with respect to the frequency. It easily follows from the equalities,

$$\begin{aligned} \frac{d}{d\omega} \left(\ln(S_{22}(j\omega)) \right) \bigg|_{\omega=\omega_0} &= j \text{ang}[S_{22}](j\omega_0), \\ \frac{d}{d\omega} \left(\ln(S_{22}(j\omega)) \right) \bigg|_{\omega=\omega_0} &= j \frac{d}{d\omega} \left(\text{Im}(\ln(S_{22}(j\omega))) \right) \bigg|_{\omega=\omega_0}, \end{aligned}$$

where $\text{Im}(\cdot)$ represents the imaginary part. Thus, we have the following equalities for the angular derivative of S_{22} at $\alpha_0 = j\omega_0$,

$$\begin{aligned} \text{ang}[S_{22}](\alpha_0) &= -j \frac{d}{d\omega} \left(\ln(S_{22}(j\omega)) \right) \bigg|_{\omega=\omega_0} \\ &= \frac{d}{d\omega} \left(\text{Im}(\ln(S_{22}(j\omega))) \right) \bigg|_{\omega=\omega_0} = \frac{d}{d\omega} \left(\arg(S_{22}(j\omega)) \right) \bigg|_{\omega=\omega_0}. \end{aligned} \quad (4.2)$$

A useful property of the angular derivatives easily follows from the first equality in (4.2),

$$\text{ang}[S_{11}S_{22}](\alpha_0) = \text{ang}[S_{11}](\alpha_0) + \text{ang}[S_{22}](\alpha_0). \quad (4.3)$$

From Cauchy-Riemann equations, we also have,

$$\left. \frac{d}{d\omega} \left(\operatorname{Im}(\ln(S_{22}(j\omega))) \right) \right|_{\omega=\omega_0} = \left. \frac{d}{dx} \left(\operatorname{Re}(\ln(S_{22}(x + j\omega_0))) \right) \right|_{x=0},$$

where $\operatorname{Re}(\cdot)$ represents the real part. Thus we have the following equation for angular derivative of S_{22} at $\alpha_0 = j\omega_0$ as well,

$$\operatorname{ang}[S_{22}](\alpha_0) = \left. \frac{d}{dx} \left(\operatorname{Re}(\ln(S_{22}(x + j\omega_0))) \right) \right|_{x=0} = \left. \frac{d}{dx} \left(\ln(|S_{22}(x + j\omega_0)|) \right) \right|_{x=0}.$$

Now, we will state and prove an important property of angular derivatives. For a detailed review about the angular derivatives we refer the reader to Chapter 21, [8].

Proposition 4.1.3. *The angular derivative of reflection coefficient of a lossless system at a simple transmission zero, $\alpha_0 = j\omega_0$ on the imaginary axis,*

$$\operatorname{ang}[S_{22}](\alpha_0) = \overline{S_{22}(\alpha_0)} S'_{22}(\alpha_0),$$

is a non-positive real quantity (i.e $\operatorname{ang}[S_{22}](\alpha_0) \leq 0$). Moreover, if S_{22} is not a uni-modular constant function, it satisfies,

$$\operatorname{ang}[S_{22}](\alpha_0) < 0.$$

Proof. Let us denote $m(\omega) = |S_{22}(j\omega)|^2 = \overline{S_{22}(j\omega)} S_{22}(j\omega)$, where ω is real. Since by assumption, $\alpha_0 = j\omega_0$ is a transmission zero of S on the imaginary axis, $m(\omega)$ attains a maximum at $\omega = \omega_0$. So, we have,

$$\begin{aligned} 0 &= \frac{dm}{d\omega}(\omega_0) \\ &= -j \overline{S'_{22}(j\omega_0)} S_{22}(j\omega_0) + j \overline{S_{22}(j\omega_0)} S'_{22}(j\omega_0) \\ &= j \left(\overline{S_{22}(j\omega_0)} S'_{22}(j\omega_0) - \overline{S'_{22}(j\omega_0)} S_{22}(j\omega_0) \right) \\ &= -2 \operatorname{Im}(\overline{S_{22}(j\omega_0)} S'_{22}(j\omega_0)). \end{aligned}$$

This implies $\overline{S_{22}(\alpha_0)} S'_{22}(\alpha_0) = \operatorname{ang}[S_{22}](\alpha_0)$ is a real quantity. Now, in order to prove that $\operatorname{ang}[S_{22}](\alpha_0)$ is non-positive, let us define, for $r > 0$,

$$M(r) = \frac{|S_{22}(j\omega_0 + r)|^2 - 1}{2r}.$$

By definition $M(r) < 0$ for any $r > 0$. We can compute the limit as $r \rightarrow 0$ to get,

$$\begin{aligned} 0 &\geq \lim_{r \rightarrow 0} M(r) \\ &= \lim_{r \rightarrow 0} \frac{|S_{22}(j\omega_0 + r)|^2 - 1}{2r} \\ &= \lim_{r \rightarrow 0} \frac{\overline{S_{22}(j\omega_0 + r)} S'_{22}(j\omega_0 + r) + \overline{S'_{22}(j\omega_0 + r)} S_{22}(j\omega_0 + r)}{2} \\ &= \operatorname{Re}(\overline{S_{22}(j\omega_0)} S'_{22}(j\omega_0)) \\ &= \overline{S_{22}(j\omega_0)} S'_{22}(j\omega_0). \end{aligned}$$

The last equality follows since we have already proved $\overline{S_{22}(\alpha_0)}S'_{22}(\alpha_0) = \text{ang}[S_{22}](\alpha_0)$ is a real quantity. So, we have,

$$\text{ang}[S_{22}](\alpha_0) \leq 0.$$

Now, let us assume that S_{22} is not a uni-modular constant function. We claim that $\text{ang}[S_{22}](\alpha_0) = \overline{S_{22}(\alpha_0)}S'_{22}(\alpha_0) < 0$. Suppose that $S'_{22}(\alpha_0) = 0$. Since we assume that S_{22} is not a uni-modular constant, there should exist atleast one non-vanishing derivative of S_{22} at α_0 . Let us say $S_{22}^{(k)}$, where $k > 1$ is the first non-vanishing derivative. For any point in an open neighbourhood of α_0 , it follows from the Taylor series expansion,

$$\begin{aligned} S_{22}(\alpha_0 + s) &= S_{22}(\alpha_0) + S_{22}^{(k)}(\alpha_0)s^k + o(s^k) \\ &= S_{22}(\alpha_0)(1 + \overline{S_{22}(\alpha_0)}S_{22}^{(k)}(\alpha_0)s^k) + o(s^k). \end{aligned}$$

Since $|S_{22}| \leq 1$ in Π^+ , it follows that,

$$\forall r \geq 0 \text{ and } \forall \theta \in [-\pi/2, \pi/2], \quad |1 + \overline{S_{22}(\alpha_0)}S_{22}^{(k)}(\alpha_0)r^k e^{jk\theta}| \leq 1.$$

This is a contradiction since the argument $k\theta$ ranges over $[0, 2\pi]$ in this case. This yields $S'_{22}(\alpha) \neq 0$ and hence, $\text{ang}[S_{22}](\alpha_0) < 0$ when S_{22} is not a uni-modular constant. \square

Remark 4.1.4. *The angular derivative of S_{22} can be related to the derivative of modulus square of S_{22} in the x -direction. We have,*

$$\begin{aligned} \left. \frac{d}{dx} \left(S_{22}(j\omega_0 + x) \overline{S_{22}(j\omega_0 + x)} \right) \right|_{x=0} &= 2 \text{Re}(\overline{S_{22}(j\omega_0)} S'_{22}(j\omega_0)) \\ &= 2 \text{Re}(\text{ang}[S_{22}](\alpha_0)) \\ &= 2 \text{ang}[S_{22}](\alpha_0). \end{aligned}$$

The last equality follows from proposition 4.1.3.

4.2 Matching Problem in Real Setting

We will use the same setting of matching circuit chained to the given load as introduced in chapter 2 (figure 4.1) to formulate the matching problem in the real setting. Let the given passband of interest be denoted by the union of frequency ranges $a_1\text{Hz}$ to $b_1\text{Hz}$, $a_2\text{Hz}$ to $b_2\text{Hz}$, ..., $a_m\text{Hz}$ to $b_m\text{Hz}$, where $0 \leq a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$. It will be normalized to be between 0 and $1j$ on the imaginary axis and we denote $\mathbb{I} = [\frac{a_1}{b_m}j, \frac{b_1}{b_m}j] \cup [\frac{a_2}{b_m}j, \frac{b_2}{b_m}j] \cup \dots \cup [\frac{a_m}{b_m}j, 1j]$. The reflection coefficient of the load is denoted by a non-constant Schur function, $L_{11} \in \mathbb{B}$ and we assume its modulus to be not uniformly equal to one for all frequencies in the passband. Throughout the chapter, we also assume

$$L_{11}(-j\omega) = \overline{L_{11}(j\omega)}, \quad \omega \in \mathbb{R}. \quad (4.4)$$

It guarantees that the time domain impedance of the load is real (Equation 4.22, [77]). We are interested in finding the scattering matrix of the matching network, F which minimize

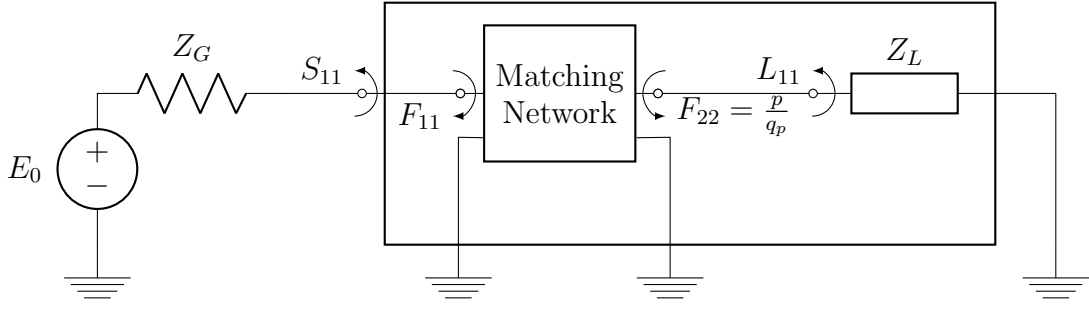


Figure 4.1: Matching Network Connected to One-Port Load

in \mathbb{I} the modulus of input reflection coefficient of the system obtained by chaining the matching network to the load. Belevitch theorem will be used to represent the scattering matrix F of the matching network to be synthesized. Let us denote,

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \frac{1}{q} \begin{pmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{pmatrix},$$

where ϵ is a unimodular constant and p, q, r are polynomials of degree at most N with real coefficients (hereinafter denoted by $\mathbb{P}_{\mathbb{R},N}$) satisfying the Feldtkeller equation, $qq^* = pp^* + rr^*$. It should be noted that we restrict to this subspace of \mathbb{P}_N since we are interested in realizing the matching circuit with real L, C components. Furthermore, we assume the transmission polynomial $R = rr^*$ of the matching circuit is fixed and $r \neq 0$ has no zeroes inside the passband \mathbb{I} . As mentioned in chapter 2, we have the modulus of the input reflection of the chained system equal to the pseudo-hyperbolic distance between F_{22} and $\overline{L_{11}}$ on the imaginary axis,

$$|S_{11}(s)| = \left| \frac{F_{22}(s) - \overline{L_{11}(s)}}{1 - F_{22}(s)L_{11}(s)} \right| = \left| \frac{\frac{p(s)}{q(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q(s)}L_{11}(s)} \right|.$$

Let us denote by $\mathbb{SB}_{\mathbb{R},N}$ the polynomials of degree at most N with real coefficients which are stable in the broad sense (no zeros in Π^+). It should be noted that in figure 4.1, we have used the notation, $F_{22} = \frac{p}{q_p}$. The subscript in q is used to denote that for any given polynomial $p \in \mathbb{P}_{\mathbb{R},N}$, the polynomial $q \in \mathbb{SB}_{\mathbb{R},N}$ satisfying the spectral equation can be determined from p . To be precise, this follows from the following proposition proved in [10],

Proposition 4.2.1. *To any non zero $Q \in \mathbb{P}_{2N}^+$, one can associate $q \in \mathbb{SB}_N$ such that for any $s = j\omega$, $\omega \in \mathbb{R}$, we have,*

$$Q(s) = |q(s)|^2 = q(s)q^*(s). \quad (4.5)$$

The polynomial $q(s)$ is unique upto a unimodular constant and if Q has exact degree $2N$ then q has exact degree N . Moreover, for fixed $z_0 > 0$, the map

$$f_{z_0} : \mathbb{P}_{2N}^+ \setminus \{0\} \rightarrow \mathbb{SB}_N$$

with $f_{z_0}(Q)$ the unique solution to (4.5) meeting $q(z_0) > 0$ is continuous.

So, when the polynomial $r \neq 0$ in $\mathbb{P}_{\mathbb{R},N}$ is fixed, it follows from proposition 4.2.1 that for any $p \in \mathbb{P}_{\mathbb{R},N}$ one can associate $q_p \in \mathbb{SB}_{\mathbb{R},N}$ as follows

$$p \xrightarrow{g} pp^* + rr^* \stackrel{\text{def}}{=} Q \xrightarrow{f_{z_0}} q_p.$$

The map $f_{z_0} \circ g : \mathbb{P}_{\mathbb{R},N} \rightarrow \mathbb{SB}_{\mathbb{R},N}$ is continuous. From here on we will use q_p instead of q to mark the above association of q with p . Now, we are in a position to formulate the matching problem in the real setting,

Problem. \mathcal{P} . *Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4) and a polynomial $r \in \mathbb{P}_{\mathbb{R},N}$ which doesn't vanish in \mathbb{I} , where N is the target degree of the matching circuit,*

$$\text{Find: } \hat{l} = \min_{p \in \mathbb{P}_{\mathbb{R},N}} \max_{s \in \mathbb{I}} \left| \frac{\frac{p(s)}{q_p(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q_p(s)} L_{11}(s)} \right|$$

$$\text{where: } q_p q_p^* = pp^* + rr^* \quad \text{and } q_p \in \mathbb{SB}_{\mathbb{R},N}.$$

It should be noted that $\hat{l} \neq 0$ since $\overline{L_{11}(s)}$ corresponds to the evaluation on the imaginary axis of $L_{11}^*(s)$ which is an anti-analytic function and hence it is not possible to find $F_{22} = \frac{p}{q_p} \in \mathbb{B}$ satisfying, for all $s \in \mathbb{I}$, $F_{22}(s) = \overline{L_{11}(s)}$.

4.2.1 Existence of Solution

Theorem 4.2.2. *(Solution to \mathcal{P}). There exist $\hat{p} \in \mathbb{P}_{\mathbb{R},N}$ which solves problem \mathcal{P} .*

Proof. Let us denote the cost function in problem \mathcal{P} as $\Psi(p) = \max_{s \in \mathbb{I}} \left| \frac{\frac{p(s)}{q_p(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q_p(s)} L_{11}(s)} \right|$. The continuity of the cost function $\psi : \mathbb{P}_{\mathbb{R},N} \rightarrow \mathbb{R}$ follows from the following facts. The map $f_{z_0} \circ g$ associating q_p to a given p is continuous (proposition 4.2.1) and the evaluation map $p \rightarrow \frac{p(s)}{q_p(s)}$ for $s \in \mathbb{I}$ is continuous since the polynomial r doesn't vanish in \mathbb{I} . This implies that the function $\phi : \mathbb{P}_{\mathbb{R},N} \times \mathbb{I} \rightarrow \mathbb{R}$ defined as $\phi(p, s) = \left| \frac{\frac{p(s)}{q_p(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q_p(s)} L_{11}(s)} \right|$ is continuous and hence the cost function $\Psi(p) = \max_{s \in \mathbb{I}} \phi(p, s)$ is continuous (Corollary 5.4.2, [74]). We have $\{\Psi(p) : p \in \mathbb{P}_{\mathbb{R},N}\}$ to be a non-empty set of real numbers that is bounded below by zero and hence it has an infimum, which is a finite real number. Let us denote $\hat{m} = \inf_{p \in \mathbb{P}_{\mathbb{R},N}} \Psi(p)$.

We can consider a sequence, $(p_n) \in \mathbb{P}_{\mathbb{R},N}$ such that $(\Psi(p_n))$ form a decreasing sequence of real numbers satisfying $\lim_{n \rightarrow \infty} (\Psi(p_n)) = \hat{m}$, say,

$$\forall n \in \mathbb{N}, \Psi(p_n) \leq \hat{m} + \frac{1}{n}. \quad (4.6)$$

Now, we can prove that such a sequence of polynomials, (p_n) , itself is a bounded sequence. If (p_n) is unbounded (norm defined in A.1.1 in Appendix), it follows that for all $s \in \mathbb{I}$, $\lim_{n \rightarrow \infty} \left| \frac{p_n(s)}{q_{p_n}(s)} \right|^2 = \lim_{n \rightarrow \infty} (1 + \left| \frac{r(s)}{p_n(s)} \right|^2)^{-1} = 1$ and hence $\lim_{n \rightarrow \infty} \Psi(p_n) = 1$. This contradicts the fact that (p_n) is a minimizing sequence since for $p \equiv 0$, we have, $\psi(p) = \max_{s \in \mathbb{I}} |L_{11}(s)| < 1$. So,

we have the sequence (p_n) to be bounded. The set $\mathbb{P}_{\mathbb{R},N}$ being closed, we have a subsequence of (p_n) in $\mathbb{P}_{\mathbb{R},N}$ converging to an element \hat{p} in $\mathbb{P}_{\mathbb{R},N}$ and the continuity of the function Ψ ensures $\Psi(\hat{p}) = \hat{m}$. Since \hat{m} is the infimum of $\{\Psi(p) : p \in \mathbb{P}_{\mathbb{R},N}\}$, we have $\hat{m} = \hat{l}$ and so $\Psi(\hat{p}) = \hat{l}$. This verifies the existence of $\hat{p} \in \mathbb{P}_{\mathbb{R},N}$ at which \hat{l} is attained. \square

4.2.2 Minimal Number of Critical Points

In this section, we will provide an interesting result characterising the optimal solution in problem \mathcal{P} . It states that if the solution to problem \mathcal{P} is denoted by $\hat{p} \in \mathbb{P}_{\mathbb{R},N}$, then $\delta(\frac{\hat{p}(s)}{q_{\hat{p}}(s)}, \overline{L_{11}(s)}) = \hat{l}$ atleast $\lceil \frac{N}{2} \rceil + 1$ times in \mathbb{I} , where $\lceil \cdot \rceil$ represent the ceiling function. A formal statement of the result and its proof will be given after some preparatory work. We will introduce a theorem discussing about the directional directive of max functions and a lemma concerning the pseudo-hyperbolic distance. We will also state a problem discussed in [10] concerning pointwise matching which will be useful in the proof of the main result of this section. Theorem 5.4.7 in [74] states the directional differentiability properties of max functions involving continua.

Theorem 4.2.3. *Consider the function*

$$\Psi(x) = \max_{y \in Y} \phi(x, y). \quad (4.7)$$

Suppose that

- $\phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous,
- $\nabla_x \phi(\cdot, \cdot)$ exists and is continuous, and
- $Y \subset \mathbb{R}^m$ is compact.

Then the directional derivative $d\Psi(x; h)$ exist for all $x, h \in \mathbb{R}^n$ and is given by,

$$d\Psi(x; h) = \max_{y \in \hat{Y}(x)} \langle \nabla_x \phi(x, y), h \rangle, \quad (4.8)$$

where

$$\hat{Y}(x) = \{y \in Y : \Psi(x) = \phi(x, y)\}. \quad (4.9)$$

A more complete version of theorem 4.2.3 and its proof can be found in Theorem 5.4.7, [74]. Now we will state and prove a lemma concerning the pseudo-hyperbolic distance function.

Lemma 4.2.4. *For any $a \in \mathbb{D}$, if we consider the function, $f : \mathbb{D} \rightarrow \mathbb{R}$,*

$$f(z) = \left| \frac{z - a}{1 - \bar{a}z} \right|^2,$$

unless $z = a$, there exists a strict descent direction for f at z .

Proof. Let us denote the differential $dz = h$. The differential of f at $z \in \mathbb{D}$ can be calculated as follows

$$\begin{aligned} df &= \left(\frac{z-a}{1-\bar{a}z} \right) \left(\frac{\bar{h}}{1-a\bar{z}} + \frac{(\bar{z}-\bar{a})\bar{h}}{(1-a\bar{z})^2} \right) + \left(\frac{\bar{z}-\bar{a}}{1-a\bar{z}} \right) \left(\frac{h}{1-\bar{a}z} + \frac{(z-a)h}{(1-\bar{a}z)^2} \right) \\ &= 2\operatorname{Re} \left[\left(\frac{\bar{z}-\bar{a}}{1-a\bar{z}} \right) \left(\frac{h}{1-\bar{a}z} + \frac{(z-a)h}{(1-\bar{a}z)^2} \right) \right] \\ &= \frac{2}{|1-\bar{a}z|^2} \operatorname{Re} \left[\left((\bar{z}-\bar{a}) + \frac{|z-a|^2}{(1-\bar{a}z)} \right) h \right] \\ &= \frac{2}{|1-\bar{a}z|^2} \operatorname{Re} \left[\left(\bar{z}-\bar{a} + \left| \frac{z-a}{1-\bar{a}z} \right|^2 (1-a\bar{z}) \right) h \right]. \end{aligned} \quad (4.10)$$

So, if we take $h_0 = (a-z) + \left| \frac{z-a}{1-\bar{a}z} \right|^2 (\bar{a}z-1)$, using (4.10), we have the differential of f for this particular h_0 , denoted as df_0 to be

$$df_0 = \frac{-2}{|1-\bar{a}z|^2} \left| (z-a) + \left| \frac{z-a}{1-\bar{a}z} \right|^2 (1-\bar{a}z) \right|^2 \leq 0. \quad (4.11)$$

If we assume $df_0 = 0$, we have

$$\begin{aligned} (z-a) \left(1 + \frac{\bar{z}-\bar{a}}{|1-\bar{a}z|^2} (1-\bar{a}z) \right) &= 0 \\ \text{i.e. } (z-a)(1-\bar{a}z)(1-a\bar{z} + \bar{z}-\bar{a}) &= 0. \end{aligned}$$

This gives $z = a$, $z = \frac{1}{\bar{a}}$ or $z = \frac{a-1}{1-\bar{a}}$. Since $|\frac{1}{\bar{a}}| > 1$ and $|\frac{a-1}{1-\bar{a}}| = 1$, we have $df_0 < 0$ for all $z \in \mathbb{D} \setminus \{a\}$ and hence the result. \square

Now, we will introduce a special version of problem $\hat{\mathcal{P}}$ discussed in [10]. In [10], it is discussed in the lower half plane setting, but we will continue in the right half plane setting as done throughout the thesis. The discussion that is following will play an important role in parameterising the set of polynomials $p \in \mathbb{P}_{\mathbb{R},N}$ in problem \mathcal{P} in the main theorem of this subsection.

Problem. $\hat{\mathcal{P}}_{\mathbb{R}}$. Given $\lceil \frac{N}{2} \rceil$ distinct points $(s_1, s_2, \dots, s_{\lceil \frac{N}{2} \rceil})$ each in $j\mathbb{R}$, $\lceil \frac{N}{2} \rceil$ interpolation conditions $(c_1, c_2, \dots, c_{\lceil \frac{N}{2} \rceil})$ in $\mathbb{D}^{\lceil \frac{N}{2} \rceil}$ and a polynomial $r \neq 0$ in $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$, find (p, q_p) a couple of polynomials of degree at most $2\lceil \frac{N}{2} \rceil - 1$ such that,

$$\begin{aligned} \frac{p}{q_p}(s_k) &= c_k, \quad k = 1, 2, \dots, \left\lceil \frac{N}{2} \right\rceil \\ \frac{p}{q_p}(\bar{s}_k) &= \bar{c}_k, \quad k = 1, 2, \dots, \left\lceil \frac{N}{2} \right\rceil \\ q_p q_p^* &= p p^* + r r^* \end{aligned}$$

where q_p is stable in the broad sense and for fixed $z_0 > 0$, $q_p(z_0) > 0$.

The reader can refer to [10] for a general version of problem $\hat{\mathcal{P}}_{\mathbb{R}}$ and a proof about the existence and uniqueness of a solution to problem $\hat{\mathcal{P}}_{\mathbb{R}}$. Since we are interested in the real setting, we have considered this special case of problem $\hat{\mathcal{P}}$ where the interpolation conditions are conjugate at the conjugate points. In this setting, we can show that the couple of polynomials (p, q_p) have real coefficients. If we assume that the solution to problem $\hat{\mathcal{P}}_{\mathbb{R}}$ is $(\hat{p}, q_{\hat{p}})$, a pair of complex polynomials of degree at most $2\lceil \frac{N}{2} \rceil - 1$, it follows from the conditions in the problem $\hat{\mathcal{P}}_{\mathbb{R}}$ that

$$\begin{aligned} \frac{\bar{\hat{p}}}{q_{\bar{\hat{p}}}}(s_k) &= c_k, \quad k = 1, 2, \dots, \left\lceil \frac{N}{2} \right\rceil \\ \frac{\bar{\hat{p}}}{q_{\bar{\hat{p}}}}(\bar{s}_k) &= \bar{c}_k, \quad k = 1, 2, \dots, \left\lceil \frac{N}{2} \right\rceil \\ q_{\bar{\hat{p}}} q_{\hat{p}}^* &= \bar{\hat{p}} \hat{p}^* + rr^*, \end{aligned}$$

where $\bar{\hat{p}}$ represent the polynomial obtained by conjugating the coefficients of \hat{p} . It should be noted that $q_{\bar{\hat{p}}} = \bar{q}_{\hat{p}}$ is stable in the broad sense and satisfy $q_{\bar{\hat{p}}}(z_0) > 0$. If the coefficients of the polynomial \hat{p} are not real, this contradicts the uniqueness of solution $(\hat{p}, q_{\hat{p}})$ to problem $\hat{\mathcal{P}}_{\mathbb{R}}$. Remark 4.1 applied to Theorem 10 in [10] states this result. This implies that the polynomial \hat{p} is in $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ and so the polynomial $q_{\hat{p}}$ associated to \hat{p} following proposition 4.2.1 is in $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ as well. The key result upon which the existence and uniqueness of the solution to problem $\hat{\mathcal{P}}_{\mathbb{R}}$ is proved in [10] is the following (Theorem 10, [10]).

Theorem 4.2.5. *The evaluation map E defined from $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ to $\mathbb{D}^{2\lceil \frac{N}{2} \rceil}$,*

$$p \xrightarrow{E} \begin{bmatrix} \frac{p(s_1)}{q_p(s_1)} \\ \vdots \\ \frac{p(s_{\lceil \frac{N}{2} \rceil})}{q_p(s_{\lceil \frac{N}{2} \rceil})} \\ \frac{p(\bar{s}_1)}{q_p(\bar{s}_1)} \\ \vdots \\ \frac{p(\bar{s}_{\lceil \frac{N}{2} \rceil})}{q_p(\bar{s}_{\lceil \frac{N}{2} \rceil})} \end{bmatrix}$$

is a homeomorphism from $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ onto $\mathbb{D}^{2\lceil \frac{N}{2} \rceil}$ and the restriction of E to those p in $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ having no common imaginary zero with r is a diffeomorphism onto its image.

Now, we are in a position to state and prove the important result of this section concerning the number of critical points of problem \mathcal{P} . Let $\hat{p} \in \mathbb{P}_{\mathbb{R}, N}$ denote the optimum for problem \mathcal{P} at which $\hat{l} \neq 0$ is attained. We define the set of critical points for problem \mathcal{P} as the following,

$$\hat{\mathbb{I}}(\hat{p}) = \left\{ s \in \mathbb{I} : \delta \left(\frac{\hat{p}(s)}{q_{\hat{p}}(s)}, \overline{L_{11}(s)} \right) = \hat{l} \right\}. \quad (4.12)$$

Theorem 4.2.6. *Let $\hat{p} \in \mathbb{P}_{\mathbb{R},N}$ denote the solution to problem \mathcal{P} and $\hat{\mathbb{I}}(\hat{p})$ represent the set of critical points of problem \mathcal{P} as defined in (4.12). The number of elements in the set $\hat{\mathbb{I}}(\hat{p})$ is atleast $\lceil \frac{N}{2} \rceil + 1$, where $\lceil \cdot \rceil$ represent the ceiling function.*

Proof. We follow proof by contradiction by assuming the cardinality of set $\hat{\mathbb{I}}(\hat{p})$ is strictly less than $\lceil \frac{N}{2} \rceil + 1$. In this case, we will show that there exists $p_0 \in \mathbb{P}_{\mathbb{R},N}$ such that

$$\forall s \in \mathbb{I}, \delta\left(\frac{p_0(s)}{q_{p_0}(s)}, \overline{L_{11}(s)}\right) < \hat{l}. \quad (4.13)$$

This will contradict the optimality of \hat{p} for problem \mathcal{P} and hence the result follows. In order to prove (4.13), let us assume the set $\hat{\mathbb{I}}(\hat{p})$ has only $\lceil \frac{N}{2} \rceil$ elements and denote them as $s_1, s_2, \dots, s_{\lceil \frac{N}{2} \rceil}$ in \mathbb{I} . Since the polynomial $r \in \mathbb{P}_{\mathbb{R},N}$ doesn't vanish in \mathbb{I} , it follows from theorem 4.2.5 that the map E defined from $\mathbb{P}_{\mathbb{R},2\lceil \frac{N}{2} \rceil-1}$ to $\mathbb{D}^{2\lceil \frac{N}{2} \rceil}$,

$$p \xrightarrow{E} \begin{bmatrix} \frac{p(s_1)}{q_p(s_1)} \\ \vdots \\ \frac{p(s_{\lceil \frac{N}{2} \rceil})}{q_p(s_{\lceil \frac{N}{2} \rceil})} \\ \frac{p(\bar{s}_1)}{q_p(\bar{s}_1)} \\ \vdots \\ \frac{p(\bar{s}_{\lceil \frac{N}{2} \rceil})}{q_p(\bar{s}_{\lceil \frac{N}{2} \rceil})} \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_{\lceil \frac{N}{2} \rceil} \\ c_{\lceil \frac{N}{2} \rceil+1} \\ \vdots \\ c_{2\lceil \frac{N}{2} \rceil} \end{bmatrix} \quad (4.14)$$

is a diffeomorphism from $\mathbb{P}_{\mathbb{R},2\lceil \frac{N}{2} \rceil-1}$ onto its image. It should be noted that, for $i = 1, 2, \dots, \lceil \frac{N}{2} \rceil$, $c_{\lceil \frac{N}{2} \rceil+i} = \bar{c}_i$ and if we define $c = (c_1, c_2, \dots, c_{2\lceil \frac{N}{2} \rceil})^t$, we have $E^{-1}(c) = p$. For the ease of notation, we denote

$$u_i = \begin{cases} s_i, & 1 \leq i \leq \lceil \frac{N}{2} \rceil \\ \bar{s}_i, & \lceil \frac{N}{2} \rceil + 1 \leq i \leq 2\lceil \frac{N}{2} \rceil. \end{cases}$$

If the normalized pass band on the imaginary axis is $\mathbb{I} = [\frac{a_1}{b_m}j, \frac{b_1}{b_m}j] \cup [\frac{a_2}{b_m}j, \frac{b_2}{b_m}j] \cup \dots \cup [\frac{a_m}{b_m}j, 1j]$, where $0 \leq a_1 < b_1 < a_2 < b_2 < \dots < a_m < b_m$, we denote by $\mathbb{I}^* = [-1j, -\frac{a_m}{b_m}j] \cup [-\frac{b_{m-1}}{b_m}j, -\frac{a_{m-1}}{b_m}j] \cup \dots \cup [-\frac{b_1}{b_m}j, -\frac{a_1}{b_m}j]$. For $p \in \mathbb{P}_{\mathbb{R},2\lceil \frac{N}{2} \rceil-1}$, we define,

$$\Delta_u(p) \stackrel{\text{def}}{=} \delta\left(\frac{p(u)}{q_p(u)}, \overline{L_{11}(u)}\right), \quad u \in \mathbb{I} \cup \mathbb{I}^*. \quad (4.15)$$

If we denote $\hat{c}_i = \frac{\hat{p}(u_i)}{q_{\hat{p}}(u_i)}$, $1 \leq i \leq 2\lceil \frac{N}{2} \rceil$ where \hat{p} is the solution to problem \mathcal{P} , then we have,

$$\hat{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{2\lceil \frac{N}{2} \rceil})^t \xrightarrow{E^{-1}} \hat{p} \quad (4.16)$$

and also since $L_{11}(\bar{u}_i) = \overline{L_{11}(u_i)}$,

$$\Delta_{u_i}(\hat{p}) = \delta(\hat{c}_i, \overline{L_{11}(u_i)}) = \hat{l}, \quad 1 \leq i \leq 2\lceil \frac{N}{2} \rceil. \quad (4.17)$$

We have the following results,

- $\Phi : \mathbb{D}^{2\lceil \frac{N}{2} \rceil} \times (\mathbb{I} \cup \mathbb{I}^*) \rightarrow \mathbb{R}^+$, defined by

$$\Phi(c, u) \stackrel{\text{def}}{=} \Delta_u \circ E^{-1}(c) = \Delta_u(p) = \delta\left(\frac{p(u)}{q_p(u)}, \overline{L_{11}(u)}\right), \quad u \in \mathbb{I} \cup \mathbb{I}^*. \quad (4.18)$$

is continuous. Since the evaluation map E is a homeomorphism, E^{-1} , the map c giving p is continuous in $\mathbb{D}^{2\lceil \frac{N}{2} \rceil}$. Δ_u is continuous at $E^{-1}(c) = p$ as well since p giving q_p is continuous in $\mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ (proposition 4.2.1), the evaluation map $p \rightarrow \frac{p(u)}{q_p(u)}$ for $u \in \mathbb{I} \cup \mathbb{I}^*$ is continuous since the polynomial r doesn't vanish in \mathbb{I} and $\delta(\frac{p(u)}{q_p(u)}, \overline{L_{11}(u)})$ is the pseudo-hyperbolic distance between $\frac{p(u)}{q_p(u)}$ and $\overline{L_{11}(u)}$. So, we have Φ to be the composition of two continuous maps and hence continuous.

- $\nabla_c \Phi(., .)$ exists and is continuous. Given any $(c, u) \in \mathbb{D}^{2\lceil \frac{N}{2} \rceil} \times (\mathbb{I} \cup \mathbb{I}^*)$, from definition 4.18, we have, $\Phi(c, u) = \Delta_u \circ E^{-1}(c)$. Since we assume the polynomial r doesn't vanish in \mathbb{I} , we have E^{-1} to be differentiable at c with respect to c and for any $u \in (\mathbb{I} \cup \mathbb{I}^*)$, Δ_u is also differentiable at p with respect to the coefficients of p . This yields for any $(c, u) \in \mathbb{D}^{2\lceil \frac{N}{2} \rceil} \times (\mathbb{I} \cup \mathbb{I}^*)$,

$$\nabla_c \Phi(c, u) = \nabla_p \Delta_u(p) \nabla_c E^{-1}(c). \quad (4.19)$$

We have $\nabla_c \Phi$ to be continuous at (c, u) because we have E to be C^∞ smooth around p (Corollary 3, [10]) and E is a diffeomorphism as well. This implies $\nabla_c E^{-1}$ is continuous at c and we also have $\nabla_p \Delta_u$ continuous at p as well.

- $(\mathbb{I} \cup \mathbb{I}^*) \subset j\mathbb{R}$ is compact.

