Some results on the McKean–Vlasov optimal control and mean field games : Limit theorems, dynamic programming principle and numerical approximations

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Abstract

This thesis is formulated in three parts with eight chapters and presents a research topic dealing with controlled processes/particles/agents in interaction.

In the first part of the dissertation, we focus our attention on the study of interacting controlled processes representing a cooperative equilibrium, also called Pareto equilibrium. A cooperative equilibrium can be seen as a situation where there is no way to improve the preference criterion of one agent without lowering the preference criterion of at least one other agent. It is now known that this kind of optimization problem is related, when the number of agents goes to infinity, to the optimal control of McKean–Vlasov processes. In the first three chapters of this thesis, we provide a precise mathematical answer to the connection between these two optimization problems in various frameworks improving the existing literature, in particular by taking into account the law of control while allowing a common noise situation.

More precisely, in Chapter 2, we first give a weak formulation of the McKean–Vlasov optimal control with common noise and law of control, which is necessary to understand the optimization problem of McKean–Vlasov processes and to analyze the behavior of cooperative equilibria when the number of agents goes to infinity. With the help of this necessary weak formulation, using a notion of relaxed controls, in a non–Markovian framework with a common noise, a complete analysis of the convergence of (approximate) cooperative equilibria is given in Chapter 3. Some properties of the optima of the McKean–Vlasov optimal control problem are also proved.

Then, we pursue in Chapter 4 the analysis of the convergence of cooperative equilibria in a case where the empirical distribution of controls is considered while allowing a common noise. Still using the weak formulation of Chapter 2, by introducing a notion of measure–valued controlled process motivated by the Fokker–Planck equation verified by the McKean–Vlasov processes, the characterization of the limits of the cooperative equilibria is provided within this framework similar to the Chapter 3.

After studying the behavior of the cooperative equilibria, we conclude the first part in Chapter 5 where we spend times in the analysis of the limit problem i.e. the McKean–Vlasov optimal control, through the establishing of the Dynamic Programming Principle (DPP) for this stochastic control problem. Thanks to one more time the weak formulation of Chapter 2 and an adaptation of measurable selection arguments, we give a DPP for various forms of the McKean–Vlasov optimal control improving the existing literature in particular by considering weaker assumptions on the coefficients and reward functions.

The second part of this thesis is devoted to the study of the interacting controlled processes now representing a Nash equilibrium, also called competitive equilibrium. A Nash equilibrium situation in a game is a situation in which no one has anything to gain by moving unilaterally from his own position. Since the pioneering works of Lasry and Lions [111] and Huang, Caines, and Malhamé [84], the behavior of Nash equilibria when the number of agents goes to infinity has been intensively studied and the associated limit game is known as Mean Field Games (MFG). In this second part, we analyze first the convergence of the competitive equilibrium to the MFG in a framework with the law of control and with control of volatility, then, the issue of the existence of MFG equilibrium in this context is studied.

In Chapter 6, by adapting the techniques used to study the behavior of the cooperative equilibria, with the introduction of a new weak form of MFG equilibrium, that we coin measure–valued MFG equilibrium, and a notion of approximate strong MFG equilibrium, we explore the convergence of competitive equilibria by taking into account the empirical distribution of controls in a common noise case while controlling the non–common noise volatility.

We finish this second part in Chapter 7 by proving existence results of the measure–valued MFG equilibrium. The proof is achieved by the use of a fixed point theorem, especially by Kakutani's fixed point theorem. This existence result induces the existence of approximate Nash equilibrium and approximate strong MFG equilibrium.

Finally, the last part which includes only Chapter 8, is dedicated to some numerical methods to solve the McKean–Vlasov limit problem. Inspired by the proof of the convergence of cooperative equilibrium, we give in Chapter 8 a numerical algorithm to solve the McKean–Vlasov optimal control problem and we prove its convergence. Then, we implement our algorithm using neural networks and test its efficiency on some application examples, namely the mean–variance portfolio selection, the inter–bank systemic risk model and the optimal liquidation with market impact.

Keywords: Stochastic differential equation, stochastic control problem, Pareto equilibria, Nash equilibria, mean field game, McKean–Vlasov equations, common noise, interacting particles, law of control, propagation of chaos, limit theory, dynamic programming principle, measurable selection, HJB equation, numerical approximations, Monte–Carlo, neural networks.

Resumé

Cette thèse est formulée en trois parties avec huit chapitres et présente un sujet de recherche traitant des particules/ agents/ processus controllés en interactions.

Dans la première partie de cette dissertation, nous portons notre attention sur l'étude de processus controllés en interaction représentant un équilibre coopératif, aussi appelé équilibre de Pareto. Un équilibre coopératif peut être vu comme une situation où il n'existe pas de moyen d'améliorer la préférence d'un agent sans rabaisser la préference d'au moins un autre agent. Il est maintenant bien connu que ce type de problème d'optimisation est lié, quand le nombre d'agents tend vers l'infini, au contrôle optimal de l'équation de McKean–Vlasov. Dans les trois premiers chapitres de cette thèse, nous donnons une réponse mathématique précise à la connection entre ces deux problèmes d'optimisation dans differents cadres améliorant la littérature existante, en particulier en prenant en compte la loi du contrôle tout en considérant un bruit commun.

Plus précisement, dans le Chapitre 2, d'abord nous donnons une formulation faible du contrôle optimal de l'équation de McKean–Vlasov avec bruit commun et loi du contrôle, qui est nécessaire pour comprendre le problème d'optimisation de McKean–Vlasov and analyser le comportement des équilibres coopératifs quand le nompbre d'agents va vers l'infini. Grâce à cette indispensable faible formulation, en utilisant une notion de contrôle relaxé, dans un cadre non-Markovien avec bruit commun, une analyse complète de la convergence des équilibres coopératifs est donnée au Chapitre 3. Quelques propriétés des optima du problème de contrôle de McKean–Vlasov sont aussi prouvées.

Ensuite, nous poursuivons dans le Chapitre 4 l'analyse de la convergence des équilibres coopératifs dans un cas où la distribution empirique des contrôles est considérée tout en permettant un bruit commun. Toujours en utilisant la formulation faible du Chapitre 2, en introduisant une notion de processus contrôlés à valeur mesure motivé par l'équation de Fokker-Planck vérifiée par les processus McKean-Vlasov, la caractérisation des limites des équilibres coopératifs est fournie dans ce cadre similaire au Chapitre 3.

Après avoir étudié le comportement des équilibres coopératifs, nous concluons la première partie dans le Chapitre 5 où nous passons du temps à l'analyse du problème limite c'est à dire le contrôle optimal McKean–Vlasov, en établissant le Principe de Programmation Dynamique (PPD) pour ce problème de contrôle stochastique. Grâce une fois de plus à la formulation faible du Chapitre 2 et une adaptation des arguments de sélection mesurable, nous donnons un PPD pour différentes formes du contrôle optimal McKean–Vlasov améliorant la littérature existante notamment en considérant des hypothèses plus faibles sur les coefficients et les fonctions de récompense.

La seconde partie de cette thèse est consacrée à l'étude des processus contrôlés en interaction représentant désormais un équilibre de Nash, également appelé équilibre compétitif. Une situation d'équilibre de Nash dans un jeu est une situation dans laquelle personne n'a rien à gagner en quittant unilatéralement sa propre position. Depuis les travaux pionniers de Lasry and Lions [111] et Huang, Caines, and Malhamé [84], le comportement des équilibres de Nash lorsque le nombre d'agents va à l'infini a été intensivement étudié et le jeu limite associé est connu sous le nom de jeux à champ moyen (MFG). Dans cette seconde partie, nous analysons d'abord la convergence des équilibres compétitifs vers les MFG dans un cadre avec la loi de contrôle et avec le contrôle de la volatilité, puis, la question de l'existence de l'équilibre MFG dans ce contexte est étudiée.

Dans le Chapitre 6, en adaptant les techniques utilisées pour étudier le comportement des équilibres coopératifs, avec l'introduction d'une nouvelle forme faible d'équilibre MFG, que nous nommons equilibre MFG à valeur mesure, et une notion d'équilibre MFG fort approximatif, nous explorons la convergence des équilibres compétitifs en prenant en compte la distribution empirique des contrôles dans un cas de bruit commun tout en contrôlant la volatilité non commune.

Nous terminons cette seconde partie dans le Chapitre 7 en prouvant des résultats d'existence de l'équilibre MFG à valeur mesure. La preuve est obtenue par l'utilisation d'un théorème de point fixe, en particulier par le théorème de point fixe

de Kakutani. Ce résultat d'existence induit l'existence d'un équilibre de Nash approximatif et d'un équilibre MFG fort approximatif.

Enfin, la dernière partie qui ne comprend que le Chapitre 8, est dédiée à quelques méthodes numériques pour résoudre le problème de contôle de McKean-Vlasov. Inspiré de la preuve de la convergence d'équilibre coopératif, nous donnons dans le Chapitre 8, un algorithme numérique pour résoudre le problème de contrôle optimal McKean-Vlasov et nous prouvons sa convergence. Ensuite, nous implémentons notre algorithme à partir des réseaux de neurones et testons son efficacité sur quelques exemples d'application, à savoir la sélection de portefeuille par moyenne-variance, le modèle de risque systémique interbancaire et la liquidation optimale avec impact de marché.

Mots-clés: Équation différentielle stochastique, problème de contrôle stochastique, équilibres de Pareto, équilibres de Nash, jeux à champ moyen, équations de McKean–Vlasov, bruit commun, particules en interaction, loi de contrôle, propagation du chaos, théorie des limites, principe de programmation dynamique, sélection mesurable, équation HJB, approximations numériques, Monte–Carlo, réseaux de neurones.

Frequently used notation

(i) Given a metric space (E,ρ), we denote by $\mathcal{B}(E)$ its Borel σ–algebra, and by $\mathcal{P}(E)$ the collection of all Borel probability measures on E. For every constant $p \geq 1$, we denote by $\mathcal{P}_p(E)$ the set of $\mu \in \mathcal{P}(E)$ such that $\int_E \rho(e,e_0)^p \mu(\mathrm{d}e) < \infty$ for some (and thus for all) $e_0 \in E$. We equip $\mathcal{P}_p(E)$ with the Wasserstein distance $\mathcal{W}_p$ defined by

$$\mathcal{W}_p(\mu,\mu') := \Big(\inf_{\lambda \in \Pi(\mu,\mu')} \int_E \int_E \rho(e,e')^p \lambda(\mathrm{d}e,\mathrm{d}e') \Big)^{1/p}, \ (\mu,\mu') \in \mathcal{P}_p(E) \times \mathcal{P}_p(E),$$

where $\Pi(\mu,\mu')$ denotes the set of all probability measures λ on $E \times E$ such that $\lambda(\mathrm{d}e, E) = \mu$ and $\lambda(E, \mathrm{d}e') = \mu'(\mathrm{d}e')$. Let $\mu \in \mathcal{P}(E)$ and $\varphi : E \longrightarrow \mathbb{R}$ be a μ–integrable function, we write

$$\langle \varphi, \mu \rangle := \langle \mu, \varphi \rangle := \mathbb{E}^\mu[\varphi] := \int_E \varphi(e) \mu(\mathrm{d}e).$$

Let (E',ρ') be another metric space and $\mu' \in \mathcal{P}(E')$. We denote by $\mu \otimes \mu' \in \mathcal{P}(E \times E')$ their product probability measure. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a sub–σ–algebra $\mathcal{G} \subset \mathcal{F}$, we denote by $(\mathbb{P}^{\mathcal{G}}_\omega)_{\omega \in \Omega}$ the conditional probability measure on $\mathbb{P}$ knowing $\mathcal{G}$ (whenever it exists). For a random variable $\xi : \Omega \longrightarrow E$, we write $\mathcal{L}^{\mathbb{P}}(\xi) := \mathbb{P} \circ \xi^{-1}$ the law of ξ under $\mathbb{P}$, and for any $\omega \in \Omega$, $\mathcal{L}^{\mathbb{P}}(\xi|\mathcal{G})(\omega) := \mathbb{P}^{\mathcal{G}}_\omega \circ \xi^{-1}$ the conditional distribution of ξ knowing $\mathcal{G}$ under $\mathbb{P}$.

(ii) For any (E,Δ) and (E',Δ') two Polish spaces, we shall refer to $C_b(E,E')$ to designate the set of continuous functions f from E into E' such that $\sup_{e \in E} \Delta'(f(e), e'_0) < \infty$ for some $e'_0 \in E'$. Let $\mathbb{N}^*$ be the notation of the set of positive integers. Given non-negative integers m and n, we denote by $\mathbb{S}^{m \times n}$ the collection of all $m \times n$–dimensional matrices with real entries, equipped with the standard Euclidean norm, which we denote by $|\cdot|$ regardless of the dimensions, for notational simplicity. We also denote $\mathbb{S}^n := \mathbb{S}^{n \times n}$, and denote by $0_{m \times n}$ the element in $\mathbb{S}^{m \times n}$ whose entries are all 0, and by I_n the identity matrix in $\mathbb{S}^n$. For any matrix $a \in \mathbb{S}^n$ which is symmetric positive semi-definite, we write $a^{1/2}$ the unique symmetric positive semi-definite square root of the matrix a. Let k be a positive integer, we denote by $C^k_b(\mathbb{R}^n;\mathbb{R})$ the set of bounded maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, having bounded continuous derivatives of order up to and including k. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be twice differentiable, we denote by ∇f and $\nabla^2 f$ the gradient and Hessian of f.

(iii) Let $T > 0$, and (Σ,ρ) be a Polish space, we denote by $C([0,T],\Sigma)$ the space of all continuous functions on $[0,T]$ taking values in Σ. Then $C([0,T],\Sigma)$ is a Polish space under the uniform convergence topology, and we denote by $\|\cdot\|$ the uniform norm. When $\Sigma = \mathbb{R}^k$ for some $k \in \mathbb{N}$, we simply write $\mathcal{C}^k := C([0,T],\mathbb{R}^k)$, also we shall denote by $\mathcal{C}^k_{\mathcal{W}} := C([0,T],\mathcal{P}(\mathbb{R}^k))$, and for $p \geq 1$, $\mathcal{C}^{k,p}_{\mathcal{W}} := C([0,T],\mathcal{P}_p(\mathbb{R}^k))$.

Contents

Chapter 1

Introduction

1.1 Intuitions and preliminaries

The uncontrolled McKean–Vlasov SDE One of the main objectives of this thesis is to understand (in a certain sense) the behavior of N–interacting controlled particles when N goes to infinity. However, let us begin by describing the N–interacting particles (without controls) to fix the ideas, and show the main differences and difficulties with the controlled case. Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space supporting $(W^i)_{i\in\mathbb{N}^*}$ a sequence of independent $\mathbb{R}^d$–valued $\mathbb{F}$–Brownian motion, and $(\xi^i)_{i\in\mathbb{N}^*}$ a sequence of i.i.d. $\mathcal{F}_0$–measurable random variables.

We start with the no–common noise situation. What we call N–interacting particles (without common noise) are the $\mathbb{F}$–adapted continuous process $(\mathbf{X}^1, \dots, \mathbf{X}^N)$ governed by the following stochastic differential equation (SDE)

$$\mathbf{X}_t^i = \xi^i + \int_0^t b\big(s, \mathbf{X}^i_{s\wedge\cdot}, \varphi_s^{N,\mathbf{X}}\big)\mathrm{d}s + \int_0^t \sigma\big(s, \mathbf{X}^i_{s\wedge\cdot}, \varphi_s^{N,\mathbf{X}}\big)\mathrm{d}W_s^i,\ t\in[0,T],\ \mathbb{P}\text{–a.s.} \tag{1.1.1}$$
$$\varphi_t^{N,\mathbf{X}} := \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{X}^i_{t\wedge\cdot}}.$$

An important observation in this equation is the fact that the map $(b,\sigma) : [0,T]\times C([0,T];\mathbb{R}^n)\times\mathcal{P}(C([0,T];\mathbb{R}^n)) \longrightarrow \mathbb{R}^n\times\mathbb{R}^{n\times d}$ are independent of i and N, and the sequence $(W^i,\xi^i)_{1\le i\le N}$ are i.i.d. Therefore, these particles are exchangeable in the sense that: for any permutation $p : \{1,\dots,N\} \longrightarrow \{1,\dots,N\}$, the law of $(\mathbf{X}^{p(1)},\dots,\mathbf{X}^{p(N)})$ is independent of p. This is a key remark because if such a system is tractable mathematically when N goes to infinity, it is because this system has this kind of symmetry.

Since the seminal papers of McKean Jr. [121] and Kac [91], and the monograph of Snitzman [149], the behavior when N goes to infinity of these kind of systems has been intensively studied and is now well known and understood. Let us give some intuition of the limit of such a system. The only interaction between the particles is through the empirical distribution $\varphi^{N,\mathbf{X}}$. If for a moment, we assume that the map (b,σ) is independent of $\varphi^{N,\mathbf{X}}$ i.e. defined on $[0,T]\times C([0,T];\mathbb{R}^n)$, we can notice that the sequence $(\mathbf{X}^i)_{i\in\mathbb{N}^*}$ is i.i.d, so that by the law of large numbers (with integrability conditions), $\mathbb{P}$–a.e. when $N\longrightarrow\infty$ the empirical distribution $\varphi_t^{N,\mathbf{X}}$ converges to $\mathcal{L}^{\mathbb{P}}(X^1_{t\wedge\cdot})$, which is a deterministic value. Now, back to our original (b,σ) with dependence w.r.t. $\varphi^{N,\mathbf{X}}$, motivated by the previous observation if we have in our mind that $\varphi_t^{N,\mathbf{X}}$ converges to a deterministic value μ_t, so when N is large enough, the sequence $(\mathbf{X}^i)_{i\in\mathbb{N}^*}$ is *quasi* i.i.d, in fact, if we replace μ_t by $\varphi_t^{N,\mathbf{X}}$ in Equation (1.1.1), we find a new sequence $(\widehat{\mathbf{X}}^i)_{i\in\mathbb{N}^*}$ which is an i.i.d sequence of distribution X where

$$X_t = \xi + \int_0^t b\big(s, X_{s\wedge\cdot}, \mu_s\big)\mathrm{d}s + \int_0^t \sigma\big(s, X_{s\wedge\cdot}, \mu_s\big)\mathrm{d}W_s,\ t\in[0,T],\ \mathbb{P}\text{–a.e.}, \tag{1.1.2}$$

with (W,ξ) having the same law as (W^1,ξ^1) and independent of $(W^i,\xi^i)_{i\in\mathbb{N}^*}$. Formally speaking, by expecting that the empirical distributions $\varphi_t^{N,\mathbf{X}}$ and $\varphi_t^{N,\widehat{\mathbf{X}}}$ are close enough when $N\to\infty$, so share the same limit, the deterministic value μ_t must verify: for a continuous function f,

$$\mathbb{E}^{\mathbb{P}}[f(X_{t\wedge\cdot})] = \mathbb{E}^{\mathbb{P}}[f(\widehat{\mathbf{X}}^1_{t\wedge\cdot})] = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}}[f(\widehat{\mathbf{X}}^i_{t\wedge\cdot})] = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}}[f(\mathbf{X}^i_{t\wedge\cdot})] = \langle f,\mu_t\rangle \Longrightarrow \mathcal{L}^{\mathbb{P}}(X_{t\wedge\cdot}) = \mu_t. \tag{1.1.3}$$

With the condition $\mathcal{L}^{\mathbb{P}}(X_{t\wedge\cdot}) = \mu_t$, the Equation (1.1.2) is known as the McKean–Vlasov equation (without common noise). In the case where the previous equations are well defined and admit unique solutions (for example when (b,σ) are Lipschitz in $(\mathbf{x},\mu)$ uniformly in t), the convergence result $\lim_{N\to\infty} \varphi_t^{N,\mathbf{X}} = \mu_t$ has been proved in many frameworks see for instance Oelschläger [133], Gärtner [71], Graham and Méléard [74] and [149]. This result is called *propagation of chaos* in the literature, a terminology coming from Mark Kac.

The use of these type of particles is to model the interaction/interdependence which can arise in the study of certain systems in Physics, Biology, Economics, Finance, ... The independence assumption between the sources of noise $(W^i)_{i\in\mathbb{N}^*}$ is not pertinent most of the time. Consequently, to be more accurate in our modelling, adding another source of correlation can be very useful in practice. This motivated the research in this subject to consider the interacting particles with common noise. Formally speaking, in Equation (1.1.1), the idea is to add a source of noise via another Brownian motion B called *common noise*, independent of $(\xi^i, W^i)_{i\in\mathbb{N}^*}$ and impacting all the N–particles. More precisely, $(\mathbf{X}^1,\cdots,\mathbf{X}^N)$ follows now the dynamic

$$\mathbf{X}_t^i = \xi^i + \int_0^t b\big(s,\mathbf{X}_{s\wedge\cdot}^i,\varphi_s^{N,\mathbf{X}}\big)\mathrm{d}s + \int_0^t \sigma\big(s,\mathbf{X}_{s\wedge\cdot}^i,\varphi_s^{N,\mathbf{X}}\big)\mathrm{d}W_s^i + \int_0^t \sigma_0\big(s,\mathbf{X}_{s\wedge\cdot}^i,\varphi_s^{N,\mathbf{X}}\big)\mathrm{d}B_s,\ t\in[0,T],\ \mathbb{P}\text{–a.e.} \tag{1.1.4}$$
$$\varphi_t^{N,\mathbf{X}} := \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{X}_{t\wedge\cdot}^i}.$$

In this situation, we notice that the interactions between the particles are through the empirical distribution $\varphi^{N,\mathbf{X}}$ and the Brownian motion B. The system still keeps a certain symmetry. Indeed, the map (b,σ,σ_0) are independent of (i,N) and for any permutation $p:\{1,\dots,N\}\longrightarrow\{1,\dots,N\}$, the law of $(\mathbf{X}^{p(1)},\dots,\mathbf{X}^{p(N)},B)$ is independent of p. Hence, the question of the behavior of this correlated system when $N\to\infty$ can be asked mathematically. However, in contrast with the no–common noise situation, here, we cannot expect that the limit of the empirical distribution $\varphi_t^{N,\mathbf{X}}$ is deterministic. Indeed, if the map (b,σ,σ_0) is independent of $\varphi^{N,\mathbf{X}}$, the sequence $(\mathbf{X}_{t\wedge\cdot}^i)_{i\in\mathbb{N}^*}$ is conditionally independent and shares the same conditional distribution given the σ–field $\mathcal{G}_t := \sigma\{B_s,\ 0\le s\le t\}$. By the law of large numbers (or at least an easy extension taking into account of the conditional i.i.d property), one has

$$\lim_{N\to\infty}\varphi_t^{N,\mathbf{X}} = \mathcal{L}^{\mathbb{P}}(\mathbf{X}_{t\wedge\cdot}^1\big|\mathcal{G}_t),\ \mathbb{P}\text{–a.e.}$$

The limit is then $\mathcal{G}_t$–measurable. Therefore, in the case of dependence w.r.t. $\varphi^{N,\mathbf{X}}$ of (b,σ,σ_0), a limit of $\varphi^{N,\mathbf{X}}$ cannot be deterministic and will be at least a $\mathcal{G}_t$–measurable random probability measure. In the case the limit is precisely a $\mathcal{G}_t$–measurable random probability measure μ_t, the same intuition used previously in the no–common noise setting allows to say that when N is large enough, we can assume that in Equation (1.1.4) $\varphi^{N,\mathbf{X}}$ is *equal* to μ_t, then $(\mathbf{X}^i)_{i\in\mathbb{N}^*}$ is *quasi* conditionally i.i.d., and the new sequence $(\widehat{\mathbf{X}}^i)_{i\in\mathbb{N}^*}$ is conditionally i.i.d., where for each i, $(\widehat{\mathbf{X}}^i, B, W^i, \xi^i)$ has the same distribution as (X,B,W,ξ) and

$$X_t = \xi + \int_0^t b\big(s,X_{s\wedge\cdot},\mu_s\big)\mathrm{d}s + \int_0^t \sigma\big(s,X_{s\wedge\cdot},\mu_s\big)\mathrm{d}W_s + \int_0^t \sigma_0\big(s,X_{s\wedge\cdot},\mu_s\big)\mathrm{d}B_s,\ t\in[0,T],\ \mathbb{P}\text{–a.e.} \tag{1.1.5}$$

The same line of heuristic argument as (1.1.3) leads to

$$\mu_t = \mathcal{L}^{\mathbb{P}}(X_{t\wedge\cdot}\big|\mathcal{G}_t).$$

Equation (1.1.5) is sometimes called *conditional* McKean–Vlasov equaiton, and the corresponding convergence result *conditional* propagation of chaos. The presence of a random probability measure makes it less easy to transform the previous formal derivations into a rigorous proof. But the rigorous proof latter has been established in Kurtz and Xiong [101] and Dawson and Vaillancourt [53].

The controlled McKean–Vlasov SDE In a spirit of more relevant and practical modeling research, besides the exogenous parameters (b,σ,σ_0) and the noise sources $((\xi^i,W^i)_{i\in\mathbb{N}^*},B)$, it can be very useful to add endogenous parameters like controls $(\alpha^i)_{i\in\mathbb{N}^*}$. The idea can be seen as the desire to allow particles to adapt to a particular situation, endogenously.

For instance, in Physics, allowing the particles to move in a certain environment while minimizing their energy, in Finance, maximizing a portfolio's expectation value containing many assets while minimizing its variance. Mathematically speaking, this leads to consider the $\big(\sigma\{\xi^i, W^i_{t\wedge\cdot}, B_{t\wedge\cdot},\ 1\le i\le N\}\big)_{t\in[0,T]}$–predictable process $\overline{\alpha} := (\alpha^{1,N},\dots,\alpha^{N,N})$ in order to control the process $(\mathbf{X}^{\overline{\alpha},1},\dots,\mathbf{X}^{\overline{\alpha},N})$ governed by the SDE

$$\mathbf{X}^{\overline{\alpha},i}_t = \xi^i + \int_0^t b\big(s,\mathbf{X}^{\overline{\alpha},i}_{s\wedge\cdot},\varphi^N_s,\alpha^{i,N}_s\big)\mathrm{d}s + \int_0^t \sigma\big(s,\mathbf{X}^{\overline{\alpha},i}_{s\wedge\cdot},\varphi^N_s,\alpha^{i,N}_s\big)\mathrm{d}W^i_s + \int_0^t \sigma_0\big(s,\mathbf{X}^{\overline{\alpha},i}_{s\wedge\cdot},\varphi^N_s,\alpha^{i,N}_s\big)\mathrm{d}B_s,\ t\in[0,T], \quad (1.1.6)$$
$$\varphi^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{\big(\mathbf{x}^i_{t\wedge\cdot},\,\alpha^{i,N}_t\big)}.$$

Here, in addition to the empirical distribution of states $\varphi^{N,\mathbf{X}}$ and the common noise B, the interactions potentially also appear through the controls $(\alpha^{1,N},\dots,\alpha^{N,N})$. Consequently, although the map (b,σ,σ_0) does not depend on (i,N), the presence of controls $(\alpha^{1,N},\dots,\alpha^{N,N})$ makes the system less symmetrical than before. The question of the mathematical tractability of the behavior of this system when N goes to infinity, becomes much more delicate. However, following the intuition of the uncontrolled situation i.e. the propagation of chaos, a realistic conjecture is to consider that the controlled system (1.1.6) is related to the controlled process X^α where

$$X^\alpha_t = \xi + \int_0^t b\big(s,X^\alpha_{s\wedge\cdot},\overline{\mu}_s,\alpha_s\big)\mathrm{d}s + \int_0^t \sigma\big(s,X^\alpha_{s\wedge\cdot},\overline{\mu}_s,\alpha_s\big)\mathrm{d}W_s + \int_0^t \sigma_0\big(s,X^\alpha_{s\wedge\cdot},\overline{\mu}_s,\alpha_s\big)\mathrm{d}B_s,\ t\in[0,T],\ \mathbb{P}\text{–a.e.}, \quad (1.1.7)$$
$$\overline{\mu}_t = \mathcal{L}^{\mathbb{P}}(X^\alpha_{t\wedge\cdot},\alpha_t|\mathcal{G}_t),$$

with α a $(\sigma\{\xi,W_{t\wedge\cdot},B_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable process, seen as a control. We want to work with controls as general as possible for more flexibility, which, in this case, means only $(\sigma\{\xi,W_{t\wedge\cdot},B_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable processes. Then a convergence result of type " $\lim_{N\to\infty}\varphi^{N,\mathbf{X}}_t = \mu_t$, $\mathbb{P}$–a.e. " is clearly not possible.

As we said in the preamble, the control of the system (or the particles) makes it possible to refine the modeling via an optimization procedure which creates endogenous parameters. When we use the term *optimization*, it raises the natural question of the criterion of optimization/equilibrium to consider. In this thesis, we will focus our analysis on two criteria: *cooperative* equilibrium and *competitive* equilibrium.

1.1.1 Cooperative equilibrium or Pareto optimum

The cooperative equilibrium refers to the economic concept of *Pareto* optimum associated to Vilfredo Pareto, an Italian engineer and economist. A Pareto optimum can be described as a situation where it does not exist an alternative allocation where improvements can be made to at least one agent's well–being without reducing any other agent's well–being. In our framework, this is translated mathematically by saying that $\overline{\alpha}^\star := (\alpha^{\star,1,N},\dots,\alpha^{\star,N,N})$ is a Pareto optimum if: for any control $\overline{\beta} := (\beta^{1,N},\dots,\beta^{N,N})$, one has

$$\sum_{i=1}^N \mathbb{E}^{\mathbb{P}}\Big[\int_0^T L\big(t,\mathbf{X}^{\overline{\alpha}^\star,i}_{t\wedge\cdot},\varphi^N_t,\alpha^{\star,i,N}_t\big)\mathrm{d}t + g\big(\mathbf{X}^{\overline{\alpha}^\star,i}_{T\wedge\cdot},\varphi^{N,\mathbf{X}}_T\big)\Big] \ge \sum_{i=1}^N \mathbb{E}^{\mathbb{P}}\Big[\int_0^T L\big(t,\mathbf{X}^{\overline{\beta},i}_{t\wedge\cdot},\varphi^N_t,\beta^{i,N}_t\big)\mathrm{d}t + g\big(\mathbf{X}^{\overline{\beta},i}_{T\wedge\cdot},\varphi^{N,\mathbf{X}}_T\big)\Big]. \quad (1.1.8)$$

Our goal is to understand the behavior when $N\to\infty$ of the *Pareto* optimum. It is now known that this *Pareto* optimum is related to the optimal control $\alpha^\star$ satisfying: for any control β

$$\mathbb{E}\Big[\int_0^T L\big(t,X^{\alpha^\star}_{t\wedge\cdot},\mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot},\alpha^\star_t|\mathcal{G}_t\big),\alpha^\star_t\big)\mathrm{d}t + g\big(X^{\alpha^\star}_{T\wedge\cdot},\mathcal{L}\big(X^{\alpha^\star}_{T\wedge\cdot}|\mathcal{G}_T\big)\big)\Big]$$
$$\ge \mathbb{E}\Big[\int_0^T L\big(t,X^{\beta}_{t\wedge\cdot},\mathcal{L}\big(X^{\beta}_{t\wedge\cdot},\beta_t|\mathcal{G}_t\big),\beta_t\big)\mathrm{d}t + g\big(X^{\beta}_{T\wedge\cdot},\mathcal{L}\big(X^{\beta}_{T\wedge\cdot}|\mathcal{G}_T\big)\big)\Big].$$

This optimization problem is known as the stochastic control of McKean–Vlasov processes or Mean Field Control problem (MFC). There are not many articles in the literature which treat rigorously the connection of these two problems, also

called *limit theory*. When this question is raised, only special cases of (b, σ, σ_0) are taken account without common noise i.e. $\sigma_0 = 0$. Let us cite Fischer and Livieri [67] who studied a mean–variance optimization problem stemming from mathematical finance, and obtained a convergence result allowing to understand the behavior of the Pareto optimum in a particular situation. For general setting, such a study has been made in Lacker [104] in a context without common noise ($\sigma_0 = 0$) and without the law of control, where an essential tool is a compactness argument, which is made accessible by formulating an appropriate relaxed control, in the spirit of El Karoui, Huu Nguyen, and Jeanblanc-Picqué [63], and by introducing suitable martingale problems, similar to those of Stroock and Varadhan [150]. The same formulation and arguments have also been used in Bahlali, Mezerdi, and Mezerdi [15; 16; 17; 18] and Chala [50] to study stability and approximation problems.

In the first part of this thesis, we will put our attention on the study of this connection, extending the results already obtained on this subject. Namely, in a first step, in a general framework, we will extend the results of [104] in the case with common noise and including the law of control. We will make a link between the (approximate) *Pareto* optimum and the (approximate) optimum of the McKean–Vlasov optimal control problem. Next, following the work of Pham and Wei [139; 138], we will study the *limit* problem i.e. the McKean–Vlasov optimal control problem by establishing a Dynamic Programming Principle (DPP) with the fewest possible assumptions.

While drawing inspiration from techniques already used in the literature (mostly [104] for the limit theory, and [139; 138] for the DPP), our techniques will turn out to be very different, and will allow us to deal with cases that could not be taken into account before. Furthermore, the techniques will be general enough to help us understand the behavior when $N \to \infty$ in the situation of *competitive* equilibrium.

1.1.2 Competitive equilibrium or Nash equilibrium

The *Nash* equilibrium, named after the mathematician John Forbes Nash Jr., is well known in the economic literature. The idea behind this notion of equilibrium is the desire to define a strategy for a game in which no player has anything to gain by changing only their own strategy. Here, in more mathematical terms, $\overline{\alpha}^\star := (\alpha^{\star,1,N}, \dots, \alpha^{\star,N,N})$ is a Nash equilibrium if: for any control β, and any $i \in \{1, \dots, N\}$, if we introduce the control $\overline{\alpha}^{i,\beta}$ by

$$\overline{\alpha}^{i,\beta} := \big(\alpha^{\star,1,N}, \dots, \alpha^{\star,i-1,N}, \beta, \alpha^{\star,i+1,N}, \dots, \alpha^{\star,N,N}\big),$$

one has

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T L\big(t, \mathbf{X}^{\overline{\alpha}^\star,i}_{t\wedge\cdot}, \varphi^N_t, \alpha^{\star,i,N}_t\big)\mathrm{d}t + g\big(\mathbf{X}^{\overline{\alpha}^\star,i}_{T\wedge\cdot}, \varphi^{N,\mathbf{X}}_T\big)\bigg] \geq \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T L\big(t, \mathbf{X}^{\overline{\alpha}^{i,\beta},i}_{t\wedge\cdot}, \varphi^N_t, \beta_t\big)\mathrm{d}t + g\big(\mathbf{X}^{\overline{\alpha}^{i,\beta},i}_{T\wedge\cdot}, \varphi^{N,\mathbf{X}}_T\big)\bigg]. \tag{1.1.9}$$

As previously, our goal is to understand the behavior when $N \to \infty$ of the *Nash* equilibrium. Since the pioneering work of Lasry and Lions [111] and Huang, Caines, and Malhamé [84], the associated limit problem is largely studied in the literature and is known under the name of *Mean Field Games* (MFG). It can be described, loosely speaking, as follows: $\alpha^\star$ is an MFG equilibria if for any β

$$\begin{aligned}&\mathbb{E}\bigg[\int_0^T L\big(t, X^{\alpha^\star}_{t\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot}, \alpha^\star_t|\mathcal{G}_t\big), \alpha^\star_t\big)\mathrm{d}t + g\big(X^{\alpha^\star}_{T\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{T\wedge\cdot}|\mathcal{G}_T\big)\big)\bigg]\\ &\geq \mathbb{E}\bigg[\int_0^T L\big(t, X_{t\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot}, \alpha^\star_t|\mathcal{G}_t\big), \beta_t\big)\mathrm{d}t + g\big(X_{T\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{T\wedge\cdot}|\mathcal{G}_T\big)\big)\bigg],\end{aligned} \tag{1.1.10}$$

where

$$\mathrm{d}X_t = b\big(t, X_{t\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot}, \alpha^\star_t|\mathcal{G}_t\big), \beta_t\big)\mathrm{d}t + \sigma\big(t, X_{t\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot}, \alpha^\star_t|\mathcal{G}_t\big), \beta_t\big)\mathrm{d}W_t + \sigma_0\big(t, X_{t\wedge\cdot}, \mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot}, \alpha^\star_t|\mathcal{G}_t\big), \beta_t\big)\mathrm{d}B_t.$$

This structure means that, when the process $\big(\mathcal{L}\big(X^{\alpha^\star}_{t\wedge\cdot}, \alpha^\star_t|\mathcal{G}_t\big)\big)_{t\in[0,T]}$ is fixed, a single representative player solves an optimal control problem. A representation property of the entire population is given to the optimal control $\alpha^\star$ by the McKean–Vlasov equation verified by $X^{\alpha^\star}$. $\alpha^\star$ can be seen as an equilibrium. In contrast to the stochastic control of McKean–Vlasov processes, the MFG literature is very large. As a main reference, let us cite the book of Carmona and

Delarue [43; 44] and that of Cardaliaguet, Delarue, Lasry, and Lions [38] which deal with many questions related to this problem (existence, uniqueness, convergence,...) with no control in the volatility (σ, σ_0) and without law of controls. However, most of the time, the results focus on the properties of the limit problem. The connection between $Nash$ equilibria and MFG is rarely treated. As major papers used for our study, we can evoke Fisher [68] and the general analysis of Lacker [103] using the notion of weak solution without the law of control and without control of the volatility (σ, σ_0).

By an adequate adaptation of techniques used in the *cooperative* framework, in the second part of this thesis, we will study the convergence of $Nash$ equilibrium. Especially, we will extend the results of [103] by taking into account the law of control while allowing the volatility σ to be controlled. Also, we will consider an approximate MFG equilibrium by conceding a small error in the optimization (1.1.10), and we will show that this solution requires fewer assumptions to be well defined and is related to the approximate $Nash$ equilibrium. This will be possible by considering a new notion of MFG equilibrium. We will come back to our approach later in Section 1.3 for some details.

1.2 McKean–Vlasov optimal control with common noise

1.2.1 Limits of cooperative equilibrium

1.2.1.1 Motivation

As evoked in Section 1.1, in a general framework, we cannot expect a convergence result of the type

$$\lim_{N\to\infty} \varphi_t^{N,\mathbf{X}} = \mu_t, \ \mathbb{P}\text{–a.e.}$$

For lack of a better way, the weak convergence or convergence in distribution is the best we can hope for a possible relationship between (approximate) cooperative equilibria and optima of the McKean–Vlasov optimal control problem (this is also true for the case of (approximate) Nash equilibrium and MFG). Most of the time this is largely enough. Indeed, as we have seen, the idea is to optimize a quantity which is the expectation of a functional of the particles. Thus, taking into account the distribution of particles appears naturally, and getting a weak convergence provides a first answer to the understanding of the behavior of such a controlled system.

A first general result is brought by Lacker [104] in the case with no–common noise ($\sigma_0 = 0$) and without law of controls i.e. with $\varphi^{N,\mathbf{X}}$ instead of φ^N in the coefficients (b, σ, L, g). It can be formulated as follows (with simplifications and avoiding some technical aspects): when $\sigma_0 = 0$, so $\mathcal{G}_t := \{\emptyset, \Omega\}$, and the coefficients $(b, \sigma)(t, \mathbf{x}, \mu, a)$ are continuous in $(\mathbf{x}, \mu, a)$ uniformly in t, Lipschitz in $(\mathbf{x}, \mu)$ uniformly in (t, a), and with linear growth, then the sequence $\left(\mathbb{P} \circ \left(\varphi_T^{N,\mathbf{X}}\right)^{-1}\right)_{N\in\mathbb{N}^*}$ associated to the ϵ_N[1]–Pareto equilibrium $(\alpha^{\star,1,N}, \dots, \alpha^{\star,N,N})$ is relatively compact when $\lim_{N\to\infty} \epsilon_N = 0$, and for any convergent sub–sequence $\left(\mathbb{P}\circ\left(\varphi_T^{N_k,\mathbf{X}}\right)^{-1}\right)_{k\in\mathbb{N}^*}$, there exists a sequence of $(\sigma\{\xi, W_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable controls $(\alpha^k)_{k\in\mathbb{N}^*}$ s.t. α^k is a δ_k–optimum for the MFC problem with $\lim_{k\to\infty} \delta_k = 0$, and if X^k denotes the solution of Equation (1.1.7) associated to the controls α^k, and

$$\text{if } \widehat{\mathbb{P}} := \lim_{k\to\infty} \mathbb{P} \circ \left(\varphi_T^{N_k,\mathbf{X}}\right)^{-1} \text{ then one has } \lim_{k\to\infty} \mathbb{P} \circ \left(X^k_{T\wedge\cdot}\right)^{-1} \in \text{supp}\big(\widehat{\mathbb{P}}\big), \tag{1.2.1}$$

where $\text{supp}\big(\widehat{\mathbb{P}}\big)$ is the support of the probability measure $\widehat{\mathbb{P}}$ which is a probability on $\mathcal{P}(C([0,T];\mathbb{R}^n))$. The idea of the proof of this result is to put the interacting particles/ ϵ_N–Pareto optimum especially the empirical distribution in a good space. Then, obtaining some estimates which ensure the relative compactness of the sequence of empirical distributions. And finish by characterizing the limits with a martingale problem and a relaxed controls, we will come back to these concepts a bit later. The proof is technical, and uses tools from proofs of propagation of chaos and stochastic control. Notwithstanding seeing the proof, an important observation of the convergence in (1.2.1) is the fact that the sequence of empirical distributions of controlled interacting particles of Equation (1.1.6) and the sequence of distributions of controlled McKean–Vlasov processes of Equation (1.1.7) share the same accumulating points. Thus, as we said in Section 1.1.1, the

[1] It means in the optimization (1.1.8) verifying by the Pareto optimum, a small error ϵ_N is conceded.

cooperative equilibrium and the optimum of the MFC problem are related. Hence, trying to understand the behavior of the controlled interacting particles is similar to trying to understand the behavior of controlled McKean–Vlasov processes.

From the result (1.2.1), a natural question is whether this type of result is true in the case with common noise i.e. σ_0 not necessarily null. Despite the appearances, this is not an easy question. In the proof of (1.2.1), the weak convergence is used everywhere. To simplify, the fact that in the coefficients (b, σ) of the associated McKean–Vlasov process appears a deterministic probability $\mathcal{L}^{\mathbb{P}}\big(X^{\alpha}_{t\wedge\cdot}\big)$ allows to use this convergence through the continuous coefficients (b, σ), and this is an important fact in this proof. In the situation where we have the conditional probability $\mathcal{L}^{\mathbb{P}}\big(X^{\alpha}_{t\wedge\cdot}\big|\mathcal{G}_t\big)$ instead of $\mathcal{L}^{\mathbb{P}}\big(X^{\alpha}_{t\wedge\cdot}\big)$, this difference generates a huge problem. The weak convergence of sequence of controlled processes X^{α} can no longer be used through the coefficients (b, σ). Indeed, the weak convergence of sequence of type $\mathcal{L}^{\mathbb{P}}(X^{\alpha}_{t\wedge\cdot})$ does not induce in general the convergence of sequence of random probabilities of type $\mathcal{L}^{\mathbb{P}}\big(X^{\alpha}_{t\wedge\cdot}\big|\mathcal{G}_t\big)$. We then need to have another approach.

Another important observation with a view to extend this result in the case with a common noise is to notice that the weak convergence is independent of the choice of the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$. The distribution of the random variable does not depend on the probability space in which the random variable resides. Therefore, the question of the writing of Equation (1.1.6) and Equation (1.1.7) in all spaces possible appears naturally. In the literature, this is called *weak* formulation of (controlled) SDE, while writing of Equation (1.1.6) and Equation (1.1.7) on a fix probability space with a point–wise uniqueness, is usually called *strong* formulation. The idea of the *weak* formulation is to keep the fundamental properties of processes involved in the *strong* formulation of the SDE while allowing any strong solution to be *seen* as a weak solution and any weak SDE to be *approximated* by a sequence of strong SDE when we have some additional assumptions on (b, σ, σ_0). Usually, the well–posedness of weak SDE requires weaker assumptions on the coefficients (b, σ, σ_0), hence the qualifier of *weak*. A weak formulation for Equation (1.1.6) is standard in the literature and not really crucial for our goal here. But a weak formulation for Equation (1.1.7) i.e. the Mckean–Vlasov equation in the case of common noise is not classic and not a trivial extension. The randomness of the probability measure μ_t and the randomness of the control α may be incompatible in the sense that a naïve weak formulation will not have the good properties previously mentioned. The noise generated by the control α_t and the random probability measure μ_t need to be handled carefully.

In the first part of this thesis, our goal will be to extend the result (1.2.1) in a more general setting. Thus, Chapter 2 will give a *good* weak formulation of the controlled conditional McKean–Vlasov SDE (1.1.7) essential to try to answer the question of weak convergence. After, in Chapter 3, we will study the characterization of the limits of Pareto equilibria with common noise in the case where σ_0 is uncontrolled and with no law of control. And finally, in Chapter 4, in a Markovian framework with σ_0 constant, we will study the natural extension consisting in replacing the empirical distribution of states $\varphi^{N,\mathbf{X}}_t$ by the empirical distribution of states and controls $\frac{1}{N}\sum_{i=1}^{N}\delta_{(\mathbf{X}^i_t,\alpha^{i,N}_t)}$ in the coefficients (b, σ, L) i.e. the case with law of controls. This extension, although rarely studied in the literature, is quite natural and can be very useful in modeling in Finance for example. In this setting, we will characterize the limit of controlled interacting particles. This will turn out to be a non–trivial case.

We want to emphasize that, while sharing some similarities with the techniques used by [104] for the proof of (1.2.1), our techniques deviate considerably from it. Adding a common noise and later taking into account the empirical distribution of states and controls turn out to be a difficult extension requiring radically different tools.

1.2.1.2 Towards a characterization of the limits: an essential weak formulation

Literature and motivations The notion of weak solution of SDE is standard in the literature. As mentioned previously, the idea can be understood as the desire to be able to write the SDE on all the possible probability spaces. In the case of control of classical SDE, in our situation it means no Mckean–Vlasov SDE (no $\overline{\mu}_t$ or μ_t), this notion is largely discussed in El Karoui and Tan [62]. To be simple, the weak formulation is obtained by weakening the measurability requirements of the main random variables X and α (in the controlled case) while keeping the equation verified by these variables. Thus, X and α become adapted to the general filtration of the probability space considered instead of $(\sigma\{\xi, W_{t\wedge\cdot}\})_{t\in[0,T]}$ in the strong formulation (we recall that we have taken here $\sigma_0 = 0$). Besides, the equation verified by (X, α), the Brownian motion W and the initial value ξ is still true in the probability space considered. Therefore, any strong controlled SDE can be seen as a weak controlled SDE, and any weak controlled SDE can be reached by a sequence of strong controlled SDE when (b, σ) are Lipschitz for instance (see [62]).

In the case of McKean–Vlasov SDE without common noise, the associated weak formulation is quite close to the notion of weak formulation in the no–McKean–Vlasov case, and does not present any difficulty (see [104]). The case with common noise is on the other hand much more delicate. Indeed, here besides X and α, we have to take into account the randomness of $\overline{\mu}$. Following the previous ideas of weak formulation, a naïve approach is to enlarge the filtration to which $\overline{\mu}$ is adapted, that is to say letting $\overline{\mu}$ be adapted to the general filtration of the space and not adapted to $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$. But, this approach is senseless in the Mckean–Vlasov setting with common noise because $\overline{\mu}$ is adapted to the common noise B (if we think of the interacting particles), so in general the filtration of the common noise, necessarily a sub–filtration, cannot be replaced by the general current filtration. Besides, in general, it is not possible to approximate a $\mathbb{F}$–adapted process by a sequence of $\mathbb{G}$–adapted processes, even for weak convergence. Notice also that the randomness of the control α can interfere with the randomness of $\overline{\mu}$. The approximation of any weak controlled SDE by a sequence of strong controlled SDE is achieved by an approximation of the control α, but α appears also in $\overline{\mu}$ (through X^α and α), therefore dealing these two randomness can be delicate. Consequently, the weakening of the adaptability of $\overline{\mu}$ must be done carefully with some additional constraint. Motivated by the weak formulation considered in Carmona, Delarue, and Lacker [49] and the techniques of [62], we will formulate an adequate weak formulation.

Main results By borrowing some ideas of the weak formulation of [49], thanks to a notion of compatible filtration in particular the (H) Hypothesis, and the martingale problem, we will formulate a *good* weak formulation for our problem that is to say: any strong control will be a weak control, the weak formulation will be well defined with fewer assumptions, the set of weak controls enjoys some convexity property and any weak control will be reached by a sequence of strong controls. In these properties, the most delicate is the last one. To show this result, we will use an approximation by discretization considered in [62] combined with the (H) Hypothesis properties of the weak control. Moreover, our proof allows to fill a subtle technical gap in the literature related to the notion of independence and that of measurability (see Remark 2.3.14 for more details).

1.2.1.3 Relaxed formulation and charaterization of the limits

Literature and motivations As we have seen in Section 1.2.1.1, the Pareto optimum when $N \to \infty$, and the optimum of the MFC problem share in some sense the same accumulating points. A candidate for this set of accumulating points is the set of relaxed controls.

In the classical optimal control theory, the set of relaxed controls has been introduced to recover a closed and convex set, while ensuring that its elements could be appropriately approximated by strong or weak controls. The point was that it then becomes easier in this formulation to deduce the existence and stability properties of the optimal solution, while ensuring under mild conditions that the value of the problem is not modified.

In the no Mckean–Vlasov case, the relaxed formulation in the stochastic control problem is usually obtained by interpreting the controlled process and the control as a probability measure on a *good* canonical space. This *good* canonical space in this situation is $C([0,T];\mathbb{R}^n) \times \mathbb{M}(A)$ where A is the set where the control take its values, and $\mathbb{M}(A)$ is the set of Borel measure q on $A \times [0,T]$ such that the marginal distribution of q on $[0,T]$ is the Lebesgue measure i.e. $q(A \times [0,t]) = t$. By using a martingale problem characterizing the SDE, a relaxed formulation is then obtained in this classical case (see El Karoui, Huu Nguyen, and Jeanblanc-Picqué [63]).

Similarly to the previous weak formulation (or weak control), a relaxed formulation in the McKean–Vlasov setting without common noise and without law of control is very close to the classical setting (see [104]). However, as mentioned in Section 1.2.1.1, the case with common noise is radically different. The presence of the common noise generates some significant technical hurdles, especially due to the appearance of the conditional distribution terms, which are generally not continuous with respect to the joint distribution.

Main results Inspired by ideas from [63], [49] and [104], when the volatility σ_0 is uncontrolled and without the presence of the law of control, in Chapter 3 of this first part, we will provide a relaxed formulation which will be necessary to link later the Pareto optimum when $N \to \infty$, and the optimum of the MFC problem.

This set of relaxed controls will be found by considering an enlarged canonical space for the interpretation of the controlled process and the control as a probability measure. The canonical space will bring a particular attention to the conditional

probability measure $\mathcal{L}^{\mathbb{P}}\big(X^{\alpha}\big|\mathcal{G}_T\big)$. We will also need to add a compatibility property of filtrations ((H)–Hypothesis condition) as in the weak formulation, and a *conditional* martingale problem or a family of martingale problem.

We will prove that, in this setting:

- any strong control can be seen as a relaxed control;
- put in a good space, any sequence of (approximate) Pareto optima, when $N \to \infty$, is relatively compact and its limits are all relaxed controls;
- any relaxed control can be approximated by a sequence of strong controls and a sequence of Pareto optima when $N \to \infty$.

Therefore we link the Pareto optimum when $N \to \infty$, and the optimum of the MFC problem. All of this will be possible by using the weak formulation, a technical proof of approximation involving (conditional) martingale measures, compactness results, and a propagation of chaos proved in a slightly different context.

1.2.1.4 Measure–valued processes and limits with law of control

Literature and motivations Despite the general aspect of the previous relaxed formulation, taking into account the empirical distribution of the controlled states and controls $\frac{1}{N}\sum_{i=1}^{N}\delta_{(\mathbf{X}_t^i,\alpha_t^{i,N})}$ is not possible. Considering the general forms of control as we do, generates some continuity problem. Indeed, the continuity of the map $t \longrightarrow \varphi_t^{N,\mathbf{X}}$ is crucial. But, in the case where we consider the empirical distribution $\frac{1}{N}\sum_{i=1}^{N}\delta_{(\mathbf{X}_t^i,\alpha_t^{i,N})}$, this type of continuity is no longer true.

Therefore, our tool of relaxed controls can not be used and we need to do something else. We want to emphasize that this extension is quite natural and can be very useful in the application (see [126]). However, there are not many papers in the literature which study the mean field control problem with law of control and its connection with a cooperative equilibrium. To the best of our knowledge, only the recent papers of Laurière and Tangpi [113] (with strong assumptions) and Motte and Pham [127] (for Mean-field Markov decision processes) treat the convergence question. Our approach on this subject is very different of these two papers and is motivated by the Fokker–Planck equation verified by the process $\big(\mathcal{L}^{\mathbb{P}}\big(X_t^{\alpha}\big|\mathcal{G}_t\big)\big)_{t\in[0,T]}$.

Main results Recall that when $N \to \infty$, the Pareto optimum with the empirical distribution of controlled states and controls and the optimum of the MFC problem with law of control must have the same accumulating points. By keeping in mind this fact, in Chapter 4 of this first part, in a Markovian setting with σ_0 constant and σ non–degenerate, we will formulate a set of measure–valued controls which will play the same role as the set of relaxed controls in the case without law of control.

In a first step, we will show that

- any strong control can be interpreted as a measure–valued control;
- put in a good space, any sequence of (approximate) Pareto optima, when $N \to \infty$, is relatively compact and its limits are all measure–valued controls;
- any measure–valued control can be approximated by a sequence of strong controls and a sequence of Pareto optima when $N \to \infty$.

Consequently, we characterize the sequence of Pareto optima and show its link with the MFC problem. The characterization is possible by the appropriate use of (controlled) Fokker–Planck equations. Inspired by the techniques developed in the proofs of Gyöngy [76], especially [76, Lemma 2.1] (an adaptation of Krylov [98]) and [76, Proposition 4.3] which are regularization results, we will determine the desire set of measure–valued controls thanks to a Fokker–Planck equation. The conditions used on the coefficients are general, except the non-degeneracy of the volatility σ. This assumption is capital to deal with the Fokker–Planck equation.

In a second step, in the case without common noise but still with the law of control, when σ is uncontrolled of type $\sigma(t,x,\bar{\nu},a)=\sigma(t,x)$, we will show that it is enough to work only with the closed–loop controls $\big(\alpha(t,X_t^\alpha),\mathcal{L}^{\mathbb{P}}\big(X_t^\alpha,\alpha(t,X_t^\alpha)\big)\big)$ instead of open–loop controls $\big(\alpha(t,\xi,W_{t\wedge\cdot}),\mathcal{L}^{\mathbb{P}}\big(X_t^\alpha,\alpha(t,\xi,W_{t\wedge\cdot})\big)\big)$. Indeed, we will prove that any open–loop control can be reached by a sequence of closed–loop controls and a sequence of closed–loop Pareto optima. This result is proved by using technical estimates on the density of the Fokker–Planck equation verified by $\big(\mathcal{L}^{\mathbb{P}}\big(X_t^\alpha\big)\big)_{t\in[0,T]}$ and proved in Bogachev, Krylov, Röckner, and Shaposhnikov [31]. We want to emphasize that the classical way in the literature to prove an equivalence between the closed–loop and open–loop does not work here. The Markovian projection with the convexity assumption as in [104] and Lacker, Shkolnikov, and Zhang [108] can not be apply in the presence of the law of control. The law of control cannot be recovered after projection.

1.2.2 Dynamic Programming Principle for the limit problem

1.2.2.1 Literature and motivation

In the first three chapters of the first part of this thesis, we have intensively studied the convergence of cooperative equilibria and its link with the optima of its associated limit problem i.e. the stochastic control of McKean–Vlasov processes. Now, we will focus on the analysis of the limit problem from a specific approach. As for stochastic control problems, there are two known classical techniques which can be used to analyze the mean–field control problem: the approach by the Pontryagin stochastic maximum principle and the approach by the dynamical programming principle (DPP).

The first one, the Pontryagin stochastic maximum principle, is a strategy which allows to derive necessary and/or sufficient conditions characterizing the optimal solution of the control problem, through a pair of processes (Y,Z) satisfying a backward stochastic differential equation (BSDE for short), also called adjoint equation in this case, coupled with a forward SDE, corresponding to the optimal path. Andersson and Djehiche [13] and Buckdahn, Djehiche, and Li [34] use this approach for a specific case of the mean–field control problem, corresponding to the case where the coefficients of the equation and the reward functions only depend on some moments of the distribution of the state and without common noise. Still without common noise, a more general analysis of this approach is made by Carmona and Delarue [42] thanks to the notion of differentiability in the space of probability measure introduced by Lions in his Collège de France course [118] (see also the lecture notes of Cardaliaguet [36]). Related results were also obtained by Acciaio, Backhoff Veraguas, and Carmona [1] for so–called extended mean–field control problem involving the law of the controls, and where a link with causal optimal transport was also highlighted.

The second one, the dynamic programming principle, is a technique which can be simplified as the desire to transform the global optimization problem into a recursive resolution of successive local optimization problems. This fact is an intuitive result, which is often used as some sort of meta–theorem, but is not so easy to prove rigorously in general. Note also that, in contrast to the Pontryagin maximum principle approach, this approach in general requires fewer assumptions, though it can be applied in less situations. Notwithstanding these advantages, the DPP approach has long been unexplored for the control of McKean–Vlasov equations. One of the main reasons is actually a very bleak one for us: due to the non–linear dependency with respect to the law of process, the problem is actually a time inconsistent control problem (like the classical mean–variance optimisation problem in finance, see the recent papers by Björk and Murgoci [28], Björk, Khapko, and Murgoci [29], and [80] for a more thorough discussion of this topic), and Bellman's optimality principle does not hold in this case. However, though the problem itself is time–inconsistent, one can recover some form of the DPP by extending the state space of the problem. This was first achieved by Laurière and Pironneau [112], and later by Bensoussan, Frehse, and Yam [24; 25; 26], who assumed the existence at all times of a density for the marginal distribution of the state process, and reformulated the problem as a deterministic density control problem, with a family of deterministic control terms. Under this reformulation, they managed to prove a DPP and deduce a dynamic programming equation in the space of density functions.

Following similar ideas, but without the assumptions of the existence of density, and allowing the coefficients and reward functions to not only depend on the distribution of the state, but on the joint distribution of the state and the control, Pham and Wei [139] also deduced, in a Markovian context, a DPP by looking at a set of closed loop (or feedback) controls, in a no–common noise context. It is one of the first general result on this subject. Later, they extended this strategy to

a common noise setting (where the control process is adapted to the common noise filtration) in [138]. Their result can be presented as follows: if we denote by $X^{t,\nu,\alpha}$ the solution of Equation (1.1.7) defined on $[t,T]$, starting from $X^{t,\nu,\alpha}=\xi$ where $\mathcal{L}^{\mathbb{P}}(\xi)=\nu$ and controlled by α, for any $\tau:\Omega\to[t,T]$ a $\mathbb{G}$–stopping time, when (b,σ,σ_0) are Lipschitz in (x,ν) uniformly in (t,a) and continuous in a, and the reward functions (L,g) satisfy

$$\big|L(t,x,\nu,a)-L(t,x',\nu',a)\big|+\big|g(x,\nu)-g(x',\nu')\big|\leq C\Big[1+|x|+|x'|+\langle|\mathrm{Id}|^2,\nu\rangle^{1/2}+\langle|\mathrm{Id}|^2,\nu'\rangle^{1/2}\Big]\Big[|x-x'|+\mathcal{W}_2(\nu,\nu')\Big], \tag{1.2.2}$$

one has

$$V(t,\nu)=\sup_{\alpha\in\mathcal{A}^{\mathbb{B}}}\mathbb{E}\bigg[\int_t^\tau L\big(s,X_s^{t,\nu,\alpha},\mathcal{L}\big(X_s^{t,\nu,\alpha}\big|\mathcal{G}_s\big),\alpha_s\big)\mathrm{d}s+V\big(\tau,\mathcal{L}\big(X_\tau^{t,\nu,\alpha}\big|\mathcal{G}_\tau\big)\big)\big)\bigg], \tag{1.2.3}$$

where $V:[0,T]\times\mathcal{P}(\mathbb{R}^n)\to\mathbb{R}$ is a Borel measurable function defined by

$$V(t,\nu):=\sup_{\alpha\in\mathcal{A}^{\mathbb{B}}}\mathbb{E}\bigg[\int_t^T L\big(s,X_s^{t,\nu,\alpha},\mathcal{L}\big(X_s^{t,\nu,\alpha}\big|\mathcal{G}_s\big),\alpha_s\big)\mathrm{d}s+g\big(X_T^{t,\nu,\alpha},\mathcal{L}\big(X_T^{t,\nu,\alpha}\big|\mathcal{G}_T\big)\big)\big)\bigg],$$

with $\mathcal{A}^{\mathbb{B}}$ the set of $\mathbb{G}$–predictable and square integrable processes. In Chapter 5 of this first part, we will use the DPP approach to deal with the mean–field control problem. Specifically, we will extend the previous result of [138] i.e. result (1.2.3) in a more general setting with assumptions weaker than (1.2.2). Indeed, in the Dynamic Programming Principle result (1.2.3), two direct questions appear: first, are the Assumption (1.2.2) necessary, can we replace them with something weaker? Second, is it possible to consider a more general set of controls and not only a $\mathbb{G}$–predictable processes?

The first question is justified by the fact that in the proof of [138], Assumption (1.2.2) is used essentially to guarantee the measurability of the value function $V:[0,T]\times\mathcal{P}(\mathbb{R}^n)\to\mathbb{R}$. Indeed, one of the major difficulties of this problem is the question of the measurability of the value function. This problem is not really related to the McKean–Vlasov control problem. It is in all DPP questions because of the expectation of V which appears in the right side of equation (1.2.3).

The second one appears because of the particularity of the mean–field control problem. In our setting, the natural expected set of controls is the set of $(\sigma\{\xi,W_{t\wedge\cdot},B_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable processes. But, considering this kind of controls is complicated in the setting of MFC with common noise. One of the reason the $\mathbb{G}$–predictable processes were used in [138] is the fact that the classical *conditioning* argument in DPP does not work with the $(\sigma\{\xi,W_{t\wedge\cdot},B_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable processes (see Remark 5.3.6).

To bypass these difficulties, we will use the classical measurable selection arguments for the measurability issues, see for instance El Karoui and Tan [61; 62], Dellacherie [56], Bertsekas and Shreve in [27; 146; 147; 148], and Shreve [143; 144; 145]). And for the problem related to the set of controls, the weak formulation presented in Chapter 2 will be very useful.

1.2.2.2 Main results

In a first step, using measurable selection arguments as in [61; 62], we prove the universal measurability of the associated value function, and derive the stability of controls with respect to conditioning and concatenation, and finally deduce the DPP for the weak formulation under very general assumptions on (b,σ,σ_0,L,g). In a second step, we address the DPP for the classical strong formulation. Using the DPP in weak formulation, and adding standard Lipschitz conditions on the drift and diffusion coefficients, as in [138], but without any regularity assumptions on the reward functions, and in a non–Markovian context, we obtain the DPP for the strong formulation of McKean–Vlasov control problems with common noise, where the control is adapted to the "common noise" filtration $\mathbb{G}$ i.e. the filtration generated by the Brownian motion B. Also, for the more general strong formulation, where the control is adapted to both ξ, W and B, we obtain the DPP under some additional regularity conditions on the reward functions. These regularity conditions may seem unexpected at first sight, but they seem unavoidable due to the non–linear dependency of the drift and volatility coefficients with respect to the conditional distribution of (X,α) (see Remark 5.3.6 for a more thorough discussion). Finally, the DPP results in the general non–Markovian context induces the same results in the Markovian one.

1.3 Mean field games of controls with common noise and volatility σ controlled

Understanding the behavior of Nash equilibria as the number of players goes to infinity is an old and natural question in game theory. We refer to Aumann [14] for one of the first study of markets with continuum (infinity) traders, and in the same spirit to Schmeidler [141] who investigated equilibrium points in nonatomic non–cooperative games. Later Mas-colell [120] extended the results of [141] by considering strategies in terms of distributions rather than measurable functions.

In the past decades, a modern analysis in the case of differential games has been provided simultaneously by Lasry and Lions [111] and Huang, Caines, and Malhamé [84]. Since then, this subject has undergone intensive study: Bensoussan and Frehse [23], Carmona and Delarue [40], Huang, Caines, and Malhamé [85; 86; 87], Kolokoltsov, Li, and Yang [96], Ahuja [7], Cardaliaguet, Delarue, Lasry, and Lions [38], Fisher [68], Lacker [103; 45; 105; 102]. It is worth emphasizing that most of the aforementioned papers focus on the analysis of the limit problem i.e. existence and uniqueness of the MFG problem or construction of the approximate Nash equilibria from a MFG solution, only a few papers, namely [111; 38; 105; 103; 68] treat the question of the convergence of Nash equilibria which is our main purpose on this part. The techniques used to deal with these questions can be classified into two approaches: the PDE techniques and the probabilistic techniques.

The PDE techniques are based on the introduction of dynamic value functions which characterize the optima of the problem. Indeed, it is possible to characterize the solutions of Nash–equilibria (1.1.9) and solutions of MFG problem (1.1.10) via value functions satisfying PDE equations. In a Markovian case, without the law of control and with σ and σ_0 constants (to simplify), the PDE system associated to the N–Nash equilibria is the following N functions $(v^{N,i} : [0,T] \times (\mathbb{R}^n)^N \to \mathbb{R})_{i\in\{1,\dots,N\}}$ satisfying: for $t \in [0,T)$ and $[x] := (x_1, \dots, x_N) \in (\mathbb{R}^n)^N$,

$$-\partial_t v^{N,i}(t,[x]) - H\big(x_i, m^N_{[x]}, D_{x_i} v^{N,i}(t,[x])\big) - \sum_{j=1, j\neq i}^{N} D_{x_j} v^{N,j}(t,[x])^\top b^\star\big(x_j, m^N_{[x]}, D_{x_j} v^{N,j}(t,[x])\big)$$
$$- \frac{1}{2}\sum_{j=1}^{N} \mathrm{Tr}\Big[D^2_{x_j,x_j} v^{N,i}(t,[x])\sigma\sigma^\top\Big] - \frac{1}{2}\sum_{j,k=1}^{N} \mathrm{Tr}\Big[D^2_{x_j,x_k} v^{N,i}(t,[x])\sigma_0\sigma_0^\top\Big] = 0,$$

with $v^{N,i}(T,[x]) = g(x_i, m^N_{[x]})$, where $m^N_{[x]} := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, and $H : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^N) \times \mathbb{R}^n \to \mathbb{R}$ is the Hamiltonian defined for all $(x,m) \in \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n)$ by

$$H(x,m,y) := b(x,m,\alpha^\star(x,m,y))^\top y + L(x,m,\alpha^\star(x,m,y)) \text{ where } \alpha^\star(x,m,y) \in \arg\max_{a\in A}\big[b(x,m,a)^\top y + L(x,m,a)\big],$$

and $b^\star(x,m,y) := b(x,m,\alpha^\star(x,m,y))$. From the functions $(v^{N,i})_{i\{1,\cdots,N\}}$, a N–Nash equilibrium is constructed as follows: let $\mathbf{X} := (\mathbf{X}^1, \cdots, \mathbf{X}^N)$ be the solution of

$$\mathbf{X}^i_t = \xi^i + \int_0^t b^\star(\mathbf{X}^i_s, \varphi^{N,\mathbf{X}}_s, D_{x_i} v^{N,i}(s,\mathbf{X}_s))\mathrm{d}s + \sigma W^i_t + \sigma_0 B_t, \text{ for all } t \in [0,T],$$

then

$$\big(\alpha^{\star,1}, \cdots, \alpha^{\star,N}\big) \text{ is a closed–loop}^2 \text{ Nash equilibrium, where for all } (t,i),\ \alpha^{\star,i}(t,\mathbf{X}_t) := \alpha^\star(\mathbf{X}^i_t, \varphi^{N,\mathbf{X}}_t, D_{x_i} v^{N,i}(t,\mathbf{X}_t)).$$

Under this viewpoint, the understanding of the behavior of the N–Nash equilibrium is reduced to understanding the behavior of the functions $(v^{N,i})_{i\in\{1,\dots,N\}}$. An answer is brought by [111; 38] by introducing U the solution of the Master

equation associated to the limit MFG problem: $U : [0,T]\times\mathbb{R}^n\times\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ satisfies: for all $(t,x,m) \in [0,T)\times\mathbb{R}^n\times\mathcal{P}(\mathbb{R}^n)$

$$\begin{aligned}
&-\partial_t U(t,x,m) - H\big(x,m,D_xU(t,x,m)\big) - \frac{1}{2}\mathrm{Tr}\Big[\big(\sigma\sigma^\top + \sigma_0\sigma_0^\top\big)D_x^2U(t,x,m)\Big] \\
&\quad - \int_{\mathbb{R}^n} b^\star\big(z,m,D_xU(t,z,m)\big)^\top D_mU(t,x,m)(z)m(\mathrm{d}z) \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}\Big[\big(\sigma\sigma^\top + \sigma_0\sigma_0^\top\big)D_zD_mU(t,x,m)(z)\Big]m(\mathrm{d}z) \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathrm{Tr}\Big[\sigma_0\sigma_0^\top D_m^2U(t,x,m)(z,z')\Big]m(\mathrm{d}z)m(\mathrm{d}z') \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^n} \mathrm{Tr}\Big[\sigma_0\sigma_0^\top D_xD_mU(t,x,m)(z)\Big]m(\mathrm{d}z) = 0,
\end{aligned}$$

with $U(T,x,m) = g(x,m)$. We refer to [38, Section 2.2] for the exact meaning of the derivatives D_mU and D_m^2U. With this PDE, [38] proves under strong assumptions that

$$\frac{1}{N}\sum_{i=1}^N \Big|v^{N,i}(t,[x]) - U(t,x_i,m^N_{[x]})\Big| \le CN^{-1} \text{ and } \mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbf{X}_t^i - \mathbf{Y}_t^i\big|\Big] \le CN^{-\frac{1}{n+8}}, \tag{1.3.1}$$

where $\mathbf{Y} := (\mathbf{Y}^1,\cdots,\mathbf{Y}^N)$ is a tuple such that $\mathcal{L}^{\mathbb{P}}\big(\mathbf{Y}^i,W^i,\xi^i,B\big) = \mathcal{L}^{\mathbb{P}}\big(Y,W,\xi,B\big)$ with Y solution of

$$Y_t = \xi + \int_0^t b^\star(Y_s,\mu_s,D_xU(s,Y_s,\mu_s))\mathrm{d}s + \sigma W_t + \sigma_0 B_t, \quad \mu_t = \mathcal{L}^{\mathbb{P}}(Y_t), \text{ for all } t\in[0,T].$$

Here $\alpha^\star$ where $\alpha^\star(t,Y_t) := \alpha^\star(Y_t,\mu_t,D_xU(t,Y_t,\mu_t))$ is a closed–loop MFG equilibrium.

This approach by PDE for assessing the convergence of Nash–equilibria can be powerful and provides estimates for quantifying the rate of convergence. One of the important limitations of these techniques is the necessity to consider strong assumptions on the coefficients (b,L,g) (especially the Lasry–Lions monotonicity). Considering more general coefficients (b,σ,σ_0,L,g) is very complicated in this context, and besides, all the PDEs require a uniqueness property. But, it is well known that in game theory the uniqueness of equilibria is an exception not the rule. The probabilistic approach does not suffer (most of the time) from the requiring of very strong assumptions, and the necessity of uniqueness can be avoided.

The probabilistic approach is more diverse and uses sometimes techniques from the PDE approach. Indeed, a probabilistic method quite close to the proof of [38] has been proposed by Carmona and Delarue [44, Chapter 6 Section 6.3]. Also, inspired by the method of [38], Delarue, Lacker, and Ramanan [54] get more general estimates related to (1.3.1) using techniques of large deviations and concentration of measure. One of the probabilistic approach which moves away from the PDE method and which gives results in a more general framework is provided for the analysis of open–loop Nash equilibria by [68] subsequently improved by [103]. Their idea is to use the notion of relaxed controls in the spirit of El Karoui, Huu Nguyen, and Jeanblanc-Picqué [63], combined with some compactness results. In the situation without law of control and without control of volatility, they introduce a notion weak MFG equilibrium especially in [103] for the case with common noise (see also [49]), and prove: if $(\alpha^{1,N},\ldots,\alpha^{N,N})$ is an ϵ_N–approximate Nash equilibrium with $\lim_{N\to\infty}\epsilon_N = 0$, and

$$\overline{\mathbb{P}}_N := \frac{1}{N}\sum_{i=1}^N \mathbb{P}\circ\Big(\xi^i,B,\mathbf{W}^i,\Lambda^i,\mathbf{X}^i,\widehat{\mu}^N\Big)^{-1}, \text{ where } \Lambda^i = \delta_{\alpha_t^{i,N}}(\mathrm{d}a)\mathrm{d}t \text{ and } \widehat{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}^i,\mathbf{W}^i,\Lambda^i\right)},$$

then under suitable assumptions,

$$\text{the sequence } (\overline{\mathbb{P}}_N)_{N\in\mathbb{N}^*} \text{ is relatively compact and each limit point is a weak MFG equilibrium.} \tag{1.3.2}$$

From any weak MFG equilibrium, it is possible to construct approximate Nash equilibrium. Besides, the solution formulated in (1.1.10) can be seen as a weak MFG equilibrium, and with some strong assumptions especially the Lasry–Lions monotonicity assumption, (1.1.10) is the unique weak MFG equilibrium. Despite the absence of rate of convergence as in (1.3.1), this probabilistic approach gives a characterization of all possible limits of Nash equilibrium and does not require any a priori uniqueness result. In addition, the framework can be more general that the PDE approach, σ and σ_0 are not necessarily constants. In the second part of this thesis we will use this approach and improve the result (1.3.2).

Motivated by the probabilistic techniques, in Chapter 6, we will deal with the question of convergence of (approximate) Nash equilibria in a more general context. We will analyze this problem in the framework of law of control while allowing the volatility σ to be controlled. In the proof of (1.3.2), it is clearly not possible to take into account the law of control, the empirical distribution of controls generates some discontinuity problems. Besides, the absence of control in σ (and σ_0) plays a crucial role in the determination of the limit, and removing this assumption is not an easy task. With the help of the Fokker–Planck equation associated to the McKean–Vlasov process and the techniques used in the proof of the convergence of Pareto equilibria, by introducing a new weak notion of MFG equilibrium, we will characterize all the limits of approximate open–loop Nash equilibira with the law of control and with σ controlled. We will come back on our techniques in Section 1.3.1 below.

After having explored the issue of the convergence of (approximate) Nash equilibria, we will spend some time analyzing the limit problem, dealing in particular with the issue of the existence of the MFG equilibrium i.e. find $\alpha^\star$ solution of (1.1.10). There are various articles in the literature that address this issue with PDE and probabilistic approach [25; 23; 40; 96; 38; 105; 49]. In the Markovian context, without the law of control and when $\sigma_0 = 0$ (no common noise) and σ constant, the question of existence of (1.1.10) is sometimes formulated in terms of searching (v, μ) the solution of the system of forward Kolmogorov equation coupled with backward Hamilton–Jacobi–Bellman equation which can be described as follows: $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the HJB equation, for all $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$-\partial_t v(t,x) - \sup_{a \in A} \Big[b(x, \mu_t, a)^\top \nabla v(t,x) + L(x, \mu_t, a) \Big] + \frac{1}{2} \mathrm{Tr}\big[\sigma\sigma^\top \nabla^2 v(t,x)\big] = 0,$$

with $v(T,x) = g(x, \mu_T)$. And $\mu : [0,T] \to \mathcal{P}(\mathbb{R}^n)$ is a deterministic function satisfying in the weak sense the Fokker–Planck equation

$$\partial_t \mu_t = -\mathrm{div}\big[b\big(x, \mu_t, \alpha^\star(t,x,\mu_t, \nabla v(t,x))\big)\mu_t\big] + \frac{1}{2}\mathrm{Tr}\big[\nabla^2 \mu_t \sigma\sigma^\top\big],$$

where

$$\alpha^\star(t,x,\mu_t,\nabla v(t,x)) \in \arg\max_{a \in A} \Big[b(x,\mu_t,a)^\top \nabla v(t,x) + L(x,\mu_t,a) \Big].$$

Then if we define $X^\star$ solution of the SDE

$$X_t^\star = \xi + \int_0^t b\big(X_s^\star, \mu_s, \alpha^\star(s, X_s^\star, \mu_s, \nabla v(s, X_s^\star))\big)\mathrm{d}s + \sigma W_t, \text{ for all } t \in [0,T], \text{ with } \mathcal{L}^{\mathbb{P}}(X_t^\star) = \mu_t,$$

$\big(\alpha^\star(t, X_t^\star, \mu_t, \nabla v(t, X_t^\star))\big)_{t\in[0,T]}$ is a closed–loop MFG equilibrium. This approach has been initiated by [111] and [84], and more papers later followed this method. An alternative way to address this question was initiated by Carmona and Delarue [40] by the use of the Pontryagin maximum principle to reduce the MFG problem to a forward–backward SDE of McKean–Vlasov type as follows: (X,Y,Z) solves the FBSDE

$$\begin{aligned} X_t &= \xi + \int_0^t b(X_s, \mu_s, \alpha^\star(s, X_s, \mu_s, Y_s))\mathrm{d}s + W_t \\ Y_t &= \nabla g(X_T, \mu_T) - \int_t^T D_x H(X_s, \mu_s, Y_s, \alpha^\star(s, X_s, \mu_s, Y_s))\mathrm{d}s + \int_t^T Z_s \mathrm{d}W_s, \text{ for all } t \in [0,T], \end{aligned}$$

where $\mu_t = \mathcal{L}^{\mathbb{P}}(X_t)$ and $H(x,m,y,a) := b(x,\mu_t,a)^\top \nabla v(t,x) + L(x,\mu_t,a)$. $\big(\alpha^\star(t, X_t, \mu_t, Y_t)\big)_{t\in[0,T]}$ is then a MFG equilibrium. All these approaches are very technical, require strong assumptions and are difficult to use in a more

general context especially to consider the case with common noise and the law of control. A much more flexible notion of MFG equilibrium has been initiated by Carmona, Delarue, and Lacker [49] in the context of common noise by using the notion of relaxed controls and by discretizing the common noise filtration. This notion generalizes the classical notion and allows to work in a general framework. In Chapter 7, we will extend the result of [49] in the case of law of control with control of volatility σ, by introducing a notion of measure–valued MFG equilibrium. Besides, we will provide a notion of approximate MFG equilibrium and will give a correspondence between this notion, the approximate Nash equilibria and the measure–valued MFG equilibrium. We will explain our methods with some details in Section 1.3.2 below.

1.3.1 Limits of competitive equilibrium

Literature and motivation As mentioned in Section 1.2.1.1 i.e the convergence of Pareto optimum, in our general setting, the weak convergence or convergence in distribution is the best we can hope for a possible connection between (approximate) competitive equilibrium and Mean Field Games. Our approach will be essentially based on the techniques used in the case of convergence of cooperative equilibrium.

Fisher [68] and Lacker [103] are one of the first papers which deal with this question in a general setting. However, their analyzes do not allow to take into account the dependence w.r.t the (conditional) distribution of controls and the control of the volatility (σ, σ_0).

Except Carmona and Lacker [45] which constructs an approximate Nash equilibrium with an uncontrolled and non-degenerative volatility σ ($\sigma > 0$) by a weak formulation, and the recent work of Laurière and Tangpi [113] which treats the convergence of Nash equilibria by probabilistic methods (via FBSDEs), to the best of our knowledge, there are no other papers using probabilistic or PDE methods that answer the question of the relation between the approximate Nash equilibria and the MFG solution in the context of law of control also called in the literature MFG of controls or *extended* MFG. Indeed, the techniques used so far to treat the question of study of the limit problem turn out to be too *rigid* to deal with the problem of the convergence of Nash equilibria, all the limits of approximate Nash equilibrium can not be described by the notion considered in the literature up to now. The approach that will be developed in this chapter is very different from those previously mentioned, and will take into account very general assumptions. Despite many differences, the approach is in the same spirit as [103] and [68], which are, in the framework without law of control, the most significant papers investigating the connection between the Nash equilibria and the MFG under very general assumptions. We want to emphasize that the interesting techniques developed in [103] and [68] do not work in the case of MFG of controls, in the presence of the law of control, since the assumptions of continuity on the coefficients are no longer verified (see also the discussion in Section 1.2.1.4).

Our analysis will be in the same spirit as [68] and [103], but in the case of law of controls and the volatility σ can be controlled. We will use many techniques mentioned in the case of cooperative equilibrium particularly the measure–valued solution.

Main results In order to solve the difficulty generated by the empirical distribution of controls and the control of the volatility σ, in a Markovian setting, we introduce the notion of measure–valued MFG equilibrium. The idea of our notion comes from the (stochastic) Fokker–Planck equation verified by $\left(\mathcal{L}^{\mathbb{P}}\left(X_t^{\alpha^\star}|\mathcal{G}_t\right)\right)_{t\in[0,T]}$. This notion of MFG solution is very close to the classical notion, the main difference is that the optimization is taken over all solutions of specific Fokker–Planck equations and not over solutions of an SDE. This notion has already been considered in the literature by Cardaliaguet, Delarue, Lasry, and Lions [38, Section 3.7] and in some way by Lacker [105]. Borrowing techniques from [103], under suitable assumptions, and working in a suitable space, we prove that the sequence of Nash equilibria is tight, and with the help of techniques used in the cooperative equillibrium, we show that every limit in distribution is a measure–valued mean field equilibrium. And conversely, for each measure–valued mean field equilibrium, we construct an approximate Nash equilibrium which has this measure–valued mean field equilibrium as limit. In other words, the measure–valued solutions are the accumulating points of (approxiamte)–Nash equilibria.

In addition to these convergence results, we will provide another approximation not taken into account until now. Similarly to approximate Nash equilibria, by considering an ϵ–strong solution of mean field games which is the classical strong solution where the optimality is obtained by admitting a small error ϵ, we prove that the measure–valued solutions are the accumulating points of this type of solutions when ϵ goes to zero.

1.3.2 Existence of approximate Nash equilibrium and approximate strong MFG

Literature and motivation In contrast to the cooperative case where the existence of the approximate Pareto equilibria is relatively immediate, in the competitive case the question of the existence of approximate Nash equilibria is not an obvious one. Indeed, in the problem (1.1.9), the optimization is not a standard one. Therefore, obtaining an approximate Nash equilibria $(\alpha^{\star,1,N},\dots,\alpha^{\star,N,N})$ can be difficult. As we have seen, this problem is related to the Mean Field Games, or more precisely to a weaker form of the solution of MFG (see [49], [103], or the measure–valued solution of MFG). Proving that there exists a solution of this form of MFG problem is equivalent to proving that there exists an approximate Nash equilibria.

For the MFG of controls or *extended* MFG, the literature on this topic is quite small and usually without common noise i.e. $\sigma_0 = 0$. Gomes and Voskanyan [72], by using PDE methods, study these types of interactions in the deterministic case i.e. $\sigma = \sigma_0 = 0$. Strong assumptions of continuity and convexity make it possible to obtain the existence and the regularity of the solutions. In order to explore a problem of optimal liquidation in finance, Cardaliaguet and Lehalle [37] apply similar PDE techniques for this problem in the case without common noise, while allowing σ non–zero. With the same philosophy, Kobeissi [94] provides some results and discusses properties of existence and uniqueness in examples. Let us also mention Achdou and Kobeissi [3] which gives numerical approximations via finite difference for the PDE system arising in the MFG of controls.

Probability techniques have also been used to give some results for the existence of solutions of the limit problem. Without common noise, using a weak formulation of the MFG of controls, Carmona and Lacker [45] obtains the existence and uniqueness of the MFG of controls by imposing an uncontrolled and non–degenerative volatility σ ($\sigma > 0$). They illustrate their results on the price impact models (which share some similarities with those considered in [37]) and the flocking model. Similarly, Graber [73] for the studies of models of production of an exhaustible resource, solves related existence and uniqueness problems. Alasseur, Taher, and Matoussi [9] also focused their attention on these questions for the study of a model for a power network with distributed local power generation and storage.

Main results With the help of the techniques used in the case of cooperative equilibria and the discretization procedure used in [49], with a separability condition on (b,σ,L) (see Assumption 5.5.5) and a non–degeneracy condition of type $\sigma\sigma^\top > 0$, we will provide an existence result for the notion of measure–valued solution of MFG that we used for the characterization of the approximate Nash equilibria and approximate strong solution of MFG. We want to emphasize the fact that this existence result also proves the existence of the approximate strong solution of the MFG (of controls). As mentioned in the previous literature, in the MFG theory, the existence of a strong MFG solution is very difficult to obtain and requires strong assumptions. Admitting a small error $\epsilon > 0$, it is possible to get the existence of an ϵ–strong MFG equilibrium (approximate solution) under general assumptions. It is worth emphasizing that our results allow to handle the case where σ is controlled i.e. the control α appears in the function σ. There are not many works that look at the situation where the volatility is controlled.

1.4 Numerical approximation

Literature and motivation The first chapters of this thesis focused on the mathematical analysis of the behavior of sequence of controlled interacting particles and the study of the associated limit problems. Based on this work, in the last chapter in this thesis, we will provide some numerical algorithms to solve the mean–field control problem.

The idea in this chapter is to provide a concrete procedure to be followed in order to solve the mean field control problem via a computer in practical example. The numerical procedure is based on the optimal control problem of the N–interacting controlled particles when N is fixed but large. There are two classical techniques used in the literature to solve numerically a stochastic control problem: PDE approach by finite difference and probabilistic approach by Monte Carlo.

The PDE approach by finite difference is motivated by the characterization of the solution of optimal control problem as the solution of the (parabolic) PDE. Then, the goal is to give a scheme to solve approximately this PDE. This will therefore allow to solve the problem of optimal control (see for instance Achdou and Pironneau [5]). This approach has

been used to propose some numerical methods for solving the mean–field control problem in Laurière and Pironneau [112], Achdou and Laurière [4], Pfeiffer [136]. While being very efficient in some cases, this approach has an important limitation related to the dimensions of the data of the problem. Indeed, it is well known that when the problem's dimensions (here $N \times n$ the dimension of particles $(\mathbf{X}^1, \dots, \mathbf{X}^N)$) increase, the efficiency of the finite difference scheme becomes decreases. Besides, if we want to allow the control of the volatility σ, this will considerably affect the efficiency. In the mean field control problem, from our viewpoint, the PDE approach can not be used. To be accurate, we need to choose N large, therefore the dimensions are very large, then the efficiency of such an algorithm is bad. Because of this kind of curse of dimensionality, we will use a probabilistic approach by Monte–Carlo.

The Monte–Carlo approach is straightforward, and uses the stochastic control problem without (or few) transformations. The idea is to keep the stochastic control problem but replace the expectation $\mathbb{E}$ by an empirical sample motivated by the classical law of large number (see Pagès [135]). This approach can be very efficient even if the dimensions are large. Balata, Huré, Laurière, Pham, and Pimentel [19], Fouque and Zhang [70], Carmona and Laurière [46] numerically solve the mean–field control problem via some Monte–Carlo techniques. However, even if usually the Monte Carlo method suffers less from the problems of dimension, in our case, as N is supposed to go to infinity, this can be a brake on the effectiveness of this approach, especially in the optimization part.

To bypass the difficulty related to the dimension even with our Monte Carlo approach, we will use neural networks. The past decades have shown the effectiveness of neural networks in the optimization problems involving large dimensions in many cases (see LeCun, Bottou, Bengio, and Haffner [114], Bengio [22], LeCun, Bengio, and Hinton [115]). The idea behind the use of the neural networks is to look for the optimum, a function most of the time, in a set of particular functions. These particular functions are compositions of affine functions composed with a non-linearity. The consideration of these functions is justified by the universal approximation theorem which states that any function can be approximated by these kind of functions with the *good* metric. The optimization procedure becomes then much more accessible with the help of a system of back propagation of gradients.

In this last chapter, in a spirit close to Han and E [78], Fouque and Zhang [70] and Carmona and Laurière [46], we will provide an algorithm using the Monte Carlo approach with the help of (deep) neural network.

Note that although the methods we will propose in this chapter solve the MFC problem, they can be used to numerically solve the MFG by using some equivalence representation result as in Carmona and Delarue [44, Chapter 6]. Some numerical methods has been already proposed to solve the MFG in the literature by Achdou and Capuzzo-Dolcetta [2], Achdou, Camilli, and Capuzzo-Dolcetta [6], Carlini and Silva [39], Chassagneux et al. [51].

Main results Inspired by the connection between the Pareto equilibria and the mean field control problem, we will give a numerical scheme for solving the optimal control of Mckean–Vlasov problem with common noise and including the law of control. It should be emphasized that we also allow the control of the volatility. We prove the convergence of this scheme under general conditions. With the help of the open–source neural–network library Keras (written in Python), we will implement our algorithm and compare the outputs to solutions of benchmark models obtained by analytical formulas.

Part I

McKean–Vlasov optimal control

We introduce here some additional notations and then formulate the assumptions that we will use in this part. Denote by $\mathbb{M}(E)$ the space of all Borel measures $q(\mathrm{d}t,\mathrm{d}e)$ on $[0,T]\times E$, whose marginal distribution on $[0,T]$ is the Lebesgue measure $\mathrm{d}t$, that is to say $q(\mathrm{d}t,\mathrm{d}e)=q(t,\mathrm{d}e)\mathrm{d}t$ for a family $(q(t,\mathrm{d}e))_{t\in[0,T]}$ of Borel probability measures on E. Let Λ denote the canonical element on $\mathbb{M}(E)$, we define

$$\Lambda^t(\mathrm{d}s,\mathrm{d}e):=\Lambda(\mathrm{d}s,\mathrm{d}e)\big|_{[0,t]\times E}+\delta_{e_0}(\mathrm{d}e)\mathrm{d}s\big|_{(t,T]\times E},\text{ for some fixed } e_0\in E. \tag{1.4.1}$$

This part share a certain number of functions which we now introduce. Let n, and d be two positive integers, and ℓ a non–negative one, which will be fixed throughout this part. The controlled diffusion processes have the following coefficient functions

$$(b,\sigma,\sigma_0):[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A\longrightarrow\mathbb{R}^n\times\mathbb{S}^{n\times d}\times\mathbb{S}^{n\times\ell},$$

and the reward value is defined with the coefficient functions

$$L:[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A\longrightarrow\mathbb{R},\text{ and } g:\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n)\longrightarrow\mathbb{R}.$$

We assume the following regularity and growth conditions on the coefficient functions.

Assumption 1.4.1. *The maps (b,σ,σ_0,L,g) are Borel measurable and non–anticipative, in the sense that*

$$\big(b,\sigma,\sigma_0,L\big)(t,\mathbf{x},\bar\nu,a)=\big(b,\sigma,\sigma_0,L\big)(t,\mathbf{x}(t\wedge\cdot),\bar\nu(t),a),\ \textit{for all } (t,\mathbf{x},\bar\nu,a)\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A.$$

Moreover, there exist positive constants C, p, p' and $\hat p$, such that $p'>p\geq 2\geq\hat p\geq 0$, and

(i) the function (b,σ,σ_0) is continuous in $(\mathbf{x},\bar\nu,a)$ and uniformly Lipschitz in $(\mathbf{x},\bar\nu)$, i.e. for all $(t,\mathbf{x},\bar\nu,a,\mathbf{x}',\bar\nu')\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)$

$$\big|(b,\sigma,\sigma_0)(t,\mathbf{x},\bar\nu,a)-(b,\sigma,\sigma_0)(t,\mathbf{x}',\bar\nu',a)\big|\leq C\big(\|\mathbf{x}-\mathbf{x}'\|+\mathcal{W}_p(\bar\nu,\bar\nu')\big);$$

(ii) for all $(t,\mathbf{x},\bar\nu,a)\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A$

$$\begin{aligned}|b(t,\mathbf{x},\bar\nu,a)|&\leq C\bigg(1+\|\mathbf{x}\|+\bigg(\int_{\mathcal{C}^n\times A}\big(\|\mathbf{x}'\|^p+\rho(a_0,a')^p\big)\bar\nu(\mathrm{d}\mathbf{x}',\mathrm{d}a')\bigg)^{\frac{1}{p}}+\rho(a_0,a)\bigg),\\ |(\sigma,\sigma_0)(t,\mathbf{x},\bar\nu,a)|^2&\leq C\bigg(1+\|\mathbf{x}\|^{\hat p}+\bigg(\int_{\mathcal{C}^n\times A}\big(\|\mathbf{x}'\|^p+\rho(a_0,a')^p\big)\bar\nu(\mathrm{d}\mathbf{x}',\mathrm{d}a')\bigg)^{\frac{\hat p}{p}}+\rho(a_0,a)^{\hat p}\bigg);\end{aligned}$$

(iii) the function g is lower semi–continuous, for every $t\in[0,T]$, the function L is lower semi–continuous in $(\mathbf{x},\bar\nu,a)$, and for an additional constant $C_L>0$, we have for all $(t,\mathbf{x},\bar\nu,\nu,a)\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times\mathcal{P}(\mathcal{C}^n)\times A$

$$\begin{aligned}|g(\mathbf{x},\nu)|&\leq C\bigg(1+\|\mathbf{x}\|^p+\int_{\mathcal{C}^n}\|\mathbf{x}'\|^p\nu(\mathrm{d}\mathbf{x}')\bigg),\\ L(t,\mathbf{x},\bar\nu,a)&\leq C\bigg(1+\|\mathbf{x}\|^p+\int_{\mathcal{C}^n\times A}\big(\|\mathbf{x}'\|^p+\rho(a_0,a')^p\big)\bar\nu(\mathrm{d}\mathbf{x}',\mathrm{d}a')\bigg)-C_L\rho(a_0,a)^{p'},\\ L(t,\mathbf{x},\bar\nu,a)&\geq -C\bigg(1+\|\mathbf{x}\|^p+\int_{\mathcal{C}^n\times A}\big(\|\mathbf{x}'\|^p+\rho(a_0,a')^p\big)\bar\nu(\mathrm{d}\mathbf{x}',\mathrm{d}a')\bigg).\end{aligned} \tag{1.4.2}$$

Remark 1.4.2. *Most of the integrability conditions in Assumption 1.4.1 are consistent with (or simply adapted from) those in Lacker [104, Assumption A]. Basically, they are here to ensure that everything remains sufficiently integrable to apply weak convergence techniques. In particular, (i) and (ii) are used to ensure the well–posedness of the controlled* SDEs*, while the coercivity condition in Item (iii) is used to ensure the (pre–)compactness of the set of all optimal relaxed control rules.*

Chapter 2

Strong and weak formulations

2.1 Introduction

The analysis of McKean–Vlasov optimal control problems has, in the recent years, drawn the attention of the applied mathematics community. One of the main reasons is their close proximity mean–field games (MFGs for short), introduced in the pioneering work of Lasry and Lions [109; 110; 111] and Huang, Caines, and Malhamé [83; 84; 85; 86; 87], as way to describe Nash equilibria for a large population of symmetric players, interacting through their empirical distribution. We will discuus this question in the second part of this thesis. However, we refer the interested readers to Carmona, Delarue, and Lachapelle [47] for a more thorough discussion about the similarities and differences between these two theories.

As we said in the introduction, an important step to characterize the limits of Pareto equilibria is to provide first a *good* weak formulation of the McKean–Vlasov control problem with common noise. In this chapter, our ultimate goal is to accomplish this task by introducing the strong and the weak formulations. The strong one is given in a fixed probability space equipped with two Brownian motions, as well as their natural filtrations. By considering more general probability spaces and filtrations, but imposing a technical (H)–hypothesis type condition, we obtain a weak formulation of the control problem. Our weak formulation is consistent with that of the classical optimal control problems, and enjoys some convexity and stability properties. More importantly, by considering them as probability measures on the canonical space, we show that any weak control rule can be approximated by strong control rules in the sense of weak convergence, which implies the equivalence between the strong and weak formulations. We emphasise that this first result is a crucial technical step in the proof of the DPP in Chapter 5.

The presence of the common noise generates some significant technical hurdles, especially due to the appearance of the conditional distribution terms, which are generally not continuous with respect to the joint distribution. In the context of MFG, this difficulty has been tackled by Carmona, Delarue, and Lacker [49], and Lacker [103]. In the context of McKean–Vlasov optimal control problem however, we need to formulate appropriate notions of weak control rules, and develop new techniques to ensure the approximation property. Another technical difficulty comes from the presence of the conditional law of the control process α in the coefficient functions (for the strong and weak formulations), a situation which has been rarely studied in the literature (see for instance Graber [73], Élie, Mastrolia, and Possamaï [65], Zalashko [154], Pham and Wei [139], Acciaio, Backhoff Veraguas, and Carmona [1], and Basei and Pham [20]). Our equivalence results between the strong and weak formulations is very general, and its proof is quite different from that in the case without common noise. It allows in particular to fill a subtle technical gap in the related literature (see Remark 2.3.14 for more details).

The rest of the chapter is structured as follows. We provide in Section 2.2 the notions of strong and weak formulations for the McKean–Vlasov stochastic control problem in a common noise and non–Markovian setting The main results of the chapter are presented in Theorem 2.2.3 including the equivalence between the strong and weak. Most of the technical proofs are completed in Section 2.3.

Throughout the chapter, we fix a nonempty Polish space (A, ρ) and an element $a_0 \in A$, and denote $\mathbb{M} := \mathbb{M}(A)$. Finally, consider the canonical space $\mathcal{C}^n \times \mathbb{M}$ (resp. $\mathcal{C}^n \times A$), with canonical element (X, Λ) (resp. (X, α)), and $\widehat{\nu} \in \mathcal{P}(\mathcal{C}^n \times \mathbb{M})$ (resp. $\bar{\nu} \in \mathcal{P}(\mathcal{C}^n \times A)$). We define, for each $t \in [0, T]$

$$\widehat{\nu}(t) := \widehat{\nu} \circ (X_{t\wedge\cdot}, \Lambda^t)^{-1}, \ \big(\text{resp. } \bar{\nu}(t) := \bar{\nu} \circ (X_{t\wedge\cdot}, \alpha)^{-1}\big). \tag{2.1.1}$$

2.2 McKean–Vlasov optimal control: different formulations

We introduce here a strong and a weak formulation of the McKean–Vlasov optimal control problem.

2.2.1 A strong formulation

To give a strong formulation of the McKean–Vlasov optimal control problem, we first introduce a fixed probability space equipped with an initial random variable X_0, and two independent Brownian motions W and B. Precisely, let us consider the canonical space

$$\Omega := \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell,$$

equipped with its Borel σ–algebra $\mathcal{F} := \mathcal{B}(\Omega)$ and canonical element (X_0, W, B). Let $\mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}$ and $\mathbb{G} = (\mathcal{G}_t)_{0\le t\le T}$ be two filtrations on $(\Omega, \mathcal{F})$ defined by

$$\mathcal{F}_t := \sigma\big((X_0, W_s, B_s) : s \in [0,t]\big), \text{ and } \mathcal{G}_t := \sigma\big(B_s : s \in [0,t]\big),\ t \in [0,T].$$

Let p be the constant in Assumption 1.4.1 and $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. We denote by $\mathbb{P}_\nu$ the probability measure on $(\Omega, \mathcal{F})$, under which $X_0 \sim \nu$ and (W, B) is a standard $\mathbb{R}^{d+\ell}$–dimensional Brownian motion, independent of X_0. Recall that a_0 is a fixed point in A. We denote by $\mathcal{A}_p(\nu)$ the collection of all $\mathbb{F}$–predictable, A–valued processes $\alpha = (\alpha_s)_{0\le s\le T}$ satisfying

$$\mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \big(\rho(\alpha_s, a_0)\big)^p \mathrm{d}s\Big] < \infty. \tag{2.2.1}$$

Then given a control process $\alpha \in \mathcal{A}_p(\nu)$, the controlled McKean–Vlasov SDE

$$X_t^\alpha = X_0 + \int_0^t b\big(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s\big)\mathrm{d}s + \int_0^t \sigma\big(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s\big)\mathrm{d}W_s + \int_0^t \sigma_0\big(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s\big)\mathrm{d}B_s,\ t \in [0,T],\ \mathbb{P}_\nu\text{–a.s.}, \tag{2.2.2}$$

with $\overline{\mu}^\alpha_s := \mathcal{L}^{\mathbb{P}_\nu}\big(X^\alpha_{s\wedge\cdot}, \alpha_s \big| \mathcal{G}_s\big)$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}_\nu$–a.e., has a unique strong solution, that is, there is a unique $\mathbb{F}$–adapted continuous process X^α on $(\Omega, \mathcal{F})$ satisfying Equation (2.2.2) and $\mathbb{E}^{\mathbb{P}_\nu}\big[\sup_{t\in[0,T]} |X_t^\alpha|^p\big] < \infty$ (see for instance Theorem 5.5.3).

Denote also $\mu_t^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(X^\alpha_{t\wedge\cdot} \big| \mathcal{G}_t\big)$ for all $t \in [0,T]$. The strong formulation of the McKean–Vlasov control problem is then given by

$$V_S(\nu) := \sup_{\alpha\in\mathcal{A}_p(\nu)} \mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T L\big(t, X^\alpha_{t\wedge\cdot}, \overline{\mu}^\alpha_t, \alpha_t\big)\mathrm{d}t + g\big(X^\alpha_{T\wedge\cdot}, \mu^\alpha_T\big)\Big]. \tag{2.2.3}$$

2.2.2 A weak formulation

As in the classical SDE theory, one can consider all possible probability spaces to define a weak solution of the controlled SDE (2.2.2).

Definition 2.2.1 (Weak control). *Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, we say that a term*

$$\gamma := \big(\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma, \mathbb{F}^\gamma := (\mathcal{F}^\gamma_t)_{0\le t\le T}, \mathbb{G}^\gamma := (\mathcal{G}^\gamma_t)_{0\le t\le T}, X^\gamma, W^\gamma, B^\gamma, \overline{\mu}^\gamma, \mu^\gamma, \alpha^\gamma\big),$$

is a weak control associated with the initial (distribution) condition ν if

(i) $(\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma)$ is a probability space, equipped with two filtrations $\mathbb{F}^\gamma$ and $\mathbb{G}^\gamma$ such that, for all $t \in [0,T]$

$$\mathcal{G}^\gamma_t \subseteq \mathcal{F}^\gamma_t, \text{ and } \mathbb{E}^{\mathbb{P}^\gamma}\big[1_D \big| \mathcal{G}^\gamma_t\big] = \mathbb{E}^{\mathbb{P}^\gamma}\big[1_D \big| \mathcal{G}^\gamma_T\big],\ \mathbb{P}^\gamma\text{–a.s., for all } D \in \mathcal{F}^\gamma_t \vee \sigma(W^\gamma); \tag{2.2.4}$$

(ii) $X^\gamma := (X^\gamma_s)_{s\in[0,T]}$ is an $\mathbb{R}^n$–valued $\mathbb{F}^\gamma$–adapted continuous process and $\alpha^\gamma := (\alpha^\gamma_s)_{0\le s\le T}$ is an A–valued $\mathbb{F}^\gamma$–predictable process such that $\mathbb{E}^{\mathbb{P}^\gamma}\big[\|X^\gamma\|^p + \int_0^T \big(\rho(\alpha^\gamma_s, a_0)\big)^p \mathrm{d}s\big] < \infty$;

(iii) (W^γ, B^γ) *is an* $\mathbb{R}^d \times \mathbb{R}^\ell$*–valued standard Brownian motion with respect to* $\mathbb{F}^\gamma$*,* B^γ *is in addition adapted to* $\mathbb{G}^\gamma$*,* $\mathcal{F}_0^\gamma \vee \sigma(W^\gamma)$ *is independent of* $\mathcal{G}_T^\gamma$*, and* μ^γ *(resp.* $\overline{\mu}^\gamma$*) is a* $\mathcal{P}(\mathcal{C}^n)$*–valued (resp.* $\mathcal{P}(\mathcal{C}^n \times A)$*–valued)* $\mathbb{G}^\gamma$*–predictable process such that*

$$\mu_t^\gamma = \mathcal{L}^{\mathbb{P}^\gamma}\big(X_{t\wedge\cdot}^\gamma \big| \mathcal{G}_t^\gamma\big), \text{ and } \overline{\mu}_t^\gamma = \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X_{t\wedge\cdot}^\gamma, \alpha_t^\gamma\big)\big| \mathcal{G}_t^\gamma\big), \ \mathrm{d}\mathbb{P}^\gamma \otimes \mathrm{d}t\text{–a.s.;}$$

(iv) X^γ *satisfies* $\mathbb{P}^\gamma \circ (X_0^\gamma)^{-1} = \nu$ *and*

$$X_t^\gamma = X_0^\gamma + \int_0^t b(s, X_{s\wedge\cdot}^\gamma, \overline{\mu}_s^\gamma, \alpha_s^\gamma)\mathrm{d}s + \int_0^t \sigma(s, X_{s\wedge\cdot}^\gamma, \overline{\mu}_s^\gamma, \alpha_s^\gamma)\mathrm{d}W_s^\gamma + \int_0^t \sigma_0(s, X_{s\wedge\cdot}^\gamma, \overline{\mu}_s^\gamma, \alpha_s^\gamma)\mathrm{d}B_s^\gamma, \ t \in [0,T], \ \mathbb{P}^\gamma\text{–a.s.}$$

Remark 2.2.2. *In* Definition 2.2.1*,* $\mathbb{G}^\gamma$ *plays the role of the common noise filtration, to which* B^γ *is adapted and from which* (X_0, W^γ) *is independent. In the literature on enlargement of filtrations (see* Jacod [88] *for instance), the* (H)*–hypothesis states that for all* $t \in [0,T]$

$$\mathbb{E}^{\mathbb{P}^\gamma}\big[1_D \big| \mathcal{G}_t^\gamma\big] = \mathbb{E}^{\mathbb{P}^\gamma}\big[1_D \big| \mathcal{G}_T^\gamma\big], \text{ for all } D \in \mathcal{F}_t^\gamma.$$

It is generally different from Condition (2.2.4)*, since the independence of the increment* $(W_s^\gamma - W_t^\gamma)_{s\in[t,T]}$ *from* $\mathcal{F}_t^\gamma$ *and* $\mathcal{G}_T^\gamma$ *does not imply the independence between* $(W_s^\gamma - W_t^\gamma)_{s\in[t,T]}$ *and* $\mathcal{F}_t^\gamma \vee \mathcal{G}_T^\gamma$*. In particular,* Condition (2.2.4) *will be reformulated later on as* (2.3.4) *and* (2.3.11)*, which are in turn crucially used in the approximation of a weak control by strong control rules in* Lemma 2.3.10 *and* Lemma 2.3.11*.*

Let us denote by $\Gamma_W(\nu)$ the collection of all weak controls associated with the initial condition ν, and introduce the weak formulation of the control problem by

$$V_W(\nu) := \sup_{\gamma\in\Gamma_W(\nu)} J(\gamma), \text{ with } J(\gamma) := \mathbb{E}^{\mathbb{P}^\gamma}\bigg[\int_0^T L(s, X_{s\wedge\cdot}^\gamma, \overline{\mu}_s^\gamma, \alpha_s^\gamma)\mathrm{d}s + g(X_{T\wedge\cdot}^\gamma, \mu_T^\gamma)\bigg]. \tag{2.2.5}$$

2.2.3 Equivalence of formulations

Let us now provide the main results of this chapter i.e. the equivalence between the strong and different formulations of the McKean–Vlasov control problem.

For γ a weak control, we introduce

$$Y^\gamma := X_\cdot^\gamma - \int_0^\cdot \sigma_0(s, X_{s\wedge\cdot}^\gamma, \overline{\mu}_s^\gamma, \alpha_s^\gamma)\mathrm{d}B_s^\gamma \text{ and } \widehat{\mu}^\gamma := \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma, Y^\gamma, \delta_{\alpha_s^\gamma}(\mathrm{d}a)\mathrm{d}s, W^\gamma\big)\big| \mathcal{G}_T^\gamma\big),$$

for $\alpha \in \mathcal{A}_p(\nu)$, with X^α solution of Equation (2.2.2), we define

$$Y_\cdot^\alpha = X_\cdot^\alpha - \int_0^\cdot \sigma_0\big(s, X_{s\wedge\cdot}^\alpha, \overline{\mu}_s^\alpha, \alpha_s\big)\mathrm{d}B_s \text{ and } \widehat{\mu}^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(\big(X^\alpha, Y^\alpha, \delta_{\alpha_s}(\mathrm{d}a)\mathrm{d}s, W\big)\big| \mathcal{G}_T\big),$$

and consider the following subset of probability measure on $\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathbb{M} \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathbb{M})$

$$\overline{\mathcal{P}}_W(\nu) := \big\{\mathbb{P}^\gamma \circ \big(X^\gamma, Y^\gamma, \delta_{\alpha_s^\gamma}(\mathrm{d}a)\mathrm{d}s, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1} : \gamma \text{ a weak control with initial condition } \nu\big\} \tag{2.2.6}$$

and

$$\overline{\mathcal{P}}_S(\nu) := \big\{\mathbb{P}_\nu \circ \big(X^\alpha, Y^\alpha, \delta_{\alpha_s}(\mathrm{d}a)\mathrm{d}s, W, B, \widehat{\mu}^\alpha\big)^{-1} : \alpha \in \mathcal{A}_p(\nu)\big\}. \tag{2.2.7}$$

Theorem 2.2.3. *Let* Assumption 1.4.1 *hold true and* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$*. (i) The set* $\overline{\mathcal{P}}_W(\nu)$ *is non–empty and convex.*

(ii) We have

$$V_S(\nu) = V_W(\nu).$$

If in addition $\ell \neq 0$*, then every weak control rule in* $\overline{\mathcal{P}}_W(\nu)$ *is the limit of a sequence of strong control rules in* $\overline{\mathcal{P}}_S(\nu)$*, under the Wasserstein distance* $\mathcal{W}_p$ *on* $\mathcal{P}_p(\overline{\Omega})$*.*

Remark 2.2.4. *When $\ell = 0$, or $\ell \neq 0$ and $\sigma_0 = 0$, the (strong formulation of the) McKean–Vlasov control problem (2.2.3) or (2.3.6), reduces to the non–common noise context. However, in the weak formulation (2.3.7), the (conditional) distribution term $\widehat{\mu}^\gamma$ may still be random under a weak control rule $\gamma \in \Gamma_W(\nu)$. In the case $\ell \neq 0$ and $\sigma_0 = 0$, the Brownian motion B can be seen as an external noise in (2.3.6), which allows to track the randomness of $\widehat{\mu}^\gamma$ and approximate a weak control rule by strong control rules. This is also the main reason why we consider the case $\ell \neq 0$ separately in* Theorem 2.2.3.*(ii)*.

Remark 2.2.5. *The results in* Theorem 2.2.3 *extend those in the no–common noise setting in* Lacker [104]. *Nevertheless, we insist on the fact that the equivalence results, the formulation of the strong and weak control rules, and the technical proofs below are not merely extensions of those in* [104], *and are in fact quite different. The main reason is that with the presence of the common noise, the $\overline{\mu}^\alpha$ term in (2.2.2)–(2.2.3) is a conditional distribution term, which, in general, is not continuous with respect to the joint distribution of $(X^\alpha, \alpha, W^\alpha, B^\alpha)$. Moreover, the equivalence result $V_S = V_W$ is also crucially used to establish the dynamic programming principle in Theorem 5.3.4.*

2.3 Proof of equivalence between strong and weak formulations

2.3.1 Strong and weak formulations on the canonical space

The above strong and weak control problem can be reformulated on a canonical space, by considering an appropriate martingale problem.

2.3.1.1 The canonical space and admissible control rules

Recall that A is a fixed nonempty Polish space, $\mathbb{M} := \mathbb{M}(A)$ denotes the space of all positive Borel measures q on $[0,T] \times A$ such that the marginal distribution of q on $[0,T]$ is the Lebesgue measure, implying that we can always write $q(\mathrm{d}t, \mathrm{d}a) = q_t(\mathrm{d}a)\mathrm{d}t$, where $(q_t(\mathrm{d}a))_{t\in[0,T]}$ is a Borel measurable kernel from $[0,T]$ to $\mathcal{P}(A)$. We also introduce a subset $\mathbb{M}_0 \subset \mathbb{M}$, which is the collection of all $q \in \mathbb{M}$ such that $q(\mathrm{d}t, \mathrm{d}a) = \delta_{\psi(t)}(\mathrm{d}a)\mathrm{d}t$ for some Borel measurable function $\psi : [0,T] \longrightarrow A$. We will consider two canonical spaces

$$\widehat{\Omega} := \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d, \text{ and } \overline{\Omega} := \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathcal{P}\big(\widehat{\Omega}\big).$$

The canonical space $\widehat{\Omega}$ is equipped with the corresponding canonical element $\big(\widehat{X}, \widehat{Y}, \widehat{\Lambda}, \widehat{W}\big)$, its Borel σ–algebra $\widehat{\mathcal{F}} := \mathcal{B}(\widehat{\Omega})$, and its canonical filtration $\widehat{\mathbb{F}} := \big(\widehat{\mathcal{F}}_t\big)_{t\in[0,T]}$ defined by

$$\widehat{\mathcal{F}}_t := \sigma\Big(\big(\widehat{X}_s, \widehat{Y}_s, \widehat{\Lambda}([0,s] \times D), \widehat{W}_s\big) : D \in \mathcal{B}(A),\ s \in [0,t]\Big),\ t \in [0,T].$$

Notice that one can choose a version of the disintegration $\widehat{\Lambda}(\mathrm{d}t, \mathrm{d}a) = \widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t$ such that $(\widehat{\Lambda}_t)_{t\in[0,T]}$ is a $\mathcal{P}(A)$–valued, $\widehat{\mathbb{F}}$–predictable process (see e.g. [102, Lemma 3.2.]).

Similarly, we equip the canonical space $\overline{\Omega}$ with the canonical element $(X, Y, \Lambda, W, B, \widehat{\mu})$, and its Borel σ–algebra $\overline{\mathcal{F}} := \mathcal{B}(\overline{\Omega})$. Moreover, based on $\widehat{\mu}$, let us define three processes $(\mu_t)_{t\in[0,T]}$, $(\overline{\mu}_t)_{t\in[0,T]}$ and $(\widehat{\mu}_t)_{t\in[0,T]}$ on $\overline{\Omega}$ by (recall (3.4.7) for the definition of $\widehat{\Lambda}^t$)

$$\mu_t := \widehat{\mu} \circ \big(\widehat{X}_{t\wedge\cdot}\big)^{-1},\ \overline{\mu}_t(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \mathbb{E}^{\widehat{\mu}}\Big[\delta_{\widehat{X}_{t\wedge\cdot}}(\mathrm{d}\mathbf{x})\widehat{\Lambda}_t(\mathrm{d}a)\Big],\ \text{and } \widehat{\mu}_t := \widehat{\mu} \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{Y}_{t\wedge\cdot}, \widehat{\Lambda}^t, \widehat{W}\big)^{-1},\ t \in [0,T]. \tag{2.3.1}$$

We then introduce two filtrations $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{t\in[0,T]}$ and $\overline{\mathbb{G}} := (\overline{\mathcal{G}}_t)_{t\in[0,T]}$ on $(\overline{\Omega}, \overline{\mathcal{F}})$ by

$$\overline{\mathcal{F}}_t := \sigma\Big((X_s, Y_s, \Lambda([0,s] \times D), W_s, B_s, \langle\widehat{\mu}_s, \phi\rangle) : D \in \mathcal{B}(A), \phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d),\ s \in [0,t]\Big).$$

and

$$\overline{\mathcal{G}}_t := \sigma\Big((B_s, \langle\widehat{\mu}_s, \phi\rangle) : \phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d), s \in [0,t]\Big).$$

To interpret the strong or weak controls as probability measures on the canonical space $\overline{\Omega}$, we will consider a controlled martingale problem. Let us define the maps $\bar{b} : [0,T] \times \mathcal{C}^n \times A \times \mathcal{P}(\mathcal{C}^n \times A) \longrightarrow \mathbb{R}^{n+n+d+\ell}$, and $\bar{a} : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A \longrightarrow \mathbb{S}^{n+n+d+\ell}$, such that for any $(t, \mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{b}, \bar{\nu}, a) \in [0,T] \times \mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times A) \times A$

$$\bar{b}\big(t, \mathbf{x}, \mathbf{w}, \mathbf{b}, \bar{\nu}, a\big) := \begin{pmatrix} b(t, \mathbf{x}, \bar{\nu}, a) \\ b(t, \mathbf{x}, \bar{\nu}, a) \\ 0_d \\ 0_\ell \end{pmatrix}, \; \bar{a}\big(t, \mathbf{x}, \mathbf{w}, \mathbf{b}, \bar{\nu}, a\big) := \begin{pmatrix} \sigma(t, \mathbf{x}, \bar{\nu}, a) & \sigma_0(t, \mathbf{x}, \bar{\nu}, a) \\ \sigma(t, \mathbf{x}, \bar{\nu}, a) & 0_{n\times\ell} \\ \mathrm{I}_{d\times d} & 0_{d\times\ell} \\ 0_{\ell\times d} & \mathrm{I}_{\ell\times\ell} \end{pmatrix} \begin{pmatrix} \sigma(t, \mathbf{x}, \bar{\nu}, a) & \sigma_0(t, \mathbf{x}, \bar{\nu}, a) \\ \sigma(t, \mathbf{x}, \bar{\nu}, a) & 0_{n\times\ell} \\ \mathrm{I}_{d\times d} & 0_{d\times\ell} \\ 0_{\ell\times d} & \mathrm{I}_{\ell\times\ell} \end{pmatrix}^\top.$$

Next, for all $t \in [0,T]$ and $\varphi \in C_b^2(\mathbb{R}^{n+n+d+\ell})$, we define the generator $\overline{\mathcal{L}}_t$ by

$$\overline{\mathcal{L}}_t\varphi\big(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{b}, \bar{\nu}, a\big) := \bar{b}(t, \mathbf{x}, \bar{\nu}, a) \cdot \nabla\varphi(\mathbf{x}(t), \mathbf{y}(t), \mathbf{w}(t), \mathbf{b}(t)) + \frac{1}{2}\mathrm{Tr}\big[\bar{a}(t, \mathbf{x}, \bar{\nu}, a)\nabla^2\varphi(\mathbf{x}(t), \mathbf{y}(t), \mathbf{w}(t), \mathbf{b}(t))\big]. \tag{2.3.2}$$

This allows to define, for any $\varphi \in C_b^2(\mathbb{R}^{n+n+d+\ell})$, $\overline{S}^\varphi := (\overline{S}_t^\varphi)_{t\in[0,T]}$ on $\overline{\Omega}$ by

$$\overline{S}_t^\varphi := \varphi(X_t, Y_t, W_t, B_t) - \iint_{[0,t]\times A} \overline{\mathcal{L}}_s\varphi\big(X_s, Y_s, W_s, B_s, \overline{\mu}_s, a\big)\Lambda_s(\mathrm{d}a)\mathrm{d}s, \; t \in [0,T], \tag{2.3.3}$$

where for a borel function $\phi : [0,T] \to \mathbb{R}$, $\int_0^\cdot \phi(s)\mathrm{d}s := \int_0^\cdot \phi^+(s)\mathrm{d}s - \int_0^\cdot \phi^-(s)\mathrm{d}s$ with the convention $\infty - \infty = -\infty$.

Definition 2.3.1. *Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. A probability $\overline{\mathbb{P}}$ on $(\overline{\Omega}, \overline{\mathcal{F}})$ is an admissible control rule with initial condition ν if*

(i) $\overline{\mathbb{P}}\big[X_0 = Y_0, \; W_0 = 0, \; B_0 = 0\big] = 1$, $\overline{\mathbb{P}} \circ X_0^{-1} = \nu$, *and* (X, Λ) *satisfy* $\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X\|^p + \iint_{[0,T]\times A}\big(\rho(a_0, a)\big)^p\Lambda_t(\mathrm{d}a)\mathrm{d}t\big] < \infty$;

(ii) the pair (X_0, W) is independent of $\overline{\mathcal{G}}_T$ under $\overline{\mathbb{P}}$, and for all $t \in [0,T]$

$$\widehat{\mu}_t(\bar{\omega}) = \overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_T} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W)^{-1}, \text{ for } \overline{\mathbb{P}}\text{–a.e. } \bar{\omega} \in \overline{\Omega}; \tag{2.3.4}$$

(iii) the process $\big(\overline{S}_t^\varphi\big)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–martingale for all $\varphi \in C_b^2\big(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell\big)$.

Let us then define for any $\nu \in \mathcal{P}_p(\mathbb{R}^n)$,

$$\overline{\mathcal{P}}_A(\nu) := \big\{\text{All admissible control rules } \overline{\mathbb{P}} \text{ with initial condition } \nu\big\}.$$

Remark 2.3.2. *(i) Under an admissible control rule $\overline{\mathbb{P}}$, B and W are standard Brownian motions, Λ is the $\mathcal{P}(A)$–valued process induced by the control process, X is the controlled process, and $\overline{\mu}$ is the conditional distribution of the control and controlled process. The process Y will only be really used to introduce the relaxed formulation (see* Chapter 3 *). In particular, when $\sigma_0 = 0$ or $\ell = 0$, we have $Y = X$.*

(ii) Notice that $\widehat{\mu}_t$ is $\overline{\mathcal{G}}_t$–measurable, hence, it follows that (2.3.4) is equivalent to

$$\widehat{\mu}_t(\bar{\omega}) = \overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_t} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W)^{-1} = \overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_T} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W)^{-1}, \text{ for } \overline{\mathbb{P}}\text{–a.e. } \bar{\omega} \in \overline{\Omega}. \tag{2.3.5}$$

Proposition 2.3.3. *Let $\nu \in \mathcal{P}_p(\mathcal{C}^n)$ and $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$. Then for $\overline{\mathbb{P}}$–almost every $\bar{\omega} \in \overline{\Omega}$, $\widehat{W}$ is an $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$–Brownian motion.*

Proof. Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$, $0 \le s \le t \le T$, $\phi \in C_b(\mathbb{R}^d)$, $\varphi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)$ and $\psi \in C_b(\mathcal{C}^\ell \times C([0,T]; \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d))$. Notice that W is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–Brownian motion, independent of $\overline{\mathcal{G}}_T$ under $\overline{\mathbb{P}}$. Therefore, it follows that

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(W_t - W_s)\varphi(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot})\psi(B_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot})\big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(W_t - W_s)\big]\mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}}\big[\varphi(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot})\big|\overline{\mathcal{G}}_s\big]\psi(B_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot})\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(W_t - W_s)\big|\overline{\mathcal{G}}_s\big]\mathbb{E}^{\overline{\mathbb{P}}}\big[\varphi(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot})\big|\overline{\mathcal{G}}_s\big]\psi(B_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot})\Big].
\end{aligned}$$

This implies that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(W_t - W_s)\varphi(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot})\big|\overline{\mathcal{G}}_s\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(W_t - W_s)\big|\overline{\mathcal{G}}_s\big]\mathbb{E}^{\overline{\mathbb{P}}}\big[\varphi(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot})\big|\overline{\mathcal{G}}_s\big],\ \overline{\mathbb{P}}\text{–a.s.}$$

By (2.3.4) in Definition 2.3.1, it follows that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$

$$\mathbb{E}^{\hat{\mu}(\bar{\omega})}\big[\phi(\widehat{W}_t - \widehat{W}_s)\varphi(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{\Lambda}^s, \widehat{W}_{s\wedge\cdot})\big] = \mathbb{E}^{\hat{\mu}(\bar{\omega})}\big[\phi(\widehat{W}_t - \widehat{W}_s)\big]\ \mathbb{E}^{\hat{\mu}(\bar{\omega})}\big[\varphi(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{\Lambda}^s, \widehat{W}_{s\wedge\cdot})\big].$$

In other words, $\widehat{W}$ has independent increments with respect to $\widehat{\mathbb{F}}$ under $\hat{\mu}(\bar{\omega})$, for $\overline{\mathbb{P}}$–almost every $\bar{\omega} \in \overline{\Omega}$.

Further, notice that under $\overline{\mathbb{P}}$, W is a Brownian motion independent of $(B, \hat{\mu})$, then W is still a Brownian motion under the conditional law of $\overline{\mathbb{P}}$ knowing $\mathcal{G}_T$. It follows that the continuous process $\widehat{W}$ has independent and (Gaussian) stationary increment w.r.t. $(\widehat{\mathbb{F}}, \hat{\mu}(\bar{\omega}))$, and hence it is an $(\widehat{\mathbb{F}}, \hat{\mu}(\bar{\omega}))$–Brownian motion, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$. □

2.3.1.2 The strong formulation on the canonical space

To reformulate the strong formulation (2.2.3) of the control problem on the canonical space $\overline{\Omega}$, it is enough to consider the class of measures induced by the controls and the controlled processes on the canonical space. Recall that for each $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, $\mathbb{P}_\nu$ is defined in Section 2.2.1 as a probability measure on $(\Omega, \mathcal{F})$, and that for any $\alpha \in \mathcal{A}_p(\nu)$, the controlled McKean–Vlasov SDE (2.2.2) has a unique strong solution X^α. Let us recall

$$Y_t^\alpha := X_t^\alpha - \int_0^t \sigma_0(s, X_{s\wedge\cdot}^\alpha, \overline{\mu}_s^\alpha, \alpha_s)\mathrm{d}B_s,\ t \in [0,T],\ \Lambda_t^\alpha(\mathrm{d}a)\mathrm{d}t := \delta_{\alpha_t}(\mathrm{d}a)\mathrm{d}t,\ \ \widehat{\mu}^\alpha := (\mathbb{P}_\nu)^{\overline{\mathcal{G}}_T}\big(X^\alpha, Y^\alpha, \Lambda^\alpha, W\big)^{-1},$$

and

$$\overline{\mathcal{P}}_S(\nu) := \big\{\mathbb{P}_\nu \circ \big(X^\alpha, Y^\alpha, \Lambda^\alpha, W, B, \widehat{\mu}^\alpha\big)^{-1} : \alpha \in \mathcal{A}_p(\nu)\big\}.$$

It is straightforward to see that

$$V_S(\nu) = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S(\nu)} J\big(\overline{\mathbb{P}}\big),\ \text{with } J\big(\overline{\mathbb{P}}\big) := \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\iint_{[0,T]\times A} L\big(t, X_{t\wedge\cdot}, \overline{\mu}_t, a\big)\Lambda_t(\mathrm{d}a)\mathrm{d}t + g\big(X_{T\wedge\cdot}, \mu_T\big)\bigg]. \tag{2.3.6}$$

Let

$$\mathcal{L}_0[A] := \big\{\text{All Borel measurable functions } \phi : [0,T]\times\mathbb{R}^n\times\mathcal{C}^d\times\mathcal{C}^\ell \longrightarrow A\big\}.$$

The following Proposition is the analog of Lemma 5.4.4 (see also Definition 5.4.3) in a different canonical space.

Proposition 2.3.4. *We have, for all $\nu \in \mathcal{P}_p(\mathbb{R}^n)$*

$$\overline{\mathcal{P}}_S(\nu) = \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu) : \exists\ \phi \in \mathcal{L}_0[A],\ \overline{\mathbb{P}}\big[\Lambda_t(\mathrm{d}a)\mathrm{d}t = \delta_{\phi(t, X_0, W_{t\wedge\cdot}, B_{t\wedge\cdot})}(\mathrm{d}a)\mathrm{d}t\big] = 1\Big\}.$$

Remark 2.3.5. *Notice that the map $\mathcal{P}(\mathcal{C}^n\times\mathcal{C}^n\times\mathcal{C}^d\times\mathbb{M}) \ni \widehat{\mu} \longmapsto \delta_{\overline{\mu}_t}(\mathrm{d}\bar{\nu})\mathrm{d}t \in \mathbb{M}(\mathcal{C}^n\times A)$ is generically not continuous. Consequently, $\overline{\mathbb{P}} \longmapsto J(\overline{\mathbb{P}})$ is not continuous in general, even if L and g are both bounded and continuous.*

2.3.1.3 The weak formulation on the canonical space

Now we also represent the set of weak control rules as a subset of $\overline{\mathcal{P}}_A$. Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and $\gamma \in \Gamma_W(\nu)$. Recall that for any $t \in [0,T]$

$$Y_t^\gamma := X_t^\gamma - \int_0^t \sigma_0(s, X_{s\wedge\cdot}^\gamma, \overline{\mu}_s^\gamma, \alpha_s^\gamma)\mathrm{d}B_s^\gamma,\ \Lambda_t^\gamma(\mathrm{d}a)\mathrm{d}t := \delta_{\alpha_t^\gamma}(\mathrm{d}a)\mathrm{d}t,\ \text{and } \widehat{\mu}^\gamma := \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma, Y^\gamma, \Lambda^\gamma, W^\gamma\big)\big|\mathcal{G}_T^\gamma\big).$$

Proposition 2.3.6. *Then with J defined in (2.3.6), we have*

$$\big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu) : \overline{\mathbb{P}}\big[\Lambda \in \mathbb{M}_0\big] = 1\big\} = \big\{\mathbb{P}^\gamma \circ \big(X^\gamma, Y^\gamma, \Lambda^\gamma, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1} : \gamma \in \Gamma_W(\nu)\big\},\ \text{and } V_W(\nu) = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)} J\big(\overline{\mathbb{P}}\big). \tag{2.3.7}$$

Proof. With a slight extension of Lemma 5.4.4 (taking into account the process Y and the small changes in the presentation of the definition of weak controls $\Gamma_W(\nu)$), every weak control rule $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$, together with the canonical space $\overline{\Omega}$ and canonical processes, can be viewed as a weak control $\gamma \in \Gamma_W(\nu)$. Conversely, every weak control γ induces a weak control rule $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$ on the canonical space. It follows that (2.3.7) holds true (see also Corollary 5.4.6). □

Remark 2.3.7. *By* Proposition 2.3.4*, it is straightforward to see that for all* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$

$$\overline{\mathcal{P}}_S(\nu) = \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu) : \exists\, \phi \in \mathcal{L}_0[A],\ \overline{\mathbb{P}}\big[\Lambda_t(\mathrm{d}a)\mathrm{d}t = \delta_{\phi(t,X_0,W_{t\wedge\cdot},B_{t\wedge\cdot})}(\mathrm{d}a)\mathrm{d}t\big] = 1\big\}.$$

In particular, as expected, any strong control rule is also a weak control rule, i.e. $\overline{\mathcal{P}}_S(\nu) \subset \overline{\mathcal{P}}_W(\nu)$.

2.3.2 Approximating weak control rules by strong control rules

This part is devoted to the approximation of weak control rules by strong controls. Indeed, to prove Theorem 2.2.3, the crucial steps consist in first approximating weak control rules by strong control rules.

We first provide a moment estimate of the solution to the controlled SDEs, which will be repeatedly used in the upcoming proofs. This is in fact an easy extension of Lacker [104, Lemmata 3.1. and 3.3.] (which are a succession of application of Gronwall Lemma), then for brevity we omit the proof.

Lemma 2.3.8. *Let* Assumption 1.4.1 *hold true, and* $q \geq p$*. Then there exists a constant* $K > 0$ *such that, for each* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$*, we have*

$$\mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{t\in[0,T]}|X_t|^q\Big] + \mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{t\in[0,T]}|Y_t|^q\Big] \leq K\bigg(1 + \int_{\mathbb{R}^n}|x'|^q\nu(\mathrm{d}x') + \mathbb{E}^{\overline{\mathbb{P}}}\Big[\iint_{[0,T]\times A}\rho(a_0,a)^q\Lambda_t(\mathrm{d}a)\mathrm{d}t\Big]\bigg).$$

Remark 2.3.9. Notice that by a classical existence result (see Theorem 5.5.3 for the McKean-Vlasov case), we know that: under Assumption 1.4.1, for each $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$, $\mathbb{E}^{\overline{\mathbb{P}}}\big[\sup_{t\in[0,T]}|X_t|^q\big] + \mathbb{E}^{\overline{\mathbb{P}}}\big[\sup_{t\in[0,T]}|Y_t|^q\big] < \infty$. Lemma 2.3.8 essentially gives precise estimations of these quantities.

Let Assumption 1.4.1 hold true, $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$. From the martingale problem in Definition 2.3.1 and by using Stroock and Varadhan [150, Theorem 4.5.2], on the filtered probability space $(\overline{\Omega}, \overline{\mathbb{F}}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, (W, B) are standard Brownian motions, (W, X_0) are independent of $(B, \widehat{\mu})$, and there exists a $\overline{\mathbb{F}}$–predictable A–valued process $(\alpha_t)_{t\in[0,T]}$, such that, $\overline{\mathbb{P}}$–a.s.,

$$\begin{aligned}
X_t &= X_0 + \int_0^t b\big(r, X, \overline{\mu}_r, \alpha_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, X, \overline{\mu}_r, \alpha_r\big)\mathrm{d}W_r + \int_0^t \sigma_0\big(r, X, \overline{\mu}_r, \alpha_r\big)\mathrm{d}B_r,\ t \in [0,T],\\
Y_t &= X_t - \int_0^t \sigma_0\big(r, X, \overline{\mu}_r, \alpha_r\big)\mathrm{d}B_r,\ t \in [0,T],
\end{aligned}$$

with $\Lambda_t(\mathrm{d}a)\mathrm{d}t = \Lambda_t^\alpha(\mathrm{d}a)\mathrm{d}t := \delta_{\alpha_t}(\mathrm{d}a)\mathrm{d}t$ and

$$\widehat{\mu}_t = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big|B_{t\wedge\cdot}, \widehat{\mu}_{t\wedge\cdot}\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big|B, \widehat{\mu}\big),\ \overline{\mu}_t(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \mathbb{E}^{\hat{\mu}}\big[\delta_{\widehat{X}_{t\wedge\cdot}}(\mathrm{d}\mathbf{x})\widehat{\Lambda}_t(\mathrm{d}a)\big]. \tag{2.3.8}$$

Let us take a sequence $\big((t_i^m)_{0\leq i\leq m}\big)_{m\geq 1}$ of partitions of $[0,T]$, with $0 = t_0^m < t_1^m < \cdots < t_m^m = T$, and such that

$$\sup_{0\leq i\leq m-1}|t_{i+1}^m - t_i^m| \underset{m\to\infty}{\longrightarrow} 0.$$

For any integer $m \geq 1$, define for simplicity the map $[0,T] \ni t \longmapsto [t]^m := \sum_{i=0}^{m-1} t_i^m \mathbf{1}_{[t_i^m, t_{i+1}^m)}(t)$, as well as $\varepsilon_m := t_1^m$. Let $W_\cdot^m := W_{\varepsilon_m\vee\cdot} - W_{\varepsilon_m}$ and $B_\cdot^m := B_{\varepsilon_m\vee\cdot} - B_{\varepsilon_m}$, we define also two filtrations $\overline{\mathbb{F}}^m := (\overline{\mathcal{F}}_t^m)_{t\in[0,T]}$ and $\overline{\mathbb{G}}^m = (\overline{\mathcal{G}}_t^m)_{t\in[0,T]}$ by

$$\overline{\mathcal{F}}_t^m := \sigma\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W_{t\wedge\cdot}^m, B_{t\wedge\cdot}^m, \widehat{\mu}_{t\wedge\cdot}\big),\ \text{and}\ \overline{\mathcal{G}}_t^m := \sigma\big(B_{t\wedge\cdot}^m, \widehat{\mu}_{t\wedge\cdot}\big),\ t \in [0,T].$$

Lemma 2.3.10 (Approximation with piecewise constant controls). *In the filtered probability space $(\overline{\Omega}, \overline{\mathbb{F}}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, there exists a sequence of $\overline{\mathbb{F}}$–predictable processes $(\alpha^m)_{m\geq 1}$, and a sequence a $\overline{\mathbb{F}}$–adapted continuous processes $(X^m)_{m\geq 1}$ such that for any $m \geq 1$*

$$\alpha_0^m = a_0,\ \alpha_t^m = \alpha_{\lfloor t \rfloor_m}^m,\ \text{on } [0,T],\ \lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^T \rho(\alpha_t, \alpha_t^m)^p \mathrm{d}t\Big] = 0,\ \text{and}\ \lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{s\in[0,T]} |X_s - X_s^m|^p\Big] = 0, \tag{2.3.9}$$

where for each $m \geq 1$, $(X_t^m)_{t\in[0,T]}$ is the unique strong solution of

$$X_t^m = X_0 + \int_{\varepsilon_m}^{t\vee\varepsilon_m} b\big(r, X_{r\wedge\cdot}^m, \overline{\mu}_r^m, \alpha_r^m\big)\mathrm{d}r + \int_{\varepsilon_m}^{t\vee\varepsilon_m} \sigma\big(r, X_{r\wedge\cdot}^m, \overline{\mu}_r^m, \alpha_r^m\big)\mathrm{d}W_r^m + \int_{\varepsilon_m}^{t\vee\varepsilon_m} \sigma_0\big(r, X_{r\wedge\cdot}^m, \overline{\mu}_r^m, \alpha_r^m\big)\mathrm{d}B_r^m, \tag{2.3.10}$$

with $\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X^m\|^p\big] < \infty$ and $\overline{\mu}_t^m := \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, \alpha_t^m \big| \overline{\mathcal{G}}_t^m\big)$. Moreover, if we denote $\Lambda_t^m(\mathrm{d}a)\mathrm{d}t := \delta_{\alpha_t^m}(\mathrm{d}a)\mathrm{d}t$, as well as

$$\widehat{\mu}_t^m := \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, Y_{t\wedge\cdot}^m, (\Lambda^m)^t, W^m \big| \overline{\mathcal{G}}_t^m\big) \text{ and } Y_t^m := X_t^m - \int_{\varepsilon_m}^{t\vee\varepsilon_m} \sigma_0\big(r, X_{r\wedge\cdot}^m, \overline{\mu}_r^m, \alpha_r^m\big)\mathrm{d}B_r^m,\ \text{for all } t\in[0,T],$$

then (X_0, W^m) is $\overline{\mathbb{P}}$–independent of $(B^m, \widehat{\mu}^m)$, $\widehat{\mu}_t^m = \widehat{\mu}_T^m \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{Y}_{t\wedge\cdot}, \widehat{\Lambda}^t, \widehat{W}\big)^{-1}$, and

$$\widehat{\mu}_t^m = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, Y_{t\wedge\cdot}^m, (\Lambda^m)^t, W^m \big| B_{t\wedge\cdot}^m, \widehat{\mu}_{t\wedge\cdot}^m\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, Y_{t\wedge\cdot}^m, (\Lambda^m)^t, W^m \big| B^m, \widehat{\mu}^m\big),\ \overline{\mathbb{P}}\text{–a.s., for all } t\in[0,T]. \tag{2.3.11}$$

Proof. First, we claim that for each $m \geq 1$,

$$\widehat{\mu}_t = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W \big| B_{t\wedge\cdot}^m, \widehat{\mu}_{t\wedge\cdot}\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W \big| B^m, \widehat{\mu}\big),\ \overline{\mathbb{P}}\text{–a.s., for all } t\in[0,T]. \tag{2.3.12}$$

Indeed, for all $\phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)$ and $\psi \in C_b(\mathcal{C}^\ell \times C([0,T], \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)))$, it follows by (2.3.4) that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\langle\phi, \widehat{\mu}_t\rangle\psi(B^m, \widehat{\mu})\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W)\psi(B^m, \widehat{\mu})\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\big\langle\phi, \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W | B^m, \widehat{\mu}\big)\big\rangle\psi(B^m, \widehat{\mu})\big].$$

This implies (2.3.12) by arbitrariness of (ϕ, ψ). We further observe that $(\overline{\mathbb{F}}^m, \overline{\mathbb{G}}^m)$ satisfies

$$\mathbb{E}^{\overline{\mathbb{P}}}[1_D | \overline{\mathcal{G}}_t^m] = \mathbb{E}^{\overline{\mathbb{P}}}[1_D | \overline{\mathcal{G}}_T^m],\ \text{for all } D \in \overline{\mathcal{F}}_t^m \vee \sigma(W^m) \text{ and } t\in[0,T]. \tag{2.3.13}$$

Next, as $\mathbb{E}^{\overline{\mathbb{P}}}\big[\int_0^T \rho(\alpha_t, a_0)^p \mathrm{d}t\big] < \infty$, it follows (this is a straightforward extension of for instance Liptser and Shiryaev [119, Lemma 4.4.]) that there exists a sequence of piecewise constant and $\overline{\mathbb{F}}$–predictable process α^m satisfying the first two properties in Equation (2.3.9). Without loss of generality, let us also set $\alpha_T^m := \alpha_{t_{m-1}^m}^m$.

Then given α^m, let X^m be the unique $\overline{\mathbb{F}}^m$–adapted solution of the McKean–Vlasov SDE (2.3.10) (see also Theorem 5.5.3 for its well–posedness), with $\overline{\mu}_t^m := \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, \alpha_t^m | \overline{\mathcal{G}}_t^m\big)$. Let $\widehat{\mu}^m$, $\Lambda_t^m(\mathrm{d}a)\mathrm{d}t$ and Y^m be defined as in the statement of Lemma 2.3.10.

The independence between (X_0, W^m) and $(B^m, \widehat{\mu}^m)$ follows directly from the independence of (X_0, W) and $\overline{\mathcal{G}}_T$. Further, by Proposition 2.3.3, W is a Brownian motion under the conditional law of $\overline{\mathbb{P}}$ knowing $\overline{\mathcal{G}}_T$. It follows that, for each $t\in[0,T]$, $(W_{t+s}^m - W_t^m)_{s\in[0,T-t]} = (W_{(t+s)\vee\varepsilon_m} - W_{t\vee\varepsilon_m})_{s\in[0,T-t]}$ and $(X_{t\wedge\cdot}^m, Y_{t\wedge\cdot}^m, (\Lambda^m)^t, W_{t\wedge\cdot}^m)$ are independent under the conditional law of $\overline{\mathbb{P}}$ knowing $\overline{\mathcal{G}}_t^m$ (or $\overline{\mathcal{G}}_T^m$). Together with (2.3.13), it follows that

$$\widehat{\mu}_t^m = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, Y_{t\wedge\cdot}^m, (\Lambda^m)^t, W^m \big| \overline{\mathcal{G}}_t^m\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot}^m, Y_{t\wedge\cdot}^m, (\Lambda^m)^t, W^m \big| \overline{\mathcal{G}}_T^m\big),\ \text{and } \overline{\mu}_t^m(\mathrm{d}\mathbf{x}, \mathrm{d}a) = \mathbb{E}^{\hat{\mu}^m}\Big[\delta_{\widehat{X}_{t\wedge\cdot}}(\mathrm{d}\mathbf{x})\widehat{\Lambda}_t(\mathrm{d}a)\Big],$$

and therefore

$$\widehat{\mu}_t^m = \widehat{\mu}_T^m \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{Y}_{t\wedge\cdot}, \widehat{\Lambda}^t, \widehat{W}\big)^{-1},\ \overline{\mathbb{P}}\text{–a.s., for all } t\in[0,T].$$

Since $(\widehat{\mu}_t^m)_{t\in[0,T]}$ is a function of $\widehat{\mu}_T^m$, and $(B^m, \widehat{\mu}^m)$ and (X_0, W^m) are $\overline{\mathbb{P}}$–independent, it follows by using the definition of $\overline{\mathbb{G}}^m$ that (2.3.11) holds true.

To conclude, it is enough to prove that $\lim_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}}\big[\sup_{s\in[0,T]}|X_s-X_s^m|^p\big]=0$. For any $t\in[\varepsilon_m,T]$, one has

$$\begin{aligned}X_t-X_t^m&=X_{\varepsilon_m}-X_0+\int_{\varepsilon_m}^t\big(b(r,X_{r\wedge\cdot},\overline{\mu}_r,\alpha_r)-b(r,X^m_{r\wedge\cdot},\overline{\mu}^m_r,\alpha^m_r)\big)\mathrm{d}r\\&\quad+\int_{\varepsilon_m}^t\big(\sigma(r,X_{r\wedge\cdot},\overline{\mu}_r,\alpha_r)-\sigma(r,X^m_{r\wedge\cdot},\overline{\mu}^m_r,\alpha^m_r)\big)\mathrm{d}W_r+\int_{\varepsilon_m}^t\big(\sigma_0(r,X_{r\wedge\cdot},\overline{\mu}_r,\alpha_r)-\sigma_0(r,X^m_{r\wedge\cdot},\overline{\mu}^m_r,\alpha^m_r)\big)\mathrm{d}B_r.\end{aligned}$$

Next, using Jensen's inequality, Burkholder–Davis–Gundy inequality, the Lipschitz property of (b,σ,σ_0), and the inequality

$$\mathcal{W}_p\big(\overline{\mu}_t,\overline{\mu}^m_t\big)^p=\mathcal{W}_p\Big(\mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot},\alpha_t\big|\overline{\mathcal{G}}^m_t\big),\mathcal{L}^{\overline{\mathbb{P}}}\big(X^m_{t\wedge\cdot},\alpha^m_t\big|\overline{\mathcal{G}}^m_t\big)\Big)^p\le\mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{s\in[0,t]}\big|X_s-X_s^m\big|^p+\rho\big(\alpha^m_t,\alpha_t\big)^p\Big|\overline{\mathcal{G}}^m_t\Big],$$

there exists a positive constant K, which may vary from line to line, such that

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{s\in[\varepsilon_m,t]}|X_s-X_s^m|^p\Big]&\le K\mathbb{E}^{\overline{\mathbb{P}}}\Big[|X_{\varepsilon_m}-X_0|^p+\int_{\varepsilon_m}^t\big|(b,\sigma,\sigma_0)(r,X_{r\wedge\cdot},\overline{\mu}_r,\alpha_r)-(b,\sigma,\sigma_0)(r,X^m_{r\wedge\cdot},\overline{\mu}^m_r,\alpha^m_r)\big|^p\mathrm{d}r\Big]\\&\le K\Big(\mathbb{E}^{\overline{\mathbb{P}}}\big[|X_{\varepsilon_m}-X_0|^p\big]+\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_{\varepsilon_m}^t\sup_{u\in[\varepsilon_m,r]}\big|X_u-X^m_u\big|^p\mathrm{d}r\Big]+C_m\Big),\end{aligned}$$

where

$$C_m:=\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^T\Big(\big|(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha_r)-(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha^m_r)\big|^p+\rho\big(\alpha^m_r,\alpha_r\big)^p\Big)\mathrm{d}r\Big].$$

By Gronwall's lemma (recall that all expectations appearing here are finite), we deduce that for all $t\in[\varepsilon_m,T]$

$$\mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{s\in[\varepsilon_m,t]}|X_s-X_s^m|^p\Big]\le K\Big(\mathbb{E}^{\overline{\mathbb{P}}}\big[|X_{\varepsilon_m}-X_0|^p\big]+C_m\Big),$$

so that

$$\mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{s\in[0,T]}|X_s-X_s^m|^p\Big]\le K\Big(\mathbb{E}^{\overline{\mathbb{P}}}\big[|X_{\varepsilon_m}-X_0|^p\big]+\mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{r\in[0,\varepsilon_m]}|X_r-X_0|^p\Big]+C_m\Big).$$

By Assumption 1.4.1, we have, for all $r\in[0,T]$,

$$\begin{aligned}\big|(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha_r)-(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha^m_r)\big|^p&\le K\Big(\big\|X_{r\wedge\cdot}\big\|^p+\mathbb{E}^{\overline{\mathbb{P}}}\Big[\big\|X_{r\wedge\cdot}\big\|^p+\rho\big(a_0,\alpha_r\big)^p\Big|\overline{\mathcal{G}}_T\Big]+\rho\big(a_0,\alpha_r\big)^p\Big)\\&\quad+K\rho\big(\alpha^m_r,\alpha_r\big)^p.\end{aligned}$$

By dominated convergence and the continuity of coefficients (b,σ,σ_0), it follows that for all $K>0$,

$$\lim_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^T\big|(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha_r)-(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha^m_r)\big|^p1_{\{\rho(\alpha^m_r,\alpha_r)\le K\}}\mathrm{d}r\Big]=0.$$

In addition, since $\big(\big\|X_{r\wedge\cdot}\big\|^p+\rho\big(a_0,\alpha_r\big)^p\big)1_{\{\rho(\alpha^m_r,\alpha_r)\ge K\}}\le\big(\big\|X_{r\wedge\cdot}\big\|^p+\rho\big(a_0,\alpha_r\big)^p\big)$, which is $\overline{\mathbb{P}}$–integrable, using the uniform integrability of the sequence $(\alpha^m)_{m\in\mathbb{N}^\star}$, one obtains that

$$\begin{aligned}&\limsup_{K\to\infty}\limsup_{m\to\infty}\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^T\big|(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha_r)-(b,\sigma,\sigma_0)(r,X,\overline{\mu}_r,\alpha^m_r)\big|^p1_{\{\rho(\alpha^m_r,\alpha_r)>K\}}\mathrm{d}r\Big]\\&\le\limsup_{K\to\infty}\limsup_{m\to\infty}K\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^T\Big(\big(\big\|X_{r\wedge\cdot}\big\|^p+\rho\big(a_0,\alpha_r\big)^p\big)+\rho\big(\alpha^m_r,\alpha_r\big)^p\Big)1_{\{\rho(\alpha^m_r,\alpha_r)>K\}}\mathrm{d}r\Big]\\&\le\limsup_{K\to\infty}\sup_{m>0}K\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_0^T\rho\big(\alpha^m_r,\alpha_r\big)^p1_{\{\rho(\alpha^m_r,\alpha_r)>K\}}\mathrm{d}r\Big]=0.\end{aligned}$$

This implies that $\lim_{m\to\infty}C_m=0$, and hence (2.3.9) does indeed hold. □

Lemma 2.3.11. *In the context of* Lemma 2.3.10, *let* $m \geq 1$. *In the (possibly enlarged) filtered probability space* $(\overline{\Omega}, \overline{\mathbb{F}}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$, *there exists a sequence of i.i.d. random variables* $U^m = (U^m_i)_{i\geq 1}$, *with uniform distribution on* $[0,1]$, *and* $\overline{\mathbb{P}}$*–independent of* (X_0, B^m, W), *together with a* $(\sigma(U^m, X_0, W_{t\wedge\cdot}, B^m_{t\wedge\cdot}))_{t\in[0,T]}$*–predictable process* $(\widetilde{\gamma}^m_t)_{t\in[0,T]}$, *such that if we let* $(\widetilde{X}^m_t)_{t\in[0,T]}$ *be the unique strong solution of*

$$\widetilde{X}^m_t = X_0 + \int_{\varepsilon_m}^{t\vee\varepsilon_m} b\big(r, \widetilde{X}^m_{r\wedge\cdot}, \overline{\zeta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}r + \int_{\varepsilon_m}^{t\vee\varepsilon_m} \sigma\big(r, \widetilde{X}^m_{r\wedge\cdot}, \overline{\zeta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}W^m_r + \int_{\varepsilon_m}^{t\vee\varepsilon_m} \sigma_0\big(r, \widetilde{X}^m_{r\wedge\cdot}, \overline{\zeta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}B^m_r,$$

with $\overline{\zeta}^m_t := \mathcal{L}^{\mathbb{P}}\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{\gamma}^m_t \big| B^m, U^m\big)$, *and define further* $\widetilde{\Lambda}^m_t(\mathrm{d}a)\mathrm{d}t := \delta_{\widetilde{\gamma}^m_t}(\mathrm{d}a)\mathrm{d}t$, *as well as*

$$\widetilde{Y}^m_t := \widetilde{X}^m_t - \int_{\varepsilon_m}^{t\vee\varepsilon_m} \sigma_0\big(r, \widetilde{X}^m_{r\wedge\cdot}, \overline{\zeta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}B^m_r, \text{ and } \widehat{\zeta}^m_t := \mathcal{L}^{\mathbb{P}}\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{Y}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W^m \big| B^m, U^m\big),$$

then, with $(X^m, Y^m, \Lambda^m, W^m, B^m, \widehat{\mu}^m)$ *defined in* Lemma 2.3.10, *we have*

$$\mathcal{L}^{\overline{\mathbb{P}}}\Big(\widetilde{X}^m, \widetilde{Y}^m, \widetilde{\Lambda}^m, W^m, B^m, \widehat{\zeta}^m_T\Big) = \mathcal{L}^{\overline{\mathbb{P}}}\Big(X^m, Y^m, \Lambda^m, W^m, B^m, \widehat{\mu}^m\Big). \tag{2.3.14}$$

Finally, when $\ell = 0$ *and* $\widehat{\mu}^m$ *is deterministic, then one can take* $(\widetilde{\gamma}^m_t)_{t\in[0,T]}$ *to be* $(\sigma(X_0, W_{t\wedge\cdot}))_{t\in[0,T]}$*–predictable.*

Proof. Let us fix $m \geq 1$, and introduce $\{W^m\}_0 = \{B^m\}_0 := 0$, and then for $i \in \{1, \ldots, m\}$,

$$\{B^m\}_i := \big(B^{m,(k-1)}\big)_{1\leq k\leq i},\ \{W^m\}_i := \big(W^{m,(k-1)}\big)_{1\leq k\leq i},\ \{\widehat{\mu}^m\}_i := \big(\widehat{\mu}^m_{t^m_k}\big)_{0\leq k\leq i}, \text{ and } \{\alpha^m\}_i := (\alpha^m_k)_{0\leq k\leq i},$$

where $B^{m,(k-1)}_t := B^m_{(t\vee t^m_{k-1})\wedge t^m_k} - B^m_{t^m_{k-1}}$ and $W^{m,(k-1)}_t := W^m_{(t\vee t^m_{k-1})\wedge t^m_k} - W^m_{t^m_{k-1}}$, $t \in [0,T]$.

Step 1. For each $i \in \{1, \ldots, m\}$, there exists (see Kurtz [99, Lemma 1.3.]) a Borel measurable function $G^\mu_i : \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)^i \times [0,1] \longrightarrow \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)$ such that, for any uniform random variable U^m_i independent of $\big(\{B^m\}_i, \{\widehat{\mu}^m\}_{i-1}\big)$ and $\widehat{G}^\mu_i := G^\mu_i\big(\{B^m\}_i, \{\widehat{\mu}^m\}_{i-1}, U^m_i\big)$, we have

$$\mathcal{L}^{\overline{\mathbb{P}}}\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_{i-1}, \widehat{\mu}^m_{t^m_i}\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_{i-1}, \widehat{G}^\mu_i\big), \tag{2.3.15}$$

Above, G^μ_i is a function of $(\{B^m\}_i, \{\widehat{\mu}^m\}_{i-1}, U^m_i)$ rather than of $(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_{i-1}, U^m_i)$, since $\widehat{\mu}^m_{t^m_i}$ is actually $\overline{\mathbb{P}}$–independent of (X_0, W^m). We can apply a similar argument to find a Borel measurable function $G^\alpha_i : \mathbb{R}^n \times (\mathcal{C}^\ell \times \mathcal{C}^d)^i \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)^{(i+1)} \times A^i \times \mathbb{R} \longrightarrow A$, and uniform random variable V^m_i independent of the variables $\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_{(i-1)}\big)$ such that

$$\mathcal{L}^{\overline{\mathbb{P}}}\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_{(i-1)}, \alpha^m_i\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_{(i-1)}, \widetilde{G}^\alpha_i\big), \tag{2.3.16}$$

where

$$\widetilde{G}^\alpha_i := G^\alpha_i\Big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_{(i-1)}, V^m_i\Big).$$

Observe that one can take $(U^m_1, \ldots, U^m_m)$ to be independent of $(V^m_1, \ldots, V^m_m)$. We can then find a Borel function $\kappa^d : \mathbb{R}^d \longrightarrow [0,1]$ such that $\mathcal{L}^{\overline{\mathbb{P}}}\big(\kappa^d\big(W_{i\varepsilon_m/m} - W_{(i-1)\varepsilon_m/m}\big)\big)$ is a uniform distribution. Define next $\widetilde{\gamma}^m_0 := \alpha^m_0 = a_0$, $\widehat{\zeta}_0 := \widehat{\mu}_{t^m_0\wedge\cdot}$, and for any $i \in \{1, \ldots, m-1\}$

$$\widehat{\zeta}^m_i := G^\mu_i\Big(X_0, \{B^m\}_i, \{\widehat{\zeta}^m\}_{(i-1)}, U^m_i\Big),\ \widetilde{\gamma}^m_i := G^\alpha_i\Big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\zeta}^m\}_i, \{\widetilde{\gamma}^m\}_{(i-1)}, \kappa^d\big(W_{i\varepsilon_m/m} - W_{(i-1)\varepsilon_m/m}\big)\Big).$$

Then, for each $i \in \{0, \ldots, m\}$, $\widehat{\zeta}^m_i$ is $\sigma(U^m_1, \ldots, U^m_i, \{B^m\}_i)$–measurable, and

$$\widetilde{\gamma}^m_i \text{ is } \sigma\big(X_0, W_{\varepsilon_m\wedge\cdot}, \{W^m\}_i, \{\widehat{\zeta}^m\}_i, \{B^m\}_i\big)\text{–measurable.} \tag{2.3.17}$$

When $\ell = 0$ and $\widehat{\mu}^m$ is deterministic, the previous construction implies that $\{\widehat{\zeta}^m\}_m = \{\widehat{\mu}^m\}_m$ is deterministic and $\widetilde{\gamma}_i^m$ is $\sigma\big(X_0, W_{\varepsilon_m\wedge\cdot}, \{W^m\}_i\big)$–measurable.

Step 2. We next prove by induction that, for each $i \in \{0, \dots, m\}$

$$\mathcal{L}^{\overline{\mathbb{P}}}\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X_0, \{B^m\}_i, \{W^m\}_i, \{\widehat{\zeta}^m\}_i, \{\widetilde{\gamma}^m\}_i\big). \tag{2.3.18}$$

When $i = 0$, (2.3.18) holds true since α_0^m and $\widehat{\mu}_{t_0^m}$ are deterministic constants.