If we define

$$\Psi(c) = \max_{u \in \mathbb{I} \cup \mathbb{I}^*} \Phi(c, u), \quad (4.20)$$

according to theorem 4.2.3, we have the directional derivative $d\Psi(c; h)$ existing for all $c, h \in \mathbb{D}^{2\lceil \frac{N}{2} \rceil}$ and is given by

$$d\Psi(c; h) = \max_{u \in \hat{\mathbb{I}}_0(c)} \nabla_c \Phi(c, u) \cdot h, \quad (4.21)$$

where, $\hat{\mathbb{I}}_0(c) = \{u \in (\mathbb{I} \cup \mathbb{I}^*) : \Psi(c) = \Phi(c, u)\}$. For $u_i, i = 1, 2, \dots, 2\lceil \frac{N}{2} \rceil$, since $\hat{l} \neq 0$, i.e., $\hat{c}_i \neq \overline{L_{11}(u_i)}$, it follows from lemma 4.2.4 that there exists a direction at u_i ,

$$dc_i = \overline{L_{11}(u_i)} - \hat{c}_i + (\hat{l})^2 (L_{11}(u_i) \hat{c}_i - 1), \quad 1 \leq i \leq 2\left\lceil \frac{N}{2} \right\rceil \quad (4.22)$$

such that

$$\nabla_{c_i} \Phi(\hat{c}, u_i) dc_i < 0, \quad i = 1, 2, \dots, 2\left\lceil \frac{N}{2} \right\rceil, \quad (4.23)$$

where ∇_{c_i} denote the derivative with respect to c_i . So, if we take the direction $h_0 = (dc_1, dc_2, \dots, dc_{2\lceil \frac{N}{2} \rceil})^t$, then we have,

$$\begin{aligned} d\Psi(\hat{c}; h_0) &= \max_{u \in \hat{\mathbb{I}}_0(\hat{c})} \nabla_c \Phi(\hat{c}, u) \cdot h_0 \\ &= \max_{u \in \hat{\mathbb{I}}_0(\hat{c})} \nabla_c \Phi(\hat{c}, u) \cdot (dc_1, dc_2, \dots, dc_{2\lceil \frac{N}{2} \rceil})^t \\ &= \max \left\{ \nabla_{c_i} \Phi(\hat{c}, u_i) dc_i, i = 1, 2, \dots, 2\left\lceil \frac{N}{2} \right\rceil \right\} \\ &< 0. \end{aligned} \tag{4.24}$$

The last inequality follows from equation (4.23). It shows that there exists $c_0 = \hat{c} + \epsilon h_0 \in \mathbb{D}^{2\lceil \frac{N}{2} \rceil}$ such that $c_0 \xrightarrow{E^{-1}} p_0 \in \mathbb{P}_{\mathbb{R}, 2\lceil \frac{N}{2} \rceil - 1}$ which satisfies for sufficiently small positive ϵ ,

$$\Psi(c_0) = \Psi(\hat{c}) + d\Psi(\hat{c}; \epsilon h_0) + o(\|\epsilon h\|) < \Psi(\hat{c}). \tag{4.25}$$

It follows from the definition 4.20 of $\Psi(c)$ that

$$\max_{u \in \mathbb{I} \cup \mathbb{I}^*} \Phi(c_0, u) < \hat{l}.$$

The definition 4.18 of $\Phi(c, u)$ implies that the polynomial $p_0 \in \mathbb{P}_{\mathbb{R}, N}$ satisfy

$$\max_{s \in \mathbb{I}} \delta \left(\frac{p_0(s)}{q_{p_0}(s)}, \overline{L_{11}(s)} \right) < \hat{l}.$$

This contradicts the optimality of \hat{p} and hence completes the proof. \square

4.2.3 Numerical Implementation of Problem \mathcal{P}

In this section, we will describe the numerical implementation of problem \mathcal{P} . Some of the matlab notations will be used throughout this chapter for the purpose of simplified explanation: $.*$ for element-wise multiplication, $./$ for element-wise right division, $*$ for matrix multiplication and $\text{ones}(m, n)$ for matrix of size $m \times n$ whose entries are all one. We recall the matching problem in the real setting,

Problem. \mathcal{P} . *Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4) and a polynomial $r \in \mathbb{P}_{\mathbb{R}, N}$ which doesn't vanish in \mathbb{I} , where N is the target degree of the matching circuit,*

$$\begin{aligned} \text{Find: } \hat{l} &= \min_{p \in \mathbb{P}_{\mathbb{R}, N}} \max_{s \in \mathbb{I}} \left| \frac{\frac{p(s)}{q_p(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q_p(s)} L_{11}(s)} \right| \\ \text{where: } q_p q_p^* &= p p^* + r r^* \quad \text{and } q_p \in \mathbb{SB}_{\mathbb{R}, N}. \end{aligned}$$

We will be using a simplified criterion to implement the problem numerically. The square of the pseudo-hyperbolic distance in the criterion of problem \mathcal{P} can be written as,

$$\left| \frac{\frac{p(s)}{q_p(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q_p(s)} L_{11}(s)} \right|^2 = \left| \frac{p(s) - q_p(s) \overline{L_{11}(s)}}{q_p(s) - p(s) L_{11}(s)} \right|^2. \tag{4.26}$$

The numerator of the expression in the right side of the equality can be expressed as the following at any $s \in \mathbb{I}$,

$$\begin{aligned} |p - q_p \overline{L_{11}}|^2 &= |p|^2 + |q_p L_{11}|^2 - p \overline{q_p} L_{11} - \overline{p} q_p \overline{L_{11}} \\ &= |p|^2 + |p L_{11}|^2 + |r L_{11}|^2 - p \overline{q_p} L_{11} - \overline{p} q_p \overline{L_{11}} \\ &= |q_p - p L_{11}|^2 - |r|^2 (1 - |L_{11}|^2). \end{aligned}$$

The last equality follows since $|q_p|^2 = |p|^2 + |r|^2$ on $j\mathbb{R}$. Substituting this in equation (4.26), we have,

$$\left| \frac{\frac{p(s)}{q_p(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q_p(s)} L_{11}(s)} \right|^2 = 1 - \left| \frac{r(s)}{q_p(s) - p(s) L_{11}(s)} \right|^2 (1 - |L_{11}(s)|^2). \quad (4.27)$$

Making use of equation (4.27), we can write down problem \mathcal{P} with simplified criterion as follows,

Problem. \mathcal{P}_{sc} . Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4) and a polynomial $r \in \mathbb{P}_{\mathbb{R},N}$ which doesn't vanish in \mathbb{I} , where N is the target degree of the matching circuit,

$$\begin{aligned} \text{Find: } \hat{l}_{sc} &= \min_{p \in \mathbb{P}_{\mathbb{R},N}} \max_{s \in \mathbb{I}} \left| \frac{q_p(s) - p(s) L_{11}(s)}{r(s)} \right|^2 \frac{1}{1 - |L_{11}(s)|^2} \\ \text{where: } q_p q_p^* &= p p^* + r r^* \text{ and } q_p \in \mathbb{SB}_{\mathbb{R},N}. \end{aligned}$$

The solution to problem \mathcal{P}_{sc} provides the optimal $p \in \mathbb{P}_{\mathbb{R},N}$ for problem \mathcal{P} and it can be easily deduced that,

$$\hat{l} = \sqrt{1 - (\hat{l}_{sc})^{-1}}. \quad (4.28)$$

So, we will be discussing the numerical implementation of problem \mathcal{P}_{sc} . Throughout the numerical implementations discussed in this chapter, we will consider the polynomial r to be fixed of the form $r = s^n$, $0 \leq n \leq N$, where N is the target degree of the matching network to be synthesized. We consider transmission polynomial of this form since it corresponds to some of the fundamental topologies of LC circuits that can be realized easily for practical applications. Let us denote the points in the passband of interest (after normalization) where the reflection coefficient measurement of the load, L_{11} is given as $\{s_1, s_2, \dots, s_k\}$. We assume that the measurement data of L_{11} is available at sufficiently enough points in the passband \mathbb{I} and problem \mathcal{P}_{sc} is implemented numerically using this set of discrete points in \mathbb{I} . A rational approximation of the scattering parameter of the load L_{11} is not necessary in the numerical implementation. Nevertheless, it should be noted that the problem is not convex and what we are discussing below is the implementation scheme to find a local minima. The problem \mathcal{P}_{sc} can be solved practically by defining an extra variable $\Gamma \in \mathbb{R}$, satisfying,

$$\Gamma \geq \left| \frac{q_p(s_i) - p(s_i) L_{11}(s_i)}{r(s_i)} \right|^2 \frac{1}{1 - |L_{11}(s_i)|^2}, \quad i = 1, 2, \dots, k \quad (4.29)$$

and then solving the following problem,

$$\begin{aligned} & \min_{(p, \Gamma) \in \mathbb{P}_N \times \mathbb{R}} \Gamma \\ \text{sub : } & \left| \frac{q_p(s_i) - p(s_i)L_{11}(s_i)}{r(s_i)} \right|^2 \frac{1}{1 - |L_{11}(s_i)|^2} - \Gamma \leq 0, \quad i = 1, 2, \dots, k. \end{aligned}$$

Derivatives of Criterion and Constraint

The polynomial $p \in \mathbb{P}_{\mathbb{R}, N}$ can be parametrised using its coefficients, say $c_p = [p_N, \dots, p_1, p_0]^t \in \mathbb{R}^{N+1}$ in the standard basis $\{s^N, s^{N-1}, \dots, s^2, s, 1\}$, i.e, we have,

$$\forall s \in j\mathbb{R}, \quad p(s) = p_N s^N + p_{N-1} s^{N-1} + \dots + p_2 s^2 + p_1 s + p_0. \quad (4.30)$$

Thus, the gradient of the criterion with respect to the optimisation variable, $[p_N, \dots, p_0, \Gamma]^t \in \mathbb{R}^{N+2}$ is trivial and is equal to $[0, 0, \dots, 0, 1]^t \in \mathbb{R}^{N+2}$. To evaluate the gradient of the constraint with respect to the optimisation variable, let us denote the constraint function as, for $s \in j\mathbb{R}$,

$$C(s) = \frac{|G(s)|^2}{1 - |L_{11}(s)|^2} - \Gamma, \quad (4.31)$$

where, $G(s) = \frac{q_p(s) - p(s)L_{11}(s)}{r(s)}$. The polynomial q_p is obtained as the minimum phase spectral factor of the positive polynomial $Q \stackrel{\text{def}}{=} pp^* + rr^*$. A Newton-Raphson iterative algorithm as proposed in [66] is used to calculate this. It should be noted that the factorisation $Q = q_p q_p^*$ is unique upto a uni-modular constant and so for q_p to be uniquely defined, we impose the constraint $q_p(1) > 0$ as well. In order to express the gradient of constraint at the data points, let us define the following vectors,

$$Lval = [L_{11}(s_1), L_{11}(s_2), \dots, L_{11}(s_k)]^t \quad (4.32)$$

$$rval = [r(s_1), r(s_2), \dots, r(s_k)]^t \quad (4.33)$$

$$Gval = [G(s_1), G(s_2), \dots, G(s_k)]^t. \quad (4.34)$$

The vectors $Lval$ and $rval$ are available directly from the given data and fixed r respectively and the vector $Gval$ can be obtained after the polynomial q_p is obtained. It should be noted that the objective Γ can be obtained as $\max(|Gval|^2 / (1 - |Lval|^2))$ and the evaluation of the constraint $C(s)$ at the data points as $|Gval|^2 / (1 - |Lval|^2) - \Gamma$. The derivative of polynomial p with respect to its coefficients at the data points is denoted by dp and it is equal to the following Vandermonde matrix of size $k \times (N + 1)$,

$$dp = \begin{bmatrix} s_1^N & s_1^{N-1} & \dots & s_1 & 1 \\ s_2^N & s_2^{N-1} & \dots & s_2 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ s_k^N & s_k^{N-1} & \dots & s_k & 1 \end{bmatrix}. \quad (4.35)$$

So, we have the gradient of constraint with respect to optimisation variable evaluated at the data points,

$$\nabla C = [der, -ones(k, 1)]^t,$$

$(N+2) \times k$ matrix, where, $der = 2 * \text{Re}(dG * \overline{Gval}) ./ (1 - Lval * \overline{Lval})$ is of size $k \times (N+1)$ and

$$dG = (dq_p - dp * Lval) ./ rval,$$

derivative with respect to coefficients of p evaluated at the data points. So, it remains to calculate the derivative of polynomial q with respect to the coefficients of p at the given data points. In order to calculate this, denote the positive polynomial pp^* as $P \in \mathbb{P}_{2N}^+$. It can be noted that the coefficients of odd powers of P in the standard basis are zero and so, let us denote the coefficients of even powers as $c_P = [P_N, P_{N-1}, \dots, P_1, P_0] \in \mathbb{R}^{N+1}$. That is, we have,

$$P = P_N s^{2N} + P_{N-1} s^{2N-2} + \dots + P_1 s^2 + P_0. \quad (4.36)$$

Let us define a map between the vector $c_p \in \mathbb{R}^{N+1}$ and $c_P \in \mathbb{R}^{N+1}$ as,

$$\mathcal{F} : \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1}, \mathcal{F}(c_p) = c_P. \quad (4.37)$$

The Jacobian matrix of the map \mathcal{F} for a given polynomial $p \in \mathbb{P}_{\mathbb{R},N}$ can be computed to have the following structure,

- If N is even,

$$J_{\mathcal{F}}(p) = \begin{bmatrix} p_N & & & & & & & & & \\ p_{N-2} & -p_{N-1} & p_N & & & & & & & \\ p_{N-4} & -p_{N-3} & p_{N-2} & -p_{N-1} & p_N & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\ p_2 & -p_3 & p_4 & -p_5 & p_6 & \dots & \dots & p_N & & \\ p_0 & -p_1 & p_2 & -p_3 & p_4 & \dots & \dots & p_{N-2} & -p_{N-1} & p_N \\ & & p_0 & -p_1 & p_2 & \dots & \dots & p_{N-4} & -p_{N-3} & p_{N-2} \\ & & & & p_0 & \dots & \dots & p_{N-6} & -p_{N-5} & p_{N-4} \\ & & & & & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & p_0 & -p_1 & p_2 \\ & & & & & & & & & p_0 \end{bmatrix}$$

- If N is odd,

$$J_{\mathcal{F}}(p) = \begin{bmatrix} -p_N & & & & & & & & & \\ -p_{N-2} & p_{N-1} & -p_N & & & & & & & \\ -p_{N-4} & p_{N-3} & -p_{N-2} & p_{N-1} & -p_N & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\ -p_3 & p_4 & -p_5 & p_6 & -p_7 & \dots & -p_N & & & \\ -p_1 & p_2 & -p_3 & p_4 & -p_5 & \dots & \dots & p_{N-1} & -p_N & \\ & p_0 & -p_1 & p_2 & -p_3 & \dots & \dots & p_{N-3} & -p_{N-2} & p_{N-1} \\ & & & p_0 & -p_1 & \dots & \dots & p_{N-5} & -p_{N-4} & p_{N-3} \\ & & & & & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & p_0 & -p_1 & p_2 \\ & & & & & & & & & p_0 \end{bmatrix}$$

It should be noted that the Jacobian matrix of the map \mathcal{F} for polynomial q_p , $J_{\mathcal{F}}(q_p)$ follows the same equation as described above in odd and even degree cases where we have the corresponding coefficients of polynomial q_p instead of that of p . Let us denote the Jacobian matrix of the function p giving q_p by J . We have the spectral equation, $q_p q_p^* = pp^* + rr^*$ and the polynomial r is fixed in our optimisation problem. Considering the maps $p \rightarrow q_p \rightarrow q_p q_p^*$ and the one which directly computes $p \rightarrow pp^* + rr^*$, we have,

$$J_{\mathcal{F}}(q_p) * J = J_{\mathcal{F}}(p) \quad (4.38)$$

and hence $J = (J_{\mathcal{F}}(q_p))^{-1} * J_{\mathcal{F}}(p)$. This yields, $dq_p = dp * J$, where dp is defined in (4.35) and we have the complete calculation of derivatives of the criterion and constraint $C(s)$. In addition, we impose constraints to fix the sign of $\frac{p}{q_p}$ at infinity or zero depending on the fixed transmission polynomial r as well. The objective gradient and constraint gradient is then provided to the matlab non-linear programming solver `fmincon` to find the solution for problem \mathcal{P}_{sc} .

Example : Superdirective Antenna

We present in figure 4.2 the same example of superdirective antenna presented in chapter 2 and 3 with reflection coefficient denoted by L_{11} to illustrate the result obtained by solving problem \mathcal{P}_{sc} . The passband of interest is 870 MHz - 900 MHz, the target degree N of matching circuit was fixed to be equal to 5 and all the transmission zeros were fixed at zero ($r = s^5$).

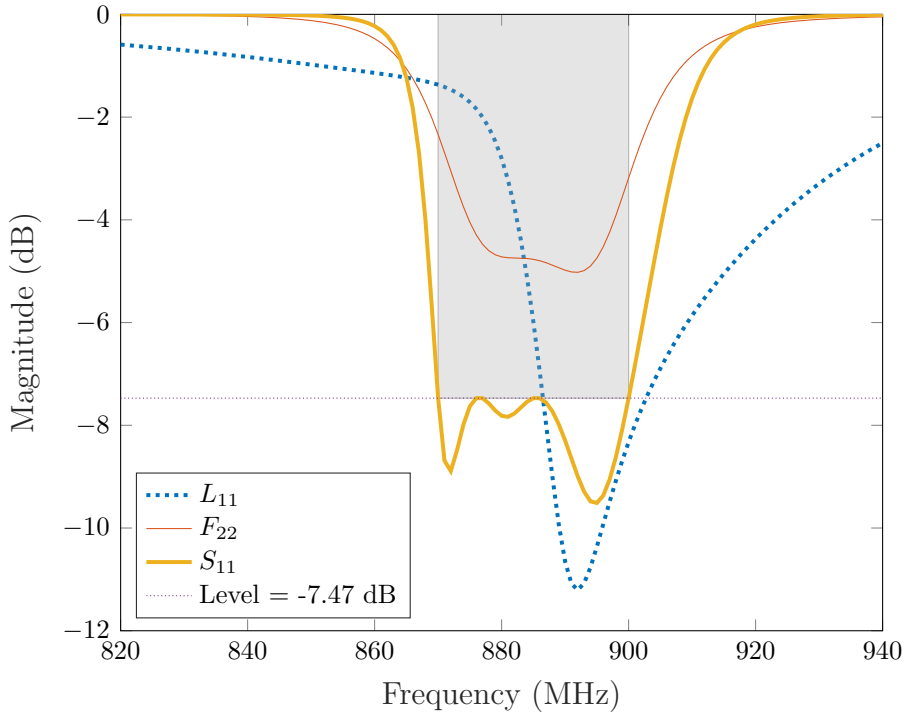


Figure 4.2: Result of solving problem \mathcal{P}_{sc} with $N = 5$ and $r = s^5$

It should be noted that we were able to significantly reduce the maximal mismatch value of the load in the given passband from around -1.37 dB (L_{11} at 870 MHz) to -7.47 dB. The 4 critical points ($\lceil \frac{5}{2} \rceil + 1$) at which S_{11} attains -7.47 dB which is clear from the figure certifies the optimality of the obtained response according to theorem 4.2.6. So, the obtained F_{22} provides a characterisation of the matching circuit to be synthesized and the next important step is to obtain the electronic circuit realizing the scattering matrix F .

4.3 Synthesis of LC Matching Circuits

In this section, we will discuss the realization of optimal matching network by circuits consisting of inductors, capacitors and possibly an impedance transformer. Initially, we will derive the scattering matrix of basic L, C sections that will be useful in our optimisation schemes.

4.3.1 Elementary L, C Sections

In this subsection we will derive the scattering matrix of two sections : a series impedance Z and a parallel admittance Y . In particular, this will provide us the scattering matrix of an inductor or capacitor connected in series or parallel. Initially, we will derive the scattering matrix of a series impedance Z by considering the figure 4.3.

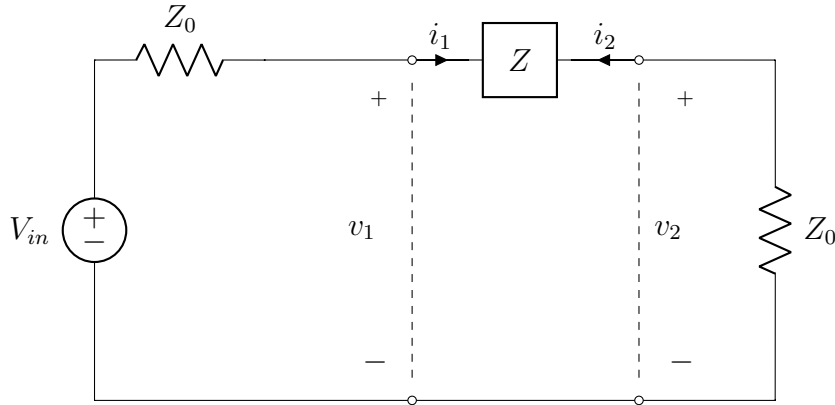


Figure 4.3: Series Impedance Z connected to a Voltage Source

From the circuit in figure 4.3, we have the following equations,

$$i_1 = -i_2 \quad (4.39)$$

$$v_1 - v_2 = Zi_1 = -Zi_2. \quad (4.40)$$

We can derive the scattering matrix of a series impedance Z with positive real reference impedance Z_0 by making use of the above equations and the definition of power waves.

From the definition of power waves in chapter 1 (equations 1.18 and 1.19), we have,

$$a_1 = \frac{v_1 + Z_0 i_1}{2\sqrt{Z_0}}, \quad b_1 = \frac{v_1 - Z_0 i_1}{2\sqrt{Z_0}}, \quad (4.41)$$

$$a_2 = \frac{v_2 + Z_0 i_2}{2\sqrt{Z_0}}, \quad b_2 = \frac{v_2 - Z_0 i_2}{2\sqrt{Z_0}}. \quad (4.42)$$

It follows from these equations that

$$a_1 - b_1 = i_1 \sqrt{Z_0}, \quad (4.43)$$

$$a_2 - b_2 = i_2 \sqrt{Z_0}. \quad (4.44)$$

Considering $b_1 = \frac{v_1 - Z_0 i_1}{2\sqrt{Z_0}}$ and substituting for v_1 and i_1 from equations (4.39) and (4.40), we have,

$$\begin{aligned} b_1 &= \frac{v_2 + Z i_1 + Z_0 i_2}{2\sqrt{Z_0}} \\ &= a_2 + \frac{Z i_1}{2\sqrt{Z_0}}. \end{aligned}$$

The second equality follows from the definition of a_2 in equation (4.42). Now, substituting for i_1 from equation (4.43) and rearranging, we have,

$$b_1 = \left(\frac{Z}{2Z_0 + Z} \right) a_1 + \left(\frac{2Z_0}{2Z_0 + Z} \right) a_2. \quad (4.45)$$

Similarly, considering $b_2 = \frac{v_2 - Z_0 i_2}{2\sqrt{Z_0}}$ and substituting for v_2 and i_2 from equations (4.39) and (4.40) respectively, we have,

$$\begin{aligned} b_2 &= \frac{v_1 + Z i_2 + Z_0 i_1}{2\sqrt{Z_0}} \\ &= a_1 + \frac{Z i_2}{2\sqrt{Z_0}}. \end{aligned}$$

The second equality follows from the definition of a_1 in equation (4.41). Now, substituting for i_2 from equation (4.44) and rearranging, we have,

$$b_2 = \left(\frac{2Z_0}{2Z_0 + Z} \right) a_1 + \left(\frac{Z}{2Z_0 + Z} \right) a_2. \quad (4.46)$$

So, from equations (4.45) and (4.46), we have,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{Z}{2Z_0 + Z} & \frac{2Z_0}{2Z_0 + Z} \\ \frac{2Z_0}{2Z_0 + Z} & \frac{Z}{2Z_0 + Z} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad (4.47)$$

and thus the S -matrix of a series impedance Z is

$$S = \begin{bmatrix} \frac{Z}{2Z_0 + Z} & \frac{2Z_0}{2Z_0 + Z} \\ \frac{2Z_0}{2Z_0 + Z} & \frac{Z}{2Z_0 + Z} \end{bmatrix}. \quad (4.48)$$

4.3. SYNTHESIS OF LC MATCHING CIRCUITS

It can be expressed as the S -matrix of a series admittance Y with reference admittance Y_0 as

$$S = \begin{bmatrix} \frac{Y_0}{2Y+Y_0} & \frac{2Y}{2Y+Y_0} \\ \frac{2Y}{2Y+Y_0} & \frac{Y_0}{2Y+Y_0} \end{bmatrix}. \quad (4.49)$$

Thus, using (4.48) and (4.49), we have the S -matrix of a series inductor having impedance, $Z = Ls$ and that of series capacitor having admittance, $Y = Cs$, to be :

$$S^{serL} = \begin{bmatrix} \frac{Ls}{2Z_0+Ls} & \frac{2Z_0}{2Z_0+Ls} \\ \frac{2Z_0}{2Z_0+Ls} & \frac{Ls}{2Z_0+Ls} \end{bmatrix} = \frac{1}{s + K_{serL}} \begin{bmatrix} s & K_{serL} \\ K_{serL} & s \end{bmatrix}, \quad (4.50)$$

$$S^{serC} = \begin{bmatrix} \frac{Y_0}{2Cs+Y_0} & \frac{2Cs}{2Cs+Y_0} \\ \frac{2Cs}{2Cs+Y_0} & \frac{Y_0}{2Cs+Y_0} \end{bmatrix} = \frac{1}{s + K_{serC}} \begin{bmatrix} K_{serC} & s \\ s & K_{serC} \end{bmatrix}, \quad (4.51)$$

where, $K_{serL} \stackrel{\text{def}}{=} \frac{2Z_0}{L}$ and $K_{serC} \stackrel{\text{def}}{=} \frac{Y_0}{2C}$. Now, in a similar manner we can derive the scattering matrix of a parallel admittance Y by considering figure 4.4.

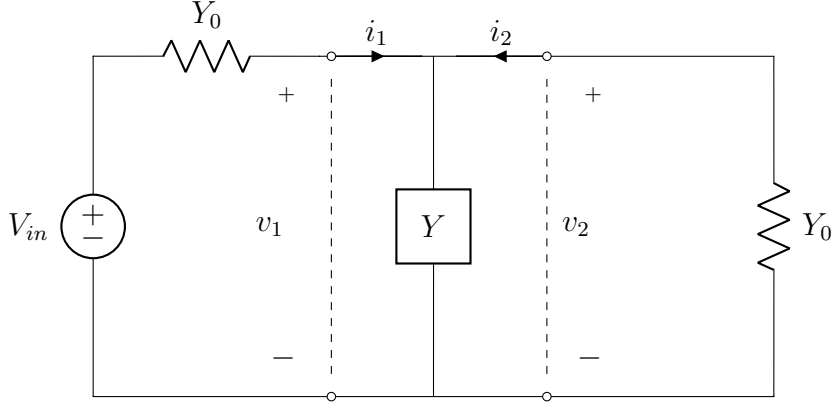


Figure 4.4: Parallel Admittance Y connected to a Voltage Source

From the circuit in figure 4.4, we have the following equations,

$$v_1 = v_2 \quad (4.52)$$

$$i_1 + i_2 = v_1 Y = v_2 Y. \quad (4.53)$$

We can derive the scattering matrix of a parallel admittance Y with reference admittance Y_0 by making use of the above equations and the definition of power waves. From the definition of power waves, we have,

$$a_1 = \frac{v_1 Y_0 + i_1}{2\sqrt{Y_0}}, \quad b_1 = \frac{v_1 Y_0 - i_1}{2\sqrt{Y_0}}, \quad (4.54)$$

$$a_2 = \frac{v_2 Y_0 + i_2}{2\sqrt{Y_0}}, \quad b_2 = \frac{v_2 Y_0 - i_2}{2\sqrt{Y_0}}. \quad (4.55)$$

It follows from these equations that

$$a_1 + b_1 = v_1 \sqrt{Y_0} \quad (4.56)$$

$$a_2 + b_2 = v_2 \sqrt{Y_0} \quad (4.57)$$

Considering $b_1 = \frac{v_1 Y_0 - i_1}{2\sqrt{Y_0}}$ and substituting for v_1 and i_1 from equations (4.52) and (4.53) respectively, we have,

$$\begin{aligned} b_1 &= \frac{v_2 Y_0 + i_2 - v_1 Y}{2\sqrt{Y_0}} \\ &= a_2 - \frac{v_1 Y}{2\sqrt{Y_0}}. \end{aligned}$$

The second equality follows from the definition of a_2 in equation (4.55). Now, substituting for v_1 from equation (4.56) and rearranging, we have,

$$b_1 = \left(\frac{-Y}{2Y_0 + Y} \right) a_1 + \left(\frac{2Y_0}{2Y_0 + Y} \right) a_2. \quad (4.58)$$

Similarly, considering $b_2 = \frac{v_2 Y_0 - i_2}{2\sqrt{Y_0}}$ and substituting for v_2 and i_2 from equations (4.52) and (4.53) respectively, we have,

$$\begin{aligned} b_2 &= \frac{v_1 Y_0 + i_1 - v_2 Y}{2\sqrt{Y_0}} \\ &= a_1 - \frac{v_2 Y}{2\sqrt{Y_0}}. \end{aligned}$$

The second equality follows from the definition of a_1 in equation (4.54). Now, substituting for v_2 from equation (4.57) and rearranging, we have,

$$b_2 = \left(\frac{2Y_0}{2Y_0 + Y} \right) a_1 + \left(\frac{-Y}{2Y_0 + Y} \right) a_2. \quad (4.59)$$

So, we have,

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{-Y}{2Y_0 + Y} & \frac{2Y_0}{2Y_0 + Y} \\ \frac{2Y_0}{2Y_0 + Y} & \frac{-Y}{2Y_0 + Y} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad (4.60)$$

and thus the S -matrix of a parallel admittance Y is

$$S = \begin{bmatrix} \frac{-Y}{2Y_0 + Y} & \frac{2Y_0}{2Y_0 + Y} \\ \frac{2Y_0}{2Y_0 + Y} & \frac{-Y}{2Y_0 + Y} \end{bmatrix}. \quad (4.61)$$

It can be expressed as the S -matrix of a parallel impedance Z with reference impedance Z_0 as,

$$S = \begin{bmatrix} \frac{-Z_0}{2Z + Z_0} & \frac{2Z}{2Z + Z_0} \\ \frac{2Z}{2Z + Z_0} & \frac{-Z_0}{2Z + Z_0} \end{bmatrix}. \quad (4.62)$$

Thus, using equations (4.62) and (4.61), we have the S -matrix of a parallel inductor having impedance $Z = Ls$ and that of parallel capacitor having admittance $Y = Cs$ to be :

$$S^{parL} = \begin{bmatrix} \frac{-Z_0}{2Ls + Z_0} & \frac{2Ls}{2Ls + Z_0} \\ \frac{2Ls}{2Ls + Z_0} & \frac{-Z_0}{2Ls + Z_0} \end{bmatrix} = \frac{1}{s + K_{parL}} \begin{bmatrix} -K_{parL} & s \\ s & -K_{parL} \end{bmatrix}, \quad (4.63)$$

$$S^{parC} = \begin{bmatrix} \frac{-Cs}{2Y_0 + Cs} & \frac{2Y_0}{2Y_0 + Cs} \\ \frac{2Y_0}{2Y_0 + Cs} & \frac{-Cs}{2Y_0 + Cs} \end{bmatrix} = \frac{1}{s + K_{parC}} \begin{bmatrix} -s & K_{parC} \\ K_{parC} & -s \end{bmatrix}, \quad (4.64)$$

where $K_{parL} \stackrel{\text{def}}{=} \frac{Z_0}{2L}$ and $K_{parC} \stackrel{\text{def}}{=} \frac{2Y_0}{C}$.

4.3.2 Circuit with Impedance Transformer

The solution to problem \mathcal{P}_{sc} obtained via the numerical implementation discussed in the previous subsection provides us the optimal reflection polynomial p_{opt} for the matching problem in the real setting. In this subsection we discuss the realization of the optimal matching network as an LC circuit having the reflection coefficient $F_{22} = \frac{p_{opt}}{q_{opt}}$. Elementary sections of inductor or capacitor having S -matrix as discussed in subsection 4.3.1 can be de-chained from F_{22} sequentially (refer definition 3.2.2 for de-chaining). From subsection 4.3.1 it should be clear that a transmission zero at zero can be realized by a series capacitor or a parallel inductor and they have the S -matrices as follows respectively,

$$S^{serC} = \begin{bmatrix} \frac{Y_0}{2Cs+Y_0} & \frac{2Cs}{2Cs+Y_0} \\ \frac{2Cs}{2Cs+Y_0} & \frac{Y_0}{2Cs+Y_0} \end{bmatrix} = \frac{1}{s + K_{serC}} \begin{bmatrix} K_{serC} & s \\ s & K_{serC} \end{bmatrix}, \quad (4.65)$$

$$S^{parL} = \begin{bmatrix} \frac{-Z_0}{2Ls+Z_0} & \frac{2Ls}{2Ls+Z_0} \\ \frac{2Ls}{2Ls+Z_0} & \frac{-Z_0}{2Ls+Z_0} \end{bmatrix} = \frac{1}{s + K_{parL}} \begin{bmatrix} -K_{parL} & s \\ s & -K_{parL} \end{bmatrix}, \quad (4.66)$$

where $K_{serC} = \frac{Y_0}{2C}$ and $K_{parL} = \frac{Z_0}{2L}$. Similarly a transmission zero at infinity can be realized by a series inductor or a parallel capacitor and they have the S -matrices as follows respectively,

$$S^{serL} = \begin{bmatrix} \frac{Ls}{2Z_0+Ls} & \frac{2Z_0}{2Z_0+Ls} \\ \frac{2Z_0}{2Z_0+Ls} & \frac{Ls}{2Z_0+Ls} \end{bmatrix} = \frac{1}{s + K_{serL}} \begin{bmatrix} s & K_{serL} \\ K_{serL} & s \end{bmatrix}, \quad (4.67)$$

$$S^{parC} = \begin{bmatrix} \frac{-Cs}{2Y_0+Cs} & \frac{2Y_0}{2Y_0+Cs} \\ \frac{2Y_0}{2Y_0+Cs} & \frac{-Cs}{2Y_0+Cs} \end{bmatrix} = \frac{1}{s + K_{parC}} \begin{bmatrix} -s & K_{parC} \\ K_{parC} & -s \end{bmatrix}, \quad (4.68)$$

where, $K_{serL} = \frac{2Z_0}{L}$ and $K_{parC} = \frac{2Y_0}{C}$. In order to explain the de-chaining of a transmission zero at zero or infinity from the optimal F_{22} , let us denote,

$$F_{22} = \frac{p_{opt}}{q_{opt}} = \frac{p_N s^N + p_{N-1} s^{N-1} + \dots + p_1 s + p_0}{q_N s^N + q_{N-1} s^{N-1} + \dots + q_1 s + q_0}. \quad (4.69)$$

Let us initially consider the case when F_{22} has a transmission zero at zero (we have, $|F_{22}(0)| = 1$). Following the definition of angular derivative in definition 4.1.1, we have

$$\begin{aligned} \text{ang}[F_{22}](0) &= \frac{F'_{22}(0)}{F_{22}(0)} = \frac{1}{F_{22}(0)} \left(\frac{q_{opt}(0)p'_{opt}(0) - p_{opt}(0)q'_{opt}(0)}{(q_{opt}(0))^2} \right) \\ &= \frac{q_0}{p_0} \left(\frac{p'_{opt}(0)}{q_0} - \frac{p_0}{q_0} \frac{q'_{opt}(0)}{q_0} \right) \\ &= \frac{p_1}{p_0} - \frac{q_1}{q_0}. \end{aligned}$$

In the case of transmission zero at infinity, after doing a change of variable from $s \rightarrow \frac{1}{s}$ and then doing the calculation similar to the above yields

$$\text{ang}[F_{22}](\infty) = \frac{p_{N-1}}{p_N} - \frac{q_{N-1}}{q_N}.$$

So, we have the angular derivative of F_{22} at zero and infinity as follows:

$$\text{ang}[F_{22}](0) = \frac{p_1}{p_0} - \frac{q_1}{q_0} \quad (4.70)$$

$$\text{ang}[F_{22}](\infty) = \frac{p_{N-1}}{p_N} - \frac{q_{N-1}}{q_N}. \quad (4.71)$$

Now, depending on the value of F_{22} at the transmission zero, we can realize the transmission zero using a series or parallel L or C .

- Transmission zero at zero and $F_{22}(0) = 1$, i.e $p_0 = q_0$.
In this case, a series capacitor with S -matrix as shown in equation (4.65) can be used to de-chain the transmission zero at zero from F_{22} . A simple calculation yields the angular derivative of S_{22} of the series capacitor at zero as

$$\text{ang}[S_{22}^{serC}](0) = \frac{-1}{K_{serC}}. \quad (4.72)$$

The value of K_{serC} is thus chosen by equating $\text{ang}[S_{22}^{serC}]$ to the angular derivative of F_{22} at zero mentioned in equation (4.70). So, we have,

$$K_{serC} = \frac{-1}{\text{ang}[F_{22}](0)} = -\left(\frac{p_1}{p_0} - \frac{q_1}{q_0}\right)^{-1}.$$

It also gives the value of capacitance of the capacitor in Farad after de-normalising (the factor b_m Hz, highest frequency of the passband) the frequency,

$$C = \frac{-Y_0 \text{ang}[F_{22}](0)}{4\pi b_m} \text{ Farad} = \frac{-Y_0}{4\pi b_m} \left(\frac{p_1}{p_0} - \frac{q_1}{q_0}\right) \text{ Farad}.$$

- Transmission zero at zero and $F_{22}(0) = -1$, i.e $p_0 = -q_0$.
In this case, a parallel inductor with S -matrix as shown in equation (4.66) can be used to de-chain the transmission zero at zero from F_{22} . The angular derivative of S_{22} of a parallel inductor at zero can be calculated to be

$$\text{ang}[S_{22}^{parL}](0) = \frac{-1}{K_{parL}}. \quad (4.73)$$

The value of K_{parL} is thus chosen by equating $\text{ang}[S_{22}^{parL}](0)$ to the angular derivative of F_{22} at zero mentioned in equation (4.70). So, we have,

$$K_{parL} = \frac{-1}{\text{ang}[F_{22}](0)} = -\left(\frac{p_1}{p_0} - \frac{q_1}{q_0}\right)^{-1}.$$

It gives the value of inductor in Henry as

$$L = \frac{-Z_0 \text{ang}[F_{22}](0)}{4\pi b_m} \text{ Henry} = \frac{-Z_0}{4\pi b_m} \left(\frac{p_1}{p_0} - \frac{q_1}{q_0}\right) \text{ Henry}.$$

- Transmission zero at infinity and $F_{22}(\infty) = 1$, i.e $p_N = q_N$.

In this case, a series inductor with S -matrix as shown in equation (4.67) can be used to de-chain the transmission zero at infinity from F_{22} . The angular derivative of S_{22} of a series inductor at infinity can be calculated to be

$$\text{ang}[S_{22}^{serL}](\infty) = -K_{serL}. \quad (4.74)$$

The value of K_{serL} is thus chosen by equating $\text{ang}[S_{22}^{serL}](\infty)$ to the angular derivative of F_{22} at infinity mentioned in equation (4.71). So, we have,

$$K_{serL} = -\text{ang}[F_{22}](\infty) = \frac{q_{N-1}}{q_N} - \frac{p_{N-1}}{p_N}.$$

It gives the value of inductor in Henry as

$$L = \frac{-Z_0}{\pi b_m \text{ang}[F_{22}](\infty)} \text{ Henry} = \frac{-Z_0}{\pi b_m} \left(\frac{p_{N-1}}{p_N} - \frac{q_{N-1}}{q_N} \right)^{-1} \text{ Henry}.$$

- Transmission zero at infinity and $F_{22}(\infty) = -1$, i.e $p_N = -q_N$.

In this case, a parallel capacitor with S -matrix as shown in equation (4.68) can be used to de-chain the transmission zero at infinity from F_{22} . The angular derivative of S_{22} of a parallel capacitor at infinity can be calculated to be

$$\text{ang}[S_{22}^{parC}](\infty) = -K_{parC}. \quad (4.75)$$

The value of K_{parC} is thus chosen by equating $\text{ang}[S_{22}^{parC}](\infty)$ to the angular derivative of F_{22} at infinity as mentioned in equation (4.71). So, we have,

$$K_{parC} = -\text{ang}[F_{22}](\infty) = \frac{q_{N-1}}{q_N} - \frac{p_{N-1}}{p_N}.$$

It gives the value of inductor in Henry as

$$C = \frac{-Y_0}{\pi b_m \text{ang}[F_{22}](\infty)} \text{ Farad} = \frac{-Y_0}{\pi b_m} \left(\frac{p_{N-1}}{p_N} - \frac{q_{N-1}}{q_N} \right)^{-1} \text{ Farad}.$$

So, we have the complete S -matrix description of sections that can be used to realize a transmission zero of the optimal matching network with scattering matrix F . We can make use of the de-chaining formula mentioned in equation 3.4 to de-chain elementary sections from F_{22} . Let us denote the S -matrix of first elementary section to be de-chained from F_{22} by E (E represent one of the scattering matrices S^{serC} , S^{parL} , S^{serL} or S^{parC}). We can calculate the new reflection coefficient which is denoted by $F_{22}^{[1]}$ obtained after de-chaining this elementary section from F_{22} as follows,

$$F_{22}^{[1]} = \frac{F_{22} - E_{22}}{F_{22}E_{11} - \det(E)}.$$

We can do a sequential de-chaining of the remaining $(N - 1)$ transmission zeros using elementary sections with scattering matrices denoted by $E^{[1]}, E^{[2]}, \dots, E^{[N-1]}$. The reflection coefficient at each stage after de-chaining elementary section can be calculated as,

$$F_{22}^{[i+1]} = \frac{F_{22}^{[i]} - E_{22}^{[i]}}{F_{22}^{[i]} E_{11}^{[i]} - \det(E^{[i]})}, \quad i = 1, 2, \dots, N - 1.$$

This yields us N sections of inductors or capacitors each in series or parallel depending upon the fixed transmission zeros of the matching circuit and a final resistor. The terminating resistor value Z can be evaluated from the reflection coefficient $F_{22}^{[N]}$ which is obtained after de-chaining all N transmission zeros. We have,

$$Z = Z_0 \left(\frac{1 + F_{22}^{[N]}}{1 - F_{22}^{[N]}} \right).$$

In the practical applications where it is desirable to be terminating with the reference impedance Z_0 , we would require an impedance transformer. The value of scalar $F_{22}^{[N]}$ unless equal to zero, an impedance transformer at the end of LC circuit converting the terminating impedance Z to Z_0 can satisfy this requirement. In the next subsection, we will discuss about the possibilities to avoid this impedance transformer at the end of the LC matching circuit since it can be undesirable in some practical applications. Along with the increased total cost of the matching circuit, the presence of impedance transformer can also sometimes lead to other practical difficulties in the realisation of the matching circuit especially when designing matching circuits for small sized PCB antennas.

Example : Superdirective Antenna

The matching circuit providing the response in figure 4.2 is shown in figure 4.5. The element values of the components in the matching circuit are calculated using the de-chaining process described in subsection 4.3.2. It should be noted that the circuit is terminating in $166.6\mu\Omega$ and hence for the antenna to be matched to 50Ω , we require an impedance transformer at the end of the matching circuit with impedance ratio, $167 * 10^{-6} : 50$ (equivalently turn ratio of 1:547).

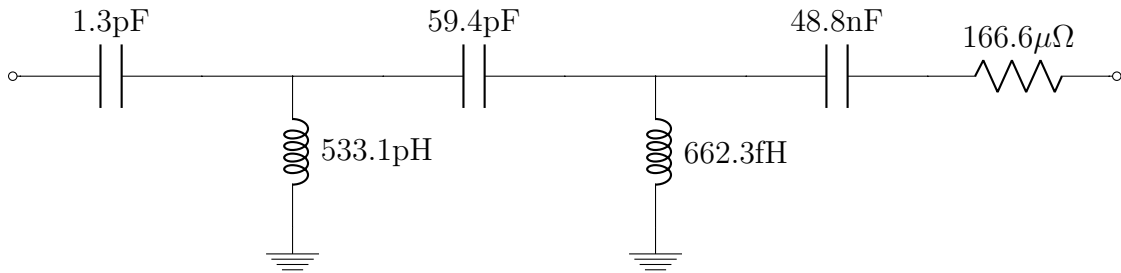


Figure 4.5: Matching circuit providing the result in fig 4.2

4.3.3 Transformerless Synthesis

In this subsection, we will derive the necessary and sufficient conditions to be imposed on the coefficients of the reflection polynomial p of the matching network in order to ensure

a circuit with no transformer at the end. These conditions depend on the chosen topology of the matching circuit, which in turn is determined by the location of transmission zeros of the filter. We will show that these conditions are straightforward and easily derivable in the case when all transmission zeros of the matching circuit are chosen at zero or at infinity. Let us consider a degree N matching circuit with transmission zeros possible at infinity and zero. We will discuss the transformerless conditions by dividing the possible combinations of location of transmission zeros into three general types : matching circuit with N transmission zeros at infinity, N transmission zeros at zero, matching circuit having atleast one transmission zero at both zero and infinity.