Now, assume that (2.3.18) is true for some $i \in \{0, \dots, m-1\}$. First, take $\phi \in C_b(\mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^\ell \times \mathcal{C}^d \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^n)^i \times A^i)$, $\psi \in C_b(\mathcal{C}^d)$, $\varphi \in C_b(\mathcal{C}^\ell)$ and $h \in C_b(\mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d))$. Using the independence of the increments of the Brownian motion W^m, together with (2.3.11) and (2.3.15), we have

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big)\psi\big(W^{m,(i)}\big)\varphi\big(B^{m,(i)}\big)h\big(\widehat{\mu}^m_{t_{i+1}^m}\big)\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}}\Big[\psi\big(W^{m,(i)}\big)\Big]\mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big)\Big|B^m_{t_i^m\wedge\cdot}, \widehat{\mu}^m_{t_i^m\wedge\cdot}\Big]\varphi\big(B^{m,(i)}\big)h\big(\widehat{\mu}^m_{t_{i+1}^m}\big)\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}}\Big[\psi\big(W^{m,(i)}\big)\Big]\mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big)\Big|B^m, \widehat{\mu}^m\Big]\varphi\big(B^{m,(i)}\big)h\big(\widehat{G}^\mu_{i+1}\big)\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big)\psi\big(W^{m,(i)}\big)\Big|B^m, \widehat{\mu}^m\Big]\varphi\big(B^{m,(i)}\big)h\big(\widehat{G}^\mu_{i+1}\big)\Big].
\end{aligned} \tag{2.3.19}$$

Further, let $\varphi_1 \in C_b(\mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)^i)$ and $\varphi_2 \in C_b([0,1])$, using the independence of U^m_{i+1} and that of the increments of the Brownian motions (B^m, W^m), and the induction hypothesis, we obtain

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big)\psi\big(W^{m,(i)}\big)\Big|B^m, \widehat{\mu}^m\Big]\varphi_1\big(B^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\mu}^m\}_i\big)\varphi_2(U^m_{i+1})\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, \{\alpha^m\}_i\big)\psi\big(W^{m,(i)}\big)\varphi_1\big(B^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\mu}^m\}_i\big)\Big]\mathbb{E}^{\overline{\mathbb{P}}}\Big[\varphi_2(U^m_{i+1})\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\zeta}^m\}_i, \{\widetilde{\gamma}^m\}_i\big)\psi\big(W^{m,(i)}\big)\varphi_1\big(B^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\zeta}^m\}_i\big)\varphi_2(U^m_{i+1})\Big].
\end{aligned} \tag{2.3.20}$$

Using the arbitrariness of (φ_1, φ_2), and a classical density argument, we can replace $\varphi_1\big(B^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\zeta}^m\}_i\big)\varphi_2(U^m_{i+1})$ by $\varphi(B^{m,(i)})h\big(G^\mu_{i+1}\big(B^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\mu}^m\}_i, U^m_{i+1}\big)\big)$, for arbitrary continuous and bounded functions φ and h, in (2.3.20), leading to

$$(2.3.19) = \mathbb{E}^{\overline{\mathbb{P}}}\Big[\phi\big(X_0, B^m_{t_i^m\wedge\cdot}, W^m_{t_i^m\wedge\cdot}, \{\widehat{\zeta}^m\}_i, \{\widetilde{\gamma}^m\}_i\big)\psi\big(W^{m,(i)}\big)\varphi\big(B^{m,(i)}\big)h\big(\widehat{\zeta}^m_{i+1}\big)\Big],$$

and hence

$$\mathcal{L}^{\overline{\mathbb{P}}}\Big(X_0, B^m_{t_{i+1}^m\wedge\cdot}, W^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\mu}^m\}_{(i+1)}, \{\alpha^m\}_i\Big) = \mathcal{L}^{\overline{\mathbb{P}}}\Big(X_0, B^m_{t_{i+1}^m\wedge\cdot}, W^m_{t_{i+1}^m\wedge\cdot}, \{\widehat{\zeta}^m\}_{(i+1)}, \{\widetilde{\gamma}^m\}_i\Big).$$

Together with the result (2.3.16), and by the independence of V^m_{i+1} with the other variables, it follows that

$$\mathcal{L}^{\overline{\mathbb{P}}}\Big(X_0, \{B^m\}_{(i+1)}, \{W^m\}_{(i+1)}, \{\widehat{\mu}^m\}_{(i+1)}, \{\alpha^m\}_i, \alpha^m_{i+1}\Big) = \mathcal{L}^{\overline{\mathbb{P}}}\Big(X_0, \{B^m\}_{(i+1)}, \{W^m\}_{(i+1)}, \{\widehat{\zeta}^m\}_{(i+1)}, \{\widetilde{\gamma}^m\}_i, \widetilde{\gamma}^m_{i+1}\Big),$$

which concludes the proof of (2.3.18) by induction.

Step 3. Under Assumption 1.4.1, the solution of SDE (2.3.10) can be expressed as function of $(X_0, W^m, B^m, (\Lambda^m), \widehat{\mu}^m)$. More precisely, there exists a Borel function $H^m : [0,T] \times \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathbb{M} \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d) \longrightarrow \mathcal{C}^n \times \mathcal{C}^n$ such that

$$(X^m_t, Y^m_t) = H^m_t\Big(X_0, W^m_{t\wedge\cdot}, B^m_{t\wedge\cdot}, (\Lambda^m)^t, \widehat{\mu}^m_T\Big),\ t \in [0,T],\ \overline{\mathbb{P}}\text{–a.s.}$$

Moreover, by Lemma 2.3.10, the processes $(\widehat{\mu}^m_t)_{t\in[0,T]}$ and $(\overline{\mu}^m_t)_{t\in[0,T]}$ are actually functions of $\widehat{\mu}^m_T$.

Define $\widetilde{\gamma}^m_t := \widetilde{\gamma}^m_i$ for $t \in [t_i^m, t_{i+1}^m)$, $i \in \{0, \dots, m-1\}$, $\widetilde{\Lambda}^m_t(\mathrm{d}a)\mathrm{d}t := \delta_{\widetilde{\gamma}^m_t}(\mathrm{d}a)\mathrm{d}t$, and

$$\widehat{\zeta}^m_t := \widehat{\zeta}^m_m \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{Y}_{t\wedge\cdot}, \widehat{\Lambda}^t, \widehat{W}\big)^{-1},\ \text{and}\ \overline{\zeta}^m_t(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \mathbb{E}^{\widehat{\zeta}^m_m}\Big[\delta_{\widehat{X}_{t\wedge\cdot}}(\mathrm{d}\mathbf{x})\widehat{\Lambda}_t(\mathrm{d}a)\Big],\ \text{for all } t \in [0,T],$$

and then

$$(\widetilde{X}^m_t, \widetilde{Y}^m_t) := H^m_t\big(X_0, W^m_{t\wedge\cdot}, B^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, \widehat{\zeta}^m_T\big).$$

It follows from Equation (2.3.18) that (2.3.14) holds true, and $\big(\widetilde{X}^m, \widetilde{Y}^m\big)$ satisfies the SDE in the statement of Lemma (2.3.11). It remains to prove that

$$\widehat{\zeta}^m_t = \mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{Y}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W^m \big| B^m, U^m\big), \overline{\mathbb{P}}\text{–a.s., for all } t \in [0,T]. \tag{2.3.21}$$

Recall that $\widehat{\mu}^m_t = \widehat{\mu}^m_T \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{Y}_{t\wedge\cdot}, \widehat{\Lambda}^t, \widehat{W}\big)^{-1}$ for all $t \in [0,T]$. Let $\phi \in C_b(\mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d))$, $\varphi \in C_b(\mathcal{C}^\ell \times C([0,T]; \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)))$. By Equation (2.3.18), we have

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\big[\langle \phi, \widehat{\zeta}^m_t\rangle \varphi\big(B^m, \widehat{\zeta}^m_{T\wedge\cdot}\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\langle f, \widehat{\mu}^m_t\rangle \varphi\big(B^m, \widehat{\mu}^m_{T\wedge\cdot}\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[f\big(X^m_{t\wedge\cdot}, Y^m_{t\wedge\cdot}, (\Lambda^m)^t, W^m\big)\varphi\big(B^m, \widehat{\mu}^m_{T\wedge\cdot}\big)\big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\big[f\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{Y}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W^m\big)\varphi\big(B^m, \widehat{\zeta}^m_{T\wedge\cdot}\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\langle f, \mathcal{L}^{\mathbb{P}}\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{Y}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W^m \big| B^m, \widehat{\zeta}^m_{T\wedge\cdot}\big)\rangle \varphi\big(B^m, \widehat{\zeta}^m_{T\wedge\cdot}\big)\big].
\end{aligned}$$

This implies that

$$\widehat{\zeta}^m_t = \mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{Y}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W^m \big| B^m, \widehat{\zeta}^m_{T\wedge\cdot}\big).$$

Recall from (2.3.17) that $\widetilde{\gamma}^m_i$ is $\sigma(X_0, W_{\varepsilon_m\wedge\cdot}, \{W^m\}_i, \{\widehat{\zeta}^m\}_i, \{B^m\}_i)$–measurable, $\widehat{\zeta}^m_i$ is $\sigma(\{B^m\}_i, U^m)$–measurable for each $i \in \{0, \dots, m-1\}$, and U^m is independent of $(X_0, B^m, W_{\varepsilon_m\wedge\cdot}, W^m)$ under $\overline{\mathbb{P}}$. It follows that (2.3.21) holds true. □

For Proposition 2.3.12 below, let us denote by $(\alpha_t)_{t\in[0,T]}$ an A–valued $\overline{\mathbb{F}}$–predictable process on the canonical space $\overline{\Omega}$, satisfying that $\Lambda_t(\mathrm{d}a)\mathrm{d}t = \delta_{\alpha_t}(\mathrm{d}a)\mathrm{d}t$, $\overline{\mathbb{P}}$–a.e., for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$.

Proposition 2.3.12. *Let* Assumption 1.4.1 *hold true,* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ *and* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$.

(i) When $\ell \neq 0$, *there exists a sequence* $(\overline{\mathbb{P}}^m)_{m\geq 1} \subset \overline{\mathcal{P}}_S(\nu)$ *such that*

$$\lim_{m\to\infty} \mathcal{L}^{\overline{\mathbb{P}}^m}\big(X, Y, \Lambda, W, B, \widehat{\mu}, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X, Y, \Lambda, W, B, \widehat{\mu}, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\big), \text{ in } \mathcal{W}_p. \tag{2.3.22}$$

(ii) When $\ell = 0$, *there exists a family* $(\overline{\mathbb{P}}^m_u)_{u\in[0,1], m\geq 1} \subset \overline{\mathcal{P}}_S(\nu)$, *such that* $u \longmapsto \overline{\mathbb{P}}^m_u$ *is Borel measurable, and*

$$\lim_{m\to\infty} \int_0^1 \mathcal{L}^{\overline{\mathbb{P}}^m_u}\big(X, Y, \Lambda, W, B, \widehat{\mu}, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\big)\mathrm{d}u = \mathcal{L}^{\overline{\mathbb{P}}}\Big(X, Y, \Lambda, W, B, \widehat{\mu}, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\Big), \text{ in } \mathcal{W}_p. \tag{2.3.23}$$

Proof. First, let $(\widetilde{X}^m, \widetilde{Y}^m, B^m, W^m, \widehat{\zeta}^m, \overline{\zeta}^m, \widetilde{\gamma}^m, \widetilde{\Lambda}^m)$ be given as in Lemma 2.3.11. Using Lemma 2.3.10 and Lemma 2.3.11, we have

$$\lim_{m\to\infty} \mathcal{L}^{\overline{\mathbb{P}}}\Big(\widetilde{X}^m, \widetilde{Y}^m, B^m, W^m, \widehat{\zeta}^m_T, \delta_{(\overline{\zeta}^m_t, \widetilde{\gamma}^m_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\Big) = \mathcal{L}^{\overline{\mathbb{P}}}\Big(X, Y, B, W, \widehat{\mu}, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\Big), \text{ in } \mathcal{W}_p.$$

(i) When $\ell \neq 0$, since B_{ε_m} is independent of (X_0, W, B^m), one can take $U^m := \kappa\big(B_{\varepsilon_m}\big)$ for some measurable function $\kappa : \mathbb{R} \longrightarrow [0,1]^m$. Consequently, we have $\widehat{\zeta}^m_t = \mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{X}^m_{t\wedge\cdot}, \widetilde{Y}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W^m \big| B\big)$, $\overline{\mathbb{P}}$–a.s., for all $t \in [0,T]$. Let us then define $(\widetilde{S}^m_t)_{t\in[0,T]}$ as the unique strong solution of

$$\widetilde{S}^m_t = X_0 + \int_0^t b\big(r, \widetilde{S}^m, \overline{\beta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widetilde{S}^m, \overline{\beta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}W_r + \int_0^t \sigma_0\big(r, \widetilde{S}^m, \overline{\beta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}B_r,$$

with $\overline{\beta}^m_t := \mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{S}^m_{t\wedge\cdot}, \widetilde{\gamma}^m_t | B_{t\wedge\cdot}\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{S}^m_{t\wedge\cdot}, \widetilde{\gamma}^m_t | B\big)$. Denote, for all $t \in [0,T]$

$$\widetilde{Z}^m_t := \widetilde{S}^m_t - \int_0^t \sigma_0\big(r, \widetilde{S}^m, \overline{\beta}^m_r, \widetilde{\gamma}^m_r\big)\mathrm{d}B_r, \text{ and } \widehat{\beta}^m_t := \mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{S}^m_{t\wedge\cdot}, \widetilde{Z}^m_{t\wedge\cdot}, (\widetilde{\Lambda}^m)^t, W \big| B\big).$$

Using almost the same arguments as in the proof of (2.3.9) in Lemma 2.3.10, we can deduce that

$$\lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{P}}}\Big[\sup_{t\in[0,T]} |\widetilde{S}^m_t - \widetilde{X}^m_t|^p\Big] = 0,$$

and moreover

$$\lim_{m\to\infty}\mathcal{L}^{\overline{\mathbb{P}}}\Big(\widetilde{S}^m,\widetilde{Z}^m,B,W,\widehat{\beta}^m_T,\delta_{(\overline{\beta}^m_t,\widetilde{\gamma}^m_t)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\Big)=\lim_{m\to\infty}\mathcal{L}^{\overline{\mathbb{P}}}\Big(\widetilde{X}^m,\widetilde{Y}^m,B^m,W^m,\widehat{\zeta}^m_T,\delta_{(\overline{\zeta}^m_t,\widetilde{\gamma}^m_t)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\Big)$$
$$=\lim_{m\to\infty}\mathcal{L}^{\overline{\mathbb{P}}}\Big(X^m,Y^m,B^m,W^m,\widehat{\mu}^m_T,\delta_{(\overline{\mu}^m_t,\alpha^m_t)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\Big)=\mathcal{L}^{\overline{\mathbb{P}}}\Big(X,Y,B,W,\widehat{\mu},\delta_{(\overline{\mu}_t,\alpha_t)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\Big),\ \text{in } \mathcal{W}_p. \tag{2.3.24}$$

Then it is enough to denote $\overline{\mathbb{P}}^m:=\overline{\mathbb{P}}\circ(\widetilde{S}^m,\widetilde{Z}^m,\widetilde{\Lambda}^m,W,B,\widehat{\beta}^m_T)^{-1}$ to conclude the proof of (i).

(ii). When $\ell=0$, so that the process B disappears, one has $\widehat{\zeta}^m_t=\mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{X}^m_{t\wedge\cdot},\widetilde{Y}^m_{t\wedge\cdot},(\widetilde{\Lambda}^m)^t,W^m\big|U^m\big)$, $t\in[0,T]$, $\overline{\mathbb{P}}$–a.s., where U^m is independent of $\big(\widetilde{X}^m,\widetilde{Y}^m,\widetilde{\Lambda}^m,W\big)$. Let us define $(\widetilde{S}^m_t)_{t\in[0,T]}$ as the unique strong solution of

$$\widetilde{S}^m_t=X_0+\int_0^t b\big(r,\widetilde{S}^m,\overline{\beta}^m_r,\widetilde{\gamma}^m_r\big)\mathrm{d}r+\int_0^t\sigma\big(r,\widetilde{S}^m,\overline{\beta}^m_r,\widetilde{\gamma}^m_r\big)\mathrm{d}W_r,$$

with

$$\overline{\beta}^m_t:=\mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{S}^m_{t\wedge\cdot},\widetilde{\gamma}^m_t|U^m\big)=\mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{S}^m_{t\wedge\cdot},\widetilde{\gamma}^m_t|U^m\big),\ \widetilde{Z}^m_t:=\widetilde{S}^m_t,\ \text{and}\ \widehat{\beta}^m_t:=\mathcal{L}^{\overline{\mathbb{P}}}\big(\widetilde{S}^m_{t\wedge\cdot},\widetilde{Z}^m_{t\wedge\cdot},(\widetilde{\Lambda}^m)^t,W\big|U^m\big),$$

As in (i), we can apply almost the same arguments as in the proof of Lemma 2.3.10 to deduce that

$$\lim_{m\to\infty}\mathcal{L}^{\overline{\mathbb{P}}}\Big(\widetilde{S}^m,\widetilde{Z}^m,B,W,\widehat{\beta}^m_T,\delta_{(\overline{\beta}^m_t,\widetilde{\gamma}^m_t)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\Big)=\mathcal{L}^{\overline{\mathbb{P}}}\Big(X,Y,B,W,\widehat{\mu},\delta_{(\overline{\mu}_t,\alpha_t)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\Big),\ \text{in } \mathcal{W}_p.$$

Beside, it is easy to check that

$$\mathcal{L}^{\overline{\mathbb{P}}}\Big(\widetilde{S}^m,\widetilde{Z}^m,\widetilde{\Lambda}^m,W,B,\widehat{\beta}^m\Big|U^m\Big)\in\overline{\mathcal{P}}_S(\nu),\ \overline{\mathbb{P}}\text{–a.s.},$$

which concludes the proof of (ii). □

Remark 2.3.13. *When $\ell=0$, if we assume in addition that $\widehat{\mu}$ is deterministic under $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(\nu)$, we can omit the term U^m in the proof of* Proposition 2.3.12.(ii) *by* Lemma 2.3.11*, and hence there is no need to consider the conditional law of $(\widetilde{S}^m,\widetilde{Z}^m,\widetilde{\Lambda}^m,W,B,\widehat{\beta}^m)$ knowing U^m. It follows that we can find a sequence $(\overline{\mathbb{P}}^m)_{m\geq1}\subset\overline{\mathcal{P}}_S(\nu)$ such that (2.3.22) holds.*

Remark 2.3.14. *In summary, our proof for approximating weak control by strong control rules consists in three main steps*

- (i) *approximate the (weak) control process by piecewise constant processes and freeze the controlled process on $[0,\varepsilon]$;*
- (ii) *represent the piecewise constant control process as functionals of the Brownian motions and some independent randomness using the (H)–hypothesis type condition (3.2.1);*
- (iii) *replace the independent randomness by the increment of the Brownian motions on $[0,\varepsilon]$, so that the control processes becomes functionals of the Brownian motions only.*

This is quite different from the steps in Lacker [104] *for McKean–Vlasov control problem without common noise, and in spirit closer to the technical steps in* El Karoui and Tan [62, Theorem 4.5.]*, which approximates weak control rule by strong control rules for classical stochastic control problems. In particular, our approach allows to avoid a subtle gap in the proof of* [103, Lemma 6.7.]*. In that proof, a key technical step uses implicitly the following erroneous argument (see the paragraph after* (6.19) *in* [103]*): let W and U be two independent random variables on a probability space $(\Omega^*,\mathcal{F}^*,\mathbb{P}^*)$, and $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ be such that $Z:=f(W,U)$ is independent of W, then Z is measurable with respect to the (completed) σ–algebra generated by U. For a counter–example, let us consider the case that $W\sim N(0,1)$ and $U\sim\mathcal{U}[-1,1]$ and that W is independent of U, then $Z:=U1_{\{W\geq0\}}-U1_{\{W<0\}}$ is independent of W, but not measurable w.r.t. $\sigma(U)$.*

2.3.3 Proof of equivalence

(i) Let Assumption 1.4.1 hold true, and take $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. The non–emptiness of $\overline{\mathcal{P}}_W(\nu)$ follows by a stability result for the martingale problem in Assumption 1.4.1. We provide a detailed proof in Theorem 2.4.2.

For the convexity of $\overline{\mathcal{P}}_W(\nu)$, we first prove that $\overline{\mathcal{P}}_A(\nu)$ is convex. Let us consider $(\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_2) \in \overline{\mathcal{P}}_A(\nu) \times \overline{\mathcal{P}}_A(\nu)$, $\theta \in [0,1]$ and $\overline{\mathbb{P}} := \theta\overline{\mathbb{P}}_1 + (1-\theta)\overline{\mathbb{P}}_2$, and show that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$. First, it is direct to check that $\overline{\mathbb{P}}$ satisfies Conditions (i) and (iii) in Definition 2.3.1. To check Condition (ii) in Definition 2.3.1, we consider $t \in [0,T]$, $f \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)$, $\psi \in C_b(\mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d))$, $\varphi \in C_b(\mathbb{R}^n \times \mathcal{C}^d)$. Notice that under both $\overline{\mathbb{P}}_1$ and $\overline{\mathbb{P}}_2$, (X_0, W) has the same distribution and is independent of $(B, \widehat{\mu})$, it follows that

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}}\big[\varphi(X_0,W)\psi\big(B,\widehat{\mu}\big)\big] &= \theta\mathbb{E}^{\overline{\mathbb{P}}_1}\big[\varphi(X_0,W)\psi\big(B,\widehat{\mu}\big)\big] + (1-\theta)\mathbb{E}^{\overline{\mathbb{P}}_2}\big[\varphi(X_0)\beta(W)\psi\big(B,\widehat{\mu}\big)\big] \\ &= \mathbb{E}^{\overline{\mathbb{P}}}\big[\varphi(X_0)\beta(W)\big]\big(\theta\mathbb{E}^{\overline{\mathbb{P}}_1}\big[\psi\big(B,\widehat{\mu}\big)\big] + (1-\theta)\mathbb{E}^{\overline{\mathbb{P}}_2}\big[\psi\big(B,\widehat{\mu}\big)\big]\big) = \mathbb{E}^{\overline{\mathbb{P}}}\big[\varphi(X_0,W)\big]\mathbb{E}^{\overline{\mathbb{P}}}\big[\psi\big(B,\widehat{\mu}\big)\big],\end{aligned}$$

which implies the independence of (X_0, W) and $(B, \widehat{\mu})$ under $\overline{\mathbb{P}}$. Furthermore, one has, for each $i \in \{1,2\}$

$$\mathbb{E}^{\overline{\mathbb{P}}_i}\big[\langle f, \widehat{\mu}_t\rangle\psi\big(B,\widehat{\mu}\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}_i}\big[f\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big)\psi\big(B,\widehat{\mu}\big)\big],$$

then it is straightforward to obtain that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\langle f, \widehat{\mu}_t\rangle\psi\big(B,\widehat{\mu}\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[f\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big)\psi\big(B,\widehat{\mu}\big)\big].$$

This implies that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$

$$\widehat{\mu}_t(\bar{\omega}) = \overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_t} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, W, \Lambda^t)^{-1} = \overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_T} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, W, \Lambda^t)^{-1}.$$

Then $\overline{\mathbb{P}}$ also satisfies Condition (ii) in Definition 2.3.1, and hence $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$. This proves that $\overline{\mathcal{P}}_A(\nu)$ is convex.

Next, assume in addition that $(\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_2) \in \overline{\mathcal{P}}_W(\nu) \times \overline{\mathcal{P}}_W(\nu)$, that is to say $(\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_2) \in \overline{\mathcal{P}}_A(\nu) \times \overline{\mathcal{P}}_A(\nu)$ and $\overline{\mathbb{P}}_i\big[\Lambda \in \mathbb{M}_0\big] = 1$ for $i \in \{1,2\}$. It follows that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$ and $\overline{\mathbb{P}}\big[\Lambda \in \mathbb{M}_0\big] = 1$, so that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$.

(ii) Fix $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. First, one has clearly $V_S(\nu) \le V_W(\nu)$. Furthermore, for any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$, by Proposition 2.3.12 and under condition $\ell \ge 1$, there is a sequence of probability measures $(\overline{\mathbb{P}}^m)_{m\ge1} \subset \overline{\mathcal{P}}_S(\nu)$ such that

$$\lim_{m\to\infty} \mathcal{L}^{\overline{\mathbb{P}}^m}\big(X, Y, \Lambda, W, B, \widehat{\mu}, \delta_{(\overline{\mu}_t,\alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big(X, Y, \Lambda, W, B, \widehat{\mu}, \delta_{(\overline{\mu}_t,\alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\big), \tag{2.3.25}$$

in $\mathcal{P}_p\big(\overline{\Omega} \times \mathbb{M}(\mathcal{P}(\mathcal{C}^n \times A) \times A)\big)$ under $\mathcal{W}_p$. This implies in particular that $\overline{\mathbb{P}}^m \longrightarrow \overline{\mathbb{P}}$ in $\mathcal{P}_p(\overline{\Omega})$ under $\mathcal{W}_p$.

Besides, although $\overline{\mathbb{P}} \longmapsto J(\overline{\mathbb{P}})$ is not continuous in general (see Remark 2.3.5), the convergence in (2.3.25) is stronger than simply $\overline{\mathbb{P}}^m \longrightarrow \overline{\mathbb{P}}$. With the growth and lower semi–continuity conditions of L and g in Assumption 1.4.1, and by a slight extension of [104, Lemma 4.1], the convergence (2.3.25) implies that

$$V_S(\nu) \ge \lim_{m\to\infty} J(\overline{\mathbb{P}}^m) \ge J(\overline{\mathbb{P}}).$$

It follows that $V_S(\nu) = V_W(\nu)$.

When $\ell = 0$, using Proposition 2.3.12, it is enough to consider a convex combination of strong control rules and apply the same argument as above to conclude the proof.

2.4 Appendix: existence of weak solution to the McKean–Vlasov equations

We provide here an existence result of weak solution to the McKean-Vlasov equation.

Assumption 2.4.1. *The constants $(p,\hat{p})$ satisfy: $p \geq (1\vee\hat{p})$, $2 \geq \hat{p} \geq 0$, and for all $(t,\mathbf{x},\bar{\nu},a) \in [0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A$,*

$$|b(t,\mathbf{x},\bar{\nu},a)| \leq C\bigg(1+\|\mathbf{x}\|+\Big(\int_{\mathcal{C}^n\times A}\big(\|\mathbf{x}'\|^p+\rho(a_0,a')^p\big)\bar{\nu}(\mathrm{d}\mathbf{x}',\mathrm{d}a')\Big)^{\frac{1}{p}}+\rho(a_0,a)\bigg),$$
$$|(\sigma,\sigma_0)(t,\mathbf{x},\bar{\nu},a)|^2 \leq C\bigg(1+\|\mathbf{x}\|^{\hat{p}}+\Big(\int_{\mathcal{C}^n\times A}\big(\|\mathbf{x}'\|^p+\rho(a_0,a')^p\big)\bar{\nu}(\mathrm{d}\mathbf{x}',\mathrm{d}a')\Big)^{\frac{\hat{p}}{p}}+\rho(a_0,a)^{\hat{p}}\bigg),$$

Theorem 2.4.2. *Let $\nu\in\mathcal{P}(\mathbb{R}^n)$, $p'>p$, $p'\geq 2$, and (b,σ,σ_0) be continuous in $(\mathbf{x},\bar{\nu},a)$.*

- *If (b,σ,σ_0) is bounded then there exists a probability measure $\overline{\mathbb{P}}\in\mathcal{P}(\overline{\Omega})$ verifying the points (i), (ii), and (iii) of definition 2.3.1 expect the integrability condition.*
- *If Assumption 2.4.1 is satisfied with $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ then there exists $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(\nu)$ satisfying $\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X\|^{p'}\big]<\infty$.*

Proof. Without loss of generality, we can assume that A is a singleton given by $A=\{a_0\}$ (otherwise, we can use a constant control process equals to a_0).

First, recall that the filtered probability space $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P}_\nu)$ is defined in Section 2.2.1 and equipped with an initial random variable X_0, together with Brownian motions (W,B). For each $m\geq 1$, we consider the solution $(X_t^m)_{t\in[0,T]}$ of the Euler scheme of McKean–Vlasov equation (2.2.2), that is

$$X_t^m := X_0+\int_0^t b\big(r,X^m_{[r]^m\wedge\cdot},\overline{\mu}^m_{[r]^m},a_0\big)\mathrm{d}r+\int_0^t\sigma\big(r,X^m_{[r]^m\wedge\cdot},\overline{\mu}^m_{[r]^m},a_0\big)\mathrm{d}W_r+\int_0^t\sigma_0\big(r,X^m_{[r]^m\wedge\cdot},\overline{\mu}^m_{[r]^m},a_0\big)\mathrm{d}B_r,\ \mathbb{P}_\nu\text{–a.s.},$$

where $\overline{\mu}_t^m := \mathcal{L}^{\mathbb{P}_\nu}(X^m_{t\wedge\cdot}|\mathcal{G}_t)\otimes\delta_{a_0}$, $[t]^m = iT2^{-m}$ for all $t\in\big[iT2^{-m},(i+1)T2^{-m}\big)$ and $i\in\{0,\dots,2^m-1\}$. It is straightforward to verify that for each $m\in\mathbb{N}^*$: if Assumption 2.4.1 is satisfied with $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ $\mathbb{E}^{\mathbb{P}_\nu}\big[\sup_{t\in[0,T]}|X_t^m|^{p'}\big]<\infty$ and if (b,σ,σ_0) is bounded $\mathbb{E}^{\mathbb{P}_\nu}\big[\sup_{t\in[0,T]}|X_t^m-X_0|^q\big]<\infty$ for all $q\geq 0$.

By classical application of Gronwall Lemma like that used in Lemma 3.4.1 (see also Lacker [104, Lemmata 3.1. and 3.3.]), we can deduce that, for some constant $C>0$ independent of m such that: if Assumption 2.4.1 is satisfied with $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$

$$\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{t\in[0,T]}|X_t^m|^{p'}\Big]\leq C\bigg(1+\int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x)\bigg),\tag{2.4.1}$$

and if (b,σ,σ_0) is bounded

$$\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{t\in[0,T]}|X_t^m-X_0|^{p'}\Big]\leq C.\tag{2.4.2}$$

Let us denote $Y_\cdot^m := X_\cdot^m-\int_0^\cdot\sigma_0\big(r,S^m_{[r]^m\wedge\cdot},\overline{\mu}^m_{[r]^m},a_0\big)\mathrm{d}B_r$, and

$$\overline{\mathbb{P}}^m := \mathbb{P}_\nu\circ\Big(X^m,Y^m,\Lambda^\circ,W,B,\widehat{\mu}^m\Big)^{-1},\ \text{with } \Lambda_t^\circ(\mathrm{d}a)\mathrm{d}t := \delta_{a_0}(\mathrm{d}a)\mathrm{d}t,\ \text{and } \widehat{\mu}^m := \mathcal{L}^{\mathbb{P}_\nu}\big(X^m,Y^m,\Lambda^\circ,W\big|\mathcal{G}_T\big).$$

Therefore if (b,σ,σ_0) is bounded, by (2.4.2), the sequence $(\overline{\mathbb{P}}^m)_{m\geq1}$ is relatively compact under the weak convergence topology. Otherwise, if $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ with Assumption 2.4.1 satisfies, as $p'>p$, then with the estimate (2.4.1), it follows by [49, Proposition A.2] that $(\overline{\mathbb{P}}^m)_{m\geq1}$ is relatively compact under $\mathcal{W}_p$. By a sub-sequence, let us assume that $\overline{\mathbb{P}}^m\longrightarrow\overline{\mathbb{P}}$, as m goes to ∞, for some $\overline{\mathbb{P}}\in\mathcal{P}(\overline{\Omega})$, and then show that $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(\nu)$.

Recall that, for each $\varphi\in C_b^2(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d\times\mathcal{C}^\ell)$, the process $\overline{S}^\varphi$ is defined on $\overline{\Omega}$ by (5.4.4). Similarly, we define the processes $\overline{S}^{\varphi,m}=(\overline{S}_t^{\varphi,m})_{t\in[0,T]}$ on $\overline{\Omega}$ by

$$\overline{S}_t^{\varphi,m} := \varphi(X_t,Y_t,W_t,B_t)-\int_0^t\mathcal{L}_r\varphi\big(X_{[r]^m\wedge\cdot},Y_{[r]^m\wedge\cdot},W_{[r]^m\wedge\cdot},B_{[r]^m\wedge\cdot},a_0,\overline{\mu}_{[r]^m}\big)\mathrm{d}r.$$

By the continuity of the coefficient functions (b,σ,σ_0), then uniform continuity on a compact set, it is straightforward to check that on each compact subset $\overline{\Omega}_c \subset \overline{\Omega}$, one has

$$\lim_{m\to\infty} \sup_{\bar\omega\,\in\,\overline{\Omega}_c} \big|\overline{S}_t^{\varphi,m}(\bar\omega) - \overline{S}_t^{\varphi}(\bar\omega)\big| = 0, \text{ for every } t\in[0,T]. \tag{2.4.3}$$

Further, whatever if (b,σ,σ_0) is bounded or if (b,σ,σ_0) satisfies Assumption 2.4.1 with $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, we have

$$\sup_{m\geq 1} \mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big|\overline{S}_t^{\varphi,m}\big|^{p'}\big] < \infty, \text{ for all } t\in[0,T],\ \varphi\in C_b^2(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d\times\mathcal{C}^\ell). \tag{2.4.4}$$

Now, as $(\overline{\mathbb{P}}^m)_{m\in\mathbb{N}^*}$ is relatively compact, for each $\varepsilon>0$, we can find a compact subset $\overline{\Omega}_\varepsilon\subset\overline{\Omega}$ such that $\overline{\mathbb{P}}^m[\overline{\Omega}_\varepsilon]\geq 1-\varepsilon$ for all $m\geq 1$. For any bounded continuous function $\phi\in C_b(\mathcal{C}^n\times\mathcal{C}^n\times\mathcal{C}^d\times\mathcal{C}^\ell\times\mathcal{P}(\mathcal{C}^n\times\mathcal{C}^n\times\mathcal{C}^d))$ and $s\leq t$, we denote $\Phi_s := \phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_s\big)$, with (b,σ,σ_0) bounded or with (b,σ,σ_0) satisfying Assumption 2.4.1 and $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, it follows that

$$\begin{aligned}
\big|\mathbb{E}^{\overline{\mathbb{P}}}\big[\big(\overline{S}_t^{\varphi}-\overline{S}_s^{\varphi}\big)\Phi_s\big]\big| &= \lim_{m\to\infty}\big|\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big(\overline{S}_t^{\varphi}-\overline{S}_s^{\varphi}\big)\Phi_s\big]\big| \\
&\leq \limsup_{m\to\infty}\big|\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big(\overline{S}_t^{\varphi}-\overline{S}_s^{\varphi}\big)\Phi_s 1_{\overline{\Omega}_\varepsilon}\big]\big| + \limsup_{m\to\infty}\big|\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big(\overline{S}_t^{\varphi}-\overline{S}_s^{\varphi}\big)\Phi_s 1_{\overline{\Omega}_\varepsilon^c}\big]\big| \\
&\leq \limsup_{m\to\infty}\Big[\big|\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big(\overline{S}_t^{\varphi,m}-\overline{S}_s^{\varphi,m}\big)\Phi_s\big]\big| + \big|\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big(\overline{S}_t^{\varphi,m}-\overline{S}_s^{\varphi,m}\big)\Phi_s 1_{\overline{\Omega}_\varepsilon^c}\big]\big| + \big|\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\big(\overline{S}_t^{\varphi}-\overline{S}_s^{\varphi}\big)\Phi_s 1_{\overline{\Omega}_\varepsilon^c}\big]\big|\Big] \\
&\leq C\varepsilon^{\frac{p'-1}{p'}},
\end{aligned}$$

where the last inequality follows by Hölder's inequality together with (2.4.4) and the fact that $\overline{\mathbb{P}}^m[\overline{\Omega}_\varepsilon^c]\leq\varepsilon$, for all $m\geq 1$. Let $\varepsilon\longrightarrow 0$, it follows that $\overline{S}^\varphi$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–martingale for all $\varphi\in C_b^2(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d\times\mathcal{C}^\ell)$.

Moreover, by almost the same arguments as in the proof of Proposition 3.4.6, we have for all $t\in[0,T]$,

$$\widehat{\mu}_t = \mathcal{L}^{\overline{\mathbb{P}}}(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big|\overline{\mathcal{G}}_T) = \mathcal{L}^{\overline{\mathbb{P}}}(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big|\overline{\mathcal{G}}_t),\ \overline{\mathbb{P}}^\infty\text{–a.s.}$$

Finally, it is easy to see that $\overline{\mathbb{P}}\circ(X_0)^{-1}=\nu$ and $(B,\widehat{\mu})$ is independent of (W,X_0) under $\overline{\mathbb{P}}$, then the first statement of the theorem follows. When Assumption 2.4.1 is satisfied with $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, one has by estimate (2.4.1) combined with Fatou Lemma, $\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X\|^{p'}\big]<\infty$. Therefore we deduce the second statement. □

Chapter 3

Relaxed formulation and characterization of the limits

3.1 Introduction

In this chapter, we are interested in establishing the limit theory for the McKean–Vlasov optimal control problem. In other words, we wish to rigorously prove that the stochastic ontrol problem of McKean-Vlasov equation naturally arises as the limit of a large population optimal control problem. In the uncontrolled case, this property is by now extremely well–known, and usually referred to as "propagation of chaos". Much effort has been devoted to it since the seminal works of Kac [91] and McKean Jr. [121], see also the illuminating lecture notes of Snitzman [149]. Without any claim to comprehensiveness, we refer to Oelschläger [133], and Gärtner [71] for models in the Markovian context without common noise, to Budhiraja, Dupuis, and Fischer [35] for a large deviation principle associated to the limit theory, and also to Méléard and Roelly-Coppoletta [125], Jourdain and Méléard [89], and Oelschläger [134] for the case of 'strong' and 'moderate' interactions, to Shkolnikov [142], Jourdain and Reygner [90] for rank–based models, and finally to Méléard [124], and Graham and Méléard [74] for Boltzmann–type models.

In the controlled case, Fischer and Livieri [67] studied a mean–variance optimisation problem stemming from mathematical finance, and obtained results in this direction. For general McKean–Vlasov controlled equations, such a limit theory has been proved in Lacker [104] in a context without common noise, where an essential tool is a compactness argument, which is made accessible by formulating an appropriate relaxed control for McKean–Vlasov equations, in the spirit of El Karoui, Huu Nguyen, and Jeanblanc-Picqué [63], and by introducing suitable martingale problems, similar to those of Stroock and Varadhan [150]. The same formulation and arguments have also been used in Bahlali, Mezerdi, and Mezerdi [15; 16; 17; 18] and Chala [50] to study stability and approximation problems.

In the present chapter, our first main objective is to establish the corresponding limit theory. To this end, in the restricted case where the common noise part σ_0 is not controlled, and the dependence of the coefficient functions b, σ, and σ_0 in $\mathcal{L}(X^\alpha_{t\wedge\cdot}, \alpha_t|B)$ is through $\mathcal{L}(X^\alpha_{t\wedge\cdot}|B)$ only (in words, the conditional law of the control process is not included in the coefficient function), we introduce a relaxed formulation. We subsequently prove that any relaxed control rule can be approximated by strong control rules, in the sense of weak convergence of probability measures on the canonical space. Besides, the relaxed formulation enjoys an additional closedness property, implying the existence of optimal control rules under mild additional technical conditions. The closedness property and the equivalence results between the relaxed and the strong formulations (hence with the weak formulation also see Chapter 2) are also crucially used to obtain the limit theory.

Our main contribution lies in the fact that we are generalising several fundamental results for McKean–Vlasov control problems to a context with common noise, including the formulation of the weak and relaxed problems, their equivalence, and the corresponding limit theory. The presence of the common noise creates some significant technical difficulties (see Introduction and Chapter 2). We then need to formulate appropriate notions of relaxed control rules, and develop new techniques to ensure the approximation property. Another technical difficulty comes from the presence of the conditional law of the control process α in the coefficient functions (for the strong formulation), this situation has been rarely studied in the literature, and will be the subject of the next chapter.

An important point is that our approach also bypasses a technical issue in the literature considering relaxed formulations for McKean–Vlasov control problems without common noise, namely [104; 17], and which proves equivalence results between several formulations. Indeed, their proofs are based on an incorrect technical result in an unpublished, and actually inaccessible paper [122][1], see Remark 3.3.8 for more details. We instead adapt the approximation arguments in [63] to remedy this technical gap.

The rest of the chapter is structured as follows. We introduce in Section 3.2 the notion of relaxed formulation for the McKean–Vlasov stochastic control problem in a common noise and non–Markovian setting, and define also an N–particles (strong) control problem. The main results of the chapter are presented in Section 3.2.3, including the existence of optimal control, the equivalence between the strong, weak and relaxed formulations and the limit theory. Most of the technical proofs are completed in Section 3.3.

In this chapter, we work with the following additional assumptions:

Assumption 3.1.1. *There exist Borel measurable functions* $(b^\circ, \sigma^\circ, L^\circ) : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \times A \longrightarrow \mathbb{R}^n \times \mathbb{S}^{n\times d}$ *and* $\sigma_0^\circ : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \longrightarrow \mathbb{S}^{n\times \ell}$ *such that, for all* $(t, \mathbf{x}, \bar\nu, a) \in [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A$, *with* $\nu(\mathrm{d}\mathbf{x}) := \bar\nu(\mathrm{d}\mathbf{x}, A)$

$$(b, \sigma, L)(t, \mathbf{x}, a, \bar\nu) = (b^\circ, \sigma^\circ, L^\circ)(t, \mathbf{x}, a, \nu), \text{ and } \sigma_0(t, \mathbf{x}, a, \bar\nu) = \sigma_0^\circ(t, \mathbf{x}, \nu).$$

By abuse of notations, we still write (b, σ, L, σ_0) *in lieu of* $(b^\circ, \sigma^\circ, L^\circ, \sigma_0^\circ)$.

3.2 Relaxed formulation and limit theory without law of controls

3.2.1 The relaxed formulation

In the classical optimal control theory, the relaxed control/relaxed formulation has been introduced to recover some closed and convex properties, while ensuring that each relaxed control could be appropriately approximated by strong or weak control rules. The point was that it then becomes easier in this formulation to deduce the existence and stability properties of the optimal solution, while ensuring under mild conditions that the value of the problem is not modified. In the context of McKean–Vlasov optimal control with common noise, an appropriate relaxed formulation is not easy to find especially because of the common noise and the presence of a control term in the volatility σ.

Usually, a relaxed formulation is presented on a canonical space with the help of martingale problem. Here, for sake of clarity and brevity, and an easy reading, we choose to present this formulation as the weak formulation Definition 2.2.1. This point of view is of course equivalent to the formulation on a canonical space as we will see in Section 3.3.1.

Definition 3.2.1 (relaxed control)**.** *Let* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, *we say that a term*

$$\beta := \big(\Omega^\beta, \mathcal{F}^\beta, \mathbb{P}^\beta, \mathbb{F}^\beta := (\mathcal{F}_t^\beta)_{0\le t\le T}, \mathbb{G}^\beta := (\mathcal{G}_t^\beta)_{0\le t\le T}, X^\beta, Y^\beta, W^\beta, B^\beta, N^\beta, \mu^\beta, \Lambda^\beta\big),$$

is a relaxed control associated with the initial (distribution) condition ν *if*

(*i*) $(\Omega^\beta, \mathcal{F}^\beta, \mathbb{P}^\beta)$ *is a probability space, equipped with two filtrations* $\mathbb{F}^\beta$ *and* $\mathbb{G}^\beta$ *such that, for all* $t \in [0,T]$

$$\mathcal{G}_t^\beta \subseteq \mathcal{F}_t^\beta, \text{ and } \mathbb{E}^{\mathbb{P}^\beta}\big[1_D \big| \mathcal{G}_t^\beta\big] = \mathbb{E}^{\mathbb{P}^\beta}\big[1_D \big| \mathcal{G}_T^\beta\big], \ \mathbb{P}^\beta\text{–a.s., for all } D \in \mathcal{F}_t^\beta \vee \sigma(W^\beta); \tag{3.2.1}$$

(*ii*) $X^\beta := (X_s^\beta)_{s\in[0,T]}$ *and* $Y^\beta := (Y_s^\beta)_{s\in[0,T]}$ *are two* $\mathbb{R}^n$*–valued* $\mathbb{F}^\beta$*–adapted continuous process and* $\Lambda^\beta := (\Lambda_s^\beta)_{0\le s\le T}$ *is an* $\mathcal{P}(A)$*–valued* $\mathbb{F}^\beta$*–predictable process such that:* $\mathbb{E}^{\mathbb{P}^\beta}\Big[\|X^\beta\|^p + \|Y^\beta\|^p + \int_0^T \int_A \big(\rho(a, a_0)\big)^p \Lambda_s^\beta(\mathrm{d}a)\mathrm{d}s\Big] < \infty$;

[1] Through personal communications with S. Méléard, it was confirmed to us that she and her co–authors discovered a mistake soon after finishing the paper, and hence abandoned it. Nevertheless, although the original manuscript is now nowhere accessible, some of its results have been announced in the conference proceedings [123]. More specifically, the problematic result is [123, Corollary on pages 196–197], which has been crucially used in [17, Proposition 2.2.], and [104, Lemma 7.1.].

(iii) (W^β, B^β) *is an* $\mathbb{R}^d \times \mathbb{R}^\ell$*–valued standard Brownian motion with respect to* $\mathbb{F}^\beta$*,* B^β *is in addition adapted to* $\mathbb{G}^\beta$*,* $\mathcal{F}_0^\beta \vee \sigma(W^\beta)$ *is independent of* $\mathcal{G}_T^\beta$*, and* μ^β *is a* $\mathcal{P}(\mathcal{C}^n)$*–valued* $\mathbb{G}^\beta$*–predictable process such that*

$$\mu_t^\beta = \mathcal{L}^{\mathbb{P}^\beta}\big(X^\beta_{t\wedge\cdot}\big|\mathcal{G}_t^\beta\big),\ \mathbb{P}^\beta\text{–a.s.;}$$

(iv) (X^β, Y^β) *satisfy:* $\mathbb{P}^\beta[X_0^\beta = Y_0^\beta] = 1$*, with* $\mathbb{P}^\beta \circ (X_0^\beta)^{-1} = \nu$*, and* $Y_\cdot^\beta = X_\cdot^\beta - \int_0^\cdot \sigma_0(s, X^\beta_{s\wedge\cdot}, \mu_s^\beta)\mathrm{d}B_s^\beta$*,* $\mathbb{P}^\beta$*–a.e.*

(v) *for* $\mathbb{P}^\beta$*–a.e.*$\omega \in \Omega^\beta$*,* $N^\beta = \big(N^{1,\beta}, \dots, N^{d,\beta}\big)$ *is an* $\Big(\mathbb{F}^\beta, \mathbb{P}_\omega^{\beta,\mathcal{G}_T^\beta}\Big)$*–martingale measure with intensity* $\Lambda_t^\beta(\mathrm{d}a)\mathrm{d}t$*, the martingales* $(\widehat{N}^{i,\beta})_{1\le i\le d}$ *are orthogonal, and satisfy:* $\mathbb{P}_\omega^{\beta,\mathcal{G}_T^\beta}$*–a.s., for all* $t \in [0,T]$*,* $W_t^\beta = \iint_{[0,t]\times A} \widehat{N}^\beta(\mathrm{d}a, \mathrm{d}s)$*, and*

$$Y_t^\beta = Y_0^\beta + \int_0^t\int_A b(s, X^\beta_{s\wedge\cdot}, \mu_s^\beta, a)\Lambda_s^\beta(\mathrm{d}a)\mathrm{d}s + \int_0^t\int_A \sigma(s, X^\beta_{s\wedge\cdot}, \mu_s^\beta, a)N^\beta(\mathrm{d}a, \mathrm{d}s),\ \textit{for all}\ t\in[0,T],\ \mathbb{P}_\omega^{\beta,\mathcal{G}_T^\beta}\text{–a.s.} \tag{3.2.2}$$

Let β be a relaxed control and introduce

$$\widehat{\mu}^\beta := \mathcal{L}^{\mathbb{P}^\beta}\big((X^\beta, Y^\beta, \Lambda^\beta, W^\beta)\big|\mathcal{G}_T^\beta\big),$$

we define $\overline{\mathcal{P}}_R$ the set of control rules as the following subset of probability measure on $\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathbb{M} \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathbb{M})$

$$\overline{\mathcal{P}}_R(\nu) := \big\{\mathbb{P}^\beta \circ \big(X^\beta, Y^\beta, \Lambda^\beta, W^\beta, B^\beta, \widehat{\mu}^\beta\big)^{-1} : \beta \text{ a relaxed control with initial condition } \nu\big\}. \tag{3.2.3}$$

The relaxed formulation of the McKean–Vlasov control problem is then defined by, with $J\big(\overline{\mathbb{P}}\big)$ given in (2.3.6),

$$V_R(\nu) := \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_R(\nu)} J\big(\overline{\mathbb{P}}\big).$$

Remark 3.2.2. *Under* Assumption 3.1.1*, the reward function* L *depends on* ν *(instead of* $\bar\nu$*). In this case, and in contrast to the general situation in* Remark 2.3.5*, the map* $\mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathbb{M}) \ni \widehat{\mu} \longmapsto \delta_{\mu_t}(\mathrm{d}\nu)\mathrm{d}t \in \mathbb{M}(\mathcal{C}^n)$ *is continuous, so that* $\overline{\mathbb{P}} \longmapsto J(\overline{\mathbb{P}})$ *is lower semi–continuous (resp. continuous) as soon as* L *and* g *are lower semi–continuous and bounded from below (resp. continuous and bounded).*

Remark 3.2.3. (i) *The martingale problem under* $\overline{\mathbb{P}}$ *in* Definition 2.3.1 *involves conditional distributions in the coefficient functions, which creates some regularity problem in the approximation procedure, since conditional distributions are not continuous with respect to joint distributions. By considering the conditional equality* (3.2.2) *(or equivalently the conditional martingale problem) under* $\mathbb{P}_\omega^{\beta,\mathcal{G}_T^\beta}$*–a.s., the* $\mu^\beta(\omega)$ *term in the coefficient functions becomes deterministic, which in turn allows to avoid the regularity problem. This (conditional) equality or martingale problem is partially inspired from a technical proof of* [104]*, but in our context with common noise, we need to consider a family of martingale problems (see the formulation on the canonical space in* Proposition 3.3.2 *), and deal with some non–trivial measurability issues. Notice also that the process* Y^γ *does not play an essential role in the strong or weak formulations, but he is crucially used in the conditional martingale problem in* Proposition 3.3.2*.*

(ii) *With our techniques, we are only able to prove the equivalence* $V_W = V_R$ *(c.f.* Theorem 3.2.4*), as well as the desired approximation results, under* Assumption 3.1.1*. For more general cases, it seems to be a very challenging problem that we would like to leave for future research. We nonetheless point out the fact that the great majority of the extant literature on either mean–field games or McKean–Vlasov control problems with common noise, does not allow for* σ_0 *or* σ *to be controlled as well, see for instance* Ahuja [8], Bensoussan, Frehse, and Yam [25], Cardaliaguet, Delarue, Lasry, and Lions [38], Carmona, Fouque, and Sun [48], Carmona, Delarue, and Lacker [49], Graber [73], Guéant, Lasry, and Lions [75], Kolokoltsov and Troeva [95], Lacker [103]*, and* Lacker and Webster [107]*. Notable exceptions are* Carmona and Delarue [41]*, though the discussion in the general setting remains at a rather informal level there, the monograph by* Carmona and Delarue [44]*, although all the main results given have uncontrolled common noise,* Pham and Wei [138]*, though*

the problem is considered in a Markovian setting, with feedback controls, and no limit theory is explored, Pham [137] *and* Yong [152] *where only linear quadratic problems are considered,* Bayraktar, Cosso, and Pham [21]*, though no limit theory is addressed there as well. We would also like to highlight the recent work of* Acciaio, Backhoff Veraguas, and Carmona [1] *which derives a general stochastic Pontryagin maximum principle for McKean–Vlasov control problems in strong formulation without common noise, where the coefficients depend on the joint law of the control and the state process. The authors also consider a weak formulation for their problem, but with uncontrolled volatility and for a drift which does not depend on the law of the controls, deriving again a stochastic maximum principle. Finally* Élie, Mastrolia, and Possamaï [65] *considers a contract theory problem with a principal and mean–field agents, without common noise and where only the drift is controlled but can depend on the law of the controls, as well as* Élie, Hubert, Mastrolia, and Possamaï [64] *which also considers a contract theory problem, but with common noise and volatility controls.*

3.2.2 A large population stochastic control problem with common noise

One of the main objectives of this chapter is to provide the limit theory for the McKean–Vlasov control problem, that is, the problem $V_S(\nu)$ in (2.2.3) can be seen as the limit of a large population problem. Let N be a positive integer, we consider the canonical space

$$\Omega^N := \big(\mathbb{R}^n \times \mathcal{C}^d\big)^N \times \mathcal{C}^\ell,$$

with canonical process $\big((X_0^1, \dots, X_0^N), (W^1, \dots, W^N), B\big)$ and canonical filtration $\mathbb{F}^N := (\mathcal{F}_t^N)_{0 \le t \le T}$ defined by

$$\mathcal{F}_t^N := \sigma\big((X_0^i, W_s^i, B_s) : i \in \{1, \dots, N\},\ s \in [0,t]\big),\ t \in [0,T].$$

Fix some $(\nu^1, \dots, \nu^N) \in \mathcal{P}_p(\mathbb{R}^n)^N$, and define $\nu_N := \nu^1 \otimes \cdots \otimes \nu^N$ the corresponding product measure. We consider the probability measure $\mathbb{P}_\nu^N$ on $\big(\Omega^N, \mathcal{F}^N\big)$ with $\mathcal{F}^N := \mathcal{B}(\Omega^N)$, under which $X_0 := (X_0^1, \dots, X_0^N)$ has distribution ν_N, and $(W^1, \dots, W^N, B)$ is a standard Brownian motion, independent of X_0. Let us denote by $\mathcal{A}_p^N(\nu_N)$ the collection of all processes $\alpha := (\alpha^i)_{i=1,\dots,N}$, where each $\alpha^i := (\alpha_t^i)_{0 \le t \le T}$ is an A–valued, $\mathbb{F}^N$–predictable process satisfying

$$\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T \big(\rho(\alpha_s^i, a_0)\big)^p \mathrm{d}s\right] < \infty.$$

Then under standard Lipschitz conditions on the coefficient functions (see Assumption 1.4.1), for every fixed $(\alpha^1, \dots, \alpha^N) \in \mathcal{A}_p^N(\nu_N)$, there is a unique $(\mathbb{R}^n)^N$–valued $\mathbb{F}^N$–adapted continuous process $(X^{\alpha,1}, \dots, X^{\alpha,N})$ satisfying: for $i \in \{1, \dots, N\}$, $\mathbb{E}^{\mathbb{P}_\nu^N}\big[\|X^{\alpha,i}\|^p\big] < \infty$ and

$$X_t^{\alpha,i} = X_0^i + \int_0^t b\big(s, X_{s\wedge\cdot}^{\alpha,i}, \varphi_s^N, \alpha_s^i\big)\mathrm{d}s + \int_0^t \sigma\big(s, X_{s\wedge\cdot}^{\alpha,i}, \varphi_s^N, \alpha_s^i\big)\mathrm{d}W_s^i + \int_0^t \sigma_0\big(s, X_{s\wedge\cdot}^{\alpha,i}, \varphi_s^N, \alpha_s^i\big)\mathrm{d}B_s,\ t \in [0,T], \tag{3.2.4}$$

where

$$\varphi_s^N(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(X_{s\wedge\cdot}^{\alpha,i}, \alpha_s^i)}(\mathrm{d}\mathbf{x}, \mathrm{d}a),\ \mathrm{d}s \otimes \mathrm{d}\mathbb{P}_\nu^n\text{–a.e., and } \varphi_s^{N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{X_{s\wedge\cdot}^{\alpha,i}}(\mathrm{d}\mathbf{x}),\ \text{for all } s \in [0,T].$$

The value function of the large population stochastic control problem is then defined by

$$V_S^N(\nu^1, \dots, \nu^N) := \sup_{\alpha \in \mathcal{A}_p^N(\nu_N)} J_N(\alpha),\ \text{where } J_N(\alpha) := \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T L\big(t, X_{t\wedge\cdot}^{\alpha,i}, \varphi_t^N, \alpha_t^i\big)\mathrm{d}t + g\big(X_{T\wedge\cdot}^{\alpha,i}, \varphi_T^{N,X}\big)\right]. \tag{3.2.5}$$

3.2.3 Equivalence with the relaxed formulation and limit theory

Let us now provide the main results of this chapter. The first one consists in the equivalence between the different formulations of the McKean–Vlasov control problem. Recall that the constants p, p', and $\hat{p}$ are fixed in Assumption 1.4.1.

Theorem 3.2.4. *Let* Assumption 1.4.1 *hold true. Suppose in addition that* Assumption 3.1.1 *are verified.*

(i) Then, for every $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, $\overline{\mathcal{P}}_R(\nu)$ is a non–empty convex closed subset of $\mathcal{P}_p(\overline{\Omega})$, under the Wasserstein topology $\mathcal{W}_p$.

(ii) For $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, and $A \subset \mathbb{R}^j$ for some $j \geq 1$. Then the set $\overline{\mathcal{P}}_W(\nu)$ is dense in the closed set $\overline{\mathcal{P}}_R(\nu)$ under $\mathcal{W}_p$, and consequently

$$V_S(\nu) = V_W(\nu) = V_R(\nu).$$

If L and g are continuous in all their arguments, there exists some $\overline{\mathbb{P}}^\star \in \overline{\mathcal{P}}_R(\nu)$ such that $V_R(\nu) = J\big(\overline{\mathbb{P}}^\star\big)$.

Remark 3.2.5. *As* Theorem 2.2.3*, the results in* Theorem 3.2.4 *extend those in the no–common noise setting in* Lacker [104] *and used different techniques of approximations because of the presence of the common noise.*

Remark 3.2.6. *A natural question that we have not addressed is that of the existence of so–called feedback controls, since* Theorem 3.2.4.*(iii) only gives existence of an optimal relaxed control. It is known in classical control theory that* Filippov*'s condition* [66]*, which was notably used by* Haussmann and Lepeltier [79]*, and by* Lacker [102; 104] *for* MFGs *and McKean–Vlasov control problems without common noise, is usually sufficient to obtain, from any relaxed control, a control depending on the trajectories of X only, and which achieves no worse value. In the common noise context, things become slightly more subtle. The intuitive result is that one should be able to obtain a similar result but with controls depending on the trajectories of both X and μ. In* Lacker, Shkolnikov, and Zhang [108] *will exactly prove such a result, with the additional desirable property that the feedback controls preserve the marginal laws of (X, μ).*

We next provide some results related to the limit theory, that is, the large population control problem converges to the McKean–Vlasov control problem under technical conditions. For every $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, we denote by $\overline{\mathcal{P}}^\star_R(\nu)$ the set of optimal relaxed controls

$$\overline{\mathcal{P}}^\star_R(\nu) := \big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu) : V_R(\nu) = J(\overline{\mathbb{P}})\big\}.$$

Let $(\nu^1, \dots, \nu^N) \in \mathcal{P}_p(\mathbb{R}^n)$, $\nu_N := \nu^1 \otimes \cdots \otimes \nu^N$ and $\alpha = (\alpha^1, \dots, \alpha^N) \in \mathcal{A}_p^N(\nu_N)$, we define

$$\mathbb{P}^N(\alpha^1, \dots, \alpha^N) := \frac{1}{N}\sum_{i=1}^N \mathcal{L}^{\mathbb{P}^N_\nu}\big(X^{\alpha,i}, Y^{\alpha,i}, \delta_{\alpha^i_t}(\mathrm{d}a)\mathrm{d}t, W^i, B, \overline{\varphi}_N\big) \in \mathcal{P}(\overline{\Omega}), \tag{3.2.6}$$

where $Y^{\alpha,i}_\cdot := X^{\alpha,i}_\cdot - \int_0^\cdot \sigma_0(s, X^{\alpha,i}, \varphi^N_s, \alpha^i_s)\mathrm{d}B_s$ and $\overline{\varphi}_N := \frac{1}{N}\sum_{i=1}^N \delta_{\left(X^{\alpha,i}, Y^{\alpha,i}, \delta_{\alpha^i_t}(\mathrm{d}a)\mathrm{d}t, W^i\right)}$.

Theorem 3.2.7. *Let* Assumption 1.4.1 *and* Assumption 3.1.1 *hold true, assume that $A \subset \mathbb{R}^j$ for some $j \geq 1$, and that L and g are continuous in all their arguments. With the constants p and p' given in* Assumption 1.4.1*, let $(\nu^i)_{i\geq 1} \subset \mathcal{P}_{p'}(\mathbb{R}^n)$ be such that $\sup_{N\geq 1}\frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x|^{p'}\nu^i(\mathrm{d}x) < \infty$.*

(i) Let $\big(\overline{\mathbb{P}}^N\big)_{N\geq 1}$ be given by $\overline{\mathbb{P}}^N := \mathbb{P}^N(\alpha^{N,1}, \dots, \alpha^{N,N})$, where $(\alpha^{N,1}, \dots, \alpha^{N,N}) \in \mathcal{A}_p^N(\nu_N)$ satisfies

$$J(\alpha^{N,1}, \dots, \alpha^{N,N}) \geq V_S^N(\nu^1, \dots, \nu^N) - \varepsilon_N, \text{ for all } N \geq 1, \tag{3.2.7}$$

for a sequence $(\varepsilon_N)_{N\geq 1} \subset \mathbb{R}_+$ satisfying $\lim_{N\to\infty}\varepsilon_N = 0$. Then the sequence $(\overline{\mathbb{P}}^N)_{N\geq 1}$ is relatively compact under $\mathcal{W}_p$, and for any converging subsequence $\big(\overline{\mathbb{P}}^{N_m}\big)_{m\geq 1}$, we have

$$\lim_{m\to\infty}\mathcal{W}_p\bigg(\frac{1}{N_m}\sum_{i=1}^{N_m}\nu^i, \nu\bigg) = 0, \text{ for some } \nu \in \mathcal{P}_p(\mathbb{R}^n), \text{ and } \lim_{m\to\infty}\mathcal{W}_p\big(\overline{\mathbb{P}}^{N_m}, \overline{\mathbb{P}}^\infty\big) = 0, \text{ for some } \overline{\mathbb{P}}^\infty \in \overline{\mathcal{P}}^\star_R(\nu).$$

(ii) Assume in addition that $\mathcal{W}_p\big(N^{-1}\sum_{i=1}^N \nu^i, \nu\big) \underset{N\to\infty}{\longrightarrow} 0$, for some $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, and let $\overline{\mathbb{P}}^\star \in \overline{\mathcal{P}}^\star_R(\nu)$. Then we can construct a sequence $(\overline{\mathbb{P}}^N)_{N\geq 1}$, together with $(\alpha^{N,1}, \dots, \alpha^{N,N})_{N\geq 1}$ satisfying (3.2.7), such that $\mathcal{W}_p\big(\overline{\mathbb{P}}^N, \overline{\mathbb{P}}^\star\big) \underset{N\to\infty}{\longrightarrow} 0$.

(iii) Finally, we have

$$\lim_{N\to\infty}\bigg|V_S^N\big(\nu^1, \dots, \nu^N\big) - V_S\bigg(\frac{1}{N}\sum_{i=1}^N \nu^i\bigg)\bigg| = 0. \tag{3.2.8}$$

Proposition 3.2.8. *Let* Assumption 1.4.1 *and* Assumption 3.1.1 *hold true, suppose in addition that $A \subset \mathbb{R}^j$ for some $j \geq 1$, and that L and g are continuous in all their arguments. With the constants p and p' given in* Assumption 1.4.1*, let $(\nu^m)_{m\geq 1} \subset \mathcal{P}_{p'}(\mathbb{R}^n)$ and $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ be such that*

$$\sup_{m\geq 1} \int_{\mathbb{R}^n} |x|^{p'} \nu^m(\mathrm{d}x) < \infty, \text{ and } \lim_{m\to\infty} \mathcal{W}_p\big(\nu^m, \nu\big) = 0.$$

Then

$$\lim_{m\to\infty} V_S(\nu^m) = V_S\big(\nu\big). \tag{3.2.9}$$

In particular, the map $V_S : \mathcal{P}_{p'}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is continuous.

Remark 3.2.9. *(i) As far as we know, the above results are new in the setting with presence of common noise. Even without taking into account the common noise, our results in* Theorem 3.2.7 *and* Proposition 3.2.8 *are also more general than the existing ones. In particular, by taking $\ell = 0$, we recover the most essential results in* Lacker [104] *for the case without common noise. But in* Theorem 3.2.7*, the initial distribution does not need to be convergent, it is only required to have finite moments in a uniform way, and the initial condition for each agents are allowed to have different distributions. Moreover, the continuity result of the value function $V_S(\nu)$ in* Proposition 3.2.8 *requires less technical conditions (such as the Lipschitz assumptions on L and g) than in* [138, Lemma 3.3.].

(ii) Theorem 3.2.7 *shows that any ε_N–optimal control of the large population stochastic control problem converges towards an optimal control of the McKean-Vlasov stochastic control problem. In particular, when there exists a unique strong optimal control of the McKean–Vlasov control problem, any ε_N–optimal control of the large population control problem converges towards this control.*

3.3 Proof of equivalence with the relaxed formulation

3.3.1 Relaxed formulation on the canonical space

As in the strong and weak formulation, we formulate the relaxed formulation on the canonical space. To do this as mentioned in Chapter 2, we use the notion of "admissible control". We recall briefly this kind of controls (defined on the canonical space). We have two canonical spaces

$$\widehat{\Omega} := \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d, \text{ and } \overline{\Omega} := \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathcal{P}\big(\widehat{\Omega}\big),$$

equipped with the corresponding canonical element $\big(\widehat{X}, \widehat{Y}, \widehat{\Lambda}, \widehat{W}\big)$, its Borel σ–algebra $\widehat{\mathcal{F}} := \mathcal{B}(\widehat{\Omega})$, and its canonical filtration $\widehat{\mathbb{F}} := \big(\widehat{\mathcal{F}}_t\big)_{t\in[0,T]}$ defined by

$$\widehat{\mathcal{F}}_t := \sigma\Big(\big(\widehat{X}_s, \widehat{Y}_s, \widehat{\Lambda}([0,s]\times D), \widehat{W}_s\big) : D \in \mathcal{B}(A),\ s \in [0,t]\Big),\ t \in [0,T].$$

and the canonical space $\overline{\Omega}$ with the canonical element $(X, Y, \Lambda, W, B, \widehat{\mu})$, and its Borel σ–algebra $\overline{\mathcal{F}} := \mathcal{B}(\overline{\Omega})$. Based on $\widehat{\mu}$, we recall the three processes $(\mu_t)_{t\in[0,T]}$, $(\overline{\mu}_t)_{t\in[0,T]}$ and $(\widehat{\mu}_t)_{t\in[0,T]}$ on $\overline{\Omega}$ by (recall (1.4.1) for the definition of $\widehat{\Lambda}^t$)

$$\mu_t := \widehat{\mu} \circ \big(\widehat{X}_{t\wedge\cdot}\big)^{-1},\ \overline{\mu}_t(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \mathbb{E}^{\widehat{\mu}}\Big[\delta_{\widehat{X}_{t\wedge\cdot}}(\mathrm{d}\mathbf{x})\widehat{\Lambda}_t(\mathrm{d}a)\Big], \text{ and } \widehat{\mu}_t := \widehat{\mu} \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{Y}_{t\wedge\cdot}, \widehat{\Lambda}^t, \widehat{W}\big)^{-1},\ t \in [0,T]. \tag{3.3.1}$$

The two filtrations $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{t\in[0,T]}$ and $\overline{\mathbb{G}} := (\overline{\mathcal{G}}_t)_{t\in[0,T]}$ on $(\overline{\Omega}, \overline{\mathcal{F}})$ are defined by

$$\overline{\mathcal{F}}_t := \sigma\Big((X_s, Y_s, \Lambda([0,s]\times D), W_s, B_s, \langle\widehat{\mu}_s, \phi\rangle) : D \in \mathcal{B}(A), \phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d),\ s \in [0,t]\Big).$$

and

$$\overline{\mathcal{G}}_t := \sigma\Big((B_s, \langle\widehat{\mu}_s, \phi\rangle) : \phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d), s \in [0,t]\Big).$$

For all $t \in [0,T]$ and $\varphi \in C_b^2(\mathbb{R}^{n+n+d+\ell})$, the generator generator $\overline{\mathcal{L}}_t$ are defined by

$$\overline{\mathcal{L}}_t\varphi\big(\mathbf{x},\mathbf{y},\mathbf{w},\mathbf{b},\bar{\nu},a\big) := \bar{b}(t,\mathbf{x},\bar{\nu},a)\cdot\nabla\varphi(\mathbf{x}(t),\mathbf{y}(t),\mathbf{w}(t),\mathbf{b}(t)) + \frac{1}{2}\mathrm{Tr}\big[\bar{a}(t,\mathbf{x},\bar{\nu},a)\nabla^2\varphi(\mathbf{x}(t),\mathbf{y}(t),\mathbf{w}(t),\mathbf{b}(t))\big], \tag{3.3.2}$$

where

$$\bar{b}\big(t,\mathbf{x},\mathbf{w},\mathbf{b},\bar{\nu},a\big) := \begin{pmatrix} b(t,\mathbf{x},\bar{\nu},a) \\ b(t,\mathbf{x},\bar{\nu},a) \\ 0_d \\ 0_\ell \end{pmatrix},\ \bar{a}\big(t,\mathbf{x},\mathbf{w},\mathbf{b},\bar{\nu},a\big) := \begin{pmatrix} \sigma(t,\mathbf{x},\bar{\nu},a) & \sigma_0(t,\mathbf{x},\bar{\nu},a) \\ \sigma(t,\mathbf{x},\bar{\nu},a) & 0_{n\times\ell} \\ \mathrm{I}_{d\times d} & 0_{d\times\ell} \\ 0_{\ell\times d} & \mathrm{I}_{\ell\times\ell} \end{pmatrix}\begin{pmatrix} \sigma(t,\mathbf{x},\bar{\nu},a) & \sigma_0(t,\mathbf{x},\bar{\nu},a) \\ \sigma(t,\mathbf{x},\bar{\nu},a) & 0_{n\times\ell} \\ \mathrm{I}_{d\times d} & 0_{d\times\ell} \\ 0_{\ell\times d} & \mathrm{I}_{\ell\times\ell} \end{pmatrix}^\top.$$

Then, for any $\varphi \in C_b^2(\mathbb{R}^{n+n+d+\ell})$, $\overline{S}^\varphi := (\overline{S}_t^\varphi)_{t\in[0,T]}$ on $\overline{\Omega}$ by

$$\overline{S}_t^\varphi := \varphi(X_t,Y_t,W_t,B_t) - \iint_{[0,t]\times A} \overline{\mathcal{L}}_s\varphi\big(X_s,Y_s,W_s,B_s,\overline{\mu}_s,a\big)\Lambda_s(\mathrm{d}a)\mathrm{d}s,\ t\in[0,T], \tag{3.3.3}$$

where for a borel function $\phi : [0,T] \to \mathbb{R}$, $\int_0^\cdot \phi(s)\mathrm{d}s := \int_0^\cdot \phi^+(s)\mathrm{d}s - \int_0^\cdot \phi^-(s)\mathrm{d}s$ with the convention $\infty-\infty=-\infty$.

Definition 3.3.1. *Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. A probability $\overline{\mathbb{P}}$ on $(\overline{\Omega},\overline{\mathcal{F}})$ is an admissible control rule with initial condition ν if*

(i) $\overline{\mathbb{P}}\big[X_0 = Y_0,\ W_0 = 0,\ B_0 = 0\big] = 1$, $\overline{\mathbb{P}}\circ X_0^{-1} = \nu$, *and* (X,Λ) *satisfy* $\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X\|^p + \iint_{[0,T]\times A}\big(\rho(a_0,a)\big)^p\Lambda_t(\mathrm{d}a)\mathrm{d}t\big] < \infty$;

(ii) the pair (X_0,W) is independent of $\overline{\mathcal{G}}_T$ under $\overline{\mathbb{P}}$, and for all $t\in[0,T]$

$$\widehat{\mu}_t(\bar{\omega}) = \overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_T}\circ(X_{t\wedge\cdot},Y_{t\wedge\cdot},\Lambda^t,W)^{-1},\ \text{for } \overline{\mathbb{P}}\text{–a.e. } \bar{\omega}\in\overline{\Omega}; \tag{3.3.4}$$

(iii) the process $\big(\overline{S}_t^\varphi\big)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–martingale for all $\varphi\in C_b^2\big(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^\ell\big)$.

Let us then define for any $\nu\in\mathcal{P}_p(\mathbb{R}^n)$,

$$\overline{\mathcal{P}}_A(\nu) := \big\{\text{All admissible control rules } \overline{\mathbb{P}} \text{ with initial condition } \nu\big\}.$$

Relaxed formulation on the canonical space In our context, when the coefficient functions (b,σ,σ_0,L,g) do not depend on the marginal distribution $\bar{\nu}$ or ν, so that the control problem degenerates to the classical one, the relaxed control rule coincides with the admissible control rule $\overline{\mathcal{P}}_A(\nu)$ (see above Definition 3.3.1). For general McKean–Vlasov control problems, it is not hard to prove that $\overline{\mathcal{P}}_A(\nu)$ is closed and convex. However, in general, it is not the closure of the set of strong or weak control rules in the context with common noise (see Example 3.3.3 below). This motivated us to consider a more restrictive case (see Definition 3.2.1 for instance), where the common noise is not controlled, for which we are able to provide an appropriate relaxed control rule set as a subset of $\overline{\mathcal{P}}_A(\nu)$, which is both convex and the closure of $\overline{\mathcal{P}}_S(\nu)$ or $\overline{\mathcal{P}}_W(\nu)$.

Let us introduce a martingale problem on $(\widehat{\Omega},\widehat{\mathcal{F}})$. For any $(t,\mathbf{x},\nu,a)\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n)\times A$

$$\hat{b}\big(t,\mathbf{x},\nu,a\big) := \begin{pmatrix} b(t,\mathbf{x},\nu,a) \\ 0_d \end{pmatrix},\ \hat{a}\big(t,\mathbf{x},\nu,a\big) := \begin{pmatrix} \sigma(t,\mathbf{x},a,\nu) \\ \mathrm{I}_d \end{pmatrix}\begin{pmatrix} \sigma(t,\mathbf{x},a,\nu) \\ \mathrm{I}_d \end{pmatrix}^\top,$$

and for all $\varphi\in C_b^2(\mathbb{R}^{n+d})$ and $(t,\mathbf{x},\mathbf{y},\mathbf{w},\nu,a)\in[0,T]\times\mathcal{C}^n\times\mathcal{C}^n\times\mathcal{C}^d\times\mathcal{P}(\mathcal{C}^n)\times A$

$$\widehat{\mathcal{L}}_t\varphi\big(\mathbf{x},\mathbf{y},\mathbf{w},\nu,a\big) := \hat{b}(t,\mathbf{x},\nu,a)\cdot\nabla\varphi(\mathbf{y}(t),\mathbf{w}(t)) + \frac{1}{2}\mathrm{Tr}\big[\hat{a}(t,\mathbf{x},\nu,a)\nabla^2\varphi(\mathbf{y}(t),\mathbf{w}(t))\big]. \tag{3.3.5}$$

Then given a family $(\nu(t))_{0\le t\le T}$ of probability measures in $\mathcal{P}(\mathcal{C}^n)$ such that $[0,T]\ni t\longmapsto\nu(t)\in\mathcal{P}(\mathcal{C}^n)$ is Borel measurable, and $\varphi\in C_b^2(\mathbb{R}^{n+d})$, we introduce a process $(\widehat{M}_t^{\varphi,\nu})_{t\in[0,T]}$ on $(\widehat{\Omega},\widehat{\mathcal{F}})$ by

$$\widehat{M}_t^{\varphi,\nu} := \varphi\big(\widehat{Y}_t,\widehat{W}_t\big) - \varphi(\widehat{Y}_0,\widehat{W}_0) - \iint_{[0,t]\times A}\widehat{\mathcal{L}}_s\varphi\big(\widehat{X},\widehat{Y},\widehat{W},\nu(s),a\big)\widehat{\Lambda}_s(\mathrm{d}a)\mathrm{d}s,\ (t,\varphi)\in[0,T]\times C_b^2(\mathbb{R}^{n+d}). \tag{3.3.6}$$

Recall that for a borel function $\phi : [0,T]\to\mathbb{R}$, we write $\int_0^\cdot\phi(s)\mathrm{d}s := \int_0^\cdot\phi^+(s)\mathrm{d}s - \int_0^\cdot\phi^-(s)\mathrm{d}s$ with the convention $\infty-\infty=-\infty$.

Proposition 3.3.2. *Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. A probability measure $\overline{\mathbb{P}} \in \mathcal{P}(\overline{\Omega})$ is a relaxed control rule with initial condition ν i.e. $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$ if and only if $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$, and moreover, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, the process $\widehat{M}^{\varphi,\mu(\bar{\omega})}$ is an $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$–martingale for each $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$, where $\mu(\bar{\omega}) := (\mu_t(\bar{\omega}))_{t\in[0,T]}$ is defined from $\widehat{\mu}(\bar{\omega})$ as in (3.3.1).*
Therefore,

$$\overline{\mathcal{P}}_R(\nu) = \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu) : \overline{\mathbb{P}} \text{ –a.e. } \bar{\omega} \in \overline{\Omega},\ \big(\widehat{M}_t^{\varphi,\mu(\bar{\omega})}\big)_{t\in[0,T]} \text{ is an } \big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)\text{–martingale for each } \varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)\Big\}. \quad (3.3.7)$$

Proof. Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$. By Itô's formula and Proposition 3.3.9, it is straightforward to check that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$ and $\overline{\mathbb{P}}$ –a.e. $\bar{\omega} \in \overline{\Omega}$, $\big(\widehat{M}_t^{\varphi,\mu(\bar{\omega})}\big)_{t\in[0,T]}$ is an $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$–martingale for each $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$.
Now, we take $\overline{\mathbb{P}}$ belonging to the set of the right side of the equality (3.3.7). By Proposition 3.3.5 (see below), there exists a family of measure–valued processes $(\widehat{N}^{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ such that, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, on an extension $(\widehat{\Omega}^\star, \widehat{\mathbb{F}}^\star, \widehat{\mu}(\bar{\omega}) \otimes \lambda)$ of $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega}) \otimes \lambda)$ where λ is the Lebesgue measure on $[0,1]$, $\widehat{N}^{\bar{\omega}} = \big(\widehat{N}^{1,\bar{\omega}}, \dots, \widehat{N}^{d,\bar{\omega}}\big)$ is an $\big(\widehat{\mathbb{F}}^\star, \widehat{\mu}(\bar{\omega}) \otimes \lambda\big)$–martingale measure with intensity $\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t$, the martingales $(\widehat{N}^{i,\bar{\omega}})_{1\le i\le d}$ are orthogonal, and satisfy $\widehat{\mu}(\bar{\omega}) \otimes \lambda$–a.s.

$$\widehat{Y}_t = \widehat{X}_0 + \iint_{[0,t]\times A} b\big(r, \widehat{X}, \mu(\bar{\omega}), a\big)\widehat{\Lambda}_r(\mathrm{d}a)\mathrm{d}r + \iint_{[0,t]\times A} \sigma\big(r, \widehat{X}, \mu(\bar{\omega}), a\big)\widehat{N}^{\bar{\omega}}(\mathrm{d}a, \mathrm{d}r),\ \widehat{W}_t = \iint_{[0,t]\times A} \widehat{N}^{\bar{\omega}}(\mathrm{d}a, \mathrm{d}s), \quad (3.3.8)$$

Moreover, let $\widehat{\mathbb{H}}^\star = (\widehat{\mathcal{H}}_t^\star)_{t\in[0,T]}$ with $\widehat{\mathcal{H}}_t^\star := \overline{\mathcal{G}}_t \otimes \widehat{\mathcal{F}}_t^\star$ be a filtration on $\overline{\Omega} \times \widehat{\Omega}^\star$, denote by $\mathcal{P}^{\widehat{\mathbb{H}}^\star}$ the predictable σ–algebra on $[0,T] \times \overline{\Omega} \times \widehat{\Omega}^\star$ with respect to $\widehat{\mathbb{H}}^\star$. Then for all bounded $\mathcal{P}^{\widehat{\mathbb{H}}^\star} \otimes \mathcal{B}(A)$–measurable function $f : [0,T] \times \overline{\Omega} \times \widehat{\Omega}^\star \times A \longrightarrow \mathbb{R}$, one can define the stochastic integral $\iint_{[0,t]\times A} f^{\bar{\omega}}(s,a)\widehat{N}^{\bar{\omega}}(\mathrm{d}s, \mathrm{d}a)$ in such a way that

$$(t, \bar{\omega}, \hat{\omega}^\star) \longmapsto \Big(\iint_{[0,t]\times A} f^{\bar{\omega}}(s,a)\widehat{N}^{\bar{\omega}}(\mathrm{d}a, \mathrm{d}s)\Big)(\hat{\omega}^\star) \text{ is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

We define $\widetilde{\Omega} := \widehat{\Omega}^\star \times \overline{\Omega}$, $\widetilde{\mathcal{F}} := \widehat{\mathcal{F}}^\star \otimes \overline{\mathcal{G}}$, $\widetilde{\mathcal{F}}_t := \widehat{\mathcal{F}}_t^\star \otimes \overline{\mathcal{G}}_t$, for all $t \in [0,T]$, and for $\tilde{\omega} := (\hat{\omega}^\star, \bar{\omega}) \in \widetilde{\Omega}$,

$$\widetilde{X}(\tilde{\omega}) := \widehat{X}(\hat{\omega}^\star),\ \widetilde{Y}(\tilde{\omega}) := \widehat{Y}(\hat{\omega}^\star),\ \widetilde{W}(\tilde{\omega}) := \widehat{W}(\hat{\omega}^\star),\ \widetilde{B}(\tilde{\omega}) := B(\bar{\omega}),\ \widetilde{\Lambda}(\tilde{\omega}) := \widehat{\Lambda}(\hat{\omega}^\star),\ \widetilde{N}(\tilde{\omega}) := \widehat{N}^{\bar{\omega}}(\hat{\omega}^\star).$$

We equip $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ with the probability $\widetilde{\mathbb{P}}$ defined as follow:

$$\widetilde{\mathbb{P}}(K \times A) := \int_A \widehat{\mu}(\bar{\omega}) \otimes \lambda(K)\overline{\mathbb{P}}(\mathrm{d}\bar{\omega}), \text{ for all } K \in \widehat{\mathcal{F}}^\star \text{ and } A \in \overline{\mathcal{G}}.$$

Let $(P_t)_{t\in[0,T]}$ be the process $P_t(\tilde{\omega}) := (B_t(\bar{\omega}), \widehat{\mu}_t(\bar{\omega}))$, we introduce $(\widetilde{\mathcal{G}}_t := \sigma\{P_s,\ s \le t\})_{t\in[0,T]}$. As $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$, by eq. (3.3.4), we deduce that $\widetilde{\mathbb{P}} \circ \big(\widetilde{X}, \widetilde{Y}, \widetilde{W}, \widetilde{B}, \widetilde{\Lambda}, \widetilde{\mathbb{P}}^{\widetilde{\mathcal{G}}_T} \circ \big(\widetilde{X}, \widetilde{Y}, \widetilde{W}, \widetilde{\Lambda}\big)^{-1}\big)^{-1} = \overline{\mathbb{P}} \circ \big(X, Y, W, B, \Lambda, \widehat{\mu}\big)^{-1}$, and using eq. (3.3.8), one has

$$\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, \widetilde{\mathbb{F}} := (\widetilde{\mathcal{F}}_t)_{0\le t\le T}, \widetilde{\mathbb{G}} := (\widetilde{\mathcal{G}}_t)_{0\le t\le T}, \widetilde{X}, \widetilde{Y}, \widetilde{W}, \widetilde{B}, \widetilde{N}, (\widetilde{\mathbb{P}}^{\widetilde{\mathcal{G}}_t} \circ (\widetilde{X})^{-1})_{t\in[0,T]}, \widetilde{\Lambda}\big)$$

is a relaxed control.

□

We observe that $\overline{\mathcal{P}}_R(\nu) \subset \overline{\mathcal{P}}_A(\nu)$ by Proposition 3.3.2. The next example shows that $\overline{\mathcal{P}}_R(\nu)$ is a proper subset of $\overline{\mathcal{P}}_A(\nu)$.

Example 3.3.3. *Let us consider the case where: $n = d = \ell = 1$, $\nu = \delta_0$, $A = \{a_1, a_2\} \subset \mathbb{R}$, $b = 0$, $\sigma(t, \mathbf{x}, a, \bar{\nu}) = a\mathrm{I}_n$, and $\sigma_0 = \mathrm{I}_n$. Consider a filtered probability space $(\Omega^\star, \mathcal{F}^\star, \mathbb{F}^\star, \mathbb{P}^\star)$ supporting an $\mathbb{R}^{d+d+\ell}$–valued standard Brownian motion $(W^1, W^2, B^\star)$, let*

$$X_t^\star := a_1\frac{\sqrt{2}}{2}W_t^1 + a_2\frac{\sqrt{2}}{2}W_t^2 + B_t^\star,\ W_t^\star := \frac{\sqrt{2}}{2}W_t^1 + \frac{\sqrt{2}}{2}W_t^2,\ \overline{W}_t^\star := \frac{\sqrt{2}}{2}W_t^1 - \frac{\sqrt{2}}{2}W_t^2,\ \mathcal{G}_t^\star := \sigma\big((B_s^\star, \overline{W}_s^\star) : 0 \le s \le t\big).$$

By setting $Y^\star_\cdot := X^\star_\cdot - B^\star_\cdot$ *and* $\Lambda^\star_t(\mathrm{d}a)\mathrm{d}t := \frac{1}{2}\delta_{a_1}(\mathrm{d}a)\mathrm{d}t + \frac{1}{2}\delta_{a_2}(\mathrm{d}a)\mathrm{d}t$, *it is direct to check that*

$$\overline{\mathbb{P}} := \mathcal{L}^{\mathbb{P}^\star}\Big(X^\star, Y^\star, \Lambda^\star, W^\star, B^\star, \mathcal{L}^{\mathbb{P}^\star}\big(X^\star, Y^\star, \Lambda^\star, W^\star, B^\star \big| \mathcal{G}^\star_T\big)\Big) \in \overline{\mathcal{P}}_A(\nu).$$

However, one observes that

$$Y^\star_\cdot = \Big(\frac{1}{2}a_1 + \frac{1}{2}a_2\Big)W^\star_\cdot + \Big(\frac{1}{2}a_1 - \frac{1}{2}a_2\Big)\overline{W}^\star_\cdot,$$

is not an Itō process under the conditional law $\mathbb{P}^\star$ *knowing* $\mathcal{G}^\star_T$. *Consequently, one has* $\overline{\mathbb{P}} \notin \overline{\mathcal{P}}_R(\nu)$.

We next show that $\overline{\mathcal{P}}_W(\nu) \subset \overline{\mathcal{P}}_R(\nu)$ (see Proposition 2.3.6 for a representation of $\overline{\mathcal{P}}_W(\nu)$), where we use crucially the fact that $\widehat{\mu}_t$ is the conditional law of $(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W)$, and not only of $(X_{t\wedge\cdot}, \Lambda^t)$.

Proposition 3.3.4. *Let* $\nu \in \mathcal{P}_p(\mathcal{C}^n)$, *every* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(\nu)$ *belongs to* $\overline{\mathcal{P}}_R(\nu)$.

Proof. It follows by Proposition 2.3.3 that the continuous process $\widehat{W}$ is an $(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega}))$–Brownian motion, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$. Using the definition of weak control rules in Proposition 2.3.6 (see also proof of Proposition 3.3.9), it is direct to deduce that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $\widehat{W}$ is an $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$–Brownian motion, and

$$\widehat{Y}_t = \widehat{X}_0 + \int_0^t b\big(s, \widehat{X}_\cdot, \hat{\alpha}_s, \mu_s(\bar{\omega})\big)\mathrm{d}s + \int_0^t \sigma\big(s, \widehat{X}_\cdot, \hat{\alpha}_s, \mu_s(\bar{\omega})\big)\mathrm{d}\widehat{W}_s, \quad t \in [0,T],\ \widehat{\mu}(\bar{\omega})\text{–a.s.},$$

where $(\hat{\alpha}_t)_{t\in[0,T]}$ is an $\widehat{\mathbb{F}}$–predictable process satisfying $\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}a = \delta_{\hat{\alpha}_t}(\mathrm{d}a)\mathrm{d}t$. It follows that, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $\widehat{M}^{\varphi,\mu(\bar{\omega})}$ is an $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$–martingale for each $\varphi \in C^2_b(\mathbb{R}^n \times \mathbb{R}^d)$, and hence $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$. □

3.3.2 Approximating relaxed controls by weak control rules

We provide here an approximation result of relaxed control rules by weak control rules. The Definition 3.2.1 although simple to present, hides some measurability problems associated to the martingale measure N^β. To achieving our approximation, we need to use the canonical representation, and construct a martingale measure which share some "good" measurability properties (see Section 3.3.4.2 for a brief reminder on the notions/properties of martingale measure as introduced by El Karoui and Méléard [58].)

Recall that $\widehat{\Omega} := \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d$. Let us also introduce an abstract filtered probability space $(\Omega^\star, \mathcal{F}^\star, \mathbb{F}^\star := (\mathcal{F}^\star_t)_{t\in[0,T]}, \mathbb{P}^\star)$, equipped with $2(n+d)$ i.i.d. martingale measures $(N^{\star,i})_{1\le i\le 2(n+d)}$, with intensity $\nu_0(\mathrm{d}a)\mathrm{d}t$, for some diffuse probability measure ν_0 on A, and a sequence of i.i.d. standard d–dimensional Brownian motions $(W^{\star,i})_{i\ge1}$. Let us define

$$\widehat{\Omega}^\star := \widehat{\Omega} \times \Omega^\star,\ \widehat{\mathcal{F}}^\star := \widehat{\mathcal{F}} \otimes \mathcal{F}^\star,\ \widehat{\mathcal{F}}^\star_t := \widehat{\mathcal{F}}_t \otimes \mathcal{F}^\star_t,\ \widehat{\mathbb{P}}_{\bar{\omega}} := \widehat{\mu}(\bar{\omega}) \otimes \mathbb{P}^\star, \text{ for all } t \in [0,T], \text{ and } \bar{\omega} \in \overline{\Omega}.$$

The random elements $(\widehat{X}, \widehat{Y}, \widehat{\Lambda}, \widehat{W})$ and $(N^\star, W^{\star,i}, i \ge 1)$ can then naturally be extended to $\widehat{\Omega}^\star$. Let us first provide an improved version of [58, Theorem IV–2], whose proof is completed in Appendix 3.3.4.2.

Proposition 3.3.5. *Let* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ *and* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$. *Then there exists a family of measure–valued processes* $(\widehat{N}^{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ *such that, for* $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $\widehat{N}^{\bar{\omega}} = \big(\widehat{N}^{1,\bar{\omega}}, \dots, \widehat{N}^{d,\bar{\omega}}\big)$ *is an* $\big(\widehat{\mathbb{F}}^\star, \widehat{\mathbb{P}}_{\bar{\omega}}\big)$–*martingale measure with intensity* $\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t$, *the martingales* $(\widehat{N}^{i,\bar{\omega}})_{1\le i\le d}$ *are orthogonal, and satisfy*

$$\widehat{Y}_t = \widehat{X}_0 + \iint_{[0,t]\times A} b\big(r, \widehat{X}, \mu(\bar{\omega}), a\big)\widehat{\Lambda}_r(\mathrm{d}a)\mathrm{d}r + \iint_{[0,t]\times A} \sigma\big(r, \widehat{X}, \mu(\bar{\omega}), a\big)\widehat{N}^{\bar{\omega}}(\mathrm{d}a, \mathrm{d}r),\ \widehat{W}_t = \iint_{[0,t]\times A} \widehat{N}^{\bar{\omega}}(\mathrm{d}a, \mathrm{d}s),\ \widehat{\mathbb{P}}_{\bar{\omega}}\text{–a.s.} \tag{3.3.9}$$

Moreover, let $\widehat{\mathbb{H}}^\star = (\widehat{\mathcal{H}}^\star_t)_{t\in[0,T]}$ *with* $\widehat{\mathcal{H}}^\star_t := \overline{\mathcal{G}}_t \otimes \widehat{\mathcal{F}}^\star_t$ *be a filtration on* $\overline{\Omega} \times \widehat{\Omega}^\star$, *denote by* $\mathcal{P}^{\widehat{\mathbb{H}}^\star}$ *the predictable* σ–*algebra on* $[0,T] \times \overline{\Omega} \times \widehat{\Omega}^\star$ *with respect to* $\widehat{\mathbb{H}}^\star$. *Then for all bounded* $\mathcal{P}^{\widehat{\mathbb{H}}^\star} \otimes \mathcal{B}(A)$–*measurable function* $f : [0,T] \times \overline{\Omega} \times \widehat{\Omega}^\star \times A \longrightarrow \mathbb{R}$, *one can define the stochastic integral* $\iint_{[0,t]\times A} f^{\bar{\omega}}(s,a)\widehat{N}^{\bar{\omega}}(\mathrm{d}s, \mathrm{d}a)$ *in such a way that*

$$(t, \bar{\omega}, \hat{\omega}^\star) \longmapsto \Big(\iint_{[0,t]\times A} f^{\bar{\omega}}(s,a)\widehat{N}^{\bar{\omega}}(\mathrm{d}a, \mathrm{d}s)\Big)(\hat{\omega}^\star) \text{ is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.} \tag{3.3.10}$$

Proposition 3.3.6. *Let* Assumption 1.4.1 *and* Assumption 3.1.1 *hold, assume that* $A \subset \mathbb{R}^j$ *for* $j \geq 1$, *and that* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$ *with the constant* p' *given in* Assumption 1.4.1. *Then for every* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$, *there exists a sequence* $\big(\overline{\mathbb{P}}^m\big)_{m\geq 1} \subset \overline{\mathcal{P}}_W(\nu)$ *such that*

$$\lim_{m\to\infty} \mathcal{W}_p\big(\overline{\mathbb{P}}^m, \overline{\mathbb{P}}\big) = 0.$$

Proof. We only provide here the proof with the additional condition that σ_0 is a constant, which illustrates better our main ideas. We refer to Section 3.3.4.3 for a proof in the general case.

First, let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$, recall from Proposition 3.3.5 that on the enlarged filtered space $\big(\widehat{\Omega}^\star, \widehat{\mathcal{F}}^\star, \widehat{\mathbb{F}}^\star\big)$, we have a family $(\widehat{N}^{\bar\omega})_{\bar\omega\in\overline{\Omega}}$ such that $\widehat{N}^{\bar\omega}$ is a martingale measure with intensity $\Lambda_t(\mathrm{d}a)\mathrm{d}t$ under the probability measure $\widehat{\mathbb{P}}_{\bar\omega} := \widehat{\mu}(\bar\omega) \otimes \mathbb{P}^\star$, for $\overline{\mathbb{P}}$–a.e. $\bar\omega \in \overline{\Omega}$, and

$$\widehat{X}_t = \widehat{X}_0 + \iint_{[0,t]\times A} b\big(r, \widehat{X}, \mu(\bar\omega), a\big)\widehat{\Lambda}_r(\mathrm{d}a)\mathrm{d}r + \iint_{[0,t]\times A} \sigma\big(r, \widehat{X}, \mu(\bar\omega), a\big)\widehat{N}^{\bar\omega}(\mathrm{d}a, \mathrm{d}r) + \sigma_0 B_t(\bar\omega),\ t \in [0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.},$$
$$\widehat{W}_t = \iint_{[0,t]\times A} \widehat{N}^{\bar\omega}(\mathrm{d}a, \mathrm{d}s),\ t \in [0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}$$

By Lemma 3.3.7 below, there exists, on $(\widehat{\Omega}^\star, \widehat{\mathcal{F}}^\star)$, a sequence $(\widehat{\mathbb{F}}^{\star,m})_{m\geq 1}$ of sub–filtrations of $\widehat{\mathbb{F}}^\star$, together with a sequence of family of processes $\big(\hat\alpha^m, (\widehat{W}^{\bar\omega,m})_{\bar\omega\in\overline{\Omega}}, (\widehat{X}^{\bar\omega,m})_{\bar\omega\in\overline{\Omega}}\big)_{m\geq 1}$, where $\hat\alpha^m$ is an A–valued $\widehat{\mathbb{F}}^m$–predictable process for each $m \geq 1$, and for $\overline{\mathbb{P}}$–a.e. $\bar\omega \in \overline{\Omega}$, $\widehat{W}^{\bar\omega,m}$ is an $(\widehat{\mathbb{F}}^{\star,m}, \widehat{\mathbb{P}}_{\bar\omega})$–Brownian motion, and

$$\widehat{\mathbb{P}}_{\bar\omega}\Big[\lim_{m\to\infty} \widehat{\Lambda}^m(\mathrm{d}a, \mathrm{d}t) = \widehat{\Lambda}(\mathrm{d}a, \mathrm{d}t)\Big] = 1,\ \lim_{m\to\infty} \mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}^{\bar\omega,m}, \widehat{\Lambda}^m(\mathrm{d}a, \mathrm{d}t), \widehat{W}^{\bar\omega,m}\big),\ \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}, \widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t, \widehat{W}\big)\Big) = 0, \quad (3.3.11)$$

with $\widehat{\Lambda}^m(\mathrm{d}a, \mathrm{d}t) = \delta_{\hat\alpha^m_t}(\mathrm{d}a)\mathrm{d}t$, and

$$\widehat{X}^{\bar\omega,m}_t = \widehat{X}_0 + \int_0^t b\big(r, \widehat{X}^{\bar\omega,m}, \widehat{\mathbb{P}}_{\bar\omega} \circ (\widehat{X}^{\bar\omega,m})^{-1}, \hat\alpha^m_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{X}^{\bar\omega,m}, \widehat{\mathbb{P}}_{\bar\omega} \circ (\widehat{X}^{\bar\omega,m})^{-1}, \hat\alpha^m_r\big)\mathrm{d}\widehat{W}^{\bar\omega,m}_r + \sigma_0 B_t(\bar\omega), \quad (3.3.12)$$

and for each $m \geq 1$

$$(t, \bar\omega, \hat\omega^\star) \longmapsto \big(\widehat{X}^{\bar\omega,m}_{t\wedge\cdot}(\hat\omega^\star), (\widehat{\Lambda}^m)^t(\hat\omega^\star), \widehat{W}^{\bar\omega,m}_{t\wedge\cdot}(\hat\omega^\star)\big) \text{ is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable}, \quad (3.3.13)$$

so that, with the predictable σ–algebra $\mathcal{P}^{\overline{\mathbb{G}}}$ on $[0,T]\times\overline{\Omega}$ with respect to $\overline{\mathbb{G}}$

$$(t, \bar\omega) \longmapsto \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}^{\bar\omega,m}_{t\wedge\cdot}, (\widehat{\Lambda}^m)^t, \widehat{W}^{\bar\omega,m}\big) \text{ is } \mathcal{P}^{\overline{\mathbb{G}}}\text{–measurable}.$$

Further, let us denote $\widehat{Y}^{\bar\omega,m} := \widehat{X}^{\bar\omega,m} - \sigma_0 B(\bar\omega)$ and

$$\overline{\mathbb{P}}^m := \int_{\overline{\Omega}} \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big(\widehat{X}^{\bar\omega,m}, \widehat{Y}^{\bar\omega,m}, \widehat{\Lambda}^m, \widehat{W}^{\bar\omega,m}, B(\bar\omega), \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}^{\bar\omega,m}, \widehat{Y}^{\bar\omega,m}, \widehat{\Lambda}^m, \widehat{W}^{\bar\omega,m}\big)\Big)\overline{\mathbb{P}}(\mathrm{d}\bar\omega).$$

It follows by (3.3.11) that $\lim_{m\to\infty} \mathcal{W}_p(\overline{\mathbb{P}}^m, \overline{\mathbb{P}}) = 0$. To conclude, it is enough to show that $\overline{\mathbb{P}}^m \in \overline{\mathcal{P}}_W(\nu)$. Since, by construction, $\overline{\mathbb{P}}^m[\Lambda \in \mathbb{M}_0] = 1$, then it is enough to show that $\overline{\mathbb{P}}^m \in \overline{\mathcal{P}}_R(\nu)$. To this end, let us check that $\overline{\mathbb{P}}^m$ satisfies all the conditions in Proposition 3.3.9.

It is easy to check that $\overline{\mathbb{P}}^m\big[\widehat{\mu}\circ(\widehat{X}_0)^{-1} = \nu, X_0 = Y_0, W_0 = 0, B_0 = 0\big] = 1$ and $\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\|X\|^p + \iint_{[0,T]\times A} |a - a_0|^p \Lambda_t(\mathrm{d}a)\mathrm{d}t\big] < \infty$. Furthermore, for every $\phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)$, $\varphi \in C_b(\mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d))$ and $t \in [0,T]$, we have

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\langle\phi, \widehat{\mu}_t\rangle\varphi\big(B, \widehat{\mu}\big)\big] &= \int_{\overline{\Omega}} \mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\big[\phi\big(\widehat{X}^{\bar\omega,m}_{t\wedge\cdot}, \widehat{Y}^{\bar\omega,m}_{t\wedge\cdot}, (\widehat{\Lambda}^m)^t, \widehat{W}^{\bar\omega,m}\big)\big]\varphi\Big(B(\bar\omega), \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}^{\bar\omega,m}, \widehat{Y}^{\bar\omega,m}, \widehat{\Lambda}^m, \widehat{W}^{\bar\omega,m}\big)\Big)\overline{\mathbb{P}}(\mathrm{d}\bar\omega) \\ &= \mathbb{E}^{\overline{\mathbb{P}}^m}\big[\phi\big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big)\varphi\big(B, \widehat{\mu}\big)\big].\end{aligned}$$

This implies that

$$\widehat{\mu}_t = (\overline{\mathbb{P}}^m)^{\overline{\mathcal{G}}_t} \circ \big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big)^{-1} = (\overline{\mathbb{P}}^m)^{\overline{\mathcal{G}}_T} \circ \big(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, \Lambda^t, W\big)^{-1},\ \overline{\mathbb{P}}^m\text{–a.s.}$$

Next, for all $\phi \in C_b(\mathbb{R}^\ell)$, $\psi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d \times \mathcal{C}^\ell \times C([0,T]; \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d)))$, and $s \in [0,t]$, we have

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\phi(B_t - B_s)\psi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot}\big)\big] \\
&= \int_{\overline{\Omega}} \phi\big(B_t(\bar\omega) - B_s(\bar\omega)\big)\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big[\psi\big(\widehat{X}^{\bar\omega,m}_{s\wedge\cdot}, \widehat{Y}^{\bar\omega,m}_{s\wedge\cdot}, (\widehat{\Lambda}^m)^s, \widehat{W}^{\bar\omega,m}_{s\wedge\cdot}, B_{s\wedge\cdot}(\bar\omega), \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}^{\bar\omega,m}_{s\wedge\cdot}, \widehat{Y}^{\bar\omega,m}_{s\wedge\cdot}, (\widehat{\Lambda}^m)^s, \widehat{W}^{\bar\omega,m}\big)\big)\Big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega) \\
&= \mathbb{E}^{\overline{\mathbb{P}}^m}\big[\phi\big(B_t(\bar\omega) - B_s(\bar\omega)\big)\big]\int_{\overline{\Omega}} \mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big[\psi\big(\widehat{X}^{\bar\omega,m}_{s\wedge\cdot}, \widehat{Y}^{\bar\omega,m}_{s\wedge\cdot}, (\widehat{\Lambda}^m)^s, \widehat{W}^{\bar\omega,m}_{s\wedge\cdot}, B_{s\wedge\cdot}(\bar\omega), \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\big(\widehat{X}^{\bar\omega,m}_{s\wedge\cdot}, \widehat{Y}^{\bar\omega,m}_{s\wedge\cdot}, (\widehat{\Lambda}^m)^s, \widehat{W}^{\bar\omega,m}\big)\big)\Big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega) \\
&= \mathbb{E}^{\overline{\mathbb{P}}^m}\big[\phi(B_t - B_s)\big]\mathbb{E}^{\overline{\mathbb{P}}^m}\big[\psi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot}\big)\big],
\end{aligned}$$

which implies that B has independent increments with respect to $(\overline{\mathbb{P}}^m, \overline{\mathbb{F}})$. Besides, since $\overline{\mathbb{P}}^m \circ B^{-1}$ is the Wiener measure, it follows that B is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}}^m)$–Brownian motion. Also, as $Z = X - Y = \sigma_0 B$, one has immediately that S^f (defined in (3.3.24)) is an $(\overline{\mathbb{F}}^\circ, \overline{\mathbb{P}}^m)$–martingale for all $f \in C_b^2(\mathbb{R}^{n+\ell})$. Finally, by construction, Condition (iii) in Proposition 3.3.9 is also satisfied. Therefore, $\overline{\mathbb{P}}^m \in \overline{\mathcal{P}}_R(\nu)$, and hence $\overline{\mathbb{P}}^m \in \overline{\mathcal{P}}_W(\nu)$. □

Lemma 3.3.7. *Let us stay in the context of* Proposition 3.3.6*, and assume in addition that σ_0 is a constant. Then on the space $(\widehat{\Omega}^\star, \widehat{\mathcal{F}}^\star)$, there exists a sequence $(\widehat{\mathbb{F}}^{\star,m})_{m\geq 1}$ of sub–filtrations of $\widehat{\mathbb{F}}^\star$, together with a sequence of family of processes $\big(\hat\alpha^m, (\widehat{W}^{\bar\omega,m})_{\bar\omega\in\overline{\Omega}}, (\widehat{X}^{\bar\omega,m})_{\bar\omega\in\overline{\Omega}}\big)_{m\geq 1}$, where $\hat\alpha^m$ is an A–valued $\widehat{\mathbb{F}}^{\star,m}$–predictable process for each $m \geq 1$, and for $\overline{\mathbb{P}}$–a.e. $\bar\omega \in \overline{\Omega}$, $\widehat{W}^{\bar\omega,m}$ is an $(\widehat{\mathbb{F}}^{\star,m}, \widehat{\mathbb{P}}_{\bar\omega})$–Brownian motion, and with $\widehat{\Lambda}^m(\mathrm{d}a, \mathrm{d}t) = \delta_{\hat\alpha^m_t}(\mathrm{d}a)\mathrm{d}t$ and $\widehat{X}^{\bar\omega,m}$ be defined in (3.3.12), the convergence and measurability results in (3.3.11) and (3.3.13) hold true.*

Proof. We will adapt the arguments in [63, Theorem 4.9.] to approximate, under each $\widehat{\mathbb{P}}_{\bar\omega}$, the process

$$\widehat{X}_t = \widehat{X}_0 + \iint_{[0,t]\times A} b\big(r, \widehat{X}, \mu(\bar\omega), a\big)\widehat{\Lambda}_r(\mathrm{d}a, \mathrm{d}r) + \iint_{[0,t]\times A} \sigma\big(r, \widehat{X}, \mu(\bar\omega), a\big)\widehat{N}^{\bar\omega}(\mathrm{d}a, \mathrm{d}r) + \sigma_0 B_t(\bar\omega),\ t \in [0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.},$$

and at the same time check the measurability property at each step.

Step 1. We first show that one can assume w.l.o.g. that $A \subset \mathbb{R}^j$ is a compact set. Indeed, for each $e \geq 1$, let us denote $A_e := A \cap [-e, e]^j$, $\pi_e : A \longrightarrow A_e$ the projection from A to A_e, and then define $\widehat{\Lambda}^e$ and for all $\bar\omega \in \overline{\Omega}$, $\widehat{N}^{\bar\omega,e}$ by

$$\iint_{[0,T]\times A} \phi(s,a)\widehat{\Lambda}^e(\mathrm{d}a, \mathrm{d}r) := \iint_{[0,T]\times A} \phi_e(s,a)\widehat{\Lambda}(\mathrm{d}a, \mathrm{d}r),\ \iint_{[0,T]\times A} \phi(s,a)\widehat{N}^{\bar\omega,e}(\mathrm{d}a, \mathrm{d}r) := \iint_{[0,T]\times A} \phi_e(s,a)\widehat{N}^{\bar\omega}(\mathrm{d}a, \mathrm{d}r),$$

for all $\phi \in C_b([0,T]\times A)$ and $\phi_e(s,a) := \phi(s, \pi_e(a))$. Denote also $(b_e, \sigma_e)(t, \mathbf{x}, \nu, a) := (b, \sigma)(t, \mathbf{x}, \nu, \pi_e(a))$. For $\bar\omega \in \overline{\Omega}$, let $\widehat{X}^{\bar\omega,e}$ be the unique solution to

$$\begin{aligned}
\widehat{X}^{\bar\omega,e}_t = \widehat{X}_0 &+ \iint_{[0,t]\times A} b\big(r, \widehat{X}^{\bar\omega,e}, \mu(\bar\omega), a\big)\widehat{\Lambda}^e_r(\mathrm{d}a)\mathrm{d}r + \iint_{[0,t]\times A} \sigma\big(r, \widehat{X}^{\bar\omega,e}, \mu(\bar\omega), a\big)\widehat{N}^{\bar\omega,e}(\mathrm{d}a, \mathrm{d}r) \\
&+ \sigma_0 B_t(\bar\omega),\ t \in [0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}
\end{aligned}$$

Then by similar arguments as in the proof of Lemma 2.3.10, it is standard to deduce that, for some constant $C > 0$ independent of $e \geq 1$ and $\bar\omega$, and which may change value from line to line

$$\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big[\sup_{t\in[0,T]} |\widehat{X}^{\bar\omega,e}_t - \widehat{X}_s|^p\Big] \leq C\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big[\iint_{[0,t]\times A} \big|\big((b,\sigma) - (b_e,\sigma_e)\big)(t, \widehat{X}, \widehat{\mathbb{P}}_{\bar\omega}\circ(\widehat{X})^{-1}, a)\big|^p\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t\Big].$$

Using the growth conditions on (b,σ) in Assumption 1.4.1, we have

$$\iint_{[0,T]\times A} \big|\big((b,\sigma) - (b_e,\sigma_e)\big)\big(t, \widehat{X}, \widehat{\mathbb{P}}_{\bar\omega}\circ(\widehat{X})^{-1}, a\big)\big|^p\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t \leq C\Big(\big\|\widehat{X}\big\|^p + \mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\big[\big\|\widehat{X}\big\|^p\big]\iint_{[0,T]\times A} |a|^p\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t\Big).$$

It follows by the dominated convergence theorem that: for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$

$$\lim_{e\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big[\sup_{t\in[0,T]} \big|\widehat{X}_t^{\bar{\omega},e} - \widehat{X}_s\big|^p\Big] = 0.$$

Moreover, as A is a Polish subspace of $\mathbb{R}^j$, then A is closed, and hence A_e is compact. This allows to reduce the problem to the case where A is compact.

Step 2. We now assume in addition that A is compact and proceed the proof. By compactness of A, there is a sequence of positive reel numbers $(\delta_e)_{e\geq 1}$ such that $\lim_{e\to\infty}\delta_e = 0$, and for each $e \geq 1$, one can find a partition $(A_1^e, \dots, A_e^e)$ of A and $(a_1^e, \dots, a_e^e)$ satisfying $a_i^e \in A_i^e$ and $|a_i^e - a| < \delta_e$ for all $a \in A_i^e$, $i \in \{1, \dots, e\}$. For $\bar{\omega} \in \overline{\Omega}$, let $\widehat{X}^{\bar{\omega},e}$ be the unique solution to the SDE

$$\widehat{X}_t^{\bar{\omega},e} = \widehat{X}_0 + \sum_{i=1}^e \int_0^t b\big(r, \widehat{X}^{\bar{\omega},e}, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},e})^{-1}, a_i^e\big)\widehat{\Lambda}_r(A_i^e)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{X}^{\bar{\omega},e}, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},e})^{-1}, a_i^e\big)\mathrm{d}\widehat{N}_r^{\bar{\omega}}(A_i^e) + \sigma_0 B_t(\bar{\omega}). \quad (3.3.14)$$

Using again standard arguments as in the proof of Lemma 2.3.10, we obtain that, for some constant $C > 0$ (independent of e and $\bar{\omega}$), which may change from line to line

$$\mathbb{E}^{\widehat{\mathbb{P}}_\omega}\Big[\sup_{t\in[0,T]} \big|\widehat{X}_t^{\bar{\omega},e} - \widehat{X}_t\big|^p\Big] \leq C\mathbb{E}^{\widehat{\mathbb{P}}_\omega}\Big[\sum_{i=1}^e \iint_{[0,T]\times A_i^e} \big|(b,\sigma)(r,\widehat{X},\widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X})^{-1},a) - (b,\sigma)(r,\widehat{X},\overline{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X})^{-1},a_i^e)\big|^p\Lambda_r(\mathrm{d}a)\mathrm{d}r\Big].$$

For every fixed $(r, \mathbf{x}, \nu)$, the map $a \longmapsto (b,\sigma)(r,\mathbf{x},\nu,a)$ is continuous and hence uniformly continuous. Using dominated convergence, it follows that, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$,

$$\lim_{e\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big[\sup_{t\in[0,T]} \big|\widehat{X}_t^{\bar{\omega},e} - \widehat{X}_t\big|^p\Big] = 0. \quad (3.3.15)$$

Recall that the space $\widehat{\Omega}^\star$ is equipped with a sequence of i.i.d. Brownian motion $(W^{\star,i})_{i\geq 1}$. Let us define, for each $i = 1, \dots, e$,

$$\widehat{Z}_t^{\bar{\omega},e,i} := \int_0^t (q_s^{e,i})^{-1/2}\mathbf{1}_{\{q_s^{e,i}\neq 0\}}\mathrm{d}\widehat{N}_s^{\bar{\omega}}(A_i^e) + \int_0^t \mathbf{1}_{\{q_s^{e,i}=0\}}\mathrm{d}W_s^{*,i}, \text{ with } q_s^{e,i} := \widehat{\Lambda}_s(A_i^e), \text{ for all } t \in [0,T].$$

Then it is direct to see that: for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $(\widehat{Z}^{\bar{\omega},e,1}, \dots, \widehat{Z}^{\bar{\omega},e,e})$ is an e–dimensional $\big(\widehat{\mathbb{F}}, \widehat{\mathbb{P}}_{\bar{\omega}}\big)$–Brownian motion, and one can rewrite (3.3.14), for any $t \in [0,T]$

$$\widehat{X}_t^{\bar{\omega},e} = \widehat{X}_0 + \sum_{i=1}^e \int_0^t b\big(r, \widehat{X}^{\bar{\omega},e}, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},e})^{-1}, a_i^e\big)q_r^{e,i}\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{X}^{\bar{\omega},e}, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},e})^{-1}, a_i^m\big)\sqrt{q_r^{e,i}}\mathrm{d}\widehat{Z}_r^{\bar{\omega},e,i} + \sigma_0 B_t(\bar{\omega}), \quad (3.3.16)$$

and

$$\widehat{W}_t = \widehat{Z}_t^{\bar{\omega},e} := \sum_{i=1}^e \int_0^t \sqrt{q_r^{e,i}}\mathrm{d}\widehat{Z}_r^{\bar{\omega},e,i},\ t \in [0,T],\ \widehat{\mathbb{P}}_{\bar{\omega}}\text{–a.s.} \quad (3.3.17)$$

Furthermore, by considering the process $(q_r^{e,i}, i = 1, \dots, e)_{r\in[0,T]}$ as a control process, and using Lemma 2.3.10, we can assume w.l.o.g. that $q^{e,i}$ is an $\widetilde{\mathbb{F}}$–predictable process, and is in addition constant on each interval $[t_k, t_{k+1})$, for $0 = t_0 < t_1 < \cdots < t_K = T$. Let $\widehat{\Lambda}^e(\mathrm{d}a,\mathrm{d}t) := \sum_{i=1}^e q_t^{e,i}\delta_{a_i^e}(\mathrm{d}a)\mathrm{d}t$, it follows by (3.3.15) and (3.3.17) that

$$\widehat{\mathbb{P}}_{\bar{\omega}}\Big[\lim_{e\to\infty}\widehat{\Lambda}^e(\mathrm{d}a,\mathrm{d}t) = \widehat{\Lambda}(\mathrm{d}a,\mathrm{d}t)\Big] = 1,\ \lim_{e\to\infty}\mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\big(\widehat{X}^{\bar{\omega},e}, \widehat{\Lambda}^e(\mathrm{d}a,\mathrm{d}t), \widehat{Z}^{\bar{\omega},e}\big),\ \mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\big(\widehat{X}, \widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t, \widehat{W}\big)\Big) = 0. \quad (3.3.18)$$

Moreover, for every $e \geq 1$, it follows by (3.3.10) in Proposition 3.3.5 that one can choose $(\widehat{Z}^{\bar{\omega},e,i})_{i=1,\dots,e}$ such that

$$(t,\bar{\omega},\hat{\omega}^\star) \longmapsto \big(\widehat{Z}_{t\wedge\cdot}^{\bar{\omega},e,1}(\hat{\omega}^\star), \dots, \widehat{Z}_{t\wedge\cdot}^{\bar{\omega},e,e}(\hat{\omega}^\star)\big) \text{ is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

Recall that the solution $\widehat{X}^{\bar\omega,e}$ of SDE (3.3.16) can be defined by Picard iterations (see e.g. Theorem 5.5.3), then by similar arguments as in [140, Lemma 2.6.], one can choose $\widehat{X}^{\bar\omega,e}$ such that

$$(t,\bar\omega,\hat\omega^\star)\longmapsto\big(\widehat{X}^{\bar\omega,e}_{t\wedge\cdot}(\hat\omega^\star),(\widehat\Lambda^e)^t(\hat\omega^\star),\widehat{Z}^{\bar\omega,e,1}_{t\wedge\cdot}(\hat\omega^\star),\dots,\widehat{Z}^{\bar\omega,e,e}_{t\wedge\cdot}(\hat\omega^\star)\big)\text{ is }\mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}\tag{3.3.19}$$

Step 3. We now consider the approximation of $(\widehat{X}^{\bar\omega,e})_{\bar\omega\in\overline\Omega}$, $\widehat\Lambda^e$ and $\widehat{Z}^{\bar\omega,e}$ for a fixed $e\ge1$. For simplicity of presentation, we consider the case $e=2$, $K=2$ and $t_1=T/2$, so that

$$\widehat\Lambda^2(\mathrm{d}a,\mathrm{d}t):=q_t^{2,1}\delta_{a_1^2}(\mathrm{d}a)\mathrm{d}t+q_t^{2,2}\delta_{a_2^2}(\mathrm{d}a)\mathrm{d}t,\ q_t^{2,1}+q_t^{2,2}=1,\ (q_t^{2,1},q_t^{2,2})=\begin{cases}(q_0^{2,1},q_0^{2,2}),\ \text{for } t\in[0,t_1),\\(q_{t_1}^{2,1},q_{t_1}^{2,2}),\ \text{for } t\in[t_1,T],\end{cases}$$

where $(q_0^{2,1},q_0^{2,2})\in[0,1]^2$ are two deterministic constants and $(q_{t_1}^{2,1},q_{t_1}^{2,2})$ are $[0,1]$–valued $\widehat{\mathcal{F}}^\star_{t_1}$–measurable random variables.

First, we consider a further discretisation of $[0,t_1]$: $0=t_0^1<t_1^1<\cdots<t_m^1=t_1$ with $t_i^1:=i\Delta t$, $\Delta t:=t_1/m$, and then define two d–dimensional processes $(\widehat{W}^{\bar\omega,m,1},\widehat{W}^{\bar\omega,m,2})$. Let $\widehat{W}_0^{\bar\omega,m,1}=\widehat{W}_0^{\bar\omega,m,2}=0$, and then for each $i=0,\dots,m-1$, let

$$\widehat{W}_t^{\bar\omega,m,1}:=\begin{cases}\widehat{W}_{t_i^1}^{\bar\omega,m,1}+\sqrt{q_0^{2,1}}\Big(\widehat{Z}^{\bar\omega,2,1}_{t_i^1+(t-t_i^1)/q_0^{2,1}}-\widehat{Z}^{\bar\omega,2,1}_{t_i^1}\Big),\ t\in[t_i^1,\theta_i^1],\\ \widehat{W}_{\theta_i^1}^{\bar\omega,m,1},\ t\in(\theta_i^1,t_{i+1}^1],\end{cases}\quad\text{with }\theta_i^1:=t_i^1+q_0^{2,1}\Delta t\in[t_i^1,t_{i+1}^1],$$

and

$$\widehat{W}_t^{\bar\omega,m,2}:=\begin{cases}\widehat{W}_{t_i^1}^{\bar\omega,m,2},\ t\in[t_i^1,\theta_i^1],\\ \widehat{W}_{\theta_i^1}^{\bar\omega,m,2}+\sqrt{q_0^{2,2}}\Big(\widehat{Z}^{\bar\omega,2,2}_{t_i^1+(t-\theta_i^1)/q_0^{2,2}}-\widehat{Z}^{\bar\omega,2,2}_{t_i^1}\Big),\ t\in(\theta_i^1,t_{i+1}^1].\end{cases}$$

Namely, one "compresses" the increment of the Brownian motion $\widehat{Z}^{\bar\omega,2,1}$ from $[t_i^1,t_{i+1}^1]$ to $[t_i^1,\theta_i^1]$ to obtain $\widehat{W}^{\bar\omega,m,1}$, and "compresses" the increment of the Brownian motion $\widehat{Z}^{\bar\omega,2,2}$ from $[t_i^1,t_{i+1}^1]$ to $[\theta_i^1,t_{i+1}^1]$ to obtain $\widehat{W}^{\bar\omega,m,2}$.

Next, on $[t_1,T]$, we take the discretisation $t_1=t_0^2<\ldots,t_m^2=T$ with $t_i^2:=t_1+i\Delta t$, $\Delta t:=t_1/m=(T-t_1)/m$, and for each $i=0,\dots,m-1$, let $\theta_i^2:=t_i^2+q_{t_1}^{2,2}\Delta t\in[t_i^2,t_{i_1}^2]$. Notice that $q_{t_1}^{2,2}$ is an $\widehat{\mathcal{F}}^\star_{t_1}$–random variable. It follows that the $(\theta_i^2)_{0\le i\le m-a}$ are also random. By rewriting its definition on $[0,t_1]$ in an equivalent way, we define $(\widehat{W}^{\bar\omega,m,1},\widehat{W}^{\bar\omega,m,2})$ on $[t_1,T]$ by

$$\begin{cases}\widehat{W}_t^{\bar\omega,m,1}:=\widehat{W}_{t_i^2}^{\bar\omega,m,1}+\sqrt{q_0^{2,2}}\Big(\widehat{Z}^{\bar\omega,2,1}_{t_i^2+(t\wedge\theta_i^2-t_i^2)/q_0^{2,2}}-\widehat{Z}^{\bar\omega,2,1}_{t_i^2}\Big),\\ \widehat{W}_t^{\bar\omega,m,2}:=\widehat{W}_{t_i^2}^{\bar\omega,m,2}\mathbf{1}_{\{t\in[t_i^2,\theta_i^2]\}}+\Big(\widehat{W}_{\theta_i^2}^{\bar\omega,m,2}+\sqrt{q_0^{2,2}}\Big(\widehat{Z}^{\bar\omega,2,2}_{t_i^2+(t-\theta_i^2)/q_0^{2,2}}-\widehat{Z}^{\bar\omega,2,2}_{t_i^2}\Big)\Big)\mathbf{1}_{\{t\in(\theta_i^2,t_{i+1}^2]\}},\end{cases}\quad\text{for } t\in(t_i^2,t_{i+1}^2].$$

Next, let us define $I_1^m:=\cup_{i=1}^{m-1}([t_i^1,\theta_i^1)\cup[t_i^2,\theta_i^2))$ and $I_2^m:=\cup_{i=0}^{m-1}([\theta_i^1,t_{i+1}^1)\cup[\theta_i^2,t_{i+1}^2))$

$$\widehat{W}_t^{\bar\omega,m}:=\big(\widehat{W}_{t-\Delta t}^{\bar\omega,m,1}+\widehat{W}_{t-\Delta t}^{\bar\omega,m,2}\big)\mathbf{1}_{\{t\in[\Delta t,T]\}},\text{ and }\widehat\Lambda^{2,m}(\mathrm{d}a,\mathrm{d}t):=\delta_{\alpha_t^m}(\mathrm{d}a)\mathrm{d}t,\text{ with }\alpha_t^m:=a_1^2\mathbf{1}_{\{t\in I_1^m\}}+a_2^2\mathbf{1}_{\{t\in I_2^m\}}.$$

Notice that $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline\Omega$, $\widehat{W}^{\bar\omega,m,1}$ and $\widehat{W}^{\bar\omega,m,2}$ are $\widehat{\mathbb{P}}_{\bar\omega}$–martingales w.r.t. their natural filtrations with quadratic variation $c_\cdot^{m,1}:=\int_0^\cdot\mathbf{1}_{I_1^m}(r)\mathrm{d}r$ and $c_\cdot^{m,2}:=\int_0^\cdot\mathbf{1}_{I_2^m}(r)\mathrm{d}r$ respectively. Further, with the time shift appearing in its definition, the process $\widehat{W}^{\bar\omega,m}$ is $\widehat{\mathbb{F}}^\star$–adapted. Moreover, $\widehat{W}^{\bar\omega,m}$ is a $\widehat{\mathbb{P}}_{\bar\omega}$–Brownian motion on $[\Delta t,T]$ with respect to its natural filtration (but not $\widehat{\mathbb{F}}^\star$), and

$$\big(\widehat\Lambda^{2,m},\widehat{c}_\cdot^{m,1},\widehat{c}_\cdot^{m,2},\widehat{W}_\cdot^{\bar\omega,m,1},\widehat{W}_\cdot^{\bar\omega,m,2}\big)\underset{m\to\infty}{\longrightarrow}\Big(\widehat\Lambda^2,\int_0^\cdot q_r^{2,1}\mathrm{d}r,\int_0^\cdot q_r^{2,2}\mathrm{d}r,\int_0^\cdot\sqrt{q_r^{2,1}}\mathrm{d}\widehat{Z}_r^{\bar\omega,2,1},\int_0^\cdot\sqrt{q_r^{2,2}}\mathrm{d}\widehat{Z}_r^{\bar\omega,2,2}\Big),\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}\tag{3.3.20}$$

Let us define $\widehat{X}^{\bar{\omega},2,m} = (\widehat{X}^{\bar{\omega},2,m}_t)_{t\in[0,T]}$ as the unique solution, under $\widehat{\mathbb{P}}_{\bar{\omega}}$, to

$$\begin{aligned}\widehat{X}^{\bar{\omega},2,m}_t &= \widehat{X}_0 + \int_{\Delta t}^{t\vee\Delta t} b(r,\widehat{X}^{\bar{\omega},2,m}_r, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},2,m})^{-1}, \alpha^m_r)\mathrm{d}r + \int_{\Delta t}^{t\vee\Delta t}\sigma(r,\widehat{X}^{\bar{\omega},2,m}_r, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},2,m})^{-1}, \alpha^m_r)\mathrm{d}\widehat{W}^{\bar{\omega},m}_r + \sigma_0 B_t(\bar{\omega})\\ &= \widehat{X}_0 + \sum_{i=1}^{2}\Big(\int_{\Delta t}^{t\vee\Delta t} b(r,\widehat{X}^{\bar{\omega},2,m}_r, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},2,m})^{-1}, a_i^2)\mathrm{d}\widehat{c}^{m,i}_r + \int_{\Delta t}^{t\vee\Delta t}\sigma(r,\widehat{X}^{\bar{\omega},2,m}_r, \widehat{\mathbb{P}}_{\bar{\omega}}\circ(\widehat{X}^{\bar{\omega},2,m})^{-1}, a_i^2)\mathrm{d}\widehat{W}^{\bar{\omega},m,i}_{r-\Delta t}\Big)\\ &\quad + \sigma_0 B_t(\bar{\omega}).\end{aligned} \tag{3.3.21}$$

Besides, as in Lemma 3.4.1, it is standard to obtain the following estimate, for some constant $C>0$

$$\sup_{m\geq 1}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big[\sup_{t\in[0,T]}|\widehat{X}^{\bar{\omega},2,m}_t|^{p'}\Big] \leq C\Big(1+\int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x)\Big) < \infty.$$

Using [49, Proposition B.1], it follows that: for $\overline{\mathbb{P}}$–a.e. $\bar{\omega}\in\overline{\Omega}$,

$$\Big(\mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\big(\widehat{c}^{m,1}_\cdot,\widehat{c}^{m,2}_\cdot,\widehat{W}^{\bar{\omega},m,1}_\cdot,\widehat{W}^{\bar{\omega},m,2}_\cdot,\widehat{X}^{\bar{\omega},2,m}_\cdot\big)\Big)_{m\geq 1} \text{ is tight under } \mathcal{W}_p.$$

Then along an arbitrary convergent sub-sequence $(m_k)_{k\geq 1}$ (which can potentially depend on $\bar{\omega}$), one has

$$\mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\big(\widehat{c}^{m_k,1}_\cdot,\widehat{c}^{m_k,2}_\cdot,\widehat{W}^{\bar{\omega},m_k,1}_\cdot,\widehat{W}^{\bar{\omega},m_k,2}_\cdot,\widehat{X}^{\bar{\omega},2,m_k}_\cdot\big) \underset{k\to\infty}{\longrightarrow} \mathcal{L}^{\mathbb{P}^\star}\big(\widehat{c}^{\star,1}_\cdot,\widehat{c}^{\star,2}_\cdot,\widehat{W}^{\star,1}_\cdot,\widehat{W}^{\star,2}_\cdot,\widehat{X}^{\star}_\cdot\big) \text{ weakly and under } \mathcal{W}_p,$$

for some random elements $\big(\widehat{c}^{\star,1},\widehat{c}^{\star,2},\widehat{W}^{\star,1}_\cdot,\widehat{W}^{\star,2}_\cdot,\widehat{X}^{\star}_\cdot\big)$ in $(\Omega^\star,\mathcal{F}^\star,\mathbb{P}^\star)$. By considering the martingale problem associated with the SDE (3.3.21), it is standard to check that $\widehat{X}^\star$ satisfies

$$\widehat{X}^\star = \widehat{X}_0 + \sum_{i=1}^{2}\int_0^t b(r,\widehat{X}^\star_r,\mathbb{P}^\star\circ(\widehat{X}^\star)^{-1},a_i^2)\mathrm{d}\widehat{c}^{\star,i}_r + \int_0^t\sigma(r,\widehat{X}^\star_r,\mathbb{P}^\star\circ(\widehat{X}^\star)^{-1},a_i^2)\mathrm{d}\widehat{W}^{\star,i}_r + \sigma_0 B_t(\bar{\omega}),\ \mathbb{P}^\star\text{–a.s.}$$

Besides, by the convergence result in Equation (3.3.20), one has

$$\mathcal{L}^{\mathbb{P}^\star}\big(\widehat{X}_0,\widehat{c}^{\star,1},\widehat{c}^{\star,2},\widehat{W}^{\star,1}_\cdot,\widehat{W}^{\star,2}_\cdot\big) = \mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big(\widehat{X}_0,\int_0^\cdot q^{2,1}_r\mathrm{d}r,\int_0^\cdot q^{2,2}_r\mathrm{d}r,\int_0^\cdot\sqrt{q^{2,1}_r}\mathrm{d}\widehat{Z}^{\bar{\omega},2,1}_r,\int_0^\cdot\sqrt{q^{2,2}_r}\mathrm{d}\widehat{Z}^{\bar{\omega},2,2}_r\Big).$$

Then it follows by the strong uniqueness (hence uniqueness in law) of the solution to SDE (3.3.16) that

$$\mathcal{L}^{\mathbb{P}^\star}\big(\widehat{c}^{\star,1},\widehat{c}^{\star,2},\widehat{W}^{\star,1}_\cdot,\widehat{W}^{\star,2}_\cdot,\widehat{X}^{\star}\big) = \mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big(\int_0^\cdot q^{2,1}_r\mathrm{d}r,\int_0^\cdot q^{2,2}_r\mathrm{d}r,\int_0^\cdot\sqrt{q^{2,1}_r}\mathrm{d}\widehat{Z}^{\bar{\omega},2,1}_r,\int_0^\cdot\sqrt{q^{2,2}_r}\mathrm{d}\widehat{Z}^{\bar{\omega},2,2}_r,\widehat{X}^{\bar{\omega},2}\Big).$$

Since the limit is unique, and hence does not depend on the sub-sequence, we obtain that: for $\overline{\mathbb{P}}$–a.e. $\bar{\omega}\in\overline{\Omega}$

$$\mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\big(\widehat{\Lambda}^{2,m},\widehat{W}^{\bar{\omega},m,1}_\cdot,\widehat{W}^{\bar{\omega},m,2}_\cdot,\widehat{W}^{\bar{\omega},m}_\cdot,\widehat{X}^{\bar{\omega},2,m}_\cdot\big) \underset{m\to\infty}{\longrightarrow} \mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big(\widehat{\Lambda}^2,\int_0^\cdot\sqrt{q^{2,1}_r}\mathrm{d}\widehat{Z}^{\bar{\omega},2,1}_r,\int_0^\cdot\sqrt{q^{2,2}_r}\mathrm{d}\widehat{Z}^{\bar{\omega},2,2}_r,\widehat{Z}^{\bar{\omega},2}_\cdot,\widehat{X}^{\bar{\omega},2}\Big).$$

Further, using (3.3.19) and the explicit construction of $\widehat{W}^{\bar{\omega},m}$ and the fact that the solution $\widehat{X}^{\bar{\omega},2,m}$ of SDE (3.3.21) can be defined by a Picard iteration, it follows that one can choose $\widehat{X}^{\bar{\omega},2,m}$ such that

$$(t,\bar{\omega},\hat{\omega}^\star)\longmapsto\Big(\widehat{X}^{\bar{\omega},2,m}_{t\wedge\cdot}(\hat{\omega}^\star),(\widehat{\Lambda}^{2,m})^t(\hat{\omega}^\star),\widehat{W}^{\bar{\omega},m}_{t\wedge\cdot}(\hat{\omega}^\star)\Big) \text{ is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

Finally, we observe that $\widehat{X}^{\bar{\omega},2,m}$ is only defined by SDE (3.3.21) on $[\Delta t,T]$, with $\Delta t = t_1/e\longrightarrow 0$ when $e\longrightarrow\infty$. Thus, we can easily extend it to an SDE on $[0,T]$ as (3.3.12) and preserve the same convergence and measurability properties. □

Remark 3.3.8. *Our definition of the relaxed formulation and the proof on the approximation of relaxed control rules by weak control rules is quite different from those used by* Lacker [104] *in the non–common noise context. In particular, it allows to fill in a subtle technical gap in* [104, Proof of Theorem 2.4]*, where the approximation procedure relies on the erroneous martingale measure approximation result of* Méléard [122]*, as explained in* Footnote 1*. Notice however that* [104, Paragraph before Theorem 2.4.] *does mention the possibility of an alternative proof in the spirit of* [63] *and* [62]*, but without more details. This is exactly the program we have carried out.*

3.3.3 Proof of equivalence

$(i.1)$ Let Assumption 1.4.1 and Assumption 3.1.1 hold true, we next show that $\overline{\mathcal{P}}_R(\nu)$ is convex for $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. Let $(\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_2) \in \overline{\mathcal{P}}_R(\nu) \times \overline{\mathcal{P}}_R(\nu)$, $\theta \in [0,1]$, and $\overline{\mathbb{P}} := \theta\overline{\mathbb{P}}_1 + (1-\theta)\overline{\mathbb{P}}_2$. Then $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu)$ since $(\overline{\mathbb{P}}_1, \overline{\mathbb{P}}_2) \in \overline{\mathcal{P}}_R(\nu) \times \overline{\mathcal{P}}_R(\nu) \subset \overline{\mathcal{P}}_A(\nu) \times \overline{\mathcal{P}}_A(\nu)$. Let also $\varphi \in C_b(\mathbb{R}^n \times \mathbb{R}^d)$, $0 \le s \le t$ and $\zeta : \widehat{\Omega} \longrightarrow \mathbb{R}$ a bounded $\widehat{\mathcal{F}}_s$–measurable variable, then

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\big|\mathbb{E}^{\hat{\mu}}\big[\widehat{M}_t^{\varphi,\mu}\zeta\big] - \mathbb{E}^{\hat{\mu}}\big[\widehat{M}_s^{\varphi,\mu}\zeta\big]\big|\big] = \theta\mathbb{E}^{\overline{\mathbb{P}}_1}\big[\big|\mathbb{E}^{\hat{\mu}}\big[\widehat{M}_t^{\varphi,\mu}\zeta\big] - \mathbb{E}^{\hat{\mu}}\big[\widehat{M}_s^{\varphi,\mu}\zeta\big]\big|\big] + (1-\theta)\mathbb{E}^{\overline{\mathbb{P}}_2}\big[\big|\mathbb{E}^{\hat{\mu}}\big[\widehat{M}_t^{\varphi,\mu}\zeta\big] - \mathbb{E}^{\hat{\mu}}\big[\widehat{M}_s^{\varphi,\mu}\zeta\big]\big|\big] = 0.$$

By considering a countable dense family of φ, $0 \le s \le t$ and ζ, it follows that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $(\widehat{M}_t^{\varphi,\mu(\bar{\omega})})_{t\in[0,T]}$ is an $(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega}))$–martingale for all $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$. This proves that $\mathbb{P} \in \overline{\mathcal{P}}_R(\nu)$.

$(i.2)$ Take $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, we now show that $\overline{\mathcal{P}}_R(\nu)$ is closed under the $\mathcal{W}_p$–topology. First, from Lemma 3.4.1, we have $\overline{\mathcal{P}}_R(\nu) \subset \mathcal{P}_p(\overline{\Omega})$. Let $(\overline{\mathbb{P}}_m)_{m\ge1} \subset \overline{\mathcal{P}}_R(\nu)$, and $\overline{\mathbb{P}} \in \mathcal{P}(\overline{\Omega})$ be such that $\lim_m \mathcal{W}_p(\overline{\mathbb{P}}_m, \overline{\mathbb{P}}) = 0$. Then $\overline{\mathbb{P}} \in \mathcal{P}_p(\overline{\Omega})$.

Let $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$ and $f \in C_b^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell)$, by Assumption 1.4.1, there exists some constant $C > 0$ such that for all $(\bar{\omega}, \hat{\omega}) \in \overline{\Omega} \times \widehat{\Omega}$ and $t \in [0,T]$

$$\big|\overline{S}_t^f(\bar{\omega})\big| \le C\bigg(1 + \|X_{t\wedge\cdot}(\bar{\omega})\|^p + \int_{\mathcal{C}^n} \|\mathbf{x}\|^p \mu(\bar{\omega})(\mathrm{d}\mathbf{x}) + \iint_{[0,T]\times A} \rho(a_0,a)^p \Lambda_r(\bar{\omega})(\mathrm{d}a)\mathrm{d}r\bigg), \tag{3.3.22}$$

and

$$\big|\widehat{M}_t^{\varphi,\mu(\bar{\omega})}(\hat{\omega})\big| \le C\bigg(1 + \big\|\widehat{Y}_{t\wedge\cdot}(\hat{\omega})\big\|^p + \int_{\mathcal{C}^n} \|\mathbf{x}\|^p \mu(\bar{\omega})(\mathrm{d}\mathbf{x}) + \iint_{[0,T]\times A} \rho(a_0,a)^p \widehat{\Lambda}_r(\tilde{\omega})(\mathrm{d}a)\mathrm{d}r\bigg). \tag{3.3.23}$$

Let $0 \le s \le t$, $\zeta : \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d \longrightarrow \mathbb{R}$ and $\phi : \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathcal{P}(\overline{\Omega}) \longrightarrow \mathbb{R}$ be two bounded continuous functions. Using the regularity of the coefficient functions (b, σ, σ_0), together with (3.3.22) and (3.3.23), it follows that

$$\begin{aligned} 0 &= \lim_{m\to\infty} \mathbb{E}^{\overline{\mathbb{P}}_m}\big[\big|\mathbb{E}^{\hat{\mu}}\big[\widehat{M}_t^{\varphi,\mu}\zeta\big(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{\Lambda}^s, \widehat{W}_{s\wedge\cdot}\big)\big] - \mathbb{E}^{\hat{\mu}}\big[\widehat{M}_s^{\varphi,\mu}\zeta\big(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{\Lambda}^s, \widehat{W}_{s\wedge\cdot}\big)\big]\big|\big] \\ &= \mathbb{E}^{\overline{\mathbb{P}}}\big[\big|\mathbb{E}^{\hat{\mu}}\big[\widehat{M}_t^{\varphi,\mu}\zeta\big(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{\Lambda}^s, \widehat{W}_{s\wedge\cdot}\big)\big] - \mathbb{E}^{\hat{\mu}}\big[\widehat{M}_s^{\varphi,\mu}\zeta\big(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{\Lambda}^s, \widehat{W}_{s\wedge\cdot}\big)\big]\big|\big], \end{aligned}$$

and

$$\begin{aligned} 0 &= \lim_{m\to\infty} \big|\mathbb{E}^{\overline{\mathbb{P}}_m}[\overline{S}_t^f \phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_s\big)] - \mathbb{E}^{\overline{\mathbb{P}}_m}[\overline{S}_s^f \phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_s\big)]\big| \\ &= \big|\mathbb{E}^{\mathbb{P}}[\overline{S}_t^f \phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_s\big)] - \mathbb{E}^{\overline{\mathbb{P}}}[\overline{S}_s^f \phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, \Lambda^s, W_{s\wedge\cdot}, B_{s\wedge\cdot}, \widehat{\mu}_s\big)]\big|. \end{aligned}$$

This implies that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $(\widehat{M}_t^{\varphi,\mu(\bar{\omega})})_{t\in[0,T]}$ is an $(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega}))$–martingale for all $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$, and $(\overline{S}_t^f)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–martingale for all $f \in C_b^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell)$.

Finally, it is straightforward to check all the other conditions in Proposition 3.3.2, and we can conclude that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$.

(ii) We assume here that $A \subset \mathbb{R}^j$, $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$. It is enough to use Proposition 3.3.6 to deduce that $\overline{\mathcal{P}}_W(\nu)$ is dense in $\overline{\mathcal{P}}_R(\nu)$ with respect to $\mathcal{W}_p$. Next, under Assumption 3.1.1, together with the growth condition of L and g in Assumption 1.4.1, $\overline{\mathbb{P}} \longmapsto J(\overline{\mathbb{P}})$ is lower semi–continuous (see Remark 3.2.2) on $\mathcal{P}_p(\overline{\Omega})$. This is enough to prove that $V_W(\nu) = V_R(\nu)$.

Finally, when L and g are continuous, under Assumption 1.4.1 and Assumption 3.1.1, $\overline{\mathbb{P}} \longmapsto J(\overline{\mathbb{P}})$ is continuous on $\mathcal{P}_p(\overline{\Omega})$. Let $(\overline{\mathbb{P}}^m)_{m\ge1} \subset \overline{\mathcal{P}}_R(\nu)$ be a sequence such that

$$\lim_{m\to\infty} J(\overline{\mathbb{P}}^m) = V_R(\nu) < \infty.$$

The coercivity condition (1.4.2) in Assumption 1.4.1 ensures that $(\overline{\mathbb{P}}^m)_{m\ge1}$ is relatively compact w.r.t. $\mathcal{W}_p$ (see also Proposition 3.4.6 below for a more detailed argument). By the closedness of $\overline{\mathcal{P}}_R(\nu)$, it follows that there exists $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\nu)$, such that $\mathcal{W}_p(\overline{\mathbb{P}}^m, \overline{\mathbb{P}}) \longrightarrow 0$, possibly along a subsequence. Together with the continuity of $J : \mathcal{P}_p(\overline{\Omega}) \longrightarrow \mathbb{R}$, this implies that $\overline{\mathbb{P}}$ is an optimal relaxed control rule. □

3.3.4 Appendix: proof of some technical results

We finally provide here the proof of the approximation result (of relaxed control by weak control rules) in Proposition 3.3.6, and some related technical results. Recall that Assumption 1.4.1 and Assumption 3.1.1 hold true, and A is a subset of $\mathbb{R}^j$ for some $j \geq 1$.

3.3.4.1 An equivalent reformulation for relaxed control rules on the canonical space

On $\overline{\Omega}$, let us introduce a filtration $\overline{\mathbb{F}}^\circ = (\mathcal{F}_t^\circ)_{t\in[0,T]}$ and a process $S^f = (S_t^f)_{t\in[0,T]}$, for every $f \in C_b^2(\mathbb{R}^{n+\ell})$, by

$$\overline{\mathcal{F}}_t^\circ := \sigma(X_{t\wedge\cdot}, Y_{t\wedge\cdot}, B_{t\wedge\cdot}, \mu_t), \text{ and } S_t^f := f\big(Z_t, B_t\big) - \varphi(Z_0, B_0) - \int_0^t \frac{1}{2}\mathrm{Tr}\big[a_0(s, X, \mu)\nabla^2\varphi(Z_s, B_s)\big]\mathrm{d}s, \ t \in [0,T], \tag{3.3.24}$$

where $Z := X - Y$ and

$$a_0\big(t, \mathbf{x}, \nu\big) := \begin{pmatrix} \sigma_0(t, \mathbf{x}, \nu) \\ \mathrm{I}_\ell \end{pmatrix} \begin{pmatrix} \sigma_0(t, \mathbf{x}, \nu) \\ \mathrm{I}_\ell \end{pmatrix}^\top, \text{ for each } (t, \mathbf{x}, \nu) \in [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n). \tag{3.3.25}$$

Denote by

$$\overline{\mathcal{P}}_R^c(\nu) := \Big\{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_A(\nu) : \overline{\mathbb{P}} \text{–a.e. } \bar{\omega} \in \overline{\Omega}, \ \big(\widehat{M}_t^{\varphi,\mu(\bar{\omega})}\big)_{t\in[0,T]} \text{ is an } \big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)\text{–martingale for each } \varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)\Big\},$$

the relaxed control rules on the canonical space. We recall that $\overline{\mathcal{P}}_A(\nu)$ is defined in (3.3.1).

Proposition 3.3.9. *Let $\nu \in \mathcal{P}(\mathbb{R}^n)$, then a probability measure $\overline{\mathbb{P}} \in \mathcal{P}(\overline{\Omega})$ belongs to $\overline{\mathcal{P}}_R^c(\nu)$ if and only if*

(i) $\overline{\mathbb{P}}\big[\widehat{\mu}_0 \circ (\widehat{X}_0)^{-1} = \nu, Y_0 = X_0, W_0 = 0, B_0 = 0\big] = 1$, $\mathbb{E}^{\overline{\mathbb{P}}}\Big[\|X\|^p + \int_{[0,T]\times A}\big(\rho(a_0, a)\big)^p \Lambda_t(\mathrm{d}a)\mathrm{d}t\Big] < \infty$, *and for any* $t \in [0,T]$

$$\widehat{\mu}_t(\bar{\omega}) = \overline{\mathbb{P}}_{\bar{\omega}}^{\mathcal{G}_t} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, W, \Lambda^t)^{-1} = \overline{\mathbb{P}}_{\bar{\omega}}^{\mathcal{G}_T} \circ (X_{t\wedge\cdot}, Y_{t\wedge\cdot}, W, \Lambda^t)^{-1}, \text{ for } \overline{\mathbb{P}} \text{–a.e. } \bar{\omega} \in \overline{\Omega};$$

(ii) $(B_t)_{t\in[0,T]}$ *is an* $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$*–Brownian motion, and for each* $f \in C_b^2(\mathbb{R}^n \times \mathbb{R}^\ell)$, *the process* $\big(S_t^f\big)_{t\in[0,T]}$ *is an* $(\overline{\mathbb{F}}^\circ, \overline{\mathbb{P}})$*–martingale*;

(iii) *for* $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, *the process* $\big(\widehat{M}_t^{\varphi,\mu(\bar{\omega})}\big)_{t\in[0,T]}$ *is an* $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$*–martingale for each* $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$.

Proof. First, let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R^c(\nu)$, then $\overline{S}^\varphi$ (recall (3.3.3)) is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–martingale for all $\varphi \in C_b^2\big(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell\big)$, which implies immediately that $(B_t)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–Brownian motion and S^f is an $(\overline{\mathbb{F}}^\circ, \overline{\mathbb{P}})$–martingale for all $f \in C_b^2(\mathbb{R}^n \times \mathbb{R}^\ell)$. It follows that $\overline{\mathbb{P}}$ satisfies Conditions *(i)*–*(iii)* in the statement.

Next, let $\overline{\mathbb{P}} \in \mathcal{P}(\overline{\Omega})$ satisfying Conditions *(i)*–*(iii)* in the statement. To prove that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R^c(\nu)$, we first prove that $\overline{S}^\varphi$ is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–martingale for all $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell)$. To this end, let us introduce, for every $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$, a process $M^\varphi = (M_t^\varphi)_{t\in[0,T]}$ on $(\overline{\Omega}, \overline{\mathcal{F}})$ by (recall also the definitions of $\widehat{\mathcal{L}}$ and $\widehat{M}^{\varphi,\nu}$ in (3.3.5) and (3.3.6))

$$M_t^\varphi := \varphi\big(Y_t, W_t\big) - \varphi(Y_0, W_0) - \iint_{[0,t]\times A} \widehat{\mathcal{L}}_s\varphi\big(X, Y, W, \mu_s, a\big)\Lambda_s(\mathrm{d}a)\mathrm{d}s. \tag{3.3.26}$$

Since B is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–Brownian motion, we have, for all $\theta \in \mathbb{R}^\ell$, $0 \leq s \leq t$, $\phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathbb{M})$, and $\psi \in C_b(\mathcal{C}^\ell \times \mathcal{P}(\widehat{\Omega}))$,

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\Big[M_t^\varphi \exp\big(\theta \cdot B_t - |\theta|^2 t/2\big)\phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, W_{s\wedge\cdot}, \Lambda^s\big)\psi\big(B_{s\wedge\cdot}, \widehat{\mu}_s\big)\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\widehat{\mu}}\Big[\widehat{M}_t^{\varphi,\mu}\phi\big(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{W}_{s\wedge\cdot}, \widehat{\Lambda}^s\big)\Big] \exp\big(\theta \cdot B_t - |\theta|^2 t/2\big)\psi\big(B_{s\wedge\cdot}, \widehat{\mu}_s\big)\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\widehat{\mu}}\Big[\widehat{M}_s^{\varphi,\mu}\phi\big(\widehat{X}_{s\wedge\cdot}, \widehat{Y}_{s\wedge\cdot}, \widehat{W}_{s\wedge\cdot}, \widehat{\Lambda}^s\big)\Big] \exp\big(\theta \cdot B_t - |\theta|^2 t/2\big)\psi\big(B_{s\wedge\cdot}, \widehat{\mu}_s\big)\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}}\Big[M_s^\varphi \exp\big(\theta \cdot B_s - |\theta|^2 s/2\big)\phi\big(X_{s\wedge\cdot}, Y_{s\wedge\cdot}, W_{s\wedge\cdot}, \Lambda^s\big)\psi\big(B_{s\wedge\cdot}, \widehat{\mu}_s\big)\Big].
\end{aligned}$$

In other words, $(M_t^\varphi \exp(\theta B_s - \frac{1}{2}\theta\langle B\rangle_s\theta^\top))_{t\in[0,T]}$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–martingale for any $\varphi \in C_b^2(\mathbb{R}^{n+d})$ and $\theta \in \mathbb{R}^\ell$.

Furthermore, from Condition (ii) in the statement, we know that B is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–Brownian motion, and

$$Y_\cdot = X_\cdot - \int_0^\cdot \sigma_0(s,X,\mu)\mathrm{d}B_s,\ \overline{\mathbb{P}}\text{–a.s.}$$

It follows by [150, Theorems 4.2.1 and 8.1.1] that $\overline{S}^\varphi$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–martingale for all $\varphi \in C_b^2\big(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^\ell\big)$. Moreover, it is straightforward to check that $\mathcal{L}^{\overline{\mathbb{P}}}\big(X_0,W,B,\widehat{\mu}\big) = \mathcal{L}^{\overline{\mathbb{P}}}(X_0)\otimes\mathcal{L}^{\overline{\mathbb{P}}}(W)\otimes\mathcal{L}^{\overline{\mathbb{P}}}(B,\widehat{\mu})$, i.e. X_0, W and $(B,\widehat{\mu})$ are mutually independent under $\overline{\mathbb{P}}$, and therefore, $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R^c(\nu)$. □

3.3.4.2 Proof of Proposition 3.3.5

3.3.4.2.1 Martingale measure, stochastic integral and their measurability We recall here the definition of the martingale measures from El Karoui and Méléard [58], but in a special context, and then discuss the associated stochastic integration and some measurability issues. Let us consider the Polish space A, and an abstract filtered probability space $(\Omega^\star,\mathcal{F}^\star,\mathbb{F}^\star,\mathbb{P}^\star)$, equipped with a random measure $\nu_t(\mathrm{d}a)\mathrm{d}t$ on $[0,T]\times A$, where $t \longmapsto \nu_t(\mathrm{d}a)$ is $\mathcal{P}(A)$–valued predictable process. Denote by $\mathcal{P}^{\mathbb{F}^\star}$ the predictable σ–field w.r.t. the filtration $\mathbb{F}^\star$.

Definition 3.3.10. *We will say $(N_t(\mathrm{d}a))_{t\in[0,T]}$ is an $(\mathbb{F}^\star,\mathbb{P}^\star)$–martingale measure of intensity $\nu_t(\mathrm{d}a)\mathrm{d}t$ if*

(i) for all $B\in\mathcal{B}(A)$, $(N_t(B))_{t\in[0,T]}$ is a $(\mathbb{F}^\star,\mathbb{P}^\star)$–martingale with quadratic variation $\int_0^\cdot \nu_s(B)\mathrm{d}s$, and with $N_0(B)=0$;

(ii) let $B_1,B_2\in\mathcal{B}(A)$ be such that $B_1\cap B_2=\emptyset$, then $(N_t(B_1))_{t\in[0,T]}$ and $(N_t(B_2))_{t\in[0,T]}$ are two orthogonal martingales.

Given an $(\mathbb{F}^\star,\mathbb{P}^\star)$–martingale measure $(N_t(\mathrm{d}a))_{t\in[0,T]}$ of intensity $\nu_t(\mathrm{d}a)\mathrm{d}t$, and a $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)$–measurable function $f:[0,T]\times\Omega^\star\times A\longrightarrow\mathbb{R}$ such that

$$\mathbb{E}^{\mathbb{P}^\star}\left[\iint_{[0,T]\times A}|f(s,a)|^2\nu_s(\mathrm{d}a)\mathrm{d}s\right]<\infty,$$

one can first approximate f by a sequence $(f^m)_{m\geq1}$ of simple functions of the form $f^m(s,a) := \sum_{k=1}^m f_k^m 1_{(s_k^m,t_k^m]}(s)1_{B_k^m}(a)$, where $B_k^m\subset\mathcal{B}(A)$,

$$s_k^m<t_k^m,\ f_k^m \text{ is } \mathcal{F}^\star_{s_k^m}\text{–measurable, for all } k=1,\dots,m,\ \text{and } \lim_{m\to\infty}\mathbb{E}^{\mathbb{P}^\star}\left[\iint_{[0,T]\times A}\big|f(s,a)-f^m(s,a)\big|^2\nu_s(\mathrm{d}a)\mathrm{d}s\right]=0.$$

Then one can define the stochastic integral, for $t\in\mathbb{Q}$,

$$N_t(f)=\iint_{[0,t]\times A}f(s,a)N(\mathrm{d}a,\mathrm{d}s) := \lim_{m\to\infty}N_t(f^m) := \lim_{m\to\infty}\sum_{k=1}^m f_k^m\big(N_{t_k^m\wedge t}(B_k^m)-N_{s_k^m\wedge t}(B_k^m)\big),\ \text{with the limit in } \mathbb{L}^2,$$

and then, for all $t\in[0,T]$

$$N_t(f)=\iint_{[0,T]\times A}f(s,a)N(\mathrm{d}a,\mathrm{d}s) := \limsup_{\mathbb{Q}\ni s\nearrow t}N_s(f).$$

Notice that $(N_t(f))_{t\in[0,T]}$ is an $(\mathbb{F}^\star,\mathbb{P}^\star)$–continuous martingale with quadratic variation $\int_0^\cdot\int_A f(s,a)\nu_s(\mathrm{d}a)\mathrm{d}s$, and it is in fact independent of the approximating sequence $(f^m)_{m\geq1}$ (see e.g. [58, Section 1]).