(i) **All Transmission zeros of the S -matrix of matching circuit at infinity**

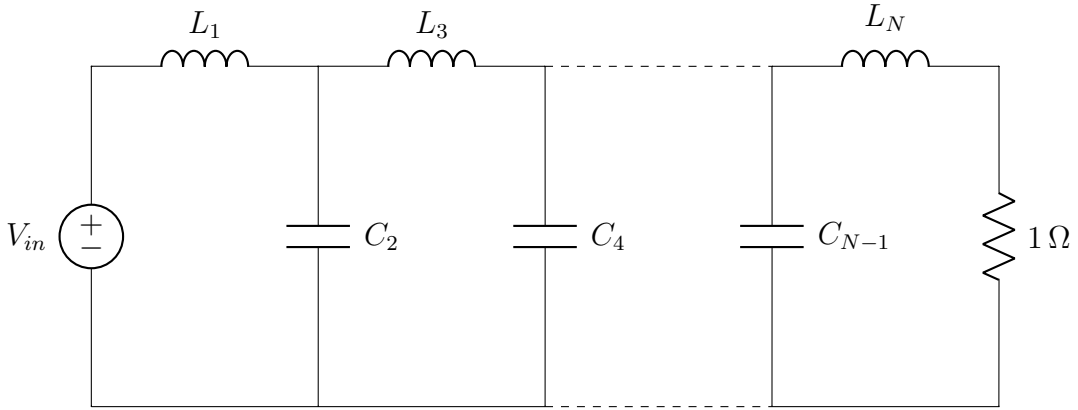


Figure 4.6: Degree N low pass ladder mid-series

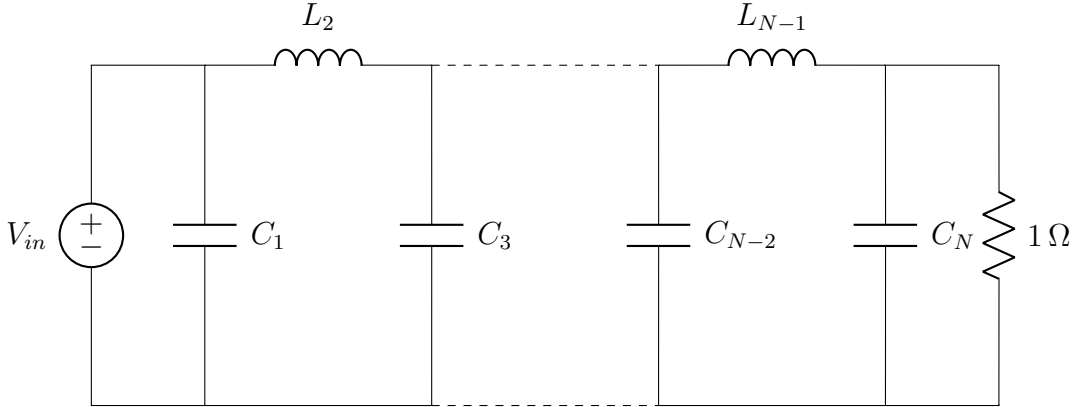


Figure 4.7: Degree N low pass ladder mid-shunt

Figure 4.6 and figure 4.7 represent the two possible realization circuits of a matching circuit with degree N scattering matrix having all transmission zeros at infinity. These are known as low pass ladder circuits since all the transmission zeros of the scattering matrix of the circuit are at infinity. The terminating resistance unlike the previous section, now is imposed to be fixed to a given value (usually the reference impedance 50 ohm in practical applications). This allows the realization of matching networks

only using inductors and capacitors, avoiding the requirement of an impedance transformer. Throughout this subsection, we will use the normalized impedance value of 1 ohm for the terminating resistance. The transformerless condition in this case is discussed in the next theorem. The theorem describes the condition that ensures that an impedance transformer is not required in the realization of the matching circuit.

Theorem 4.3.1. *Let $S = \frac{1}{q} \begin{bmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{bmatrix}$ denote the scattering matrix of a degree N lossless two port in the Belevitch form. The polynomials $p(s)$ and $q(s)$ are assumed to have real coefficients and the polynomial $r(s)$ is assumed to be equal to 1. The necessary and sufficient condition for the two-port to be realizable as a lossless low pass ladder circuit terminating at a unit resistor is*

$$p(0) = 0, \quad (4.76)$$

i.e, if, $p(s) = a_N s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0$, we have $a_0 = 0$.

Proof. In order to prove the necessity, let us consider de-chaining N transmission zeros at infinity from the given scattering matrix, S . From equations (4.50) and (4.64), we have the general form of scattering matrix of a low pass element to be :

$$S^{lp} = \frac{1}{s + \rho} \begin{bmatrix} \gamma s & \rho \\ \rho & \gamma s \end{bmatrix}, \quad (4.77)$$

where, $\rho = \frac{2Z_0}{L}$, $\gamma = 1$ in the case of series inductor with inductance L Henry and $\rho = \frac{2Y_0}{C}$, $\gamma = -1$ in the case of parallel capacitor with capacitance C Farad. Let us denote by p_i , q_i and r_i , the polynomials of S matrix at the i th stage of de-chaining. i.e after de-chaining i transmission zeros at infinity from the given S matrix, where $1 \leq i \leq N$. Using de-chaining formula, after first extraction of transmission zero at infinity, we have,

$$\frac{p_1}{q_1} = \frac{\frac{p}{q} - \frac{\gamma s}{s+\rho}}{\frac{p}{q} \frac{\gamma s}{s+\rho} - \det(S^{lp})} \quad (4.78)$$

$$= \frac{\frac{p}{q} - \frac{\gamma s}{s+\rho}}{\frac{p}{q} \frac{\gamma s}{s+\rho} - \frac{s-\rho}{s+\rho}} \quad (4.79)$$

$$= \frac{(p - \gamma q)s + \rho p}{(p\gamma - q)s + \rho q}. \quad (4.80)$$

So, we have,

$$\frac{p_1(0)}{q_1(0)} = \frac{p(0)}{q(0)} \quad (4.81)$$

and it easily follows that,

$$\frac{p(0)}{q(0)} = \frac{p_1(0)}{q_1(0)} = \frac{p_2(0)}{q_2(0)} = \dots = \frac{p_{N-1}(0)}{q_{N-1}(0)} = \frac{p_N(0)}{q_N(0)}. \quad (4.82)$$

Since we assume there is no transformer at the end of the circuit, we have $\frac{p_N}{q_N} = 0$ and hence from (4.82), we have $\frac{p(0)}{q(0)} = 0$, where $q(0) < \infty$ implying $p(0) = a_0 = 0$ proving the necessity. The sufficiency also follows easily from the equalities in (4.82). Under the assumption $a_0 = 0$, we have from (4.82) the last section at the end of de-chaining N transmission zeros at infinity to satisfy

$$\frac{p_N(0)}{q_N(0)} = 0. \quad (4.83)$$

Since the last section is composed of constant real polynomials, we immediately have, $p_N = 0$. Since $r_N = 1$, using spectral equation $q_N q_N^* = p_N p_N^* + r_N r_N^*$ and normalisation, $q_N(1) > 0$, we have, $q_N = 1$. This implies the S -matrix at the end of de-chaining N transmission zeros at infinity is, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, completing the proof. \square

Example : Superdirective Antenna

The result of solving problem \mathcal{P}_{sc} with $N = 5$ when all the transmission zeros of the matching circuit were fixed at infinity ($r = 1$) and the transformerless condition $a_0 = 0$ was imposed is presented in figure 4.8. The matching circuit providing the response in figure 4.8 is shown in figure 4.9. We have provided the value of elements using reference impedance 50 ohm, instead of the normalized reference impedance 1 ohm as discussed in the above theorem.

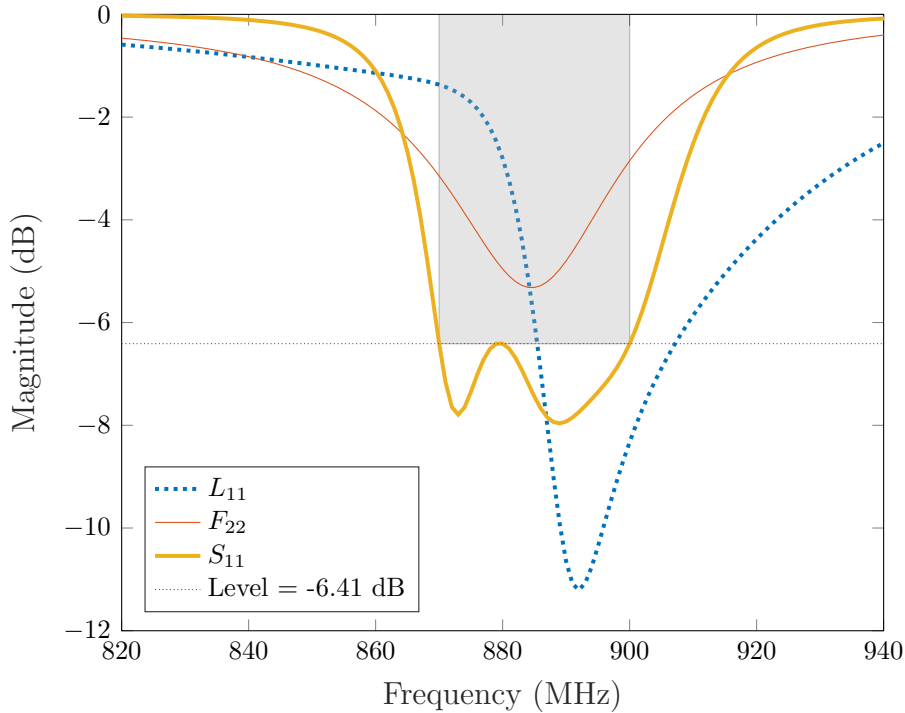


Figure 4.8: Result of solving problem \mathcal{P}_{sc} with $N = 5$, $r(s) = 1$ and $a_0 = 0$

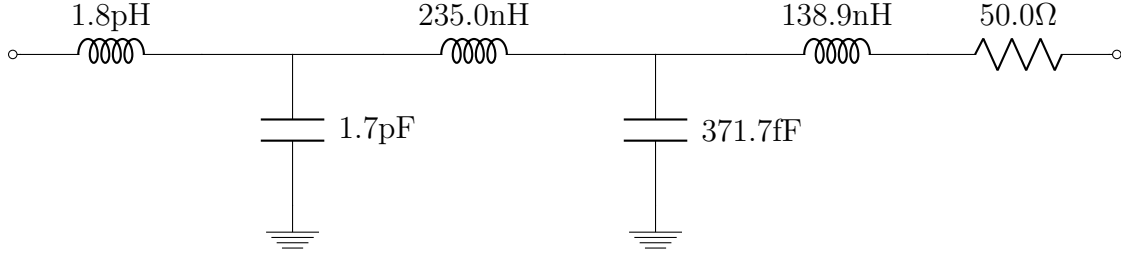


Figure 4.9: Matching circuit providing the result in fig 4.8

It should be noted that the matching criterion provides a significant reduction in the maximal mismatch value of the load in the given passband which was around -1.37 dB (L_{11} at 870 MHz) before matching to -6.41 dB. Also, the additional constraint has ensured that the matching circuit is now terminating in 50 ohm and hence we have avoided the use of an impedance transformer. Now, we will discuss the transformerless condition when all the transmission zeros of the matching circuit are fixed at zero.

(ii) **All Transmission zeros of the S -matrix of matching circuit at zero**

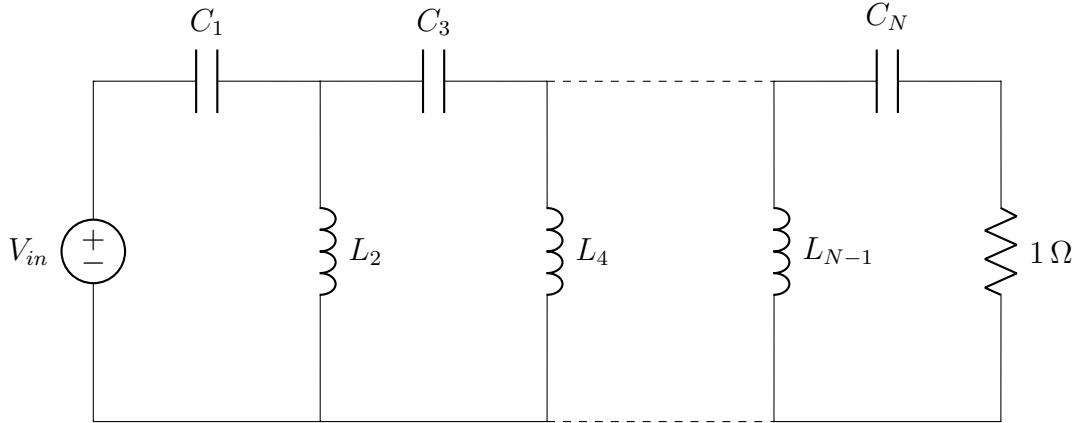


Figure 4.10: Degree N high pass ladder mid-series

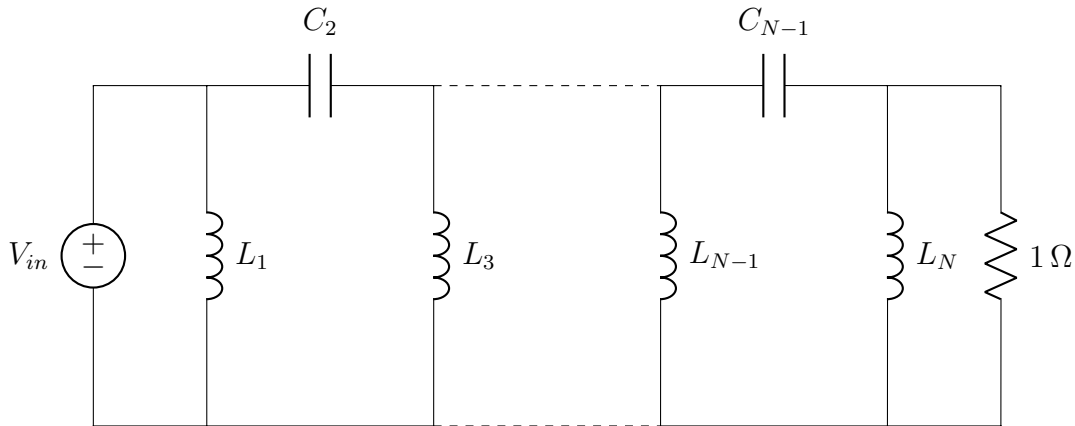


Figure 4.11: Degree N high pass ladder mid-shunt

Figure 4.10 and figure 4.11 represent the two possible realization circuits of a matching circuit with degree N scattering matrix having all transmission zeros at zero. These are known as high pass ladder circuits since all the transmission zeros of the scattering matrix of the circuit are at zero. The transformerless condition in this case is described in the next theorem. It provides the necessary and sufficient condition for a matching circuit with all transmission zeros at zero to be realizable without using an impedance transformer.

Theorem 4.3.2. *Let $S = \frac{1}{q} \begin{bmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{bmatrix}$ denote the scattering matrix of a degree N lossless two port in the Belevitch form. The polynomials $p(s)$ and $q(s)$ are assumed to have real coefficients and the polynomial $r(s)$ is assumed to be equal to s^N . The necessary and sufficient condition for the two-port to be realizable as a lossless high pass ladder terminating at a unit resistor is :*

$$a_N = 0, \quad (4.84)$$

where, $p(s) = a_N s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0$.

Proof. In order to prove the necessity, let us consider de-chaining N transmission zeros at zero from the given scattering matrix S . From equations (4.51) and (4.63), we have the general form of scattering matrix of a high pass element to be :

$$S^{hp} = \frac{1}{s + \rho} \begin{bmatrix} \gamma \rho & s \\ s & \gamma \rho \end{bmatrix}, \quad (4.85)$$

where, $\rho = \frac{Y_0}{2C}$, $\gamma = 1$ in the case of series capacitor with capacitance C Farad and $\rho = \frac{Z_0}{2L}$, $\gamma = -1$ in the case of parallel inductor with inductance L Henry. Let us denote by p_i , q_i and r_i , the polynomials of S matrix at the i th stage of de-chaining. i.e after de-chaining i transmission zeros at zero from the given S matrix, where $1 \leq i \leq N$. Using de-chaining formula, after first extraction of transmission zero at zero, we have,

$$\frac{p_1}{q_1} = \frac{\frac{p}{q} - \frac{\gamma \rho}{s + \rho}}{\frac{p}{q} \frac{\gamma \rho}{s + \rho} - \det(S^{hp})} \quad (4.86)$$

$$= \frac{\frac{p}{q} - \frac{\gamma \rho}{s + \rho}}{\frac{p}{q} \frac{\gamma \rho}{s + \rho} - \frac{p - s}{s + \rho}} \quad (4.87)$$

$$= \frac{ps + \rho(p - \gamma q)}{qs + \rho(\gamma p - q)} \quad (4.88)$$

So, we have,

$$\frac{p_1}{q_1}(\infty) = \frac{p}{q}(\infty), \quad (4.89)$$

and it easily follows that,

$$\frac{p}{q}(\infty) = \frac{p_1}{q_1}(\infty) = \frac{p_2}{q_2}(\infty) = \dots = \frac{p_{N-1}}{q_{N-1}}(\infty) = \frac{p_N}{q_N}(\infty). \quad (4.90)$$

Since we assume there is no transformer at the end, we have, $\frac{p_N}{q_N} = 0$, and hence from (4.90), we have, $\frac{p}{q}(\infty) = 0$, i.e.,

$$a_N = 0, \quad (4.91)$$

proving the necessity. The sufficiency also follows easily from the equality (4.90). Under the assumption $a_N = 0$, we have from (4.90), the last section at the end of de-chaining N transmission zeros at infinity to satisfy

$$\frac{p_N}{q_N}(\infty) = 0. \quad (4.92)$$

Since the last section is composed of constant real polynomials, we immediately have $p_N = 0$. Since $r_N = 1$, using spectral equation, $q_N q_N^* = p_N p_N^* + r_N r_N^*$ and normalisation, $q_N(1) > 0$, we have, $q_N = 1$. This implies the S -matrix at the end of de-chaining N transmission zeros at infinity is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, completing the proof. \square

Example : Superdirective Antenna

The result of solving problem \mathcal{P}_{sc} with $N = 5$ when all the transmission zeros of the matching circuit were fixed at zero ($r = s^5$) and the transformerless condition $a_5 = 0$ was imposed is presented in figure 4.12. The matching circuit providing the response in figure 4.12 is shown in figure 4.13. The reference impedance used is 50 ohm.

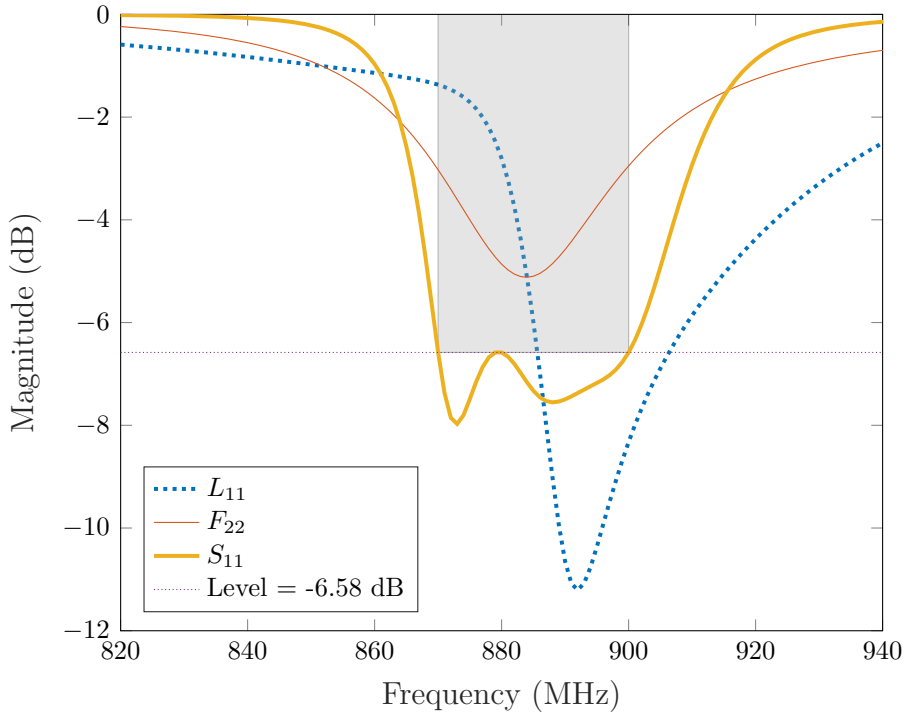


Figure 4.12: Result of solving problem \mathcal{P}_{sc} with $N = 5$, $r(s) = s^5$ and $a_5 = 0$

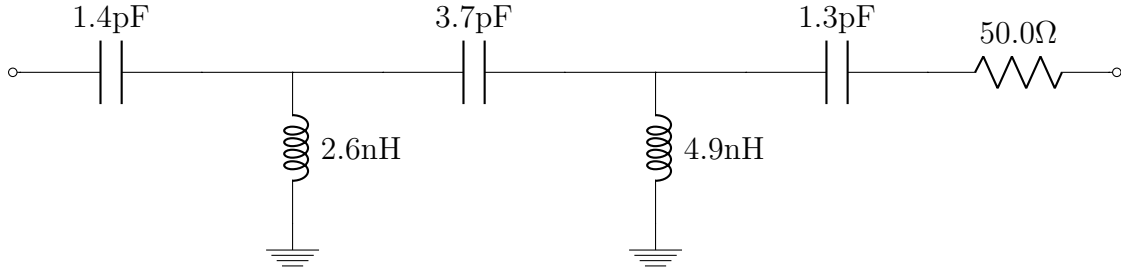


Figure 4.13: Matching circuit providing the result in fig 4.12

It should be noted that the optimal criterion obtained, -6.58 dB is lesser than the one in figure 4.2 (-7.47 dB) which was obtained by solving problem \mathcal{P}_{sc} with $N = 5$ and $r(s) = s^5$. This was expected because of the additional transformerless constraint we have imposed but it still provides significant reduction in the power mismatch. The matching circuit is now terminating in 50 ohm and we do not require an impedance transformer in the circuit. Now, we will discuss the transformerless condition when the transmission zeros of the matching circuit are present at both zero and infinity.

(iii) **Transmission zeros of the S -matrix of matching circuit at zero and infinity**

The case when there are transmission zeros present at both zero and infinity, the constraint to be imposed on the coefficients of p becomes less straightforward since it depends upon the order of extraction of transmission zeros. The number of possible combinations increase rapidly with each increasing degree. The case when there are only two transmission zeros; one at infinity and one at zero is presented below. A general condition on the coefficients of p in the case when there are N transmission zeros where there is atleast one at zero and one at infinity even though possible is not done here since its numerical implementation can be less straight forward as well. An alternative synthesis scheme based on the chaining of elementary L, C sections will be provided in the next section where in the problem of transformerless synthesis is solved out more naturally. We will conclude this subsection by describing the transformerless condition for a degree 2 matching circuit with one transmission zero at infinity and one at zero.

Degree 2 matching circuit : 1 transmission zero at infinity and 1 at zero

As mentioned, since the transformerless condition depends upon the order of extraction of transmission zeros and also the sign of the value of the reflection coefficient at each transmission zero, we can subdivide the degree 2 case into four main types. This includes LC resonator in series, LC resonator in parallel, circuit with only inductors and circuit with only capacitors. It should be clear at the end of this subsection that even in this case there are two different constraints depending upon the chosen topology of the matching circuit.

(a) Series LC resonator

Let us denote the polynomials in the Belevitch form of degree 2 scattering matrix by

$$\begin{aligned} p(s) &= a_2 s^2 + a_1 s + a_0, \\ q(s) &= b_2 s^2 + b_1 s + b_0, \\ r(s) &= s. \end{aligned}$$

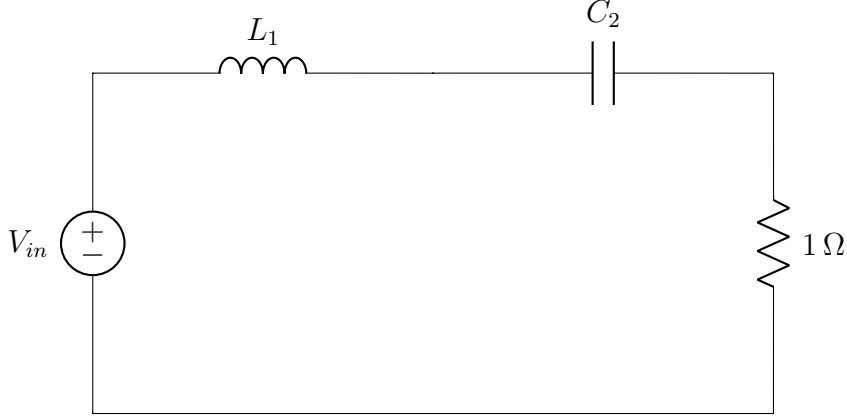


Figure 4.14: Series LC Resonator connected to a Voltage Source

Since we consider the transmission zeros at infinity and zero to be realised by a series inductor and series capacitor respectively as shown in figure 4.14, we have :

$$a_2 = b_2, \quad a_0 = b_0. \quad (4.93)$$

The driving point input impedance of the loss-less two port when the output port is terminated in 1Ω can be calculated as follows,

$$z = \frac{q + p}{q - p} \quad (4.94)$$

$$= \frac{(b_2 + a_2)s^2 + (b_1 + a_1)s + (b_0 + a_0)}{(b_1 - a_1)s} \quad (4.95)$$

$$= \left(\frac{b_2 + a_2}{b_1 - a_1} \right) s + \left(\frac{b_1 + a_1}{b_1 - a_1} \right) + \left(\frac{b_0 + a_0}{b_1 - a_1} \right) \frac{1}{s} \quad (4.96)$$

This implies, we have,

$$L_1 = \frac{b_2 + a_2}{b_1 - a_1}, \quad C_2 = \frac{b_1 - a_1}{b_0 + a_0}, \quad \frac{b_1 + a_1}{b_1 - a_1} = 1. \quad (4.97)$$

So, the necessary condition for the circuit to be transformerless is,

$$b_1 + a_1 = b_1 - a_1,$$

that is,

$$a_1 = 0. \quad (4.98)$$

(b) Parallel LC resonator

Let us denote the polynomials in the Belevitch form of degree 2 scattering matrix by

$$\begin{aligned} p(s) &= a_2 s^2 + a_1 s + a_0, \\ q(s) &= b_2 s^2 + b_1 s + b_0, \\ r(s) &= s. \end{aligned}$$

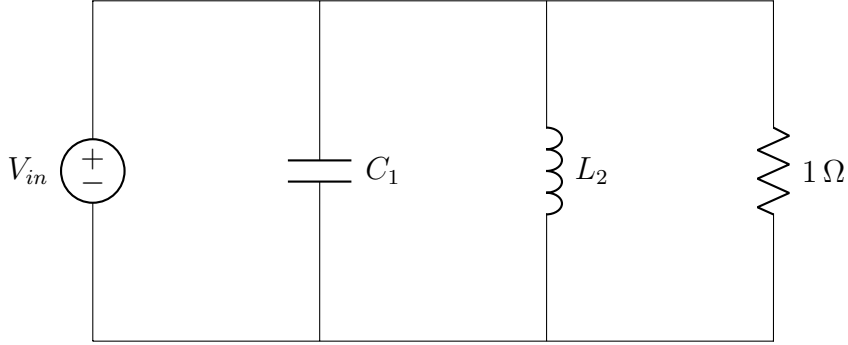


Figure 4.15: Parallel LC Resonator connected to a Voltage Source

Since we consider the transmission zeros at infinity and zero to be realised by a parallel capacitor and parallel inductor respectively as shown in figure 4.15, we have :

$$a_2 = -b_2, \quad a_0 = -b_0. \quad (4.99)$$

The input admittance of the two port when the output port is terminated in a unit resistor can be calculated as follows,

$$y = \frac{q - p}{q + p} \quad (4.100)$$

$$= \frac{(b_2 - a_2)s^2 + (b_1 - a_1)s + (b_0 - a_0)}{(b_1 + a_1)s} \quad (4.101)$$

$$= \left(\frac{b_2 - a_2}{b_1 + a_1} \right) s + \left(\frac{b_1 - a_1}{b_1 + a_1} \right) + \left(\frac{b_0 - a_0}{b_1 + a_1} \right) \frac{1}{s} \quad (4.102)$$

This implies, we have,

$$C_1 = \frac{b_2 - a_2}{b_1 + a_1}, \quad L_2 = \frac{b_1 + a_1}{b_0 - a_0}, \quad \frac{b_1 - a_1}{b_1 + a_1} = 1. \quad (4.103)$$

So, the necessary condition for the circuit to be transformerless is,

$$b_1 - a_1 = b_1 + a_1,$$

that is,

$$a_1 = 0. \quad (4.104)$$

(c) Circuit with only inductors

Let us denote the polynomials in the Belevitch form of degree 2 scattering matrix by

$$\begin{aligned} p(s) &= a_2 s^2 + a_1 s + a_0, \\ q(s) &= b_2 s^2 + b_1 s + b_0, \\ r(s) &= s. \end{aligned}$$

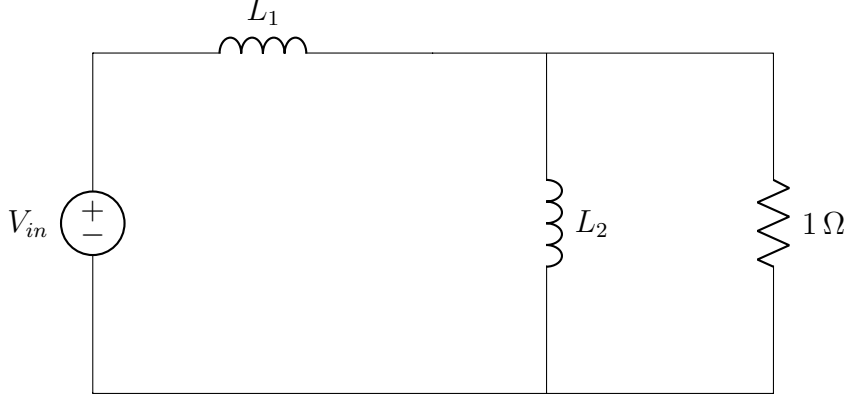


Figure 4.16: Inductor only circuit

Since we consider the transmission zeros at infinity and zero to be realised by a series inductor and parallel inductor respectively as shown in figure 4.16, we have :

$$a_2 = b_2, \quad a_0 = -b_0. \quad (4.105)$$

The input impedance of the two port when the output port is terminated in a unit resistor can be calculated as follows,

$$z = \frac{q + p}{q - p} \quad (4.106)$$

$$= \frac{(b_2 + a_2)s^2 + (b_1 + a_1)s}{(b_1 - a_1)s + (b_0 - a_0)}. \quad (4.107)$$

We have the partial fraction expansion of the above expression to be,

$$z = \left(\frac{b_2 + a_2}{b_1 - a_1} \right) s + \frac{\beta}{b_1 - a_1} + \frac{\gamma}{(b_1 - a_1)s + (b_0 - a_0)}, \quad (4.108)$$

where,

$$\beta = (b_1 + a_1) - \left(\frac{b_2 + a_2}{b_1 - a_1} \right) (b_0 - a_0), \quad (4.109)$$

$$\gamma = -\beta \left(\frac{b_0 - a_0}{b_1 - a_1} \right). \quad (4.110)$$

So, we can extract the transmission zero at infinity using series inductor with inductance, $L_1 = \frac{b_2 + a_2}{b_1 - a_1}$ Henry and we have the impedance

$$z^{(1)} = \frac{\beta}{b_1 - a_1} + \frac{\gamma}{(b_1 - a_1)s + (b_0 - a_0)}, \quad (4.111)$$

the remainder after extraction. The corresponding admittance, $y^{(1)} = \frac{1}{z^{(1)}}$ can be calculated to be,

$$\begin{aligned} y^{(1)} &= \frac{(b_1 - a_1)^2 s + (b_1 - a_1)(b_0 - a_0)}{\beta(b_1 - a_1)s + \beta(b_0 - a_0) + \gamma(b_1 - a_1)} \\ &= \frac{(b_1 - a_1)^2 s + (b_1 - a_1)(b_0 - a_0)}{\beta(b_1 - a_1)s} \\ &= \left(\frac{b_1 - a_1}{\beta} \right) + \left(\frac{b_0 - a_0}{\beta} \right) \frac{1}{s}. \end{aligned}$$

The second equality follows from equation (4.110). We can extract the transmission zero at zero using parallel inductor with inductance, $L_2 = \frac{\beta}{b_0 - a_0}$ Henry and the remainder after extraction is $\frac{b_1 - a_1}{\beta}$. So, the necessary condition for the circuit to be terminating in a unit resistor with no transformer is,

$$b_1 - a_1 = \beta \quad (4.112)$$

$$= (b_1 + a_1) - \left(\frac{b_2 + a_2}{b_1 - a_1} \right) (b_0 - a_0). \quad (4.113)$$

The second equality follows from equation (4.109). If we make use of the spectral equation $qq^* = pp^* + rr^*$ as well, the condition can be expressed just in terms of coefficients of p . After some basic calculation it can be shown that the condition simplifies to

$$a_1^2 - 4a_0^2 a_2^2 = 0,$$

that is,

$$a_1 = \pm 2a_0 a_2. \quad (4.114)$$

(d) Circuit with only capacitors

Let us denote the polynomials in the Belevitch form of degree 2 scattering matrix by

$$p(s) = a_2 s^2 + a_1 s + a_0,$$

$$q(s) = b_2 s^2 + b_1 s + b_0,$$

$$r(s) = s.$$

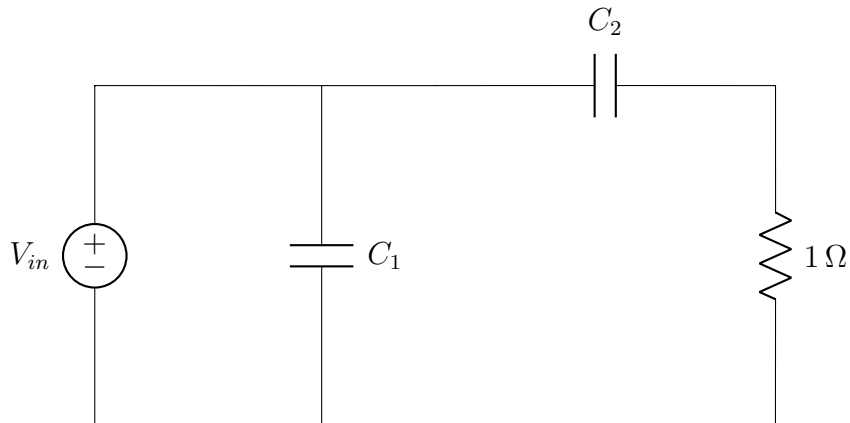


Figure 4.17: Capacitor only circuit

Since we consider the transmission zeros at infinity and zero to be realised by a parallel capacitor and series capacitor respectively as shown above, we have :

$$a_2 = -b_2, \quad a_0 = b_0. \quad (4.115)$$

The input admittance of the two port when the output port is terminated in a unit resistor can be calculated as follows,

$$y = \frac{q - p}{q + p} \quad (4.116)$$

$$= \frac{(b_2 - a_2)s^2 + (b_1 - a_1)s}{(b_1 + a_1)s + (b_0 + a_0)} \quad (4.117)$$

We have the partial fraction expansion of the above expression to be,

$$y = \left(\frac{b_2 - a_2}{b_1 + a_1} \right) s + \frac{\beta}{b_1 + a_1} + \frac{\gamma}{(b_1 + a_1)s + (b_0 + a_0)}, \quad (4.118)$$

where,

$$\beta = (b_1 - a_1) - \left(\frac{b_2 - a_2}{b_1 + a_1} \right) (b_0 + a_0) \quad (4.119)$$

$$\gamma = -\beta \left(\frac{b_0 + a_0}{b_1 + a_1} \right) \quad (4.120)$$

So, we can extract the transmission zero at infinity using parallel capacitor with capacitance, $C_1 = \frac{b_2 - a_2}{b_1 + a_1}$ Farad and we have the admittance,

$$y^{(1)} = \frac{\beta}{b_1 + a_1} + \frac{\gamma}{(b_1 + a_1)s + (b_0 + a_0)}, \quad (4.121)$$

the remainder after extraction. The corresponding impedance, $z^{(1)} = \frac{1}{y^{(1)}}$ can be calculated to be,

$$\begin{aligned} z^{(1)} &= \frac{(b_1 + a_1)^2 s + (b_1 + a_1)(b_0 + a_0)}{\beta(b_1 + a_1)s + \beta(b_0 + a_0) + \gamma(b_1 + a_1)} \\ &= \frac{(b_1 + a_1)^2 s + (b_1 + a_1)(b_0 + a_0)}{\beta(b_1 + a_1)s} \\ &= \left(\frac{b_1 + a_1}{\beta} \right) + \left(\frac{b_0 + a_0}{\beta} \right) \frac{1}{s}. \end{aligned}$$

The second equality follows from equation (4.120). We can extract the transmission zero at zero using series capacitor with capacitance, $C_2 = \frac{b_0 + a_0}{\beta}$ Farad and the remainder is $\frac{b_1 + a_1}{\beta}$. So, the necessary condition for the circuit to be terminating in a unit resistor with no transformer is,

$$b_1 + a_1 = \beta \quad (4.122)$$

$$= (b_1 - a_1) - \left(\frac{b_2 - a_2}{b_1 + a_1} \right) (b_0 + a_0). \quad (4.123)$$

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The second equality follows from equation (4.119). If we make use of the spectral equation $qq^* = pp^* + rr^*$ as well, the condition can be expressed just in terms of coefficients of p . After some basic calculation it can be shown that the condition simplifies to

$$a_1^2 - 4a_0^2a_2^2 = 0,$$

that is,

$$a_1 = \pm 2a_0a_2. \quad (4.124)$$

We conclude this section with the remark that in the degree two case, even if we change the order of extraction (element 1 becomes the second element and element 2 becomes the first element), a calculation similar to done above will yield the same condition for transformerless synthesis in each case. In the case of series and parallel LC resonators (iii.a and iii.b), even if we change the order of extraction, we have the same circuit and the condition for no impedance transformer in the realisation of the circuit is $a_1 = 0$. In the case of other two circuits (iii.c and iii.d), if we change the order of extraction (in iii.c, parallel inductor becomes the first element followed by series inductor and similarly in iii.d, now series capacitor becomes first element followed by parallel capacitor), the condition for no impedance transformer in the realisation of the circuit is same as iii.c and iii.d : $a_1 = \pm 2a_0a_2$. It should also be noted that all the conditions mentioned for degree 2 case are not only necessary for transformerless synthesis but sufficient as well. Even though it has not been described explicitly, it can be seen that under the respective conditions for each circuit, the element values of inductors and capacitors as mentioned in each case will allow a transformerless synthesis, proving the sufficiency. The number of equivalent circuit topologies increase rapidly with the increase in the number of components allowed in the matching network (degree N in problem \mathcal{P}). Due to this reason rather than deriving a general transformerless condition on the coefficients of p for any degree, we introduce in the next section an approach based on the chaining of elementary sections allowing transformerless synthesis more directly.

4.4 Synthesis of L, C Matching Circuits : Elementary Chaining Approach

In this section, we will discuss the formulation of the matching problem in the real setting using an elementary chaining approach. We will make use of the angular derivatives of the reflection coefficients of the basic LC sections at the transmission zeros in order to parametrise the polynomials involved in the problem; namely p and q . In order to describe this, let us recall the scattering matrix of basic LC sections realizing the transmission zeros at zero and infinity and the angular derivatives of the reflection coefficient at the transmission zeros. As explained in subsection 4.3.2, we have the scattering matrix of elementary sections as follows. Let us denote the S -matrix of a low pass section (transmission zero at infinity) which can be realised using a series inductor or parallel capacitor by S^{lp} and the S -matrix of a high pass section (transmission zero at zero) which can be realised using a

series capacitor or parallel inductor by S^{hp} . We have,

$$S^{lp} = \frac{1}{\rho^{lp}s + 1} \begin{bmatrix} \gamma\rho^{lp}s & 1 \\ 1 & \gamma\rho^{lp}s \end{bmatrix}, \quad S^{hp} = \frac{1}{s + \rho^{hp}} \begin{bmatrix} \gamma\rho^{hp} & s \\ s & \gamma\rho^{hp} \end{bmatrix}, \quad (4.125)$$

where, in the case of low pass section (transmission zero at infinity),

- series inductor with inductance L Henry : $\rho^{lp} = \frac{L}{2Z_0} = -(\text{ang}[S_{22}^{lp}](\infty))^{-1}$, $\gamma = 1$
- parallel capacitor with capacitance C Farad : $\rho^{lp} = \frac{C}{2Y_0} = -(\text{ang}[S_{22}^{lp}](\infty))^{-1}$, $\gamma = -1$

and in the case of high pass section (transmission zero at zero),

- series capacitor with capacitance C Farad : $\rho^{hp} = \frac{Y_0}{2C} = -(\text{ang}[S_{22}^{hp}](0))^{-1}$, $\gamma = 1$
- parallel inductor with inductance L Henry : $\rho^{hp} = \frac{Z_0}{2L} = -(\text{ang}[S_{22}^{hp}](0))^{-1}$, $\gamma = -1$.

In this approach, we will be following a method of chaining of elementary sections, each of which is characterised by their respective ρ 's to build the matching network. The topology of the matching network will be fixed by fixing the sequence of chaining of elementary sections. The formulation of the matching problem remains the same as problem \mathcal{P} implemented in subsection 4.2.3, except that now polynomials $p \in \mathbb{P}_{\mathbb{R},N}$ and $q \in \mathbb{SB}_{\mathbb{R},N}$ will be parametrised using $\rho = [\rho_1, \rho_2, \dots, \rho_N]^t \in \mathbb{R}_+^N$ and also the fact that the underlying set of reflection coefficients of the matching circuit, $\frac{p_\rho}{q_\rho} \in \mathbb{B}$ is built by the chaining of elementary LC sections. So, compared to problem \mathcal{P} , the possible set of responses of the matching circuit in this approach is more restrictive but provides the advantage of transformerless matching circuit directly. Each ρ_i , $1 \leq i \leq N$ represent ρ^{lp} or ρ^{hp} described above of the elementary sections. The chosen topology of matching circuit determines the formula of each ρ_i . Now, the matching problem can be formulated over the set of these vectors of strictly positive elements.

Problem. \mathcal{B} . *Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4) and a polynomial $r = s^n$, $0 \leq n \leq N$, where N is the target degree of the matching circuit,*

$$\text{Find: } \hat{l} = \min_{\rho \in \mathbb{R}_+^N} \max_{s \in \mathbb{I}} \left| \frac{\frac{p_\rho(s)}{q_\rho(s)} - \overline{L_{11}(s)}}{1 - \frac{p_\rho(s)}{q_\rho(s)} L_{11}(s)} \right|$$

$$\text{where: } q_\rho q_\rho^* = p_\rho p_\rho^* + r r^*, \quad p_\rho \in \mathbb{P}_{\mathbb{R},N}, \quad q_\rho \in \mathbb{SB}_{\mathbb{R},N}$$

and $\frac{p_\rho}{q_\rho}$ is the output reflection coefficient of the matching circuit consisting of N elementary L, C sections chained in a given fixed order.

4.4.1 Numerical Implementation of Problem \mathcal{B}

Similar to what was done in the numerical implementation of problem \mathcal{P} described in subsection 4.2.3, we will be solving problem \mathcal{B} numerically using a simplified criterion. We can write down problem \mathcal{B} with simplified criterion as,

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Problem. \mathcal{B}_{sc} . Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4) and a polynomial $r = s^n$, $0 \leq n \leq N$, where N is the target degree of the matching circuit,

$$\text{Find: } \hat{l}_{sc} = \min_{\rho \in \mathbb{R}_+^N} \max_{s \in \mathbb{I}} \left| \frac{q_\rho(s) - p_\rho(s)L_{11}(s)}{r(s)} \right|^2 \frac{1}{1 - |L_{11}(s)|^2}$$

where : $q_\rho q_\rho^* = p_\rho p_\rho^* + r r^*$, $p_\rho \in \mathbb{P}_{\mathbb{R}, N}$, $q_\rho \in \mathbb{SB}_{\mathbb{R}, N}$

and $\frac{p_\rho}{q_\rho}$ is the output reflection coefficient of the matching circuit consisting of N elementary L, C sections chained in a given fixed order.

The solution to problem \mathcal{B}_{sc} provides the optimal $\rho \in \mathbb{R}_+^N$ for problem \mathcal{B} and it can be easily deduced that

$$\hat{l} = \sqrt{1 - (\hat{l}_{sc})^{-1}}. \quad (4.126)$$

Similar to problem \mathcal{P} , we assume the reflection coefficient measurement of the load, L_{11} is given at sufficiently many points $\{s_1, s_2, \dots, s_k\} \in \mathbb{I}$ and the problem \mathcal{B}_{sc} is implemented on these discrete set of points. The problem \mathcal{B}_{sc} can be solved practically by defining an extra variable $\Gamma \in \mathbb{R}$, satisfying,

$$\Gamma \geq \left| \frac{q_\rho(s_i) - p_\rho(s_i)L_{11}(s_i)}{r(s_i)} \right|^2 \frac{1}{1 - |L_{11}(s_i)|^2}, \quad i = 1, 2, \dots, k$$

and then solving the following problem,

$$\begin{aligned} & \min_{(\rho, \Gamma) \in \mathbb{R}_+^{N+1}} \Gamma \\ \text{sub : } & \left| \frac{q_\rho(s_i) - p_\rho(s_i)L_{11}(s_i)}{r(s_i)} \right|^2 \frac{1}{1 - |L_{11}(s_i)|^2} - \Gamma \leq 0, \quad i = 1, 2, \dots, k. \end{aligned} \quad (4.127)$$

The algorithm to solve problem \mathcal{B}_{sc} will be developed based on an iterative scheme on the polynomials p, q and r of the scattering matrix obtained by chaining elementary LC sections. We will make use of the chain matrices of elementary sections to derive the iterations of these polynomials. From the correspondance relationship between S -matrix and T -matrix (equation 1.29 in chapter 1), the chain matrices of S^{lp} and S^{hp} denoted by T^{lp} and T^{hp} respectively can be easily shown to be the following :

$$T^{lp} = \begin{bmatrix} 1 - \rho^{lp}s & \gamma \rho^{lp}s \\ -\gamma \rho^{lp}s & 1 + \rho^{lp}s \end{bmatrix}, \quad T^{hp} = \frac{1}{s} \begin{bmatrix} s - \rho^{hp} & \gamma \rho^{hp} \\ -\gamma \rho^{hp} & s + \rho^{hp} \end{bmatrix}. \quad (4.128)$$

The iteration formulas of the polynomials can be derived by considering the general scattering matrix of a lossless two-port,

$$S = \frac{1}{q} \begin{bmatrix} \epsilon p^* & -\epsilon r^* \\ r & p \end{bmatrix}. \quad (4.129)$$

The corresponding chain matrix T of S is

$$T = \frac{1}{r} \begin{bmatrix} -\epsilon q^* & \epsilon p^* \\ -p & q \end{bmatrix}. \quad (4.130)$$

The iteration formulas for the chaining of elementary sections to S can be developed by the multiplication of chain matrix of the chained elementary section (T^{lp} or T^{hp}) on the right to T . Then the scattering matrix of the chained system can be obtained using the correspondance relationship from T to S -matrix. This gives us the polynomials in the Belevitch form of chained system as follows:

- Chaining a low pass section (transmission zero at infinity)

$$T_{\uparrow} = T * T^{lp} \quad (4.131)$$

$$p_{\uparrow} = p(-\rho^{lp}s + 1) + \gamma\rho^{lp}sq \quad (4.132)$$

$$q_{\uparrow} = -\gamma\rho^{lp}sp + q(\rho^{lp}s + 1) \quad (4.133)$$

$$r_{\uparrow} = r \quad (4.134)$$

- Chaining a high pass section (transmission zero at zero)

$$T_{\uparrow} = T * T^{hp} \quad (4.135)$$

$$p_{\uparrow} = p(s - \rho^{hp}) + \gamma\rho^{hp}q \quad (4.136)$$

$$q_{\uparrow} = -\gamma\rho^{hp}p + q(s + \rho^{hp}) \quad (4.137)$$

$$r_{\uparrow} = sr \quad (4.138)$$

So, once the sequence of chaining of N elementary sections is fixed (in the direction such that last section will be chained to the load), we can make use of the above iteration formulas to calculate the polynomials p_{ρ} and q_{ρ} in the Belevitch form of degree N scattering matrix formed by chaining these elementary sections. If we denote the chain matrix of fixed elementary sections in the order of chaining as $T^{E_1}, T^{E_2}, \dots, T^{E_N}$, where each T^{E_i} represent T^{lp} or T^{hp} depending upon the chosen topology of matching network, we can define the iterations,

$$\begin{aligned} T_1 &= T^{E_1} \\ T_2 &= T_1 T^{E_2} \\ T_3 &= T_2 T^{E_3} \\ &\vdots \\ T_N &= T_{N-1} T^{E_N} \end{aligned}$$

The chain matrices T_i , $1 \leq i \leq N$ are represented by

$$T_i = \frac{1}{r_i} \begin{bmatrix} -\epsilon q_i^* & \epsilon p_i^* \\ -p_i & q_i \end{bmatrix}. \quad (4.139)$$

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At each stage of iteration of the elementary chain matrix multiplication, the iterations of polynomials follow the set of formulas (4.132), (4.133), (4.134) or (4.136), (4.137), (4.138) depending upon the chaining section at each stage. At the end of chaining of N sections, the polynomials $p_N = p_\rho$ and $q_N = q_\rho$ of the degree N scattering matrix in the Belevitch form is obtained. The polynomial r_N will be equal to the fixed polynomial r .

Derivatives of Criterion and Constraint

After deriving the polynomials p_ρ and q_ρ using the iterations formulas mentioned above, we are in a position to calculate the gradient of criterion and constraint of problem 4.127 at the data points $\{s_1, s_2, \dots, s_k\}$. The gradient of the criterion with respect to the optimisation variable $(\rho, \Gamma) = [\rho_1, \rho_2, \dots, \rho_{N-1}, \rho_N, \Gamma]^t \in \mathbb{R}_+^{N+1}$ is trivial and is equal to $[0, 0, \dots, 0, 1]^t \in \mathbb{R}^{N+1}$. To evaluate the gradient of the constraint with respect to the optimisation variable, let us denote the constraint function as, for $s \in j\mathbb{R}$,

$$C(s) = \frac{|G(s)|^2}{1 - |L_{11}(s)|^2} - \Gamma, \quad (4.140)$$

where, $G(s) = \frac{q_\rho(s) - p_\rho(s)L_{11}(s)}{r(s)}$. In order to express the gradient of constraint at the data points, let us define the following vectors,

$$Lval = [L_{11}(s_1), L_{11}(s_2), \dots, L_{11}(s_k)]^t \quad (4.141)$$

$$rval = [r(s_1), r(s_2), \dots, r(s_k)]^t \quad (4.142)$$

$$Gval = [G(s_1), G(s_2), \dots, G(s_k)]^t. \quad (4.143)$$

It should be noted that the vectors $Lval$ and $rval$ are available directly from the given data and fixed r respectively and the vector $Gval$ can be obtained after the polynomials p_ρ and q_ρ are obtained using the iterations as described before. The objective Γ can be obtained as $\max(|Gval|^2 / (1 - |Lval|^2))$ and the evaluation of the constraint $C(s)$ at the data points as $|Gval|^2 / (1 - |Lval|^2) - \Gamma$. We have the gradient of constraint with respect to optimisation variable, evaluated at the data points,

$$\nabla C = [der, -ones(k, 1)]^t,$$

$(N + 1) \times k$ matrix, where, $der = 2 \cdot \text{Re}(dG \cdot \overline{Gval}) / (1 - Lval \cdot \overline{Lval})$ is of size $k \times N$. The derivative of G with respect to ρ evaluated at the data points, denoted by dG is

$$dG = (dq_\rho - dp_\rho \cdot Lval) / rval,$$

where dq_ρ and dp_ρ denote the derivative of q_ρ and p_ρ with respect to ρ respectively evaluated at the data points. The matrices dq_ρ , dp_ρ and dG are of size $k \times N$. So, it remains to calculate the matrices dp_ρ and dq_ρ to complete the derivative calculations. The derivatives dp_ρ and dq_ρ can be evaluated iteratively by taking the derivative with respect to ρ of the chaining equations mentioned in (4.132), (4.133), (4.136) and (4.137). Let us denote by $\mathcal{U}_p(s)$, the upper triangle matrix with (i, j) -th element, $j \geq i$, representing the derivative of p_j with

respect to ρ_i at a data point $s \in \{s_1, s_2, \dots, s_k\}$:

$$\mathcal{U}_p(s) = \begin{bmatrix} \frac{dp_1}{d\rho_1}(s) & \frac{dp_2}{d\rho_1}(s) & \dots & \frac{dp_{N-1}}{d\rho_1}(s) & \frac{dp_N}{d\rho_1}(s) \\ 0 & \frac{dp_2}{d\rho_2}(s) & \dots & \frac{dp_{N-1}}{d\rho_2}(s) & \frac{dp_N}{d\rho_2}(s) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{dp_{N-1}}{d\rho_{N-1}}(s) & \frac{dp_N}{d\rho_{N-1}}(s) \\ 0 & 0 & \dots & 0 & \frac{dp_N}{d\rho_N}(s) \end{bmatrix},$$

Similarly, $\mathcal{U}_q(s)$ denote the upper triangle matrix with (i, j) -th element, $j \geq i$, representing the derivative of q_j with respect to ρ_i at a data point $s \in \{s_1, s_2, \dots, s_k\}$:

$$\mathcal{U}_q(s) = \begin{bmatrix} \frac{dq_1}{d\rho_1}(s) & \frac{dq_2}{d\rho_1}(s) & \dots & \frac{dq_{N-1}}{d\rho_1}(s) & \frac{dq_N}{d\rho_1}(s) \\ 0 & \frac{dq_2}{d\rho_2}(s) & \dots & \frac{dq_{N-1}}{d\rho_2}(s) & \frac{dq_N}{d\rho_2}(s) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{dq_{N-1}}{d\rho_{N-1}}(s) & \frac{dq_N}{d\rho_{N-1}}(s) \\ 0 & 0 & \dots & 0 & \frac{dq_N}{d\rho_N}(s) \end{bmatrix},$$

The matrices $\mathcal{U}_p(s)$ and $\mathcal{U}_q(s)$ can be calculated at any data point $\{s_1, s_2, \dots, s_k\}$ using the following iterations:

- Chaining a low pass section (transmission zero at infinity)

$$\frac{dp_1}{d\rho_1}(s) = \gamma s, \quad \frac{dq_1}{d\rho_1}(s) = s. \quad (4.144)$$

for $i = 2 : N$,

$$\frac{dp_i}{d\rho_i}(s) = -sp_{i-1}(s) + \gamma_{i-1}sq_{i-1}(s); \quad (4.145)$$

$$\frac{dq_i}{d\rho_i}(s) = -\gamma_{i-1}sp_{i-1}(s) + sq_{i-1}(s); \quad (4.146)$$

end

for $i = 1 : N-1$

for $j = i+1 : N$

$$\frac{dp_j}{d\rho_i}(s) = (-\rho_j s + 1) \frac{dp_{j-1}}{d\rho_i}(s) + \gamma_{j-1} \rho_j s \frac{dq_{j-1}}{d\rho_i}(s); \quad (4.147)$$

$$\frac{dq_j}{d\rho_i}(s) = -\gamma_{j-1} \rho_j s \frac{dp_{j-1}}{d\rho_i}(s) + (\rho_j s + 1) \frac{dq_{j-1}}{d\rho_i}(s); \quad (4.148)$$

end

end

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- Chaining a high pass section (transmission zero at zero)

$$\frac{dp_1}{d\rho_1}(s) = 1, \quad \frac{dq_1}{d\rho_1}(s) = 1. \quad (4.149)$$

for $i = 2 : N$,

$$\frac{dp_i}{d\rho_i}(s) = -p_{i-1}(s) + \gamma_{i-1}q_{i-1}(s); \quad (4.150)$$

$$\frac{dq_i}{d\rho_i}(s) = -\gamma_{i-1}p_{i-1}(s) + q_{i-1}(s); \quad (4.151)$$

end

for $i = 1 : N-1$

for $j = i+1 : N$

$$\frac{dp_j}{d\rho_i}(s) = (s - \rho_j) \frac{dp_{j-1}}{d\rho_i}(s) + \gamma_{j-1}\rho_j \frac{dq_{j-1}}{d\rho_i}(s); \quad (4.152)$$

$$\frac{dq_j}{d\rho_i}(s) = -\gamma_{j-1}\rho_j \frac{dp_{j-1}}{d\rho_i}(s) + (s + \rho_j) \frac{dq_{j-1}}{d\rho_i}(s); \quad (4.153)$$

end

end

The last column of matrices $\mathcal{U}_p(s)$ and $\mathcal{U}_q(s)$ evaluated at s_i provides the i -th row of matrices dp_ρ and dq_ρ . So, we have the derivatives of p_ρ and q_ρ with respect to ρ at the data points,

$$dp_\rho = \begin{bmatrix} \frac{dp_N}{d\rho_1}(s_1) & \frac{dp_N}{d\rho_2}(s_1) & \dots & \frac{dp_N}{d\rho_{N-1}}(s_1) & \frac{dp_N}{d\rho_N}(s_1) \\ \frac{dp_N}{d\rho_1}(s_2) & \frac{dp_N}{d\rho_2}(s_2) & \dots & \frac{dp_N}{d\rho_{N-1}}(s_2) & \frac{dp_N}{d\rho_N}(s_2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{dp_N}{d\rho_1}(s_k) & \frac{dp_N}{d\rho_2}(s_k) & \dots & \frac{dp_N}{d\rho_{N-1}}(s_k) & \frac{dp_N}{d\rho_N}(s_k) \end{bmatrix},$$

$$dq_\rho = \begin{bmatrix} \frac{dq_N}{d\rho_1}(s_1) & \frac{dq_N}{d\rho_2}(s_1) & \dots & \frac{dq_N}{d\rho_{N-1}}(s_1) & \frac{dq_N}{d\rho_N}(s_1) \\ \frac{dq_N}{d\rho_1}(s_2) & \frac{dq_N}{d\rho_2}(s_2) & \dots & \frac{dq_N}{d\rho_{N-1}}(s_2) & \frac{dq_N}{d\rho_N}(s_2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{dq_N}{d\rho_1}(s_k) & \frac{dq_N}{d\rho_2}(s_k) & \dots & \frac{dq_N}{d\rho_{N-1}}(s_k) & \frac{dq_N}{d\rho_N}(s_k) \end{bmatrix}.$$

Thus, we have the complete calculation of derivatives of the criterion and constraint $C(s)$ of problem in 4.127. In addition, we impose the constraint $\rho_i > 0$ for $i = 1, 2, \dots, N$ as well. The objective gradient and constraint gradient is then provided to the matlab non-linear programming solver `fmincon` to find the solution for problem \mathcal{B} . This completes the description of implementation of problem \mathcal{B} using iterations of polynomials p and q parametrised using the angular derivatives. The solution to problem \mathcal{B} yields the optimal

$\rho \in \mathbb{R}_+^N$ denoted by $\hat{\rho} = [\hat{\rho}_1, \hat{\rho}_2, \dots, \hat{\rho}_N]^t$. This directly provides the optimal matching network terminating with reference impedance Z_0 . The values of inductors or capacitors of the matching network can be obtained from $\hat{\rho}$ depending upon the chosen topology of the circuit. After de-normalising the frequency (factor b_m Hz : highest frequency of passband), we have the following equations which follow from the equations in 4.125 . In the case of low pass section (transmission zero at infinity) at the i -th position ($1 \leq i \leq N$),

- series inductor : $L = \frac{Z_0 \hat{\rho}_i}{\pi b_m}$ Henry
- parallel capacitor : $C = \frac{Y_0 \hat{\rho}_i}{\pi b_m}$ Farad

and in the case of high pass section (transmission zero at zero) at the i -th position,

- series capacitor: $C = \frac{Y_0}{4\pi b_m \hat{\rho}_i}$ Farad
- parallel inductor : $L = \frac{Z_0}{4\pi b_m \hat{\rho}_i}$ Henry.

Example : Superdirective Antenna

The result of solving problem \mathcal{B}_{sc} with $N = 5$ when 3 transmission zeros of the matching circuit were fixed at zero and the remaining 2 at infinity ($r = s^3$) is presented in figure 4.18. The order of chaining of transmission zeros of the matching circuit (starting from the load) were fixed to be $\infty, 0, \infty, 0, 0$ and the elementary sections realising the transmission zeros: parallel C , series C , series L , parallel L and series C respectively. The matching circuit providing the response in figure 4.18 is shown in figure 4.19.

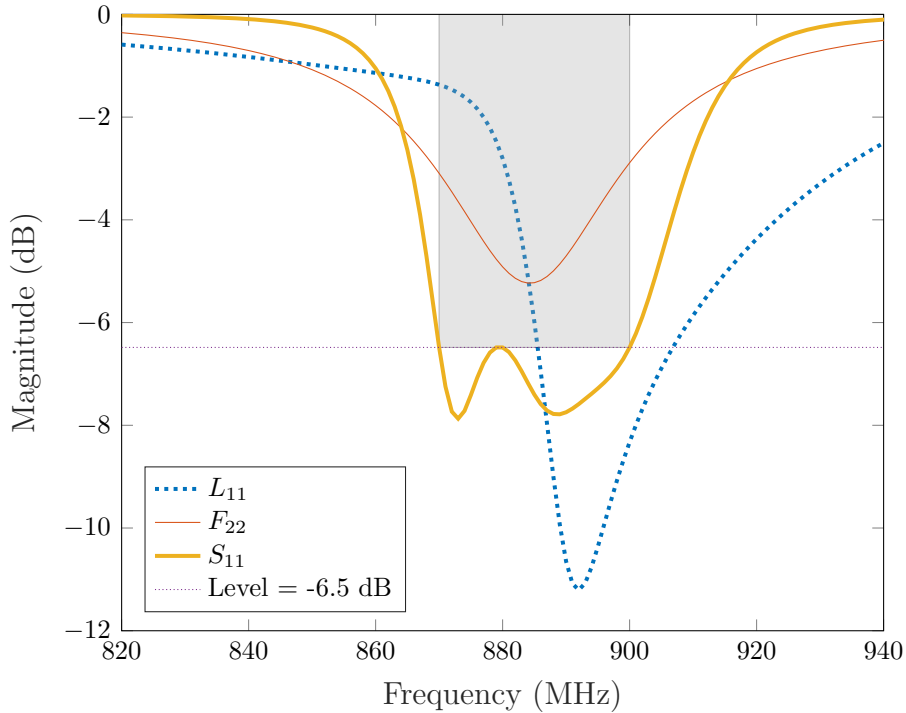


Figure 4.18: Result of solving problem \mathcal{B}_{sc} with $N = 5$ and $r(s) = s^3$

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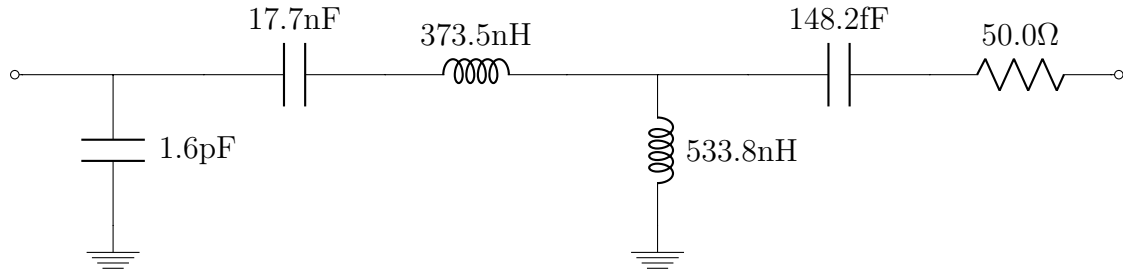


Figure 4.19: Matching circuit providing the result in fig 4.18

The matching circuit provides a significant reduction in the maximal mismatch value of the load in the given passband from around -1.37 dB (L_{11} at 870 MHz) to -6.5 dB and also the elementary chaining approach has ensured matching circuit with no impedance transformer as shown in figure 4.19. An iteration scheme to solve problem \mathcal{B}_{sc} for all the possible combinations of chaining of elementary sections for a fixed degree N was implemented as well. Some other matching circuits providing response very close to S_{11} in figure 4.18 for superdirective antenna is shown below. Matching circuits 2, 3 and 4 shown in figures 4.20, 4.21 and 4.22 have $r(s)$ fixed to be equal to s^2 , s^5 and s^4 respectively.

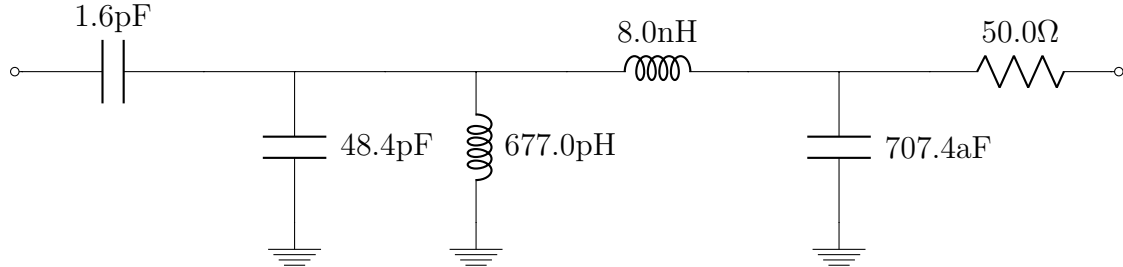


Figure 4.20: Matching circuit 2 providing response close to S_{11} in fig 4.18

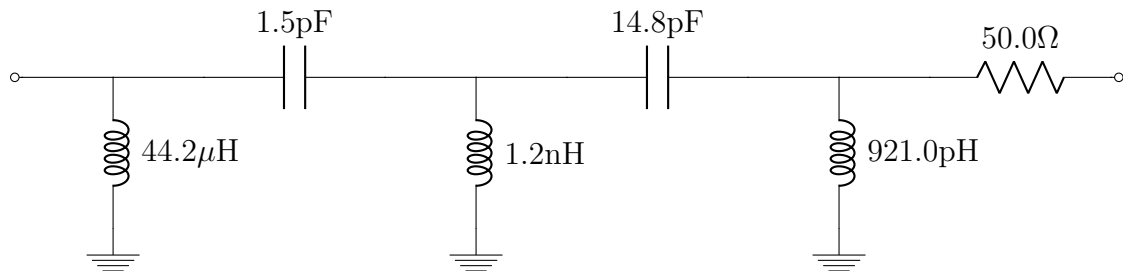


Figure 4.21: Matching circuit 3 providing response close to S_{11} in fig 4.18

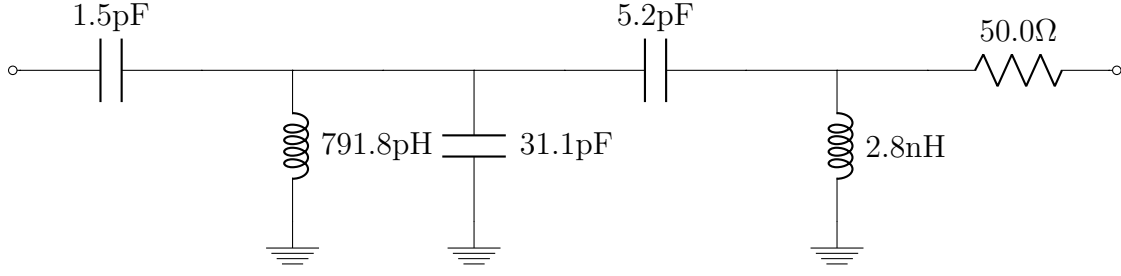


Figure 4.22: Matching circuit 4 providing response close to S_{11} in fig 4.18

It should be noted that alternative to the way the iterations and derivatives are defined in this section, they can be defined directly using chain matrix as well. Motivated by the requirement of implementing microstrip lines in between the LC sections, the implementation directly using chain matrix is discussed in the next section. The chain matrix formalism is important since the scattering matrix of microstrip lines does not have a rational form and we no longer have a scheme of iterating polynomials as described in equations (4.132), (4.133), (4.134), (4.136), (4.137) and (4.138).

4.5 Synthesis of L, C Matching Circuits with Fixed Microstrip Lines

In this section, we will discuss the formulation and implementation of the matching problem in the real setting in such a way as to include in the optimisation fixed microstrip lines in between the L, C sections of matching circuit. In the realization of matching networks using lumped elements on PCB, microstrip lines between the lumped inductors and capacitors can have an effect on the expected results of matching. This signifies the practical importance of this section. The effective dielectric constant ϵ of a microstrip line is given approximately by (equation 3.195, [77])

$$\epsilon = \frac{\epsilon_r + 1}{2} + \frac{\epsilon_r - 1}{2} \frac{1}{\sqrt{1 + 12 \frac{d}{w}}}, \quad (4.154)$$

where w is the width of the microstrip line, d is the thickness of the dielectric substrate having relative permittivity ϵ_r . The characteristic impedance, Z_c of the microstrip line can be calculated as (equation 3.196, [77])

$$Z_c = \begin{cases} \frac{60}{\sqrt{\epsilon}} \ln \left(8 \frac{d}{w} + \frac{w}{4d} \right) & \text{for } \frac{w}{d} \leq 1 \\ \frac{120\pi}{\sqrt{\epsilon} \left(\frac{w}{d} + 1.393 + 0.667 \ln \left(\frac{w}{d} + 1.444 \right) \right)} & \text{for } \frac{w}{d} > 1. \end{cases} \quad (4.155)$$

Let us consider a microstrip line with charactersitic impedance equal to the fixed reference impedance Z_0 . The scattering matrix of this microstrip line with effective dielectric

4.5. SYNTHESIS OF L, C MATCHING CIRCUITS WITH FIXED MICROSTRIP LINES

constant ϵ and length l for the reference impedance Z_0 is

$$S^{ML} = \begin{bmatrix} 0 & \exp(-\frac{2\pi l\sqrt{\epsilon}}{c_0}s) \\ \exp(-\frac{2\pi l\sqrt{\epsilon}}{c_0}s) & 0 \end{bmatrix}, \quad (4.156)$$

where c_0 denote the speed of light. The corresponding chain matrix is

$$T^{ML} = \begin{bmatrix} \exp(-\frac{2\pi l\sqrt{\epsilon}}{c_0}s) & 0 \\ 0 & \exp(\frac{2\pi l\sqrt{\epsilon}}{c_0}s) \end{bmatrix} \quad (4.157)$$

We will continue to use the elementary chaining approach discussed in the last section with the difference that we have to take care of the chaining of microstrip lines in between each elementary section. The parameterisation of the reflection coefficient of the matching circuit will be done using $\rho = [\rho_1, \rho_2, \dots, \rho_N]^t$, the angular derivatives of the reflection coefficient of elementary L, C sections that was introduced in the last section. The matching problem can be formulated as:

Problem. \mathcal{H} . *Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4),*

$$\text{Find: } \hat{l} = \min_{\rho \in \mathbb{R}_+^N} \max_{s \in \mathbb{I}} \left| \frac{S_{22}^\rho(s) - \overline{L_{11}(s)}}{1 - S_{22}^\rho(s)L_{11}(s)} \right|,$$

where S_{22}^ρ is the output reflection coefficient of matching network consisting of N elementary L, C sections chained in a given fixed order and having microstrip lines of fixed electrical and physical properties in between each sections.

For simplicity of derivative calculations, we solve problem \mathcal{H} after squaring $\delta(S_{22}^\rho(s), \overline{L_{11}(s)})$ in the criterion. As in the previous section, let us denote the points in the passband of interest where the measured reflection coefficient of the load L_{11} is given as $\{s_1, s_2, \dots, s_k\}$. The problem \mathcal{H} can be solved practically by defining an extra variable $\Gamma \in \mathbb{R}$, satisfying,

$$\Gamma \geq \left| \frac{S_{22}^\rho(s_i) - \overline{L_{11}(s_i)}}{1 - S_{22}^\rho(s_i)L_{11}(s_i)} \right|^2, \quad i = 1, 2, \dots, k \quad (4.158)$$

and then solving the following problem,

$$\begin{aligned} & \min_{(\rho, \Gamma) \in \mathbb{R}_+^{N+1}} \Gamma \\ \text{sub : } & \left| \frac{S_{22}^\rho(s_i) - \overline{L_{11}(s_i)}}{1 - S_{22}^\rho(s_i)L_{11}(s_i)} \right|^2 - \Gamma \leq 0, \quad i = 1, 2, \dots, k. \end{aligned} \quad (4.159)$$

4.5.1 Numerical Implementation of Problem \mathcal{H}

For a fixed topology of the matching circuit, the evaluation of S_{22}^ρ at the data points can be easily done with the help of chain matrices of elementary L, C sections and microstrip line. If we denote the chain matrices of N elementary sections in the matching circuit by

$T^{\rho_1}, T^{\rho_2}, \dots, T^{\rho_N}$ (in the order of chaining) and the chain matrix of microstrip line of fixed length and width present between the sections by T_{ML} , the overall chain matrix of the matching circuit, denoted by T can be evaluated at any frequency as,

$$T^\rho = T^{\rho_1} * T^{ML} * T^{\rho_2} * T^{ML} * T^{\rho_3} * \dots * T^{\rho_{N-1}} * T^{ML} * T^{\rho_N}. \quad (4.160)$$

Each T^{ρ_i} represent the chain matrix of elementary L, C sections. The value of ρ_i in each case depend upon the fixed sequence of chaining of sections. We have the chain matrix of elementary sections

$$T^{lp} = \begin{bmatrix} 1 - \rho^{lp}s & \gamma\rho^{lp}s \\ -\gamma\rho^{lp}s & 1 + \rho^{lp}s \end{bmatrix}, \quad T^{hp} = \frac{1}{s} \begin{bmatrix} s - \rho^{hp} & \gamma\rho^{hp} \\ -\gamma\rho^{hp} & s + \rho^{hp} \end{bmatrix} \quad (4.161)$$

where ρ^{lp} , ρ^{hp} and γ satisfy the equations mentioned in (4.125). Then the scattering matrix of matching circuit denoted by S^ρ can be evaluated using the conversion,

$$S^\rho = \frac{1}{T_{22}^\rho} \begin{bmatrix} T_{12}^\rho & -\det(T^\rho) \\ 1 & -T_{21}^\rho \end{bmatrix}, \quad (4.162)$$

So, once we have the chain matrix T^ρ , we have the evaluation of S_{22}^ρ at any data point,

$$S_{22}^\rho(s) = -\frac{T_{21}^\rho(s)}{T_{22}^\rho(s)}. \quad (4.163)$$

Now, we are in a position to calculate the gradient of criterion and constraint of problem 4.159 at the data points $\{s_1, s_2, \dots, s_k\}$.

Derivatives of Criterion and Constraint

The gradient of the criterion with respect to the optimisation variable, $(\rho, \Gamma) = [\rho_1, \rho_2, \dots, \rho_{N-1}, \rho_N, \Gamma]^t \in \mathbb{R}_+^{N+1}$ is trivial and is equal to $[0, 0, \dots, 0, 1]^t \in \mathbb{R}^{N+1}$. To evaluate the gradient of constraint with respect to the optimisation variable, let us denote the constraint function as, for $s \in j\mathbb{R}$,

$$C(s) = |G(s)|^2 - \Gamma, \quad (4.164)$$

where, $G(s) = \frac{S_{22}^\rho(s) - \overline{L_{11}(s)}}{1 - S_{22}^\rho(s)L_{11}(s)}$. In order to express the gradient of constraint at the data points, let us define the following vectors,

$$Lval = [L_{11}(s_1), L_{11}(s_2), \dots, L_{11}(s_k)]^t \quad (4.165)$$

$$Sval = [S_{22}^\rho(s_1), S_{22}^\rho(s_2), \dots, S_{22}^\rho(s_k)]^t \quad (4.166)$$

$$Gval = [G(s_1), G(s_2), \dots, G(s_k)]^t. \quad (4.167)$$

It should be noted that the vector $Lval$ is available directly from the given data and the vectors $Sval$ and $Gval$ can be obtained after calculating S_{22}^ρ at the data points using equation 4.163. The objective Γ can be obtained as $\max(|Gval|^2 / (1 - |Lval|^2))$ and the evaluation of the constraint $C(s)$ at the data points as $|Gval|^2 / (1 - |Lval|^2) - \Gamma$. We have

4.5. SYNTHESIS OF L, C MATCHING CIRCUITS WITH FIXED MICROSTRIP LINES

the gradient of the constraint with respect to optimisation variable evaluated at the data points,

$$\nabla C = [der, -ones(k, 1)]^t,$$

$(N + 1) \times k$ matrix, where, $der = 2 \cdot \text{Re}(dG \cdot \overline{Gval})$ is of size $k \times N$. the derivative of G with respect to ρ evaluated at the data points, denoted by dG is

$$dG = ((1 - Sval \cdot Lval) \cdot dS_{22}^\rho + (Sval - \overline{Lval}) \cdot Lval \cdot dS_{22}^\rho) / (1 - Sval \cdot Lval).^2$$

where dS_{22}^ρ denote the derivative of S_{22}^ρ with respect to ρ evaluated at the data points and is of size $k \times N$. So, it remains to calculate the matrix dS_{22}^ρ . From equation (4.163), we have

$$dS_{22}^\rho = \frac{T_{21}^\rho val \cdot dT_{22}^\rho - T_{22}^\rho val \cdot dT_{21}^\rho}{(T_{22}^\rho val)^2}$$

where $T_{21}^\rho val = [T_{21}^\rho(s_1), T_{21}^\rho(s_2), \dots, T_{21}^\rho(s_k)]^t$ and $T_{22}^\rho val = [T_{22}^\rho(s_1), T_{22}^\rho(s_2), \dots, T_{22}^\rho(s_k)]^t$ can be obtained directly from equation (4.160) of T^ρ . In order to calculate the matrices dT_{22}^ρ and dT_{21}^ρ , it should be noted that we have the derivatives of elementary chain matrices T^{lp} and T^{hp} with respect to ρ^{lp} and ρ^{hp} respectively at any data point $s \in \{s_1, s_2, \dots, s_k\}$:

$$\frac{dT^{lp}}{d\rho^{lp}}(s) = \begin{bmatrix} -s & \gamma s \\ -\gamma s & s \end{bmatrix}, \quad \frac{dT^{hp}}{d\rho^{hp}}(s) = \frac{1}{s} \begin{bmatrix} -1 & \gamma \\ -\gamma & 1 \end{bmatrix}. \quad (4.168)$$

So, the derivative of matrix T^ρ with respect to $\rho = [\rho_1, \rho_2, \dots, \rho_N]$ at any point $s \in \{s_1, s_2, \dots, s_k\}$ can be calculated using the following derivative calculations:

$$\begin{aligned} \frac{dT^\rho}{d\rho_1}(s) &= \frac{dT^{\rho_1}}{d\rho_1}(s) * T^{ML} * T^{\rho_2} * T^{ML} * T^{\rho_3} * \dots * T^{\rho_{N-1}} * T^{ML} * T^{\rho_N} \\ \frac{dT^\rho}{d\rho_2}(s) &= T^{\rho_1} * T^{ML} * \frac{dT^{\rho_2}}{d\rho_2}(s) * T^{ML} * T^{\rho_3} * \dots * T^{\rho_{N-1}} * T^{ML} * T^{\rho_N} \\ &\vdots \\ \frac{dT^\rho}{d\rho_N}(s) &= T^{\rho_1} * T^{ML} * T^{\rho_2} * T^{ML} * T^{\rho_3} * \dots * T^{\rho_{N-1}} * T^{ML} * \frac{dT^{\rho_N}}{d\rho_N}(s) \end{aligned}$$

where $\frac{dT^{\rho_i}}{d\rho_i}(s)$, $1 \leq i \leq N$ follow either of the equations mentioned in (4.168) depending upon the elementary section at i -th position. So, we have the matrices dT_{22}^ρ and dT_{21}^ρ from the above calculations:

$$dT_{22}^\rho = \begin{bmatrix} \frac{dT_{22}^\rho}{d\rho_1}(s_1) & \frac{dT_{22}^\rho}{d\rho_2}(s_1) & \dots & \frac{dT_{22}^\rho}{d\rho_{N-1}}(s_1) & \frac{dT_{22}^\rho}{d\rho_N}(s_1) \\ \frac{dT_{22}^\rho}{d\rho_1}(s_2) & \frac{dT_{22}^\rho}{d\rho_2}(s_2) & \dots & \frac{dT_{22}^\rho}{d\rho_{N-1}}(s_2) & \frac{dT_{22}^\rho}{d\rho_N}(s_2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{dT_{22}^\rho}{d\rho_1}(s_k) & \frac{dT_{22}^\rho}{d\rho_2}(s_k) & \dots & \frac{dT_{22}^\rho}{d\rho_{N-1}}(s_k) & \frac{dT_{22}^\rho}{d\rho_N}(s_k) \end{bmatrix},$$

$$dT_{21}^\rho = \begin{bmatrix} \frac{dT_{21}^\rho}{d\rho_1}(s_1) & \frac{dT_{21}^\rho}{d\rho_2}(s_1) & \dots & \frac{dT_{21}^\rho}{d\rho_{N-1}}(s_1) & \frac{dT_{21}^\rho}{d\rho_N}(s_1) \\ \frac{dT_{21}^\rho}{d\rho_1}(s_2) & \frac{dT_{21}^\rho}{d\rho_2}(s_2) & \dots & \frac{dT_{21}^\rho}{d\rho_{N-1}}(s_2) & \frac{dT_{21}^\rho}{d\rho_N}(s_2) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{dT_{21}^\rho}{d\rho_1}(s_k) & \frac{dT_{21}^\rho}{d\rho_2}(s_k) & \dots & \frac{dT_{21}^\rho}{d\rho_{N-1}}(s_k) & \frac{dT_{21}^\rho}{d\rho_N}(s_k) \end{bmatrix}.$$

Thus, we have the complete calculation of derivatives of the criterion and constraint $C(s)$ of problem 4.159. Just as in the last section, we impose the constraint $\rho_i > 0$ for $i = 1, 2, \dots, N$ as well and then the matlab non-linear programming solver `fmincon` is used to find the solution for problem \mathcal{H} . This completes the description of implementation of problem \mathcal{H} . Once the optimal ρ of problem \mathcal{H} is obtained, the computation of inductance and capacitor values of the optimal matching network follows the same equations mentioned at the end of section 4.4.