Let us now consider another abstract measurable space $(E,\mathcal{E})$, a family of probability measures $(\mathbb{P}_e^\star)_{e\in E}$ on $(\Omega^\star,\mathcal{F}^\star,\mathbb{F}^\star)$ under which N is a martingale measure with intensity $\nu_t(\mathrm{d}a)\mathrm{d}t$, and the random measure $\nu_t(\mathrm{d}a)\mathrm{d}t$ has the same distribution under each $\mathbb{P}_e^\star$. In addition, the family $(\mathbb{P}_e^\star)_{e\in E}$ verifies that for all Borel function $\varphi:\Omega^\star\times E\longrightarrow\mathbb{R}$ such that for each e, $\varphi(\cdot,e)$ is $\mathbb{P}_e$–integrable, we have

$$E\ni e\longmapsto\int_{\Omega^\star}\varphi(\omega^\star,e)\mathbb{P}_e^\star(\mathrm{d}\omega^\star)\in\mathbb{R}\text{ is }\mathcal{E}\text{–measurable.}\tag{3.3.27}$$

Let $f:[0,T]\times\Omega^\star\times A\times E\to\mathbb{R}$ be $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)\times\mathcal{E}$–measurable function such that

$$\mathbb{E}^{\mathbb{P}_e^\star}\left[\iint_{[0,T]\times A}|f^e(s,a)|^2\nu_s(\mathrm{d}a)\mathrm{d}s\right]<\infty,\ \text{for each } e\in E.\tag{3.3.28}$$

Lemma 3.3.11. *One can construct a family of processes* $\{(N_t(f^e))_{t\in[0,T]}\}_{e\in E}$ *such that*

$$(t,\omega^\star,e)\longmapsto N_t(f^e,\omega^\star) \text{ is } \mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{E}\text{–measurable,} \tag{3.3.29}$$

and

$$N_t(f^e,\omega^\star)=\Big(\iint_{[0,t]\times A} f^e(s,a)N(\mathrm{d}a,\mathrm{d}s)\Big)(\omega^\star),\ t\in[0,T],\ \mathbb{P}^\star_e\text{–a.s. for each } e\in E. \tag{3.3.30}$$

Proof. Let us first consider the simple functions $f:[0,T]\times\Omega^\star\times A\times E\to\mathbb{R}$ in form $f(s,a):=\sum_{k=1}^m f_k 1_{(s_k,t_k]}(s)1_{B_k}(a)$, where for each $k=1,\dots,m$,

$$s_k<t_k,\ f_k:\Omega^\star\times E\to\mathbb{R} \text{ is } \mathcal{F}^\star_{s_k}\otimes\mathcal{E}\text{–measurable, and } B_k\in\mathcal{B}(A).$$

Then it is clear that

$$(t,\omega^\star,e)\longmapsto N_t(f^e,\omega^\star):=\sum_{k=1}^m f_k(\omega^\star,e)\big(N_{t_k\wedge t}(\omega^\star,B_k)-N_{s_k\wedge t}(\omega^\star,B_k)\big) \text{ is } \mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{E}\text{–measurable.}$$

Next, let $f_1,f_2:[0,T]\times\Omega^\star\times A\times E\to\mathbb{R}$ be two bounded $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)\times\mathcal{E}$–measurable functions. Assume that both f_1 and f_2, one can construct the stochastic integrals satisfying (3.3.29) and (3.3.30), then it is clear that for $f:=f_1\pm f_2$, $N_t(f):=N_t(f_1)\pm N_t(f_2)$ satisfies also (3.3.29) and (3.3.30).

Further, let $(f_m)_{m\geq1}$ be a sequence of positive bounded functions increasely converging to bounded function f pointwisely, all f,f_m are $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)\times\mathcal{E}$–measurable functions, and for each $m\geq1$, one can construct $N_t(f_m)$ satisfying (3.3.29) and (3.3.30). Then it is clear that for each $e\in E$,

$$N_t(f^e_m)\longrightarrow\iint_{[0,t]\times A} f^e(s,a)N(\mathrm{d}a,\mathrm{d}s) \text{ in } \mathbb{L}^2(\mathbb{P}^\star_e),\ \text{as } m\longrightarrow\infty.$$

Following [129, Lemma 3.2.] combined with Condition (3.3.27), one can find a family of sub-sequence $(m_k(e))_{k\geq1,e\in E}$ which is $\mathcal{E}$–measurable and

$$N_t(f^e):=\limsup_{k\to\infty}N_t(f^e_{m_k(e)})=\iint_{[0,t]\times A} f^e(s,a)N(\mathrm{d}a,\mathrm{d}s),\ \mathbb{P}^\star_e\text{–a.s., for each } e\in E.$$

In other words, one can choose a version $N_t(f)$ satisfying (3.3.29) and (3.3.30). By the monotone class theorem, it follows that the statement holds true for all bounded functions $f:[0,T]\times\Omega^\star\times A\times E\to\mathbb{R}$ which is $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)\times\mathcal{E}$–measurable.

Finally, let $f:[0,T]\times\Omega^\star\times A\times E\to\mathbb{R}$ be a $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)\times\mathcal{E}$–measurable function satisfying (3.3.28). For each $m\in\mathbb{N}^*$, define $f_m:=f1_{|f|\leq m}$, then $(f_m)_{m\in\mathbb{N}^*}$ is a sequence of $\mathcal{P}^{\mathbb{F}^\star}\otimes\mathcal{B}(A)\times\mathcal{E}$–measurable functions satisfying

$$\mathbb{E}^{\mathbb{P}^\star_e}\Big[\iint_{[0,T]\times A}|f^e(s,a)-f^e_m(s,a)|^2\nu_s(\mathrm{d}a)\mathrm{d}s\Big]<\infty,\ \text{for each } e\in E.$$

Then it is enough to use the arguments in [129, Lemma 3.2] with the condition (3.3.27) again to define $N_t(f^e)$ as limit of $N_t(f^e_m)$, which satisfies (3.3.29) and (3.3.30). □

Proof of Proposition 3.3.5 Recall that the probability space $(\Omega^\star,\mathcal{F}^\star,\mathbb{P}^\star)$ is equipped with $2(n+d)$ i.i.d. martingale measures $(N^{\star,i})_{i=1,\dots,2(n+d)}$ with intensity $\nu_0(\mathrm{d}a)\mathrm{d}t$ for some diffuse probability measure ν_0 on A, which is extended on $(\widehat{\Omega}^\star,\widehat{\mathcal{F}}^\star,\widehat{\mathbb{P}}_{\bar\omega})$ for every $\bar\omega\in\overline{\Omega}$. We will now follow the technical steps in El Karoui and Méléard [58] to construct the family of martingale measures $(\widehat{N}^{\bar\omega})_{\bar\omega\in\overline{\Omega}}$ satisfying (3.3.9), and then check the measurability property in (3.3.10).

Let us first denote

$$\Sigma(t,\mathbf{x},a,\nu):=\begin{pmatrix}\sigma(t,\mathbf{x},a,\nu) & 0_{n\times n}\\ \mathrm{I}_d & 0_{d\times n}\end{pmatrix},\ \text{and } \big(\Sigma\Sigma^\top\big)^+(t,\mathbf{x},a,\nu):=\lim_{\varepsilon\searrow0}\big(\varepsilon\mathrm{I}_{d+n}+(\Sigma\Sigma^\top)(t,\mathbf{x},a,\nu)\big)^{-1}, \tag{3.3.31}$$

where $(\Sigma\Sigma^\top)^+$ is the pseudo–inverse of $\Sigma\Sigma^\top$. Then for all bounded Borel measurable function $f : [0,T]\times A \longrightarrow \mathbb{R}$, let

$$\Gamma^{\bar\omega}(s,f) := \int_A \Sigma\Sigma^\top(s,\widehat X, a, \mu_s(\bar\omega)) f(s,a)\widehat\Lambda_s(\mathrm{d}a), \text{ and its pseudo–inverse } \Gamma^{\bar\omega,+}(s,f) := \limsup_{\varepsilon\searrow 0}\big(\varepsilon \mathrm{I}_{d+n} + \Gamma(s,\bar\omega,f)\big)^{-1}.$$

Denote also by $\mathbf{1}$ the constant function on $[0,T]\times A$ which equals to 1. Furthermore, let $\pi_i : \mathbb{R}^{n+d} \longrightarrow \mathbb{R}$, $i = 1,\dots,n+d$, be the projection function defined by $\pi_i((z) := z_i$ for every $z := (z_1,\dots,z^{n+d})$, and $\widehat M^{\bar\omega,i} := \widehat M^{\pi_i,\mu(\bar\omega)}$ be the martingale defined in (3.3.6), whose quadratic variation process is given by

$$\big\langle \widehat M^{\bar\omega,i}, \widehat M^{\bar\omega,j}\big\rangle_t = \int_0^t \Gamma^{\bar\omega}_{i,j}(s,\mathbf{1})\mathrm{d}s,\ t\in[0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}$$

(i) By [58, Theorem III-2.], there exists a $\mathcal{P}^{\widehat{\mathbb{F}}}\otimes\mathcal{B}(A)$–measurable function $\varphi : [0,T]\times\widehat\Omega\times A \longrightarrow A$ such that

$$\widehat\Lambda_s(\hat\omega, B) = \int_A 1_B(\varphi(s,\hat\omega,a))\nu_0(\mathrm{d}a),\ \text{for all } (s,\hat\omega)\in[0,T]\times\widehat\Omega,\ B\in\mathcal{B}(A).$$

This allows to define, for every $\bar\omega\in\overline\Omega$, two independent martingale measures $(\widetilde N^{\star,\bar\omega,i})_{1\le i\le n+d}$ and $(\widetilde N^{\star,\bar\omega,i})_{n+d+1\le i\le 2(n+d)}$ from $(N^{\star,i})_{1\le i\le n+d}$ and $(N^{\star,i})_{n+d+1\le i\le 2(n+d)}$ as follows. For each $\bar\omega\in\overline\Omega$, let us define for all $B\in\mathcal{B}(A)$, $i=1,\dots,n+d$,

$$\widetilde N^{\star,\bar\omega,i}_t(B) := \sum_{k=1}^{n+d}\iint_{[0,t]\times A} 1_B(\varphi(s,a))\Sigma_{ik}(s,\widehat X,\varphi(s,a),\mu_s(\bar\omega))N^{\star,k}(\mathrm{d}a,\mathrm{d}s),\ t\in[0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.},$$

and for all $B\in\mathcal{B}(A)$, $i = n+d+1,\dots,2(n+d)$,

$$\widetilde N^{\star,\bar\omega,i}_t(B) := \sum_{k=n+d+1}^{2(n+d)}\iint_{[0,t]\times A} 1_B(\varphi(s,a))\Sigma_{ik}(s,\widehat X,\varphi(s,a),\mu_s(\bar\omega))N^{\star,k}(\mathrm{d}a,\mathrm{d}s),\ t\in[0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}$$

By [58, Theorem III-3.], $(\widetilde N^{\star,\bar\omega,i})_{1\le i\le n+d}$ and $(\widetilde N^{\star,\bar\omega,i})_{n+d+1\le i\le 2(n+d)}$ are two independent martingale measures with intensity $\widehat\Lambda^{\Sigma,\bar\omega}_t(\mathrm{d}a)\times\mathrm{d}t$ defined by $\widehat\Lambda^{\Sigma,\bar\omega}_t(B) := \Gamma^{\bar\omega}(t,\mathbf{1}_B)$ for all $B\in\mathcal{B}(A)$.

Next, we define the martingale measure $(\widetilde N^{\bar\omega,i})_{i=1,\dots,n+d}$, from $(\widetilde N^{\star,\bar\omega,i},\widehat M^{\bar\omega,i})_{i=1,\dots,n+d}$ as follows. For each bounded $\mathcal{P}^{\widehat{\mathbb{F}}^\star}\otimes\mathcal{B}(A)$–measurable function $f : [0,T]\times\widehat\Omega^\star\times A\longrightarrow\mathbb{R}$, and $i=1,\dots,n+d$, let

$$\begin{aligned}\iint_{[0,t]\times A} f(s,a)\widetilde N^{\bar\omega,i}(\mathrm{d}a,\mathrm{d}s) :=& \sum_{k=1}^{n+d}\int_0^t \Big(\Gamma^{\bar\omega}(s,f)\big(\Gamma^{\bar\omega,+}\Gamma^{\bar\omega}\Gamma^{\bar\omega,+}\big)(s,\mathbf{1})\Big)^{i,k}\mathrm{d}\widehat M^{\bar\omega,k}_s\\ &+\sum_{k=1}^{n+d}\int_0^t\int_A \Big(f(s,a)I_{n+d} - \Gamma^{\bar\omega}(s,f)\big(\Gamma^{\bar\omega,+}\Gamma^{\bar\omega}\Gamma^{\bar\omega,+}\big)(s,\mathbf{1})\Big)^{i,k}\widetilde N^{\star,\bar\omega,k}(\mathrm{d}a,\mathrm{d}s).\end{aligned}$$

Let us refer to the proof of [58, Proposition III-9., Theorem III-10.]) for the fact the above does define a martingale measure $(\widetilde N^{\star,\bar\omega,i})_{i=1,\dots,n+d}$ with intensity $\widehat\Lambda^{\Sigma,\bar\omega}_t(\mathrm{d}a)\times\mathrm{d}t$, and that it satisfies $\widetilde N^{i,\bar\omega}_t(A) = \widehat M^{\bar\omega,i}_t$, for each $i=1,\dots,n+d$.

Finally, let

$$\Sigma^{-1}(s,\bar\omega,\widehat\omega,a) := \Sigma(\Sigma\Sigma^\top)^+\Sigma\Sigma^\top(\Sigma\Sigma^\top)^+(s,\bar\omega,\widehat\omega,a),$$

we define $(\widehat N^{\bar\omega,i})_{i=1,\dots,n+d}$ as follows. For every bounded $\mathcal{P}^{\widehat{\mathbb{F}}^\star}\otimes\mathcal{B}(A)$–measurable function $f : [0,T]\times\widehat\Omega^\star\times A\longrightarrow\mathbb{R}$, $i=1,\dots,n+d$, let

$$\widehat N^{\bar\omega,i}_t(f) := \sum_{k=1}^{n+d}\iint_{[0,t]\times A} f(s,a)\Sigma^{-1}_{ik}(s,\bar\omega,a)\widetilde N^{\bar\omega,k}(\mathrm{d}a,\mathrm{d}s) + \iint_{[0,t]\times A}\big(\mathrm{I}_{n+d} - \Sigma\Sigma^\top(\Sigma\Sigma^\top)^+\big)(s,\bar\omega,a)f(s,a)\widetilde N^{\star,\bar\omega,n+d+i}(\mathrm{d}a,\mathrm{d}s),$$

where we notice that $\Sigma\Sigma^\top(\Sigma\Sigma^\top)^+$ is the projection from $\mathbb{R}^{n+d}$ to the range of $\Sigma\Sigma^\top$. It follows then $(\widehat{N}^{\bar\omega,i})_{i=1,\dots,d}$ is a martingale measure with intensity $\widehat{\Lambda}_t(\mathrm{d}a)\times\mathrm{d}t$ and satisfies (3.3.9).

(ii) Let us now consider a bounded $\mathcal{P}^{\widehat{\mathbb{H}}^\star}\otimes\mathcal{B}(A)$–measurable function $f:[0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star\times A\longrightarrow\mathbb{R}$. By the above explicit construction of $\widehat{N}^{\bar\omega}$, it is clear that one can rewrite the stochastic integral

$$\iint_{[0,t]\times A} f^{\bar\omega}(s,a)\widehat{N}^{\bar\omega}(\mathrm{d}s,\mathrm{d}a)=\sum_{i=1}^{2(n+d)}\iint_{[0,t]\times A}\phi^{\bar\omega}(s,a)N^{\star,i}(\mathrm{d}a,\mathrm{d}s)+\sum_{i=1}^{n+d}\int_0^t\psi_s^{\bar\omega}\mathrm{d}\widehat{M}_s^{\bar\omega,i},\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.},$$

for some $\mathcal{P}^{\widehat{\mathbb{H}}^\star}\otimes\mathcal{B}(A)$–measurable function $\phi:[0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star\times A\longrightarrow\mathbb{R}$, and $\mathcal{P}^{\widehat{\mathbb{H}}^\star}$–measurable function $\psi:[0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star\longrightarrow\mathbb{R}$. Then one can apply the same arguments as in Lemma 3.3.11 to choose a good version of the stochastic integral s.t.

$$(t,\bar\omega,\hat\omega^\star)\longmapsto\Big(\iint_{[0,t]\times A}f^{\bar\omega}(s,a)\widehat{N}^{\bar\omega}(\mathrm{d}a,\mathrm{d}s)\Big)(\hat\omega^\star)\text{ is }\mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable},$$

i.e. (3.3.10) holds true. □

3.3.4.3 Proof of Proposition 3.3.6 (general case)

We provide here the proof of the Proposition 3.3.6 when the coefficients (b,σ,σ_0) verify

$$(b,\sigma)(t,\mathbf{x},\bar\nu,a)=(b,\sigma)(t,\mathbf{x},\nu,a),\text{ and }\sigma_0(t,\mathbf{x},\bar\nu,a):=\sigma_0(t,\mathbf{x},\nu),\tag{3.3.32}$$

for all $(t,\mathbf{x},\bar\nu,a)\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times A)\times A$ with $\nu(\mathrm{d}\mathbf{x}):=\bar\nu(\mathrm{d}\mathbf{x},A)$.

The idea of this part is to present how to extend the techniques used when σ_0 is constant to this general case. We now give the outline of the proof. In a nutshell, we want to approximate the relaxed control $\overline{\mathbb{P}}$ by weak control rules when, on $\overline{\Omega}$, $(X,Y,B,\widehat{\mu})$ verifies

$$Y_\cdot=X_\cdot-\int_0^\cdot\sigma_0(s,X,\mu)\mathrm{d}B_s,\ \overline{\mathbb{P}}\text{–a.s.},\tag{3.3.33}$$

and for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, under $\widehat{\mathbb{P}}_{\bar\omega}$, the canonical processes $(\widehat{X},\widehat{Y},\widehat{\Lambda},\widehat{W})$ verifies $\widehat{W}_\cdot=\int_0^\cdot\int_A\widehat{N}^{\bar\omega}(\mathrm{d}a,\mathrm{d}s)$, $\widehat{\mathbb{P}}_{\bar\omega}$–a.s., and

$$\widehat{Y}_t=\widehat{X}_0+\iint_{[0,t]\times A}b\big(r,\widehat{X},\mu(\bar\omega),a\big)\widehat{\Lambda}_r(\mathrm{d}a,\mathrm{d}r)+\iint_{[0,t]\times A}\sigma\big(r,\widehat{X},\mu(\bar\omega),a\big)\widehat{N}^{\bar\omega}(\mathrm{d}a,\mathrm{d}r),\text{ for all }t\in[0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}\tag{3.3.34}$$

Step 1: In this first step, we rewrite (3.3.34) as an equation that takes into account only $\widehat{X}$, and not $\widehat{Y}$. To do this, observe that, we can find a Borel measurable function $\mathcal{I}:(t,\mathbf{x},\pi,\mathbf{b})\in[0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n)\times\mathcal{C}^\ell\longrightarrow\mathcal{I}(t,\mathbf{x},\pi,\mathbf{b})\in\mathbb{R}^n$ verifying $\mathcal{I}(t,\mathbf{x},\pi,\mathbf{b})=\mathcal{I}(t,\mathbf{x}_{t\wedge\cdot},\pi\circ(\widehat{X}_{t\wedge\cdot})^{-1},\mathbf{b}_{t\wedge\cdot})$ and

$$\mathcal{I}\big(t,X,\mu,B\big)=\int_0^t\sigma_0(r,X,\mu)\mathrm{d}B_r,\ \overline{\mathbb{P}}\text{–a.s.}\tag{3.3.35}$$

Therefore, using the (H)–property, i.e. for all $t\in[0,T]$, $\widehat{\mu}_t(\bar\omega)=\overline{\mathbb{P}}_{\bar\omega}^{\overline{\mathcal{G}}_T}\circ(X_{t\wedge\cdot},Y_{t\wedge\cdot},W,\Lambda^t)^{-1}$, for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, we get an equivalent formulation of (3.3.33) on $\widehat{\Omega}$,

$$\widehat{Y}_\cdot=\widehat{X}_\cdot-\mathcal{I}\big(\cdot,\widehat{X},\mu(\bar\omega),B(\bar\omega)\big),\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s, for }\overline{\mathbb{P}}\text{ –a.e. }\bar\omega\in\overline{\Omega}.$$

and then, a reformulation of (3.3.34) involving only $\widehat{X}$. We can see $\mathcal{I}\big(\cdot,\widehat{X},\mu(\bar\omega),B(\bar\omega)\big)$ as a 'conditional' stochastic integral w.r.t B given the σ–field $\overline{\mathcal{G}}_T$.

Moreover, for any $\mathbb{R}^n$–valued $\widehat{\mathbb{H}}^\star$–adapted continuous process $(\widehat{S}_t)_{t\in[0,T]}$, we can find a measurable function $S : \overline{\Omega} \times \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M} \times \mathcal{C}^d \times \Omega^\star \longrightarrow \mathcal{C}^n$ such that $\widehat{S}^{\bar\omega}_t(\hat\omega,\omega^\star) = S_t\big(\bar\omega, \widehat{X}_{t\wedge\cdot}(\hat\omega), \widehat{Y}_{t\wedge\cdot}(\hat\omega), \widehat{\Lambda}^t(\hat\omega), \widehat{W}_{t\wedge\cdot}(\hat\omega), \omega^\star\big)$, for all $(t,\bar\omega,\hat\omega,\omega^\star) \in [0,T]\times\overline{\Omega}\times\widehat{\Omega}\times\Omega^\star$. And, since for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, $\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\big[\sup_{t\in[0,T]}|\widehat{S}^{\bar\omega}_t|^p\big] < \infty$, thanks to the (H)–property, we easily verify that for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$

$$\mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big(\mathcal{I}\big(\cdot,\widehat{S}^{\bar\omega},\beta(\bar\omega),B(\bar\omega)\big),\widehat{S}^{\bar\omega},\beta(\bar\omega),B(\bar\omega)\Big) = \overline{\mathbb{P}}^{\mathcal{G}_T}_{\bar\omega}\otimes\mathbb{P}^\star\circ\left(\int_0^\cdot \sigma_0\big(s,S\big(\bar\omega,X,Y,\Lambda,W\big),\beta\big)\mathrm{d}B_s, S\big(\bar\omega,X,Y,\Lambda,W\big),\beta,B\right)^{-1}, \tag{3.3.36}$$

for any Borel measurable function $\beta : \overline{\Omega} \longrightarrow \mathcal{P}(\mathcal{C}^n)$ such that $(\beta\circ(\widehat{X}_{t\wedge\cdot})^{-1})_{t\in[0,T]}$ is a $\overline{\mathbb{G}}$–predictable process satisfying $\mathbb{E}^{\overline{\mathbb{P}}}[\int_{\mathcal{C}^n}\|\mathbf{x}\|^p\beta(\mathrm{d}\mathbf{x})] < \infty$.

Step 2: Now, we approximate $\widehat{X}$ thanks to $\widehat{Y}$ by a sequence of special processes. This approximation is obtained through the same arguments used in the proof of Proposition 3.3.6 when σ_0 is constant. More precisely, for any $k\in\mathbb{N}^\star$, there exists $(a^k_1,\dots,a^k_k)\in A^k$, a sub–division $t^k_0 = 0 < \cdots < t^k_k = T$, as well as k $\mathcal{P}(A)$–valued $\widehat{\mathbb{F}}^\star$–predictable processes $(\widehat{\Lambda}^{k,1},\dots,\widehat{\Lambda}^{k,k})$, which are constant on each interval $[t^k_i,t^k_{i+1}]$, and for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, k $(\widehat{\mathbb{F}},\widehat{\mathbb{P}}^\star_{\bar\omega})$–independent Brownian motions $(\widehat{Z}^{\bar\omega,k,1},\dots,\widehat{Z}^{\bar\omega,k,k})$ such that if we define

$$\widehat{Y}^{\bar\omega,k}_t := \widehat{X}_0 + \sum_{i=1}^k\int_0^t b(r,\widehat{X},\mu(\bar\omega),a^k_i)\widehat{\Lambda}^{k,i}_r\mathrm{d}r + \int_0^t\sigma(r,\widehat{X},\mu(\bar\omega),a^k_i)\sqrt{\widehat{\Lambda}^{k,i}_r}\mathrm{d}\widehat{Z}^{\bar\omega,k,i}_r,\ t\in[0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.},$$

and $\widehat{X}^{\bar\omega,k}_\cdot := \widehat{Y}^{\bar\omega,k}_\cdot + \mathcal{I}\big(\cdot,\widehat{X},\mu(\bar\omega),B(\bar\omega)\big)$, we have

$$\lim_{k\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\left[\sup_{t\in[0,T]}\big|\widehat{Y}^{\bar\omega,k}_t - \widehat{Y}_t\big|^p\right] = 0,\ \text{then}\ \lim_{k\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\left[\sup_{t\in[0,T]}\big|\widehat{X}^{\bar\omega,k}_t - \widehat{X}_t\big|^p\right] = 0.$$

Moreover, for each $k\in\mathbb{N}^\star$

$$(t,\bar\omega,\tilde\omega^\star)\longmapsto\Big(\widetilde{Y}^{\bar\omega,k}_{t\wedge\cdot}(\hat\omega^\star),\widehat{X}^{\bar\omega,k}_{t\wedge\cdot}(\hat\omega^\star),(\widehat{\Lambda}^{1,k})^t(\hat\omega^\star),\dots,(\widehat{\Lambda}^{k,k})^t(\hat\omega^\star),\widehat{Z}^{\bar\omega,k,1}_{t\wedge\cdot}(\hat\omega^\star),\dots,\widehat{Z}^{\bar\omega,k,k}_{t\wedge\cdot}(\hat\omega^\star)\Big)\ \text{is}\ \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

Next, let us introduce $X^{k,\circ}$ an $\mathbb{R}^n$–valued $\widehat{\mathbb{H}}^\star$–adapted continuous process satisfying for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, $X^{\bar\omega,k,\circ}$ is the unique strong solution of: $\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\big[\|X^{\bar\omega,k,\circ}\|^p\big] < \infty$ and for all $t\in[0,T]$,

$$\begin{aligned}\widehat{X}^{\bar\omega,k,\circ}_t = \widehat{X}_0 &+ \sum_{i=1}^k\int_0^t b(r,\widehat{X}^{\bar\omega,k,\circ},\widehat{\mathbb{P}}_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,\circ})^{-1},a^k_i)\widehat{\Lambda}^{k,i}_r\mathrm{d}r + \int_0^t\sigma(r,\widehat{X}^{\bar\omega,k,\circ},\widehat{\mathbb{P}}_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,\circ})^{-1},a^k_i)\sqrt{\widehat{\Lambda}^{k,i}_r}\mathrm{d}\widehat{Z}^{\bar\omega,k,i}_r\\ &+\mathcal{I}\big(t,\widehat{X}^{\bar\omega,k,\circ},\widehat{\mathbb{P}}^\star_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,\circ})^{-1},B(\bar\omega)\big),\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}\end{aligned} \tag{3.3.37}$$

The existence and uniqueness $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$ of (3.3.37) is just an extension of the classical Picard iteration scheme, adapted in this context. Indeed, thanks to the definition of $\mathcal{I}$ and the fact that B is a $(\overline{\mathbb{P}},\overline{\mathbb{F}})$–Brownian motion, it is enough to adapted the proof of Theorem 5.5.3 to show this result.

We also define $\widehat{Y}^{\bar\omega,k,\circ}_\cdot := \widehat{X}^{\bar\omega,k,\circ}_\cdot - \mathcal{I}\big(\cdot,\widehat{X}^{\bar\omega,k,\circ},\widehat{\mathbb{P}}^\star_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,\circ})^{-1},B(\bar\omega)\big)$. Notice that, with the same arguments used for the constant σ_0 case (property (3.3.10) and Picard iteration argument), we can deduce that, for each $k\in\mathbb{N}^\star$,

$$(t,\bar\omega,\hat\omega^\star)\longmapsto\big(\widehat{X}^{\bar\omega,k,\circ}_{t\wedge\cdot}(\hat\omega^\star),\widehat{Y}^{\bar\omega,k,\circ}_{t\wedge\cdot}(\hat\omega^\star),(\widehat{\Lambda}^{1,k})^t(\hat\omega^\star),\dots,(\widehat{\Lambda}^{k,k})^t(\hat\omega^\star),\widehat{Z}^{\bar\omega,k,1}_{t\wedge\cdot}(\hat\omega^\star),\dots,\widehat{Z}^{\bar\omega,k,k}_{t\wedge\cdot}(\hat\omega^\star)\big)\ \text{is}\ \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

By using the definition of $\mathcal{I}$ (see (3.3.35)) and the observation (3.3.36), it is straightforward to check that

$$\begin{aligned}&\int_{\overline{\Omega}}\mathbb{E}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}\Big[\Big|\mathcal{I}\big(t,\widehat{X}^{\omega,k,\circ},\mathcal{L}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}(\widehat{X}^{\omega,k,\circ}),B(\bar\omega)\big) - \mathcal{I}\big(t,\widehat{X},\mu(\bar\omega),B(\bar\omega)\big)\Big|^p\Big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\\ \le\ &\int_{\overline{\Omega}}\int_0^t\mathbb{E}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}\Big[\Big|\sigma_0\big(r,\widehat{X}^{\omega,k,\circ},\mathcal{L}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}(\widehat{X}^{\omega,k,\circ})\big) - \sigma_0\big(r,\widehat{X},\mu(\bar\omega)\big)\Big|^p\Big]\mathrm{d}r\overline{\mathbb{P}}(\mathrm{d}\bar\omega),\end{aligned} \tag{3.3.38}$$

then thanks to this inequality, using classical techniques, we get that for some $K > 0$

$$\int_{\overline{\Omega}} \mathbb{E}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}\Big[\sup_{s\in[0,T]} |\widehat{X}^{\bar\omega,k,\circ}_s - \widehat{X}^{\bar\omega,k}_s|^p\Big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega) \le K\int_0^T\int_{\overline{\Omega}} \mathbb{E}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}\Big[\sup_{s\in[0,t]} |\widehat{X}^{\bar\omega,k}_s - \widehat{X}_s|^p\Big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\mathrm{d}t,$$

and we can deduce that $\lim_{k\to\infty}\int_{\overline{\Omega}} \mathbb{E}^{\widehat{\mathbb{P}}^\star_{\bar\omega}}\big[\sup_{s\in[0,T]} |\widehat{X}^{\bar\omega,k,\circ}_s - \widehat{X}_s|^p\big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega) = 0$.

Step 3: Finally, we provide the weak process that we are looking for, and prove a last convergence result. Again, we use the arguments from the constant σ_0 case. For $k\in\mathbb{N}^\star$ fixed, there exists, for each $m\in\mathbb{N}^\star$, Borel sets $(I^{k,1}_m,\dots,I^{k,k}_m)$ such that $\cup_{i=1}^k I^{k,i}_m = [0,T]$, and for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, $(\widehat{W}^{\bar\omega,m,1},\dots,\widehat{W}^{\bar\omega,m,k})$ are k $(\widehat{\mathbb{F}}^\star,\widehat{\mathbb{P}}^\star_{\bar\omega})$–martingales with quadratic variation $\langle\widehat{W}^{\bar\omega,m,i}\rangle_\cdot = \hat{c}^{m,i}_\cdot := \int_0^\cdot 1_{I^{k,i}_m}(r)\mathrm{d}r,$, $i\in\{1,\dots,k\}$. Furthermore, all these processes verify

$$\lim_{m\to\infty}\Big(\widehat{W}^{\bar\omega,m,i},\hat{c}^{m,i}\Big) = \Big(\int_0^\cdot\sqrt{\widehat{\Lambda}^{k,i}_r}\mathrm{d}\widehat{Z}^{\bar\omega,m,i}_r, \int_0^\cdot\widehat{\Lambda}^{k,i}_r\mathrm{d}r\Big),\ \text{for each } i\in\{1,\dots,k\},\ \widehat{\mathbb{P}}^\star_{\bar\omega}\text{–a.e.} \tag{3.3.39}$$

Consider $\widehat{X}^{k,m}$, an $\mathbb{R}^n$–valued $\widehat{\mathbb{H}}^\star$–adapted continuous process s.t. for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, $\widehat{X}^{\bar\omega,k,m}$ is the unique strong solution of

$$\begin{aligned}\widehat{X}^{\bar\omega,k,m}_t = \widehat{X}_0 + \sum_{i=1}^k\int_0^t b(r,\widehat{X}^{\bar\omega,k,m},\widehat{\mathbb{P}}_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,m})^{-1},a^k_i)\mathrm{d}\hat{c}^{m,i}_r + \int_0^t\sigma(r,\widehat{X}^{\bar\omega,k,m},\widehat{\mathbb{P}}_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,m})^{-1},a^k_i)\mathrm{d}\widehat{W}^{\bar\omega,m,i}_r\\ + \mathcal{I}\big(\cdot,\widehat{X}^{\bar\omega,k,m},\widehat{\mathbb{P}}^\star_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,m})^{-1},B(\bar\omega)\big),\ t\in[0,T],\ \widehat{\mathbb{P}}_{\bar\omega}\text{–a.s.}\end{aligned}$$

Define $\widehat{Y}^{\bar\omega,k,m}_\cdot := \widehat{X}^{\bar\omega,k,m}_\cdot - \mathcal{I}\big(\cdot,\widehat{X}^{\bar\omega,k,m},\widehat{\mathbb{P}}^\star_{\bar\omega}\circ(\widehat{X}^{\bar\omega,k,m})^{-1},B(\bar\omega)\big)$. We have that

$$(t,\bar\omega,\hat\omega^\star)\longmapsto\Big(\widehat{Y}^{\bar\omega,k,m}_{t\wedge\cdot}(\hat\omega^\star),\widehat{X}^{\bar\omega,k,m}_{t\wedge\cdot}(\hat\omega^\star),\widehat{W}^{\bar\omega,m,1}_{t\wedge\cdot}(\hat\omega^\star),\dots,\widehat{W}^{\bar\omega,m,k}_{t\wedge\cdot}(\hat\omega^\star)\Big)\ \text{is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

Define, for each $m\in\mathbb{N}^\star$, the probability on $\overline{\Omega}^{\star,k} := \mathcal{C}^n\times\mathcal{C}^n\times(\mathcal{C})^k\times(\mathcal{C}^d)^k\times\mathcal{C}^\ell\times\mathcal{P}(\mathcal{C}^n\times\mathcal{C}^n\times(\mathcal{C})^k\times(\mathcal{C}^d)^k)$,

$$\overline{\mathbb{P}}^{\star,m} := \int_{\overline{\Omega}}\mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big(\widehat{X}^{\bar\omega,k,m},\widehat{Y}^{\bar\omega,k,m},\hat{c}^{m,1},\dots,\hat{c}^{m,k},\widehat{W}^{\bar\omega,m,1},\dots,\widehat{W}^{\bar\omega,m,k},B(\bar\omega),\widehat{\mu}^m(\bar\omega)\Big)\overline{\mathbb{P}}(\mathrm{d}\bar\omega),$$

where $\widehat{\mu}^m(\bar\omega) := \mathcal{L}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big(\widehat{X}^{\bar\omega,k,m},\hat{c}^{m,1},\dots,\hat{c}^{m,k},\widehat{W}^{\bar\omega,m,1},\dots,\widehat{W}^{\bar\omega,m,k}\Big)$.

Similarly to Lemma 3.4.1, by using an inequality of type (3.3.38), we get, for some constant $C>0$

$$\sup_{m\ge1}\int_{\overline{\Omega}}\mathbb{E}^{\widehat{\mathbb{P}}_{\bar\omega}}\Big[\sup_{t\in[0,T]}|\widehat{X}^{\bar\omega,k,m}_t|^{p'} + \sup_{t\in[0,T]}|\widehat{Y}^{\bar\omega,k,m}_t|^{p'}\Big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega) \le C\Big(1+\int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x)\Big) < \infty.$$

Then, it is straightforward to deduce that $(\overline{\mathbb{P}}^{\star,m})_{m\in\mathbb{N}^\star}$ is relatively compact for the Wasserstein metric $\mathcal{W}_p$. Therefore, via a convergent subsequence $(m_j)_{j\ge1}$, we have

$$\lim_{j\to\infty}\overline{\mathbb{P}}^{\star,m_j} = \mathcal{L}^{\overline{\mathbb{P}}^\star}\Big(X^\star,Y^\star,c^{1,\star},\dots,c^{k,\star},W^{1,\star},\dots,W^{k,\star},B^\star,\widehat{\mu}^\star\Big),\text{under } \mathcal{W}_p,$$

for some random elements $\big(X^\star,Y^\star,c^{1,\star},\dots,c^{k,\star},W^{1,\star},\dots,W^{k,\star},B^\star,\widehat{\mu}^\star\big)$ in $(\overline{\Omega}^\star,\overline{\mathbb{F}}^\star,\overline{\mathbb{P}}^\star)$. By a simple use of the observation (3.3.36), we deduce

$$Y^\star_\cdot = X^\star_\cdot - \int_0^\cdot\sigma_0(s,X^\star,\mathcal{L}^{\mathbb{P}^\star}(X^\star|B^\star,\widehat{\mu}^\star))\mathrm{d}B_s,\ \overline{\mathbb{P}}^\star\text{–a.s.}$$

If, for all $t\in[0,T]$, we define

$$\widehat{\mu}^\star_t := \widehat{\mu}^\star\circ\big(\widehat{X}^\star_{t\wedge\cdot},\widehat{Y}^\star_{t\wedge\cdot},\hat{c}^{1,\star}_{t\wedge\cdot},\dots,\hat{c}^{k,\star}_{t\wedge\cdot},\widehat{W}^{1,\star}_{t\wedge\cdot},\dots,\widehat{W}^{k,\star}_{t\wedge\cdot}\big)^{-1},$$

where $(\widehat{X}^\star, \widehat{Y}^\star, \hat{c}^{1,\star}, \dots, \hat{c}^{k,\star}, \widehat{W}^{1,\star}, \dots, \widehat{W}^{k,\star})$ the canonical processes on $\mathcal{C}^n \times \mathcal{C}^n \times (\mathcal{C})^k \times (\mathcal{C}^d)^k$, we get

$$\widehat{\mu}_t^\star = \mathcal{L}^{\mathbb{P}^\star}\Big(X^\star_{t\wedge\cdot}, Y^\star_{t\wedge\cdot}, c^{1,\star}_{t\wedge\cdot}, \dots, c^{k,\star}_{t\wedge\cdot}, W^{1,\star}_{t\wedge\cdot}, \dots, W^{k,\star}_{t\wedge\cdot} \big| \widehat{\mu}^\star, B^\star\Big), \ \mathbb{P}^\star\text{–a.s.}$$

In addition, using the (conditional) martingale problem, we have, for $\overline{\mathbb{P}}^\star$–a.e. $\bar{\omega} \in \overline{\Omega}^\star$, and for all $t \in [0,T]$

$$\widehat{Y}_t^\star = \widehat{X}_0^\star + \sum_{i=1}^k \int_0^t b(r, \widehat{X}^\star, \widehat{\mu}^\star(\bar{\omega}) \circ (\widehat{X}^\star)^{-1}, a_i^k)\mathrm{d}\hat{c}_r^{i,\star} + \int_0^t \sigma(r, \widehat{X}^\star, \widehat{\mu}^\star(\bar{\omega}) \circ (\widehat{X}^\star)^{-1}, a_i^k)\mathrm{d}\widehat{W}_r^{i,\star}, \ \widehat{\mu}^\star(\bar{\omega})\text{–a.s.}$$

With the help of the (H)–property, one gets $\widehat{Y}_\cdot^\star = \widehat{X}_\cdot^\star - \mathcal{I}\big(\cdot, \widehat{X}^\star, \widehat{\mu}^\star(\bar{\omega}) \circ (\widehat{X}^\star)^{-1}, B^\star(\bar{\omega})\big)$, $\overline{\mathbb{P}}^\star_{\bar{\omega}}$–a.s, for $\overline{\mathbb{P}}^\star$ –a.e. $\bar{\omega} \in \overline{\Omega}^\star$. By (3.3.39), we deduce that

$$\mathcal{L}^{\overline{\mathbb{P}}}\Big(B, \widehat{\beta} \circ \big(\widehat{X}_0, \hat{c}^{1,\star}, \dots, \hat{c}^{k,\star}, \widehat{W}^{1,\star}, \dots, \widehat{W}^{k,\star}\big)^{-1}\Big) = \mathcal{L}^{\overline{\mathbb{P}}^\star}\Big(B^\star, \widehat{\mu}^\star \circ \big(\widehat{X}_0^\star, \hat{c}^{1,\star}, \dots, \hat{c}^{k,\star}, \widehat{W}^{1,\star}, \dots, \widehat{W}^{k,\star}\big)^{-1}\Big),$$

where

$$\widehat{\beta}(\bar{\omega}) := \mathcal{L}^{\widehat{\mathbb{P}}^\star_{\bar{\omega}}}\Big(\widehat{X}^{\bar{\omega},k,\circ}, \widehat{Y}^{\bar{\omega},k,\circ}, \int_0^\cdot \widehat{\Lambda}_r^{k,1}\mathrm{d}r, \dots, \int_0^\cdot \widehat{\Lambda}_r^{k,k}\mathrm{d}r, \int_0^\cdot \sqrt{\widehat{\Lambda}_r^{k,k}}\mathrm{d}\widehat{Z}_r^{\bar{\omega},m,i}, \dots, \int_0^\cdot \sqrt{\widehat{\Lambda}_r^{k,k}}\mathrm{d}\widehat{Z}_r^{\bar{\omega},m,i}\Big).$$

We can conclude that $\lim_{j\to\infty} \overline{\mathbb{P}}^{\star,m_j}$ is equal to

$$\int_{\overline{\Omega}} \mathcal{L}^{\widehat{\mathbb{P}}^\star_{\bar{\omega}}}\Big(\widehat{X}^{\bar{\omega},k,\circ}, \widehat{Y}^{\bar{\omega},k,\circ}, \int_0^\cdot \widehat{\Lambda}_r^{k,1}\mathrm{d}r, \dots, \int_0^\cdot \widehat{\Lambda}_r^{k,k}\mathrm{d}r, \int_0^\cdot \sqrt{\widehat{\Lambda}_r^{k,k}}\mathrm{d}\widehat{Z}_r^{\bar{\omega},m,i}, \dots, \int_0^\cdot \sqrt{\widehat{\Lambda}_r^{k,k}}\mathrm{d}\widehat{Z}_r^{\bar{\omega},m,i}, B(\bar{\omega}), \widehat{\beta}(\bar{\omega})\Big)\overline{\mathbb{P}}(\mathrm{d}\bar{\omega}). \quad (3.3.40)$$

Since this is true for any subsequence $(\overline{\mathbb{P}}^{\star,m_j})_{j\in\mathbb{N}^\star}$, we deduce the convergence for the whole sequence.

By mimicking the techniques mentioned in the proof in the case where σ_0 is constant, combined with (3.3.36), we conclude that $(\overline{\mathbb{Q}}^{\star,k,m})_{(k,m)\in\mathbb{N}^\star\times\mathbb{N}^\star}$ is a sequence of weak control rules where

$$\overline{\mathbb{Q}}^{\star,k,m} := \int_{\overline{\Omega}} \mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big(\widehat{X}^{\bar{\omega},k,m}, \widehat{Y}^{\bar{\omega},k,m}, \widehat{\Lambda}^{k,m}, \widehat{W}^{\bar{\omega},k,m}, B(\bar{\omega}), \mathcal{L}^{\widehat{\mathbb{P}}_{\bar{\omega}}}\Big(\widehat{X}^{\bar{\omega},k,m}, \widehat{Y}^{\bar{\omega},k,m}, \widehat{\Lambda}^{k,m}, \widehat{W}^{\bar{\omega},k,m}\Big)\Big)\overline{\mathbb{P}}(\mathrm{d}\bar{\omega}),$$

with $\widehat{W}^{\bar{\omega},k,m} := \sum_{i=1}^k \widehat{W}^{\bar{\omega},m,i}$, and $\widehat{\Lambda}^{k,m}(\mathrm{d}a, \mathrm{d}t) := \sum_{i=1}^k \delta_{a_i^k} 1_{I_m^{k,i}}(t)(\mathrm{d}a)\mathrm{d}t$. Moreover, using (3.3.40), we have

$$\lim_{k\to\infty} \lim_{m\to\infty} \mathcal{W}_p\big(\overline{\mathbb{Q}}^{\star,k,m}, \overline{\mathbb{P}}\big) = 0.$$

This concludes the proof. □

3.4 Proof of limit theory

Based on the equivalence result and the closedness property of $\overline{\mathcal{P}}_R(\nu)$ in Theorem 3.2.4, we can provide the proof of the limit theory result in Theorem 3.2.7 and the continuity result in Proposition 3.2.8.

3.4.1 Approximation of McKean–Vlasov SDEs by large population SDEs

We show in this section that, for any control $\alpha \in \mathcal{A}_p(\nu)$ and the controlled process X^α defined in (2.2.2), they can be approximated by a large population controlled SDE $(X^{\alpha,1}, \dots, X^{\alpha,N})$ as in (3.2.4). Let us enforce Assumption 1.4.1, and assume that $A \subset \mathbb{R}^j$ for some $j \geq 1$.

We first provide a moment estimate of the solution to the N–controlled SDEs as in Lemma 2.3.8. this is in fact an easy extension of Lacker [104, Lemmata 3.1. and 3.3.] (which are a succession of application of Gronwall Lemma), then for brevity we omit the proof.

Lemma 3.4.1. *Let* Assumption 1.4.1 *hold true, and $q \geq p$. Then there exists a constant $K > 0$ such that, for all $N \geq 1$, $(\nu, \nu^1, \dots, \nu^N) \in \big(\mathcal{P}_q(\mathbb{R}^n)\big)^{N+1}$ and $(\alpha^1, \dots, \alpha^N) \in \mathcal{A}_q^N(\nu_N)$,*

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{t\in[0,T]}|X_t^{\alpha,i}|^q\Big] \leq K\bigg(1 + \int_{\mathbb{R}^n}|x'|^q\frac{1}{N}\sum_{i=1}^N\nu^i(\mathrm{d}x') + \frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T\rho(a_0,\alpha_t^i)^q\mathrm{d}t\Big]\bigg).$$

Recall from Section 2.2.1 that $\Omega := \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell$ is equipped with the canonical element (X_0, W, B), the canonical filtration $\mathbb{F}$ and a sub–filtration $\mathbb{G}$. We consider a probability measure $\mathbb{P}_\star$, under which X_0, W, B are mutually independent, (W, B) is an $\mathbb{F}$–Brownian motion, and $X_0 \sim \mathcal{U}[0,1]$. In particular, the probability space $(\Omega, \mathcal{F}_0, \mathbb{P}_\star)$ is rich enough to support an $\mathbb{R}^n$–valued random variable of any distribution. Let ξ be an $\mathcal{F}_0$–measurable random variable such that $\mathbb{E}[|\xi|^p] < \infty$, α be an $\mathbb{F}$–predictable process satisfying the integrability condition (2.2.1). We denote by $X^{\xi,\alpha}$ the unique strong solution of the controlled McKean–Vlasov SDE

$$X_t^{\xi,\alpha} = \xi + \int_0^t b\big(r, X^{\xi,\alpha}, \overline{\mu}_r^{\xi,\alpha}, \alpha_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, X^{\xi,\alpha}, \overline{\mu}_r^{\xi,\alpha}, \alpha_r\big)\mathrm{d}W_r + \int_0^t \sigma_0\big(r, X^{\xi,\alpha}, \overline{\mu}_r^{\xi,\alpha}, \alpha_r\big)\mathrm{d}B_r,\ \mathbb{P}_\star\text{–a.s.}, \tag{3.4.1}$$

with $\overline{\mu}_r^{\xi,\alpha} := \mathcal{L}^{\mathbb{P}_\star}(X_{r\wedge\cdot}^{\xi,\alpha}, \alpha_r)|\mathcal{G}_r)$, $\mathbb{P}_\star$–a.s. and satisfying $\mathbb{E}^{\mathbb{P}_\star}\big[\|X^{\xi,\alpha}\|^p\big] < \infty$. As for (2.2.2), $X^{\xi,\alpha}$ is an $\mathbb{F}^\star$–adapted continuous process.

Given in addition a $\mathbb{G}$–optional $\mathcal{P}(\mathcal{C}^n \times A)$–valued process $\overline{\mu} = (\overline{\mu}_t)_{t\in[0,T]}$ satisfying the integrability condition

$$\mathbb{E}^{\mathbb{P}_\star}\bigg[\iiint_{[0,T]\times\mathcal{C}^n\times A}(\|\mathbf{x}\|^p + \|a - a_0\|^p)\overline{\mu}_t(\mathrm{d}\mathbf{x}, \mathrm{d}a)\mathrm{d}t\bigg] < \infty, \tag{3.4.2}$$

we denote by $X^{\xi,\overline{\mu},\alpha}$ the unique solution of the standard SDE

$$X_t^{\xi,\overline{\mu},\alpha} = \xi + \int_0^t b\big(r, X^{\xi,\overline{\mu},\alpha}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, X^{\xi,\overline{\mu},\alpha}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}W_r + \int_0^t \sigma_0\big(r, X^{\xi,\overline{\mu},\alpha}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}B_r,\ \mathbb{P}_\star\text{–a.s.}, \tag{3.4.3}$$

with $\mathbb{E}^{\mathbb{P}_\star}\big[\|X^{\xi,\overline{\mu},\alpha}\|^p\big] < \infty$. Above, $X^{\xi,\overline{\mu},\alpha}$ is defined as an $\mathbb{F}$–adapted continuous process. In particular, one has $X^{\xi,\overline{\mu}^{\xi,\alpha},\alpha} = X^{\xi,\alpha}$, $\mathbb{P}_\star$–a.s. and

$$\mathcal{L}^{\mathbb{P}_\star}(X^{\xi,\alpha}, W, B) = \mathcal{L}^{\mathbb{P}_\star}(X^{\xi',\alpha}, W, B), \text{ and } \mathcal{L}^{\mathbb{P}_\star}(X^{\xi,\overline{\mu},\alpha}, W, B) = \mathcal{L}^{\mathbb{P}_\star}(X^{\xi',\overline{\mu},\alpha}, W, B), \text{ whenever } \mathcal{L}^{\mathbb{P}_\star}(\xi) = \mathcal{L}^{\mathbb{P}_\star}(\xi').$$

Lemma 3.4.2. *Let $(\xi^m)_{m\geq 0}$ be a sequence of $\mathcal{F}_0$–measurable random variables such that*

$$\lim_{m\to\infty}\mathcal{W}_p(\mathbb{P}_\star\circ(\xi^m)^{-1}, \mathbb{P}_\star(\xi^0)^{-1}) = 0,$$

and $\sup_{m\geq 0}\mathbb{E}^{\mathbb{P}_\star}[|\xi^m|^{p'}] < \infty$. Let $\phi : [0,T]\times\mathbb{R}^n\times\mathcal{C}^d\times\mathcal{C}^\ell \longrightarrow A$ be a bounded continuous function, and $(\alpha^m)_{m\geq 0}$ be defined by $\alpha_t^m := \phi(t, \xi^m, W_{t\wedge\cdot}, B_{t\wedge\cdot})$ for all $t \in [0,T]$. Then, for each $t \in [0,T]$, we have

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_\star}\Big[\mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi^m,\alpha^m}, \alpha_t^m\big|\mathcal{G}_t\big), \mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi^0,\alpha^0}, \alpha_t^0\big|\mathcal{G}_t\big)\Big)\Big] = 0,$$

and, for any fixed $\overline{\mu} = (\overline{\mu}_t)_{t\in[0,T]}$ satisfying (3.4.2),

$$\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_\star}\Big[\mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi^m,\overline{\mu},\alpha^m}, \alpha_t^m\big|\mathcal{G}_t\big), \mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi^0,\overline{\mu},\alpha^0}, \alpha_t^0\big|\mathcal{G}_t\big)\Big)\Big] = 0.$$

Proof. We will only prove the first convergence result, since the second follows by almost the same arguments.

First, without loss of generality, one can use Skorokhod's representation theorem and assume that $\lim_{n\to\infty}\xi^n = \xi^0$, $\mathbb{P}_\star$–a.s. Then, using the Lipschitz properties and the polynomial growth of the coefficient functions, we have using classical arguments (see notably *Step* 1 of the proof of Lemma 3.3.7), that there exists a constant $K > 0$ such that, for $m \geq 1$,

$$\mathbb{E}^{\mathbb{P}_\star}\Big[\sup_{t\in[0,T]}|X_t^{\xi^m,\alpha^m} - X_t^{\xi^0,\alpha^0}|^p\Big] \leq K\bigg(\mathbb{E}^{\mathbb{P}_\star}\big[|\xi^m - \xi^0|^p\big] + \mathbb{E}^{\mathbb{P}_\star}\Big[\int_0^T\big|\alpha_t^m - \alpha_t^0\big|^p\mathrm{d}t\Big] + C_m\bigg), \tag{3.4.4}$$

where

$$C_m := \mathbb{E}^{\mathbb{P}_\star}\Big[\int_0^T \big|(b,\sigma,\sigma_0)\big(r, X^{\xi^0,\alpha^0}, \overline{\mu}_r^{\xi^0,\alpha^0}, \alpha_r^m\big) - (b,\sigma,\sigma_0)\big(r, X^{\xi^0,\alpha^0}, \overline{\mu}_r^{\xi^0,\alpha^0}, \alpha_r^0\big)\big|^p \mathrm{d}r\Big].$$

Next, since $\sup_m \mathbb{E}^{\mathbb{P}_\star}[|\xi^m|^{p'}] < \infty$, for some $p' > p$, then $(|\xi^m - \xi^0|^p)_{m\geq 1}$ is $\mathbb{P}_\star$–uniformly integrable and it follows that $\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_\star}\big[|\xi^m - \xi^0|^p\big] = 0$. Moreover, since $\phi : [0,T]\times\mathbb{R}^n\times\mathcal{C}^d\times\mathcal{C}^\ell \longrightarrow A$ is bounded continuous, we obtain that

$$\lim_{m\to\infty} |\alpha_t^m - \alpha_t^0| = \lim_{m\to\infty} C_m = 0, \text{ and hence } \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_\star}\Big[\sup_{t\in[0,T]} \big|X_t^{\xi^m,\alpha^m} - X_t^{\xi^0,\alpha^0}\big|^p\Big] = 0.$$

To conclude, it is enough to notice that, as $m \longrightarrow 0$,

$$\mathbb{E}^{\mathbb{P}_\star}\Big[\mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi^m,\alpha^m}, \alpha_t^m\big|\mathcal{G}_t\big), \mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi^0,\alpha^0}, \alpha^0\big|\mathcal{G}_t\big)\Big)\Big] \leq \mathbb{E}^{\mathbb{P}_\star}\Big[\big|X_{t\wedge\cdot}^{\xi^m,\alpha^m} - X_{t\wedge\cdot}^{\xi^0,\alpha^0}\big|^p\Big]^{1/p} + \mathbb{E}^{\mathbb{P}_\star}\big[\big|\alpha_t^m - \alpha_t^0\big|^p\big]^{1/p} \longrightarrow 0.$$

□

To proceed, let us consider, for each $N \geq 1$, the space $\Omega^N := (\mathbb{R}^n)^N \times (\mathcal{C}^d)^N \times \mathcal{C}^\ell$ defined in Section 3.2.2, equipped with canonical elements $(X_0^1, \ldots, X_0^N, W^1, \ldots, W^N)$ and canonical filtration $\mathbb{F}^N$. On Ω^N, we also introduce a sub–filtration

$$\mathbb{G}^N := (\mathcal{G}_t^N)_{t\in[0,T]}, \text{ with } \mathcal{G}_t^N := \sigma(B_s : s \in [0,t]).$$

Given $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and a sequence $(\nu^i)_{i\geq 1} \subset \mathcal{P}_p(\mathbb{R}^n)$, we take the first N elements to define $\mathbb{P}_\nu^N$ on Ω^N, under which $X_0^i \sim \nu^i$, and B, W^i are standard Brownian motions, and $(X_0^1, \ldots, X_0^N, W^1, \ldots, W^N, B)$ are mutually independent.

Further, in Lemma 3.4.2, we keep using the bounded continuous function ϕ to define the control process α. Together with an initial random variable $\xi \sim \nu$, one obtain a $\mathbb{G}$–optional process $\overline{\mu}^{\xi,\alpha}$ in Ω. Notice that in Ω, the process $\overline{\mu}^{\xi,\alpha}$ is a functional of the common noise process B, one can then extend it as a $\mathbb{G}^N$–optional process in Ω^N while keeping the same notation for simplicity.

Finally, with the same bounded continuous function $\phi : [0,T]\times\mathbb{R}^n\times\mathcal{C}^d\times\mathcal{C}^\ell \longrightarrow A$ in Lemma 3.4.2, we introduce the control processes $(\alpha^1, \ldots, \alpha^N)$ by $\alpha_t^i := \phi(t, X_0^i, W_{t\wedge\cdot}^i, B_{t\wedge\cdot})$, and then define a sequence of processes $\overline{X}^{\alpha^i,i}$, $i = 1, \ldots, N$, by

$$\overline{X}_t^{\alpha^i,i} = X_0^i + \int_0^t b\big(r, \overline{X}^{\alpha^i,i}, \overline{\mu}_r^{\xi,\alpha}, \alpha_r^i\big)\mathrm{d}r + \int_0^t \sigma\big(r, \overline{X}^{\alpha^i,i}, \overline{\mu}_r^{\xi,\alpha}, \alpha_r^i\big)\mathrm{d}W_r^i + \int_0^t \sigma_0\big(r, \overline{X}^{\alpha^i,i}, \overline{\mu}_r^{\xi,\alpha}, \alpha_r^i\big)\mathrm{d}B_r, \ \mathbb{P}_\nu^N\text{–a.s.} \quad (3.4.5)$$

Notice that the above SDE is almost the same as (3.4.3), except that we use here $(X_0^i, \overline{\mu}^{\xi,\alpha}, W^i)$ instead of $(\xi, \overline{\mu}, W)$.

Lemma 3.4.3. *Assume that ν and $(\nu^i)_{i\geq 1}$ satisfy*

$$\lim_{N\to\infty} \mathcal{W}_p\Big(\frac{1}{N}\sum_{i=1}^N \nu^i, \nu\Big) = 0, \text{ and } \sup_{N\geq 1} \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x|^{p'} \nu^i(\mathrm{d}x) < \infty.$$

Then

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \mathcal{W}_p\big(\overline{\varphi}_t^N, \overline{\mu}_t^{\xi,\alpha}\big)\mathrm{d}t\Big] = 0, \text{ with } \overline{\varphi}_t^N(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{\big(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i\big)}(\mathrm{d}\mathbf{x}, \mathrm{d}a). \quad (3.4.6)$$

Proof. Notice that to prove (3.4.6), it is enough to prove that, in the space $\big(\mathbb{M}\big(\mathcal{P}(\mathcal{C}^n\times A)\times\mathcal{P}(\mathcal{C}^n\times A)\big), \mathcal{W}_p\big)$,

$$\overline{\Lambda}^N(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t) := \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\delta_{\big(\overline{\varphi}_t^N, \overline{\mu}_t^{\xi,\alpha}\big)}(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}')\mathrm{d}t\Big] \underset{N\to\infty}{\longrightarrow} \overline{\Lambda}^0(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t) := \mathbb{E}^{\mathbb{P}_\star}\Big[\delta_{\overline{\mu}_t^{\xi,\alpha}}(\mathrm{d}\bar{\nu})\delta_{\overline{\mu}_t^{\xi,\alpha}}(\mathrm{d}\bar{\nu}')\mathrm{d}t\Big].$$

First, by a trivial extension of Lemma 3.4.1 and Lemma 2.3.8, there exists a constant K independent of $i \geq 1$, s.t.

$$\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\frac{1}{N}\sum_{i=1}^N \sup_{[0,T]}|\overline{X}_t^{\alpha^i,i}|^{p'} + \int_0^T |a_0 - \alpha_t^i|^{p'}\mathrm{d}t\Big] \leq \frac{1}{N}\sum_{i=1}^N K\bigg(1 + \mathbb{E}^{\mathbb{P}_\nu^N}[|X_0^i|^{p'}] + \mathbb{E}^{\mathbb{P}_\star}[|\xi|^{p'}] + \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T |a_0 - \alpha_t^i|^{p'}\mathrm{d}t\Big]\bigg)$$
$$\leq K\bigg(1 + \int_{\mathbb{R}^n}|x|^{p'}\frac{1}{N}\sum_{i=1}^N \nu^i(\mathrm{d}x)\bigg) < \infty,$$

where the second inequality follows by the fact that ϕ is bounded. Since $p' > p$, it follows by [49, Proposition-A.2.] and [49, Proposition-B.1.] that $(\overline{\Lambda}^N)_{N\in\mathbb{N}}$ is relatively compact in $(\mathbb{M}\big(\mathcal{P}(\mathcal{C}^n \times A) \times \mathcal{P}(\mathcal{C}^n \times A)\big), \mathcal{W}_p)$.

Let $(N_m)_{m\geq 1}$ be a subsequence such that $\overline{\Lambda}^{N_m} \longrightarrow_{m\to\infty} \overline{\Lambda}^\infty$ under $\mathcal{W}_p$. We only need to show that $\overline{\Lambda}^\infty = \overline{\Lambda}^0$, or equivalently (see Proposition 3.5.1), that for every $k \geq 1$, $g_1, \ldots, g_k \in C_b(\mathcal{C}^n \times A)$, $f \in C_b([0,T] \times \mathcal{P}(\mathcal{C}^n \times A))$, we have

$$\int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)^2} \prod_{i=1}^k \langle g_i, \bar{\nu}\rangle f(t,\bar{\nu}')\overline{\Lambda}^\infty(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t) = \int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)^2} \prod_{i=1}^k \langle g_i, \bar{\nu}\rangle f(t,\bar{\nu}')\overline{\Lambda}^0(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t). \tag{3.4.7}$$

In the following, we provide the proof of (3.4.7) for the case $k = 2$, since the proof for the general case is identical.

Notice that $\overline{\mu}^{\xi,\alpha}$ is $\mathbb{G}^N$–adapted, and $X^{\alpha^i,i}$ depends only on (X_0^i, W^i, B). It therefore follows that $(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i)$ and $(\overline{X}_{t\wedge\cdot}^{\alpha^j,j}, \alpha_t^j)$ are conditionally independent given the σ–algebra $\mathcal{G}_t^N$, for all $t \in [0,T]$. Thus for $i \neq j$,

$$\mathbb{E}^{\mathbb{P}_\nu^N}\big[g_1\big(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i\big)g_2\big(\overline{X}_{t\wedge\cdot}^{\alpha^j,j}, \alpha_t^j\big)f\big(t, \overline{\mu}_t^{\xi,\alpha}\big)\big] = \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\mathbb{E}^{\mathbb{P}_\nu^N}\big[g_1(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i)\big|\mathcal{G}_t^N\big]\mathbb{E}^{\mathbb{P}_\nu^N}\big[g_2(\overline{X}_{t\wedge\cdot}^{\alpha^j,j}, \alpha_t^j)\big|\mathcal{G}_t^N\big]f\big(t, \overline{\mu}_t^{\xi,\alpha}\big)\Big].$$

Since f, g_1, and g_2 are bounded, it follows that

$$\begin{aligned}
&\int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)^2} \langle g_1, \bar{\nu}\rangle\langle g_2, \bar{\nu}\rangle f(t,\bar{\nu}')\overline{\Lambda}^\infty(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t)\\
&= \lim_{m\to\infty}\int_0^T \frac{1}{N_m^2}\sum_{i,j=1}^{N_m} \mathbb{E}^{\mathbb{P}_\nu^{N_m}}\Big[g_1\big(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i\big)g_2\big(\overline{X}_{t\wedge\cdot}^{\alpha^j,j}, \alpha_t^j\big)f\big(t, \overline{\mu}_t^{\xi,\alpha}\big)\Big]\mathrm{d}t\\
&= \lim_{m\to\infty}\int_0^T \frac{1}{N_m^2}\sum_{i,j=1}^{N_m} \mathbb{E}^{\mathbb{P}_\nu^{N_m}}\Big[\mathbb{E}^{\mathbb{P}_\nu^{N_m}}\big[g_1(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i)\big|\mathcal{G}_t^N\big]\ \mathbb{E}^{\mathbb{P}_\nu^{N_m}}\big[g_2(\overline{X}_{t\wedge\cdot}^{\alpha^j,j}, \alpha_t^j)\big|\mathcal{G}_t^N\big]\ f(t, \overline{\mu}_t^{\xi,\alpha})\Big]\mathrm{d}t\\
&= \lim_{m\to\infty}\int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)^2} \langle g_1, \bar{\nu}\rangle\langle g_2, \bar{\nu}\rangle f(t,\bar{\nu}')\mathbb{E}^{\mathbb{P}_\nu^{N_m}}\Big[\delta_{\left(\frac{1}{N_m}\sum_{i=1}^{N_m}\mathcal{L}^{\mathbb{P}_\nu^{N_m}}(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i|\mathcal{G}_t^N), \overline{\mu}_t^{\xi,\alpha}\right)}(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}')\Big]\mathrm{d}t.
\end{aligned}$$

Let U^N be a random variable on $(\Omega, \mathcal{F}_0, \mathbb{P}_\star)$ such that $\mathcal{L}^{\mathbb{P}_\star}(U^N) = \frac{1}{N}\sum_{i=1}^N \nu^i$. If we note $\alpha_t^{\star,N} := \phi(t, U^N, W_{t\wedge\cdot}, B_{t\wedge\cdot})$, we have, from Lemma 3.4.2 that, for all $t \in [0,T]$

$$\begin{aligned}
&\lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_\nu^{N_m}}\bigg[\mathcal{W}_p\Big(\frac{1}{N_m}\sum_{i=1}^{N_m}\mathcal{L}^{\mathbb{P}_\nu^{N_m}}\big(\overline{X}_{t\wedge\cdot}^{\alpha^i,i}, \alpha_t^i|\mathcal{G}_t^N\big), \overline{\mu}_t^{\xi,\alpha}\Big)\bigg]\\
&= \lim_{m\to\infty}\mathbb{E}^{\mathbb{P}_\star}\Big[\mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{U^N,\overline{\mu}^{\xi,\alpha},\alpha^{\star,N}}, \alpha_t^{\star,N}|\mathcal{G}_t\big),\ \mathcal{L}^{\mathbb{P}_\star}\big(X_{t\wedge\cdot}^{\xi,\overline{\mu}^{\xi,\alpha},\alpha}, \alpha_t|\mathcal{G}_t\big)\Big)\Big] = 0.
\end{aligned}$$

Consequently

$$\int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)^2} \langle g_1, \bar{\nu}\rangle\langle g_2, \bar{\nu}\rangle f(t,\bar{\nu}')\overline{\Lambda}^\infty(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t) = \int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)^2} \langle g_1, \bar{\nu}\rangle\langle g_2, \bar{\nu}\rangle f(t,\bar{\nu}')\overline{\Lambda}^0(\mathrm{d}\bar{\nu}, \mathrm{d}\bar{\nu}', \mathrm{d}t),$$

and the proof is concluded. □

Given a probability measure $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and a sequence $(\nu^i)_{i\geq 1} \subset \mathcal{P}_p(\mathbb{R}^n)$, we consider the probability spaces $(\Omega, \mathcal{F}, \mathbb{P}_\nu)$ and $(\Omega^N, \mathcal{F}^N, \mathbb{P}_\nu^N)$, introduced respectively in Section 2.2.1 and Section 3.2.2. Let us fix a bounded continuous function $\phi : [0,T] \times \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \longrightarrow A$, and define a control process $\alpha := (\alpha_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F})$, and control processes $(\alpha^1, \dots, \alpha^N)$ on $(\Omega^N, \mathcal{F}^N)$ by

$$\alpha_t := \phi(t, X_0, W_{t\wedge\cdot}, B_{t\wedge\cdot}),\ \alpha_t^i := \phi(t, X_0^i, W^i_{t\wedge\cdot}, B_{t\wedge\cdot}),\ t \in [0,T],\ i = 1, \dots, N. \tag{3.4.8}$$

Using the control process α, $(X^\alpha, \overline{\mu}^\alpha)$ is defined by (2.2.2) under $\mathbb{P}_\nu$. In particular, in the probability space $(\Omega, \mathcal{F}, \mathbb{P}_\star)$, let $\xi \sim \nu$, and $(X^{\xi,\alpha}, \overline{\mu}^{\xi,\alpha})$ be defined by (3.4.1). We have $\mathbb{P}_\star \circ (\overline{\mu}^{\xi,\alpha})^{-1} = \mathbb{P}_\nu \circ (\mu^\alpha)^{-1}$. Next, let ξ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_\star)$ satisfying $\mathbb{P}_\star \circ \xi^{-1} = \nu$. We also naturally extend the $\mathbb{G}$–optional process $\overline{\mu}^{\xi,\alpha}$ on Ω into a $\mathbb{G}^N$–optional process on Ω^N. Then with the bounded control processes $(\alpha^1, \dots, \alpha^N)$, $(X^{\alpha,i})_{i=1,\dots,N}$ is defined by (3.2.4) under $\mathbb{P}_\nu^N$, and $(\overline{X}^{\alpha^i,i})_{i=1,\dots,N}$ is defined by (3.4.5). Recall also that

$$\varphi_t^{N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{(X^{\alpha,i}_{t\wedge\cdot})}(\mathrm{d}\mathbf{x}),\ \varphi_t^N(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(X^{\alpha,i}_{t\wedge\cdot}, \alpha_t^i)}(\mathrm{d}\mathbf{x}, \mathrm{d}a),\ \text{and } \overline{\varphi}_t^N(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\overline{X}^{\alpha^i,i}_{t\wedge\cdot}, \alpha_t^i\right)}(\mathrm{d}\mathbf{x}, \mathrm{d}a).$$

Proposition 3.4.4. *Let α and $(\alpha^i)_{1\leq i\leq N}$ be defined in (3.4.8), together with the Borel measurable function $\phi : [0,T] \times \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \longrightarrow A$. Assume that*

$$\alpha \in \mathcal{A}_p(\nu),\quad \sup_{N\geq 1}\frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x|^{p'} \nu^i(\mathrm{d}x) < \infty,\quad \text{and } \lim_{N\to\infty} \mathcal{W}_p\left(\frac{1}{N}\sum_{i=1}^N \nu^i, \nu\right) = 0.$$

Then

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T \mathcal{W}_p(\varphi_t^N, \overline{\mu}_t^{\xi,\alpha})\mathrm{d}t\right] = 0,\ \text{and } \lim_{N\to\infty} \mathcal{L}^{\mathbb{P}_\nu^N}\left(\delta_{\varphi_t^N}(\mathrm{d}\bar{\nu})\mathrm{d}t, \varphi^{N,X}\right) = \mathcal{L}^{\mathbb{P}_\nu}\left(\delta_{\overline{\mu}_t^\alpha}(\mathrm{d}\bar{\nu})\mathrm{d}t, \mu^\alpha\right) \text{ under } \mathcal{W}_p. \tag{3.4.9}$$

Consequently

$$V_S(\nu) \leq \liminf_{N\to\infty} V_S^N(\nu^1, \dots, \nu^N).$$

Proof. (i) Using Assumption 1.4.1, together with Burkholder–Davis–Gundy inequality and Gronwall's emma, it follows by classical arguments that there exist positive constants K, and K' such that for all $N \geq 1$, $i = 1, \dots, N$ and $t \in [0,T]$

$$\mathbb{E}^{\mathbb{P}_\nu^N}\left[\sup_{r\in[0,t]} \left|X_r^{\alpha,i} - \overline{X}_r^{\alpha^i,i}\right|^p\right] \leq K\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^t \mathcal{W}_p(\varphi_r^N, \overline{\mu}_r^{\xi,\alpha})^p \mathrm{d}r\right] \leq K'\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^t \left(\mathcal{W}_p(\varphi_r^N, \overline{\varphi}_r^N)^p + \mathcal{W}_p(\overline{\varphi}_r^N, \overline{\mu}_r^{\xi,\alpha})^p\right)\mathrm{d}r\right].$$

Further, notice that

$$\mathbb{E}^{\mathbb{P}_\nu^N}\left[\mathcal{W}_p(\varphi_t^N, \overline{\varphi}_t^N)^p\right] \leq \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\left[\sup_{r\in[0,t]} |X_r^{\alpha,i} - \overline{X}_r^{\alpha^i,i}|^p\right] \leq K\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^t \left(\mathcal{W}_p(\varphi_r^N, \overline{\varphi}_r^N)^p + \mathcal{W}_p(\overline{\varphi}_r^N, \overline{\mu}_r^{\xi,\alpha})^p\right)\mathrm{d}r\right],$$

it follows by Gronwall's lemma and then by Lemma 3.4.3 that

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_\nu^N}\left[\mathcal{W}_p(\varphi_t^N, \overline{\varphi}_t^N)^p\right] \leq \lim_{N\to\infty} K\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^t \mathcal{W}_p(\overline{\varphi}_r^N, \overline{\mu}_r^{\xi,\alpha})^p \mathrm{d}r\right] = 0,\ \text{and thus } \lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T \mathcal{W}_p(\varphi_t^N, \overline{\mu}_t^{\xi,\alpha})^p \mathrm{d}t\right] = 0.$$

As an immediate consequence, we also have

$$\lim_{N\to\infty} \mathcal{L}^{\mathbb{P}_\nu^N}\left(\delta_{\varphi_t^N}(\mathrm{d}\bar{\nu})\mathrm{d}t, \varphi^{N,X}\right) = \mathcal{L}^{\mathbb{P}_\nu}\left(\delta_{\overline{\mu}_t^\alpha}(\mathrm{d}\bar{\nu})\mathrm{d}t, \mu^\alpha\right),\ \text{under } \mathcal{W}_p.$$

(ii) Let us now consider an arbitrary control process $\alpha \in \mathcal{A}_p(\nu)$, so that there exists a Borel measurable function $\phi : [0,T] \times \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \longrightarrow A$ such that $\alpha_t = \phi(t, \xi, W_{t\wedge}, B_{t\wedge\cdot})$ for all $t \in [0,T]$, $\mathbb{P}_\nu$–a.s. Then there exists (see e.g. [49, Proposition C.1.]) a sequence of bounded continuous functions $(\phi^m)_{m\geq 1} : [0,T] \times \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \longrightarrow A$ such that

$$\lim_{m\to\infty} \alpha_t^m := \lim_{m\to\infty} \phi^m(t, \xi, W_{t\wedge}, B_{t\wedge\cdot}) = \phi(t, \xi, W_{t\wedge}, B_{t\wedge\cdot}) = \alpha_t,\ \mathrm{d}\mathbb{P}_\nu \otimes \mathrm{d}t \text{–a.e.}$$

Then, in the probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P}_\nu)$, it follows by standard arguments (see e.g. the proof of Proposition 2.3.12 or Lemma 3.3.7) that

$$\lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{t\in[0,T]} \big|X_t^{\alpha^m} - X_t^{\alpha}\big|^p\Big] = 0, \text{ and } \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \mathcal{W}_p\big(\overline{\mu}_t^{\alpha^m}, \overline{\mu}_t^{\alpha}\big)^p \mathrm{d}t\Big] = 0.$$

Finally, for each $m \geq 1$, consider the bounded continuous function ϕ^m. For each $N \geq 1$, on the space $(\Omega^N, \mathcal{F}^N, \mathbb{P}_\nu^N)$, we can define control processes $(\alpha^{m,i})_{1\leq i\leq N}$ by $\alpha_t^{m,i} := \phi^m(t, X_0^i, W_{t\wedge\cdot}^i, B_{t\wedge\cdot})$, $t \in [0,T]$, $i \in \{1,\dots,N\}$, and then define $(X^{\alpha^m,1}, \dots, X^{\alpha^m,N})$ as the unique solution of Equation (3.2.4) with control processes $(\alpha^{m,i})_{i=1,\dots,N}$.

Define then

$$\varphi_t^{m,N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{(X_{t\wedge\cdot}^{\alpha^m,i})}(\mathrm{d}\mathbf{x}) \text{ and } \varphi_t^{m,N}(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(X_{t\wedge\cdot}^{\alpha^m,i}, \alpha_t^{m,i})}(\mathrm{d}\mathbf{x}, \mathrm{d}a).$$

We have, thanks to Equation (3.4.9)

$$\lim_{N\to\infty} \mathcal{L}^{\mathbb{P}_\nu^N}\big(\delta_{\varphi_t^{m,N}}(\mathrm{d}m)\mathrm{d}t, \varphi^{m,N,X}\big) = \mathcal{L}^{\mathbb{P}_\nu}\big(\delta_{\mu_t^{\alpha^m}}(\mathrm{d}m)\mathrm{d}t, \mu^{\alpha^m}\big), \text{ under } \mathcal{W}_p.$$

It follows then

$$\begin{aligned}
J(\alpha) &= \mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \big\langle L(t,\cdot,\overline{\mu}_t^\alpha), \overline{\mu}_t^\alpha\big\rangle \mathrm{d}t + \big\langle g(\cdot,\mu_T^\alpha), \mu_T^\alpha\big\rangle\Big] \\
&\leq \lim_{m\to\infty} \mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \big\langle L(t,\cdot,\overline{\mu}_t^{\alpha^m}), \overline{\mu}_t^{\alpha^m}\big\rangle \mathrm{d}t + \big\langle g(\cdot,\mu_T^{\alpha^m}), \mu_T^{\alpha^m}\big\rangle\Big] \\
&\leq \lim_{m\to\infty}\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \big\langle L(t,\cdot,\varphi_t^{m,N}), \varphi_t^{m,N}\big\rangle \mathrm{d}t + \big\langle g(\cdot,\varphi_T^{m,N,X}), \varphi_T^{m,N,X}\big\rangle\Big] \\
&\leq \lim_{m\to\infty}\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T L\big(t, X^{\alpha^m,i}, \alpha_t^{m,i}, \varphi_t^{m,N}\big)\mathrm{d}t + g\big(X^{\alpha^m,i}, \varphi_T^{m,N,X}\big)\Big] \leq \liminf_{N\to\infty} V_S^N(\nu^1,\dots,\nu^N).
\end{aligned}$$

By arbitrariness of $\alpha \in \mathcal{A}_p(\nu)$, it follows that $V_S(\nu) \leq \liminf_{N\to\infty} V_S^N(\nu^1, \cdot, \nu^N)$. □

Using exactly the same arguments and Lemma 3.4.2 we can obtain the following result, whose proof is therefore omitted.

Proposition 3.4.5. *Assume that*

$$\sup_{m\geq 1}\int_{\mathbb{R}^n} |x|^{p'}\nu^m(\mathrm{d}x) < \infty, \text{ and } \lim_{m\to\infty} \mathcal{W}_p(\nu^m, \nu) = 0.$$

Then with the control process α defined in (3.4.8), we have

$$\lim_{m\to\infty} \mathcal{L}^{\mathbb{P}_{\nu^m}}\big(\delta_{\overline{\mu}_t^\alpha}(\mathrm{d}\bar{\nu})\mathrm{d}t, \mu^\alpha\big) = \mathcal{L}^{\mathbb{P}_\nu}\big(\delta_{\overline{\mu}_t^\alpha}(\mathrm{d}\bar{\nu})\mathrm{d}t, \mu^\alpha\big), \text{ under } \mathcal{W}_p, \text{ and consequently } V_S(\nu) \leq \liminf_{m\to\infty} V_S(\nu^m).$$

3.4.2 Tightness of the optimal control rules

Let us now stay in the context of Theorem 3.2.7 and prove that the set of optimal or ε–optimal control rules is tight. Recall that Assumption 1.4.1 and Assumption 3.1.1 hold true, $A \subset \mathbb{R}^j$ for some $j \geq 1$, and both L and g are continuous in all their arguments. Let $N \geq 1$, $(\nu, \nu^1, \dots, \nu^N) \subset \mathcal{P}_p(\mathbb{R}^n)$, $\alpha \in \mathcal{A}(\nu)$ and $(\alpha^1, \dots, \alpha^N) \in \mathcal{A}^N(\nu_N)$. $\mathbb{P}^N(\alpha^1, \dots, \alpha^N)$ is a probability measure on $\overline{\Omega}$ defined by (3.2.6).

Proposition 3.4.6. *(i) In the context of* Theorem 3.2.7*, Let* $(\nu^i)_{i\geq 1} \subset \mathcal{P}_p(\mathbb{R}^n)$ *satisfy* $\sup_{N\geq 1} \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x|^{p'} \nu^i(\mathrm{d}x) < \infty$ *and* $(\overline{\mathbb{P}}^N)_{N\geq 1} \subset \mathcal{P}_p(\overline{\Omega})$ *satisfy* (3.2.7)*, then both* $(\frac{1}{N}\sum_{i=1}^N \nu^i)_{N\geq 1}$ *and* $(\overline{\mathbb{P}}^N)_{N\geq 1}$ *are relatively compact under* $\mathcal{W}_p$*. Moreover, for any converging subsequence* $(\overline{\mathbb{P}}^{N_m})_{m\geq 1}$*, we have*

$$\lim_{m\to\infty} \mathcal{W}_p\Big(\frac{1}{N_m}\sum_{i=1}^{N_m} \nu^i, \nu\Big) = 0, \text{ for some } \nu \in \mathcal{P}_p(\mathbb{R}^n), \text{ and } \lim_{m\to\infty} \mathcal{W}_p\big(\overline{\mathbb{P}}^{N_m}, \overline{\mathbb{P}}^\infty\big) = 0, \text{ for some } \overline{\mathbb{P}}^\infty \in \overline{\mathcal{P}}_R(\nu).$$

(ii) In the context of Proposition 3.2.8*, let* $(\varepsilon_m)_{m\geq 1} \subset \mathbb{R}_+$ *be such that* $\lim_{m\to\infty} \varepsilon_m = 0$*,* $(\overline{\mathbb{P}}^m)_{m\geq 1}$ *be a sequence such that*

$$\overline{\mathbb{P}}^m \in \overline{\mathcal{P}}_R(\nu^m), \text{ and } J(\overline{\mathbb{P}}^m) \geq V_S(\nu^m) - \varepsilon_m.$$

Then the sequence $(\overline{\mathbb{P}}^m)_{m\geq 1}$ *is relatively compact, and moreover, any cluster point of* $(\overline{\mathbb{P}}^m)_{m\geq 1}$ *belongs to* $\overline{\mathcal{P}}_R(\nu)$*.*

Proof. We will only consider (i), since the proof of (ii) is identical.

Tightness: To prove the tightness of $(\overline{\mathbb{P}}^N)_{N\geq 1}$ under $\mathcal{W}_p$, it is enough to adapt the proof of [104, Proposition 3.5.] to our context. First, let us define control processes $(\alpha^{0,i})_{i\geq 1}$ by $\alpha_t^{0,i} \equiv a_0$ for all $t \in [0,T]$ and $i \geq 1$, and denote $\overline{\mathbb{P}}_0^N := \mathbb{P}^N(\alpha^{0,1}, \dots, \alpha^{0,N})$. By Lemma 3.4.1, there exist some constants K, $K' > 0$, such that for all $N \geq 1$

$$J\big(\overline{\mathbb{P}}_0^N\big) \geq -K\Big(1 + \mathbb{E}^{\overline{\mathbb{P}}_0^N}\Big[\sup_{t\in[0,T]} |X_t|^p\Big]\Big) = -K\Big(1 + \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{t\in[0,T]} \big|X_t^{\alpha^0,i}\big|^p\Big]\Big) \geq -K'\Big(1 + \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x|^p \nu^i(\mathrm{d}x)\Big).$$

Since by (3.2.7)

$$J\big(\overline{\mathbb{P}}^N\big) \geq V_S^N(\nu^1, \dots, \nu^N) - \varepsilon_N \geq J\big(\overline{\mathbb{P}}_0^N\big) - \varepsilon_N,$$

it follows that $J\big(\overline{\mathbb{P}}^N\big) \geq -C$, for some constant C independent of N. Using again Lemma 3.4.1, the coercivity condition (1.4.2), and the growth conditions in Assumption 1.4.1, it follows that

$$J\big(\overline{\mathbb{P}}^N\big) \leq K\Big(1 + \int_{\mathbb{R}^n} |x'|^p \frac{1}{N}\sum_{i=1}^N \nu^i(\mathrm{d}x') + \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \big|\alpha_t^{i,N} - a_0\big|^p \mathrm{d}t\Big]\Big) - C_L \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \big|\alpha_t^{i,N} - a_0\big|^{p'} \mathrm{d}t\Big].$$

Then, there exists some constant $C > 0$, independent of N, such that

$$C_L \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \big|\alpha_t^{i,N} - a_0\big|^{p'} \mathrm{d}t\Big] - K\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \big|\alpha_t^{i,N} - a_0\big|^p \mathrm{d}t\Big] < C.$$

Since $p' > p$, it follows that

$$\sup_{N\geq 1} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^T \big|\alpha_t^{i,N} - a_0\big|^{p'} \mathrm{d}t\Big] < \infty. \tag{3.4.10}$$

With the condition $\sup_{N\geq 1} \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x|^{p'} \nu^i(\mathrm{d}x) < \infty$, and by similar arguments as in [104, Proposition 3.5.], it is easy to deduce that both $(\frac{1}{N}\sum_{i=1}^N \nu^i)_{N\geq 1}$ and $(\overline{\mathbb{P}}^N)_{N\geq 1}$ are relatively compact under $\mathcal{W}_p$.

Identification of the limit: Up to a subsequence, let us assume w.l.o.g. that

$$\lim_{N\to\infty} \mathcal{W}_p\big(\overline{\mathbb{P}}^N, \overline{\mathbb{P}}\big) = 0, \text{ for some } \overline{\mathbb{P}} \in \mathcal{P}_p(\overline{\Omega}), \text{ so that } \lim_{N\to\infty} \mathcal{W}_p\Big(\frac{1}{N}\sum_{i=1}^N \nu^i, \nu\Big) = 0, \text{ with } \nu := \overline{\mathbb{P}} \circ X_0^{-1} \in \mathcal{P}_p(\mathbb{R}^n),$$

and then prove that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\overline{\Omega})$. To this end, it is enough, by Proposition 3.3.9, to prove that $\overline{\mathbb{P}}$ satisfies the following properties

(*i*) $\overline{\mathbb{P}}\big[\widehat{\mu}\circ(X_0)^{-1}=\nu, X_0=Y_0, W_0=0, B_0=0\big]=1$;

(*ii*) $\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X\|^p+\int_{[0,T]\times A}\big(\rho(a_0,a)\big)^p\Lambda_t(\mathrm{d}a)\mathrm{d}t\big]<\infty$;

(*iii*) $\widehat{\mu}$ satisfies (3.3.4) under $\overline{\mathbb{P}}$;

(*iv*) $(B_t)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–Brownian motion;

(*v*) the process $(S^f_t)_{t\in[0,T]}$ (defined in (3.3.24)) is an $(\overline{\mathbb{F}}^\circ,\overline{\mathbb{P}})$–martingale with filtration $\overline{\mathbb{F}}^\circ=(\mathcal{F}^\circ_t)_{t\in[0,T]}$ defined by $\overline{\mathcal{F}}^\circ_t:=\sigma(X_{t\wedge\cdot},Y_{t\wedge\cdot},B_{t\wedge\cdot},\mu_t)$ for all $f\in C^2_b(\mathbb{R}^n\times\mathbb{R}^\ell)$;

(*vi*) finally, for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, $\big(\widehat{M}^{\varphi,\mu(\bar\omega)}_t\big)_{t\in[0,T]}$ (defined in (3.3.6)) is an $\big(\widehat{\mathbb{F}},\widehat{\mu}(\bar\omega)\big)$–martingale for all $\varphi\in C^2_b(\mathbb{R}^n\times\mathbb{R}^d)$.

First, let us consider two bounded continuous functions h^1, h^2 in $C_b(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}}\big[\langle h^1,\widehat{\mu}\circ(X_0)^{-1}\rangle\langle h^2,\widehat{\mu}\circ(X_0)^{-1}\rangle\big]&=\lim_{N\to\infty}\frac{1}{N^2}\sum_{i,j=1}^N\mathbb{E}^{\mathbb{P}^N_\nu}\big[h^1(X^i_0)h^2(X^j_0)\big]\\&=\lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^N\langle h^1h^2,\nu^i\rangle+\lim_{N\to\infty}\frac{1}{N^2}\sum_{i\neq j}^N\langle h^1,\nu^i\rangle\langle h^2,\nu^j\rangle\\&=\lim_{N\to\infty}\Big\langle h^1,\frac{1}{N}\sum_{i=1}^N\nu^i\Big\rangle\Big\langle h^2,\frac{1}{N}\sum_{i=1}^N\nu^i\Big\rangle=\langle h^1,\nu\rangle\langle h^2,\nu\rangle.\end{aligned}$$

Using similar arguments, we can deduce that for all $k\geq1$ and bounded continuous functions $h^1,\dots,h^k\in C_b(\mathbb{R}^n)$

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\Pi^i_{i=1}\langle h^i,\widehat{\mu}\circ(X_0)^{-1}\rangle\big]=\Pi^k_{i=1}\langle h^i,\nu\rangle,\ \text{and hence}\ \overline{\mathbb{P}}[\widehat{\mu}\circ(X_0)^{-1}=\nu]=1.$$

Besides, with the definition of $\mathbb{P}^N_\nu$ in Section 3.2.2, and then by (3.4.10), it is easy to deduce that

$$\overline{\mathbb{P}}\big[X_0=Y_0,W_0=0,B_0=0\big]=1,\ \text{and}\ \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\|X\|^p+\iint_{[0,T]\times A}(\rho(a_0,a))^p\Lambda_t(\mathrm{d}a)\mathrm{d}t\bigg]<\infty.$$

Next, notice that, for all $\phi\in C_b(\mathcal{C}^n\times\mathcal{C}^n\times\mathbb{M}\times\mathcal{C}^d)$ and $\psi\in C_b(\mathcal{C}^\ell\times\mathcal{P}(\widehat{\Omega}))$,

$$\begin{aligned}&\mathbb{E}^{\overline{\mathbb{P}}}\big[\phi\big(X_{t\wedge\cdot},Y_{t\wedge\cdot},\Lambda^t,W\big)\psi\big(B,\widehat{\mu}\big)\big]\\&=\lim_{N\to\infty}\mathbb{E}^{\overline{\mathbb{P}}^N}\big[\phi\big(X_{t\wedge\cdot},Y_{t\wedge\cdot},\Lambda^t,W\big)\psi\big(B,\widehat{\mu}\big)\big]=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}^N_\nu}\big[\phi\big(X^i_{t\wedge\cdot},Y^i_{t\wedge\cdot},(\delta_{\alpha^{i,N}_s}(\mathrm{d}a)\mathrm{d}s)^t,W^i\big)\psi\big(B,\overline{\varphi}_N\big)\big]\\&=\lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\mathbb{E}^{\overline{\varphi}_N}\big[\phi\big(\widehat{X}_{t\wedge\cdot},\widehat{Y}_{t\wedge\cdot},(\widehat{\Lambda})^t,\widehat{W}\big)\big]\psi\big(B,\overline{\varphi}_N\big)\Big]=\mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\hat{\mu}}\big[\phi\big(\widehat{X}_{t\wedge\cdot},\widehat{Y}_{t\wedge\cdot},\widehat{\Lambda}^t,\widehat{W}\big)\big]\psi\big(B,\widehat{\mu}\big)\Big],\end{aligned}$$

which implies that $\widehat{\mu}$ satisfies (3.3.4) under $\overline{\mathbb{P}}$ that is, for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$,

$$\widehat{\mu}_t(\bar\omega)=\overline{\mathbb{P}}^{\overline{\mathcal{G}}_T}_{\bar\omega}\circ\big(X_{t\wedge\cdot},Y_{t\wedge\cdot},\Lambda^t,W\big)^{-1}=\overline{\mathbb{P}}^{\overline{\mathcal{G}}_t}_{\bar\omega}\circ\big(X_{t\wedge\cdot},Y_{t\wedge\cdot},\Lambda^t,W\big)^{-1}.$$

We next show that $(B_t)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–Brownian motion. First, since $\mathbb{P}^N_\nu\circ B^{-1}$ is the Wiener measure, it is clear that $\overline{\mathbb{P}}\circ B^{-1}$ is also the Wiener measure. Next, let $\phi\in C_b(\overline{\Omega})$, for all $s\in[0,T]$, we define the random variables

$$\Phi_s:=\phi\big(X_{s\wedge\cdot},Y_{s\wedge\cdot},\Lambda^s,W_{s\wedge\cdot},B_{s\wedge\cdot},\widehat{\mu}_{s\wedge\cdot}\big)\ \text{on}\ \overline{\Omega},\ \text{and}\ \Phi^i_s:=\phi\big(X^{\alpha,i}_{s\wedge\cdot},Y^{\alpha,i}_{s\wedge\cdot},(\delta_{\alpha^{i,N}_t}(\mathrm{d}a)\mathrm{d}t)^s,W^i_{s\wedge\cdot},B_{s\wedge\cdot},\overline{\varphi}^N_{s\wedge\cdot}\big)\ \text{on}\ (\Omega^N,\mathcal{F}^N).$$

On $(\Omega^N, \mathcal{F}^N)$, we introduce the σ–algebra $\mathcal{F}^{N,W} := \sigma\{W^1, \dots, W^N\}$. Then, for all $\psi \in C_b(\mathbb{R}^\ell)$ and $t \geq s$

$$\begin{aligned}\mathbb{E}^{\overline{\mathbb{P}}}\big[\psi(B_t - B_s)\Phi_s\big] &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\big[\psi(B_t - B_s)\Phi_s^i\big] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\mathbb{E}^{\mathbb{P}_\nu^N}\big[\psi(B_t - B_s)\Phi_s^i\big|\mathcal{F}^{N,W}\big]\Big] \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\mathbb{E}^{\mathbb{P}_\nu^N}\big[\psi(B_t - B_s)\big|\mathcal{F}^{N,W}\big]\mathbb{E}^{\mathbb{P}_\nu^N}\big[\Phi_s^i\big|\mathcal{F}^{N,W}\big]\Big] \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\big[\psi(B_t - B_s)\big]\mathbb{E}^{\mathbb{P}_\nu^N}\big[\Phi_s^i\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\psi(B_t - B_s)\big]\mathbb{E}^{\overline{\mathbb{P}}}\big[\Phi_s\big].\end{aligned}$$

This implies that B is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$–Brownian motion.

We finally consider the two martingale problems in Proposition 3.3.9, for which we can adapt the proofs in [104, Proposition 5.1.]. Let $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d)$, $f \in C_b(\mathbb{R}^n \times \mathbb{R}^\ell)$, $\psi \in C_b(\widehat{\Omega})$, $\phi \in C_b(\mathcal{P}(\widehat{\Omega}))$ and $\beta \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n))$. In addition, on $(\Omega^N, \mathcal{F}^N)$, we define the processes $M^{\varphi,i}$ for $i = 1, \dots, N$ by

$$M_t^{\varphi,i} := \varphi\big(Y_t^{\alpha,i}, W_t^i\big) - \varphi(Y_0^{\alpha,i}, W_0^i) - \int_0^t \widehat{\mathcal{L}}_s\varphi\big(X^{\alpha,i}, Y^{\alpha,i}, W^i, \alpha_s^{i,N}, \varphi_s^{N,X}\big)\mathrm{d}s,$$

where $\widehat{\mathcal{L}}$ is defined in (3.3.5). Then $(M^{\varphi,i})_{i\in\{1,\dots,N\}}$ are $(\mathbb{P}_\nu^N, \mathbb{F}^N)$–orthogonal martingales with quadratic variation

$$\left(\int_0^t \big|\sigma\big(r, X^{\alpha,i}, \varphi_r^{N,X}, \alpha_s^{i,N}\big)\nabla\varphi(X_r^{\alpha,i})\big|^2\mathrm{d}r\right)_{t\in[0,T]}, \ i = 1, \dots, N.$$

Denote

$$\big\langle\big(\widehat{M}_t^{\varphi,\mu} - \widehat{M}_r^{\varphi,\mu}\big)\Psi_r, \widehat{\mu}\big\rangle := \mathbb{E}^{\widehat{\mu}}\Big[\big(\widehat{M}_t^{\varphi,\mu} - \widehat{M}_r^{\varphi,\mu}\big)\psi(\widehat{X}_{r\wedge\cdot}, \widehat{Y}_{r\wedge\cdot}, \widehat{\Lambda}^r, \widehat{W}_{r\wedge\cdot})\Big],$$

it follows by direct computation that, for some constant $C > 0$ whose value may vary from line to line

$$\begin{aligned}&\Big|\mathbb{E}^{\overline{\mathbb{P}}}\big[\phi(\widehat{\mu})\ \big\langle\big(\widehat{M}_t^{\varphi,\mu} - \widehat{M}_r^{\varphi,\mu}\big)\Psi_r, \widehat{\mu}\big\rangle\big]\Big| = \lim_{N\to\infty}\Big|\mathbb{E}^{\overline{\mathbb{P}}^N}\big[\phi(\widehat{\mu})\ \big\langle\big(\widehat{M}_t^{\varphi,\mu} - \widehat{M}_r^{\varphi,\mu}\big)\Psi_r, \widehat{\mu}\big\rangle\big]\Big| \\ \leq\ &\limsup_{N\to\infty} \mathbb{E}^{\overline{\mathbb{P}}^N}\big[\big|\phi(\widehat{\mu})\big|^2\big]^{1/2}\mathbb{E}^{\overline{\mathbb{P}}^N}\Big[\big|\big\langle\big(\widehat{M}_t^{\varphi,\mu} - \widehat{M}_r^{\varphi,\mu}\big)\Psi_r, \widehat{\mu}\big\rangle\big|^2\Big]^{1/2} \\ =\ &\limsup_{N\to\infty} C\mathbb{E}^{\overline{\mathbb{P}}}\big[\big|\phi(\widehat{\mu})\big|^2\big]^{1/2}\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\Big|\frac{1}{N}\sum_{i=1}^N (M_t^{\varphi,i} - M_r^{\varphi,i})\psi\big(X_{r\wedge\cdot}^{\alpha,i}, Y_{r\wedge\cdot}^{\alpha,i}, (\Lambda^i)^r, W_{r\wedge\cdot}^i\big)\Big|^2\Big]^{1/2} \\ \leq\ &\limsup_{N\to\infty} C\mathbb{E}^{\overline{\mathbb{P}}}\big[\big|\phi(\widehat{\mu})\big|^2\big]^{1/2}\Big(\frac{1}{N^2}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_r^t \big|\sigma\big(s, X^{\alpha,i}, \varphi_s^{N,X}, \alpha_s^{i,N}\big)\nabla\varphi(X_s^{\alpha,i})\big|^2\mathrm{d}s\Big]\Big)^{1/2} \\ \leq\ &\limsup_{N\to\infty} C\mathbb{E}^{\overline{\mathbb{P}}}\big[\big|\phi(\widehat{\mu})\big|^2\big]^{1/2}\Big(\frac{1}{N^2}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_r^t \big|X_{s\wedge\cdot}^{\alpha,i}\big|^p + \rho(a_0, \alpha_s^{i,N})^p\mathrm{d}s\Big]\Big)^{1/2} \leq \limsup_{N\to\infty}\frac{C}{\sqrt{N}} = 0.\end{aligned}$$

This implies that, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$

$$\big\langle\big(\widehat{M}_t^{\varphi,\mu(\bar{\omega})} - \widehat{M}_r^{\varphi,\mu(\bar{\omega})}\big)\Psi_r, \widehat{\mu}(\bar{\omega})\big\rangle = 0.$$

Similarly, with $\hat{a}_0$ defined in equation (3.3.25) and $Z^i := X^{\alpha,i} - Y^{\alpha,i}$, let us introduce $(S_t^{f,i})_{t\in[0,T]}$ on Ω^N by

$$S_t^{f,i} := f\big(Z_t^i, B_t\big) - \varphi(Z_0^i, B_0) - \int_0^t \frac{1}{2}\mathrm{Tr}\big[\hat{a}_0(s, X^{\alpha,i}, \varphi_s^{N,X})\nabla^2\varphi(Z_s^i, B_s)\big]\mathrm{d}s, \ \mathbb{P}_\nu^N\text{–a.s., for all } t \in [0,T].$$

Denoting $\Lambda^i(\mathrm{d}a)\mathrm{d}t := \delta_{\alpha_t^{i,N}}(\mathrm{d}a)\mathrm{d}t$, and applying the same arguments as above, it follows that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\big(S_t^f - S_r^f\big)\beta\big(X_{r\wedge\cdot}, Y_{r\wedge\cdot}, B_{r\wedge\cdot}, \mu_r\big)\big] = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\Big[\big(S_t^{f,i} - S_r^{f,i}\big)\beta\big(X_{r\wedge\cdot}^{\alpha,i}, Y_{r\wedge\cdot}^{\alpha,i}, B_{r\wedge\cdot}, \varphi_r^{N,X}\big)\Big] = 0.$$

Finally, by considering (r,t,ψ,ϕ) in a countable dense subset of $[0,T]\times[0,T]\times C_b(\widehat{\Omega})\times C_b(\mathcal{P}(\widehat{\Omega}))$, it follows that the process $\big(S_t^f\big)_{t\in[0,T]}$ is an $(\overline{\mathbb{F}}^\circ,\overline{\mathbb{P}})$–martingale for all $f\in C_b^2(\mathbb{R}^n\times\mathbb{R}^\ell)$, and for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$, $(\widehat{M}_t^{\varphi,\mu(\bar\omega)})_{t\in[0,T]}$ is an $\big(\widehat{\mathbb{F}},\widehat{\mu}(\bar\omega)\big)$–martingale for all $\varphi\in C_b^2(\mathbb{R}^n\times\mathbb{R}^d)$. We then conclude that $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_R(\nu)$. □

3.4.3 Proof of continuity of value function

Let $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ and $(\nu^m)_{m\geq1}\subset\mathcal{P}_{p'}(\mathbb{R}^n)$ be such that $\sup_{m\geq1}\int_{\mathbb{R}^n}\|x'\|^{p'}\nu^m(\mathrm{d}x')<\infty$ and $\lim_{m\to\infty}\mathcal{W}_p(\nu^m,\nu)=0$.

We first consider two sequences $(\varepsilon_m)_{m\geq1}\subset\mathbb{R}_+$ and $(\overline{\mathbb{P}}^m)_{m\geq1}$ such that

$$\lim_{m\to\infty}\varepsilon^m=0,\ \overline{\mathbb{P}}^m\in\overline{\mathcal{P}}_R(\nu^m),\ \text{and}\ J(\overline{\mathbb{P}}^m)\geq V_R(\nu^m)-\varepsilon^m,\ \text{for all}\ m\geq1.$$

It follows by Proposition 3.4.6 that $(\overline{\mathbb{P}}^m)_{m\in\mathbb{N}}$ is relatively compact under $\mathcal{W}_p$. Via a subsequence, let us assume that $\overline{\mathbb{P}}^m\to\overline{\mathbb{P}}^\infty$ under $\mathcal{W}_p$, so that $\overline{\mathbb{P}}^\infty\in\overline{\mathcal{P}}_R(\nu)$. Using the continuity and growth conditions of (L,g) in Assumption 1.4.1 and Assumption 3.1.1, it follows that $\lim_{m\to\infty}J(\overline{\mathbb{P}}^m)=J(\overline{\mathbb{P}}^\infty)$, and therefore

$$\limsup_{m\to\infty}V_R(\nu^m)\leq\lim_{m\to\infty}J(\overline{\mathbb{P}}^m)=J(\overline{\mathbb{P}}^\infty)\leq V_R(\nu)=V_S(\nu).$$

Together with the inequality from Proposition 3.4.5, we then conclude the proof. □

3.4.4 Proof of limit theory

(i) By Proposition 3.4.6, the sequence $(\overline{\mathbb{P}}^N)_{N\geq1}$ is relatively compact under $\mathcal{W}_p$. Further, for any convergent subsequence $(\overline{\mathbb{P}}^{N_m})_{m\geq1}$, one has

$$\lim_{m\to\infty}\mathcal{W}_p\bigg(\frac{1}{N_m}\sum_{i=1}^{N_m}\nu^i,\nu\bigg)=0,\ \text{for some}\ \nu\in\mathcal{P}_p(\mathbb{R}^n),\ \text{and}\ \lim_{m\to\infty}\mathcal{W}_p\big(\overline{\mathbb{P}}^{N_m},\overline{\mathbb{P}}^\infty\big)=0,\ \text{for some}\ \overline{\mathbb{P}}^\infty\in\overline{\mathcal{P}}_R(\nu).$$

Moreover, under Assumption 1.4.1 and Assumption 3.1.1, it follows by (3.2.7) and Remark 3.2.2 that

$$\limsup_{N\to\infty}V_S^N(\nu^1,\dots,\nu^N)\leq\lim_{m\to\infty}J(\overline{\mathbb{P}}^{N_m})=J(\overline{\mathbb{P}}^\infty)\leq V_R(\nu)=V_S(\nu).$$

Together with Proposition 3.4.4, one obtains that

$$\lim_{N\to\infty}V_S^N(\nu^1,\dots,\nu^N)=J(\overline{\mathbb{P}}^\infty)=V_R(\nu)=V_S(\nu),\tag{3.4.11}$$

and hence $\overline{\mathbb{P}}^\infty\in\overline{\mathcal{P}}_R^\star(\nu)$.

(ii) The second item is in fact a direct consequence of Proposition 2.3.12, Proposition 3.3.6 and Proposition 3.4.4.

(iii) Finally, let $(N_m)_{m\in\mathbb{N}}$ be a sequence such that

$$\limsup_{N\to\infty}\bigg|V_S^N(\nu^1,\dots,\nu^N)-V_S\bigg(\frac{1}{N}\sum_{i=1}^N\nu^i\bigg)\bigg|=\lim_{m\to\infty}\bigg|V_S^{N_m}(\nu^1,\dots,\nu^{N_m})-V_S\bigg(\frac{1}{N}\sum_{i=1}^{N_m}\nu^i\bigg)\bigg|.$$

One more time, through a subsequence, we can assume that

$$\frac{1}{N_m}\sum_{i=1}^{N_m}\nu^i\underset{m\to\infty}{\longrightarrow}\nu,\ \text{under}\ \mathcal{W}_p,\ \text{for some}\ \nu\in\mathcal{P}_p(\mathbb{R}^n).$$

Using (3.4.11) and Proposition 3.2.8, we obtain that

$$\limsup_{N\to\infty}\bigg|V_S^N(\nu^1,\dots,\nu^N)-V_S\bigg(\frac{1}{N}\sum_{i=1}^N\nu^i\bigg)\bigg|\leq\lim_{m\to\infty}\bigg|V_S^{N_m}(\nu^1,\dots,\nu^{N_m})-V_S(\nu)\bigg|+\lim_{m\to\infty}\bigg|V_S(\nu)-V_S\bigg(\frac{1}{N}\sum_{i=1}^{N_m}\nu^i\bigg)\bigg|=0,$$

and thus (3.2.8) holds true. □

3.5 Appendix: characterisation of probability measures on a set of probability measures

Proposition 3.5.1. *Let E be a Polish space, $(\Upsilon_1, \Upsilon_2) \in \mathcal{P}(\mathcal{P}(E)) \times \mathcal{P}(\mathcal{P}(E))$ be such that*

$$\int_{\mathcal{P}(E)} \prod_{i=1}^{k} \langle \varphi_i, \nu \rangle \Upsilon_1(\mathrm{d}\nu) = \int_{\mathcal{P}(E)} \prod_{i=1}^{k} \langle \varphi_i, \nu \rangle \Upsilon_2(\mathrm{d}\nu), \text{ for all } k \geq 1, \text{ and } (\varphi_i)_{i \in \{1,\dots,k\}} \subset C_b(E). \tag{3.5.1}$$

Then $\Upsilon_1 = \Upsilon_2$.

Proof. First, using (3.5.1), we have, for all $k \geq 1$, for every family of polynomial functions $(\psi^i)_{i \in \{1,\dots,k\}}$, and every $(\varphi_i)_{i \in \{1,\dots,k\}} \subset C_b(E; \mathbb{R})$,

$$\int_{\mathcal{P}(E)} \prod_{i=1}^{k} \psi^i(\langle \varphi_i, \nu \rangle) \Upsilon_1(\mathrm{d}\nu) = \int_{\mathcal{P}(E)} \prod_{i=1}^{k} \psi^i(\langle \varphi_i, \nu \rangle) \Upsilon_2(\mathrm{d}\nu). \tag{3.5.2}$$

Since we can approximate any continuous function by polynomial functions, uniformly on compact sets, it follows that (3.5.2) still holds true for all $k \geq 1$, $(\psi^i)_{i \in \{1,\dots,k\}} \subset C_b(\mathbb{R}; \mathbb{R})$ and $(\varphi_i)_{i \in \{1,\dots,k\}} \subset C_b(E; \mathbb{R})$. This further implies that, for all $(r_1, \dots, r_k) \in \mathbb{R}^k$

$$\int_{\mathcal{P}(E)} \prod_{i=1}^{k} 1_{\{\nu : \langle \varphi_i, \nu \rangle < r^i\}} \Upsilon_1(\mathrm{d}\nu) = \int_{\mathcal{P}(E)} \prod_{i=1}^{k} 1_{\{\nu : \langle \varphi_i, \nu \rangle < r^i\}} \Upsilon_2(\mathrm{d}\nu).$$

In other words, $\Upsilon_1[A] = \Upsilon_2[A]$ for all $A \in \Psi$, where

$$\Psi := \Big\{ A[r_1, \dots, r_m; \varphi_1, \dots, \varphi_m] : m \geq 1,\ (r_1, \dots, r_m) \in \mathbb{R}^m \text{ and } (\varphi_1, \dots, \varphi_m) \in C_b(E)^m \Big\},$$

with

$$A[r_1, \dots, r_m; \varphi_1, \dots, \varphi_m] := \big\{ \lambda \in \mathcal{P}(E) : \langle \varphi_i, \lambda \rangle < r_i,\ i = 1, \dots, m \big\}.$$

Notice that the weak convergence topology on $\mathcal{P}(E)$ is generated by the open sets in Ψ, it follows by the monotone class theorem that $\Upsilon_1 = \Upsilon_2$ on the Borel σ–field of $\sigma(\Psi)$. □

Chapter 4

Measure–valued controls and limit theory with law of control

4.1 Introduction

The aim of this chapter is to provide a rigorous connection between two stochastic control problems: the stochastic control problem of large population (or particles) interacting through the empirical distribution of their states and controls on the one hand, and on the other hand the problem of control of stochastic dynamics depending upon the joint (conditional) distribution of the controlled state and the control, also called *extended* mean field control problem.

The connection we are investigating i.e. that the stochastic control problem of large population *converges* towards the mean field control problem, is often called *limit theory* or (controlled) *propagation of chaos*. In contrast with the classical framework of McKean–Vlasov stochastic control problem which only considers the conditional distribution of X_t, here, there is in addition the presence of the conditional distribution of (X_t, α_t). Indeed, when there is no law of control i.e. no $\mathcal{L}(X_t, \alpha_t|B)$ but only $\mathcal{L}(X_t|B)$ in (b, σ, L, g), these problems have been studied in the literature. Let us mention the work of Snitzman [149] which shows for particular coefficients (b, σ) in the absence of control (and the law of control), via some compactness arguments, a connection of this type. See also the papers of Oelschläger [133] and Gärtner [71], with no control and no law of control as well, which use martingale problem in the sense of Stroock and Varadhan [150] adapted in the context of Mckean–Vlasov equation to prove similar results under minimal assumptions.

In the controlled dynamic case but no *extended* type, that is to say when the dynamic depends on the control but not its law, Fischer and Livieri [67] gets a connection between the large population stochastic control problem and the (*extended*) mean field control problem for the study of a mean–variance problem arising in finance. Another interesting work is that of Budhiraja, Dupuis, and Fischer [35], where they study the behavior of empirical measures of controlled interacting diffusion in order to prove a large deviation principle in a McKean–Vlasov framework. Still without touching the case with law of control, the first work that deals with the case with control under general assumptions are Lacker [104] and Chapter 3 of this thesis. Thanks to an (extension of) martingale problem of [150], as well as relaxed controls initiated by Fleming and Nisio [69], and developed by El Karoui, Huu Nguyen, and Jeanblanc-Picqué [63], combined with compactness arguments adapted to the McKean–Vlasov setting, [104] proves the connection between the two problems under general conditions on (b, σ, L, g) without common noise. Following upon these ideas, in the Chapter 3, we develop a general overview of McKean–Vlasov or mean field control problem, and treat the case with common noise, which turns out to be a non trivial extension.

In the presence of the law of control, this propagation of chaos result is a natural expectation. In spite of appearances. The aforementioned techniques do not work in this context. Two main reasons can explain the unsuitable aspect of the techniques mentioned above. Firstly, the continuity of the application $t \longmapsto \mathcal{L}(X_t|B)$ (or $t \longmapsto \varphi_t^{N,\mathbf{X}}$) plays a crucial role. Indeed, the classical idea is to put this application in a canonical space, which is here the space $C([0,T]; \mathcal{P}(\mathbb{R}^n))$ of continuous functions from $[0,T]$ into the space of probability measures on $\mathbb{R}^n$, and via compactness arguments and martingale problem get this connection (see [104], and Chapter 3 for the non–Markovian case with common noise). In our situation, this type of continuity is lost because we must take into account the application $t \longmapsto \mathcal{L}(X_t, \alpha_t|B)$ (or $t \longmapsto \varphi_t^N$) which does not have this property since the presence of control α can generate some discontinuities. Secondly, as highlighted in Chapter 3, proving a result of *propagation of chaos* is extremely related to the search of the closure

of the set of all probabilities that are the image measure of the controlled state process, the control and the conditional distribution of the controlled state process and control i.e. $\mathcal{L}\big(X, \delta_{\alpha_t}(\mathrm{d}u)\mathrm{d}t, \mathcal{L}(X, \delta_{\alpha_t}(\mathrm{d}u)\mathrm{d}t|B)\big)$. Unfortunately, the natural space one might think to answer this question is not a closed set due to another problem of continuity (see Remark 4.3.1 for a more thorough discussion).

There are not many papers in the literature which study the mean field control problem with law of control and its connection with a large population stochastic control problem. To the best of our knowledge, only the recent papers of Laurière and Tangpi [113] (with strong assumptions) and Motte and Pham [127] (for Mean–field Markov decision processes) treat the limit theory question. Most papers focus on the questions of existence and uniqueness of optimal control. Acciaio, Backhoff Veraguas, and Carmona [1], with the help of Pontryagin's maximum principle, obtain necessary and sufficient conditions to characterize the optimum with strong assumptions on the coefficients in a no common noise framework. Pham and Wei [138] (without common noise, with closed loop controls) and the Chapter 5 of this thesis establish the Dynamic Programming Principle (DPP for short) and give a Hamilton–Jacobi equation on a space of probability measures verified by the value function. Let us also mention Carmona and Lacker [45], Élie, Mastrolia, and Possamaï [65], Cardaliaguet and Lehalle [37] and [113] who study similar problem in the mean field games framework called mean field game of controls or *extended* mean field game, as well as our Chapter 6 and Chapter 7 adapt the arguments of this chapter to the context of mean field game of controls.

In this chapter, our goal is to give some properties on the *extended* mean field control problem and to show its connection with the large population stochastic control problem under general assumptions on (b, σ, L, g) (see Theorem 4.5.3 and Theorem 4.5.1). To bypass the difficulties highlighted above, we follow the idea mentioned in Chapter 3 which is to introduce a new optimization problem by considering a suitable set of controls. This set must be the closure of some set of probability measures. In this framework, the appropriate space is the closure of all the probabilities that are the distributions of the conditional distribution of the state controlled process and the conditional distribution of the state controlled process and the control i.e. $\mathcal{L}\big(\mathcal{L}(X_t|B))_{t\in[0,T]}, \delta_{\mathcal{L}(X_t,\alpha_t|B)}(\mathrm{d}m)\mathrm{d}t\big)$ (for more details see Section 4.4). Taking into account this type of probability turns out to be the key to solve the main difficulties. The characterization of its closure is possible by the appropriate use of (controlled) Fokker–Planck equation. Inspired by the techniques developed in the proofs of Gyöngy [76], especially [76, Lemma 2.1] (an adaptation of Krylov [98]) and [76, Proposition 4.3] which are regularization results, we can determine the desired set thanks to a Fokker–Planck equation. The conditions used on the coefficients are general, except the non–degeneracy of the volatility σ. This assumption is capital to prove our main results. Apart from this assumption, our result appears to be one of the first to establish some general properties on *extended* mean field control problem and to show its connection with the large population stochastic problem. Lacker [105] used similar techniques in the context of convergence of closed loop Nash equilibria, but his analysis focuses mainly on an adequate manipulation of [76, Theorem 4.6], while ours focuses on the techniques used for the proofs. Also, let us mention Lacker, Shkolnikov, and Zhang [108] which establishes a correspondence between Fokker–Planck equations and solutions of SDE in a McKean–Vlasov framework with common noise.

Also, if we restrict our study in the case without common noise and with a particular form of coefficient σ (see Assumption 4.6.1) we prove that the stochastic control of McKean–Vlasov process with law of control over the set of open loop control is equivalent to the same stochastic control problem over the set of closed–loop control. We want to emphasize that unlike the classical literature on the equivalence between the open loop controls and the closed loop controls, our approach does not use any convexity assumptions. The projection argument combined with the convexity assumptions can not be applied in the situation of law of control. The projection can not allow to recover the law of the control. Our techniques to prove the equivalence between closed loop and open loop controls use essentially the density of the associated the Fokker–Planck equation, and more precisely, some estimates obtained in Bogachev, Krylov, Röckner, and Shaposhnikov [31].

The rest of the chapter is structured as follows. Section 4.2, Section 4.3 and Section 4.4 carefully formulate first the N–agents stochastic control problem, then the strong formulation of the extended mean field control problem and finally the stochastic control of measure–valued processes. Next, in Section 4.5, we present the main results of this chapter: the equivalence between the strong formulation of extended mean field control problem and the stochastic control of measure-valued processes, and the propagation of chaos result i.e. the extended mean field control problem is, when N goes to infinity, the limit of N–agents stochastic control problem in presence of interactions through the empirical distribution of state and control processes. Section 4.6, in a restrictive situation, deals with the equivalence between the

closed loop and the open loop control, and give a limit theory result in the setting of closed loop controls. Finally, Section 4.7 provides some approximation results related to the Fokker–Planck equation and Section 4.8 is devoted to the proof of our main results.

With a Polish space E, we denote by $\mathbb{M}(E)$ the space of all Borel measures $q(\mathrm{d}t,\mathrm{d}e)$ on $[0,T]\times E$, whose marginal distribution on $[0,T]$ is the Lebesgue measure $\mathrm{d}t$, that is to say $q(\mathrm{d}t,\mathrm{d}e)=q(t,\mathrm{d}e)\mathrm{d}t$ for a family $(q(t,\mathrm{d}e))_{t\in[0,T]}$ of Borel probability measures on E. We also consider the subset $\mathbb{M}_0(E)\subset\mathbb{M}(E)$ which is the collection of all $q\in\mathbb{M}(E)$ such that $q(\mathrm{d}t,\mathrm{d}e)=\delta_{\psi(t)}(\mathrm{d}e)\mathrm{d}t$ for some Borel measurable function $\psi:[0,T]\to E$. Let Λ denote the canonical element on $\mathbb{M}(E)$, we define

$$\Lambda_{t\wedge\cdot}(\mathrm{d}s,\mathrm{d}e):=\Lambda(\mathrm{d}s,\mathrm{d}e)\big|_{[0,t]\times E}+\delta_{e_0}(\mathrm{d}e)\mathrm{d}s\big|_{(t,T]\times E},\ \text{for some fixed } e_0\in E. \tag{4.1.1}$$

For $p\geq 1$, we write $\mathbb{M}_p(E)$ to designate the elements of $q\in\mathbb{M}(E)$ such that $q/T\in\mathcal{P}_p(E\times[0,T])$.

Let $(\ell,n)\in\mathbb{N}\times\mathbb{N}^\star$, (U,ρ) be a nonempty Polish space and $\mathcal{P}_U^n$ denote the space of all Borel probability measures on $\mathbb{R}^n\times U$ i.e. $\mathcal{P}_U^n:=\mathcal{P}(\mathbb{R}^n\times U)$. We give ourselves the following Borel measurable functions

$$\big[b,\sigma,L\big]:[0,T]\times\mathbb{R}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{P}_U^n\times U\longrightarrow\mathbb{R}^n\times\mathbb{S}^{n\times n}\times\mathbb{R}\ \text{and}\ g:\mathbb{R}^n\times\mathcal{C}_{\mathcal{W}}^n\longrightarrow\mathbb{R}.$$

Assumption 4.1.1. *The functions $[b,\sigma,L]$ are non–anticipative in the sense that, for all $(t,x,\pi,m,u)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{P}_U^n\times U$*

$$\big[b,\sigma,L\big](t,x,\pi,m,u)=\big[b,\sigma,L\big](t,x,\pi_{t\wedge\cdot},m,u).$$

Moreover, there exist positive constants C and p such that $p\geq 2$ and
(i) U is a compact space;
(ii) b and σ are continuous bounded functions, and $\sigma_0\in\mathbb{S}^{n\times\ell}$ constant;
(iii) one has for all $(t,x,x',\pi,\pi',m,m',u)\in[0,T]\times(\mathbb{R}^n)^2\times(\mathcal{C}_{\mathcal{W}}^n)^2\times(\mathcal{P}_U^n)^2\times U$

$$\big|[b,\sigma](t,x,\pi,m,u)-[b,\sigma](t,x',\pi',m',u)\big|\ \leq\ C\big(|x-x'|+\sup_{s\in[0,T]}\mathcal{W}_p(\pi_s,\pi'_s)+\mathcal{W}_p(m,m')\big);$$

(iv) for some constant $\theta>0$, one has, for all $(t,x,\pi,m,u)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{P}_U^n\times U$,

$$\theta\mathrm{I}_n\leq\ \sigma\sigma^\top(t,x,\pi,m,u);$$

(v) the reward functions L and g are continuous, and for all $(t,x,\pi,m,u)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{P}_U^n\times U$, one has

$$\big|L(t,x,\pi,m,u)\big|+|g(x,\pi)|\leq C\bigg[1+|x|^p+\sup_{s\in[0,T]}\mathcal{W}_p(\pi_s,\delta_0)^p+\int_{\mathbb{R}^n}|x'|^p m(\mathrm{d}x',U)\bigg].$$

Remark 4.1.2. *These assumptions are standard and in the same spirit as those used in* [104] *and* Assumption 1.4.1, *but with some specific modifications adapted to the context of this chapter. They ensure the well-posedness of the objects used throughout this chapter. Due to the technical aspect of our chapter, the point (i) is considered essentially to simplify (the presentation of) the proofs. But, using the classical uniform integrability condition as in* [104] *and* Assumption 1.4.1, *it is possible to work with U a non–bounded set of $\mathbb{R}^n$ for instance. The point (iv) is the least classical assumption in the study in this problem. This is an important assumption for the proofs of our results, in particular to deal with the Fokker–Planck equations and the different* SDEs *considered in the proofs (see* Section 4.7*).*

4.2 The N–agents stochastic control problem

In this section, we present the N–agents stochastic control problem or large population control problem. The study of this control problem when N goes to infinity is one of the main objective of this chapter.

For a fixed $(\nu^1,\ldots,\nu^N)\in\mathcal{P}_p(\mathbb{R}^n)^N$, let

$$\Omega^N\ :=\ (\mathbb{R}^n)^N\times(\mathcal{C}^n)^N\times\mathcal{C}^\ell,$$

be the canonical space, with canonical variable $\mathbf{X}_0 = (\mathbf{X}_0^1, \dots, \mathbf{X}_0^N)$, canonical processes $\mathbf{W} = (\mathbf{W}_s^1, \dots, \mathbf{W}_s^N)_{0\le s\le T}$ and $B = (B_s)_{0\le s\le T}$, and probability measure $\mathbb{P}_\nu^N$ under which $\mathbf{X}_0 \sim \nu_N := \nu^1 \otimes \cdots \otimes \nu^N$ and $(\mathbf{W}, B)$ are standard Brownian motions independent of $\mathbf{X}_0$. Let $\mathbb{F}^N = (\mathcal{F}_s^N)_{0\le s\le T}$ be defined by

$$\mathcal{F}_s^N := \sigma\big\{\mathbf{X}_0, \mathbf{W}_r, B_r,\ r \in [0,s]\big\},\ s \in [0,T].$$

Let us denote by $\mathcal{A}_N(\nu_N)$ the collection of all U–valued $\mathbb{F}^N$–predictable processes. Then given $\alpha := (\alpha^1, \dots, \alpha^N) \in (\mathcal{A}_N(\nu_N))^N$, denote by $\mathbf{X}^\alpha := (\mathbf{X}^{\alpha,1}_\cdot, \dots, \mathbf{X}^{\alpha,N}_\cdot)$ the unique strong solution of the following system of SDEs, for each $i \in \{1, \dots, N\}$, $\mathbb{E}^{\mathbb{P}_\nu^N}\big[\|\mathbf{X}^{\alpha,i}\|^p\big] < \infty$,

$$\mathbf{X}_t^{\alpha,i} = \mathbf{X}_0^i + \int_0^t b\big(r, \mathbf{X}_r^{\alpha,i}, \varphi^{N,\mathbf{X}}_{r\wedge\cdot}, \varphi_r^N, \alpha_r^i\big)\mathrm{d}r + \int_0^t \sigma\big(r, \mathbf{X}_r^{\alpha,i}, \varphi^{N,\mathbf{X}}_{r\wedge\cdot}, \varphi_r^N, \alpha_r^i\big)\mathrm{d}\mathbf{W}_r^i + \sigma_0 B_t,\ \text{for all } t \in [0,T], \tag{4.2.1}$$

with

$$\varphi_t^{N,\mathbf{X}}(\mathrm{d}x) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}_t^{\alpha,i}\right)}(\mathrm{d}x) \ \text{ and } \ \varphi_t^N(\mathrm{d}x, \mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}_t^{\alpha,i},\, \alpha_t^i\right)}(\mathrm{d}x, \mathrm{d}u),\ \text{for all } \ t \in [0,T].$$

Then, the value function $V_S^N(\nu^1, \dots, \nu^N)$ is defined by

$$V_S^N(\nu^1, \dots, \nu^N) := \sup_{(\alpha^1, \dots, \alpha^N)} J^N(\alpha) \text{ where } J^N(\alpha) := \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\bigg[\int_0^T L\big(t, \mathbf{X}_t^{\alpha,i}, \varphi^{N,\mathbf{X}}_{t\wedge\cdot}, \varphi_t^N, \alpha_t^i\big)\mathrm{d}t + g\big(\mathbf{X}_T^{\alpha,i}, \varphi^{N,\mathbf{X}}_{T\wedge\cdot}\big)\bigg], \tag{4.2.2}$$

which is well–posed under Assumption 4.1.1.

Remark 4.2.1. (*i*) *Our formulation allows for coefficients depending on the path of the empirical distribution of* $\mathbf{X}^\alpha$, *but can only accommodate a Markovian dependence with respect to* $\mathbf{X}^\alpha$ *itself. In some sense, we work on a non-Markovian framework w.r.t. the empirical distribution of* $\mathbf{X}^\alpha$. *Indeed, as we will see in* Section 4.4, *our point of view is to write the entire problem as an optimization involving mainly the empirical distribution of* $\mathbf{X}^\alpha$ *i.e.* $\varphi^{N,\mathbf{X}}$. *Therefore our key variable is* $\varphi^{N,\mathbf{X}}$ (*not* $\mathbf{X}^\alpha$) *and we can deal with its path, hence the non–Markovian aspect.*

(*ii*) *Sometimes, the probability on* $\mathcal{C}_{\mathcal{W}}^n \times \mathbb{M}(\mathcal{P}_U^n) \times \mathcal{C}^\ell$

$$\mathbb{P}(\alpha^1, ..., \alpha^N) := \mathbb{P}_\nu^N \circ \Big((\varphi_t^{N,\mathbf{X}})_{t\in[0,T]}, \delta_{(\varphi_s^N)}(\mathrm{d}m)\mathrm{d}s, (B_t)_{t\in[0,T]}\Big)^{-1} \tag{4.2.3}$$

will be used to refer to $(\alpha^1, \dots, \alpha^N) \in (\mathcal{A}_N(\nu_N))^N$. *The notation* $\mathcal{P}_S^N(\nu^1, \dots, \nu^N)$ *will designate all probabilities of this type. The need for this space will become clearer in the following.*

4.3 The extended mean field control problem

On a fix probability space, we formulate the classical McKean–Vlasov control problem with common noise including the (conditional) law of control.

For a fixed $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, let

$$\Omega \ := \ \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^\ell,$$

be the canonical space, with canonical variable ξ, canonical processes $W = (W_t)_{0\le t\le T}$ and $B = (B_t)_{0\le t\le T}$, and probability measure $\mathbb{P}_\nu$ under which $\xi \sim \nu$ and (W, B) are standard Brownian motions independent of ξ. Let $\mathbb{F} = (\mathcal{F}_s)_{0\le s\le T}$ and $\mathbb{G} = (\mathcal{G}_s)_{0\le s\le T}$ be defined by: for all $s \in [0,T]$,

$$\mathcal{F}_s := \sigma\big\{\xi, W_r, B_r,\ r \in [0,s]\big\} \qquad \text{and} \qquad \mathcal{G}_s := \sigma\big\{B_r,\ r \in [0,s]\big\}.$$

Let us denote by $\mathcal{A}(\nu)$ the collection of all U-valued processes $\alpha = (\alpha_s)_{0\le s\le T}$ which are $\mathbb{F}$-predictable. Then given $\alpha \in \mathcal{A}(\nu)$, let X^α be the unique strong solution of the SDE (see for instance Theorem 5.5.3): $\mathbb{E}^{\mathbb{P}_\nu}\big[\|X^\alpha\|^p\big] < \infty$, $X_0^\alpha = \xi$, and for $t \in [0,T]$,

$$X_t^\alpha = X_0^\alpha + \int_0^t b\big(r, X_r^\alpha, \mu_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^\alpha, \mu_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}W_r + \sigma_0 B_t, \tag{4.3.1}$$

with $\mu_r^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r^\alpha \big| \mathcal{G}_r\big)$ and $\overline{\mu}_r^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r^\alpha, \alpha_r \big| \mathcal{G}_r\big)$, for all $r \in [0,T]$.

Let us now introduce the following McKean–Vlasov control problem by

$$V_S(\nu) \;:=\; \sup_{\alpha\in\mathcal{A}(\nu)} \Phi(\alpha) \text{ where } \Phi(\alpha) := \mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T L(t, X_t^\alpha, \mu_{t\wedge\cdot}^\alpha, \overline{\mu}_t^\alpha, \alpha_t)\mathrm{d}t + g(X_T^\alpha, \mu^\alpha)\bigg]. \tag{4.3.2}$$

Remark 4.3.1 (Discussion on a possible relaxed extended mean–field control problem)**.** *An adequate way to study the properties of V_S and/or to give a limit theory is to find the closure $\overline{\mathcal{S}}(\nu)$ of some particular space $\mathcal{S}(\nu)$ for the Wasserstein topology. To simplify, let us take $\ell = 0$ (without common noise), according to the classical ideas of relaxed controls, $\mathcal{S}(\nu) := \big\{\mathbb{P}_\nu \circ \big(X^\alpha,\ \delta_{\alpha_t}(\mathrm{d}u)\mathrm{d}t\big)^{-1},\ \alpha \in \mathcal{A}(\nu)\big\}$ (see discussion* Chapter 3 *and also* Lacker [104]*).*

Following [104] *and* Chapter 3*, let us give an example to see why the "natural" expected relaxed controls is not a "good" set. Let $n = 1$, $U = [1,2]$, $\nu = \delta_0$, $\sigma(t,x,\pi,m,u) := \big|\int_U u'\, m(\mathbb{R}^n, \mathrm{d}u')\big|$ and $b = 0$. Notice that $\mathcal{S}(\nu) \subset \mathcal{P}\big(\mathcal{C}^n \times \mathbb{M}(U)\big)$, then the canonical space is $\overline{\Omega}_R := \mathcal{C}^n \times \mathbb{M}(U)$. Denote $(X, \Lambda_t(\mathrm{d}u)\mathrm{d}t)$ the canonical process and $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{t\in[0,T]}$ the canonical filtration. A naive relaxed controls is $\mathcal{P}_R(\nu) \subset \mathcal{P}(\mathcal{C}^n \times \mathbb{M}(U))$ defined by*

$$\mathcal{P}_R(\nu) := \Big\{\overline{\mathbb{P}} :\ \overline{\mathbb{P}}(X_0 = 0) = 1,\ (M_t^{\overline{\mathbb{P}},f})_{t\in[0,T]} \text{ is a } (\overline{\mathbb{P}}, \overline{\mathbb{F}})\text{–martingale } \forall f \in C_b^2(\mathbb{R})\Big\},$$

where $M_t^{\overline{\mathbb{P}},f} := f(X_t) - \frac{1}{2}\int_0^t \nabla^2 f(X_s)\mathbb{E}^{\overline{\mathbb{P}}}\big[\int_U u\, \Lambda_s(\mathrm{d}u)\big]^2\mathrm{d}s$.

But, $\mathcal{P}_R(\nu)$ defined in this way is not a closed set. Indeed the map $q \in \mathbb{M}(U) \longrightarrow q_t \in \mathcal{P}(U)$ is not continuous for the Wasserstein topology. Therefore $\mathcal{P}_R(\nu)$ can not be the closure of $\mathcal{S}(\nu)$. Due to this type of lack of continuity, this approach cannot work. We need then to change the framework.

4.4 Stochastic control of measure-valued processes

As previously mentioned, the classical approach of relaxed controls is not appropriate. To bypass the difficulty generated by the (conditional) distribution of control in this study, especially to prove the limit theory result or (controlled) propagation of chaos, we introduce a new stochastic control problem. Motivated by the Fokker–Planck equation verified by the couple $(\mu^\alpha, \overline{\mu}^\alpha)$ from (4.3.1), we give in this part an equivalent formulation of the extended mean-field control problem which is less "rigid".

4.4.1 Measure–valued rules

Recall that $\mathbb{M} := \mathbb{M}\big(\mathcal{P}_U^n\big)$ denotes the collection of all finite (Borel) measures $q(\mathrm{d}t, \mathrm{d}m)$ on $[0,T] \times \mathcal{P}_U^n$, whose marginal distribution on $[0,T]$ is the Lebesgue measure $\mathrm{d}s$, i.e. $q(\mathrm{d}s, \mathrm{d}m) = q(s, \mathrm{d}m)\mathrm{d}s$ for a measurable family $(q(s,\mathrm{d}m))_{s\in[0,T]}$ of Borel probability measures on $\mathcal{P}_U^n$. Let Λ be the canonical element on $\mathbb{M}$. We then introduce a canonical filtration $\mathbb{F}^\Lambda = (\mathcal{F}_t^\Lambda)_{0\le t\le T}$ on $\mathbb{M}$ by

$$\mathcal{F}_t^\Lambda := \sigma\big\{\Lambda(C \times [0,s]) : \forall s \le t, C \in \mathcal{B}(\mathcal{P}_U^n)\big\}.$$

For each $q \in \mathbb{M}$, one has a disintegration property: $q(\mathrm{d}t, \mathrm{d}m) = q(t, \mathrm{d}m)\mathrm{d}t$, and there is a version of disintegration such that $(t,q) \longmapsto q(t, \mathrm{d}m)$ is $\mathbb{F}^\Lambda$–predictable.

(μ, Λ, B) denotes the canonical element on $\overline{\Omega} := \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell$. The canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in[0,T]}$ is then defined by: for all $t \in [0,T]$

$$\overline{\mathcal{F}}_t := \sigma\big\{\mu_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, B_{t\wedge\cdot}\big\},$$

where $\Lambda_{t\wedge\cdot}$ denotes the restriction of Λ on $\mathcal{P}^n_U \times [0,t]$ (see notation 4.1.1). Notice that we can choose a version of disintegration $\Lambda(\mathrm{d}m, \mathrm{d}t) = \Lambda_t(\mathrm{d}m)\mathrm{d}t$ with $(\Lambda_t)_{t\in[0,T]}$ a $\mathcal{P}(\mathcal{P}^n_U)$–valued $\overline{\mathbb{F}}$–predictable process.

Let us consider $\mathcal{L}$ the following generator: for all $(t,x,\pi,m,u) \in [0,T]\times\mathbb{R}^n \times \mathcal{C}^n_{\mathcal{W}} \times \mathcal{P}^n_U \times U$ and any $\varphi \in C^2(\mathbb{R}^n)$

$$\mathcal{L}_t\varphi(x,\pi,m,u) := \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\top(t,x,\pi,m,u)\nabla^2\varphi(x)\big] + b(t,x,\pi,m,u)^\top\nabla\varphi(x),$$

also we introduce, for every $f \in C^2(\mathbb{R}^n)$, $N_t(f)$:

$$N_t(f) := \langle f(\cdot - \sigma_0 B_t), \mu_t\rangle - \langle f, \mu_0\rangle - \int_0^t \int_{\mathcal{P}^n_U}\int_{\mathbb{R}^n\times U} \mathcal{L}_r[f(\cdot - \sigma_0 B_r)]\big(x,\mu,m,u\big)m(\mathrm{d}x,\mathrm{d}u)\Lambda_r(\mathrm{d}m)\mathrm{d}r. \tag{4.4.1}$$

Notice that, under Assumption 4.1.1, the integral in the definition $N(f)$ is well–posedness. For each $\pi \in \mathcal{P}(\mathbb{R}^n)$, one considers the Borel set $\mathbb{Z}_\pi$ which is the set of probability measures m on $\mathbb{R}^n \times U$ with marginal on $\mathbb{R}^n$ equal to π i.e.

$$\mathbb{Z}_\pi := \Big\{m \in \mathcal{P}^n_U : m(\mathrm{d}x, U) = \pi(\mathrm{d}x)\Big\}.$$

Definition 4.4.1. *For every $\nu \in \mathcal{P}(\mathbb{R}^n)$, $\mathrm{P} \in \mathcal{P}(\overline{\Omega})$ is a measure–valued rule if:*

- $\mathrm{P}\big(\mu_0 = \nu\big) = 1$.
- $(B_t)_{t\in[0,T]}$ *is a* $(\mathrm{P}, \overline{\mathbb{F}})$ *Wiener process starting at zero and for* P*–almost every* $\omega \in \overline{\Omega}$, $N_t(f) = 0$ *for all* $f \in C^2_b(\mathbb{R}^n)$ *and every* $t \in [0,T]$.
- *For* $\mathrm{dP}\otimes\mathrm{d}t$ *almost every* $(t,\omega) \in [0,T]\times\Omega$, $\Lambda_t\big(\mathbb{Z}_{[\mu_t]}\big) = 1$.

We shall denote by $\overline{\mathcal{P}}_V(\nu)$ the set of all measure–valued rules with initial value ν.

4.4.2 Optimization problem

Let us define, for all $(\pi, q) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}^n_U)$,

$$J(\pi, q) := \int_0^T \int_{\mathcal{P}^n_U}\int_{\mathbb{R}^n\times U} L\big(t,x,\pi,m,u\big)m(\mathrm{d}x,\mathrm{d}u)q_t(\mathrm{d}m)\mathrm{d}t + \int_{\mathbb{R}^n} g\big(x,\pi\big)\pi_T(\mathrm{d}x).$$

Notice that under Assumption 4.1.1, the map $J : \mathcal{C}^{n,p}_{\mathcal{W}} \times \mathbb{M}_p(\mathcal{P}^n_U) \to \mathbb{R}$ is continuous (see for instance Lemma 4.7.1). We can now define the measure–valued control problem: for each $\nu \in \mathcal{P}(\mathbb{R}^n)$,

$$V_V(\nu) := \sup_{\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)} \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda)\big]. \tag{4.4.2}$$

Remark 4.4.2. (i) Definition 4.4.1 *is partly inspired by the Fokker–Planck equation verified by* $(\mu^\alpha_t, \overline{\mu}^\alpha_t)_{t\in[0,T]}$ *(see* (4.3.1) *and* Proposition 4.4.3*), in particular the last two points characterize this Fokker–Planck aspect. Indeed,* (μ,Λ) *satisfy: for all* (t,f)

$$\langle f(\cdot - \sigma_0 B_t), \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \int_{\mathcal{P}^n_U}\int_{\mathbb{R}^n\times U} \mathcal{L}_r[f(\cdot - \sigma_0 B_r)]\big(x,\mu,m,u\big)m^x(\mathrm{d}u)\mu_r(\mathrm{d}x)\Lambda_r(\mathrm{d}m)\mathrm{d}r,$$

where for each $m \in \mathcal{P}^n_U$, *the Borel measurable function* $\mathbb{R}^n \ni x \to m^x \in \mathcal{P}(U)$ *verifies* $m^x(\mathrm{d}u)m(\mathrm{d}x,U) = m(\mathrm{d}x,\mathrm{d}u)$. *This kind of control turns out to be less "rigid". Especially,* $\overline{\mathcal{P}}_V(\nu)$ *is a compact set for the Wasserstein topology* (*see* Theorem 4.5.1).

(ii) Working with these variables seems to be the key to better understand the problem and solve the principal difficulties. Mainly, to prove a limit theory result in this context, we make an approximation of the distribution of (μ, Λ) *thanks to the distribution of variables of type* $(\mu^\alpha, \delta_{\overline{\mu}^\alpha_t}(\mathrm{d}m)\mathrm{d}t)$ *and not thanks to the approximation of the law of* X*. This approximation is achieved by using Fokker–Planck equations. To the best of our knowledge, looking at this kind of variable or "control" has never been studied in the literature (except in* Chapter 2 *and* Chapter 3*, only for technical reasons).*

As stated in the preamble of this part, the measure–valued control problem is motivated by the Fokker–Planck equation verified by the couple $(\mu^\alpha, \overline{\mu}^\alpha)$ of the strong formulation. Therefore, the strong controls i.e. $(\mu^\alpha, \overline{\mu}^\alpha)_{\alpha\in\mathcal{A}(\nu)}$ can be seen as a special case of measure-valued rules. By applying Itô's formula, it is straightforward to deduce the following proposition.

Proposition 4.4.3. *For each* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$*, let us introduce*

$$\overline{\mathcal{P}}_S(\nu) := \Big\{ \mathbb{P}_\nu \circ \big((\mu^\alpha_t)_{t\in[0,T]}, \delta_{\overline{\mu}^\alpha_r}(\mathrm{d}m)\mathrm{d}r, (B_t)_{t\in[0,T]}\big)^{-1},\ \alpha \in \mathcal{A}(\nu) \Big\}.$$

one has $\overline{\mathcal{P}}_S(\nu) \subset \overline{\mathcal{P}}_V(\nu)$ *and*

$$V_S(\nu) = \sup_{\mathrm{Q}\in\overline{\mathcal{P}}_S(\nu)} \mathbb{E}^{\mathrm{Q}}\big[J(\mu,\Lambda)\big].$$

Proof. Let $f \in C^2(\mathbb{R}^n)$ and $t \in [0,T]$, denote by $N_t(\mu,\Lambda,B)(f) := N_t(f)$. For any $\alpha \in \mathcal{A}(\nu)$, it is obvious that $\mathbb{P}_\nu(\mu^\alpha_0 = \nu) = 1$ and $\delta_{\overline{\mu}^\alpha_t}(\mathbb{Z}_{\mu^\alpha_t}) = 1$ $\mathrm{d}\mathbb{P}_\nu \otimes \mathrm{d}t$ a.e.. After applying Itô's formula with the process $X^\alpha_\cdot - \sigma_0 B_\cdot$, and taking the conditional expectation w.r.t. the σ–field $\mathcal{G}_T$, one has $N_t(\mu^\alpha, \delta_{\overline{\mu}^\alpha_t}(\mathrm{d}m)\mathrm{d}t, B)(f) = 0$, $\mathbb{P}_\nu$–a.e. for all (t,f). Then $\mathbb{P}_\nu \circ \big(\mu^\alpha, \delta_{\overline{\mu}^\alpha_t}(\mathrm{d}m)\mathrm{d}t, B\big)^{-1} \in \overline{\mathcal{P}}_V(\nu)$. Therefore $\overline{\mathcal{P}}_S(\nu) \subset \overline{\mathcal{P}}_V(\nu)$. In addition, notice that

$$\Phi(\alpha) = \mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T \int_{\mathcal{P}^n_U} \langle L(t,\cdot,\mu^\alpha_{t\wedge\cdot},m,\cdot),m\rangle \delta_{\overline{\mu}^\alpha_t}(\mathrm{d}m)\mathrm{d}t + \langle g(\cdot,\mu^\alpha),\mu^\alpha_T\rangle\bigg],$$

consequently $V_S(\nu) = \sup_{\mathrm{Q}\in\overline{\mathcal{P}}_S(\nu)} \mathbb{E}^{\mathrm{Q}}\big[J(\mu,\Lambda)\big]$. □

4.5 Equivalence results and limit theory

Now, we formulate the main results of this chapter.

Theorem 4.5.1 (Equivalence)**.** *Let* Assumption 4.1.1 *hold true and* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$*, with* $p' > p$*. Then* $\overline{\mathcal{P}}_V(\nu)$ *is convex and compact for the Wasserstein metric* $\mathcal{W}_p$*. Moreover*

(i) When $\ell \neq 0$*, for* $\mathcal{W}_p$*, the set* $\overline{\mathcal{P}}_S(\nu)$ *is dense in* $\overline{\mathcal{P}}_V(\nu)$*.*

(ii) When $\ell = 0$*, for any* $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$*, there exists a family* $(\mathrm{P}^k_z)_{(k,z)\in\mathbb{N}^*\times[0,1]} \subset \overline{\mathcal{P}}_S(\nu)$ *such that for each* $k \in \mathbb{N}^*$*,* $[0,1] \ni z \to \mathrm{P}^k_z \in \mathcal{P}(\overline{\Omega})$ *is Borel measurable and one gets* $\lim_{k\to\infty} \mathcal{W}_p\Big(\int_0^1 \mathrm{P}^k_z\,\mathrm{d}z,\ \mathrm{P}\Big) = 0$*.*

Consequently

$$V_V(\nu) = V_S(\nu),$$

and there exists $\mathrm{P}^\star \in \overline{\mathcal{P}}_V(\nu)$ *such that* $V_S(\nu) = \mathbb{E}^{\mathrm{P}^\star}\big[J(\mu,\Lambda)\big]$*.*

Remark 4.5.2. *(i) As in* Chapter 3 *(see also* [103] *and* Chapter 6 *for the mean field game context), there are some specificities when* $\ell = 0$*. Indeed, when* $\ell = 0$*,* $(\mu^\alpha, \overline{\mu}^\alpha)$ *are deterministic, but* (μ,Λ) *can still be random, therefore, except in particular situation, it is not possible to approximate the non atomic measure* P *by a sequence of atomic measure of type* $\delta_{(\mu^\alpha, \delta_{\overline{\mu}^\alpha_s}(\mathrm{d}m)\mathrm{d}s)}$*. However, a randomisation is possible as mentioned in (ii) of* Theorem 4.5.1*.*

(ii) Theorem 4.5.1 *and the following* Theorem 4.5.3 *are in the same spirit that* Theorem 3.2.4 *and* Theorem 3.2.7 *in* Chapter 3*. The main difference is the presence of the distribution of controlled state and control, and this particularity turns to be a non trivial extension (see discussion in section 4.3).*

Theorem 4.5.3 (Propagation of chaos). *Let* Assumption 4.1.1 *hold true,* $p' > p$ *and* $(\nu^i)_{i\in\mathbb{N}^*} \subset \mathcal{P}_{p'}(\mathbb{R}^n)$ *satisfying* $\sup_{N\geq 1} \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^n} |x'|^{p'} \nu^i(\mathrm{d}x') < \infty$. *Then*

$$\lim_{N\to\infty} \Big| V_S^N\big(\nu^1,\dots,\nu^N\big) - V_S\Big(\frac{1}{N}\sum_{i=1}^N \nu^i\Big)\Big| = 0.$$

Finally, we provide some properties of optimal control of our problem. For any $\nu \in \mathcal{P}(\mathbb{R}^n)$, denote by $\overline{\mathcal{P}}_V^\star(\nu)$ the set of optimal control i.e. $\mathrm{P}^\star \in \overline{\mathcal{P}}_V^\star(\nu)$ if $\mathrm{P}^\star \in \overline{\mathcal{P}}_V(\nu)$ and $V_V(\nu) = \mathbb{E}^{\mathrm{P}^\star}\big[J\big(\mu,\Lambda\big)\big]$.

Proposition 4.5.4. *Suppose that the conditions of* Theorem 4.5.3 *hold. Let* $\lim_{N\to\infty} \mathcal{W}_p\big(\frac{1}{N}\sum_{i=1}^N \nu^i, \nu\big) = 0$ *with* $\nu \in \mathcal{P}_p(\mathbb{R}^n)$.

(*i*) *For any sequence of non negative numbers* $(\epsilon_N)_{N\in\mathbb{N}^*}$ *verifying* $\lim_{N\to\infty} \epsilon_N = 0$, *if* $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ *is the sequence satisfying* $\mathrm{P}^N := \mathbb{P}(\alpha^1,\dots,\alpha^N)$ (*see definition* (4.2.3)) *with*

$$\text{for each } N\in\mathbb{N}^*,\ \alpha^i \in \mathcal{A}_N(\nu_N)\ \forall i \in [\![1,N]\!] \text{ and } V_S^N(\nu^1,\dots,\nu^N) - \epsilon_N \leq \mathbb{E}^{\mathrm{P}^N}\big[J\big(\mu,\Lambda\big)\big], \tag{4.5.1}$$

then

$$\lim_{N\to\infty} \inf_{\mathrm{P}^\star\in\overline{\mathcal{P}}_V^\star(\nu)} \mathcal{W}_p\big(\mathrm{P}^N,\mathrm{P}^\star\big) = 0.$$

(*ii*) *Moreover, for each* $\mathrm{P}^\star \in \overline{\mathcal{P}}_V^\star(\nu)$, *there exist* $(\epsilon_N)_{N\in\mathbb{N}^*} \subset (0,\infty)$ *verifying* $\lim_{N\to\infty} \epsilon_N = 0$ *and a sequence* $(\mathrm{P}^{\star,N})_{N\in\mathbb{N}^*}$ *satisfying* $\mathrm{P}^{\star,N} := \mathbb{P}(\alpha^{\star,1},\dots,\alpha^{\star,N})$ *and condition 4.5.1 s.t.* $\lim_{N\to\infty} \mathcal{W}_p(\mathrm{P}^{\star,N},\mathrm{P}^\star) = 0$.

Remark 4.5.5. *The previous proposition shows that any* ϵ_N*–optimal control of the large population stochastic control problem converges towards an optimal control of the McKean–Vlasov stochastic control problem in distribution sense. In particular when there exists a unique strong optimal control of the McKean–Vlasov control problem, any* ϵ_N*–optimal control of the large population control problem converges towards this control.*

The next corollary is just a combination of Theorem 4.5.3 and Proposition 3.4.4. It states, if a strong control is close enough to the optimum value of the mean field control problem, from this control, we can construct N–agents which are close to the optimum of the N–agents stochastic control problem.

Corollary 4.5.6. *Let* Assumption 4.1.1 *hold true. Let* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, *with* $p' > p$, $(\epsilon_N)_{N\in\mathbb{N}^*}$ *be a sequence of non-negative real such that* $\lim_{N\to\infty} \epsilon_N = 0$. *Also, for each* $N \in \mathbb{N}^*$, *let* $\alpha^N \in \mathcal{A}(\nu)$ *satisfying* $\alpha_t^N = \phi^N(t,\xi,W_{t\wedge\cdot},B_{t\wedge\cdot})$ $\mathbb{P}_\nu$ *a.e. for all* $t\in[0,T]$ *with a Borel function* $\phi^N : [0,T]\times\mathbb{R}^n\times\mathcal{C}^d\times\mathcal{C}^\ell \to U$, *and*

$$V_S(\nu) - \epsilon_N \leq \Phi(\alpha^N).$$

Then, there exists $(\delta_N)_{N\in\mathbb{N}^*} \subset (0,\infty)$ *s.t.* $\lim_{N\to\infty} \delta_N = 0$ *and* $(\alpha^{1,N},\dots,\alpha^{N,N}) \in \mathcal{A}_N(\nu_N)^N$ *with* $\nu_N := \nu\otimes\cdots\otimes\nu$ *satisfying*

$$\alpha_t^{i,N} = \phi^N(t,\mathbf{X}_0^i,\mathbf{W}_{t\wedge\cdot}^i,B_{t\wedge\cdot}),\ \mathbb{P}_\nu^N \text{ a.e. for all } t\in[0,T] \text{ and } V_S^N(\nu,\dots,\nu) - \delta_N \leq J^N(\alpha^{1,N},\dots,\alpha^{N,N}).$$

4.6 Case of closed loop controls (without convexity assumptions)

In this part, we show, if $\ell = 0$ and the form of coefficients are restricted (see Assumption 4.6.1 below), our McKean–Vlasov optimal control problem can be focused only on the closed loop controls instead of open loop controls considered so far.

Assumption 4.6.1. *Here, we assume that* $\ell = 0$, *and*

$$\sigma(t,x,\pi,m,u) = \tilde{\sigma}(t,x), \text{ for all } (t,x,\pi,m,u) \in [0,T]\times\mathbb{R}^n\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{P}^n_U\times U.$$

For simplicity, we will note σ *instead of* $\tilde{\sigma}$.

Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and $(\Omega^\star,\mathbb{F}^\star,\mathcal{F}^\star,\mathbb{P}^\star)$ be a filtered probability space supporting W a $\mathbb{R}^d$–valued $\mathbb{F}^\star$–Brownian motion and ξ a $\mathcal{F}^\star_0$–random variable such that $\mathcal{L}^{\mathbb{P}^\star}(\xi) = \nu$. We denote by $\mathcal{A}^m$ the collection of all Borel measurable function $\alpha : [0,T]\times\mathbb{R}^n \to U$. Given a $\alpha \in \mathcal{A}^m$, X^α is a[1] solution of: $X^\alpha_0 = \xi$, and for $t \in [0,T]$,

$$X^\alpha_t = X^\alpha_0 + \int_0^t b\big(r, X^\alpha_r, \mu^\alpha_{r\wedge\cdot}, \overline{\mu}^\alpha_r, \alpha(r,X^\alpha_r)\big)\mathrm{d}r + \int_0^t \sigma\big(r,X^\alpha_r\big)\mathrm{d}W_r, \tag{4.6.1}$$

with $\mu^\alpha_r := \mathcal{L}^{\mathbb{P}^\star}\big(X^\alpha_r\big)$ and $\overline{\mu}^\alpha_r := \mathcal{L}^{\mathbb{P}^\star}\big(X^\alpha_r, \alpha(r,X^\alpha_r)\big)$, for all $r \in [0,T]$. Let us define

$$\overline{\mathcal{P}}^{\mathrm{M}}(\nu) := \Big\{\mathbb{P}^\star\circ\big((\mu^\alpha_t)_{t\in[0,T]}, \delta_{\overline{\mu}^\alpha_r}(\mathrm{d}m)\mathrm{d}r\big)^{-1},\ \alpha : [0,T]\times\mathbb{R}^n\to U \text{ a Borel measurable map}\Big\}.$$

Theorem 4.6.2. *Let Assumptions 4.1.1 and 4.6.1 hold true and* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, *with* $p' > p$. *Then, any element of* $\overline{\mathcal{P}}_S(\nu)$ *is the limit for* $\mathcal{W}_p$ *of a sequence of elements of* $\overline{\mathcal{P}}^{\mathrm{M}}(\nu)$. *Consequently*

$$V_S(\nu) = \sup_{\alpha\in\mathcal{A}^m}\mathbb{E}^{\mathbb{P}^\star}\left[\int_0^T L(t,X^\alpha_t,\mu^\alpha_{t\wedge\cdot},\overline{\mu}^\alpha_t,\alpha(t,X^\alpha_t))\mathrm{d}t + g(X^\alpha_T,\mu^\alpha)\right].$$

Let us denote by $\mathcal{A}^m_N$ the collection of all Borel measurable function $\alpha : [0,T]\times(\mathbb{R}^n)^N \to U$. On $(\Omega^\star,\mathbb{F}^\star,\mathcal{F}^\star,\mathbb{P}^\star)$, we consider $(W^i)_{i\in\mathbb{N}^*}$ a sequence of independent $\mathbb{R}^d$–valued Brownian motion, and $(\xi^i)_{i\in\mathbb{N}^*}$ a sequence of iid $\mathcal{F}^\star_0$–random variables of law ν. Then given $\alpha := (\alpha^1,\dots,\alpha^N) \in (\mathcal{A}^m_N)^N$, denote by $\mathbf{X}^\alpha := (\mathbf{X}^{\alpha,1}_\cdot,\dots,\mathbf{X}^{\alpha,N}_\cdot)$ the solution of the following system of SDEs, for each $i \in \{1,\dots,N\}$, $\mathbb{E}^{\mathbb{P}^\star}\big[\|\mathbf{X}^{\alpha,i}\|^p\big] < \infty$,

$$\mathbf{X}^{\alpha,i}_t = \xi^i + \int_0^t b\big(r,\mathbf{X}^{\alpha,i}_r,\varphi^{N,\mathbf{X}}_{r\wedge\cdot},\varphi^N_r,\alpha^i(r,\mathbf{X}^\alpha_r)\big)\mathrm{d}r + \int_0^t\sigma\big(r,\mathbf{X}^{\alpha,i}_r\big)\mathrm{d}\mathbf{W}^i_r, \text{ for all } t\in[0,T], \tag{4.6.2}$$

with

$$\varphi^{N,\mathbf{X}}_t(\mathrm{d}x) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}^{\alpha,i}_t\right)}(\mathrm{d}x) \text{ and } \varphi^N_t(\mathrm{d}x,\mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N\delta_{\left(\mathbf{X}^{\alpha,i}_t,\,\alpha^i(t,\mathbf{X}^\alpha_t)\right)}(\mathrm{d}x,\mathrm{d}u), \text{ for all } t\in[0,T].$$

Theorem 4.6.3. [Propagation of chaos closed loop] *Let* Assumption 4.1.1 *hold true and* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$ *with* $p' > p$. *Then*

$$V_S(\nu) = \lim_{N\to\infty}\sup_{\alpha\in\mathcal{A}^m_N}\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}^\star}\left[\int_0^T L\big(t,\mathbf{X}^{\alpha,i}_t,\varphi^{N,\mathbf{X}}_{t\wedge\cdot},\varphi^N_t,\alpha^i(t,\mathbf{X}^\alpha_t)\big)\mathrm{d}t + g\big(\mathbf{X}^{\alpha,i}_T,\varphi^{N,\mathbf{X}}_{T\wedge\cdot}\big)\right].$$

Remark 4.6.4. Theorem 4.6.2 *and* Theorem 4.6.3 *show that we can focus our research of optimal controls over the set of closed loop controls. An important point is to notice that we do not need to use any convexity assumptions as in classical literature on this subject (see for instance* [104] *or* [108]). *This kind of techniques can not work in this framework. Indeed, in the presence of law of control, the projection techniques do not allow to recover the law of the control, consequently these techniques can no longer be used.*

4.7 Approximation of Fokker–Planck equations

In this section, we give an approximation of a particular Fokker–Planck equation via a sequence of measure–valued processes constructed from classical SDE processes interacting through the empirical distribution of their states and controls. This result is a crucial part for the proof of Theorem 4.5.1 and Theorem 4.5.3.

[1]The Equation (4.6.1) does not have necessary a unique strong solution. However, this equation has a unique weak–solution (see [106] for instance).

4.7.1 Regularization procedures and its consequences

4.7.1.1 Regularization procedures

Some additional notations are needed for our subsequent proofs. These are mainly regularization procedures through convolutions.

Let us take $G \in C^\infty(\mathbb{R}^n;\mathbb{R})$ satisfying $G \geq 0$, $G(x) = G(-x)$ for all $x \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} G(y)\mathrm{d}y = 1$, then define the regularizing kernel $G_\epsilon(x) := \epsilon^{-n}G(\epsilon^{-1}x)$ for each $\epsilon > 0$. Throughout this chapter

$$G(x) := (1+|x|^2)^{-n}\bigg(\int_{\mathbb{R}^n}(1+|x'|^2)^{-n}\mathrm{d}x'\bigg)^{-1}, \text{for all } x \in \mathbb{R}^n. \tag{4.7.1}$$

Let $\psi : [0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^2\times(\mathcal{P}^n_U)^2\times U \longrightarrow \mathbb{R}^j$ be a Borel function, with $j \in \mathbb{N}^*$. For each $\epsilon > 0$, one defines the function $\psi^\epsilon : \mathcal{C}^\ell \times (\mathcal{C}^n_{\mathcal{W}})^2 \times \mathcal{P}(\mathcal{P}^n_U) \times [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^j$ as follows: for every $(t,x,\mathbf{b},\pi,\beta,q) \in [0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^2\times\mathcal{P}((\mathcal{P}^n_U)^2)$

$$\psi^\epsilon[\mathbf{b},\pi,\beta,q](t,x) := \int_{(\mathcal{P}^n_U)^2}\int_{\mathbb{R}^n}\int_U \psi(t,y,\mathbf{b}_{t\wedge\cdot},\pi_{t\wedge\cdot},\beta_{t\wedge\cdot},m,\bar\nu,u)\frac{G_\epsilon(x-y)}{(m(\mathrm{d}z,U))^{(\epsilon)}(x)}m(\mathrm{d}u,\mathrm{d}y)q(\mathrm{d}m,\mathrm{d}\bar\nu), \tag{4.7.2}$$

where for every $m \in \mathcal{P}^n_U$, $(m(\mathrm{d}z,U))^{(\epsilon)}(x) := \int_{\mathbb{R}^n} G_\epsilon(x-y)m(\mathrm{d}y,U)$.

Observe that $\big|\psi^\epsilon[\mathbf{b},\pi,\beta,q](t,x)\big| \leq \sup_{z',\mathbf{b}',\zeta',m',\nu',u'}\big|\psi(t,z',\mathbf{b}',\zeta',m',\bar\nu',u')\big|$, for all $(\mathbf{b},\pi,\beta,q,t,x)$. Then if ψ is bounded ψ^ϵ is uniformly bounded w.r.t $\epsilon > 0$. Also notice that, given $(t,\mathbf{b},\pi,\beta,q)$, for each $\epsilon > 0$, the function $\mathbb{R}^n \ni x \to \psi^\epsilon[\mathbf{b},\pi,\beta,q](t,x) \in \mathbb{R}^j$ belongs to $C^\infty_b(\mathbb{R}^n)$, hence the name of *regularization*.

Under additional conditions on π, q and initial function ψ, one has, in some sense, "$\lim_{\epsilon\to 0}\psi^\epsilon = \psi$" (see Proposition 4.9.2 for more details).

Lemma 4.7.1. *For all $\psi : [0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^2\times(\mathcal{P}^n_U)^2\times U \longrightarrow \mathbb{R}$, and $\phi : [0,T]\times\mathbb{R}^n \to \mathbb{R}$ two bounded continuous functions. For each $\epsilon > 0$, the function*

$$\big(\mathbf{b},\vartheta,\pi,\beta,q\big) \in \mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^3\times\mathbb{M}\big((\mathcal{P}^n_U)^2\big) \longrightarrow \int_0^T\int_{\mathbb{R}^n}\psi^\epsilon[\mathbf{b},\pi,\beta,q_t](t,x)\phi(t,x)\vartheta_t(\mathrm{d}x)\mathrm{d}t \in \mathbb{R},$$

is continuous.

Proof. Let $(\mathbf{b}^k,\vartheta^k,\pi^k,\beta^k,q^k)_{k\in\mathbb{N}} \subset \mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^3\times\mathbb{M}((\mathcal{P}^n_U)^2)$ and $(\mathbf{b},\vartheta,\pi,\beta,q) \in \mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^3\times\mathbb{M}((\mathcal{P}^n_U)^2)$ verifying $\lim_k(\mathbf{b}^k,\vartheta^k,\pi^k,\beta^k,q^k) = (\mathbf{b},\vartheta,\pi,\beta,q)$. Notice that,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n}\psi^\epsilon[\mathbf{b},\pi,\beta,q_t](t,x)\phi(t,x)\vartheta_t(\mathrm{d}x)\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{R}^n}\int_{(\mathcal{P}^n_U)^2}\int_{\mathbb{R}^n\times U}\psi(t,y,\mathbf{b}_{t\wedge\cdot},\pi_{t\wedge\cdot},\beta_{t\wedge\cdot},m,\bar\nu,u)\frac{G_\epsilon(x-y)}{(m(\mathrm{d}z,U))^{(\epsilon)}(x)}m(\mathrm{d}u,\mathrm{d}y)q_t(\mathrm{d}m,\mathrm{d}\bar\nu)\phi(t,x)\vartheta_t(\mathrm{d}x)\mathrm{d}t\\
&= \int_0^T\int_{\mathbb{R}^n}\int_{\mathcal{C}^\ell\times(\mathcal{C}^n_{\mathcal{W}})^2}\int_{(\mathcal{P}^n_U)^2}\int_{\mathbb{R}^n\times U}\psi(t,y,g,e,e',m,\bar\nu,u)\phi(t,x)H^\epsilon(x,m)(\mathrm{d}u,\mathrm{d}y)q_t(\mathrm{d}m,\mathrm{d}\bar\nu)\vartheta_t(\mathrm{d}x)\Psi_t(\mathrm{d}g,\mathrm{d}e,\mathrm{d}e')\mathrm{d}t,
\end{aligned}$$

where

$$H^\epsilon(x,m)(\mathrm{d}u,\mathrm{d}y) := \frac{G_\epsilon(x-y)}{(m(\mathrm{d}z,U))^{(\epsilon)}(x)}m(\mathrm{d}u,\mathrm{d}y) \text{ and } \Psi_t(\mathrm{d}g,\mathrm{d}e,\mathrm{d}e')\mathrm{d}t := \delta_{(\mathbf{b}_{t\wedge\cdot},\pi_{t\wedge\cdot},\beta_{t\wedge\cdot})}(\mathrm{d}g,\mathrm{d}e,\mathrm{d}e')\mathrm{d}t.$$

Next, we define

$$Z^k(\mathrm{d}u,\mathrm{d}y,\mathrm{d}m,\mathrm{d}\bar\nu,\mathrm{d}g,\mathrm{d}e,\mathrm{d}e',\mathrm{d}x,\mathrm{d}t) := \frac{1}{T}H^\epsilon(x,m)(\mathrm{d}u,\mathrm{d}y)q^k_t(\mathrm{d}m,\mathrm{d}\bar\nu)\vartheta^k_t(\mathrm{d}x)\delta_{(\mathbf{b}^k_{t\wedge\cdot},\pi^k_{t\wedge\cdot},\beta^k_{t\wedge\cdot})}(\mathrm{d}g,\mathrm{d}e,\mathrm{d}e')\mathrm{d}t,$$

and

$$Z(\mathrm{d}u,\mathrm{d}y,\mathrm{d}m,\mathrm{d}\bar{\nu},\mathrm{d}g,\mathrm{d}e,\mathrm{d}e',\mathrm{d}x,\mathrm{d}t) := \frac{1}{T}H^{\epsilon}(x,m)(\mathrm{d}u,\mathrm{d}y)q_t(\mathrm{d}m,\mathrm{d}\bar{\nu})\vartheta_t(\mathrm{d}x)\Psi_t(\mathrm{d}g,\mathrm{d}e,\mathrm{d}e')\mathrm{d}t,$$

then $(Z^k)_{k\in\mathbb{N}}$ is a sequence of probability measures belongs to $\mathcal{P}\big(U\times\mathbb{R}^n\times(\mathcal{P}_U^n)^2\times\mathcal{C}^\ell\times(\mathcal{C}_{\mathcal{W}}^n)^2\times\mathbb{R}^n\times[0,T]\big)$. As $\lim_{k\to\infty}(\mathbf{b}^k,\vartheta^k,\pi^k,\beta^k,q^k)=(\mathbf{b},\vartheta,\pi,\beta,q)$, it is straightforward to see that $(Z^k)_{k\in\mathbb{N}}$ is relatively compact in $\mathcal{P}\big(U\times\mathbb{R}^n\times(\mathcal{P}_U^n)^2\times\mathcal{C}^\ell\times(\mathcal{C}_{\mathcal{W}}^n)^2\times\mathbb{R}^n\times[0,T]\big)$ and each sub-sequence converges to Z, therefore $(Z^k)_{k\in\mathbb{N}}$ converges to Z in a weak sense. As the function $(t,y,\mathbf{b},e,e',m,\bar{\nu},u,x)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times(\mathcal{C}_{\mathcal{W}}^n)^2\times(\mathcal{P}_U^n)^2\times U\times\mathbb{R}^n\to\psi(t,y,\mathbf{b}_{t\wedge\cdot},e,e',m,\bar{\nu},u)\phi(t,x)\in\mathbb{R}^n$ is bounded continuous, we can conclude. □

4.7.1.2 Regularization of the Fokker–Planck equation

In this part, with the help of the regularization procedure, we show it can be possible to approximate a particular solution of a Fokker–Planck equation with "non–smooth" coefficients by Fokker–Planck equation with "smooth" coefficients, this part is largely inspired by the proof of [76, Lemma 2.1].

Let $p'>p$, $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, $\mathbf{b}\in\mathcal{C}^\ell$, $(\mathbf{n}_t)_{t\in[0,T]}$ and $(\mathbf{z}_t)_{t\in[0,T]}$ be two $\mathcal{P}(\mathbb{R}^n)$–valued continuous processes and also $\hat{\mathbf{q}}_t(\mathrm{d}m,\mathrm{d}m')\mathrm{d}t\in\mathbb{M}((\mathcal{P}_U^n)^2)$. Moreover, $(\mathbf{n},\mathbf{z},\hat{\mathbf{q}},\mathbf{b})$ satisfy the following equation:

$$\langle f(t,.),\mathbf{n}_t\rangle \;=\langle f(0,.),\nu\rangle+\int_0^t\Big[\langle\partial_t f(r,.),\mathbf{n}_r\rangle+\int_{(\mathcal{P}_U^n)^2}\langle\mathcal{A}_r[f(r,\cdot)](.,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar{\nu},.),m\rangle\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\Big]\mathrm{d}r,$$

for all $(t,f)\in[0,T]\times C_b^{1,2}([0,T]\times\mathbb{R}^n)$, where the generator $\mathcal{A}$ is defined by

$$\mathcal{A}_t\varphi(x,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar{\nu},u) := \frac{1}{2}\mathrm{Tr}\big[\hat{\sigma}\hat{\sigma}^\top(t,x,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar{\nu},u)\nabla^2\varphi(x)\big]+\hat{b}(t,x,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar{\nu},u)^\top\nabla\varphi(x), \tag{4.7.3}$$

with $(\hat{b},\hat{\sigma}):[0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times(\mathcal{C}_{\mathcal{W}}^n)^2\times(\mathcal{P}_U^n)^2\times U\to\mathbb{R}^n\times\mathbb{S}^n$ is bounded and continuous function in all arguments, and for each $\bar{\nu}\in\mathcal{P}_U^n$, the map $(\hat{b},\hat{\sigma})(\cdot,\cdot,\mathbf{b},\cdot,\mathbf{z},\cdot,\bar{\nu},\cdot)$ satisfies Assumtpion 4.1.1 with constant θ independent of $\bar{\nu}$.

Now, let us introduce the generator of the "*regularized*" Fokker-Planck equation $\mathcal{A}^\epsilon$: for all $(t,\hat{q},x)\in[0,T]\times\mathcal{P}((\mathcal{P}_U^n)^2)\times\mathbb{R}^n$

$$\mathcal{A}_t^\epsilon\varphi[\mathbf{b},\mathbf{n},\mathbf{z},\hat{q}](x) := \frac{1}{2}\mathrm{Tr}\big[\hat{a}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{q}](t,x)\nabla^2\varphi(x)\big]+\hat{b}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{q}](t,x)^\top\nabla\varphi(x), \tag{4.7.4}$$

where for $(t,x,\gamma,\pi,\beta,m,\bar{\nu},u)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times(\mathcal{C}_{\mathcal{W}}^n)^2\times(\mathcal{P}_U^n)^2\times U$, $\hat{a}(t,x,\gamma,\pi,\beta,m,\bar{\nu},u):=\hat{\sigma}\hat{\sigma}^\top(t,x,\gamma,\pi,\beta,m,\bar{\nu},u)$ and $(\hat{a}^\epsilon,\hat{b}^\epsilon)$ are defined as (4.7.2) with the functions $(\hat{a},\hat{b})$.

We are now ready to formulate our regularization/approximation result of Fokker-Planck equation. The following proposition is proved in Appendix 4.9.1.

Lemma 4.7.2 (Regularization of Fokker–Planck equation)**.** *Let $\nu\in\mathcal{P}_p(\mathbb{R}^n)$, for each $\epsilon>0$, there exists a unique solution $(\mathbf{n}_t^\epsilon)_{t\in[0,T]}\in\mathcal{C}_{\mathcal{W}}^{n,p}$ of: for all $f\in C_b^{1,2}([0,T]\times\mathbb{R}^n)$ and $t\in[0,T]$,*

$$\langle f(t,.),\mathbf{n}_t^\epsilon\rangle \;=\; \int_{\mathbb{R}^n}f(0,y)\nu(\mathrm{d}y)+\int_0^t\Big[\int_{\mathbb{R}^n}\partial_t f(r,y)\mathbf{n}_r^\epsilon(\mathrm{d}y)+\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f(r,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,y)\mathbf{n}_r^\epsilon(\mathrm{d}y)\Big]\mathrm{d}r. \tag{4.7.5}$$

Moreover, if $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ and $\hat{\mathbf{q}}_t(\mathbb{Z}_{[\mathbf{n}_t]}\times\mathcal{P}_U^n)=1$ $\mathrm{d}t$–for almost every $t\in[0,T]$, then

$$\lim_{\epsilon\to0}\sup_{t\in[0,T]}\mathcal{W}_p(\mathbf{n}_t^\epsilon,\mathbf{n}_t)=0. \tag{4.7.6}$$

Remark 4.7.3. *(i) Let $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\mathbb{P})$ be a probability space supporting W a $\widehat{\mathbb{F}}$–Wiener process of dimension $\mathbb{R}^n$ and ξ a $\mathcal{F}_0$–random variable such that $\mathcal{L}^{\mathbb{P}}(\xi)(\mathrm{d}y)=\nu(\mathrm{d}y)$. Given $\epsilon>0$, let Y^ϵ be the unique strong solution (well defined, see Appendix* Section 4.9.1 *(more precisely the Proof of Lemma* Lemma 4.7.2*))*

$$Y^\epsilon_s=\xi+\int_0^s \hat{b}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,Y^\epsilon_r)\mathrm{d}r+\int_0^s(\hat{a}^\epsilon)^{1/2}[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,Y^\epsilon_r)\mathrm{d}W_r,\ \textit{for all } s\in[0,T], \tag{4.7.7}$$

one has, by uniqueness of (4.7.5)*, $\mathcal{L}^{\mathbb{P}}(Y^\epsilon_s)=\mathbf{n}^\epsilon_s$ for all $s\in[0,T]$ where $\mathbf{n}^\epsilon$ is the solution of* (4.7.5)*.*

(ii) We will sometimes use the previous lemma with Proposition 4.9.2*, in which $\mathbf{n}^\epsilon$ must be obtainable through a diffusion process that has a volatility term which verifies $\hat{a}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,Y^\epsilon_r)\geq\theta\mathrm{I}_{n\times n}$.* SDE (4.7.7) *allows to say $\mathbf{n}^\epsilon$ satisfies these conditions. Also, from Lemma* Lemma 4.7.2 *and the* SDE *representation* (4.7.7)*, it is straightforward to see that the measure $\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t$ is equivalent to the Lebesgue measure on $\mathbb{R}^n\times[0,T]$ (see for instance* Proposition 4.9.1 *).*

Remark 4.7.4. *Combining* Remark 4.7.3 *(diffusion form* (4.7.7) *of $\mathbf{n}^\epsilon$) with* Lemma 4.7.2 *(convergence result* (4.7.6)*), as (b,σ) are bounded, there exists a constant $C>0$, depending only of coefficients (b,σ), p and p', such that*

$$\sup_{r\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\mathbf{n}_r(\mathrm{d}x)\leq C\left(1+\int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x)\right)\ \textit{and}\ \mathcal{W}_p\big(\mathbf{n}_s,\mathbf{n}_t\big)^p\leq C|t-s|,\ \textit{for all } (t,s)\in[0,T]\times[0,T].$$

The next lemma is a useful result for the following, it is just a combination of Lemma 4.7.2 and Proposition 4.9.2.

Lemma 4.7.5. *Let us stay in the context of* Lemma 4.7.2 *with $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$. One has*

$$\lim_{\epsilon\to0}\left[\int_0^T\int_{(\mathcal{P}^n_U)^2}\left[\int_{\mathbb{R}^n}\big|K^\epsilon(r,x,m,m')\big|^p\mathbf{n}^\epsilon_r(\mathrm{d}x)+\mathcal{W}_p\Big(H^\epsilon(z,m)(\mathrm{d}u)\mathbf{n}^\epsilon_r(\mathrm{d}z),m(\mathrm{d}u,\mathrm{d}z)\Big)^p\right]\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}m')\mathrm{d}r\right]=0,$$

where

$$K^\epsilon(s,x,m,\bar{\nu}):=\int_{\mathbb{R}^n\times U}\big[\hat{b},\hat{\sigma}\hat{\sigma}^\top\big]\big(s,y,\mathbf{n},\mathbf{z},m,\bar{\nu},u\big)\overline{H}^\epsilon(x,m)(\mathrm{d}u,\mathrm{d}y)-\int_U\big[\hat{b},\hat{\sigma}\hat{\sigma}^\top\big]\big(s,x,\mathbf{n},\mathbf{z},m,\bar{\nu},u\big)H^\epsilon(x,m)(\mathrm{d}u),$$

with $\overline{H}^\epsilon(x,m)(\mathrm{d}u,\mathrm{d}y):=m(\mathrm{d}u,\mathrm{d}y)\frac{G_\epsilon(x-y)}{(m(U,\mathrm{d}z))^{(\epsilon)}(x)}$ and $H^\epsilon(x,m)(\mathrm{d}u):=\int_{\mathbb{R}^n}\overline{H}^\epsilon(x,m)(\mathrm{d}u,\mathrm{d}y)$.

Proof. By Lemma 4.7.2, $\lim_{\epsilon\to0}\sup_{t\in[0,T]}\mathcal{W}_p(\mathbf{n}^\epsilon_t,\mathbf{n}_t)=0$. As $\hat{\mathbf{q}}_t(\mathbb{Z}_{[\mathbf{n}_t]}\times\mathcal{P}^n_U)=1$ $\mathrm{d}t$–almost surely $t\in[0,T]$, using convex inequality and Proposition 4.9.2,

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_0^T\int_{(\mathcal{P}^n_U)^2}\int_{\mathbb{R}^n}\big|K^\epsilon(r,x,m,\bar{\nu})\big|^p\mathbf{n}^\epsilon_r(\mathrm{d}x)\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r\\
&\leq\limsup_{\epsilon\to0}\int_0^T\int_{(\mathcal{P}^n_U)^2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\times U}\Big|\big[\hat{b},\hat{\sigma}\hat{\sigma}^\top\big](r,x,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar{\nu},u)\\
&\qquad\qquad-\big[\hat{b},\hat{\sigma}\hat{\sigma}^\top\big](r,y,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar{\nu},u)\Big|^p\frac{G_\epsilon(x-y)}{(\mathbf{n}_r)^{(\epsilon)}(x)}m^y(\mathrm{d}u)\mathbf{n}_r(\mathrm{d}y)\mathbf{n}^\epsilon_r(\mathrm{d}x)\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r=0.
\end{aligned}$$

For all bounded continuous function $h:\mathbb{R}^n\times U\to\mathbb{R}$, using Proposition 4.9.2 again,

$$\begin{aligned}
&\lim_{\epsilon\to0}\int_0^T\int_{(\mathcal{P}^n_U)^2}\Big|\int_{\mathbb{R}^n\times U}h(x,u)H^\epsilon(x,m)(\mathrm{d}u)\mathbf{n}^\epsilon_r(\mathrm{d}x)-\int_{\mathbb{R}^n\times U}h(z,u)m(\mathrm{d}z,\mathrm{d}u)\Big|\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r\\
&\leq\lim_{\epsilon\to0}\int_0^T\int_{(\mathcal{P}^n_U)^2}\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n\times U}h(x,u)m^y_r(\mathrm{d}u)\frac{G_\epsilon(x-y)}{(\mathbf{n}_r)^{(\epsilon)}(x)}\mathbf{n}_r(\mathrm{d}y)-\int_{\mathbb{R}^n\times U}h(z,u)m(\mathrm{d}z,\mathrm{d}u)\Big|\mathbf{n}^\epsilon_r(\mathrm{d}x)\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r=0,
\end{aligned}$$

similarly to [150, Theorem 1.1.2.], one finds a countable family of bounded continuous functions $(h^k)_{k\in\mathbb{N}^*}$ characterizing the weak convergence, therefore by Lebesgue's dominated convergence,

$$\lim_{\epsilon\to 0}\sum_{k\geq 0}\int_0^T\int_{(\mathcal{P}_U^n)^2}\frac{1}{2^k}\Big|\int_{\mathbb{R}^n\times U}h^k(x,u)H^\epsilon(x,m)(\mathrm{d}u)\mathbf{n}_r^\epsilon(\mathrm{d}x)-\int_{\mathbb{R}^n\times U}h^k(z,u)m(\mathrm{d}z,\mathrm{d}u)\Big|\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r=0,$$

then $\lim_{\epsilon\to 0}\int_0^T\int_{(\mathcal{P}_U^n)^2}\Delta\Big(H^\epsilon(z,m)(\mathrm{d}u)\mathbf{n}_r^\epsilon(\mathrm{d}z),m(\mathrm{d}u,\mathrm{d}z)\Big)\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r=0$, where Δ is the metric characterizing the weak convergence on $\mathcal{P}_U^n$. As $[\hat{b},\hat{\sigma}]$ are bounded and $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, for $(r,m)\in[0,T]\times\mathcal{P}_U^n$,

$$\lim_{K\to\infty}\sup_{\epsilon>0}\int_{|z|+\rho(u_0,u)\geq K}|z|^p+\rho(u_0,u)^p\;\;H^\epsilon(z,m)(\mathrm{d}u)\mathbf{n}_r^\epsilon(\mathrm{d}z)=0.$$

This is enough to conclude that, $\lim_{\epsilon\to 0}\int_0^T\int_{(\mathcal{P}_U^n)^2}\mathcal{W}_p\Big(H^\epsilon(z,m)(\mathrm{d}u)\mathbf{n}_r^\epsilon(\mathrm{d}z),m(\mathrm{d}u,\mathrm{d}z)\Big)\hat{\mathbf{q}}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r=0.$

□

4.7.2 Approximation by N–agents

Now, let us formulate the approximation result of Fokker–Planck equation by N–interacting SDE equations. In order to achieve this, we first describe the associated framework.

Let $p'>p$, $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ and $\big(\Omega^{\mathbf{q}},\mathcal{F}^{\mathbf{q}},\mathbb{F}^{\mathbf{q}},\mathbb{Q}\big)$ be a filtered probability space supporting $(B_t)_{t\in[0,T]}$ a $\mathbb{R}^\ell$–valued $\mathbb{F}^{\mathbf{q}}$–adapted continuous process, $(\mu_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ two $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{F}^{\mathbf{q}}$–continuous processes, $\overline{\Lambda}$ a $\mathbb{M}\big((\mathcal{P}_U^n)^2\big)$–valued variable such that $(\overline{\Lambda}_t)_{t\in[0,T]}$ is $\mathbb{F}^{\mathbf{q}}$–predictable. Besides, $(\mu,B,\zeta,\overline{\Lambda})$ satisfy: $\overline{\Lambda}_t\big(\mathbb{Z}_{[\mu_t]}\times\mathcal{P}_U^n\big)=1$, $\mathrm{d}\mathbb{Q}\otimes\mathrm{d}t$–almost surely, and for $\mathbb{Q}$–a.e. $\omega\in\Omega^{\mathbf{q}}$,

$$\langle f,\mu_t\rangle=\langle f,\nu\rangle+\int_0^t\Big[\int_{\mathcal{P}_U^n\times\mathcal{P}_U^n}\int_{\mathbb{R}^n\times U}\mathcal{A}_rf(y,B,\phi(\mu),\zeta,m,\bar{\nu},u)m(\mathrm{d}y,\mathrm{d}u)\overline{\Lambda}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\Big]\mathrm{d}r,\tag{4.7.8}$$

for all $t\in[0,T]$ and $f\in C_b^2(\mathbb{R}^n)$, where

$$\mathcal{A}_t\varphi(x,\mathbf{b},\pi,\beta,m,\bar{\nu},u):=\frac{1}{2}\mathrm{Tr}\big[\hat{\sigma}\hat{\sigma}^\top(t,x,\mathbf{b},\pi,\beta,m,\bar{\nu},u)\nabla^2\varphi(x)\big]+\hat{b}(t,x,\mathbf{b},\pi,\beta,m,\bar{\nu},u)^\top\nabla\varphi(x),\tag{4.7.9}$$

with, as in (4.7.3), $(\hat{b},\hat{\sigma})$ is continuous in all arguments and bounded, and the map $(\hat{b},\hat{\sigma})(\cdot,\cdot,\mathbf{b},\cdot,\beta,\cdot,\bar{\nu},\cdot)$ satisfies Assumtpion 4.1.1 with constant C and θ independent of $(\mathbf{b},\beta,\bar{\nu})$ (see Assumtpion 4.1.1). Besides, $\phi:\mathcal{C}_{\mathcal{W}}^n\to\mathcal{C}_{\mathcal{W}}^n$ is a Lipschitz function s.t. for all $t\in[0,T]$, $\phi_t(\pi)=\phi_t(\pi_{t\wedge\cdot})$.

Remark 4.7.6. *Notice that, (4.7.8) is an equation. Indeed, with the condition $\overline{\Lambda}_t\big(\mathbb{Z}_{[\mu_t]}\times\mathcal{P}_U^n\big)=1$, $\mathrm{d}\mathbb{Q}\otimes\mathrm{d}t$–almost surely, the process μ appears on both sides on the equality. Under the general* Assumption 4.1.1*, it is not difficult to show there are processes $(\mu,\overline{\Lambda})$ verify equation (4.7.8) (see for instance* Theorem 2.4.2 *). However, without additional assumptions, a uniqueness result cannot be expected.*

Let $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{\mathbb{P}})$ be another filtered probability space supporting:

- $(W^i)_{i\in\mathbb{N}^*}$ a sequence of $\mathbb{R}^n$–valued independent $\widehat{\mathbb{F}}$–Brownian motions and $(\xi^i)_{i\in\mathbb{N}^*}$ a sequence of independent $\widehat{\mathcal{F}}_0$–random variables s.t. $\mathcal{L}^{\widehat{\mathbb{P}}}(\xi_i)=\nu_i\in\mathcal{P}_{p'}(\mathbb{R}^n)$,
- $(\mu^N)_{N\in\mathbb{N}^*}$ and $(\zeta^N)_{N\in\mathbb{N}^*}$ two sequences of $\mathcal{P}(\mathbb{R}^n)$–valued $\widehat{\mathbb{F}}$–adapted continuous processes, and $(B^N)_{N\in\mathbb{N}^*}$ a sequence of $\mathbb{R}^\ell$–valued $\widehat{\mathbb{F}}$–adapted continuous processes,
- $(m^N)_{N\in\mathbb{N}^*}$ and $(\bar{\nu}^N)_{N\in\mathbb{N}^*}$ two sequences of $\mathcal{P}_U^n$–valued $\widehat{\mathbb{F}}$–predictable processes,

satisfying:

$$\lim_{N\to\infty} \mathcal{W}_{p'}\Big(\frac{1}{N}\sum_{i=1}^{N}\nu^i, \nu\Big) = 0 \text{ and } \lim_{N\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\phi(\mu^N), \zeta^N, \overline{\Lambda}^N,\ B^N\Big) = \mathcal{L}^{\mathbb{Q}}\big(\phi(\mu), \zeta, \overline{\Lambda}, B\big),\ \text{ in } \mathcal{W}_p, \tag{4.7.10}$$

where $\overline{\Lambda}^N_t(\mathrm{d}m, \mathrm{d}\bar{\nu})\mathrm{d}t := \delta_{(m^N_t,\ \bar{\nu}^N_t)}(\mathrm{d}m, \mathrm{d}\bar{\nu})\mathrm{d}t$.

Furthermore, let $(Z^i)_{i\in\mathbb{N}^*}$ be a sequence of independent $[0,1]$–valued $\widehat{\mathcal{F}}$–measurable uniform variables independent of other variables, and for each $(i,N)\in\mathbb{N}^*\times\mathbb{N}^*$, denote by $\widehat{\mathbb{F}}^{i,N} := (\widehat{\mathcal{F}}^{i,N}_t)_{t\in[0,T]}$ the filtration defined by:

$$\widehat{\mathcal{F}}^{i,N}_t := \sigma\Big\{\xi^i, \overline{\Lambda}^N_{t\wedge\cdot}, \phi_{t\wedge\cdot}(\mu^N), \zeta^N_{t\wedge\cdot}, W^i_{t\wedge\cdot}, B^N_{t\wedge\cdot}, Z^i\Big\},\ \text{for each } t\in[0,T]. \tag{4.7.11}$$

Proposition 4.7.7. *There exists a sequence of processes $(\alpha^{i,N})_{(i,N)\in\mathbb{N}^*\times\mathbb{N}^*}$ satisfying for each $(i,N)\in\mathbb{N}^*\times\mathbb{N}^*$, $\alpha^{i,N}$ is $\widehat{\mathbb{F}}^{i,N}$–predictable, s.t. if we let $(\widehat{X}^1_t, ..., \widehat{X}^N_t)_{t\in[0,T]}$ be the continuous processes unique strong solution of: for each $i\in\{1,...,N\}$, $\mathbb{E}^{\widehat{\mathbb{P}}}[\|\widehat{X}^i\|^{p'}]<\infty$, for all $t\in[0,T]$*

$$\widehat{X}^i_t = \xi^i + \int_0^t \hat{b}\big(r, \widehat{X}^i_r, B^N, \phi(\widehat{\mu}^N), \zeta^N, \widehat{m}^N_r, \bar{\nu}^N_r, \alpha^{i,N}_r\big)\mathrm{d}r + \int_0^t \hat{\sigma}\big(r, \widehat{X}^i_r, B^N, \phi(\widehat{\mu}^N), \zeta^N, \widehat{m}^N_r, \bar{\nu}^N_r, \alpha^{i,N}_r\big)\mathrm{d}W^i_r,\ \widehat{\mathbb{P}}\text{–a.e.} \tag{4.7.12}$$

where $\widehat{m}^N_t(\mathrm{d}x,\mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{X}^i_t,\ \alpha^{i,N}_t)}(\mathrm{d}x,\mathrm{d}u)$, $\widehat{\mu}^N_t(\mathrm{d}x) := \widehat{m}^N_t(\mathrm{d}x, U)$, then, one has, for a sub-sequence $(N_k)_{k\in\mathbb{N}^*}\subset\mathbb{N}^*$,

$$\lim_{k\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\big(\widehat{m}^{N_k}_t, m^{N_k}_t\big)^p\mathrm{d}t + \sup_{t\in[0,T]}\mathcal{W}_p\Big(\phi_t(\widehat{\mu}^{N_k}),\ \phi_t(\mu^{N_k})\Big)\Big] = 0,$$

and

$$\lim_{k\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{\mu}^{N_k}, \zeta^{N_k}, \widehat{\Lambda}^{N_k}, B^{N_k}\Big) = \mathcal{L}^{\mathbb{Q}}\big(\mu, \zeta, \overline{\Lambda}, B\big),\ \textit{in } \mathcal{W}_p\ \textit{ with }\ \widehat{\Lambda}^{N_k}_s(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s := \delta_{(\widehat{m}^{N_k}_s, \bar{\nu}^{N_k}_s)}(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s. \tag{4.7.13}$$

Remark 4.7.8. (*i*) Proposition 4.7.7 *as well as* Proposition 4.7.9 (*see below*) *can be considered as a general characterization of Fokker-Planck equation of type* (4.7.8) *via a sequence of* SDE *processes interacting through the empirical distribution of the states and "controls". These results are very useful both in the study of extended mean field control problem* (*see* Proposition 4.8.3) *and in mean field game of controls* (*see* Chapter 6).

(*ii*) *Because of non-uniqueness of Fokker-Planck equation* (4.7.8), *the condition* (4.7.10) *is a crucial and essential assumption. Furthermore, notice that, the condition* (4.7.10) *does not require any equation verified by the sequence* $\big(\phi(\mu^N), \zeta^N, \overline{\Lambda}^N,\ B^N\big)_{N\in\mathbb{N}^*}$. *Only the convergence result* (4.7.10) *is necessary.*

(*iii*) *Observe that, the sequence $(\Lambda^N)_{N\in\mathbb{N}^*}$ is a subset of $\mathbb{M}_0\big((\mathcal{P}^n_U)^2\big)$ and not a general subset of $\mathbb{M}\big((\mathcal{P}^n_U)^2\big)$. For an understandable and easy presentation, we consider this type of sequence, but a general subset of $\mathbb{M}\big((\mathcal{P}^n_U)^2\big)$ is possible* (*see* Proposition 4.7.10 *below*).

(*iv*) *The presence of the map ϕ, notably in* (4.7.10), *specifies the required non-linearity. In particular, if ϕ is null, it means that no assumption of convergence towards μ is necessary to find a sequence of* SDE *processes converging to μ.*

Proof of Proposition 4.7.7. The proof is divided in three steps for a better understanding.

Step 1 : **Approximation by regularization of** F-P **equation**: Let $\epsilon>0$, recall that $\mathcal{A}^\epsilon$ is defined in (4.7.4). For all $\omega\in\Omega^{\mathfrak{q}}$, by Lemma 4.7.2, there exists a continuous process $(\mu^\epsilon_t(\omega))_{t\in[0,T]}$ verifying

$$\langle f, \mu^\epsilon_t(\omega)\rangle \ = \ \langle f,\nu\rangle + \int_0^t\int_{\mathbb{R}^n}\mathcal{A}^\epsilon_r f\big[B(\omega), \phi(\mu(\omega)), \zeta(\omega), \overline{\Lambda}_r(\omega)\big](x)\mu^\epsilon_r(\omega)(\mathrm{d}x)\mathrm{d}r,$$

for all $(f,t) \in C_b^2(\mathbb{R}^n;\mathbb{R}) \times [0,T]$, and for $\mathbb{Q}$–a.e. $\omega \in \Omega^{\mathfrak{q}}$, $\lim_{\epsilon\to 0} \sup_{t\in[0,T]} \mathcal{W}_p(\mu_t^\epsilon(\omega), \mu_t(\omega)) = 0$. Also, by Lemma 4.9.4, there is a function $\Phi^\epsilon : \mathcal{C}^\ell \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M}\big((\mathcal{P}_U^n)^2\big) \to \mathcal{C}_{\mathcal{W}}^n$ such that $\mathbb{Q}$–a.e. $\omega \in \Omega^{\mathfrak{q}}$

$$\mu_t^\epsilon(\omega) = \Phi_t^\epsilon\Big(B_{t\wedge\cdot}(\omega), \phi_{t\wedge\cdot}(\mu(\omega)), \zeta_{t\wedge\cdot}(\omega), \overline{\Lambda}_{t\wedge\cdot}(\omega)\Big), \text{ for all } t\in[0,T]. \tag{4.7.14}$$

Step 2 : **Approximation by discretization**: Now, let us define for all $(x,m) \in \mathbb{R}^n \times \mathcal{P}_U^n$, the probability

$$H^\epsilon(x,m)(\mathrm{d}u) := \int_{\mathbb{R}^n} m(\mathrm{d}u,\mathrm{d}y) \frac{G_\epsilon(x-y)}{(m(U,\mathrm{d}z))^{(\epsilon)}(x)}.$$

By [30], there exists a Borel application $N^\epsilon : (x,m,v) \in \mathbb{R}^n \times \mathcal{P}_U^n \times [0,1] \to N^\epsilon(x,m)(v) \in U$ s.t. for all $(x,m) \in \mathbb{R}^n \times \mathcal{P}_U^n$ and any $[0,1]$–valued uniform random variable F,

$$\widehat{\mathbb{P}} \circ \big(N^\epsilon(x,m)(F)\big)^{-1}(\mathrm{d}u) = H^\epsilon(x,m)(\mathrm{d}u).$$

Step 2.1 : *Construction of scheme of discretization*: Let us consider the partition $(t_k^N)_{1\le k\le 2^N}$ with $t_k^N = \frac{kT}{2^N}$, and take a sequence of $\mathbb{R}^n$–valued independent Brownian motions $(Z^i)_{i\in\mathbb{N}^*}$, independent of all of other variables. Let $\varphi : [0,T]\times\mathbb{R}^n \to [0,1]$ be a Borel function such that, for all $t\in[0,T]$, $\mathcal{L}^{\widehat{\mathbb{P}}}(\varphi(t-t_k^N, Z_t^i - Z_{t_k^N}^i))$ is the uniform law when $t > t_k^N$. For all $i\in\{1,...,N\}$, denote by $V_t^{i,N} := \varphi(t-t_k^N, Z_t^i - Z_{t_k^N}^i)$, when $t\in[t_k^N, t_{k+1}^N)$, and given $\epsilon>0$, we define on $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$, by Euler scheme, $X^{\epsilon,i,N} := X^i$ as follows: $X_0^i := \xi^i$ and

$$\begin{aligned} X_t^i = X_0^i &+ \int_0^t \widehat{B}\Big(s, X_{[s]^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, N^\epsilon(X_{[s]^N}^i, m_s^N)(V_s^{i,N})\Big)\mathrm{d}s \\ &+ \int_0^t \widehat{\Sigma}\Big(s, X_{[s]^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, N^\epsilon(X_{[s]^N}^i, m_s^N)(V_s^{i,N})\Big)\mathrm{d}W_s^i, \text{ for all } t\in[0,T],\ i\in\{1,...,N\}, \end{aligned} \tag{4.7.15}$$

where $[s]^N = t_k^N$ if $t_k^N \le s < t_{k+1}^N$, and, for $s\in[t_k^N, t_{k+1}^N)$,

$$\begin{aligned} &\widehat{B}\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, N^\epsilon(X_{t_k^N}^i, m_s^N)(V_s^{i,N})\Big) \\ &\quad := \hat b\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, N^\epsilon(X_{t_k^N}^i, m_s^N)(V_s^{i,N})\Big) + B\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N\Big), \end{aligned}$$

and

$$\begin{aligned} &\widehat{\Sigma}\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, N^\epsilon(X_{t_k^N}^i, m_s^N)(V_s^{i,N})\Big) \\ &\quad := \Sigma\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N\Big)\hat\sigma\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, N^\epsilon(X_{t_k^K}^i, m_s^N)(V_s^{i,N})\Big), \end{aligned}$$

with

$$\begin{aligned} &B\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N\Big) \\ &:= \left[\hat b^\epsilon\big[B^N, \phi(\mu^N), \zeta^N, \overline{\Lambda}_s^N\big](s, X_{t_k^N}^i) - \int_U \hat b\big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, u\big) H^\epsilon(X_{t_k^N}^i, m_s^N)(\mathrm{d}u)\right], \end{aligned}$$

and

$$\begin{aligned} &\Sigma\Big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N\Big) \\ &:= \left[\hat a^\epsilon\big[B^N, \phi(\mu^N), \zeta^N, \overline{\Lambda}_s^N\big](s, X_{t_k^N}^i)^{1/2}\left(\int_U \hat a\big(s, X_{t_k^N}^i, B^N, \phi(\mu^N), \zeta^N, m_s^N, \bar\nu_s^N, u\big) H^\epsilon(X_{t_k^N}^i, m_s^N)(\mathrm{d}u)\right)^{-1/2}\right], \end{aligned} \tag{4.7.16}$$

recall that $\overline{\Lambda}^N_s(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s := \delta_{m^N_s}(\mathrm{d}m)\delta_{\bar{\nu}^N_s}\mathrm{d}\bar{\nu})\mathrm{d}s$.

Notice that, there exists a Borel function $F^N : \mathbb{R}^n \times \mathbb{M}\big((\mathcal{P}^n_U)^2\big) \times \mathcal{C}^n_{\mathcal{W}} \times \mathcal{C}^n_{\mathcal{W}} \times \mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^\ell \to \mathcal{C}^n$ s.t. for each $i \in \{1,...,N\}$,

$$X^i_t = F^N_t\big(\xi^i, \overline{\Lambda}^N_{t\wedge\cdot}, \phi_{t\wedge\cdot}(\mu^N), \zeta^N_{t\wedge\cdot}, W^i_{t\wedge\cdot}, Z^i_{t\wedge\cdot}, B^N_{t\wedge\cdot}\big), \text{ for all } t\in[0,T],\ \widehat{\mathbb{P}}\text{–a.e.} \tag{4.7.17}$$

Step 2.2 : *Compactness and identification of the limit*: At this stage, we want to show a compactness result and identify the limit of a certain sequence of probability measures constructed from the SDE process $(X^1,...,X^N)$.
Using the assumptions imposed on coefficients $(\hat{b},\hat{\sigma})$ (see the definition of the generator $\mathcal{A}$ in (4.7.9)), especially the fact that $\hat{\sigma}\hat{\sigma}^\top \geq \theta \mathrm{I}_n$ and $(\hat{b},\hat{\sigma})$ are bounded, one has that $[\widehat{B},\widehat{\Sigma}]$ are bounded and there exists a constant $\mathrm{D}>0$, such that for all ϵ and N

$$\sup_{i\in\{1,\dots,N\}} \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\big|X^{\epsilon,i,N}_t - X^{\epsilon,i,N}_s\big|^p\Big] \leq \mathrm{D}|t-s|, \text{ for all } (t,s)\in[0,T]\times[0,T]. \tag{4.7.18}$$

Moreover, by using the fact that $\sup_{N\geq 1}\sum_{i=1}^N \int_{\mathbb{R}^n}|x|^{p'}\nu^i(\mathrm{d}x) < \infty$ (see condition (4.7.10)), it is straightforward to verify that: $\sup_{N\geq1}\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\sup_{t\in[0,T]}|X^{\epsilon,i,N}_t|^{p'}\Big] < \infty$. Then, by [49, Proposition A.2] or/and [49, Proposition-B.1], for each $\epsilon>0$, the sequence $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p$, where

$$\mathrm{P}^N := \widehat{\mathbb{P}}\circ\Big(\vartheta^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N,B^N\Big)^{-1} \in \mathcal{P}\Big(\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\big((\mathcal{P}^n_U)^2\big)\times\mathcal{C}^\ell\Big)$$

with $\vartheta^N_t(\mathrm{d}x) := \frac{1}{N}\sum_{i=1}^N \delta_{X^{\epsilon,i,N}_t}(\mathrm{d}x)$.

Let us identify the limit of any convergent sub-sequence of $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$. For sake of clarity, denote X^i instead of $X^{\epsilon,i,N}$. For each $N\in\mathbb{N}^*$, $i\in\{1,...,N\}$, and $(s,u)\in[0,T]\times U$, let $\big[\hat{b}^{\epsilon,i,N}_s,\hat{a}^{\epsilon,i,N}_s\big] := [\hat{b}^\epsilon,\hat{a}^\epsilon]\big[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N_s\big](s,X^i_{[s]^N})$ and

$$\Big[\hat{b}^{i,N}_s,\hat{a}^{i,N}_s,\widehat{B}^{i,N}_s,\widehat{\Sigma}^{i,N}_s,\widehat{A}^{i,N}_s\Big](u) := \Big[\hat{b},\hat{a},\widehat{B},\widehat{\Sigma},\widehat{\Sigma}\widehat{\Sigma}^\top\Big]\big(s,X^i_{[s]^N},B^N,\phi(\mu^N),\zeta^N,m^N_s,\bar{\nu}^N_s,u\big).$$

By Itô's formula, for all $f\in C^\infty_b(\mathbb{R}^n)$ and $t\in[0,T]$

$$\begin{aligned}
\langle f,\vartheta^N_t\rangle &= \langle f,\vartheta^N_0\rangle + \frac{1}{N}\sum_{i=1}^N\int_0^t \nabla f(X^i_s)\widehat{\Sigma}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)\mathrm{d}W^i_s\\
&\quad+\frac{1}{N}\sum_{i=1}^N\int_0^t\Big[\nabla f(X^i_s)\widehat{B}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big) + \frac{1}{2}\mathrm{Tr}\Big[\widehat{A}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)\nabla^2 f(X^i_s)\Big]\Big]\mathrm{d}s\\
&= \langle f,\vartheta^N_0\rangle + \frac{1}{N}\sum_{i=1}^N\int_0^t \nabla f(X^i_s)\widehat{\Sigma}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)\mathrm{d}W^i_s\\
&\quad+\frac{1}{N}\sum_{i=1}^N\int_0^t\Big[\nabla f(X^i_{[s]^N})\widehat{B}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big) + \frac{1}{2}\mathrm{Tr}\Big[\widehat{A}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)\nabla^2 f(X^i_{[s]^N})\Big]\Big]\mathrm{d}s\\
&\quad+\frac{1}{N}\sum_{i=1}^N\int_0^t\big[\nabla f(X^i_s)-\nabla f(X^i_{[s]^N})\big]\widehat{B}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)\mathrm{d}s\\
&\qquad+\frac{1}{N}\sum_{i=1}^N\int_0^t\frac{1}{2}\mathrm{Tr}\Big[\widehat{A}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)\big[\nabla^2 f(X^i_s)-\nabla^2 f(X^i_{[s]^N})\big]\Big]\mathrm{d}s.
\end{aligned}$$

Observe that, for $s\in(t^N_k,t^N_{k+1})$, for each $i\neq j$, $[\widehat{B}]^{i,j}_s = [\widehat{A}]^{i,j}_s = 0$, where

$$[\widehat{B}]^{i,j}_s := \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\nabla f(X^i_{[s]^N})\Big\{\widehat{B}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N},m^N_s)(V^{i,N}_s)\big)-\hat{b}^{\epsilon,i,N}_s\Big\}\nabla f(X^j_{[s]^N})\Big\{\widehat{B}^{j,N}_s\big(N^\epsilon(X^j_{[s]^N},m^N_s)(V^{j,N}_s)\big)-\hat{b}^{\epsilon,j,N}_s\Big\}\Big]$$

and

$$[\widehat{A}]^{i,j}_s := \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big\{\widehat{A}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big) - \hat{a}^{\epsilon,i,N}_s\Big\}\nabla^2 f(X^i_{[s]^N})\Big\{\widehat{A}^{j,N}_s\big(N^\epsilon(X^j_{[s]^N}, m^N_s)(V^{j,N}_s)\big) - \hat{a}^{\epsilon,j,N}_s\Big\}\nabla^2 f(X^j_{[s]^N})\Big].$$

Indeed, by using the fact that: for all $(x, m, e) \in \mathbb{R}^n \times \mathcal{P}^n_U \times \{1, ..., N\}$, $\widehat{\mathbb{P}} \circ \big(N^\epsilon(x, m)(V^{e,K}_s)\big)^{-1}(\mathrm{d}u) = H^\epsilon(x, m)(\mathrm{d}u)$, and (V^i_s, V^j_s) are independent and independent of other variables, one has

$$\begin{aligned}[\widehat{B}]^{i,j}_s = \mathbb{E}^{\widehat{\mathbb{P}}}\Big[&\nabla f(X^i_{[s]^N})\Big\{\hat{b}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big) - \int_U \hat{b}^{i,N}_s(u)H^\epsilon(X^i_{[s]^N}, m^N_s)(\mathrm{d}u)\Big\}\\ &\nabla f(X^j_{[s]^N})\Big\{\hat{b}^{i,N}_s\big(N^\epsilon(X^j_{[s]^N}, m^N_s)(V^{j,N}_s)\big) - \int_U \hat{b}^{i,N}_s(u)H^\epsilon(X^j_{[s]^N}, m^N_s)(\mathrm{d}u)\Big\}\Big] = 0. \end{aligned} \tag{4.7.19}$$

By similar way, if we denote by $\Sigma^{i,N}_s := \Sigma\Big(s, X^i_{[s]^N}, B^N, \phi(\mu^N), \zeta^N, m^N_s, \bar{\nu}^N_s\Big)$, one finds

$$\begin{aligned}[\widehat{A}]^{i,j}_s &= \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\nabla^2 f(X^i_{[s]^N})\Big\{\Sigma^{i,N}_s\ \hat{a}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big)\ (\Sigma^{i,N}_s)^\top - \hat{a}^{\epsilon,i,N}_s\Big\}\\ &\qquad\qquad \nabla^2 f(X^j_{[s]^N})\Big\{\Sigma^{j,N}_s\ \hat{a}^{j,N}_s\big(N^\epsilon(X^j_{[s]^N}, m^N_s)(V^{j,N}_s)\big)\ (\Sigma^{j,N}_s)^\top - \hat{a}^{\epsilon,j,N}_s\Big\}\Big]\\ &= \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\nabla^2 f(X^i_{[s]^N})\Big\{\Sigma^{i,N}_s\ \int_U \hat{a}^{i,N}_s(u)H^\epsilon(X^i_{[s]^N}, m^N_s)(\mathrm{d}u)\ (\Sigma^{i,N}_s)^\top - \hat{a}^{\epsilon,i,N}_s\Big\}\\ &\qquad\qquad \nabla^2 f(X^j_{[s]^N})\Big\{\Sigma^{j,N}_s\ \int_U \hat{a}^{j,N}_s(u)H^\epsilon(X^j_{[s]^N}, m^N_s)(\mathrm{d}u)\ (\Sigma^{j,N}_s)^\top - \hat{a}^{\epsilon,j,N}_s\Big\}\Big]\\ &= \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\nabla^2 f(X^i_{[s]^N})\Big\{\hat{a}^{\epsilon,i,N}_s - \hat{a}^{\epsilon,i,N}_s\Big\}\nabla^2 f(X^j_{[s]^N})\Big\{\hat{a}^{\epsilon,j,N}_s - \hat{a}^{\epsilon,j,N}_s\Big\}\Big] = 0. \end{aligned} \tag{4.7.20}$$

By simple calculations,

$$\begin{aligned}&\langle f, \vartheta^N_t\rangle - \langle f, \vartheta^N_0\rangle - \int_0^t \int_{\mathbb{R}^n} \mathcal{A}^\epsilon_r f\big[B^N, \phi(\mu^N), \zeta^N, \overline{\Lambda}^N_r\big](x)\vartheta^N_{[r]^N}(\mathrm{d}x)\mathrm{d}r\\ &= \frac{1}{N}\sum_{i=1}^N \int_0^t \nabla f(X^i_{[s]^N})\widehat{\Sigma}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big)\mathrm{d}W^i_s + \int_0^t \Big[\nabla f(X^i_{[s]^N})\Big\{\widehat{B}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big) - \hat{b}^{\epsilon,i,N}_s\Big\}\\ &+ \frac{1}{2}\mathrm{Tr}\Big[\Big\{\widehat{A}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big) - \hat{a}^{\epsilon,i,N}_s\Big\}\nabla^2 f(s, X^i_{[s]^N})\Big] + \big[\nabla f(X^i_s) - \nabla f(X^i_{[s]^N})\big]\widehat{B}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big)\\ &\qquad + \frac{1}{2}\mathrm{Tr}\Big[\widehat{A}^{i,N}_s\big(N^\epsilon(X^i_{[s]^N}, m^N_s)(V^{i,N}_s)\big)\big[\nabla^2 f(X^i_s) - \nabla^2 f(X^i_{[s]^N})\big]\Big]\Big]\,\mathrm{d}s, \end{aligned}$$

consequently, there exists a constant $C>0$ (independent of N) such that

$$\begin{aligned}
&\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\langle f,\vartheta_t^N\rangle-\langle f,\vartheta_0^N\rangle-\int_0^t\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f\big[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}_r^N\big](x)\vartheta^N_{[r]^N}(\mathrm{d}x)\mathrm{d}r\Big|^2\Big]\\
&\le C\,\Bigg(\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\frac{1}{N}\sum_{i=1}^N\int_0^t\nabla f(X^i_{[s]^N})\widehat{\Sigma}_s^{i,N}\big(N^\epsilon(X^i_{[s]^N},m_s^N)(V_s^{i,N})\big)\mathrm{d}W_s^i\Big|^2\Big]\\
&\qquad\qquad+\int_0^t\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\frac{1}{N}\sum_{i=1}^N\nabla f(X^i_{[s]^N})\Big\{\widehat{B}_s^{i,N}\big(N^\epsilon(X^i_{[s]^N},m_s^N)(V_s^{i,N})\big)-\hat{b}_s^{\epsilon,i,N}\Big\}\Big|^2\Big]\mathrm{d}s\\
&+\int_0^t\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\frac{1}{N}\sum_{i=1}^N\frac{1}{2}\mathrm{Tr}\Big[\Big\{\widehat{A}_s^{i,N}\big(N^\epsilon(X^i_{[s]^N},m_s^N)(V_s^{i,N})\big)-\hat{a}_s^{\epsilon,i,N}\Big\}\nabla^2f(X^i_{[s]^N})\Big]\Big|^2\Big]\mathrm{d}s\\
&+\int_0^t\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\big[\nabla f(X_s^i)-\nabla f(X^i_{[s]^N})\big]\Big|^2+\Big|\frac{1}{2}\big[\nabla^2f(X_s^i)-\nabla^2f(X^i_{[s]^N})\big]\Big|^2\Big]\mathrm{d}s\Bigg).
\end{aligned}$$

By successively applying the results (4.7.19) and (4.7.20) and inequality (4.7.18), one gets a constant $M>0$ depending of (f,b,σ) (which changes from line to line) s.t.

$$\begin{aligned}
&\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\langle f,\vartheta_t^N\rangle-\langle f,\vartheta_0^N\rangle-\int_0^t\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f\big[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}_r^N\big](x)\vartheta^N_{[r]^N}(\mathrm{d}x)\mathrm{d}r\Big|^2\Big]\\
&\le M\Bigg(\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\frac{1}{N}\sum_{i=1}^N\int_0^t\nabla f(X_s^i)\widehat{\Sigma}_s^{i,N}\big(N^\epsilon(X^i_{[s]^N},m_s^N)(V_s^{i,N})\big)\mathrm{d}W_s^i\Big|^2\Big]+\frac{1}{2^N}+\frac{1}{N}\Bigg)\\
&\le M\Bigg(\frac{1}{N^2}\sum_{i=1}^N\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\nabla f(X_s^i)\widehat{\Sigma}_s^{i,N}\big(N^\epsilon(X^i_{[s]^N},m_s^N)(V_s^{i,N})\big)\Big|^2\mathrm{d}s\Big]+\frac{1}{2^N}+\frac{1}{N}\Bigg).
\end{aligned}\tag{4.7.21}$$

Remark that as ∇f and $\widehat{\Sigma}$ are bounded,

$$\frac{1}{N^2}\sum_{i=1}^N\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\nabla f(X_s^i)\widehat{\Sigma}_s^{i,N}\big(N^\epsilon(X^i_{[s]^N},m_s^N)(V_s^{i,N})\big)\Big|^2\mathrm{d}s\Big]\le M\frac{1}{N}.\tag{4.7.22}$$

Thanks to inequality (4.7.18), it is straightforward to verify that

$$\lim_{N\to\infty}\mathcal{W}_p\Big(\widehat{\mathcal{L}^{\mathbb{P}}}\big(\vartheta^N,\vartheta^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N,B^N\big),\widehat{\mathcal{L}^{\mathbb{P}}}\big(\vartheta^N,(\vartheta^N_{[t]^N})_{t\in[0,T]},\phi(\mu^N),\zeta^N,\overline{\Lambda}^N,B^N\big)\Big)=0.\tag{4.7.23}$$

Let $\mathrm{P}^\infty\in\mathcal{P}\big(\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\big((\mathcal{P}^n_U)^2\big)\times\mathcal{C}^\ell\big)$ be the limit of any sub-sequence $(\mathrm{P}^{N_k})_{k\in\mathbb{N}^*}$ of $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$, and denote by $(\beta^\vartheta,\beta^\mu,\beta^\zeta,\overline{\beta},B)$ the canonical process on $\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\big((\mathcal{P}^n_U)^2\big)\times\mathcal{C}^\ell$. By combining inequalities (4.7.21) and (4.7.22) with result (4.7.23), by passing to the limit, using continuity of coefficients, given $\epsilon>0$: for all $(t,f)\in[0,T]\times C_b^\infty(\mathbb{R}^n)$

$$\begin{aligned}
&\lim_k\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\Big|\langle f,\vartheta_t^{N_k}\rangle-\langle f,\vartheta_0^{N_k}\rangle-\int_0^t\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f\big[B^{N_k},\phi(\mu^{N_k}),\zeta^{N_k},\overline{\Lambda}_r^{N_k}\big](x)\vartheta^{N_k}_{[r]^{N_k}}(\mathrm{d}x)\mathrm{d}r\Big|^2\Big]\\
&=\mathbb{E}^{\mathrm{P}^\infty}\Big[\Big|\langle f,\beta_t^\vartheta\rangle-\langle f,\nu\rangle-\int_0^t\Big[\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f\big[B,\beta^\mu,\beta^\zeta,\overline{\beta}_r\big](x)\beta_r^\vartheta(\mathrm{d}x)\Big]\mathrm{d}r\Big|^2\Big]=0.
\end{aligned}$$

Therefore, after taking a countable family of (f,t), one gets: for all $(t,f)\in[0,T]\times C_b^\infty(\mathbb{R}^n)$

$$\langle f,\beta_t^\vartheta\rangle=\langle f,\nu\rangle+\int_0^t\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f\big[B,\beta^\mu,\beta^\zeta,\overline{\beta}_r\big](x)\beta_r^\vartheta(\mathrm{d}x)\mathrm{d}r,\ \text{for all } t\in[0,T],\ \mathrm{P}^\infty\text{–a.e.}$$

from this equality, we can show the previous equality holds true for all $f \in C_b^2(\mathbb{R}^n)$. For each $\epsilon > 0$, by uniqueness $\beta^\vartheta := \Phi^\epsilon\big(B, \beta^\mu, \beta^\zeta, \overline{\beta}\big)$ with $\Phi^\epsilon : \mathcal{C}^\ell \times \mathcal{C}^n_{\mathcal{W}} \times \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}\big((\mathcal{P}^n_U)^2\big) \to \mathcal{C}^n_{\mathcal{W}}$ a Borel function used in (4.7.14). Notice that, by assumptions (4.7.10),

$$\mathrm{P}^\infty \circ \big(\beta^\mu, \beta^\zeta, \overline{\beta}, B\big)^{-1} = \lim_k \widehat{\mathbb{P}} \circ \big(\phi(\mu^{N_k}), \zeta^{N_k}, \overline{\Lambda}^{N_k}, B^{N_k}\big)^{-1} = \mathbb{Q} \circ \big(\phi(\mu), \zeta, \overline{\Lambda}, B\big)^{-1} \text{ in } \mathcal{W}_p.$$

This result is enough to deduce that $\mathrm{P}^\infty = \mathbb{Q} \circ \big(\mu^\epsilon, \phi(\mu), \zeta, \overline{\Lambda}, B\big)^{-1}$. This is true for any limit P^∞ for any sub-sequence of $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$, therefore

$$\lim_{N\to\infty} \widehat{\mathbb{P}} \circ \big(\vartheta^N, \phi(\mu^N), \zeta^N, \overline{\Lambda}^N, B^N\big)^{-1} = \mathbb{Q} \circ \big(\mu^\epsilon, \phi(\mu), \zeta, \overline{\Lambda}, B\big)^{-1} \text{ in } \mathcal{W}_p. \tag{4.7.24}$$

Step 3 : **Last approximation**: To finish, now, let us define $\widehat{X}^{\epsilon,i,N} := \widehat{X}^i$ the strong solution of

$$\widehat{X}^i_t = \xi^i + \int_0^t \hat{b}\big(r, \widehat{X}^i_r, B^N, \phi(\widehat{\mu}^N), \zeta^N, \widehat{m}^N_r, \bar{\nu}^N_r, \alpha^i_r\big)\mathrm{d}r + \int_0^t \hat{\sigma}\big(r, \widehat{X}^i_r, B^N, \phi(\widehat{\mu}^N), \zeta^N, \widehat{m}^N_r, \bar{\nu}^N_r, \alpha^i_r\big)\mathrm{d}W^i_r, \text{ for all } t \in [0,T]$$

where

$$\alpha^i_t := N^\epsilon(X^i_{t^N_k}, m^N_t)(V^{i,N}_t) \text{ for all } t \in [t^N_k, t^N_{k+1}[,\ \widehat{m}^N_t(\mathrm{d}x, \mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{X}^i_t,\, \alpha^i_t)}(\mathrm{d}x, \mathrm{d}u) \text{ and } \widehat{\mu}^N_t(\mathrm{d}x) := \widehat{m}^N_t(\mathrm{d}x, U),$$

recall that $(X^1, ..., X^N)$ are defined in (4.7.15). It is straightforward to check that: there exists a constant $\mathrm{D} > 0$ (independent of ϵ and N)

$$\sup_{i\in\{1,\dots,N\}} \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\big|\widehat{X}^i_t - \widehat{X}^i_s\big|^p\Big] \le \mathrm{D}|t-s|, \text{ for all } (t,s) \in [0,T]\times[0,T]. \tag{4.7.25}$$

By Bukholder-Davis-Gundy inequality, lipschitz property of coefficients and previous inequality (4.7.25),

$$
\begin{aligned}
&\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{s\in[0,t]}|\widehat{X}^i_s - X^i_s|^p\Big] \\
&\le \hat{D}\Bigg(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\hat{b}\big(r,\widehat{X}^i_r,B^N,\phi(\widehat{\mu}^N),\zeta^N,\widehat{m}^N_r,\bar{\nu}^N_r,\alpha^i_r\big)-\widehat{B}\Big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,N^\epsilon(X^i_{[r]^N},m^N_r)(V^{i,N}_r)\Big)\Big|^p\mathrm{d}r\Big] \\
&\quad+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\hat{\sigma}\big(r,\widehat{X}^i_r,B^N,\phi(\widehat{\mu}^N),\zeta^N,\widehat{m}^N_r,\bar{\nu}^N_r,\alpha^i_r\big)-\widehat{\Sigma}\Big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,N^\epsilon(X^i_{[r]^N},m^N_r)(V^{i,N}_r)\Big)\Big|^p\mathrm{d}r\Big]\Bigg) \\
&\le \hat{D}\Bigg(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\hat{b}\big(r,\widehat{X}^i_r,B^N,\phi(\widehat{\mu}^N),\zeta^N,\widehat{m}^N_r,\bar{\nu}^N_r,\alpha^i_r\big)-\hat{b}(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,\alpha^i_r)\Big|^p\Big] \\
&\quad+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|b(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,\alpha^i_r)-\widehat{B}\Big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,N^\epsilon(X^i_{[r]^N},m^N_r)(V^{i,N}_r)\Big)\Big|^p\mathrm{d}r\Big] \\
&\quad+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\hat{\sigma}\big(r,\widehat{X}^i_r,B^N,\phi(\widehat{\mu}^N),\zeta^N,\widehat{m}^N_r,\bar{\nu}^N_r,\alpha^i_r\big)-\hat{\sigma}(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,\alpha^i_r)\Big|^p\mathrm{d}r\Big] \\
&\quad+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\hat{\sigma}(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,\alpha^i_r)-\widehat{\Sigma}\Big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,N^\epsilon(X^i_{[r]^N},m^N_r)(V^{i,N}_r)\Big)\Big|^p\mathrm{d}r\Big]\Bigg) \\
&\le \hat{D}\Bigg(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|\hat{b}^\epsilon\big[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N_r\big](r,X^i_{[r]^N})-\int_U\hat{b}\big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,u\big)H^\epsilon(m^N,X^i_{[s]^N})(\mathrm{d}u)\Big|^p\mathrm{d}r\Big] \\
&\quad+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\Big|1-\Sigma\Big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r\Big)\Big|^p\mathrm{d}r+\int_0^t\mathcal{W}_p\Big(\frac{1}{N}\sum_{i=1}^N\delta_{\left(X^i_{[r]^N},\,\alpha^i_r\right)}(\mathrm{d}x,\mathrm{d}u),m^N_r(\mathrm{d}x,\mathrm{d}u)\Big)^p\mathrm{d}r\Big] \\
&\quad+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p(\phi_{e'}(\vartheta^N),\phi_{e'}(\mu^N))+\int_0^t\sup_{e\in[0,r]}\big|\widehat{X}^i_e-X^i_e\big|^p\mathrm{d}r\Big]+\frac{1}{2^N}\Bigg),
\end{aligned}
$$

then by Gronwall lemma

$$
\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{t\in[0,T]}|\widehat{X}^i_t-X^i_t|^p\Big]\le\hat{D}\Bigg(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p\Big(\phi_{e'}(\vartheta^N),\phi_{e'}(\mu^N)\Big)\Big]+\frac{1}{2^N}+E^{\epsilon,i,N}+C^{\epsilon,N}\Bigg)
$$

where $C^{\epsilon,N}:=\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\Big(\frac{1}{N}\sum_{i=1}^N\delta_{\left(X^i_{[r]^N},\,\alpha^i_r\right)}(\mathrm{d}x,\mathrm{d}u),m^N_r(\mathrm{d}x,\mathrm{d}u)\Big)^p\mathrm{d}r\Big]$, and

$$
\begin{aligned}
&E^{\epsilon,i,N} \\
&:=\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Bigg[\int_0^T\Big|\big[\hat{b}^\epsilon,\hat{a}^\epsilon\big]\big[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N_r\big](r,X^i_{[r]^N})-\int_U\big[\hat{b},\hat{a}\big]\big(r,X^i_{[r]^N},B^N,\phi(\mu^N),\zeta^N,m^N_r,\bar{\nu}^N_r,u\big)H^\epsilon(X^i_{[r]^N},m^N_r)(\mathrm{d}u)\Big|^p\mathrm{d}r\Bigg].
\end{aligned}
$$

Firstly, thanks to results (4.7.24) and Lemma 4.7.5, one gets

$$
\lim_{\epsilon\to0}\lim_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p\Big(\phi_{e'}(\vartheta^N),\phi_{e'}(\mu^N)\Big)\Big]=\lim_{\epsilon\to0}\mathbb{E}^{\mathbb{Q}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p\Big(\phi_{e'}(\mu^\epsilon),\phi_{e'}(\mu)\Big)\Big]=0. \tag{4.7.26}
$$

Secondly, after calculations, it is straightforward to deduce that

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^N E^{\epsilon,i,N}=\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Bigg[\int_0^T\int_{\mathbb{R}^n}\int_{(\mathcal{P}^n_U)^2}\Big|\int_{U\times\mathbb{R}^n}\big[\hat{b},\hat{a}\big]\big(r,y,B^N,\phi(\mu^N),\zeta^N,m,\bar{\nu},u\big)\frac{G_\epsilon(x-y)}{(m(U,\mathrm{d}z))^{(\epsilon)}(x)}m(\mathrm{d}u,\mathrm{d}y) \\
-\int_{U\times\mathbb{R}^n}\big[\hat{b},\hat{a}\big]\big(r,x,B^N,\phi(\mu^N),\zeta^N,m,\bar{\nu},u\big)\frac{G_\epsilon(x-y)}{(m(U,\mathrm{d}z))^{(\epsilon)}(x)}m(\mathrm{d}u,\mathrm{d}y)\Big|^p\overline{\Lambda}^N_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\vartheta^N_{[r]^N}(\mathrm{d}x)\mathrm{d}r\Bigg].
\end{aligned}
$$

By regularity of coefficients (Assumption 4.1.1 and $(\hat{b},\hat{\sigma})$ bounded), the results (4.7.24) and (4.7.23) allow to get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}E^{\epsilon,i,N}\leq\mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T\int_{\mathbb{R}^n}\int_{(\mathcal{P}_U^n)^2}\int_{U\times\mathbb{R}^n}\Big|\big[\hat{b},\hat{a}\big]\big(r,y,B,\phi(\mu),\zeta,m,\bar{\nu},u\big)\\ -\big[\hat{b},\hat{a}\big]\big(r,x,B,\phi(\mu),\zeta,m,\bar{\nu},u\big)\Big|^p\frac{G_\epsilon(x-y)}{(m(U,\mathrm{d}z))^{(\epsilon)}(x)}m(\mathrm{d}u,\mathrm{d}y)\overline{\Lambda}_r(\mathrm{d}m,\mathrm{d}\bar{\nu})\mu_r^\epsilon(\mathrm{d}x)\mathrm{d}r\Bigg],$$

then, by Lemma 4.7.5, $\lim\limits_{\epsilon\to0}\lim\limits_{N\to\infty}\dfrac{1}{N}\sum\limits_{i=1}^{N}E^{\epsilon,i,N}=0$.

Next, let us define the variable

$$\Upsilon_r^N(\mathrm{d}e',\mathrm{d}e)\mathrm{d}r:=\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\delta_{\left(\overline{m}_r^N,m_r^N\right)}(\mathrm{d}e',\mathrm{d}e)\mathrm{d}r\Big]\in\mathbb{M}\big((\mathcal{P}_U^n)^2\big),\ \text{where}\ \overline{m}_r^N(\mathrm{d}x,\mathrm{d}u):=\frac{1}{N}\sum_{i=1}^{N}\delta_{\left(X^i_{[r]^N},\,\alpha_r^i\right)}(\mathrm{d}x,\mathrm{d}u),$$

It is easy to check that the sequence $(\Upsilon^N)_{N\in\mathbb{N}^*}$ is relatively compact for the Wasserstein metric $\mathcal{W}_p$. Denote by Υ^∞ the limit of any sub-sequence $(\Upsilon^{N_k})_{k\in\mathbb{N}^*}$. Let $Q\in\mathbb{N}^*$, $(f^q)_{q\in\{1,\dots Q\}}:\mathbb{R}^n\times U\to\mathbb{R}^Q$ be bounded continuous functions and $g:[0,T]\times\mathcal{P}_U^n\to\mathbb{R}$. One has

$$\int_0^T\int_{(\mathcal{P}_U^n)^2}\prod_{q=1}^{Q}\langle f^q,e'\rangle g(t,e)\Upsilon_t^\infty(\mathrm{d}e',\mathrm{d}e)\mathrm{d}t=\mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T\int_{\mathcal{P}_U^n}\prod_{q=1}^{Q}\int_{\mathbb{R}^n}\langle f^q(x,\cdot),H^\epsilon(x,m)\rangle\mu_t^\epsilon(\mathrm{d}x)g(t,m)\overline{\Lambda}_t(\mathrm{d}m,\mathcal{P}_U^n)\mathrm{d}t\Bigg].$$

We prove this equality when $Q=2$, the case $Q\in\mathbb{N}^*$ follows immediately. Indeed,

$$\int_0^T\int_{(\mathcal{P}_U^n)^2}\langle f^1,e'\rangle\langle f^2,e\rangle g(t,e)\Upsilon_t^\infty(\mathrm{d}e',\mathrm{d}e)\mathrm{d}t=\lim_k\frac{1}{N_k}\frac{1}{N_k}\sum_{i,j=1}^{N_k}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\big(X^i_{[t]^{N_k}},\alpha_t^i\big)f^2\big(X^j_{[t]^{N_k}},\alpha_t^j\big)g(t,m_t^{N_k})\mathrm{d}t\Big]$$

$$=\lim_k\Bigg(\frac{1}{N_k}\frac{1}{N_k}\sum_{i\neq j}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\int_U f^1\Big(X^i_{[t]^{N_k}},u\Big)H^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(\mathrm{d}u)\int_U f^2\Big(X^j_{[t]^{N_k}},u\Big)H^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(\mathrm{d}u)g(t,m_t^{N_k})\mathrm{d}t\Big]$$

$$+\frac{1}{N_k}\frac{1}{N_k}\sum_{i=1}^{N_k}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\Big(X^i_{[t]^{N_k}},N^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(V_t^{i,N_k})\Big)f^2\Big(X^j_{[t]^{N_k}},N^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(V_t^{i,N_k})\Big)g(t,m_t^{N_k})\mathrm{d}t\Big]\Bigg)$$

$$=\lim_k\Bigg(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\int_{\mathbb{R}^n}\int_U f^1(x,u)H^\epsilon(x,m_t^{N_k})(\mathrm{d}u)\vartheta^{N_k}_{[t]^{N_k}}(\mathrm{d}x)\int_{\mathbb{R}^n}\int_U f^2(y,u)H^\epsilon(y,m_t^{N_k})(\mathrm{d}u)\vartheta^{N_k}_{[t]^{N_k}}(\mathrm{d}y)g(t,m_t^{N_k})\mathrm{d}t\Big]$$

$$-\frac{1}{N_k}\frac{1}{N_k}\sum_{i=1}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\int_U f^1\Big(X^i_{[t]^{N_k}},u\Big)H^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(\mathrm{d}u)\int_U f^2\Big(X^i_{[t]^{N_k}},u\Big)H^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(\mathrm{d}u)g(t,m_t^{N_k})\mathrm{d}t\Big]$$

$$+\frac{1}{N_k}\frac{1}{N_k}\sum_{i=1}^{N_k}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\Big(X^i_{[t]^{N_k}},N^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(V_t^{i,N_k})\Big)f^2\Big(X^i_{[t]^{N_k}},N^\epsilon(X^i_{[t]^{N_k}},m_t^{N_k})(V_t^{i,N_k})\Big)g(t,m_t^{N_k})\mathrm{d}t\Big]\Bigg)$$

$$=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T\int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n}\int_U f^1(x,u)H^\epsilon(x,m)(\mathrm{d}u)\mu_t^\epsilon(\mathrm{d}x)\int_{\mathbb{R}^n}\int_U f^2(y,u)H^\epsilon(y,m)(\mathrm{d}u)\mu_t^\epsilon(\mathrm{d}y)g(t,m)\overline{\Lambda}_t(\mathrm{d}m,\mathcal{P}_U^n)\mathrm{d}t\Big],$$

where the fourth equality is true because of the same argument used in (4.7.19) and (4.7.20) i.e. for all $(s,v)\in(t_k^{N_l},t_{k+1}^{N_l})\times\{1,\dots,N_l\}$, $\widehat{\mathbb{P}}\circ\big(N^\epsilon(x,m)(V_s^{v,N_l})\big)^{-1}(\mathrm{d}u)=H^\epsilon(x,m)(\mathrm{d}u)$, and for $i\neq j$ (V_s^i,V_s^j) are independent and independent of other variables, and the last equality follows from (4.7.24) and (4.7.23), and the terms starting with $\frac{1}{(N_l)^2}\sum_{i=1}^{N_l}$ go to zero because (f^1,f^2,g) are bounded. Hence,

$$\Upsilon_t^\infty(\mathrm{d}e',\mathrm{d}e)\mathrm{d}t=\widehat{\Upsilon}_t(\mathrm{d}e',\mathrm{d}e)\mathrm{d}t,\ \text{where}\ \widehat{\Upsilon}_t(\mathrm{d}e',\mathrm{d}e)\mathrm{d}t:=\mathbb{E}^{\mathbb{Q}}\Big[\delta_{\left(H^\epsilon(x,e)(\mathrm{d}u)\mu_t^\epsilon(\mathrm{d}x)\right)}(\mathrm{d}e')\overline{\Lambda}_t(\mathrm{d}e,\mathcal{P}_U^n)\mathrm{d}t\Big],$$

this is true for any limit Υ^∞ of any-sub-sequence. Therefore the sequence $(\Upsilon^N)_{N\in\mathbb{N}^*}$ converges towards $\widehat{\Upsilon}$ for the wasserstein metric $\mathcal{W}_p$. Then, to finish, by Lemma 4.7.5,

$$\begin{aligned}\lim_{\epsilon\to 0}\lim_{N\to\infty} C^{\epsilon,N} &= \lim_{\epsilon\to 0}\lim_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\Big(\frac{1}{N}\sum_{i=1}^N \delta_{\left(X^i_{[r]^N},\,\alpha^i_r\right)}(\mathrm{d}x,\mathrm{d}u), m^N_r\Big)^p \mathrm{d}r\Big]\\ &= \lim_{\epsilon\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T\int_{\mathcal{P}^n_U}\mathcal{W}_p\Big(H^\epsilon(x,m)(\mathrm{d}u)\mu^\epsilon_t(\mathrm{d}x), m\Big)^p\overline{\Lambda}_t(\mathrm{d}m,\mathcal{P}^n_U)\mathrm{d}t\Big] = 0.\end{aligned}$$

All these results allow to deduce that $\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{t\in[0,T]}\big|\widehat{X}^{\epsilon,i,N}_t - X^{\epsilon,i,N}_t\big|^p\Big] = 0$. As

$$\begin{aligned}&\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p(\widehat{m}^N_t, m^N_t)^p\mathrm{d}r\Big]\\ &\le \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\big(\widehat{m}^N_t(\mathrm{d}x,\mathrm{d}u), \frac{1}{N}\sum_{i=1}^N\delta_{\left(X^i_{[t]^N},\,\alpha^i_t\right)}(\mathrm{d}x,\mathrm{d}u)\big)^p\mathrm{d}r\Big] + \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\big(\frac{1}{N}\sum_{i=1}^N\delta_{\left(X^i_{[t]^N},\,\alpha^i_t\right)}(\mathrm{d}x,\mathrm{d}u), m^N_t(\mathrm{d}x,\mathrm{d}u)\big)^p\mathrm{d}r\Big]\\ &\le \frac{1}{N}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\big|\widehat{X}^{\epsilon,i,N,K}_t - X^{\epsilon,i,N}_{[t]^N}\big|^p\mathrm{d}t\Big] + \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\big(\frac{1}{N}\sum_{i=1}^N\delta_{\left(X^i_{[t]^N},\,\alpha^i_t\right)}(\mathrm{d}x,\mathrm{d}u), m^N_t(\mathrm{d}x,\mathrm{d}u)\big)^p\mathrm{d}r\Big]\\ &\le \frac{1}{N}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\big|\widehat{X}^{\epsilon,i,N}_t - X^{\epsilon,i,N}_t\big|^p\mathrm{d}t\Big] + \frac{1}{2^N} + \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\big(\frac{1}{N}\sum_{i=1}^N\delta_{\left(X^i_{[t]^N},\,\alpha^i_t\right)}(\mathrm{d}x,\mathrm{d}u), m^N_t(\mathrm{d}x,\mathrm{d}u)\big)^p\mathrm{d}r\Big]\end{aligned}$$

then $\lim_{\epsilon\to 0}\lim_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p(\widehat{m}^N_t, m^N_t)^p\mathrm{d}r\Big] = 0$, similarly, using (4.7.26),

$$\begin{aligned}&\lim_{\epsilon\to 0}\lim_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p(\phi_{e'}(\widehat{\mu}^N),\phi_{e'}(\mu^N))\Big]\\ &\qquad\le \lim_{\epsilon\to 0}\lim_{N\to\infty}\Big(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p(\phi_{e'}(\widehat{\mu}^N),\phi_{e'}(\vartheta^N))\Big] + \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p(\phi_{e'}(\vartheta^N),\phi_{e'}(\mu^N))\Big]\Big)\\ &\qquad\le K\lim_{\epsilon\to 0}\lim_{N\to\infty}\Big(\frac{1}{N}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{t\in[0,T]}\big|\widehat{X}^{\epsilon,i,N}_t - X^{\epsilon,i,N}_t\big|^p\Big] + \frac{1}{2^N} + \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{e'\in[0,T]}\mathcal{W}_p(\phi_{e'}(\vartheta^N),\phi_{e'}(\mu^N))\Big]\Big) = 0.\end{aligned}$$

All previous result combined with measurability property (4.7.17) allowed to say $(\alpha^1,...,\alpha^N)$ and $(\widehat{X}^1,...,\widehat{X}^N)$ are the controls and the processes we are looking for.

□

In Proposition 4.7.7, in fact, instead of interaction processes of type (4.7.12), it is possible to use a sequence of McKean-Vlasov processes and obtain similar result. Let us assume conditions and inputs previously mentioned for Proposition 4.7.7 are satisfied. Let W be a $(\widehat{\mathbb{P}},\widehat{\mathbb{F}})$–Brownian motion, ξ be a $\widehat{\mathcal{F}}_0$–random variable with $\mathcal{L}^{\widehat{\mathbb{P}}}(\xi)=\nu$, and Z be a uniform variable independent of (ξ,W). In addition,

$$\big(\psi(\mu^N),\zeta^N,\overline{\Lambda}^N,B^N\big)_{N\in\mathbb{N}^*}\ \text{ are }\ \widehat{\mathbb{P}}\text{–independent of }\big(W,\xi,Z\big). \tag{4.7.27}$$

For each $N\in\mathbb{N}^*$, define the filtrations $\widehat{\mathbb{F}}^N := (\widehat{\mathcal{F}}^N_t)_{t\in[0,T]}$ and $\widehat{\mathbb{G}} := (\widehat{\mathcal{G}}^N_t)_{t\in[0,T]}$ by

$$\widehat{\mathcal{F}}^N_t := \sigma\Big\{\xi,\overline{\Lambda}^N_{t\wedge\cdot},\phi_{t\wedge\cdot}(\mu^N),\zeta^N_{t\wedge\cdot},W_{t\wedge\cdot},B^N_{t\wedge\cdot},Z\Big\}\ \text{ and }\ \widehat{\mathcal{G}}^N_t := \sigma\big\{\psi_{t\wedge\cdot}(\mu^N),\zeta^N_{t\wedge\cdot},\overline{\Lambda}^N_{t\wedge\cdot},B^N_{t\wedge\cdot}\big\},\ \text{for all }\ t\in[0,T].$$

We provide, now, approximations by McKean-Vlasov processes. The proofs of the next Proposition 4.7.9 and Proposition 4.7.10 are left in Appendix 4.9.1.

Proposition 4.7.9. *There exists a sequence of processes $(\alpha^N)_{N\in\mathbb{N}^*}$ satisfying: for each $N\in\mathbb{N}^*$, α^N is $\widehat{\mathbb{F}}^N$–predictable, such that if X^N is the unique strong solution of: $\mathbb{E}^{\widehat{\mathbb{P}}}[\|X^N\|^{p'}]<\infty$, for all $t\in[0,T]$,*

$$X_t^N=\xi+\int_0^t\hat{b}\big(r,X_r^N,B^N,\phi(\widehat{\mu}^N),\zeta^N,\widehat{m}_r^N,\bar{\nu}_r^N,\alpha_r^N\big)\mathrm{d}r+\int_0^t\hat{\sigma}\big(r,X_r^N,B^N,\phi(\widehat{\mu}^N),\zeta^N,\widehat{m}_r^N,\bar{\nu}_r^N,\alpha_r^N\big)\mathrm{d}W_r,\ \widehat{\mathbb{P}}\text{–a.e.},\tag{4.7.28}$$

where $\widehat{m}_t^N:=\mathcal{L}^{\widehat{\mathbb{P}}}\big(X_t^N,\alpha_t^N\big|\widehat{\mathcal{G}}_t^N\big)$ and $\widehat{\mu}_t^N:=\mathcal{L}^{\widehat{\mathbb{P}}}\big(X_t^N\big|\widehat{\mathcal{G}}_t^N\big)$, then for the sub-sequence $(N_k)_{k\in\mathbb{N}^*}$ given in Proposition 4.7.7,

$$\lim_{k\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\widehat{m}_t^{N_k},m_t^{N_k}\big)^p\mathrm{d}t+\sup_{t\in[0,T]}\mathcal{W}_p\Big(\phi_t(\widehat{\mu}^{N_k}),\phi_t(\mu^{N_k})\Big)\Big]=0,$$

and if $\widehat{\Lambda}_s(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s:=\delta_{(\widehat{m}_s^{N_k},\bar{\nu}_s^{N_k})}(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s$,

$$\lim_{k\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{\mu}^{N_k},\zeta^{N_k},\widehat{\Lambda},B^{N_k}\Big)=\mathcal{L}^{\mathbb{Q}}\big(\mu,\zeta,\overline{\Lambda},B\big),\ \textit{in}\ \mathcal{W}_p.\tag{4.7.29}$$

Another useful approximation Using roughly the same arguments as those used in the proof of the Proposition 4.7.7, another approximation result can be provided. This can be seen as another version of Proposition 4.7.9 where the sequence $(\overline{\Lambda}^N)_{N\in\mathbb{N}^*}$ is not necessarily a subset of $\mathbb{M}_0\big((\mathcal{P}_U^n)^2\big)$ and the control that achieves the approximation is a probability measure.

Proposition 4.7.10. *Let us stay in the context of* Proposition 4.7.9 *with $(\overline{\Lambda}^N)_{N\in\mathbb{N}^*}$ not necessarily a subset of $\mathbb{M}_0\big((\mathcal{P}_U^n)^2\big)$. There exists $(\beta^N)_{N\in\mathbb{N}^*}$ a sequence of $\mathcal{P}(U)$–valued $(\widehat{\mathcal{F}}_t\otimes\mathcal{B}(\mathcal{P}_U^n))_{t\in[0,T]}$–predictable processes such that if $(X_t^N)_{t\in[0,T]}:=(X_t)_{t\in[0,T]}$ is the unique strong solution of: $\mathbb{E}^{\widehat{\mathbb{P}}}[\|X^N\|^{p'}]<\infty$, for all $t\in[0,T]$*

$$\begin{aligned}X_t=\xi&+\int_0^t\int_{(\mathcal{P}_U^n)^2}\int_U\hat{b}\big(r,X_r,B^N,\phi(\eta^N),\zeta^N,\widehat{\mathbf{m}}_r^N[m],\bar{\nu},u\big)\beta_r^N(m)(\mathrm{d}u)\ \overline{\Lambda}_r^N(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r\\&+\int_0^t\Big(\int_{(\mathcal{P}_U^n)^2}\int_U\hat{\sigma}\hat{\sigma}^\top\big(r,X_r,B^N,\phi(\eta^N),\zeta^N,\widehat{\mathbf{m}}_r^N[m],\bar{\nu},u\big)\beta_r^N(m)(\mathrm{d}u)\ \overline{\Lambda}_r^N(\mathrm{d}m,\mathrm{d}\bar{\nu})\Big)^{1/2}\mathrm{d}W_r,\ \widehat{\mathbb{P}}\text{–a.e.},\end{aligned}$$

where

$$\widehat{\mathbf{m}}_t^N[m](\mathrm{d}x,\mathrm{d}u):=\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\beta_t^N(m)(\mathrm{d}u)\delta_{X_t^N}(\mathrm{d}x)\Big|\widehat{\mathcal{G}}_t^N\Big]\ \textit{and}\ \widehat{\mu}_t^N:=\mathcal{L}^{\widehat{\mathbb{P}}}(X_t^N|\widehat{\mathcal{G}}_t^N)\ \ \textit{for all}\ t\in[0,T],$$

then, one has, for a sub-sequence $(N_j)_{j\in\mathbb{N}^*}\subset\mathbb{N}^*$,

$$\lim_{j\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^T\int_{\mathcal{P}_U^n}\mathcal{W}_p\big(\widehat{\mathbf{m}}_r^{k_j}[m],m\big)\overline{\Lambda}_r^{N_j}(\mathrm{d}m,\mathcal{P}_U^n)\mathrm{d}r\Big]=0\ \textit{and}\ \lim_{j\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}}\Big[\sup_{s\in[0,T]}\mathcal{W}_p\Big(\phi_s(\widehat{\mu}^{N_j}),\phi_s(\mu^{N_j})\Big)\Big]=0,$$

in addition if $\widehat{\Lambda}_s^N(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s:=\int_{\mathcal{P}_U^n}\delta_{\widehat{\mathbf{m}}_s^N[e]}(\mathrm{d}m)\overline{\Lambda}_s^N(\mathrm{d}e,\mathrm{d}\bar{\nu})\mathrm{d}s$,

$$\lim_{j\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{\mu}^{N_j},\zeta^{N_j},\widehat{\Lambda}^{N_j},B^{N_j}\Big)=\mathcal{L}^{\mathbb{Q}}\Big(\mu,\phi(\mu),\zeta,\overline{\Lambda},B\Big),\ \textit{in}\ \mathcal{W}_p.\tag{4.7.30}$$

Remark 4.7.11. *With exactly the same proof, an important observation is the following: if the coefficients functions $(\hat{b},\hat{\sigma})$ are of the form of type*

$$\big(\hat{b},\hat{\sigma}\hat{\sigma}^\top\big)(t,x,\mathbf{b},\pi,\beta,m,\bar{\nu},u):=\big(\hat{b}^\star,\hat{a}^\star\big)(t,\mathbf{b},\pi,\beta,\bar{\nu})+\big(\hat{b}^\circ,\hat{a}^\circ\big)(t,x,\mathbf{b},\pi,\beta,m,u),$$

where $(\hat{b}^\star,\hat{a}^\star,\hat{b}^\circ,\hat{a}^\circ)$ are bounded continuous functions, we can replace the convergence assumptions (4.7.10) by

$$\lim_{N\to\infty}\mathcal{W}_{p'}\Big(\frac{1}{N}\sum_{i=1}^N\nu^i,\nu\Big)=0\ \textit{and}\ \lim_{N\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\Big(\phi(\mu^N),\zeta^N,\Lambda^{\circ,N},\Lambda^{\star,N},\ B^N\Big)=\mathcal{L}^{\mathbb{Q}}\big(\phi(\mu),\zeta,\Lambda^\circ,\Lambda^\star,B\big),\ \ \textit{in}\ \mathcal{W}_p,\tag{4.7.31}$$

with $\Lambda^{\circ,N} := \overline{\Lambda}^N_t(\mathrm{d}m, \mathcal{P}^n_U)\mathrm{d}t$, $\Lambda^{\star,N} := \overline{\Lambda}^N_t(\mathcal{P}^n_U, \mathrm{d}\bar{\nu})\mathrm{d}t$, $\Lambda^{\circ} := \overline{\Lambda}_t(\mathrm{d}m, \mathcal{P}^n_U)\mathrm{d}t$, *and* $\Lambda^{\star} := \overline{\Lambda}_t(\mathcal{P}^n_U, \mathrm{d}\bar{\nu})\mathrm{d}t$. *And then, in* Proposition 4.7.7, Proposition 4.7.9 *and* 4.7.10, *the convergence results* (4.7.13), (4.7.29) *and* (4.7.30) *are replaced by*

$$\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{\mu}^{N_j}, \zeta^{N_j}, \widehat{\Lambda}^{N_j}_t(\mathrm{d}m, \mathcal{P}^n_U)\mathrm{d}t, \widehat{\Lambda}^{N_j}_t(\mathcal{P}^n_U, \mathrm{d}\bar{\nu})\mathrm{d}t, B^{N_j}\Big) = \mathcal{L}^{\mathbb{Q}}\Big(\mu, \phi(\mu), \zeta, \Lambda^{\circ}, \Lambda^{\star}, B\Big), \text{ in } \mathcal{W}_p.$$

In other words, when the variables $(m, \bar{\nu})$ *of* $(\hat{b}, \hat{\sigma}\hat{\sigma}^\top)$ *are "separated", we just need separated condition on* $(\overline{\Lambda}^N)_{N\in\mathbb{N}^*}$ *of type* (4.7.31).

4.7.3 Some compactness results

Assumption 4.7.12. *Here, we assume that* $\ell = 0$, *and*

$$\sigma(t, x, \pi, m, u) = \tilde{\sigma}(t, x), \text{ for all } (t, x, \pi, m, u) \in [0, T] \times \mathbb{R}^n \times \mathcal{C}^n_{\mathcal{W}} \times \mathcal{P}^n_U \times U.$$

For simplicity, we will note σ *instead of* $\tilde{\sigma}$.

Let $p' > p$, $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$. We consider the sequence $(\mathbf{n}^k, \mathbf{z}^k, \hat{\mathbf{q}}^k)_{k\in\mathbb{N}^*}$ where for each $k \in \mathbb{N}^*$, $(\mathbf{n}^k_t)_{t\in[0,T]}$ and $(\mathbf{z}^k_t)_{t\in[0,T]}$ are two $\mathcal{P}(\mathbb{R}^n)$–valued continuous processes and $\hat{\mathbf{q}}^k_t(\mathrm{d}m, \mathrm{d}m')\mathrm{d}t \in \mathbb{M}((\mathcal{P}^n_U)^2)$. Moreover, $(\mathbf{n}^k, \mathbf{z}^k, \hat{\mathbf{q}}^k)$ satisfy the following equation: $\hat{\mathbf{q}}^k_t(\mathbb{Z}_{\mathbf{n}^k_t} \times \mathcal{P}^n_U) = 1$, $\mathrm{d}t$–a.e., and

$$\begin{aligned}
&\langle f(t,.), \mathbf{n}^k_t\rangle \\
&= \langle f(0,.), \nu\rangle + \int_0^t \Big[\langle \partial_t f(r,.), \mathbf{n}^k_r\rangle + \int_{(\mathcal{P}^n_U)^2} \int_{\mathbb{R}^n\times U} \mathcal{A}_r[f(r,\cdot)](x, \mathbf{n}^k, \mathbf{z}^k, m, \bar{\nu}^k, u) m^x(\mathrm{d}u)\mathbf{n}^k_r(\mathrm{d}x)\hat{\mathbf{q}}^k_r(\mathrm{d}m, \mathrm{d}\bar{\nu})\Big]\mathrm{d}r, \quad (4.7.32)
\end{aligned}$$

for all $(t, f) \in [0, T] \times C^{1,2}_b([0, T] \times \mathbb{R}^n)$, where the generator $\mathcal{A}$ is defined by

$$\mathcal{A}_t\varphi(x, \pi, \zeta, m, \bar{\nu}, u) := \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\top(t, x)\nabla^2\varphi(x)\big] + b(t, x, \pi, \zeta, m, \bar{\nu}, u)^\top\nabla\varphi(x). \quad (4.7.33)$$

Let us introduce for each $\gamma \in (0, 1)$, $\mathcal{C}^\gamma([s, t] \times A)$ the space of γ–Hölder functions on $[s, t] \times A$ where $(s, t) \in [0, T] \times [0, T]$ and $A \subset \mathbb{R}^n$. We define the norm

$$\|u\|_{\mathcal{C}^\gamma([s,t]\times A)} := \sup_{(r,a)\in[s,t]\times A} |u(r, a)| + \sup_{(r,a)\neq(r',a')} \frac{|u(r, a) - u(r', a')|}{|r - r'|^\alpha|a - a'|^\alpha}, \text{ for all } u \in \mathcal{C}^\gamma([s, t] \times A)$$

Lemma 4.7.13. *With the previous considérations, for each* $k \in \mathbb{N}^*$, *the solution of* Equation (4.7.32) $\mathbf{n}^k$ *is such that* $\mathbf{n}^k_t(\mathrm{d}x)\mathrm{d}t = u^k(t, x)\mathrm{d}x\mathrm{d}t$, *where* u^k *is a locally continuous functions and satisfies: for all* $(t, s) \in (0, T)$, *and* $A \subset \mathbb{R}^n$ *an open set with a compact closure, there exists a constant* $N[n, s, t, A, \theta]$ *depending on* $[n, s, t, A, \theta]$ *and independent of* $k \in \mathbb{N}^*$ *such that*

$$\|u^k\|_{\mathcal{C}^\gamma([s,t]\times A)} \leq N[n, s, t, A, \theta], \text{ for all } k \in \mathbb{N}^*$$

Proof. For each $k \in \mathbb{N}^*$, $\mathbf{n}^k_t(\mathrm{d}x)\mathrm{d}t = u^k(t, x)\mathrm{d}x\mathrm{d}t$ has a density w.r.t. to the Lebesgue measure on $\mathbb{R}^n \times [0, T]$ (see [31, Corollary 6.3.2]). Denote by $W^{p,1}$ the Sobolev space of functions on $\mathbb{R}^n$ that are square integrable together with their gradient. Let us define the norm $\|\|_{\mathbb{H}^{p,1}([s,t]\times A)}$

$$\|u\|_{\mathbb{H}^{p,1}([s,t]\times A)} := \Big(\int_s^t \|u(t, \cdot)\|^2_{W^{p,1}}\mathrm{d}t\Big)^{1/2}.$$

We note $\mathbb{H}^{p',-1}([s, t] \times A)$ the dual of $\mathbb{H}^{p,1}([s, t] \times A)$, where p' is the conjugate of p. Next, we define

$$\|u\|_{\mathcal{H}^{p,1}([s,t]\times A)} := \|u\|_{\mathbb{H}^{p,1}([s,t]\times A)} + \|\partial_t u\|_{\mathbb{H}^{p,-1}([s,t]\times A)}.$$

By [31, Theorem 6.2.2], there exists $N[n,p,s,t,A]$ such that for each $u \in \mathcal{H}^{p,1}$

$$\|u^k\|_{\mathcal{C}^\gamma([s,t]\times A)} \leq N[n,p,s,t,A]\|u\|_{\mathcal{H}^{p,1}([s,t]\times A)}.$$

Also, by Rademacher's theorem we know that $\sigma\sigma^\top(t,\cdot) \in W^{p,1}$, and as the Lipschitz constant is independent of t, one has $\sup_{t\in[0,T]} \|\sigma\sigma^\top(t,\cdot)\|_{W^{p,1}} < \infty$. Now, by [31, Corollary 6.4.5], there exists a constant $C[n,p,s,t,A] > 0$

$$\|u^k\|_{\mathbb{H}^{p,1}([s,t]\times A)} \leq C[n,p,s,t,A]\int_0^T\int_A |u(r,x)|\mathrm{d}x\mathrm{d}r.$$

By [31, Theorem 6.5.3], there exists another constant $\tilde{C}[n,p,s,t,A] > 0$

$$\int_0^T\int_A |u(r,x)|^p\mathrm{d}x\mathrm{d}r \leq \tilde{C}[n,p,s,t,A].$$

$\|\partial_t u\|_{\mathbb{H}^{p,-1}([s,t]\times A)}$ is also bounded by elements only depending on (n,p,s,t,A) (see for instance [31, Corollary 6.4.3] or the proof of [31, Theorem 6.4.1]). We can conclude.

□

Lemma 4.7.14. *In the context of* Lemma 4.7.13*, there exists $u \in C([0,T]\times\mathbb{R}^n)$, such that for each $(t,s) \in (0,T)\times(0,T)$ and $\bar{A} \subset \mathbb{R}^n$ a compact set, one has*

$$\lim_{k\to\infty} \sup_{(r,x)\in[s,t]\times\bar{A}} \left|u^k(r,x) - u(r,x)\right| = 0.$$

Proof. For each $M \in \mathbb{N}^*$, denote by $B(M) := \{x \in \mathbb{R}^n : |x| \leq M\}$. By Lemma 4.7.13, there exists $N[n,M,\theta]$

$$\|u^k\|_{\mathcal{C}^\gamma\left([T/M,T(1-1/M)]\times B(M)\right)} \leq N[n,M,\theta], \text{ for all } k \in \mathbb{N}^*.$$

Then by Arzelà–Ascoli theorem, for each $M \in \mathbb{N}^*$, there exists $u^M \in C([0,T]\times\mathbb{R}^n)$ and a sub-sequence $(k_l^M)_{l\in\mathbb{N}^*}$ such that

$$\lim_{l\to\infty} \sup_{(r,x)\in[T/M,T(1-1/L)]\times B(M)} \left|u^{k_l^M}(r,x) - u^M(r,x)\right| = 0.$$

By Cantor's diagonal argument, we can find $u \in C([0,T]\times\mathbb{R}^n)$ and a sub-sequence $(k_l)_{l\in\mathbb{N}^*}$ such that u^{k_l} converges uniformly to u on each compact set of $[s,t]\times\bar{A} \subset [0,T]\times\mathbb{R}^n$. □

For each density of probability f on $\mathbb{R}^n$, and $\mathbf{q} \in \mathbb{M}((\mathcal{P}_U^n)^2)$, we denote by

$$\mathbf{q}_t[f](\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}t := \int \delta_{\left(e^x(\mathrm{d}u')f(x)\mathrm{d}x\right)}(\mathrm{d}m)\mathbf{q}_t(\mathrm{d}e,\mathrm{d}\bar{\nu})\mathrm{d}t$$

Recall that G is defined in Equation (4.7.1). Assume that:

$$\lim_{k\to\infty} \mathbf{q}_t^k[G](\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}t := \mathbf{q}_t^\infty[G](\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}t, \text{ in weakly sense,} \tag{4.7.34}$$

for some $\mathbf{q}^\infty \in \mathbb{M}((\mathcal{P}_U^n)^2)$. Then, we have the following result

Lemma 4.7.15. *Let us stay in the context of* Lemma 4.7.14 *and* Lemma 4.7.13*, one has*

$$\lim_{k\to\infty} \mathbf{q}_t^k[u^k](\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}t := \mathbf{q}_t^\infty[u](\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}t \textit{ in the weak sense.}$$

Proof. It is straightforward to show that $(\mathbf{q}^k[u^k])_{k\in\mathbb{N}^*}$ is relatively compact for the weak convergence topology. If we note $\mathbf{q}^\star$ the limit of any sub–sequence, to prove this result, it is enough to show that: for all $q \in \mathbb{N}^*$, for any functions $(\phi^1, ..., \phi^q) \in C(\mathbb{R}^n \times U)$ bounded continuous with compact support and $\beta \in C([0,T] \times \mathcal{P}_U^n)$, one has

$$\int_0^T \prod_{i=1}^q \langle \phi^i, m\rangle \beta(t,\bar\nu)\mathbf{q}_t^\star(\mathrm{d}m,\mathrm{d}\bar\nu)\mathrm{d}t = \int_0^T \prod_{i=1}^q \langle \phi^i, m\rangle \beta(t,\bar\nu)\mathbf{q}_t^\infty[u](\mathrm{d}m,\mathrm{d}\bar\nu)\mathrm{d}t.$$

This equality is true because

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^T \prod_{i=1}^q \int_{U\times R^n} \phi^i(u,x)m^x(\mathrm{d}u)u^k(t,x)\mathrm{d}x\beta(t,\bar\nu)\mathbf{q}_t^k(\mathrm{d}m,\mathrm{d}\bar\nu)\mathrm{d}t\\
&= \lim_{k\to\infty} \int_0^T \prod_{i=1}^q \int_{U\times R^n} \phi^i(u,x)m^x(\mathrm{d}u)u^k(t,x)\frac{1}{G(x)}G(x)\mathrm{d}x\beta(t,\bar\nu)\mathbf{q}_t^k(\mathrm{d}m,\mathrm{d}\bar\nu)\mathrm{d}t\\
&= \lim_{k\to\infty} \int_0^T \prod_{i=1}^q \int_{U\times R^n} \phi^i(u,x)u(t,x)\frac{1}{G(x)}m^x(\mathrm{d}u)G(x)\mathrm{d}x\beta(t,\bar\nu)\mathbf{q}_t^k(\mathrm{d}m,\mathrm{d}\bar\nu)\mathrm{d}t\\
&= \lim_{k\to\infty} \int_0^T \prod_{i=1}^q \int_{U\times R^n} \phi^i(u,x)u(t,x)\frac{1}{G(x)}m^x(\mathrm{d}u)G(x)\mathrm{d}x\beta(t,\bar\nu)\mathbf{q}_t^\infty(\mathrm{d}m,\mathrm{d}\bar\nu)\mathrm{d}t,
\end{aligned}$$

where we use for the third equality the uniform convergence of u^k to u on each compact set, and for the last equality, the Assumption 4.7.34 and the fact that $(t,x) \to \phi^i(u,x)u(t,x)\frac{1}{G(x)}$ is continuous bounded. We can conclude. □

4.8 Proofs of equivalence and limit theory

4.8.1 Equivalence result

This section is devoted to the proof of Theorem 4.5.1. To achieve this proof, we provide an approximation of measure-valued rule by McKean-Vlasov processes. Before starting the proofs, by shifting some probabilities, let us give a reformulation of measure-valued rules. For all $(t,\mathbf{b},\pi,m) \in [0,T]\times\mathcal{C}^\ell\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{P}_U^n$,

$$\pi_t[\mathbf{b}](\mathrm{d}y) := \int_{\mathbb{R}^n} \delta_{\left(y'+\sigma_0\mathbf{b}_t\right)}(\mathrm{d}y)\pi_t(\mathrm{d}y'), \quad m[\mathbf{b}_t](\mathrm{d}u,\mathrm{d}y) := \int_{\mathbb{R}^n\times U} \delta_{(y'+\sigma_0\mathbf{b}_t)}(\mathrm{d}y)m(\mathrm{d}u,\mathrm{d}y') \tag{4.8.1}$$

and any $q \in \mathbb{M}$,

$$q_t[\mathbf{b}](\mathrm{d}m) := \int_{\mathcal{P}_U^n} \delta_{\left(m'[\mathbf{b}_t]\right)}(\mathrm{d}m)q_t(\mathrm{d}m'). \tag{4.8.2}$$

In the same way, let us consider the "*shift*" generator $\widehat{\mathcal{L}}$,

$$\widehat{\mathcal{L}}_t[\varphi](y,\mathbf{b},\pi,m,u) := \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma^\top(t,y+\sigma_0\mathbf{b}_t,\pi_t[\mathbf{b}_t],m[\mathbf{b}_t],u)\nabla^2\varphi(y)\right] + b(t,y+\sigma_0\mathbf{b}_t,\pi_t[\mathbf{b}_t],m[\mathbf{b}_t],u)^\top\nabla\varphi(y). \tag{4.8.3}$$

Next, on the canonical filtered space $(\overline\Omega,\overline{\mathbb{F}})$ (see Section 4.4), let $(\vartheta_t)_{t\in[0,T]}$ be the $\mathcal{P}(\mathbb{R}^n)$–valued $\overline{\mathbb{F}}$–adapted continuous process and $(\Theta_t)_{t\in[0,T]}$ be the $\mathcal{P}_U^n$–valued $\overline{\mathbb{F}}$–predictable process defined by

$$\vartheta_t(\bar\omega) := \mu_t(\bar\omega)[-B(\bar\omega)] \text{ and } \Theta_t(\bar\omega)(\mathrm{d}m) := \Lambda_t(\bar\omega)[-B(\bar\omega)](\mathrm{d}m), \text{ for all } (t,\bar\omega)\in[0,T]\times\overline\Omega. \tag{4.8.4}$$

The next result follows immediately, so we omit the proof.

Lemma 4.8.1. *Let* $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$. *Then,* $\Theta_t(\mathbb{Z}_{\vartheta_t}) = 1$, $\mathrm{dP}\otimes\mathrm{d}t$, *a.e.* $(t,\bar\omega)\in[0,T]\times\overline\Omega$, *and* P*–a.e.* $\bar\omega\in\overline\Omega$, *for all* $(f,t)\in C_b^2(\mathbb{R}^n)\times[0,T]$,

$$N_t(f) = \langle f,\vartheta_t\rangle - \langle f,\nu\rangle - \int_0^t \int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n\times U} \widehat{\mathcal{L}}_r f(y,B,\vartheta,m,u)m(\mathrm{d}u,\mathrm{d}y)\Theta_r(\mathrm{d}m)\mathrm{d}r.$$

Next, let us provide some estimates for the different controls. The first result is standard, the second is just an application of Lemma 4.7.2 (see also Remark 4.7.4) combined with Lemma 4.8.1.

Lemma 4.8.2 (Estimates). *Under* Assumption 4.1.1, *for any* $(\nu, \nu^1, \dots, \nu^N) \in \mathcal{P}_{p'}(\mathbb{R}^n)^{N+1}$ *with* $p' > p$, *there exists* $K > 0$, *depending only of coefficients* (b, σ) *and* p', *such that: for every* $(\alpha^1, \dots, \alpha^N) \in (\mathcal{A}_N(\nu_N))^N$ *one has*

$$\mathbb{E}^{\mathrm{P}^N}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\mu_t(\mathrm{d}x)\right] \le K\left[1+\int_{\mathbb{R}^n}|x'|^{p'}\frac{1}{N}\sum_{i=1}^N\nu^i(\mathrm{d}x')\right],$$

where $\mathrm{P}^N := \mathbb{P}(\alpha^1, ..., \alpha^N) \in \mathcal{P}(\overline{\Omega})$ (*see definition* (4.2.3)), *and for each* $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$ *or* $\alpha \in \mathcal{A}(\nu)$ *with* $\mathrm{P} = \mathbb{P}_\nu \circ \big(\mu^\alpha, \delta_{\overline{\mu}^\alpha_t}(\mathrm{d}m)\mathrm{d}t, B\big)^{-1}$

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\vartheta_t(\omega)(\mathrm{d}x) + \mathbb{E}^{\mathrm{P}}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\mu_t(\mathrm{d}x)\right] \le K\left[1+\int_{\mathbb{R}^n}|x'|^{p'}\nu(\mathrm{d}x')\right],\ \mathrm{P}\text{-}a.e.\ \omega\in\overline{\Omega}.$$

In addition

$$\mathcal{W}_p\big(\vartheta_s(\omega),\vartheta_t(\omega)\big)^p \le K|t-s|,\ \textit{for all}\ (t,s)\in[0,T]\times[0,T],\ \mathrm{P}\text{-}a.e.\ \omega\in\overline{\Omega},$$

where ϑ *is the process given in equation* (4.8.4).

4.8.1.1 Technical lemmata

In this part, from a measure–valued rule, we will build a sequence of processes that approximate the measure–valued rule and that are close enough to strong control rules. This part is fundamental for the proof of Theorem 4.5.1.

let $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$, and $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be a filtered probability space supporting W $\mathbb{R}^n$–valued $\widetilde{\mathbb{F}}$–Brownian motion and let ξ be a $\widetilde{\mathcal{F}}_0$–random variable s.t. $\mathcal{L}^{\widetilde{\mathbb{P}}}(\xi) = \nu$. Define the filtered probability space $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ which is an extension of the canonical space $(\overline{\Omega}, \overline{\mathbb{F}}, \mathrm{P})$: $\widehat{\Omega} := \widetilde{\Omega}\times\overline{\Omega}$, $\widehat{\mathbb{F}} := (\widetilde{\mathcal{F}}_t \otimes \overline{\mathcal{F}}_t)_{t\in[0,T]}$ and $\widehat{\mathbb{P}} := \widetilde{\mathbb{P}} \otimes \mathrm{P}$. The variables (ξ, W) of $\widetilde{\Omega}$ and (B, μ, Λ) of $\overline{\Omega}$ are naturally extended on the space $\widehat{\Omega}$ while keeping the same notation $(\xi, W, B, \mu, \Lambda)$ for simplicity. Also, let us consider the filtration $(\widehat{\mathcal{G}}_t)_{t\in[0,T]}$ defined by

$$\widehat{\mathcal{G}}_t := \sigma\big\{B_{t\wedge\cdot}, \mu_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}\big\},\ \text{for all}\ t\in[0,T].$$

Proposition 4.8.3. *Under* Assumption 4.1.1, *for any* $[0,1]$*–valued uniform variable* Z $\widehat{\mathbb{P}}$*–independent of* $(\xi, W, B, \mu, \Lambda)$, *there exists a sequence of* $\widehat{\mathbb{F}}$*–predictable processes* $(\alpha^k)_{k\in\mathbb{N}^*}$ *satisfying: for each* $k \in \mathbb{N}^*$,

$$\alpha^k_t := G^k(t, \xi, \mu_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, W_{t\wedge}, B_{t\wedge}, Z),\ \widehat{\mathbb{P}}\text{–}a.e.,\ \textit{for all}\ t\in[0,T],$$

with a Borel function $G^k : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}^n_U)\times\mathcal{C}^n\times\mathcal{C}^\ell\times[0,1]\to U$ *such that if we let* $\widehat{X}^k$ *be the unique strong solution of:* $\mathbb{E}^{\widehat{\mathbb{P}}}[\|\widehat{X}^k\|^{p'}] < \infty$, *for all* $t\in[0,T]$

$$\widehat{X}^k_t = \xi + \int_0^t b(r, \widehat{X}^k_r, \mu^k, \overline{\mu}^k_r, \alpha^k_r)\mathrm{d}r + \int_0^t \sigma(r, \widehat{X}^k_r, \mu^k, \overline{\mu}^k_r, \alpha^k_r)\mathrm{d}W_r + \sigma_0 B_t,\ \widehat{\mathbb{P}}\text{–a.e.}$$

where $\mu^k_t := \mathcal{L}^{\mathbb{P}}(\widehat{X}^k_t|\widehat{\mathcal{G}}_t)$ *and* $\overline{\mu}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_t, \alpha^k_t|\widehat{\mathcal{G}}_t)$ *then*

$$\lim_{k\to\infty}\left[\mathcal{W}_p\Big(\delta_{\overline{\mu}^k_s}(\mathrm{d}m)\mathrm{d}s, \Lambda_s(\mathrm{d}m)\mathrm{d}s\Big) + \sup_{t\in[0,T]}\mathcal{W}_p(\mu^k_t, \mu_t)\right] = 0,\ \widehat{\mathbb{P}}\text{–a.e.} \tag{4.8.5}$$

Therefore

$$\lim_{k\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\Big((\mu^k_t)_{t\in[0,T]}, \delta_{\overline{\mu}^k_s}(\mathrm{d}m)\mathrm{d}s, (B_t)_{t\in[0,T]}\Big) = \mathrm{P},\ \textit{for the Wasserstein metric}\ \mathcal{W}_p.$$

Proof. As $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$, by definition, P–a.e. $\omega \in \overline{\Omega}$, $N_t(f) = 0$ for all $f \in C_b^2(\mathbb{R}^n)$ and $t \in [0,T]$. By Lemma 4.8.1, by taking into account the extension of all variables on $\widehat{\Omega}$, recall that $(\vartheta_t)_{t\in[0,T]}$ and $(\Theta_t)_{t\in[0,T]}$ are defined in (4.8.4), one has $\Theta_t(\mathbb{Z}_{\vartheta_t}) = 1$, for $\mathrm{d}\widehat{\mathbb{P}} \otimes \mathrm{d}t$–a.e. $(t,\omega) \in [0,T] \times \widehat{\Omega}$, and $\widehat{\mathbb{P}}$–a.e. $\omega \in \widehat{\Omega}$, for all $(f,t) \in C_b^2(\mathbb{R}^n) \times [0,T]$,

$$N_t(f) = \langle f, \vartheta_t\rangle - \langle f, \nu\rangle - \int_0^t \int_{\mathcal{P}_U^n} \int_{\mathbb{R}^n \times U} \widehat{\mathcal{L}}_r f(y, B, \vartheta, m, u) m(\mathrm{d}u, \mathrm{d}y) \Theta_r(\mathrm{d}m)\mathrm{d}r.$$

Define

$$\Gamma := \Big\{ m \in \mathcal{P}_U^n : \int_{\mathbb{R}^n} |y|^{p'} m(\mathrm{d}y, U) \le \hat{K} \Big\},$$

where $\hat{K} > 0$ is such that $\hat{K} > K\Big[1 + \int_{\mathbb{R}^n} |x'|^{p'} \nu(\mathrm{d}x')\Big]$, with K is the constant used in Lemma 4.8.2. Notice that Γ is a compact set of $\mathcal{P}_p(\mathbb{R}^n \times U)$ and by Lemma 4.8.2, one has $\Theta_t(\Gamma) = 1$, $\mathrm{d}\widehat{\mathbb{P}} \otimes \mathrm{d}t$, a.e. $(t,\omega) \in [0,T] \times \widehat{\Omega}$.
As Γ is a compact set of $\mathcal{P}_p(\mathbb{R}^n \times U)$, there exists a family of measurable functions $(h^k)_{k\in\mathbb{N}^*}$ with $h^k : [0,T] \times \mathbb{M} \to \mathcal{P}_U^n$, s.t.

$$\lim_{k\to\infty} \delta_{h^k(t,\Theta_{t\wedge\cdot})}(\mathrm{d}m)\mathrm{d}t = \Theta_t(\mathrm{d}m)\mathrm{d}t, \ \widehat{\mathbb{P}}\text{–a.e. then } \lim_k \widehat{\mathcal{L}^{\mathbb{P}}}\big(\vartheta, \delta_{h^k(t,\Theta_{t\wedge\cdot})}(\mathrm{d}m)\mathrm{d}t, B\big) = \widehat{\mathcal{L}^{\mathbb{P}}}(\vartheta, \Theta, B), \ \text{ in } \mathcal{W}_p.$$

In the same spirit of notations (4.8.3), introduce

$$[\hat{b}, \hat{\sigma}](t, y, \mathbf{b}, \pi, m, u) := [b, \sigma](t, y + \sigma_0 \mathbf{b}_t, \pi[\mathbf{b}], m[\mathbf{b}_t], u), \tag{4.8.6}$$

notice that $[\hat{b}, \hat{\sigma}] : [0,T] \times \mathbb{R}^n \times \mathcal{C}^\ell \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n \times U \to \mathbb{R}^n \times \mathbb{S}^{n\times n}$ is continuous and for $\mathbf{b} \in \mathcal{C}^\ell$, $[\hat{b}, \hat{\sigma}](\cdot,\cdot,\mathbf{b},\cdot,\cdot,\cdot)$ verify the Assumption 4.1.1 with constant C and θ independent of $\mathbf{b}$ (see Assumption 4.1.1).

Now, let us apply Proposition 4.7.9 (see also Proposition 4.7.7). As $\big(\vartheta, \delta_{h^k(t,\Theta_{t\wedge\cdot})}(\mathrm{d}m)\mathrm{d}t, B\big)_{k\in\mathbb{N}^*}$ is $\widehat{\mathbb{P}}$ independent of (ξ, W) and

$$\lim_k \widehat{\mathcal{L}^{\mathbb{P}}}\Big(\vartheta, \delta_{h^k(s,\Theta_{s\wedge\cdot})}(\mathrm{d}m)\mathrm{d}s, B\Big) = \widehat{\mathcal{L}^{\mathbb{P}}}\big(\vartheta, \Theta_s(\mathrm{d}m)\mathrm{d}s, B\big), \ \text{in } \mathcal{W}_p,$$

by Proposition 4.7.9, there exists $G^k : [0,T] \times \mathbb{R}^n \times \mathbb{M} \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{C}^n \times \mathcal{C}^\ell \times [0,1] \to U$ a Borel function such that if X^k is the unique strong solution of: for all $t \in [0,T]$

$$X_t^k = \xi + \int_0^t \hat{b}\big(r, X_r^k, B, \vartheta^k, \overline{\vartheta}_r^k, \alpha_r^k\big)\mathrm{d}r + \int_0^t \hat{\sigma}\big(r, X_r^k, B, \vartheta^k, \overline{\vartheta}_r^k, \alpha_r^k\big)\mathrm{d}W_r, \ \widehat{\mathbb{P}}\text{–a.e.}, \tag{4.8.7}$$

where

$$\alpha_t^k := G^k\big(t, \xi, \Theta_{t\wedge\cdot}^k, \vartheta_{t\wedge\cdot}, W_{t\wedge\cdot}, B_{t\wedge\cdot}, Z\big), \ \ \overline{\vartheta}_t^k := \widehat{\mathcal{L}^{\mathbb{P}}}\big(X_t^k, \alpha_t^k | \mathcal{G}_t^k\big) \ \text{ and } \ \vartheta_t^k := \widehat{\mathcal{L}^{\mathbb{P}}}\big(X_t^k | \mathcal{G}_t^k\big),$$

with $\Theta_t^k(\mathrm{d}m)\mathrm{d}t := \delta_{\left(h^k(t,\Theta_{t\wedge\cdot})\right)}(\mathrm{d}m)\mathrm{d}t$, and $\mathbb{G}^k := (\mathcal{G}_s^k)_{s\in[0,T]} := (\sigma\{\vartheta_{s\wedge\cdot}, \Theta_{s\wedge\cdot}^k, B_{s\wedge\cdot}\})_{s\in[0,T]}$, then

$$\lim_{j\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\big(\overline{\vartheta}_t^{k_j}, \mathbf{m}_t^{k_j}\big)^p \mathrm{d}t + \sup_{t\in[0,T]} \mathcal{W}_p\big(\vartheta_t^{k_j}, \vartheta_t\big)\Big] = 0 \ \text{ and } \ \lim_{j\to\infty} \widehat{\mathcal{L}^{\mathbb{P}}}\big(\vartheta^{k_j}, \Theta^{k_j}, B\big) = \widehat{\mathcal{L}^{\mathbb{P}}}\big(\vartheta, \Theta, B\big), \ \text{ in } \mathcal{W}_p,$$

where $\mathbf{m}_t^k := h^k(t, \Theta_{t\wedge\cdot})$ and $(k_j)_{j\in\mathbb{N}^*} \subset N^*$ is a sub-sequence. Notice that, as $\mathbb{G}^k \subset \widehat{\mathbb{G}}$, and (ξ, W, Z) are $\widehat{\mathbb{P}}$ independent of $\widehat{\mathbb{G}}$, one has $\widehat{\mathcal{L}^{\mathbb{P}}}\big(X_t^k, \alpha_t^k | \mathcal{G}_t^k\big) = \widehat{\mathcal{L}^{\mathbb{P}}}\big(X_t^k, \alpha_t^k | \widehat{\mathcal{G}}_t\big)$, $\widehat{\mathbb{P}}$–a.e. for all $t \in [0,T]$. Using equation (4.8.6),

$$\begin{aligned} X_t^k = \xi &+ \int_0^t b\big(r, X_r^k + \sigma_0 B_r, (\widehat{\mathcal{L}^{\mathbb{P}}}(X_s^k + \sigma_0 B_s | \widehat{\mathcal{G}}_s))_{s\in[0,T]}, \widehat{\mathcal{L}^{\mathbb{P}}}(X_r^k + \sigma_0 B_r, \alpha_r^k | \widehat{\mathcal{G}}_r), \alpha_r^k\big)\mathrm{d}r \\ &+ \int_0^t \sigma\big(r, X_r^k + \sigma_0 B_r, (\widehat{\mathcal{L}^{\mathbb{P}}}(X_s^k + \sigma_0 B_s | \widehat{\mathcal{G}}_s))_{s\in[0,T]}, \widehat{\mathcal{L}^{\mathbb{P}}}(X_r^k + \sigma_0 B_r, \alpha_r^k | \widehat{\mathcal{G}}_r), \alpha_r^k\big)\mathrm{d}W_r, \ \text{for all } t \in [0,T], \ \widehat{\mathbb{P}}\text{–a.e.} \end{aligned}$$

Denote by $\widehat{X}^k := X^k + \sigma_0 B$, one finds

$$\widehat{X}^k_t = \xi + \int_0^t b\big(r, \widehat{X}^k_r, (\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_s|\widehat{\mathcal{G}}_s))_{s\in[0,T]}, \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}_r, \alpha^k_r|\widehat{\mathcal{G}}_r), \alpha^k_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{X}^k_r, (\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_s|\widehat{\mathcal{G}}_s))_{s\in[0,T]}, \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_r, \alpha^k_r|\widehat{\mathcal{G}}_r), \alpha^k_r\big)\mathrm{d}W_r + \sigma_0 B_t, \text{ for all } t\in[0,T], \ \widehat{\mathbb{P}}\text{–a.e.}$$

With the notation introduce in (4.8.1) and (4.8.2), it is straightforward to check that the map

$$(\pi, q, \mathbf{b}) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell \to \big(\pi[\mathbf{b}], q_t[\mathbf{b}](\mathrm{d}m)\mathrm{d}t, \mathbf{b}\big) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell$$

is continuous. Consequently, one has

$$\lim_{j\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\Big(\overline{\vartheta}^{k_j}_t[B_t], \mathbf{m}^{k_j}_t[B_t]\Big)^p \mathrm{d}t + \sup_{t\in[0,T]} \mathcal{W}_p(\vartheta^{k_j}_t[B], \vartheta_t[B])\Big] = 0,$$

therefore, in $\mathcal{W}_p$

$$\begin{aligned}\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\Big((\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^{k_j}_t|\widehat{\mathcal{G}}_t))_{t\in[0,T]}, \delta_{(\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^{k_l}_s, \alpha^{k_l}_s|\widehat{\mathcal{G}}_s))}(\mathrm{d}m)\mathrm{d}s, B\Big) &= \lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\big(\vartheta^{k_j}[B], \Theta^{k_j}_t[B](\mathrm{d}m)\mathrm{d}t, B\big)\\ &= \mathcal{L}^{\widehat{\mathbb{P}}}\big(\vartheta[B], \Theta_t[B](\mathrm{d}m)\mathrm{d}t, B\big)\end{aligned}$$

After simple calculations, $(\vartheta[B], \Theta_t[B](\mathrm{d}m)\mathrm{d}t, B) = (\mu, \Lambda, B)$, $\widehat{\mathbb{P}}$–a.e. Then

$$\lim_{j\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\int_0^T \mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^{k_j}_t, \alpha^{k_j}_t|\widehat{\mathcal{G}}_t), \mathbf{m}^{k_j}_t[B_t]\Big)^p \mathrm{d}t + \sup_{t\in[0,T]} \mathcal{W}_p\big(\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^{k_j}_t|\widehat{\mathcal{G}}_t), \mu_t\big)\Big] = 0$$

and hence

$$\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\Big((\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^{k_j}_t|\widehat{\mathcal{G}}_t))_{t\in[0,T]}, \delta_{(\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^{k_j}_s, \alpha^{k_l}_s|\widehat{\mathcal{G}}_s))}(\mathrm{d}m)\mathrm{d}s, B\Big) = \mathcal{L}^{\widehat{\mathbb{P}}}(\mu, \Lambda, B) = \mathrm{P}, \text{ in } \mathcal{W}_p.$$

After extraction from $(\widehat{X}^{k_j}, \alpha^{k_j})_{j\in\mathbb{N}^*}$, one has also the $\widehat{\mathbb{P}}$–a.e. convergence (4.8.5).

□

4.8.1.2 Proof of Theorem 4.5.1

First, for $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, under Assumption 4.1.1, let us prove that $\overline{\mathcal{P}}_V(\nu)$ is a compact set for the Wasserstein topology $\mathcal{W}_p$. Let $(\mathrm{P}_k)_{k\in\mathbb{N}^*} \subset \overline{\mathcal{P}}_V(\nu)$, by Proposition 3.4.6, $(\mathrm{P}_k)_{k\in\mathbb{N}^*}$ is relatively compact for the Wassertein topology $\mathcal{W}_p$ and any limit P_∞ of any sub-sequence belongs to $\overline{\mathcal{P}}_V(\nu)$. Therefore $\overline{\mathcal{P}}_V(\nu)$ is compact. By similar techniques used in Theorem 3.2.4 and Theorem 2.2.3, it is straightforward to show that $\overline{\mathcal{P}}_V(\nu)$ is convex.

Next, we prove the items (i) and (ii) of Theorem 4.5.1. By applying Proposition 4.8.3, with the same notations, for any $[0,1]$–valued uniform variable Z $\widehat{\mathbb{P}}$–independent of $(\xi, W, B, \mu, \Lambda)$, there exists a sequence of $\widehat{\mathbb{F}}$–predictable processes $(\alpha^k)_{k\in\mathbb{N}^*}$ satisfying: for each $k \in \mathbb{N}^*$,

$$\alpha^k_t := G^k(t, \xi, \mu_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, W_{t\wedge}, B_{t\wedge}, Z), \ \widehat{\mathbb{P}}\text{–a.e., for all } t\in[0,T],$$

with $G^k : [0,T]\times\mathbb{R}^n \times \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}^n_U) \times \mathcal{C}^n \times \mathcal{C}^\ell \times [0,1] \to U$ is a Borel function such that if $\widehat{X}^k$ is the unique strong solution of: for all $t\in[0,T]$

$$\widehat{X}^k_t = \xi + \int_0^t b(r, \widehat{X}^k_r, \mu^k, \overline{\mu}^k_r, \alpha^k_r)\mathrm{d}r + \int_0^t \sigma(r, \widehat{X}^k_r, \mu^k, \overline{\mu}^k_r, \alpha^k_r)\mathrm{d}W_r + \sigma_0 B_t, \ \widehat{\mathbb{P}}\text{–a.e.}$$

where $\mu_t^k := \mathcal{L}^{\mathbb{P}}(\widehat{X}_t^k|\widehat{\mathcal{G}}_t)$ and $\overline{\mu}_t^k := \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}_t^k, \alpha_t^k|\widehat{\mathcal{G}}_t)$ then

$$\lim_{k\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\Big((\mu_t^k)_{t\in[0,T]}, \delta_{\overline{\mu}_s^k}(\mathrm{d}m)\mathrm{d}s, (B_t)_{t\in[0,T]}\Big) = \mathrm{P}, \text{ for the Wasserstein metric } \mathcal{W}_p.$$

For each $k \in \mathbb{N}^*$, $\widehat{X}_t^k = H_t^k(\xi, W_{t\wedge\cdot}, \mu_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, B_{t\wedge\cdot}, Z)$, for all $t \in [0,T]$, $\widehat{\mathbb{P}}$–a.e. with $H^k : \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell \times [0,1] \to \mathcal{C}^n$ a Borel function. Then, as (ξ, W, Z) are $\widehat{\mathbb{P}}$–independent of (μ, Λ, B), one gets for all $t \in [0,T]$, $\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_{t\wedge\cdot}, \alpha_t^k|\widehat{\mathcal{G}}_t) = \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_{t\wedge\cdot}, \alpha_t^k|\widehat{\mathcal{G}}_T)$, $\widehat{\mathbb{P}}$–a.e.. Let us introduce the process $(\widehat{\mu}_t^k)_{t\in[0,T]}$,

$$\widehat{\mu}_t^k := \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_{t\wedge\cdot}, \widehat{X}^k_{t\wedge\cdot} - \sigma_0 B_{t\wedge\cdot}, W, \Lambda^k_{t\wedge\cdot}|\widehat{\mathcal{G}}_t), \text{ for all } t \in [0,T],$$

with $\Lambda_t^k(\mathrm{d}u)\mathrm{d}t := \delta_{\alpha_t^k}(\mathrm{d}u)\mathrm{d}t$. For each $k \in \mathbb{N}^*$, $\widehat{\mu}_t^k \in \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M}(U))$, for all $t \in [0,T]$ and if $(\widetilde{X}, \widetilde{Y}, \widetilde{W}, \widetilde{\Lambda})$ is the canonical process on $\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M}(U)$, one has $\mu_t^k = \mathcal{L}^{\widehat{\mu}_t^k}(\widetilde{X}_t)$, $\widehat{\mathbb{P}}$–a.e., and $\mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}_t^k, \alpha_t^k|\widehat{\mathcal{G}}_t)(\mathrm{d}x, \mathrm{d}u) = \mathbb{E}^{\widehat{\mu}_t^k}[\delta_{\widetilde{X}_t}(\mathrm{d}x)\widetilde{\Lambda}_t(\mathrm{d}u)]$, $\widehat{\mathbb{P}}$–a.e. for all $t \in [0,T]$. It is straightforward to see that $\widehat{\mu}_t^k = \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_{t\wedge\cdot}, \widehat{X}^k_{t\wedge\cdot} - \sigma_0 B_{t\wedge\cdot}, W, \Lambda^k_{t\wedge\cdot}|\widehat{\mathcal{G}}_T)$, for each $k \in \mathbb{N}^*$, then

$$\widehat{\mu}_t^k = \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_{t\wedge\cdot}, \widehat{X}^k_{t\wedge\cdot} - \sigma_0 B_{t\wedge\cdot}, W, \Lambda^k_{t\wedge\cdot}|B_{t\wedge\cdot}, \widehat{\mu}^k_{t\wedge\cdot}) = \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^k_{t\wedge\cdot}, \widehat{X}^k_{t\wedge\cdot} - \sigma_0 B_{t\wedge\cdot}, W, \Lambda^k_{t\wedge\cdot}|B, \widehat{\mu}^k), \ \widehat{\mathbb{P}}\text{–a.e., for all } t \in [0,T],$$

and $(B, \widehat{\mu}^k)$ are $\widehat{\mathbb{P}}$–independent of (ξ, W). For all $k \in \mathbb{N}^*$, denote

$$\overline{\mathbb{Q}}^k := \widehat{\mathbb{P}} \circ \big(\widehat{X}^k, \widehat{X}^k - \sigma_0 B, \Lambda^k, W, B, \widehat{\mu}^k\big)^{-1} \in \mathcal{P}\Big(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M}(U) \times \mathcal{C}^n \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M}(U))\Big),$$

then $\overline{\mathbb{Q}}^k$ is a weak control (according to Proposition 2.3.6). Then by (a slight extension of) Proposition 2.3.12,
(1) when $\ell \neq 0$, there exists a sequence $\alpha^{j,k} \in \mathcal{A}(\nu)$, and $X^{\alpha^{j,k}}$ the strong solution of (5.2.11) with control $\alpha^{j,k}$ such that

$$\lim_{j\to\infty} \mathbb{P}_\nu \circ \Big(X^{\alpha^{j,k}}, W, B, \delta_{(\overline{\mu}_s^{\alpha^{j,k}}, \alpha_s^{j,k})}(\mathrm{d}m, \mathrm{d}u)\mathrm{d}s\Big)^{-1} = \widehat{\mathbb{P}} \circ \Big(\widehat{X}^k, W, B, \delta_{(\overline{\mu}_s^k, \alpha_s^k)}(\mathrm{d}m, \mathrm{d}u)\mathrm{d}s\Big)^{-1}, \text{ in } \mathcal{W}_p,$$

(2) when $\ell = 0$, there exists a family of Borel functions $(\kappa_j^k)_{k,j}$ with $\kappa_j^k : [0,T] \times \mathbb{R}^n \times \mathcal{C}^n \times [0,1] \to U$, such that if $\alpha_t^{j,k}[z] := \kappa_j^k(t, \xi, W_{t\wedge\cdot}, z)$, for $z \in [0,1]$, one gets $(\alpha_t^{j,k}[z])_{t\in[0,T]} \in \mathcal{A}(\nu)$ and

$$\lim_{j\to\infty} \int_0^1 \mathbb{P}_\nu \circ \Big(X^{\alpha^{j,k}[z]}, W, B, \delta_{(\overline{\mu}_s^{\alpha^{j,k}[z]}, \alpha_s^{j,k}[z])}(\mathrm{d}m, \mathrm{d}u)\mathrm{d}s\Big)^{-1}\mathrm{d}z = \widehat{\mathbb{P}} \circ \Big(\widehat{X}^k, W, B, \delta_{(\overline{\mu}_s^k, \alpha_s^k)}(\mathrm{d}m, \mathrm{d}u)\mathrm{d}s\Big)^{-1}, \text{ in } \mathcal{W}_p,$$

All these results are enough to deduce the items (i) and (ii) of Theorem 4.5.1, and conclude that: for $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, $V_S(\nu) = V_V(\nu)$ and there exists $\mathrm{P}^\star \in \overline{\mathcal{P}}_V(\nu)$ such that $V_V(\nu) = \mathbb{E}^{\mathrm{P}^*}\big[J\big(\mu, \Lambda\big)\big]$.

4.8.2 Propagation of chaos

With the help of Theorem 4.5.1, in this section we provide one of the main objective of this chapter, which is to prove the limit theory result or (controlled) propagation of chaos.

4.8.2.1 Technical results: study of the behavior of processes when N goes to infinity

In this part, the properties of some sequences of probability measures on the canonical space $\overline{\Omega}$ are given. Mainly, the behavior when N goes to infinity of sequences of type $(\mathbb{P}(\alpha^1, \ldots, \alpha^N))_{N\in\mathbb{N}^*}$ construct from the formulation of N–agents stochastic control problem are studied. (see Section 4.2 and Remark 4.2.1).

Proposition 4.8.4. *Let* Assumption 4.1.1 *hold true and* $(\nu^i)_{i\in\mathbb{N}^*} \subset \mathcal{P}_{p'}(\mathbb{R}^n)$. *Recall that* $\nu_N := \nu^1 \otimes \cdots \otimes \nu^N$, *for each* $N \in \mathbb{N}^*$.
(i) *Let* $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ *be the sequence satisfying* $\mathrm{P}^N := \mathbb{P}(\alpha^{1,N}, \ldots, \alpha^{N,N})$ *(see definition* (4.2.3)*) with* $\alpha^{i,N} \in \mathcal{A}_N(\nu_N)$ $\forall i \in [\![1, N]\!]$, *for each* $N \in \mathbb{N}^*$. *If*

$$\sup_{N\geq 1} \frac{1}{N} \sum_{i=1}^N \int_{\mathbb{R}^n} |x'|^{p'} \nu^i(\mathrm{d}x') < \infty$$

then $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ *is pre-compact in* $\mathcal{P}_p(\overline{\Omega})$ *for the metric* $\mathcal{W}_p$ *and for every* $\mathrm{P}^\infty \in \mathcal{P}(\overline{\Omega})$ *the limit of any sub–sequence* $(\mathrm{P}^{N_j})_{j\in\mathbb{N}}$, $\mathrm{P}^\infty \in \overline{\mathcal{P}}_V\big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i\big)$.

(*ii*) *Let us consider the sequence* $(\mathrm{P}_k)_{k\in\mathbb{N}^*}$ *of probability measures such that* $\mathrm{P}_k \in \overline{\mathcal{P}}_V(\nu^k)$ *for each* $k\in\mathbb{N}^*$. *If*

$$\sup_{k\geq 1}\int_{\mathbb{R}^n}|x'|^{p'}\nu^k(\mathrm{d}x') < \infty$$

then $(\mathrm{P}_k)_{k\in\mathbb{N}^*}$ *is pre-compact in* $\mathcal{P}_p(\overline{\Omega})$ *for the metric* $\mathcal{W}_p$ *and for every* $\mathrm{P}_\infty \in \mathcal{P}(\overline{\Omega})$ *the limit of any sub-sequence* $(\mathrm{P}_{k_j})_{j\in\mathbb{N}^*}$, $\mathrm{P}_\infty \in \overline{\mathcal{P}}_V\big(\lim_{j\to\infty}\nu^{k_j}\big)$.

Proof. (*i*) Thanks to Proposition A.2 or/and Proposition B.1 of [49], as U is compact, it is easy to check that $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ is precompact on $\mathcal{P}_p(\overline{\Omega})$ for the metric $\mathcal{W}_p$.

Let P^∞ be a limit and $(\mathrm{P}^{N_j})_{j\in\mathbb{N}^*}$ the corresponding sub–sequence. For sake of simplicity, denote $(\mathrm{P}^{N_j})_{j\in\mathbb{N}^*} = (\mathrm{P}^N)_{N\in\mathbb{N}^*}$ and $\nu := \lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i$.

Now, let us show $\mathrm{P}^\infty \in \overline{\mathcal{P}}_V(\nu)$. Let $f\in C_b^2(\mathbb{R}^n)$. For each $t\in[0,T]$, denote $N_t(B_{t\wedge\cdot},\Lambda_{t\wedge\cdot},\mu_{t\wedge\cdot})(f) = N_t(f)$ to specify the dependence w.r.t. (B,μ,Λ) (see definition (4.4.1)). Notice that the function $(t,\mathbf{b},\pi,q)\in[0,T]\times\mathcal{C}^\ell\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\to N_t(\mathbf{b}_{t\wedge\cdot},q_{t\wedge\cdot},\pi_{t\wedge\cdot})(f)\in\mathbb{R}$ is continuous and bounded. It is straightforward to check that: for all $t\in[0,T]$

$$N_t\big(B_{t\wedge\cdot},(\delta_{\varphi^N_s}(\mathrm{d}m)\mathrm{d}s)_{t\wedge\cdot},\varphi^{N,\mathbf{X}}_{t\wedge\cdot}\big)(f) = \frac{1}{N}\sum_{i=1}^N\int_0^t\nabla f(\mathbf{X}^{\alpha,i}_r-\sigma_0 B_r)\sigma(r,\mathbf{X}^{\alpha,i}_r,\varphi^{N,\mathbf{X}}_{r\wedge\cdot},\varphi^N_r,\alpha^i_r)\mathrm{d}\mathbf{W}^i_r,\ \mathbb{P}^N_\nu\text{–a.e.}$$

With the same techniques used in the proof of [104, Proposition 5.1] or Proposition 3.4.6 in Chapter 3, one has

$$\begin{aligned}
\mathbb{E}^{\mathrm{P}^\infty}\Big[\big|(N_t(f)\big|^2\Big] &= \mathbb{E}^{\mathrm{P}^\infty}\Big[\big|(N_t(B_{t\wedge\cdot},\Lambda_{t\wedge\cdot},\mu_{t\wedge\cdot})(f)\big|^2\Big] = \lim_{N\to\infty}\mathbb{E}^{\mathrm{P}^N}\Big[\big|(N_t(B_{t\wedge\cdot},\Lambda_{t\wedge\cdot},\mu_{t\wedge\cdot})(f)\big|^2\Big]\\
&= \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\big|(N_t\big(B_{t\wedge\cdot},(\delta_{\varphi^N_s}(\mathrm{d}m)\mathrm{d}s)_{t\wedge\cdot},\varphi^{N,\mathbf{X}}_{t\wedge\cdot}\big)(f)\big|^2\Big]\\
&= \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\Big|\frac{1}{N}\sum_{i=1}^N\int_0^t\nabla f(\mathbf{X}^{\alpha,i}_r-\sigma_0 B_r)\sigma(r,\mathbf{X}^{\alpha,i}_r,\varphi^{N,\mathbf{X}},\varphi^N_r,\alpha^i_r)\mathrm{d}\mathbf{W}^i_r\Big|^2\Big]\\
&= \lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\int_0^t\Big|\nabla f(\mathbf{X}^{\alpha,i}_r-\sigma_0 B_r)\sigma(r,\mathbf{X}^{\alpha,i}_r,\varphi^{N,\mathbf{X}},\varphi^N_r,\alpha^i_r)\Big|^2\mathrm{d}r\Big] = 0.
\end{aligned}$$

By taking (t,f) under a countable set of $[0,T]\times C_b^2(\mathbb{R}^n)$ then P^∞–a.e. $\omega\in\overline{\Omega}$, $N_t(f)=0$ for all $(t,f)\in[0,T]\times C_b^2(\mathbb{R}^n)$. For all $h\in C_b(\mathbb{R}^n)$, the map $(q,\pi)\in\mathbb{M}\times\mathcal{C}^n_{\mathcal{W}}\longrightarrow\int_0^T\int_{\mathcal{P}^n_U}\big|\langle h,m(\mathrm{d}z,U)\rangle-\langle h,\pi_t(\mathrm{d}z)\rangle\big|^2 q_t(\mathrm{d}m)\mathrm{d}t\in\mathbb{R}$ is bounded and continuous (see for instance Lemma 4.7.1), one finds

$$\begin{aligned}
&\mathbb{E}^{\mathrm{P}^\infty}\Big[\int_0^T\int_{\mathcal{P}^n_U}\big|\langle h,m(\mathrm{d}z,U)\rangle-\langle h,\mu_t(\mathrm{d}z)\rangle\big|^2\Lambda_t(\mathrm{d}m)\mathrm{d}t\Big]\\
&= \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\int_0^T\int_{\mathcal{P}^n_U}\big|\langle h,m(\mathrm{d}z,U)\rangle-\langle h,\varphi^{N,\mathbf{X}}_t(\mathrm{d}z)\rangle\big|^2\delta_{\varphi^N_t}(\mathrm{d}m)\mathrm{d}t\Big] = \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\int_0^T\Big|\frac{1}{N}\sum_{i=1}^N[h(\mathbf{X}^{\alpha,i}_t)-h(\mathbf{X}^{\alpha,i}_t)]\Big|^2\mathrm{d}t\Big] = 0,
\end{aligned}$$

by taking h under a countable set of $C_b(\mathbb{R}^n)$, one concludes $\Lambda_t\big(\mathbb{Z}_{\mu_t}\big)=1$ $\mathrm{P}^\infty\otimes\mathrm{d}t$–a.e. It is obvious that $(B_t)_{t\in[0,T]}$ is a $(\mathrm{P}^\infty,\overline{\mathbb{F}})$ Wiener process. Let $Q\in\mathbb{N}^*$, and $(h^q)_{q\in\{1,..,Q\}}:\mathbb{R}^n\to\mathbb{R}^Q$ be bounded functions, one has

$$\mathbb{E}^{\mathrm{P}^\infty}\Big[\prod_{q=1}^Q\langle h^q,\mu_0\rangle\Big] = \prod_{q=1}^Q\langle h^q,\nu\rangle.$$

Let us show this result when $Q = 2$, when $Q \in \mathbb{N}^*$, the proof is similar.

$$\mathbb{E}^{\mathrm{P}^\infty}\left[\langle h^1, \mu_0\rangle\langle h^2, \mu_0\rangle\right] = \lim_N \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}^{\mathbb{P}_\nu^N}\left[h^1(\mathbf{X}_0^{\alpha,i})h^2(\mathbf{X}_0^{\alpha,j})\right] = \lim_N \frac{1}{N^2}\sum_{i=1}^N \langle h^1, \nu^i\rangle\langle h^2, \nu^i\rangle + \frac{1}{N^2}\sum_{i\neq j}^N \langle h^1, \nu^i\rangle\langle h^2, \nu^j\rangle$$
$$= \lim_N \langle h^1, \frac{1}{N}\sum_{i=1}^N \nu^i\rangle\langle h^2, \frac{1}{N}\sum_{i=1}^N \nu^i\rangle = \langle h^1, \nu\rangle\langle h^2, \nu\rangle,$$

by Proposition 3.5.1, $\mathrm{P}^\infty \circ (\mu_0)^{-1} = \delta_\nu$, then $\mu_0 = \nu$, P^∞–a.e. All these results allow to deduce the first statement of this proposition.

(ii) For the second part of this proposition, notice that, thanks to Lemma 4.8.2,

$$\sup_{k\in\mathbb{N}^*} \mathbb{E}^{\mathrm{P}_k}\left[\sup_{t\in[0,T]}\int_{\mathbb{R}^n} |x|^{p'}\vartheta_t(\mathrm{d}x)\right] \leq K\left[1 + \sup_{k\in\mathbb{N}^*}\int_{\mathbb{R}^n} |x'|^{p'}\nu^k(\mathrm{d}x')\right] < \infty$$

and

$$\limsup_{\delta\to 0}\sup_{k\in\mathbb{N}^*}\sup_{\tau} \mathbb{E}^{\mathrm{P}^k}\left[\mathcal{W}_p\big(\vartheta_{(\tau+\delta)\wedge T}, \vartheta_\tau\big)\right] = 0,$$

where τ is a $[0,T]$–valued $\overline{\mathbb{F}}$–stopping time, and recall that $(\vartheta)_{t\in[0,T]}$ is the $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{F}$–adapted continuous process defined in equation (4.8.4). Then by Aldous' criterion [92, Lemma 16.12] (see also proof of [49, Proposition-B.1]), $\big(\mathrm{P}_k \circ \big((\vartheta_t)_{t\in[0,T]}\big)^{-1}\big)_{k\in\mathbb{N}^*}$ is relatively compact for the metric $\mathcal{W}_p$. Then, using the fact that $\mathrm{P}_k \in \overline{\mathcal{P}}_V(\nu^k)$ for each $k \in \mathbb{N}^*$ and the relation between (ϑ, Θ) and the canonical processes (μ, Λ) (see equation (4.8.4)), we deduce that $(\mathrm{P}_k)_{k\in\mathbb{N}^*} = \big(\mathrm{P}_k \circ (\mu, \Lambda, B)^{-1}\big)_{k\in\mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p$. The rest of the proof is similar to the previous proof. □

Proposition 4.8.5. *Let* Assumption 4.1.1 *hold true, $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$ with $p' > p$ and $(\nu^i)_{i\in\mathbb{N}} \subset \mathcal{P}_{p'}(\mathbb{R}^n)$ such that*

$$\sup_{i\in\mathbb{N}}\int_{\mathbb{R}^n} |x'|^{p'}\nu^i(\mathrm{d}x') < \infty \text{ and } \nu^i \xrightarrow[i\to\infty]{\mathcal{W}_p} \nu, \text{ then } \lim_{i\to\infty} V_S(\nu^i) = V_S\big(\nu\big).$$

In particular, the map $V_S : \mathcal{P}_{p'}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ is continuous.

Proof. By Theorem 4.5.1, one has $V_S(\nu) = V_V(\nu)$, thanks to this result, the proof is similar to the proof of Proposition 3.2.8. Let $(\delta^k)_{k\in\mathbb{N}} \subset \mathbb{N}^*$ with $\lim_k \delta^k = 0$ and $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ be a sequence such that $\mathrm{P}^k \in \overline{\mathcal{P}}_V(\nu^k)$ and $V_V(\nu^k) - \delta^k \leq \mathbb{E}^{\mathrm{P}^k}[J(\mu,\Lambda)]$. By Proposition 4.8.4, $(\mathrm{P}^k)_{k\in\mathbb{N}}$ is relatively compact on $(\mathcal{P}_p(\overline{\Omega}), \mathcal{W}_p)$ and if $\mathrm{P} \in \mathcal{P}(\overline{\Omega})$ is the limit of any sub-sequence $(\mathrm{P}^{k_j})_{j\in\mathbb{N}^*}$ then $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$. Using Assumption 4.1.1, by convergence of $(\mathrm{P}^{k_j})_{j\in\mathbb{N}^*}$, one has $\lim_j |\mathbb{E}^{\mathrm{P}^{k_j}}[J(\mu,\Lambda)] - \mathbb{E}^{\mathrm{P}}[J(\mu,\Lambda)]| = 0$. Therefore, one gets

$$\limsup_{k\to\infty} V_V(\nu^k) \leq \lim_{j\to\infty} \mathbb{E}^{\mathrm{P}^{k_j}}[J(\mu,\Lambda)] = \mathbb{E}^{\mathrm{P}}[J(\mu,\Lambda)] \leq V_V(\nu) = V_S(\nu).$$

By Proposition 3.4.4, $V_S(\nu) \leq \liminf_{j\to\infty} V_S(\nu^{k_j})$, this is enough to conclude that $\lim_{k\to\infty} V_S(\nu^k) = V_S(\nu)$, and deduce the result. □

4.8.2.2 Proof of Theorem 4.5.3

By combining Theorem 4.5.1, Proposition 4.8.4 and Proposition 4.8.5, this proof turns to be the same used in the proof of Theorem 3.2.7. For the sake of completeness, we repeat the proof.

(i) By Proposition 4.8.4 (with the same notations), if the sequence $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ is such that: $V_S^N(\nu^1, ..., \nu^N) - \epsilon_N \leq \mathbb{E}^{\mathrm{P}^N}[J(\mu,\Lambda)]$, where $(\epsilon^N)_{N\in\mathbb{N}^*}$ is sequence with $\lim_N \epsilon^N = 0$, then $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ is relatively compact on $(\mathcal{P}_p(\overline{\Omega}), \mathcal{W}_p)$ and for every $\mathrm{P}^\infty \in \mathcal{P}(\overline{\Omega})$ the limit of the sub-sequence $(\mathrm{P}^{N_j})_{j\in\mathbb{N}^*}$, $\mathrm{P}^\infty \in \overline{\mathcal{P}}_V\big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\big)$, therefore

$$\limsup_{N\to\infty} V_S^N(\nu^1, ..., \nu^N) \leq \lim_{j\to\infty} \mathbb{E}^{\mathrm{P}^{N_j}}[J(\mu,\Lambda)] = \mathbb{E}^{\mathrm{P}}[J(\mu,\Lambda)] \leq V_V\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big).$$

Then, as $\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\in\mathcal{P}_{p'}(\mathbb{R}^n)$ and Assumption 4.1.1 holds true one can deduce that $V_V\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big)=V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big)$. By Proposition 3.4.4, $V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big)\leq\liminf_{j\to\infty}V_S^{N_j}(\nu^1,...,\nu^{N_j})$. To recap

$$V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big)\leq\liminf_{j\to\infty}V_S^{N_j}(\nu^1,...,\nu^{N_j})\leq\limsup_{j\to\infty}V_S^{N_j}(\nu^1,...,\nu^{N_j})\leq V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big).$$

(ii) Let $(N_j)_{j\in\mathbb{N}}$ be the sequence corresponding to :

$$\limsup_{N\to\infty}\Big|V_S^N(\nu^1,...,\nu^N)-V_S\Big(\frac{1}{N}\sum_{i=1}^{N}\nu^i\Big)\Big|=\lim_{j\to\infty}\Big|V_S^{N_j}(\nu^1,...,\nu^{N_j})-V_S\Big(\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i\Big)\Big|.$$

By the previous proof, $\lim_{j\to\infty}V_S^{N_j}(\nu^1,...,\nu^{N_j})=V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i\Big)$, as $(\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i)_{j\in\mathbb{N}^*}$ is bounded in $(\mathcal{P}_{p'}(\mathbb{R}^n),\mathcal{W}_{p'})$ and converges in $(\mathcal{P}_p(\mathbb{R}^n),\mathcal{W}_p)$, by Proposition 4.8.5,

$$\lim_{j\to\infty}V_S\Big(\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i\Big)=V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu^i\Big),$$

this is enough to conclude the proof. □

4.8.3 Proof of Proposition 4.5.4

Notice that, for $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, by Theorem 4.5.1, $\overline{\mathcal{P}}_V^\star(\nu)$ is nonempty. Let us define the distance function to the set $\overline{\mathcal{P}}_V^\star(\nu)$, for each $Q\in\mathcal{P}(\overline{\Omega})$, $\Psi^\star(Q):=\inf_{\mathrm{P}^\star\in\overline{\mathcal{P}}_V^\star(\nu)}\mathcal{W}_p\big(Q,\mathrm{P}^\star\big)$. It is well know that, as $\overline{\mathcal{P}}_V^\star(\nu)$ is nonempty, the function $\Psi^\star:Q\in\mathcal{P}_p(\overline{\Omega})\to\mathbb{R}$ is continuous. Then by Proposition 4.8.4, $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ is pre-compact in $\mathcal{P}_p(\overline{\Omega})$ for the metric $\mathcal{W}_p$ and if $\mathrm{P}\in\mathcal{P}(\overline{\Omega})$ is the limit of any sub-sequence $(\mathrm{P}^{N_j})_{j\in\mathbb{N}^*}$, one have $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$. Under assumption 4.1.1, $\lim_{j\to\infty}\mathbb{E}^{\mathrm{P}^{N_j}}[J(\mu,\Lambda)]=\mathbb{E}^{\mathrm{P}}[J(\mu,\Lambda)]$. Combining Theorem 4.5.3 and Proposition 4.8.5, one has

$$\lim_{j\to\infty}V_S^{N_j}(\nu^1,...,\nu^{N_j})=V_S\Big(\lim_{j\to\infty}\frac{1}{N_j}\sum_{i=1}^{N_j}\nu_i\Big)=V_S(\nu)=V_V(\nu)\leq\mathbb{E}^{\mathrm{P}}[J(\mu,\Lambda)],$$

then $\mathrm{P}\in\overline{\mathcal{P}}_V^\star(\nu)$. Hence each limit of any sub-sequence of $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ belongs to $\overline{\mathcal{P}}_V^\star(\nu)$. Consequently, if $(\mathrm{P}^{N_j})_{j\in\mathbb{N}}$ is the sub-sequence corresponding to $\limsup_{N\to\infty}\Psi^\star(\mathrm{P}^N)=\lim_{j\to\infty}\Psi^\star(\mathrm{P}^{N_j})$, by continuity of $\Psi^\star$ and the fact that any limit is an optimal control, $\limsup_{N\to\infty}\Psi^\star(\mathrm{P}^N)=0$. The second part of this proposition is just a combination of Theorem 4.5.1, Proposition 3.4.4 and Theorem 4.5.3. This is enough to conclude the result.

4.8.4 Proofs of the case of closed loop controls

Assumption 4.8.6. *Here, we assume that $\ell=0$, and*

$$\sigma(t,x,\pi,m,u)=\tilde{\sigma}(t,x),\ \textit{for all}\ (t,x,\pi,m,u)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{P}^n_U\times U.$$

For simplicity, we will note σ instead of $\tilde{\sigma}$.