Example : Superdirective Antenna

The result of solving problem \mathcal{H} with $N = 5$ when all the transmission zeors of the elementary sections in the matching circuit were fixed at zero is presented in figure 4.23. The elementary sections in the order of chaining to the load were fixed to be series C , parallel L , series C , parallel L and series C and microstrip lines of length equal to 1.5 mm were fixed to be present between the elementary sections. For the purpose of illustration, the width of the microstrip line was fixed to be 1.5 mm and dielectric substrate thickness of 0.8 mm with relative permittivity $\epsilon_r = 4.4$. This provides effective dielectric constant, $\epsilon = 3.39$ and characteristic impedance of the microstrip line equal to 50Ω according to the equations in 4.154 and 4.155. The matching circuit providing the response in figure 4.23 is shown in figure 4.24.

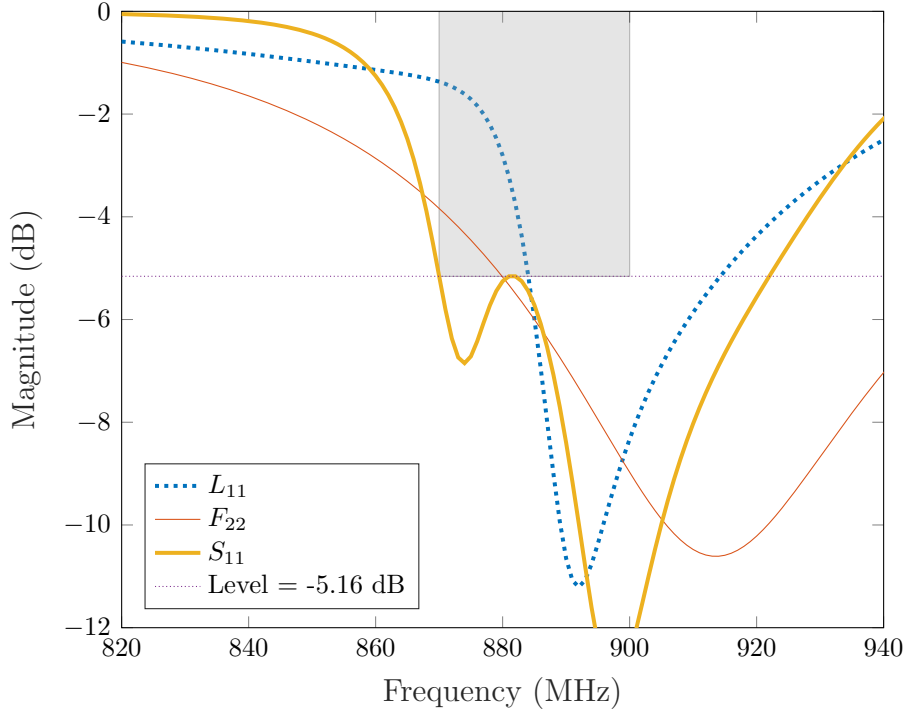


Figure 4.23: Result of solving problem \mathcal{H} with $N = 5$ and $r(s) = s^5$

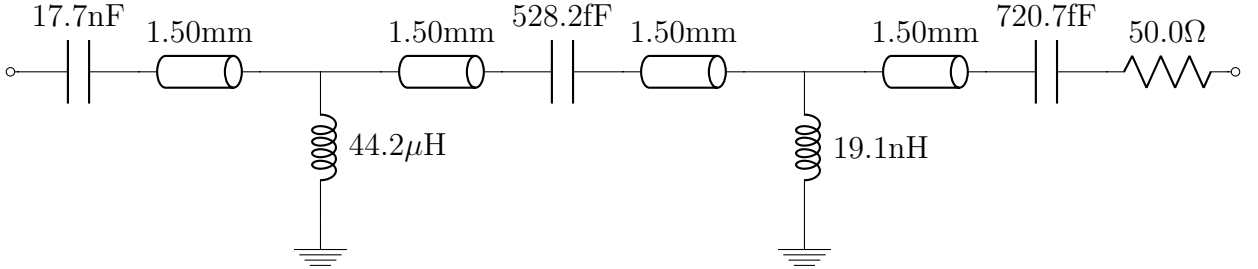


Figure 4.24: Matching circuit providing the result in fig 4.23

The optimal criterion obtained, -5.16 dB can be noticed to be not as good as the one in figure 4.12 (-6.58 dB) which was obtained by solving problem \mathcal{P}_{sc} with $N = 5$ and $r(s) = s^5$ and transformerless condition $a_5 = 0$. The topology of the circuits in both cases (figure 4.13 and figure 4.24) are the same except the addition of fixed microstrip lines in between the L, C elements in the circuit in figure 4.24. The provided example clearly illustrates the impact of introducing microstrip lines on the element values of the optimal matching circuit and the optimal matching criterion and hence signifies the practical importance of problem \mathcal{H} .

In figure 4.25, we have provided a comparison of the results obtained on superdirective antenna example by solving different problems discussed in the thesis using degree 5 matching circuit : problem \mathfrak{P}_C in chapter 3 (convex optimisation approach) and problems \mathcal{P}, \mathcal{B}

(matching circuit with real components having impedance transformer and constrained to have no impedance transformer respectively) and problem \mathcal{H} (where microstrip lines are included between the LC components). The figures are placed in the order : top left, top right, bottom left, bottom right respectively.

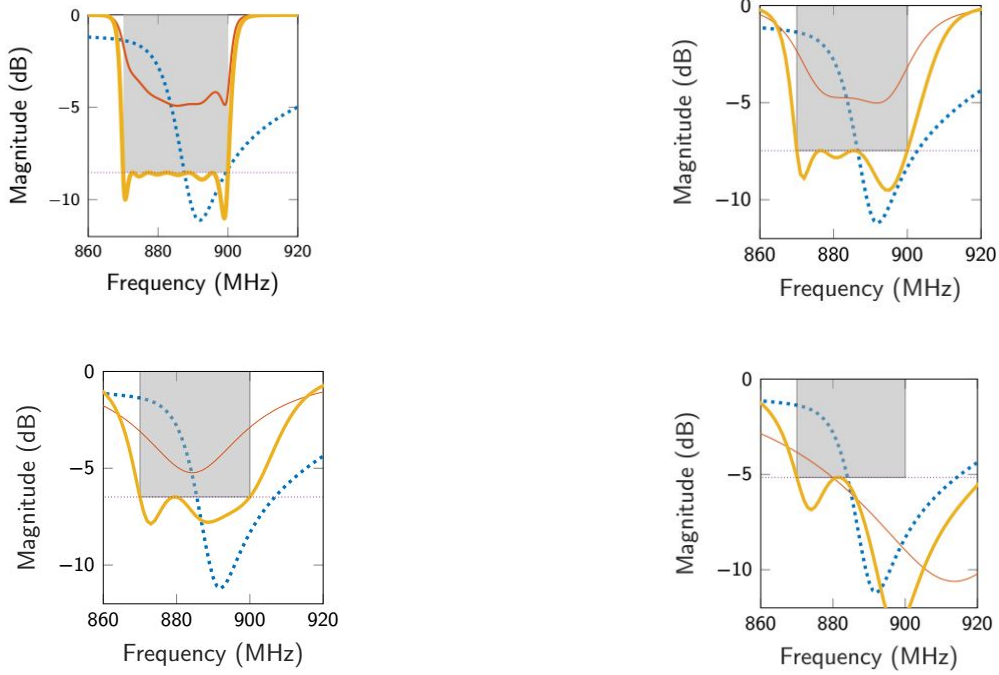


Figure 4.25: Result of problems $\mathfrak{P}_C, \mathcal{P}, \mathcal{B}$ and \mathcal{H} : -8.5 dB, -7.5 dB, -6.6 dB, -5.2 dB.

We conclude this section by stating that the elementary chaining approach (where parametrisation is done using angular derivatives) for the matching problem described in sections 4.4 and 4.5 provides a number of advantages compared to the parametrisation over coefficients of p described in subsection 4.2.3 :

- The computation of $q \in \mathbb{SB}_{\mathbb{R}, N}$ from $p \in \mathbb{P}_{\mathbb{R}, N}$, which involves the spectral factorisation can be avoided.
- It avoids the impedance transformer at the end of the matching circuit without imposing any extra constraints since the construction of the circuit is done by elementary chaining of L,C sections.
- It facilitates the option of adding fixed microstrip lines in between the L,C components in the optimisation and provide the optimal L,C values taking into consideration these transmission lines.
- If required, it provides an easy and efficient way to bound the values of L,C components in the optimisation to any range required as per the application or user requirements. This can be done by adding extra constraints of upper bound and lower bound on ρ in the optimisation. If we denote the lower bound and upper bound of inductance values as L_{lb} and L_{ub} Henry and that of capacitance values as C_{lb} and C_{ub}

Farad respectively, we can add the following additional constraint on ρ_i , $1 \leq i \leq N$ in the optimisation. In the case of low pass section (transmission zero at infinity) at the i -th position,

- (a) series inductor : $\frac{\pi b_m L_{lb}}{Z_0} \leq \rho_i \leq \frac{\pi b_m L_{ub}}{Z_0}$
- (b) parallel capacitor : $\frac{\pi b_m C_{lb}}{Y_0} \leq \rho_i \leq \frac{\pi b_m C_{ub}}{Y_0}$

and in the case of high pass section (transmission zero at zero) at the i -th position,

- (a) series capacitor: $\frac{Y_0}{4\pi b_m C_{ub}} \leq \rho_i \leq \frac{Y_0}{4\pi b_m C_{lb}}$
- (b) parallel inductor : $\frac{Z_0}{4\pi b_m L_{ub}} \leq \rho_i \leq \frac{Z_0}{4\pi b_m L_{lb}}$.

We would like to make the remark that all the problems discussed in this chapter are non-convex and so the numerical implementation schemes discussed do not guarantee a unique optimum. This is the main difference in comparison to the chapter 3 convex optimisation approach, but nevertheless the problems discussed in this chapter allows us to find local optima that satisfy the necessary practical constraints. Finally, before concluding this chapter, we will provide an interesting remark concerning the possible number of equivalent circuit topologies that can realize a given scattering matrix of lossless two-port. It should be noted that even though in this chapter, we have not discussed the realization of a transmission zero of an S -matrix at a finite non-zero point on the imaginary axis, this can be easily included in the optimisation schemes discussed as well. This will provide matching circuits consisting of LC series resonators connected in parallel and LC parallel resonators connected in series. In the remark provided below, we consider circuits consisting of only elementary L, C sections that was described in section 4.3, that is series or parallel inductor or capacitor.

Remark 4.5.1. *There exist $4 * 3^{N-1}$ (with some circuit repetitions for $N \geq 2$) possible number of equivalent circuit topologies that can realize a given degree N lossless scattering matrix whose transmission zeros are allowed to be at zero or infinity.*

The proof of remark 4.5.1 easily follows since the first element of N element circuit has 4 possibilities (series or parallel inductor/capacitor) and all the remaining $N - 1$ positions have 3 options each. It should be noted that in the above count, for example with $N = 2$, there are 2 circuit repetitions : one circuit with a series inductor and a series capacitor and another circuit with a parallel inductor and a parallel capacitor. This repetition increases with N and for a general $N \geq 2$, to have the exact number of equivalent circuit topologies with no repetition of circuits within them, we have to subtract this repetition count from $4 * 3^{N-1}$. This is not presented here since the main interest of remark 4.5.1 was to provide an insight into the exponential increase of equivalent circuit topologies with the number of components N . We conclude this section here and in the next section, we will provide a conclusion to the chapter.

4.6 Conclusion

In this chapter, we initially introduced the matching problem in the real setting and its numerical implementation. The natural parameterisation based on the coefficients of the

polynomial variable, $p \in \mathbb{P}_{\mathbb{R},N}$ in the optimisation problem was used for the implementation of the problem. Some important theoretical results including the existence of solution and characterisation of the solution based on the number of critical points were also made. A study of the constraints that need to be imposed on the coefficients of polynomial p was done in order to ensure a transformerless synthesis of the matching circuit. A section was also devoted to an elementary chaining approach based on the angular derivatives of the reflection coefficients of the basic L, C sections. This provided a more direct way of transformerless synthesis of the matching circuit possible, along with the option to bound the element values of L, C elements to specific ranges required by the user if required. Finally, motivated by the requirement to include microstrip lines in between the elementary L, C sections, a section devoted to the formulation of the problem based on chain matrices was also made. In general, in this chapter, we have discussed some of the practical constraints that are important in the matching network synthesis like transformerless circuit, bounding the L, C values as per the user requirement, taking into consideration the microstrip lines between lumped inductors and capacitors in the matching circuit and showed that they can be easily included in the optimisation schemes.

In the next chapter, we will give an illustration of the numerical implementations of the different optimisation schemes developed throughout the thesis on some concrete antenna examples. The results obtained from part I of the thesis and the current chapter will provide an idea of the theoretical bounds of the matching criterion for the given load in the desired passband and the criterion obtained after considering the necessary practical constraints. The realization of the optimal response using matching circuit consisting of lumped inductors and capacitors will be done. The illustrations of some basic prototypes of antenna designed together with these matching circuits and their reflection coefficient measured in the Vector network Analyzer will be provided as well.

CHAPTER 5

Realisation : Lumped Element Matching Networks

5.1 Introduction

In this chapter we will present different prototypes of PCB antennas designed together with the matching circuit obtained from the optimisation schemes developed in the thesis. Matching of antenna prototypes at different ranges in the frequency band are chosen to test the robustness of the developed matching techniques. A dual band matching example will also be provided. The results of the numerical implementation of different optimisation schemes introduced in the thesis will be illustrated for the presented prototypes. This will provide a comparison of the matching criterion obtained by solving the generalised matching problem in chapter 3 with the matching problems in real setting introduced in chapter 4. For a given degree of the matching circuit, the first part of the thesis provides hard bounds on the possible matching level obtainable. The solution to the matching problems in the real setting in the second part of the thesis take into consideration different practical necessities of the realization of the circuits and hence provides the optimum under these constraints.

The combined results obtained from both parts of the thesis provides a complete description of the matching criterion possible for the antenna prototype in the targeted passband. The theoretical bounds obtained from the global system approach in chapter 3 helps to give an estimate of the level of drop in the matching criterion after imposing the necessary practical constraints. The electromagnetic simulation software High Frequency Structure Simulator (Ansys HFSS [2]) will be used to validate the matching circuit responses and also to simulate the scattering measurement of antenna prototypes together with the matching circuits calculated. The S_{11} measurement obtained from the Vector Network Analyzer will be illustrated for all the prototypes presented. The measurements before and after adding a matching circuit to the antenna will be presented, certifying the robustness of the developed matching procedures.

5.2 Prototype 1 : 169 MHz Antenna

In the first example, we present a prototype of a simple monopole antenna targeted to be matched at 169 MHz ISM band. This ISM band is ideal for sensors or devices that are located in hard to reach places since it provides good RF performance in terms of coverage

and building penetrations. The S_{11} measurement of the antenna without any matching circuit obtained using a network analyzer (reference impedance = $50\ \text{ohm}$) is depicted in figure 5.1. The targeted passband for matching, 167 MHz to 171 MHz is represented using the rectangle in the figure. It can be noticed that the antenna is very reflective in the prescribed passband and to be precise, the S_{11} measurement at the end points of the passband are -0.077 dB and -0.084 dB.

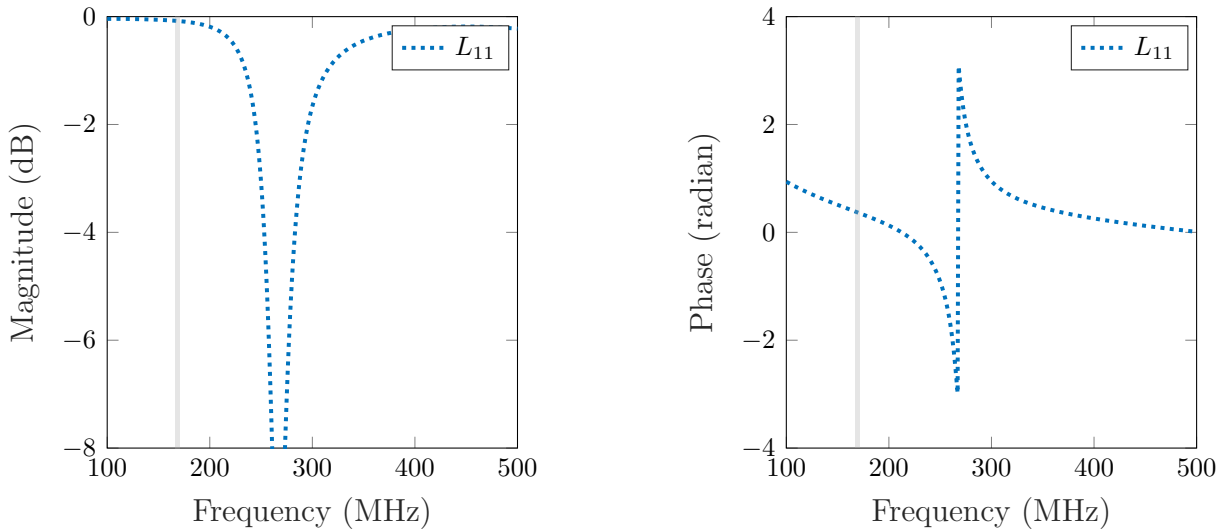


Figure 5.1: Scattering parameter of the load

In accordance with the spirit of the thesis, we will solve this matching problem initially using the global system approach developed in chapter 3 and then present the results obtained in the case of matching problem formulation in the real setting described in chapter 4. This will provide a description about the bounds for the matching criterion possible for any chosen degree of the matching circuit and also the effect of the practical constraints imposed.

In figure 5.2 the result of solving matching problem formulated using the global system approach (problem \mathfrak{P}_C in chapter 3) is presented. The rational approximation of given load data was done using a degree one system in the frequency band of interest. The target degree of the global system was fixed to be three ($N = 3$) and all the transmission zeros of the matching circuit were fixed at infinity, in particular $R_F = 1$. It can be noted that we were able to significantly reduce the maximal mismatch value of the load in the passband from -0.077 dB to -6.86 dB. The 4 critical points ($N+1$) in the passband at which S_{11} attains -6.86 dB characterize the optimality of the obtained response. Since the McMillan degree of the load is 1, the matching criterion obtained corresponds to the theoretical matching limit possible with degree 2 matching circuit.

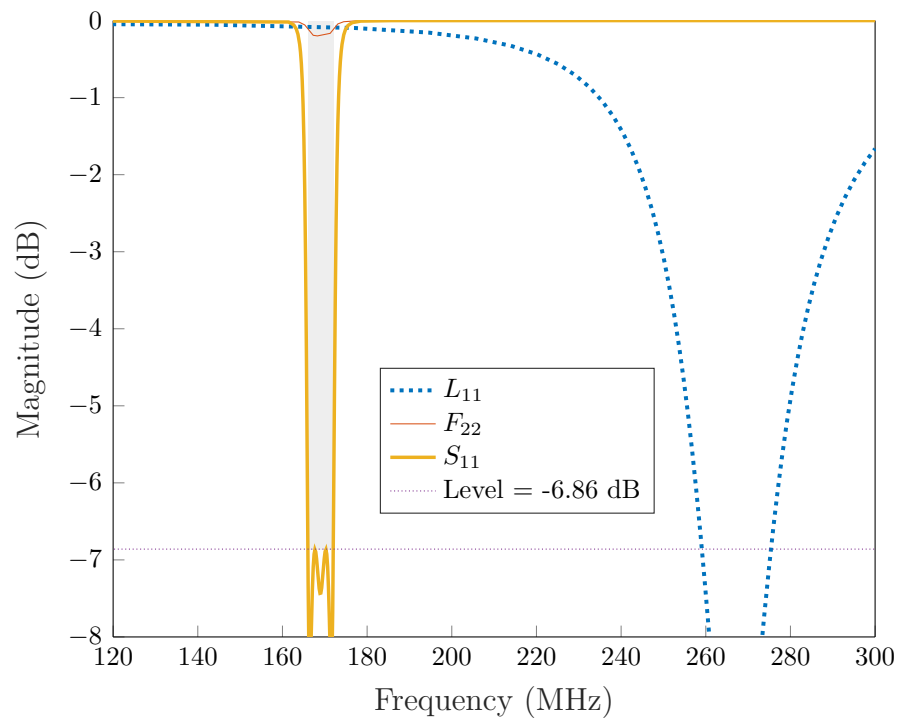


Figure 5.2: Result of global system approach (problem \mathfrak{P}_C) with $N = 3$

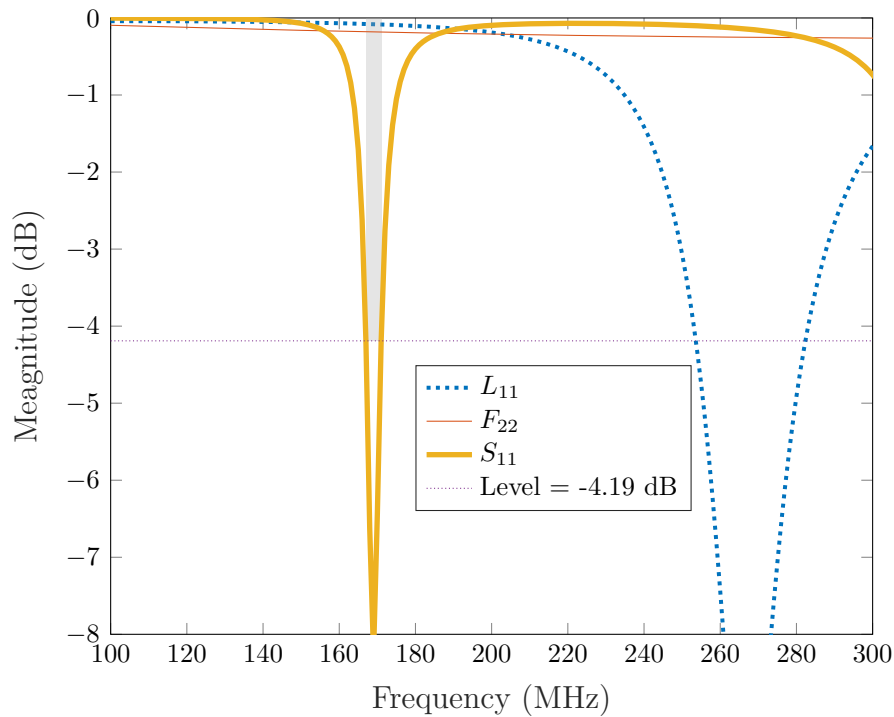


Figure 5.3: Result of matching problem in real setting (problem \mathcal{P}) with $N = 2$

The result of solving matching problem in the real setting (problem \mathcal{P} in chapter 4) is

presented in figure 5.3. We did not impose the transformerless condition and the target degree of the matching circuit was fixed to be two ($N = 2$) with transmission zeros one at zero and one at infinity ($r(s) = s$). The optimal matching criterion obtained, -4.19 dB is lesser than the one obtained by solving problem \mathfrak{P}_C as expected but still provides a significant improvement compared to L_{11} . The 2 critical points ($\lceil \frac{N}{2} \rceil + 1$) at which S_{11} attains -4.19 dB in the passband characterize the optimality of the obtained response. In order to calculate the element values of the optimal matching circuit, the de-chaining of the transmission zeros were performed in the order: one at zero and then the one at infinity. The matching circuit thus obtained, providing the result in figure 5.3 is depicted in figure 5.4. It should be noted that the circuit is not terminating in 50 ohm since we have not imposed any transformerless condition in problem \mathcal{P} .

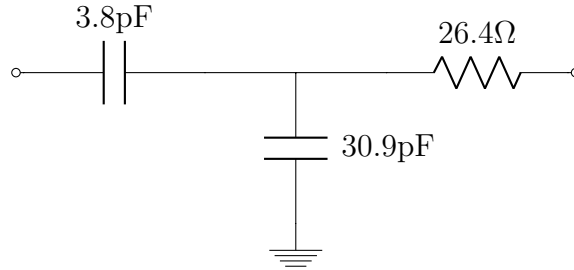


Figure 5.4: Matching circuit providing the result in fig 5.3

Now, the result of solving matching problem using the elementary chaining approach, where parametrisation of F_{22} is done using the angular derivatives is depicted in figure 5.5. The problem under consideration is problem \mathcal{H} where fixed microstrip lines are included between the elementary sections as well. Again the target degree of the matching circuit is fixed to be two but with microstrip lines of length 1.5 mm and effective dielectric constant 3.3 included between the elements. The elementary sections in the order of chaining were fixed to be series capacitor (transmission zero at zero) and then parallel capacitor (transmission zero at infinity) to be in accordance with the circuit topology in figure 5.4. It can be noticed in figure 5.5 that we are able to attain almost the same criterion, -4.18 dB compared to the solution of problem \mathcal{P} in figure 5.3.

A validation of the matching response obtained was performed using Ansys Designer for circuit simulation in HFSS. It was done by connecting a port containing antenna L_{11} data at the left of the circuit in figure 5.7 and simulating the response at a 50 Ω port connected after the last element. The response obtained, denoted by S_{11} HFSS CS is plotted in figure 5.5 and can be observed to be exactly agreeing with S_{11} .

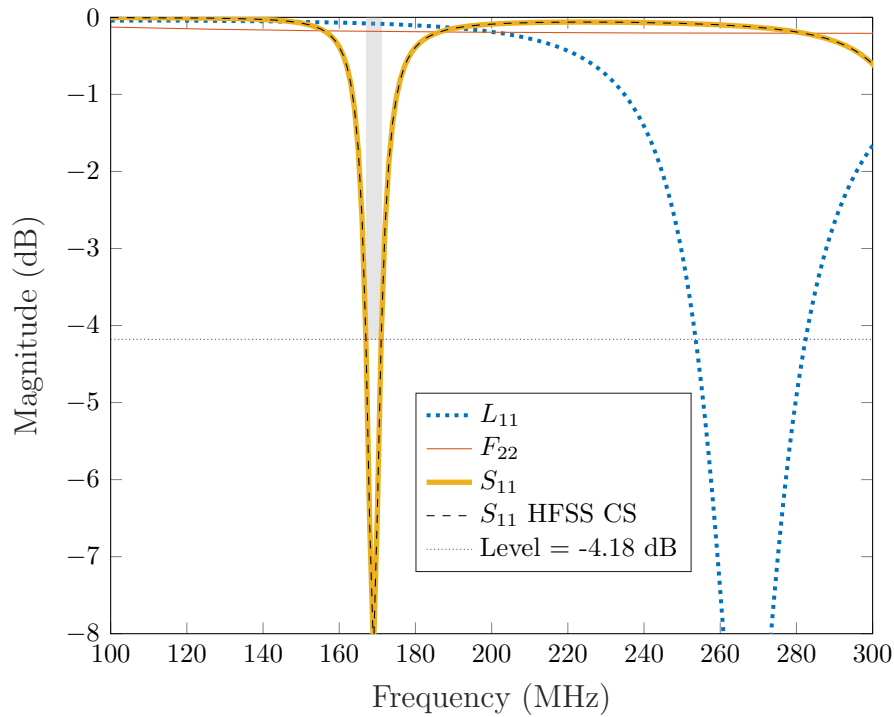


Figure 5.5: Result of matching problem in real setting (problem \mathcal{H}) with $N = 2$

The Smith chart representation of S_{11} in figure 5.5 is depicted in figure 5.6.

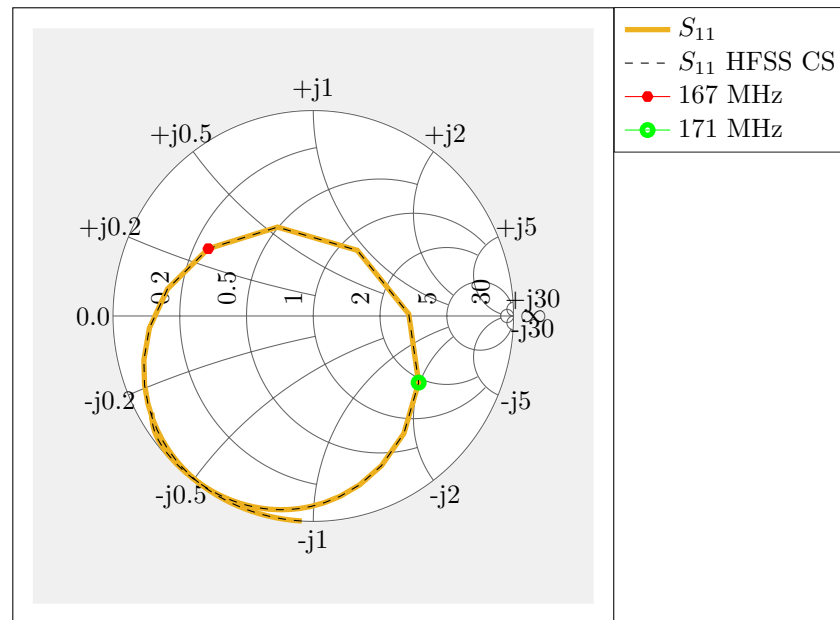


Figure 5.6: Smith chart representation of S_{11} in figure 5.5

The matching circuit providing the result in figure 5.5 is depicted below. It should be noted that now the circuit is terminating in 50 ohm and microstrip line is included between

the elements as well.

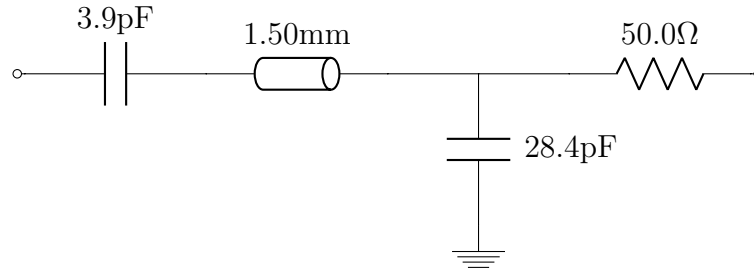


Figure 5.7: Matching circuit providing the result in fig 5.5

Before realizing the matching circuit on the pcb, a monte carlo kind of analysis was performed as well to test the sensitivity of the circuit in figure 5.7. The LC element values and transmission line length was sampled uniformly in a range of original values (values in the circuit in figure 5.7) plus or minus a small percentage of variation and the resulting response of circuit chained to the antenna was obtained for each sample. The result of this analysis using a maximum of 3 percent variation on the L, C element values of the above circuit and transmission line length with 20 samples is presented below.

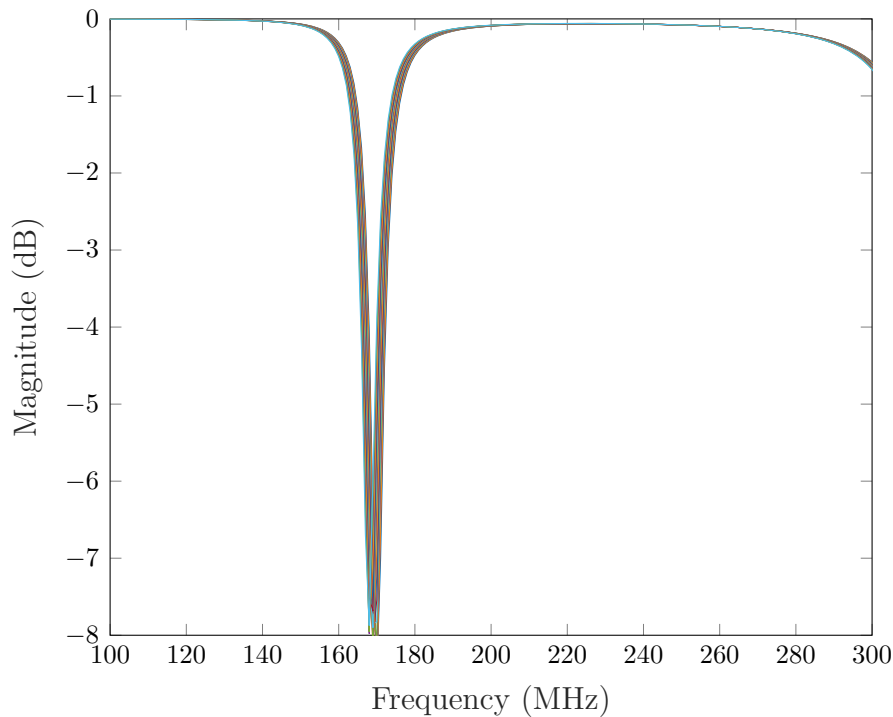


Figure 5.8: Monte Carlo analysis of global response in fig 5.5

It can be noted from the sensitivity analysis that the circuit in figure 5.7 is not very sensitive to small variations in the component values and transmission line length. We would like to point out to the reader that this need not be the case in general, some circuits can be extremely sensitive as well. From the different possible combinations of matching

5.2. PROTOTYPE 1 : 169 MHZ ANTENNA

circuit topologies, in this chapter, we have chosen to consider only the circuits that are not very sensitive since they are supposed to be realized practically. Realizing extremely sensitive circuits can be practically very challenging.

Now, we are in a position to include the matching circuit in the pcb design of the antenna. The nearest element values available in the Johanson capacitor kit 0603 0.3pF-82pF was chosen (3.9pF, 28pF). The pcb antenna with 2 component matching circuit soldered is shown in figure 5.9. The dimensions are length = 255 mm, width = 155 mm and FR-4 epoxy substrate thickness of 1.6 mm.



Figure 5.9: 169 MHz PCB Antenna with 2 component matching circuit

The result of S_{11} measurement of the antenna measured using a VNA analyzer after adding the 2 component matching circuit shown in figure 5.7 is presented in figure 5.10. The corresponding Smith Chart representation is provided in figure 5.11. It can be noticed that even though the matching criterion obtained is not in exact agreement that to the simulated result in figure 5.5, we are able to provide matching in the target band. The main reason behind the deviation in the obtained response compared to simulation is the fact that we have considered lossless matching circuit throughout the theory developed, whereas in reality in the practical realisation, there are losses in the LC components and in the matching circuit. When plotting the reflection coefficient of the antenna together with the matching circuit, these losses contribute to the fact that matching criterion is better than the simulated ones using lossless circuit.

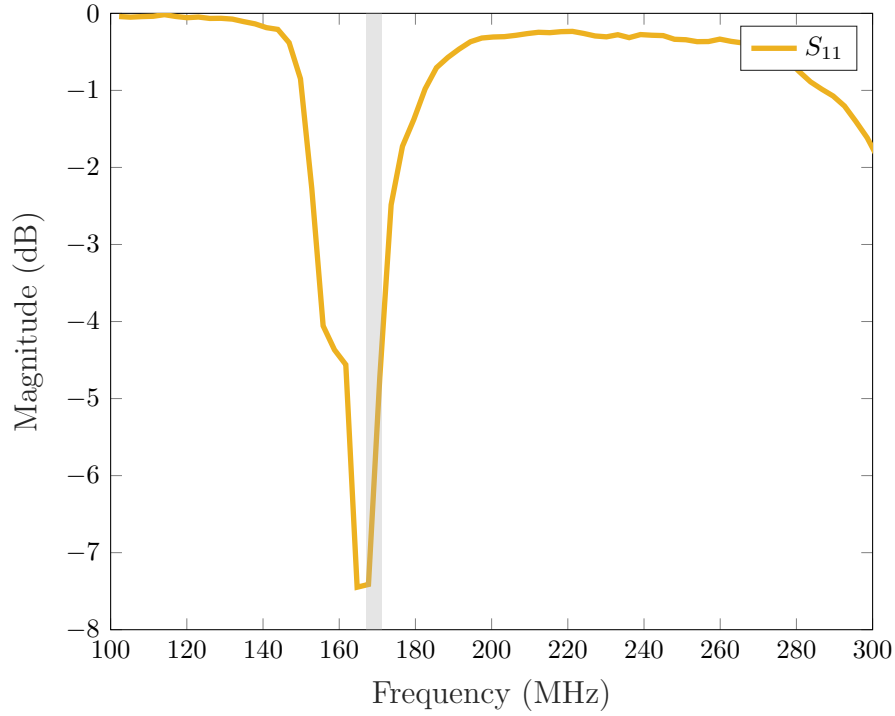


Figure 5.10: S_{11} measurement of the designed antenna with 2 component matching circuit

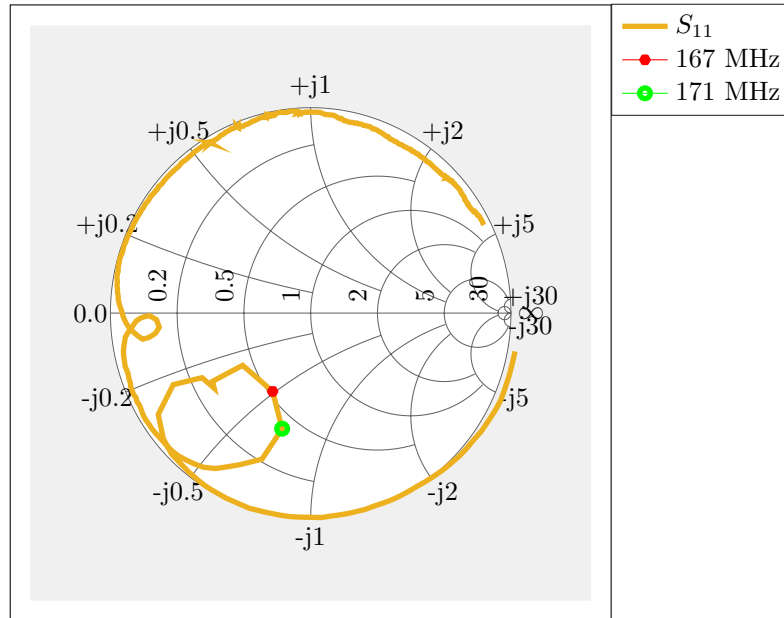


Figure 5.11: S parameter chart of the designed antenna with 2 component matching circuit

Nevertheless, compared to the initial S_{11} measurement of the antenna depicted in figure 5.1, we have been able to remarkably improve the matching criterion in the targeted passband. This concludes the study of degree 2 matching circuit.