4.8.4.1 Proof of Theorem 4.6.2

Let $p'>p$, $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$. For $\alpha\in\mathcal{A}(\nu)$, we note $\mathbf{n}_t:=\mathcal{L}^{\mathbb{P}_\nu}(X_t^\alpha)$ and $\overline{\mathbf{n}}_t:=\mathcal{L}^{\mathbb{P}_\nu}(X_t^\alpha,\alpha_t)$ for each $t\in[0,T]$. then the pair $(\mathbf{n},\overline{\mathbf{n}})$ satisfies the following equation: $\mathrm{d}t$–a.e., and

$$\langle f(t,.),\mathbf{n}_t\rangle\ =\ \langle f(0,.),\nu\rangle+\int_0^t\Big[\langle\partial_t f(r,.),\mathbf{n}_r\rangle+\int_{\mathbb{R}^n\times U}\mathcal{A}_r[f(r,\cdot)](x,\mathbf{n},\overline{\mathbf{n}}_r,u)\overline{\mathbf{n}}_r^x(\mathrm{d}u)\mathbf{n}_r(\mathrm{d}x)\Big]\mathrm{d}r,$$

for all $(t,f) \in [0,T] \times C_b^{1,2}([0,T]\times\mathbb{R}^n)$, where the generator $\mathcal{A}$ is defined by

$$\mathcal{A}_t\varphi(x,\pi,\zeta,m,\bar{\nu},u) := \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\top(t,x)\nabla^2\varphi(x)\big] + b(t,x,\pi,m,u)^\top\nabla\varphi(x).$$

By [49, Proposition C.1], there exists a sequence of measurable function $(\alpha^k)_{k\in\mathbb{N}^*}$ with for each $k\in\mathbb{N}^*$, $\alpha^k : [0,T]\times\mathbb{R}^n \to U$ and for each $t\in[0,T]$,

$$\lim_{k\to\infty} \delta_{\alpha^k(t,x)}(\mathrm{d}u)G(x)\mathrm{d}x = \overline{\mathbf{n}}_t^x(\mathrm{d}u)(\mathrm{d}u)G(x)\mathrm{d}x, \text{ in the weak sense.} \tag{4.8.8}$$

Now, we define X^k a solution of

$$X_t^k = \xi + \int_0^t b\big(r, X_r^k, \mathbf{n}_{r\wedge\cdot}^k, \overline{\mathbf{n}}_r^k, \alpha^k(r,X_r^k)\big)\mathrm{d}r + \int_0^t \sigma\big(r,X_r^k\big)\mathrm{d}W_r,$$

with $\mathbf{n}_r^k := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r^k\big)$ and $\overline{\mathbf{n}}_r^k := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r^k, \alpha^k(r,X_r^k)\big)$, for all $r\in[0,T]$. By Lemma 4.7.14 and Lemma 4.7.13, there exists a sequence of continuous functions $(u^k)_{k\in\mathbb{N}^*}$ and u such that: $\mathbf{n}_t^k(\mathrm{d}x)\mathrm{d}t := u^k(t,x)\mathrm{d}x\mathrm{d}t$, and for each $(t,s)\in[0,T]\times[0,T]$ and $\bar{A}\subset\mathbb{R}^n$ compact

$$\lim_{k\to\infty}\sup_{(r,x)\in[s,t]\times\bar{A}} \big|u^k(r,x) - u(r,x)\big| = 0.$$

By Equation (4.8.8) and Lemma 4.7.15, we have

$$\lim_{k\to\infty} \delta_{\alpha^k(t,x)}(\mathrm{d}u)u^k(t,x)\mathrm{d}x = \overline{\mathbf{n}}_t^x(\mathrm{d}u)u(t,x)\mathrm{d}x, \text{ in the weak sense, } \mathrm{d}t\text{–a.e.}$$

If we note for each $t\in[0,T]$, $\mathbf{n}_t^\infty(\mathrm{d}x) := u(t,x)\mathrm{d}x$ We can deduce that

$$\langle f(t,\cdot),\mathbf{n}_t^\infty\rangle = \langle f(0,.),\nu\rangle + \int_0^t\Big[\langle\partial_t f(r,.),\mathbf{n}_r^\infty\rangle + \int_{\mathbb{R}^n\times U}\mathcal{A}_r[f(r,\cdot)](x,\mathbf{n}^\infty,\overline{\mathbf{n}}_r^{x'}(\mathrm{d}u')\mathbf{n}_r^\infty(\mathrm{d}x'),u)\overline{\mathbf{n}}_r^x(\mathrm{d}u)\mathbf{n}_r^\infty(\mathrm{d}x)\Big]\mathrm{d}r.$$

If we define

$$b^\star(t,x,\pi) := \int_U b(t,x,\pi,\overline{\mathbf{n}}_r^{x'}(\mathrm{d}u')\pi(\mathrm{d}x'),u)\overline{\mathbf{n}}_r^x(\mathrm{d}u).$$

By [106, Theorem 2.3], the SDE associated to the coefficients $(b^\star,\sigma)$ has a unique law. We can deduce that $\mathbf{n}^\infty = \mathbf{n}$. Consequently

$$\lim_{k\to\infty}\big(\mathbf{n}^k, \delta_{\overline{\mathbf{n}}_r^k}(\mathrm{d}m)\mathrm{d}r\big) = \big(\mathbf{n}, \delta_{\overline{\mathbf{n}}_r}(\mathrm{d}m)\mathrm{d}r\big), \text{ in } \mathcal{W}_p.$$

4.8.4.2 Proof of Theorem 4.6.3

First of all, it is obvious that: for each $N\in\mathbb{N}^*$,

$$\sup_{\alpha\in\mathcal{A}_N^m}\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}^\star}\Big[\int_0^T L\big(t,\mathbf{X}_t^{\alpha,i},\varphi_{t\wedge\cdot}^{N,\mathbf{X}},\varphi_t^N,\alpha^i(t,\mathbf{X}_t^\alpha)\big)\mathrm{d}t + g\big(\mathbf{X}_T^{\alpha,i},\varphi_{T\wedge\cdot}^{N,\mathbf{X}}\big)\Big] \le V_S^N(\nu,\dots,\nu).$$

Therefore, one has by Theorem 4.5.3,

$$\limsup_{N\to\infty}\sup_{\alpha\in\mathcal{A}_N^m}\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}^\star}\Big[\int_0^T L\big(t,\mathbf{X}_t^{\alpha,i},\varphi_{t\wedge\cdot}^{N,\mathbf{X}},\varphi_t^N,\alpha^i(t,\mathbf{X}_t^\alpha)\big)\mathrm{d}t + g\big(\mathbf{X}_T^{\alpha,i},\varphi_{T\wedge\cdot}^{N,\mathbf{X}}\big)\Big] \le V_S(\nu).$$

Next, let $\alpha\in\mathcal{A}^m$, and X^α solution of: for all $t\in[0,T]$

$$X_t = \xi + \int_0^t b\big(r,X_r,\mathbf{n}_{r\wedge\cdot},\overline{\mathbf{n}}_r,\alpha(r,X_r)\big)\mathrm{d}r + \int_0^t\sigma\big(r,X_r\big)\mathrm{d}W_r,\ \mathbb{P}^\star\text{–a.e.}$$

with $\mathbf{n}_r := \mathcal{L}^{\mathbb{P}_\nu}(X_r)$ and $\overline{\mathbf{n}}_r := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r, \alpha(r, X_r)\big)$, for all $r \in [0,T]$. Let us define $\mathbf{X}^\alpha := (\mathbf{X}^{\alpha,1}_\cdot, \dots, \mathbf{X}^{\alpha,N}_\cdot)$ the solution of the following system of SDEs, for each $i \in \{1, \dots, N\}$, $\mathbb{E}^{\mathbb{P}^N_\nu}\big[\|\mathbf{X}^{\alpha,i}\|^p\big] < \infty$,

$$\mathbf{X}^{\alpha,i}_t = \xi^i + \int_0^t b\big(r, \mathbf{X}^{\alpha,i}_r, \varphi^{N,\mathbf{X}}_{r\wedge\cdot}, \varphi^N_r, \alpha(r, \mathbf{X}^{\alpha,i}_r)\big)\mathrm{d}r + \int_0^t \sigma\big(r, \mathbf{X}^{\alpha,i}_r\big)\mathrm{d}\mathbf{W}^i_r, \text{ for all } t \in [0,T], \tag{4.8.9}$$

with

$$\varphi^{N,\mathbf{X}}_t(\mathrm{d}x) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}^{\alpha,i}_t\right)}(\mathrm{d}x) \text{ and } \varphi^N_t(\mathrm{d}x, \mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}^{\alpha,i}_t,\, \alpha(t, \mathbf{X}^{\alpha,i}_t)\right)}(\mathrm{d}x, \mathrm{d}u), \text{ for all } t \in [0,T].$$

By [106, Theorem 2.5], one has $(\mathcal{L}^{\mathbb{P}^\star}(\varphi^{N,\mathbf{X}}))_{N\in\mathbb{N}^*}$ converges to $\mathcal{L}^{\mathbb{P}^\star}(\mathbf{n})$ for $\mathcal{W}_p$, and for all $t \in [0,T]$ and all function measurable bounded $\phi : \mathbb{R}^n \to \mathbb{R}$

$$\lim_{N\to\infty} \langle \phi, \varphi^{N,\mathbf{X}}_t \rangle = \langle \phi, \mathbf{n}_t \rangle, \text{ in probability sense.}$$

Using the fact that $\alpha : [0,T] \times \mathbb{R}^n \to U$ is measurable bounded, we deduce that $\lim_{N\to\infty} \mathcal{L}^{\mathbb{P}^\star}\big(\varphi^{N,\mathbf{X}}, \delta_{\varphi^N_t}(\mathrm{d}m)\mathrm{d}t\big) = \mathcal{L}^{\mathbb{P}^\star}\big(\mathbf{n}, \delta_{\overline{\mathbf{n}}_t}(\mathrm{d}m)\mathrm{d}t\big)$ for $\mathcal{W}_p$. Consequently, by using Theorem 4.6.2, one gets

$$V_S(\nu) \le \liminf_{N\to\infty} \sup_{\alpha\in\mathcal{A}^m_N} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}^\star}\bigg[\int_0^T L\big(t, \mathbf{X}^{\alpha,i}_t, \varphi^{N,\mathbf{X}}_{t\wedge\cdot}, \varphi^N_t, \alpha^i(t, \mathbf{X}^\alpha_t)\big)\mathrm{d}t + g\big(\mathbf{X}^{\alpha,i}_T, \varphi^{N,\mathbf{X}}_{T\wedge\cdot}\big)\bigg].$$

We can conclude.

4.9 Appendix: some technical results

4.9.1 Technical proofs

Here we shall successively give the proofs of Lemma 4.7.2, Proposition 4.7.9 and Proposition 4.7.10.

Proof of Lemma 4.7.2. By taking, for $\delta > 0$,

$$\mathbf{q}^\delta_t(\mathrm{d}m, \mathrm{d}m') := \frac{1}{\delta}\int_{(t-\delta)\vee 0}^t \hat{\mathbf{q}}^\delta_s(\mathrm{d}m, \mathrm{d}m')\mathrm{d}s, \text{ for all } t \in [0,T],$$

using similar approach to [119, Lemma 4.4], the sequence $(\hat{\mathbf{q}}^\delta)_{\delta>0}$ satisfying: for each $\delta > 0$, $\hat{\mathbf{q}}^\delta_t(\mathrm{d}m, \mathrm{d}m')\mathrm{d}t \in \mathbb{M}((\mathcal{P}^n_U)^2)$, $\hat{\mathbf{q}}^\delta : t \in [0,T] \to \hat{\mathbf{q}}^\delta_t(\mathrm{d}m, \mathrm{d}m') \in (\mathcal{P}^n_U)^2$ is continuous, and $\lim_{\delta\to 0} \hat{\mathbf{q}}^\delta_t = \hat{\mathbf{q}}_t$, in weakly sense for $\mathrm{d}s$ almost every $t \in [0,T]$.

Let us fix $t_0 \in [0,T]$, $\phi \in C^2_b(\mathbb{R}^n)$, by [97, Chapter 2 Section 9 Theorem 10], there exists $v^{\epsilon,\delta} \in C^{1,2}_b([0,t_0]\times\mathbb{R}^n)$ satisfying:

$$\partial_t v^{\epsilon,\delta}(t,x) + \mathcal{A}^\epsilon_t[v^{\epsilon,\delta}(t,.)][\mathbf{b}, \mathbf{n}, \mathbf{z}, \hat{\mathbf{q}}^\delta_t](x) = 0 \text{ for all } (t,x) \in [0,t_0)\times\mathbb{R}^n \text{ and } v^{\epsilon,\delta}(t_0,x) = \phi(x). \tag{4.9.1}$$

Notice that, under Assumption 4.1.1, for each $\epsilon > 0$, $\hat{a}^\epsilon[\mathbf{b}, \mathbf{n}, \mathbf{z}, \kappa](t,x) \ge \theta \mathrm{I}_{n\times n}$ for all $(t,x,\kappa) \in [0,T]\times\mathbb{R}^n\times\mathcal{P}((\mathcal{P}^n_U)^2)$. By Proposition 4.9.3, for all $t \in [0,T]$, $x \in \mathbb{R}^n \to (\hat{a}^\epsilon)^{1/2}[\mathbf{n}, \mathbf{z}, \kappa](t,x) \in \mathbb{S}^{n\times n}$ is Lipschitz (with Lipschitz constant independent of $(t, \mathbf{n}, \mathbf{z}, \kappa)$).

Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a probability space supporting W a $\mathbb{R}^n$–valued $(\mathbb{P}, \mathbb{F})$–Wiener process, and ξ a $\mathcal{F}_0$–random variable such that $\mathcal{L}^{\mathbb{P}}(\xi) \in \mathcal{P}_p(\mathbb{R}^n)$. Now, for every $t \in [0,t_0]$, denote by $X^{\epsilon,\delta,t,\xi} := X$ the continuous process unique strong solution of:

$$X_s = \xi + \int_t^s \hat{b}^\epsilon[\mathbf{b}, \mathbf{n}, \mathbf{z}, \hat{\mathbf{q}}^\delta_r](r, X_r)\mathrm{d}r + \int_t^s (\hat{a}^\epsilon)^{1/2}[\mathbf{b}, \mathbf{n}, \mathbf{z}, \hat{\mathbf{q}}^\delta_r](r, X_r)\mathrm{d}W_r \text{ for all } s \in [t,T], \ \mathbb{P}\text{–a.e..}$$

By applying Itô formula, one has (Feynman Kac's formula)

$$v^{\epsilon,\delta}(t,x)=\mathbb{E}^{\mathbb{P}}\Big[\phi(X_{t_0}^{\epsilon,\delta,t,\xi})\big|\xi=x\Big]=\mathbb{E}^{\mathbb{P}}\Big[\phi(X_{t_0}^{\epsilon,\delta,t,x})\Big]\text{ for all }(t,x)\in[0,t_0]\times\mathbb{R}^n. \tag{4.9.2}$$

By definition of $\hat{a}^\epsilon$ and $\hat{b}^\epsilon$ (see (4.7.2)), and by using the fact that $\hat{\mathbf{q}}^\delta\in\mathbb{M}((\mathcal{P}_U^n)^2)$, there exists a constant C_ϵ (independent of $\delta>0$) such that: for all $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$\big|\nabla^2\big(\hat{b}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta],\hat{a}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta]\big)(t,x)\big|+\big|\nabla\big(\hat{b}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta],\hat{a}^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta]\big)(t,x)\big|\leq C_\epsilon,$$

then by [97, Chapter 2 Section 8 Theorem 8, Theorem 7], for two unit vectors $(w^1,w^2)\in\mathbb{R}^n\times\mathbb{R}^n$, there exist two $\mathbb{R}^n$–valued $\mathbb{F}$–adapted continuous processes $Y^{\epsilon,\delta,t,x,w^1}:=Y$ and $Z^{\epsilon,\delta,t,x,w^1,w^2}:=Z$ such that

$$\lim_{h\to0}\mathbb{E}^{\mathbb{P}}\Big[\sup_{s\in[t,t_0]}\Big|\frac{X_s^{\epsilon,\delta,t,x+hw^1}-X_s^{\epsilon,\delta,t,x}}{h}-Y_s\Big|\Big]=0\text{ and }\lim_{h\to0}\mathbb{E}^{\mathbb{P}}\Big[\sup_{s\in[t,t_0]}\Big|\frac{Y_s^{\epsilon,\delta,t,x+hw^2,w^1}-Y_s^{\epsilon,\delta,t,x,w^1}}{h}-Z_s\Big|\Big]=0,$$

formally speaking, Y can be seen as the "derivative" (given a direction w^1) of $x\to X^x$, and Z the "derivative" (given w^1 and another direction w^2) of Y. In addition $\mathbb{E}^{\mathbb{P}}\big[\sup_{s\in[t,t_0]}|Y_s|+|Z_s|\big]\leq K_\epsilon$, with K_ϵ depending on ϵ but not of δ.
As $\phi\in C_b^2(\mathbb{R}^n)$, by using the previous results and equation (4.9.2), there exists $\hat{K}_\epsilon>0$ (independent of δ) satisfying: for all $(t,x)\in[0,T]\times\mathbb{R}^n$

$$\big|\nabla^2v^{\epsilon,\delta}(t,x)\big|+\big|\nabla v^{\epsilon,\delta}(t,x)\big|+\big|v^{\epsilon,\delta}(t,x)\big|\leq\hat{K}_\epsilon. \tag{4.9.3}$$

Therefore, for all $\epsilon>0$,

$$\big|\mathcal{A}_t^\epsilon v^{\epsilon,\delta}(t,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t](x)-\mathcal{A}_t^\epsilon v^{\epsilon,\delta}(t,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta](x)\big|\leq\hat{K}_\epsilon\big(\big|[\hat{b}^\epsilon,\hat{a}^\epsilon][\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t](t,x)-[\hat{b}^\epsilon,\hat{a}^\epsilon][\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta](t,x)\big|\big),$$

by definition (4.7.2), as $\lim_{\delta\to0}\hat{\mathbf{q}}_t^\delta=\hat{\mathbf{q}}_t$, for $\mathrm{d}s$ almost every $t\in[0,T]$, one gets:

$$\lim_\delta\big|\mathcal{A}_t^\epsilon v^{\epsilon,\delta}(t,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t](x)-\mathcal{A}_t^\epsilon v^{\epsilon,\delta}(t,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_t^\delta](x)\big|=0, \tag{4.9.4}$$

for each $\epsilon>0$ and $x\in\mathbb{R}^n$, for $\mathrm{d}s$ almost every $t\in[0,T]$.

Uniqueness: For each $\epsilon>0$ fixed, let us prove the uniqueness of $(\mathbf{n}_t^\epsilon)_{t\in[0,T]}$ solution of equation (4.7.5). Let $\mathbf{n}^{1,\epsilon}$ and $\mathbf{n}^{2,\epsilon}$ be two solutions of the Fokker-Planck equation (4.7.5) mentioned in the Lemma, for any $t_0\in[0,T]$ and $\phi\in C_b^2(\mathbb{R}^n)$, denote by $v:=v^{\epsilon,\delta,\phi,t_0}$ solution of (4.9.1) associated to (t_0,ϕ). One finds

$$\begin{aligned}
&\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{1,\epsilon}(\mathrm{d}y)-\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{2,\epsilon}(\mathrm{d}y)\\
&=\int_0^{t_0}\langle\partial_tv(r,.),\mathbf{n}_r^{1,\epsilon}\rangle-\langle\partial_tv(r,.),\mathbf{n}_r^{2,\epsilon}\rangle+\langle\mathcal{A}_r^\epsilon v[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](.),\mathbf{n}_r^{1,\epsilon}\rangle-\langle\mathcal{A}_r^\epsilon v[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](.),\mathbf{n}_r^{2,\epsilon}\rangle\mathrm{d}r\\
&=\int_0^{t_0}\langle\mathcal{A}_r^\epsilon v[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](\cdot)-\mathcal{A}_r^\epsilon v[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r^\delta](\cdot),\mathbf{n}_r^{1,\epsilon}\rangle+\langle\mathcal{A}_r^\epsilon v[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](\cdot)-\mathcal{A}_r^\epsilon v[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r^\delta](\cdot),\mathbf{n}_r^{2,\epsilon}\rangle\ \mathrm{d}r,
\end{aligned}$$

by (4.9.4), given $\epsilon>0$, after taking $\delta\to0$, by Lebesgue's dominated convergence theorem, $\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{1,\epsilon}(\mathrm{d}y)=\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{2,\epsilon}(\mathrm{d}y)$, this is true for all $(t_0,\phi)\in[0,T]\times C_b^2(\mathbb{R}^n)$, then $\mathbf{n}^{1,\epsilon}=\mathbf{n}^{2,\epsilon}$.

Convergence of $\mathbf{n}^\epsilon$: Now, we show the second assertion of our lemma. Using the fact that $\hat{\mathbf{q}}_t(\mathbb{Z}_{[\mathbf{n}_t]} \times \mathcal{P}_U^n) = 1$ $\mathrm{d}t$–almost surely $t \in [0,T]$, one gets for all $t \in [0,T]$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} v^{\epsilon,\delta}(t,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\mathbf{n}_t(\mathrm{d}z)\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} v^{\epsilon,\delta}(t,z-y)\mathbf{n}_t(\mathrm{d}z)G_\epsilon(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Big[\int_{\mathbb{R}^n} \partial_t v^{\epsilon,\delta}(s,z-y)\mathbf{n}_s(\mathrm{d}z) \\
&\qquad + \int_{(\mathcal{P}_U^n)^2} \int_{\mathbb{R}^n\times U} \mathcal{A}_s[v^{\epsilon,\delta}(s,\cdot-y)](z,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar\nu,u) m(\mathrm{d}z,\mathrm{d}u)\hat{\mathbf{q}}_s(\mathrm{d}m,\mathrm{d}\bar\nu)\Big] G_\epsilon(y)\ \mathrm{d}y\ \mathrm{d}s \\
&= \int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[\partial_t v^{\epsilon,\delta}(s,z-y) \\
&\qquad + \int_{(\mathcal{P}_U^n)^2} \int_{U} \mathcal{A}_s[v^{\epsilon,\delta}(s,\cdot-y)](z,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar\nu,u) m^z(\mathrm{d}u)\hat{\mathbf{q}}_s(\mathrm{d}m,\mathrm{d}\bar\nu)\Big] G_\epsilon(y)\ \mathbf{n}_s(\mathrm{d}z)\ \mathrm{d}y\ \mathrm{d}s \\
&= \int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Big[\partial_t v^{\epsilon,\delta}(s,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\ \mathbf{n}_s(\mathrm{d}z) \\
&\qquad + \int_{(\mathcal{P}_U^n)^2} \int_{\mathbb{R}^n\times U} \hat b(s,z,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar\nu,u)\nabla v^{\epsilon,\delta}(s,y) G_\epsilon(z-y) m^z(\mathrm{d}u)\mathbf{n}_s(\mathrm{d}z)\hat{\mathbf{q}}_s(\mathrm{d}m,\mathrm{d}\bar\nu) \\
&\qquad + \int_{(\mathcal{P}_U^n)^2} \int_{\mathbb{R}^n\times U} \frac{1}{2}\mathrm{Tr}\big[\hat a(s,z,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar\nu,u)\nabla^2 v^{\epsilon,\delta}(s,y)\big] G_\epsilon(z-y) m^z(\mathrm{d}u)\mathbf{n}_s(\mathrm{d}z)\hat{\mathbf{q}}_s(\mathrm{d}m,\mathrm{d}\bar\nu)\Big]\ \mathrm{d}y\ \mathrm{d}s \\
&= \int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} \Big[\partial_t v^{\epsilon,\delta}(s,y)(\mathbf{n}_s)^{(\epsilon)}(y) \\
&\qquad + \int_{(\mathcal{P}_U^n)^2} \int_{\mathbb{R}^n\times U} \hat b(s,z,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar\nu,u) \frac{G_\epsilon(z-y)}{(m(\mathrm{d}z',U))^{(\epsilon)}(y)} m(\mathrm{d}z,\mathrm{d}u)\hat{\mathbf{q}}_s(\mathrm{d}m,\mathrm{d}\bar\nu)\nabla v^{\epsilon,\delta}(s,y)(\mathbf{n}_s)^{(\epsilon)}(y) \\
&\qquad + \frac{1}{2}\mathrm{Tr}\Big[\int_{(\mathcal{P}_U^n)^2} \int_{\mathbb{R}^n\times U} \hat a(s,z,\mathbf{b},\mathbf{n},\mathbf{z},m,\bar\nu,u) \frac{G_\epsilon(z-y)}{(m(\mathrm{d}z',U))^{(\epsilon)}(y)} m(\mathrm{d}z,\mathrm{d}u)\hat{\mathbf{q}}_s(\mathrm{d}m,\mathrm{d}\bar\nu)\nabla^2 v^{\epsilon,\delta}(s,y)\Big](\mathbf{n}_s)^{(\epsilon)}(y)\Big]\mathrm{d}y\ \mathrm{d}s \\
&= \int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y) \int_{\mathbb{R}^n} G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y + \int_0^t \int_{\mathbb{R}^n} [\partial_t v^{\epsilon,\delta}(r,y) + \mathcal{A}_r^\epsilon[v^{\epsilon,\delta}(r,\cdot)][\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,y)](\mathbf{n}_r)^{(\epsilon)}(y)\ \mathrm{d}y\ \mathrm{d}r,
\end{aligned}
$$

where for each $\pi \in \mathcal{P}(\mathbb{R}^n)$, we write $\pi^{(\epsilon)}(x) := \int_{\mathbb{R}^n} G_\epsilon(x-z)\pi(\mathrm{d}z)$, for all $x \in \mathbb{R}^n$.

Then by (4.9.1)

$$
\begin{aligned}
&\int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y)\nu^{(\epsilon)}(y)\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \phi(y)(\mathbf{n}_{t_0})^{(\epsilon)}(y)\mathrm{d}y + \int_0^{t_0}\int_{\mathbb{R}^n} [\mathcal{A}_r^\epsilon[v^{\epsilon,\delta}(r,\cdot)][\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r^\delta](y) - \mathcal{A}_r^\epsilon[v^{\epsilon,\delta}(r,\cdot)][\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](y)](\mathbf{n}_r)^{(\epsilon)}(y)\mathrm{d}y\mathrm{d}r.
\end{aligned}
$$

By equation (4.9.2), one has

$$
\int_{\mathbb{R}^n} v^{\epsilon,\delta}(0,y)\int_{\mathbb{R}^n} G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y = \int_{\mathbb{R}^n} \mathbb{E}\Big[\phi(X_{t_0}^{\epsilon,\delta,0,\xi})\big|\xi = y\Big]\nu^{(\epsilon)}(y)\mathrm{d}y = \int_{\mathbb{R}^n}\phi(x)\mathbf{n}_{t_0}^{\epsilon,\delta}(\mathrm{d}x),
$$

where $\mathbf{n}_t^{\epsilon,\delta} := \mathcal{L}^{\mathbb{P}}(X_t^{\epsilon,\delta,0,\xi^\epsilon})$ for $t \in [0,T]$, with $\mathcal{L}^{\mathbb{P}}(\xi^\epsilon)(\mathrm{d}y) = \nu^{(\epsilon)}(y)\mathrm{d}y$.

Combining the previous equality,

$$\begin{aligned}\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) &= \int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) - \int_{\mathbb{R}^n}\phi(y)\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathbf{n}_{t_0}(\mathrm{d}z)\mathrm{d}y + \int_{\mathbb{R}^n}\phi(y)\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathbf{n}_{t_0}(\mathrm{d}z)\mathrm{d}y\\ &= \int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) - \int_{\mathbb{R}^n}\phi(y)\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathbf{n}_{t_0}(\mathrm{d}z)\mathrm{d}y + \int_{\mathbb{R}^n}v^\epsilon(0,y)\int_{\mathbb{R}^n}G_\epsilon(z-y)\nu(\mathrm{d}z)\mathrm{d}y\\ &\quad + \int_0^{t_0}\int_{\mathbb{R}^n}\Big[\mathcal{A}_r^\epsilon v^{\epsilon,\delta}(r,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](y) - \mathcal{A}_r^\epsilon v^{\epsilon,\delta}(r,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r^\delta](y)\Big]\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathbf{n}_r(\mathrm{d}z)\ \mathrm{d}y\ \mathrm{d}r\\ &= \int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{\epsilon,\delta}(\mathrm{d}y) + \int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) - \int_{\mathbb{R}^n}\phi(y)\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathrm{d}y\ \mathbf{n}_{t_0}(\mathrm{d}z)\\ &\quad + \int_0^{t_0}\int_{\mathbb{R}^n}\Big[\mathcal{A}_r^\epsilon v^{\epsilon,\delta}(r,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](y) - \mathcal{A}_r^\epsilon v^{\epsilon,\delta}(r,\cdot)[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r^\delta](y)\Big]\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathbf{n}_r(\mathrm{d}z)\ \mathrm{d}y\ \mathrm{d}r.\end{aligned}$$

Consequently, for each $\epsilon>0$,

$$\limsup_{\delta\to0}\Big|\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) - \int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{\epsilon,\delta}(\mathrm{d}y)\Big| \le \Big|\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) - \int_{\mathbb{R}^n}\phi(y)\int_{\mathbb{R}^n}G_\epsilon(z-y)\mathrm{d}y\ \mathbf{n}_{t_0}(\mathrm{d}z)\Big|.$$

Finally

$$\lim_{\epsilon\to0}\limsup_{\delta\to0}\Big|\int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}(\mathrm{d}y) - \int_{\mathbb{R}^n}\phi(y)\mathbf{n}_{t_0}^{\epsilon,\delta}(\mathrm{d}y)\Big| = 0, \tag{4.9.5}$$

for any $\phi\in C_b^2(\mathbb{R}^n)$ and $t_0\in[0,T]$, where we used $\lim_{\varepsilon\to0}|\int_{\mathbb{R}^n}\phi(y)G_\epsilon(z-y)\mathrm{d}y-\phi(z)|=0$, for all $z\in\mathbb{R}^n$.

Notice that $\nu^{(\epsilon)}(y)(\mathrm{d}y)$ converges weakly to $\nu(\mathrm{d}y)$. By Skorokhod's representation theorem, one can find a probability space $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ supporting $(\xi^\epsilon)_{\epsilon>0}$ and ξ such that $\mathcal{L}^{\tilde{\mathbb{P}}}(\xi^\epsilon)=\nu^{(\epsilon)}(y)(\mathrm{d}y)$ and $\mathcal{L}^{\tilde{\mathbb{P}}}(\xi)=\nu(\mathrm{d}y)$, and $\lim_{\epsilon\to0}\xi^\epsilon=\xi$ $\tilde{\mathbb{P}}$ a.e.. And when $\mathcal{L}^{\mathbb{P}}(\xi)=\nu\in\mathcal{P}_{p'}(\nu)$, one has $\sup_{\epsilon>0}\mathbb{E}^{\tilde{\mathbb{P}}}[|\xi^\epsilon|^{p'}]=\sup_{\epsilon>0}\int_{\mathbb{R}^n}|y|^{p'}\nu^{(\epsilon)}(y)(\mathrm{d}y)\le C(1+\int_{\mathbb{R}^n}|y|^{p'}\nu(\mathrm{d}y))<\infty$, by using standard techniques of uniform integrability, $\lim_{\epsilon\to0}\mathbb{E}^{\tilde{\mathbb{P}}}[|\xi^\epsilon-\xi|^p]=0$, recall that $p'>p$. If necessary, it is possible to enlarge the initial space, by sake of clarity and without technical problems, let us assume $(\tilde\Omega,\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ is equal to the initial space $(\Omega,\mathcal{F},\mathbb{P})$. For each $\epsilon>0$, Let X^ϵ be the continuous process unique strong solution of

$$X_s^\epsilon = \xi + \int_0^s\hat b^\epsilon[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,X_r^\epsilon)\mathrm{d}r + \int_0^s(\hat a^\epsilon)^{1/2}[\mathbf{b},\mathbf{n},\mathbf{z},\hat{\mathbf{q}}_r](r,X_r^\epsilon)\mathrm{d}W_r \text{ for all } s\in[0,T],\ \mathbb{P}\text{–a.e.}$$

By using the regularity of $(\hat b^\epsilon,\hat a^\epsilon)$ for ϵ fixed, it is straightforward to find

$$\lim_{\epsilon\to0}\lim_{\delta\to0}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\big|X_t^\epsilon - X_t^{\epsilon,\delta,0,\xi^\epsilon}\big|^p\Big] = 0.$$

By Itô formula and uniqueness of the Fokker–Planck equation (4.7.5), $\mathbf{n}_t^\epsilon=\mathcal{L}^{\mathbb{P}}(X_t^\epsilon)$ for each $t\in[0,T]$, thanks to (4.9.5) and previous result, one gets, in weakly convergence sense,

$$\lim_\epsilon\mathbf{n}_t^\epsilon = \lim_\epsilon\lim_\delta\mathbf{n}_t^{\epsilon,\delta} = \mathbf{n}_t \text{ for each } t\in[0,T].$$

Therefore we proved: for each $t\in[0,T]$, $\mathbf{n}_t^\epsilon$ converges weakly to $\mathbf{n}_t$. To deduce the Wasserstein convergence $\mathcal{W}_p$, notice that: $\sup_{\epsilon>0}\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\mathbf{n}_t^\epsilon(\mathrm{d}x)\le C(1+\int_{\mathbb{R}^n}|y|^{p'}\nu(\mathrm{d}y))<\infty$, and

$$\begin{aligned}\limsup_{\delta'\to0}\sup_{\epsilon>0}\sup_{s\in[0,T]}\mathcal{W}_p\big(\mathbf{n}^\epsilon_{(s+\delta')\wedge T},\mathbf{n}^\epsilon_s\big)^p &= \limsup_{\delta'\to0}\sup_{\epsilon>0}\sup_{s\in[0,T]}\mathcal{W}_p\big(\mathcal{L}^{\mathbb{P}}(X^\epsilon_{(s+\delta')\wedge T}),\mathcal{L}^{\mathbb{P}}(X^\epsilon_s)\big)^p\\ &\le \limsup_{\delta'\to0}\sup_{\epsilon>0}\sup_{s\in[0,T]}\mathbb{E}^{\mathbb{P}}\big[\big|X^\epsilon_{(s+\delta')\wedge T}-X^\epsilon_s\big|^p\big] \le \hat C\limsup_{\delta'\to0}\delta' = 0,\end{aligned}$$

where the last equality follows from the Holder's property of trajectories of X^ϵ with a constant independent of ϵ (essentially because $(\hat{b},\hat{\sigma})$ are bounded). By Aldous' criterion [92, Lemma 16.12] (see also proof of [49, Proposition-B.1]), $(\mathbf{n}^\epsilon)_{\epsilon>0}$ is relatively compact in $C([0,T];\mathcal{P}_p(\mathbb{R}^n))$ with the metric $\Delta(\nu,\nu') := \sup_{t\in[0,T]} \mathcal{W}_p(\nu_t,\nu'_t)$ for all $(\nu,\nu') \in C([0,T];\mathcal{P}_p(\mathbb{R}^n))\times C([0,T];\mathcal{P}_p(\mathbb{R}^n))$. As for each $t\in[0,T]$, $\mathbf{n}^\epsilon_t$ converges weakly to $\mathbf{n}_t$, then the limit of each sub-sequence of $(\mathbf{n}^\epsilon)_{\epsilon>0}$ is $\mathbf{n}$, consequently $\lim_{\epsilon\to 0} \sup_{t\in[0,T]} \mathcal{W}_p(\mathbf{n}^\epsilon_t,\mathbf{n}_t) = 0$.

□

Proof of Proposition 4.7.9. Before starting, let us mention that many parts of this proof use Proposition 4.7.7 and its associated proof.

Let us take the sequence of processes $(\alpha^{i,N})_{(i,N)\in\mathbb{N}^*\times\mathbb{N}^*}$ given in Proposition 4.7.7 with $\widehat{\mathcal{L}^{\mathbb{P}}}(\xi^i) = \nu^i = \nu$ for each i, and define the unique strong solution $X^{i,N}$ of: for all $t\in[0,T]$

$$X^{i,N}_t = \xi^i + \int_0^t \hat{b}\big(r, X^{i,N}_r, B^N, \phi(\widehat{\mu}^{i,N}), \zeta^N, \widehat{m}^{i,N}_r, \bar{\nu}^N_r, \alpha^{i,N}_r\big)\mathrm{d}r + \int_0^t \hat{\sigma}\big(r, X^{i,N}_r, B^N, \phi(\widehat{\mu}^{i,N}), \zeta^N, \widehat{m}^{i,N}_r, \bar{\nu}^N_r, \alpha^{i,N}_r\big)\mathrm{d}W^i_r,$$

with $\widehat{m}^{i,N}_t := \widehat{\mathcal{L}^{\mathbb{P}}}\big(X^{i,N}_t, \alpha^{i,N}_t \big| \widehat{\mathcal{G}}^N_t\big)$ and $\widehat{\mu}^{i,N}_t := \widehat{\mathcal{L}^{\mathbb{P}}}\big(X^{i,N}_t \big| \widehat{\mathcal{G}}^N_t\big)$. As $\alpha^{i,N}$ is $\widehat{\mathbb{F}}^{i,N}$–predictable ($\widehat{\mathbb{F}}^{i,N}$ is defined in (4.7.11)), there exists a Borel function $G : [0,T]\times\mathbb{R}^n\times\mathbb{M}\big((\mathcal{P}^n_U)^2\big)\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{C}^n\times\mathcal{C}^\ell\times[0,1]\to U$ satisfying $\alpha^{i,N}_t = G\big(t,\xi^i,\overline{\Lambda}^N_{t\wedge\cdot},\phi_{t\wedge\cdot}(\mu^N),\zeta^N_{t\wedge\cdot},W^i_{t\wedge\cdot},B^N_{t\wedge\cdot},Z^i\big)$, $\mathrm{d}t\otimes\mathrm{d}\widehat{\mathbb{P}}$–a.e. Define $\alpha^N_t := G\big(t,\xi,\overline{\Lambda}^N_{t\wedge\cdot},\phi_{t\wedge\cdot}(\mu^N),\zeta^N_{t\wedge\cdot},W_{t\wedge\cdot},B^N_{t\wedge\cdot},Z\big)$. Let X^N be the unique strong solution of equation (4.7.28) (associated to α^N). By independence Assumption (4.7.27), recall that $\widehat{m}^N$ is given in equation (4.7.28),

$$\widehat{m}^{i,N}_t = \widehat{m}^N_t,\ \widehat{\mathbb{P}}\text{–a.e., and given the } \sigma\text{–field } \widehat{\mathcal{G}}^N_t,\ \text{for } i\neq j,\ (X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t) \text{ are independent of } (X^{j,N}_{t\wedge\cdot},\alpha^{j,N}_t) \tag{4.9.6}$$

and $\widehat{\mathcal{L}^{\mathbb{P}}}\big(X^{i,N},\xi^i,\overline{\Lambda}^N,\phi(\mu^N),\zeta^N,W^i,B^N,Z^i\big) = \widehat{\mathcal{L}^{\mathbb{P}}}\big(X^N,\xi,\overline{\Lambda}^N,\phi(\mu^N),\zeta^N,W,B^N,Z\big)$.

Let us introduce for each $N\in\mathbb{N}^*$, the measure on $[0,T]\times\mathcal{P}(\mathcal{C}^n\times U)\times\mathcal{P}(\mathcal{C}^n\times U)$

$$\Gamma^N_t(\mathrm{d}e,\mathrm{d}e')\mathrm{d}t := \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\delta_{\left(\overline{\beta}^N_t,\ \mathcal{L}^{\mathbb{P}}(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t|\mathcal{G}^N_t)\right)}(\mathrm{d}e,\mathrm{d}e')\mathrm{d}t\Big],\ \text{with } \overline{\beta}^N_t(\mathrm{d}\mathbf{x},\mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t)}(\mathrm{d}\mathbf{x},\mathrm{d}u).$$

As $(\hat{b},\hat{\sigma})$ are bounded and $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, it is straightforward to check that $\sup_{N\geq 1}\sup_{i\in\{1,\dots,N\}}\mathbb{E}^{\widehat{\mathbb{P}}}\big[\sup_{t\in[0,T]}\big|X^{i,N}_t\big|^{p'}\big] < \infty$, and hence $(\Gamma^N)_{N\in\mathbb{N}^*}$ is relatively compact for the Wasserstein metric $\mathcal{W}_p$. Denote by Γ^∞ the limit of any sub-sequence of $(\Gamma^N)_{N\in\mathbb{N}^*}$. For simplicity, we will use the same name for the sequence and the sub-sequence. One gets

$$\Gamma^\infty_t(\mathrm{d}e,\mathrm{d}e')\mathrm{d}t = \delta_e(\mathrm{d}e')\Gamma^\infty_t\big(\mathrm{d}e,\mathcal{P}(\mathcal{C}^n\times U)\big)\mathrm{d}t. \tag{4.9.7}$$

It is enough to show that: for all $Q\in\mathbb{N}^*$, any bounded functions $(f^q)_{d\in\{1,\dots,Q\}} : \mathcal{C}^n\times U\to\mathbb{R}^Q$ and $g : [0,T]\times\mathcal{P}(\mathcal{C}^n\times U)\to\mathbb{R}$

$$\int_0^T\int_{\mathcal{P}(\mathcal{C}^n\times U)^2}\prod_{q=1}^Q\langle f^q,e\rangle g(t,e')\Gamma^\infty_t(\mathrm{d}e,\mathrm{d}e')\mathrm{d}t = \int_0^T\int_{\mathcal{P}(\mathcal{C}^n\times U)}\prod_{q=1}^Q\langle f^q,e\rangle g(t,e)\Gamma^\infty_t\big(\mathrm{d}e,\mathcal{P}(\mathcal{C}^n\times U)\big)\mathrm{d}t.$$

Let us prove this result when $Q=2$, the case $Q\in\mathbb{N}^*$ is true by similar way.

$$\int_0^T\int_{\mathcal{P}(\mathcal{C}^n\times U)^2}\prod_{q=1}^Q\langle f^q,e\rangle g(t,e')\Gamma_t^\infty(\mathrm{d}e,\mathrm{d}e')\mathrm{d}t=\lim_{N\to\infty}\frac{1}{N}\sum_{i,j=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)f^2\big(X^{j,N}_{t\wedge\cdot},\alpha^{j,N}_t\big)g(t,\widehat{m}^N_t)\mathrm{d}t\Big]$$

$$=\lim_{N\to\infty}\frac{1}{N^2}\Bigg(\sum_{i\neq j}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)f^2\big(X^{j,N}_{t\wedge\cdot},\alpha^{j,N}_t\big)g(t,\widehat{m}^N_t)\mathrm{d}t\Big]+\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)f^2\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)g(t,\widehat{m}^N_t)\mathrm{d}t\Big]\Bigg)$$

$$=\lim_{N\to\infty}\Bigg(\frac{1}{N^2}\sum_{i\neq j}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\big[f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)\big|\widehat{\mathcal{G}}^N_t\big]\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\big[f^2\big(X^{j,N}_{t\wedge\cdot},\alpha^{j,N}_t\big)\big|\widehat{\mathcal{G}}^N_t\big]g(t,\widehat{m}^N_t)\mathrm{d}t\Big]$$

$$+\frac{1}{N^2}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)f^2\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)g(t,\widehat{m}^N_t)\mathrm{d}t\Big]\Bigg)$$

$$=\lim_{N\to\infty}\Bigg(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\langle f^1,\widehat{m}^N_t\rangle\langle f^2,\widehat{m}^N_t\rangle g(t,\widehat{m}^N_t)\mathrm{d}t\Big]-\frac{1}{N^2}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\big[f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)\big|\widehat{\mathcal{G}}^N_t\big]\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\big[f^2\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)g(t,\widehat{m}^N_t)\big|\widehat{\mathcal{G}}^N_t\big]\mathrm{d}t\Big]$$

$$+\frac{1}{N^2}\sum_{i=1}^N\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T f^1\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)f^2\big(X^{i,N}_{t\wedge\cdot},\alpha^{i,N}_t\big)g(t,\widehat{m}^N_t)\mathrm{d}t\Big]\Bigg)=\int_0^T\int_{\mathcal{P}(\mathbb{R}^n\times U)}\langle f^1,e\rangle\langle f^2,e\rangle g(t,e)\Gamma_t^\infty\big(\mathrm{d}e,\mathcal{P}(\mathcal{C}^n\times U)\big)\mathrm{d}t,$$

where we used result (4.9.6) and the fact that the terms starting with $\frac{1}{(N_l)^2}\sum_{i=1}^{N_l}$ go to zero because (f^1,f^2,g) are bounded.
Next, for all $t\in[0,T]$, using Lipshitz property, there exists a constant $C>0$ (which changes from line to line)

$$\begin{aligned}&\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{s\in[0,t]}\big|X^{i,N}_s-\widehat{X}^i_s\big|^p\Big]\\&\leq C\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\sup_{r\in[0,s]}\big|X^{i,N}_r-\widehat{X}^i_r\big|^p+\sup_{r\in[0,s]}\mathcal{W}_p\big(\widehat{\mu}^N_r,\frac{1}{N}\sum_{i=1}^N\delta_{X^{i,N}_r}\big)^p+\mathcal{W}_p\big(\widehat{m}^N_s,\frac{1}{N}\sum_{i=1}^N\delta_{(X^{i,N}_s,\alpha^{i,N}_s)}\big)^p\mathrm{d}s\Big]\\&\leq C\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^t\sup_{r\in[0,s]}\big|X^{i,N}_r-\widehat{X}^i_r\big|^p+\mathcal{W}_p\big(\mathcal{L}^{\mathbb{P}}\big((X^{i,N}_{s\wedge\cdot},\alpha^{i,N}_s\big|\widehat{\mathcal{G}}^N_s\big),\overline{\beta}^N_s\big)^p\mathrm{d}s\Big],\end{aligned}$$

recall that $(\widehat{X}^1,...,\widehat{X}^N)$ are defined in equation (4.7.12) (in Proposition 4.7.7), and $\widehat{m}^N_t:=\mathcal{L}^{\widehat{\mathbb{P}}}\big(X^N_t,\alpha^N_t\big|\widehat{\mathcal{G}}^N_t\big)$ and $\widehat{\mu}^N_t:=\mathcal{L}^{\widehat{\mathbb{P}}}\big(X^N_t\big|\widehat{\mathcal{G}}^N_t\big)$.

Then by Gronwall Lemma $\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{s\in[0,T]}\big|X^{i,N}_s-\widehat{X}^i_s\big|^p\Big]\leq C\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\mathcal{L}^{\widehat{\mathbb{P}}}\big((X^{i,N}_{s\wedge\cdot},\alpha^{i,N}_s\big|\widehat{\mathcal{G}}^N_s\big),\overline{\beta}^N_s\big)^p\mathrm{d}s\Big]$. As,

$$\begin{aligned}&\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\widehat{m}^N_s,\frac{1}{N}\sum_{i=1}^N\delta_{(\widehat{X}^{i,N}_s,\alpha^{i,N}_s)}\big)^p\mathrm{d}s\Big]\\&\leq\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\widehat{m}^N_s,\frac{1}{N}\sum_{i=1}^N\delta_{(X^{i,N}_s,\alpha^{i,N}_s)}\big)^p\mathrm{d}s\Big]+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\frac{1}{N}\sum_{i=1}^N\delta_{(X^{i,N}_s,\alpha^{i,N}_s)},\frac{1}{N}\sum_{i=1}^N\delta_{(\widehat{X}^{i,N}_s,\alpha^{i,N}_s)}\big)^p\mathrm{d}s\Big]\\&\leq C\Big(\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\mathcal{L}^{\widehat{\mathbb{P}}}\big((X^{i,N}_{s\wedge\cdot},\alpha^{i,N}_s\big|\widehat{\mathcal{G}}^N_s\big),\overline{\beta}^N_s\big)^p\mathrm{d}s\Big]+\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\big|X^{i,N}_s-\widehat{X}^i_s\big|^p\mathrm{d}s\Big]\Big)\\&\leq C\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\mathcal{L}^{\widehat{\mathbb{P}}}\big((X^{i,N}_{s\wedge\cdot},\alpha^{i,N}_s\big|\widehat{\mathcal{G}}^N_s\big),\overline{\beta}^N_s\big)^p\mathrm{d}s\Big],\end{aligned}$$

therefore, by taking the sub-sequence corresponding to the lim sup, by result (4.9.7),

$$\limsup_{l\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\mathcal{W}_p\big(\widehat{m}^{N_l}_s,\frac{1}{N_l}\sum_{i=1}^{N_l}\delta_{(\widehat{X}^{i,N}_s,\alpha^{i,N}_s)}\big)^p\mathrm{d}s+\sup_{t\in[0,T]}\mathcal{W}_p\big(\phi_t(\widehat{\mu}^{N_l}),\phi_t(\mu^{N_l})\big)\Big]=0.$$

From all previous results, it is straightforward to check that

$$\lim_{N\to\infty}\mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}}\big(\widehat{\mu}^N,\zeta^N,\delta_{(\hat{m}_s^N,\bar{\nu}_s^N)}(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s,B^N\big),\mathcal{L}^{\widehat{\mathbb{P}}}\big(\widehat{\gamma}^N,\zeta^N,\delta_{(\hat{\theta}_s^N,\bar{\nu}_s^N)}(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s,B^N\big)\Big)=0,$$

where $\widehat{\gamma}_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{(\widehat{X}_t^{i,N})}$ and $\widehat{\theta}_t:=\frac{1}{N}\sum_{i=1}^N\delta_{(\widehat{X}_t^{i,N},\alpha_t^{i,N})}$. Consequently, by Proposition 4.7.7

$$\lim_{k\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\big(\widehat{\mu}^{N_k},\zeta^{N_k},\delta_{(\hat{m}_s^{N_k},\bar{\nu}_s^{N_k})}(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s,B^{N_k}\big)=\lim_{k\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\big(\widehat{\gamma}^{N_k},\zeta^{N_k},\delta_{(\hat{\theta}_s^{N_k},\bar{\nu}_s^{N_k})}(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}s,B^{N_k}\big)=\mathcal{L}^{\mathbb{Q}}\big(\mu,\zeta,\overline{\Lambda},B\big),$$

recall that $\widehat{m}_t^N:=\mathcal{L}^{\widehat{\mathbb{P}}}\big(X_t^N,\alpha_t^N\big|\widehat{\mathcal{G}}_t^N\big)$ and $\widehat{\mu}_t^N:=\mathcal{L}^{\widehat{\mathbb{P}}}\big(X_t^N\big|\widehat{\mathcal{G}}_t^N\big)$. □

Proof of Proposition 4.7.10. The proof of this Proposition is exactly the same as Proposition 4.7.7, we essentially recall the main step.

Approximation by SDE : Tightness and identification of the limit process: Let us define the unique strong solution $Z^{\epsilon,N}$ of:

$$Z_t^{\epsilon,N}=\xi+\int_0^t\hat{b}^\epsilon[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}_r^N](r,Z_r^{\epsilon,N})\mathrm{d}r+\int_0^t(\hat{a}^\epsilon)^{1/2}[B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}_r^N](r,Z_r^{\epsilon,N})\mathrm{d}W_r,\ t\in[0,T],\ \widehat{\mathbb{P}}\text{–a.e.}.$$

And for all $(t,\omega)\in[0,T]\times\Omega$, denote $\vartheta_t^{\epsilon,N}(\omega):=\mathcal{L}^{\widehat{\mathbb{P}}}\big(Z_t^{\epsilon,N}\big|\widehat{\mathcal{G}}_t^N\big)(\omega)$, and

$$\mathrm{P}^{\epsilon,N}:=\mathcal{L}^{\widehat{\mathbb{P}}}\Big(\vartheta^{\epsilon,N},B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N\Big)\in\mathcal{P}\big(\mathcal{C}_{\mathcal{W}}^n\times\mathcal{C}^\ell\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathbb{M}\big((\mathcal{P}_U^n)^2\big)\big).$$

As $[\hat{b}^\epsilon,\hat{a}^\epsilon]$ are bounded, again it is straightforward to check that $(\mathrm{P}^{\epsilon,N})_{N\in\mathbb{N}^*}$ is relatively compact for the Wasserstein metric $\mathcal{W}_p$. Denote by $\mathrm{P}^{\epsilon,\infty}$ the limit of any sub-sequence of $(\mathrm{P}^{\epsilon,N})_{N\in\mathbb{N}^*}$. Therefore, under Assumption 4.1.1, by applying similar techniques to those used in *step* 2.2 of proof of Proposition 4.7.7, one gets for all $(f,t)\in C_b^2(\mathbb{R}^n;\mathbb{R})\times[0,T]$, one gets

$$\langle f,\beta_t\rangle=\int_{\mathbb{R}^n}f(y)\nu(\mathrm{d}y)+\int_0^t\int_{\mathbb{R}^n}\mathcal{A}_r^\epsilon f\big[B,\beta^\mu,\beta^\zeta,\overline{\beta}\big](x)\beta_r(\mathrm{d}x)\mathrm{d}r,\quad\mathrm{P}^{\epsilon,\infty}\text{–a.e.},\tag{4.9.8}$$

where $(\beta,B,\beta^\mu,\beta^\zeta,\overline{\beta})$ is the canonical element on $\mathcal{C}_{\mathcal{W}}^n\times\mathcal{C}^\ell\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathbb{M}\big((\mathcal{P}_U^n)^2$. Using a countable family of (f,t), we can deduce $\mathrm{P}^{\epsilon,\infty}$–a.e. equation (4.9.8) is true for all $(f,t)\in C_b^2(\mathbb{R}^n;\mathbb{R})\times[0,T]$.
By Lemma 4.9.4, one has $\beta=\Phi^\epsilon\big(B,\beta^\mu,\beta^\zeta,\overline{\beta}\big)$ where Φ^ϵ is the function used in equation (4.7.14). Also

$$\mathcal{L}^{\mathrm{P}^{\epsilon,\infty}}\big(B,\beta^\mu,\beta^\zeta,\overline{\beta}\big)=\lim_{N\to\infty}\mathcal{L}^{\mathrm{P}^{\epsilon,N}}\big(B,\beta^\mu,\beta^\zeta,\overline{\beta}\big)=\lim_{N\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\big(B,\phi(\mu),\zeta^N,\overline{\Lambda}^N\big)=\mathcal{L}^{\mathbb{Q}}\big(B,\phi(\mu),\zeta,\overline{\Lambda}\big).$$

Then $\mathcal{L}^{\mathrm{P}^{\epsilon,\infty}}\big(\beta,B,\beta^\mu,\beta^\zeta,\overline{\beta}\big)=\mathcal{L}^{\mathbb{Q}}\big(\mu^\epsilon,B,\phi(\mu),\zeta,\overline{\Lambda}\big)$. This result is true for any limit of any sub-sequence of $(\mathrm{P}^{\epsilon,N})_{N\in\mathbb{N}^*}$, consequently $(\mathrm{P}^{\epsilon,N})_{N\in\mathbb{N}^*}$ converges and

$$\lim_{N\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}}\big(\vartheta^{\epsilon,N},B^N,\phi(\mu^N),\zeta^N,\overline{\Lambda}^N\big)=\mathcal{L}^{\mathbb{Q}}\big(\mu^\epsilon,B,\phi(\mu),\zeta,\overline{\Lambda}\big).$$

Last approximation: Let us consider for all $(\epsilon,N)\in(0,\infty)\times\mathbb{N}^*$, the $\widehat{\mathbb{F}}$–adapted $\mathbb{R}^n$–valued continuous process $X^{\epsilon,N}:=X$ strong solution of : for all $s\in[0,T]$

$$\begin{aligned}X_s=\xi&+\int_0^s\int_{(\mathcal{P}_U^n)^2}\int_U\hat{b}(r,X_r,B^N,\phi(\widehat{\mu}^{\epsilon,N}),\zeta^N,\widehat{\mathbf{m}}_r^{\epsilon,N}[m],\bar{\nu},u)H^\epsilon(Z_r^{\epsilon,N},m)(\mathrm{d}u)\overline{\Lambda}_r^N(\mathrm{d}m,\mathrm{d}\bar{\nu})\mathrm{d}r\\&+\int_0^s\Big(\int_{(\mathcal{P}_U^n)^2}\int_U\hat{\sigma}\hat{\sigma}^\top(r,X_r,B^N,\phi(\widehat{\mu}^{\epsilon,N}),\zeta^N,\widehat{\mathbf{m}}_r^{\epsilon,N}[m],\bar{\nu},u)H^\epsilon(Z_r^{\epsilon,N},m)(\mathrm{d}u)\overline{\Lambda}_r^N(\mathrm{d}m,\mathrm{d}\bar{\nu})\Big)^{1/2}\mathrm{d}W_r,\ \widehat{\mathbb{P}}\text{–a.e.}\end{aligned}$$

where recall that $H^\epsilon(x,m)(\mathrm{d}u) := \int_{\mathbb{R}^n} m(\mathrm{d}u,\mathrm{d}y)\frac{G_\epsilon(x-y)}{(m(U,\mathrm{d}z))^{(\epsilon)}(x)}$ and

$$\widehat{\mathbf{m}}_r^{\epsilon,N}[m](\mathrm{d}z,\mathrm{d}u) := \widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[H^\epsilon(Z_r^{\epsilon,N},m)(\mathrm{d}u)\delta_{X_r^{\epsilon,N}}(\mathrm{d}z)\Big|\widehat{\mathcal{G}}_r^N\Big] \text{ and } \widehat{\mu}_r^{\epsilon,N} := \mathcal{L}^{\widehat{\mathbb{P}}}(X_r^{\epsilon,N}|\widehat{\mathcal{G}}_r^N).$$

Combining Proposition 4.9.3 and the techniques applied in *step* 3 of Proof of Proposition 4.7.7, one gets

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{t\in[0,T]}|X_t^{\epsilon,N}-Z_t^{\epsilon,N}|^p\Big]=0 \text{ and } \lim_{\epsilon\to 0}\lim_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\int_0^T\int_{\mathcal{P}_U^n}\mathcal{W}_p(\widehat{\mathbf{m}}_r^{\epsilon,N}[m],m)\overline{\Lambda}_r^N(\mathrm{d}m,\mathcal{P}_U^n)\mathrm{d}r\Big]=0.$$

Similarly, $\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\widehat{\mathbb{E}}^{\widehat{\mathbb{P}}}\Big[\sup_{s\in[0,T]}\mathcal{W}_p\big(\phi(\widehat{\mu}^{\epsilon,N}),\phi_s(\mu^N)\big)\Big]=0$. $X^{\epsilon,N}$ is the process we are looking for.

□

4.9.2 Convolution approximation

Let $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P})$ be a filtered probability space supporting W a $\mathbb{R}^n$–valued $\mathbb{F}$–Brownian motion and ξ a $\mathcal{F}_0$–random variable verifying $\mathbb{E}^{\mathbb{P}}[|\xi|^p]<\infty$, $(b_t,\sigma_t)_{t\in[0,T]}$ $\mathbb{R}^n\times\mathbb{S}^n$ bounded predictable process such that there exists $\theta>0$ satisfying $[\sigma_t][\sigma_t]^\top\geq\theta\mathrm{I}_{n\times n}$. For all $t\in[0,T]$, denote by

$$X_t=\xi+\int_0^t b_s\mathrm{d}s+\int_0^t\sigma_s\mathrm{d}W_s,\ \ \mathbb{P}\text{–a.e.}$$

the following proposition is just an application of [76, Proposition 4.2] (see also [97])

Proposition 4.9.1 (equivalence of measures)**.** *With the previous considerations, the measure* $\mathbf{n}$ *on* $\mathbb{R}^n\times[0,T]$ *defined by*

$$\mathbf{n}(\mathrm{d}x,\mathrm{d}t):=\mathbb{P}\circ(X_t)^{-1}(\mathrm{d}x)\mathrm{d}t$$

is equivalent to the Lebesque measure on $\mathbb{R}^n\times[0,T]$.

Next, let $(\epsilon_k)_{k\in\mathbb{N}^*}\subset(0,\infty)$ such that $\lim_{k\to\infty}\epsilon_k=0$. Let $G\in C^\infty(\mathbb{R}^n;\mathbb{R})$ satisfying $G\geq 0$, $G(x)=G(-x)$ for $x\in\mathbb{R}^n$, and $\int_{\mathbb{R}^n}G(y)\mathrm{d}y=1$, and define $G_k(x):=\epsilon_k{}^{-n}G(\epsilon_k{}^{-1}x)$ and for all $\pi\in\mathcal{P}(\mathbb{R}^n)$, $\pi^{(k)}(x):=\int_{\mathbb{R}^n}G_k(x-y)\pi(\mathrm{d}y)$ for all $x\in\mathbb{R}^n$. Also, denote by X^k the process defined by

$$X_t^k=\xi+\int_0^t b_r^k\mathrm{d}r+\int_0^t\sigma_r^k\mathrm{d}W_r \text{ for all } t\in[0,T],\ \mathbb{P}\text{–a.e.},$$

where there exists $\mathrm{D}>0$ s.t. for all k and t, $|\sigma_t^k|+|b_t^k|\leq\mathrm{D}$, $\mathbb{P}$–a.e., $[\sigma_t^k][\sigma_t^k]^\top\geq\theta\mathrm{I}_{n\times n}$, $\mathbb{P}$–a.e.. In addition $\mathbb{E}^{\mathbb{P}}[|\xi|^p]<\infty$ where $p\geq 1$.

Let $(\mathbf{n}_t)_{t\in[0,T]}$ be a $\mathcal{P}(\mathbb{R}^n)$–valued continuous process such that $\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t$ is equivalent to the Lebesgue measure on $[0,T]\times\mathbb{R}^n$, and for the weak topology,

$$\lim_{k\to\infty}\mathcal{L}^{\mathbb{P}}(X_t^k)=\mathbf{n}_t \text{ for each } t\in[0,T].$$

The following proposition shows that it is possible to approach some bounded measurable functions via smooth functions (bounded derivative functions) by using the marginal distributions of X^k.

Proposition 4.9.2 (regularization by convolution)**.** *For all bounded Borel measurable function* $\varphi:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^q$, *such that for all* $(t,z)\in[0,T]\times\mathbb{R}^n$, $\varphi(t,.,z):y\in\mathbb{R}^n\to\varphi(t,y,z)\in\mathbb{R}^q$ *is continuous, one has*

$$\lim_{k\to\infty}\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\varphi(t,x,y)\frac{G_k(t,x-y)}{(\mathbf{n}_t)^{(k)}(x)}\mathbf{n}_t(\mathrm{d}y)-\varphi(t,x,x)\Big|\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t=0 \tag{4.9.9}$$

and

$$\lim_{k\to\infty}\int_0^T\Big|\mathbb{E}^{\mathbb{P}}\Big[\int_{\mathbb{R}^n}\varphi(t,X_t^k,y)\frac{G_l(t,X_t^k-y)}{(\mathbf{n}_t)^{(k)}(X_t^k)}\mathbf{n}_t(\mathrm{d}y)\Big]-\int_{\mathbb{R}^n}\varphi(t,x,x)\mathbf{n}_t(\mathrm{d}x)\Big|\mathrm{d}t=0.$$

Proof. Mention that, as $\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t$ is equivalent to the Lebesgue measure on $[0,T]\times\mathbb{R}^n$, there exists Borel measurable function $c:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ such that $c(s,z)>0$ $\mathrm{d}t\otimes\mathrm{d}x$ a.e. $(s,z)\in[0,T]\times\mathbb{R}^n$, and $\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t=c(t,x)\mathrm{d}x\mathrm{d}t$.
First, let us prove the result (4.9.9). If

$$A_k:=\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\varphi(t,x,y)\frac{G_k(x-y)}{(\mathbf{n}_t)^{(k)}(x)}\mathbf{n}_t(\mathrm{d}y)-\varphi(t,x,x)\Big|\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t,$$

one finds

$$\begin{aligned}
&\Big|A_k-\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\big\{\varphi(t,x,y)-\varphi(t,x,x)\big\}G_k(x-y)c(t,y)\mathrm{d}y\Big|\mathrm{d}x\mathrm{d}t\Big|\\
&=\Big|\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\big\{\varphi(t,x,y)-\varphi(t,x,x)\big\}G_k(x-y)c(t,y)\mathrm{d}y\Big|\Big(\frac{c(t,x)}{(\mathbf{n}_t)^{(k)}(x)}-1\Big)\mathrm{d}x\mathrm{d}t\Big|\\
&\le K\Big|\int_0^T\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G_k(x-y)c(t,y)\mathrm{d}y\Big|\frac{c(t,x)}{(\mathbf{n}_t)^{(k)}(x)}-1\Big|\mathrm{d}x\mathrm{d}t\Big|=K\Big|\int_0^T\int_{\mathbb{R}^n}(\mathbf{n}_t)^{(k)}(x)\Big|\frac{c(t,x)}{(\mathbf{n}_t)^{(k)}(x)}-1\Big|\mathrm{d}x\mathrm{d}t\Big|\\
&\le K\Big|\int_0^T\int_{\mathbb{R}^n}\Big|c(t,x)-(\mathbf{n}_t)^{(k)}(x)\Big|\mathrm{d}x\mathrm{d}t\Big|=K\Big|\int_0^T\int_{\mathbb{R}^n}\Big|c(t,x)-\int_{\mathbb{R}^n}G_k(x-y)c(t,y)\mathrm{d}y\Big|\mathrm{d}x\mathrm{d}t\Big|\to_{k\to\infty}=0,
\end{aligned}\tag{4.9.10}$$

where for the first inequality is true because φ is bounded and the last result is obtained by the classical result of approximation by convolution.
Now, for all $(t,y,\delta)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}_+^*$, $v(t,y,\delta):=\sup_{z||y-z|\le\delta}|\varphi(t,y,y)-\varphi(t,z,y)|$, notice that $\lim_{\delta\to0}v(t,y,\delta)=0$. Observe that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\big\{\varphi(t,x,y)-\varphi(t,y,y)\big\}G_k(x-y)c(t,y)\mathrm{d}y\Big|\mathrm{d}x\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\big\{\varphi(t,x,y)-\varphi(t,y,y)\big\}\big(1_{|x-y|\le\delta}+1_{|x-y|>\delta}\big)G_k(x-y)c(t,y)\mathrm{d}y\Big|\mathrm{d}x\mathrm{d}t\\
&\le\int_0^T\int_{\mathbb{R}^n}v(t,y,\delta)\int_{\mathbb{R}^n}1_{|x-y|\le\delta}G_k(x-y)c(t,y)\mathrm{d}y\mathrm{d}x\mathrm{d}t+K\int_0^T\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}1_{|x-y|>\delta}G_k(x-y)c(t,y)\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\le\int_0^T\int_{\mathbb{R}^n}v(t,y,\delta)\int_{\mathbb{R}^n}G_k(x-y)c(t,y)\mathrm{d}y\mathrm{d}x\mathrm{d}t+K\ T\int_{\mathbb{R}^n}1_{|z|>\delta}G_k(z)\mathrm{d}z\\
&\le\int_0^T\int_{\mathbb{R}^n}v(t,y,\delta)c(t,y)\mathrm{d}y\mathrm{d}t+K\ T\int_{\mathbb{R}^n}1_{|z|>\delta}G_k(z)\mathrm{d}z,
\end{aligned}$$

it is well know, for each $\delta>0$, $\lim_{k\to\infty}\int_{\mathbb{R}^n}1_{|z|>\delta}G_k(z)\mathrm{d}z=0$, one gets

$$\limsup_{k\to\infty}\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n}\{\varphi(t,x,y)-\varphi(t,y,y)\}G_k(x-y)c(t,y)\mathrm{d}y\Big|\mathrm{d}x\mathrm{d}t\le\lim_{\delta\to0}\int_0^T\int_{\mathbb{R}^n}v(t,x,\delta)c(t,x)\mathrm{d}x\mathrm{d}t=0,\tag{4.9.11}$$

the last inequality is true because of Lebesgue's dominated convergence theorem.

Finally, one has

$$\begin{aligned}
\limsup_{k\to\infty} A_k &= \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \{\varphi(t,x,y)-\varphi(t,x,x)\} G_k(x-y)c(t,y)\mathrm{d}y \Big| \mathrm{d}x\mathrm{d}t \\
&= \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi(t,x,y)c(t,y)G_k(x-y)\mathrm{d}y - \int_{\mathbb{R}^n} \varphi(t,x,x)G_k(x-y)c(t,y)\mathrm{d}y \Big| \mathrm{d}x\mathrm{d}t \\
&= \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi(t,y,y)c(t,y)G_k(x-y)\mathrm{d}y - \int_{\mathbb{R}^n} \varphi(t,x,x)G_k(x-y)c(t,y)\mathrm{d}y \Big| \mathrm{d}x\mathrm{d}t \\
&\le \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi(t,y,y)c(t,y)G_k(x-y)\mathrm{d}y - \varphi(t,x,x)c(t,x) \Big| \mathrm{d}x\mathrm{d}t \\
&\qquad + \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \varphi(t,x,x)c(t,x) - \int_{\mathbb{R}^n} \varphi(t,x,x)G_k(x-y)c(t,y)\mathrm{d}y \Big| \mathrm{d}x\mathrm{d}t \\
&\le \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi(t,y,y)c(t,y)G_k(x-y)\mathrm{d}y - \varphi(t,x,x)c(t,x) \Big| \mathrm{d}x\mathrm{d}t \\
&\qquad + \limsup_{k\to\infty} K \int_0^T \int_{\mathbb{R}^n} \Big| c(t,x) - \int_{\mathbb{R}^n} G_k(x-y)c(t,y)\mathrm{d}y \Big| \mathrm{d}x\mathrm{d}t = 0,
\end{aligned}$$

where the first equality derived from (4.9.10), the third equality follows from (4.9.11) and we find 0 because of approximation by convolution result. Therefore $\lim_{k\to\infty} A_k = 0$, then the first assertion is proved.

For the second point, let $k_0 \in \mathbb{N}^*$ one has

$$\begin{aligned}
S^k(\varphi) &:= \int_0^T \left| \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^n} \varphi(t,X_t^k,y) \frac{G_k(t,X_t^k-y)}{(\mathbf{n}_t)^{(k)}(X_t^k)} \mathbf{n}_t(\mathrm{d}y) \right] - \int_{\mathbb{R}^n} \varphi(t,x,x)\mathbf{n}_t(\mathrm{d}x) \right| \mathrm{d}t \\
&\le \int_0^T \left| \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^n} \varphi(t,X_t^k,y) \frac{G_k(t,X_t^k-y)}{(\mathbf{n}_t)^{(k)}(X_t^k)} \mathbf{n}_t(\mathrm{d}y) - \int_{\mathbb{R}^n} \varphi(t,X_t^k,y) \frac{G_{k_0}(t,X_t^k-y)}{(\mathbf{n}_t)^{(k_0)}(X_t^k)} \mathbf{n}_t(\mathrm{d}y) \right] \right| \mathrm{d}t \\
&\quad + \int_0^T \left| \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^n} \varphi(t,X_t^k,y) \frac{G_{k_0}(t,X_t^k-y)}{(\mathbf{n}_t)^{(k_0)}(X_t^k)} \mathbf{n}_t(\mathrm{d}y) \right] - \int_{\mathbb{R}^n} \varphi(t,x,x)\mathbf{n}_t(\mathrm{d}x) \right| \mathrm{d}t.
\end{aligned}$$

By [97, Chapter 2 Section 3 Theorem 4] and Markov inequality, for each $R>0$, there exists a constant $C>0$ depending only on (D,θ,T,R) satisfying

$$\begin{aligned}
S^k(\varphi) \le C \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi(t,x,y) \frac{G_k(t,x-y)}{(\mathbf{n}_t)^{(k)}(x)} \mathbf{n}_t(\mathrm{d}y) - \int_{\mathbb{R}^n} \varphi(t,x,y) \frac{G_{k_0}(t,x-y)}{(\mathbf{n}_t)^{(k_0)}(x)} \mathbf{n}_t(\mathrm{d}y) \Big|^n 1_{|x|\le R} \mathrm{d}x\mathrm{d}t \\
+ T\frac{\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,T]}|X_t^k|^p]}{R^p} + \int_0^T \left| \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^n} \varphi(t,X_t^k,y) \frac{G_{k_0}(t,X_t^k-y)}{(\mathbf{n}_t)^{(k_0)}(X_t^k)} \mathbf{n}_t(\mathrm{d}y) \right] - \int_{\mathbb{R}^n} \varphi(t,x,x)\mathbf{n}_t(\mathrm{d}x) \right| \mathrm{d}t.
\end{aligned}$$

By using the first statement of the proposition (see proof above), then there exists $(k_j)_{j\in\mathbb{N}^*} \subset \mathbb{N}^*$ a sub-sequence such that:

$$\lim_{j\to\infty} \left| \int_{\mathbb{R}^n} \varphi(s,z,y) \frac{G_{k_j}(t,x-y)}{(\mathbf{n}_t)^{(k_j)}(x)} \mathbf{n}_t(\mathrm{d}y) - \varphi(s,z,z) \right| = 0, \ \mathbf{n}_t(\mathrm{d}x)\mathrm{d}t \ \text{ a.e. } (s,z) \in [0,T]\times\mathbb{R}^n.$$

As $\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t$ is equivalent to the Lebesgue measure on $[0,T]\times\mathbb{R}^n$, $\lim_{j\to\infty} \left| \int_{\mathbb{R}^n} \varphi(s,z,y) \frac{G_{k_j}(t,x-y)}{(\mathbf{n}_t)^{(k_j)}(x)} \mathbf{n}_t(\mathrm{d}y) - \varphi(s,z,z) \right| = 0$, $\mathrm{d}t\otimes\mathrm{d}x$ a.e. $(s,z)\in[0,T]\times\mathbb{R}^n$. All these observations allows us to say, by Lebesgue's dominated convergence theorem

$$\limsup_{k_0\to\infty} \limsup_{k\to\infty} \int_0^T \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \varphi(t,x,y) \frac{G_k(t,x-y)}{(\mathbf{n}_t)^{(k)}(x)} \mathbf{n}_t(\mathrm{d}y) - \int_{\mathbb{R}^n} \varphi(t,x,y) \frac{G_{k_0}(t,x-y)}{(\mathbf{n}_t)^{(k_0)}(x)} \mathbf{n}_t(\mathrm{d}y) \Big|^n 1_{|x|\le R} \mathrm{d}x\mathrm{d}t = 0.$$

Finally, combining the previous result with the weak convergence, $\lim_{k\to\infty}\mathcal{L}^{\mathbb{P}}(X^k_t)=\mathbf{n}_t$ for each $t\in[0,T]$, and an obvious application of the first statement of the proposition, one gets

$$\begin{aligned}\limsup_{k\to\infty} S^k(\varphi) \le \limsup_{k_0,k\to\infty} C\int_0^T\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} h(t,x,y)\frac{G_k(t,x-y)}{(\mathbf{n}_t)^{(k)}(x)}\mathbf{n}_t(\mathrm{d}y)-\int_{\mathbb{R}^n} h(t,x,y)\frac{G_{k_0}(t,x-y)}{(\mathbf{n}_t)^{(k_0)}(x)}\mathbf{n}_t(\mathrm{d}y)\Big|^n 1_{|x|\le R}\mathrm{d}x\mathrm{d}t\\ +\limsup_{l_0\to\infty}\int_0^T\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h(t,x,y)\frac{G_{l_0}(t,x-y)}{(\mathbf{n}_t)^{(l_0)}(x)}\mathbf{n}_t(\mathrm{d}y)\mathbf{n}_t(\mathrm{d}x)\mathrm{d}t-\int_0^T\int_{\mathbb{R}^n} h(t,x,x)\mathbf{n}_t(\mathrm{d}x)\Big|\mathrm{d}t\\ +T\frac{\sup_{k>0}\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,T]}|X^k_t|^p]}{R^p}\le\ T\frac{\sup_{k>0}\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,T]}|X^k_t|^p]}{R^p},\end{aligned}$$

as $\sup_{k>0}\mathbb{E}^{\mathbb{P}}[\sup_{t\in[0,T]}|X^k_t|^p]<\infty$, by taking $R\to\infty$, we deduce the result.

□

4.9.3 Some properties of Fokker-Planck equation

Let us recall a useful result on square root of matrices. Denote by $\mathbb{S}^+_n$ the set of symmetric positive definite matrices of dimension $n\in\mathbb{N}^*$. The principal square root function is denoted by: $f: Q\in\mathbb{S}^+_n\mapsto f(Q):=Q^{1/2}\in\mathbb{S}^+_n$.

Proposition 4.9.3. [81, Theorem 6.2] *There exists a constant $C(n)$ depending only of the dimension $n\in\mathbb{N}^*$ such that for any $(A,B)\in\mathbb{S}^+_n\times\mathbb{S}^+_n$*

$$|f(A)-f(B)|\le C(n)\big[\lambda_{\min}(A)^{1/2}+\lambda_{\min}(B)^{1/2}\big]^{-1}|A-B|,$$

where $\lambda_{\min}(\cdot)$ is the smallest eigenvalue.

Let E and E' be two Polish spaces and $[\overline{b},\overline{a}]:[0,T]\times\mathbb{R}^n\times C([0,T];E)\times\mathbb{M}(E')\to\mathbb{R}^n\times\mathbb{S}^{n\times n}$ be a bounded Borel functions s.t.: for all $(t,\pi,\hat{q})\in[0,T]\times C([0,T];E)\times\mathbb{M}(E')$,

$$\text{the function } x\in\mathbb{R}^n\to[\overline{b},\overline{a}](t,x,\pi_{t\wedge\cdot},\hat{q}_{t\wedge\cdot})\in\mathbb{R}^n\times\mathbb{S}^{n\times n}\ \text{ belongs to }\ C^2_b(\mathbb{R}^n)\text{ and }\overline{a}\ge\rho\mathrm{I}_n,\tag{4.9.12}$$

for a certain $\rho>0$.

Also, let us introduce, for all $\varphi\in C^2(\mathbb{R}^n)$, $\overline{\mathcal{L}}_t\varphi[\pi,\hat{q}](x):=\frac{1}{2}\mathrm{Tr}\big[\overline{a}(t,x,\pi,\hat{q}_{t\wedge\cdot})\nabla^2\varphi(x)\big]+\overline{b}(t,x,\pi,\hat{q}_{t\wedge\cdot})^\top\nabla\varphi(x)$.

Lemma 4.9.4. *Let $\nu\in\mathcal{P}_p(\mathbb{R}^n)$. There exists a Borel function $Z: C([0,T];E)\times\mathbb{M}(E')\to\mathcal{C}^n_{\mathcal{W}}$ s.t. if $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P})$ is a filtered probability space supporting $(\mu_t)_{t\in[0,T]}$ a E–valued $\mathbb{F}$–adapted continuous process and $(\hat{\Lambda}_t)_{t\in[0,T]}$ a $\mathcal{P}(E')$–valued $\mathbb{F}$–predictable process, then, the unique $\mathcal{P}(\mathbb{R}^n)$–valued $(\sigma\{\mu_{t\wedge\cdot},\hat{\Lambda}_{t\wedge\cdot}\})_{t\in[0,T]}$–adapted continuous process $(\vartheta_t)_{t\in[0,T]}$ solution of: $\vartheta\in\mathcal{C}^{n,p}_{\mathcal{W}}$, and for all $(t,f)\in[0,T]\times C^2_b(\mathbb{R}^n)$,*

$$\langle f,\vartheta_t\rangle\ =\ \int_{\mathbb{R}^n} f(y)\nu(\mathrm{d}y)+\int_0^t\int_{\mathbb{R}^n}\overline{\mathcal{L}}_r f[\mu,\hat{\Lambda}](x)\vartheta_r(\mathrm{d}x)\mathrm{d}r,\ \mathbb{P}\text{–a.e.}\tag{4.9.13}$$

satisfies

$$\vartheta_t=Z_t(\mu_{t\wedge\cdot},\hat{\Lambda}_{t\wedge\cdot}),\ \textit{for all } t\in[0,T],\ \mathbb{P}\text{–a.e.}$$

Proof. For the uniqueness of (4.9.13), as the coefficients $[\overline{b},\overline{a}]$ verify (4.9.12), by a slight extension of (proof of) Lemma 4.7.2, one gets that equation (4.9.13) has at most one solution.

Let W be a $\mathbb{R}^n$–valued $(\mathbb{P},\mathbb{F})$ Brownian motion and ξ be a $\mathcal{F}_0$–random variable of law ν, in addition, (ξ,W) are $\mathbb{P}$–independent of $(\mu,\hat{\Lambda})$. Next, let us show the existence and find the function Z. Combining (4.9.12) and Proposition 4.9.3, for any $(t,\pi,\hat{q})$, the application $x\in\mathbb{R}^n\to\big(\overline{a}(t,x,\pi_{t\wedge\cdot},\hat{q}_{t\wedge\cdot})\big)^{1/2}\in\mathbb{S}^{n\times n}$ is Lipshitz, with a Lipschitz constant depends only on $\overline{a}$. Therefore, there exists the $\mathbb{R}^n$–valued $\mathbb{F}$–adapted process X unique strong solution of

$$X_s=\xi+\int_0^s\overline{b}(r,X_r,\mu,\hat{\Lambda})\mathrm{d}r+\int_0^s\big(\overline{a}(r,X_r,\mu,\hat{\Lambda})\big)^{1/2}\mathrm{d}W_r\text{ for all }s\in[0,T].$$

It is well know that $X_t = H_t(\xi, W_{t\wedge\cdot}, \mu_{t\wedge\cdot}, \hat{\Lambda}_{t\wedge\cdot})$, for all $t \in [0,T]$, $\mathbb{P}$–a.e. where $H : \mathbb{R}^n \times \mathcal{C}^n \times C([0,T];E) \times \mathbb{M}(E') \to \mathcal{C}^n$ is a Borel function (independent of $\mathbb{P}$).

Denote by $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$ the filtration defined by $\mathcal{G}_t := \sigma\{\mu_{t\wedge\cdot}, \hat{\Lambda}_{t\wedge\cdot}\}$, for all $t \in [0,T]$. As (ξ, W) are $\mathbb{P}$–independent of $(\mu, \hat{\Lambda})$, one has: for all $t \in [0,T]$, $\mathcal{L}^{\mathbb{P}}(X_{t\wedge\cdot}|\mathcal{G}_t) = \mathcal{L}^{\mathbb{P}}(X_{t\wedge\cdot}|\mathcal{G}_T)$, $\mathbb{P}$–a.e. then by Lemma 5.5.1, the process $(\beta_t)_{t\in[0,T]}$ is a $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{G}$–adapted continuous process where $\beta : (t,\omega) \in [0,T] \times \Omega \to \mathcal{L}^{\mathbb{P}}(X_t|\mathcal{G}_t)(\omega) \in \mathcal{P}(\mathbb{R}^n)$, and by Itô's formula $(\beta_t)_{t\in[0,T]}$ is solution of equation (4.9.13). In addition, there exists a Borel function (independent of $\mathbb{P}$) $Z : C([0,T];E) \times \mathbb{M}(E') \to \mathcal{C}^n_{\mathcal{W}}$ such that: $\mathbb{P}$–a.e., for all $t \in [0,T]$, $\beta_t = Z_t(\mu_{t\wedge\cdot}, \hat{\Lambda}_{t\wedge\cdot})$. □

Chapter 5

The dynamic programming principle

5.1 Introduction

Our interest in this chapter is in the optimal control of McKean–Vlasov stochastic equations, and more precisely in the rigorous establishment of the dynamic programming principle (DPP for short), under conditions as general as possible. The optimal control of McKean–Vlasov dynamics is a rather recent problem in the literature. In a nutshell, the idea behind the DPP is that the global optimization problem can be solved by a recursive resolution of successive local optimization problems. This fact is an intuitive result, which is often used as some sort of meta–theorem, but is not so easy to prove rigorously in general.

Note that this approach in general requires fewer assumptions, though it can be applied in less situations. Notwithstanding these advantages, the DPP approach has long been unexplored for the control of McKean–Vlasov equations. One of the main reasons is actually a very bleak one for us: due to the non–linear dependency with respect to the law of process, the problem is actually a time inconsistent control problem (like the classical mean–variance optimisation problem in finance, see the recent papers by Björk and Murgoci [28], Björk, Khapko, and Murgoci [29], and Hernández and Possamaï [80] for a more thorough discussion of this topic), and Bellman's optimality principle does not hold in this case. However, though the problem itself is time–inconsistent, one can recover some form of the DPP by extending the state space of the problem. This was first achieved by Laurière and Pironneau [112], and later by Bensoussan, Frehse, and Yam [24; 25; 26], who assumed the existence at all times of a density for the marginal distribution of the state process, and reformulated the problem as a deterministic density control problem, with a family of deterministic control terms. Under this reformulation, they managed to prove a DPP and deduce a dynamic programming equation in the space of density functions. Following similar ideas, but without the assumptions of the existence of density, and allowing the coefficients and reward functions to not only depend on the distribution of the state, but to the joint distribution of the state and the control, Pham and Wei [139] also deduced a DPP by looking at a set of closed loop (or feedback) controls, in a non common noise context. They then extended this strategy to a common noise setting (where the control process is adapted to common noise filtration) in [138]. Specialisations to linear–quadratic settings were also explored by Pham [137], Li, Sun, and Yong [116], Li, Sun, and Xiong [117], Huang, Li, and Yong [82], Yong [152], and Basei and Pham [20].

Our approach to obtaining the DPP is very different. One common drawback of all the results we mentioned above, is that they generically require some Markovian[1] property of the system or its distribution, as well as strong regularity assumptions on coefficient and reward functions considered. This should appear somehow surprising to people familiar with the classical DDP theory. Indeed, for stochastic control problems, it is possible to use measurable selection arguments to obtain the DPP, in settings requiring nothing beyond mild measurability assumptions. As a rule of thumb, one needs two essential ingredients to prove the dynamic programming principle: first ensuring the stability of the controls with respect to conditioning and concatenation, and second the measurability of the associated value function. The use of measurable selection argument makes it possible to provide an adequate framework for verifying the conditioning, the concatenation and the measurability requirements of the associated value function without strong assumptions. This technique was followed by Dellacherie [56], by Bertsekas and Shreve in [27; 146; 147; 148], and by Shreve [143; 144; 145]

[1] An exception is the work of Djehiche and Hamadène [57], which considers optimal control (and also a zero–sum game) of a non–Markovian McKean–Vlasov equation, and obtains both a characterisation of the value function and the optimal control using BSDE techniques, reminiscent of the classical results of Hamadène and Lepeltier [77] and El Karoui and Quenez [59; 60] for the non–McKean–Vlasov case. However, their approach does not allow for common noise, and is limited to control on the drift of the state process only.

for discrete–time stochastic control problems. Later, El Karoui, Huu Nguyen, and Jeanblanc-Picqué in [63] presented a framework for stochastic control problem in continuous time (accommodating general Markovian processes). Thanks to the notion of relaxed control, that is to say the interpretation of a control as a probability measure on some canonical space, and thanks to the use of the notion of martingale problems, they proved a DPP by simple and clear arguments. El Karoui and Tan [61; 62] extended this approach to the non–Markovian case. Similar results were obtained by several authors, among which we mention Nutz and Soner [131], Neufeld and Nutz [128; 130], Nutz and van Handel [132], Žitković [156], and Possamaï, Tan, and Zhou [140].

Following the framework in [61; 63], we develop in this chapter a general analysis based upon the measurable selection argument for the non–Markovian optimal control of McKean–Vlasov equations with common noise. In particular, we investigate the case where the drift and diffusion coefficients, as well as the reward functions, are allowed to depend on the joint conditional distribution of the path of the state process and of the control, see [139] for the case of the joint distribution of the state process and of feedback controls (see also Yong [152] for a more specific situation) in a non–common noise case.

Motivated by the notion of weak solution of classical SDEs, and similarly to the ideas used by El Karoui and Tan [61], and Carmona, Delarue, and Lacker [49] in a mean–field games context, our first task is to provide an appropriate "relaxation" of the problem. We therefore introduce a notion of weak solution of controlled McKean–Vlasov equation with common noise similar as that used in chapter one. Notice that this is by no means a straightforward task. In standard McKean–Vlasov stochastic control problems, the controls (open loop in that case) are adapted with respect to the filtration generated by both the Brownian motion (W, B) (with B being the common noise) and the initial random variable ξ (serving as an initial condition for the problem). Then, the conditional distributions considered are associated to the filtration of B, in other words the "common noise" filtration, that is, $\mathcal{L}(X_{t\wedge\cdot}, \alpha_t|B)$, where X is the state and α the control. We call this the strong formulation. The strong formulation does not enjoy a good stability condition. To see this, it is enough to notice that the conditional distribution is not continuous with respect to the joint distribution $\big($for instance the function $\mathcal{L}(X_t, B) \longmapsto \mathbb{E}\big[\big|\mathbb{E}\big[X_t\big|B\big]\big|^2\big]$ is not continuous$\big)$. To overcome this difficulty, we introduce a notion of weak solution by considering a more general filtration $\mathbb{F}$ describing the adaptability of the controls, and an extended common noise filtration $\mathbb{G}$ as in Chapter 2 (see also [49]). Nevertheless, more conditions on $\mathbb{F}$ and $\mathbb{G}$ are needed to ensure that the formulation remains first compatible with the notion of strong solutions, then enjoys good stability properties for fixed control processes, and finally ensures that weak controls can be approximated sufficiently well by strong controls.

With the help of this notion, we can then provide a weak formulation for McKean–Vlasov control problems with common noise. By interpreting controls as probability measures on an appropriate canonical space, and using measurable selection arguments as in [61; 62], we then move on to prove the universal measurability of the associated value function, and derive the stability of controls with respect to conditioning and concatenation, and finally deduce the DPP for the weak formulation under very general assumptions. Our next result addresses the DPP for the classical strong formulation. Using the DPP in weak formulation, and by adding standard Lipschitz conditions on the drift and diffusion coefficients, as in [138], but without any regularity assumptions on reward functions, and in a non–Markovian context, we obtain the DPP for the strong formulation of McKean–Vlasov control problems with common noise, where the control is adapted to the "common noise" filtration (B in this case of strong formulation). Also, for general strong formulation, where the control is adapted to both ξ, W and B, we obtain the DPP under some additional regularity conditions on the reward functions. These regularity conditions may seem unexpected at first sight, but they seem unavoidable due to the non–linear dependency of the drift and volatility coefficients with respect to the conditional distribution of X (see Remark 5.3.6 for a more thorough discussion). Finally, the DPP results in the general non–Markovian context induces the same results in the Markovian one.

The rest of the chapter is organized as follows. After recalling briefly some notations and introducing the probabilistic structure to give an adequate and precise definition of the tools that are used throughout the chapter, we introduce in Section 5.2 several notions of weak and strong formulation (in a fixed probability space or in the canonical space) for the McKean–Vlasov stochastic control problem with common noise in a non–Markovian framework, and prove some equivalence results. Next, in Section 5.3, we present the main result of this chapter, the DPP for three formulations: weak formulation, strong formulation, and a $\mathbb{B}$–strong formulation where the control is adapted with respect to the "common noise" filtration. We first provide all our results in the non–Markovian setting, and then in a Markovian framework. Finally, Section 5.4 is devoted to the proof of our main results.

Notations. (i) Given a measurable space $(\Omega, \mathcal{F})$, we denote by $\mathcal{P}(\Omega)$ the collection of all probability measures on $(\Omega, \mathcal{F})$. For any probability measure $\mathbb{P} \in \mathcal{P}(\Omega)$, we denote by $\mathcal{F}^{\mathbb{P}}$ the $\mathbb{P}$–completion of the σ–field $\mathcal{F}$, and by $\mathcal{F}^U := \bigcap_{\mathbb{P}\in\mathcal{P}(\Omega)} \mathcal{F}^{\mathbb{P}}$ the universal completion of $\mathcal{F}$. Let $\xi : \Omega \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ be a random variable and $\mathbb{P} \in \mathcal{P}(\Omega)$, we define

$$\mathbb{E}^{\mathbb{P}}\big[\xi\big] := \mathbb{E}^{\mathbb{P}}[\xi_+] - \mathbb{E}^{\mathbb{P}}[\xi_-], \text{ where } \xi_+ := \xi \vee 0,\ \xi_- := (-\xi) \vee 0, \text{ with the convention } \infty - \infty := -\infty.$$

We also use the following notation to denote the expectation of ξ under $\mathbb{P}$

$$\mathbb{E}^{\mathbb{P}}[\xi] = \langle \mathbb{P}, \xi \rangle = \langle \xi, \mathbb{P} \rangle.$$

When Ω is a Polish space, a subset $A \subseteq \Omega$ is called an analytic set if there is another Polish space E, and a Borel subset $B \subseteq \Omega \times E$ such that $A = \{\omega \in \Omega : \exists e \in E,\ (\omega, e) \in B\}$. A function $f : \Omega \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is called upper semi–analytic (u.s.a. for short) if $\{\omega \in \Omega : f(\omega) > c\}$ is analytic for every $c \in \mathbb{R}$. Any upper semi–analytic function is universally measurable (see e.g. [27, Chapter 7]).

(ii) Let Ω be a metric space, $\mathcal{F}$ its Borel σ–field and $\mathcal{G} \subset \mathcal{F}$ be a sub–σ–field which is countably generated. Following [150], we say that $(\mathbb{P}^{\mathcal{G}}_\omega)_{\omega\in\Omega}$ is a family of r.c.p.d. (regular conditional probability distributions) of $\mathbb{P}$ knowing $\mathcal{G}$ if it satisfies

- the map $\omega \longmapsto \mathbb{P}^{\mathcal{G}}_\omega$ is $\mathcal{G}-$measurable, and for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, one has $\mathbb{P}[A \cap B] = \int_B \mathbb{P}^{\mathcal{G}}_\omega[A]\mathbb{P}(\mathrm{d}\omega)$;
- $\mathbb{P}^{\mathcal{G}}_\omega\big[[\omega]_{\mathcal{G}}\big] = 1$ for all $\omega \in \Omega$, where $[\omega]_{\mathcal{G}} := \bigcap \big\{A \in \mathcal{F} : A \in \mathcal{G} \text{ and } \omega \in A\big\}$.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]})$ be a filtered probability space, $\mathcal{G} \subset \mathcal{F}$ be a sub–σ–field, and E a metric space. Then given a random element $\xi : \Omega \longrightarrow E$, we use both the notations $\mathcal{L}^{\mathbb{P}}(\xi|\mathcal{G})(\omega)$ and $\mathbb{P}^{\mathcal{G}}_\omega \circ (\xi)^{-1}$ to denote the conditional distribution of ξ knowing $\mathcal{G}$ under $\mathbb{P}$. Moreover, given a measurable process $X : [0, T] \times \Omega \longrightarrow E$, we can always define $\mu_t := \mathcal{L}^{\mathbb{P}}(X_t|\mathcal{G}_t)$ to be a $\mathcal{P}(E)$–valued $\mathbb{G}$–optional process (see for instance Lemma 5.5.1).

(iii) Let $\mathcal{C}^n_{s,t} := C([s, t]; \mathbb{R}^n)$ be the space of all $\mathbb{R}^n$–valued continuous functions on $[s, t]$, for $0 \le s \le t \le T$. When $n = 0$, the space $\mathcal{C}^n$ and $\mathcal{C}^n_{s,t}$ both degenerate to a singleton.

(iv) Throughout the chapter, we fix a constant $p \ge 0$, a nonempty Polish space (U, ρ) and a point $u_0 \in U$. Notice that a Polish space is always isomorphic to a Borel subset of $[0, 1]$. Let us denote by π such one (isomorphic) bijection between U and $\pi(U) \subseteq [0, 1]$. We further extend the definition of π^{-1} to $\mathbb{R} \cup \{-\infty, \infty\}$ by setting $\pi^{-1}(x) := \partial$ for all $x \notin \pi(U)$ and let $\overline{U} := U \cup \{\partial\}$, where ∂ is the usual cemetery point. Let $\nu \in \mathcal{P}(\mathcal{C}^k)$ (resp. $\bar\nu \in \mathcal{P}(\mathcal{C}^k \times U)$) be a Borel probability measure on the canonical space $\mathcal{C}^k$ (resp. $\mathcal{C}^k \times U$) equipped with the canonical process X (resp. (X, α)). We denote, for each $t \in [0, T]$,

$$\nu(t) := \nu \circ X^{-1}_{t\wedge\cdot} \ \big(\text{resp. } \bar\nu(t) := \bar\nu \circ (X_{t\wedge\cdot}, \alpha)^{-1}\big).$$

5.2 Weak and strong formulations of the McKean–Vlasov control problem

The main objective of this chapter is to study the following (non–Markovian) McKean–Vlasov control problem, in both strong and weak formulations, of the form

$$\text{“}\sup_\alpha \mathbb{E}\bigg[\int_0^T L\big(t, X^\alpha_{t\wedge\cdot}, \mathcal{L}\big(X^\alpha_{t\wedge\cdot}, \alpha_t\big|\mathcal{G}_t\big), \alpha_t\big)\mathrm{d}t + g\big(X^\alpha_\cdot, \mathcal{L}\big(X^\alpha_\cdot\big|\mathcal{G}_T\big)\big)\bigg], \text{”}$$

where $\mathbb{G} := (\mathcal{G}_t)_{0\le t\le T}$ is a filtration modeling the common noise, supporting a Brownian motion B, $\mathcal{L}\big(X^\alpha_{t\wedge\cdot}, \alpha_t\big|\mathcal{G}_t\big)$ denotes the joint conditional distribution of $(X^\alpha_{t\wedge\cdot}, \alpha_t)$ knowing $\mathcal{G}_t$, and $(X^\alpha_t)_{t\in[0,T]}$ is a McKean–Vlasov type process, controlled by $\alpha = (\alpha_t)_{0\le t\le T}$ and generated by W together with an independent Brownian motion B

$$\text{“}\mathrm{d}X^\alpha_t = b\big(t, X^\alpha_{t\wedge\cdot}, \mathcal{L}\big(\big(X^\alpha_{t\wedge\cdot}, \alpha_t\big)\big|\mathcal{G}_t\big), \alpha_t\big)\mathrm{d}t + \sigma\big(t, X^\alpha_{t\wedge\cdot}, \mathcal{L}\big(\big(X^\alpha_{t\wedge\cdot}, \alpha_t\big)\big|\mathcal{G}_t\big), \alpha_t\big)\mathrm{d}W_t + \sigma_0\big(t, X^\alpha_{t\wedge\cdot}, \mathcal{L}\big(\big(X^\alpha_{t\wedge\cdot}, \alpha_t\big)\big|\mathcal{G}_t\big), \alpha_t\big)\mathrm{d}B_t. \text{”} \tag{5.2.1}$$

We will provide in the following a precise definition to the above controlled SDE, depending on the strong/weak formulation considered. Let us first specify the dimensions and some basic conditions on the coefficient functions. Let $(n, \ell, d) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The coefficient functions

$$b : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longrightarrow \mathbb{R}^n, \ \sigma : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longrightarrow \mathbb{S}^{n\times d},$$

$$\sigma_0 : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longrightarrow \mathbb{S}^{n\times \ell}, \ L : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longrightarrow \mathbb{R}, \ g : \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \longrightarrow \mathbb{R},$$

are all assumed to be Borel measurable, and non–anticipative in the sense that

$$\big(b, \sigma, \sigma_0, L\big)(t, \mathbf{x}, \bar{\nu}, u) = \big(b, \sigma, \sigma_0, L\big)(t, \mathbf{x}_{t\wedge\cdot}, \bar{\nu}(t), u), \text{ for all } (t, \mathbf{x}, \bar{\nu}, u) \in [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U.$$

5.2.1 A weak formulation

A weak formulation of the control problem is obtained by considering all weak solutions of the controlled McKean–Vlasov SDE (5.2.1). Here the word "weak" refers to the fact that the probability space, as well as the equipped Brownian motion, is not assumed to be fixed, but is a part of the solution itself. This is of course consistent with the notion of the weak solution in the classical SDE theory.

Definition 5.2.1. *Let $(t, \nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$. We say that a term*

$$\gamma := \big(\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma, \mathbb{F}^\gamma = (\mathcal{F}^\gamma_s)_{0\le s\le T}, \mathbb{G}^\gamma = (\mathcal{G}^\gamma_s)_{0\le s\le T}, X^\gamma, W^\gamma, B^\gamma, \overline{\mu}^\gamma, \mu^\gamma, \alpha^\gamma\big),$$

is a weak control associated with the initial condition (t, ν) if the following conditions are satisfied:

(i) $(\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma)$ is a probability space, equipped with two filtrations $\mathbb{F}^\gamma$ and $\mathbb{G}^\gamma$ such that, for all $s \in [0,T]$,

$$\mathcal{G}^\gamma_s \subseteq \mathcal{F}^\gamma_s, \text{ and } \mathbb{E}^{\mathbb{P}^\gamma}\big[\mathbf{1}_D \big| \mathcal{G}^\gamma_s\big] = \mathbb{E}^{\mathbb{P}^\gamma}\big[\mathbf{1}_D \big| \mathcal{G}^\gamma_T\big], \ \mathbb{P}^\gamma\text{–a.s., for all } D \in \mathcal{F}^\gamma_s \vee \sigma(W^\gamma); \tag{5.2.2}$$

(ii) $X^\gamma = (X^\gamma_s)_{s\in[0,T]}$ is an $\mathbb{R}^n$–valued, $\mathbb{F}^\gamma$–adapted continuous process, $\alpha^\gamma := (\alpha^\gamma_s)_{t\le s\le T}$ is an U–valued, $\mathbb{F}^\gamma$–predictable process, and with the fixed constant $p \ge 0$, one has

$$\mathbb{E}^{\mathbb{P}^\gamma}\big[\|X^\gamma\|^p\big] + \mathbb{E}^{\mathbb{P}^\gamma}\bigg[\int_t^T \big(\rho(\alpha^\gamma_s, u_0)\big)^p \mathrm{d}s\bigg] < \infty; \tag{5.2.3}$$

(iii) (W^γ, B^γ) is an $\mathbb{R}^d \times \mathbb{R}^\ell$–valued, $\mathbb{F}^\gamma$–adapted continuous process; $(W^{\gamma,t}, B^{\gamma,t}) := \big((W^{\gamma,t}_s)_{0\le s\le T}, (B^{\gamma,t}_s)_{0\le s\le T}\big)$, defined by $W^{\gamma,t}_s := W^\gamma_{s\vee t} - W^\gamma_t$, and $B^{\gamma,t}_s := B^\gamma_{s\vee t} - B^\gamma_t$, $s \in [t,T]$, is a standard $(\mathbb{F}^\gamma, \mathbb{P}^\gamma)$–Brownian motion on $[t,T]$; $B^{\gamma,t}$ is $\mathbb{G}^\gamma$–adapted; $\mathcal{F}^\gamma_t \vee \sigma(W^\gamma)$ is $\mathbb{P}^\gamma$–independent of $\mathcal{G}^\gamma_T$, and $\mu^\gamma = (\mu^\gamma_s)_{t\le s\le T}$ (resp. $\overline{\mu}^\gamma = (\overline{\mu}^\gamma_s)_{t\le s\le T}$) is a $\mathbb{G}^\gamma$–predictable $\mathcal{P}(\mathcal{C}^n)$–valued (resp. $\mathcal{P}(\mathcal{C}^n \times U)$–valued) process satisfying

$$\mu^\gamma_s = \mathcal{L}^{\mathbb{P}^\gamma}\big(X^\gamma_{s\wedge\cdot} \big| \mathcal{G}^\gamma_s\big) \ \big(\text{resp. } \overline{\mu}^\gamma_s = \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma_{s\wedge\cdot}, \alpha^\gamma_s\big) \big| \mathcal{G}^\gamma_s\big)\big), \text{ for } \mathrm{d}\mathbb{P}^\gamma \otimes \mathrm{d}s\text{–a.e. } (s,\omega) \in [t,T] \times \Omega^\gamma;$$

(iv) X^γ satisfies $\mathbb{P}^\gamma \circ (X^\gamma_{t\wedge\cdot})^{-1} = \nu(t)$ and

$$X^\gamma_s = X^\gamma_t + \int_t^s b(r, X^\gamma_\cdot, \overline{\mu}^\gamma_r, \alpha^\gamma_r)\mathrm{d}r + \int_t^s \sigma(r, X^\gamma_\cdot, \overline{\mu}^\gamma_r, \alpha^\gamma_r)\mathrm{d}W^\gamma_r + \int_t^s \sigma_0(r, X^\gamma_\cdot, \overline{\mu}^\gamma_r, \alpha^\gamma_r)\mathrm{d}B^\gamma_r, \ s \in [t,T], \ \mathbb{P}^\gamma\text{–a.s.}, \tag{5.2.4}$$

where the integrals are implicitly assumed to be well–defined.

For all $(t, \nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$, let us denote

$$\Gamma_W(t, \nu) := \big\{\text{All weak controls with initial condition } (t, \nu)\big\}.$$

Then with the reward functions $L : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longrightarrow \mathbb{R}$ and $g : \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \longrightarrow \mathbb{R}$, we introduce the value function of our McKean–Vlasov optimal control problem by

$$V_W(t, \nu) := \sup_{\gamma \in \Gamma_W(t,\nu)} J(t, \gamma), \text{ where } J(t, \gamma) := \mathbb{E}^{\mathbb{P}^\gamma}\bigg[\int_t^T L(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s)\mathrm{d}s + g(X^\gamma_{T\wedge\cdot}, \mu^\gamma_T)\bigg]. \tag{5.2.5}$$

Remark 5.2.2. *In a weak control γ, the filtration $\mathbb{G}^\gamma$ is used to model the common noise. In particular, $B^{\gamma,t}$ is adapted to $\mathbb{G}^\gamma$, and $W^{\gamma,t}$ is independent of $\mathbb{G}^\gamma$. In the classical strong formulation, $\mathbb{G}^\gamma$ is fixed as the filtration $\mathbb{F}^{B^{\gamma,t}}$ generated by $B^{\gamma,t}$, but for a general weak control, $\mathbb{G}^\gamma$ may be larger than $\mathbb{F}^{B^{\gamma,t}}$. This will be the main difference between the strong and weak formulations in our approach. Meanwhile, $(\mathbb{G}^\gamma, \mathbb{F}^\gamma)$ satisfies a (H)–hypothesis type condition in (5.2.2), which is consistent with the classical strong formulation (see* Section 5.2.2 *below). This property will be crucial in our proof of the* DPP *result for the strong formulation of the control problem, as well as in the limit theory of the McKean–Vlasov control problem in* Chapter 3 *and* Chapter 4.

Remark 5.2.3. *(i) At this stage, the integrability condition (5.2.3) could be construed as artificial. Depending on more concrete property of the coefficient functions (b, σ, σ_0), it would play the role of an admissibility condition for the control process, and ensure that the stochastic integrals in (5.2.4) are well–defined. For a more concrete example, consider the case where $U = \mathbb{R}$ and $u_0 = 0$. When b, σ, and σ_0 are all uniformly bounded, one can choose $p = 0$ so that all $\mathbb{R}$–valued predictable processes would then be admissible. When $\sigma(t, x, u, \bar{\nu}) = u$, one may choose $p = 2$ to ensure that the stochastic integral $\int_t^T \alpha_s^\gamma \mathrm{d}W_s^\gamma$ is well–defined and is a square–integrable martingale. It is also possible to consider more general types of integrability conditions, such as*

$$\mathbb{E}^{\mathbb{P}^\gamma}\left[\Phi\Big(\int_t^T \Psi\big(\rho(u_0, \alpha_s^\gamma)\big)\mathrm{d}s\Big)\right] < \infty,$$

for given maps $\Phi : [0,\infty) \longrightarrow [0,\infty)$ and $\Psi : [0,\infty) \longrightarrow [0,\infty)$. This would for instance allow to consider exponential integrability requirements. For the sake of simplicity, we have chosen the condition in (5.2.3), but insist that it plays no essential role in the proof of the dynamic programming principle.

(ii) The set $\Gamma_W(t,\nu)$ could be empty, in which case $V_W(t,\nu) = -\infty$ by convention. For example, when $\int_{\mathcal{C}^n} \|\mathbf{x}\|^p \nu(d\mathbf{x}) = \infty$, then $\Gamma_W(t,\nu) = \emptyset$, since (5.2.3) cannot be satisfied. Nevertheless, $\Gamma_W(t,\nu)$ is non–empty under either one of the following conditions (see for instance Theorem 2.4.2 *for a brief proof)*

- *(b, σ, σ_0) are bounded and continuous in $(\mathbf{x}, \bar{\nu}, u)$ and $\nu \in \mathcal{P}(\mathcal{C}^n)$;*
- *(b, σ, σ_0) are continuous in $(\mathbf{x}, \bar{\nu}, u)$, and with the fixed constant p and other positive constants C, p', $\hat{p}$ such that $p' > p \geq 1 \vee \hat{p}$, and $p' \geq 2 \geq \hat{p} \geq 0$, we have $\nu \in \mathcal{P}_{p'}(\mathcal{C}^n)$ and*

$$\big|b(t,\mathbf{x},\bar{\nu},u)\big| \leq C\bigg(1 + \|\mathbf{x}\| + \Big(\int_{\mathcal{C}^n\times U} \big(\|\mathbf{x}'\|^p + \rho(u_0,u')^p\big)\bar{\nu}(\mathrm{d}\mathbf{x}',\mathrm{d}u')\Big)^{\frac{1}{p}} + \rho(u_0,u)\bigg),$$

$$\big|(\sigma,\sigma_0)(t,\mathbf{x},\bar{\nu},u)\big|^2 \leq C\bigg(1 + \|\mathbf{x}\|^{\hat{p}} + \Big(\int_{\mathcal{C}^n\times U} \big(\|\mathbf{x}'\|^p + \rho(u_0,u')^p\big)\bar{\nu}(\mathrm{d}\mathbf{x}',\mathrm{d}u')\Big)^{\frac{\hat{p}}{p}} + \rho(u_0,u)^{\hat{p}}\bigg),$$

for all $(t,\mathbf{x},\bar{\nu},u) \in [0,T]\times\mathcal{C}^n\times\mathcal{P}(\mathcal{C}^n\times U)\times U$.

Remark 5.2.4. *For technical reasons, we introduce an $\mathbb{R}$–valued, $\mathbb{F}^\gamma$–adapted continuous process $A^\gamma := (A_s^\gamma)_{0\leq s\leq T}$ by (recall that $\pi : U \to [0,1]$ is a fixed isomorphic bijection between U and $\pi(U)$)*

$$A_s^\gamma := \int_t^{s\vee t} \pi(\alpha_r^\gamma)\mathrm{d}r,\ s\in[0,T],\ \text{so that } \pi\big(\alpha_s^\gamma\big) = \overline{\lim_{n\to\infty}}\, n\big(A_s^\gamma - A^\gamma_{(s-1/n)\vee 0}\big),\ \mathrm{d}\mathbb{P}^\gamma\otimes\mathrm{d}s\text{–a.e. on } \Omega^\gamma\times[t,T]. \tag{5.2.6}$$

We further define a $\mathcal{P}(\mathcal{C}^n\times\mathcal{C}\times\mathcal{C}^d\times\mathcal{C}^\ell)$–valued process $\widehat{\mu}^\gamma = (\widehat{\mu}_s^\gamma)_{0\leq s\leq T}$ by

$$\widehat{\mu}_s^\gamma := \mathcal{L}^{\mathbb{P}^\gamma}\big(X^\gamma_{s\wedge\cdot}, A^\gamma_{s\wedge\cdot}, W^\gamma, B^\gamma_{s\wedge\cdot}\big)\mathbf{1}_{\{s\in[0,t]\}} + \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma_{s\wedge\cdot}, A^\gamma_{s\wedge\cdot}, W^\gamma, B^\gamma_{s\wedge\cdot}\big)\big|\mathcal{G}_s^\gamma\big)\mathbf{1}_{\{s\in(t,T]\}}. \tag{5.2.7}$$

The process $\widehat{\mu}^\gamma$ can be defined to be a $\mathbb{G}^\gamma$–adapted and $\mathbb{P}^\gamma$–a.s. continuous process (equipping $\mathcal{P}(\mathcal{C}^n\times\mathcal{C}\times\mathcal{C}^d\times\mathcal{C}^\ell)$ with the weak convergence topology). Indeed, by (5.2.2), we have

$$\widehat{\mu}_s^\gamma = \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma_{s\wedge\cdot}, A^\gamma_{s\wedge\cdot}, W^\gamma, B^\gamma_{s\wedge\cdot}\big)\big|\mathcal{G}_T^\gamma\big),\ \mathbb{P}^\gamma\text{–a.e., for all } s\in(t,T].$$

Then by Lemma 5.5.1, *$\widehat{\mu}^\gamma$ can be defined to be $\mathbb{P}^\gamma$–a.s. continuous on both $[0,t]$ and $(t,T]$. Moreover, using the independence property between $\mathcal{F}_t^\gamma \vee \sigma(W^\gamma)$ and $\mathcal{G}_T^\gamma$, we have, $\mathbb{P}^\gamma$–a.s.*

$$\lim_{r\searrow t}\widehat{\mu}_r^\gamma = \lim_{r\searrow t}\mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma_{r\wedge\cdot}, A^\gamma_{r\wedge\cdot}, W^\gamma, B^\gamma_{r\wedge\cdot}\big)\big|\mathcal{G}_T^\gamma\big) = \mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma_{t\wedge\cdot}, A^\gamma_{t\wedge\cdot}, W^\gamma, B^\gamma_{t\wedge\cdot}\big)\big|\mathcal{G}_T^\gamma\big) = \mathcal{L}^{\mathbb{P}^\gamma}\big(X^\gamma_{t\wedge\cdot}, A^\gamma_{t\wedge\cdot}, W^\gamma, B^\gamma_{t\wedge\cdot}\big) = \widehat{\mu}_t^\gamma.$$

Remark 5.2.5. *It is perfectly possible for us to consider a slightly more general class of control problems allowing for exponential discounting. More precisely, we could have an additional Borel map $k : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longrightarrow \mathbb{R}$ and consider, for fixed $(t,\nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$, the problem of maximising over $\gamma \in \Gamma_W(t,\nu)$ the functional*

$$\tilde{J}(t,\gamma) := \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^T \mathrm{e}^{-\int_t^s k(u,X^\gamma_{u\wedge\cdot},\overline{\mu}^\gamma_u,\alpha^\gamma_u)\mathrm{d}u} L(s,X^\gamma_{s\wedge\cdot},\overline{\mu}^\gamma_s,\alpha^\gamma_s)\mathrm{d}s + \mathrm{e}^{-\int_t^T k(u,X^\gamma_{u\wedge\cdot},\overline{\mu}^\gamma_u,\alpha^\gamma_u)\mathrm{d}u} g(X^\gamma_{T\wedge\cdot},\mu^\gamma_T)\Big].$$

We refrained from working at that level of generality for notational simplicity, but our results extend directly to this context.

5.2.2 A strong formulation

To obtain a strong formulation of the control problem, the usual approach is to consider a fixed probability space, equipped with fixed Brownian motions and fixed Brownian filtrations. In fact, this is equivalent to fix the filtrations, in the weak control γ, to be the Brownian filtrations (see Proposition 5.2.10 below). We will therefore present the two equivalent definitions one after the other.

5.2.2.1 Strong formulation as a special case of weak formulation

Let us start with the main definition.

Definition 5.2.6. *Let $(t,\nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$. A term $\gamma = \big(\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{F}^\gamma, \mathbb{P}^\gamma, \mathbb{G}^\gamma, X^\gamma, W^\gamma, B^\gamma, \overline{\mu}^\gamma, \mu^\gamma, \alpha^\gamma\big)$ is called a strong control associated with the initial condition (t,ν), if $\gamma \in \Gamma_W(t,\nu)$ and the filtrations $\mathbb{G}^\gamma$ and $\mathbb{F}^\gamma$ are $\mathbb{P}^\gamma$-augmented filtrations of $\mathbb{G}^{\gamma,\circ} := (\mathcal{G}^{\gamma,\circ}_s)_{s\in[0,T]}$ and $\mathbb{F}^{\gamma,\circ} := (\mathcal{F}^{\gamma,\circ}_s)_{s\in[0,T]}$, which are defined by*

$$\mathcal{G}^{\gamma,\circ}_s := \begin{cases} \{\emptyset,\Omega^\gamma\}, \text{ if } 0 \le s < t, \\ \sigma\big(B^{\gamma,t}_r : r \in [t,s]\big), \text{ if } 0 \le t \le s \le T, \end{cases} \quad \text{and } \mathcal{F}^{\gamma,\circ}_s := \begin{cases} \sigma\big(X^\gamma_{s\wedge\cdot}\big), \text{ if } 0 \le s < t, \\ \sigma\big((X^\gamma_{t\wedge\cdot}, B^{\gamma,t}_r, W^{\gamma,t}_r) : r \in [t,s]\big), \text{ if } 0 \le t \le s \le T. \end{cases}$$

If, in addition, the control process α^γ is $\mathbb{G}^\gamma$–predictable, then γ is called a $\mathbb{B}$–strong control.

Let us denote by $\Gamma_S(t,\nu)$ the collection of all strong controls with initial condition (t,ν), and by $\Gamma^{\mathbb{B}}_S(t,\nu)$ the collection of all $\mathbb{B}$–strong controls with initial condition (t,ν), i.e.

$$\Gamma^{\mathbb{B}}_S(t,\nu) := \big\{\gamma \in \Gamma_S(t,\nu) : \alpha^\gamma \text{ is } \mathbb{G}^\gamma\text{–predictable}\big\}.$$

Remark 5.2.7. *For a strong control $\gamma \in \Gamma_S(t,\nu)$, the filtration $\mathbb{G}^\gamma$ is generated by $B^{\gamma,t}$, and $\mathbb{F}^\gamma$ is generated by the initial variable $X^\gamma_{t\wedge\cdot}$ and the Brownian motion $W^{\gamma,t}_\cdot$ and $B^{\gamma,t}_\cdot$. Consequently, the control process α^γ is adapted to the filtration generated by $(X^\gamma_{t\wedge\cdot}, W^{\gamma,t}_\cdot, B^{\gamma,t}_\cdot)$, and the common noise comes only from $B^{\gamma,t}$. We will show in* Proposition 5.2.10 *that this is equivalent to the case with a fixed probability space equipped with fixed Brownian motions and the initial random variable. We here define the strong control rules as special cases of weak control rules in order to avoid repeating all the technical conditions in* Definition 5.2.1.

Our proof of the dynamic programming principle for the strong formulation of the McKean–Vlasov problem relies essentially on its equivalence to the weak formulation, which requires the following standard Lipschitz condition on the coefficient functions. Moreover, this condition ensures the existence and uniqueness of the solution to SDE (5.2.4) (see e.g. Theorem 5.5.3).

Assumption 5.2.8. *Let the constant in (5.2.3) be $p = 2$. There exists a constant $C > 0$ such that, for all $(t,\mathbf{x},\mathbf{x}',\bar{\nu},\bar{\nu}',u) \in [0,T] \times \mathcal{C}^n \times \mathcal{C}^n \times \mathcal{P}_2(\mathcal{C}^n \times U) \times \mathcal{P}_2(\mathcal{C}^n \times U) \times U$, one has*

$$\big\|\big(b,\sigma,\sigma_0\big)(t,\mathbf{x},\bar{\nu},u) - \big(b,\sigma,\sigma_0\big)(t,\mathbf{x}',\bar{\nu}',u)\big\| \le C\big(\|\mathbf{x}-\mathbf{x}'\| + \mathcal{W}_2(\bar{\nu},\bar{\nu}')\big), \tag{5.2.8}$$

$$\big\|(b,\sigma,\sigma_0)(t,\mathbf{x},\bar{\nu},u)\big\|^2 \le C\bigg(1 + \|\mathbf{x}\|^2 + \int_{\mathcal{C}^n\times U}\big(\|\mathbf{y}\|^2 + \rho(\hat{u},u_0)^2\big)\bar{\nu}(\mathrm{d}\mathbf{y},\mathrm{d}\hat{u}) + \rho(u,u_0)^2\bigg).$$

Under Assumption 5.2.8, the set $\Gamma_S(t,\nu)$ is nonempty for all $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$ (see e.g. Theorem 5.5.3). We then introduce the following strong formulation (resp. $\mathbb{B}$–strong formulation) of the McKean–Vlasov control problem

$$V_S(t,\nu) := \sup_{\gamma \in \Gamma_S(t,\nu)} J(t,\gamma), \text{ and } V_S^{\mathbb{B}}(t,\nu) := \sup_{\gamma \in \Gamma_S^{\mathbb{B}}(t,\nu)} J(t,\gamma). \tag{5.2.9}$$

Remark 5.2.9 (The case without common noise: $\ell = 0$)**.** *In the literature, the McKean–Vlasov control problem without common noise has also largely been studied. This contained as a special case in our setting. Indeed, when $\ell = 0$, the process $B^{\gamma,t}$ degenerates to be a singleton and hence $\mathcal{G}_s^\gamma = \{\emptyset, \Omega^\gamma\}$ for all $s \in [0,T]$. It follows that $\overline{\mu}^\gamma$ appearing in (5.2.4) turns out to satisfy*

$$\overline{\mu}_s^\gamma = \mathcal{L}^{\mathbb{P}^\gamma}\big(X_{s\wedge\cdot}^\gamma, \alpha_s^\gamma\big), \text{ for } \mathrm{d}\mathbb{P}^\gamma \otimes \mathrm{d}t\text{–a.e. } (s,\omega) \in [t,T] \times \Omega^\gamma,$$

and the value function $V_S(t,\nu)$ in (5.2.9) is the standard formulation of the control problem without common noise (see e.g. [43]).

5.2.2.2 Strong formulation on a fixed probability space

For $0 \le s \le t \le T$, let $\mathcal{C}_{s,t}^n := C([s,t],\mathbb{R}^n)$ denote the space of all $\mathbb{R}^n$–valued continuous paths on $[s,t]$, we then introduce a first canonical space, for every $t \in [0,T]$

$$\Omega^t := \mathcal{C}_{0,t}^n \times \mathcal{C}_{t,T}^d \times \mathcal{C}_{t,T}^\ell, \ \mathcal{F}^t := \mathcal{B}(\Omega^t), \tag{5.2.10}$$

with corresponding canonical processes $\zeta := (\zeta_s)_{0\le s\le t}$, $W := (W_s)_{t\le s\le T}$, and $B := (B_s)_{t\le s\le T}$. Let $W_s^t := W_{s\vee t} - W_t$ and $B_s^t := B_{s\vee t} - B_t$ for all $s \in [0,T]$, and define $\mathbb{F}^{t,\circ} = (\mathcal{F}_s^{t,\circ})_{0\le s\le T}$ and $\mathbb{G}^{t,\circ} = (\mathcal{G}_s^{t,\circ})_{0\le s\le T}$ by

$$\mathcal{F}_s^{t,\circ} := \begin{cases} \sigma\big(\zeta_{s\wedge\cdot}\big), \text{ if } 0 \le s < t, \\ \sigma\big((\zeta_{t\wedge\cdot}, W_r^t, B_r^t) : r \in [t,s]\big), \text{ if } 0 \le t \le s \le T, \end{cases} \text{ and } \mathcal{G}_s^{t,\circ} := \begin{cases} \{\emptyset, \Omega^t\}, \text{ if } 0 \le s < t, \\ \sigma\big(B_r^t : r \in [t,s]\big), \text{ if } 0 \le t \le s \le T. \end{cases}$$

Let $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$, we fix a probability measure $\mathbb{P}_\nu^t$ on Ω^t, such that $\mathcal{L}^{\mathbb{P}_\nu^t}\big(\zeta_{t\wedge\cdot}\big) = \nu(t)$, and (W^t, B^t) are standard Brownian motions on $[t,T]$, independent of ζ. Let $\mathbb{F}^t = (\mathcal{F}_s^t)_{0\le s\le T}$ and $\mathbb{G}^t = (\mathcal{G}_s^t)_{0\le s\le T}$ be the $\mathbb{P}_\nu^t$–augmented filtration of $\mathbb{F}^{t,\circ}$ and $\mathbb{G}^{t,\circ}$, we denote by $\mathcal{A}_2(t,\nu)$ (resp. $\mathcal{A}_2^{\mathbb{B}}(t,\nu)$) the collection of all U–valued processes $\alpha = (\alpha_s)_{t\le s\le T}$ which are $\mathbb{F}^t$–predictable (resp. $\mathbb{G}^t$–predictable) and such that

$$\mathbb{E}^{\mathbb{P}_\nu^t}\bigg[\int_t^T \big(\rho(u_0,\alpha_s)\big)^2 \mathrm{d}s\bigg] < \infty.$$

Then, given $\alpha \in \mathcal{A}_2(t,\nu)$, let X^α be the unique strong solution (in sense of Definition 5.5.2) of the SDE, with initial condition $X_{t\wedge\cdot}^\alpha = \zeta_{t\wedge\cdot}$,

$$X_s^\alpha = X_t^\alpha + \int_t^s b\big(r, X_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}r + \int_t^s \sigma\big(r, X_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}W_r^t + \int_t^s \sigma_0\big(r, X_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}B_r^t, \ t \le s \le T, \ \mathbb{P}_\nu^t\text{–a.s.}, \tag{5.2.11}$$

where $\overline{\mu}_r^\alpha = \mathcal{L}^{\mathbb{P}_\nu^t}\big((X_{r\wedge\cdot}^\alpha, \alpha_r)\big|\mathcal{G}_r^t\big)$, $\mathrm{d}\mathbb{P}_\nu^t \times \mathrm{d}r$–a.e. on $\Omega^t \times [t,T]$. Notice that the existence and uniqueness of a solution to SDE (5.2.11) is ensured by Assumption 5.2.8 (see Theorem 5.5.3). Finally, we denote for any $\alpha \in \mathcal{A}_2(t,\nu)$, $\mu_s^\alpha := \mathcal{L}^{\mathbb{P}_\nu^t}\big(X_{s\wedge\cdot}^\alpha\big|\mathcal{G}_s^t\big)$, $s \in [t,T]$.

We next show that the above strong formulation of the control problem with fixed probability space is equivalent to that in Definition 5.2.6 as a special case of the weak control rules.

Proposition 5.2.10. *Let* Assumption 5.2.8 *hold true. Then for all $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$, one has*

$$V_S(t,\nu) = \sup_{\alpha \in \mathcal{A}_2(t,\nu)} J(t,\nu,\alpha), \text{ and } V_S^{\mathbb{B}}(t,\nu) = \sup_{\alpha \in \mathcal{A}_2^{\mathbb{B}}(t,\nu)} J(t,\nu,\alpha), \tag{5.2.12}$$

where

$$J(t,\nu,\alpha) := \mathbb{E}^{\mathbb{P}_\nu^t}\bigg[\int_t^T L(s, X_{s\wedge\cdot}^\alpha, \overline{\mu}_s^\alpha, \alpha_s)\mathrm{d}s + g(X_\cdot^\alpha, \mu_T^\alpha)\bigg].$$

Proof. We will only consider the case of V_S, since the arguments for the case of $V_S^{\mathbb{B}}$ are exactly the same. First, given $\alpha \in \mathcal{A}_2(t,\nu)$, let us define

$$\gamma := \big(\Omega^t, \mathcal{F}^t, \mathbb{P}^t_\nu, \mathbb{F}^t, \mathbb{G}^t, X^\alpha, W^t, B^t, \overline{\mu}^\alpha, \mu^\alpha, \alpha\big).$$

Then it is straightforward to check that γ is a strong control rule (i.e. $\gamma \in \Gamma_S(t,\nu)$) such that $J(t,\gamma) = J(t,\nu,\alpha)$.

Next, let $\gamma \in \Gamma_S(t,\nu)$. Notice that $(X^\gamma, \alpha^\gamma)$ is $\mathbb{F}^\gamma$–predictable, and $(\mu^\gamma, \overline{\mu}^\gamma)$ is $\mathbb{G}^\gamma$–predictable. Using for instance Claisse, Talay, and Tan [52, Proposition 10] (with a slight extension consisting simply in having a larger $\mathcal{F}_0$), there exists two Borel measurable functions $\Psi_1 : [0,T] \times \Omega^t \longrightarrow \mathbb{R}^n \times U$ and $\Psi_2 : [0,T] \times \mathcal{C}^\ell \longrightarrow \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\mathcal{C}^n \times U)$ such that

$$\big(X^\gamma_s, \alpha^\gamma_s\big) = \Psi_1\big(s, X^\gamma_{t\wedge\cdot}, W^{\gamma,t}_{s\wedge\cdot}, B^{\gamma,t}_{s\wedge\cdot}\big),\ \big(\mu^\gamma_s, \overline{\mu}^\gamma_s\big) = \Psi_2\big(s, B^{\gamma,t}_{s\wedge\cdot}\big),\ s \in [0,T],\ \mathbb{P}^\gamma\text{–a.s.}$$

Then, on Ω^t, let us define $(X^\star_s, \alpha^\star_s) := \Psi_1(s, \zeta t \wedge \cdot, W^t_{s\wedge\cdot}, B^t_{s\wedge\cdot})$ and $(\mu^\star_s, \overline{\mu}^\star_s) := \Psi_2(s, B^t_{s\wedge\cdot})$, so that

$$\alpha^\star \in \mathcal{A}_2(t,\nu),\ \text{and}\ \mathbb{P}^t_\nu \circ \big(X^\star, W^t, B^t, \alpha^\star, \overline{\mu}^\star, \mu^\star\big)^{-1} = \mathbb{P}^\gamma \circ \big(X^\gamma, W^{\gamma,t}, B^{\gamma,t}, \alpha^\gamma, \overline{\mu}^\gamma, \mu^\gamma\big)^{-1}.$$

This implies that $X^\star$ is the unique strong solutions to SDE (5.2.11) with control $\alpha^\star$, such that $\mu^\star_s = \mathcal{L}^{\mathbb{P}^t_\nu}(X^\star_{s\wedge\cdot}|\mathcal{G}^t_s)$, $\overline{\mu}^\star_s = \mathcal{L}^{\mathbb{P}^t_\nu}((X^\star_{s\wedge\cdot}, \alpha^\star_s)|\mathcal{G}^t_s)$ and $J(t,\nu,\alpha^\star) = J(t,\gamma)$. □

5.3 The dynamic programming principle

The main results of our chapter consist in the dynamic programming principle (DPP) for the previously introduced formulations of the McKean–Vlasov control problem. We will first prove the DPP for the general strong and weak control problems introduced in Section 5.2, and then show how they naturally induce the associated results in the Markovian case. Finally, we also discuss heuristically the Hamilton–Jacobi–Bellman (HJB for short) equations which can be deduced for each formulation.

5.3.1 The dynamic programming principle in the general case

5.3.1.1 Dynamic programming principle for the weak control problem

To provide the dynamic programming principle of the McKean–Vlasov control problem (5.2.5), let us introduce another canonical space

$$\Omega^\star := \mathcal{C}^\ell \times C\big([0,T], \mathcal{P}(\mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C}^\ell)\big),\ \text{with canonical process}\ (B^\star, \widehat{\mu}^\star) := (B^\star_s, \widehat{\mu}^\star_s)_{s\in[0,T]},$$

and canonical filtration $\mathbb{G}^\star := (\mathcal{G}^\star_s)_{0\le s\le T}$ defined by $\mathcal{G}^\star_s := \sigma\big\{(\widehat{\mu}^\star_r, B^\star_r) : r \in [0,s]\big\}$, $s \in [0,T]$. Then, for every $\mathbb{G}^\star$–stopping time $\tau^\star$ (which can then be written as a function of B and $\widehat{\mu}$), for all $(t,\nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$ and $\gamma \in \Gamma_W(t,\nu)$, we define (recall that $\widehat{\mu}^\gamma$ is defined by (5.2.7))

$$\tau^\gamma := \tau^\star\big(B^{\gamma,t}_\cdot, \widehat{\mu}^\gamma_\cdot\big). \tag{5.3.1}$$

Theorem 5.3.1. *The value function $V_W : [0,T] \times \mathcal{P}(\mathcal{C}^n) \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ of the weak McKean–Vlasov control problem (5.2.5) is upper semi–analytic. Moreover, let $(t,\nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$, $\tau^\star$ be a $\mathbb{G}^\star$–stopping time taking values in $[t,T]$, and τ^γ be defined in (5.3.1), one has*

$$V_W(t,\nu) = \sup_{\gamma\in\Gamma_W(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\bigg[\int_t^{\tau^\gamma} L(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s)\mathrm{d}s + V_W\big(\tau^\gamma, \mu^\gamma_{\tau^\gamma}\big)\bigg]. \tag{5.3.2}$$

5.3.1.2 Dynamic programming for the strong control problems

We now consider the two strong formulations of the control problems introduced in (5.2.9), or equivalently in (5.2.12). To formulate the DPP results, we will rather use the fixed probability space context in (5.2.12). Recall that, given initial condition $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$, a fixed probability space $(\Omega^t, \mathcal{F}^t, \mathbb{P}^t_\nu)$ is defined in and below (5.2.10). Let us first consider the strong control problem $V_S^{\mathbb{B}}$.

Theorem 5.3.2. *Let* Assumption 5.2.8 *hold. Then the value function* $V_S^{\mathbb{B}} : [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ *is upper semi–analytic. Moreover, let* $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$*, and* τ *be a* $\mathbb{G}^{t,\circ}$*–stopping time on* $(\Omega^t, \mathcal{F}^t, \mathbb{P}_\nu^t)$*, taking values in* $[t,T]$*, one has*

$$V_S^{\mathbb{B}}(t,\nu) = \sup_{\alpha \in \mathcal{A}_2^{\mathbb{B}}(t,\nu)} \mathbb{E}^{\mathbb{P}_\nu^t}\Big[\int_t^\tau L(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s)\mathrm{d}s + V_S^{\mathbb{B}}\big(\tau, \mu^\alpha_\tau\big)\Big]. \tag{5.3.3}$$

For the strong control problem V_S, we need some additional regularity conditions on the coefficient functions.

Assumption 5.3.3. *For all* $t \in [0,T]$*, the functions*

$$(b,\sigma,\sigma_0) : (\mathbf{x}, \bar{\nu}, u) \in \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times U) \times U \longmapsto (b,\sigma,\sigma_0)(t,\mathbf{x},\bar{\nu},u) \in \mathbb{R}^n \times \mathbb{S}^{n\times d} \times \mathbb{S}^{n\times \ell},$$

are continuous, and there exists a constant $C > 0$ *such that, for all* $(t,\mathbf{x},u,\bar{\nu}) \in [0,T] \times \mathcal{C}^n \times U \times \mathcal{P}(\mathcal{C}^n \times U)$*,*

$$\big|(L,g)(t,\mathbf{x},\bar{\nu},u)\big|^2 \le C\Big(1 + \|\mathbf{x}\|^2 + \int_{\mathcal{C}^n\times U} \big(\|\mathbf{y}\|^2 + \rho(u',u_0)^2\big)\bar{\nu}(\mathrm{d}\mathbf{y},\mathrm{d}u') + \rho(u,u_0)^2\Big).$$

Moreover, the map

$$(\mathbf{x},\bar{\nu},u) \in \mathcal{C}^n \times \mathcal{P}_2(\mathcal{C}^n \times U) \times U \longmapsto (L,g)(t,\mathbf{x},\bar{\nu},u) \in \mathbb{R}\times\mathbb{R},$$

is lower semi–continuous for all $t \in [0,T]$*.*

Theorem 5.3.4. *Let* Assumption 5.2.8 *and* Assumption 5.3.3 *hold true. Let* $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$*, and* τ *be a* $\mathbb{G}^{t,\circ}$*–stopping time on* $(\Omega^t, \mathcal{F}^t, \mathbb{P}_\nu^t)$ *taking values in* $[t,T]$*. Then*

$$V_S(t,\nu) = V_W(t,\nu),$$

so that the value function $V_S : [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \longrightarrow \mathbb{R} \cup \{-\infty,\infty\}$ *is upper semi–analytic, and one has*

$$V_S(t,\nu) = \sup_{\alpha \in \mathcal{A}_2(t,\nu)} \mathbb{E}^{\mathbb{P}_\nu^t}\Big[\int_t^\tau L(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s)\mathrm{d}s + V_S\big(\tau, \mu^\alpha_\tau\big)\Big]. \tag{5.3.4}$$

Remark 5.3.5. (*i*) *Our results for the dynamic programming for* V_W *and* V_S *in* Theorem 5.3.1 *and* Theorem 5.3.4 *are new in this general framework. For the result in* Theorem 5.3.2*, where the control is adapted to the common noise* B*, the same* DPP *result has been obtained in* Pham and Wei [138, Proposition 3.1]*. However, our result is more general for two reasons. First, we do not require any regularity conditions on the reward functions* L *and* g*, thanks to our use of measurable selection arguments. Second, we are able to stay in a generic non–Markovian framework with interaction terms given by the law of both control and controlled processes, while the results of* [138] *are given in a Markovian context with interaction terms given by the law of controlled process.*

(*ii*) *From our point of view, the formulations* V_W *and* V_S *in* (5.2.5) *and* (5.2.9) *seem to be more natural, because they should be the ones arising naturally as limit of finite population control problems (see* Lacker [104] *for the case without common noise, and* Chapter 3 *and* Chapter 4 *for the context with common noise and law of control). Indeed, for the problem with a finite population* N*, when the controller observes the evolution of the empirical distribution of* $(X^1,\dots,X^N)$*, it is more reasonable to assume that he/she uses the information generated by both* $(X_{t\wedge\cdot}, W, B)$ *(as in the definition of* V_S*), rather than just the information from* B *(as in the definition* $V_S^{\mathbb{B}}$*), to control the system. In this sense, the formulation* $V_S^{\mathbb{B}}$ *may not be the most natural strong formulation for McKean–Vlasov control problems with common noise.*

Remark 5.3.6. *The* DPP *result for* V_S *in* Theorem 5.3.4 *has been proved under additional regularity conditions, namely the ones given in* Assumption 5.3.3*. This should appear as a surprise to readers familiar with the measurable selection approach to the* DPP *for classical stochastic control problems. We will try here to give some intuition on why, at least if one uses our method of proof, there does not seem to be any way to make do without these aditional assumptions.*

Let us consider the classical conditioning argument in the proof of the DPP*. Given a control process* $\alpha := (\alpha_s)_{s\in[t,T]} \in \mathcal{A}_2(t,\nu)$*, which is adapted to the filtration generated by* $(X_{t\wedge\cdot}, W_s^t, B_s^t)_{s\in[t,T]}$*, we consider some time* $t_o \in (t,T]$*, and the filtration* $\widetilde{\mathbb{G}} := (\widetilde{\mathcal{G}}_s)_{s\in[t,T]}$*, generated by* B^t*. Then, under the* r.c.p.d. *of* $\mathbb{P}_\nu^t$ *knowing* $\widetilde{\mathcal{G}}_{t_o}$*, the process* $(\alpha_s)_{s\in[t_o,T]}$ *will be*

adapted to the filtration generated by $(X_{t_o\wedge\cdot}, W^{t_o}_s, B^{t_o}_s)_{s\in[t_o,T]}$ *together with* $(W^t_s)_{s\in[t,t_o]}$. *Because of the randomness of* $(W^t_s)_{s\in[t,t_o]}$, *we cannot consider* $(\alpha_s)_{s\in[t_o,T]}$ *as a 'strong' control process under the* r.c.p.d. *of* $\mathbb{P}^t_\nu$ *knowing* $\widetilde{\mathcal{G}}_{t_o}$.

To bypass this difficulty, we will need to use the equivalence result $V_S = V_W$ *together with the* DPP *results for* V_W *given by* Theorem 5.3.1. *The equivalence result will be proved in* Chapter 2 *under the integrability and regularity conditions in* Assumption 5.2.8 *and* Assumption 5.3.3.

5.3.2 Dynamic programming principle in the Markovian case

With the DPP results in the general non–Markovian context of Theorem 5.3.1, Theorem 5.3.2 and Theorem 5.3.4, we can easily establish the DPP results for the control problems in the Markovian setting. In fact, we will consider a framework which is slightly more general than the classical Markovian formulation, by considering the so–called updating functions, as in Brunick and Shreve [33].

Let E be a non–empty Polish space. A Borel measurable function $\Phi : \mathcal{C}^n \longrightarrow C([0,T],E)$ is called an updating function if it satisfies

$$\Phi_t(\mathbf{x}) = \Phi_t(\mathbf{x}(t\wedge\cdot)),\ \text{for all } (t,\mathbf{x})\in[0,T]\times\mathcal{C}^n,$$

and for all $0\le s\le t\le T$

$$\big(\Phi_r(\mathbf{x})\big)_{r\in[s,t]} = \big(\Phi_r(\mathbf{x}')\big)_{r\in[s,t]},\ \text{whenever } \Phi_s(\mathbf{x}) = \Phi_s(\mathbf{x}'),\ \text{and } \big(\mathbf{x}(r)-\mathbf{x}(s)\big)_{r\in[s,t]} = \big(\mathbf{x}'(r)-\mathbf{x}'(s)\big)_{r\in[s,t]}.$$

The intuition of the updating function Φ is the following: the value of $\Phi_t(\mathbf{x})$ depends only on the path of $\mathbf{x}$ up to time t, and for $0\le s<t$, $\Phi_t(\mathbf{x})$ depends only on $\Phi_s(\mathbf{x})$ and the increments of $\mathbf{x}$ between s and t. On the canonical space $\mathcal{C}^n$, let $X := (X_t)_{t\in[0,T]}$ be the canonical process. We also define a new process $Z_t := \Phi_t(X)$, $t\in[0,T]$. Let us borrow some examples of updating functions from [33].

Example 5.3.7. (*i*) *The most simple updating function is the running process itself, that is*

$$\Phi_t(\mathbf{x}) := \mathbf{x}(t),\ \text{with } E=\mathbb{R}^n.$$

(*ii*) *Let* $M^i_t(\mathbf{x}) := \max_{0\le s\le t}\mathbf{x}^i(s)$ *for* $i=1,\cdots,n$, $t\in[0,T]$, *and* $A_t(\mathbf{x}) := \int_0^t \mathbf{x}(s)\mathrm{d}s$, $t\in[0,T]$. *Then the running process, together with the running maximum and running average process, is also an example of updating functions*

$$\Phi_t(\mathbf{x}) := \big(\mathbf{x}(t), M_t(\mathbf{x}), A_t(\mathbf{x})\big),\ \text{with } E=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n.$$

Throughout this subsection, we fix an update function Φ. In this context, one can in fact define the value function on $[0,T]\times\mathcal{P}(E)$ under some additional conditions. Given $\bar\nu\in\mathcal{P}(\mathcal{C}^n\times U)$ (resp. $\nu\in\mathcal{P}(\mathcal{C}^n)$), let us consider X (resp. (X,α)) as canonical element on the canonical space $\mathcal{C}^n$ (resp. $\mathcal{C}^n\times U$), and then define

$$[\bar\nu]^\circ_t := \bar\nu\circ(\Phi_t(X),\alpha)^{-1}\in\mathcal{P}(E\times U)\ \big(\text{resp. } [\nu]^\circ_t := \nu\circ(\Phi_t(X))^{-1}\in\mathcal{P}(E)\big),\ t\in[0,T].$$

Assumption 5.3.8. *For a fixed updating function* $\Phi : \mathcal{C}^n\longrightarrow C([0,T],E)$, *there exist Borel measurable functions* $(b^\circ,\sigma^\circ,\sigma^\circ_0,L^\circ,g^\circ) : [0,T]\times E\times U\times\mathcal{P}(E\times U)\longrightarrow\mathbb{R}^n\times\mathbb{S}^{n\times d}\times\mathbb{S}^{n\times\ell}\times\mathbb{R}\times\mathbb{R}$, *such that*

$$\big(b,\sigma,\sigma_0,L,g\big)(t,\mathbf{x},\bar\nu,u) = \big(b^\circ,\sigma^\circ,\sigma^\circ_0,L^\circ,g^\circ\big)(t,\Phi_t(\mathbf{x}),[\bar\nu]^\circ_t,u),\ \text{for all } (t,\mathbf{x},u,\bar\nu)\in[0,T]\times\mathcal{C}^n\times U\times\mathcal{P}(\mathcal{C}^n\times U).$$

Let $t\in[0,T]$, and $\nu^\circ\in\mathcal{P}(E)$, we define first the following sets

$$\mathcal{V}(t,\nu^\circ) := \big\{\nu\in\mathcal{P}(\mathcal{C}^n) : [\nu]^\circ_t = \nu^\circ\big\},$$

$$\Gamma^\circ_W(t,\nu^\circ) := \bigcup_{\nu\in\mathcal{V}(t,\nu^\circ)}\Gamma_W(t,\nu),\ \Gamma^\circ_S(t,\nu^\circ) := \bigcup_{\nu\in\mathcal{V}(t,\nu^\circ)}\Gamma_S(t,\nu),\ \Gamma^{\mathbb{B},\circ}_S(t,\nu^\circ) := \bigcup_{\nu\in\mathcal{V}(t,\nu^\circ)}\Gamma^{\mathbb{B}}_S(t,\nu),$$

as well as the value functions, with $J(t,\gamma)$ defined in (5.2.5),

$$V^\circ_W(t,\nu^\circ) := \sup_{\gamma\in\Gamma^\circ_W(t,\nu^\circ)}J(t,\gamma),\ V^\circ_S(t,\nu^\circ) := \sup_{\gamma\in\Gamma^\circ_S(t,\nu^\circ)}J(t,\gamma)\ \text{and } V^{\mathbb{B},\circ}_S(t,\nu^\circ) := \sup_{\gamma\in\Gamma^{\mathbb{B},\circ}_S(t,\nu^\circ)}J(t,\gamma).$$

Remark 5.3.9. *When the updating function is the running process given by $\Phi_t(\mathbf{x}) := \mathbf{x}(t)$, the problems V_W°, V_S° and $V_S^{\mathbb{B},\circ}$ are of course exactly the classical Markovian formulation of the control problems.*

Lemma 5.3.10. *Let* Assumption 5.3.8 *hold true, and fix some $t \in [0,T]$. Then, for any $(\nu_1,\nu_2) \in \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\mathcal{C}^n)$ such that $[\nu_1]_t^\circ = [\nu_2]_t^\circ$, one has*

$$V_W(t,\nu_1) = V_W(t,\nu_2),\ V_S(t,\nu_1) = V_S(t,\nu_2) \text{ and } V_S^{\mathbb{B}}(t,\nu_1) = V_S^{\mathbb{B}}(t,\nu_2).$$

Consequently, for all $\nu \in \mathcal{P}(\mathcal{C}^n)$, one has

$$V_W(t,\nu) = V_W^\circ(t,[\nu]_t^\circ),\ V_S(t,\nu) = V_S^\circ(t,[\nu]_t^\circ), \text{ and } V_S^{\mathbb{B}}(t,\nu) = V_S^{\mathbb{B},\circ}(t,[\nu]_t^\circ).$$

Proof. We will only consider the equality for V_W, the arguments for V_S and $V_S^{\mathbb{B}}$ will be the same. First, we can consider ν_2 as a probability measure defined on the canonical space $\mathcal{C}^n$ with canonical process X, and containing the random variable $Z_t := \Phi_t(X)$. Then, on (a possible enlarged) probability space $(\mathcal{C}^n, \mathcal{B}(\mathcal{C}^n), \nu_2)$, there exists a Borel measurable function $\psi : E \times [0,1] \longrightarrow \mathcal{C}^n$, together with a random variable η with uniform distribution on $[0,1]$, which is independent of Z_t, such that $\nu_2 \circ \big(Z_t, X_\cdot\big)^{-1} = \nu_2 \circ \big(Z_t, \psi(Z_t,\eta)\big)^{-1}$. Next, consider an arbitrary $\gamma_1 := \big(\Omega^1, \mathcal{F}^1, \mathbb{P}^1, \mathbb{F}^1, \mathbb{G}^1, X^1, W^1, B^1, \overline{\mu}^1, \mu^1, \alpha^1\big) \in \Gamma_W(t,\nu_1)$. Without loss of generality (that is up to enlargement of the space), we assume that there exists a random variable η with uniform distribution on $[0,1]$ in the probability space $(\Omega^1, \mathcal{F}_0^{\mathbf{1}}, \mathbb{P}^1)$, and which is independent of the random variables $(X^1, W^1, B^1, \overline{\mu}^1, \mu^1, \alpha^1)$.

We then define γ^2 as follows. Let $Z_s^{\mathbf{1}} := \Phi_s(X^1)$, for all $s \in [0,T]$, so that, by definition, $\mathbb{P}^1 \circ \big(Z_t^{\mathbf{1}}\big)^{-1} = [\nu_1]_t^\circ = [\nu_2]_t^\circ$. Next, let

$$X_s^2 := \begin{cases} \psi_s\big(Z_t^{\mathbf{1}}, \eta\big), \text{ if } s \in [0,t], \\ X_t^2 + X_s^{\mathbf{1}} - X_t^{\mathbf{1}}, \text{ if } s \in (t,T]. \end{cases}$$

It follows by the properties of ψ and those of the updating function Φ that

$$\mathbb{P}^1 \circ \big(X^2_{t\wedge\cdot}\big) = \nu_2(t), \text{ and } \Phi_s(X^2) = \Phi_s(X^1),\ s \in [t,T]. \tag{5.3.5}$$

Let $\overline{\mu}_s^2 := \mathcal{L}^{\mathbb{P}^1}\big((X^2_{s\wedge\cdot}, \alpha_s^{\mathbf{1}}) | \mathcal{G}_s^{\mathbf{1}}\big)$, $\mu_s^2 := \mathcal{L}^{\mathbb{P}^1}\big(X^2_{s\wedge\cdot} | \mathcal{G}_s^{\mathbf{1}}\big)$, for $s \in [t,T]$, and $\gamma_2 := \big(\Omega^1, \mathcal{F}^1, \mathbb{F}^1, \mathbb{P}^1, \mathbb{G}^1, X^2, W^1, B^1, \overline{\mu}^2, \mu^2, \alpha^1\big)$. Using Assumption 5.3.8 and (5.3.5), we have $\gamma_2 \in \Gamma_W(t,\nu_2)$ and $J(t,\gamma_2) = J(t,\gamma_1)$, implying $V_W(t,\nu_1) = V_W(t,\nu_2)$. □

Now we provide the dynamic programming principle for the Markovian control problem under Assumption 5.3.8.

Corollary 5.3.11. *Let* Assumption 5.3.8 *hold true, $t \in [0,T]$ and $\nu^\circ \in \mathcal{P}(E)$. Let $\tau^\star$ be a $\mathbb{G}^\star$–stopping time taking values in $[t,T]$ on Ω° and $(\tau^\gamma)_{\gamma \in \Gamma_W^\circ(t,\nu^\circ)}$ be defined from $\tau^\star$ as in* (5.3.1)*, and τ be a $\mathbb{G}^{t,\circ}$–stopping time taking values in $[t,T]$ on Ω^t. Then one has the following dynamic programming results.*

(i) The function $V_W^\circ : [0,T] \times \mathcal{P}(E) \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is upper semi–analytic and, with $Z_s^\gamma := \Phi_s(X_\cdot^\gamma)$,

$$V_W^\circ(t,\nu^\circ) = \sup_{\gamma \in \Gamma_W^\circ(t,\nu^\circ)} \mathbb{E}^{\mathbb{P}^\gamma}\left[\int_t^{\tau^\gamma} L^\circ\big(s, Z_s^\gamma, [\overline{\mu}^\gamma]_s^\circ, \alpha_s^\gamma\big)\mathrm{d}s + V_W^\circ\big(\tau^\gamma, [\mu^\gamma]_{\tau^\gamma}^\circ\big)\right]. \tag{5.3.6}$$

(ii) Let Assumption 5.2.8 *hold true, then $V_S^{\mathbb{B},\circ} : [0,T] \times \mathcal{P}(E) \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is upper semi–analytic, and with $Z_s^\alpha := \Phi_s(X_\cdot^\alpha)$, one has*

$$V_S^{\mathbb{B},\circ}(t,\nu^\circ) = \sup_{\alpha \in \mathcal{A}_2^{\mathbb{B}}(t,\nu^\circ)} \mathbb{E}^{\mathbb{P}_\nu^t}\left[\int_t^{\tau} L\big(s, Z_s^\alpha, [\overline{\mu}^\alpha]_s^\circ, \alpha_s\big)\mathrm{d}s + V_S^{\mathbb{B},\circ}\big(\tau, [\mu^\alpha]_\tau^\circ\big)\right]. \tag{5.3.7}$$

(iii) Let Assumption 5.2.8 *and* Assumption 5.3.3 *hold, then $V_S^\circ(t,\nu^\circ) = V_W^\circ(t,\nu^\circ)$, and with $Z_s^\alpha := \Phi_s(X_\cdot^\alpha)$,*

$$V_S^\circ(t,\nu^\circ) = \sup_{\alpha \in \mathcal{A}_2(t,\nu^\circ)} \mathbb{E}\left[\int_t^{\tau} L\big(s, Z_s^\alpha, [\overline{\mu}^\alpha]_s^\circ, \alpha_s\big)\mathrm{d}s + V_S^\circ\big(\tau, [\mu^\alpha]_\tau^\circ\big)\right]. \tag{5.3.8}$$

Proof. We will only consider the case V_W, the arguments for V_S and $V_S^{\mathbb{B}}$ will be the same.

Let $[\![\mathcal{V}]\!] := \{(t,\nu,\nu^\circ) \in [0,T]\times\mathcal{P}(\mathcal{C}^n)\times\mathcal{P}(E) : [\nu]_t^\circ = \nu^\circ\}$. Notice that $\Phi : \mathcal{C}^n \longrightarrow C([0,T],E)$ is Borel, then $(t,\nu) \longmapsto [\nu]_t^\circ$ is also Borel, and hence $[\![\mathcal{V}]\!]$ is a Borel subset of $[0,T]\times\mathcal{P}(\mathcal{C}^n)\times\mathcal{P}(E)$. Further, one has $V_W^\circ(t,\nu^\circ) = \sup_{(t,\nu,\nu^\circ)\in[\![\mathcal{V}]\!]} V_W(t,\nu)$ from Lemma 5.3.10, and V_W is upper semi–analytic by Theorem 5.3.1. It follows by the measurable selection theorem (e.g. [61, Proposition 2.17]) that $V_W^\circ : (t,\nu^\circ) \in [0,T]\times\mathcal{P}(E) \longrightarrow V_W^\circ(t,\nu^\circ) \in \mathbb{R}\cup\{-\infty,\infty\}$ is also upper semi–analytic. Finally, using the DPP results in Theorem 5.3.1, it follows that

$$\begin{aligned} V_W^\circ(t,\nu^\circ) = \sup_{\nu\in\mathcal{V}(t,\nu^\circ)} V_W(t,\nu) &= \sup_{\nu\in\mathcal{V}(t,\nu^\circ)} \sup_{\gamma\in\Gamma_W(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + V_W\big(\tau^\gamma, \mu^\gamma_{\tau^\gamma}\big)\Big] \\ &= \sup_{\nu\in\mathcal{V}(t,\nu^\circ)} \sup_{\gamma\in\Gamma_W(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L^\circ\big(s, Z^\gamma_s, [\overline{\mu}^\gamma]^\circ_s, \alpha^\gamma_s\big)\mathrm{d}s + V_W^\circ\big(\tau^\gamma, [\mu^\gamma]^\circ_{\tau^\gamma}\big)\Big] \\ &= \sup_{\gamma\in\Gamma_W^\circ(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L^\circ\big(s, Z^\gamma_s, [\overline{\mu}^\gamma]^\circ_s, \alpha^\gamma_s\big)\mathrm{d}s + V_W^\circ\big(\tau^\gamma, [\mu^\gamma]^\circ_{\tau^\gamma}\big)\Big]. \end{aligned}$$

□

5.3.3 Discussion: from dynamic programming to the HJB equation

One of the classical applications of the DPP consists in giving some local characterisation of the value function, such as in proving that it is the viscosity solution of an HJB equation. This was achieved in Pham and Wei [139] for the control problem V_S° in the setting with $\sigma_0 \equiv 0$), and in Pham and Wei [138] for the control problem $V_S^{\circ,\mathbb{B}}$ (for $\Phi_t(\mathbf{x}) := \mathbf{x}(t)$, with $E = \mathbb{R}^n$). It relies essentially on the notion of differentiability with respect to probability measures due to Lions (see e.g. [118] and Cardaliaguet's notes [36, Section 6]), and Itō's formula along a measure (see e.g. Carmona and Delarue [41, Proposition 6.5 and Proposition 6.3]). We will now provide some heuristic arguments to derive the HJB equation from our DPP results for both $V_S^{\mathbb{B},\circ}$ and V_S°, with updating function $\Phi_t(\mathbf{x}) = \mathbf{x}(t)$.

Let us first recall briefly the notion of the derivative, in sense of Fréchet, $\partial_\nu V(\nu)$ for a function $V : \mathcal{P}_2(\mathbb{R}^n) \longrightarrow \mathbb{R}$. Consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ rich enough so that, for any $\nu \in \mathcal{P}_2(\mathbb{R}^n)$, there exists a random variable $Z : \Omega \longrightarrow \mathbb{R}^n$ such that $\mathcal{L}^{\mathbb{P}}(Z) = \nu$. We denote by $\mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ the space of square–integrable random variables on $(\Omega,\mathcal{F},\mathbb{P})$. Let $V : \mathcal{P}_2(\mathbb{R}^n) \longrightarrow \mathbb{R}$, we consider $\widetilde{V} : \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P}) \longrightarrow \mathbb{R}^n$, the lifted version of V, defined by $\widetilde{V}(X) := V(\mathcal{L}^{\mathbb{P}}(X))$. Recall that $\widetilde{V}$ is said to be continuously Fréchet differentiable, if there exists a unique continuous application $D\widetilde{V} : \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P}) \longrightarrow \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$, such that, for all $Z \in \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$,

$$\lim_{\|Y\|_2\to 0} \frac{\big|\widetilde{V}(Z+Y) - \widetilde{V}(Z) - \mathbb{E}\big[Y^\top D\widetilde{V}(Z)\big]\big|}{\|Y\|_2},$$

where $\|Y\|_2 := \mathbb{E}[|Y|^2]^{1/2}$ for any $Y \in \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$. We say that V is of class C^1 if $\widetilde{V}$ is continuously Fréchet differentiable, and denote for any $\nu \in \mathcal{P}_2(\mathbb{R}^n)$, $\partial_\nu V(\nu)(Z) := D\widetilde{V}(Z)$, $\mathbb{P}$–a.s., for any $Z \in \mathcal{L}^2(\Omega,\mathcal{F},\mathbb{P})$ such that $\mathcal{L}^{\mathbb{P}}(Z) = \nu$. Notice that one has $\partial_\nu V(\nu) : \mathbb{R}^n \ni y \longmapsto \partial_\nu V(\nu)(y) \in \mathbb{R}^n$ and this function belongs to $\mathcal{L}^2(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n),\nu)$. Besides, the law of $D\widetilde{V}(Z)$ is independent of the choice of Z. Similarly, we also define the derivatives $\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^n \ni (\nu,y) \longmapsto \partial_y\partial_\nu V(\nu)(y) \in \mathbb{S}^n$ and

$$\mathcal{P}_2(\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n \ni (\nu,y,y') \longmapsto \partial^2_\nu V(\nu)(y,y') := \partial_\nu\big[\partial_\nu V(\nu)(y)\big](y') \in \mathbb{S}^n.$$

In the following, we say V is a "smooth function", if all the above Fréchet derivatives are well defined and are continuous.