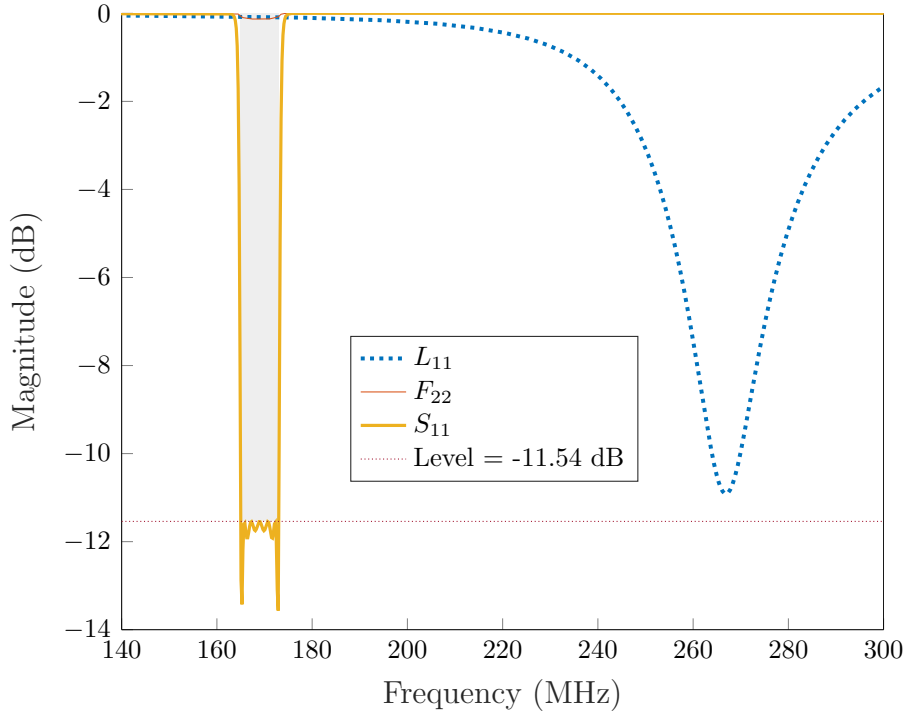


Figure 5.12: Result of global system approach (problem \mathfrak{P}_C) with $N = 6$

Next, the result of matching using a circuit with 5 components is presented for the same antenna in the same target passband, 167 MHz-171 MHz. Initially we present in figure 5.12 the result of solving matching problem formulated using the global system approach (problem \mathfrak{P}_C). The target degree of the global system was fixed to be six ($N = 6$) and all the transmission zeros of the matching circuit were fixed at infinity, in particular $R_F = 1$. It can be noted that we are able to significantly reduce the maximal mismatch value of the load in the passband from -0.077 dB to -11.54 dB. The 7 critical points ($N + 1$) in the passband at which S_{11} attains -11.54 dB characterize the optimality of the obtained response. Since the McMillan degree of the load is 1, the matching criterion obtained corresponds to the theoretical matching limit possible with degree 5 matching circuit.

The result of solving matching problem \mathcal{H} where parametrisation of F_{22} is done using the angular derivatives is depicted in figure 5.13. The capacitor and inductor values in the matching circuit were bounded to be in the ranges 0.3pF-82pF and 1.5nH-220nH by fixing bounds on the angular derivatives and fixed microstrip lines of length 1.5 mm and effective dielectric constant 3.3 were included between the elements as well. The target degree of the matching circuit was fixed to be five ($N = 5$) and the elementary sections in the order of chaining were fixed to be series capacitor, parallel inductor, series inductor, parallel capacitor and series capacitor.

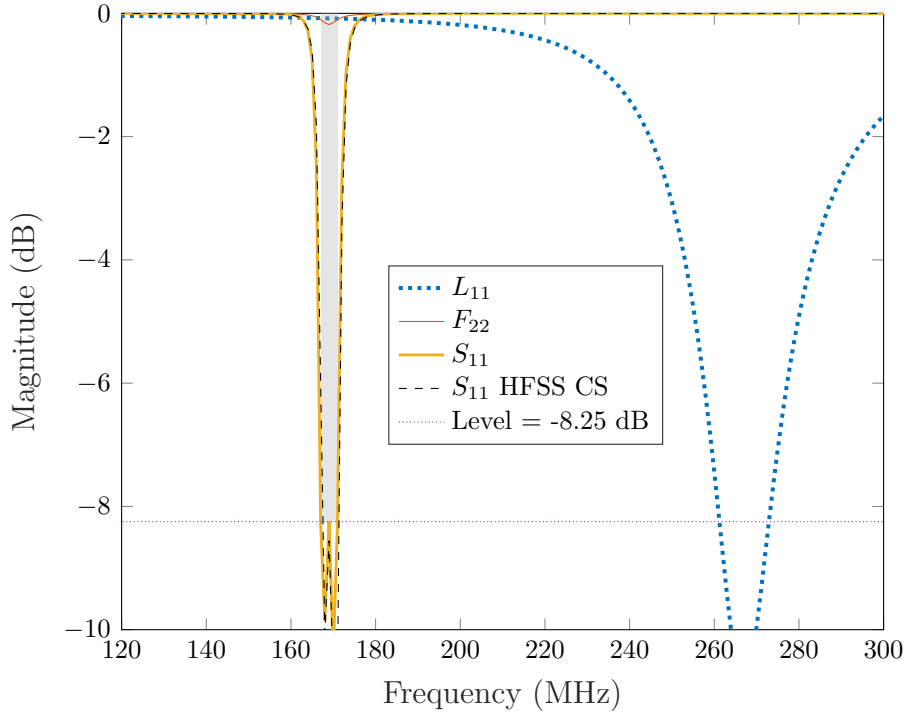


Figure 5.13: Result of matching problem in real setting (problem \mathcal{H}) with $N = 5$

The Smith chart representation of S_{11} in figure 5.13 is depicted in figure 5.14.

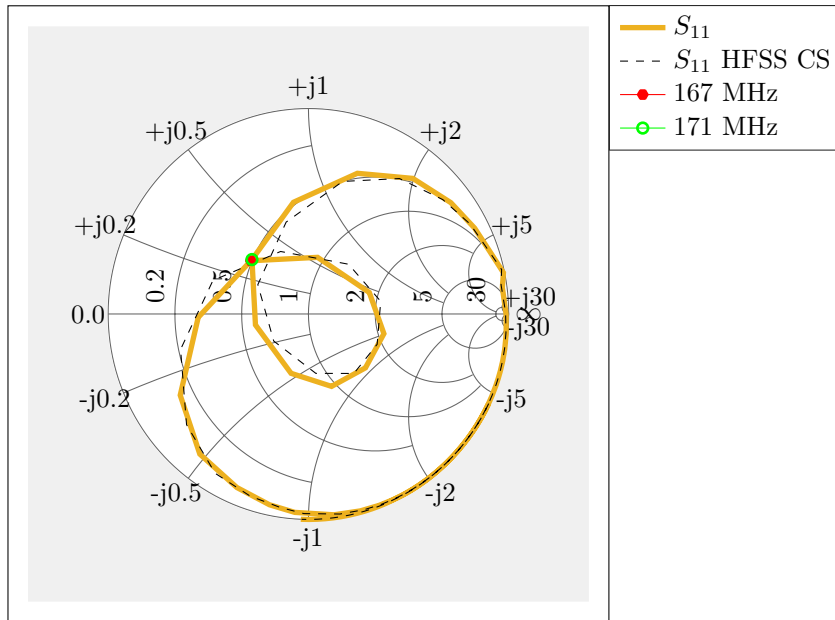


Figure 5.14: Smith chart representation of S_{11} in figure 5.13

The matching circuit providing the result in 5.13 is depicted in figure 5.15. A validation of the matching response obtained was performed using Ansys Designer for circuit simula-

5.2. PROTOTYPE 1 : 169 MHZ ANTENNA

tion. The response obtained, denoted by S_{11} HFSS CS is plotted in figure 5.13 and can be observed to be agreeing with S_{11} .

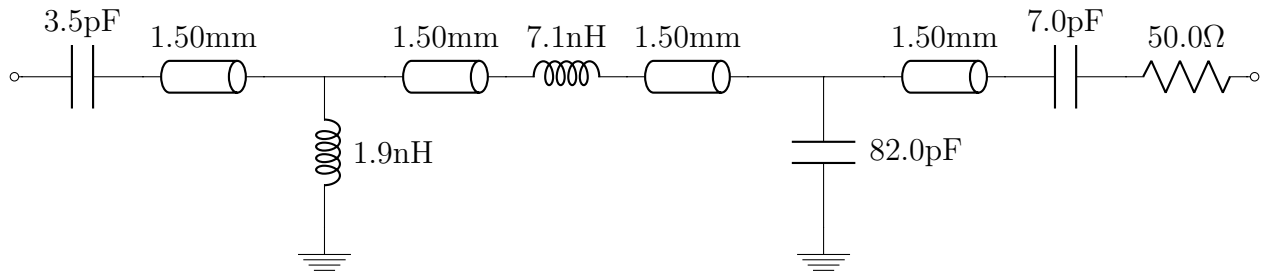


Figure 5.15: Matching circuit providing the result in fig 5.13

A monte carlo kind of analysis to test the sensitivity of the circuit on transmission length was done. It was done using a maximum of 5 percent variation on the transmission line length with 20 samples of transmission line length considered uniformly in this range. The result is presented below and it can be observed that the circuit response is not very sensitive to small variation in the transmission line length.

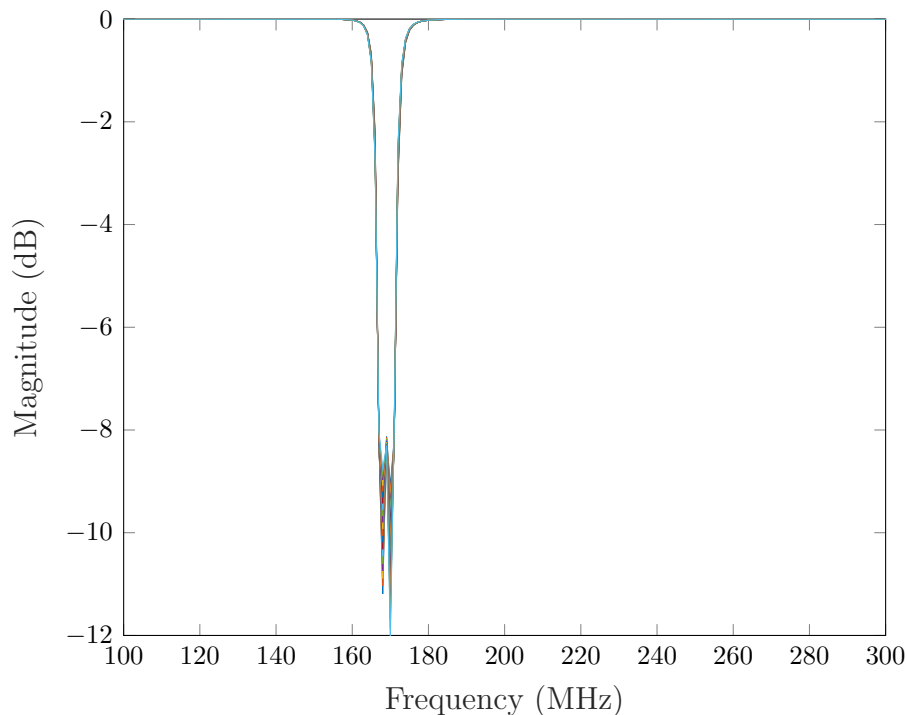


Figure 5.16: Monte Carlo analysis of global response (S_{11}) in fig 5.13

Now the matching circuit in figure 5.15 was included in the pcb design of antenna by element wise soldering. The nearest element values available in the Johanson capacitor kit 0603 0.3pF-82pF and TDK chip inductors kit 0603 1.5nH-220nH was chosen. The pcb antenna after soldering these elements is shown below.



Figure 5.17: 169 MHz PCB Antenna with 5 component matching circuit

The result of S_{11} measurement of the antenna measured using a VNA analyzer after adding the 5 element matching circuit is presented in figure 5.18. The corresponding Smith Chart representation is provided in figure 5.19.

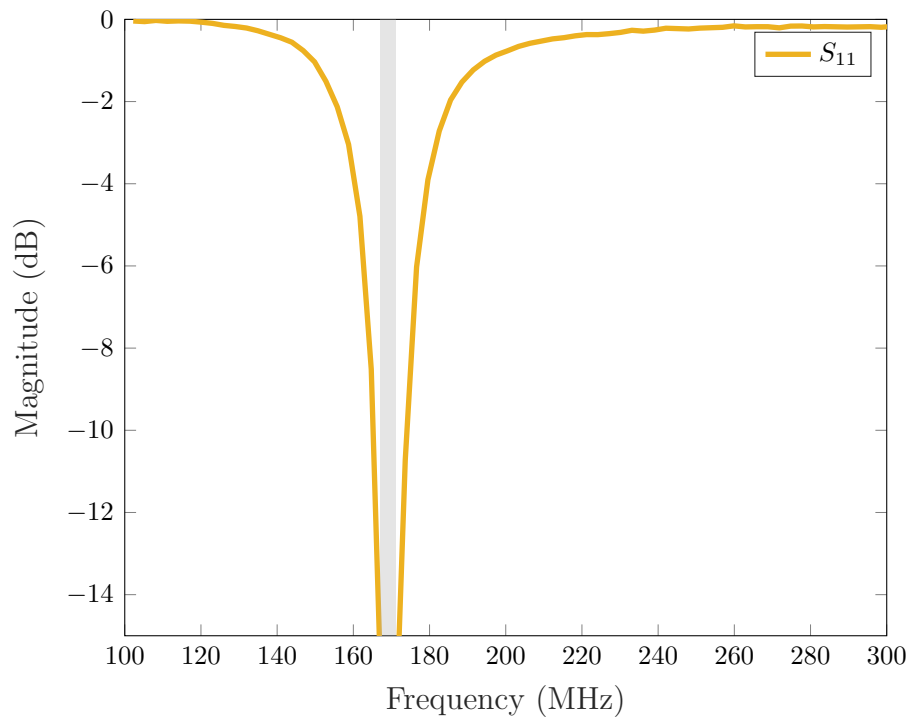


Figure 5.18: S_{11} measurement of the designed antenna with 5 component matching circuit

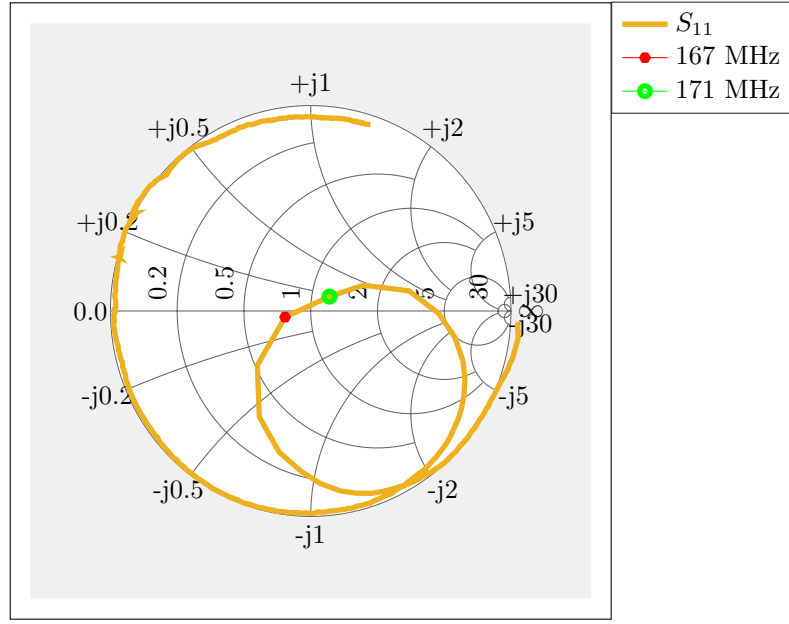


Figure 5.19: S parameter chart of the designed antenna with 5 component matching circuit

As mentioned in the degree 2 matching circuit case, the losses in the components of the matching circuit contributes to the deviation in the response obtained in reality compared to the simulation. Nevertheless, compared to the initial S_{11} measurement of the antenna depicted in figure 5.1, we have been able to improve the matching criterion significantly. Also, it provides a better result than the one compared to matching circuit with 2 components as well (figure 5.10). This concludes the study of matching of 169 MHz antenna and now we move to the second example of matching 433 MHz antenna.

5.3 Prototype 2 : 433 MHz Antenna

In the second example, we present a prototype of a simple monopole antenna targeted to be matched at 433 MHz ISM band. The S_{11} measurement of this antenna obtained using a network analyzer is depicted in figure 5.20. The targeted pass band 427 MHz to 439 MHz is represented using the rectangle in the figure. The S_{11} measurement at the end points of the passband are -0.110 dB and -0.116 dB.

Similar to the example studied in the previous section, we will solve the matching problem initially using the global system approach developed in chapter 3 and then present the results obtained in the case of matching problem formulation in the real setting described in chapter 4. In figure 5.21 the result of solving matching problem formulated using the global system approach (problem \mathfrak{P}_C) is presented. The rational approximation of the given load data was done using a degree one system in the frequency band of interest. The target degree of the global system was fixed to be three ($N = 3$) and all the transmission zeros of the matching circuit were fixed at infinity ($R_F = 1$). It can be noted that we are able to significantly reduce the maximal mismatch value of the load in the passband from -0.11 dB to -7.47 dB. Since the McMillan degree of the load is 1, the matching criterion obtained corresponds to the theoretical matching limit possible with degree 2 matching circuit.

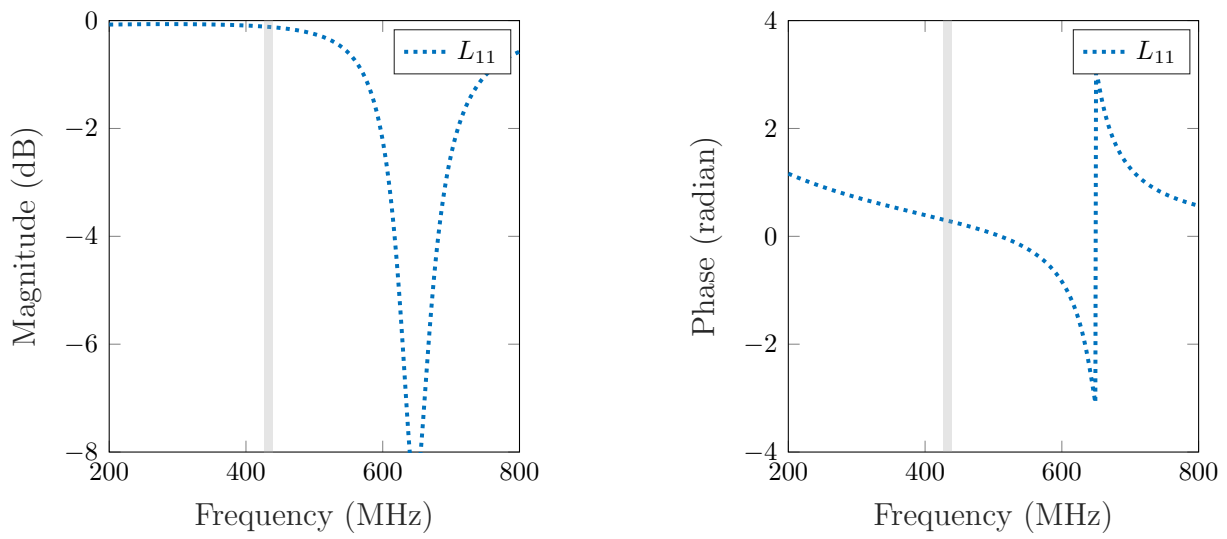


Figure 5.20: Scattering parameter of the load

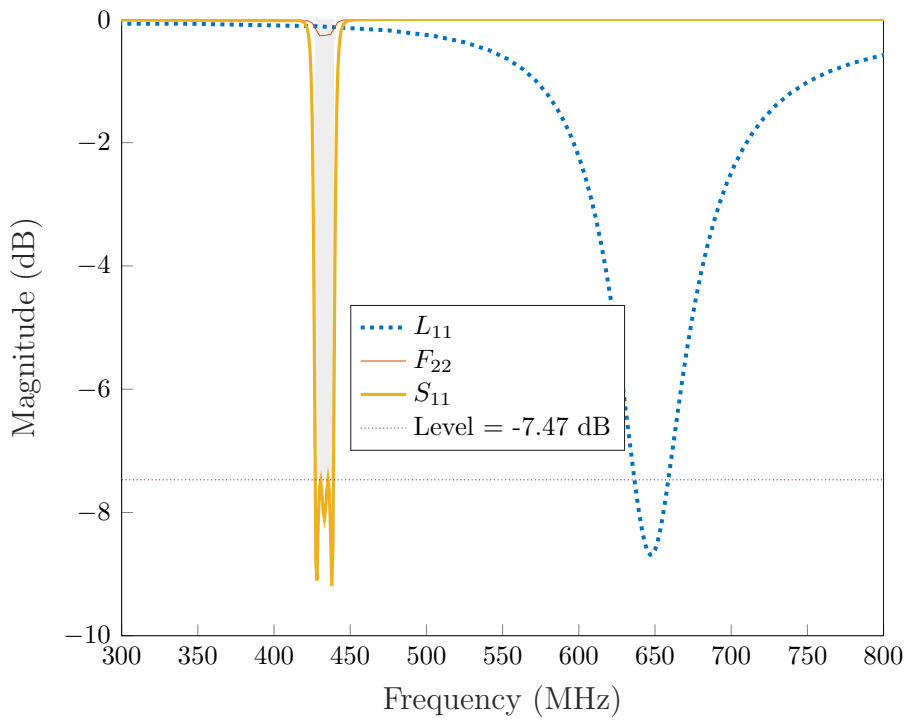


Figure 5.21: Result of global system approach (problem \mathfrak{P}_C) with $N = 3$

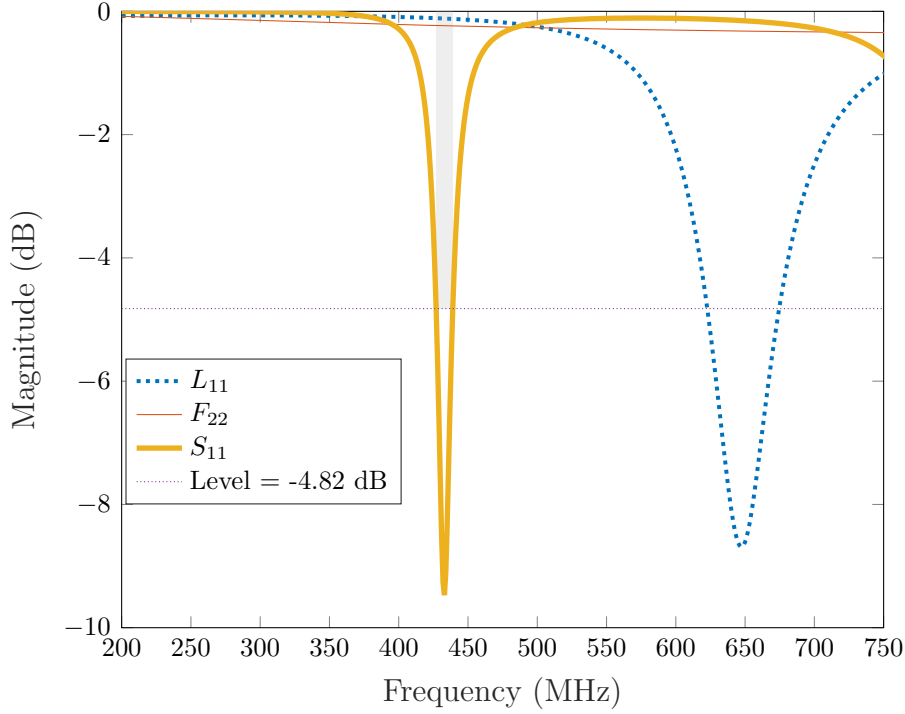


Figure 5.22: Result of matching problem in real setting (problem \mathcal{P}) with $N = 2$

The result of solving matching problem in the real setting (problem \mathcal{P}) is presented in figure 5.22. We did not impose the transformerless condition and the target degree of the matching circuit was fixed to be two ($N = 2$) with transmission zeros one at zero and one at infinity ($r(s) = s$). The optimal matching criterion obtained, -4.82 dB provides a significant improvement compared to L_{11} . The 2 critical points ($\lceil \frac{N}{2} \rceil + 1$) at which S_{11} attains -4.82 dB in the passband characterize the optimality of the obtained response. In order to calculate the element values of the optimal matching circuit, the de-chaining of the transmission zeros were performed in the order: one at zero and then the one at infinity. The matching circuit thus obtained, providing the result in figure 5.22 is depicted in figure 5.23. It should be noted that the circuit is not terminating in 50 ohm since we have not imposed any transformerless condition in problem \mathcal{P} .

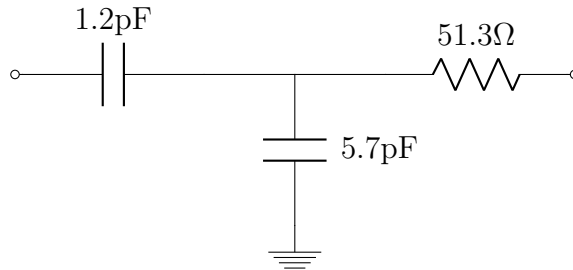


Figure 5.23: Matching circuit providing the result in fig 5.22

Now, the result of solving the matching problem in the real setting using the elemen-

tary chaining approach where parametrisation of F_{22} is done using the angular derivatives is depicted in figure 5.24. The problem under consideration is problem \mathcal{H} where fixed microstrip lines are included between the elementary sections as well. The target degree of the matching circuit was fixed to be two but with microstrip lines of length 1.5 mm and effective dielectric constant 3.23 included between the elements. The elementary sections in the order of chaining were fixed to be series capacitor (transmission zero at zero) and then parallel capacitor (transmission zero at infinity) to be in accordance with the circuit topology in figure 5.23. It can be noticed in figure 5.5 that we are able to attain almost the same criterion, -4.82 dB compared to the solution of problem \mathcal{P} in figure 5.23.

A validation of the matching response obtained was performed using Ansys Designer for circuit simulation and is represented by S_{11} HFSS CS in figure 5.24. It was obtained by connecting a port containing antenna L_{11} data at the left of the circuit in figure 5.26 and simulating the response at a 50 Ω port connected after the last element. The response obtained can be observed to be exactly agreeing with S_{11} .

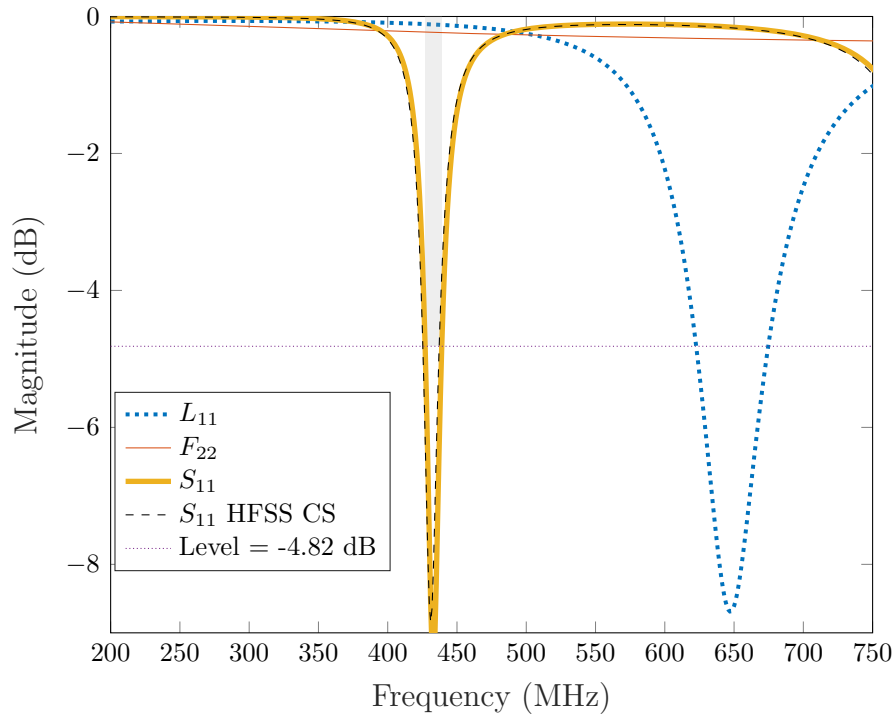


Figure 5.24: Result of matching problem in real setting (problem \mathcal{H}) with $N = 2$

The Smith chart representation of S_{11} in figure 5.24 is depicted in figure 5.25.

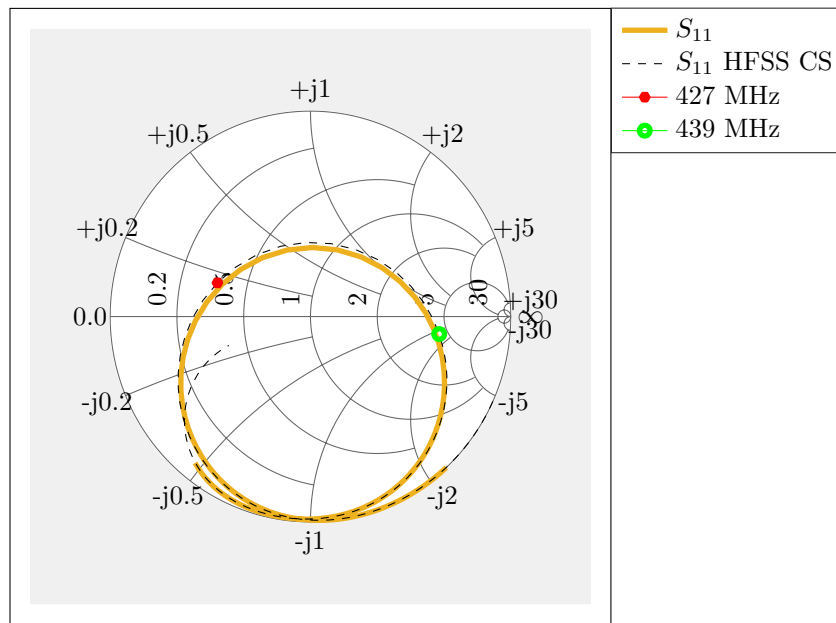


Figure 5.25: Smith chart representation of S_{11} in figure 5.24

The matching circuit providing the above result is depicted below. It should be noted that now the circuit is terminating in 50 ohm and microstrip line is included between the elements as well.

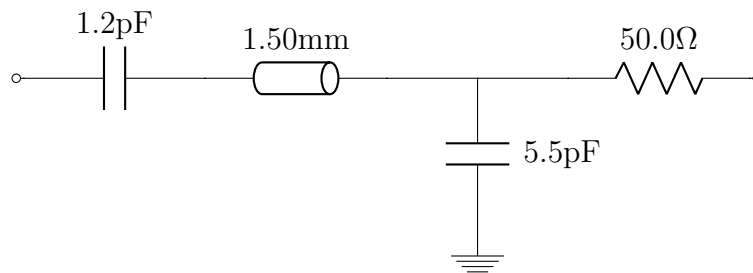


Figure 5.26: Matching circuit providing the result in fig 5.24

Before realizing the matching circuit on the pcb, a monte carlo analysis was performed as well. The result of this analysis using a maximum of 3 percent variation on the L, C element values of the above circuit and transmission line length with 20 samples is presented below.

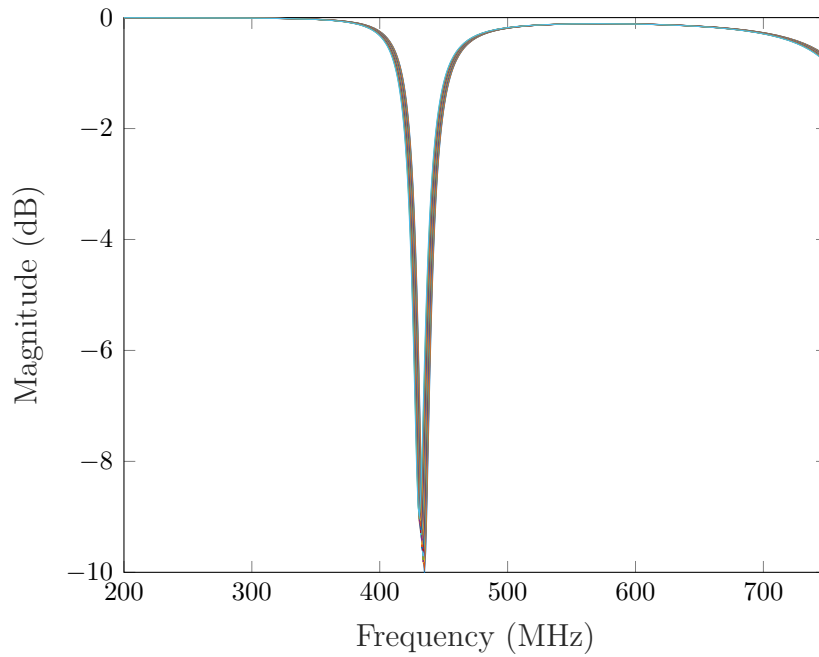


Figure 5.27: Monte Carlo analysis of global response in fig 5.24

The nearest element values available in the Johanson capacitor kit 0603, 0.3pF-82pF was chosen to solder the matching circuit components into the pcb antenna (1.2pF, 5.6pF). The pcb antenna with 2 component matching circuit soldered is shown in figure 5.28. The dimensions are length = 100 mm, width = 60 mm and FR-4 epoxy substrate thickness of 0.8 mm.



Figure 5.28: 433 MHz PCB Antenna with 2 component matching circuit

5.3. PROTOTYPE 2 : 433 MHZ ANTENNA

The result of S_{11} measurement of the antenna measured using a VNA analyzer after adding the 2 component matching circuit in figure 5.26 is presented in figure 5.29. The corresponding Smith Chart representation of figure 5.29 is provided in figure 5.30.

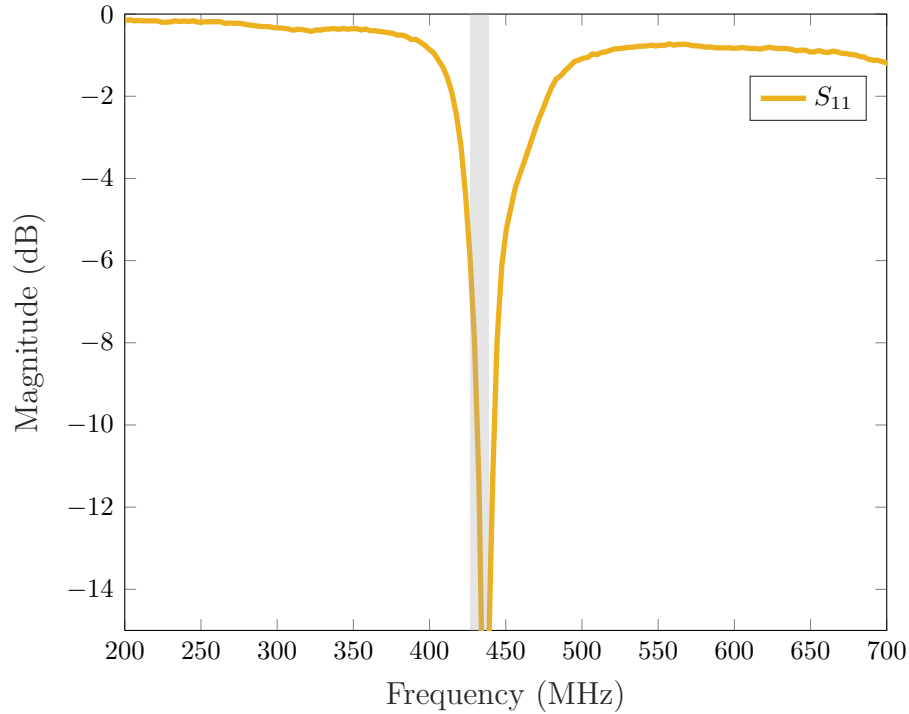


Figure 5.29: S_{11} measurement of the designed antenna with 2 component matching circuit

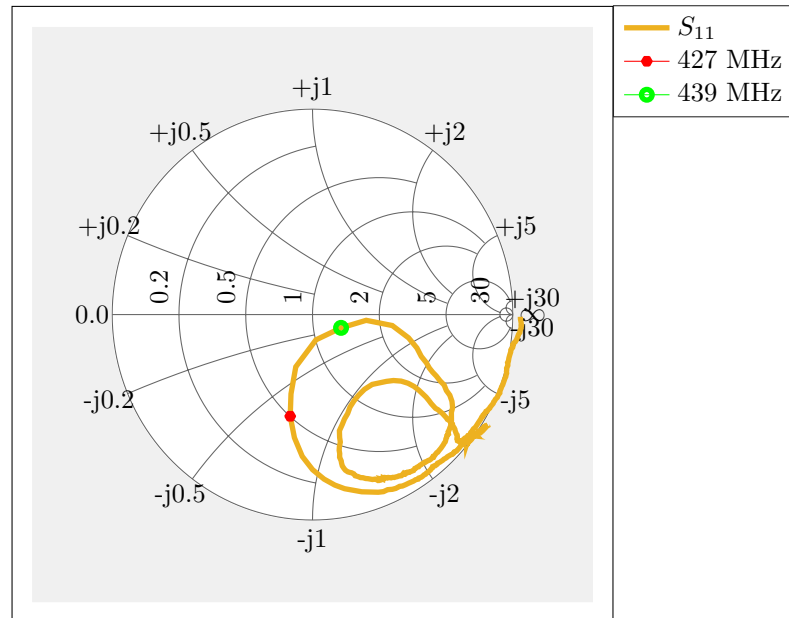


Figure 5.30: S parameter chart of the designed antenna with 2 component matching circuit

As mentioned in prototype 1 realization, the losses in the matching circuit components

has lead to the matching criterion in realization being better than the simulation. However, it can be noticed that the matching criterion obtained is close to the simulated result in figure 5.24 and compared to the initial S_{11} measurement of the antenna depicted in figure 5.20, we have been able to remarkably improve the matching criterion in the targeted passband. This concludes the study of matching of 433 MHz antenna.

5.4 Prototype 3 : 433 MHz and 868 MHz Dual Band Antenna

In this section, we present dual band matching example for the monopole antenna prototype described in section 5.3. The S_{11} measurement of this antenna obtained using a network

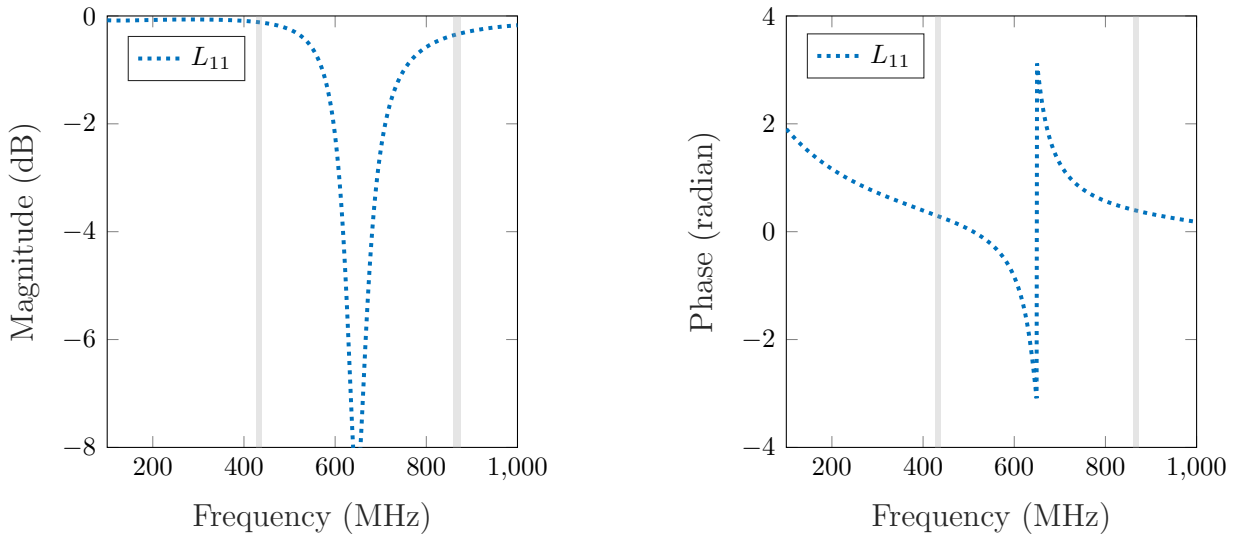


Figure 5.31: Scattering parameter of the load

analyzer is depicted in figure 5.31. The targeted pass band 427 MHz to 439 MHz and 860 MHz to 876 MHz is represented using the rectangle in the figure. The S_{11} measurement at the end points of the passband are -0.110 dB, -0.116 dB and -0.356 dB, -0.321 dB.

In figure 5.32 the result of solving matching problem formulated using the global system approach (problem \mathfrak{P}_C) is presented. The rational approximation of the given load data was done using a degree two system in the frequency band of interest. The target degree of the global system was fixed to be five ($N = 5$) and all the transmission zeros of the matching circuit were fixed at infinity ($R_F = 1$). It can be noted that we are able to significantly reduce the maximal mismatch value of the load in the passband from -0.11 dB to -8.8 dB.

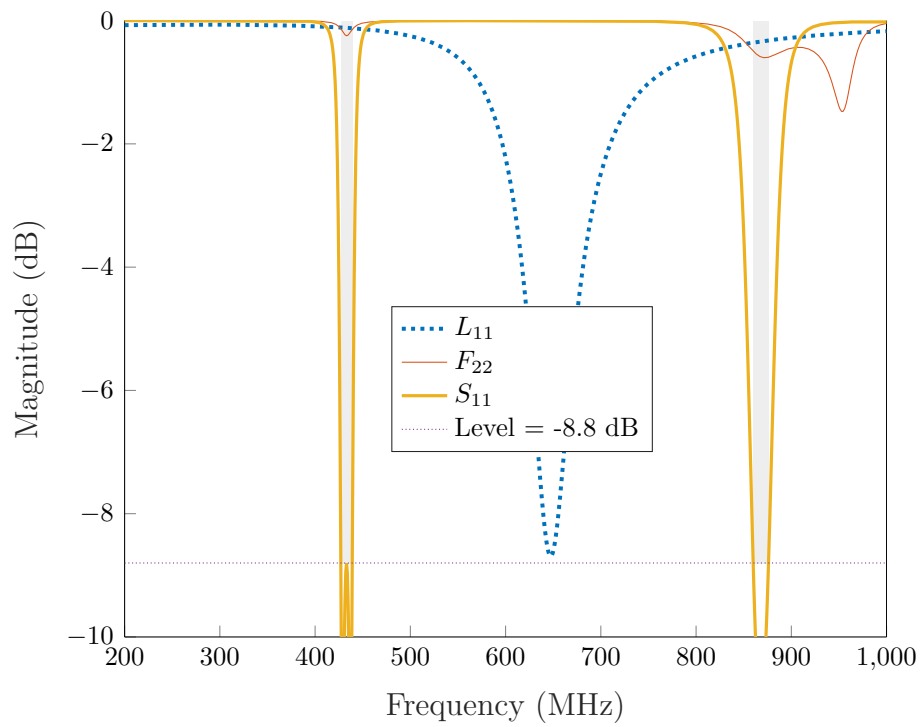


Figure 5.32: Result of global system approach (problem \mathfrak{P}_C) with $N = 5$

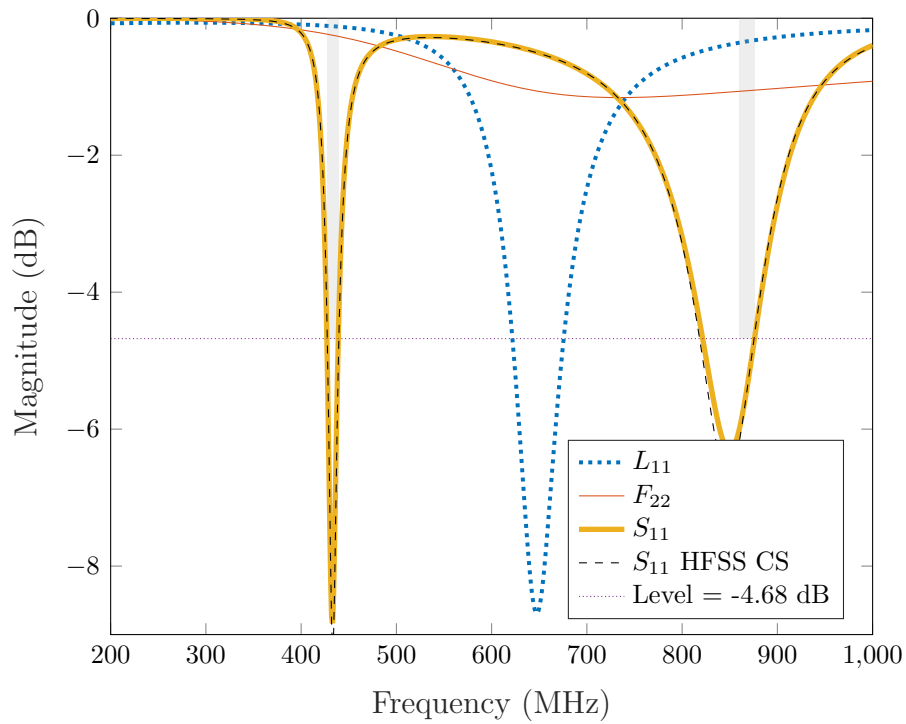
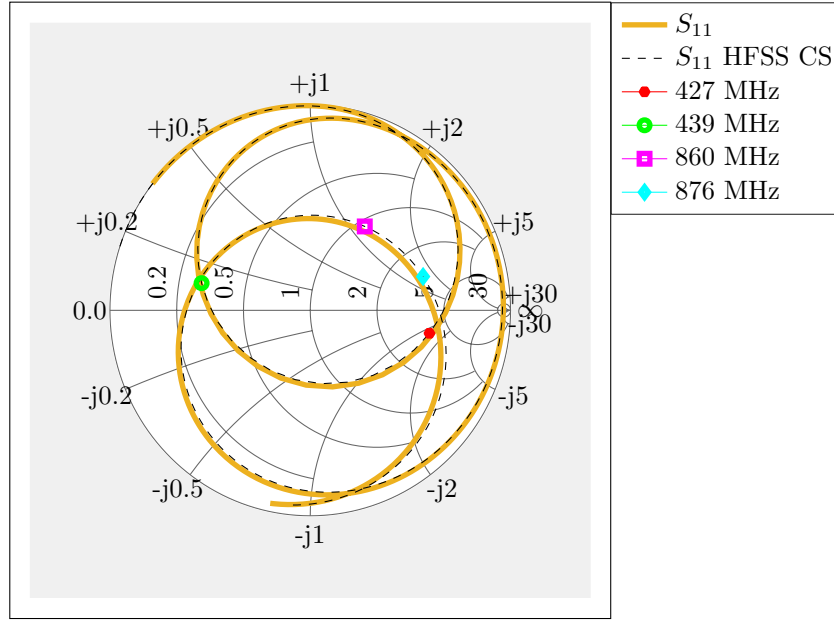


Figure 5.33: Result of matching problem in real setting (problem \mathcal{H}) with $N = 4$


 Figure 5.34: Smith chart representation of S_{11} in figure 5.33

The result of solving matching problem in the real setting using the elementary chaining approach where parametrisation of F_{22} is done using the angular derivatives is depicted in figure 5.33. The Smith chart representation of S_{11} in figure 5.33 is depicted in figure 5.34. The problem under consideration is problem \mathcal{H} where fixed microstrip lines are included between the elementary sections as well. The target degree of the matching circuit was fixed to be four ($N = 4$) with microstrip lines of length 1.5 mm and effective dielectric constant 3.23 included between the elements. The elementary sections in the order of chaining were fixed to be series capacitor, parallel capacitor, series inductor and parallel inductor. The capacitor and inductor values in the matching circuit were bounded to be in the ranges 0.3pF-82pF and 1.5nH-220nH by fixing bounds on the angular derivatives. The matching circuit providing the result in figure 5.33 is depicted in figure 5.35. S_{11} HFSS CS in figure 5.33 represent the response obtained in Ansys Designer in HFSS for circuit simulation of circuit in figure 5.35 connected to load L_{11} .

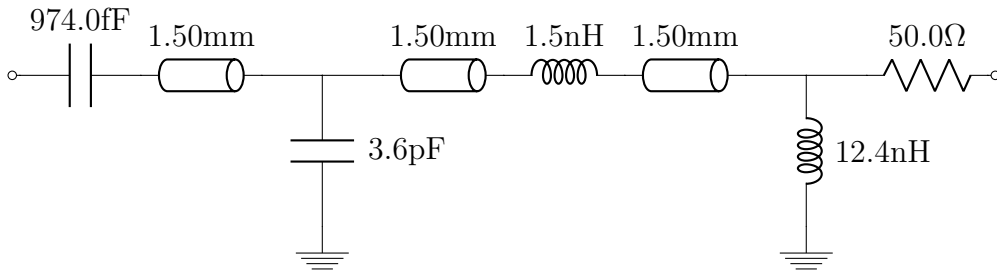


Figure 5.35: Matching circuit providing the result in fig 5.33

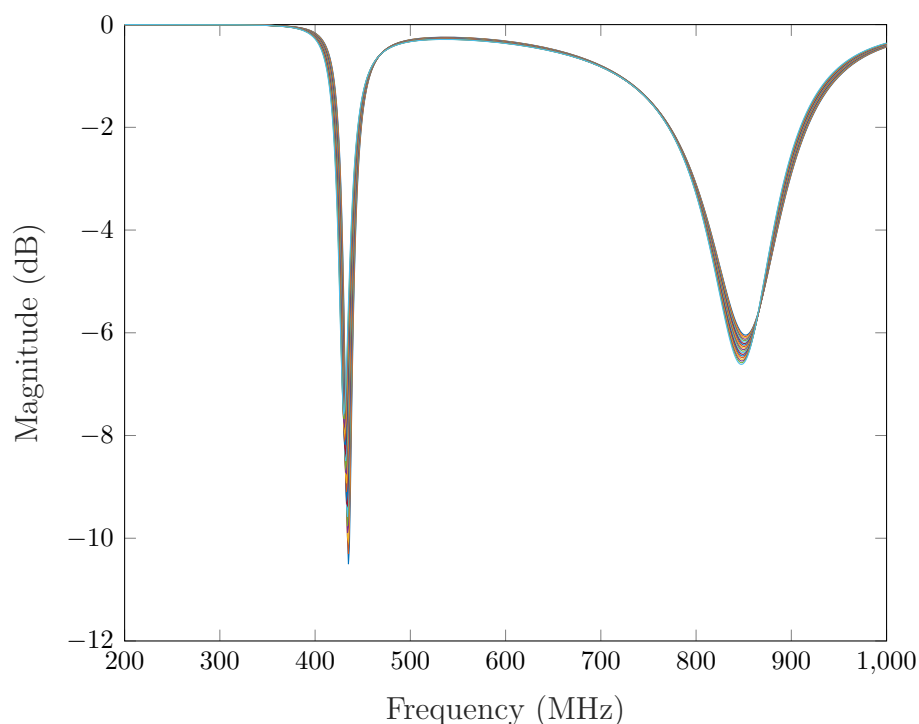


Figure 5.36: Monte Carlo analysis of global response in fig 5.33

The result of monte carlo analysis using a maximum of 3 percent variation on the L, C element values of the matching circuit and transmission line length with 20 samples is presented in figure 5.36. The nearest element values available in the Johanson capacitor kit 0603, 0.3pF-82pF was chosen to solder the matching circuit components into the pcb antenna. The pcb antenna with 4 component matching circuit soldered is shown in figure 5.37.

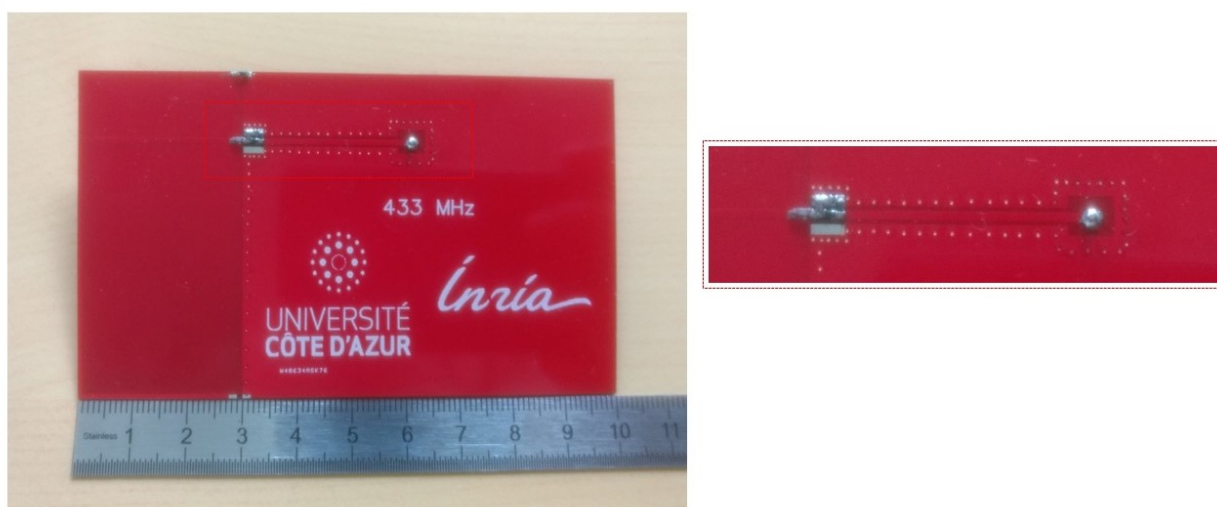
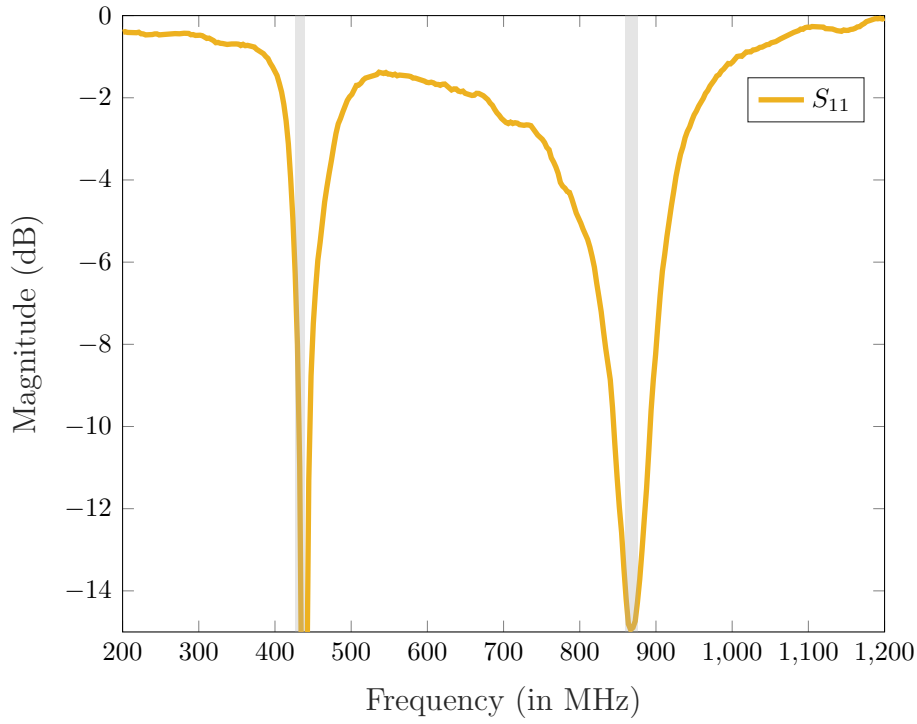
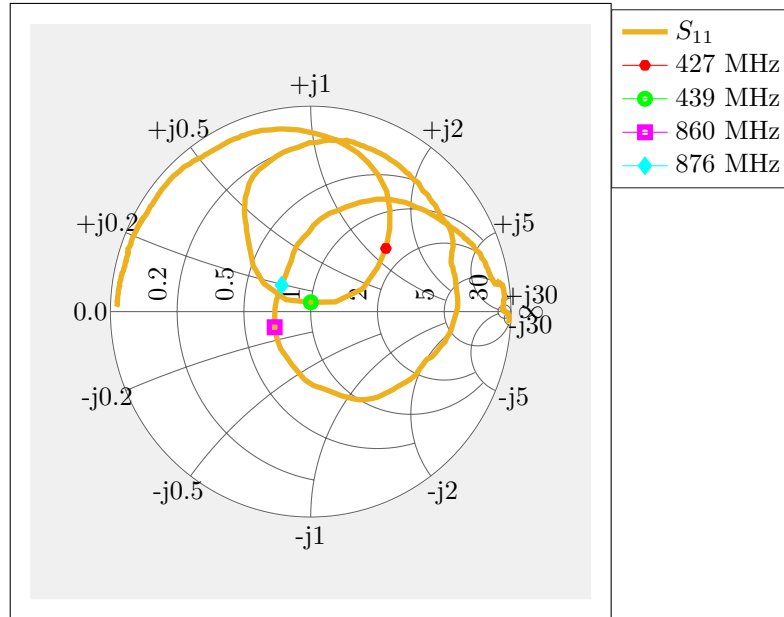


Figure 5.37: 433 MHz and 868 MHz Dual band Antenna with 4 component matching circuit


 Figure 5.38: S_{11} measurement of the designed antenna with matching circuit

 Figure 5.39: S parameter chart of the designed antenna with matching circuit

The result of S_{11} measurement of the antenna measured using a VNA analyzer after adding the matching circuit in figure 5.35 is presented in figure 5.38. The corresponding Smith Chart representation of figure 5.38 is provided in figure 5.39. As mentioned in other prototype realizations, the losses in the matching circuit components has lead to the

matching criterion in realization being better than the simulation. However, the matching criterion obtained is close to the simulated result in figure 5.33 and compared to the initial S_{11} measurement of the antenna depicted in figure 5.31, we have been able to remarkably improve the matching criterion in the targeted passband. This concludes the study of dual band matching example.

5.5 Conclusion

In this chapter, different optimisation schemes introduced in the thesis have been tested on concrete antenna examples. We have presented two PCB antennas designed to be matched in single bands around 169 MHz and 433 MHz and also a dual band antenna to be matched around 433 MHz and 868 MHz. A systematic study of the implementation of the matching problems described in part I and part II of the thesis have been done. The implementation of global system approach discussed in chapter 3 (problem \mathfrak{P}_C) provided theoretical matching bounds for each antenna in the specific passband when considering finite degree matching circuits. The implementation of the matching problems in the real setting (problems considered in chapter 4) provided optimal matching criterion after taking into account various practical constraints necessary for the realization of the matching circuit. The presentation of the results obtained after taking into consideration the practical necessities along with the results of global system approach provided an idea of how far the performance of realized matching circuits are from the hard bounds of matching level possible for the given degree.

The next chapter, which is the final chapter of the thesis provides a brief conclusion to the thesis along with some future perspectives of the work presented.

Conclusion and Future Perspectives

In this chapter, we will give a brief conclusion to the thesis along with a discussion about some of the possible directions of future work related to the topic. The thesis has made a detailed study of one of the classical problems in RF circuit design, the problem of impedance matching. Along with the theoretical aspects of broadband matching problem, practical applicability of the developed approaches to solve the problem were given due importance in the thesis. Part I of the thesis included an introductory chapter to the thesis along with two other chapters which studied the broadband matching problem using two different yet closely related approaches.

- In chapter 1 of the thesis a brief introduction to the broadband matching problem was made along with the necessary prerequisites of functional analysis and the concepts of RF engineering necessary for the problem.
- In chapter 2, a formulation of the finite degree matching problem was provided and an approach to solve the problem using Nehari theory was described. The gain functions were fixed to be from a specific family and its modulus parametrised using a one dimensional real parameter. The degree bounds of the matching circuit obtained using the approach were discussed and some illustrations of the numerical implementation of the developed approach on concrete antenna examples were provided as well.
- In chapter 3, a generalized formulation of the finite degree matching problem was provided based on the de-embedding theory of Fano-Youla. The possible set of responses of the global system were generalized to a set of “realizable” rational Schur functions of fixed maximal degree and having fixed transmission polynomial and the realizability constraint was imposed using the positivity of Pick matrix. The chapter formulated the matching problem as a convex optimisation problem over a subset of positive polynomials of fixed maximal degree and discussed various theoretical results concerning the problem. The numerical implementation of the problem as a non-linear semi-definite programming problem and some illustrations of the implementation of the developed approach on antenna examples were presented as well. A section was devoted at the end of this chapter to study the connection between two approaches discussed in part I of the thesis to solve the matching problem. A possible future direction of this work is to discuss the case of loads having transmission zeros on the imaginary axis and possibly generalize the theory presented in the chapter with the

help of boundary Nevanlinna Pick interpolation. An important step in this direction, generalizing the de-chaining conditions and de-embedding theorem when there are transmission zeros of the load present on the imaginary axis is presented in theorem B.2.1 in Appendix.

Part II of the thesis included two chapters which discussed the realization of the matching circuits as ladder networks and the current chapter concluding the thesis.

- Chapter 4 of the thesis discussed various practical constraints concerning the realization of the matching circuits. The transmission zeros of the matching circuit to be synthesized were fixed at either zero or infinity (or both) and a series of different problems were solved in order to better describe the realizable circuits. The realization of the matching circuits as ladder networks consisting of inductors and capacitors was presented. A matching circuit synthesis scheme allowing transformerless synthesis, including microstrip lines in between the lumped elements and bounding the element values of lumped inductors and capacitors to any desirable range as required by the user were also presented. A possible future direction of the work presented in the chapter is to study the implementation of matching circuit having transmission zero at finite non-zero frequency. This kind of transmission zero can be realized using series LC resonator in parallel or parallel LC resonator in series. Especially in the case of matching antennas in the frequency range where it is highly reflective, optimal placement of some transmission zeros which are finite and non-zero outside the targeted passband could provide better matching criterion compared to the case of transmission zeros located at zero or infinity. Some numerical implementations in this direction were carried out by optimizing the location of this kind of a transmission zero and the remaining transmission zeros of the matching circuit fixed at zero or infinity. The problem was formulated as follows:

Problem. \mathcal{P}_{FTz} . Given a passband \mathbb{I} , non-constant reflection coefficient $L_{11} \in \mathbb{B}$ of the load which is strictly contractive in \mathbb{I} and satisfying equation (4.4) and a polynomial $r_0 = s^k$ where $0 \leq k \leq N-2$, where $N \geq 2$ is the target degree of the matching circuit

$$\text{Find: } \hat{l} = \min_{(p, \alpha) \in \mathbb{P}_{\mathbb{R}, N} \times (j\mathbb{R} \setminus \mathbb{I})} \max_{s \in \mathbb{I}} \left| \frac{\frac{p(s)}{q(s)} - \overline{L_{11}(s)}}{1 - \frac{p(s)}{q(s)} L_{11}(s)} \right|$$

$$\text{where: } r = (s - \alpha)(s + \alpha)r_0,$$

$$qq^* = pp^* + rr^* \quad \text{and } q \in \mathbb{SB}_{\mathbb{R}, N}.$$

The numerical implementation of this problem can be carried out similar to problem \mathcal{P} in chapter 4. The derivative calculation of the constraint with respect to the non-fixed part of the polynomial r can be performed similarly to that of with respect to polynomial p .

(1). In the first example, we consider the 169 MHz antenna (passband : 167 MHz-171 MHz) presented in section 5.2 in chapter 5 and solve problem \mathcal{P}_{FTz} with $N = 3$ and $r_0 = s$. The optimisation places the transmission zero at finite non-zero frequency at $\alpha = 0.85j$ which corresponds to a transmission zero at 145 MHz and produces the result as depicted in figure 6.1.

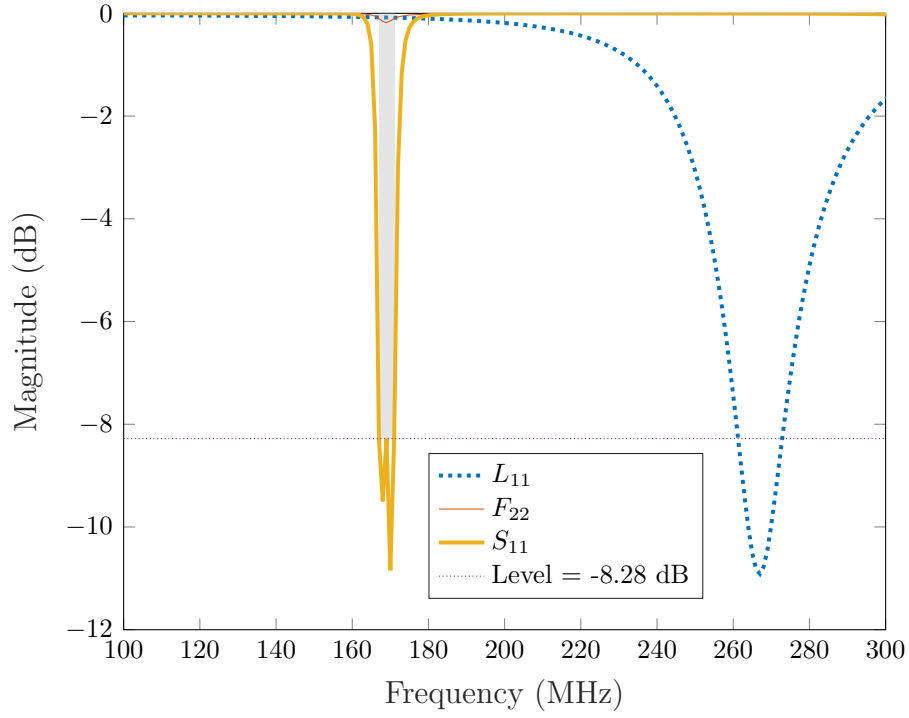


Figure 6.1: Result of solving \mathcal{P}_{FTz} with $N = 3$ and $r_0 = s$

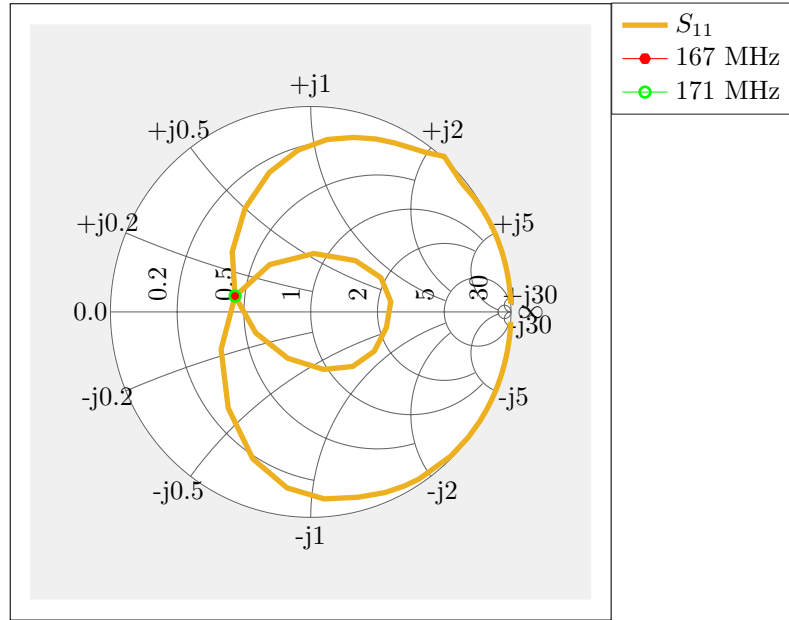
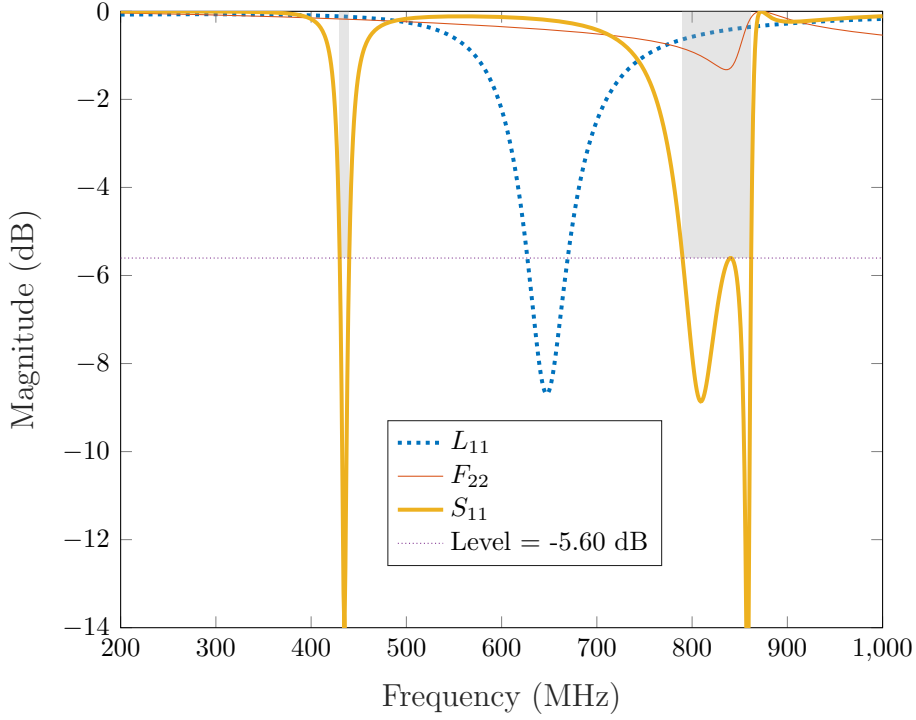
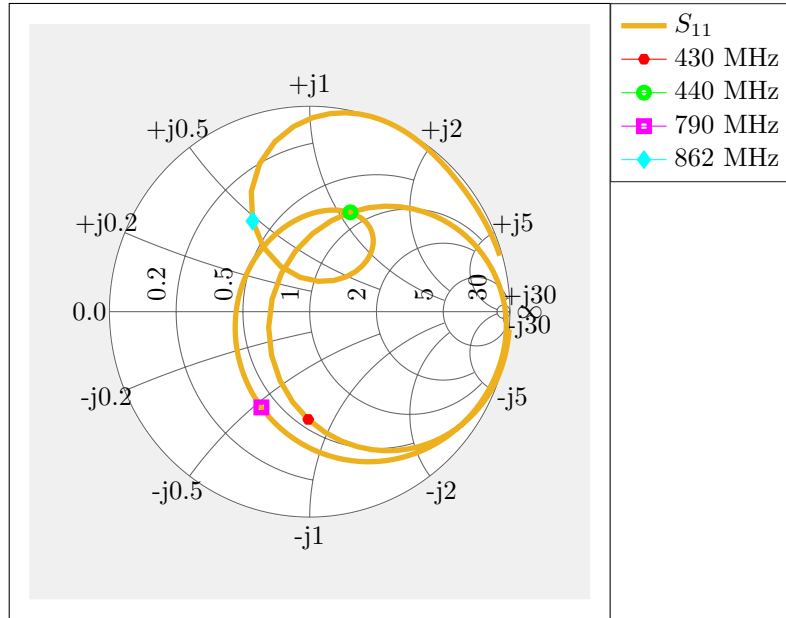


Figure 6.2: Smith chart representation of S_{11} in figure 6.1

The Smith chart representation of S_{11} in figure 6.1 is depicted in figure 6.2.

(2). In the second example, we consider the 433 MHz antenna presented in section 5.3 in chapter 5.


 Figure 6.3: Result of solving \mathcal{P}_{FTz} with $N = 3$ and $r_0 = s$

 Figure 6.4: Smith chart representation of S_{11} in figure 6.3

In this example, we solve problem \mathcal{P}_{FTz} for passband 430 MHz - 440 MHz and 790 MHz- 862 MHz, $N = 3$ and $r_0 = s$. The optimisation places the transmission zero at finite non-zero frequency at $\alpha = 1.012j$ which corresponds to a transmission zero

at 872 MHz and produces the result as depicted in figure 6.3. The Smith chart representation of S_{11} in figure 6.3 is depicted in figure 6.4. So, the results presented in the case of matching circuits including some finite transmission zeros on the imaginary axis are promising and can be specifically useful in the case of matching antennas in the frequency bands where it is highly reflective. A further study in this direction by testing more examples and realizing the matching circuits can be practically very useful.

- Chapter 5 of the thesis provided illustrations of the results obtained by solving different matching problems formulated in the thesis and made a comparison between them. The illustrations of some prototypes of PCB antennas designed together with the matching circuits obtained from the developed approach were also made.
- The current chapter has provided a brief conclusion to the thesis along with some possible future perspectives related to the work presented.

APPENDIX A

Schur Functions and Nevanlinna-Pick Theorem

A.1 Polynomial Norm and Convergence of Rational Schur Functions

Definition A.1.1. Given a set of $(N+1)$ data points, $\{(a_0, b_0), (a_1, b_1), \dots, (a_N, b_N)\}$, $P(s)$ defined as follows is the unique polynomial of degree N which interpolates the given data,

$$P(s) = \sum_{i=0}^N b_i L_i(s) \quad (\text{A.1})$$

where,

$$L_i(s) := \prod_{j=0, j \neq i}^N \frac{s - a_j}{a_i - a_j}. \quad (\text{A.2})$$

The set of Lagrange polynomials,

$$\mathbb{L} = \{L_0(s), L_1(s), \dots, L_N(s)\} \quad (\text{A.3})$$

form a basis for the set of all polynomials of degree at most equal to N . So for any polynomial $P(s)$ of degree less than or equal to N , we have,

$$P(s) = \sum_{i=0}^N b_i L_i(s), b_i \in \mathbb{R} \quad (\text{A.4})$$

and the norm can be defined as,

$$\|P\| = \left(\sum_{i=0}^N |b_i|^2 \right)^{\frac{1}{2}} = \left(\sum_{i=0}^N |P(a_i)|^2 \right)^{\frac{1}{2}}. \quad (\text{A.5})$$

We consider the interpolation points a_i on the imaginary axis ($a_i = j\omega_i$, $\omega_i \in \mathbb{R}$, $0 \leq i \leq N$), in particular in the interval \mathbb{I} (passband of interest).

Proposition A.1.2. *For every sequence of rational Schur functions having a uniform degree bound, there exists a subsequence which converges to a rational Schur function on every compact set inside the right half plane and the convergence holds pointwise on the imaginary axis as well except a set of finite points.*

Proof. Let $(f_n) = (\frac{P_n}{Q_n})$ be a sequence of rational Schur functions with uniform degree bound. Now, using Definition A.1.1 for the norm, let us normalize the sequence (f_n) as follows,

$$(f_n) = \left(\frac{\frac{P_n}{\|Q_n\|}}{\frac{Q_n}{\|Q_n\|}} \right) \quad (\text{A.6})$$

Since $\left(\frac{Q_n}{\|Q_n\|} \right)$ is a sequence of polynomials with unit norm, we have a subsequence $\left(\frac{Q_{n_d}}{\|Q_{n_d}\|} \right)$ which converges to a polynomial \tilde{Q} on every compact set inside the right half plane and the convergence holds pointwise on the imaginary axis as well. The zeros of $\left(\frac{Q_n}{\|Q_n\|} \right)$ being in the open left half plane implies \tilde{Q} can have finite number of zeros on the imaginary axis and the rest in the open left half plane. Let,

$$\mathbb{E}_1 = \{s \in j\mathbb{R} : \tilde{Q}(s) = 0\} \quad (\text{A.7})$$

Since (f_n) forms a sequence of Schur functions, we have,

$$\forall s \in j\mathbb{R}, \left| \frac{P_n}{Q_n}(s) \right| \leq 1 \quad (\text{A.8})$$

$$\Rightarrow \forall s \in j\mathbb{R}, |P_n(s)| \leq |Q_n(s)| \quad (\text{A.9})$$

$$\Rightarrow \forall s \in j\mathbb{R}, \frac{|P_n(s)|}{\|Q_n\|} \leq \frac{|Q_n(s)|}{\|Q_n\|} \quad (\text{A.10})$$

Now, we can prove that $\frac{P_n}{\|Q_n\|}$ is bounded in the defined norm. Using (A.10), we have,

$$\forall s \in j\mathbb{R}, \frac{|P_n(s)|}{\|Q_n\|} \leq \frac{|Q_n(s)|}{\|Q_n\|} = \frac{|Q_n(s)|}{\left(\sum_{j=0}^N |Q_n(a_j)|^2 \right)^{\frac{1}{2}}} \quad (\text{A.11})$$

So, we have,

$$\forall s \in j\mathbb{R}, \frac{|P_n(s)|}{\|Q_n\|} \leq \frac{|Q_n(s)|}{\left(\sum_{i=0}^N |Q_n(j\omega_i)|^2 \right)^{\frac{1}{2}}} \quad (\text{A.12})$$

$$\Rightarrow \forall s = j\omega_i, \frac{|P_n(s)|}{\|Q_n\|} \leq 1 \quad (\text{A.13})$$

This yields the boundedness of sequence $\frac{P_n}{\|Q_n\|}$ in the defined norm, since we have,

$$\left\| \frac{P_n}{\|Q_n\|} \right\| = \left(\sum_{i=0}^N \left| \frac{P_n(j\omega_i)}{\|Q_n\|} \right|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{i=0}^N 1 \right)^{\frac{1}{2}} = \sqrt{N+1}$$

The inequality follows from (A.13). So, we have a subsequence $(\frac{P_{n_d}}{\|Q_{n_d}\|})$ of $(\frac{P_n}{\|Q_n\|})$ which converges to a polynomial \tilde{P} on every compact set inside the right half plane and converging pointwise on the imaginary axis as well. Let,

$$\mathbb{E}_2 = \{s \in j\mathbb{R} : \tilde{P}(s) = 0\} \quad (\text{A.14})$$

From (A.10), it follows that $\mathbb{E}_2 \subseteq \mathbb{E}_1$ and so after cancelling the common zeros of \tilde{P} and \tilde{Q} on the imaginary axis (if it exists), and renaming them as P and Q respectively, (A.8) yields,

$$\forall s \in j\mathbb{R}, \quad \left| \frac{P}{Q}(s) \right| \leq 1 \quad (\text{A.15})$$

So, we have the function $\frac{P}{Q}$ to be Schur. Now for any compact set K inside the right half plane, we have,

$$\lim_{n \rightarrow \infty} \sup_{s \in K} \left| \frac{P_{n_d}}{Q_{n_d}}(s) - \frac{P}{Q}(s) \right| = 0 \quad (\text{A.16})$$

This proves the existence of a subsequence $(\frac{P_{n_d}}{Q_{n_d}})$ of (f_n) which converges to Schur function $\frac{P}{Q}$ on every compact set inside the right half plane. We also have,

$$\lim_{n \rightarrow \infty} \left| \frac{P_{n_d}}{Q_{n_d}}(s) - \frac{P}{Q}(s) \right| = 0, \quad s \in j\mathbb{R} \setminus \mathbb{E}_1 \quad (\text{A.17})$$

So, we have the convergence holding pointwise on the imaginary axis as well, except the set \mathbb{E}_1 . This completes the proof. \square

A.2 Nevanlinna-Pick Theorem

The importance of Nevanlinna-Pick theorem (stated in 3.2.5) in characterising the constraint set \mathbb{H}_R^N of problem \mathfrak{P}_C should have been clear from subsection 3.3.1 in chapter 3. We devote this section to give a proof to the necessity part of Nevanlinna-Pick theorem for matrix valued functions. We switch to the more general version of Nevanlinna-Pick theorem which deals with matrix valued functions since it was useful in proving the concavity of Pick matrix in chapter 3. Before getting into the details, we recall some already used notations and also define some new notations.

- For any given $k \in \mathbb{N}$, I_k represent the identity matrix of order k .
- Let $\mathbb{B}_{p \times q}(\Pi^+)$ denote the set of $p \times q$ matrix valued functions $F(\omega)$ which are analytic in Π^+ and contractive, $(\overline{F(j\omega)})^t F(j\omega) \preceq I_q$, $\omega \in \mathbb{R}$.
- For all $z \in \overline{\Pi^+}$, let us denote $\rho_\gamma(z) = \overline{\gamma} + z$. We have, $\Pi^+ = \{\gamma \in \mathbb{C} : \rho_\gamma(\gamma) > 0\}$ and $j\mathbb{R} = \{\gamma \in \mathbb{C} : \rho_\gamma(\gamma) = 0\}$. Also, $(\rho_\gamma(z)) = \gamma + \overline{z} = \rho_z(\gamma)$.
- $H^p(\Pi^+)$ denotes the space of scalar valued Hardy functions of class p over Π^+ , where, $1 \leq p \leq \infty$.