5.3.3.1 HJB equation for the common noise strong formulation

Let us consider the control problem $V_S^{\mathbb{B},\circ}$ and repeat the arguments in [138] in a heuristic way. Given a "smooth function" $V : [0,T] \times \mathcal{P}_2(\mathbb{R}^n) \longrightarrow \mathbb{R}$, $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^n)$, and $\gamma \in \Gamma_S^{\mathbb{B}}(t,\nu)$, it follows from Itô's formula that, for $s \in [t,T]$,

$$\begin{aligned} V\big(s,\mu_s^\gamma\big) = V\big(t,\nu\big) &+ \int_t^s \int_{\mathbb{R}^n} \Big(\partial_t V\big(r,\mu_r^\gamma\big) + \partial_\nu V\big(r,\mu_r^\gamma\big)(y) \cdot b\big(r,y,\mu_r^\gamma \otimes \delta_{\alpha_r^\gamma},\alpha_r^\gamma\big)\Big)\mu_r^\gamma(\mathrm{d}y)\mathrm{d}r \\ &+ \frac{1}{2}\int_t^s \int_{\mathbb{R}^n} \mathrm{Tr}\big[\partial_x\partial_\nu V\big(r,\mu_r^\gamma\big)(y)\big(\sigma^\top\sigma + \sigma_0^\top\sigma_0\big)\big(r,y,\mu_r^\gamma \otimes \delta_{\alpha_r^\gamma},\alpha_r^\gamma\big)\big]\mu_r^\gamma(\mathrm{d}y)\mathrm{d}r \\ &+ \frac{1}{2}\int_t^s \int_{(\mathbb{R}^n)^2} \mathrm{Tr}\big[\partial_\nu^2 V\big(r,\mu_r^\gamma\big)(y,y')\sigma_0^\top\big(r,y,\mu_r^\gamma \otimes \delta_{\alpha_r^\gamma},\alpha_r^\gamma\big)\sigma_0\big(r,y',\mu_r^\gamma \otimes \delta_{\alpha_r^\gamma},\alpha_r^\gamma\big)\big]\mu_r^\gamma(\mathrm{d}y)\mu_r^\gamma(\mathrm{d}y')\mathrm{d}r \\ &+ \int_t^s \int_{\mathbb{R}^n \times U} \partial_\nu V\big(r,\mu_r^\gamma\big)(x) \cdot \sigma_0\big(r,y,\mu_r^\gamma \otimes \delta_{\alpha_r^\gamma},\alpha_r^\gamma\big)\mu_r^\gamma(\mathrm{d}y)\mathrm{d}B_r. \end{aligned} \tag{5.3.9}$$

As $\gamma \in \Gamma_S^{\circ,\mathbb{B}}(t,\nu)$, for Lebesgue–almost every $r \in [t,T]$, α_r^γ is a measurable function of $(B_u - B_t)_{u\in[t,r]}$. Considering piecewise constant control process, α^γ would be a deterministic constant on a small time horizon $[t,t+\varepsilon]$. By replacing V in (5.3.9) by $V_S^{\mathbb{B},\circ}$ and taking supremum as in DDP (5.3.3) (but over constant control processes), this leads to the Hamiltonian

$$H^{\mathbb{B}}[V]\big(t,\nu\big) := \sup_{u\in U}\Big\{\int_{\mathbb{R}^n}\Big(\big(L + [V]^1\big)\big(t,y,\nu\otimes\delta_u,u\big)\Big)\nu(\mathrm{d}y) + \int_{(\mathbb{R}^n)^2}[V]^2\big(t,y,u,y',\nu\otimes\delta_u,u\big)\nu(\mathrm{d}y)\nu(\mathrm{d}y')\Big\},$$

where for any $(r,y,u,y',u',\bar\nu) \in [0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times U\times\mathcal{P}(\mathbb{R}^n\times U)$

$$[V]^1(r,y,\bar\nu,u) := \partial_\nu V(r,\nu)(y)\cdot b(r,y,\bar\nu,u) + \frac{1}{2}\mathrm{Tr}\big[\partial_x\partial_\nu V(r,\nu)(y)(\sigma^\top\sigma+\sigma_0^\top\sigma_0)(r,y,\bar\nu,u)\big],$$

and

$$[V]^2(r,y,u,y',u',\bar\nu) := \frac{1}{2}\mathrm{Tr}\big[\partial_\nu^2 V\big(r,\nu\big)(y,y')\sigma_0^\top(r,y,\bar\nu,u)\sigma_0(r,y',\bar\nu,u')\big].$$

Heuristically, $V^{\mathbb{B},\circ}$ should satisfy the HJB equation

$$-\partial_t V_S^{\mathbb{B},\circ}(t,\nu) - H^{\mathbb{B}}\big[V_S^{\mathbb{B},\circ}\big]\big(t,\nu\big) = 0,\ (t,\nu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^n),\ V_S^{\mathbb{B},\circ}(T,\cdot) = g(\cdot).$$

We refer to [138] for a detailed rigorous proof of the fact that $V^{\mathbb{B},\circ}$ is a viscosity solution of the above HJB equation under some technical regularity conditions.

5.3.3.2 HJB equation for the general strong formulation

Similarly, for the control problem V_S°, we consider a strong control rule $\gamma \in \Gamma_S^\circ(t,\nu)$, where for Lebesgue–almost every $r \in [t,T]$, the control process α_r^γ is a measurable function of both $(B_u - B_t)_{u\in[t,r]}$, $(W_u - W_t)_{u\in[t,r]}$ and $X_{t\wedge\cdot}$. As the control process α^γ is adapted to the filtration generated by $(X_{t\wedge\cdot}, W^{\gamma,t}, B^{\gamma,t})$, by considering adapted piecewise constant control processes, the control process on the first small interval $[t,t+\varepsilon)$ should be a measurable function of $X_{t\wedge\cdot}$. Similarly to Pham and Wei [139] in a non–common noise setting and by considering the Itô formula (5.3.9), this would formally lead to the Hamiltonian

$$H[V]\big(t,\nu\big) := \sup_{a\in\mathbb{L}^2_\nu}\Big\{\int_{\mathbb{R}^n}\big(L+[V]^1\big)\big(t,y,\nu\circ(\hat a)^{-1},a(y)\big)\nu(\mathrm{d}y) + \int_{(\mathbb{R}^n)^2}[V]^2\big(t,y,a(y),y',a(y'),\nu\circ(\hat a)^{-1}\big)\nu(\mathrm{d}y)\nu(\mathrm{d}y')\Big\},$$

where $\hat a : \mathbb{R}^n \ni x \longmapsto (x,a(x)) \in \mathbb{R}^n\times U$, and $\mathbb{L}^2_\nu$ of all ν–square integrable functions $a : (\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n),\nu) \longrightarrow U$. Heuristically, V_S° should be a solution of the HJB equation

$$-\partial_t V_S^\circ(t,\nu) - H[V_S^\circ]\big(t,\nu\big) = 0,\ (t,\nu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^n),\ V_S^{\mathbb{B},\circ}(T,\cdot) = g(\cdot).$$

As explained above, the difference between the HJB equations for $V_S^{\mathbb{B},\circ}$ and V_S° comes mainly from the fact that the control process α^γ, for $\gamma \in \Gamma_S^\circ(t,\nu)$, depends on the initial random variable condition, which in turn modifies the Hamiltonian function which appears in the PDE. Finally, we also refer to Wu and Zhang [151] for a discussion of the McKean–Vlasov control problem in a non–Markovian framework without common noise.

5.4 Proofs of dynamic programming principle

We now provide the proofs of our main DPP results in Theorems 5.3.1, 5.3.2 and 5.3.4, where a key ingredient is the measurable selection argument. We will first reformulate the control problems on an appropriate canonical space in Section 5.4.1, and then provide some technical lemmata for the problems formulated on the canonical space in Section 5.4.2, and finally give the proofs of the main results themselves in Section 5.4.3.

5.4.1 Reformulation of the control problems on the canonical space

5.4.1.1 Canonical space

In order to prove the dynamic programming results in Section 5.3, we first reformulate the controlled McKean–Vlasov SDE problems on an appropriate canonical space. This is going to be achieved by the usual way, that is to say by considering appropriately defined controlled martingale problems. Recall that n, d and ℓ are the dimensions of the spaces in which X, W and B take values, $\overline{U} = U \cup \{\partial\}$ and that π^{-1} maps $\mathbb{R} \cup \{\infty, -\infty\}$ to $\overline{U}$. Let us introduce a first canonical space by

$$\widehat{\Omega} := \mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C}^\ell \text{ with canonical process } (\widehat{X}, \widehat{A}, \widehat{W}, \widehat{B}), \text{ and } \widehat{\alpha}_t := \pi^{-1}\Big(\overline{\lim_{n\to+\infty}} n\big(\widehat{A}_t - \widehat{A}_{0\vee(t-1/n)}\big)\Big),\ t \in [0,T].$$

Denote by $C([0,T], \mathcal{P}(\widehat{\Omega}))$ be the space of all continuous paths on $[0,T]$ taking values in $\mathcal{P}(\widehat{\Omega})$, which is a Polish space for the uniform topology (see e.g. [10, Lemmata 3.97, 3.98, and 3.99]), we introduce a second canonical space by

$$\overline{\Omega} := \widehat{\Omega} \times C\big([0,T], \mathcal{P}(\widehat{\Omega})\big), \text{ with canonical process } (X, A, W, B, \widehat{\mu}) \text{ and canonical filtration } \overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\le t\le T}.$$

Let $\overline{\mathcal{F}} := \mathcal{B}(\overline{\Omega})$ be the Borel σ–field on $\overline{\Omega}$. Notice that for any $t \in [0,T]$, $\widehat{\mu}_t$ is a probability measure on $\widehat{\Omega}$, we then define two processes $\mu = (\mu_t)_{0\le t\le T}$ and $\overline{\mu} = (\overline{\mu}_t)_{0\le t\le T}$ on $\overline{\Omega}$ by

$$\overline{\mu}_t := \widehat{\mu}_t \circ \big(\widehat{X}_{t\wedge\cdot}, \widehat{\alpha}_t\big)^{-1},\ \mu_t := (\widehat{\mu}_t) \circ \big(\widehat{X}_{t\wedge\cdot}\big)^{-1}.$$

Define also the $\overline{U}$–valued process $\overline{\alpha} = (\overline{\alpha}_t)_{0\le t\le T}$ on $\overline{\Omega}$ by

$$\overline{\alpha}_t := \pi^{-1}\Big(\overline{\lim_{n\to+\infty}} n\big(A_t - A_{0\vee(t-1/n)}\big)\Big),\ t \in [0,T].$$

Finally, for any $t \in [0,T]$, we introduce the processes $W^t := (W^t_s)_{s\in[0,T]}$ and $B^t := (B^t_s)_{s\in[0,T]}$ by

$$B^t_s := B_{s\vee t} - B_t, \text{ and } W^t_s := W_{s\vee t} - W_t,\ s \in [0,T],$$

and the filtration $\overline{\mathbb{G}}^t := (\overline{\mathcal{G}}^t_s)_{0\le s\le T}$ by

$$\overline{\mathcal{G}}^t_s := \begin{cases} \{\emptyset, \Omega\}, \text{ if } s \in [0,t), \\ \sigma\big((B^t_r, \widehat{\mu}_r) : r \in [0,s]\big), \text{ if } s \in [t,T]. \end{cases} \tag{5.4.1}$$

5.4.1.2 Controlled martingale problems on the canonical space

We now reformulate the strong/weak control problem as a controlled martingale problem on the canonical space $\overline{\Omega}$, where a control (term) can be considered as a probability measure on $\overline{\Omega}$. To this end, let us first introduce the corresponding generator. Given the coefficient functions b, σ and σ_0, for all $(t, \mathbf{x}, \mathbf{w}, \mathbf{b}, \bar{\nu}, u) \in [0,T] \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \times \mathcal{P}(\mathcal{C}^n \times U) \times U$, let

$$\bar{b}\big(t, (\mathbf{x}, \mathbf{w}, \mathbf{b}), \bar{\nu}, u\big) := \big(b, 0_d, 0_\ell\big)(t, \mathbf{x}, \bar{\nu}, u), \tag{5.4.2}$$

and

$$\bar{a}\big(t, (\mathbf{x}, \mathbf{w}, \mathbf{b}), \bar{\nu}, u\big) := \begin{pmatrix} \sigma & \sigma_0 \\ I_{d\times d} & 0_{d\times\ell} \\ 0_{\ell\times d} & I_{\ell\times\ell} \end{pmatrix} \begin{pmatrix} \sigma & \sigma_0 \\ I_{d\times d} & 0_{d\times\ell} \\ 0_{\ell\times d} & I_{\ell\times\ell} \end{pmatrix}^\top (t, \mathbf{x}, \bar{\nu}, u), \tag{5.4.3}$$

and then introduce the generator $\mathcal{L}$, for all $\varphi \in C_b^2(\mathbb{R}^{n+d+\ell})$,

$$\mathcal{L}_t\varphi\big(\mathbf{x},\mathbf{w},\mathbf{b},\bar\nu,u\big) := \sum_{i=1}^{n+d+\ell} \bar b_i(t,(\mathbf{x},\mathbf{w},\mathbf{b}),\bar\nu,u)\partial_i\varphi(\mathbf{x}_t,\mathbf{w}_t,\mathbf{b}_t) + \frac{1}{2}\sum_{i,j=1}^{n+d+\ell} \bar a_{i,j}(t,(\mathbf{x},\mathbf{w},\mathbf{b}),\bar\nu,u)\partial^2_{i,j}\varphi(\mathbf{x}_t,\mathbf{w}_t,\mathbf{b}_t).$$

We next define a process $|\overline{S}| = \big(|\overline{S}|_t\big)_{0\le t\le T}$ by

$$|\overline{S}|_t := \int_0^t \big(|\bar b| + |\bar a|\big)(s,X,W,B,\overline{\mu}_s,\overline{\alpha}_s)\mathrm{d}s,$$

and then, for all $\varphi \in C_b^2(\mathbb{R}^{n+d+\ell})$, let $\overline{S}^\varphi = (\overline{S}^\varphi_t)_{t\in[0,T]}$ be defined by

$$\overline{S}^\varphi_t := \varphi(X_t,W_t,B_t) - \int_0^t \mathcal{L}_s\varphi\big(X,W,B,\overline{\mu}_s,\overline{\alpha}_s\big)\mathrm{d}s,\ t\in[0,T], \tag{5.4.4}$$

where for $\phi : [0,T]\to\mathbb{R}$, $\int_0^t\phi(s)\mathrm{d}s := \int_0^t\phi^+(s)\mathrm{d}s - \int_0^t\phi^-(s)\mathrm{d}s$ with the convention $\infty-\infty=-\infty$. Notice that on $\{|\overline{S}|_T<\infty\}$, the process $\overline{S}^\varphi$ is $\mathbb{R}$–valued. To localise the process $\overline{S}^\varphi$, we also introduce, for each $m\ge1$,

$$\tau_m := \inf\big\{t : |\overline{S}|_t \ge m\big\},\ \text{and } S^{\varphi,m}_t := \overline{S}^\varphi_{t\wedge\tau_m} = \overline{S}^\varphi_t\mathbf{1}_{\{\tau_m\ge t\}} + \overline{S}^\varphi_{\tau_m}\mathbf{1}_{\{\tau_m<t\}},\ t\in[0,T]. \tag{5.4.5}$$

Notice that the process $|\overline{S}|$ is left–continuous, τ_m is a $\mathbb{F}^+$–stopping time on $\overline{\Omega}$, and $S^{\varphi,m}$ is an $\mathbb{F}$–adapted uniformly bounded process.

Definition 5.4.1. *Let $(t,\widehat{\nu})\in[0,T]\times\mathcal{P}(\widehat{\Omega})$. A probability $\overline{\mathbb{P}}$ on $(\overline{\Omega},\overline{\mathcal{F}})$ is called a weak control rule with initial condition $(t,\widehat{\nu})$ if*

(i) the process $\overline{\alpha}=(\overline{\alpha}_s)_{t\le s\le T}$ satisfies

$$\overline{\mathbb{P}}\big[\overline{\alpha}_s\in U\big]=1,\ \text{for Lebesgue–a.e. } s\in[t,T],\ \text{and } \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\int_t^T\big(\rho(u_0,\overline{\alpha}_s)\big)^p\mathrm{d}s\bigg]<\infty;$$

(ii) the process $\widehat{\mu}=(\widehat{\mu}_s)_{0\le s\le T}$ satisfies

$$\widehat{\mu}_s = \overline{\mathbb{P}}\circ(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot})^{-1}\mathbf{1}_{\{s\in[0,t]\}} + \overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_T}\circ(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot})^{-1}\mathbf{1}_{\{s\in(t,T]\}},\ \overline{\mathbb{P}}-\text{a.s.} \tag{5.4.6}$$

with $\overline{\mathbb{P}}\circ(X_{t\wedge\cdot},A_{t\wedge\cdot},W_{t\wedge\cdot},B_{t\wedge\cdot})^{-1}=\widehat{\nu}(t)$;

(iii) $\mathbb{E}^{\overline{\mathbb{P}}}[\|X\|^p]<\infty$, $\overline{\mathbb{P}}[|\overline{S}|_T<\infty]=1$, the process $\overline{S}^\varphi$ is an $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–local martingale on $[t,T]$, for all $\varphi\in C_b^2\big(\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^\ell\big)$.

Given $\nu\in\mathcal{P}(\mathcal{C}^n)$, we denote by $\mathcal{V}(\nu)$ the collection of all probability measures $\widehat{\nu}\in\mathcal{P}(\widehat{\Omega})$ such that $\widehat{\nu}\circ\widehat{X}^{-1}=\nu$, and let

$$\widehat{\mathcal{P}}_W(t,\widehat{\nu}) := \big\{\text{All weak control rules } \mathbb{P} \text{ with initial condition } (t,\widehat{\nu})\big\},\ \text{and } \overline{\mathcal{P}}_W(t,\nu) := \bigcup_{\widehat{\nu}\in\mathcal{V}(\nu)}\widehat{\mathcal{P}}_W(t,\widehat{\nu}).$$

Remark 5.4.2. *Let $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\nu)$ for some $t\in[0,T]$ and $\nu\in\mathcal{P}(\mathcal{C}^n)$. Notice that for $s\in(t,T]$, $\widehat{\mu}_s$ is $\overline{\mathcal{G}}^t_s$–measurable, then by (5.4.6), one has*

$$\widehat{\mu}_s = \overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_s}\circ(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot})^{-1},\ \overline{\mathbb{P}}\text{–a.s.}$$

Further, as the canonical process $(\widehat{\mu}_s)_{s\in[0,T]}$ is continuous, it follows that

$$\mathcal{L}^{\overline{\mathbb{P}}}\big(X_{t\wedge\cdot},A_{t\wedge\cdot},W,B_{t\wedge\cdot}\big)=\widehat{\mu}_t=\lim_{s\searrow t}\widehat{\mu}_s=\lim_{s\searrow t}\mathcal{L}^{\overline{\mathbb{P}}}\big((X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot})\big|\overline{\mathcal{G}}^t_T\big)=\mathcal{L}^{\overline{\mathbb{P}}}\big((X_{t\wedge\cdot},A_{t\wedge\cdot},W,B_{t\wedge\cdot})\big|\overline{\mathcal{G}}^t_T\big),\ \overline{\mathbb{P}}\text{–a.s.}$$

This implies that $\mathcal{F}_t \vee \sigma(W) = \sigma(X_{t\wedge\cdot}, A_{t\wedge\cdot}, W, B_{t\wedge\cdot})$ *is independent of* $\overline{\mathcal{G}}^t_T$*, which is consistent with the conditions in* Definition 5.2.1.

Finally, under $\overline{\mathbb{P}}$, $(\widehat{\mu}_s)_{s\in[0,t]}$ *is completely determined by* $\widehat{\nu}(t)$*. More precisely, one has*

$$\widehat{\mu}_t(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}\mathbf{w},\mathrm{d}\mathbf{b}) = \int_{\widehat{\Omega}\times\mathcal{C}^d} \delta_{(\mathbf{x}',a',\mathbf{w}'\oplus_t\mathbf{w}^\star,\mathbf{b}')}(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}\mathbf{w},\mathrm{d}\mathbf{b})\,\widehat{\nu}(t)(\mathrm{d}\mathbf{x}',\mathrm{d}a',\mathrm{d}\mathbf{w}',\mathrm{d}\mathbf{b}')\mathcal{L}^{\overline{\mathbb{P}}}\big(W^t\big)(\mathrm{d}\mathbf{w}^\star),\ \overline{\mathbb{P}}\text{–a.s.}$$

where W^t *is a* $(\overline{\mathbb{F}},\overline{\mathbb{P}})$*–Brownian motion on* $[t,T]$*, by the martingale problem property in* Definition 5.4.1.

Definition 5.4.3. *Let* $(t,\nu)\in[0,T]\times\mathcal{P}_2(\mathcal{C}^n)$*. A probability* $\overline{\mathbb{P}}$ *on* $(\overline{\Omega},\overline{\mathcal{F}})$ *is called a strong control rule (resp.* $\mathbb{B}$*–strong control rule) with initial condition* (t,ν)*, if* $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\nu)$ *and moreover there exists some Borel measurable function* $\phi:[0,T]\times\Omega^t\longrightarrow U$ *(resp.* $\phi:[0,T]\times\mathcal{C}^\ell_{t,T}\longrightarrow U$*) such that*

$$\overline{\alpha}_s = \phi\big(s, X_{t\wedge\cdot}, W^t_{s\wedge\cdot}, B^t_{s\wedge\cdot}\big)\ \big(\text{resp. } \phi\big(s, B^t_{s\wedge\cdot}\big)\big),\ \overline{\mathbb{P}}\text{–a.s., for all } s\in[t,T].$$

Let us then denote by $\overline{\mathcal{P}}_S(t,\nu)$ (resp. $\overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$) the collection of all strong (resp. $\mathbb{B}$–strong) control rules with initial condition (t,ν).

5.4.1.3 Equivalence of the reformulation

We now show that every strong/weak control (term) induces a strong/weak control rule on the canonical space, and any strong/weak control rule on the canonical space can be induced by a strong/weak control (term).

Lemma 5.4.4. *(i) Let* $t\in[0,T]$ *and* $\nu\in\mathcal{P}(\mathcal{C}^n)$*. Then for every* $\gamma\in\Gamma_W(t,\nu)$*, one has*

$$\overline{\mathbb{P}}^\gamma := \mathbb{P}^\gamma\circ\big(X^\gamma, A^\gamma, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1}\in\overline{\mathcal{P}}_W(t,\nu). \tag{5.4.7}$$

Conversely, given $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\nu)$*, there exists some* $\gamma\in\Gamma_W(t,\nu)$ *such that* $\mathbb{P}^\gamma\circ\big(X^\gamma, A^\gamma, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1} = \overline{\mathbb{P}}$*.*

(ii) Let $t\in[0,T]$*,* $\nu\in\mathcal{P}_2(\mathcal{C}^n)$*, and* Assumption 5.2.8 *hold true. Then for every* $\gamma\in\Gamma_S(t,\nu)$ *(resp.* $\Gamma^{\mathbb{B}}_S(t,\nu)$*), one has*

$$\overline{\mathbb{P}}^\gamma := \mathbb{P}^\gamma\circ\big(X^\gamma, A^\gamma, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1}\in\overline{\mathcal{P}}_S(t,\nu)\ \big(\text{resp. } \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)\big). \tag{5.4.8}$$

Conversely, given $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_S(t,\nu)$ $\big(\text{resp. } \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)\big)$*, there exists some* $\gamma\in\Gamma_S(t,\nu)$ $\big(\text{resp. } \Gamma^{\mathbb{B}}_S(t,\nu)\big)$ *such that* $\mathbb{P}^\gamma\circ\big(X^\gamma, A^\gamma, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1} = \overline{\mathbb{P}}$*.*

Proof. *(i)* First, let $\gamma\in\Gamma_W(t,\nu)$ and $\overline{\mathbb{P}}^\gamma := \mathbb{P}^\gamma\circ\big(X^\gamma, A^\gamma, W^\gamma, B^\gamma, \widehat{\mu}^\gamma\big)^{-1}$. First, it is straightforward to check that

$$\overline{\mathbb{P}}^\gamma[\overline{\alpha}_s\in U] = 1,\ \text{for } \mathrm{d}t\text{–a.e. } s\in[t,T],\ \mathbb{E}^{\overline{\mathbb{P}}^\gamma}\big[\|X\|^p\big]<\infty \text{ and } \mathbb{E}^{\overline{\mathbb{P}}^\gamma}\bigg[\int_t^T(\rho(u_0,\overline{\alpha}_s))^p\mathrm{d}s\bigg]<\infty.$$

Further, as the integrals in (5.2.4) are well defined, one has $|\overline{S}|_T<\infty$, $\overline{\mathbb{P}}^\gamma$–a.s. Moreover, by Itô's formula, the process $\big(\overline{S}^\varphi_s\big)_{s\in[t,T]}$ defined in (5.4.4) is an $(\overline{\mathbb{F}},\overline{\mathbb{P}}^\gamma)$–local martingale, for every $\varphi\in C^2_b\big(\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^\ell\big)$.

Next, notice that $B^{t,\gamma}$ and $\widehat{\mu}^\gamma$ are adapted to $\mathbb{G}^\gamma$, one has, for all $(s,\beta,\psi)\in(t,T]\times C_b\big(\widehat{\Omega}\big)\times C_b\big(\mathcal{C}^\ell\times C([0,T];\mathcal{P}(\widehat{\Omega}))\big)$,

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}^\gamma}\big[\langle\beta,\widehat{\mu}_s\rangle\psi\big(B^t_{T\wedge\cdot},\widehat{\mu}_{T\wedge\cdot}\big)\big] &= \mathbb{E}^{\mathbb{P}^\gamma}\big[\langle\beta,\widehat{\mu}^\gamma_s\rangle\psi\big(B^{\gamma,t}_{T\wedge\cdot},\widehat{\mu}^\gamma_{T\wedge\cdot}\big)\big] = \mathbb{E}^{\mathbb{P}^\gamma}\big[\langle\beta,\mathcal{L}^{\mathbb{P}^\gamma}\big(\big(X^\gamma_{s\wedge\cdot},A^\gamma_{s\wedge\cdot},W^\gamma,B^\gamma_{s\wedge\cdot}\big)\big|\mathcal{G}^\gamma_T\big)\rangle\psi\big(B^{\gamma,t}_{T\wedge\cdot},\widehat{\mu}^\gamma_{T\wedge\cdot}\big)\big]\\
&= \mathbb{E}^{\mathbb{P}^\gamma}\big[\beta\big(X^\gamma_{s\wedge\cdot},A^\gamma_{s\wedge\cdot},W^\gamma,B^\gamma_{s\wedge\cdot}\big)\psi\big(B^{\gamma,t}_{T\wedge\cdot},\widehat{\mu}^\gamma_{T\wedge\cdot}\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}^\gamma}\big[\beta\big(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot}\big)\psi\big(B^t_{T\wedge\cdot},\widehat{\mu}_{T\wedge\cdot}\big)\big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^\gamma}\big[\langle\beta,\mathcal{L}^{\overline{\mathbb{P}}^\gamma}\big(\big(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot}\big)\big|\overline{\mathcal{G}}^t_T\big)\rangle\psi\big(B^t_{T\wedge\cdot},\widehat{\mu}_{T\wedge\cdot}\big)\big].
\end{aligned}$$

This implies that $\widehat{\mu}_s = \mathcal{L}^{\overline{\mathbb{P}}^\gamma}\big(\big(X_{s\wedge\cdot},A_{s\wedge\cdot},W^\gamma,B^\gamma_{s\wedge\cdot}\big)\big|\overline{\mathcal{G}}^t_T\big)$, $\overline{\mathbb{P}}^\gamma$–a.s. for all $s\in(t,T]$. By the same argument and using the fact that $\mathcal{F}^\gamma_t\vee\sigma(W^\gamma)$ is independent of $\mathcal{G}^\gamma_T$, one can easily check that $\widehat{\mu}_s = \mathcal{L}^{\overline{\mathbb{P}}^\gamma}(X_{s\wedge\cdot},A_{s\wedge\cdot},W_\cdot,B_{s\wedge\cdot})$ for $s\in[0,t]$, and that $\overline{\mathbb{P}}^\gamma\circ X^{-1}_{t\wedge\cdot} = \nu$. This implies that $\overline{\mathbb{P}}^\gamma\in\overline{\mathcal{P}}_W(t,\nu)$.

Assume in addition that $\gamma \in \Gamma_S(t,\nu)$ so that α^γ is $\mathbb{F}^\gamma$–predictable. Then there exists a Borel measurable function $\phi : [t,T] \times \Omega^t \longrightarrow U$ such that $\alpha^\gamma_s = \phi\big(s, X^\gamma_{t\wedge\cdot}, W^{\gamma,t}_{s\wedge\cdot}, B^{\gamma,t}_{s\wedge\cdot}\big)$, for all $s \in [t,T]$, $\mathbb{P}^\gamma$–a.s. (see e.g. Claisse, Talay, and Tan [52, Proposition 10]). This implies that $\overline{\alpha}_s = \phi\big(s, X_{t\wedge\cdot}, W^t_{s\wedge\cdot}, B^t_{s\wedge\cdot}\big)$, $\overline{\mathbb{P}}^\gamma$–a.s. for all $s \in [t,T]$, and it follows that $\overline{\mathbb{P}}^\gamma \in \overline{\mathcal{P}}_S(t,\nu)$.

(ii) Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)$ for some $\nu \in \mathcal{P}(\mathcal{C}^n)$. By Stroock and Varadhan [150, Theorem 4.5.2], one knows that (W,B) are $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–Brownian motions on $[t,T]$, and

$$X_s = X_t + \int_t^s b(r, X_\cdot, \overline{\mu}_r, \overline{\alpha}_r)\mathrm{d}r + \int_t^s \sigma(r, X_\cdot, \overline{\mu}_r, \overline{\alpha}_r)\mathrm{d}W_r + \int_t^s \sigma_0(r, X_\cdot, \overline{\mu}_r, \overline{\alpha}_r)\mathrm{d}B_r,\ \overline{\mathbb{P}}\text{–a.s.},$$

Moreover, with the filtration $\overline{\mathbb{G}}^t$ defined in (5.4.1), and in view of Remark 5.4.2, it is straightforward to check that

$$\gamma := \big(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{F}}, \overline{\mathbb{P}}, \overline{\mathbb{G}}^t, X, W, B, \overline{\mu}, \mu, \overline{\alpha}\big) \in \Gamma_W(t,\nu).$$

If, in addition, $\mathbb{P} \in \overline{\mathcal{P}}_S(t,\nu)$, so that A is a $\big(\sigma\big(X_{t\wedge r\wedge\cdot}, W^t_{r\wedge\cdot}, B^t_{r\wedge\cdot}\big)\big)_{r\in[t,T]}$–adapted continuous process. Using Corollary 5.5.4 and the fact that $\widehat{\mu}_s = \mathcal{L}^{\mathbb{P}}\big((X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\big|B^t_{r\in[t,s]}, \widehat{\mu}_{s\wedge\cdot}\big)$, $\mathbb{P}$–a.s., for all $s \in [t,T]$, one can deduce that $\widehat{\mu}_s = \mathcal{L}^{\mathbb{P}}\big((X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\big|B^t_{r\in[t,s]}\big)$, $\mathbb{P}$–a.s., for all $s \in [t,T]$. Let $\tilde{\mathbb{G}}^t$ be the filtration generated by B^t, $\tilde{\mathbb{F}}^t$ be the filtration generated by $(X_{t\wedge\cdot}, W^t, B^t)$, and $\tilde{\mathbb{G}}^{t,\overline{\mathbb{P}}}$, $\tilde{\mathbb{F}}^{t,\overline{\mathbb{P}}}$ be the corresponding $\overline{\mathbb{P}}$–augmented filtrations. Then $\widehat{\mu}$ is $\tilde{\mathbb{G}}^{t,\overline{\mathbb{P}}}$–predictable, and X is $\tilde{\mathbb{F}}^{t,\overline{\mathbb{P}}}$–predictable. Then it follows that

$$\gamma' := \big(\overline{\Omega}, \overline{\mathcal{F}}, \tilde{\mathbb{F}}^{t,\overline{\mathbb{P}}}, \overline{\mathbb{P}}, \tilde{\mathbb{G}}^{t,\overline{\mathbb{P}}}, X, W^t, B^t, \overline{\mu}, \mu, \overline{\alpha}\big) \in \Gamma_S(t,\nu).$$

(iii) Finally, the results related to $\overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$ and $\Gamma^{\mathbb{B}}_S(t,\nu)$ can be deduced by almost the same arguments as for $\overline{\mathcal{P}}_S(t,\nu)$ and $\Gamma_S(t,\nu)$. □

Remark 5.4.5. *From* Lemma 5.4.4*, we can easily deduce that under* Assumption 5.2.8*, for* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S(t,\nu)$ *or* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$*, the canonical process* $\widehat{\mu}$ *satisfies*

$$\widehat{\mu}_s = \mathcal{L}^{\overline{\mathbb{P}}}\big((X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\big|B^t_{s\wedge\cdot}\big) = \mathcal{L}^{\overline{\mathbb{P}}}\big((X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\big|B^t\big),\ \overline{\mathbb{P}}\text{–a.s., for all } s \in [0,T].$$

A direct consequence of Lemma 5.4.4 is that we can now reformulate equivalently the weak/strong formulation of the McKean–Vlasov control problem on the canonical space.

Corollary 5.4.6. *Let* $(t,\nu) \in [0,T] \times \mathcal{P}(\mathcal{C}^n)$*, one has*

$$V_W(t,\nu) = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)} J(t,\overline{\mathbb{P}}),\ \textit{where}\ J(t,\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T L\big(s, X, \overline{\mu}_s, \overline{\alpha}_s\big)\mathrm{d}s + g\big(X, \mu_T\big)\Big]. \tag{5.4.9}$$

Moreover, when Assumption 5.2.8 *holds true and* $\nu \in \mathcal{P}_2(\mathcal{C}^n)$*, one has*

$$V_S(t,\nu) = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S(t,\nu)} J(t,\overline{\mathbb{P}}),\ \text{and}\ V^{\mathbb{B}}_S(t,\nu) = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)} J(t,\overline{\mathbb{P}}).$$

5.4.2 Technical lemmata

We provide in this section some technical results related to the sets $\widehat{\mathcal{P}}_W(t,\widehat{\nu})$.

Lemma 5.4.7. *Both graph sets*

$$[\![\widehat{\mathcal{P}}_W]\!] := \big\{(t,\widehat{\nu},\overline{\mathbb{P}}) : \overline{\mathbb{P}} \in \widehat{\mathcal{P}}_W(t,\widehat{\nu})\big\} \text{ and } [\![\overline{\mathcal{P}}_W]\!] := \big\{(t,\nu,\overline{\mathbb{P}}) : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)\big\}$$

are analytic subsets of, respectively, $[0,T] \times \mathcal{P}(\widehat{\Omega}) \times \mathcal{P}(\overline{\Omega})$ *and* $[0,T] \times \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega})$*. Moreover, the value function*

$$V_W : [0,T] \times \mathcal{P}(\mathcal{C}^n) \longrightarrow \mathbb{R} \cup \{-\infty, \infty\},$$

is upper semi–analytic.

Proof. For $0 \le r \le s \le T$, $m \ge 1$, $\chi \in C_b(\overline{\Omega})$, $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell)$, $\phi \in C_b(\widehat{\Omega})$, $\psi \in C_b(\mathcal{C}^\ell \times C([0,T];\mathcal{P}(\widehat{\Omega})))$, we define

$$\xi_{r\wedge\cdot} := \chi\big(X_{r\wedge\cdot}, A_{r\wedge\cdot}, W_{r\wedge\cdot}, B_{r\wedge\cdot}, \widehat{\mu}_{r\wedge\cdot}\big),$$

and the following Borel measurable subsets of $[0,T] \times \mathcal{P}(\widehat{\Omega}) \times \mathcal{P}(\overline{\Omega})$:

$$\begin{aligned}
K^1 &:= \Big\{(t,\widehat{\nu},\overline{\mathbb{P}}) : \int_t^T \overline{\mathbb{P}}[\overline{\alpha}_\theta \in U]\mathrm{d}\theta = T-t,\ \mathbb{E}^{\mathbb{P}}\big[\|X\|^p\big] < \infty,\ \mathbb{E}^{\mathbb{P}}\Big[\int_t^T (\rho(u_0,\overline{\alpha}_\theta))^p \mathrm{d}\theta\Big] < \infty\Big\},\\
K^{2,m}_{r,s}[\chi,\varphi] &:= \big\{(t,\widehat{\nu},\mathbb{P}) : \mathbb{E}^{\mathbb{P}}\big[\overline{S}^{\varphi,m}_r \xi_{r\wedge\cdot}\big] = \mathbb{E}^{\mathbb{P}}\big[\overline{S}^{\varphi,m}_s \xi_{r\wedge\cdot}\big]\big\},\\
K^3_s[\phi] &:= \Big\{(t,\widehat{\nu},\mathbb{P}) : \mathbb{E}^{\mathbb{P}}\big[\big|\langle\phi,\widehat{\mu}_{t\wedge s}\rangle - \mathbb{E}^{\mathbb{P}}\big[\phi(X_{[t\wedge s]\wedge\cdot}, A_{[t\wedge s]\wedge\cdot}, W, B_{[t\wedge s]\wedge\cdot})\big]\big|\big] = \mathbb{E}^{\mathbb{P}}\big[\big|\langle\phi,\widehat{\mu}_t(t)\rangle - \langle\phi,\widehat{\nu}(t)\rangle\big|\big] = 0\Big\},\\
K^4_s[\phi,\psi] &:= \Big\{(t,\widehat{\nu},\mathbb{P}) : \mathbb{E}^{\mathbb{P}}\big[\langle\phi,\widehat{\mu}_{t\vee s}\rangle\psi(B^t,\widehat{\mu})\big] = \mathbb{E}^{\mathbb{P}}\big[\phi(X_{[t\vee s]\wedge\cdot}, A_{[t\vee s]\wedge\cdot}, W, B_{[t\vee s]\wedge\cdot})\psi(B^t,\widehat{\mu})\big]\Big\}.
\end{aligned}$$

The above Borel measurable sets allow to characterise the graph set $[\![\widehat{\mathcal{P}}_W]\!]$. Indeed, K^1 contains the probabilities on $\overline{\Omega}$ such that the canonical element $\overline{\alpha}$ takes its values in U and not in $U \cup \{\partial\}$, $K^{2,m}_{r,s}[\chi,\varphi]$ reduces the set $\mathcal{P}(\overline{\Omega})$ to the set of probabilities on $\overline{\Omega}$ that solves a (local) martingale problem, while the probabilities which satisfy the "fixed point property", i.e. the canonical process $\widehat{\mu}$ is equal to the conditional distribution of canonical process (X,A,W,B), are contained in $K^3_s[\phi]$ and $K^4_s[\phi,\psi]$.

Let us consider a countable dense subset $\mathbb{X}$ of $\big(r,s,m,\chi,\varphi,\phi,\psi\big)$ in

$$[0,T]^2 \times \mathbb{N} \times C_b(\overline{\Omega}) \times C_b^2(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell) \times C_b(\widehat{\Omega}) \times \mathcal{C}_b(\mathcal{C}^\ell \times C([0,T];\mathcal{P}(\widehat{\Omega}))),$$

where $0 \le r \le s \le T$. By the above remarks, it is straightforward to check that

$$[\![\widehat{\mathcal{P}}_W]\!] = \bigcap_{\mathbb{X}} \Big(K^1[h] \cap K^{2,m}_{r,s}[\chi,\varphi] \cap K^3_s[\phi] \cap K^4_s[\phi,\psi] \cap K^5_s[\phi,\psi]\Big),$$

and hence it is a Borel subset of $[0,T] \times \mathcal{P}(\widehat{\Omega}) \times \mathcal{P}(\overline{\Omega})$. Furthermore, since $\mathcal{P}(\widehat{\Omega}) \ni \widehat{\nu} \longmapsto \widehat{\nu} \circ (X)^{-1} \in \mathcal{P}(\mathcal{C}^n)$ is continuous, the set

$$[\![\overline{\mathcal{P}}_W]\!] = \big\{(t,\nu,\overline{\mathbb{P}}) : (t,\widehat{\nu},\overline{\mathbb{P}}) \in [\![\widehat{\mathcal{P}}_W]\!],\ \widehat{\nu} \circ (X)^{-1} = \nu\big\},$$

is an analytic subset of $[0,T] \times \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega})$. Finally, use the (analytic) measurable selection theorem (see e.g. El Karoui and Tan [61, Proposition 2.17]), it follows that

$$V_W(t,\nu) = \sup_{(t,\nu,\overline{\mathbb{P}}) \in [\![\overline{\mathcal{P}}_W]\!]} J(t,\overline{\mathbb{P}}),$$

is upper semi–analytic as desired. □

We next prove a stability result w.r.t. the "conditioning" of $\widehat{\mathcal{P}}_W(t,\widehat{\nu})$.

Lemma 5.4.8. *Let $(t,\widehat{\nu}) \in [0,T] \times \mathcal{P}(\widehat{\Omega})$, $\overline{\mathbb{P}} \in \widehat{\mathcal{P}}_W(t,\widehat{\nu})$, $\bar{\tau}$ be a $\overline{\mathbb{G}}^t$–stopping time taking values in $[t,T]$, and $\big(\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar{\tau}}}_{\bar{\omega}}\big)_{\bar{\omega}\in\overline{\Omega}}$ be a family of* r.c.p.d. *of $\overline{\mathbb{P}}$ knowing $\overline{\mathcal{G}}^t_{\bar{\tau}}$. Then*

$$\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar{\tau}}}_{\bar{\omega}} \in \widehat{\mathcal{P}}_W\big(\bar{\tau}(\bar{\omega}), \widehat{\mu}_{\bar{\tau}(\bar{\omega})}(\bar{\omega})\big),\ \text{for } \overline{\mathbb{P}}\text{–a.e. } \bar{\omega} \in \overline{\Omega}.$$

Proof. Let $\overline{\mathbb{P}} \in \widehat{\mathcal{P}}_W(t,\widehat{\nu})$. First, it is easy to check that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, one has $\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar{\tau}}}_{\bar{\omega}}[\overline{\alpha}_s \in U] = 1$, for Lebesgue–almost every $s \in [\bar{\tau}(\bar{\omega}),T]$, and $\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar{\tau}}}_{\bar{\omega}}}\big[\int_{\bar{\tau}(\bar{\omega})}^T \big(\rho(u_0,\overline{\alpha}_s)\big)^p \mathrm{d}s\big] < \infty$.

Next, notice that for all $s \in [0,T]$, $\beta \in C_b(\mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C}^\ell)$, $\psi \in C_b(\mathcal{C}^\ell \times C([0,T],\mathcal{P}(\widehat{\Omega})))$ and $Z \in \overline{\mathcal{G}}^t_{\bar{\tau}}$,

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\langle\beta,\widehat{\mu}_s\rangle\psi(B^{\bar{\tau}},\widehat{\mu})\mathbf{1}_{Z\cap\{\bar{\tau}\le s\}}\big] = \mathbb{E}^{\overline{\mathbb{P}}}\big[\beta(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\psi(B^{\bar{\tau}},\widehat{\mu})\mathbf{1}_{Z\cap\{\bar{\tau}\le s\}}\big],$$

so that, for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$ and any $\bar\tau(\bar\omega)\le s\le T$,

$$\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}\big[\langle\beta,\widehat{\mu}_s\rangle\psi(B^{\bar\tau}_{s\wedge\cdot},\widehat{\mu}_{s\wedge\cdot})\big]=\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}\big[\beta(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot})\psi(B^{\bar\tau}_{s\wedge\cdot},\widehat{\mu}_{s\wedge\cdot})\big].$$

By considering a countable dense set of maps $(\beta,\psi)\in C_b(\mathcal{C}^n\times\mathcal{C}\times\mathcal{C}^d\times\mathcal{C}^\ell)\times C_b(\mathcal{C}^\ell\times C([0,T],\mathcal{P}(\widehat{\Omega})))$, it follows that

$$\widehat{\mu}_s=\mathcal{L}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}\big(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot}|\overline{\mathcal{G}}^{\bar\tau(\bar\omega)}_T\big),\ \overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}\text{–a.s., for all } s\ge\bar\tau(\bar\omega),\ \text{for } \overline{\mathbb{P}}\text{–a.e. } \bar\omega\in\overline{\Omega}.$$

Similarly, one can prove that, for $\overline{\mathbb{P}}$–a.e. $\bar\omega$, and $s\le\bar\tau(\bar\omega)$,

$$\widehat{\mu}_s=\mathcal{L}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot}),\ \overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}\text{–a.s.},$$

and hence, for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$,

$$\widehat{\mu}_s=\mathcal{L}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot})\mathbf{1}_{\{s\in[0,\bar\tau(\bar\omega)]\}}+\mathcal{L}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}(X_{s\wedge\cdot},A_{s\wedge\cdot},W,B_{s\wedge\cdot}|\overline{\mathcal{G}}^{\bar\tau(\bar\omega)}_T)\mathbf{1}_{\{s\in(\bar\tau(\bar\omega),T]\}},\ \overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}\text{–a.s.}$$

Finally, it is clear that for $\overline{\mathbb{P}}$–a.e. $\bar\omega$, one has $\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}\big[\|X\|^p\big]<\infty$ and $\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}\big[|\overline{S}|_T<\infty\big]=1$. Moreover, let $\varphi\in C^2_b(\mathbb{R}^n\times\mathbb{R}^d\times\mathbb{R}^\ell)$, so that the localised process $S^{\varphi,m}=\overline{S}^\varphi_{\tau_m\wedge\cdot}$ is a $(\overline{\mathbb{F}},\overline{\mathbb{P}})$–martingale on $[t,T]$. Fix $T\ge r>s\ge t$, $J\in\overline{\mathcal{F}}_s$ and $K\in\overline{\mathcal{G}}^t_{\bar\tau}$, we have

$$\mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\cdot}}\big[\overline{S}^\varphi_{\tau_m\wedge r}\mathbf{1}_J\big]\mathbf{1}_{K\cap\{\bar\tau\le s\}}\Big]=\mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{\tau_m\wedge r}\mathbf{1}_{J\cap K\cap\{\bar\tau\le s\}}\big]=\mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{\tau_m\wedge s}\mathbf{1}_{J\cap K\cap\{\bar\tau\le s\}}\big]=\mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\cdot}}\big[\overline{S}^\varphi_{\tau_m\wedge s}\mathbf{1}_J\big]\mathbf{1}_{K\cap\{\bar\tau\le s\}}\Big].$$

This implies that

$$\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}\big[\overline{S}^\varphi_{\tau_m\wedge r}\mathbf{1}_J\big]=\mathbb{E}^{\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}}\big[\overline{S}^\varphi_{\tau_m\wedge s}\mathbf{1}_J\big],\ \text{for } \overline{\mathbb{P}}\text{–a.e. } \bar\omega.$$

By considering countably many s,r,J, it follows that $\overline{S}^\varphi$ is a $\big(\overline{\mathbb{F}},\overline{\mathbb{P}}^{\overline{\mathcal{G}}^t_{\bar\tau}}_{\bar\omega}\big)$–local martingale on $[\bar\tau(\bar\omega),T]$ for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline{\Omega}$. We hence conclude the proof. □

We next provide a stability result for $\overline{\mathcal{P}}_W$ under concatenation. For any constant $M>0$, let us introduce

$$\overline{\mathcal{P}}^M_t:=\Big\{\overline{\mathbb{P}}\in\mathcal{P}(\overline{\Omega})\ :\mathbb{E}^{\overline{\mathbb{P}}}\big[\|X\|^p\big]+\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T\big(\rho(u_0,\overline{\alpha}_s)\big)^p\mathrm{d}s\Big]\le M\Big\},\ \overline{\mathcal{P}}^M_W(t,\nu):=\overline{\mathcal{P}}_W(t,\nu)\cap\overline{\mathcal{P}}^M_t,\ \widehat{\mathcal{P}}^M_W(t,\widehat{\nu}):=\widehat{\mathcal{P}}_W(t,\widehat{\nu})\cap\overline{\mathcal{P}}^M_t,$$

and

$$V^M_W(t,\nu):=\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^M_W(t,\nu)}J(t,\overline{\mathbb{P}}).$$

Notice that $V^M_W(t,\nu)\nearrow V_W(t,\nu)$ as $M\nearrow\infty$. Moreover, as in Lemma 5.4.7, the graph set

$$\big\{(t,\nu,M,\overline{\mathbb{P}})\ :\overline{\mathbb{P}}\in\overline{\mathcal{P}}^M_W(t,\nu)\big\}\text{ is analytic, and }(t,\nu,M)\longmapsto V^M_W(t,\nu)\in\mathbb{R}\cup\{-\infty,\infty\}\text{ is upper semi–analytic.}\quad(5.4.10)$$

Lemma 5.4.9. *Let $t\in[0,T]$, $\widehat{\nu}_1,\widehat{\nu}_2\in\mathcal{P}(\widehat{\Omega})$ and $\nu\in\mathcal{P}(\mathcal{C}^n)$ be such that $\widehat{\nu}_1\circ X^{-1}_{t\wedge\cdot}=\widehat{\nu}_2\circ X^{-1}_{t\wedge\cdot}=\nu(t)$. Then for all $\overline{\mathbb{P}}_1\in\widehat{\mathcal{P}}_W(t,\widehat{\nu}_1)$, there exists $\overline{\mathbb{P}}_2\in\widehat{\mathcal{P}}_W(t,\widehat{\nu}_2)$ satisfying*

$$\overline{\mathbb{P}}_1\circ\big(X,A^t,W^t,B^t\big)^{-1}=\overline{\mathbb{P}}_2\circ\big(X,A^t,W^t,B^t\big)^{-1},$$

where $A^t_\cdot:=A_{\cdot\vee t}-A_t$, so that $J(t,\overline{\mathbb{P}}_1)=J(t,\overline{\mathbb{P}}_2)$. Consequently, one has

$$V_W(t,\nu)=\sup_{\overline{\mathbb{P}}\in\widehat{\mathcal{P}}_W(t,\widehat{\nu}_1)}J(t,\overline{\mathbb{P}}),\ \text{and}\ V^M_W(t,\nu)=\sup_{\overline{\mathbb{P}}\in\widehat{\mathcal{P}}^M_W(t,\widehat{\nu}_1)}J(t,\overline{\mathbb{P}}).$$

The proof is almost the same as that of Lemma 5.3.10, and hence it is omitted.

Lemma 5.4.10. *Let $(t,\nu)\in[0,T]\times\mathcal{P}(\mathcal{C}^n)$, $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\nu)$, $\bar\tau$ be a $\overline{\mathbb{G}}^t$–stopping time taking values in $[t,T]$, $\varepsilon>0$. Then there exists a family of probability measures $(\bar Q^\varepsilon_{t,\widehat\nu,M})_{(t,\widehat\nu,M)\in[0,T]\times\mathcal{P}(\widehat\Omega)\times\mathbb{R}_+}$ such that $(t,\widehat\nu,M)\longmapsto\bar Q^\varepsilon_{t,\widehat\nu,M}$ is universally measurable, and for every $(t,\widehat\nu,M)$ s.t. $\widehat{\mathcal{P}}^M_W(t,\widehat\nu)\neq\emptyset$, one has*

$$\bar Q^\varepsilon_{t,\widehat\nu,M}\in\widehat{\mathcal{P}}^M_W(t,\widehat\nu)\text{ and }J\big(t,\bar Q^\varepsilon_{t,\widehat\nu,M}\big)\ge\begin{cases}V^M_W(t,\nu)-\varepsilon,\ \text{when } V^M_W(t,\nu)<\infty,\\ 1/\varepsilon,\ \text{when } V^M_W(t,\nu)=\infty,\end{cases}\quad\text{with }\nu:=\widehat\nu\circ\widehat X^{-1}.\tag{5.4.11}$$

Moreover, there exists a $\overline{\mathbb{P}}$–integrable, $\overline{\mathcal{G}}^t_{\bar\tau}$–measurable r.v. $\widehat M:\overline\Omega\to\mathbb{R}_+$ such that for all constant $M\ge0$, one can find a probability measure $\overline{\mathbb{P}}^{M,\varepsilon}\in\overline{\mathcal{P}}_W(t,\nu)$ satisfying $\overline{\mathbb{P}}^{M,\varepsilon}|_{\overline{\mathcal{F}}_{\bar\tau}}=\overline{\mathbb{P}}|_{\overline{\mathcal{F}}_{\bar\tau}}$ and

$$\big(\bar Q^\varepsilon_{\bar\tau(\bar\omega),\widehat\mu_{\bar\tau(\bar\omega)}(\bar\omega),M+\widehat M(\bar\omega)}\big)_{\bar\omega\in\overline\Omega}\text{ is a version of r.c.p.d. of }\overline{\mathbb{P}}^{M,\varepsilon}\text{ knowing }\overline{\mathcal{G}}^t_{\bar\tau}.$$

Proof. The existence of the family of probability measures $((\bar Q^\varepsilon_{t,\widehat\nu,M})_{(t,\widehat\nu,M)\in[0,T]\times\mathcal{P}(\widehat\Omega)\times\mathbb{R}_+})$ satisfying (5.4.11) follows by (5.4.10) and Lemma 5.4.9, together with the measurable selection theorem (see e.g. [61, Proposition 2.21]).

With $\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\nu)$, we consider a family of r.c.p.d. $(\overline{\mathbb{P}}_{\bar\omega})_{\bar\omega\in\overline\Omega}$ of $\overline{\mathbb{P}}$ knowing $\overline{\mathcal{G}}^t_{\bar\tau}$, and define

$$\widehat M(\bar\omega):=\mathbb{E}^{\overline{\mathbb{P}}_{\bar\omega}}\Big[\|X\|^p+\int_{\bar\tau}^T\big(\rho(\overline\alpha_s,u_0)\big)^p\mathrm{d}s\Big],$$

so that $\overline{\mathbb{P}}_{\bar\omega}\in\widehat{\mathcal{P}}^{\widehat M(\bar\omega)}_W\big(\bar\tau(\bar\omega),\widehat\mu_{\bar\tau(\bar\omega)}(\bar\omega)\big)$ for $\overline{\mathbb{P}}$–a.e. $\bar\omega$, by Lemma 5.4.8. In particular, $\widehat{\mathcal{P}}^{\widehat M(\bar\omega)}_W\big(\bar\tau(\bar\omega),\widehat\mu_{\bar\tau(\bar\omega)}(\bar\omega)\big)$ is nonempty for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline\Omega$. For a fixed constant $M\ge0$, let

$$\bar Q^\varepsilon_{\bar\omega}:=\bar Q^\varepsilon_{\bar\tau(\bar\omega),\widehat\mu_{\bar\tau(\bar\omega)}(\bar\omega),\widehat M(\bar\omega)+M}.$$

Notice that, for $\overline{\mathbb{P}}$–a.e. $\bar\omega\in\overline\Omega$,

$$\widehat\mu_{\bar\tau(\bar\omega)}(\bar\omega)=\overline{\mathbb{P}}_{\bar\omega}\circ(X_{\bar\tau(\bar\omega)\wedge\cdot},A_{\bar\tau(\bar\omega)\wedge\cdot},W,B_{\bar\tau(\bar\omega)\wedge\cdot})^{-1}=\bar Q^\varepsilon_{\bar\omega}\circ(X_{\bar\tau(\bar\omega)\wedge\cdot},A_{\bar\tau(\bar\omega)\wedge\cdot},W,B_{\bar\tau(\bar\omega)\wedge\cdot})^{-1},$$

then

$$\begin{aligned}\mathcal{L}^{\bar Q^\varepsilon_{\bar\omega}}\big(X_{\bar\tau(\bar\omega)\wedge\cdot},A_{\bar\tau(\bar\omega)\wedge\cdot},W_{\bar\tau(\bar\omega)\wedge\cdot},B_{\bar\tau(\bar\omega)\wedge\cdot},\widehat\mu_{\bar\tau(\bar\omega)\wedge\cdot}\big)&=\mathcal{L}^{\bar Q^\varepsilon_{\bar\omega}}\big(X_{\bar\tau(\bar\omega)\wedge\cdot},A_{\bar\tau(\bar\omega)\wedge\cdot},W_{\bar\tau(\bar\omega)\wedge\cdot},B_{\bar\tau(\bar\omega)\wedge\cdot}\big)\otimes\mathcal{L}^{\bar Q^\varepsilon_{\bar\omega}}\big(\widehat\mu_{\bar\tau(\bar\omega)\wedge\cdot}\big)\\&=\mathcal{L}^{\overline{\mathbb{P}}_{\bar\omega}}\big(X_{\bar\tau(\bar\omega)\wedge\cdot},A_{\bar\tau(\bar\omega)\wedge\cdot},W_{\bar\tau(\bar\omega)\wedge\cdot},B_{\bar\tau(\bar\omega)\wedge\cdot},\widehat\mu_{\bar\tau(\bar\omega)\wedge\cdot}\big).\end{aligned}\tag{5.4.12}$$

In particular, one has

$$\bar Q^\varepsilon_{\bar\omega}\big[B^t_{\bar\tau(\bar\omega)\wedge\cdot}=(\bar\omega^b)^t_{\bar\tau(\bar\omega)\wedge\cdot},\ \widehat\mu_{\bar\tau(\bar\omega)\wedge\cdot}=\bar\omega^{\hat\nu}_{\bar\tau(\bar\omega)\wedge\cdot}\big]=1,\ \text{for }\overline{\mathbb{P}}\text{–a.e. }\bar\omega=(\bar\omega^x,\bar\omega^a,\bar\omega^w,\bar\omega^b,\bar\omega^{\hat\nu})\in\overline\Omega.\tag{5.4.13}$$

Let us then define a probability measure $\overline{\mathbb{P}}^{M,\varepsilon}$ on $\overline\Omega$ by

$$\overline{\mathbb{P}}^{M,\varepsilon}[K]:=\int_{\overline\Omega}\bar Q_{\bar\omega}(K)\overline{\mathbb{P}}(\mathrm{d}\bar\omega),\ \text{for all }K\in\overline{\mathcal{F}}.$$

By (5.4.12), one has $\overline{\mathbb{P}}^{M,\varepsilon}=\overline{\mathbb{P}}$ on $\overline{\mathcal{F}}_{\bar\tau}$, and moreover, $(\bar Q^\varepsilon_{\bar\omega})_{\bar\omega\in\overline\Omega}$ is a family of r.c.p.d. of $\overline{\mathbb{P}}^{M,\varepsilon}$ knowing $\overline{\mathcal{G}}^t_{\bar\tau}$. To conclude the proof, it is enough to check that $\overline{\mathbb{P}}^{M,\varepsilon}\in\overline{\mathcal{P}}_W(t,\nu)$.

First, it is clear that $\overline{\mathbb{P}}^{M,\varepsilon}[\overline\alpha_s\in U]=1$, for Lebesgue–almost every $s\in[t,T]$, and

$$\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\Big[\int_t^T\big(\rho(u_0,\overline\alpha_s)\big)^p\mathrm{d}s\Big]\le\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^{\bar\tau}\big(\rho(u_0,\overline\alpha_s)\big)^p\mathrm{d}s\Big]+\mathbb{E}^{\overline{\mathbb{P}}}\big[\widehat M\big]<\infty.$$

Next, for each $\beta \in C_b(\widehat{\Omega})$, $\psi \in C_b(\mathcal{C}^\ell \times \mathcal{P}(\widehat{\Omega}))$, $h \in C_b(\mathcal{C}^\ell)$ and $s \in [t,T]$, one has

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\beta(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\psi(B^{\bar\tau}, \widehat{\mu})h(B^t_{\bar\tau\wedge\cdot})\big] = \int_{\bar\omega} \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\beta(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\psi(B^{\bar\tau}, \widehat{\mu})h(B^t_{\bar\tau\wedge\cdot})\big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\\
&= \int_{\bar\omega} \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\beta(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\psi(B^{\bar\tau}, \widehat{\mu})\big]h(B^t_{\bar\tau(\bar\omega)\wedge\cdot}(\bar\omega))\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\\
&= \int_{\bar\omega} \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\beta(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot})\big|\overline{\mathcal{G}}_T^{\bar\tau(\bar\omega)}\big]\psi(B^{\bar\tau}, \widehat{\mu})\big]h(B^t_{\bar\tau(\bar\omega)\wedge\cdot}(\bar\omega))\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\\
&= \int_{\bar\omega} \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\langle\beta, \widehat{\mu}_s\rangle\psi(B^{\bar\tau}, \widehat{\mu})\big]h(B^t_{\bar\tau(\bar\omega)\wedge\cdot}(\bar\omega)))\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\\
&= \int_{\bar\omega} \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\langle\beta, \widehat{\mu}_s\rangle\psi(B^{\bar\tau}, \widehat{\mu})h(B^t_{\bar\tau\wedge\cdot}))\big]\overline{\mathbb{P}}(\mathrm{d}\bar\omega)\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\langle\beta, \widehat{\mu}_s\rangle\psi(B^{\bar\tau}, \widehat{\mu})h(B^t_{\bar\tau\wedge\cdot})\big],
\end{aligned}$$

where the second and fifth equalities are due to Equation (5.4.13), and the fourth follows by the fact that $\bar{Q}^\varepsilon_{\bar\omega} \in \widehat{\mathcal{P}}_W(\bar\tau(\bar\omega), \widehat{\mu}_{\bar\tau(\bar\omega)}(\bar\omega))$. Notice that $B^t_u = B^t_{\bar\tau\wedge u} + B^{\bar\tau}_u$ for any $u \in [t,T]$, the above equality implies that

$$\widehat{\mu}_s = \mathcal{L}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W, B_{s\wedge\cdot}\big|\overline{\mathcal{G}}^t_T\big),\ \overline{\mathbb{P}}^{M,\varepsilon}\text{–a.s.}$$

Finally, we easily check that $\overline{\mathbb{P}}^{M,\varepsilon}[|\overline{S}|_T < \infty] = 1$ and $\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}[\|X\|^p] \le \mathbb{E}^{\overline{\mathbb{P}}}[\|X\|^p] + M < \infty$. For a fixed test function $\varphi \in C^2_b(\mathbb{R}^{n+d+\ell})$, we consider the localised stopping times τ_m defined in (5.4.5) and $\tau^{\bar\omega}_k(\bar\omega') := \bar\tau(\bar\omega) \vee \tau_k(\bar\omega')$ for each $\bar\omega \in \overline{\Omega}$. We know that $\tau^{\bar\omega}_k \le \tau^{\bar\omega}_{k+1}$, for any $k \in \mathbb{N}$, that $\tau^{\bar\omega}_k \underset{k\to\infty}{\longrightarrow} \infty$, and that $(\overline{S}^\varphi_{s\wedge\tau^{\bar\omega}_k})_{s\in[\tau(\bar\omega),T]}$ is an $(\overline{\mathbb{F}}, \bar{Q}^\varepsilon_{\bar\omega})$–martingale for all $k \in \mathbb{N}$. Notice that for all $s \in [t,T]$ and $A \in \overline{\mathcal{F}}_s$, the map

$$\bar\omega \longmapsto \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\Big[\overline{S}^\varphi_{s\wedge\tau_m\wedge\tau^{\bar\omega}_k}\mathbf{1}_A\mathbf{1}_{s>\bar\tau(\bar\omega)}\Big] \text{ is } \overline{\mathcal{G}}^t_{\bar\tau}\text{–measurable.}$$

Then for $s \le r \le T$,

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{s\wedge\tau_m}\mathbf{1}_A\big] = \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{s\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{s\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s>\bar\tau}\big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{s\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\big] + \lim_{k\to\infty}\int_{\overline{\Omega}} \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\omega}}\big[\overline{S}^\varphi_{s\wedge\tau_m\wedge\tau^{\bar\omega}_k}\mathbf{1}_A\mathbf{1}_{s>\bar\tau(\bar\omega)}\big]\overline{\mathbb{P}}(d\bar\omega) = \mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{s\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s>\bar\tau}\big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{\bar\tau\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\mathbf{1}_{r<\bar\tau}\big] + \mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{\bar\tau\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\mathbf{1}_{\bar\tau\le r}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s>\bar\tau}\big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\mathbf{1}_{r<\bar\tau}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{\bar\tau\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\mathbf{1}_{\bar\tau\le r}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s>\bar\tau}\big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\mathbf{1}_{r<\bar\tau}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s\le\bar\tau}\mathbf{1}_{\bar\tau\le r}\big] + \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\mathbf{1}_{s>\bar\tau}\big] = \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\overline{S}^\varphi_{r\wedge\tau_m}\mathbf{1}_A\big],
\end{aligned}$$

which means that $(\overline{S}^\varphi_u)_{u\in[t,T]}$ is an $(\overline{\mathbb{F}}, \overline{\mathbb{P}}^{M,\varepsilon})$–local martingale, and hence $\overline{\mathbb{P}}^{M,\varepsilon} \in \overline{\mathcal{P}}_W(t,\nu)$. □

5.4.3 Proof of the main results

5.4.3.1 Proof of Theorem 5.3.1

First, V_W is upper semi–analytic by Lemma 5.4.7. Further, let $\bar\tau$ be a $\overline{\mathbb{G}}^t$–stopping time taking value in $[t,T]$, it follows by Lemma 5.4.8 that, for every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)$

$$\begin{aligned}
J(t,\overline{\mathbb{P}}) &= \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\int_{\bar\tau}^T L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)ds + g\big(X_{T\wedge\cdot}, \mu_T\big)\Big|\overline{\mathcal{G}}^t_\tau\bigg]\bigg]\\
&\le \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + V_W\big(\bar\tau, \mu_{\bar\tau}\big)\bigg]\\
&\le \sup_{\overline{\mathbb{P}}'\in\overline{\mathcal{P}}_W(t,\nu)} \mathbb{E}^{\overline{\mathbb{P}}'}\bigg[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + V_W\big(\bar\tau, \mu_{\bar\tau}\big)\bigg].
\end{aligned}$$

Notice that a $\overline{\mathbb{G}}^t$–stopping time on $\overline{\Omega}$ can be considered as a $\mathbb{G}^\star$–stopping time $\tau^\star$ on $\Omega^\star$. Then by the way how τ^γ is defined from $\tau^\star$ in (5.3.1) and Lemma 5.4.4, we obtain the inequality

$$V_W(t,\nu) \leq \sup_{\gamma\in\Gamma_W(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s)\mathrm{d}s + V_W\big(\tau^\gamma, \mu^\gamma_{\tau^\gamma}\big)\Big]. \tag{5.4.14}$$

We now consider the reverse inequality, for which one can assume w.l.o.g. that

$$V_W(t,\nu) < \infty, \text{ and } \sup_{\overline{\mathbb{P}}'\in\overline{\mathcal{P}}_W(t,\nu)} \mathbb{E}^{\overline{\mathbb{P}}'}\Big[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + V_W\big(\bar\tau, \mu_{\bar\tau}\big)\Big] > -\infty. \tag{5.4.15}$$

Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)$ be a weak control rule, then by Lemma 5.4.10, for some $\overline{\mathcal{F}}_{\bar\tau}$–measurable $\overline{\mathbb{P}}$–integrable r.v. $\widehat{M}:\overline{\Omega}\to\mathbb{R}_+$, one has a family of probability measures $(\overline{\mathbb{P}}^{M,\varepsilon})_{M\geq 0}$ in $\overline{\mathcal{P}}_W(t,\nu)$ such that

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + \big(V_W^{M+\widehat{M}(\bar\omega)}(\bar\tau, \mu_{\bar\tau}) - \varepsilon\big)\mathbf{1}_{\{V_W^{M+\widehat{M}(\bar\omega)}(\bar\tau,\mu_{\bar\tau})<\infty\}}\Big] + \frac{1}{\varepsilon}\overline{\mathbb{P}}\big[V_W^{M+\widehat{M}(\bar\omega)}(\bar\tau,\mu_{\bar\tau}) = \infty\big] \\
&\leq \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, , \overline{\alpha}_s)\mathrm{d}s + \mathbb{E}^{\bar{Q}^\varepsilon_{\bar\tau,\hat\mu,M+\widehat{M}}}\Big[\int_{\bar\tau}^T L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + g\big(X_{T\wedge\cdot}, \mu_T\big)\Big]\Big] \\
&= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\Big[\int_t^T L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + g\big(X_{T\wedge\cdot}, \mu_T\big)\Big] \leq V_W(t,\nu).
\end{aligned}$$

If $\overline{\mathbb{P}}\big[V_W^{M+\widehat{M}}(\bar\tau,\mu_{\bar\tau}) = \infty\big)\big] > 0$ for some $M \geq 0$, then by taking $\varepsilon \longrightarrow 0$, one finds $V_W(t,\nu) = \infty$ which is in contradiction to (5.4.15). When $\overline{\mathbb{P}}\big[V_W^{M+\widehat{M}}(\bar\tau,\mu_{\bar\tau}) = \infty\big] = 0$ for all $M \geq 0$, let $M \longrightarrow \infty$ and then take the supremum over all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)$, it follows that

$$\sup_{\overline{\mathbb{P}}'\in\overline{\mathcal{P}}_W(t,\nu)} \mathbb{E}^{\overline{\mathbb{P}}'}\Big[\int_t^{\bar\tau} L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + V_W\big(\bar\tau, \mu_{\bar\tau}\big)\Big] - \varepsilon \leq V_W(t,\nu).$$

Notice that $\varepsilon > 0$ is arbitrary, and again by the way how τ^γ is defined from $\tau^\star$ (equivalent to $\bar\tau$ on $\overline{\Omega}$) and Lemma 5.4.4, we can conclude the proof with (5.4.14). □

5.4.3.2 Proof of Theorem 5.3.4

Let $(t,\nu) \in [0,T]\times\mathcal{P}_2(\mathcal{C}^n)$ and Assumption 5.3.3 hold, by Theorem 2.2.3 (letting $\hat p = p = 2$ in the assumptions), one has

$$V_S(t,\nu) = V_W(t,\nu).$$

Therefore $V_S : [0,T]\times\mathcal{P}_2(\mathcal{C}^n) \longrightarrow \mathbb{R}\cup\{-\infty,\infty\}$ has the same measurability as $V_W : [0,T]\times\mathcal{P}_2(\mathcal{C}^n) \longrightarrow \mathbb{R}\cup\{-\infty,\infty\}$.

Next, let τ be a $\mathbb{G}^{t,\circ}$–stopping time on $(\Omega^t, \mathcal{F}^t, \mathbb{P}^t_\nu)$ taking value in $[t,T]$, we denote $\tau^\gamma := \tau(B^{\gamma,t})$ for $\gamma\in\Gamma_S(t,\nu)$. Then by the formulation equivalence result in Proposition 5.2.10, the DPP result (5.3.4) is equivalent to

$$V_S(t,\nu) = \sup_{\gamma\in\Gamma_S(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + V_S(\tau^\gamma, \mu^\gamma)\Big].$$

Recall that under Assumption 5.2.8 and by Theorem 5.5.3, one has

$$\mathbb{E}^{\mathbb{P}^\gamma}\Big[\sup_{s\in[0,T]}|X^\gamma_s|^2\Big] < \infty.$$

Then by Lemma 5.4.8 and the fact that $V_S = V_W$, it follows that

$$V_S(t,\nu) = \sup_{\gamma\in\Gamma_S(t,\nu)} J(t,\gamma) = \sup_{\gamma\in\Gamma_S(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + \int_{\tau^\gamma}^T L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + g\big(X^\gamma_{T\wedge\cdot}, \mu^\gamma_T\big)\Big]$$
$$\leq \sup_{\gamma\in\Gamma_S(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + V_S(\tau^\gamma, \mu^\gamma)\Big].$$

Further, by Theorem 5.3.1, we have

$$V_S(t,\nu) = V_W(t,\nu) = \sup_{\gamma\in\Gamma_W(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + V_W(\tau^\gamma, \mu^\gamma)\Big]$$
$$\geq \sup_{\gamma\in\Gamma_S(t,\nu)} \mathbb{E}^{\mathbb{P}^\gamma}\Big[\int_t^{\tau^\gamma} L\big(s, X^\gamma_{s\wedge\cdot}, \overline{\mu}^\gamma_s, \alpha^\gamma_s\big)\mathrm{d}s + V_S(\tau^\gamma, \mu^\gamma)\Big],$$

and hence the proof is concluded. □

5.4.3.3 Proof of Theorem 5.3.2

In this part, we use the results and techniques of Theorem 5.3.1 to show the DPP for $V^{\mathbb{B}}_S$. We start by proving the universal measurability of $V^{\mathbb{B}}_S$. For this, we consider an equivalent formulation of $V^{\mathbb{B}}_S$, which is more appropriate for our purpose.

5.4.3.3.1 An equivalent reformulation for $V^{\mathbb{B}}_S$ Let $\widetilde{\Omega}^\star := \mathcal{C}^\ell$ be the canonical space with canonical process $\widetilde{B}^\star$, and $\widetilde{\mathbb{P}}^\star$ be the Wiener measure, under which $\widetilde{B}^\star$ is an ℓ–dimensional standard Brownian motion. Let $\widetilde{\mathbb{F}}^\star = (\widetilde{\mathcal{F}}^\star_t)_{t\in[0,T]}$ be the canonical filtration. Recall that we consider a fixed Borel map $\pi : U\cup\{\partial\} \to \mathbb{R}\cup\{-\infty,\infty\}$. We denote by $\mathcal{U}$ the set of $\widetilde{\mathbb{F}}^\star$–predictable processes θ taking values in $\mathbb{R}$, such that $\mathbb{E}^{\widetilde{\mathbb{P}}^\star}\big[\int_0^T |\theta_t|^2\mathrm{d}t\big] < \infty$. Define a metric $d^\star$ on $\mathcal{U}$ by

$$d^\star(\eta,\theta)^2 := \mathbb{E}^{\widetilde{\mathbb{P}}^\star}\Big[\int_0^T |\eta_t - \theta_t|^2\mathrm{d}t\Big], \text{ for all } (\eta,\theta)\in\mathcal{U}\times\mathcal{U},$$

so that $(\mathcal{U}, d^\star)$ is a Polish space (see e.g. Brezis [32, Theorems 4.8 and 4.13]). Next, let $\theta\in\mathcal{U}$, and define $A^\theta_t := \int_0^t \theta_s\mathrm{d}s$, $t\in[0,T]$. We consider then the map $\Upsilon : \mathcal{U} \longrightarrow \mathcal{P}(\mathcal{C}^\ell\times\mathcal{C})$ defined by

$$\Upsilon(\theta) := \widetilde{\mathbb{P}}^\star \circ \Big(\widetilde{B}^\star_\cdot, A^{\theta(\widetilde{B}^\star_\cdot)}_\cdot\Big)^{-1},\ \theta\in\mathcal{U}.$$

Let us introduce, for all $t\in[0,T]$, $\nu\in\mathcal{P}_2(\mathcal{C}^n)$ and $\widehat{\nu}\in\mathcal{P}_2(\widehat{\Omega})$ such that $\nu = \widehat{\nu}\circ\widehat{X}^{-1}$,

$$\overline{\mathcal{P}}^\star_S(t,\nu) := \big\{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\nu) : \overline{\mathbb{P}}\circ\big(B_\cdot, A_\cdot\big)^{-1}\in\Upsilon\big(\mathcal{U}\big), \text{ and } B_{t\wedge\cdot} \text{ is } \overline{\mathbb{P}}\text{–independent of } (B^t, A)\big\},$$

and

$$\widehat{\mathcal{P}}^\star_S(t,\widehat{\nu}) := \widehat{\mathcal{P}}_W(t,\widehat{\nu})\cap\overline{\mathcal{P}}^\star_S(t,\nu).$$

Lemma 5.4.11. *Let $(t,\nu)\in[0,T]\times\mathcal{P}_2(\mathcal{C}^n)$ and $\widehat{\nu}\in\mathcal{P}(\widehat{\Omega})$ be such that $\widehat{\nu}\circ\widehat{X}^{-1} = \nu$. Then under* Assumption 5.2.8*, one has $\overline{\mathcal{P}}^\star_S(t,\nu)\subseteq\overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$ and*

$$V^{\mathbb{B}}_S(t,\nu) = \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^\star_S(t,\nu)} J(t,\overline{\mathbb{P}}). \tag{5.4.16}$$

Proof. First, take $\gamma\in\Gamma^{\mathbb{B}}_S(t,\nu)$. W.l.o.g., we can assume that there exists an independent Brownian motion $\widetilde{B}$ in the space $(\Omega^\gamma.\mathcal{F}^\gamma, \mathbb{P}^\gamma)$, and let $B^{\star,\gamma}_\cdot := B^\gamma_{t\vee\cdot} - B^\gamma_t + \widetilde{B}_{t\wedge\cdot}$, then

$$\gamma' := \big(\Omega^\gamma, \mathcal{F}^\gamma, \mathbb{P}^\gamma, \mathbb{F}^\gamma, \mathbb{G}^\gamma, X^\gamma, W^\gamma, B^{\star,\gamma}, \overline{\mu}^\gamma, \mu^\gamma, \alpha^\gamma\big)\in\overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu).$$

Recall that α^γ is $\mathbb{G}^\gamma$–predictable and $\mathbb{G}^\gamma$ is the augmented filtration generated by $B^{\gamma,t}$, then for some Borel function $\phi : [t,T] \times \mathcal{C}^\ell \to U$, one has $\alpha^\gamma_s = \phi(s, B^{\gamma,t}_{s\wedge\cdot})$, $s \in [t,T]$, $\mathbb{P}^\gamma$–a.s. Let $A^{\gamma'}_\cdot := \int_0^\cdot \pi\big(\phi(s, B^{\gamma,t}_{s\wedge\cdot})\big)\mathrm{d}s$ and $\widehat{\mu}^{\gamma'}$ be defined as in (5.2.7), it follows that $\overline{\mathbb{P}}' := \mathbb{P}^\gamma \circ \big(X^\gamma, A^{\gamma'}, W^\gamma, B^{\gamma'}, \widehat{\mu}^{\gamma'}\big)^{-1} \in \overline{\mathcal{P}}_W(t,\nu)$ satisfies $\overline{\mathbb{P}}' \circ (B,A)^{-1} \in \Gamma(\mathcal{U})$ and $J(t,\overline{\mathbb{P}}') = J(t,\gamma)$. Then $J(t,\gamma) = J(t,\overline{\mathbb{P}}') \le \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^\star_S(t,\nu)} J(t,\overline{\mathbb{P}})$ and hence $V^{\mathbb{B}}_S(t,\nu) \le \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^\star_S(t,\nu)} J(t,\overline{\mathbb{P}})$.

Next, given $\overline{\mathbb{P}} \in \overline{\mathcal{P}}^\star_S(t,\nu)$, since $\overline{\mathbb{P}} \circ \big(B, A\big)^{-1} \in \Upsilon\big(\mathcal{U}\big)$, there exists $\theta^\star \in \mathcal{U}$ such that $\overline{\mathbb{P}} \circ \big(B_\cdot, A_\cdot\big)^{-1} = \widetilde{\mathbb{P}}^\star \circ \big(\widetilde{B}^\star_\cdot, A^{\theta^\star(\widetilde{B}^\star_\cdot)}_\cdot\big)\big)^{-1}$. Thus $\pi\big(\overline{\alpha}_s(\omega)\big) = \theta^\star_s(B_{s\wedge\cdot}(\omega))$, for $\mathrm{d}\overline{\mathbb{P}} \otimes \mathrm{d}t$–a.e. $(s,\bar\omega) \in [t,T] \times \overline{\Omega}$. As $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\nu)$, we know $\overline{\mathbb{P}}[\overline{\alpha}_s \in U] = 1$ for $\mathrm{d}t$–a.e. $s \in [0,T]$, therefore $\pi\big(\overline{\alpha}_s(\bar\omega)\big) = \theta^\star_s(B_{s\wedge\cdot}(\bar\omega)) \in \pi(U)$ and $\overline{\alpha}_s(\bar\omega) = \pi^{-1}\big(\theta^\star_s(B_{s\wedge\cdot}(\bar\omega))\big) \in U$, for $\mathrm{d}\overline{\mathbb{P}} \otimes \mathrm{d}t$–a.e. $(s,\bar\omega) \in [0,T] \times \overline{\Omega}$. Further, since (B^t, A) is $\overline{\mathbb{P}}$–independent of $B_{t\wedge\cdot}$, it follows that there is a Borel measurable function $\phi : [0,T] \times \mathcal{C}^\ell \longrightarrow \mathbb{R}$ such that $A_s = \phi(s, B^t_{s\wedge\cdot})$, $s \in [0,T]$, $\overline{\mathbb{P}}$–a.s., and therefore $\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$. This implies that $\overline{\mathcal{P}}^\star_S(t,\nu) \subseteq \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$, and the equality (5.4.16). □

We are now ready to prove the measurability of $V^{\mathbb{B}}_S$.

Lemma 5.4.12. *The graph sets*

$$[\![\overline{\mathcal{P}}^\star_S]\!] := \big\{(t,\nu,\overline{\mathbb{P}}) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega}) : \overline{\mathbb{P}} \in \overline{\mathcal{P}}^\star_S(t,\nu)\big\}, \text{ and } [\![\widehat{\mathcal{P}}^\star_S]\!] := \big\{(t,\widehat{\nu},\overline{\mathbb{P}}) : \overline{\mathbb{P}} \in \widehat{\mathcal{P}}^\star_S(t,\widehat{\nu})\big\},$$

are analytic sets in respectively $[0,T] \times \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega})$ *and* $[0,T] \times \mathcal{P}_2(\widehat{\Omega}) \times \mathcal{P}(\overline{\Omega})$. *Consequently,* $V^{\mathbb{B}}_S : [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ *is upper semi–analytic.*

Proof. We will only consider the case of $\overline{\mathcal{P}}^\star_S$, while the proof is almost the same for $\widehat{\mathcal{P}}^\star_S$. First, notice that

$$\Upsilon : \mathcal{U} \longrightarrow \mathcal{P}(\mathcal{C}^\ell \times \mathcal{C}),$$

is continuous and injective, so that $\Upsilon\big(\mathcal{U}\big)$ is a Borel subset of $\mathcal{P}(\mathcal{C}^\ell \times \mathcal{C})$ (see e.g. Kechris [93, Theorem 15.1]). It follows that

$$\mathbb{D}^1 := \big\{(t,\nu,\overline{\mathbb{P}}) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega}) : \overline{\mathbb{P}} \circ \big(B_\cdot, A_\cdot\big)^{-1} \in \Upsilon\big(\mathcal{U}\big)\big\},$$

is a Borel subset of $[0,T] \times \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega})$, as the map

$$\Gamma_1 : [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega}) \ni (t,\nu,\overline{\mathbb{P}}) \longmapsto \overline{\mathbb{P}} \circ \big(B_\cdot, A_\cdot\big)^{-1} \in \mathcal{P}(\mathcal{C}^\ell \times \mathcal{C}),$$

is Borel measurable. Similarly

$$\mathbb{D}^2 := \big\{(t,\nu,\overline{\mathbb{P}}) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega}) : B_{t\wedge\cdot} \text{ is } \overline{\mathbb{P}}\text{–independent of } (B^t, A, \widehat{\mu})\big\},$$

is also a Borel subset of $[0,T] \times \mathcal{P}(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega})$. Indeed, for all $(h,\psi) \in C_b(\mathcal{C}^\ell) \times C_b\big(\mathcal{C}^\ell \times \mathcal{C}\big)$, the function

$$\Gamma_{h,\psi} : [0,T] \times \mathcal{P}_2(\mathcal{C}^n) \times \mathcal{P}(\overline{\Omega}) \ni (t,\nu,\overline{\mathbb{P}}) \longmapsto \big(\mathbb{E}^{\overline{\mathbb{P}}}[h(B_{t\wedge\cdot})\psi(B^t,A)] - \mathbb{E}^{\overline{\mathbb{P}}}[h(B_{t\wedge\cdot})]\mathbb{E}^{\overline{\mathbb{P}}}[\psi(B^t,A)]\big) \in \mathbb{R},$$

is continuous. By consider a countable dense subset $\mathcal{R} \subset C_b(\mathcal{C}^\ell) \times C_b\big(\mathcal{C}^\ell \times \mathcal{C}\big)$, it follows that

$$\mathbb{D}^2 = \bigcap_{(h,\psi)\in\mathcal{R}} \Gamma^{-1}_{\varphi,\psi}\{0\},$$

is a Borel set. Finally, notice that

$$[\![\mathcal{P}^\star_S]\!] = [\![\overline{\mathcal{P}}_W]\!] \cap \mathbb{D}^1 \cap \mathbb{D}^2,$$

and we then conclude the proof by Lemma 5.4.7 and Lemma 5.4.11. □

Recall that for each $M > 0$, $\mathcal{P}_t^M$ is defined in Section 5.4.2, we similarly introduce, for $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$ and $\widehat{\nu} \in \mathcal{P}_2(\widehat{\Omega})$ such that $\nu = \widehat{\nu} \circ \widehat{X}^{-1}$,

$$\overline{\mathcal{P}}_S^{\mathbb{B},M}(t,\nu) := \overline{\mathcal{P}}_S^{\mathbb{B}}(t,\nu) \cap \mathcal{P}_t^M,\ \overline{\mathcal{P}}_S^{\star,M}(t,\nu) := \overline{\mathcal{P}}_S^{\star} \cap \overline{\mathcal{P}}_S^{\mathbb{B},M}(t,\nu),\ \widehat{\mathcal{P}}_S^{\mathbb{B},M}(t,\widehat{\nu}) := \overline{\mathcal{P}}_S^{\mathbb{B}}(t,\widehat{\nu}) \cap \mathcal{P}_t^M,\ \widehat{\mathcal{P}}_S^{\star,M}(t,\widehat{\nu}) := \overline{\mathcal{P}}_S^{\star} \cap \widehat{\mathcal{P}}_W^{\mathbb{B},M}(t,\widehat{\nu}).$$

By Lemma 5.4.9, it is clear that

$$V_S^{\mathbb{B},M}(t,\nu) := \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S^{\mathbb{B},M}(t,\nu)} J(t,\overline{\mathbb{P}}) = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S^{\star,M}(t,\nu)} J(t,\overline{\mathbb{P}}) = \sup_{\overline{\mathbb{P}} \in \widehat{\mathcal{P}}_S^{\mathbb{B},M}(t,\widehat{\nu})} J(t,\overline{\mathbb{P}}) = \sup_{\overline{\mathbb{P}} \in \widehat{\mathcal{P}}_S^{\star,M}(t,\widehat{\nu})} J(t,\overline{\mathbb{P}}) \nearrow V_S^{\mathbb{B}}(t,\nu), \text{ as } M \nearrow \infty.$$

Lemma 5.4.13. *(i) Let $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$, $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S^{\mathbb{B}}(t,\nu)$, $\bar{\tau}$ a $\overline{\mathbb{G}}^t$–stopping time taking values in $[t,T]$, and $\big(\overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_{\bar{\tau}}^t}\big)_{\bar{\omega} \in \overline{\Omega}}$ be a family of* r.c.p.d. *of $\overline{\mathbb{P}}$ knowing $\overline{\mathcal{G}}_{\bar{\tau}}^t$. Then $\overline{\mathbb{P}}_{\bar{\omega}}^{\overline{\mathcal{G}}_{\bar{\tau}}^t} \in \overline{\mathcal{P}}_S^{\mathbb{B}}\big(\bar{\tau}(\bar{\omega}), \mu_{\bar{\tau}(\bar{\omega})}(\bar{\omega})\big)$, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$.*

(ii) The graph set $\big\{(t,\widehat{\nu},M,\overline{\mathbb{P}}) \ : \overline{\mathbb{P}} \in \widehat{\mathcal{P}}_S^{\star,M}(t,\widehat{\nu})\big\}$ is analytic. Further, let $(t,\nu) \in [0,T] \times \mathcal{P}_2(\mathcal{C}^n)$, $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S^{\mathbb{B}}(t,\nu)$, $\bar{\tau}$ be a $\overline{\mathbb{G}}^t$–stopping time taking values in $[t,T]$, and $\varepsilon > 0$. Then there exists a family of probability measures and a family of probability measures $(\bar{Q}^{\varepsilon}_{t,\widehat{\nu},M})_{(t,\widehat{\nu},M) \in [0,T] \times \mathcal{P}(\widehat{\Omega}) \times \mathbb{R}_+}$ such that $(t,\widehat{\nu},M) \longmapsto \bar{Q}^{\varepsilon}_{t,\widehat{\nu},M}$ is universally measurable, and for every $(t,\widehat{\nu},M)$ s.t. $\widehat{\mathcal{P}}_S^{\mathbb{B},M}(t,\widehat{\nu}) \neq \emptyset$, one has

$$\bar{Q}^{\varepsilon}_{t,\widehat{\nu},M} \in \widehat{\mathcal{P}}_S^{\mathbb{B},M}(t,\widehat{\nu}), \text{ and } J\big(t,\bar{Q}^{\varepsilon}_{t,\widehat{\nu},M}\big) \geq \begin{cases} V_S^{\mathbb{B},M}(t,\nu) - \varepsilon, & \text{when } V_S^{\mathbb{B},M}(t,\nu) < \infty, \\ \frac{1}{\varepsilon}, & \text{when; } V_S^{\mathbb{B},M}(t,\nu) = \infty, \end{cases} \quad \text{for } \nu = \widehat{\nu} \circ \widehat{X}^{-1}. \tag{5.4.17}$$

Moreover, there is a $\overline{\mathcal{G}}_{\bar{\tau}}^t$–measurable and $\overline{\mathbb{P}}$–integrable r.v. $\widehat{M} : \overline{\Omega} \to \mathbb{R}_+$ such that for all constant $M > 0$, there exists $\overline{\mathbb{P}}^{M,\varepsilon} \in \overline{\mathcal{P}}_S^{\mathbb{B}}(t,\nu)$ such that $\overline{\mathbb{P}}^{M,\varepsilon}|_{\overline{\mathcal{F}}_{\bar{\tau}}} = \overline{\mathbb{P}}|_{\overline{\mathcal{F}}_{\bar{\tau}}}$ and

$$\Big(\bar{Q}^{\varepsilon}_{\bar{\tau}(\bar{\omega}),\widehat{\mu}(\bar{\omega}),M+\widehat{M}(\bar{\omega})}\Big)_{\bar{\omega} \in \overline{\Omega}} \text{ is a version of the r.c.p.d. of } \overline{\mathbb{P}}^{\varepsilon,M} \text{ knowing } \overline{\mathcal{G}}_{\bar{\tau}}^t.$$

Proof. *(i)* Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S^{\mathbb{B}}(t,\nu)$, then there exists a Borel measurable function $\phi : [t,T] \times \mathcal{C}^{\ell} \longrightarrow U$ such that

$$\bar{\alpha}_s = \phi\big(s, B^t_{s\wedge\cdot}\big), \text{ for all } s \in [t,T], \ \overline{\mathbb{P}}\text{–a.s.}$$

Let us consider the concatenated path $(\bar{\omega} \otimes_t \bar{\mathsf{w}})_s := \bar{\omega}_{t\wedge s} + \bar{\mathsf{w}}_{s\vee t} - \bar{\mathsf{w}}_t$ and define a Borel measurable function $\phi^{\bar{\omega}}$ by

$$\phi^{\bar{\omega}}\big(s,\bar{\mathsf{w}}^b\big) := \phi\big(s, \bar{\omega}^b \otimes_{\bar{\tau}(\bar{\omega})} \bar{\mathsf{w}}^b\big), \text{ for } s \in [\bar{\tau}(\bar{\omega}),T], \ \bar{\omega} = (\bar{\omega}^x,\bar{\omega}^a,\bar{\omega}^w,\bar{\omega}^b,\bar{\omega}^{\widehat{\mu}}), \ \mathsf{w} = (\mathsf{w}^x,\mathsf{w}^a,\mathsf{w}^w,\mathsf{w}^b,\mathsf{w}^{\widehat{\mu}}) \in \overline{\Omega}.$$

Then by a classical conditioning argument, it is easy to check that for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$,

$$\bar{\alpha}_s = \phi^{\bar{\omega}}\big(s, B^{\bar{\tau}(\bar{\omega})}_{s\wedge\cdot}\big), \text{ for all } s \in [\bar{\tau}(\bar{\omega}),T], \ \overline{\mathbb{P}}^{\overline{\mathcal{G}}_{\bar{\tau}}^t}_{\bar{\omega}}\text{–a.s.}$$

Using Lemma 5.4.8 and Definition 5.4.3, it follows that $\overline{\mathbb{P}}^{\overline{\mathcal{G}}_{\bar{\tau}}^t}_{\bar{\omega}} \in \overline{\mathcal{P}}_S^{\mathbb{B}}\big(\bar{\tau}(\bar{\omega}), \mu_{\bar{\tau}(\bar{\omega})}(\bar{\omega})\big)$, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$.

(ii) Using Lemma 5.4.12, it is easy to see that the graph set $\big\{(t,\nu,M,\overline{\mathbb{P}}) \ : \overline{\mathbb{P}} \in \overline{\mathcal{P}}_S^{\star,M}(t,\nu)\big\}$ is analytic. Then one can apply the same arguments as in Lemma 5.4.10 to obtain a measurable family $(\bar{Q}^{\varepsilon}_{t,\nu,M})_{(t,\nu,M) \in [0,T] \times \mathcal{P}(\mathcal{C}^n) \times \mathbb{R}_+}$ such that

$$\bar{Q}^{\varepsilon}_{t,\nu,M} \in \overline{\mathcal{P}}_S^{\star,M}(t,\nu) \text{ and } J(t,\bar{Q}^{\varepsilon}_{t,\nu,M}) \geq (V_S^{\mathbb{B},M}(t,\nu) - \varepsilon)\mathbf{1}_{\{V_S^{\mathbb{B},M} < \infty\}} + \frac{1}{\varepsilon}\mathbf{1}_{\{V_S^{\mathbb{B},M} = \infty\}}.$$

To proceed, we will define a family $(\bar{Q}^{\varepsilon}_{t,\widehat{\nu},M})_{(t,\widehat{\nu},M) \in [0,T] \times \mathcal{P}(\widehat{\Omega}) \times \mathbb{R}_+}$ from the family $(\bar{Q}^{\varepsilon}_{t,\nu,M})_{(t,\nu,M) \in [0,T] \times \mathcal{P}(\mathcal{C}^n) \times \mathbb{R}_+}$ as follows. For all $(t,\widehat{\nu}) \in [0,T] \times \mathcal{P}(\widehat{\Omega})$, let $\nu := \widehat{\nu} \circ \widehat{X}^{-1}$. Then on the probability space $(\overline{\Omega}, \overline{\mathcal{F}}_T, \bar{Q}^{\varepsilon}_{t,\nu,M})$, we consider a $\overline{\mathcal{F}}_t$–measurable random element $(A'_s, W'_s, B'_s)_{s \in [0,t]}$ such that

$$\bar{Q}^{\varepsilon}_{t,\nu,M} \circ (X_{t\wedge\cdot}, A'_{t\wedge\cdot}, W'_{t\wedge\cdot}, B'_{t\wedge\cdot})^{-1} = \widehat{\nu}(t).$$

Define

$$A'_s := A'_t + A_s - A_t,\ W'_s := W'_t + W^t_s,\ B'_s := B'_t + B^t_s,\ \text{for } s \in [t,T],$$
$$\widehat{\mu}'_s := \mathcal{L}^{\bar{Q}^\varepsilon_{t,\nu,M}}(X_{s\wedge}, A'_{s\wedge}, W', B'_{s\wedge})\mathbf{1}_{\{s\in[0,t]\}} + \mathcal{L}^{\bar{Q}^\varepsilon_{t,\nu,M}}(X_{s\wedge}, A'_{s\wedge}, W', B'_{s\wedge}|\overline{\mathcal{G}}^t_T)\mathbf{1}_{\{s\in(t,T]\}}.$$

Let

$$\bar{Q}^\varepsilon_{t,\widehat{\nu},M} := \bar{Q}^\varepsilon_{t,\nu,M} \circ \big(X, A', W', B', \widehat{\mu}'\big),\ \text{so that } J(t,\bar{Q}^\varepsilon_{t,\widehat{\nu},M}) = J(t,\bar{Q}^\varepsilon_{t,\nu,M}) \text{ and hence satisfies (5.4.17)}.$$

Let $\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$, as in Lemma 5.4.10, for $M>0$, we let

$$\widehat{M}(\bar{\omega}) := \mathbb{E}\bigg[\|X\|^p + \int_{\bar{\tau}}^T \big(\rho(\overline{\alpha}_s, u_0)\big)^p \mathrm{d}s \bigg| \overline{\mathcal{G}}^t_{\bar{\tau}}\bigg](\bar{\omega}),\ \text{and } \bar{Q}^\varepsilon_{\bar{\omega}} := \bar{Q}^\varepsilon_{\bar{\tau}(\bar{\omega}),\widehat{\mu}_{\bar{\tau}(\bar{\omega})}(\bar{\omega}),\widehat{M}(\bar{\omega})+M}.$$

Again, as in the proof of Lemma 5.4.10, one has $\bar{Q}^\varepsilon_{\bar{\omega}}$ satisfies (5.4.12) and (5.4.13), which allows defining $\overline{\mathbb{P}}^{M,\varepsilon}$ by

$$\overline{\mathbb{P}}^{M,\varepsilon}[K] \in \int_{\overline{\Omega}} \bar{Q}_{\bar{\omega}}[K]\overline{\mathbb{P}}(d\bar{\omega}),\ \text{for all } K \in \overline{\mathcal{F}},$$

so that $\overline{\mathbb{P}}^{M,\varepsilon} \in \overline{\mathcal{P}}_W(t,\nu)$, $\overline{\mathbb{P}}^{M,\varepsilon} = \overline{\mathbb{P}}$ on $\overline{\mathcal{F}}_{\bar{\tau}}$ and $(\bar{Q}^\varepsilon_{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ is a family of r.c.p.d. of $\overline{\mathbb{P}}^{M,\varepsilon}$ knowing $\overline{\mathcal{G}}^t_{\bar{\tau}}$.

Finally, it is enough to prove that $\overline{\mathbb{P}}^{M,\varepsilon} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$. Let $s\in[t,T]$, $h\in C_b(\mathbb{R})$ and $\psi \in C_b(\mathcal{C}^\ell)$, then

$$\begin{aligned}
\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s h(A_s)\psi(B^t_{s\wedge\cdot})\mathbf{1}_{s>\bar{\tau}}\big] &= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\mathbb{E}^{\bar{Q}^\varepsilon_\cdot}\big[A_s h(A_s)\psi(B^t_{s\wedge\cdot})\big]\mathbf{1}_{s>\bar{\tau}}\big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\Big[\mathbb{E}^{\bar{Q}^\varepsilon_\cdot}\big[\mathbb{E}^{\bar{Q}^\varepsilon_\cdot}\big[A_s\big|B^t_{s\wedge\cdot}\big]h(A_s)\psi(B^t_{s\wedge\cdot})\big]\mathbf{1}_{s>\bar{\tau}}\Big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\Big[\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|\overline{\mathcal{G}}^t_{\bar{\tau}} \vee \sigma(B^t_{s\wedge\cdot})\big]h(A_s)\psi(B^t_{s\wedge\cdot})\big|\overline{\mathcal{G}}^t_{\bar{\tau}}\big]\mathbf{1}_{s>\bar{\tau}}\Big]\\
&= \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|B^t_{s\wedge\cdot}\big]h(A_s)\psi(B^t_{s\wedge\cdot})\mathbf{1}_{s>\bar{\tau}}\big],
\end{aligned}$$

where the second equality follows by the fact that $\bar{Q}^\varepsilon_{\bar{\omega}} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(\bar{\tau}(\bar{\omega}),\nu')$ for some $\nu' \in \mathcal{P}(\mathcal{C}^n)$ and hence $h(A_s) = h(\phi(B^{\bar{\tau}(\bar{\omega})}_{s\wedge\cdot}))$, $\bar{Q}^\varepsilon_{\bar{\omega}}$–a.s., for some Borel measurable function ϕ, and the last quality follows by the fact that on $\{s>\bar{\tau}\}$, $\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|\overline{\mathcal{G}}^t_{\bar{\tau}} \vee \sigma(B^t_{s\wedge\cdot})\big] = \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|B^t_{s\wedge\cdot}\big]$. Further, as $\overline{\mathbb{P}}^{M,\varepsilon}|_{\overline{\mathcal{F}}_{\bar{\tau}}} = \overline{\mathbb{P}}|_{\overline{\mathcal{F}}_{\bar{\tau}}}$ and $\overline{\mathbb{P}} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$, one can use similarly argument to find that

$$\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s h(A_s)\psi(B^t_{s\wedge\cdot})\mathbf{1}_{s\le\bar{\tau}}\big] = \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|B^t_{s\wedge\cdot}\big]h(A_s)\psi(B^t_{s\wedge\cdot})\mathbf{1}_{s\le\bar{\tau}}\big].$$

This implies that

$$\mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\Big[\Big(A_s - \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|B^t_{s\wedge\cdot}\big]\Big)h(A_s)\psi(B^t_{s\wedge\cdot})\Big] = 0,\ \text{and hence } A_s = \mathbb{E}^{\overline{\mathbb{P}}^{M,\varepsilon}}\big[A_s\big|B^t_{s\wedge\cdot}\big],\ \overline{\mathbb{P}}^{M,\varepsilon}\text{–a.s.}$$

In other words, A is a continuous process, adapted to the $\overline{\mathbb{P}}^{M,\varepsilon}$–augmented filtration generated by B^t, then there exists a Borel measurable function $\hat{\phi} : [t,T]\times\mathcal{C}^\ell \longrightarrow U$ such that $A_s = \hat{\phi}(s, B^t_{s\wedge\cdot})$, for all $s\in[t,T]$, $\overline{\mathbb{P}}^{M,\varepsilon}$–a.s., and hence $\overline{\mathbb{P}}^{M,\varepsilon} \in \overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)$, which concludes the proof. □

5.4.3.3.2 Proof of Theorem 5.3.2 The proof is almost the same as that of Theorem 5.3.1. First, one has the measurability of $V^{\mathbb{B}}_S$ by Lemma 5.4.12. Next, notice that a $\mathbb{G}^{t,\circ}$–stopping time τ on Ω^t can be considered as a special $\overline{\mathbb{G}}^t$–stopping time $\bar{\tau}$ on $\overline{\Omega}$. then using the conditioning argument in Lemma 5.4.13, it follows that

$$V^{\mathbb{B}}_S(t,\nu) \le \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)} \mathbb{E}\bigg[\int_t^\tau L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + V^{\mathbb{B}}_S\big(\tau,\mu_\tau\big)\bigg].$$

Finally, it is enough to use the concatenation argument in Lemma 5.4.13 and sending $M\to\infty$ to obtain the reverse inequality

$$V^{\mathbb{B}}_S(t,\nu) \ge \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^{\mathbb{B}}_S(t,\nu)} \mathbb{E}^{\overline{\mathbb{P}}}\bigg[\int_t^\tau L(s, X_{s\wedge\cdot}, \overline{\mu}_s, \overline{\alpha}_s)\mathrm{d}s + V^{\mathbb{B}}_S\big(\tau,\mu_\tau\big)\bigg].$$

□

5.5 Appendix: some technical results on controlled McKean–Vlasov SDEs

Let us first recall a technical optional projection result.

Lemma 5.5.1. *Let E be a Polish space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, equipped with a complete filtration $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$.*

(i) Given an E–valued measurable process $(X_t)_{t\in[0,T]}$, there exists a $\mathcal{P}(E)$–valued $\mathbb{G}$–optional process β such that

$$\beta_\tau = \mathcal{L}^{\mathbb{P}}\big(X_\tau \big| \mathcal{G}_\tau\big), \ \mathbb{P}\text{–a.s., for all } \mathbb{G}\text{–stopping times } \tau.$$

(ii) Assume in addition that X is a continuous process, and that the $\mathbb{G}$–optional σ–field is identical to the $\mathbb{G}$–predictable σ–field. Then one can choose β to be an a.s. continuous process.

Proof. *(i)* The existence of such process β is ensured by, e.g. Kurtz [100, Theorem A.3] or Yor [153, Proposition 1].

(ii) When X is a continuous process, it follows again by [100, Theorem A.3] (or [153, Proposition 1]) that β is càdlàg $\mathbb{P}$–a.s. Further, let $\varphi \in C_b(E)$ and $(\tau_n)_{n\geq 1}$ be a increasing sequence of uniformly bounded $\mathbb{G}$–stopping times[2]. One has $\langle \varphi, \beta_{\tau_n}\rangle = \mathbb{E}^{\mathbb{P}}[\varphi(X_{\tau_n})|\mathcal{G}_{\tau_n}]$, $\mathbb{P}$–a.s., and hence $\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}}[\langle \varphi, \beta_{\tau_n}\rangle] = \mathbb{E}^{\mathbb{P}}[\langle \varphi, \beta_{\lim_n \tau_n}\rangle]$. Then it follows by Dellacherie [55, Theorem IV–T24] that $(\langle \varphi, \beta_t\rangle)_{t\in[0,T]}$ is left–continuous, $\mathbb{P}$–a.s. By considering a countable dense family of functions φ in $C_b(E)$, one concludes that β is also left–continuous a.s. □

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ a complete filtration, supporting two independent $\mathbb{F}$–Brownian motions $B^\star$ and $W^\star$, which are respectively $\mathbb{R}^d$– and $\mathbb{R}^\ell$–valued. Let us fix a $\mathbb{R}^n$–valued, $\mathbb{F}$–adapted continuous process $(\xi_s)_{s\geq 0}$, a U–valued $\mathbb{F}$–predictable process $(\alpha_s)_{s\geq 0}$, a complete sub–filtration $\mathbb{G} = (\mathcal{G}_s)_{s\geq 0}$ of $\mathbb{F}$, and denote $W^{*,t}_s := W^\star_{s\vee t} - W^\star_t$ and $B^{*,t}_s := B^\star_{s\vee t} - B^\star_t$. We will study the following SDE with data $(t, \xi, \alpha, \mathbb{G})$: $X_s = \xi_s$ for all $s \in [0, t]$, and with $\overline{\mu}_r := \mathcal{L}^{\mathbb{P}}(X_{r\wedge\cdot}, \alpha_r|\mathcal{G}_r)$,

$$X_s = \xi_t + \int_t^s b\big(r, X_{r\wedge\cdot}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}r + \int_t^s \sigma\big(r, X_{r\wedge\cdot}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}W^\star_r + \int_t^s \sigma_0\big(r, X_{r\wedge\cdot}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}B^\star_r, \text{ for all } s \geq t, \ \mathbb{P}\text{–a.s.} \quad (5.5.1)$$

Definition 5.5.2. *A strong solution of* SDE (5.5.1)*, with data $(t, \xi, \alpha, \mathbb{G})$, on $[0, T]$, is an $\mathbb{R}^n$–valued $\mathbb{F}$–adapted continuous process $X = (X_t)_{t\geq 0}$ such that $\mathbb{E}\big[\sup_{s\in[0,T]} |X_s|^2\big] < \infty$ and (5.5.1) holds true.*

Theorem 5.5.3. *Let $0 \leq t \leq T$,* Assumption 5.2.8 *hold true, $\mathbb{E}\big[\sup_{s\in[0,t]} |\xi_s|^p\big] < \infty$ and $\mathbb{E}\big[\int_t^T \rho(u_0, \alpha_s)^p \mathrm{d}s\big] < \infty$ for some $p \geq 2$. Then*

(i) there exists a unique strong solution $X^{t,\xi,\alpha}$ of (5.5.1) on $[0, T]$ with data $(t, \xi, \alpha, \mathbb{G})$. Moreover, it holds that $\mathbb{E}\big[\sup_{s\in[0,T]} \big|X^{t,\xi,\alpha}_s\big|^p\big] < \infty$;

(ii) assume in addition that $(\xi_{t\wedge\cdot}, W^\star, B^\star_{t\wedge\cdot})$ is independent of $\mathbb{G}$, and B^t is $\mathbb{G}$–adapted, and there exists a Borel measurable function $\phi : [0, T] \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \longrightarrow U$ such that

$$\alpha_s = \phi\big(s, \xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{\star,t}_{s\wedge\cdot}\big), \ \mathbb{P}\text{–a.s., for all } s \in [0, T].$$

Then, with $A_s := \int_t^{s\vee t} \pi(\alpha_r)\mathrm{d}r$, there exists a continuous process $(\widehat{\mu}_t)_{t\in[0,T]}$ such that for all $s \in [0, T]$

$$\widehat{\mu}_s = \mathcal{L}^{\mathbb{P}}\big((X^{t,\xi,\alpha}_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot})\big|\mathcal{G}_T\big) = \mathcal{L}^{\mathbb{P}}\big((X^{t,\xi,\alpha}_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot})\big|\mathcal{G}_s\big) = \mathcal{L}^{\mathbb{P}}\big((X^{t,\xi,\alpha}_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot})\big|B^{*,t}_{s\wedge\cdot}\big), \ \mathbb{P}\text{–a.s.}$$

Proof. *(i)* We follow [150, Theorem 5.1.1] to prove the existence and uniqueness of a strong solution to (5.5.1). Let $\mathcal{S}^p$ be defined by

$$\mathcal{S}^p := \big\{Y := (Y_s)_{s\in[0,T]} : \mathbb{R}^n\text{–valued and } \mathbb{F}\text{–adapted and continuous process such that } \mathbb{E}^{\mathbb{P}}\big[\|Y\|^p_T\big] < \infty\big\},$$

[2] Which is $\mathbb{G}$–predictable time as soon as the $\mathbb{G}$–optional σ–field is identical to the $\mathbb{G}$–predictable σ–field.

where $\|\mathbf{x}\|_s := \sup_{r\in[0,s]}|\mathbf{x}_r|$ for $s\in[0,T]$ and $\mathbf{x}\in\mathcal{C}^n$. For all $Y\in\mathcal{S}^p$, we define, with $\overline{\mu}_s := \mathcal{L}^{\mathbb{P}}\big(Y_{s\wedge\cdot},\alpha_s\big|\mathcal{G}_T\big)$, $\Psi(Y) := (\Psi(Y)_s)_{0\le s\le T}$ by

$$\Psi(Y)_s := \xi_{t\wedge s} + \int_t^{t\vee s} b\big(r,Y_{r\wedge\cdot},\overline{\mu}_r,\alpha_r\big)\mathrm{d}r + \int_t^{t\vee s}\sigma\big(r,Y_{r\wedge\cdot},\overline{\mu}_r,\alpha_r\big)\mathrm{d}W_r^\star + \int_t^{t\vee s}\sigma_0\big(r,Y_{r\wedge\cdot},\overline{\mu}_r,\alpha_r\big)\mathrm{d}B_r^\star.$$

Then, for $(Y^1,Y^2)\in\mathcal{S}^p\times\mathcal{S}^p$, with $\overline{\mu}^i_s := \mathcal{L}^{\mathbb{P}}\big(Y^i_{s\wedge\cdot},\alpha_s\big|\mathcal{G}_T\big)$, $i\in\{1,2\}$, one has

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\big\|\Psi(Y^1)-\Psi(Y^2)\big\|_s^p\Big] &\le 3^p\mathbb{E}^{\mathbb{P}}\Big[\sup_{v\in[t,s]}\Big|\int_t^v\big(\sigma\big(r,Y^{\mathbf{1}}_{r\wedge\cdot},\overline{\mu}^{\mathbf{1}}_r,\alpha_r\big)-\sigma\big(r,Y^2_{r\wedge\cdot},\overline{\mu}^2_r,\alpha_r\big)\big)\mathrm{d}W_r^\star\Big|^p\Big]\\ &\quad+3^p\mathbb{E}^{\mathbb{P}}\Big[\sup_{v\in[t,s]}\Big|\int_t^v\big(\sigma_0\big(r,Y^{\mathbf{1}}_{r\wedge\cdot},\overline{\mu}^{\mathbf{1}}_r,\alpha_r\big)-\sigma_0\big(r,Y^2_{r\wedge\cdot},\overline{\mu}^2_r,\alpha_r\big)\big)\mathrm{d}B_r^\star\Big|^p\Big]\\ &\quad+3^p\mathbb{E}^{\mathbb{P}}\Big[\Big|\int_t^s\big|b\big(r,Y^{\mathbf{1}}_{r\wedge\cdot},\overline{\mu}^{\mathbf{1}}_r,\alpha_r\big)-b\big(r,Y^2_{r\wedge\cdot},\overline{\mu}^2_r,\alpha_r\big)\big|\mathrm{d}r\Big|^p\Big].\end{aligned}$$

Notice that, for all $r\in[t,T]$,

$$\mathcal{W}_2\big(\overline{\mu}^{\mathbf{1}}_r,\overline{\mu}^2_r\big)^p \le \mathcal{W}_p\big(\overline{\mu}^{\mathbf{1}}_r,\overline{\mu}^2_r\big)^p = \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}}((Y^{\mathbf{1}}_{r\wedge\cdot},\alpha_r)|\mathcal{G}_r),\mathcal{L}^{\mathbb{P}}((Y^2_{r\wedge\cdot},\alpha_r)|\mathcal{G}_r)\Big)^p \le \mathbb{E}^{\mathbb{P}}\Big[\|Y^{\mathbf{1}}_{r\wedge\cdot}-Y^2_{r\wedge\cdot}\|^p\Big|\mathcal{G}_r\Big].$$

Then by Burkholder–Davis–Gundy inequality, Jensen's inequality and Assumption 5.2.8, there is some constant $C_T>0$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\|\Psi(Y^1)-\Psi(Y^2)\|_s^p\Big] \le C_T\int_t^s\mathbb{E}^{\mathbb{P}}\Big[\|Y^{\mathbf{1}}_{r\wedge\cdot}-Y^2_{r\wedge\cdot}\|_r^p\Big]\mathrm{d}r. \tag{5.5.2}$$

Besides, by Assumption 5.2.8

$$\mathbb{E}\big[\|\Psi(0)\|^p\big] \le C\bigg(1+\mathbb{E}^{\mathbb{P}}\Big[\sup_{r\in[0,t]}|\xi_r|^p\Big]+\mathbb{E}^{\mathbb{P}}\Big[\int_t^T\rho(u_0,\alpha_r)^p\mathrm{d}r\Big]\bigg).$$

Then by taking $Y^2=0$, (5.5.2) implies that $\Psi(Y)\in\mathcal{S}^p$ whenever $Y\in\mathcal{S}^p$. Moreover, for any positive integer n

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big[\|\Psi^n(Y^1)-\Psi^n(Y^2)\|_s^p\Big] &\le C_T\int_t^s\mathbb{E}^{\mathbb{P}}\Big[\|\Psi^{n-1}(Y^1)-\Psi^{n-1}(Y^2)\|_r^p\Big]\mathrm{d}r\\ &\le (C_T)^2\int_t^s\int_t^r\mathbb{E}^{\mathbb{P}}\Big[\|\Psi^{n-2}(Y^1)-\Psi^{n-2}(Y^2)\|_v^p\Big]\mathrm{d}v\mathrm{d}r\\ &\le (C_T)^n\int\mathbf{1}_{\{s\ge v_1\ge v_2\ge\ldots\ge v_n\ge t\}}\mathbb{E}^{\mathbb{P}}\Big[\|Y^1-Y^2\|_{v_n}^p\Big]\mathrm{d}v_1\ldots\mathrm{d}v_n\\ &\le (C_T)^n\mathbb{E}^{\mathbb{P}}\Big[\|Y^1-Y^2\|_s^p\Big]\frac{(s-t)^n}{n!}.\end{aligned}$$

Let $Y\in\mathcal{S}^p$, $X^0:=Y$, and $X^n:=\Psi^n(Y)$, for $n\ge1$, it follows that

$$\mathbb{E}^{\mathbb{P}}\big[\|X^n-X^{n+1}\|_s^p\big] \le (C_T)^n\mathbb{E}^{\mathbb{P}}\Big[\|Y-\Psi(Y)\|_s^p\Big]\frac{(s-t)^n}{n!}, \text{ and hence } \mathbb{E}^{\mathbb{P}}\Big[\sum_{n\ge1}\|X^n-X^{n+1}\|_T^p\Big]<\infty,$$

which implies that the sequence $(X^n)_{n\ge1}$ converges uniformly, $\mathbb{P}$–a.s., to some $X\in\mathcal{S}^p$. Finally, it is straightforward to see that X is the unique strong solution of (5.5.1) with data $(t,\xi,\alpha,\mathbb{G})$.

(ii) Let $\nu := \mathbb{P}\circ(\xi_{t\wedge\cdot})^{-1}$. We recall that the canonical space $\Omega^t := \mathcal{C}^n_{0,t}\times\mathcal{C}^d_{t,T}\times\mathcal{C}^\ell_{t,T}$ was introduced in Section 5.2.2.2, with corresponding canonical processes $\zeta=(\zeta_s)_{0\le s\le t}$, $W=(W_s)_{t\le s\le T}$, and $B=(B_s)_{t\le s\le T}$. and filtration $\mathbb{F}^{t,\circ}$, $\mathbb{G}^{t,\circ}$. Moreover, under $\mathbb{P}^t_\nu$, $W^t_s := W_{s\vee t}-W_t$ and $B^t_s := B_{s\vee t}-B_t$ are standard Brownian motion on $[t,T]$ independent of ζ, and $\mathbb{F}^t$ and $\mathbb{G}^t$ are the $\mathbb{P}^t_\nu$–augmented filtration of $\mathbb{F}^{t,\circ}$ and $\mathbb{G}^{t,\circ}$.

Define $\widetilde{\alpha}_s := \phi(s, \zeta_{t\wedge\cdot}, W^t_{s\wedge\cdot}, B^t_{s\wedge\cdot})$, for all $s \in [t,T]$. There exists a unique solution Y^α of Equation (5.5.1) on Ω^t, associated with $(t, \zeta_{t\wedge\cdot}, \widetilde{\alpha}, \mathbb{G}^t)$. As Y^α is an $\mathbb{F}^t$–adapted continuous process, there exists an Borel measurable function $\Psi : [0,T] \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \to \mathbb{R}^n$ such that

$$Y^\alpha_s = \Psi_s(\zeta_{t\wedge\cdot}, W^t_{s\wedge\cdot}, B^t_{s\wedge\cdot}),\ s \in [0,T],\ \mathbb{P}^t_\nu\text{–a.s.}$$

Next, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us define

$$X^\alpha_s := \Psi_s(\xi_{t\wedge\cdot}, W^{*,t}_{s\wedge\cdot}, B^{*,t}_{s\wedge\cdot}).$$

Then it is clear that X^α is the unique solution, on $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$, of Equation (5.5.1), associated with $(t, \xi_{t\wedge\cdot}, \alpha, \mathbb{F}^{B^{\star,t}})$, where $\mathbb{F}^{B^{\star,t}} := (\mathcal{F}^{B^{\star,t}}_s)_{s\in[0,T]}$ is the $\mathbb{P}$–augmented filtration generated by $B^{\star,t}$. Moreover, as $(\xi_{t\wedge\cdot}, W^\star, B^\star_{t\wedge\cdot})$ is independent of $\mathbb{G}$, and $B^{\star,t}$ is $\mathbb{G}$–adapted, one has, for all $s \in [t,T]$,

$$\begin{aligned}
\overline{\mu}_s = \mathcal{L}^{\mathbb{P}}\big((X^\alpha_{s\wedge\cdot}, \alpha_s)\big|B^{\star,t}_{s\wedge\cdot}\big) &= \mathcal{L}^{\mathbb{P}}\big((\Psi_{s\wedge\cdot}(\xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{\star,t}_{s\wedge\cdot}), \phi(s, \xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{*,t}_{s\wedge\cdot}))\big|B^{*,t}_{s\wedge\cdot}\big)\\
&= \mathcal{L}^{\mathbb{P}}\big((\Psi_{s\wedge\cdot}(\xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{\star,t}_{s\wedge\cdot}), \phi(s, \xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{\star,t}_{s\wedge\cdot}))\big|\mathcal{G}_s\big)\\
&= \mathcal{L}^{\mathbb{P}}\big((X^\alpha_{s\wedge\cdot}, \alpha_s)\big|\mathcal{G}_s\big),\ \mathbb{P}\text{–a.s.}
\end{aligned}$$

This implies that X^α is also a solution of Equation (5.5.1) associated with $(t, \xi_{t\wedge\cdot}, \alpha, \mathbb{G})$, and hence $X^{t,\xi,\alpha} = X^\alpha$ by uniqueness of solution to Equation (5.5.1).

Further, as A_s is a Borel measurable function of $(\xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{\star,t}_{s\wedge\cdot})$, it follows by the same argument that, for all $s \in [0,T]$,

$$\widehat{\mu}_s := \mathcal{L}^{\mathbb{P}}\big(X^{t,\xi,\alpha}_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot}\big|\mathcal{G}_T\big) = \mathcal{L}^{\mathbb{P}}\big(X^{t,\xi,\alpha}_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot}\big|\mathcal{G}_s\big) = \mathcal{L}^{\mathbb{P}}\big(X^{t,\xi,\alpha}_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot}\big|B^{\star,t}_{s\wedge\cdot}\big),\ \mathbb{P}-\text{a.s.}$$

Finally, using Lemma 5.5.1, one can choose the process $\widehat{\mu}$ to be continuous. □

Let us consider the following system of SDE, where a solution is a couple of $\mathbb{F}$–adapted continuous processes $(X, \widehat{\mu})$ such that: for some $\widehat{\nu}_0 \in \mathcal{P}(\mathcal{C}^n \times \mathcal{C} \times \mathcal{C}^d \times \mathcal{C}^\ell)$, $\mathbb{E}^{\mathbb{P}}\big[\|X\|^2 + \mathcal{W}_2(\widehat{\mu}, \widehat{\nu}_0)^2\big] < \infty$, $X_s = \xi_s$ for $s \in [0,t]$, and

$$X_s = \xi_t + \int_t^s b\big(r, X_{r\wedge\cdot}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}r + \int_t^s \sigma\big(r, X_{r\wedge\cdot}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}W^\star_r + \int_t^s \sigma_0\big(r, X_{r\wedge\cdot}, \overline{\mu}_r, \alpha_r\big)\mathrm{d}B^\star_r,\ s \in [t,T],\ \mathbb{P}\text{–a.s.}, \quad (5.5.3)$$

and $\overline{\mu}_r := \mathcal{L}^{\mathbb{P}}(X_{r\wedge\cdot}, \alpha_r | B^{*,t}_{r\wedge\cdot}, \widehat{\mu}_{r\wedge\cdot})$ for all $r \geq t$, and with $A_s := \int_t^{s\wedge t} \pi(\alpha_r)\mathrm{d}r$, $\widehat{\mu}_s = \mathcal{L}^{\mathbb{P}}\big(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot}\big|B^{t,*}_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot}\big)$ for all $s \geq 0$, and finally $(\xi_{t\wedge\cdot}, W^\star, B^\star_{t\wedge\cdot})$ is independent of $(B^{*,t}, \widehat{\mu})$.

Corollary 5.5.4. *Let* Assumption 5.2.8 *hold true, and assume that there exists a Borel measurable function* $\phi : [0,T] \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{C}^\ell \longrightarrow U$ *such that*

$$\alpha_s = \phi\big(s, \xi_{t\wedge\cdot}, W^{\star,t}_{s\wedge\cdot}, B^{\star,t}_{s\wedge\cdot}\big),\ \text{for } \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{ –a.e. } (s,\omega) \in [t,T]\times\Omega,\ \textit{and } \mathbb{E}\bigg[\int_t^T \rho(u_0, \alpha_s)^2\mathrm{d}s\bigg] < \infty.$$

Then, Equation (5.5.3) *has a unique solution* $(X, \widehat{\mu})$*, where* X *is the strong solution of* Equation (5.5.1) *with data* $(t, \xi, \alpha, \mathbb{F}^{B^{\star,t}})$*, with* $\mathbb{F}^{B^{\star,t}}$ *being the* $\mathbb{P}$*–augmented filtration generated by* $B^{\star,t}$ *and*

$$\widehat{\mu}_s = \mathcal{L}^{\mathbb{P}}\big(X_{s\wedge\cdot}, A_{s\wedge\cdot}, W^\star, B^\star_{s\wedge\cdot}\big|B^{\star,t}_{s\wedge\cdot}\big),\ s \in [t,T],\ \mathbb{P}\text{–a.s.}$$

Proof. Given a solution $(X, \widehat{\mu})$ to Equation (5.5.3), we notice that X is a strong solution of Equation (5.5.1) associated with data $(t, \alpha, \xi, \mathbb{F}^{B^{\star,t},\widehat{\mu}})$, where $\mathbb{F}^{B^{\star,t},\widehat{\mu}} := (\mathcal{F}^{B^{\star,t},\widehat{\mu}}_s)_{s\in[0,T]}$ with $\mathcal{F}^{B^{\star,t},\widehat{\mu}}_s := \sigma(B^{\star,t}_{s\wedge\cdot}, \widehat{\mu}_{s\wedge\cdot})$. As $(\xi_{t\wedge\cdot}, W^\star, B^\star_{t\wedge\cdot})$ is independent of $(B^{\star,t}, \widehat{\mu})$, it is then enough to apply Theorem 5.5.3 to conclude that X is the strong solution of Equation (5.5.1) with data $(t, \xi, \alpha, \mathbb{F}^{B^{\star,t}})$. □

Part II

Mean Field Game of Controls with common noise and controlled volatility

The general assumptions used throughout this part are now formulated. The dimensions $(n,\ell) \in \mathbb{N}^* \times \mathbb{N}$, the nonempty Polish space (U,ρ) and the horizon time $T>0$ are fixed and $\mathcal{P}_U^n$ denote the space of all Borel probability measures on $\mathbb{R}^n \times U$ i.e. $\mathcal{P}_U^n := \mathcal{P}(\mathbb{R}^n \times U)$. We are given the following Borel measurable functions

$$\big[b,\sigma,L\big] : [0,T] \times \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n \times U \longrightarrow \mathbb{R}^n \times \mathbb{S}^{n\times n} \times \mathbb{R} \text{ and } g : \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \longrightarrow \mathbb{R}.$$

Assumption 5.5.5. *$[b,\sigma,L]$ are Borel measurable in all their variables, and non–anticipative in the sense that, for all $(t,x,u,\pi,m) \in [0,T] \times \mathbb{R}^n \times U \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n$*

$$\big[b,\sigma,L\big](t,x,u,\pi,m) = \big[b,\sigma,L\big](t,x,u,\pi_{t\wedge\cdot},m).$$

Moreover, there are positive constants C and p such that $p \geq 2$, and

(i) U is a compact nonempty polish set;
(ii) b and σ are bounded continuous functions, and $\sigma_0 \in \mathbb{S}^{n\times\ell}$ is a constant;
(iii) for all $(t,x,x',\pi,\pi',m,m',u) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n \times \mathcal{P}_U^n \times U$, one has

$$\big|[b,\sigma](t,x,\pi,m,u) - [b,\sigma](t,x',\pi',m',u)\big| \;\leq\; C\Big(|x-x'| + \sup_{s\in[0,T]} \mathcal{W}_p(\pi_s,\pi'_s) + \mathcal{W}_p(m,m')\Big);$$

(iv) Non–degeneracy condition: for some constant $\theta>0$, one has, for all $(t,x,\pi,m,u) \in [0,T] \times \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n \times U$,

$$\theta \mathrm{I}_n \leq \; \sigma\sigma^\top(t,x,\pi,m,u);$$

(v) the reward functions L and g are continuous, and for all $(t,x,\pi,m,u) \in [0,T] \times \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n \times U$, one has

$$\big|L(t,x,\pi,m,u)\big| + |g(x,\pi)| \leq C\bigg[1 + |x|^p + \sup_{s\in[0,T]} \mathcal{W}_p(\pi_s,\delta_0)^p + \int_{\mathbb{R}^n} |x'|^p m(\mathrm{d}x',U)\bigg];$$

(vi) Separability condition: There exist continuous functions $(b^\circ, b^\star, a^\circ, a^\star, L^\circ, L^\star)$ satisfying

$$[b,\sigma\sigma^\top](t,x,\pi,m,u) := [b^\star,a^\star](t,\pi,m) + [b^\circ,a^\circ](t,x,\pi,u) \;\text{ and }\; L(t,x,\pi,m,u) := L^\star(t,x,\pi,m) + L^\circ(t,x,\pi,u),$$

for all $(t,x,\pi,m,u) \in [0,T] \times \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathcal{P}_U^n \times U$.

Remark 5.5.6. *Most of these assumptions are classical in the study of mean field games and control problems (see* Lacker [103], Assumption 1.4.1 *and* Assumption 4.1.1 *). Only the "separability condition" and the "non–degeneracy condition" can be seen as non–standard. However, in the context of Mean field games of controls, these conditions are used by many authors, for instance* Cardaliaguet and Lehalle [37] *(only the separability condition),* Carmona and Lacker [45] *and* Laurière and Tangpi [113]. *These are essentially technical assumptions.*

Chapter 6

Convergence of Nash equilibria

6.1 Introduction

Since the pioneering work of Lasry and Lions [111] and Huang, Caines, and Malhamé [84], mean field games (MFG) have been the subject of intensive research in recent years. Due to the diversity of applications, particularly in models of oil production, volatility formation, population dynamics and economic growth (see Carmona and Delarue [43] for an overview), the study of MFG has attracted increasing interest in the field of applied mathematics.

The MFG can be seen as symmetric stochastic differential games with infinite many players. Indeed, a MFG solution can be used to construct approximate Nash equilibrium for the corresponding N–player games for large N, and, for each N–Nash equilibrium of the N–player games, this Nash equilibrium converges towards a solution of the MFG when N tends to infinity.

So far, this study has been conducted considering that the interactions between the players are realized only through the empirical distribution of the state processes, we refer to Lacker [103] for a general analysis of this case (see also Fisher [68]). The goal of this chapter is to give a general analysis of the case where the interactions is given through the empirical distribution of the state processes and controls.

Except the recent work of Laurière and Tangpi [113] which treats the convergence of Nash equilibria in the MFG of controls framework by probabilistic methods (via FBSDEs), to the best of our knowledge, there are no other papers using probabilistic or PDE methods that answer the question of the convergence of ϵ_N–Nash equilibria to the MFG solution in this context. Indeed, the techniques used so far to treat the question of study of the limit problem turn out to be too *rigid* to deal with the problem of the convergence of Nash equilibria, all the limits of approximate Nash equilibrium can not be described by the notion considered in the literature up to now. Although using probabilistic point of view, our approach is very different from these previously mentioned, and considers very general assumptions.

In order to solve the difficulty generated by the empirical distribution of controls, we introduce the notion of *measure-valued* MFG equilibrium. This notion is precisely defined in Section 6.3.2. The idea of our notion comes from the (stochastic) Fokker-Planck equation verified by the pair $(\mu^\star, \overline{\mu}^\star)$. This notion of MFG solution is very close to the classical notion, the main difference is that the optimization is taken over all solutions of specific Fokker-Planck equations and not to a solution of an SDE. This notion are already been considered in the literature by Cardaliaguet, Delarue, Lasry, and Lions [38] (Section 3.7.) and in some way by Lacker [105]. Borrowing techniques from [103], under suitable assumptions, we prove that the sequence of empirical measure flows $(\varphi^{N,\mathbf{X}}, \varphi^N)$ is tight in a suitable space, and with the help of techniques introduced in Chapter 4, we show that every limit in distribution is a *measure-valued* mean field equilibrium. And conversely, for each *measure–valued* mean field equilibrium, we construct an approximate Nash equilibrium which has this *measure–valued* mean field equilibrium as limit.

Consequently, there is a perfect symmetry between approximate Nash equilibrium and ϵ–strong MFG equilibrium, and our notion of *measure–valued* MFG equilibrium are the accumulating points of approximate Nash equilibrium and ϵ–strong MFG equilibrium. Therefore, if there exists a *measure-valued* MFG equilibrium or an approximated Nash equilibrium, there is necessarily an ϵ–strong MFG equilibrium with $\epsilon > 0$. The question of the existence of the measure–valued solution will be evoked in the next chapter.

It is worth emphasizing that our results allow to handle the case where σ is controlled i.e. the control α appears in the function σ. There are not many works that look at the situation where the volatility is controlled. Let us also mention, in

this chapter, despite general assumptions considered we are limited by some conditions that we must have for technical reasons, a separability condition on (b, σ, L) (see assumption 1.4.1) and a non–degeneracy volatility condition of type $\sigma\sigma^\top > 0$.

The rest of the chapter is organized as follows. We provide in Section 6.2 and Section 6.3 the definition of the N–player games and the corresponding MFG of controls, before stating in Section 6.4 the main limit Theorem 6.4.1, and its converse, Theorem 6.4.2. Most of the technical proofs are completed in Section 6.5.

As in Chapter 4, with a Polish space E, we denote by $\mathbb{M}(E)$ the space of all Borel measures $q(\mathrm{d}t, \mathrm{d}e)$ on $[0,T]\times E$, whose marginal distribution on $[0,T]$ is the Lebesgue measure $\mathrm{d}t$, that is to say $q(\mathrm{d}t, \mathrm{d}e) = q(t, \mathrm{d}e)\mathrm{d}t$ for a family $(q(t, \mathrm{d}e))_{t\in[0,T]}$ of Borel probability measures on E. Let Λ denote the canonical element on $\mathbb{M}(E)$, we define

$$\Lambda_{t\wedge\cdot}(\mathrm{d}s, \mathrm{d}e) := \Lambda(\mathrm{d}s, \mathrm{d}e)\big|_{[0,t]\times E} + \delta_{e_0}(\mathrm{d}e)\mathrm{d}s\big|_{(t,T]\times E}, \text{ for some fixed } e_0 \in E. \tag{6.1.1}$$

For $p \geq 1$, we use $\mathbb{M}_p(E)$ to designate the elements of $q \in \mathbb{M}(E)$ such that $q/T \in \mathcal{P}_p(E \times [0,T])$.

In this section, we first introduce an N–player game, and the definition of ϵ_N–Nash equilibria. Next, we formulate the notions of approximate strong and measure-valued MFG solutions which will be essential to describe the limit of the Nash equilibria.

6.2 The N–players games

For $(\nu^1, ..., \nu^N) \in \mathcal{P}_p(\mathbb{R}^n)^N$, let

$$\Omega^N := (\mathbb{R}^n)^N \times (\mathcal{C}^n)^N \times \mathcal{C}^\ell,$$

be the canonical space, with canonical variable $\mathbf{X}_0 = (\mathbf{X}_0^1, \dots, \mathbf{X}_0^N)$ and canonical processes $\mathbf{W} = (\mathbf{W}_s^1, \dots, \mathbf{W}_s^N)_{0\leq s\leq T}$ and $B = (B_s)_{0\leq s\leq T}$, and probability measure $\mathbb{P}_\nu^N$ under which $\mathbf{X}_0 \sim \nu_N := \nu^1 \otimes \cdots \otimes \nu^N$ and $(\mathbf{W}, B)$ are standard Brownian motion independent of $\mathbf{X}$. Let $\mathbb{F}^N = (\mathcal{F}_s^N)_{0\leq s\leq T}$ be defined by

$$\mathcal{F}_s^N := \sigma\big\{\mathbf{X}_0, \mathbf{W}_r, B_r,\ r \in [0,s]\big\},\ 0 \leq s \leq T.$$

Let us denote by $\mathcal{A}(\nu_N)$ the collection of all U–valued processes $\alpha = (\alpha_s)_{0\leq s\leq T}$ which are $\mathbb{F}^N$–predictable. Then given a control rule/strategy $\overline{\alpha} := (\alpha^1, \dots, \alpha^N) \in \mathcal{A}(\nu_N)^N$, denote by $\mathbf{X}_\cdot[\overline{\alpha}] := (\mathbf{X}_\cdot^1[\overline{\alpha}], \dots, \mathbf{X}_\cdot^N[\overline{\alpha}])$ the unique strong solution of the following system of SDEs: for each $i \in \{1, \dots, N\}$, $\mathbb{E}^{\mathbb{P}_\nu^N}[\|\mathbf{X}^i\|^p] < \infty$, for all $t \in [0,T]$,

$$\mathbf{X}_t^i[\overline{\alpha}] = \mathbf{X}_0^i + \int_0^t b\big(r, \mathbf{X}_r^i[\overline{\alpha}], \varphi^{N,\mathbf{X},\overline{\alpha}}, \varphi_r^{N,\overline{\alpha}}, \alpha_r^i\big)\mathrm{d}r + \int_0^t \sigma\big(r, \mathbf{X}_r^i[\overline{\alpha}], \varphi^{N,\mathbf{X},\overline{\alpha}}, \varphi_r^{N,\overline{\alpha}}, \alpha_r^i\big)\mathrm{d}\mathbf{W}_r^i + \sigma_0 B_t, \tag{6.2.1}$$

with

$$\varphi_r^{N,\overline{\alpha}}(\mathrm{d}x, \mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}_r^i[\overline{\alpha}],\, \alpha_r^i\right)}(\mathrm{d}x, \mathrm{d}u) \text{ and } \varphi_r^{N,\mathbf{X},\overline{\alpha}}(\mathrm{d}x) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\mathbf{X}_r^i[\overline{\alpha}]\right)}(\mathrm{d}x) \text{ for all } r \in [0,T].$$

The reward value of player i associated with control rule/strategy $\overline{\alpha} := (\alpha^1, \dots, \alpha^N)$ is then defined by

$$J_i[\overline{\alpha}] := \mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T L\big(t, \mathbf{X}_t^i[\overline{\alpha}], \varphi^{N,\mathbf{X},\overline{\alpha}}, \varphi_t^{N,\overline{\alpha}}, \alpha_t^i\big)dt + g\big(\mathbf{X}_T^i[\overline{\alpha}], \varphi^{N,\mathbf{X},\overline{\alpha}}\big)\right],$$

and for $\beta \in \mathcal{A}(\nu_N)$, one introduces the strategy $(\overline{\alpha}^{[-i]}, \beta) \in \mathcal{A}(\nu_N)^N$ by

$$(\overline{\alpha}^{[-i]}, \beta) := \big(\alpha^1, \dots, \alpha^{i-1}, \beta, \alpha^{i+1}, \dots, \alpha^N\big).$$

Definition 6.2.1. *For any $\epsilon := (\epsilon_1, \dots, \epsilon_N) \in (\mathbb{R}_+)^N$, $\overline{\alpha}$ is a ϵ–Nash equilibrium if*

$$J_i[\overline{\alpha}] \geq \sup_{\beta\in\mathcal{A}(\nu_N)} J_i\big((\overline{\alpha}^{[-i]}, \beta)\big) - \epsilon_i, \text{ for each } i \in \{1, \dots, N\}.$$

6.3 Mean field games of controls

6.3.1 ϵ–Strong mean field game equilibrium

On a fix probability space, we formulate the classical MFG problem with common noise including the (conditional) law of control.

For a fixed $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, let

$$\Omega \;:=\; \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^\ell$$

be the canonical space, with canonical variable ξ and canonical processes $W = (W_t)_{0\le t\le T}$ and $B = (B_t)_{0\le t\le T}$, and probability measure $\mathbb{P}_\nu$ under which $\xi \sim \nu$ and (W, B) are standard Brownian motion independent of ξ. Let $\mathbb{F} = (\mathcal{F}_s)_{0\le s\le T}$ and $\mathbb{G} = (\mathcal{G}_s)_{0\le s\le T}$ be defined by

$$\mathcal{F}_s := \sigma\big\{\xi, W_r, B_r,\ r \in [0,s]\big\} \qquad \text{and} \qquad \mathcal{G}_s := \sigma\big\{B_r,\ r \in [0,s]\big\}.$$

Let us denote by $\mathcal{A}(\nu)$ the collection of all U–valued $\mathbb{F}$–predictable processes. Then given $\alpha \in \mathcal{A}(\nu)$, let X^α be the unique strong solution of the SDE (e.g. Theorem 5.5.3): $\mathbb{E}^{\mathbb{P}_\nu}[\|X^\alpha\|^p] < \infty$, $X_0^\alpha = \xi$, and for $t \in [0,T]$,

$$X_t^\alpha = X_0^\alpha + \int_0^t b\big(r, X_r^\alpha, \mu_{r\wedge\cdot}, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^\alpha, \mu_{r\wedge\cdot}, \overline{\mu}_r^\alpha, \alpha_r\big)\mathrm{d}W_r + \sigma_0 B_t, \tag{6.3.1}$$

with $\overline{\mu}_r^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r^\alpha, \alpha_r \big| \mathcal{G}_r\big)$ for all $r \in [0,T]$, and also denote $\mu_r^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(X_r^\alpha \big| \mathcal{G}_r\big)$ with $r \in [0,T]$.

Given $\alpha \in \mathcal{A}(\nu)$, and X^α solution of (6.3.1), for every $\alpha' \in \mathcal{A}(\nu)$, let us introduce the unique strong solution $X^{\alpha,\alpha'}$ of: $\mathbb{E}^{\mathbb{P}_\nu}[\|X^{\alpha,\alpha'}\|^p] < \infty$, $X_0^{\alpha,\alpha'} = \xi$, and for $t \in [0,T]$,

$$X_t^{\alpha,\alpha'} = X_0^{\alpha,\alpha'} + \int_0^t b\big(r, X_r^{\alpha,\alpha'}, \mu_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r'\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^{\alpha,\alpha'}, \mu_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \alpha_r'\big)\mathrm{d}W_r + \sigma_0 B_t,\ \mathbb{P}_\nu\text{–a.e.}, \tag{6.3.2}$$

and the value function Ψ

$$\Psi(\alpha, \alpha') := \mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T L(t, X_t^{\alpha,\alpha'}, \mu_{t\wedge\cdot}^\alpha, \overline{\mu}_t^\alpha, \alpha_t')\mathrm{d}t + g(X_T^{\alpha,\alpha'}, \mu^\alpha)\bigg]. \tag{6.3.3}$$

Definition 6.3.1. *For any $\epsilon \in [0,\infty)$, we say α is an ϵ–strong MFG equilibrium, if*

$$\Psi(\alpha,\alpha) \ge \sup_{\alpha' \in \mathcal{A}(\nu)} \Psi(\alpha,\alpha') - \epsilon. \tag{6.3.4}$$

For all $\alpha \in \mathcal{A}(\nu)$, let us define

$$\mathrm{P}^\alpha := \mathbb{P}_\nu \circ \Big((\mu_t^\alpha)_{t\in[0,T]}, (\mu_t^\alpha)_{t\in[0,T]}, \delta_{(\overline{\mu}_r^\alpha)}(\mathrm{d}m)\mathrm{d}t, \delta_{(\overline{\mu}_r^\alpha)}(\mathrm{d}m')\mathrm{d}t, B\Big)^{-1}.$$

$\overline{\mathcal{P}}_S(\nu)$ and for each $\epsilon \in [0,\infty)$, $\overline{\mathcal{P}}_S^\star(\nu)[\epsilon]$ denote the subsets of $\mathcal{P}\big(\mathcal{C}_{\mathcal{W}}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M}(\mathcal{P}_U^n) \times \mathbb{M}(\mathcal{P}_U^n) \times \mathcal{C}^\ell\big)$ defines as follows

$$\overline{\mathcal{P}}_S(\nu) := \big\{\mathrm{P}^\alpha,\ \text{ with } \alpha \in \mathcal{A}(\nu)\big\} \quad \text{and} \quad \overline{\mathcal{P}}_S^\star(\nu)[\epsilon] := \big\{\mathrm{P}^\alpha,\ \text{ with } \alpha \text{ is an } \epsilon\text{–strong MFG equilibrium}\big\}.$$

In other words, $\overline{\mathcal{P}}_S(\nu)$ is the subset of all controlled McKean-Vlasov processes of type (6.3.1), and $\overline{\mathcal{P}}_S^\star(\nu)[\epsilon]$ consists of all ϵ–strong MFG equilibria. In what follows, the use of these forms of sets will become clearer.

6.3.2 Measure–valued MFG equilibrium

Inspired by the Fokker–planck equation satisfied by the couple $\big(\mathcal{L}^{\mathbb{P}_\nu}(X_s^{\alpha,\alpha'}|\mathcal{G}_s), \mathcal{L}^{\mathbb{P}_\nu}(X_s^{\alpha,\alpha'}, \alpha_s'|\mathcal{G}_s)\big)_{s\in[0,T]}$ (see Equation (6.3.2)) and the discussion in Chapter 4, we carefully formulate the notion of measure–valued control rules which is essential for the notion of measure–valued MFG equilibrium that will be introduced just after.

6.3.2.1 Measure–valued control rules

Denote by $\mathbb{M} := \mathbb{M}\big(\mathcal{P}_U^n\big)$ the collection of all finite (Borel) measures $q(\mathrm{d}t, \mathrm{d}e)$ on $[0,T]\times\mathcal{P}_U^n$, whose marginal distribution on $[0,T]$ is the Lebesgue measure $\mathrm{d}s$, i.e. $q(\mathrm{d}s, \mathrm{d}e) = q(s, \mathrm{d}e)\mathrm{d}s$ for a measurable family $(q(s, \mathrm{d}e))_{s\in[0,T]}$ of Borel probability measures on $\mathcal{P}_U^n$. Let Λ be the canonical element on $\mathbb{M}$. We then introduce a canonical filtration $\mathbb{F}^\Lambda = (\mathcal{F}_t^\Lambda)_{0\le t\le T}$ on $\mathbb{M}$ by

$$\mathcal{F}_t^\Lambda := \sigma\big\{\Lambda(C\times[0,s]) \ : \forall s\le t, C\in\mathcal{B}(\mathcal{P}_U^n)\big\}.$$

For each $q\in\mathbb{M}$, one has the disintegration property: $q(\mathrm{d}t,\mathrm{d}e) = q(t,\mathrm{d}e)\mathrm{d}t$, and there is a version of the disintegration such that $(t,q)\longmapsto q(t,\mathrm{d}e)$ is $\mathbb{F}^\Lambda$–predictable.

The canonical element on $\overline{\Omega} := \mathcal{C}_{\mathcal{W}}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathbb{M}\times\mathbb{M}\times\mathcal{C}^\ell$ is denoted by $(\mu,\zeta,\Lambda^\circ,\Lambda,B)$. Then, the canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\in[0,T]}$ is defined by: for all $t\in[0,T]$

$$\overline{\mathcal{F}}_t := \sigma\big\{\mu_{t\wedge\cdot},\zeta_{t\wedge\cdot},\Lambda^\circ_{t\wedge\cdot},\Lambda_{t\wedge\cdot},B_{t\wedge\cdot}\big\},$$

with $\Lambda^\circ_{t\wedge\cdot}$ and $\Lambda_{t\wedge\cdot}$ denote the restriction of Λ° and Λ on $[0,t]\times\mathcal{P}_U^n$ (see definition 6.1.1). Notice that we can choose a version of the disintegration $\Lambda(\mathrm{d}m,\mathrm{d}t) = \Lambda_t(\mathrm{d}m)\mathrm{d}t$ (resp $\Lambda^\circ(\mathrm{d}m,\mathrm{d}t) = \Lambda_t^\circ(\mathrm{d}m)\mathrm{d}t$) such that $(\Lambda_t)_{t\in[0,T]}$ (resp $(\Lambda_t^\circ)_{t\in[0,T]}$) a $\mathcal{P}(\mathcal{P}_U^n)$–valued $\overline{\mathbb{F}}$–predictable process. Let us also introduce the "*fix common noise*" filtration $(\overline{\mathcal{G}}_t)_{t\in[0,T]}$ by

$$\overline{\mathcal{G}}_t := \sigma\big\{\zeta_{t\wedge\cdot},\Lambda_{t\wedge\cdot},B_{t\wedge\cdot}\big\}.$$

We consider $\mathcal{L}$ the following generator: for $(t,x,\pi,m',u)\in[0,T]\times\mathbb{R}^n\times\mathcal{C}_{\mathcal{W}}^n\times\mathcal{P}_U^n\times U$, and $\varphi\in C^2(\mathbb{R}^n)$

$$\mathcal{L}_t\varphi(x,\pi,m',u) := \mathcal{L}_t^\circ\varphi(x,\pi,u) + \mathcal{L}_t^\star\varphi(x,\pi,m'), \tag{6.3.5}$$

where

$$\mathcal{L}_t^\circ\varphi(x,\pi,u) := \frac{1}{2}\mathrm{Tr}\big[a^\circ(t,x,\pi_{t\wedge\cdot},u)\nabla^2\varphi(x)\big] + b^\circ(t,x,\pi_{t\wedge\cdot},u)^\top\nabla\varphi(x), \tag{6.3.6}$$

and

$$\mathcal{L}_t^\star\varphi(x,\pi,m') := \frac{1}{2}\mathrm{Tr}\big[a^\star(t,\pi_{t\wedge\cdot},m')\nabla^2\varphi(x)\big] + b^\star(t,\pi_{t\wedge\cdot},m')^\top\nabla\varphi(x). \tag{6.3.7}$$

Also, for every $f\in C^2(\mathbb{R}^n)$, let us define $N_t(f) := N_t[\mu,\Lambda^\circ,\zeta,\Lambda](f)$ by

$$\begin{aligned} N_t[\mu,\Lambda^\circ,\zeta,\Lambda](f) := \langle f(\cdot-\sigma_0B_t),\mu_t\rangle - \langle f,\mu_0\rangle - \int_0^t\int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n}\mathcal{L}_r^\star[f(\cdot-\sigma_0B_r)](x,\zeta,m')\mu_r(\mathrm{d}x)\Lambda_r(\mathrm{d}m')\mathrm{d}r \\ - \int_0^t\int_{\mathcal{P}_U^n}\langle\mathcal{L}_r^\circ[f(\cdot-\sigma_0B_r)](\cdot,\zeta,\cdot),m\rangle\Lambda_r^\circ(\mathrm{d}m)\mathrm{d}r, \end{aligned} \tag{6.3.8}$$

and for each $\pi\in\mathcal{P}(\mathbb{R}^n)$, the Borel set $\mathbb{Z}_\pi$ by

$$\mathbb{Z}_\pi := \Big\{m\in\mathcal{P}_U^n : m(\mathrm{d}x,U) = \pi(\mathrm{d}x)\Big\}.$$

Definition 6.3.2 (measure–valued control rule)**.** *For every $\nu\in\mathcal{P}(\mathbb{R}^n)$, we say $\mathrm{P}\in\mathcal{P}(\overline{\Omega})$ is a measure–valued control rule if:*

- $\mathrm{P}\big(\mu_0=\nu\big) = 1$.
- $(B_t)_{t\in[0,T]}$ *is a* $(\mathrm{P},\overline{\mathbb{F}})$ *Wiener process starting at zero and for* P*–almost every* $\omega\in\overline{\Omega}$, $N_t(f) = 0$ *for all* $f\in C_b^2(\mathbb{R}^n)$ *and every* $t\in[0,T]$.

- *For all* $t \in [0,T]$,
$$\mathcal{L}^{\mathrm{P}}\big(\Lambda^{\circ}_{t\wedge\cdot}\big|\overline{\mathcal{G}}_t\big) = \mathcal{L}^{\mathrm{P}}\big(\Lambda^{\circ}_{t\wedge\cdot}\big|\overline{\mathcal{G}}_T\big),\ \mathrm{P}\text{ –a.e.} \tag{6.3.9}$$
- *For* $\mathrm{dP}\otimes\mathrm{d}t$ *almost every* $(t,\omega)\in[0,T]\times\overline{\Omega}$, $\Lambda^{\circ}_t\big(\mathbb{Z}_{\mu_t}\big)=1$.

We shall denote $\overline{\mathcal{P}}_V(\nu)$ the set of all measure–valued control rules with initial value ν.

Remark 6.3.3. *To do an analogy with* Section 6.3.1 *(the strong "point of view"), in order to give a better intuition of this definition, here,* μ *plays the role of* $(\mathcal{L}^{\mathbb{P}_\nu}(X_t^{\alpha,\alpha'}|\mathcal{G}_t))_{t\in[0,T]}$, Λ° *that of* $\delta_{\mathcal{L}^{\mathbb{P}_\nu}(X_s^{\alpha,\alpha'},\,\alpha'|\mathcal{G}_s)}(\mathrm{d}m)\mathrm{d}s$, ζ *and* Λ *represent the fixed measures* μ^α *and* $\delta_{\overline{\mu}_s^\alpha}(\mathrm{d}m')\mathrm{d}s$, *and* B *is the common noise.*

The next example shows that, because of condition (6.3.9), the set $\overline{\mathcal{P}}_V(\nu)$ cannot be closed in general. As $\overline{\mathcal{P}}_V(\nu)$ is not closed, the proofs become much more delicate (see for instance Proposition 6.5.8 and also Lemma 7.4.2).

Example 6.3.4. *Let us consider to simplify* $T=2$, $n=1$, $\ell=0$, $U=[0,1]$, $b(t,x,\pi,u)=u$, *and* $\sigma=1$. *Let* $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathbb{P}})$ *be a probability space supporting a* $[0,1]$*–uniform random variable* U, *and a* $\widehat{\mathbb{F}}$*–Brownian motion* W *independent of* U. *We consider for each integer* $k\geq 2$,
$$\alpha_t^k := U\mathbf{1}_{t\in[0,1)} + \mathbf{1}_{U\in A_k}\mathbf{1}_{t\in(1,2]} \text{ and } \beta_t^k := \mathbf{1}_{U\in A_k} \text{ for all } t\in[0,T] \text{ where } A_k := \cup_{j=0}^{k-1}\big[j/k,\ j/k+1/2k\big).$$
Let us define for any integer $k\geq 2$, *the processes* $X^k_\cdot = \int_0^\cdot \beta_s^k\mathrm{d}s + W_\cdot$, $\widehat{m}_t^k := \delta_{\alpha_t^k}(\mathrm{d}u)\pi_t(\mathrm{d}x)$ *with* $\pi\in\mathcal{C}^1_{\mathcal{W}}$ *fixed, also*
$$\mu_t^k := \widehat{\mathcal{L}^{\mathbb{P}}}(X_t^k|U),\ \Lambda_t^k(\mathrm{d}m)\mathrm{d}t := \delta_{\widehat{m}_t^k}(\mathrm{d}m)\mathrm{d}t,\ \text{and } \Lambda_t^{k,\circ}(\mathrm{d}m)\mathrm{d}t := \delta_{m_t^{k,\circ}}(\mathrm{d}m)\mathrm{d}t$$
where $m_t^{k,\circ} := \delta_{\alpha_t^k}(\mathrm{d}u)\mu_t^k(\mathrm{d}x)$. *It is straightforward to check that*
$$\widehat{\mathcal{L}^{\mathbb{P}}}\big(\mu^k,\pi,\Lambda^{k,\circ},\Lambda^k\big)\in\overline{\mathcal{P}}_V(\delta_0) \text{ for each } k\in\mathbb{N}^*,$$
and $\big(\widehat{\mathcal{L}^{\mathbb{P}}}\big(\mu^k,\pi,\Lambda^{k,\circ},\Lambda^k\big)\big)_{k\in\mathbb{N}^*}$ *is relatively compact for the weak topology. For* $\mathrm{P}^\infty\in\mathcal{P}(\overline{\Omega})$ *a limit of any sub-sequence, one notices that for all* $t\in[0,1/2]$
$$\begin{aligned}\mathbb{E}^{\mathrm{P}^\infty}\left[\int_0^t\langle\mathrm{Id},m(\mathrm{d}u,\mathbb{R})\rangle\Lambda_s^\circ(\mathrm{d}m)\mathrm{d}s\right] &= \mathbb{E}^{\mathrm{P}^\infty}\left[\int_0^t\langle\mathrm{Id},m(\mathrm{d}u,\mathbb{R})\rangle\Lambda_s^\circ(\mathrm{d}m)\mathrm{d}s\Big|\Lambda_{t\wedge\cdot}\right]\\ &\neq \mathbb{E}^{\mathrm{P}^\infty}\left[\int_0^t\langle\mathrm{Id},m(\mathrm{d}u,\mathbb{R})\rangle\Lambda_s^\circ(\mathrm{d}m)\mathrm{d}s\Big|\Lambda_{T\wedge\cdot}\right] = \int_0^t\langle\mathrm{Id},m(\mathrm{d}u,\mathbb{R})\rangle\Lambda_s^\circ(\mathrm{d}m)\mathrm{d}s,\ \mathrm{P}^\infty\text{–a.e.},\end{aligned}$$
therefore the condition (6.3.9) *is not verified, then* $\mathrm{P}^\infty\notin\overline{\mathcal{P}}_V(\delta_0)$.

Now, using the measure–valued control rules, we introduce the notion of (ϵ–) measure–valued MFG solution.

6.3.2.2 MFG solution

For all $(\pi,q^\circ,\eta,q)\in(\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M})^2$, one defines
$$J\big(\pi,q^\circ,\eta,q\big) := \int_0^T\left[\int_{\mathcal{P}^n_U}\langle L^\circ\big(t,\cdot,\eta,\cdot\big),m\rangle q_t^\circ(\mathrm{d}m) + \int_{\mathcal{P}^n_U}\langle L^\star\big(t,\cdot,\eta,m'\big),\pi_t\rangle q_t(\mathrm{d}m')\right]\mathrm{d}t + \langle g(\cdot,\eta),\pi_T\rangle.$$

Definition 6.3.5. *For all* $\nu\in\mathcal{P}(\mathbb{R}^n)$ *and* $\epsilon\in[0,\infty)$, $\mathrm{P}^\star$ *is an* ϵ*–measure–valued MFG solution if* $\mathrm{P}^\star\in\overline{\mathcal{P}}_V(\nu)$, *and for every* $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$ *such that* $\mathcal{L}^{\mathrm{P}^\star}\big(\zeta,\Lambda_t(\mathrm{d}m)\mathrm{d}t,B\big)=\mathcal{L}^{\mathrm{P}}\big(\zeta,\Lambda_t(\mathrm{d}m)\mathrm{d}t,B\big)$, *one has*
$$\mathbb{E}^{\mathrm{P}^\star}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]\geq\mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]-\epsilon, \tag{6.3.10}$$
and for $\mathrm{P}^\star$ *almost every* $\omega\in\overline{\Omega}$,
$$\Lambda_t^\circ(\mathrm{d}m)\mathrm{d}t=\Lambda_t\big(\mathrm{d}m'\big)\mathrm{d}t \quad\text{and}\quad \zeta=\mu. \tag{6.3.11}$$
When $\epsilon=0$, *we just say* $\mathrm{P}^\star$ *is a measure–valued MFG solution.*

For any $\nu \in \mathcal{P}(\mathbb{R}^n)$, $\overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$ is defined by

$$\overline{\mathcal{P}}^\star_V(\nu)[\epsilon] \ := \ \big\{\text{All } \epsilon\text{–measure–valued MFG solutions associated with the initial condition } \nu\big\},$$

again when $\epsilon = 0$, we shall denote $\overline{\mathcal{P}}^\star_V(\nu)[0]$ by $\overline{\mathcal{P}}^\star_V(\nu)$.

Remark 6.3.6. *Looking at this kind of measure-valued solution is largely inspired by the notion considered in* Chapter 4 *in the McKean-Vlasov setting. However, our notion of* (ϵ–) *measure-valued MFG solution enters completely in the framework of MFG solutions considered in* Carmona, Delarue, and Lacker [49]. *Notice in particular the presence of equality* (6.3.9), *which corresponds to the point* (4) *of* [49, Definition 3.1]. *Also called* (H)*–hypothesis, this means the fact that: at time* $t \in [0,T]$, *any additional randomization of the "control"* Λ°_t *must be conditionally independent of future information given current information at time* t. *Condition* (6.3.11) *is the analog of the well–known consistency property in the MFG framework. Without taking into account the law of control, one of the main differences of this notion of MFG solutions is the optimality conditions* (6.3.10) *and* (6.3.4). *Here, sometimes a small error* ϵ *is authorized. With this condition, the MFG solutions turn out to be more flexible* (*see the* Theorem 6.4.1 *and* Theorem 6.4.2).

Remark 6.3.7. *Notice that the previous definitions of the strong MFG equilibrium and* N*-players games cover the case without common noise. Indeed, for the non common noise case, it is enough to take* $\sigma_0 = 0$ *and* $\ell = 0$ (*see the previous chapters*). *When* $\sigma_0 = 0$ *and* $\ell \neq 0$, B *can be seen as an additional noise.*

The next proposition ensures that our measure–valued MFG solution definition using Fokker–Planck equation indeed generalizes the classical notion.

Proposition 6.3.8. *Let* $p' > p$ *and* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$. *Then for all* $\epsilon \in [0,\infty)$, $\overline{\mathcal{P}}^\star_S(\nu)[\epsilon] \subset \overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$.

Proof. Let α be an ϵ–strong MFG equilibrium, and its corresponding probability $\mathrm{P}^\alpha \in \overline{\mathcal{P}}^\star_S(\nu)[\epsilon]$. It is straightforward to check that $\mathrm{P}^\alpha \in \overline{\mathcal{P}}_V(\nu)$. Let $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$ such that $\mathcal{L}^{\mathrm{P}^\alpha}\big(\zeta, \Lambda_t(\mathrm{d}m)\mathrm{d}t, B\big) = \mathcal{L}^{\mathrm{P}}\big(\zeta, \Lambda_t(\mathrm{d}m)\mathrm{d}t, B\big)$.

By Lemma 6.5.2, there exists a sequence of Borel functions $(\gamma^k)_{k\in\mathbb{N}^*}$ satisfying for each $k \in \mathbb{N}^*$, $\gamma^k : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times[0,1] \to U$ s.t. if $\gamma^k_t(z) := \gamma^k(t,\xi,W_{t\wedge\cdot},B_{t\wedge\cdot},z)$, $\mathbb{P}_\nu$–a.e. for all $(t,z)\in[0,T]\times[0,1]$, one has $(\gamma_t(z))_{t\in[0,T]} \in \mathcal{A}(\nu)$ for each $z \in [0,1]$ and the value function $\Psi\big(\alpha,\gamma^k(z)\big)$ (see definition (6.3.3)) satisfies: $[0,1] \ni z \to \Psi\big(\alpha,\gamma^k(z)\big) \in \mathbb{R}$ is Borel and

$$\lim_{k\to\infty}\int_0^1 \Psi\big(\alpha,\gamma^k(z)\big)\mathrm{d}z = \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big].$$

Consequently,

$$\mathbb{E}^{\mathrm{P}^\alpha}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] = \Psi\big(\alpha,\alpha\big) \geq \lim_{k\to\infty}\int_0^1 \Psi\big(\alpha,\gamma^k(z)\big)\mathrm{d}z - \epsilon = \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] - \epsilon,$$

as obviously $\Lambda_t(\mathrm{d}m)\mathrm{d}t = \Lambda^\circ_t(\mathrm{d}m)\mathrm{d}t$ and $\mu = \zeta$, P^α–a.e., we can deduce that $\mathrm{P}^\alpha \in \overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$, and conclude the proof □

6.4 Limit theorems

The main results of this chapter are now given in the following two theorems.

Theorem 6.4.1 (Limit Theorem)**.** *Let* Assumption 5.5.5 *hold true,* $\epsilon \in [0,\infty)$, $(\epsilon_i)_{i\in\mathbb{N}^*} \subset (0,\infty)$, *and* $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$ *with* $p' > p$.

(*i*) *For each* $N \in \mathbb{N}^*$, *let* $\overline{\alpha}^N$ *be a* $(\epsilon_1,\dots,\epsilon_N)$*–Nash equilibrium, then the sequence* $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ *with* $\mathrm{P}^N := \mathrm{P}^N[\overline{\alpha}^N] \in \mathcal{P}\big(\overline{\Omega}\big)$ *is relatively compact in* $\mathcal{W}_p(\overline{\Omega})$ *where*

$$\mathrm{P}^N[\overline{\alpha}^N] := \mathbb{P}^N_\nu \circ \Big((\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t)_{t\in[0,T]}, (\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t)_{t\in[0,T]}, \delta_{(\varphi^{N,\overline{\alpha}^N}_s)}(\mathrm{d}m)\mathrm{d}s, \delta_{(\varphi^{N,\overline{\alpha}^N}_s)}(\mathrm{d}m')\mathrm{d}s, B\Big)^{-1},$$

and

$$\text{if } \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\epsilon_i=\epsilon, \text{ then each limit point } \mathrm{P}^\infty \text{ is an } \epsilon\text{–measure–valued MFG solution.}$$

(ii) Let $(\mathrm{P}^k)_{k\in\mathbb{N}^*}\subset\overline{\mathcal{P}}_S(\nu)$ *such that* $\mathrm{P}^k\in\overline{\mathcal{P}}_S(\nu)[\epsilon_k]$, *for each* $k\in\mathbb{N}^*$. *Then* $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ *is relatively compact in* $\mathcal{W}_p(\overline{\Omega})$, *and*

$$\text{if } \lim_{k\to\infty}\epsilon_k=\epsilon, \quad \text{then each limit point } \mathrm{P}^\infty \text{ is an } \epsilon\text{–measure–valued MFG solution.}$$

In particular when $\epsilon=0$, P^∞ *is a measure–valued MFG solution.*

Theorem 6.4.2 (Converse Limit Theorem)**.** *Let* Assumption 5.5.5 *hold true,* $\epsilon\in[0,\infty)$, $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ *with* $p'>p$, *and* $\mathrm{P}^\star\in\overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$.

(i) There exists a sequence $(\epsilon_k)_{k\in\mathbb{N}^*}\subset[0,\infty)$ *satisfying* $\limsup_{k\to\infty}\epsilon_k\in[0,\epsilon]$ *such that:*

(i.1) if $\ell\neq0$, *one can find a sequence* $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ *with* $\mathrm{P}^k\in\overline{\mathcal{P}}^\star_S(\nu)[\epsilon_k]$ *for each* $k\in\mathbb{N}^*$, *and* $\mathrm{P}^\star=\lim_{k\to\infty}\mathrm{P}^k$, *for the metric* $\mathcal{W}_p$.

(i.2) if $\ell=0$, *one can get a sequence* $(\mathrm{P}^k_z)_{(k,z)\in\mathbb{N}^*\times[0,1]}\subset\overline{\mathcal{P}}_S(\nu)$ *with for each* $k\in\mathbb{N}^*$, $z\mapsto\mathrm{P}^k_z$ *is Borel measurable and*

$$\int_0^1\mathrm{P}^k_z\mathrm{d}z\in\overline{\mathcal{P}}^\star_V(\nu)[\epsilon_k] \quad \text{and} \quad \lim_{k\to\infty}\int_0^1\mathrm{P}^k_z\mathrm{d}z=\mathrm{P}^\star, \text{ in } \mathcal{W}_p.$$

(ii) There exists a sequence of positive numbers $(\epsilon_i)_{i\in\mathbb{N}^*}$ *such that* $\limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\epsilon_i\in[0,\epsilon]$, *and for each* $N\in\mathbb{N}^*$, *a* $(\epsilon_1,\dots,\epsilon_N)$*–Nash equilibrium* $\overline{\alpha}^N=(\alpha^{1,N},\dots,\alpha^{N,N})$ *such that*

$$\mathrm{P}^\star=\lim_{N\to\infty}\mathbb{P}^N_\nu\circ\Big((\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t)_{t\in[0,T]},(\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t)_{t\in[0,T]},\delta_{(\varphi^{N,\overline{\alpha}^N}_s)}(\mathrm{d}m)\mathrm{d}s,\delta_{(\varphi^{N,\overline{\alpha}^N}_s)}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1}, \text{ for } \mathcal{W}_p.$$

Remark 6.4.3. Theorem 6.4.2 *and* Theorem 6.4.1 *give a general characterization of solutions of MFG of controls by connecting measure-valued MFG solutions, approximate Nash equilibria and approximate strong MFG solutions. In the presence of law of control or empirical distribution of controls, our limit theorem results seem to be the first which give this kind of characterizations under relative general assumptions. Especially, approximate strong MFG solutions and their convergence result have never been considered in the literature. Notice that they also contain part of the most results of the case without the distribution of controls mentioned in* Lacker [103]. *Let us emphasize there is no existence result in these theorems, all results are given after assuming existence results. In* Chapter 7 *(see below), we discuss some existence results.*

The next corollaries are just a combination of Theorems 6.4.2 and 6.4.1. The first mentions the closedness of $\overline{\mathcal{P}}^\star_V$ and the second a correspondence between approximate Nash equilibria and ϵ–strong MFG solution.

Corollary 6.4.4. *Suppose that the conditions of* Theorem 6.4.2 *and* Theorem 6.4.1 *hold. For each* $\epsilon\in[0,\infty)$, $\overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$ *is a closed set for the Wasserstein metric* $\mathcal{W}_p$.

Corollary 6.4.5. *Let us stay in the context of Theorems 6.4.2 and 6.4.1 with* $\ell\neq0$. *For any* $\overline{\alpha}^N$ *a* $(\epsilon_1,\dots,\epsilon_N)$*–Nash equilibrium, with* $\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N\epsilon_i=0$, *there exists, for each convergent sub-sequence* $(\mathrm{P}^{N_k}[\overline{\alpha}^{N_k}])_{k\in\mathbb{N}^*}$, *a sequence* $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ *such that:*

$$\text{for each } k\in\mathbb{N}^*,\ \mathrm{P}^k\in\overline{\mathcal{P}}^\star_S(\nu)[\delta_k] \text{ with } \lim_{k\to\infty}\delta_k=0, \quad \text{and} \quad \lim_{k\to\infty}\mathcal{W}_p\big(\mathrm{P}^{N_k}[\overline{\alpha}^{N_k}],\mathrm{P}^k\big)=0.$$

6.5 Proofs of limit theorems

6.5.1 Limit of Nash equilibria

In this section, we show some technical results needed to prove our first limit theorem result, namely Theorem 6.4.1. Before proceeding, let us give a reformulation of the measure-valued control rules which will be necessary for our proof. To make an analogy with the strong point of view, we want here to get a Fokker-Planck equation involving $\mathcal{L}^{\mathbb{P}_\nu}(X^\alpha - \sigma_0 B|\mathcal{G}_T)$ instead of $\mathcal{L}^{\mathbb{P}_\nu}(X^\alpha|\mathcal{G}_T)$. To do this, all coefficients must be shifted. Let us define, for all $(t,\mathbf{b},\pi,m) \in [0,T]\times\mathcal{C}^\ell\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{P}^n_U$,

$$\pi_t[\mathbf{b}](\mathrm{d}y) := \int_{\mathbb{R}^n} \delta_{\left(y'+\sigma_0\mathbf{b}_t\right)}(\mathrm{d}y)\pi_t(\mathrm{d}y'), \quad m[\mathbf{b}_t](\mathrm{d}u,\mathrm{d}y) := \int_{\mathbb{R}^n\times U} \delta_{(y'+\sigma_0\mathbf{b}_t)}(\mathrm{d}y)m(\mathrm{d}u,\mathrm{d}y'), \tag{6.5.1}$$

and any $q \in \mathbb{M}$,

$$q_t[\mathbf{b}](\mathrm{d}m)\mathrm{d}t := \int_{\mathcal{P}^n_U} \delta_{\left(m'[\mathbf{b}_t]\right)}(\mathrm{d}m)q_t(\mathrm{d}m')\mathrm{d}t. \tag{6.5.2}$$

In the same way, let us consider the "shift" generator $\widehat{\mathcal{L}}^\circ$,

$$\widehat{\mathcal{L}}^\circ_t[\varphi](y,\mathbf{b},\pi',u) := \frac{1}{2}\mathrm{Tr}\left[a^\circ(t,y+\sigma_0\mathbf{b}_t,\pi',u)\nabla^2\varphi(y)\right] + b^\circ(t,y+\sigma_0\mathbf{b}_t,\pi',u)^\top\nabla\varphi(y), \tag{6.5.3}$$

and also

$$[\hat{b}^\circ,\hat{a}^\circ](t,y,\mathbf{b},\pi',u) := [b^\circ,a^\circ](t,y+\sigma_0\mathbf{b}_t,\pi',u) \text{ and } [\hat{b},\hat{\sigma}](t,y,\mathbf{b},\pi',m',u) := [b,\sigma](t,y+\sigma_0\mathbf{b}_t,\pi',m',u).$$

Notice that the functions $[\hat{b},\hat{\sigma}] : [0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times\mathcal{C}^n_{\mathcal{W}}\times U \to \mathbb{R}^n\times\mathbb{S}^{n\times n}$ is continuous and for each $\mathbf{b}\in\mathcal{C}^\ell$, $[\hat{b},\hat{\sigma}](\cdot,\cdot,\mathbf{b},\cdot,\cdot)$ verify the Assumption 5.5.5.

Next, on the canonical filtered space $(\overline{\Omega},\overline{\mathbb{F}})$, let us define the $\mathcal{P}(\mathbb{R}^n)$–valued $\overline{\mathbb{F}}$–adapted continuous process $(\vartheta_t)_{t\in[0,T]}$ and the $\mathcal{P}^n_U$–valued $\overline{\mathbb{F}}$–predictable process $(\Theta_t)_{t\in[0,T]}$ by

$$\vartheta_t(\bar{\omega}) := \mu_t(\bar{\omega})[-B(\bar{\omega})] \text{ and } \Theta_t(\bar{\omega})(\mathrm{d}m) := \Lambda^\circ_t(\bar{\omega})[-B(\bar{\omega})](\mathrm{d}m), \text{ for all } (t,\bar{\omega})\in[0,T]\times\overline{\Omega}. \tag{6.5.4}$$

Lemma 6.5.1. *Let $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ with $p'>p$, and $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$. Then, $\Theta_t(\mathbb{Z}_{\vartheta_t}) = 1$, $\mathrm{dP}\otimes\mathrm{d}t$, a.e. $(t,\bar{\omega})\in[0,T]\times\overline{\Omega}$, and P–a.e. $\bar{\omega}\in\overline{\Omega}$, for all $(f,t)\in C^2_b(\mathbb{R}^n)\times[0,T]$,*

$$\begin{aligned} N_t(f) = \langle f,\vartheta_t\rangle - \langle f,\nu\rangle &- \int_0^t\int_{\mathcal{P}^n_U}\int_{\mathbb{R}^n\times U} \widehat{\mathcal{L}}^\circ_r[f](y,B,\zeta,u)m(\mathrm{d}u,\mathrm{d}y)\Theta_r(\mathrm{d}m)\mathrm{d}r \\ &- \int_0^t\int_{\mathcal{P}^n_U}\int_{\mathbb{R}^n} \mathcal{L}^\star_r[f](y,\zeta,m')\vartheta_r(\mathrm{d}y)\Lambda_r(\mathrm{d}m')\mathrm{d}r. \end{aligned}$$

Moreover, there exists a sequence $(G^k)_{k\in\mathbb{N}^*}$, such that for each $k\in\mathbb{N}^*$, $G^k : [0,T]\times\mathcal{C}^\ell\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}^n_U)\times[0,1]\to\mathcal{P}^n_U$ is a continuous function and

$$\lim_{k\to\infty}\int_0^1 \mathcal{L}^{\mathrm{P}}\left(\delta_{G^k\left(t,B_{t\wedge\cdot},\zeta_{t\wedge\cdot},\Lambda_{t\wedge\cdot},n\right)}(\mathrm{d}m')\mathrm{d}t,B,\zeta,\Lambda\right)\mathrm{d}n = \mathcal{L}^{\mathrm{P}}\left(\Theta_t(\mathrm{d}m')\mathrm{d}t,B,\zeta,\Lambda\right), \text{ in } \mathcal{W}_p. \tag{6.5.5}$$

Proof. The first point is just a reformulation of the process $N(f)$. For (6.5.5), as $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$, and Θ is a function of (Λ°,B) one has: for all $t\in[0,T]$

$$\mathcal{L}^{\mathbb{P}}\left(\Theta_{t\wedge\cdot}\middle|\overline{\mathcal{G}}_t\right) = \mathcal{L}^{\mathbb{P}}\left(\Theta_{t\wedge\cdot}\middle|\overline{\mathcal{G}}_T\right), \ \mathrm{P}\text{–a.e.},$$

recall that $\overline{\mathcal{G}}_t := \sigma\{\zeta_{t\wedge\cdot},\Lambda_{t\wedge\cdot},B_{t\wedge\cdot}\}$. By (an easy extension of) Lemma 4.8.2,

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\vartheta_t(\omega)(\mathrm{d}x)+\mathbb{E}^{\mathrm{P}}\Big[\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\mu_t(\mathrm{d}x)\Big]\leq K\Big[1+\int_{\mathbb{R}^n}|x'|^{p'}\nu(\mathrm{d}x')\Big],\ \mathrm{P}\text{-a.e. }\omega\in\overline{\Omega}.$$

Define $\Gamma:=\Big\{m\in\mathcal{P}_U^n:\ \int_{\mathbb{R}^n}|y|^{p'}m(\mathrm{d}y,U)\leq\hat{K}\Big\}$, where $\hat{K}>0$ is such that $\hat{K}>K\Big[1+\int_{\mathbb{R}^n}|x'|^{p'}\nu(\mathrm{d}x')\Big]$, with K is a constant previously used. Notice that Γ is a compact set of $\mathcal{P}_p(\mathbb{R}^n\times U)$, and one has $\Theta_t(\Gamma)=1$, $\mathrm{dP}\otimes\mathrm{d}t$, a.e. $(t,\omega)\in[0,T]\times\overline{\Omega}$. By Proposition 7.4.1, there exists a sequence $(G^k)_{k\in\mathbb{N}^*}$, such that for each $k\in\mathbb{N}^*$, $G^k:[0,T]\times\mathcal{C}^\ell\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}^n_U)\times[0,1]\to\mathcal{P}^n_U$ is a continuous function and

$$\lim_k\int_0^1\mathcal{L}^{\mathrm{P}}\Big(\delta_{G^k\left(t,B_{t\wedge\cdot},\zeta_{t\wedge\cdot},\Lambda_{t\wedge\cdot},n\right)}(\mathrm{d}m')\mathrm{d}t,B,\zeta,\Lambda\Big)\mathrm{d}n=\mathcal{L}^{\mathrm{P}}\Big(\Theta_t(\mathrm{d}m')\mathrm{d}t,B,\zeta,\Lambda\Big).$$

□

6.5.1.1 Technical lemmas

To take into account some additional randomness necessary to prove our result, let us introduce the filtered probability space $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}}_\nu)$ which is defined as follows: $\widehat{\Omega}:=[0,1]\times[0,1]\times\Omega$, $\widehat{\mathbb{F}}:=(\mathcal{B}([0,1]\otimes[0,1])\otimes\mathcal{F}_t)_{t\in[0,T]}$ and $\widehat{\mathbb{P}}_\nu:=\lambda\otimes\lambda\otimes\overline{\mathbb{P}}_\nu$, with λ the Lebesgue measure on $[0,1]$. Let (Z,N) be the canonical variables on $[0,1]\times[0,1]$, we extend naturally the variables (Z,N) of $[0,1]\times[0,1]$ and the variables (X_0,W,B) of Ω on the space $\widehat{\Omega}$, to simplify the same notation (Z,N,X_0,W,B) is kept. Also the filtration $(\widehat{\mathcal{G}}_t)_{t\in[0,T]}$ is defined by

$$\widehat{\mathcal{G}}_t:=\sigma\{N,B_{t\wedge\cdot}\},\ \text{for all } t\in[0,T].$$

Let us emphasize, after extension of all variables defined on $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P}_\nu)$, we keep the same notation on $(\widehat{\Omega},\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}}_\nu)$.

The following lemma establishes a result which implies that any measure–valued control rule satisfying some technical conditions can be approximated by processes of type $X^{\alpha,\alpha'}$ (see Definition 6.3.2).

Lemma 6.5.2. *Let* Assumption 5.5.5 *hold true, $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$ with $p'>p$, and $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$. For any sequence $(\alpha^k)_{k\in\mathbb{N}^*}\subset\mathcal{A}(\nu)$, there exists a sequence of U–valued $\widehat{\mathbb{F}}$–predictable processes $(\gamma^k)_{k\in\mathbb{N}^*}$ such that: if*

$$\lim_{k\to\infty}\widehat{\mathbb{P}}_\nu\circ\Big((\mu_t^{\alpha^k})_{t\in[0,T]},\delta_{(\overline{\mu}_s^{\alpha^k})}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1}=\mathrm{P}\circ\Big(\zeta,\Lambda_t(\mathrm{d}m')\mathrm{d}t,B\Big)^{-1},$$

then, with the unique strong solution $\widehat{X}$ of:

$$\widehat{X}_t^k=\xi+\int_0^t b\big(r,\widehat{X}_r^k,\mu^{\alpha^k},\overline{\mu}_r^{\alpha^k},\gamma_r^k\big)\mathrm{d}r+\int_0^t\sigma\big(r,\widehat{X}_r^k,\mu^{\alpha^k},\overline{\mu}_r^{\alpha^k},\gamma_r^k\big)\mathrm{d}W_r+\sigma_0B_t,\ \textit{for all } t\in[0,T],\ \widehat{\mathbb{P}}_\nu\text{–a.e.},$$

one has

$$\lim_{j\to\infty}\widehat{\mathbb{P}}_\nu\circ\Big(\mu^{k_j},\delta_{\overline{\mu}_s^{k_j}}(\mathrm{d}m)\mathrm{d}s,\ \mu^{\alpha^{k_j}},\ \delta_{(\overline{\mu}_s^{\alpha^{k_j}})}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1}=\mathrm{P}\circ\Big(\mu,\Lambda_s^\circ(\mathrm{d}m)\mathrm{d}s,\zeta,\Lambda_t(\mathrm{d}m')\mathrm{d}t,B\Big)^{-1},$$

where for all $t\in[0,T]$, $\mu_t^k:=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}_t^k|\widehat{\mathcal{G}}_t)$, and $\overline{\mu}_t^k:=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}_t^k,\gamma_t^k|\widehat{\mathcal{G}}_t)$, and $(k_j)_{j\in\mathbb{N}^*}\subset\mathbb{N}^*$ is a sub–sequence.

Proof. *Step 1 : Reformulation*: For $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$, by definition, P–a.e. $\omega\in\overline{\Omega}$, $N_t(f)=0$ for all $f\in C_b^2(\mathbb{R}^n)$ and $t\in[0,T]$. By Lemma 6.5.1, recall that $(\vartheta_t)_{t\in[0,T]}$ and $(\Theta_t)_{t\in[0,T]}$ is defined in (6.5.4), one has $\Theta_t(\mathbb{Z}_{\vartheta_t})=1$, $\mathrm{dP}\otimes\mathrm{d}t$, a.e. $(t,\omega)\in[0,T]\times\overline{\Omega}$, and P–a.e. $\omega\in\overline{\Omega}$, for all $(f,t)\in C_b^2(\mathbb{R}^n)\times[0,T]$,

$$0=\langle f,\vartheta_t\rangle-\langle f,\nu\rangle-\int_0^t\int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n\times U}\widehat{\mathcal{L}}_r^\circ f(y,B,\zeta,u)m(\mathrm{d}u,\mathrm{d}y)\Theta_r(\mathrm{d}m)\mathrm{d}r-\int_0^t\int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n}\mathcal{L}_r^\star f(y,\zeta,m')\vartheta_r(\mathrm{d}y)\Lambda_r(\mathrm{d}m')\mathrm{d}r.$$

Step 2 : *Approximation*: By Lemma 6.5.1, there exists a sequence $(G^l)_{l\in\mathbb{N}^*}$, such that for each $l\in\mathbb{N}^*$, $G^l:[0,T]\times\mathcal{C}^\ell\times\overline{\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}^n_U)}\times[0,1]\to\mathcal{P}^n_U$ is a continuous function and

$$\lim_{l\to\infty}\int_0^1\mathcal{L}^{\rm P}\Big(\delta_{G^l\left(t,B_{t\wedge\cdot},\zeta_{t\wedge\cdot},\Lambda_{t\wedge\cdot},n\right)}(\mathrm{d}m')\mathrm{d}t,B,\zeta,\Lambda\Big)\mathrm{d}n=\mathcal{L}^{\rm P}\Big(\overline{\Theta}_t(\mathrm{d}m')\mathrm{d}t,B,\zeta,\Lambda\Big).$$

Now, we apply Proposition 4.7.9 (see also Proposition 4.7.7). First, there exists a sub-sequence $(l_k)_{k\in\mathbb{N}^*}\subset\mathbb{N}^*$, such that if $\Lambda^k_s(\mathrm{d}m')\mathrm{d}s:=\delta_{\overline{\mu}^{\alpha^k}_s}(\mathrm{d}m')\mathrm{d}s$, and

$$\mathbf{m}^k_t:=G^{l_k}\big(t,B_{t\wedge\cdot},\mu^{\alpha^k}_{t\wedge\cdot},\Lambda^k_{t\wedge\cdot},N\big)\text{ and }\Theta^k_t(\mathrm{d}m)\mathrm{d}t:=\delta_{\mathbf{m}^k_t}(\mathrm{d}m)\mathrm{d}t,$$

one has

$$\lim_{k\to\infty}\widehat{\mathbb{P}}_\nu\circ\Big(\Theta^k_s(\mathrm{d}m)\mathrm{d}s,\mu^{\alpha^k},\Lambda^k,B\Big)^{-1}=\lim_{l\to\infty}\int_0^1\mathcal{L}^{\rm P}\Big(\delta_{G^l\left(t,B_{t\wedge\cdot},\zeta_{t\wedge\cdot},\Lambda_{t\wedge\cdot},n\right)}(\mathrm{d}m)\mathrm{d}t,\zeta,\Lambda,B\Big)\mathrm{d}n=\mathcal{L}^{\rm P}\Big(\Theta_t(\mathrm{d}m)\mathrm{d}t,\zeta,\Lambda,B\Big).$$

Next, under Assumption 5.5.5, by Proposition 4.7.9 (with separability condition see Remark 4.7.11), as (X_0,W) is $\widehat{\mathbb{P}}_\nu$ independent of $(B,\mu^{\alpha^k},\overline{\mu}^{\alpha^k})_{k\in\mathbb{N}^*}$, there exists a Borel function $R^k:[0,T]\times\mathbb{R}^n\times\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\times\mathbb{M}\times\mathcal{C}^n\times\mathcal{C}^\ell\times[0,1]\to U$, and if we let X^k be the unique strong solution of: for all $t\in[0,T]$,

$$X^k_t=X_0+\int_0^t\hat{b}\big(r,X^k_r,B,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}r+\int_0^t\hat{\sigma}\big(r,X^k_r,B,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}W_r,\ \widehat{\mathbb{P}}_\nu\text{–a.e.},$$

where $\mathcal{G}^k:=(\mathcal{G}^k_s)_{s\in[0,T]}:=(\sigma\{\mu^{\alpha^k}_{s\wedge\cdot},\Theta^k_{s\wedge\cdot},\Lambda^k_{s\wedge\cdot},B_{s\wedge\cdot}\})_{s\in[0,T]}$,

$$\gamma^k_t:=R^k\big(t,X_0,\mu^{\alpha^k}_{t\wedge\cdot},\Theta^k_{t\wedge\cdot},\Lambda^k_{t\wedge\cdot},W_{t\wedge\cdot},B_{t\wedge\cdot},Z\big),\ \overline{\vartheta}^k_t:=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(X^k_t,\gamma^k_t|\mathcal{G}^k_t\big)\text{ and }\vartheta^k_t:=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(X^k_t|\mathcal{G}^k_t\big),$$

then $\lim_{k\to\infty}\mathbb{E}^{\widehat{\mathbb{P}}_\nu}\Big[\int_0^T\mathcal{W}_p\big(\overline{\vartheta}^k_t,\mathbf{m}^k_t\big)^p\mathrm{d}t\Big]=0$, and

$$\lim_{j\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(\vartheta^{k_j},V^{k_j},\mu^{\alpha^{k_j}},\Lambda^{k_j},B\big)=\mathcal{L}^{\rm P}\big(\vartheta,\Theta,\zeta,\Lambda,B\big),\text{ in }\mathcal{W}_p,$$

where $V^k_t(\mathrm{d}m)\mathrm{d}t:=\delta_{\overline{\vartheta}^k_t}(\mathrm{d}m)\mathrm{d}t$ and $(k_j)_{j\in\mathbb{N}^*}\subset\mathbb{N}^*$ is a sub-sequence.

Step 3 : *Rewriting*: Notice that, as $\mathbb{G}^k\subset\widehat{\mathbb{G}}$, and (X_0,Z,W) are $\widehat{\mathbb{P}}_\nu$–independent of $\widehat{\mathbb{G}}$, one has $\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(X^k_t,\gamma^k_t|\mathcal{G}^k_t\big)=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(X^k_t,\gamma^k_t|\widehat{\mathcal{G}}_t\big)$, $\widehat{\mathbb{P}}_\nu$–a.e. for all $t\in[0,T]$. Using definition of $[\hat{b},\hat{\sigma}]$ (see the equations (6.5.3)),

$$X^k_t=\xi+\int_0^t b\big(r,X^k_r+\sigma_0B_r,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}r+\int_0^t\sigma\big(r,X^k_r+\sigma_0B_r,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}W_r,\text{ for all }t\in[0,T],\ \widehat{\mathbb{P}}_\nu\text{–a.e..}$$

Denote $\widehat{X}^k:=X^k+\sigma_0B$, one finds

$$\widehat{X}^k_t=\xi+\int_0^t b\big(r,\widehat{X}^k_r,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}r+\int_0^t\sigma\big(r,\widehat{X}^k_r,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}W_r+\sigma_0B_t,\text{ for all }t\in[0,T],\ \widehat{\mathbb{P}}_\nu\text{–a.e..}$$

It is straightforward to check that the function

$$(\pi',q,\mathbf{b})\in\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\times\mathcal{C}^\ell\to\big(\pi'[\mathbf{b}],q_t[\mathbf{b}](\mathrm{d}m)\mathrm{d}t,\mathbf{b}\big)\in\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}\times\mathcal{C}^\ell$$

is continuous. Consequently, one has

$$\begin{aligned}&\lim_{j\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\Big(\big(\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}^{k_j}_s|\widehat{\mathcal{G}}_s)\big)_{s\in[0,T]},\delta_{(\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}^{k_j}_s,\alpha^{k_j}_s|\widehat{\mathcal{G}}_s))}(\mathrm{d}m)\mathrm{d}s,\mu^{\alpha^{k_j}},\delta_{(\overline{\mu}^{\alpha^{k_j}}_s)}(\mathrm{d}m')\mathrm{d}s,B\Big)\\&=\lim_{j\to\infty}\mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(\vartheta^{k_j}[B],V^{k_j}_t[B](\mathrm{d}m)\mathrm{d}t,\mu^{\alpha^{k_j}},\delta_{(\overline{\mu}^{\alpha^{k_j}}_s)}(\mathrm{d}m')\mathrm{d}s,B\big)=\mathcal{L}^{\rm P}\big(\vartheta[B],\Theta_t[B](\mathrm{d}m)\mathrm{d}t,\zeta,\Lambda,B\big),\quad\text{in }\mathcal{W}_p.\end{aligned}$$

After calculations $(\vartheta[B], \Theta_t[B](\mathrm{d}m)\mathrm{d}t, B) = (\mu, \Lambda^\circ, B)$, $\widehat{\mathbb{P}}_\nu$–a.e. Then

$$\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}_\nu}\Big(\big(\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}^{k_j}_s|\widehat{\mathcal{G}}_s)\big)_{s\in[0,T]}, \delta_{(\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}^{k_j}_s,\alpha^{k_j}_s|\widehat{\mathcal{G}}_s))}(\mathrm{d}m)\mathrm{d}s, \mu^{\alpha^{k_j}}, \delta_{(\overline{\mu}^{\alpha^{k_j}}_s)}(\mathrm{d}m')\mathrm{d}s, B\Big) = \mathcal{L}^{\mathrm{P}}\big(\mu, \Lambda^\circ_t(\mathrm{d}m)\mathrm{d}t, \zeta, \Lambda, B\big), \text{ in } \mathcal{W}_p.$$

$(\widehat{X}^k, \gamma^k)_{k\in\mathbb{N}^*}$ is the sequence we are looking for.

□

Now, we consider the case of N–player games. Loosely speaking, we will show that: given the controls $\overline{\alpha}^N := \big(\alpha^1, \dots, \alpha^N\big)$, replace one control α^i by another κ^N has no effect on the empirical distribution $(\varphi^{N,\mathbf{X},\overline{\alpha}}, \varphi^{N,\overline{\alpha}})$ (see Definition 6.2.1) when N goes to infinity.

Let $\nu \in \mathcal{P}_p(\nu)$ and Assumption 5.5.5 hold true. Given $N \in \mathbb{N}^*$, $(\alpha^1, \dots, \alpha^N) \in \mathcal{A}(\nu_N)^N$ and $\kappa^N \in \mathcal{A}(\nu_N)$. Let us introduce, for each $i \in \{1, \dots, N\}$, the unique strong solution $\widetilde{X}^i$ of: for all $t \in [0, T]$,

$$\widetilde{X}^i_t = \mathbf{X}^i_0 + \int_0^t b\big(r, \widetilde{X}^i_r, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi^{N,\overline{\alpha}^N}_r, \kappa^N_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widetilde{X}^i_r, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi^{N,\overline{\alpha}^N}_r, \kappa^N_r\big)\mathrm{d}\mathbf{W}^i_r + \sigma_0 B_t, \ \mathbb{P}^N_\nu\text{-a.e.},$$

where $(\varphi^{N,\mathbf{X},\overline{\alpha}}, \varphi^{N,\overline{\alpha}})$ correspond to the empirical distributions associated with the controls $\overline{\alpha}^N := \big(\alpha^1, \dots, \alpha^N\big)$ (see Definition 6.2.1)

Lemma 6.5.3. *There exists a constant $K > 0$ (depending only on the p–moment of ν) such that: if $\overline{\alpha}^{N,-i} := \big(\overline{\alpha}^{[-i]}, \kappa^N\big)$, for each $i \in \{1, \dots, N\}$, one has*

$$\Big(\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\sup_{t\in[0,T]} \mathcal{W}_p\big(\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t, \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}_t\big)\Big] + \mathbb{E}^{\mathbb{P}^N_\nu}\Big[\sup_{t\in[0,T]} \big|\widetilde{X}^i_t - \mathbf{X}^i_t[\overline{\alpha}^{N,-i}]\big|^p\Big]\Big) \le K\frac{1}{N}.$$

Consequently, $\limsup_{N\to\infty} \mathcal{W}_p\big(\mathbb{Q}^N, \widetilde{\mathbb{Q}}^N\big) = 0$, *where*

$$\mathbb{Q}^N := \frac{1}{N}\sum_{i=1}^N \mathbb{P}^N_\nu \circ \Big(\mathbf{X}^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \delta_{\left(\kappa^N_t, \varphi^{N,\overline{\alpha}^{N,-i}}_t\right)}(\mathrm{d}u, \mathrm{d}m')\mathrm{d}t\Big)^{-1},$$

and

$$\widetilde{\mathbb{Q}}^N := \frac{1}{N}\sum_{i=1}^N \mathbb{P}^N_\nu \circ \Big(\widetilde{X}^i, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \delta_{\left(\kappa^N_t, \varphi^{N,\overline{\alpha}^N}_t\right)}(\mathrm{d}u, \mathrm{d}m')\mathrm{d}t\Big)^{-1}.$$

Proof. This proof is a successive application of the Gronwall's lemma. For $j \in \{1, \dots, N\}$ with $j \neq i$, for all $t \in [0, T]$, one finds

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\sup_{s\in[0,t]} \big|\mathbf{X}^j_s[\overline{\alpha}^{N,-i}] - \mathbf{X}^j_s[\overline{\alpha}^N]\big|^p\Big] \\
&\quad\le C\Big(\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\int_0^t \Big|\big[b,\sigma\big]\big(r, \mathbf{X}^j_r[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \varphi^{N,\overline{\alpha}^{N,-i}}_r, \alpha^j_r\big) - \big[b,\sigma\big]\big(r, \mathbf{X}^j_r[\overline{\alpha}^N], \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi^{N,\overline{\alpha}^N}_r, \alpha^j_r\big)\Big|^p \mathrm{d}r\Big]\Big) \\
&\quad\le C\Big(\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\int_0^t \sup_{s\in[0,r]} \big|\mathbf{X}^j_s[\overline{\alpha}^{N,-i}] - \mathbf{X}^j_s[\overline{\alpha}^N]\big|^p + \sup_{s\in[0,r]} \mathcal{W}_p\big(\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}_s, \varphi^{N,\mathbf{X},\overline{\alpha}^N}_s\big)^p + \mathcal{W}_p\big(\varphi^{N,\overline{\alpha}^{N,-i}}_r, \varphi^{N,\overline{\alpha}^N}_r\big)^p \mathrm{d}r\Big]\Big),
\end{aligned}$$

then by Gronwall's lemma,

$$\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\sup_{s\in[0,t]} \big|\mathbf{X}^j_s[\overline{\alpha}^{N,-i}] - \mathbf{X}^j_s[\overline{\alpha}^N]\big|^p\Big] \le C\Big(\mathbb{E}^{\mathbb{P}^N_\nu}\Big[\int_0^t \sup_{s\in[0,r]} \mathcal{W}_p\big(\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}_s, \varphi^{N,\mathbf{X},\overline{\alpha}^N}_s\big)^p + \mathcal{W}_p\big(\varphi^{N,\overline{\alpha}^{N,-i}}_r, \varphi^{N,\overline{\alpha}^N}_r\big)^p \mathrm{d}r\Big]\Big). \tag{6.5.6}$$

Next, using result (6.5.6),

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{s\in[0,t]}\mathcal{W}_p\big(\varphi_s^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi_s^{N,\mathbf{X},\overline{\alpha}^N}\big)^p+\mathcal{W}_p\big(\varphi_t^{N,\overline{\alpha}^{N,-i}},\varphi_t^{N,\overline{\alpha}^N}\big)^p\Big]\\
&\quad\le C\bigg(\frac{1}{N}\sum_{j=1}^N\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{s\in[0,t]}|\mathbf{X}_s^j[\overline{\alpha}^{N,-i}]-\mathbf{X}_s^j[\overline{\alpha}^N]|^p\Big]+\frac{\rho\big(\kappa_t^N,\alpha_t^i\big)^p}{N}\bigg)\\
&\quad\le C\bigg(\frac{1}{N}\sum_{j=1,j\neq i}^N\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^t\sup_{s\in[0,r]}\mathcal{W}_p\big(\varphi_s^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi_s^{N,\mathbf{X},\overline{\alpha}^N}\big)^p+\mathcal{W}_p\big(\varphi_r^{N,\overline{\alpha}^{N,-i}},\varphi_r^{N,\overline{\alpha}^N}\big)^p\mathrm{d}r\Big]\\
&\qquad+\frac{1}{N}\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{s\in[0,t]}\big|\mathbf{X}_s^i[\overline{\alpha}^{N,-i}]-\mathbf{X}_s^i[\overline{\alpha}^N]\big|^p\Big]+\frac{\rho\big(\kappa_t^N,\alpha_t^i\big)^p}{N}\bigg)\\
&\quad\le C\bigg(\frac{N-1}{N}\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^t\sup_{s\in[0,r]}\mathcal{W}_p\big(\varphi_s^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi_s^{N,\mathbf{X},\overline{\alpha}^N}\big)^p+\mathcal{W}_p\big(\varphi_r^{N,\overline{\alpha}^{N,-i}},\varphi_r^{N,\overline{\alpha}^N}\big)^p\mathrm{d}r\Big]\\
&\qquad+\frac{\int_{\mathbb{R}^n}|x|^p\nu(\mathrm{d}x)}{N}+\frac{\sup_{(u,u')\in U\times U}\rho\big(u,u'\big)^p}{N}\bigg),
\end{aligned}$$

by Gronwall's lemma again,

$$\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{s\in[0,t]}\mathcal{W}_p\big(\varphi_s^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi_s^{N,\mathbf{X},\overline{\alpha}^N}\big)^p+\mathcal{W}_p\big(\varphi_t^{N,\overline{\alpha}^{N,-i}},\varphi_t^{N,\overline{\alpha}^N}\big)^p\Big]\le C\bigg(\frac{\int_{\mathbb{R}^n}|x|^p\nu(\mathrm{d}x)}{N}+\frac{\sup_{(u,u')\in U\times U}\rho\big(u,u'\big)^p}{N}\bigg).\tag{6.5.7}$$

To finish,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{s\in[0,t]}\big|\mathbf{X}_s^i[\overline{\alpha}^{N,-i}]-\widetilde{X}_s^i\big|^p\Big]\\
&\quad\le C\bigg(\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^t\big|\big[b,\sigma\big]\big(r,\mathbf{X}_r^i[\overline{\alpha}^{N,-i}],\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi_r^{N,\overline{\alpha}^{N,-i}},\kappa_r^N\big)-\big[b,\sigma\big]\big(r,\widetilde{X}_r^i,\varphi^{N,\mathbf{X},\overline{\alpha}^N},\varphi_r^{N,\overline{\alpha}^N},\kappa_r^N\big)\big|^p\mathrm{d}r\Big]\bigg)\\
&\quad\le C\bigg(\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\int_0^t\sup_{s\in[0,r]}|\mathbf{X}_s^j[\overline{\alpha}^{N,-i}]-\widetilde{X}_s^i|^p+\sup_{s\in[0,r]}\mathcal{W}_p\big(\varphi_s^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi_s^{N,\mathbf{X},\overline{\alpha}^N}\big)^p+\mathcal{W}_p\big(\varphi_r^{N,\overline{\alpha}^{N,-i}},\varphi_r^{N,\overline{\alpha}^N}\big)^p\mathrm{d}r\Big]\bigg),
\end{aligned}$$

and thanks to Gronwall's lemma and result (6.5.7), one has

$$\mathbb{E}^{\mathbb{P}_\nu^N}\Big[\sup_{s\in[0,T]}|\mathbf{X}_s^i[\overline{\alpha}^{N,-i}]-\widetilde{X}_s^i|^p\Big]\le CT\bigg(\frac{\int_{\mathbb{R}^n}|x|^p\nu(\mathrm{d}x)}{N}+\frac{\sup_{(u,u')\in U\times U}\rho\big(u,u'\big)^p}{N}\bigg).$$

It is enough to conclude. □

For similar reasons to those mentioned in Lemma 6.5.2, for each $N\in\mathbb{N}^*$, the space $(\Omega^N,\mathbb{F}^N,\mathcal{F}^N,\mathbb{P}_\nu^N)$ needs to be enlarged. Let us introduce the filtered probability space $(\widehat{\Omega}^N,\widehat{\mathbb{F}}^N,\widehat{\mathcal{F}}^N,\widehat{\mathbb{P}}_\nu^N)$ as follows: $\widehat{\Omega}^N:=[0,1]\times[0,1]\times\Omega^N$, $\widehat{\mathbb{F}}^N:=(\mathcal{B}([0,1]\otimes[0,1])\otimes\mathcal{F}_t^N)_{t\in[0,T]}$ and $\widehat{\mathbb{P}}_\nu^N:=\lambda\otimes\lambda\otimes\mathbb{P}_\nu^N$, with λ the Lebesgue measure on $[0,1]$. Let (Z,N) denote the canonical variables on $[0,1]\times[0,1]$, we extend naturally the variables (Z,N) of $[0,1]\times[0,1]$ and the variables $(\mathbf{X}_0,\mathbf{W},B)$ of Ω^N on the space $\widehat{\Omega}^N$, and keep the same notion $(\mathrm{Z},\mathrm{N},\mathbf{X}_0,\mathbf{W},B)$. After extension of all variables defined on $(\Omega^N,\mathbb{F}^N,\mathcal{F}^N,\mathbb{P}_\nu^N)$, the same notation are kept on $(\widehat{\Omega}^N,\widehat{\mathbb{F}}^N,\widehat{\mathcal{F}}^N,\widehat{\mathbb{P}}_\nu^N)$.

The next result is the analog of Lemma 6.5.2 for the N–player games. To summarize, it states that any measure-valued control rule which verifies a particular constraint is the average limit of N–SDE processes of type (6.2.1).

Lemma 6.5.4. *Let Assumption 5.5.5 hold true, $\nu \in \mathcal{P}_{p'}(\nu)$ with $p' > p$, $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$ and a sequence $(\alpha^i)_{i\in\mathbb{N}^*}$ s.t. for each $N \in \mathbb{N}^*$, $(\alpha^1, \dots, \alpha^N) \subset \mathcal{A}(\nu_N)$ and*

$$\lim_{N\to\infty} \widehat{\mathbb{P}}_\nu^N \circ \Big(\varphi^{N,\mathbf{X},\overline{\alpha}}, \delta_{\left(\varphi_t^{N,\overline{\alpha}}\right)}(\mathrm{d}m')\mathrm{d}t, B\Big)^{-1} = \mathrm{P} \circ \big(\zeta, \Lambda_t(\mathrm{d}m')\mathrm{d}t, B\big)^{-1}.$$

There exists a sequence of Borel functions $(\phi^{i,N})_{(i,N)\in\{1,\dots,N\}\times\mathbb{N}^*}$ satisfying $\phi^{i,N} : [0,T] \times (\mathbb{R}^n)^N \times (\mathcal{C}^d)^N \times \mathcal{C}^\ell \times [0,1] \times [0,1] \to U$, s.t. if for all $t \in [0,T]$, $\gamma_t^{i,N}$ is defined by $\gamma_t^{i,N} := \phi^{i,N}\big(t, \mathbf{X}_0, \mathbf{W}_{t\wedge\cdot}, B_{t\wedge\cdot}, \mathrm{Z}, \mathrm{N}\big)$, one has

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\widehat{\mathbb{P}}_\nu^N}\bigg[\int_0^T L\Big(t, \mathbf{X}_t^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \varphi_t^{N,\overline{\alpha}^{N,-i}}, \gamma_t^{i,N}\Big)\mathrm{d}t + g\Big(\mathbf{X}_T^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}\Big)\bigg] = \mathbb{E}^{\mathrm{P}}\big[J(\mu, \Lambda^\circ, \zeta, \Lambda)\big],$$

where $\overline{\alpha}^{N,-i} := (\overline{\alpha}^{[-i]}, \gamma^{i,N}) = \big(\alpha^1, \dots, \alpha^{i-1}, \gamma^{i,N}, \alpha^{i+1}, \dots, \alpha^N\big)$.

Proof. By Lemma 6.5.1, there is a sequence $(G^l)_{l\in\mathbb{N}^*}$, such that for each $l \in \mathbb{N}^*$, $G^l : [0,T] \times \mathcal{C}^\ell \times \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M}(\mathcal{P}_U^n) \times [0,1] \to \mathcal{P}(\mathbb{R}^n \times U)$ is a continuous function and

$$\lim_{l\to\infty} \int_0^1 \mathcal{L}^{\mathrm{P}}\Big(\delta_{G^l\left(t, B_{t\wedge\cdot}, \zeta_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, n\right)}(\mathrm{d}m')\mathrm{d}t, B, \zeta, \Lambda\Big)\mathrm{d}n = \mathcal{L}^{\mathrm{P}}\Big(\Theta_t(\mathrm{d}m')\mathrm{d}t, B, \zeta, \Lambda\Big).$$

Now, we apply Proposition 4.7.7. One can find a sub-sequence $(l_N)_{N\in\mathbb{N}^*} \subset \mathbb{N}^*$, such that if $\Lambda_s^N(\mathrm{d}m')\mathrm{d}s := \delta_{\varphi_s^{N,\overline{\alpha}^N}}(\mathrm{d}m)\mathrm{d}s$,

$$\mathbf{m}_t^N := G^{l_N}\big(t, B_{t\wedge\cdot}, \varphi_{t\wedge\cdot}^{N,\mathbf{X},\overline{\alpha}^N}, \Lambda_{t\wedge\cdot}^N, \mathrm{N}\big) \text{ and } \Theta_t^N(\mathrm{d}m)\mathrm{d}t := \delta_{\mathbf{m}_t^N}(\mathrm{d}m)\mathrm{d}t,$$

one has

$$\lim_{N\to\infty} \widehat{\mathbb{P}}_\nu^N \circ \Big(\Theta_s^N(\mathrm{d}m)\mathrm{d}s, \varphi_s^{N,\mathbf{X},\overline{\alpha}^N}, \Lambda^N, B\Big)^{-1} = \lim_{l\to\infty} \int_0^1 \mathcal{L}^{\mathrm{P}}\Big(\delta_{G^l\left(t, B_{t\wedge\cdot}, \zeta_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, n\right)}(\mathrm{d}m)\mathrm{d}t, \zeta, \Lambda, B\Big)\mathrm{d}n = \mathrm{P} \circ \big(\Theta, \zeta, \Lambda, B\big)^{-1}.$$

Under Assumption 5.5.5, by Proposition 4.7.7 (with separability condition see Remark 4.7.11), there exists a Borel function $R^N : [0,T] \times \mathbb{R}^n \times \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M} \times \mathbb{M} \times \mathcal{C}^n \times \mathcal{C}^\ell \times [0,1] \to U$ s.t. if $(X^i)_{i\in\{1,\dots,N\}}$ is the unique strong solution of: for all $t \in [0,T]$

$$X_t^i = \mathbf{X}_0^i + \int_0^t \hat{b}\big(r, X_r^i, B, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi_r^{N,\overline{\alpha}^N}, \gamma_r^{i,N}\big)\mathrm{d}r + \int_0^t \hat{\sigma}\big(r, X_r^i, B, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi_r^{N,\overline{\alpha}^N}, \gamma_r^{i,N}\big)\mathrm{d}\mathbf{W}_r^i,\ \widehat{\mathbb{P}}_\nu^N\text{–a.e.},$$

where

$$\gamma_t^{i,N} := R^N\big(t, \mathbf{X}_0^i, \varphi_{t\wedge\cdot}^{N,\mathbf{X},\overline{\alpha}^N}, \Theta_{t\wedge\cdot}^N, \Lambda_{t\wedge\cdot}^N, \mathbf{W}_{t\wedge\cdot}^i, B_{t\wedge\cdot}, \mathrm{Z}\big),\ \ \overline{\vartheta}_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^i, \gamma_t^{i,N})},\ \text{and } \vartheta_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i},$$

then $\lim_{N\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}_\nu}\Big[\int_0^T \mathcal{W}_p\big(\overline{\vartheta}_t^N, \mathbf{m}_t^N\big)^p \mathrm{d}t\Big] = 0$, and

$$\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}_\nu^{N_j}}\big(\vartheta^{N_j}, V^{N_j}, \varphi^{N_j,\mathbf{X},\overline{\alpha}^{N_j}}, \Lambda^{N_j}, B\big) = \mathcal{L}^{\mathrm{P}}\big(\vartheta, \Theta, \zeta, \Lambda, B\big),\ \ \text{in } \mathcal{W}_p,$$

with $V_t^N(\mathrm{d}m)\mathrm{d}t := \delta_{\overline{\vartheta}_t^N}(\mathrm{d}m)\mathrm{d}t$ and $(N_j)_{j\in\mathbb{N}^*} \subset \mathbb{N}^*$ is a sub-sequence.
As in the proof Lemma 6.5.2, we can rewrite $(X^i)_{i\in\{1,\dots,N\}}$. Notice that, for all $t \in [0,T]$

$$X_t^i = \mathbf{X}_0^i + \int_0^t b\big(r, X_r^i + \sigma_0 B_r, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi_r^{N,\overline{\alpha}^N}, \gamma_r^{i,N}\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^i + \sigma_0 B_r, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi_r^{N,\overline{\alpha}^N}, \gamma_r^{i,N}\big)\mathrm{d}\mathbf{W}_r^i,\ \widehat{\mathbb{P}}_\nu^N\text{–a.e.}$$

Denote $\widehat{X}^i := X^i + \sigma_0 B$, for all $t \in [0,T]$,

$$\widehat{X}^i_t = \mathbf{X}^i_0 + \int_0^t b\big(r, \widehat{X}^i_r, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi^{N,\overline{\alpha}^N}_r, \gamma^{i,N}_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{X}^i_r, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi^{N,\overline{\alpha}^N}_r, \gamma^{i,N}_r\big)\mathrm{d}\mathbf{W}^i_r + \sigma_0 B_t,\ \widehat{\mathbb{P}}^N_\nu\text{–a.e.}$$

As the function $(\pi', q, \mathbf{b}) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell \to \big(\pi'[\mathbf{b}], q_t[\mathbf{b}](\mathrm{d}m)\mathrm{d}t, \mathbf{b}\big) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell$ is continuous, if we note $\widehat{\widehat{\vartheta}}^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{X}^i_t, \gamma^{i,N}_t)}$, and $\widehat{\vartheta}^N_t := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^i_t}$, one has

$$\begin{aligned}
&\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}^{N_j}_\nu}\Big(\widehat{\vartheta}^{N_j}, \delta_{\widehat{\widehat{\vartheta}}^{N_j}_s}(\mathrm{d}m)\mathrm{d}s, \varphi^{N_j,\mathbf{X},\overline{\alpha}^{N_j}}, \Lambda^{N_j}_s(\mathrm{d}m')\mathrm{d}s, B\Big)\\
&= \lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}^{N_j}_\nu}\big(\vartheta^{N_j}[B], V^{N_j}_t[B](\mathrm{d}m)\mathrm{d}t, \varphi^{N_j,\mathbf{X},\overline{\alpha}^{N_j}}, \Lambda^{N_j}_s(\mathrm{d}m')\mathrm{d}s, B\big) = \mathcal{L}^{\mathrm{P}}\big(\vartheta[B], \Theta_t[B](\mathrm{d}m)\mathrm{d}t, \zeta, \Lambda, B\big), \qquad \text{in } \mathcal{W}_p.
\end{aligned}$$

One knows $(\vartheta[B], \Theta_t[B](\mathrm{d}m)\mathrm{d}t, B) = (\mu, \Lambda^\circ, B)$, P–a.e. then

$$\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}^{N_j}_\nu}\Big(\widehat{\vartheta}^{N_j}, \delta_{\widehat{\widehat{\vartheta}}^{N_j}_s}(\mathrm{d}m)\mathrm{d}s, \varphi^{N_j,\mathbf{X},\overline{\alpha}^{N_j}}, \Lambda^{N_j}_s(\mathrm{d}m')\mathrm{d}s, B\Big) = \mathcal{L}^{\mathrm{P}}\big(\mu, \Lambda^\circ_t(\mathrm{d}m)\mathrm{d}t, \zeta, \Lambda, B\big), \text{ in } \mathcal{W}_p. \tag{6.5.8}$$

Let us define

$$\overline{\alpha}^{N,-i} := (\overline{\alpha}^{[-i]}, \gamma^{i,N}) = \big(\alpha^1, \ldots, \alpha^{i-1}, \gamma^{i,N}, \alpha^{i+1}, \ldots, \alpha^N\big),$$

thanks to Lemma 6.5.3, for each $i \in \{1, \ldots, N\}$,

$$\left(\mathbb{E}^{\widehat{\mathbb{P}}^N_\nu}\left[\int_0^T \mathcal{W}_p\big(\varphi^{N,\overline{\alpha}^N}_r, \varphi^{N,\overline{\alpha}^{N,-i}}_r\big)\mathrm{d}r\right] + \mathbb{E}^{\widehat{\mathbb{P}}^N_\nu}\left[\sup_{t\in[0,T]} |\widehat{X}^{i,N}_t - \mathbf{X}^i_t[\overline{\alpha}^{N,-i}]|^p\right]\right) \le K\frac{1}{N},$$

and $\limsup_{N\to\infty} \mathcal{W}_p\big(\mathbb{Q}^N, \widetilde{\mathbb{Q}}^N\big) = 0$, where $\mathbb{Q}^N := \frac{1}{N}\sum_{i=1}^N \mathcal{L}^{\widehat{\mathbb{P}}^N_\nu}\Big(\mathbf{X}^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \delta_{\left(\gamma^{i,N}_s, \varphi^{N,\overline{\alpha}^{N,-i}}_s\right)}(\mathrm{d}u, \mathrm{d}m')\mathrm{d}s\Big)$ and $\widetilde{\mathbb{Q}}^N := \frac{1}{N}\sum_{i=1}^N \mathcal{L}^{\widehat{\mathbb{P}}^N_\nu}\Big(\widehat{X}^i, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \delta_{\left(\gamma^{i,N}_s, \varphi^{N,\overline{\alpha}^N}_s\right)}(\mathrm{d}u, \mathrm{d}m')\mathrm{d}s\Big)$.
Therefore, using Assumption 5.5.5 (especially the separability condition), the previous result combined with (6.5.8) allow to get that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N J_i\big((\overline{\alpha}^{[-i]}, \gamma^{i,N},)\big)\\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\widehat{\mathbb{P}}^N_\nu}\left[\int_0^T L\big(t, \mathbf{X}^i_t[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \varphi^{N,\overline{\alpha}^{N,-i}}_t, \gamma^{i,N}_t\big)\mathrm{d}t + g\big(\mathbf{X}^i_T[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}\big)\right]\\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\widehat{\mathbb{P}}^N_\nu}\left[\int_0^T L\big(t, \widehat{X}^i_t, \varphi^{N,\mathbf{X},\overline{\alpha}^N}, \varphi^{N,\overline{\alpha}^N}_t, \gamma^{i,N}_t\big)\mathrm{d}t + g\big(\widehat{X}^i_T, \varphi^{N,\mathbf{X},\overline{\alpha}^N}\big)\right] = \mathbb{E}^{\mathrm{P}}\big[J(\mu, \Lambda^\circ, \zeta, \Lambda)\big].
\end{aligned}$$

□

6.5.1.2 Proof of Theorem 6.4.1 (Limit Theorem)

First point (i) By using Proposition 4.8.4 (a slight extension[1]), one finds $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ is relatively compact where

$$\mathrm{P}^N := \mathbb{P}^N_\nu \circ \Big((\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t)_{t\in[0,T]}, (\varphi^{N,\mathbf{X},\overline{\alpha}^N}_t)_{t\in[0,T]}, \delta_{\varphi^{N,\overline{\alpha}^N}_s}(\mathrm{d}m)\mathrm{d}s, \delta_{\varphi^{N,\overline{\alpha}^N}_s}(\mathrm{d}m')\mathrm{d}s, B\Big)^{-1},$$

[1]Consisting in taking into account a canonical space of type $\overline{\Omega} := \mathcal{C}^n_{\mathcal{W}} \times \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathbb{M} \times \mathcal{C}^\ell$ and not $\overline{\Omega} := \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M} \times \mathcal{C}^\ell$ as in Chapter 4

and each limit point P^∞ of any sub–sequence belongs to $\overline{\mathcal{P}}_V(\nu)$. Next, let us show that $\mathrm{P}^\infty \in \overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$. To simplify, the sequence $(\mathrm{P}^N)_{N\in\mathbb{N}^*}$ and its sub-sequence share the same notation.

Let $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$ such that $\mathcal{L}^{\mathrm{P}}\big(\zeta,\Lambda,B\big) = \mathcal{L}^{\mathrm{P}^\infty}\big(\zeta,\Lambda,B\big)$. By Lemma 6.5.4, there exists $(R^{i,N})_{(i,N)\in\{1,\dots,N\}\times\mathbb{N}^*}$ a sequence of Borel functions $R^{i,N} : [0,T]\times(\mathbb{R}^n)^N\times(\mathcal{C}^d)^N\times\mathcal{C}^\ell\times[0,1]\times[0,1]\to U$, s.t. if we denote by

$$\gamma^{i,N}_t := R^{i,N}\big(t,\mathbf{X}_0,\mathbf{W}_{t\wedge\cdot},B_{t\wedge\cdot},\mathrm{Z},\mathrm{N}\big),$$

for all $t\in[0,T]$, and

$$\overline{\alpha}^{N,-i} := (\overline{\alpha}^{[-i]},\kappa^{i,N}) = \big(\alpha^1,\dots,\alpha^{i-1},\kappa^{i,N},\alpha^{i+1},\dots,\alpha^N\big),$$

then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\widehat{\mathbb{P}}^N_\nu}\bigg[\int_0^T L\big(t,\mathbf{X}^i_t[\overline{\alpha}^{N,-i}],\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi^{N,\overline{\alpha}^{N,-i}}_t,\gamma^{i,N}_t\big)\mathrm{d}t + g\big(\mathbf{X}^i_T[\overline{\alpha}^{N,-i}],\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}\big)\bigg] = \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big].$$

Notice that, by independence of Z and N with of the other all variables

$$\begin{aligned}&\int_{[0,1]\times[0,1]}\frac{1}{N}\sum_{i=1}^N J_i\big((\overline{\alpha}^{[-i]},\kappa^{i,N}(\mathrm{z},\mathrm{n}))\big)\lambda(\mathrm{dz})\lambda(\mathrm{dn})\\ &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\widehat{\mathbb{P}}^N_\nu}\bigg[\int_0^T L\big(t,\mathbf{X}^i_t[\overline{\alpha}^{N,-i}],\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}},\varphi^{N,\overline{\alpha}^{N,-i}}_t,\gamma^{i,N}_t\big)\mathrm{d}t + g\big(\mathbf{X}^i_T[\overline{\alpha}^{N,-i}],\varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}\big)\bigg],\end{aligned}$$

where $\kappa^{i,N}_t(\mathrm{z},\mathrm{n}) := R^{i,N}\big(t,\mathbf{X}_0,\mathbf{W}_{t\wedge\cdot},B_{t\wedge\cdot},\mathrm{z},\mathrm{n}\big)$, for all $(t,\mathrm{z},\mathrm{n})\in[0,T]\times[0,1]\times[0,1]$. Therefore

$$\begin{aligned}\mathbb{E}^{\mathrm{P}^\infty}\big[J(\mu,\overline{\Lambda},\zeta,\Lambda)\big] &= \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N J_i[\overline{\alpha}^N]\\ &\geq \lim_{N\to\infty}\bigg(\int_{[0,1]\times[0,1]}\frac{1}{N}\sum_{i=1}^N J_i\big((\overline{\alpha}^{[-i]},\kappa^{i,N}(\mathrm{z},\mathrm{n}))\big)\lambda(\mathrm{dz})\lambda(\mathrm{dn}) - \frac{1}{N}\sum_{i=1}^N\epsilon_i\bigg) = \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]-\epsilon,\end{aligned}$$

then $\mathbb{E}^{\mathrm{P}^\infty}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] \geq \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]-\epsilon$, for any $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$ such that $\mathcal{L}^{\mathrm{P}}\big(\zeta,\Lambda,B\big) = \mathcal{L}^{\mathrm{P}^\infty}\big(\zeta,\Lambda,B\big)$. It is straightforward to deduce that for P^∞ almost every $\omega\in\overline{\Omega}$, $\Lambda^\circ_t(\omega)(\mathrm{d}m)\mathrm{d}t = \Lambda_t(\omega)\big(\mathrm{d}m'\big)\mathrm{d}t$ and $\zeta(\omega)=\mu(\omega)$. We conclude that $\mathrm{P}^\infty\in\overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$.

Second point (ii) The proof of this second part is similar to previous proof. By using Proposition 4.8.4 (a slight extension), one gets $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ is relatively compact where $\mathrm{P}^k\in\overline{\mathcal{P}}^\star_S(\nu)[\epsilon_k]$ i.e. there exists α^k a ϵ_k–strong MFG equilibrium s.t.

$$\mathrm{P}^k := \mathbb{P}_\nu\circ\Big((\mu^{\alpha^k}_t)_{t\in[0,T]},(\mu^{\alpha^k}_t)_{t\in[0,T]},\delta_{\overline{\mu}^{\alpha^k}_s}(\mathrm{d}m)\mathrm{d}s,\delta_{\overline{\mu}^{\alpha^k}_s}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1}.$$

Each limit point P^∞ of any sub-sequence belongs to $\overline{\mathcal{P}}_V(\nu)$. Let us prove that $\mathrm{P}^\infty\in\overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$. Again to simplify, $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ and its sub-sequence share the same notation. Let $\mathrm{P}\in\overline{\mathcal{P}}_V(\nu)$ such that $\mathcal{L}^{\mathrm{P}}\big(\zeta,\Lambda_t(\mathrm{d}m')\mathrm{d}t,B\big) = \mathcal{L}^{\mathrm{P}^\infty}\big(\zeta,\Lambda_t(\mathrm{d}m')\mathrm{d}t,B\big)$. By Lemma 6.5.2, there exists a sequence of U–valued $\widehat{\mathbb{F}}$–predictable processes $(\gamma^k)_{k\in\mathbb{N}^*}$ such that: if $\widehat{X}^k$ is the strong solution of

$$\widehat{X}^k_t = \xi + \int_0^t b\big(r,\widehat{X}^k_r,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}r + \int_0^t\sigma\big(r,\widehat{X}^k_r,\mu^{\alpha^k},\overline{\mu}^{\alpha^k}_r,\gamma^k_r\big)\mathrm{d}W_r + \sigma_0 B_t,\ \text{for all } t\in[0,T],\ \widehat{\mathbb{P}}_\nu\text{–a.e..}$$

then

$$\lim_{j\to\infty}\widehat{\mathbb{P}}_\nu\circ\Big(\mu^j,\delta_{\overline{\mu}_r^j}(\mathrm{d}m)\mathrm{d}r,\ \mu^{\alpha^{k_j}},\ \delta_{\overline{\mu}_s^{\alpha^{k_j}}}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1}=\mathrm{P}\circ\Big(\mu,\Lambda_r^\circ(\mathrm{d}m)\mathrm{d}r,\zeta,\Lambda_t(\mathrm{d}m')\mathrm{d}t,B\Big)^{-1},$$

where for all $t\in[0,T]$, $\mu_t^k:=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}_t^k|\widehat{\mathcal{G}}_t)$, and $\overline{\mu}_t^k:=\mathcal{L}^{\widehat{\mathbb{P}}_\nu}(\widehat{X}_t^k,\gamma_t^k|\widehat{\mathcal{G}}_t)$, and $(k_j)_{j\in\mathbb{N}^*}\subset\mathbb{N}^*$ is a sub-sequence.
Then using Assumption 5.5.5 (especially separability condition), and $\widehat{\mathbb{P}}_\nu$–independence of (N,Z) with the other variables, one gets

$$\begin{aligned}\mathbb{E}^{\mathrm{P}^\infty}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]&=\lim_{j\to\infty}\mathbb{E}^{\mathrm{P}^{k_j}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]\\&\geq\lim_{j\to\infty}\left(\mathbb{E}^{\widehat{\mathbb{P}}_\nu}\left[\int_0^T L(t,\widehat{X}_t^{k_j},\mu_{t\wedge\cdot}^{\alpha^{k_j}},\overline{\mu}_t^{\alpha^{k_j}},\gamma_t^k)\mathrm{d}t+g(\widehat{X}_T^{k_j},\mu^{\alpha^{k_l}})\right]-\epsilon_{k_j}\right)=\mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]-\epsilon.\end{aligned}$$

Obviously, for P^∞– a.e. $\omega\in\overline{\Omega}$, $\Lambda_t^\circ(\omega)(\mathrm{d}m)\mathrm{d}t=\Lambda_t(\omega)\big(\mathrm{d}m'\big)\mathrm{d}t$ and $\zeta(\omega)=\mu(\omega)$, we deduce that $\mathrm{P}^\infty\in\overline{\mathcal{P}}_V^\star(\nu)[\epsilon]$.

6.5.2 The converse limit result

This part is devoted to the proof of Theorem 6.4.2. We focus on the approximation of any measure-valued MFG solution rule by a sequence of approximate strong MFG solutions. The approximation by approximate Nash equilibria follows from this approximation.

When $\ell=0$ and so B disappears, we need some additional randomness to get our desired results. Throughout this part, in order to consider the cases $\ell=0$ or $\ell\neq0$, for each $q\in\{0,1\}$, let us consider the filtered probability space $(\widehat{\Omega}^q,\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}}_\nu^q)$ which is defined as follows: $\widehat{\Omega}:=[0,1]^q\times\Omega$, $\widehat{\mathbb{F}}:=(\mathcal{B}([0,1]^q)\otimes\mathcal{F}_t)_{t\in[0,T]}$ and $\widehat{\mathbb{P}}_\nu^q:=\lambda^{\otimes q}\otimes\mathbb{P}_\nu$, with λ the Lebesgue measure on $[0,1]$. Let H denote the canonical variables on $[0,1]^q$, the variable H of $[0,1]^q$ and the variables (X_0,W,B) of Ω are naturally extended on the space $\widehat{\Omega}$, for simplicity the notation stays (H,X_0,W,B). Denote by $(\widehat{\mathcal{G}}_t)_{t\in[0,T]}$ the filtration defined by

$$\widehat{\mathcal{G}}_t:=\sigma\{B_{t\wedge\cdot},\mathrm{H}\},\ \text{for all } t\in[0,T].$$

Again, after extension of all variables defined on $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P}_\nu)$, the same notation on $(\widehat{\Omega}^q,\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}}_\nu^q)$ are kept.

6.5.2.1 Some useful results

First, we give some results on weak McKean-Vlasov processes. This part is largely inspired by Chapter 2

Let $\nu\in\mathcal{P}_p(\mathbb{R}^n)$ and $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P})$ be a filtered probability space supporting

- a $\mathbb{R}^{d+\ell}$–valued $(\mathbb{F},\mathbb{P})$–Brownian motion (W,B) and a $\mathbb{R}^n$–valued $\mathcal{F}_0$–random variable $\mathbf{X}_0$ such that $\mathcal{L}^{\mathbb{P}}(\mathbf{X}_0)=\nu$.
- a U–valued $\mathbb{F}$–predictable process $(\alpha_t)_{t\in[0,T]}$.

Denote by Ω_G the space $\mathcal{C}^n\times\mathbb{M}(U)\times\mathcal{C}^n$, $(\widetilde{X},\widetilde{\mathcal{U}},\widetilde{W})$ the canonical variable on Ω_G, $\widetilde{\mathbb{F}}$ the associated canonical filtration. Let us consider a $\mathbb{R}^n\times\mathcal{P}(\Omega_G)$–valued $\mathbb{F}$–adapted continuous process $(X,\widehat{\mu})$ verifying:

$$X_t=\mathbf{X}_0+\int_0^t b\big(r,X_r,\mu,\overline{\mu}_r^\alpha,\alpha_r\big)\mathrm{d}r+\int_0^t\sigma\big(r,X_r,\mu,\overline{\mu}_r^\alpha,\alpha_r\big)\mathrm{d}W_r+\sigma_0B_t,\ t\in[0,T],\tag{6.5.9}$$

and

$$\widehat{\mu}_t=\mathcal{L}^{\mathbb{P}}\Big(X_{t\wedge\cdot},\mathcal{U}_{t\wedge\cdot},W\big|B_{t\wedge\cdot},\widehat{\mu}_{t\wedge\cdot}\Big)=\mathcal{L}^{\mathbb{P}}\Big(X_{t\wedge\cdot},\mathcal{U}_{t\wedge\cdot},W\big|B,\widehat{\mu}\Big)$$

with $\mu_t:=\mathcal{L}^{\widehat{\mu}}(\widetilde{X}_t)$, $\overline{\mu}_t^\alpha:=\mathcal{L}^{\widehat{\mu}}(\widetilde{X}_t,\widetilde{\alpha}_t)$, and $\mathcal{U}_s(\mathrm{d}u)\mathrm{d}s:=\delta_{\alpha_s}(\mathrm{d}u)\mathrm{d}s$, where $(\widetilde{\alpha}_t)_{t\in[0,T]}$ is the $\widetilde{\mathbb{F}}$–predictable process s.t. $\widetilde{\mathcal{U}}_s(\mathrm{d}u)\mathrm{d}s:=\delta_{\widetilde{\alpha}_s}(\mathrm{d}u)\mathrm{d}s$, in addition, $(B,\widehat{\mu})$ are $\mathbb{P}$–independent of $(\mathbf{X}_0,W)$.

Lemma 6.5.5. *Let* Assumption 5.5.5 *hold true.*
(i) If $\ell \neq 0$, then $q = 0$, H *disappears, and there exists a sequence $(\alpha^k)_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$ such that*

$$\lim_{k\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}_\nu}\Big(X^{\alpha^k}, W, B, \widehat{V}^k_T, \delta_{\left(\overline{\mu}^{\alpha^k}_s, \alpha^k_s\right)}(\mathrm{d}m,\mathrm{d}u)\mathrm{d}s\Big) = \mathcal{L}^{\mathbb{P}}\Big(X, W, B, \widehat{\mu}_T, \delta_{\left(\overline{\mu}^{\alpha}_s, \alpha_s\right)}(\mathrm{d}m,\mathrm{d}u)\mathrm{d}s\Big), \text{ in } \mathcal{W}_p, \tag{6.5.10}$$

where $\mathcal{U}^k_s(\mathrm{d}u)\mathrm{d}s := \delta_{\alpha^k_s}(\mathrm{d}u)\mathrm{d}s$, and

$$\widehat{V}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}_\nu}\Big(X^{\alpha^k}_{t\wedge\cdot}, \mathcal{U}^k_{t\wedge\cdot}, W\big|\widehat{\mathcal{G}}_t\Big) = \mathcal{L}^{\widehat{\mathbb{P}}_\nu}\Big(X^{\alpha^k}_{t\wedge\cdot}, \mathcal{U}^k_{t\wedge\cdot}, W\big|\widehat{\mathcal{G}}_T\Big), \text{ for all } t\in[0,T],\ \widehat{\mathbb{P}}_\nu\text{–a.e.}$$

In addition, for each sequence $(\gamma^k)_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$, there exists a sequence of Borel functions $(\phi^k)_{k\in\mathbb{N}^*}$ satisfying $\phi^k : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times C([0,T];\mathcal{P}(\Omega_G)) \to U$, such that if we let $X^{\alpha,k}$ be the $\mathbb{R}^n$–valued $\mathbb{F}$–adapted continuous process solution of

$$X^{\alpha,k}_t = \mathbf{X}_0 + \int_0^t b\big(r, X^{\alpha,k}_r, \mu^{\alpha^k}, \overline{\mu}^{\alpha^k}_r, \phi^k_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, X^{\alpha,k}_r, \mu, \overline{\mu}^{\alpha}_r, \phi^k_r\big)\mathrm{d}W_r + \sigma_0 B_t,\ t\in[0,T],$$

with $\phi^k_r := \phi^k(r, \mathbf{X}_0, W_{r\wedge\cdot}, B_{r\wedge\cdot}, \widehat{\mu}_{r\wedge\cdot})$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e., then one gets

$$\lim_{k\to\infty} \mathcal{W}_p\Big(\widehat{\mathbb{P}}_\nu \circ \Big(X^{\alpha^k,\gamma^k}, \widehat{V}^k, \delta_{(\overline{\mu}^{\alpha^k}_s, \overline{\gamma}^k_s, \gamma^k_s)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}s\Big)^{-1}, \mathbb{P}\circ\Big(X^{\alpha,k}, \widehat{\mu}, \delta_{(\overline{\mu}^{\alpha}_t, \overline{\phi}^k_t, \phi^k_t)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big)^{-1}\Big) = 0, \tag{6.5.11}$$

with $\overline{\gamma}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}_\nu}\big(X^{\alpha^k,\gamma^k}_t, \gamma^k_t\big|\widehat{\mathcal{G}}_t\big)$, and $\overline{\phi}^k_t := \mathcal{L}^{\mathbb{P}}\big(X^{\alpha,k}_t, \phi^k_t\big|B_{t\wedge\cdot}, \widehat{\mu}_{t\wedge\cdot}\big)$ for all $t\in[0,T]$.

(ii) If $\ell = 0$, then B disappears and the previous results (i) i.e. (6.5.10) and (6.5.11) stay true with $\widehat{\mathcal{G}}_t = \sigma\{H\}$ for all $t\in[0,T]$. Moreover when $\widehat{\mu}$ is deterministic, $q = 0$.

Remark 6.5.6. The techniques used to prove Lemma 6.5.5 are essentially borrowed from Proposition 2.3.12. The result (6.5.10) is a particular case of Proposition 2.3.12, while (6.5.11) is proved by adapting the techniques of the proof of (6.5.10). The result (6.5.11) is crucial for the transition from measure-valued MFG solution to the approximate strong MFG solution as we will see in the proof in Proposition 6.5.8.

Proof. This proof is essentially a mimicking of the proofs of Lemma 2.3.10 Lemma 2.3.11 and Proposition 2.3.12, we recall the main points used to finish our proof. Let $(\epsilon_k)_{k\in\mathbb{N}^*} \subset (0,\infty)$ such that $\lim_{k\to\infty}\epsilon_k = 0$.
There exists the unique strong solution X^k of:

$$X^k_t = \mathbf{X}_0 + \int_{\epsilon_k}^{\epsilon_k\vee t} b\big(r, X^k_r, \mu^k, \overline{\mu}^k_r, \alpha^k_r\big)\mathrm{d}r + \int_{\epsilon_k}^{\epsilon_k\vee t} \sigma\big(r, X^k_r, \mu^k, \overline{\mu}^k_r, \alpha^k_r\big)\mathrm{d}W^k_r + \sigma_0 B^k_t,\ t\in[0,T],\ \mathbb{P}\text{–a.e.},$$

with $\overline{\mu}^k_t := \mathcal{L}^{\mathbb{P}}\big(X^k_t, \alpha^k_t\big|B^k_{t\wedge\cdot}, \widehat{\mu}_{t\wedge\cdot}\big)$, $\mu^k_t := \mathcal{L}^{\mathbb{P}}\big(X^k_t\big|B^k_{t\wedge\cdot}, \widehat{\mu}_{t\wedge\cdot}\big)$, α^k is a piece wise constant control s.t. $\lim_k \alpha^k = \alpha$, $\alpha^k_t = 0$ when $t\in[0,\epsilon_k]$, and $W^k_\cdot := W_{\epsilon_k\vee\cdot} - W_{\epsilon_k}$, $B^k_\cdot := B_{\epsilon_k\vee\cdot} - B_{\epsilon_k}$. Using similar techniques as Lemma 2.3.10, one has

$$\lim_{k\to\infty} \mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]} \big|X^k_t - X_t\big|^p\Big] = 0. \tag{6.5.12}$$

Notice that for all $k\in\mathbb{N}^*$, if $\mathcal{U}^k_t(\mathrm{d}u)\mathrm{d}t := \delta_{\alpha^k_t}(\mathrm{d}u)\mathrm{d}t$, and $\widehat{\mu}^k_t := \mathcal{L}^{\mathbb{P}}\big(X^k_{t\wedge\cdot}, \mathcal{U}^k_{t\wedge\cdot}, W\big|B^k_{t\wedge\cdot}, \widehat{\mu}_{t\wedge\cdot}\big)$, for all $t\in[0,T]$, one has

$$\widehat{\mu}^k_t = \mathcal{L}^{\mathbb{P}}\big(X^k_{t\wedge\cdot}, \mathcal{U}^k_{t\wedge\cdot}, W\big|B^k_{t\wedge\cdot}, \widehat{\mu}^k_{t\wedge\cdot}\big) = \mathcal{L}^{\mathbb{P}}\big(X^k_{t\wedge\cdot}, \mathcal{U}^k_{t\wedge\cdot}, W\big|B^k, \widehat{\mu}^k\big),\ \mathbb{P}\text{–a.e.}$$

and $(B, \widehat{\mu}^k)$ are $\mathbb{P}$–independent of $(\mathbf{X}_0, W)$ and under Assumption 5.5.5, there exists a Borel measurable function $F^k : \mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times C([0,T];\mathcal{P}(\Omega_G))\times\mathbb{M}(U) \to \mathcal{C}^n$ such that $X^k = F^k(\mathbf{X}_0, W^k, B^k, \widehat{\mu}^k, \mathcal{U}^k)$, $\mathbb{P}$–a.e.

Next by (an extension of) Lemma 2.3.11, on $(\widehat{\Omega}^q, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}^q_\nu)$, there exist a $[0,1]$–valued uniform random variable V^k independent of (ξ, B^k, W), and a $(\sigma\{V^k, \xi, W_{t\wedge\cdot}, B^k_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable process $(\widehat{\gamma}^k_t)_{t\in[0,T]}$ such that: if $(\widehat{X}^k_t)_{t\in[0,T]}$ is the unique strong solution of

$$\widehat{X}^k_t = \xi + \int_{\epsilon_k}^{t\vee\epsilon_k} b\big(r, \widehat{X}^k_r, \zeta^k_r, \overline{\zeta}^k_r, \widehat{\gamma}^k_r\big)\mathrm{d}r + \int_{\epsilon_k}^{t\vee\epsilon_k} \sigma\big(r, \widehat{X}^k_r, \zeta^k_r, \overline{\zeta}^k_r, \widehat{\gamma}^k_r\big)\mathrm{d}W^k_r + \sigma_0 B^k_t, \text{ for all } t\in[0,T], \ \widehat{\mathbb{P}}^q_\nu\text{–a.e.},$$

with $\overline{\zeta}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{X}^k_t, \widehat{\gamma}^k_t \big| B^k, V^k\big)$ and $\zeta^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{X}^k_t \big| B^k, V^k\big)$, for all $t\in[0,T]$, then one has

$$\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(\widehat{X}^k, \widehat{\mathcal{U}}^k, W^k, B^k, \widehat{\zeta}^k\Big) = \mathcal{L}^{\mathbb{P}}\Big(X^k, \mathcal{U}^k, W^k, B^k, \widehat{\mu}^k\Big), \tag{6.5.13}$$

where $\widehat{\mathcal{U}}^k_t(\mathrm{d}u)\mathrm{d}t := \delta_{\widehat{\gamma}^k_t}(\mathrm{d}u)\mathrm{d}t$, and $\widehat{\zeta}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{X}^k_{t\wedge\cdot}, \widehat{\mathcal{U}}^k_{t\wedge\cdot}, W^k \big| B^k, V^k\big)$, for all $t\in[0,T]$, $\widehat{\mathbb{P}}^q_\nu$–a.e.

(i) When $\ell \neq 0$, $V^k = \phi(B_{\epsilon_k\wedge\cdot})$, where $\phi : \mathcal{C}^\ell \to [0,1]$ a Borel function s.t. $\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}(\phi(B_{\epsilon_k\wedge\cdot}))$ is a uniform law on $[0,1]$. Let us introduce the unique strong solution $\widehat{Z}^k$ of:

$$\widehat{Z}^k_t = \xi + \int_0^t b\big(r, \widehat{Z}^k_r, v^k, \overline{v}^k_r, \widehat{\alpha}^k_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{Z}^k_r, v^k, \overline{v}^k_r, \widehat{\alpha}^k_r\big)\mathrm{d}W_r + \sigma_0 B_t, \ t\in[0,T], \ \widehat{\mathbb{P}}^q_\nu\text{–a.e.},$$

with $\overline{v}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{Z}^k_t, \widehat{\alpha}^k_t \big| B_{t\wedge\cdot}\big)$ and $v^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{Z}^k_t \big| B_{t\wedge\cdot}\big)$, then one finds

$$\lim_{k\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}^q_\nu}\Big[\sup_{t\in[0,T]} \big|\widehat{Z}^k_t - \widehat{X}^k_t\big|^p\Big] = 0. \tag{6.5.14}$$

The results (6.5.12), (6.5.13) and 6.5.14 allow to deduce the first part of the lemma.

Now, let $(\psi^k : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell \to U)_{k\in\mathbb{N}^*}$ be a sequence of Borel measurable functions. Define the unique strong solution $\widehat{Z}^{\psi,k}$ of

$$\widehat{Z}^{\psi,k}_t = \xi + \int_0^t b\big(r, \widehat{Z}^{\psi,k}_r, v^k, \overline{v}^k_r, \psi^k_r\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{Z}^{\psi,k}_r, v^k, \overline{v}^k_r, \psi^k_r\big)\mathrm{d}W_r + \sigma_0 B_t, \ t\in[0,T], \ \widehat{\mathbb{P}}^q_\nu\text{–a.e.}$$

where recall that $\overline{v}^k_t = \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{Z}^k_t, \widehat{\alpha}^k_t \big| B_{t\wedge\cdot}\big)$, $v^k_t = \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{Z}^k_t \big| B_{t\wedge\cdot}\big)$, and $\psi^k_t := \psi^k(t, \xi, W_{t\wedge\cdot}, B_{t\wedge\cdot})$. Also, one introduces the unique strong solution $\widehat{S}^{\psi,k}$ of

$$\widehat{S}^{\psi,k}_t = \xi + \int_{\epsilon_k}^{\epsilon_k\vee t} b\big(r, \widehat{S}^{\psi,k}_r, \zeta^k, \overline{\zeta}^k_r, \hat{\psi}^k_r\big)\mathrm{d}r + \int_{\epsilon_k}^{\epsilon_k\vee t} \sigma\big(r, \widehat{S}^{\psi,k}_r, \zeta^k, \overline{\zeta}^k_r, \hat{\psi}^k_r\big)\mathrm{d}W^k_r + \sigma_0 B^k_t, \ t\in[0,T], \ \widehat{\mathbb{P}}^q_\nu\text{–a.e.}$$

with a piece wise constant control $\hat{\psi}^k$ satisfying $\lim_{k\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}^q_\nu}\Big[\int_0^T \rho(\hat{\psi}^k_t, \psi^k_t)\mathrm{d}t\Big] = 0$, and $\widehat{\psi}^k_t = 0$ with $t\in[0,\epsilon_k]$, and recall that $\overline{\zeta}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{X}^k_t, \widehat{\gamma}^k_t \big| B^k, V^k\big)$ and $\zeta^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{X}^k_t \big| B^k, V^k\big)$. Using (6.5.14), one gets

$$\lim_{k\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}^q_\nu}\Big[\sup_{t\in[0,T]} |\widehat{Z}^{\psi,k}_t - \widehat{S}^{\psi,k}_t|^p\Big] = 0. \tag{6.5.15}$$

Under Assumption 5.5.5, one has $\widehat{S}^{\psi,k} = H^k(\xi, W^k, B^k, \widehat{\zeta}, \widehat{\Upsilon}^k)$ with $H^k : \mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times C([0,T];\mathcal{P}(\Omega_G))\times\mathbb{M}(U) \to \mathcal{C}^n$ a Borel function and $\widehat{\Upsilon}^k_t(\mathrm{d}u)\mathrm{d}t := \delta_{\hat{\psi}^k_t}(\mathrm{d}u)\mathrm{d}t$.

There exists a Borel function $\beta^k : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times C([0,T];\mathcal{P}(\Omega_G)\times\mathcal{C}^n \to U$ such that

$$\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\xi, W^k, B^k, \widehat{\zeta}, \widehat{\Upsilon}^k\big) = \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\mathbf{X}_0, W^k, B^k, \widehat{\zeta}, \widehat{\beta}^k\big),$$

with $\widehat{\beta}^k_t(\mathrm{d}u)\mathrm{d}t := \delta_{\beta^k_t\left(t,\mathbf{X}_0,W^k_{t\wedge\cdot},B^k_{t\wedge\cdot},\widehat{\zeta}^k_{t\wedge\cdot},W_{\epsilon_k\wedge\cdot}\right)}(\mathrm{d}u)\mathrm{d}t$. Similarly to the first part (see Lemma 2.3.11 again), as we know that the law of $(\xi,W^k,B^k,\widehat{\zeta})$ under $\widehat{\mathbb{P}}^q_\nu$ and the law of $(\mathbf{X}_0,W^k,B^k,\widehat{\mu}^k)$ under $\mathbb{P}$ are the same (see equation (6.5.13)), using the independence of $W_{\epsilon_k\wedge\cdot}$ (w.r.t. $\widehat{\mathbb{P}}^q_\nu$ and $\mathbb{P}$), if $S^{\phi,k} := H^k(\mathbf{X}_0,W^k,B^k,\widehat{\mu}^k,\widehat{\Phi}^k)$, where

$$\widehat{\Phi}^k_t(\mathrm{d}u)\mathrm{d}t := \delta_{\beta^k\left(t,\mathbf{X}_0,W^k_{t\wedge\cdot},B^k_{t\wedge\cdot},\widehat{\mu}^k_{t\wedge\cdot},W_{\epsilon_k\wedge\cdot}\right)}(\mathrm{d}u)\mathrm{d}t, \text{ one finds } \mathcal{L}^{\mathbb{P}}\big(S^{\phi,k},W^k,B^k,\widehat{\mu}^k,\widehat{\Phi}^k\big) = \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{S}^{\psi,k},W^k,B^k,\widehat{\zeta}^k,\widehat{\Upsilon}^k\big), \tag{6.5.16}$$

and $S := S^{\phi,k}$ satisfies

$$S^{\phi,k}_t = \mathbf{X}_0 + \int_{\epsilon_k}^{\epsilon_k\vee t} b\big(r,S^{\phi,k}_r,\mu^k,\overline{\mu}^k_r,\phi^k_r\big)\mathrm{d}r + \int_{\epsilon_k}^{\epsilon_k\vee t}\sigma\big(r,S^{\phi,k}_r,\mu^k,\overline{\mu}^k_r,\phi^k_r\big)\mathrm{d}W^k_r + \sigma_0 B^k_t,\ t\in[0,T],\ \mathbb{P}\text{–a.e.}$$

where $\phi^k_t := \beta^k(t,\mathbf{X}_0,W^k_{t\wedge\cdot},B^k_{t\wedge\cdot},\widehat{\mu}^k_{t\wedge\cdot},W_{\epsilon_k\wedge\cdot})$. Notice that $\phi^k_s = 0$ for $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e. $(t,\omega)\in[0,\epsilon_k]\times\Omega$, and ϕ^k is a $(\sigma\{\mathbf{X}_0,W_{t\wedge\cdot},B_{t\wedge\cdot},\widehat{\mu}^k_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable process. Let $X^{\phi,k}$ be the strong solution of:

$$X^{\phi,k}_t = \mathbf{X}_0 + \int_0^t b\big(r,X^{\phi,k}_r,\mu,\overline{\mu}^\alpha_r,\phi^k_r\big)\mathrm{d}r + \int_0^t\sigma\big(r,X^{\phi,k}_r,\mu,\overline{\mu}^\alpha_r,\phi^k_r\big)\mathrm{d}W_r + \sigma_0 B_t,\ t\in[0,T],\ \text{–a.e.}$$

then one has

$$\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}}\Big[\sup_{t\in[0,T]}\big|S^{\phi,k}_t - X^{\phi,k}_t\big|^p\Big] = 0. \tag{6.5.17}$$

Denote $\overline{\phi}^k_t := \mathcal{L}^{\mathbb{P}}(X^{\phi,k}_t,\phi^k_t|B,\widehat{\mu})$, $\overline{\psi}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}(\widehat{Z}^{\psi,k}_t,\psi^k_t|B^k,V^k)$, $\overline{\kappa}^k_t := \mathcal{L}^{\mathbb{P}}(S^{\phi,k}_t,\phi^k_t|B,\widehat{\mu})$, $\overline{\theta}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}(\widehat{S}^{\psi,k}_t,\beta^k_t|B^k,V^k)$, and $\widehat{V}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(\widehat{Z}^k_{t\wedge\cdot},(\delta_{\hat{\alpha}^k_s}(\mathrm{d}u)\mathrm{d}s)_{t\wedge\cdot},W^k\big|B^k,V^k\big)$, to summarize, by combining (6.5.17), (6.5.16) and (6.5.15),

$$\begin{aligned}
&\limsup_k \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}}\Big(X^{\phi,k},W,B,\widehat{\mu},\delta_{\left(\overline{\mu}^\alpha_t,\overline{\phi}^k_t,\phi^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big),\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(\widehat{Z}^{\psi,k},W,B,\widehat{V}^k,\delta_{\left(\overline{v}^k_t,\overline{\psi}^k_t,\psi^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big)\Big)\\
&\le \limsup_k \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}}\Big(X^{\phi,k},W,B,\widehat{\mu},\delta_{\left(\overline{\mu}^\alpha_t,\overline{\phi}^k_t,\phi^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big),\mathcal{L}^{\mathbb{P}}\Big(S^{\phi,k},W^k,B^k,\widehat{\mu}^k,\delta_{\left(\overline{\mu}^k_t,\overline{\kappa}^k_t,\phi^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big)\Big)\\
&\quad+\limsup_k \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}}\Big(S^{\phi,k},W^k,B^k,\widehat{\mu}^k,\delta_{\left(\overline{\mu}^k_t,\overline{\kappa}^k_t,\phi^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big),\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(\widehat{S}^{\psi,k},W^k,B^k,\widehat{\zeta}^k,\delta_{\left(\overline{\zeta}^k_t,\overline{\theta}^k_t,\beta^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big)\Big)\\
&\quad+\limsup_k \mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(\widehat{S}^{\psi,k},W^k,B^k,\widehat{\zeta}^k,\delta_{\left(\overline{\zeta}^k_t,\overline{\theta}^k_t,\beta^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big),\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(\widehat{Z}^{\psi,k},W,B,\widehat{V}^k,\delta_{\left(\overline{v}^k_t,\overline{\psi}^k_t,\psi^k_t\right)}(\mathrm{d}m,\mathrm{d}m',\mathrm{d}u)\mathrm{d}t\Big)\Big)\\
&= 0.
\end{aligned}$$

This is enough to deduce the second part of point (i) of this lemma.

(ii) When $\ell = 0$, it is enough to use the same technique as Proposition 2.3.12 i.e. B disappears, $V^k = \mathrm{H}$, and use the variable H for the conditioning and repeat the exact proof like previously to obtain the result. In the case where $\widehat{\mu}$ is deterministic, as mentioned in Lemma 2.3.11, V^k (then H) disappears. There is no conditioning, and the proof is exactly the same.

□

Let $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$, in the next lemma, we stay on the filtered probability space introduced in Lemma 6.5.5 i.e. $(\widehat{\Omega}^q,\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}}^q_\nu)$ for $q\in\{0,1\}$. Recall that the notations on $(\widehat{\Omega}^q,\widehat{\mathbb{F}},\widehat{\mathcal{F}},\widehat{\mathbb{P}}^q_\nu)$ for all the variables defined on $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P}_\nu)$ stay identical.

Lemma 6.5.7. *Under* Assumption 5.5.5, *for any $\alpha\in\mathcal{A}(\nu)$, there exists a sequence $(\alpha^{i,N})_{(i,N)\in\{1,\dots,N\}\times\mathbb{N}^*}$ satisfying for each $N\in\mathbb{N}^*$, $(\alpha^{i,N})_{i\in\{1,\dots,N\}}\subset\mathcal{A}(\nu_N)$ s.t.*

$$\lim_{N\to\infty}\mathbb{P}^N_\nu\circ\Big(\varphi^{N,\mathbf{X},\overline{\alpha}^N},\delta_{\varphi^{N,\overline{\alpha}^N}_s}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1} = \widehat{\mathbb{P}}^q_\nu\circ\Big(\mu^\alpha,\delta_{\overline{\mu}^\alpha_s}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1},$$

where $\overline{\alpha}^N = (\alpha^{i,N}, \dots, \alpha^{i,N})$.

In addition, for any sequence $(\kappa^{i,N})_{(i,N)\in\{1,\dots,N\}\times\mathbb{N}^*}$ *satisfying for each* $N \in \mathbb{N}^*$, $(\kappa^{i,N})_{i\in\{1,\dots,N\}} \subset \mathcal{A}(\nu_N)$, *there exists a family of Borel functions* $(\hat{\phi}^{i,N,k})_{(i,N,k)\in\{1,\dots,N\}\times\mathbb{N}^*\times\mathbb{N}^*}$ *with* $\hat{\phi}^{i,N,k} : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times[0,1] \to U$ *such that if* $\phi_t^{i,N,k} := \hat{\phi}^{i,N,k}(t, \mathbf{X}_0, W_{t\wedge\cdot}, B_{t\wedge\cdot}, \mathrm{H})$, *and*

$$\mathrm{P}^{i,N,k} := \widehat{\mathbb{P}}_\nu^q \circ \Big(\mu^{\alpha,i,N,k}, \mu^\alpha, \delta_{\overline{\mu}_s^{\alpha,i,N,k}}(\mathrm{d}m)\mathrm{d}s, \delta_{\overline{\mu}_s^\alpha}(\mathrm{d}m')\mathrm{d}s, B\Big)^{-1},$$

with $\overline{\mu}_t^{\alpha,i,N,k} := \mathcal{L}^{\widehat{\mathbb{P}}_\nu^q}(X_t^{\alpha,\phi^{i,N,k}}, \phi_t^{i,N,k}|\widehat{\mathcal{G}}_t)$ *and* $\mu_t^{\alpha,i,N,k}(\mathrm{d}x) := \overline{\mu}_t^{\alpha,i,N,k}(\mathrm{d}x, U)$, *then one gets*

$$\lim_{N\to\infty}\lim_{k\to\infty}\left|\frac{1}{N}\sum_{i=1}^N J_i\big((\overline{\alpha}^{[-i]}, \kappa^{i,N})\big) - \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathrm{P}^{i,N,k}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]\right| = 0.$$

Proof. Notice that $(\mu^\alpha, \overline{\mu}^\alpha)$ are $\mathbb{G}$–adapted then B–measurable in that case, consequently these variables can be extended on $(\Omega^N, \mathbb{F}^N, \mathcal{F}^N, \mathbb{P}_\nu^N)$. The notation stays the same. Using (an extention of) Proposition 3.4.4, there exists $(\psi^{i,N})_{(i,N)}$ a sequence of Borel functions $\psi^{i,N} : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell \to U$, such that if $\alpha_t^{i,N} := \psi^{i,N}(t, \mathbf{X}_0^i, \mathbf{W}_{t\wedge\cdot}^i, B_{t\wedge\cdot})$ for all $t\in[0,T]$, and $\overline{\alpha}^N = (\alpha^{i,N}, \dots, \alpha^{i,N})$, then one has

$$\lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu^N}\left[\sup_{t\in[0,T]}\mathcal{W}_p\big(\varphi_t^{N,\mathbf{X},\overline{\alpha}^N}, \mu_t^\alpha\big) + \int_0^T \mathcal{W}_p\big(\varphi_t^{N,\overline{\alpha}^N}, \overline{\mu}_t^\alpha\big)\mathrm{d}t\right] = 0.$$

Next, by easy adaptation of Lemma 6.5.3 (successive application of Gronwall Lemma), there exists a sequence $(C_N)_{N\in\mathbb{N}^*}$ converging to zero when N goes to infinity satisfying: for each $i\in\{1,\dots,N\}$, if $\overline{\alpha}^{N,-i} := (\overline{\alpha}^{-i}, \kappa^{i,N})$,

$$\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T \mathcal{W}_p\big(\varphi_t^{N,\overline{\alpha}^{N,-i}}, \overline{\mu}_t^\alpha\big)\mathrm{d}t + \sup_{t\in[0,T]}\mathcal{W}_p\big(\varphi_t^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \mu_t^\alpha\big)\right] + \mathbb{E}^{\mathbb{P}_\nu^N}\left[\sup_{t\in[0,T]}\big|\mathbf{X}_t^i[(\overline{\alpha}^{-i}, \kappa^{i,N})] - X_t^{\alpha,\kappa^{i,N}}\big|^p\right] \le C_N,$$

where $X^{\alpha,\kappa^{i,N}}$ denote the unique strong solution of

$$X_t^{\alpha,\kappa^{i,N}} = \mathbf{X}_0^i + \int_0^t b\big(r, X_r^{\alpha,\kappa^{i,N}}, \mu_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \kappa_r^{i,N}\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^{\alpha,\kappa^{i,N}}, \mu_{r\wedge\cdot}^\alpha, \overline{\mu}_r^\alpha, \kappa_r^{i,N}\big)\mathrm{d}\mathbf{W}_r^i + \sigma_0 B_t.$$

Therefore, $\limsup_{N\to\infty}\mathcal{W}_p\big(Q^N, \widetilde{Q}^N\big) = 0$, where $Q^N := \frac{1}{N}\sum_{i=1}^N \mathcal{L}^{\mathbb{P}_\nu^N}\Big(\mathbf{X}^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \delta_{\left(\kappa_t^{i,N}, \varphi_t^{N,\overline{\alpha}^{N,-i}}\right)}(\mathrm{d}u,\mathrm{d}m')\mathrm{d}t\Big)$, and $\widetilde{Q}^N := \frac{1}{N}\sum_{i=1}^N \mathcal{L}^{\mathbb{P}_\nu^N}\Big(X^{\alpha,\kappa^{i,N}}, \mu^\alpha, \delta_{\left(\kappa_t^{i,N}, \overline{\mu}_t^\alpha\right)}(\mathrm{d}u,\mathrm{d}m')\mathrm{d}t\Big)$.
Thanks to this result and some techniques used in proof of Lemma 6.5.4 (with the separability condition), one has

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T L\big(t, \mathbf{X}_t^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}, \varphi_t^{N,\overline{\alpha}^{N,-i}}, \kappa_t^{i,N}\big)dt + g\big(\mathbf{X}_T^i[\overline{\alpha}^{N,-i}], \varphi^{N,\mathbf{X},\overline{\alpha}^{N,-i}}\big)\right]\right.$$
$$\left. - \frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}_\nu^N}\left[\int_0^T L(t, X_t^{\alpha,\kappa^{i,N}}, \mu_{t\wedge\cdot}^\alpha, \overline{\mu}_t^\alpha, \kappa_t^{i,N})\mathrm{d}t + g(X_T^{\alpha,\kappa^{i,N}}, \mu_T^\alpha)\right]\right| = 0.$$

By the same techniques used in the proof of Lemma 6.5.5, for all $q\in\{0,1\}$ (whatever $\ell = 0$, or $\ell\neq 0$, see also Proposition 2.3.12), there exists, for each (i,N), $(\hat{\phi}^{i,N,k})_{k\in\mathbb{N}^*}$ a family of Borel functions $\hat{\phi}^{i,N,k} : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times[0,1]$ such that if $\phi_t^{i,N,k} := \hat{\phi}^{i,N,k}(t,\xi, W_{t\wedge\cdot}, B_{t\wedge\cdot}, \mathrm{H})$,

$$\lim_{k\to\infty}\widehat{\mathbb{P}}_\nu^q \circ \Big(X^{\alpha,\phi^{i,N,k}}, \mu^\alpha, \delta_{(\overline{\mu}_t^\alpha, \phi_t^{i,N,k})}(\mathrm{d}m',\mathrm{d}u)\mathrm{d}t, W, B\Big)^{-1} = \mathbb{P}_\nu^N \circ \Big(X^{\alpha,\kappa^{i,N}}, \mu^\alpha, \delta_{(\overline{\mu}_t^\alpha, \kappa_t^{i,N})}(\mathrm{d}m',\mathrm{d}u)\mathrm{d}t, \mathbf{W}^i, B\Big)^{-1},$$

we can conclude the proof.

□

The following result is a combination of Lemma 6.5.5 and Lemma 7.4.2. Let $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, we continue to work on the filtered probability space introduced in Lemma 6.5.5 i.e. $(\widehat{\Omega}^q, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}^q_\nu)$ for $q \in \{0,1\}$. Recall that, for the variables defined on $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P}_\nu)$, after extension on $(\widehat{\Omega}^q, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}}^q_\nu)$, we use the same notation.

The next proposition provides first an approximation of Fokker–Planck process mentioned in Proposition 4.8.3. And in second, for a particular sequence of processes of type $X^{\alpha,\alpha'}$ (see Definition 6.3.2), it shows that there exists a sequence of measure-valued control rules that has the same limit.

Proposition 6.5.8. *Let Assumption 5.5.5 hold true and $\epsilon > 0$. For any $\mathrm{P} \in \overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$, there exists a sequence $(\alpha^k)_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$ such that*

$$\lim_{k\to\infty} \widehat{\mathbb{P}}^q_\nu \circ \Big(\mu^{\alpha^k}, \mu^{\alpha^k}, \delta_{\overline{\mu}^{\alpha^k}_s}(\mathrm{d}m)\mathrm{d}s, \delta_{\overline{\mu}^{\alpha^k}_s}(\mathrm{d}m')\mathrm{d}s, B\Big)^{-1} = \mathrm{P}, \text{ in } \mathcal{W}_p.$$

Besides, for any sequence $(\beta^k)_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$, there exists a sequence $(\mathrm{Q}^k)_{k\in\mathbb{N}^*}$ satisfying $\mathrm{Q}^k \in \overline{\mathcal{P}}_V(\nu)$ with $\mathrm{Q}^k \circ \big(\zeta, \Lambda, B\big)^{-1} = \mathrm{P} \circ \big(\zeta, \Lambda, B\big)^{-1}$ for each $k \in \mathbb{N}^*$ and

$$\lim_{k\to\infty} \mathcal{W}_p\Big(\widehat{\mathbb{P}}^q_\nu \circ \big(\mu^k, \mu^{\alpha^k}, \delta_{\overline{\mu}^k_s}(\mathrm{d}m)\mathrm{d}s, \delta_{\overline{\mu}^{\alpha^k}_s}(\mathrm{d}m')\mathrm{d}s, B\big)^{-1}, \mathrm{Q}^k\Big) = 0, \tag{6.5.18}$$

where $\mu^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(X^{\alpha^k,\beta^k}_t | \widehat{\mathcal{G}}_t\big)$ and $\overline{\mu}^k_t := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(X^{\alpha^k,\beta^k}_t, \beta^k_t | \widehat{\mathcal{G}}_t\big)$ for all $t \in [0,T]$. In addition when $\ell \neq 0$ then $q = 0$ and H disappears, and when $\ell = 0$, one has $q = 1$.

Remark 6.5.9. We emphasize that it is not easy to find a sequence of measure–valued control rules verifying (6.5.18). Indeed, notice that the set $\overline{\mathcal{P}}_V(\nu)$ is not a closed set in general. Therefore a classical compactness argument does not work here. Lemma 6.5.5 and Lemma 7.4.2 as well as the approximation result of Proposition 4.8.3 are very important for the proof of this proposition.

Proof. let $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ be a filtered probability space supporting a $\mathbb{R}^n$– Brownian motion W and a $\widetilde{\mathcal{F}}_0$–random variable ξ s.t. $\mathcal{L}^{\widetilde{\mathbb{P}}}(\xi) = \nu$. Let us introduce the filtered probability space $(\widehat{\Omega}, \widehat{\mathbb{F}}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ which is defined as follows: $\widehat{\Omega} := \widetilde{\Omega} \times \overline{\Omega}$, $\widehat{\mathbb{F}} := (\widetilde{\mathcal{F}}_t \otimes \overline{\mathcal{F}}_t)_{t\in[0,T]}$ and $\widehat{\mathbb{P}} := \widetilde{\mathbb{P}} \otimes \mathrm{P}$. The variables (ξ, W) of $\widetilde{\Omega}$ and the variables $(B, \mu, \Lambda^\circ, \zeta, \Lambda)$ of $\overline{\Omega}$ are extended on the space $\widehat{\Omega}$ while keeping the same notion $(\xi, W, B, \mu, \Lambda^\circ, \zeta, \Lambda)$. Also denote by $(\overline{\mathcal{G}}_t)_{t\in[0,T]}$ the filtration defined by

$$\overline{\mathcal{G}}_t := \sigma\big\{B_{t\wedge\cdot}, \zeta_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}\big\}, \text{ for all } t \in [0,T].$$

As $\mathrm{P} \in \overline{\mathcal{P}}^\star_V(\nu)[\epsilon]$, by Proposition 4.8.3, for any uniform variable Z $\widehat{\mathbb{P}}$–independent of $(\xi, W, B, \mu, \Lambda^\circ, \zeta, \Lambda)$, there exists a sequence of $\widehat{\mathbb{F}}$–predictable processes $(\alpha^j)_{j\in\mathbb{N}^*}$ satisfying for each $j \in \mathbb{N}^*$,

$$\alpha^j_t := G^j(t, \xi, \mu_{t\wedge\cdot}, \Lambda_{t\wedge\cdot}, W_{t\wedge}, B_{t\wedge}, Z), \ \widehat{\mathbb{P}}\text{–a.e., for all } t \in [0,T],$$

where $G^j : [0,T] \times \mathbb{R}^n \times \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}^n_U) \times \mathcal{C}^n \times \mathcal{C}^\ell \times [0,1] \to U$ is a Borel function s.t. if $\widehat{X}^j$ is the unique strong solution of: for all $t \in [0,T]$

$$\widehat{X}^j_t = \xi + \int_0^t b(r, \widehat{X}^j_r, \mu^j, \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^j_r, \alpha^j_r | \overline{\mathcal{G}}_r), \alpha^j_r)\mathrm{d}r + \int_0^t \sigma(r, \widehat{X}^j_r, \mu^j, \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^j_r, \alpha^j_r | \overline{\mathcal{G}}_r), \alpha^j_r)\mathrm{d}W_r + \sigma_0 B_t, \ \widehat{\mathbb{P}}\text{–a.e.}$$

where $\mu^j_t := \mathcal{L}^{\mathbb{P}}(\widehat{X}^j_t | \overline{\mathcal{G}}_t)$, and $\overline{\mu}^j_t := \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}^j_t, \alpha^j_t | \overline{\mathcal{G}}_t)$, then

$$\lim_{j\to\infty} \Big[\mathcal{W}_p\Big(\delta_{\overline{\mu}^j_t}(\mathrm{d}m)\mathrm{d}t, \Lambda_t(\mathrm{d}m)\mathrm{d}t\Big) + \sup_{t\in[0,T]} \mathcal{W}_p(\mu^j_t, \mu_t)\Big] = 0, \ \widehat{\mathbb{P}}\text{–a.e.}$$

and consequently

$$\lim_{j\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}}\Big((\mu^j_t)_{t\in[0,T]}, \delta_{\overline{\mu}^j_t}(\mathrm{d}m)\mathrm{d}t, (B_t)_{t\in[0,T]}\Big) = \mathrm{P} \circ \big(\mu, \Lambda, B\big)^{-1}, \text{ for the Wasserstein metric } \mathcal{W}_p.$$

Recall that, as $\mathrm{P} \in \overline{\mathcal{P}}_V^\star(\nu)[\epsilon]$, $\mu = \zeta$ and $\Lambda = \Lambda^\circ$, P–a.e.. Define for all $t \in [0,T]$, $\widehat{\mu}_t^j := \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{X}^j_{t\wedge\cdot}, \mathcal{U}^j_{t\wedge\cdot}, W \big| \overline{\mathcal{G}}_t\Big)$, and $\mathcal{U}_s^j(\mathrm{d}u)\mathrm{d}s := \delta_{\alpha_s^j}(\mathrm{d}u)\mathrm{d}s$. It is straightforward to check that: for each $t \in [0,T]$, $\widehat{\mu}_t^j = \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{X}^j_{t\wedge\cdot}, \mathcal{U}^j_{t\wedge\cdot}, W \big| \overline{\mathcal{G}}_T\Big)$, $\widehat{\mu}_t^j = \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{X}^j_{t\wedge\cdot}, \mathcal{U}^j_{t\wedge\cdot}, W \big| B_{t\wedge\cdot}, \widehat{\mu}^j_{t\wedge\cdot}\Big) = \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{X}^j_{t\wedge\cdot}, \mathcal{U}^j_{t\wedge\cdot}, W \big| B, \widehat{\mu}^j\Big)$, and $(B, \widehat{\mu})$ are $\widehat{\mathbb{P}}$–independent of (ξ, W).

Let $j \in \mathbb{N}^*$ be fixed, for each $q \in \{0,1\}$, by Lemma 6.5.5, there exists a sequence $(\alpha^{k,j})_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$ such that

$$\lim_{k\to\infty} \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(X^{\alpha^{k,j}}, W, B, \widehat{\mu}_T^{k,j}, \delta_{\left(\overline{\mu}_t^{\alpha^{k,j}}, \alpha_t^{k,j}\right)}(\mathrm{d}m, \mathrm{d}u)\mathrm{d}t\Big) = \mathcal{L}^{\widehat{\mathbb{P}}}\Big(\widehat{X}^j, W, B, \widehat{\mu}_T^j, \delta_{\left(\overline{\mu}_t^j, \alpha_t^j\right)}(\mathrm{d}m, \mathrm{d}u)\mathrm{d}t\Big), \text{ in } \mathcal{W}_p,$$

where $\mathcal{U}_s^{k,j}(\mathrm{d}u)\mathrm{d}s := \delta_{\alpha_s^{k,j}}(\mathrm{d}u)\mathrm{d}s$, and

$$\widehat{\mu}_t^{k,j} = \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(X^{\alpha^{k,j}}_{t\wedge\cdot}, \mathcal{U}^{k,j}_{t\wedge\cdot}, W \big| \widehat{\mathcal{G}}_t\Big) = \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(X^{\alpha^{k,j}}_{t\wedge\cdot}, \mathcal{U}^{k,j}_{t\wedge\cdot}, W \big| \widehat{\mathcal{G}}_T\Big), \text{ for all } t \in [0,T], \ \widehat{\mathbb{P}}^q_\nu\text{–a.e..}$$

In addition, for each sequence $(\gamma^{k,j})_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$, there exists a sequence of Borel functions $(\phi^{k,j})_{k\in\mathbb{N}^*}$ with $\phi^{k,j} : [0,T]\times\mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell\times C([0,T];\mathcal{P}(\mathcal{C}^n\times\mathbb{M}(U)\times\mathcal{C}^n)) \to U$, such that if $\widehat{X}^{k,j}$ is a $\mathbb{R}^n$–valued $\mathbb{F}$–adapted continuous process solution of

$$\widehat{X}_t^{k,j} = \xi + \int_0^t b\big(r, \widehat{X}_r^{k,j}, \mu^j, \overline{\mu}_r^j, \phi_r^{k,j}\big)\mathrm{d}r + \int_0^t \sigma\big(r, \widehat{X}_r^{k,j}, \mu^j, \overline{\mu}_r^j, \phi_r^{k,j}\big)\mathrm{d}W_r + \sigma_0 B_t, \ t \in [0,T], \ \widehat{\mathbb{P}}\text{–a.e.}$$

where $\phi_r^{k,j} := \phi^{k,j}(r, \xi, W_{r\wedge\cdot}, B_{r\wedge\cdot}, \widehat{\mu}^j_{r\wedge\cdot})$, $\mathrm{d}\widehat{\mathbb{P}}\otimes\mathrm{d}t$–a.e., then one gets

$$\lim_{k\to\infty} \mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(X^{\alpha^{k,j},\gamma^{k,j}}, \widehat{\mu}^{k,j}, \delta_{(\overline{\mu}_s^{\alpha^{k,j}}, \overline{\vartheta}_s^{k,j}, \gamma_s^{k,j})}(\mathrm{d}m, \mathrm{d}m', \mathrm{d}u)\mathrm{d}s\big), \mathcal{L}^{\widehat{\mathbb{P}}}\big(\widehat{X}^{k,j}, \widehat{\mu}^j, \delta_{(\overline{\mu}_t^j, \overline{\mu}_t^{k,j}, \phi_t^{k,j})}(\mathrm{d}m, \mathrm{d}m', \mathrm{d}u)\mathrm{d}t\big)\Big) = 0, \tag{6.5.19}$$

with $\vartheta_t^{k,j} := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(X_t^{\alpha^{k,j},\gamma^{k,j}} \big| \widehat{\mathcal{G}}_t\big)$, $\overline{\vartheta}_t^{k,j} := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\big(X_t^{\alpha^{k,j},\gamma^{k,j}}, \gamma_t^{k,j} \big| \widehat{\mathcal{G}}_t\big)$, and $\overline{\mu}_t^{k,j} := \mathcal{L}^{\widehat{\mathbb{P}}}\big(\widehat{X}_t^{k,j}, \phi_t^{k,j} \big| B_{t\wedge\cdot}, \widehat{\mu}^j_{t\wedge\cdot}\big)$ for all $t \in [0,T]$. Mention that when $\ell \neq 0$, then $q = 0$ and H disappears, and when $\ell = 0$, $q = 1$.

Next, as $\lim_{j\to\infty}\big(\delta_{\overline{\mu}_t^j}(\mathrm{d}m)\mathrm{d}t, \mu^j\big) = \big(\Lambda, \mu\big)$ $\widehat{\mathbb{P}}$ a.e., $(\widehat{\mu}_t^j, \overline{\mu}_t^j, \mu_t^j)_{t\in[0,T]}$ is $(\overline{\mathcal{G}}_t)_{t\in[0,T]}$–adapted and (Λ, μ, B) is $\widehat{\mathbb{P}}$–independent of (W, ξ), by Lemma 7.4.2 (see Appendix of Chapter 3), there exists $(\mathrm{P}^{k,j})_{(k,j)\in\mathbb{N}^*\times\mathbb{N}^*} \subset \overline{\mathcal{P}}_V(\nu)$ such that $\mathcal{L}^{\mathrm{P}^{k,j}}\big(\zeta, \Lambda, B\big) = \mathcal{L}^{\mathrm{P}}\big(\zeta, \Lambda, B\big)$ and

$$\limsup_{j\to\infty}\limsup_{k\to\infty} \mathcal{W}_p\Big(\mathrm{P}^{k,j}, \mathcal{L}^{\widehat{\mathbb{P}}}\big(\mu^{k,j}, \mu^j, \delta_{\overline{\mu}_s^{k,j}}(\mathrm{d}m)\mathrm{d}s, \delta_{\overline{\mu}_s^j}(\mathrm{d}m')\mathrm{d}s, B\big)\Big) = 0, \tag{6.5.20}$$

where $\overline{\mu}_t^{k,j} := \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}_t^{k,j} | B_{t\wedge\cdot}, \widehat{\mu}^j_{t\wedge\cdot})$ and $\mu_t^{k,j} := \mathcal{L}^{\widehat{\mathbb{P}}}(\widehat{X}_t^{k,j} | B_{t\wedge\cdot}, \widehat{\mu}^j_{t\wedge\cdot})$, for all $t \in [0,T]$.

The results (6.5.19) and (6.5.20) are enough to deduce that

$$\limsup_{j\to\infty}\limsup_{k\to\infty} \mathcal{W}_p\Big(\mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}\Big(\vartheta^{k,j}, \mu^{\alpha^{k,j}}, \delta_{\overline{\vartheta}_s^{k,j}}(\mathrm{d}m)\mathrm{d}s, \delta_{\overline{\mu}_s^{\alpha^{k,j}}}(\mathrm{d}m')\mathrm{d}s, B\Big), \mathrm{P}^{k,j}\Big) = 0,$$

and conclude the result.

□

6.5.2.2 Proof of Theorem 6.4.2 (Converse Limit Theorem)

First point (i) Let $\mathrm{P} \in \overline{\mathcal{P}}_V^\star(\nu)[\epsilon]$ be an ϵ–measure-valued solution. By Proposition 6.5.8, first, there exists a sequence $(\alpha^k)_{k\in\mathbb{N}^*} \subset \mathcal{A}(\nu)$ s.t.

$$\lim_{k\to\infty} \widehat{\mathbb{P}}^q_\nu \circ \Big(\mu^{\alpha^k}, \mu^{\alpha^k}, \delta_{\overline{\mu}_r^{\alpha^k}}(\mathrm{d}m)\mathrm{d}r, \delta_{\overline{\mu}_r^{\alpha^k}}(\mathrm{d}m')\mathrm{d}r, B\Big)^{-1} = \mathrm{P}, \text{ in } \mathcal{W}_p.$$

Let us introduce, for each $\alpha' \in \mathcal{A}(\nu)$,

$$\Psi(\alpha^k,\alpha') := \mathbb{E}^{\widehat{\mathbb{P}}^q_\nu}\Big[\int_0^T L(t,X_t^{\alpha^k,\alpha'},\mu_{t\wedge\cdot}^{\alpha^k},\overline{\mu}_t^{\alpha^k},\alpha'_t)\mathrm{d}t + g(X_T^{\alpha^k,\alpha'},\mu^{\alpha^k})\Big] \text{ and } \epsilon^k := \sup_{\alpha'\in\mathcal{A}(\nu)}\Psi(\alpha^k,\alpha') - \mathbb{E}^{\mathrm{P}^k}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big],$$

with $\mathrm{P}^k := \widehat{\mathbb{P}}^q_\nu \circ \Big(\mu^{\alpha^k},\mu^{\alpha^k},\delta_{\overline{\mu}_s^{\alpha^k}}(\mathrm{d}m)\mathrm{d}s,\delta_{\overline{\mu}_s^{\alpha^k}}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1}$.

Remark that $\epsilon^k \geq 0$, for all k. There exists a sequence $(\gamma^k)_{k\in\mathbb{N}} \subset \mathcal{A}(\nu)$ verifying

$$\Psi(\alpha^k,\gamma^k) - \mathbb{E}^{\mathrm{P}^k}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] \geq \epsilon^k - 2^{-k}.$$

By the second part of Proposition 6.5.8, there exists a sequence $(\mathrm{Q}^k)_{k\in\mathbb{N}^*} \subset \overline{\mathcal{P}}_V(\nu)$ satisfying $\mathrm{Q}^k \circ \big(\zeta,\Lambda,B\big)^{-1} = \mathrm{P}\circ\big(\zeta,\Lambda,B\big)^{-1}$ for each $k\in\mathbb{N}^*$ and $\limsup_{k\to\infty}\big|\Psi(\alpha^k,\gamma^k) - \mathbb{E}^{\mathrm{Q}^k}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]\big| = 0$. Then, as P is a ϵ–rmeasure-valued MFG solution,

$$\epsilon \geq \limsup_{k\to\infty}\mathbb{E}^{\mathrm{Q}^k}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] - \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] \geq \limsup_{k\to\infty}\Psi(\alpha^k,\gamma^k) - \mathbb{E}^{\mathrm{P}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] \geq \limsup_{k\to\infty}\epsilon^k.$$

Then $\limsup_{k\to\infty}\epsilon^k \in [0,\epsilon]$, and

$$\mathbb{E}^{\widehat{\mathbb{P}}^q_\nu}\Big[\int_0^T L(t,X_t^{\alpha^k},\mu_{t\wedge\cdot}^{\alpha^k},\overline{\mu}_t^{\alpha^k},\alpha_t^k)\mathrm{d}t + g(X_T^{\alpha^k},\mu_T^{\alpha^k})\Big] \geq \sup_{\alpha'\in\mathcal{A}(\nu)}\Phi(\alpha^k,\alpha') - \epsilon^k, \text{ for each } k,$$

we can conclude.

Second point (ii) Let $\epsilon\in[0,\infty)$ and $\mathrm{P}^\epsilon := \widehat{\mathbb{P}}^q_\nu\circ\Big(\mu^{\alpha^\epsilon},\mu^{\alpha^\epsilon},\delta_{(\overline{\mu}_t^{\alpha^\epsilon})}(\mathrm{d}m)\mathrm{d}t,\delta_{(\overline{\mu}_t^{\alpha^\epsilon})}(\mathrm{d}m')\mathrm{d}t,B\Big)^{-1} \in \overline{\mathcal{P}}_S(\nu)[\epsilon]$, by Lemma 6.5.7, there exists a sequence $(\alpha^{\epsilon,i,N})_{(i,N)}:=(\alpha^{i,N})_{(i,N)}$ such that $\alpha^{i,N}\in\mathcal{A}(\nu_N)$, and

$$\lim_{N\to\infty}\mathbb{P}^N_\nu\circ\Big(\varphi^{N,\mathbf{X},\overline{\alpha}^N},\delta_{\varphi_t^{N,\overline{\alpha}^N}}(\mathrm{d}m')\mathrm{d}t\Big)^{-1} = \widehat{\mathbb{P}}^q_\nu\Big(\mu^\alpha,\delta_{\overline{\mu}_t^\alpha}(\mathrm{d}m')\mathrm{d}t\Big)^{-1},$$

where $\overline{\alpha}^N = (\alpha^{i,N},\dots,\alpha^{i,N})$, and $\alpha := \alpha^\epsilon$.
In addition, for any sequence $(\kappa^{i,N,k})_{(i,N)\in\{1,\dots,N\}\times\mathbb{N}^*}$ satisfying for each $N\in\mathbb{N}^*$, $(\kappa^{i,N})_{i\in\{1,\dots,N\}}\subset\mathcal{A}(\nu_N)$, there exists a sequence $(\phi^{i,N,k})_{(i,N,k)\in\{1,\dots,N\}\times\mathbb{N}^*\times\mathbb{N}^*}\subset\mathcal{A}(\nu)$ such that if

$$\mathrm{P}^{i,N,k} := \widehat{\mathbb{P}}^q_\nu\circ\Big(\mu^{\alpha,i,N,k},\mu^\alpha,\delta_{\overline{\mu}_s^{\alpha,i,N,k}}(\mathrm{d}m)\mathrm{d}s,\delta_{\overline{\mu}_s^\alpha}(\mathrm{d}m')\mathrm{d}s,B\Big)^{-1},$$

where $\mu_t^{\alpha,i,N,k} := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}(X_t^{\alpha,\phi^{i,N,k}}\big|\widehat{\mathcal{G}}_t)$ and $\overline{\mu}_t^{\alpha,i,N,k} := \mathcal{L}^{\widehat{\mathbb{P}}^q_\nu}(X_t^{\alpha,\phi^{i,N,k}},\phi_t^{i,N,k}\big|\widehat{\mathcal{G}}_t)$, then one gets

$$\lim_{N\to\infty}\lim_{k\to\infty}\Big|\frac{1}{N}\sum_{i=1}^N J_i\big[((\overline{\alpha}^N)^{-i},\kappa^{i,N})\big] - \frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathrm{P}^{i,N,k}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]\Big| = 0.$$

Define

$$c^{\epsilon,i,N} := \sup_{\alpha'\in\mathcal{A}(\nu_N)} J_i\big[((\overline{\alpha^\epsilon}^N)^{-i},\alpha')\big] - J_i[\overline{\alpha^\epsilon}^N].$$

There exists a sequence of controls $(\kappa^{\epsilon,i,N})_{(i,N)\in\{1,\dots,N\}\times\mathbb{N}^*}$ satisfying $J_i\big[((\overline{\alpha^\epsilon}^N)^{-i},\kappa^{\epsilon,i,N})\big] - J_i[\overline{\alpha^\epsilon}^N] \geq c^{\epsilon,i,N} - 2^{-N}$, for each $i\in\{1,...,N\}$. Therefore, as $\mathrm{P}^\epsilon\in\overline{\mathcal{P}}^\star_S(\nu)[\epsilon]$ i.e. a ϵ–strong MFG solution,

$$\begin{aligned}\epsilon &\geq \Big(\limsup_{N\to\infty}\limsup_{k\to\infty}\frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathrm{P}^{i,N,k}}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big] - \mathbb{E}^{\mathrm{P}^\epsilon}\big[J(\mu,\Lambda^\circ,\zeta,\Lambda)\big]\Big)\\ &\geq \limsup_{N\to\infty}\Big(\frac{1}{N}\sum_{i=1}^N J_i\big[((\overline{\alpha^\epsilon}^N)^{-i},\kappa^{\epsilon,i,N})\big] - \frac{1}{N}\sum_{i=1}^N J_i[\overline{\alpha^\epsilon}^N]\Big) \geq \limsup_{N\to\infty}\frac{1}{N}\sum_{i=1}^N c^{\epsilon,i,N}.\end{aligned}$$

Combined this result with the first point (see proof above), we can conclude.

Chapter 7

Existence of ϵ–strong solution and measure–valued solution of mean–field games

7.1 Introduction

In this chapter, under some general assumptions, we study the question of the existence of the measure–valued solution used to characterize the limit of the approximate Nash–equilibria and approximate strong MFG solution in the previous chapter. Recall that the notions of approximate strong MFG solution and approximate Nash–equilibria are exactly the classical notions mentioned in the literature except the fact that the optimum is ac hived by admiting a small error ϵ (see Chapter 6). As we proved, there is a perfect symmetric between these three notions: measure–valued MFG solution, approximate strong solution and approximate Nash equilibria. Consequently, our resultat of existence of measure–valued MFG solution implies the existence of approximate Nash–equilibria and approximate strong MFG equilibria. It is well known in the MFG theory that the existence of a strong MFG solution is very difficult to obtain and requires strong assumptions. Admitting a small error $\epsilon > 0$, it is possible to get the existence of an ϵ–strong MFG equilibrium under general assumptions.

In a framework with the (conditional) law of control and control of the volatility σ, we prove an existence result using largely the same discretized techniques as in Carmona, Delarue, and Lacker [49] combined with some arguments evoked in Lacker [102]. In addition, the methods mentioned in chapter three will prove to be very useful.

The rest of the chapter is structured as follows. We provide in Section 7.2 the definition of the measure–valued solution and the different notions of the approximate strong MFG solution while mentioning the existence corresponding existence results in Theorem 7.2.4 and Theorem 7.2.6. The technical proofs are completed in Section 7.3.

7.2 Measure-valued solution and ϵ–strong solution

Definition 7.2.1 (measure–valued solution)**.** *Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, we say that a term*

$$\varrho := \big(\Omega^\varrho, \mathcal{F}^\varrho, \mathbb{P}^\varrho, \mathbb{F}^\varrho := (\mathcal{F}_t^\varrho)_{0\le t\le T}, \mathbb{G}^\varrho := (\mathcal{G}_t^\varrho)_{0\le t\le T}, B^\varrho, \mu^{\varrho,\star}, \Lambda^{\varrho,\star}\big),$$

is a measure–valued MFG solution associated with the initial (distribution) condition ν if

(i) $(\Omega^\varrho, \mathcal{F}^\varrho, \mathbb{P}^\varrho)$ is a probability space, equipped with two filtrations $\mathbb{F}^\varrho$ and $\mathbb{G}^\varrho$ such that, for all $t \in [0,T]$

$$\mathcal{G}_t^\varrho \subseteq \mathcal{F}_t^\varrho, \text{ and } \mathbb{E}^{\mathbb{P}^\varrho}\big[1_D\big|\mathcal{G}_t^\varrho\big] = \mathbb{E}^{\mathbb{P}^\varrho}\big[1_D\big|\mathcal{G}_T^\varrho\big],\ \mathbb{P}^\varrho\text{–a.s., for all } D \in \mathcal{F}_t^\varrho; \tag{7.2.1}$$

(ii) $\mu^{\varrho,\star} := (\mu_s^{\varrho,\star})_{s\in[0,T]}$ is an $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{G}^\varrho$–adapted continuous process and $\Lambda^{\varrho,\star} := (\Lambda_s^{\varrho,\star})_{0\le s\le T}$ is an $\mathcal{P}(\mathcal{P}_U^n)$–valued $\mathbb{G}^\varrho$–predictable process such that $\mathbb{E}^{\mathbb{P}^\varrho}\big[\sup_{s\in[0,T]}\int_{\mathbb{R}^n}\|x_s\|^p\mu_s^{\varrho,\star}(\mathrm{d}x)\big] < \infty$;

(iii) B^ϱ is an $\mathbb{R}^\ell$–valued standard Brownian motion with respect to $\mathbb{F}^\varrho$, B^ϱ is in addition adapted to $\mathbb{G}^\varrho$.

(*iv*) *For any* μ *a* $\mathcal{P}(\mathbb{R}^n)$*–valued* $\mathbb{F}^\varrho$*–adapted continuous process and* Λ *is an* $\mathcal{P}(\mathcal{P}_U^n)$*–valued* $\mathbb{F}^\varrho$*–predictable process such that:* $\mathbb{E}^{\mathbb{P}^\varrho}\big[\sup_{s\in[0,T]}\int_{\mathbb{R}^n}\|x_s\|^p\mu_s(\mathrm{d}x)\big]<\infty$, $\Lambda_t(\mathbb{Z}_{\mu_t})=1$ $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}^\varrho$*–*a.e.*, and* $\mathbb{P}^\varrho$–a.e, *for all* $(f,t)\in C_b^2(\mathbb{R}^n)\times[0,T]$

$$\begin{aligned}&\langle f(\cdot-\sigma_0B_t),\mu_t\rangle\\&=\langle f,\nu\rangle+\int_0^t\bigg[\int_{\mathcal{P}_U^n}\langle\mathcal{L}_r^\star[f(\cdot-\sigma_0B_r)](\cdot,\mu^{\varrho,\star},m'),\mu_r\rangle\Lambda_r^{\varrho,\star}(\mathrm{d}m')+\int_{\mathcal{P}_U^n}\langle\mathcal{L}_r^\circ[f(\cdot-\sigma_0B_r)](\cdot,\mu^{\varrho,\star},\cdot),m\rangle\Lambda_r(\mathrm{d}m)\bigg]\mathrm{d}r,\end{aligned}$$

we have

$$\mathbb{E}^{\mathbb{P}^\varrho}\big[J(\mu^{\varrho,\star},\Lambda^{\varrho,\star},\mu^{\varrho,\star},\Lambda^{\varrho,\star})\big]\geq\mathbb{E}^{\mathbb{P}^\varrho}\big[J(\mu,\Lambda,\mu^{\varrho,\star},\Lambda^{\varrho,\star})\big].$$

(*v*) *The pair* $(\mu^{\varrho,\star},\Lambda^{\varrho,\star})$ *satisfies:* $\Lambda_t^{\varrho,\star}(\mathbb{Z}_{\mu_t^{\varrho,\star}})=1$ $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}^\varrho$*–a.e., and ,* $\mathbb{P}^\varrho$*–a.e, for all* $(f,t)\in C_b^2(\mathbb{R}^n)\times[0,T]$

$$\begin{aligned}&\langle f(\cdot-\sigma_0B_t),\mu_t^{\varrho,\star}\rangle\\&=\langle f,\nu\rangle+\int_0^t\bigg[\int_{\mathcal{P}_U^n}\langle\mathcal{L}_r^\star[f(\cdot-\sigma_0B_r)](\cdot,\mu^{\varrho},m'),\mu_r^{\varrho,\star}\rangle\Lambda_r^{\varrho,\star}(\mathrm{d}m')+\int_{\mathcal{P}_U^n}\langle\mathcal{L}_r^\circ[f(\cdot-\sigma_0B_r)](\cdot,\mu^{\varrho,\star},\cdot),m\rangle\Lambda_r^{\varrho,\star}(\mathrm{d}m)\bigg]\mathrm{d}r.\end{aligned}$$

Proposition 7.2.2. *Let Assumption 5.5.5 hold true. For any* $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$*, one has*

$$\overline{\mathcal{P}}_V^\star(\nu)=\Big\{\mathbb{P}^\varrho\circ\big(\mu^{\varrho,\star},\Lambda^{\varrho,\star},\mu^{\varrho,\star},\Lambda^{\varrho,\star},B^\varrho\big)^{-1},\ \varrho\ \textit{a measure–valued MFG solution with initial condition}\ \nu\Big\}$$

Definition 7.2.3. (*Approximate strong open loop MFG solution*)
Let $\nu\in\mathcal{P}_p(\mathbb{R}^n)$*, and*

$$\Omega\ :=\ \mathbb{R}^n\times\mathcal{C}^n\times\mathcal{C}^\ell,$$

be the canonical space, with canonical variable ξ *and canonical processes* $W=(W_t)_{0\leq t\leq T}$ *and* $B=(B_t)_{0\leq t\leq T}$*, and probability measure* $\mathbb{P}_\nu$ *under which* $\xi\sim\nu$ *and* (W,B) *are standard Brownian motion independent of* ξ*. Let* $\mathbb{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ *and* $\mathbb{G}=(\mathcal{G}_t)_{0\leq t\leq T}$ *be defined by*

$$\mathcal{F}_t:=\sigma\big\{\xi,W_r,B_r,\ r\in[0,t]\big\}\qquad\textit{and}\qquad\mathcal{G}_t:=\sigma\big\{B_r,\ r\in[0,t]\big\}.$$

Let us denote by $\mathcal{A}(\nu)$ *the collection of all* U*-valued* $\mathbb{F}$*-predictable processes.*

For each $\epsilon>0$*, we say the process* $(\overline{\mu}_t)_{t\in[0,T]}$ *is an* ϵ*–strong MFG solution if:*

(*i*) $(\overline{\mu}_t)_{t\in[0,T]}$ *is a* $\mathcal{P}(\mathbb{R}^n\times U)$*–valued* $(\mathcal{G}_t)_{t\in[0,T]}$*–predictable process*

(*ii*) $\overline{\mu}$ *satisfies:* $\overline{\mu}_t=\mathcal{L}^{\mathbb{P}_\nu}(X_t^\star,\alpha_t^\star|B)$, $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}_\nu$*–a.e. where* $\alpha^\star\in\mathcal{A}(\nu)$*, and* $X^\star$ *is the unique strong solution of*

$$X_t^\star=\xi+\int_0^t b\big(r,X_r^\star,\mu_{r\wedge\cdot},\overline{\mu}_r,\alpha_r^\star\big)\mathrm{d}r+\int_0^t\sigma\big(r,X_r^\star,\mu_{r\wedge\cdot},\overline{\mu}_r,\alpha_r^\star\big)\mathrm{d}W_r+\sigma_0B_t,\ \textit{for all}\ t\in[0,T],\ \mathbb{P}_\nu\text{–a.e.},$$

where $\mu_t(\mathrm{d}x)=\overline{\mu}_t(\mathrm{d}x,U)$.

(*iii*) *For any* $\alpha\in\mathcal{A}(\nu)$*, and* X *the unique strong solution of*

$$X_t=\xi+\int_0^t b\big(r,X_r,\mu_{r\wedge\cdot},\overline{\mu}_r,\alpha_r\big)\mathrm{d}r+\int_0^t\sigma\big(r,X_r,\mu_{r\wedge\cdot},\overline{\mu}_r,\alpha_r\big)\mathrm{d}W_r+\sigma_0B_t,\ \textit{for all}\ t\in[0,T],\ \mathbb{P}_\nu\text{–a.e.},$$

one has

$$\mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T L(t,X_t^\star,\mu_{t\wedge\cdot},\overline{\mu}_t,\alpha_t^\star)\mathrm{d}t+g(X_T^\star,\mu)\bigg]\geq\mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T L(t,X_t,\mu_{t\wedge\cdot},\overline{\mu}_t,\alpha_t)\mathrm{d}t+g(X_T,\mu)\bigg]-\epsilon.$$

Theorem 7.2.4. *Under Assumption 5.5.5, for any $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, there exists at least one measure–valued MFG solution. Consequently, for each $\epsilon > 0$, there exists at least one ϵ–strong MFG equilibrium.*

Now, we consider that $\ell = 0$, and

$$\sigma(t, x, \pi, m, u) = \tilde{\sigma}(t, x), \text{ for all } (t, x, \pi, m, u) \in [0, T] \times \mathbb{R}^n \times \mathcal{C}^n_{\mathcal{W}} \times \mathcal{P}^n_U \times U.$$

For simplicity, we will note σ instead of $\tilde{\sigma}$.

Definition 7.2.5. (*Approximate strong Markovian MFG solution*) *Let $\nu \in \mathcal{P}_p(\mathbb{R}^n)$ and $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space supporting W a $\mathbb{R}^d$–valued $\mathbb{F}$–Brownian motion and ξ a $\mathcal{F}_0$–random variable such that $\mathcal{L}^{\mathbb{P}}(\xi) = \nu$.*

For each $\epsilon > 0$, we say the process $(\overline{\mu}_t)_{t\in[0,T]}$ is ϵ–strong Markovian MFG solution if:

(*i*) *$(\overline{\mu}_t)_{t\in[0,T]}$ is such that: for each $t \in [0, T]$, $\overline{\mu}_t$ is at value in $\mathcal{P}(\mathbb{R}^n \times U)$ and the map $\overline{\mu} : [0, T] \ni \to \overline{\mu}_t$ is Borel measurable.*

(*ii*) *$\overline{\mu}$ satisfies: $\overline{\mu}_t = \mathcal{L}^{\mathbb{P}}(X^\star_t, \alpha^\star(t, X^\star_t))$, $\mathrm{d}t$–a.e., where $\alpha^\star : [0, T] \to \mathbb{R}^n \to U$ is a Borel measurable function, and $X^\star$ is the solution of*

$$X^\star_t = \xi + \int_0^t b\big(r, X^\star_r, \mu_{r\wedge\cdot}, \overline{\mu}_r, \alpha^\star(r, X^\star_r)\big)\mathrm{d}r + \int_0^t \sigma\big(r, X^\star_r\big)\mathrm{d}W_r, \text{ for all } t \in [0, T], \ \mathbb{P}\text{–a.e.},$$

where $\mu_t(\mathrm{d}x) = \overline{\mu}_t(\mathrm{d}x, U)$.

(*iii*) *For any Borel measurable function $\alpha : [0, T] \times \mathbb{R}^n \to U$, and X the solution of*

$$X_t = \xi + \int_0^t b\big(r, X_r, \mu_{r\wedge\cdot}, \overline{\mu}_r, \alpha(r, X_r)\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r\big)\mathrm{d}W_r, \text{ for all } t \in [0, T], \ \mathbb{P}\text{–a.e.}, \tag{7.2.2}$$

one has

$$\mathbb{E}^{\mathbb{P}}\bigg[\int_0^T L(t, X^\star_t, \mu_{t\wedge\cdot}, \overline{\mu}_t, \alpha^\star(t, X^\star_t))\mathrm{d}t + g(X^\star_T, \mu)\bigg] \geq \mathbb{E}^{\mathbb{P}}\bigg[\int_0^T L(t, X_t, \mu_{t\wedge\cdot}, \overline{\mu}_t, \alpha(t, X_t))\mathrm{d}t + g(X_T, \mu)\bigg] - \epsilon.$$

Theorem 7.2.6. *Let $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, and Assumption 5.5.5 hold true. For any ρ a measure–valued MFG solution such that there exists $(\mathbf{n}^\star, \mathbf{q}^\star) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}^n_U)$ satisfying $(\mu^{\rho,\star}, \Lambda^{\rho,\star}) = (\mathbf{n}^\star, \mathbf{q}^\star)$, $\mathbb{P}^\rho$–a.e., there exists a sequence $(\overline{\mu}^\epsilon)_{\epsilon>0}$ s.t. for each $\epsilon > 0$, $\overline{\mu}^\epsilon$ is an ϵ–strong Markovian MFG solution and*

$$\lim_{\epsilon\to 0}\big(\mu^\epsilon, \delta_{\overline{\mu}^\epsilon_t}(\mathrm{d}m)\mathrm{d}t\big) = (\mathbf{n}^\star, \mathbf{q}^\star), \text{ in } \mathcal{W}_p, \ \mathbb{P}\text{–a.e.}$$

Consequently, for each $\epsilon > 0$, there at least one ϵ–strong Markovian MFG solution.

7.3 Proof of existence

7.3.1 Measure–valued no common noise MFG equilibrium

7.3.1.1 Technical results

In this part, we discuss of the case without common noise. Let $\sigma_0 = 0$ (or $\ell = 0$.) Given $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, with $p' > p$. In order to proof our theorem, a more adequate framework and other definitions are necessary. Let us introduce the notion of deterministic measure-valued no common noise control rule

Definition 7.3.1. *Given $(\mathbf{n}, \mathbf{q}) \in \mathcal{C}^{n,p}_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$, $(\mathbf{n}^\circ, \mathbf{q}^\circ) \in \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}^n_U)$ is a deterministic measure-valued no common noise control rule if: recall that N_t is defined in equation (6.3.8),*

- *$\mathbf{n}^\circ_0 = \nu$, and $N_t[\mathbf{n}^\circ, \mathbf{q}^\circ, \mathbf{n}, \mathbf{q}](f) = 0$ for all $f \in C^2_b(\mathbb{R}^n)$ and every $t \in [0, T]$.*

- *For* $\mathrm{d}t$ *almost every* $t \in [0,T]$, $\mathbf{q}_t^\circ\big(\mathbb{Z}_{\mathbf{n}_t^\circ}\big) = 1$.

$\mathcal{R}(\mathbf{n},\mathbf{q})$ will denote the set of all deterministic measure-valued no common noise control rules defined as previously. We also consider

$$\mathcal{R}^\star(\mathbf{n},\mathbf{q}) := \arg \max_{(\mathbf{n}^\circ,\mathbf{q}^\circ)\in\mathcal{R}(\mathbf{n},\mathbf{q})} J\big(\mathbf{n}^\circ,\mathbf{q}^\circ,\mathbf{n},\mathbf{q}\big),$$

where recall that

$$J\big(\mathbf{n}^\circ,\mathbf{q}^\circ,\mathbf{n},\mathbf{q}\big) := \int_0^T \bigg[\int_{\mathcal{P}_U^n} \langle L^\circ\big(t,\cdot,\mathbf{n},\cdot\big),m\rangle \mathbf{q}_t^\circ(\mathrm{d}m) + \int_{\mathcal{P}_U^n} \langle L^\star\big(t,\cdot,\pi,m'\big),\mathbf{n}_t^\circ\rangle \mathbf{q}_t(\mathrm{d}m')\bigg]\mathrm{d}t + \langle g(\cdot,\mathbf{n}),\mathbf{n}_T^\circ\rangle.$$

Notice that by Lemma 4.8.2, $\mathcal{R}(\mathbf{n},\mathbf{q}) \subset \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$.

Definition 7.3.2. *$(\mathbf{n}^\star,\mathbf{q}^\star) \in \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$ is a deterministic measure–valued no common noise MFG solution if $(\mathbf{n}^\star,\mathbf{q}^\star) \in \mathcal{R}^\star(\mathbf{n}^\star,\mathbf{q}^\star)$. We shall denote $\mathcal{S}^\star$ all deterministic measure–valued no common noise MFG solutions.*

Mention that in the following, it will be more convenient to look at $\mathcal{R}$ as a set valued function:

$$\mathcal{R} : (\mathbf{n},\mathbf{q}) \in \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U)) \to \mathcal{R}(\mathbf{n},\mathbf{q}) \subset \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U)).$$

Continuity of $\mathcal{R}$ In the next propositions, it is shown that $\mathcal{R}$ is both upper and lower hemicontinuous, and this is enough to conclude that $\mathcal{R}$ is continuous. We refer to [10, chapter 17] for an overview on set valued function.

Lemma 7.3.3. *(*Lemma 4.8.2*) There exists a constant $C > 0$ (depend only of coefficients $[\sigma,b]$ and ν), such that for any $(\mathbf{n},\mathbf{q}) \in \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$, and $(\mathbf{n}^\circ,\mathbf{q}^\circ) \in \mathcal{R}(\mathbf{n},\mathbf{q})$, one has*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^n} |x|^{p'} \mathbf{n}_t^\circ(\mathrm{d}x) \le C\Big(1 + \int_{\mathbb{R}^n} |x|^{p'} \nu(\mathrm{d}x)\Big).$$

Furthermore, for any $(t,s) \in [0,T]\times[0,T]$, one gets $\mathcal{W}_p\big(\mathbf{n}_t^\circ,\mathbf{n}_s^\circ\big)^p \le C|t-s|$.

Proposition 7.3.4. *(Upper Hemicontinuity) Let $(\mathbf{n},\mathbf{q}) \in \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$. $\mathcal{R}(\mathbf{n},\mathbf{q})$ is a compact set of $\mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$. In addition for any sequence $(\mathbf{n}^k,\mathbf{q}^k)_{k\in\mathbb{N}^*} \subset \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$ such that $\lim_{k\to\infty} (\mathbf{n}^k,\mathbf{q}^k) = (\mathbf{n},\mathbf{q})$, let $(\mathbf{n}^{\circ,k},\mathbf{q}^{\circ,k}) \in \mathcal{R}(\mathbf{n}^k,\mathbf{q}^k)$ for each $k \in \mathbb{N}^*$, then $(\mathbf{n}^{\circ,k},\mathbf{q}^{\circ,k})_{k\in\mathbb{N}^*}$ is relatively compact and each limit point belongs to $\mathcal{R}(\mathbf{n},\mathbf{q})$.*

Proof. By Lemma 7.3.3, one finds

$$\sup_{(\mathbf{n}^\circ,\mathbf{q}^\circ)\in\mathcal{R}(\mathbf{n},\mathbf{q})} \sup_{t\in[0,T]} \int_{\mathbb{R}^n} |x|^{p'} \mathbf{n}_t^\circ(\mathrm{d}x) < \infty \text{ and } \lim_{\delta\to 0} \sup_{(\mathbf{n}^\circ,\mathbf{q}^\circ)\in\mathcal{R}(\mathbf{n},\mathbf{q})} \sup_{t\in[0,T]} \mathcal{W}_p\big(\mathbf{n}_t^\circ,\mathbf{n}_{(t+\delta)\wedge T}^\circ\big) = 0,$$

as U is a compact set and for $\mathrm{d}t$ almost every $t \in [0,T]$, $\mathbf{q}_t^\circ\big(\mathbb{Z}_{\mathbf{n}_t^\circ}\big) = 1$, one has

$$\sup_{(\mathbf{n}^\circ,\mathbf{q}^\circ)\in\mathcal{R}(\mathbf{n},\mathbf{q})} \int_0^T \int_{\mathcal{P}_p(\mathbb{R}^n\times U)} \mathcal{W}_p\big(m,m_0\big)^{p'} \mathbf{q}_t^\circ(\mathrm{d}m)\mathrm{d}t < \infty, \text{ for any } m_0 \in \mathcal{P}_p(\mathbb{R}^n \times U).$$

Then by Aldou's criterion [92, Lemma 16.12], $\mathcal{R}(\mathbf{n},\mathbf{q})$ is a compact set of $\mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$.
By similar way, the sequence $(\mathbf{n}^{\circ,k},\mathbf{q}^{\circ,k})_{k\in\mathbb{N}^*}$ is relatively compact. By passing to the limit in equation verified by $(\mathbf{n}^{\circ,k},\mathbf{q}^{\circ,k},\mathbf{n}^k,\mathbf{q}^k)$ i.e. $N_t[\mathbf{n}^{\circ,k},\mathbf{q}^{\circ,k},\mathbf{n}^k,\mathbf{q}^k](f) = 0$, for each $(t,f) \in [0,T]\times C_b^2(\mathbb{R}^n)$ (see for instance lemma 4.7.1), it is straightforward to check that each limit belongs to $\mathcal{R}(\mathbf{n},\mathbf{q})$ (see Proposition 4.8.4). □

Proposition 7.3.5. *(Lower Hemicontinuity) Let $(\mathbf{n},\mathbf{q}) \in \mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$, $(\mathbf{n}^k,\mathbf{q}^k)_{k\in\mathbb{N}^*}$ be a sequence of elements of $\mathcal{C}_{\mathcal{W}}^{n,p} \times \mathbb{M}(\mathcal{P}_p(\mathbb{R}^n \times U))$ such that $\lim_{k\to\infty} (\mathbf{n}^k,\mathbf{q}^k) = (\mathbf{n},\mathbf{q})$, and $(\mathbf{n}^\circ,\mathbf{q}^\circ) \in \mathcal{R}(\mathbf{n},\mathbf{q})$. There exists $(\mathbf{n}^{\circ,j},\mathbf{q}^{\circ,j}) \in \mathcal{R}(\mathbf{n}^{k_j},\mathbf{q}^{k_j})$, for each $j \in \mathbb{N}^*$ where $(k_j)_{j\in\mathbb{N}^*} \subset \mathbb{N}^*$ is a sub-sequence with $\lim_{j\to\infty} (\mathbf{n}^{\circ,j},\mathbf{q}^{\circ,j}) = (\mathbf{n}^\circ,\mathbf{q}^\circ)$.*

Proof. As $\lim_{k\to\infty}\big(\mathbf{n}^k,\mathbf{q}^k,\mathbf{q}^\circ\big)=(\mathbf{n},\mathbf{q},\mathbf{q}^\circ)$, by Lemma 7.4.2 and/or Proposition 4.7.10, there exists $(\mathbf{n}^{j,\circ},\mathbf{q}^{\circ,j})\in\mathcal{R}(\mathbf{n}^{k_j},\mathbf{q}^{k_j})$, for each $j\in\mathbb{N}^*$ where $(k_j)_{j\in\mathbb{N}^*}$ is a sub-sequence with $\lim_{j\to\infty}(\mathbf{n}^{j,\circ},\mathbf{q}^{\circ,j})=(\mathbf{n}^\circ,\mathbf{q}^\circ)$. □

Theorem 7.3.6. *The set $\mathcal{S}^\star$ is nonempty and compact.*

Proof. Under assumption 5.5.5, it straightforward to verify that $J:(\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U)))^2\to\mathbb{R}$ is continuous. As $\mathcal{R}$ is continuous because it is upper and lower hemicontinuous, and has nonempty compact convex values by Berge Maximum theorem [10, Theorem 17.31], $\mathcal{R}^\star$ has nonempty compact convex values, is upper hemicontinuous and consequently its graph $\mathrm{Gr}(\mathcal{R}^\star):=\big\{\big(\mathbf{n},\mathbf{q},\widetilde{\mathbf{n}},\widetilde{\mathbf{q}}\big):(\widetilde{\mathbf{n}},\widetilde{\mathbf{q}})\in\mathcal{R}^\star(\mathbf{n},\mathbf{q})\big\}$ is closed. Let $(\mathbf{n},\mathbf{q})\in K$ if $(\mathbf{n},\mathbf{q})\in\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U))$ and:

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^n}|x|^{p'}\mathbf{n}_t(\mathrm{d}x)+\int_0^T\int_{\mathcal{P}_p(\mathbb{R}^n\times U)}\mathcal{W}_{p'}\big(m,m_0\big)^{p'}\mathbf{q}_t(\mathrm{d}m)\mathrm{d}t\ \ \le M,$$

where m_0 is an element of $\mathcal{P}_{p'}(\mathbb{R}^n\times U)$ and $M<\infty$ is defined by

$$M:=\sup\Big\{\int_0^T\int_{\mathcal{P}_p(\mathbb{R}^n\times U)}\mathcal{W}_p\big(m,m_0\big)^{p'}\mathbf{q}^\circ_t(\mathrm{d}m)\mathrm{d}t+C\Big(1+\int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x)\Big),\ (\mathbf{n}^\circ,\mathbf{q}^\circ)\in\mathcal{R}^\star\big(\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U))\big)\Big\},$$

and in addition

$$\mathcal{W}_p\big(\mathbf{n}_t,\mathbf{n}_s\big)^p\le C|t-s|,\ \text{for all}\ (t,s)\in[0,T]\times[0,T].$$

Thanks to the above techniques, it is obvious that K is a compact set of $\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U))$, and $\mathcal{R}^\star$ is a set valued function of K into himself i.e. $\mathcal{R}^\star:(\mathbf{n},\mathbf{q})\in K\to\mathcal{R}^\star(\mathbf{n},\mathbf{q})\subset K$.

Let E be a Polish space, denote $\mathcal{M}(E)$ the set of signed measure on E. Equipped of the weak convergence topology $\tau_\omega:=\sigma\big(\mathcal{M}(E),C_b(E)\big)$ generated by the bounded continuous function, $\mathcal{M}(E)$ is a locally convex Hausdorff space. Accordingly, $C([0,T];\mathcal{M}(\mathbb{R}^n))$ is a locally convex Hausdorff space. Likewise $\mathcal{M}(\mathcal{P}^n_U\times[0,T])$ is a locally convex Hausdorff space equipped of $\tau^q_\omega:=\sigma\big(\mathcal{M}(\mathcal{P}^n_U\times[0,T]),C_b(\mathcal{P}^n_U\times[0,T])\big)$. Then $C([0,T];\mathcal{M}(\mathbb{R}^n))\times\mathcal{M}(\mathcal{P}^n_U\times[0,T])$ is a locally convex Hausdorff space. One can see K as a subset of $C([0,T];\mathcal{M}(\mathbb{R}^n))\times\mathcal{M}(\mathcal{P}^n_U\times[0,T])$. As the topology of $C([0,T];\mathcal{M}(\mathbb{R}^n))\times\mathcal{M}(\mathcal{P}^n_U)\times[0,T])$ induced on K is equivalent to the topology of $\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}(\mathbb{R}^n\times U))$, we deduce that K which a compact set of $\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U))(\subset\mathcal{C}^n_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}^n_U))$ is also a compact set of $C([0,T];\mathcal{M}(\mathbb{R}^n))\times\mathcal{M}(\mathcal{P}(\mathbb{R}^n\times U)\times[0,T])$. To conclude, we apply the fixed point theorem of Kakutani–Fan–Glicksberg (see [10, Corollary 17.55]) to deduce $\mathcal{S}^\star$ is nonempty and compact. Therefore we can find $(\mathbf{n}^\star,\mathbf{q}^\star)\in\mathcal{R}^\star(\mathbf{n}^\star,\mathbf{q}^\star)$. □

7.3.1.2 Proof of existence of strong measure–valued no common noise MFG solution

Now let us prove the main result of this part. If $\mathrm{P}^\star(\mathrm{d}\pi,\mathrm{d}q,\mathrm{d}\pi',\mathrm{d}q',\mathrm{d}\mathbf{b}):=\delta_{(\mathbf{n}^\star,\mathbf{q}^\star,\mathbf{n}^\star,\mathbf{q}^\star)}(\mathrm{d}\pi,\mathrm{d}q,\mathrm{d}\pi',\mathrm{d}q')P_B(\mathrm{d}\mathbf{b})\in\mathcal{P}(\overline{\Omega})$, it is straightforward to check that $\mathrm{P}^\star$ is a strong relaxed no common noise MFG solution where P_B is the $\mathbb{R}^\ell$ Wiener measure.