- $H_k^p(\Pi^+)$ denotes the space of $(k \times 1)$ matrix valued functions, each entry of which belongs to $H^p(\Pi^+)$.
- $L_k^p(j\mathbb{R})$ denotes the space of $(k \times 1)$ matrix valued functions, each entry of which belongs to $L^p(j\mathbb{R})$.
- The symbol \underline{p} is used to denote the orthogonal projection of $L_k^2(j\mathbb{R})$ onto $H_k^2(\Pi^+)$.
- For $f, g \in L_k^2(j\mathbb{R})$, $\langle f, g \rangle$ denote the inner product,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \overline{(g(j\omega))^t} f(j\omega) d\omega.$$

Before stating the Nevanlinna-Pick theorem for matricial case, we prove a lemma which will be useful in the proof of Nevanlinna-Pick theorem.

Lemma A.2.1. *If $F \in \mathbb{B}_{p \times q}(\Pi^+)$, then,*

$$\underline{p} \left(\frac{\overline{F}^t \xi}{\rho_\gamma} \right) = \frac{(\overline{F(\gamma)})^t \xi}{\rho_\gamma}, \quad (\text{A.18})$$

for every choice of $\gamma \in \Pi^+$ and $\xi \in \mathbb{C}^p$.

Proof. Let us denote $f = \underline{p} \left(\frac{\overline{F}^t \xi}{\rho_\gamma} \right)$. For any $\eta \in \mathbb{C}^q$ and $\alpha \in \Pi^+$, we have,

$$\left\langle f, \frac{\eta}{\rho_\alpha} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\overline{\eta}^t}{(\rho_\alpha(j\omega))} f(j\omega) d\omega.$$

Since $\rho_\alpha(j\omega) = \overline{\alpha} + j\omega$, we get,

$$\begin{aligned} \left\langle f, \frac{\eta}{\rho_\alpha} \right\rangle &= \frac{\overline{\eta}^t}{2\pi} \int_{-\infty}^{+\infty} \frac{f(j\omega)}{\alpha - j\omega} d\omega \\ &= \frac{\overline{\eta}^t}{2\pi j} \int_{-\infty}^{+\infty} \frac{f(u)}{\alpha - u} du \quad (\text{where } u = j\omega) \\ &= \overline{\eta}^t f(\alpha). \end{aligned}$$

The last equality follows from Cauchy's formula for $H_q^2(\Pi^+)$. So, we have,

$$\begin{aligned} \overline{\eta}^t f(\alpha) &= \left\langle f, \frac{\eta}{\rho_\alpha} \right\rangle \\ &= \left\langle \frac{\overline{F}^t \xi}{\rho_\gamma}, \frac{\eta}{\rho_\alpha} \right\rangle \left(\because \frac{\eta}{\rho_\alpha} \in H_q^2(\Pi^+) \text{ for } \alpha \in \Pi^+ \right) \\ &= \left\langle \frac{\xi}{\rho_\gamma}, \frac{F\eta}{\rho_\alpha} \right\rangle = \left(\left\langle \frac{F\eta}{\rho_\alpha}, \frac{\xi}{\rho_\gamma} \right\rangle \right)^t. \end{aligned} \quad (\text{A.19})$$

From the definition of inner product, we have,

$$\left\langle \frac{F\eta}{\rho_\alpha}, \frac{\xi}{\rho_\gamma} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\bar{\xi}^t}{(\rho_\gamma(j\omega))} \frac{F(j\omega)\eta}{\rho_\alpha(j\omega)} d\omega.$$

Since $\rho_\gamma(j\omega) = \bar{\gamma} + j\omega$, we get,

$$\begin{aligned} \left\langle \frac{F\eta}{\rho_\alpha}, \frac{\xi}{\rho_\gamma} \right\rangle &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\bar{\xi}^t}{(\gamma - j\omega)} \frac{F(j\omega)\eta}{\rho_\alpha(j\omega)} d\omega \\ &= \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{\bar{\xi}^t}{(\gamma - u)} \frac{F(u)\eta}{\rho_\alpha(u)} du \quad (\text{where } u = j\omega) \\ &= \frac{\bar{\xi}^t F(\gamma)\eta}{\rho_\alpha(\gamma)}. \end{aligned} \tag{A.20}$$

The last equality follows from Cauchy's formula for $H_p^2(\Pi^+)$. Substituting (A.20) in (A.19), we obtain,

$$\bar{\eta}^t f(\alpha) = \frac{\bar{\eta}^t (\overline{F(\gamma)})^t \xi}{(\rho_\alpha(\gamma))} = \frac{\bar{\eta}^t (\overline{F(\gamma)})^t \xi}{\rho_\gamma(\alpha)}.$$

So, we have,

$$f(\alpha) = \frac{(\overline{F(\gamma)})^t \xi}{\rho_\gamma(\alpha)} \quad \forall \alpha \in \Pi^+.$$

Hence,

$$p\left(\frac{\overline{F}^t \xi}{\rho_\gamma}\right) = \frac{(\overline{F(\gamma)})^t \xi}{\rho_\gamma}.$$

□

Now, we are in a position to state and prove the Nevanlinna-Pick theorem for matrix valued functions. We will be proving only the necessity part of the theorem, that is the positivity of Pick matrix under the given conditions. For the proof of sufficiency and also to have a detailed review of matricial Nevanlinna-Pick interpolation and generalizations, we refer the reader to chapter 18, [8]. We are more interested in the necessity part of the theorem since it plays the crucial role in the characterisation of the constraint set in problem \mathfrak{P}_C .

Theorem A.2.2. (*Nevanlinna-Pick Theorem for Left-Interpolation Problem*). *Given n points, $\gamma_1, \gamma_2, \dots, \gamma_n$, each in Π^+ , direction vectors x_1, \dots, x_n , each in $\mathbb{C}^{1 \times p}$ and y_1, \dots, y_n , each in $\mathbb{C}^{1 \times q}$, there exists $F \in \mathbb{B}_{p \times q}(\Pi^+)$, satisfying the interpolation problem,*

$$x_i F(\gamma_i) = y_i, \quad i = 1, 2, \dots, n, \tag{A.21}$$

iff the Pick matrix $\Delta(F) = \{\Delta_{ij}(F)\}_{i,j=1}^n$, defined by,

$$\Delta_{ij}(F) = \frac{x_i \bar{x}_j^t - y_i \bar{y}_j^t}{\rho_{\gamma_j}(\gamma_i)} \tag{A.22}$$

is positive semi-definite.

Proof. Let us assume there exists $F \in \mathbb{B}_{p \times q}(\Pi^+)$, satisfying the interpolation problem (A.21). Denote $\xi_i = \bar{x}_i^t$ for $i = 1, 2, \dots, n$. We have $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}^p$. It follows from Lemma A.2.1 that, if $F \in \mathbb{B}_{p \times q}(\Pi^+)$, then, for $i = 1, 2, \dots, n$, we have,

$$\Psi_F\left(\frac{\xi_i}{\rho_{\gamma_i}}\right) \stackrel{\text{def}}{=} \underline{p}\left(\frac{\bar{F}^t \xi_i}{\rho_{\gamma_i}}\right) = (\overline{F(\gamma_i)})^t \frac{\xi_i}{\rho_{\gamma_i}} = \frac{\bar{y}_i^t}{\rho_{\gamma_i}}. \quad (\text{A.23})$$

Let us define $g \in H_p^2(\Pi^+)$ as,

$$g = \sum_{k=1}^n \frac{c_k \xi_k}{\rho_{\gamma_k}}, \quad c_k \in \mathbb{C}.$$

Using Lemma A.2.1, we have,

$$\Psi_F(g) = \underline{p}(\bar{F}^t g) = \sum_{k=1}^n \frac{(\overline{F(\gamma_k)})^t c_k \xi_k}{\rho_{\gamma_k}}. \quad (\text{A.24})$$

We have,

$$\begin{aligned} \|g\|^2 - \|\Psi_F(g)\|^2 &= \|g\|^2 - \|\underline{p}(\bar{F}^t g)\|^2 \\ &\geq 0. \end{aligned} \quad (\text{A.25})$$

The last inequality follows because, $\|\underline{p}(\bar{F}^t g)\|^2 \leq \|\bar{F}^t g\|^2$ since orthogonal projection is a contraction and $\|\bar{F}^t g\|^2 \leq \|g\|^2$ follows since $F \in \mathbb{B}_{p \times q}(\Pi^+)$. Now, for any $i, j \in \{1, 2, \dots, n\}$, let us evaluate the quadratic form associated to Ψ_F ,

$$\begin{aligned} Q\left(\frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j}{\rho_{\gamma_j}}\right) &= \left\langle \frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j}{\rho_{\gamma_j}} \right\rangle - \left\langle \Psi_F\left(\frac{\xi_i}{\rho_{\gamma_i}}\right), \Psi_F\left(\frac{\xi_j}{\rho_{\gamma_j}}\right) \right\rangle \\ &= \left\langle \frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j}{\rho_{\gamma_j}} \right\rangle - \left\langle \underline{p}\left(\frac{\bar{F}^t \xi_i}{\rho_{\gamma_i}}\right), \underline{p}\left(\frac{\bar{F}^t \xi_j}{\rho_{\gamma_j}}\right) \right\rangle \\ &= \left\langle \frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j}{\rho_{\gamma_j}} \right\rangle - \left\langle \frac{\bar{F}^t \xi_i}{\rho_{\gamma_i}}, \underline{p}\left(\frac{\bar{F}^t \xi_j}{\rho_{\gamma_j}}\right) \right\rangle \\ &= \left\langle \frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j}{\rho_{\gamma_j}} \right\rangle - \left\langle \frac{\xi_i}{\rho_{\gamma_i}}, F \frac{\overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}} \right\rangle \quad (\text{From Lemma A.2.1}) \\ &= \left\langle \frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j - F \overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}} \right\rangle \\ &= \left(\left\langle \frac{\xi_j - F \overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}}, \frac{\xi_i}{\rho_{\gamma_i}} \right\rangle \right)^t. \end{aligned} \quad (\text{A.26})$$

From the definition of inner product, we have,

$$\left\langle \frac{\xi_j - F \overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}}, \frac{\xi_i}{\rho_{\gamma_i}} \right\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\bar{\xi}_i^t}{(\rho_{\gamma_i}(j\omega))} \left(\frac{\xi_j - F(j\omega) \overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}(j\omega)} \right) d\omega.$$

Since $\rho_{\gamma_i}(j\omega) = \overline{\gamma_i} + j\omega$, we get,

$$\begin{aligned} \left\langle \frac{\xi_j - F\overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}}, \frac{\xi_i}{\rho_{\gamma_i}} \right\rangle &= \frac{\overline{\xi_i}^t}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{(\gamma_i - j\omega)} \left(\frac{\xi_j - F(j\omega)\overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}(j\omega)} \right) d\omega \\ &= \frac{\overline{\xi_i}^t}{2\pi j} \int_{-\infty}^{+\infty} \frac{1}{(\gamma_i - u)} \left(\frac{\xi_j - F(u)\overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}(u)} \right) du \\ &= \overline{\xi_i}^t \left(\frac{\xi_j - F(\gamma_i)\overline{F(\gamma_j)}^t \xi_j}{\rho_{\gamma_j}(\gamma_i)} \right) \\ &= \overline{\xi_i}^t \left(\frac{I_p - F(\gamma_i)\overline{F(\gamma_j)}^t}{\rho_{\gamma_j}(\gamma_i)} \right) \xi_j. \end{aligned}$$

Substituting this in (A.26), we obtain,

$$Q\left(\frac{\xi_i}{\rho_{\gamma_i}}, \frac{\xi_j}{\rho_{\gamma_j}}\right) = \overline{\xi_j}^t \left(\frac{I_p - F(\gamma_j)\overline{F(\gamma_i)}^t}{\rho_{\gamma_i}(\gamma_j)} \right) \xi_i.$$

So, for $g = \sum_{k=1}^n \frac{c_k \xi_k}{\rho_{\gamma_k}}$, we have,

$$\begin{aligned} \|g\|^2 - \|\underline{\underline{p}}(\overline{F}^t g)\|^2 &= \sum_{i,j=1}^n \overline{c_j} \overline{\xi_j}^t \left(\frac{I_p - F(\gamma_j)\overline{F(\gamma_i)}^t}{\rho_{\gamma_i}(\gamma_j)} \right) \xi_i c_i \\ &= \sum_{i,j=1}^n \overline{c_j} x_j \left(\frac{I_p - F(\omega_j)\overline{F(\omega_i)}^t}{\rho_{\gamma_i}(\gamma_j)} \right) \overline{x_i}^t c_i \\ &= \sum_{i,j=1}^n \overline{c_j} \left(\frac{x_j \overline{x_i}^t - y_j \overline{y_i}^t}{\rho_{\omega_i}(\omega_j)} \right) c_i \\ &= \sum_{i,j=1}^n \overline{c_j} \Delta_{ji}(F) c_i. \end{aligned}$$

It follows from equation (A.25) that,

$$\sum_{i,j=1}^n \overline{c_j} \Delta_{ji}(F) c_i \geq 0, \quad (\text{A.27})$$

which is valid for every choice of complex constants c_1, c_2, \dots, c_n . This establishes the necessity, Pick matrix, $\Delta(F) \succeq 0$. \square

It should be noted that we can have a transposed version of Theorem A.2.2 which gives us the following result (Example 18.5.2, [8]).

Theorem A.2.3. (*Nevanlinna-Pick Theorem for Right-Interpolation Problem*). *Given n points, $\gamma_1, \gamma_2 \dots \gamma_n$, each in Π^+ , direction vectors u_1, \dots, u_n , each in \mathbb{C}^p and v_1, \dots, v_n , each in \mathbb{C}^q , there exists $F \in \mathbb{B}_{p \times q}(\Pi^+)$, satisfying the interpolation problem,*

$$F(\gamma_i)u_i = v_i, \quad i = 1, 2, \dots, n, \quad (\text{A.28})$$

if and only if the Pick matrix $\Delta(F) = \{\Delta_{ij}(F)\}_{i,j=1}^n$, defined by,

$$\Delta_{ij}(F) = \frac{\bar{u}_i^t u_j - \bar{v}_i^t v_j}{\rho_{\gamma_i}(\gamma_j)} \quad (\text{A.29})$$

is positive semi-definite.

APPENDIX B

General Chaining Conditions and De-embedding

B.1 General Chaining Conditions

In this section, we will describe the general chaining conditions that are satisfied when we allow the scattering matrix L of the load to have transmission zeros on the imaginary axis (in addition to the transmission zeros in Π^+). At the end of the section, we will also introduce the general de-embedding theorem, a generalisation of Theorem 3.2.4. This is a first step towards the possible generalisation of the matching theory presented in chapter 3 based on the Nevanlinna-Pick interpolation on the boundary. Initially we will prove a proposition stating the necessary conditions that will be satisfied by the reflection coefficient of a system obtained by chaining a lossless scattering matrix L together with a reflection coefficient $F_{22} \in \mathbb{B}$. The conditions will be specified at the transmission zeros of the lossless scattering matrix representing the given load, L . In the case of a simple transmission zero on the imaginary axis, we will prove that the angular derivative at the transmission zero can only increase or remain the same during the chaining operation.

Proposition B.1.1. (*Chaining Conditions*). *Let L represent the (2×2) lossless scattering matrix of a given system and $F_{22} \in \mathbb{B}$ be any given reflection coefficient of a lossless system. We assume that the reflection coefficient L_{11} is strictly contractive at some point on the imaginary axis. Let ξ_0 and $\alpha_0 = j\omega_0$ represent simple transmission zeros of L in Π^+ and on the imaginary axis respectively. The reflection coefficient $S_{22} = F_{22} \circ L$ obtained by chaining F_{22} and L satisfies the following,*

$$\begin{aligned} S_{22}(\xi_0) &= L_{22}(\xi_0), \\ S_{22}(\alpha_0) &= L_{22}(\alpha_0), \\ \text{ang}[S_{22}](\alpha_0) &\geq \text{ang}[L_{22}](\alpha_0). \end{aligned}$$

Proof. From the chaining equation, we have,

$$S_{22} = L_{22} + \frac{L_{12}L_{21}F_{22}}{1 - F_{22}L_{11}} \quad (\text{B.1})$$

By definition, at the transmission zero ξ_0 of L in Π^+ , we have,

$$L_{12}(\xi_0)L_{21}(\xi_0) = 0. \quad (\text{B.2})$$

From equation B.2 and the fact that $1 - F_{22}L_{11}$ doesn't vanish at $\xi_0 \in \Pi^+$ under the strict contractive nature of L_{11} at some point on the imaginary axis, it follows,

$$\frac{L_{12}(\xi_0)L_{21}(\xi_0)F_{22}(\xi_0)}{1 - F_{22}(\xi_0)L_{11}(\xi_0)} = 0.$$

Thus, we have,

$$S_{22}(\xi_0) = L_{22}(\xi_0).$$

Now, let us consider the simple transmission zero $\alpha_0 = j\omega_0$ of L on the imaginary axis. It should be noted that from equations 1.47 and 1.53 in Chapter 1, for the lossless scattering matrix L , at the transmission zero α_0 on the imaginary axis, we have,

$$L_{12}(\alpha_0) = L_{21}(\alpha_0) = 0.$$

We claim that the term $1 - F_{22}L_{11}$ in the denominator of second term in the chaining equation B.1 if zero at α_0 can only have a simple zero. This is possible only if $F_{22}(\alpha_0) = \overline{L_{11}(\alpha_0)}$. In this case, $F_{22}L_{11}$ is a non-constant Schur function with uni-modular value at α_0 . By proposition 4.1.3, we have $\text{ang}[F_{22}L_{11}](\alpha_0) < 0$ and hence derivative of $1 - F_{22}L_{11}$ is non-vanishing at α_0 . This implies only one pole-zero cancellation in the second term in equation B.1 is possible. This yields,

$$S_{22}(\alpha_0) = L_{22}(\alpha_0).$$

Let us denote the second term in chaining equation by $\phi = \frac{g}{h}$, where $g = L_{12}L_{21}F_{22}$ and $h = 1 - F_{22}L_{11}$. Under the condition $F_{22}(\alpha_0) \neq \overline{L_{11}(\alpha_0)}$, ϕ has double zero at α_0 and hence $\text{ang}[S_{22}](\alpha_0) = \text{ang}[L_{22}](\alpha_0)$ follows from chaining equation B.1. Now, let us consider the case, $F_{22}(\alpha_0) = \overline{L_{11}(\alpha_0)}$. In this case, the derivative of ϕ can be calculated at α_0 with the help of L'Hospital's rule to yield,

$$\phi'(\alpha_0) = \frac{g''(\alpha_0)}{2h'(\alpha_0)}.$$

After calculating the second derivative of g and first derivative of h at α_0 , we get,

$$\phi'(\alpha_0) = \frac{L'_{12}(\alpha_0)L'_{21}(\alpha_0)\overline{L_{11}(\alpha_0)}}{-(\text{ang}[F_{22}](\alpha_0)) + \text{ang}[L_{11}](\alpha_0)}.$$

Thus, from chaining equation B.1, we have,

$$S'_{22}(\alpha_0) = L'_{22}(\alpha_0) + \frac{L'_{12}(\alpha_0)L'_{21}(\alpha_0)\overline{L_{11}(\alpha_0)}}{-(\text{ang}[F_{22}](\alpha_0)) + \text{ang}[L_{11}](\alpha_0)}.$$

Multiplying this equation throughout by $\overline{S_{22}(\alpha_0)}$ and using the second chaining condition $S_{22}(\alpha_0) = L_{22}(\alpha_0)$ yields,

$$\text{ang}[S_{22}](\alpha_0) = \text{ang}[L_{22}](\alpha_0) + \frac{L'_{12}(\alpha_0)L'_{21}(\alpha_0)\overline{L_{11}(\alpha_0)L_{22}(\alpha_0)}}{-(\text{ang}[F_{22}](\alpha_0)) + \text{ang}[L_{11}](\alpha_0)}. \quad (\text{B.3})$$

For the lossless scattering matrix L , from equation 1.51, we have,

$$L_{11}(j\omega)\overline{L_{21}(j\omega)} + L_{12}(j\omega)\overline{L_{22}(j\omega)} = 0.$$

The above equation upon differentiation with respect to ω and evaluated at α_0 yields,

$$L'_{21}(\alpha_0) = \overline{L'_{12}(\alpha_0)}L_{22}(\alpha_0)L_{11}(\alpha_0).$$

Substituting this for $L'_{21}(\alpha_0)$ in equation B.3, we have,

$$\text{ang}[S_{22}](\alpha_0) = \text{ang}[L_{22}](\alpha_0) + \frac{|L'_{12}(\alpha_0)|^2}{-(\text{ang}[F_{22}](\alpha_0)) + \text{ang}[L_{11}](\alpha_0)}.$$

The second term is non-negative from proposition 4.1.3 and hence we have,

$$\text{ang}[S_{22}](\alpha_0) \geq \text{ang}[L_{22}](\alpha_0).$$

This completes the proof. □

B.2 General De-embedding

Now, we are in a position to introduce the general de-embedding theorem. It states that the conditions mentioned in proposition 4.1.3 is necessary and sufficient for the matrix L to be de-chainable from S_{22} .

Theorem B.2.1. (*General De-embedding Theorem*). *Let L represent the (2×2) lossless scattering matrix of a given load having a reflection coefficient L_{11} which is strictly contractive at some point on the imaginary axis. Let $\xi_1, \xi_2, \dots, \xi_m$ represent the m transmission zeros of L in Π^+ with corresponding multiplicities $M(\xi_k)$ and $\alpha_1, \alpha_2, \dots, \alpha_l$ be the l transmission zeros of L on the imaginary axis (possibly at infinity) with corresponding multiplicities $M(\alpha_k)$. The matrix L is de-chainable of any given $S_{22} \in \mathbb{B}$ iff :*

$$\begin{aligned} \forall i \in \{0, 1, \dots, M(\xi_k) - 1\}, \quad S_{22}^{(i)}(\xi_k) &= L_{22}^{(i)}(\xi_k), \quad 1 \leq k \leq m, \\ \forall i \in \{0, 1, \dots, 2M(\alpha_k) - 2\}, \quad S_{22}^{(i)}(\alpha_k) &= L_{22}^{(i)}(\alpha_k), \quad 1 \leq k \leq l, \\ \text{For } i = 2M(\alpha_k) - 1, \quad \overline{L_{22}(\alpha_k)} \left(L_{22}^{(i)}(\alpha_k) - S_{22}^{(i)}(\alpha_k) \right) &\leq 0, \quad 1 \leq k \leq l. \end{aligned}$$

Even though we are not providing a complete proof of the above theorem, it should be noted that the proof of necessity is a consequence of Proposition B.1.1 in the case of higher order transmission zeros. The proof of sufficiency, after some elementary calculations can be obtained based on Rouches theorem.

Schur Recursion with Interpolation Conditions Inside the Right Half Plane

C.1 Schur Recursion

In this section, we will discuss the classical approach of Nevanlinna to describe all the functions $f \in \mathbb{B}$ that solve the following interpolation problem :

Problem 1. *Let $\xi_1, \xi_2, \dots, \xi_M$ be distinct points in Π^+ and $\gamma_1, \gamma_2, \dots, \gamma_M \in \mathbb{D}$. Find $f \in \mathbb{B}$ such that*

$$f(\xi_i) = \gamma_i, \quad 1 \leq i \leq M. \quad (\text{C.1})$$

For $m \leq M$, let us define

$$\mathbb{E}_m = \{f \in \mathbb{B} : f(\xi_i) = \gamma_i, \quad 1 \leq i \leq m\}. \quad (\text{C.2})$$

It should be noted that $\mathbb{E}_m \subset \mathbb{E}_{m-1} \dots \subset \mathbb{E}_2 \subset \mathbb{E}_1$ and \mathbb{E}_1 is non-empty if and only if $|\gamma_1| \leq 1$. In addition, if $|\gamma_1| = 1$, it follows from maximum modulus theorem that $f(s) = \gamma_1$ is the only function in \mathbb{E}_1 . We are interested in finding a parametrisation of \mathbb{E}_M . In the case when there are infinite number of interpolation conditions, the reader can refer to chapter IV, section 6 in [42] for a detailed review of Nevanlinna's parametrisation of

$$\mathbb{E}_\infty = \bigcap_m \mathbb{E}_m$$

and a proof showing that \mathbb{E}_∞ contains inner functions. We will restrict the discussion to the case of finite number of interpolation conditions.

Let us initially consider the case with one interpolation condition. If $f \in \mathbb{E}_1$,

$$\frac{f - \gamma_1}{1 - \overline{\gamma_1}f} = \frac{s - \xi_1}{s + \overline{\xi_1}} f_1 \quad (\text{C.3})$$

for some $f_1 \in \mathbb{B}$. This can be easily verified, if we invert (C.3), we have

$$f_1 = \frac{s + \overline{\xi_1}}{s - \xi_1} \frac{f - \gamma_1}{1 - \overline{\gamma_1}f} \quad (\text{C.4})$$

which implies that $|f_1(j\omega)| \leq 1$ for all $\omega \in \mathbb{R}$ because $|\frac{j\omega + \bar{\xi}_1}{j\omega - \xi_1}| = 1$ and $\delta(f(j\omega), \gamma_1) \leq 1$ for all $\omega \in \mathbb{R}$. Furthermore, it can be noted that $\frac{f - \gamma_1}{1 - \bar{\gamma}_1 f}$ is in \mathbb{B} when $f \in \mathbb{E}_1$ and in the expression for f_1 in (C.4), there occurs a pole-zero cancellation at $s = \xi_1$. This implies that $f_1 \in \mathbb{B}$. Conversely, whenever $f_1 \in \mathbb{B}$, the expression in (C.3) defines an $f \in \mathbb{E}_1$. This can be verified by rewriting (C.3) as

$$f = \frac{\gamma_1 - \frac{\xi_1 - s}{s + \bar{\xi}_1} f_1}{1 - \bar{\gamma}_1 \frac{\xi_1 - s}{s + \bar{\xi}_1} f_1}. \quad (\text{C.5})$$

Since $\frac{\xi_1 - s}{s + \bar{\xi}_1} f_1$ is in \mathbb{B} , it follows that $f \in \mathbb{B}$ and we also have $f(\xi_1) = \gamma_1$. So, we have $f \in \mathbb{E}_1$. We can rewrite (C.3) as

$$f(s) = \frac{A_1(s) + B_1(s)f_1(s)}{C_1(s) + D_1(s)f_1(s)}, \quad (\text{C.6})$$

where

$$\begin{aligned} A_1(s) &= \gamma_1(s + \bar{\xi}_1), & B_1(s) &= (s - \xi_1), \\ C_1(s) &= s + \bar{\xi}_1, & D_1(s) &= \bar{\gamma}_1(s - \xi_1). \end{aligned}$$

So, (C.6) is the parametrisation in the case of simple interpolation problem $f(\xi_1) = \gamma_1$, $f \in \mathbb{B}$.

Now let us suppose further that $f \in \mathbb{E}_2$. Then (C.3) helps to determine the value of $f_1(\xi_2)$ such that this happens. We have,

$$f_1(\xi_2) = \frac{\gamma_2 - \gamma_1}{1 - \bar{\gamma}_1 \gamma_2} \frac{\xi_2 + \bar{\xi}_1}{\xi_2 - \xi_1} \stackrel{\text{def}}{=} \gamma_2^{(1)}. \quad (\text{C.7})$$

Now \mathbb{E}_2 is described by just (C.7) instead of the original two conditions. Repeating the previous reasoning, \mathbb{E}_2 is non-empty if and only if $|\gamma_2^{(1)}| \leq 1$. Furthermore, if $|\gamma_2^{(1)}| = 1$, then the only function f_1 satisfying (C.7) is $f_1(s) = \gamma_2^{(1)}$ and in this case, \mathbb{E}_2 contains only the Blaschke product,

$$f(s) = \frac{A_1(s) + B_1(s)\gamma_2^{(1)}}{C_1(s) + D_1(s)\gamma_2^{(1)}} = \frac{\gamma_1 - \frac{\xi_1 - s}{s + \bar{\xi}_1} \gamma_2^{(1)}}{1 - \bar{\gamma}_1 \frac{\xi_1 - s}{s + \bar{\xi}_1} \gamma_2^{(1)}}.$$

Now, let us suppose $|\gamma_2^{(1)}| < 1$, i.e \mathbb{E}_2 contains more than one element. Following the same argument as before, $f \in \mathbb{E}_2$ iff $f_1(\xi_2) = \gamma_2^{(1)}$ and this happens if and only if

$$\frac{f_1 - \gamma_2^{(1)}}{1 - \bar{\gamma}_2^{(1)} f_1} = \frac{s - \xi_2}{s + \bar{\xi}_2} f_2 \quad (\text{C.8})$$

for some $f_2 \in \mathbb{B}$. Rewriting the above expression, we get

$$f_1 = \frac{\gamma_2^{(1)}(s + \bar{\xi}_2) + (s - \xi_2)f_2}{(s + \bar{\xi}_2) + \bar{\gamma}_2^{(1)}(s - \xi_2)f_2}. \quad (\text{C.9})$$

Substituting the above expression of f_1 in (C.6) and doing an elementary calculation yields,

$$f(s) = \frac{A_2(s) + B_2(s)f_2(s)}{C_2(s) + D_2(s)f_2(s)}, \quad (\text{C.10})$$

where

$$\begin{aligned} A_2(s) &= (s + \bar{\xi}_2) \left(A_1(s) + \gamma_2^{(1)} B_1(s) \right), & B_2(s) &= (s - \xi_2) \left(\overline{\gamma_2^{(1)}} A_1(s) + B_1(s) \right), \\ C_2(s) &= (s + \bar{\xi}_2) \left(C_1(s) + \gamma_2^{(1)} D_1(s) \right), & D_2(s) &= (s - \xi_2) \left(\overline{\gamma_2^{(1)}} C_1(s) + D_1(s) \right). \end{aligned}$$

So, (C.10) provides the parametrisation in the case of interpolation problem $f(\xi_1) = \gamma_1$, $f(\xi_2) = \gamma_2$, $f \in \mathbb{B}$.

We can proceed by induction, always assuming \mathbb{E}_m contain more than one function. Then, for $1 \leq k \leq m$, $f \in \mathbb{E}_m$ if and only if there are f_0, f_1, \dots, f_k in \mathbb{B} where $f_0 = f$ such that

$$f_{k-1}(\xi_i) = \gamma_i^{(k-1)}, \quad k \leq i \leq m,$$

where $\gamma_i^{(0)} = \gamma_i$ and for $l = 2, 3, \dots, m$,

$$\gamma_i^{(l-1)} \stackrel{\text{def}}{=} \left(\frac{\gamma_i^{(l-1)} - \gamma_{l-1}^{(l-1)}}{1 - \overline{\gamma_{l-1}^{(l-1)}} \gamma_i^{(l-1)}} \right) \left(\frac{\xi_i + \bar{\xi}_{l-1}}{\xi_i - \xi_{l-1}} \right), \quad i = l, l+1, \dots, m$$

satisfy $|\gamma_i^{(l-1)}| < 1$ and such that

$$\frac{f_{k-1} - \gamma_k^{(k-1)}}{1 - \overline{\gamma_k^{(k-1)}} f_{k-1}} = \frac{s - \xi_k}{s + \bar{\xi}_k} f_k. \quad (\text{C.11})$$

Rewriting (C.11), we obtain

$$f_{k-1} = \frac{\gamma_k^{(k-1)}(s + \bar{\xi}_k) + (s - \xi_k)f_k}{(s + \bar{\xi}_k) + \overline{\gamma_k^{(k-1)}}(s - \xi_k)f_k}. \quad (\text{C.12})$$

Now, by induction (C.12) and (C.6) gives

$$f(s) = \frac{A_m(s) + B_m(s)f_m(s)}{C_m(s) + D_m(s)f_m(s)}, \quad (\text{C.13})$$

where

$$\begin{aligned} A_m(s) &= (s + \bar{\xi}_m) \left(A_{m-1}(s) + \gamma_m^{(m-1)} B_{m-1}(s) \right), \\ B_m(s) &= (s - \xi_m) \left(\overline{\gamma_m^{(m-1)}} A_{m-1}(s) + B_{m-1}(s) \right), \\ C_m(s) &= (s + \bar{\xi}_m) \left(C_{m-1}(s) + \gamma_m^{(m-1)} D_{m-1}(s) \right), \\ D_m(s) &= (s - \xi_m) \left(\overline{\gamma_m^{(m-1)}} C_{m-1}(s) + D_{m-1}(s) \right). \end{aligned} \quad (\text{C.14})$$

are polynomials of degree at most m . So we have the following result (Lemma 6.1, Chap IV, [42]),

Theorem C.1.1. (*Nevanlinna Characterisation of \mathbb{E}_m*). Let $\xi_1, \xi_2, \dots, \xi_M$ be distinct points in Π^+ and $\gamma_1, \gamma_2, \dots, \gamma_M \in \mathbb{D}$. For $m \leq M$, if

$$\mathbb{E}_m = \{f \in \mathbb{B} : f(\xi_i) = \gamma_i, 1 \leq i \leq m\} \quad (\text{C.15})$$

and the polynomials A_m, B_m, C_m, D_m are defined by (C.14), then $f(s) \in \mathbb{E}_m$ if and only if $f(s)$ satisfies (C.13) for some $f_m \in \mathbb{B}$. Furthermore, if the set \mathbb{E}_m is a singleton, the unique interpolating function is a Blaschke product of degree $\tilde{m} < m$.

Remark C.1.2. By setting $f_m(s) = c$ in (C.13) where $|c| = 1$,

$$f(s) = \frac{A_m(s) + B_m(s)c}{C_m(s) + D_m(s)c},$$

is in \mathbb{E}_m and $f(s)$ is a rational function of degree at most m . In fact the reader can refer to Lemma 6.2 and Exercise 20 in chapter IV, [42] for a proof which shows that $f(s)$ is a Blaschke product of degree m in this case.

C.2 Fano-Youla Characterisation

In this section, we will give a brief outline of the physical interpretation of Schur recursion in terms of chaining of lossless scattering matrices to Schur functions. We will continue to use the notations as in the previous section.

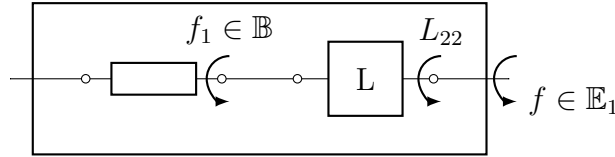


Figure C.1: Fano-Youla Characterisation of $f \in \mathbb{E}_1$

Let us consider the set \mathbb{E}_1 , the case with one interpolation condition. The equation (C.4) represents an elementary Schur reduction step. In terms of Fano-Youla characterisation, as shown below, this is equivalent to the notion that a lossless scattering matrix can be de-chained from f . Let us consider the chaining diagram in figure C.1 where L is a lossless (2×2) scattering matrix. From the de-chaining formula mentioned in (3.4), we have

$$f_1 = \frac{f - L_{22}}{L_{11}f - \det(L)}. \quad (\text{C.16})$$

In order to interpret it with respect to Schur reduction in (C.4), we can rewrite (C.4) as

$$f_1 = \frac{f - \gamma_1}{-\gamma_1 \frac{s - \xi_1}{s + \xi_1} f + \frac{s - \xi_1}{s + \xi_1}}$$

and identify

$$L_{22} = \gamma_1, \quad L_{11} = -\gamma_1 \frac{s - \xi_1}{s + \xi_1}, \quad \det(L) = -\frac{s - \xi_1}{s + \xi_1}. \quad (\text{C.17})$$

From (C.17) and since L is lossless, we have

$$L_{12}L_{21} = (1 - |\gamma_1|^2) \frac{s - \xi_1}{s + \xi_1}, \quad L_{12}L_{12}^* = L_{21}L_{21}^* = 1 - |\gamma_1|^2.$$

So, a possible choice of L , if we impose $L_{21}(\xi_1) = 0$ is

$$L = \frac{1}{s + \xi_1} \begin{bmatrix} -\overline{\gamma_1}(s - \xi_1) & \sqrt{1 - |\gamma_1|^2}(s + \xi_1) \\ \sqrt{1 - |\gamma_1|^2}(s - \xi_1) & \gamma_1(s + \xi_1) \end{bmatrix}. \quad (\text{C.18})$$

Thus if $f \in \mathbb{E}_1$, a (2×2) matrix of the form (C.18) can be de-chained from f . Conversely, if the matrix L in (C.18) is de-chainable from $f \in \mathbb{B}$, from the chaining equation in (3.3), we have

$$f = f_1 \circ L = L_{22} + \frac{L_{12}L_{21}f_1}{1 - f_1L_{11}} = \det(L) \left(\frac{L_{11}^* - f_1}{1 - f_1L_{11}} \right), \quad (\text{C.19})$$

where $f_1 \in \mathbb{B}$ and it implies that $f(\xi_1) = \gamma_1$ and hence $f \in \mathbb{E}_1$. So, we have the Fano-Youla's characterisation of \mathbb{E}_1 : $f \in \mathbb{E}_1$ if and only if there exists a lossless scattering matrix L of the form in (C.18) such that $f = f_1 \circ L$, where $f_1 \in \mathbb{B}$. It should be noted that computing from (C.19), the expression for f , we obtain

$$\begin{aligned} f = \det(L) \left(\frac{L_{11}^* - f_1}{1 - f_1L_{11}} \right) &= - \left(\frac{s - \xi_1}{s + \xi_1} \right) \left(\frac{-\gamma_1 \frac{s + \xi_1}{s - \xi_1} - f_1}{1 + \overline{\gamma_1} \frac{s - \xi_1}{s + \xi_1} f_1} \right) \\ &= \frac{\gamma_1(s + \xi_1) + (s - \xi_1)f_1}{(s + \xi_1) + \overline{\gamma_1}(s - \xi_1)f_1} \end{aligned}$$

which is the same as equation (C.6) obtained in Nevanlinna parametrisation of \mathbb{E}_1 .

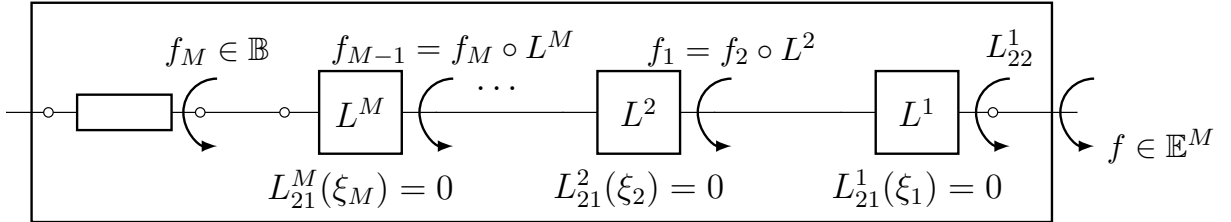


Figure C.2: Fano Youla Characterisation of $f \in \mathbb{E}_M$

In general, the Schur recursion with M interpolation condition can be physically interpreted as the chaining of elementary blocks L^1, L^2, \dots, L^M in the form (C.18), where each block L^i has transmission zero at ξ_i , $1 \leq i \leq M$. It is represented in figure (C.2). In addition, at each step of elementary chaining, we have

$$\begin{aligned} L_{22}^1(\xi_1) &= \gamma_1 \\ (f^1 \circ L^1)(\xi_2) &= \gamma_2 \\ (f^2 \circ L^2 \circ L^1)(\xi_3) &= \gamma_3 \\ &\vdots \\ (f^M \circ L^M \circ \dots \circ L^2 \circ L^1)(\xi_M) &= \gamma_M. \end{aligned}$$

It should be noted that if we denote $L = L^M \circ L^{M-1} \circ \dots \circ L^1$, then the matrix L has transmission zeros at ξ_i and $L_{22}(\xi_i) = \gamma_i$ for $1 \leq i \leq M$. So, for $f \in \mathbb{E}_M$, we have the characterisation $f = f_M \circ L$, where $f_M \in \mathbb{B}$. As mentioned in the Nevanlinna characterisation, it should be noted that, if at the m -th stage of recursion, the reflection coefficient L_{22}^m becomes uni-modular, then the recursion stops and provides a unique interpolating function f of degree $m - 1$.

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