7.3.2 Existence of Measure-valued MFG equilibrium with common noise

7.3.2.1 Technical results

We place ourselves on the probability space $(\Omega,\mathbb{F},\mathbb{P}_\nu)$ defined in 6.3.1. We use the same discretized techniques as in Carmona, Delarue, and Lacker [49].

For each $k\in\mathbb{N}^*$, let $t^k_i:=i2^{-k}T$ for $i=0,\dots,2^k$. For each positive integer k, we choose a partition $\pi^k:=\{C^k_1,\dots,C^k_k\}$ of $\mathbb{R}^\ell$ into k measurable sets of strictly positive Lebesgue measure, such that π^{k+1} is a refinement of π^k for each k, and $\mathcal{B}(\mathbb{R}^\ell):=\sigma(\cup_{k=1}^\infty\pi^k)$. For each $1\le q\le 2^k$, and $\underline{\mathrm{i}}=(i_1,\dots,i_q)\in\{1,...,k\}^q$, we define $S^{k,q}_{\underline{\mathrm{i}}}$ as follow

$$S^{k,q}_{\underline{\mathrm{i}}}:=\big\{\mathbf{b}\in\mathcal{C}^\ell:\ \mathbf{b}_{t^k_j}-\mathbf{b}_{t^k_{j-1}}\in C^k_{i_j},\ \forall j\in\{1,\dots,q\}\big\}.$$

The $S^{k,q}_{\underline{i}}$'s, $i \in \{1,\dots,k\}^q$, form a finite partition (of cardinal k^q) of $\mathcal{C}^\ell$. Now, we introduce

$$\Pi^k_q := \Big\{ S^{k,q}_{\underline{i}} : \ \underline{i} \in \{1,...,k\}^q \Big\},$$

with $\Pi^k_0 := \{\mathcal{C}^\ell\}$, notice that cardinal of Π^k_q is finite. For any $k \geq 0$, the filtration $(\sigma(\Pi^k_q))_{q=0,\dots,2^k}$ is the filtration generated by the discretization of the canonical process, we have $\sigma(\Pi^k_q) \subset \mathcal{G}_{t^k_q}$ and $\sigma(\Pi^k_q) \subset \sigma(\Pi^k_{q+1})$. For $t \in [0,T]$, we note $[t]_k := t^k_q$ when $t^k_q \leq t < t^k_{q+1}$. Let $\Pi_k(t)$ equal Π^k_q, where q is the largest integer such that $t^k_q \leq t$, and let $\mathcal{G}^k_t := \sigma(\Pi^n(t)) = \mathcal{G}^k_{[t]^k}$. $(\mathcal{G}^k_t)_{t\in[0,T]}$ is a filtration and

$$\mathcal{G}_t = \sigma\Big(\bigcup_{k=0}^{\infty} \mathcal{G}^k_t \Big).$$

Let us introduce for each $\mathbf{b} \in \mathcal{C}^\ell$,

$$[\hat{b},\hat{\sigma},\hat{L}](t,y,\mathbf{b},\pi,m,u) := [b,\sigma,L](t,y+\sigma_0\mathbf{b}_t,\pi[\mathbf{b}],m[\mathbf{b}_t],u) \text{ and } \hat{g}(y,\mathbf{b},\pi) := g(y+\sigma_0\mathbf{b}_T,\pi[\mathbf{b}]),$$

recall that $\pi[\mathbf{b}]$ and $m[\mathbf{b}_t]$ are defined in 6.5.1. Notice that $[\hat{b},\hat{\sigma},\hat{L}] : [0,T]\times\mathbb{R}^n\times\mathcal{C}^\ell\times\mathcal{C}^n_{\mathcal{W}}\times\mathcal{P}^n_U\times U \to \mathbb{R}^n\times\mathbb{S}^{n\times n}\times\mathbb{R}$ and $\hat{g} : \mathbb{R}^n\times\mathcal{C}^\ell\times\mathcal{C}^\ell\to\mathbb{R}$ are continuous and for $\mathbf{b}\in\mathcal{C}^\ell$, $[\hat{b},\hat{\sigma},\hat{L}](\cdot,\cdot,\mathbf{b},\cdot,\cdot,\cdot)$ and $\hat{g}(\cdot,\mathbf{b},\cdot)$ verify the Assumption 5.5.5 with constant C and θ independent of $\mathbf{b}$ (see Assumption 5.5.5). We also note $\hat{a} := \hat{\sigma}\hat{\sigma}^\top$.
For each $k\in\mathbb{N}^*$, let consider the continuous process $(B^k)_{t\in[0,T]}$ defined as $B^k_t := \mathbb{E}^{\mathbb{P}_\nu}[B_t|\mathcal{G}^k_T]$, then B^k takes a finite number $|\Pi^k_{2^k}|$ of value in $\mathcal{C}^\ell$.

Lemma 7.3.7. *Under the previous considerations, we have* $\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{s\in[0,T]}|B^k_s - B_s|\Big]$.

Proof. Let us define for each $k\in\mathbb{N}^*$, $\mathbb{P}^k := \mathbb{P}_\nu\circ\big(B^k,B\big)^{-1} \in \mathcal{P}(\mathcal{C}^\ell\times\mathcal{C}^\ell)$. As, B is $\mathbb{P}_\nu$–Brownian motion, by classical argument it is straightforward to show that $(\mathbb{P}^k)_{k\in[0,T]}$ is relatively compact for the Wasserstein metric $\mathcal{W}_p$. Denote by $\mathbb{P}^\infty$ the limit of any sub-sequence, and (H^1,H^2) the canonical process on $\mathcal{C}^\ell\times\mathcal{C}^\ell$. We use the same notation for the sequence and its sub-sequence.
Now, we apply the same techniques as in proof of [49, Lemma 3.6] (*thrid step.*) Given $k_0\in\mathbb{N}^*$, as $\mathcal{G}^{k_0}_T\subset\mathcal{G}^k_T$, for all $k\geq k_0$. Assume $\ell=1$. For any bounded continuous function $\phi:\mathcal{C}^\ell\to\mathbb{R}$, and $C\in\mathcal{G}^{k_0}_T$, for all $t\in[0,T]$, one has by [49, Lemma A.4]

$$\mathbb{E}^{\mathbb{P}^\infty}\big[H^1_t 1_{H^2_t\in C}\phi(H^1)\big] = \lim_{k\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\big[B^k_t 1_{B_t\in C}\phi(B^k)\big] = \lim_{k\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\big[B_t 1_{B_t\in C}\phi(B^k)\big] = \mathbb{E}^{\mathbb{P}^\infty}\big[H^2_t 1_{H^2_t\in C}\phi(H^1)\big].$$

this is true for any $t\in[0,T]$, k_0, ϕ and C, as $\mathcal{G}_t = \sigma\Big(\bigcup_{k=0}^\infty\mathcal{G}^k_t\Big)$, we conclude that $H^1 = H^2$ $\mathbb{P}^\infty$–a.e. Consequently

$$\lim_{k\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{s\in[0,T]}|B^k_s-B_s|\Big] = \lim_{k\to\infty}\mathbb{E}^{\mathbb{P}^k}\Big[\sup_{s\in[0,T]}|H^1_s-H^2_s|\Big] = \mathbb{E}^{\mathbb{P}^\infty}\Big[\sup_{s\in[0,T]}|H^1_s-H^2_s|\Big] = 0.$$

For the case $\ell>1$, we proceed coordinates by coordinates. □

For each $k\in\mathbb{N}^*$, let $\mathcal{M}^k$ denote the set of functions $(\zeta,\Lambda):\mathcal{C}^\ell\to\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U))$ that are $\mathcal{G}^k_T$–measurable such that for each $t\in[0,T]$, (ζ_t,Λ_t) is $\mathcal{G}^k_t$–measurable. Notice that any $(\zeta,\Lambda)\in\mathcal{M}^k$ is constant on S for each $S\in\Pi^k_{2^k}$. Since $\mathcal{G}^k_T := \sigma(\Pi^k_{2^k})$ is finite, the space $\mathcal{M}^k$ is homeomorphic to a closed subset of $(\mathcal{C}^{n,p}_{\mathcal{W}}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U)))^{|\Pi^k_{2^k}|}$. Hence, $\mathcal{M}^k$ is a metrizable closed convex subset of a locally convex topological vector space.

Let us introduce

$$\widehat{N}_t[\mathbf{n}^\circ,\mathbf{q}^\circ,\mathbf{n},\mathbf{q},\mathbf{b}] := \langle f,\mathbf{n}_t\rangle - \langle f,\nu\rangle - \int_0^t\int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n\times U}\widehat{\mathcal{L}}_r^\circ[f](y,\mathbf{b},\mathbf{n},u)m(\mathrm{d}u,\mathrm{d}y)\mathbf{q}_r^\circ(\mathrm{d}m)\mathrm{d}r$$
$$-\int_0^t\int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n}\mathcal{L}_r^\star[f](y,\mathbf{b},\mathbf{n},m')\mathbf{n}_r^\circ(\mathrm{d}y)\mathbf{q}_r(\mathrm{d}m')\mathrm{d}r,$$

where

$$\widehat{\mathcal{L}}_t^\circ[\varphi](y,\mathbf{b},\pi',u) := \frac{1}{2}\mathrm{Tr}\big[a^\circ(t,y+\sigma_0\mathbf{b}_t,\pi'[\mathbf{b}],u)\nabla^2\varphi(y)\big] + b^\circ(t,y+\sigma_0\mathbf{b}_t,\pi'[\mathbf{b}],u)^\top\nabla\varphi(y),$$

and $\mathcal{L}_t^\star[\varphi](y,\mathbf{b},\pi',m) := \frac{1}{2}\mathrm{Tr}\big[a^\star(t,\pi'[\mathbf{b}],m[\mathbf{b}])\nabla^2\varphi(y)\big] + b^\star(t,\pi'[\mathbf{b}],m[\mathbf{b}])^\top\nabla\varphi(y)$.

Definition 7.3.8. *Given $(\zeta,\Lambda)\in\mathcal{M}^k$, $(\mu,\Lambda^\circ)\in\mathcal{M}^k$ is a measure–valued with k–finite common noise control rule if:*

- $\mathbb{P}_\nu(\mu_0=\nu)=1$, *and* $\widehat{N}_t[\mu,\Lambda^\circ,\zeta,\Lambda,B^k](f)=0$, *for all* $f\in C_b^2(\mathbb{R}^n)$ *and every* $t\in[0,T]$, $\mathbb{P}_\nu$*–a.e.*
- *For* $\mathrm{d}t$ *almost every* $t\in[0,T]$, $\Lambda_t^\circ\big(\mathbb{Z}_{\mu_t}\big)=1$, $\mathbb{P}_\nu$*–a.e.*

$\mathcal{R}^k(\zeta,\Lambda)$ will denote the set of all measure–valued control rules with k–finite common noise defined as previously. We also consider

$$\mathcal{R}^{k,\star}(\zeta,\Lambda) := \arg\max_{(\mu,\Lambda^\circ)\in\mathcal{R}(\zeta,\Lambda)}\mathbb{E}^{\mathbb{P}_\nu}\big[\widehat{J}\big(\mu,\Lambda^\circ,\zeta,\Lambda,B^k\big)\big],$$

where

$$\widehat{J}\big(\mathbf{n}^\circ,\mathbf{q}^\circ,\mathbf{n},\mathbf{q},\mathbf{b}\big) := \int_0^T\bigg[\int_{\mathcal{P}_U^n}\langle\hat{L}^\circ\big(t,\cdot,\mathbf{b},\mathbf{n},\cdot\big),m\rangle\mathbf{q}_t^\circ(\mathrm{d}m) + \int_{\mathcal{P}_U^n}\langle L^\star\big(t,\cdot,\mathbf{b},\pi,m'\big),\mathbf{n}_t^\circ\rangle\mathbf{q}_t(\mathrm{d}m')\bigg]\mathrm{d}t + \langle\hat{g}(\cdot,\mathbf{b},\mathbf{n}),\mathbf{n}_T^\circ\rangle.$$

Definition 7.3.9. *$(\zeta^\star,\Lambda^\star)\in\mathcal{M}^k$ is a measure–valued MFG solution with k–finite common noise if $(\zeta^\star,\Lambda^\star)\in\mathcal{R}^{k,\star}(\zeta^\star,\Lambda^\star)$. We shall denote $\mathcal{S}^{k,\star}$ all measure–valued with k–finite common noise MFG solutions.*

As mention in the no common noise case, it will be more convenient to look at $\mathcal{R}^k$ as a set valued function:

$$\mathcal{R}^k : (\mathcal{C}_{\mathcal{W}}^{n,p}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U)))^{|\Pi_{2^k}^k|}\ni(\zeta,\Lambda)\to\mathcal{R}^k(\zeta,\Lambda)\subset(\mathcal{C}_{\mathcal{W}}^{n,p}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U)))^{|\Pi_{2^k}^k|}.$$

Theorem 7.3.10. *For each $k\in\mathbb{N}^*$, the set $\mathcal{S}^{k,\star}$ is nonempty and compact.*

Proof. In definition 7.3.8, $\widehat{N}_t[\mu,\Lambda^\circ,\zeta,\Lambda,B^k](f)=0$ $\mathbb{P}_\nu$–a.e. is equivalent to the $|\Pi_{2^k}^k|$–equation: for each $C\in\Pi_{2^k}^k$, with $\omega\in\{B\in C\}$ fixed, $\widehat{N}_t[\mu(\omega),\Lambda^\circ(\omega),\zeta(\omega),\Lambda(\omega),B^k(\omega)](f)=0$, and $\Lambda_t^\circ\big(\mathbb{Z}_{\mu_t}\big)=1$, $\mathbb{P}_\nu$–a.e. is equivalent to to the $|\Pi_{2^k}^k|$–equations: for each $C\in\Pi_{2^k}^k$, with $\omega\in\{B\in C\}$ fixed, $\Lambda_t^\circ(\omega)\big(\mathbb{Z}_{\mu_t(\omega)}\big)=1$. Then, by using exactly the same proof as in the no common noise (with $|\Pi_{2^k}^k|$–equations instead of one equation), we prove that $\mathcal{R}^k$ is continuous.
Next, under assumption 5.5.5, the map

$$(\mathcal{C}_{\mathcal{W}}^{n,p}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U)))^{|\Pi_{2^k}^k|}\times(\mathcal{C}_{\mathcal{W}}^{n,p}\times\mathbb{M}(\mathcal{P}_p(\mathbb{R}^n\times U)))^{|\Pi_{2^k}^k|}\ni(\mu,\Lambda^\circ,\zeta,\Lambda)\to\mathbb{E}^{\mathbb{P}_\nu}\big[\widehat{J}\big(\mu,\Lambda^\circ,\zeta,\Lambda,B^k\big)\big]\in\mathbb{R}$$

is continuous, therefore, we proceed as in the no common case (proof of Theorem 7.3.6) i.e. applying Berge Maximum Theorem and later the fixed point Theorem of Kakutani-Fan-Glicksberg to deduce that $\mathcal{S}^{k,\star}$ is nonempty and compact. □

7.3.2.2 Proof of existence of measure–valued MFG solution with common noise

For each $k \in \mathbb{N}^*$, let $(\zeta^{k,\star}, \Lambda^{k,\star}) \in \mathcal{M}^k$ be a measure–valued MFG solution with k–finite common noise. We define the sequence of probabilities $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ by: for each $k \in \mathbb{N}^*$

$$\mathrm{P}^k := \mathbb{P}_\nu \circ \big(\zeta^{k,\star}[B^k], \Lambda^{k,\star}[B^k], \zeta^{k,\star}[B^k], \Lambda^{k,\star}[B^k], B^k\big)^{-1},$$

where $\zeta^{k,\star}[B^k], \Lambda^{k,\star}[B^k]$ are defined as in (6.5.1) and (6.5.2). Under Assumption 5.5.5, as $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, it is straightforward to show that $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p$. Let P^∞ be a limit of any sub-sequence, we will show P^∞ is a measure–valued MFG solution with common noise. For simplicity, we note $(\mathrm{P}^k)_{k\in\mathbb{N}^*}$ and its sub-sequence with the same notation.

Optimality condition. Let $\mathrm{P} \in \overline{\mathcal{P}}_V(\nu)$ s.t. $\mathcal{L}^{\mathrm{P}^\infty}\big(\zeta, \Lambda_t(\mathrm{d}m)\mathrm{d}t, B\big) = \mathcal{L}^{\mathrm{P}}\big(\zeta, \Lambda_t(\mathrm{d}m)\mathrm{d}t, B\big)$. If we consider the canonical processes $(\vartheta_t)_{t\in[0,T]}$ and $(\Theta_t)_{t\in[0,T]}$ defined in (6.5.4), we can verify that:

$$\widehat{N}_t\Big[\vartheta, \Theta, \zeta[-B], \Lambda[-B], B\Big](f) = 0, \text{ for all } f \in C_b^2(\mathbb{R}^n) \text{ and every } t \in [0,T], \ \mathrm{P}\text{ –a.e,}$$

for $\mathrm{d}t$ almost every $t \in [0,T]$, $\Theta_t\big(\mathbb{Z}_{\vartheta_t}\big) = 1$, P–a.e. and $\mathcal{L}^{\mathrm{P}^\infty}\Big(\zeta[-B], \Lambda_t[-B](\mathrm{d}m)\mathrm{d}t, B\Big) = \mathcal{L}^{\mathrm{P}}\Big(\zeta[-B], \Lambda_t[-B](\mathrm{d}m)\mathrm{d}t, B\Big)$. As the map

$$(\pi', q, \mathbf{b}) \in \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M} \times \mathcal{C}^\ell \to \big(\pi'[-\mathbf{b}], q_t[-\mathbf{b}](\mathrm{d}m)\mathrm{d}t, \mathbf{b}\big) \in \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M} \times \mathcal{C}^\ell$$

is continuous, we find that

$$\begin{aligned}\lim_k \mathcal{L}^{\mathbb{P}_\nu}\Big(\zeta^{k,\star}, \Lambda_t^{k,\star}(\mathrm{d}m)\mathrm{d}t, B^k\Big) = \lim_k \mathcal{L}^{\mathrm{P}^k}\Big(\zeta[-B], \Lambda_t[-B](\mathrm{d}m)\mathrm{d}t, B\Big) &= \mathcal{L}^{\mathrm{P}^\infty}\Big(\zeta[-B], \Lambda_t[-B](\mathrm{d}m)\mathrm{d}t, B\Big)\\ &= \mathcal{L}^{\mathrm{P}}\Big(\zeta[-B], \Lambda_t[-B](\mathrm{d}m)\mathrm{d}t, B\Big).\end{aligned}$$

By Proposition 4.7.10 combined with Itô formula (see also Remark 4.7.11), under the enlarged space $(\widehat{\Omega}, \widehat{\mathbb{F}}^q, \widehat{\mathbb{P}}_\nu^q)$ of $(\Omega, \mathbb{F}, \mathbb{P}_\nu)$ defined in the preamble of Section 6.5.2, there exist the sequence of $\sigma(\zeta_{t\wedge\cdot}^{k,\star}, \Lambda_{t\wedge\cdot}^{k,\star}, B_{t\wedge\cdot}^k, \mathrm{H})_{t\in[0,T]}$–adapted continuous processes $(\vartheta^k)_{k\in\mathbb{N}^*}$ and the sequence of $\sigma(\zeta_{t\wedge\cdot}^{k,\star}, \Lambda_{t\wedge\cdot}^{k,\star}, B_{t\wedge\cdot}^k, \mathrm{H})_{t\in[0,T]}$–predictable processes $(\Theta^k)_{k\in\mathbb{N}^*}$ such that

$$\widehat{N}_t\Big[\vartheta^k, \Theta^k, \zeta^{k,\star}, \Lambda^{k,\star}, B^k\Big](f) = 0, \text{ for all } f \in C_b^2(\mathbb{R}^n) \text{ and every } t \in [0,T], \ \widehat{\mathbb{P}}_\nu^q\text{ –a.e,}$$

for $\mathrm{d}t$ almost every $t \in [0,T]$, $\Theta_t^k\big(\mathbb{Z}_{\vartheta_t^k}\big) = 1$, $\widehat{\mathbb{P}}_\nu^q$–a.e. and

$$\lim_k \mathcal{L}^{\mathbb{P}_\nu^q}\Big(\vartheta^k, \Theta^k, \zeta^{k,\star}, \Lambda_t^{k,\star}(\mathrm{d}m)\mathrm{d}t, B^k\Big) = \mathcal{L}^{\mathrm{P}}\Big(\vartheta, \Theta, \zeta[-B], \Lambda_t[-B](\mathrm{d}m)\mathrm{d}t, B\Big).$$

Under Assumption 5.5.5, using the fact that for each $k \in \mathbb{N}^*$, $(\zeta^{k,\star}, \Lambda^{k,\star}) \in \mathcal{M}^k$ is a measure–valued MFG solution with k–finite common noise and that H is independent of other variables (see definition of $(\widehat{\Omega}, \widehat{\mathbb{F}}^q, \widehat{\mathbb{P}}_\nu^q)$ in the preamble of Section 6.5.2), one finds

$$\begin{aligned}\mathbb{E}^{\mathrm{P}^\infty}\big[J(\mu, \Lambda^\circ, \zeta, \Lambda)\big] = \lim_{k\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}_\nu^q}\big[\widehat{J}(\zeta^{k,\star}, \Lambda^{k,\star}, \zeta^{k,\star}, \Lambda^{k,\star}, B^k)\big] &\geq \lim_{k\to\infty} \mathbb{E}^{\widehat{\mathbb{P}}_\nu^q}\big[\widehat{J}(\vartheta^k, \Theta^k, \zeta^{k,\star}, \Lambda^{k,\star}, B^k)\big]\\ &= \mathbb{E}^{\mathrm{P}}\big[\widehat{J}(\vartheta, \Theta, \zeta[-B], \Lambda[-B], B)\big] = \mathbb{E}^{\mathrm{P}}\big[J(\mu, \Lambda^\circ, \zeta, \Lambda)\big].\end{aligned}$$

Fixed point and F–K *equation*. Using broadly the same previous arguments, we can check that: B is a $(\overline{\mathbb{F}}, \mathrm{P}^\infty)$–Brownian motion starting at zero,

$$N_t\Big[\mu, \Lambda^\circ, \zeta, \Lambda\Big](f) = \widehat{N}_t\Big[\vartheta, \Theta, \zeta[-B], \Lambda[-B], B\Big](f) = 0, \text{ for all } f \in C_b^2(\mathbb{R}^n) \text{ and every } t \in [0,T], \ \mathrm{P}^\infty\text{ –a.e,}$$

for $\mathrm{d}t$ almost every $t \in [0,T]$, $\Lambda_t^\circ\big(\mathbb{Z}_{\mu_t}\big) = 1$, P°–a.e. and $\mathrm{P}^\infty\big(\Lambda^\circ = \Lambda, \mu = \zeta\big) = 1$. We can conclude that $\mathrm{P}^\infty \in \overline{\mathcal{P}}_V^\star(\nu)$.

7.3.3 Strong Markovian MFG equilibrium

Let $\mathrm{P}^\star \in \overline{\mathcal{P}}_V^\star(\nu)$ (see Definition 6.3.5 for the equivalence with the canonical space) be a measure–valued MFG solution such that there exists $(\mathbf{n}^\star, \mathbf{q}^\star) \in \mathcal{C}_{\mathcal{W}}^n \times \mathbb{M}(\mathcal{P}_U^n)$ satisfying $(\mu, \Lambda) = (\mathbf{n}^\star, \mathbf{q}^\star)$, $\mathrm{P}^\star$–a.e. By Theorem 4.6.2, there exists a sequence $(\alpha^{\star,k})_{k\in\mathbb{N}^*}$, such that for each $k \in \mathbb{N}^*$, $\alpha^{\star,k} : [0,T] \times \mathbb{R}^n \to U$ is a Borel function, and on a filtered probability space $(\Omega^\star, \mathbb{F}^\star, \mathbb{P}^\star)$ supporting W a $\mathbb{R}^n$–Brownian motion and ξ–a $\mathcal{F}_0$–random variable with $\mathcal{L}^{\mathbb{P}^\star}(\xi) = \nu$, $X^{\star,k}$ is a solution of

$$X_t^{\star,k} = \xi + \int_0^t b\big(r, X_r^{\star,k}, \mu_{r\wedge\cdot}^k, \overline{\mu}_r^k, \alpha^\star(r, X_r^{\star,k})\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^{\star,k}\big)\mathrm{d}W_r, \text{ for all } t \in [0,T],\ \mathbb{P}^\star\text{–a.e.},$$

$\overline{\mu}_t^k = \mathcal{L}^{\mathbb{P}^\star}(X_t^{\star,k}, \alpha^{\star,k}(t, X_t^{\star,k}))$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}^\star$–a.e, $\mu_t^k = \mathcal{L}^{\mathbb{P}^\star}(X_t^{\star,k})$, and one has

$$\lim_{k\to\infty} \big(\mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big) = \big(\mathbf{n}^\star, \mathbf{q}^\star\big), \text{ in } \mathcal{W}_p\ \mathbb{P}^\star\text{–a.e.}$$

Now, we show there is a sequence $(\epsilon_k)_{k\in\mathbb{N}^*}$, such that $\epsilon_k > 0$, $\lim_{k\to\infty} \epsilon_k = 0$, and $\overline{\mu}^k$ is an ϵ_k–strong Markovian MFG solution. Denote by $\mathcal{A}^m$ the set of Borel function $\alpha : [0,T] \times \mathbb{R}^n \to U$, and for any $\alpha \in \mathcal{A}^m$, $X^{\alpha,k}$ the associated solution

$$X_t^{\alpha,k} = \xi + \int_0^t b\big(r, X_r^{\alpha,k}, \mu_{r\wedge\cdot}^k, \overline{\mu}_r^k, \alpha(r, X_r^{\alpha,k})\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^{\alpha,k}\big)\mathrm{d}W_r, \text{ for all } t \in [0,T],\ \mathbb{P}\text{–a.e..}$$

With $\overline{\mu}_t^{\alpha,k} := \mathcal{L}^{\mathbb{P}^\star}(X_t^{\alpha,k}, \alpha(t, X_t^{\alpha,k}))$ and $\mu_t^{\alpha,k} := \mathcal{L}^{\mathbb{P}^\star}(X_t^{\alpha,k})$, we define

$$\epsilon_k := \sup_{\alpha\in\mathcal{A}^m} J\big(\mu^{\alpha,k}, \delta_{\overline{\mu}_t^{\alpha,k}}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big) - J\big(\mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big).$$

By construction for each $k \in \mathbb{N}^*$, $\epsilon_k > 0$. We choose $\alpha^k \in \mathcal{A}^m$ such that

$$J\big(\mu^{\alpha^k,k}, \delta_{\overline{\mu}_t^{\alpha^k,k}}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big) - J\big(\mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big) \geq \epsilon_k - 1/2^k.$$

Using Proposition 4.8.4 (for instance), we can show the sequence $\Big(\mathbb{P} \circ \Big(\mu^{\alpha^k,k}, \delta_{\overline{\mu}_t^{\alpha^k,k}}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\Big)^{-1}\Big)_{k\in\mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p$, and as limit of $(\mu^k, \overline{\mu}^k)_{k\in\mathbb{N}^*}$ is deterministic, one has that any limit P^∞ of any sub-sequence of $\Big(\mathbb{P} \circ \Big(\mu^{\alpha^k,k}, \delta_{\overline{\mu}_t^{\alpha^k,k}}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\Big)^{-1}\Big)_{k\in\mathbb{N}^*}$ belongs to $\overline{\mathcal{P}}_V(\nu)$ with $\mathcal{L}^{\mathrm{P}^\infty}(\zeta, \Lambda) = \mathcal{L}^{\mathrm{P}^\star}(\zeta, \Lambda)$. As $\mathrm{P}^\star$ is a measure–valued MFG equilibrium, we deduce that

$$0 \geq \limsup_{k\to\infty} J\big(\mu^{\alpha^k,k}, \delta_{\overline{\mu}_t^{\alpha^k,k}}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big) - J\big(\mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t, \mu^k, \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t\big).$$

Therefore $\lim_{k\to\infty} \epsilon_k = 0$, and $\overline{\mu}^k$ is an ϵ_k–strong Markovian MFG solution for each $k \in \mathbb{N}^*$.

7.4 Appendix: some technical results

7.4.1 Density of controls

Let E, E° and $E^\star$ be three polish spaces, and $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space supporting

- a E–valued $\mathbb{F}$–adapted continuous process $(\vartheta_t)_{t\in[0,T]}$.
- a $\mathcal{P}(E^\star)$–valued $\mathbb{F}$–predictable process $(\Phi_t)_{t\in[0,T]}$, and a $\mathcal{P}(E^\circ)$–valued $\mathbb{F}$–predictable process $(\Phi_t^\circ)_{t\in[0,T]}$.

All these variables satisfy

$$\mathcal{L}^{\mathbb{P}}\big(\Phi_{t\wedge\cdot}^\circ \big| \vartheta_{t\wedge\cdot}, \Phi_{t\wedge\cdot}\big) = \mathcal{L}^{\mathbb{P}}\big(\Phi_{t\wedge\cdot}^\circ \big| \vartheta, \Phi\big), \text{ for all } t \in [0,T],\ \mathbb{P} \text{ a.e.}, \tag{7.4.1}$$

where $\Phi_t(\mathrm{d}e^\star)\mathrm{d}t$ is considered as an element of $\mathbb{M}(E^\star)$ and $\Phi_t^\circ(\mathrm{d}e^\circ)\mathrm{d}t$ as an element of $\mathbb{M}(E^\circ)$.

Proposition 7.4.1. *With the previous considerations, there exists a $[0,1]$–valued uniform variable N independent of all variables, and a sequence $(\widehat{G}^k)_{k\in\mathbb{N}^\star}$ satisfying for each $k\in\mathbb{N}^*$, $\widehat{G}^k : [0,T]\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]\to\mathcal{P}(E^\circ)$ is a continuous function such that*

$$\lim_{k\to\infty}\mathcal{L}^{\mathbb{P}}\Big(\widehat{G}^k\big(t,\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot},N\big)(\mathrm{d}e^\circ)\mathrm{d}t,\vartheta,\Phi\Big)=\mathcal{L}^{\mathbb{P}}\Big(\Phi^\circ_t(\mathrm{d}e^\circ)\mathrm{d}t,\vartheta,\Phi\Big).$$

Moreover, if $\Gamma^\circ\subset E^\circ$ is a compact set s.t. $\Phi^\circ_t(\Gamma^\circ)=1$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e., there exists a sequence $(G^k)_{k\in\mathbb{N}^*}$ such that for each $k\in\mathbb{N}^*$, $G^k:[0,T]\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]\to E^\circ$ is a continuous function and

$$\lim_{k\to\infty}\mathcal{L}^{\mathbb{P}}\Big(\delta_{G^k\left(t,\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot},N\right)}(\mathrm{d}e^\circ)\mathrm{d}t,\vartheta,\Phi\Big)=\mathcal{L}^{\mathbb{P}}\Big(\Phi^\circ_t(\mathrm{d}e^\circ)\mathrm{d}t,\vartheta,\Phi\Big).$$

Proof. Let $t^K_0=0<\cdots<t^K_K=T$ for each $K\in\mathbb{N}^*$, with $\lim_{K\to\infty}\sup_k|t^K_k-t^K_{k-1}|=0$, denote $[t]^K=\sum_{k=1}^K t^K_k 1_{t\in[t^K_k,t^K_{k+1})}$, and define

$$\Phi^{\circ,K}_t(\mathrm{d}e^\circ):=K\int_{\left([t]^K-\frac{1}{K}\right)\vee 0}^{[t]^K}\Phi^\circ_s(\mathrm{d}e^\circ)\mathrm{d}s,\ \text{for each } t\in[0,T] \text{ and } K\in\mathbb{N}^*.$$

By mimicking the proof of [119, Lemma 4.4], the sequence of $\mathcal{P}(E^\circ)$–valued $\mathbb{F}$–predictable processes $(\Phi^{\circ,K})_{K\in\mathbb{N}^*}$ is such that: $\lim_{K\to\infty}\Phi^{\circ,K}_t(\omega)=\Phi^\circ_t(\omega)$, for the weak convergence topology, for $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$ a.e. $(t,\omega)\in[0,T]\times\Omega$, $\Phi^{\circ,K}_t=V^K(t,\Phi^\circ_{t\wedge\cdot})$ with a Borel function $V^K:[0,T]\times\mathbb{M}(E^\circ)\to\mathcal{P}(E^\circ)$ and $\Phi^{\circ,K}_t=\Phi^{\circ,K}_{t^K_k}$ when $t\in[t^K_k,t^K_{k+1})$, for every $k\in\{1,\dots,K\}$. Notice that for each K, by assumption 7.4.1, one has

$$\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t\wedge\cdot}\big|\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot}\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t\wedge\cdot}\big|\vartheta,\Phi\big),\ \text{for all } t\in[0,T],\ \mathbb{P}\text{ a.e.}\tag{7.4.2}$$

and if $\Phi^\circ_t(\Gamma^\circ)=1$ one has $\Phi^{\circ,K}_t(\Gamma^\circ)=1$.

For every $1\le k\le K$, there exists a Borel function $\mathrm{F}^k:\mathbb{M}(E^\circ)\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]\to\mathbb{M}(E^\circ)$ and uniform random variable N^k independent of $\big(\Phi^{\circ,K}_{t^K_{k-1}\wedge\cdot},\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\big)$ such that

$$\mathcal{L}^{\mathbb{P}}\big(\overline{\mathrm{F}}^k,\Phi^{\circ,K}_{t^K_{k-1}\wedge\cdot},\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t^K_k\wedge\cdot},\Phi^{\circ,K}_{t^K_{k-1}\wedge\cdot},\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\big),$$

where $\overline{\mathrm{F}}^k:=\mathrm{F}^k\big(\Phi^{\circ,K}_{t^K_{k-1}\wedge\cdot},\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot},N^k\big)$.

Now denote $\gamma^0:=\Phi^{\circ,K}_{t^K_0\wedge\cdot}$ (which can be assumed not random), and by recurrence: for all $1\le k\le K$

$$\gamma^k:=\mathrm{F}^k\big(\gamma^{k-1},\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot},N^k\big),$$

notice that γ^k is $\sigma\big\{\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot},N^1,\dots,N^k\big\}$–measurable and belongs to $\mathbb{M}(E^\circ)$, $(N^k)_k$ can be taken i.i.d and independent of the other variables.

Now let us prove that

$$\mathcal{L}^{\mathbb{P}}\big(\gamma^0,\dots,\gamma^K,\vartheta,\Phi\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_K\wedge\cdot},\vartheta,\Phi\big),\tag{7.4.3}$$

we proceed by recurrence i.e. let us prove for each k

$$\mathcal{L}^{\mathbb{P}}\big(\gamma^0,\dots,\gamma^k,\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot},\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\big).\tag{7.4.4}$$

For $k=0$, this is obvious. Assume that (7.4.4) is true for k, we verify for $(k+1)$. Let $\phi:\mathbb{M}(E^\circ)^k\to\mathbb{R}$, $G:C([0,T];E)\times\mathbb{M}(E^\star)\to\mathbb{R}$ Borel functions,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big[\phi\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot}\big)G\big(\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)\Big]=\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}\Big[\phi\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot}\big)\big|\vartheta,\Phi\Big]G\big(\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)\Big]\\
&=\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}\Big[\phi\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot}\big)\big|\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\Big]G\big(\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)\Big]\\
&=\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}\Big[\phi\big(\gamma^0,\dots,\gamma^k\big)\big|\vartheta_{t^K_k\wedge\cdot},\Phi_{t^K_k\wedge\cdot}\Big]G\big(\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)\Big]\\
&=\mathbb{E}^{\mathbb{P}}\Big[\mathbb{E}^{\mathbb{P}}\Big[\phi\big(\gamma^0,\dots,\gamma^k\big)\big|\vartheta,\Phi\Big]G\big(\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)\Big]=\mathbb{E}^{\mathbb{P}}\Big[\phi\big(\gamma^0,\dots,\gamma^k\big)G\big(\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)\Big],
\end{aligned}$$

where the result (7.4.2) is used for the second equality, the third follows from the recurrence hypothesis (7.4.4), and the fourth because of the result (7.4.2) again. This is true for all (ϕ,G), then

$$\mathcal{L}^{\mathbb{P}}\big(\gamma^0,\dots,\gamma^k,\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot},\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot}\big),$$

the previous equality allows us to get

$$\begin{aligned}
\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot},\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot},\Phi^{\circ,K}_{t^K_{k+1}\wedge\cdot}\big)&=\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K}_{t^K_0\wedge\cdot},\dots,\Phi^{\circ,K}_{t^K_k\wedge\cdot},\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot},\overline{\mathrm{F}}^{k+1}\big)\\
&=\mathcal{L}^{\mathbb{P}}\big(\gamma^0,\dots,\gamma^k,\vartheta_{t^K_{k+1}\wedge\cdot},\Phi_{t^K_{k+1}\wedge\cdot},\gamma^{k+1}\big),
\end{aligned}$$

therefore (7.4.4) is true for $(k+1)$, consequently (7.4.3) is true. As $\gamma^K\in\mathbb{M}(E^\circ)$ and the law equality (7.4.3), it is straightforward to check that

$$\gamma^K_{t\wedge\cdot}(\mathrm{d}e^\circ)=\gamma^k_{t\wedge\cdot}(\mathrm{d}e^\circ),\ \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{–a.e. }(t,\omega)\in[t^K_k,t^K_{k+1})\times\Omega,$$

then $\gamma^K_{t\wedge\cdot}$ is $\sigma\big\{\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot},N^1,\dots,N^K\big\}$–measurable, and therefore there exists a Borel measurable function $\widehat{G}^K:[0,T]\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]^K\to\mathcal{P}(E^\circ)$ s.t. $\gamma^K_t(\mathrm{d}e^\circ)\mathrm{d}t=\widehat{G}^K(t,\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot},N^1,\dots,N^K)(\mathrm{d}e^\circ)\mathrm{d}t$, $\mathbb{P}$–a.e.. And one has

$$\lim_{K\to\infty}\mathcal{L}^{\mathbb{P}}\big(\gamma^K,\vartheta,\Phi\big)=\lim_{K\to\infty}\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K},\vartheta,\Phi\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^\circ,\vartheta,\Phi\big).$$

If $\Gamma^\circ\subset E^\circ$ is a compact set s.t. $\Phi^\circ_t(\Gamma^\circ)=1$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e., we saw earlier that $\Phi^{\circ,K}_t(\Gamma^\circ)=1$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e., for all K, therefore, by law equality (7.4.3), $\gamma^K_t(\Gamma^\circ)=1$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e.. It is a classical result that for each $K\in\mathbb{N}^*$, there exists a sequence of Γ°–valued $(\sigma\{\gamma^K_{t\wedge\cdot}\})_{t\in[0,T]}$–predictable processes $(\mathbf{m}^{K,q})_{q\in\mathbb{N}^*}$ such that $\lim_q\delta_{\mathbf{m}^{K,q}_t}(\mathrm{d}m')\mathrm{d}t=\gamma^K_t(\mathrm{d}m')\mathrm{d}t$, $\mathbb{P}$–a.e.. Then, there exists a Borel function $G^{K,q}:[0,T]\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]^K\to E^\circ$ verifying

$$\mathbf{m}^{K,q}_t=G^{K,q}(t,\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot},N^1,\dots,N^K),\ \mathbb{P}\text{–a.e., for each }q\in\mathbb{N}^*.$$

Consequently

$$\lim_{K\to\infty}\lim_{q\to\infty}\mathcal{L}^{\mathbb{P}}\big(\delta_{\mathbf{m}^{K,q}_s}(\mathrm{d}m')\mathrm{d}s,\vartheta,\Phi\big)=\lim_K\mathcal{L}^{\mathbb{P}}\big(\gamma^K,\vartheta,\Phi\big)=\lim_K\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K},\vartheta,\Phi\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^\circ,\vartheta,\Phi\big).$$

Next, we will show that we can chose an approximation of $\widehat{G}^K$ and $G^{K,q}$ continuous. If

$$\mathrm{Q}^K:=\frac{1}{T}\mathbb{E}^{\mathbb{P}}\Big[\delta_{\left(\vartheta_{t\wedge\cdot},\,\Phi_{t\wedge\cdot},\,N^1,\dots,N^K\right)}(\mathrm{d}a^\vartheta,\mathrm{d}a^\Phi,\mathrm{d}n^1,\dots,\mathrm{d}n^K)\Big]\mathrm{d}t,$$

Q^K is an element of $\mathcal{P}\Big([0,T]\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]^K\Big)$, by [49, Proposition C.1.], for each K, there exists a sequence $(\widehat{G}^{K,j})_{j\in\mathbb{N}}$ such that $\widehat{G}^{K,j}:[0,T]\times C([0,T];E)\times\mathbb{M}(E^\star)\times[0,1]^K\to\mathcal{P}(E^\circ)$ is continuous and $\lim_{j\to\infty}\widehat{G}^{K,j}=\widehat{G}^K$, Q^K–a.e. If $\widetilde{\mathrm{Q}}^K:=\mathbb{P}\circ\big(\vartheta,\Phi,N^1,\dots,N^K\big)^{-1}(\mathrm{d}a,\mathrm{d}n^1,\dots,\mathrm{d}n^K)\mathrm{d}t$, it is straightforward to see that $\widetilde{\mathrm{Q}}$ is equivalent to Q, and therefore

$$\lim_{K\to\infty}\lim_{j\to\infty}\mathcal{L}^{\mathbb{P}}\Big(\widehat{G}^{K,j}(t,\vartheta_{t\wedge\cdot},\Phi_{t\wedge\cdot},N^1,\dots,N^K)(\mathrm{d}e^\circ)\mathrm{d}t,\vartheta,\Phi\Big)=\lim_{K\to\infty}\mathcal{L}^{\mathbb{P}}\big(\gamma^K,\vartheta,\Phi\big)=\lim_K\mathcal{L}^{\mathbb{P}}\big(\Phi^{\circ,K},\vartheta,\Phi\big)=\mathcal{L}^{\mathbb{P}}\big(\Phi^\circ,\vartheta,\Phi\big),$$

we deal the function $G^{K,q}$ by similar way for the case $\Gamma^\circ\subset E^\circ$ a compact set s.t. $\Phi^\circ_t(\Gamma^\circ)=1$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e., all these results are sufficient to conclude. □

7.4.2 Approximation by measure–valued control rules

In this section we provide some convergence result for a sequence of particular measure–valued control rule $\overline{\mathcal{P}}_V$ (see definition 6.3.2). This result is useful to prove the limit theorem Theorem 6.4.1. To simplify, it is shown that: for a convergent sequence of measure–valued control rules satisfying some conditions, we can find another sequence of measure–valued control rules sharing the same limit and keeping certain properties of the limit.
In order to correctly formulate our result, let us mention some notations, they are motivated by those used in Chapter 4.

Let $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$ with $p' > p$, $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a probability space supporting

- a $\mathbb{R}^{n+\ell}$ Brownian motion (W, B) and a $\mathcal{F}_0$–random variable ξ s.t. $\mathcal{L}^{\mathbb{P}}(\xi) = \nu$
- a $\mathcal{P}_U^n$–valued $\mathbb{F}$–predictable process $(\Lambda_t^\star)_{t\in[0,T]}$ and a $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{F}$–continuous process $(\zeta_t^\star)_{t\in[0,T]}$
- a sequence of Borel functions $(\phi^k)_{k\in\mathbb{N}^*}$ s.t. for each $k \in \mathbb{N}^*$, $\phi^k : [0,T] \times \mathbb{R}^n \times \mathcal{C}^n \times \mathcal{C}^\ell \times \mathcal{C}^n_{\mathcal{W}} \times \mathbb{M}(\mathcal{P}_U^n) \to U$.
- a sequence of $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{F}$–adapted continuous process $(\zeta^k)_{k\in\mathbb{N}^*}$, and a sequence of $\mathcal{P}_U^n$–valued $\mathbb{F}$–predictable processes $(\mathbf{m}^k)_{k\in\mathbb{N}^*}$.

Next, let us introduce for all $k \in \mathbb{N}^*$, the unique strong solution X^k of: for all $t \in [0,T]$

$$X_t^k = \xi + \int_0^t b\big(r, X_r^k, \zeta^k, \mathbf{m}_r^k, \phi_r^k\big)\mathrm{d}r + \int_0^t \sigma\big(r, X_r^k, \zeta^k, \mathbf{m}_r^k, \phi_r^k\big)\mathrm{d}W_r + \sigma_0 B_t,$$

with $\Lambda_t^k(\mathrm{d}m')\mathrm{d}t := \delta_{\mathbf{m}_t^k}(\mathrm{d}m')\mathrm{d}t$, and $\phi_t^k := \phi^k(t, \xi, W_{t\wedge\cdot}, B_{t\wedge\cdot}, \zeta^k_{t\wedge\cdot}, \Lambda^k_{t\wedge\cdot})$.

Denote by $\mu_t^k := \mathcal{L}^{\mathbb{P}}\big(X_t^k \big| B_{t\wedge\cdot}, \zeta^k_{t\wedge\cdot}, \Lambda^k_{t\wedge\cdot}\big)$, $\overline{\mu}_t^k := \mathcal{L}^{\mathbb{P}}\big(X_t^k, \phi_t^k \big| B_{t\wedge\cdot}, \zeta^k_{t\wedge\cdot}, \Lambda^k_{t\wedge\cdot}\big)$, for all $t \in [0,T]$, and $\Lambda_t^{\circ,k}(\mathrm{d}m)\mathrm{d}t := \delta_{\overline{\mu}_t^k}(\mathrm{d}m)\mathrm{d}t$. Also, the filtration $\mathbb{G} := (\mathcal{G}_t)_{t\in[0,T]}$ is defined by

$$\mathcal{G}_t := \sigma\{\zeta^\star_{t\wedge\cdot}, \Lambda^\star_{t\wedge\cdot}, B_{t\wedge\cdot}\}, \text{ for all } t \in [0,T].$$

Lemma 7.4.2. *If for each $k \in \mathbb{N}^*$, $(\Lambda_t^k, \zeta_t^k)_{t\in[0,T]}$ is $\mathbb{G}$–adapted, $(B, \Lambda^\star, \zeta^\star)$ is $\mathbb{P}$–independent of (W, ξ), and*

$$\lim_{k\to\infty} (\Lambda^k, \zeta^k) = (\Lambda^\star, \zeta^\star), \quad \text{in } \mathcal{W}_p,\ \mathbb{P}\text{–a.e.}, \tag{7.4.5}$$

then there exists for each $k \in \mathbb{N}^*$, a $\mathcal{P}(\mathbb{R}^n)$–valued $\mathbb{G}$–adapted continuous process $\widetilde{\mu}^k$ and a $\mathcal{P}_U^n$–valued $\mathbb{G}$–predictable process $\widetilde{\Lambda}^{\circ,k}$ solution of: for every $(t,f) \in [0,T] \times C_b^2(\mathbb{R}^n)$,

$$\begin{aligned}\langle f(\cdot - \sigma_0 B_t), \widetilde{\mu}_t^k\rangle = \langle f, \nu\rangle &+ \int_0^t \int_{\mathcal{P}_U^n} \Big[\langle \mathcal{L}_r^\circ[f(\cdot - \sigma_0 B_r)](\cdot, \zeta^\star, \cdot), m\rangle\Big]\widetilde{\Lambda}_r^{\circ,k}(\mathrm{d}m)\mathrm{d}r \\ &+ \int_0^t \int_{\mathcal{P}_U^n}\int_{\mathbb{R}^n} \mathcal{L}_r^\star[f(\cdot - \sigma_0 B_r)](x, \zeta^\star, m')\widetilde{\mu}_r^k(\mathrm{d}x)\Lambda_r^\star(\mathrm{d}m')\mathrm{d}r,\ \mathbb{P}\text{–a.e.},\end{aligned}$$

with $\widetilde{\Lambda}_t^{\circ,k}\big(\mathbb{Z}_{\widetilde{\mu}_t^k}\big) = 1$, $\mathrm{d}\mathbb{P}\otimes\mathrm{d}t$–a.e. such that: if

$$\mathrm{Q}^k := \mathcal{L}^{\mathbb{P}}\Big(\widetilde{\mu}^k, \zeta^\star, \widetilde{\Lambda}_t^{\circ,k}\big(\mathrm{d}m\big)\mathrm{d}t, \Lambda_t^\star\big(\mathrm{d}m\big)\mathrm{d}t, B\Big),$$

$\mathrm{Q}^k \in \overline{\mathcal{P}}_V(\nu)$ *for each $k \in \mathbb{N}^*$, and*

$$\lim_{k\to\infty} \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}}\big(\mu^k, \zeta^k, \Lambda_t^{\circ,k}\big(\mathrm{d}m\big)\mathrm{d}t, \Lambda_t^k\big(\mathrm{d}m'\big)\mathrm{d}t, B\big), \mathrm{Q}^k\Big).$$

Proof. Let us take a convergent sub-sequence of $\big(\mathcal{L}^{\mathbb{P}}\big(\mu^k, \zeta^k, \Lambda_t^{\circ,k}(\mathrm{d}m)\mathrm{d}t, \Lambda_t^k(\mathrm{d}m)\mathrm{d}t, B\big)\big)_{k\in\mathbb{N}^*}$ (possible because it is relatively compact see for instance Proposition 4.8.4), denote by P^∞ its limit, one uses the same notation for the sub–sequence. The limit satisfies: $N_t(f) = 0$, P^∞–a.e., for all $t \in [0,T]$ and $f \in C_b^2(\mathbb{R}^n)$, where recall that $(\mu, \zeta, \Lambda^\circ, \Lambda, B)$ is the canonical variable on $\overline{\Omega} := (\mathcal{C}_{\mathcal{W}}^n)^2 \times \mathbb{M}(\mathcal{P}_U^n)^2 \times \mathcal{C}^\ell$, and $\Lambda_t^\circ(\mathbb{Z}_{\mu_t}) = 1$, $\mathrm{dP}^\infty \otimes \mathrm{d}t$–a.e. $(t,\omega) \in [0,T] \times \overline{\Omega}$. Notice that, as $\lim_{k\to\infty} (\Lambda^k, \zeta^k) = (\Lambda^\star, \zeta^\star)$, in $\mathcal{W}_p$, $\mathbb{P}$–a.e., one has

$$\lim_{k\to\infty} \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}}\big(\zeta^k, \Lambda_t^{\circ,k}(\mathrm{d}m)\mathrm{d}t, \Lambda_t^k(\mathrm{d}m)\mathrm{d}t, B\big), \mathcal{L}^{\mathbb{P}}\big(\zeta^\star, \Lambda_t^{\circ,k}(\mathrm{d}m)\mathrm{d}t, \Lambda_t^\star(\mathrm{d}m)\mathrm{d}t, B\big)\Big) = 0.$$

Then, by taking into account the conditions (7.4.5), it is enough to apply Section 4.9.1 (see also Proposition 4.7.7) and Itô's formula to conclude the proof.

□

Part III

Numerical approximations

Chapter 8

Numerical methods

8.1 Introduction

In this chapter, we propose a numerical algorithm to solve the McKean–Vlasov control problem. Motivated by the results of Chapter 3 and Chapter 4 i.e. Theorem 3.2.7 and Theorem 4.5.3, we solve numerically the mean–field control problem. Our numerical procedure is based on a discretization of the large population stochastic control problem.

Indeed, first of all, we formulate a discretized form of the large population stochastic control problem by discretization in time with controls as function of finite samples of Brownian motions. The controls are no longer function of the trajectory of Brownian motions, only a finite value of Brownian motions matters in this situation (see Equation (8.2.3)). Then, when simultaneously the size of the discretization in time goes to zero and the number of agents goes to infinity, we prove that this discretized large population stochastic control problem has the same limit as the "normal" large population stochastic control problem i.e. the McKean–Vlasov control problem.

In the second time, based on this result, we give an algorithm to solve the McKean–Vlasov control problem using neural networks. As our problem involves a lot of data (infinite in theory), we suffer from the curse of dimensionality, therefore, using neural networks is natural and simplify the implementation. We implement our algorithm thanks to the open–source library keras, and test its efficient on three example.

Notice that, our idea is very close to Han and E [78], Fouque and Zhang [70] and Carmona and Laurière [46], which use similar methods to solve the mean–field control problem via neural networks. Despite the fact that we do not have a rate of convergence like [46], our result is more general in the sense that we work in a framework with assumptions less stronger, with law of controls and common noise while allowing to control the no–common noise volatility σ. Further, it should be emphasized that although our algorithm is intended for solving the mean–field control problem, it can be used to solve mean field games by using the equivalence result between MFG and MFC in certain contexts as formulated in Carmona and Delarue [43, Chapter 6].

The rest of the chapter is structured as follows. In Section 8.2, we recall first the strong formulation of McKean–Vlasov control problem, then provide the discretized form of the large population stochastic control problem and finally formulate the convergence result. The numerical implementation is given in Section 8.3 with the numeric examples. The technical proofs are completed in Section 8.4.

We use in this chapter some notations of Chapter 2, Chapter 3 and Chapter 4. We recall them for a better reading. Let $\mathbb{M}(E)$ bethe space of all Borel measures $q(\mathrm{d}t, \mathrm{d}e)$ on $[0,T] \times E$, whose marginal distribution on $[0,T]$ is the Lebesgue measure $\mathrm{d}t$, that is to say $q(\mathrm{d}t, \mathrm{d}e) = q(t, \mathrm{d}e)\mathrm{d}t$ for a family $(q(t, \mathrm{d}e))_{t\in[0,T]}$ of Borel probability measures on E. Let Λ denote the canonical element on $\mathbb{M}(E)$, we define

$$\Lambda^t(\mathrm{d}s, \mathrm{d}e) := \Lambda(\mathrm{d}s, \mathrm{d}e)\big|_{[0,t]\times E} + \delta_{e_0}(\mathrm{d}e)\mathrm{d}s\big|_{(t,T]\times E}, \text{ for some fixed } e_0 \in E.$$

Throughout the chapter, we fix a nonempty Polish space (A, ρ) and an element $a_0 \in A$, and denote $\mathbb{M} := \mathbb{M}(A)$. Finally, consider the canonical space $\mathcal{C}^n \times \mathbb{M}$ (resp. $\mathcal{C}^n \times A$), with canonical element (X, Λ) (resp. (X, α)), and $\widehat{\nu} \in \mathcal{P}(\mathcal{C}^n \times \mathbb{M})$ (resp. $\bar{\nu} \in \mathcal{P}(\mathcal{C}^n \times A)$). We define, for each $t \in [0,T]$

$$\widehat{\nu}(t) := \widehat{\nu} \circ (X_{t\wedge\cdot}, \Lambda^t)^{-1}, \ \big(\text{resp. } \bar{\nu}(t) := \bar{\nu} \circ (X_{t\wedge\cdot}, \alpha)^{-1}\big).$$

8.2 McKean–Vlasov optimal control and main results

Here, we recall the strong formulation of the McKean–Vlasov optimal control problem, and introduce the discretized large population control problem which will be necessary to give a numerical algorithm using neural networks to solve the McKean–Vlasov optimal control problem. In all this part, we work under the Assumption 1.4.1 with σ_0 constant.

8.2.1 A strong formulation

Let us consider the canonical space

$$\Omega := \mathbb{R}^n \times \mathcal{C}^d \times \mathcal{C}^\ell,$$

equipped with its Borel σ–algebra $\mathcal{F} := \mathcal{B}(\Omega)$ and canonical element (X_0, W, B). Let $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ and $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ be two filtrations on $(\Omega, \mathcal{F})$ defined by

$$\mathcal{F}_t := \sigma\big((X_0, W_s, B_s) : s \in [0,t]\big), \text{ and } \mathcal{G}_t := \sigma\big(B_s : s \in [0,t]\big),\ t \in [0,T].$$

Let p be the constant in Assumption 1.4.1 and $\nu \in \mathcal{P}_p(\mathbb{R}^n)$. We denote by $\mathbb{P}_\nu$ the probability measure on $(\Omega, \mathcal{F})$, under which $X_0 \sim \nu$ and (W, B) is a standard $\mathbb{R}^{d+\ell}$–dimensional Brownian motion, independent of X_0. Recall that a_0 is a fixed point in A. We denote by $\mathcal{A}_p(\nu)$ the collection of all $\mathbb{F}$–predictable, A–valued processes $\alpha = (\alpha_s)_{0\leq s\leq T}$ satisfying

$$\mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T \big(\rho(\alpha_s, a_0)\big)^p \mathrm{d}s\bigg] < \infty.$$

Then given a control process $\alpha \in \mathcal{A}_p(\nu)$, the controlled McKean–Vlasov SDE

$$X_t^\alpha = X_0 + \int_0^t b\big(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s\big)\mathrm{d}s + \int_0^t \sigma\big(s, X^\alpha_{s\wedge\cdot}, \overline{\mu}^\alpha_s, \alpha_s\big)\mathrm{d}W_s + \sigma_0 B_t,\ t \in [0,T],\ \mathbb{P}_\nu\text{–a.s.}, \tag{8.2.1}$$

with $\overline{\mu}^\alpha_s := \mathcal{L}^{\mathbb{P}_\nu}\big(X^\alpha_{s\wedge\cdot}, \alpha_s \big| \mathcal{G}_s\big)$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}_\nu$–a.e., has a unique strong solution, that is, there is a unique $\mathbb{F}$–adapted continuous process X^α on $(\Omega, \mathcal{F})$ satisfying Equation (8.2.1) and $\mathbb{E}^{\mathbb{P}_\nu}\big[\sup_{t\in[0,T]} |X_t^\alpha|^p\big] < \infty$.

Denote also $\mu_t^\alpha := \mathcal{L}^{\mathbb{P}_\nu}\big(X^\alpha_{t\wedge\cdot} \big| \mathcal{G}_t\big)$ for all $t \in [0,T]$. The strong formulation of the McKean–Vlasov control problem is then given by

$$V_S(\nu) := \sup_{\alpha\in\mathcal{A}_p(\nu)} \mathbb{E}^{\mathbb{P}_\nu}\bigg[\int_0^T L\big(t, X^\alpha_{t\wedge\cdot}, \overline{\mu}^\alpha_t, \alpha_t\big)\mathrm{d}t + g\big(X^\alpha_{T\wedge\cdot}, \mu^\alpha_T\big)\bigg]. \tag{8.2.2}$$

8.2.2 A discretized large population stochastic control problem with common noise

Now, in this section, we formulate the discretized form of the large population stochastic control problem. Let N be a positive integer, $m^N \in \mathbb{N}^*$ such that $\lim_{N\longrightarrow\infty} m^N = \infty$, $t_j^N := j\frac{1}{m^N}$ for each $j \in \{1, ..., m^N\}$, and for all $t \in [0,T]$, $[t]^N := \sum_{j=1}^{m_N} t_j^N 1_{t\in[t_j^N, t_{j+1}^N)}$.

Let $(\Omega^\star, \mathbb{F}^\star, \mathcal{F}^\star, \mathbb{P}^\star)$ be a filtered probability space supporting N–i.i.d. $\mathbb{R}^n$–valued random variables $X_0 := (X_0^1, ..., X_0^N)$ of distribution $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, the sequences of independent random variables of normal distribution $(U_k^i)_{(i,k)\in\{1,...,N\}\times\{1,...,m^N\}}$ and $(V_k^N)_{k\in\{1,...,m^N\}}$. Moreover, X_0, $(U_k^i)_{(i,k)\in\{1,...,N\}\times\{1,...,m^N\}}$, and $(V_k^N)_{k\in\{1,...,m^N\}}$ are independent.

Let us introduce the following assumption that we will use for the discretization of the SDEs:

Assumption 8.2.1. *Let* $\mathbf{H} : \mathbb{R}_+ \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A \times [0,1] \to \mathbb{R}^n$ *satisfying: for any normal random variable* U, $(h, \mathbf{x}, \bar{\nu}, a) \in \mathbb{R}_+ \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A$,

$$\mathbb{E}^{\mathbb{P}^\star}\Big[\big|\mathbf{H}(h, \mathbf{x}, \bar{\nu}, a, U)\big|^3\Big] + \Big|\mathbb{E}^{\mathbb{P}^\star}\big[\mathbf{H}(h, \mathbf{x}, \bar{\nu}, a, U)\big] - b(\mathbf{x}, \bar{\nu}, a)h\Big| + \Big|\mathbb{E}^{\mathbb{P}^\star}\big[\mathbf{H}(h, \mathbf{x}, \bar{\nu}, a, U)\mathbf{H}(h, \mathbf{x}, \nu, a, U)^\top\big] - \sigma\sigma^\top(\mathbf{x}, \bar{\nu}, a)h\Big| \leq \psi(h)$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *s.t.* $\lim_{h\to 0} h^{-1}\psi(h) = 0$, *and* $\sup_{h>0} h^{-3/2}\psi(h) < \infty$.

We denote by $\mathcal{A}_N$ the collection of all bounded measurable functions $\phi : [0,T] \times \mathbb{R}^n \times (\mathbb{R}^d \times \mathbb{R}^\ell)^{m^N} \to A$. Then, with the function $\mathbf{H}$ verifying Assumption 8.2.1, for every fixed $\phi \in \mathcal{A}_N$, let us define the continuous process $(X^{\phi,1}, \dots, X^{\phi,N})$ by: for $i \in \{1, \dots, N\}$, $X_0^{\phi,i} := X_0^i$ and for all $j \in \{0, \dots, m^N - 1\}$, $X^{\phi,i}_{t_0^N} = X_0^i$, $Y^{\phi,i}_{t_0^N} = 0$,

$$Y^{\phi,i}_{t^N_{j+1}} = Y^{\phi,i}_{t^N_j} + \mathbf{H}\big(1/m^N,\ \widehat{X}^{\phi,i},\ \varphi^N_{t^N_j},\ \phi^i_{t^N_j},\ U^i_j\big) \ \text{ and } \ X^{\phi,i}_{t^N_{j+1}} = X^{\phi,i}_{t^N_j} + Y^{\phi,i}_{t^N_{j+1}} - Y^{\phi,i}_{t^N_j} + \sigma_0\sqrt{1/m^N}V^N_j \tag{8.2.3}$$

where $\phi^i_{t^N_j} := \phi\Big(t^N_j, X^i_0, \big(\sqrt{1/m^N}U^i_{k\wedge j}, \sqrt{1/m^N}V^N_{k\wedge j}\big)_{1\le k\le m^N}\Big)$, $\big\{(U^i_j, V^N_j),\ 1 \le i \le N, 1 \le j \le m^N\big\}$ independent normal random variables, $(\widehat{X}^{\phi,1}, \dots, \widehat{X}^{\phi,N})$ denotes the linear interpolation process of $\Big\{(X^{\phi,1}_{t^N_j}, \dots, X^{\phi,N}_{t^N_j}),\ 1 \le j \le m^N\Big\}$ on interval $[0,T]$,

$$\varphi^N_{t^N_j}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i}_{t^N_j}, \phi^i_{t^N_j}\right)}(\mathrm{d}\mathbf{x}), \text{ and } \varphi^{N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^{\phi,i}}(\mathrm{d}\mathbf{x}), \text{ for all } s \in [0,T].$$

The value function of the discretized large population stochastic control problem is then defined by

$$V_S^N(\nu) := \sup_{\phi\in\mathcal{A}_N} J_N(\phi), \text{ where } J_N(\phi) := \frac{1}{N}\sum_{i=1}^N \sum_{j=0}^{m^N-1} \mathbb{E}^{\mathbb{P}^\star}\Big[L\big(t^N_j, \widehat{X}^{\phi,i}, \varphi^N_{t^N_j}, \phi^i_{t^N_j}\big)1/m^N + g\big(\widehat{X}^{\phi,i}, \varphi^{N,X}\big)\Big]. \tag{8.2.4}$$

Remark 8.2.2. *Considering a general form of the discretized diffusion (8.2.3) with the function $\mathbf{H}$ allows different possible schemes and not only the Euler's scheme. The Euler's scheme corresponds to*

$$\mathbf{H}(h, \mathbf{x}, \bar\nu, a, U) := b(\mathbf{x}, \bar\nu, a)h + \sigma(\mathbf{x}, \bar\nu, a)\sqrt{h}U.$$

In the case $n = 1$, another scheme is possible by considering the cumulative distribution function $F(h, \mathbf{x}, \bar\nu, a) : \mathbb{R} \to [0,1]$ of $b(\mathbf{x}, \bar\nu, a)h + \sigma(\mathbf{x}, \bar\nu, a)\sqrt{h}U$ and define

$$\mathbf{H}(h, \mathbf{x}, \bar\nu, a, U) := \inf\{y :\ F(h, \mathbf{x}, \bar\nu, a)(y) > \beta(U)\},$$

where β is a function s.t. the law of $\beta(U)$ is uniform.

8.2.3 Main results

Assumption 8.2.3. *There exist Borel measurable functions $(b^\circ, \sigma^\circ, L^\circ) : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n) \times A \longrightarrow \mathbb{R}^n \times \mathbb{S}^{n\times d}$ such that, for all $(t, \mathbf{x}, \bar\nu, a) \in [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A$, with $\nu(\mathrm{d}\mathbf{x}) := \bar\nu(\mathrm{d}\mathbf{x}, A)$*

$$(b, \sigma, L)(t, \mathbf{x}, a, \bar\nu) = (b^\circ, \sigma^\circ, L^\circ)(t, \mathbf{x}, a, \nu).$$

By abuse of notations, we still write (b, σ, L) in lieu of $(b^\circ, \sigma^\circ, L^\circ)$.

Assumption 8.2.4. *There exist a constant $\theta > 0$, and Borel measurable functions $(b^\star, \sigma^\star, L^\star) : [0,T] \times \mathbb{R}^n \times C([0,T]; \mathcal{P}(\mathbb{R}^n)) \times A \times \mathcal{P}(\mathbb{R}^n \times A) \longrightarrow \mathbb{R}^n \times \mathbb{S}^{n\times d}$ and $\sigma_0^\star \in \mathbb{S}^{n\times\ell}$ such that, for all $(t, \mathbf{x}, \bar\nu, \nu, a) \in [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times \mathcal{P}(\mathcal{C}^n) \times A$, with $\bar\nu^\star_t(\mathrm{d}x, \mathrm{d}a) := \bar\nu \circ \big(X_t, \alpha\big)^{-1}(\mathrm{d}x, \mathrm{d}a)$ and $\nu^\star_t(\mathrm{d}x) := \nu \circ \big(X_t\big)^{-1}(\mathrm{d}x)$*

$$(b, \sigma, L)(t, \mathbf{x}, a, \bar\nu) = (b^\star, \sigma^\star, L^\star)(t, \mathbf{x}_t, \nu^\star_{t\wedge\cdot}, \bar\nu^\star_t, a),\ \sigma_0(t, \mathbf{x}, \nu) = \sigma_0^\star \ \text{ and } \ \theta \mathrm{I}_n \le \sigma\sigma^\top(t, \mathbf{x}, a, \bar\nu).$$

By abuse of notations, we still write (b, σ, L, σ_0) in lieu of $(b^\star, \sigma^\star, L^\star, \sigma_0^\star)$.

Theorem 8.2.5. *Let* Assumption 1.4.1 *hold true. Under* Assumption 8.2.3 *or* Assumption 8.2.4*, for any $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, one has*

$$\lim_{N\to\infty} \big|V_S^N(\nu) - V_S(\nu)\big| = 0.$$

8.3 Numerical implementation through neural networks

8.3.1 Description of the implementation

Description of the neural network We use a feedforward neural network which is a composition of layer functions. The set of layer functions with input dimension d_1, output dimension d_2, and activation function $\rho : \mathbb{R} \to \mathbb{R}$ denoted $\mathrm{L}^{\rho}_{d_1,d_2}$ is defined by

$$\mathrm{L}^{\rho}_{d_1,d_2} := \Big\{ \phi : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2} \text{ s.t. } \exists\, b \in \mathbb{R}^{d_2} \text{ and } w \in \mathbb{S}^{d_2 \times d_1} \text{ satisfying } \forall\, (i,x) \in [\![1,d_2]\!] \times \mathbb{R}^{d_1},\ \phi(x)_i = \rho\Big(b_i + \sum_{j=1}^{d_1} w_{ij} x_j\Big) \Big\}.$$

The set of feedforward neural network functions with $\mathcal{H}$ hidden layers, one output layer, the activation function ρ is then defined

$$\mathrm{N}^{\rho}[d_1, ..., d_{\mathcal{H}+1}] := \Big\{ \phi : \mathbb{R}^{d_0} \to \mathbb{R}^{d_{\mathcal{H}+1}} \text{ s.t. } \forall\, i \in [\![0, \mathcal{H}-1]\!],\ \exists\, \phi_i \in \mathrm{L}^{\rho}_{d_i,d_{i+1}} \text{ and } \phi_{\mathcal{H}} \in \mathrm{L}^{\mathrm{Id}}_{d_{\mathcal{H}},d_{\mathcal{H}+1}} \text{ satisfying } \phi = \phi_0 \circ ... \circ \phi_{\mathcal{H}} \Big\}.$$

Now, we define a new optimization problem involving the feedforward neural networks. For $\nu \in \mathcal{P}_p(\mathbb{R}^n)$, let us introduce, for an activation function ρ, $(d_1, \dots, d_{\mathcal{H}}) \in (\mathbb{N}^\star)^{\mathcal{H}}$,

$$\mathcal{V}^{N,q}_S(\nu) := \sup_{\phi \,\in\, \mathrm{N}^{\rho}[(d+\ell)\times m^N+2,\ d_1,\dots,d_{\mathcal{H}},\ h]} J_N(\phi),$$

where J_N is defined in (8.2.4). Thanks to the universal approximation theorem, the following result is just an application of Theorem 8.2.5.

Proposition 8.3.1. *Let us stay in the context of* Theorem 8.2.5. *Then*

$$\lim_{N\to\infty} \lim_{q\to\infty} \big| \mathcal{V}^{N,q}_S(\nu) - V_S(\nu) \big| = 0.$$

Algorithm Based on the previous result, we give an algorithm for solving the McKean-Vlasov control problem.

Let $t^N_{k+1} - t^N_k = 1/m^N := \Delta_N$, $M \in \mathbb{N}^*$, $(\xi^{i,j}_{(i,j)\in\{1,\dots,N\}\times\{1,\dots,M\}})$ a sequence of independent random variables s.t. $\mathcal{L}^{\mathbb{P}}(\xi^{i,j}) = \nu$ for all (i,j), and $(U^{i,j}_k)_{(i,k,j)\in\{1,\dots,N\}\times\{1,\dots,m^N\}\times\{1,\dots,M\}}$ and $(V^j_k)_{(k,j)\in\{1,\dots,m^N\}\times\{1,\dots,M\}}$ a sequence of i.i.d random variables of normal distribution. For $\phi \in \mathrm{N}^{\rho}[(d+\ell)\times m^N + 2,\ q,\ h]$, let us define: $X^{\phi,i,j}_0 = \xi^{i,j}_0$, and for each $k \in \{1, \dots, m^N\}$,

$$Y^{\phi,i,j}_{t^N_{k+1}} = Y^{\phi,i,j}_{t^N_k} + \mathbf{H}\big(1/m^N,\ \widehat{X}^{\phi,i,j},\ \varphi^N_{t^N_k},\ \phi^{i,j}_{t^N_k},\ U^{i,j}_k\big) \text{ and } X^{\phi,i,j}_{t^N_{k+1}} = X^{\phi,i,j}_{t^N_k} + Y^{\phi,i,j}_{t^N_{k+1}} - Y^{\phi,i,j}_{t^N_k} + \sigma_0 \sqrt{1/m^N} V^j_k$$

where $\phi^{i,j}_{t^N_k} := \phi\Big(t^N_k, \xi^{i,j}_0, \big(\sqrt{\Delta^N} U^{i,\,j}_{k'\wedge k}, \sqrt{\Delta^N} V^j_{k'\wedge k}\big)_{1\le k'\le m^N}\Big)$,

$$\varphi^{N,j}_s(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{X}^{\phi,i,j}_s, \phi^{i,j}_s)}(\mathrm{d}\mathbf{x}, \mathrm{d}a),\ \mathrm{d}s \otimes \mathrm{d}\mathbb{P}^\star\text{–a.e., and } \varphi^{N,X,j}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^{\phi,i,j}}(\mathrm{d}\mathbf{x}),\ \text{for all } s \in [0,T].$$

The optimization problem which we implement is

$$\mathbb{V}^{N,q,M}_S(\nu^1, \dots, \nu^N) := \sup_{\phi\in\mathrm{N}^{\rho}[(d+\ell)\times m^N+2,\ d_1,\dots,d_{\mathcal{H}},\ h]} J_N(\phi),$$

where

$$J_{N,M}(\phi) := \frac{1}{M}\sum_{j=1}^M \frac{1}{N}\sum_{i=1}^N \sum_{k=1}^{m^N} L\big(t^N_k, \widehat{X}^{\phi,i,j}, \varphi^{N,j}_{t^N_k}, \phi^{i,j}_{t^N_k}\big)\Delta_N + g\big(\widehat{X}^{\phi,i,j}, \varphi^{N,X,j}\big).$$

The parameters of our model through the function ϕ are

$$\theta := \big(b_1, (w_{1,1}, \ldots, w_{1,d_1}), \ldots, (b_h), (w_{h,1}, \ldots, w_{h,d_{\mathcal{H}}})\big),$$

we will note $J_{N,M}(\theta)$ instead of $J_{N,M}(\phi)$ in the following pseudo–algorithm.

Data:

- An initial parameter θ_0. A sequence of learning rates.
- M samples of: $(\xi^{i,1})_{i\in\{1,\ldots,N\}}$ of distribution ν (initial law), $(U_k^{i,1})_{(i,k)\in\{1,\ldots,N\}\times\{1,\ldots,m^N\}}$ and $(V_k^1)_{k\in\{1,\ldots,m^N\}}$ of normal distribution

Result: Learning ϕ by the parameters θ through a SGD

```
Initialization of θ;
for q:=1,...,Q do
    Compute ∇J_{N,M}(θ_q) by Back–propagation;
    if |∇J_{N,M}(θ_q)| is small enough then
        Stop now;
    else
        Otherwise, the parameter is updating: θ_{q+1} = a_q θ_q + (1 − a_q)∇J_{N,M}(θ_q);
    end
end
```

8.3.2 Numerical results

8.3.2.1 Example 1: mean–variance portfolio selection

As first example to illustrated our algorithm, we use the mean–variance optimization problem. This problem consists in minimizing a cost functional of the form:

$$J(\alpha) := \frac{\eta}{2}\mathrm{Var}(X_T) - \mathbb{E}[X_T],$$

with $\eta > 0$, with a dynamic for the wealth process $X := X^\alpha$ controlled by the amount α_t valued in $A = \mathbb{R}$ invested in one risky stock at time $t \in [0, T]$,

$$X_t = x_0 + \int_0^t r(s)X_s\mathrm{d}s + \int_0^t \alpha_s\rho(s)\mathrm{d}s + \int_0^t \alpha_s\vartheta(s)\mathrm{d}W_s.$$

where r is the interest rate, ρ and $\vartheta > 0$ are the excess rate of return (w.r.t. the interest rate) and volatility of the stock price. This model fits the context of Linear–quadratic McKean–Vlasov problem and has been studied in with some "McKean–Vlasov" techniques Pham and Wei [139]. Also studied under another approach by Zhou and Li [155], Fischer and Livieri [67] and Andersson and Djehiche [13]. The analytic form of the optimal control is given by

$$\alpha_t^\star = \frac{\rho(t)}{\vartheta^2(t)}\left[x_0\exp\big(\int_0^t r(s)\mathrm{d}s\big) + \frac{1}{\eta}\exp\left(\int_0^T \frac{\rho^2(s)}{\vartheta^2(s)}\mathrm{d}s - \int_t^T r(s)\mathrm{d}s\right) - X_t^\star\right], \text{ for all } t \in [0, T],$$

where $X^\star$ is the optimal wealth process with portfolio strategy $\alpha^\star$. We use $\alpha^\star$ and $J(\alpha^\star)$ as benchmark to test our algorithm in Figure 8.1. We can see that the difference between the estimated value of $J(\alpha^\star)$ and the true value decrease w.r.t. the number of iterations of the SGD method of our algorithm. The two graphs are really quite close. This result is in adequacy with the theoretical result of Theorem 8.2.5.

8.3.2.2 Example 2: inter–bank systemic risk model

As second example, we consider a model of inter-bank borrowing and lending studied in Carmona, Fouque, and Sun [48], and Pham and Wei [138] where the log-monetary reserve of each bank in the asymptotics when the number of banks

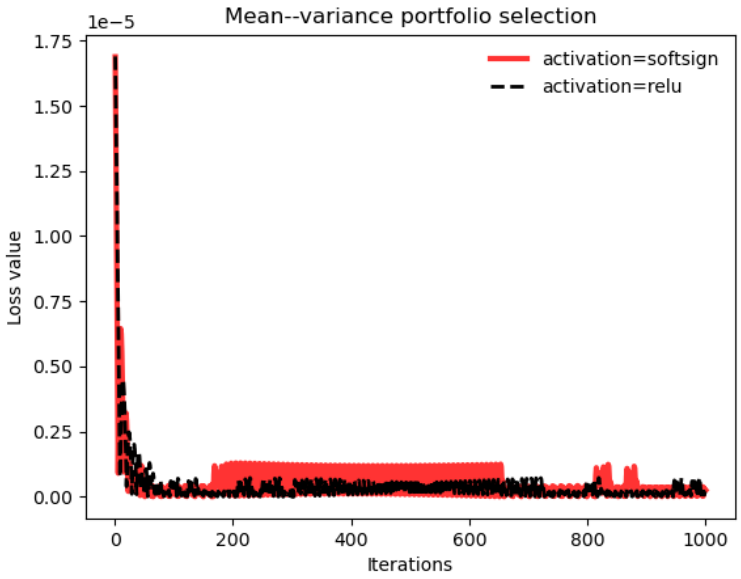

Figure 8.1: Difference between optimal value $J(\alpha^\star)$ and its estimated

tend to infinity, is governed by the McKean-Vlasov equation:

$$X_t = x_0 + \int_0^t [\kappa(\mathbb{E}[X_s|B] - X_s) + \alpha_s]\, \mathrm{d}s + \int_0^t (\sigma_0 + \sigma_1 X_s)[\sqrt{1-\rho^2}\mathrm{d}W_s + \rho \mathrm{d}B_s]., \ t \in [0,T].$$

with $\kappa \geq 0$ the rate of mean-reversion in the interaction from borrowing and lending between the banks, $\sigma_0 > 0$, $\sigma_1 \in \mathbb{R}$ are the affine coefficients of the volatility of the bank reserve, and there is a common noise B for all the banks. Moreover, all banks can control their rate of borrowing/lending to a central bank with the same policy α in order to minimize a cost functional of the form

$$J(\alpha) := \mathbb{E}\left[\int_0^T [\frac{1}{2}\alpha_t^2 - q\alpha_t(\mathbb{E}[X_t|B] - X_t) + \frac{\eta}{2}(\mathbb{E}[X_t] - X_t)^2]\mathrm{d}t + \frac{c}{2}(\mathbb{E}[X_T|B] - X_T)^2\right],$$

where $q > 0$ is a positive parameter for the incentive to borrowing ($\alpha_t > 0$) or lending ($\alpha_t < 0$), and $\eta > 0$,, $c > 0$ are positive parameters for penalizing departure from the average. The optimal control is given by

$$\alpha_t^\star = -(2\beta(t) + q)(X_t^\star - x_0 - \sigma_0 \rho B_t), \ t \in [0,T],$$

where $X^\star$ is the optimal log-monetary reserve controlled by the rate of borrowing/lending $\alpha^\star$,

$$\beta(t) := \frac{1}{2}\frac{(q-\eta^2)\big(\exp((\delta^+ - \delta^-)(T-t)) - 1\big) - c\big(\delta^+ \exp((\delta^+ - \delta^-)(T-t)) - \delta^-\big)}{\delta^- \exp((\delta^+ - \delta^-)(T-t)) - \delta^+ - c\exp((\delta^+ - \delta^-)(T-t)) - 1},$$

and

$$\delta^\pm := -(\kappa + q - \frac{\sigma_1^2}{2}) \pm \sqrt{(k + q - \frac{\sigma_1^2}{2})^2 + \eta - q^2}.$$

Similarly to the previous example, we displayed in Figure 8.2, the difference between the estimated value of $J(\alpha^\star)$ and the value of $J(\alpha^\star)$ depending on the number of iterations. We can observed the same decrease.

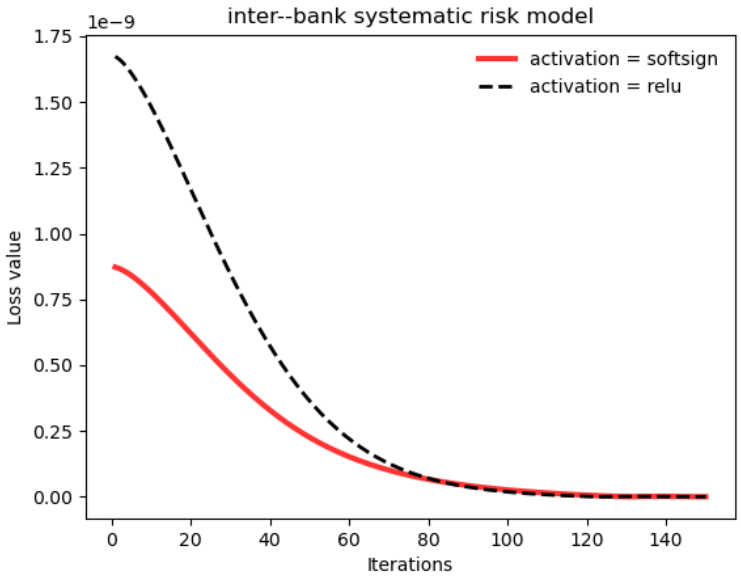

Figure 8.2: Difference between optimal value $J(\alpha^\star)$ and its estimated

8.3.2.3 Example 3: optimal liquidation with market impact

Consider a market where a group of agents have a position on a certain asset which they want to liquidate by a fixed time $T > 0$. Trades of all market participants reflect on a permanent and a temporary market impact. The optimal trades will be a result of the trade-off between trading slowly to reduce the market impact (or execution/liquidity cost), and trading fast to reduce the risk of future uncertainty in prices; This kind of model has been studied under different forms by Almgren and Chriss [11], Álvaro Cartea and Jaimungal [12], Cardaliaguet and Lehalle [37], Carmona and Lacker [45], Acciaio, Backhoff Veraguas, and Carmona [1]. The asymptotic formulation of this problem takes the following form: S is the price process modeled by

$$S_t = s_0 + \int_0^t \lambda \mathbb{E}[\alpha_s] \mathrm{d}s + \sigma W_t, \ t \in [0, T],$$

where $\lambda \geq 0$, and $\lambda \mathbb{E}[\alpha_t]$ is the permanent market impact to which all agents contribute. The inventory process Q is governed by

$$Q_t = Q_0 + \int_0^t \alpha_s \mathrm{d}s,$$

with Q_0 (possibly random) being the initial inventory to deplete by time T. The wealth process is given by

$$X_t = -\int_0^t \alpha_s (S_s + k\alpha_s) \mathrm{d}s, \ t \in [0, T],$$

where $k\alpha_t$ represents the temporary market impact which influences each agent. Using the control process α which is the trading speed, the goal is to minimize

$$J(\alpha) := \mathbb{E}\Big[\phi \int_0^T Q_t^2 \mathrm{d}t - Q_T(S_T - AQ_T) - X_T\Big],$$

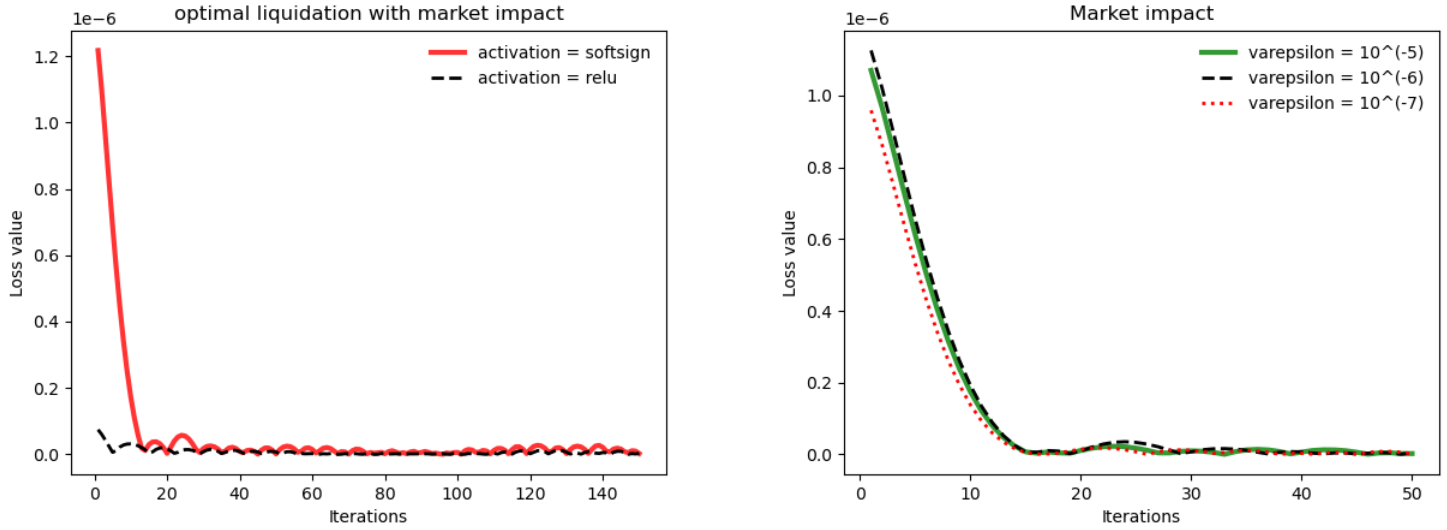

Figure 8.3: Difference between optimal value $J(\alpha^\star)$ and its estimated: on the left different activation functions and on the right different ε values

where ϕ is a risk aversion parameter, $Q_T(S_T - AQ_T)$ is the liquidation value of the remaining quantity at terminal time (with a liquidation/execution penalization). The optimal control is given by

$$\alpha_t^\star := Q_0\gamma\frac{d_1\exp(-\gamma(T-t)) - d_2\exp(\gamma(T-t))}{d_1\exp(-\gamma T) + d_2\exp(\gamma T)} + \mathbb{E}[Q_0]\frac{2\lambda\phi(\exp(-\gamma t) + \exp(\gamma t))}{(d_1\exp(-\gamma T) + d_2\exp(\gamma T))(c_1\exp(-\gamma T) + c_2\exp(\gamma T))},$$

where

$$d_1 := \sqrt{\phi k} - A,\ d_2 := \sqrt{\phi k} + A,\ c_1 := 2d_1 + \lambda,\ c_2 := 2d_2 - \lambda,\ \text{and}\ \gamma := \sqrt{\phi/k}.$$

To be able to test our theoretical result in this example, we need some modifications. Indeed, here the assumptions of Theorem 8.2.5 are no longer true. The non–degeneracy assumptions of volatility σ is not verified. To stay in the context of theorem, we replace the dynamic of Q by $\tilde{Q}_\cdot = Q_\cdot + \varepsilon\tilde{W}_\cdot$ with $\tilde{W}$ a Brownian motion independent of W and ε a positive value close to zero. We perform our algorithm with this modification and compare with the true value $J(\alpha^\star)$. We observe a decrease in the difference of the two values when the number of iterations increases. Another graph shows $J(\alpha^\star)$ and its estimated for different values of ε.

8.4 Proofs

8.4.1 Proof of Theorem 8.2.5

Lemma 8.4.1. *Let* Assumption 1.4.1 *hold true, $\nu \in \mathcal{P}_{p'}(\mathbb{R}^n)$, and $\alpha \in \mathcal{A}_{p'}(\nu)$. For any $N \in \mathbb{N}^*$, there exists a family of Borel bounded functions $(\phi^N)_{N\in\mathbb{N}^*}$ satisfying $\phi^N : [0,T] \times \mathbb{R}^n \times (\mathbb{R}^d \times \mathbb{R}^\ell)^{m^N} \to U$ s.t. if we let $(X^{\phi,1}, \dots, X^{\phi,N})$ be defined on $(\Omega^\star, \mathbb{F}^\star, \mathbb{P}^\star)$ by: for $i \in \{1, \dots, N\}$, for all $j \in \{0, ...m^N - 1\}$, $X^{\phi,i}_{t_0^N} = X_0^i$,*

$$X^{\phi,i}_{t^N_{j+1}} = X^{\phi,i}_{t^N_j} + b\big(t_j^N, \widehat{X}^{\phi,i}, \varphi^N_{t_j^N}, \phi^i_{t_j^N}\big)1/m^N + \sigma\big(t_j^N, \widehat{X}^{\phi,i}, \varphi^N_{t_j^N}, \phi^i_{t_j^N}\big)\sqrt{1/m^N}U_j^i + \sigma_0\sqrt{1/m^N}V_j^N, \tag{8.4.1}$$

where $\phi^i_{t^N_j} := \phi\Big(t^N_j, X^i_0, \big(\sqrt{1/m^N}U^i_{k\wedge j}, \sqrt{1/m^N}V^N_{k\wedge j}\big)_{1\le k\le m^N}\Big)$, $(\widehat{X}^{\phi,1},\dots,\widehat{X}^{\phi,N})$ *denotes the linear interpolation process of* $\Big\{(X^{\phi,1}_{t^N_j},\dots,X^{\phi,N}_{t^N_j}),\ 1\le j\le m^N\Big\}$ *on interval* $[0,T]$,

$$\varphi^N_s(\mathrm{d}\mathbf{x},\mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{X}^{\phi,i},\phi^i_s)}(\mathrm{d}\mathbf{x},\mathrm{d}a),\ \mathrm{d}s\otimes\mathrm{d}\mathbb{P}^N_\nu\text{–a.e., and } \varphi^{N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^{\phi,i}}(\mathrm{d}\mathbf{x}),\ \textit{for all } s\in[0,T],$$

then if $\Phi^i := \delta_{\phi^i_{[s]^N}}(\mathrm{d}a)\mathrm{d}t$, $\Phi := \delta_{\alpha_s}(\mathrm{d}a)\mathrm{d}t$,

$$\widehat{\varphi}^N_s(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i}_{s\wedge\cdot},\phi^i_{[s]^N},(\Phi^i)^s\right)}(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e)\ \textit{ and }\ \widehat{\mu}^\alpha_s(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e) := \mathcal{L}^{\mathbb{P}_\nu}\big(X^\alpha_{s\wedge\cdot},\alpha_s,\Phi^s\big|\mathcal{G}_s\big)(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e)$$

one has

$$\lim_N \mathcal{L}^{\mathbb{P}^\star}\Big(\delta_{\widehat{\varphi}^N_s}(\mathrm{d}\bar{\nu})\mathrm{d}s,\ \varphi^{N,X}\Big) = \mathcal{L}^{\mathbb{P}_\nu}\Big(\delta_{\widehat{\mu}^\alpha_s}(\mathrm{d}\bar{\nu})\mathrm{d}s,\mu^\alpha\Big),\ \textit{in } \mathcal{W}_p.$$

Proof. *step* 1: Under Assumption 1.4.1, by (an easy extension) Lemma 2.3.10, in the filtered probability space $(\Omega,\mathbb{F},\mathcal{F},\mathbb{P}_\nu)$, there exists a sequence of $\mathbb{F}$–predictable processes $(\alpha^N)_{N\ge1}$, and a sequence a $\mathbb{F}$–adapted continuous processes $(S^N)_{N\ge1}$ such that: $\sup_{N\ge1}\mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \rho(a_0,\alpha^N_t)^{p'}\mathrm{d}t\Big]<\infty$, for any $N\ge1$

$$\alpha^N_0 = a_0,\ \alpha^N_t = \alpha^N_{[t]^N},\ \text{on } [0,T],\ \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \rho(\alpha_t,\alpha^N_t)^p\mathrm{d}t\Big]=0,\ \text{and}\ \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{s\in[0,T]}|X_s-S^N_s|^p\Big]=0, \tag{8.4.2}$$

where S^N is the process defined by Euler scheme: $S^N_0 = X_0$, and for all $t\in[0,T]$,

$$S^N_t = X_0 + \int_{\varepsilon_N}^{t\vee\varepsilon_N} b\big([r]^N,\widehat{S}^N,\overline{\beta}^N_r,\alpha^N_r\big)\mathrm{d}r + \int_{\varepsilon_N}^{t\vee\varepsilon_N}\sigma\big([r]^N,\widehat{S}^N,\overline{\beta}^N_r,\alpha^N_r\big)\mathrm{d}W^N_r + \int_{\varepsilon_N}^{t\vee\varepsilon_N}\sigma_0\mathrm{d}B^N_r, \tag{8.4.3}$$

where $\widehat{S}^N$ denotes the linear interpolation process of $\big\{S^N_{t^N_j},\ 1\le j\le m^N\big\}$ on interval $[0,T]$, $\overline{\beta}^N_t := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{S}^N_{t\wedge\cdot},\alpha^N_t\big|\mathcal{G}_t\big)$, $\beta^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{S}^N\big|\mathcal{G}_T\big)$, $\varepsilon_N = t^N_1$, $W^N_\cdot := W_{\cdot\vee\varepsilon_N} - W_{\varepsilon_N}$ and $B^N_\cdot := B_{\cdot\vee\varepsilon_N} - B_{\varepsilon_N}$.

step 2: Let $N\in\mathbb{N}^*$ be fixed. For all $1\le j\le m^N$, we define $\widehat{\beta}^N_j := \mathcal{L}^{\mathbb{P}_\nu}\Big(\widehat{S}^N_{t^N_j\wedge\cdot},(\alpha^N_{t^N_{k-1}})_{1\le k\le j},\big(W^N_{t^N_k}-W^N_{t^N_{k-1}}\big)_{1\le k\le j}\Big|\mathcal{G}_T\Big)\in\mathcal{P}\big(\mathcal{C}^n\times U^j\times(\mathbb{R}^d)^j\big)$.

By similar method used in Lemma 2.3.11, there exists a sequence i.i.d uniform random variables $(U^\beta_i)_{1\le j\le m^N}$ independent of other variables and a family of Borel functions $(G^{\alpha,N}_j)_{1\le j\le m^N}$ and $(G^{\beta,N}_j)_{1\le j\le m^N}$ satisfying for all $j\in\{1,...,m^N\}$, $G^{\alpha,N}_j:\mathbb{R}^n\times\mathbb{R}^d\times(\mathbb{R}^d)^j\times(\mathbb{R}^\ell)^j\times[0,1]^j\to U$ and $G^{\beta,N}_j:(\mathbb{R}^\ell)^j\times[0,1]^j\to\mathcal{P}\big(\mathcal{C}^n\times U^j\times(\mathbb{R}^d)^j\big)$ such that

$$\begin{aligned}&\mathcal{L}^{\mathbb{P}_\nu}\Big(X_0,(\widehat{\beta}^N_j)_{1\le j\le m^N},(\alpha^N_j)_{1\le j\le m^N},(W^N_{t^N_j}-W^N_{t^N_{j-1}})_{1\le j\le m^N},(B^N_{t^N_j}-B^N_{t^N_{j-1}})_{1\le j\le m^N}\Big)\\ &=\mathcal{L}^{\mathbb{P}_\nu}\Big(X_0,(\widehat{\zeta}^N_j)_{1\le j\le m^N},(\gamma^N_j)_{1\le j\le m^N},(W^N_{t^N_j}-W^N_{t^N_{j-1}})_{1\le j\le m^N},(B^N_{t^N_j}-B^N_{t^N_{j-1}})_{1\le j\le m^N}\Big),\end{aligned} \tag{8.4.4}$$

where

$$\gamma^N_j = G^{\alpha,N}_j\Big(X_0,W^N_{t^N_1},(W^N_{t^N_k}-W^N_{t^N_{k-1}},B^N_{t^N_k}-B^N_{t^N_{k-1}},U^\beta_k)_{1\le k\le j}\Big)\text{ and }\widehat{\zeta}^N_j = G^{\beta,N}_j\Big((B^N_{t^N_k}-B^N_{t^N_{k-1}},U^\beta_k)_{1\le k\le j}\Big).$$

Notice that as $(\widehat{\beta}^N_j)_{1\le j\le m^N}$ is B–measurable, when $\ell=0$, we can remove the variables $(U^\beta_i)_{1\le j\le m^N}$. Thanks to particular form of $\widehat{S}^N$ of (2.3.10), for each $j\in\{1,...m^N\}$, there exists a Borel function $H^j:\mathbb{R}^n\times\mathcal{P}\big(\mathcal{C}^n\times U^j\times(\mathbb{R}^d)^j\big)\times(\mathbb{R}^d)^j\times(\mathbb{R}^\ell)^j\to\mathbb{R}^n$ s.t.

$$S^N_{t^N_j} = H^j\Big(X_0,\widehat{\beta}^N_j,(\alpha^N_k)_{1\le k\le j},(W^N_{t^N_k}-W^N_{t^N_{k-1}})_{1\le k\le j},(B^N_{t^N_k}-B^N_{t^N_{k-1}})_{1\le k\le j}\Big).$$

Let $Z_0^N = X_0$, and for each $j \in \{1, ..., m^N\}$, one defines $(Z^N_{t_1^N}, ..., Z^N_{t^N_{m^N}})$ by and

$$Z^N_{t_j^N} = H^j\Big(X_0, \widehat{\zeta}_j^N, (\gamma_k^N)_{1\le k\le j}, (W^N_{t_k^N} - W^N_{t_{k-1}^N})_{1\le k\le j}, (B^N_{t_k^N} - B^N_{t_{k-1}^N})_{1\le k\le j}\Big).$$

Therefore by (8.4.4), one has

$$\begin{aligned}&\mathcal{L}^{\mathbb{P}_\nu}\Big((S^N_{t_j^N})_{0\le j\le m^N}, (\widehat{\beta}_j^N, \alpha_j^N, W^N_{t_j^N} - W^N_{t_{j-1}^N}, B^N_{t_j^N} - B^N_{t_{j-1}^N})_{1\le j\le m^N}\Big)\\ &= \mathcal{L}^{\mathbb{P}_\nu}\Big((Z^N_{t_j^N})_{0\le j\le m^N}, (\widehat{\zeta}_j^N, \gamma_j^N, W^N_{t_j^N} - W^N_{t_{j-1}^N}, B^N_{t_j^N} - B^N_{t_{j-1}^N})_{1\le j\le m^N}\Big)\end{aligned} \tag{8.4.5}$$

and

$$\begin{aligned}\widehat{\zeta}_j^N &= \mathcal{L}^{\mathbb{P}_\nu}\Big(\widehat{Z}^N_{t_j^N\wedge\cdot}, (\gamma^N_{k-1})_{1\le k\le j}, \big(W^N_{t_k^N} - W^N_{t_{k-1}^N}\big)_{1\le k\le j}\Big|\big(B^N_{t_k^N} - B^N_{t_{k-1}^N}, U_k^\beta\big)_{1\le k\le j}\Big)\\ &= \mathcal{L}^{\mathbb{P}_\nu}\Big(\widehat{Z}^N_{t_j^N\wedge\cdot}, (\gamma^N_{k-1})_{1\le k\le j}, \big(W^N_{t_k^N} - W^N_{t_{k-1}^N}\big)_{1\le k\le j}\Big|B, \big(U_k^\beta\big)_{1\le k\le j}\Big),\end{aligned}$$

where $\widehat{Z}^N$ denotes the linear interpolation process of $\big\{Z^N_{t_j^N},\ 1 \le j \le m^N\big\}$ on interval $[0, T]$.

Step 3: As if $\ell = 0$, $\big(U_k^\beta\big)_{1\le k\le j}$ can be removed, then by similar methods used in Proposition 2.3.12, $\big(U_k^\beta\big)_{1\le k\le j}$ can be replaced by a function of $B^N_{t_1^N}$ and the equality (8.4.5) remains true. Let us define R^N by Euler scheme

$$R_t^N = X_0 + \int_{\varepsilon_N}^{t\vee\varepsilon_N} b\big([r]^N, \widehat{R}^N, \overline{\theta}_r^N, \gamma_r^N\big)\mathrm{d}r + \int_{\varepsilon_N}^{t\vee\varepsilon_N} \sigma\big([r]^N, \widehat{R}^N, \overline{\theta}_r^N, \gamma_r^N\big)\mathrm{d}W_r^N + \int_{\varepsilon_N}^{t\vee\varepsilon_N} \sigma_0\mathrm{d}B_r^N,$$

where $\gamma_t^N := \gamma_j^N$ for $t \in [t_j^N, t_{j+1}^N)$, $\widehat{R}^N$ denotes the linear interpolation process of $\big\{R^N_{t_j^N},\ 1 \le j \le m^N\big\}$ on interval $[0, T]$, $\overline{\theta}_t^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{R}^N_{t\wedge\cdot}, \gamma_t^N\big|\mathcal{G}_t\big)$, $\theta^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{R}^N\big|\mathcal{G}_T\big)$. Observe that: for each $N \in \mathbb{N}^*$,

$$\gamma_s^N = \phi^N\Big(s, X_0, \big(W_{[s\,\wedge\, t_k^N]^N} - W_{[s\,\wedge\, t_{k-1}^N]^N}, B_{[s\,\wedge\, t_k^N]^N} - B_{[s\,\wedge\, t_{k-1}^N]^N}\big)_{1\le k\le m^N}\Big),\ \text{for all } s \in [0, T],$$

with $\phi^N : [0, T] \times \mathbb{R}^n \times (\mathbb{R}^d \times \mathbb{R}^\ell)^{m^N} \to U$ a Borel function, $R^N_{t_j^N} = Z^N_{t_j^N}$ for all $j \in \{1, ..., m^N\}$, and

$$\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{t\in[0,T]}|R_t^N|^{p'}\Big] \le C\Big(1 + \int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x) + \sup_{N\ge1}\mathbb{E}^{\mathbb{P}_\nu}\Big[\int_0^T \rho(a_0, \gamma_t^N)^{p'}\mathrm{d}t\Big]\Big).$$

Let us introduce $\overline{\Theta}^N := \delta_{\left(\overline{\theta}_t^N, \gamma_t^N\right)}(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t$, and

$$\mathbb{Q}^N := \mathcal{L}^{\mathbb{P}_\nu}\Big(R^N, \overline{\Theta}^N, \theta^N, W^N, B^N\Big) \in \mathcal{P}\Big(\mathcal{C}^n \times \mathbb{M}\big(\mathcal{P}(\mathcal{C}^n \times U) \times U\big) \times \mathcal{P}\big(\mathcal{C}^n\big) \times \mathcal{C}^d \times \mathcal{C}^\ell\Big).$$

It is straightforward to show that $(\mathbb{Q}^N)_{N\ge1}$ is relatively compact in $\mathcal{W}_p$. Let $\mathbb{Q}^\infty$ the limit of any sub-sequence of $(\mathbb{Q}^N)_{N\ge1}$. For simplicity we keep the same notation for the sequence and the sub-sequence. Let us show

$$\mathbb{Q}^\infty = \mathcal{L}^{\mathbb{P}_\nu}\Big(X, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t, \mu_T, W, B\Big). \tag{8.4.6}$$

Let $M \in \mathbb{N}^*$, $(r_1, ..., r_M) \in [0, T]^M$, a bounded continuous function $\Phi : \big(\mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^\ell\big)^M \times \mathcal{P}(\mathcal{C}^n) \times \mathbb{M}\big(\mathcal{P}(\mathcal{C}^n \times U) \times U\big) \to \mathbb{R}$. Denote by $\overline{\zeta}_t^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{Z}^N_{t\wedge\cdot}, \gamma_t^N\big|\big(B^N_{t_j^N} - B^N_{t_{j-1}^N}\big)_{1\le j\le m^N}\big)$ and $\zeta^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{Z}^N\big|\big(B^N_{t_j^N} - B^N_{t_{j-1}^N}\big)_{1\le j\le m^N}\big)$ one has

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_\nu}\Big[\Phi\Big((X_{r_i}, W_{r_i}, B_{r_i})_{1\le i\le M}, \mu_T, \delta_{(\overline{\mu}_t, \alpha_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t\Big)\Big]\\ &= \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\Phi\Big((S^N_{[r_i]^N}, W^N_{[r_i]^N}, B^N_{[r_i]^N})_{1\le i\le M}, \beta^N, \delta_{(\overline{\beta}_t^N, \alpha_t^N)}(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t\Big)\Big]\\ &= \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\Phi\Big((Z^N_{[r_i]^N}, W^N_{[r_i]^N}, B^N_{[r_i]^N})_{1\le i\le M}, \zeta^N, \delta_{(\overline{\zeta}_t^N, \gamma_t^N)}(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t\Big)\Big]\\ &= \lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\Phi\Big((R^N_{[r_i]^N}, W^N_{[r_i]^N}, B^N_{[r_i]^N})_{1\le i\le M}, \theta^N, \overline{\Theta}^N(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t\Big)\Big] = \mathbb{E}^{\mathbb{Q}^\infty}\Big[\Phi\Big((\widetilde{X}_{r_i}, \widetilde{W}_{r_i}, \widetilde{B}_{r_i})_{1\le i\le M}, \widetilde{\mu}, \widetilde{\Lambda}(\mathrm{d}\bar{\nu}, \mathrm{d}u)\mathrm{d}t\Big)\Big],\end{aligned}$$

where $(\widetilde{X},\widetilde{\Lambda},\widetilde{\mu},\widetilde{W},\widetilde{B})$ is the canonical process on $\mathcal{C}^n\times\mathbb{M}\big(\mathcal{P}(\mathcal{C}^n\times U)\times U\big)\times\mathcal{P}\big(\mathcal{C}^n\big)\times\mathcal{C}^d\times\mathcal{C}^\ell$. The previous result is true for all $(r_1,...,r_m)$ and Φ, we can deduce (8.4.6), for any sub-sequence, then $\lim_N \mathbb{Q}^N = \mathcal{L}^{\mathbb{P}_\nu}\Big(X,\delta_{(\overline{\mu}_t,\alpha_t)}(\mathrm{d}\bar{\nu},\mathrm{d}u)\mathrm{d}t,\mu_T,W,B\Big)$ in $\mathcal{W}_p$.

Now, by Euler scheme again, let us define X^N by:

$$X_t^N = X_0 + \int_0^t b\big([r]^N,\widehat{X}^N,\overline{\mu}_r^N,\gamma_r^N\big)\mathrm{d}r + \int_0^t \sigma\big([r]^N,\widehat{X}^N,\overline{\mu}_r^N,\gamma_r^N\big)\mathrm{d}W_r + \int_0^t \sigma_0\mathrm{d}B_r, \tag{8.4.7}$$

where $\widehat{X}^N$ denotes the linear interpolation process of $\big\{X^N_{t_j^N},\ 1\le j\le m^N\big\}$ on interval $[0,T]$, $\overline{\mu}_t^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}_t^N,\gamma_t^N\big|\mathcal{G}_t\big) = \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}_t^N,\gamma_t^N\big|B_{t_1^N},\big(B^N_{t_j^N}-B^N_{t_{j-1}^N}\big)_{1\le j\le m^N}\big)$, $\mu^N := \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}^N\big|\mathcal{G}_T\big)$. By similar technique as Equation (2.3.9), it straightforward to show that $\lim_{N\to\infty}\mathbb{E}^{\mathbb{P}_\nu}\Big[\sup_{s\in[0,T]}|R_s^N - X_s^N|^p\Big] = 0$. Consequently,

$$\lim_N \mathcal{L}^{\mathbb{P}_\nu}\Big(X^N,\delta_{(\overline{\mu}_t^N,\gamma_t^N)}(\mathrm{d}\bar{\nu},\mathrm{d}u)\mathrm{d}t,\mu^N,W,B\Big) = \lim_N \mathcal{L}^{\mathbb{P}_\nu}\Big(R^N,\overline{\Theta}^N,\theta^N,W^N,B^N\Big) = \mathcal{L}^{\mathbb{P}_\nu}\Big(X,\delta_{(\overline{\mu}_t,\alpha_t)}(\mathrm{d}\bar{\nu},\mathrm{d}u)\mathrm{d}t,\mu_T,W,B\Big). \tag{8.4.8}$$

Step 4: As X^N satisfies (8.4.7), there exists $\mathbb{K}^N:\mathbb{R}^n\times\mathcal{C}^d\times\mathcal{C}^\ell\to\mathcal{C}^n$ a Borel function s.t. $X^N=\mathbb{K}^N(X_0,W,B)$, $\mathbb{P}_\nu$–a.e.. In the space $(\Omega^\star,\mathbb{F}^\star,\mathbb{P}^\star)$, for each $N\in\mathbb{N}^*$, let $(W^1,...,W^N,B)$ be a $\mathbb{F}^\star$–Brownian motion, and define the filtration on $(\Omega^\star)$, $(\mathcal{G}_t)_{t\in[0,T]} := (\sigma(B_{t\wedge\cdot})_t)_{t\in[0,T]}$, and $Z^{i,N} := \mathbb{K}^N(X_0^i,W^i,B)$. In other words $Z^{i,N}$ is the strong solution of: for all $t\in[0,T]$,

$$Z_t^{i,N} = X_0^i + \int_0^t b\big([r]^N,\widehat{Z}^{i,N},\overline{\mu}_r^N,\gamma_r^{i,N}\big)\mathrm{d}r + \int_0^t \sigma\big([r]^N,\widehat{Z}^{i,N},\overline{\mu}_r^N,\gamma_r^{i,N}\big)\mathrm{d}W_r^i + \int_0^t \sigma_0\mathrm{d}B_r,\ \mathbb{P}^\star\text{–a.e.}$$

with $\gamma_s^{i,N} := \phi^N\Big(s,X_0^i,\big(W^i_{[s\wedge t_k^N]^N}-W^i_{[s\wedge t_{k-1}^N]^N},B_{[s\wedge t_k^N]^N}-B_{[s\wedge t_{k-1}^N]^N}\big)_{1\le k\le m^N}\Big)$, $\widehat{Z}^{i,N}$ denotes the linear interpolation process of $\big\{Z^{i,N}_{t_j^N},\ 1\le j\le m^N\big\}$ on interval $[0,T]$, and $\overline{\mu}_t^N = \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}_t^N,\gamma_t^N\big|\mathcal{G}_t\big) = \mathcal{L}^{\mathbb{P}^\star}\big(\widehat{Z}_t^{i,N},\gamma_t^{i,N}\big|\mathcal{G}_t\big)$, $\mu^N = \mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}^N\big|\mathcal{G}_T\big) = \mathcal{L}^{\mathbb{P}^\star}\big(\widehat{Z}^{i,N}\big|\mathcal{G}_T\big)$. Using techniques of Lemma 3.4.3[1] and the law of large numbers, one gets

$$\lim_{N\to\infty}\mathbb{E}^{\mathbb{P}^\star}\Big[\int_0^T \mathcal{W}_p\Big(\overline{\psi}_t^N,\mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}_t^N,\gamma_t^N,(\Gamma^N)^t\big|\mathcal{G}_t\big)\Big)\mathrm{d}t + \mathcal{W}_p\big(\psi^N,\mu^N\big)\Big] = 0, \tag{8.4.9}$$

with $\Gamma^N := \delta_{\gamma_t^N}(\mathrm{d}a)\mathrm{d}t$, $\Gamma^{i,N} := \delta_{\gamma_t^{i,N}}(\mathrm{d}a)\mathrm{d}t$, $\overline{\psi}_t^N(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(Z^{i,N}_{t\wedge\cdot},\gamma_t^{i,N},\,(\Gamma^{i,N})^t\right)}(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e)$, and $\psi^N(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{Z^{i,N}}(\mathrm{d}\mathbf{x})$.

To finish, let us introduce by Euler scheme $(X^{1,N},...,X^{N,N})$: for $i\in\{1,\dots,N\}$, and all $t\in[0,T]$,

$$X^{i,N}_{[t]^N} = X_0^i + \int_0^{[t]^N} b\big([s]^N,\widehat{X}^{i,N},\varphi^N_{[s]^N},\gamma_s^{i,N}\big)\mathrm{d}s + \int_0^{[t]^N} \sigma\big([s]^N,\widehat{X}^{i,N},\varphi^N_{[s]^N},\gamma_s^{i,N}\big)\mathrm{d}W_s^i + \int_0^{[t]^N}\sigma_0\mathrm{d}B_s,$$

where $\varphi^N_{[t]^N}(\mathrm{d}\mathbf{x},\mathrm{d}u) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(X^{i,N}_{[t]^N\wedge\cdot},\gamma_t^{i,N}\right)}(\mathrm{d}\mathbf{x},\mathrm{d}a)$, $(\widehat{X}^{1,N},\dots,\widehat{X}^{N,N})$ denotes the linear interpolation process of $\Big\{(X^{1,N}_{t_j^N},\dots,X^{N,N}_{t_j^N}),\ 1\le j\le m^N\Big\}$ on interval $[0,T]$. As in Proposition 3.4.4, thanks to (8.4.9), we find

$$\lim_{N\to\infty}\ \mathbb{E}^{\mathbb{P}^\star}\Big[\int_0^T \mathcal{W}_p\Big(\overline{\varphi}_t^N,\mathcal{L}^{\mathbb{P}_\nu}\big(\widehat{X}_t^N,\gamma_t^N,(\Gamma^N)^t\big|\mathcal{G}_t\big)\Big)\mathrm{d}t\Big] + \frac{1}{N}\sum_{i=1}^N\mathbb{E}^{\mathbb{P}^\star}\Big[\sup_{s\in[0,T]}|\widehat{X}_s^{i,N}-Z_s^{i,N}|^p\Big] = 0, \tag{8.4.10}$$

where $\overline{\varphi}_t^N(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{i,N}_{t\wedge\cdot},\gamma_t^{i,N},(\Gamma^{i,N})^t\right)}(\mathrm{d}\mathbf{x},\mathrm{d}a,\mathrm{d}e)$. By noticing that we can rewrite the process $(X^{1,N},...,X^{N,N})$ under the form (8.4.1), by Combining (8.4.8), (8.4.9) and (8.4.10), we can deduce our result. □

[1] An easy extension taking into account the dependence w.r.t. N of $\mathbb{K}^N$.

Denote $\mathbb{M} := \mathbb{M}\big(\mathcal{P}(\mathcal{C}^n \times A) \times A\big)$. We will consider two canonical spaces

$$\widehat{\Omega} := \mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M}, \text{ and } \overline{\Omega} := \mathcal{C}^\ell \times \mathcal{P}\big(\widehat{\Omega}\big).$$

We equip the canonical space $\overline{\Omega}$ with the canonical element $(B, \widehat{\mu})$, and its Borel σ–algebra $\overline{\mathcal{F}} := \mathcal{B}(\overline{\Omega})$. We then introduce the filtration $\overline{\mathbb{G}} := (\overline{\mathcal{G}}_t)_{t\in[0,T]}$ on $(\overline{\Omega}, \overline{\mathcal{F}})$ by

$$\overline{\mathcal{G}}_t := \sigma\Big(\big(B_s, \langle\widehat{\mu}_s, \phi\rangle\big) : \ \phi \in C_b(\mathcal{C}^n \times \mathcal{C}^n \times \mathbb{M}), s \in [0,t]\Big).$$

The canonical space $\widehat{\Omega}$ is equipped with the corresponding canonical element $\big(\widehat{X}, \widehat{Y}, \widehat{\Lambda}\big)$, its Borel σ–algebra $\widehat{\mathcal{F}} := \mathcal{B}(\widehat{\Omega})$, and its canonical filtration $\widehat{\mathbb{F}} := \big(\widehat{\mathcal{F}}_t\big)_{t\in[0,T]}$ defined by

$$\widehat{\mathcal{F}}_t := \sigma\Big(\big(\widehat{X}_s, \widehat{Y}_s, \widehat{\Lambda}([0,s]\times D)\big) : D \in \mathcal{B}\big(\mathcal{P}(\mathcal{C}^n \times A) \times A\big),\ s \in [0,t]\Big),\ t \in [0,T].$$

Notice that one can choose a version of the disintegration $\widehat{\Lambda}(\mathrm{d}t, \mathrm{d}\bar{\nu}, \mathrm{d}a) = \widehat{\Lambda}_t(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t$ such that $(\widehat{\Lambda}_t)_{t\in[0,T]}$ is a $\mathcal{P}\big(\mathcal{P}(\mathcal{C}^n \times A) \times A\big)$–valued, $\widehat{\mathbb{F}}$–predictable process (see e.g. [102, Lemma 3.2.]).

We next introduce a martingale problem on $(\widehat{\Omega}, \widehat{\mathcal{F}})$. For all $\varphi \in C_b^2(\mathbb{R}^{n+d})$ and $(t, \mathbf{x}, \mathbf{y}, \mathbf{w}, \bar{\nu}, a) \in [0,T] \times \mathcal{C}^n \times \mathcal{C}^n \times \mathcal{C}^d \times \mathcal{P}(\mathcal{C}^n \times A) \times A$

$$\widehat{\mathcal{L}}_t\varphi\big(\mathbf{x}, \mathbf{y}, \mathbf{w}, \bar{\nu}, a\big) := b(t, \mathbf{x}, \bar{\nu}, a) \cdot \nabla\varphi(\mathbf{y}(t), \mathbf{w}(t)) + \frac{1}{2}\mathrm{Tr}\big[a(t, \mathbf{x}, \bar{\nu}, a)\nabla^2\varphi(\mathbf{y}(t), \mathbf{w}(t))\big]. \tag{8.4.11}$$

Then, we introduce a process $(\widehat{M}_t^\varphi)_{t\in[0,T]}$ on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ by

$$\widehat{M}_t^\varphi := \varphi\big(\widehat{Y}_t\big) - \varphi(\widehat{Y}_0) - \iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} \widehat{\mathcal{L}}_s\varphi\big(\widehat{X}, \widehat{Y}, \bar{\nu}, a\big)\widehat{\Lambda}_s(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}s,\ (t,\varphi) \in [0,T] \times C_b^2(\mathbb{R}^n). \tag{8.4.12}$$

Recall that for a borel function $\phi : [0,T] \to \mathbb{R}$, we write $\int_0^\cdot \phi(s)\mathrm{d}s := \int_0^\cdot \phi^+(s)\mathrm{d}s - \int_0^\cdot \phi^-(s)\mathrm{d}s$ with the convention $\infty - \infty = -\infty$.

Definition 8.4.2. *Let $\nu \in \mathcal{P}(\mathbb{R}^n)$, then a probability measure $\overline{\mathbb{P}} \in \mathcal{P}(\overline{\Omega})$ belongs to $\overline{\mathcal{P}}_R(\nu)$ if and only if*

(i) $\overline{\mathbb{P}}\big[\widehat{\mu}_0 \circ (\widehat{X}_0)^{-1} = \nu, Y_0 = X_0, B_0 = 0\big] = 1$, $\mathbb{E}^{\overline{\mathbb{P}}}\big[\mathbb{E}^{\hat{\mu}}[\|\widehat{X}\|^p]\big] + \mathbb{E}^{\overline{\mathbb{P}}}\Big[\mathbb{E}^{\hat{\mu}}\Big[\int_{[0,T]\times A}\big(\rho(a_0, a)\big)^p\widehat{\Lambda}_t(\mathrm{d}a)\mathrm{d}t\Big]\Big] < \infty$.

(ii) $(B_t)_{t\in[0,T]}$ *is an* $(\overline{\mathbb{F}}, \overline{\mathbb{P}})$*–Brownian motion, and for* $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $\widehat{Y}_\cdot = \widehat{X}_\cdot - \sigma_0 B_\cdot(\bar{\omega})$, $\widehat{\mu}(\bar{\omega})$–a.e., *and the process* $\big(\widehat{M}_t^\varphi\big)_{t\in[0,T]}$ *is an* $\big(\widehat{\mathbb{F}}, \widehat{\mu}(\bar{\omega})\big)$*–martingale for each* $\varphi \in C_b^2(\mathbb{R}^n)$.

(iii) For any Borel bounded function $f : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A \to \mathbb{R}$, one has for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$,

$$\mathbb{E}^{\hat{\mu}}\Big[\int_0^T\int_{\mathcal{P}(\mathcal{C}^n\times A)}\int_A f(t, \widehat{X}_t, \bar{\nu}, a)\widehat{\Lambda}_t(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\Big] = \int_0^T\int_{\mathcal{P}(\mathcal{C}^n\times A)}\int_{\mathcal{C}^n\times A} f(t, \mathbf{x}, \bar{\nu}, a)\bar{\nu}(\mathrm{d}\mathbf{x}, \mathrm{d}a)\widehat{\Lambda}_t(\mathrm{d}\bar{\nu}, A)\mathrm{d}t,\ \widehat{\mu}(\bar{\omega})\text{–a.e.}$$

Proposition 8.4.3 (Proposition 3.3.5)**.** *For any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R(\bar{\nu})$, on an extension $\big(\widehat{\Omega}^\star := \widehat{\Omega}\times[0,1], \widehat{\mathbb{F}}^\star := (\widehat{\mathcal{F}}_t\otimes\mathcal{B}([0,1]))_{t\in[0,T]}\big)$ of $(\widehat{\Omega}, \widehat{\mathbb{F}})$, there exists a family of measure–valued processes $(\widehat{N}^{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ such that, for* $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $\widehat{N}^{\bar{\omega}} = \big(\widehat{N}^{1,\bar{\omega}}, \dots, \widehat{N}^{d,\bar{\omega}}\big)$ *is an* $\big(\widehat{\mathbb{F}}^\star, \widehat{\mu}(\bar{\omega})\otimes\lambda\big)$*–martingale measure with intensity $\widehat{\Lambda}_t(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t$, the martingales $(\widehat{N}^{i,\bar{\omega}})_{1\le i\le d}$ are orthogonal, and satisfy*

$$\widehat{Y}_t = \widehat{X}_0 + \iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} b\big(r, \widehat{X}, m, a\big)\widehat{\Lambda}_r(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}r + \iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} \sigma\big(r, \widehat{X}, \bar{\nu}, a\big)\widehat{N}^{\bar{\omega}}(\mathrm{d}\bar{\nu}, \mathrm{d}a, \mathrm{d}r),\ \widehat{\mu}(\bar{\omega})\otimes\lambda\text{–a.s.}$$

Moreover, let $\widehat{\mathbb{H}}^\star = (\widehat{\mathcal{H}}^\star_t)_{t\in[0,T]}$ *with* $\widehat{\mathcal{H}}^\star_t := \overline{\mathcal{G}}_t \otimes \widehat{\mathcal{F}}^\star_t$ *be a filtration on* $\overline{\Omega}\times\widehat{\Omega}^\star$*, denote by* $\mathcal{P}^{\widehat{\mathbb{H}}^\star}$ *the predictable* σ*–algebra on* $[0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star$ *with respect to* $\widehat{\mathbb{H}}^\star$*. Then for all bounded* $\mathcal{P}^{\widehat{\mathbb{H}}^\star}\otimes\mathcal{B}(\mathcal{P}(\mathcal{C}^n\times A)\times A)$*–measurable function* $f:[0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star\times\mathcal{P}(\mathcal{C}^n\times A)\times A\longrightarrow\mathbb{R}$*, one can define the stochastic integral* $\iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} f^{\bar\omega}(s,\bar\nu,a)\widehat{N}^{\bar\omega}(\mathrm{d}s,\mathrm{d}\bar\nu,\mathrm{d}a)$ *in such a way that*

$$(t,\bar\omega,\hat\omega^\star)\longmapsto\left(\iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} f^{\bar\omega}(s,\bar\nu,a)\widehat{N}^{\bar\omega}(\mathrm{d}\bar\nu,\mathrm{d}a,\mathrm{d}s)\right)(\hat\omega^\star)\text{ is }\mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

Lemma 8.4.4. *Let* Assumption 1.4.1 *hold true,* $\nu\in\mathcal{P}_{p'}(\mathbb{R}^n)$*. For any* $N\in\mathbb{N}^*$*, let* $(X^{\phi,1},\dots,X^{\phi,N})$ *be defined in* (8.4.1) *on* $(\Omega^N,\mathbb{F}^N,\mathbb{P}_\nu)$*, and*

$$\mathbb{P}^N := \mathbb{P}^\star\circ\big(B^N,\widehat{\varphi}_N\big)^{-1}\in\mathcal{P}(\overline{\Omega}),\ \text{where}\ \widehat{\varphi}_N := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i},\widehat{Y}^{\phi,i},\,\delta_{(\varphi^N_t,\,\alpha^i_t)}(\mathrm{d}\bar\nu,\mathrm{d}a)\mathrm{d}t\right)},\tag{8.4.13}$$

with $B^N_t = B^N_{[t]^N} + \sqrt{t-[t]^N}V^N_{[t]^N}$ *and* $\alpha^i_t := \phi^i_{[t]^N}$*. Then the sequence* $(\mathbb{P}^N)_{N\in\mathbb{N}^*}$ *is relatively compact in* $\mathcal{W}_p$ *and any limit point belongs to* $\overline{\mathcal{P}}_R(\nu)$*.*

Proof. By recurence, we can check that: there exists a constant $C>0$ s.t. for each i

$$\mathbb{E}^{\mathbb{P}^\star}\big[\big|\widehat{Y}^{\phi,i}_t-\widehat{Y}^{\phi,i}_s\big|^3\big]\le C(m^N)^{3/2}\phi(1/m^N)|t-s|\text{ and }\mathbb{E}^{\mathbb{P}^\star}\Big[\sup_{t\in[0,T]}\big|\widehat{Y}^{\phi,i}_t\big|^{p'}\Big]\le C\bigg(1+\int_{\mathbb{R}^n}|x|^{p'}\nu(\mathrm{d}x)\bigg).$$

Using similar techniques to Chapter 3, we can show $(\mathbb{P}^N)_{N\in\mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p$ (to explicit). Let $\overline{\mathbb{P}}^\infty$ be a limit for any sub-sequence, let us show $\overline{\mathbb{P}}^\infty\in\overline{\mathcal{P}}_R(\nu)$.
It is easy to verify the first point of Definition 8.4.2 (see also Chapter 3), the fact that B is a $(\overline{\mathbb{F}},\overline{\mathbb{P}}^\infty)$ Brownian motion and that for $\overline{\mathbb{P}}^\infty$–a.e. $\bar\omega\in\overline{\Omega}$, $\widehat{Y}_\cdot=\widehat{X}_\cdot-\sigma_0 B_\cdot(\bar\omega)$, $\widehat{\mu}(\bar\omega)$–a.e. follow as well.

Let us verify the martingale problem and the point (iii). On $(\Omega^\star,\mathcal{F}^\star)$, we define the processes $(M^{\varphi,i,N},N^{\varphi,i,N})$ for $i=1,\dots,N$ by

$$M^{\varphi,i,N}_t := \varphi\big(\widehat{Y}^{\phi,i}_t\big)-\varphi\big(\widehat{Y}^{\phi,i}_0\big)-\int_0^t\widehat{\mathcal{L}}_{[s]^N}\varphi\big(\widehat{X}^{\phi,i},\widehat{Y}^{\phi,i},\varphi^N_{[s]^N},\alpha^i_s\big)\mathrm{d}s,$$

and

$$N^{\varphi,i,N}_{k+1} := \varphi\big(\widehat{Y}^{\phi,i}_{t^N_{k+1}}\big)-\varphi\big(\widehat{Y}^{\phi,i}_{t^N_k}\big)-\mathcal{L}^N_{t^N_k}\varphi\big(\widehat{X}^{\phi,i},\widehat{Y}^{\phi,i},\varphi^N_{t^N_k},\alpha^i_{t^N_k},U^i_{k+1}\big),$$

where

$$\mathcal{L}_t\varphi\big(\mathbf{x},\mathbf{y},\bar\nu,a,u\big) := \mathbf{H}(1/m^N,\mathbf{x}_{t\wedge\cdot},\bar\nu,a,u)\cdot\nabla\varphi(\mathbf{y}(t))+\frac{1}{2}\mathrm{Tr}\big[\mathbf{H}\mathbf{H}^\top(1/m^N,\mathbf{x}_{t\wedge\cdot},\bar\nu,a,u)\nabla^2\varphi(\mathbf{y}(t))\big].$$

Also, let us introduce the filtration

$$\mathcal{F}^N_k := \sigma\Big\{X^1_0,...,X^N_0,U^1_{k'},...U^N_{k'},V^N_{k'}:\ k'\le k\Big\},$$

notice that for each $i\neq j$, $\Big(N^{\varphi,i,N}_{k+1},M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\Big)$ and $\Big(N^{\varphi,j,N}_{k+1},M^{\varphi,j}_{t^N_{k+1}}-M^{\varphi,j}_{t^N_k}\Big)$ are conditionally $\mathbb{P}^\star$ independent given the σ–algebra $\mathcal{F}^N_k$. Let $\varphi\in C^2_b(\mathbb{R}^n)$, $\psi\in C_b(\widehat{\Omega})$ and $\phi\in C_b(\mathcal{P}(\widehat{\Omega}))$. Define

$$\Psi^i_r := \psi\big(X^{\alpha,i}_{r\wedge\cdot},Y^{\alpha,i}_{r\wedge\cdot},(\Lambda^i)^r\big)\text{ and }\Psi^i_r := \psi\big(\widehat{X}_{r\wedge\cdot},\widehat{Y}_{r\wedge\cdot},(\widehat{\Lambda})^r\big),$$

one has

$$\begin{aligned}
&\left|\mathbb{E}^{\overline{\mathbb{P}}^\infty}\left[\phi(\widehat{\mu})\,\left\langle\big(\widehat{M}^{\varphi}_{[t]^N}-\widehat{M}^{\varphi}_{[r]^N}\big)\Psi_{[r]^N},\widehat{\mu}\right\rangle\right]\right| = \lim_{N\to\infty}\left|\mathbb{E}^{\overline{\mathbb{P}}^N}\left[\phi(\widehat{\mu})\,\left\langle\big(\widehat{M}^{\varphi}_{[t]^N}-\widehat{M}^{\varphi}_{[r]^N}\big)\Psi_{[r]^N},\widehat{\mu}\right\rangle\right]\right| \\
&\le \limsup_{N\to\infty}\mathbb{E}^{\overline{\mathbb{P}}^N}\left[\left|\phi(\widehat{\mu})\right|^2\right]^{1/2}\mathbb{E}^{\overline{\mathbb{P}}^N}\left[\left|\left\langle\big(\widehat{M}^{\varphi}_{[t]^N}-\widehat{M}^{\varphi}_{[r]^N}\big)\Psi_{[r]^N},\widehat{\mu}\right\rangle\right|^2\right]^{1/2} \\
&= \limsup_{N\to\infty}C\mathbb{E}^{\overline{\mathbb{P}}^N}\left[\left|\phi(\widehat{\mu})\right|^2\right]^{1/2}\mathbb{E}^{\mathbb{P}^\star}\left[\left|\frac{1}{N}\sum_{i=1}^N(M^{\varphi,i}_{[t]^N}-M^{\varphi,i}_{[r]^N})\Psi^i_{[r]^N}\right|^2\right]^{1/2}=0.
\end{aligned}$$

Indeed,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^\star}\left[\left|\frac{1}{N}\sum_{i=1}^N\big(M^{\varphi,i}_{[t]^N}-M^{\varphi,i}_{[r]^N}\big)\Psi^i_{[r]^N}\right|^2\right]=\mathbb{E}^{\mathbb{P}^\star}\left[\left|\frac{1}{N}\sum_{i=1}^N\sum_k\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\right|^2\right] \\
&=\mathbb{E}^{\mathbb{P}^\star}\left[\left|\frac{1}{N}\sum_{i=1}^N\sum_k\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\right|^2\right]=\mathbb{E}^{\mathbb{P}^\star}\left[\frac{1}{N^2}\sum_{i,j=1}^N\sum_{k,q}\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\big(M^{\varphi,j}_{t^N_{q+1}}-M^{\varphi,j}_{t^N_q}\big)\Psi^j_{[r]^N}\right] \\
&=\mathbb{E}^{\mathbb{P}^\star}\left[\frac{2}{N^2}\sum_{i,j=1}^N\sum_{k<q}\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\big(M^{\varphi,j}_{t^N_{q+1}}-M^{\varphi,j}_{t^N_q}\big)\Psi^j_{[r]^N}\right] \\
&\qquad\qquad+\mathbb{E}^{\mathbb{P}^\star}\left[\frac{1}{N^2}\sum_{i,j=1}^N\sum_{k}\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\big(M^{\varphi,j}_{t^N_{k+1}}-M^{\varphi,j}_{t^N_k}\big)\Psi^j_{[r]^N}\right] \\
&=\mathbb{E}^{\mathbb{P}^\star}\left[\frac{2}{N^2}\sum_{i,j=1}^N\sum_{k<q}\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\mathbb{E}^{\mathbb{P}^N_\nu}\left[\big(M^{\varphi,j}_{t^N_{q+1}}-M^{\varphi,j}_{t^N_q}\big)\Psi^j_{[r]^N}\Big|\mathcal{F}^N_q\right]\right] \\
&\quad+\frac{1}{N^2}\sum_k\mathbb{E}^{\mathbb{P}^\star}\left[\sum_{i\neq j}\mathbb{E}^{\mathbb{P}^N_\nu}\left[\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\Big|\mathcal{F}^N_k\right]\mathbb{E}^{\mathbb{P}^N_\nu}\left[\big(M^{\varphi,j}_{t^N_{k+1}}-M^{\varphi,j}_{t^N_k}\big)\Psi^j_{[r]^N}\Big|\mathcal{F}^N_k\right]\right] \\
&\quad+\frac{1}{N^2}\sum_k\mathbb{E}^{\mathbb{P}^\star}\left[\sum_{i=1}^N\left|\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\right|^2\right] \\
&=\mathbb{E}^{\mathbb{P}^\star}\left[\frac{2}{N^2}\sum_{i,j=1}^N\sum_{q}\big(M^{\varphi,i}_{t^N_q}-M^{\varphi,i}_{[r]^N}\big)\Psi^i_{[r]^N}\mathbb{E}^{\mathbb{P}^N_\nu}\left[\big(M^{\varphi,j}_{t^N_{q+1}}-M^{\varphi,j}_{t^N_q}-N^{\varphi,j,N}_{q+1}+N^{\varphi,j,N}_{q+1}\big)\Psi^j_{[r]^N}\Big|\mathcal{F}^N_q\right]\right] \\
&\quad+\frac{1}{N^2}\sum_k\mathbb{E}^{\mathbb{P}^\star}\left[\sum_{i\neq j}\mathbb{E}^{\mathbb{P}^\star}\left[\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}-N^{\varphi,i,N}_{k+1}+N^{\varphi,i,N}_{k+1}\big)\Psi^i_{[r]^N}\Big|\mathcal{F}^N_k\right]\mathbb{E}^{\mathbb{P}^\star}\left[\big(M^{\varphi,j}_{t^N_{k+1}}-M^{\varphi,j}_{t^N_k}-N^{\varphi,j,N}_{k+1}+N^{\varphi,j,N}_{k+1}\big)\Psi^j_{[r]^N}\Big|\mathcal{F}^N_k\right]\right] \\
&\quad+\frac{1}{N^2}\sum_k\mathbb{E}^{\mathbb{P}^\star}\left[\sum_{i=1}^N\left|\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\right|^2\right] \\
&\le C\quad\sum_q\mathbb{E}^{\mathbb{P}^\star}\left[\mathbb{E}^{\mathbb{P}^\star}\left[\phi(1/m^N)+\left|\mathbf{H}\big(1/m^N,\widehat{X}^{\phi,i},\varphi^N_{t^N_q},\phi^i_{t^N_q},U^i_{q+1}\big)\right|^3\Big|\mathcal{F}^N_q\right]\right] \\
&\qquad+\frac{1}{N^2}\sum_k\mathbb{E}^{\mathbb{P}^\star}\left[\sum_{i\neq j}\mathbb{E}^{\mathbb{P}^\star}\left[\phi(1/m^N)+\left|\mathbf{H}\big(1/m^N,\widehat{X}^{\phi,i},\varphi^N_{t^N_j},\phi^i_{t^N_k},U^i_{k+1}\big)\right|^3\Big|\mathcal{F}^N_k\right]\right. \\
&\qquad\qquad\left.\mathbb{E}^{\mathbb{P}^\star}\left[\phi(1/m^N)+\left|\mathbf{H}\big(1/m^N,\widehat{X}^{\phi,j},\varphi^N_{t^N_k},\phi^j_{t^N_k},U^i_{k+1}\big)\right|^3\Big|\mathcal{F}^N_k\right]\right]+\frac{1}{N^2}\sum_k\mathbb{E}^{\mathbb{P}^\star}\left[\sum_{i=1}^N\left|\big(M^{\varphi,i}_{t^N_{k+1}}-M^{\varphi,i}_{t^N_k}\big)\Psi^i_{[r]^N}\right|^2\right] \\
&\le C\bigg(m^N\phi(1/m^N)+m^N\phi(1/m^N)^2+\frac{1}{N}\bigg).
\end{aligned}$$

For the last point (iii), let a continuous bounded function $f : [0,T] \times \mathcal{C}^n \times \mathcal{P}(\mathcal{C}^n \times A) \times A \to \mathbb{R}$,

$$\mathbb{E}^{\overline{\mathbb{P}}^\infty}\left[\left|\mathbb{E}^{\hat{\mu}}\left[\left|\mathbb{E}^{\hat{\mu}}\left[\int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)} \int_A f(t,\widehat{X}_t,\bar{\nu},a)\widehat{\Lambda}_t(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\right] - \int_0^T \int_{\mathcal{P}(\mathcal{C}^n\times A)} \int_{\mathcal{C}^n\times A} f(t,\mathbf{x},\bar{\nu},a)\bar{\nu}(\mathrm{d}\mathbf{x},\mathrm{d}a)\widehat{\Lambda}_t(\mathrm{d}\bar{\nu},A)\mathrm{d}t\right|\right]\right]$$

$$\lim_{N\to\infty} \mathbb{E}^{\mathbb{P}^\star}\left[\left|\frac{1}{N}\sum_{i=1}^N \int_0^T f(t,\widehat{X}_t^{\phi,i},\overline{\varphi}_t^N,\alpha_t^i)\mathrm{d}t - \int_0^T \frac{1}{N}\sum_{i=1}^N f(t,\widehat{X}_t^{\phi,i},\overline{\varphi}_t^N,\alpha_t^i)\mathrm{d}t\right|\right] = 0,$$

by taking a countable set of continuous functions f, we deduce the point (iii). Therefore, $\overline{\mathbb{P}}^\infty$ belongs to $\overline{\mathcal{P}}_R(\nu)$. □

Proof of Theorem 8.2.5 Let Assumption 1.4.1 hold true. For each $N \in \mathbb{N}^*$, in the space $(\Omega^\star, \mathbb{F}^\star, \mathbb{P}^\star)$, let us take a Borel function $\phi^N : [0,T] \times \mathbb{R}^n \times (\mathbb{R}^d \times \mathbb{R}^\ell)^{m^N} \to U$, satisfying for each $i \in \{1,...,m^N\}$, $\mathbb{E}^{\mathbb{P}^\star}\big[\int_0^T \rho(a_0,\phi^i_{[s]^N})^p \mathrm{d}s\big] < \infty$, where we denote by $\phi^i_{t_j^N} := \phi\Big(t_j^N, X_0^i, \big(\sqrt{1/m^N}U^i_{k\wedge j}, \sqrt{1/m^N}V^N_{k\wedge j}\big)_{1\le k\le m^N}\Big)$, and $J_N(\phi^N) \ge V_S^N(\nu) - 2^{-N}$. By Lemma 8.4.4, if we define

$$\mathbb{P}^N := \mathbb{P}^\star \circ \big(B^N, \widehat{\varphi}_N\big)^{-1} \in \mathcal{P}(\overline{\Omega}), \text{ where } \widehat{\varphi}_N := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i},\widehat{Y}^{\phi,i},\delta_{(\varphi_t,\ \alpha_t^i)}(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t\right)} \quad B_t^N = B^N_{[t]^N} + \sqrt{t-[t]^N}V^N_{[t]^N}$$

and $\alpha_t^i := \phi^i_{[t]^N}$. Then the sequence $(\mathbb{P}^N)_{N\in\mathbb{N}^*}$ is relatively compact in $\mathcal{W}_p$ and any limit point is a relaxed control. by Proposition 8.4.3, for each limit point $\overline{\mathbb{P}}$, on an extension $\big(\widehat{\Omega}^\star := \widehat{\Omega}\times[0,1], \widehat{\mathbb{F}}^\star := (\widehat{\mathcal{F}}_t \otimes \mathcal{B}([0,1]))_{t\in[0,T]}\big)$ of $(\widehat{\Omega},\widehat{\mathbb{F}})$, there exists a family of measure–valued processes $(\widehat{N}^{\bar{\omega}})_{\bar{\omega}\in\overline{\Omega}}$ such that, for $\overline{\mathbb{P}}$–a.e. $\bar{\omega} \in \overline{\Omega}$, $\widehat{N}^{\bar{\omega}} = \big(\widehat{N}^{1,\bar{\omega}},\dots,\widehat{N}^{d,\bar{\omega}}\big)$ is an $\big(\widehat{\mathbb{F}}^\star, \widehat{\mu}(\bar{\omega})\otimes\lambda\big)$–martingale measure with intensity $\widehat{\Lambda}_t(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}t$, the martingales $(\widehat{N}^{i,\bar{\omega}})_{1\le i\le d}$ are orthogonal, and satisfy

$$\widehat{Y}_t = \widehat{X}_0 + \iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} b\big(r,\widehat{X},m,a\big)\widehat{\Lambda}_r(\mathrm{d}\bar{\nu},\mathrm{d}a)\mathrm{d}r + \iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} \sigma\big(r,\widehat{X},\bar{\nu},a\big)\widehat{N}^{\bar{\omega}}(\mathrm{d}\bar{\nu},\mathrm{d}a,\mathrm{d}r),\ \widehat{\mu}(\bar{\omega})\otimes\lambda\text{–a.s.}$$

Moreover, let $\widehat{\mathbb{H}}^\star = (\widehat{\mathcal{H}}_t^\star)_{t\in[0,T]}$ with $\widehat{\mathcal{H}}_t^\star := \overline{\mathcal{G}}_t \otimes \widehat{\mathcal{F}}_t^\star$ be a filtration on $\overline{\Omega}\times\widehat{\Omega}^\star$, denote by $\mathcal{P}^{\widehat{\mathbb{H}}^\star}$ the predictable σ–algebra on $[0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star$ with respect to $\widehat{\mathbb{H}}^\star$. Then for all bounded $\mathcal{P}^{\widehat{\mathbb{H}}^\star}\otimes\mathcal{B}(\mathcal{P}(\mathcal{C}^n\times A)\times A)$–measurable function $f : [0,T]\times\overline{\Omega}\times\widehat{\Omega}^\star\times\mathcal{P}(\mathcal{C}^n\times A)\times A \longrightarrow \mathbb{R}$, one can define the stochastic integral $\iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} f^{\bar{\omega}}(s,\bar{\nu},a)\widehat{N}^{\bar{\omega}}(\mathrm{d}s,\mathrm{d}\bar{\nu},\mathrm{d}a)$ in such a way that

$$(t,\bar{\omega},\hat{\omega}^\star) \longmapsto \left(\iint_{[0,t]\times\mathcal{P}(\mathcal{C}^n\times A)\times A} f^{\bar{\omega}}(s,\bar{\nu},a)\widehat{N}^{\bar{\omega}}(\mathrm{d}\bar{\nu},\mathrm{d}a,\mathrm{d}s)\right)(\hat{\omega}^\star) \text{ is } \mathcal{P}^{\widehat{\mathbb{H}}^\star}\text{–measurable.}$$

<u>*First case*</u> : Assumption 8.2.3

If we define $\widehat{W}_\cdot = \iint_{[0,\cdot]\times A}\widehat{N}^{\bar{\omega}}(\mathrm{d}a,\mathrm{d}s)$ on $\widehat{\Omega}$, and

$$\mathrm{P} := \int_{\overline{\Omega}} \mathcal{L}^{\widehat{\mu}(\bar{\omega})\otimes\lambda}\Big(\widehat{X},\widehat{Y},\widehat{\Lambda},\widehat{W},B,\mathcal{L}^{\widehat{\mu}(\bar{\omega})\otimes\lambda}\big(\widehat{X},\widehat{Y},\widehat{\Lambda},\widehat{W}\big)\Big)\overline{\mathbb{P}}(\mathrm{d}\bar{\omega}),$$

according to Proposition 3.3.2, P is a weak control, and consequently, under Assumption 8.2.3, by Theorem 3.2.7

$$\limsup_{N\to\infty} V_S^N(\nu) \le \limsup_{N\to\infty} J_N(\phi^N) \le V_S(\nu). \tag{8.4.14}$$

<u>*Second case*</u> : 8.2.4 If we define

$$\mathrm{P} := \int_{\overline{\Omega}} \mathcal{L}^{\widehat{\mu}(\bar{\omega})}\Big(\big(\widehat{\mu}\circ(\widehat{X}_t)^{-1}\big)_{t\in[0,T]},\ \widehat{\Lambda}_t(\mathrm{d}\bar{\nu},A)\mathrm{d}t,\ (B_t)_{t\in[0,T]}\Big)\overline{\mathbb{P}}(\mathrm{d}\bar{\omega}),$$

according to Definition 4.4.1, P is a measure-valued rule, and consequently, under 8.2.4, we have the same result (8.4.14). To finish our proof, we want to show that

$$V_S(\nu) \leq \liminf_{N\to\infty} V_S^N(\nu).$$

Let $\alpha \in \mathcal{A}_{p'}(\nu)$, by Lemma 8.4.1, For any $N \in \mathbb{N}^*$, there exists a family of Borel bounded functions $(\phi^N)_{N\in\mathbb{N}^*}$ satisfying $\phi^N : [0,T] \times \mathbb{R}^n \times (\mathbb{R}^d \times \mathbb{R}^\ell)^{m^N} \to U$ s.t. if we let $(X^{\phi,1}, \dots, X^{\phi,N})$ be defined on $(\Omega^\star, \mathbb{F}^\star, \mathbb{P}^\star)$ by: for $i \in \{1, \dots, N\}$, for all $j \in \{0, \dots, m^N - 1\}$, $X^{\phi,i}_{t_0^N} = X_0^i$,

$$X^{\phi,i}_{t_{j+1}^N} = X^{\phi,i}_{t_j^N} + b\big(t_j^N, \widehat{X}^{\phi,i}, \varphi^N_{t_j^N}, \phi^i_{t_j^N}\big)1/m^N + \sigma\big(t_j^N, \widehat{X}^{\phi,i}, \varphi^N_{t_j^N}, \phi^i_{t_j^N}\big)\sqrt{1/m^N}U_j^i + \sigma_0\sqrt{1/m^N}V_j^N, \tag{8.4.15}$$

where $\phi^i_{t_j^N} := \phi\Big(t_j^N, X_0^i, \big(\sqrt{1/m^N}U^i_{k\wedge j}, \sqrt{1/m^N}V^N_{k\wedge j}\big)_{1\leq k\leq m^N}\Big)$, $(\widehat{X}^{\phi,1}, \dots, \widehat{X}^{\phi,N})$ denotes the linear interpolation process of $\Big\{(X^{\phi,1}_{t_j^N}, \dots, X^{\phi,N}_{t_j^N}),\ 1 \leq j \leq m^N\Big\}$ on interval $[0,T]$,

$$\varphi_s^N(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{X}^{\phi,i}_{s\wedge\cdot}, \phi^i_s)}(\mathrm{d}\mathbf{x}, \mathrm{d}a),\ \mathrm{d}s \otimes \mathrm{d}\mathbb{P}^\star\text{–a.e., and } \varphi^{N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{X}^{\phi,i}}(\mathrm{d}\mathbf{x}),\ \text{for all } s \in [0,T],$$

then if $\Phi^i := \delta_{\phi^i_{[s]^N}}(\mathrm{d}a)\mathrm{d}t$, $\Phi := \delta_{\alpha_s}(\mathrm{d}a)\mathrm{d}t$,

$$\widehat{\varphi}_s^N(\mathrm{d}\mathbf{x}, \mathrm{d}a, \mathrm{d}e) := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i}_{s\wedge\cdot}, \phi^i_{[s]^N}, (\Phi^i)^s\right)}(\mathrm{d}\mathbf{x}, \mathrm{d}a, \mathrm{d}e) \text{ and } \widehat{\mu}_s^\alpha(\mathrm{d}\mathbf{x}, \mathrm{d}a, \mathrm{d}e) := \mathcal{L}^{\mathbb{P}_\nu}\big(X^\alpha_{s\wedge\cdot}, \alpha_s, \Phi^s \big| \mathcal{G}_s\big)(\mathrm{d}\mathbf{x}, \mathrm{d}a, \mathrm{d}e)$$

one has

$$\lim_N \mathcal{L}^{\mathbb{P}^\star}\Big(\delta_{\widehat{\varphi}_s^N}(\mathrm{d}\bar{\nu})\mathrm{d}s,\ \varphi^{N,X}\Big) = \mathcal{L}^{\mathbb{P}_\nu}\Big(\delta_{\widehat{\mu}_s^\alpha}(\mathrm{d}\bar{\nu})\mathrm{d}s, \mu^\alpha\Big),\ \text{in } \mathcal{W}_p.$$

If we define for $i \in \{1, \dots, N\}$, $X_0^{\phi,i} := X_0^i$ and for all $j \in \{0, ..., m^N - 1\}$, $X^{\phi,i}_{t_0^N} = X_0^i$, $Y^{\phi,i}_{t_0^N} = 0$,

$$Z^{\phi,i}_{t_{j+1}^N} = Z^{\phi,i}_{t_j^N} + \mathbf{H}\big(1/m^N,\ \widehat{R}^{\phi,i},\ \widetilde{\varphi}^{N,X}_{t_j^N},\ \phi^i_{t_j^N},\ U_j^i\big) \text{ and } R^{\phi,i}_{t_{j+1}^N} = R^{\phi,i}_{t_j^N} + Z^{\phi,i}_{t_{j+1}^N} - Z^{\phi,i}_{t_j^N} + \sigma_0\sqrt{1/m^N}V_j^N$$

$(\widehat{R}^{\phi,1}, \dots, \widehat{R}^{\phi,N})$ (resp $(\widehat{Z}^{\phi,1}, \dots, \widehat{Z}^{\phi,N})$) denotes the linear interpolation process of $\Big\{(R^{\phi,1}_{t_j^N}, \dots, R^{\phi,N}_{t_j^N}),\ 1 \leq j \leq m^N\Big\}$ $\Big(\text{resp } \Big\{(Z^{\phi,1}_{t_j^N}, \dots, Z^{\phi,N}_{t_j^N}),\ 1 \leq j \leq m^N\Big\}\Big)$ on interval $[0,T]$, and

$$\widetilde{\varphi}_s^N(\mathrm{d}\mathbf{x}, \mathrm{d}a) := \frac{1}{N}\sum_{i=1}^N \delta_{(\widehat{R}^{\phi,i}_{s\wedge\cdot}, \phi^i_s)}(\mathrm{d}\mathbf{x}, \mathrm{d}a),\ \mathrm{d}s \otimes \mathrm{d}\mathbb{P}_\nu^N\text{–a.e., and } \widetilde{\varphi}^{N,X}(\mathrm{d}\mathbf{x}) := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{R}^{\phi,i}}(\mathrm{d}\mathbf{x}),\ \text{for all } s \in [0,T],$$

it is straightforward to check (similarly to Lemma 8.4.4)

$$\lim_{N\to\infty} \mathcal{W}_p\Big(\mathcal{L}^{\mathbb{P}^\star}\big(B^N, \widehat{\chi}^N\big)^{-1}, \mathcal{L}^{\mathbb{P}^\star}\big(B^N, \widehat{\psi}^N\big)^{-1}\Big),$$

where

$$\widehat{\chi}_N := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i}, \widehat{X}^{\phi,i} - \sigma_0 B^N,\ \delta_{(\varphi^N_t,\ \alpha^i_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\right)} \text{ and } \widehat{\psi}_N := \frac{1}{N}\sum_{i=1}^N \delta_{\left(\widehat{X}^{\phi,i}, \widehat{X}^{\phi,i} - \sigma_0 B^N,\ \delta_{(\widetilde{\varphi}^N_t,\ \alpha^i_t)}(\mathrm{d}\bar{\nu}, \mathrm{d}a)\mathrm{d}t\right)},$$

with $B_t^N = B^N_{[t]^N} + \sqrt{t - [t]^N}V^N_{[t]^N}$ and $\alpha_t^i := \phi^i_{[t]^N}$. Therefore,

$$V_S(\nu) \leq \liminf_{N\to\infty} V_S^N(\nu),$$

then finally $\lim_{N\to\infty} V_S^N(\nu) = V_S(\nu)$.

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RÉSUMÉ

Cette thèse est formulée en trois parties avec huit chapitres et présente un sujet de recherche traitant des particules/ agents/ processus controllés en interactions.

Dans la première partie de cette dissertation, nous portons notre attention sur l'étude de processus controllés en interaction représentant un équilibre coopératif, aussi appelé équilibre de Pareto. Un équilibre coopératif peut être vu comme une situation où il n'existe pas de moyen d'améliorer la préférence d'un agent sans rabaisser la préference d'au moins un autre agent. Il est maintenant bien connu que ce type de problème d'optimisation est lié, quand le nombre d'agents tend vers l'infini, au contrôle optimal de l'équation de McKean–Vlasov. Dans les trois premiers chapitres de cette thèse, nous donnons une réponse mathématique précise à la connection entre ces deux problèmes d'optimisation dans differents cadres améliorant la littérature existante, en particulier en prenant en compte la loi du contrôle tout en considérant un bruit commun.

La seconde partie de cette thèse est consacrée à l'étude des processus contrôlés en interaction représentant désormais un équilibre de Nash, également appelé équilibre compétitif. Une situation d'équilibre de Nash dans un jeu est une situation dans laquelle personne n'a rien à gagner en quittant unilatéralement sa propre position. Depuis les travaux pionniers de Lasry and Lions [111] et Huang, Caines, and Malhamé [84], le comportement des équilibres de Nash lorsque le nombre d'agents va à l'infini a été intensivement étudié et le jeu limite associé est connu sous le nom de jeux à champ moyen (MFG). Dans cette seconde partie, nous analysons d'abord la convergence des équilibres compétitifs vers les MFG dans un cadre avec la loi de contrôle et avec le contrôle de la volatilité, puis, la question de l'existence de l'équilibre MFG dans ce contexte est étudiée.

Enfin, la dernière partie est dédiée à quelques méthodes numériques pour résoudre le problème de contôle de McKean-Vlasov. Inspiré de la preuve de la convergence d'équilibre coopératif, nous donnons un algorithme numérique pour résoudre le problème de contrôle optimal McKean-Vlasov et nous prouvons sa convergence. Ensuite, nous implémentons notre algorithme à partir des réseaux de neurones et testons son efficacité sur quelques exemples d'application, à savoir la sélection de portefeuille par moyenne-variance, le modèle de risque systémique interbancaire et la liquidation optimale avec impact de marché.

MOTS CLÉS

Équation différentielle stochastique, problème de contrôle stochastique, équilibres de Pareto, équilibres de Nash, jeux à champ moyen, équations de McKean–Vlasov, approximations numériques, réseaux de neurones.

ABSTRACT

This thesis is formulated in three parts with eight chapters and presents a research topic dealing with controlled processes/particles/agents in interaction.

In the first part of the dissertation, we focus our attention on the study of interacting controlled processes representing a cooperative equilibrium, also called Pareto equilibrium. A cooperative equilibrium can be seen as a situation where there is no way to improve the preference criterion of one agent without lowering the preference criterion of at least one other agent. It is now known that this kind of optimization problem is related, when the number of agents goes to infinity, to the optimal control of McKean–Vlasov processes. In the first three chapters of this thesis, we provide a precise mathematical answer to the connection between these two optimization problems in various frameworks improving the existing literature, in particular by taking into account the law of control while allowing a common noise situation.

The second part of this thesis is devoted to the study of the interacting controlled processes now representing a Nash equilibrium, also called competitive equilibrium. A Nash equilibrium situation in a game is a situation in which no one has anything to gain by moving unilaterally from his own position. Since the pioneering works of Lasry and Lions [111] and Huang, Caines, and Malhamé [84], the behavior of Nash equilibria when the number of agents goes to infinity has been intensively studied and the associated limit game is known as Mean Field Games (MFG). In this second part, we analyze first the convergence of the competitive equilibrium to the MFG in a framework with the law of control and with control of volatility, then, the issue of the existence of MFG equilibrium in this context is studied.

Finally, the last part is dedicated to some numerical methods to solve the McKean–Vlasov limit problem. Inspired by the proof of the convergence of cooperative equilibrium, we give a numerical algorithm to solve the McKean–Vlasov optimal control problem and we prove its convergence. Then, we implement our algorithm using neural networks and test its efficiency on some application examples, namely the mean–variance portfolio selection, the inter–bank systemic risk model and the optimal liquidation with market impact.

KEYWORDS

Stochastic differential equation, stochastic control problem, Pareto equilibria, Nash equilibria, mean field game, McKean–Vlasov equations, numerical approximations, neural networks.