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Olivier Graf

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pour l'obtention du grade de  
**Docteur en Mathématiques**  
par  
**Olivier Graf**

**Le problème d'évolution en relativité générale**

après avis des rapporteurs

Jonathan Luk    *Stanford University*  
Arick Shao      *Queen Mary University of London*

devant le jury composé de

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<sup>1</sup>Cette citation doit bien évidemment être remise dans le contexte de [KR05, KR06a, KR06b, Sze12a, Sze12b, Sze12c, Sze16, Sze18, Sha14, Wan09].

<sup>2</sup>“Mettez-moi une footnote là avec une référence pour rassurer le lecteur.”

<sup>3</sup>“Alors là, dans une footnote on a l’impression qu’il y a 200 pages de preuve qui s’écroulent, faut me changer ça.”

<sup>4</sup>“Les footnotes faut surtout pas commencer. Moi par exemple, je suis accro aux footnotes.”

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# I Introduction



# 1. The Cauchy problem of general relativity

In this section we introduce Einstein equations and review the associated classical and characteristic Cauchy problem. We then discuss general conjectures which motivate the results of this thesis. We refer to [Wal84, Chapter 10] for further introduction on the Cauchy problem of general relativity.

## 1.1 Einstein vacuum equations

In the theory of general relativity, a *vacuum spacetime* is described by a 4-dimensional manifold  $\mathcal{M}$  endowed with a Lorentzian metric  $\mathbf{g}$ , which satisfies the following *Einstein vacuum equations*

$$\text{Ric}(\mathbf{g})_{\mu\nu} = 0, \quad \mu, \nu = 0 \dots 3, \quad (1.1)$$

where  $\text{Ric}(\mathbf{g})$  denotes the *Ricci curvature tensor* of the spacetime metric  $\mathbf{g}$ .

The prime example of a vacuum spacetime is *Minkowski space*

$$\mathcal{M} = \mathbb{R}^4, \quad \mathbf{g} = -(\text{d}x^0)^2 + (\text{d}x^1)^2 + (\text{d}x^2)^2 + (\text{d}x^3)^2 =: \boldsymbol{\eta}.$$

which is the trivial solution of (1.1).

Each component of the Ricci curvature tensor  $\text{Ric}(\mathbf{g})_{\mu\nu}$  is a second-order nonlinear differential operator on the metric components  $\mathbf{g}_{\alpha\beta}$ , for  $\alpha, \beta = 0 \dots 3$ . Therefore, Einstein equations (1.1) form a system of 10 nonlinear coupled partial differential equations on the 10 unknowns  $\mathbf{g}_{\mu\nu}$ . The equations (1.1) reduce to 6 independent equations and we have the freedom to impose 4 additional equations on the metric components  $\mathbf{g}_{\mu\nu}$ . This is related to the so-called *general covariance* of Einstein equations, *i.e.* that the system of equation (1.1) is covariant under a change of coordinates. Such an additional choice of equations is called a *gauge choice*.

To obtain a well-posed system of equations we consider the *wave coordinates* gauge, which is the following choice of four additional equations

$$\square_{\mathbf{g}}(x^\mu) := \mathbf{D}^\alpha \mathbf{D}_\alpha(x^\mu) = 0, \quad \mu = 0 \dots 3,$$

where  $\square_{\mathbf{g}}$  is called the *d'Alembertian operator* associated to the Lorentzian metric  $\mathbf{g}$ ,<sup>1</sup> and where  $\mathbf{D}$  denotes the spacetime covariant derivative associated to  $\mathbf{g}$ . Under this gauge choice, it can be shown that (1.1) rewrites as the following system of coupled nonlinear wave equations

$$\square_{\mathbf{g}}(\mathbf{g}_{\mu\nu}) = \mathcal{N}_{\mu\nu} \left( (\mathbf{g}_{\alpha\beta})_{\alpha, \beta=0 \dots 3}, (\partial_\gamma \mathbf{g}_{\delta\epsilon})_{\gamma, \delta, \epsilon=0 \dots 3} \right), \quad \mu, \nu = 0 \dots 3, \quad (1.2)$$

where  $\mathcal{N}_{\mu\nu}$  are nonlinearities, quadratic in  $\partial\mathbf{g}$ . Thus, Einstein vacuum equations (in the wave coordinates gauge) can be cast as a *system of coupled quasilinear wave equations*.

As a system of wave equations, Einstein equations (1.1) admit an initial value formulation (also called *Cauchy problem*) for which *local well-posedness* holds. In the next sections, we review well-posedness results for the *classical* and for the *characteristic* Cauchy problem.

## 1.2 The classical Cauchy problem

Initial Cauchy data for Einstein equations (1.1) are classically prescribed by a triplet  $(\Sigma, g, k)$  such that

- ◇  $(\Sigma, g)$  is a 3-dimensional Riemannian manifold,
- ◇  $k$  is a symmetric covariant 2-tensor on  $\Sigma$ ,

---

<sup>1</sup>In Minkowski space, we have

$$\square_{\boldsymbol{\eta}} = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2.$$



◇  $(g, k)$  satisfy the following *constraint equations* on  $\Sigma$

$$\begin{aligned} \mathbf{R} &= |k|_g^2 - (\operatorname{tr}_g k)^2, \\ \operatorname{div}_g k &= \nabla(\operatorname{tr}_g k), \end{aligned} \tag{1.3}$$

where  $\mathbf{R}$  denotes the scalar curvature of  $g$ ,  $\nabla$  is the covariant derivative on  $(\Sigma, g)$  and for a symmetric 2-tensor  $F$  on  $\Sigma$ ,

$$|F|_g^2 := g^{ij}g^{kl}F_{ik}F_{jl}, \quad \operatorname{tr}_g F := g^{ij}F_{ij}, \quad (\operatorname{div}_g F)_i = \nabla^j F_{ij}.$$

A vacuum spacetime  $(\mathcal{M}, \mathbf{g})$  satisfies the classical Cauchy problem for Einstein equations (1.1) with Cauchy data  $(\Sigma, g, k)$  if  $\Sigma$  properly embeds in  $\mathcal{M}$  and  $g, k$  are the *first and second fundamental forms* of  $\Sigma \subset \mathcal{M}$ . That is,  $g$  is the *induced metric* by  $\mathbf{g}$  on  $\Sigma$ , and

$$k_{ij} = -\mathbf{D}_i T_j,$$

where  $T$  is a unit normal to  $\Sigma$  in  $\mathcal{M}$ .

**Remark 1.1.** *The standard Cauchy data for Minkowski space  $(\mathbb{R}^4, \boldsymbol{\eta})$  are given by*

$$\Sigma = \mathbb{R}^3, \quad g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 =: \delta, \quad k = 0.$$

We have the following well-posedness result for the classical Cauchy problem of general relativity.

**Theorem 1.2** ([Fou52, CG69]). *Let  $(\Sigma, g, k)$  be smooth Cauchy data for Einstein vacuum equations. Then, there exists a unique smooth vacuum spacetime  $(\mathcal{M}, \mathbf{g})$  which satisfies the following properties:*

- ◇  $(\mathcal{M}, \mathbf{g})$  admits  $(\Sigma, g, k)$  as Cauchy data,
- ◇  $(\mathcal{M}, \mathbf{g})$  is globally hyperbolic and admits  $\Sigma$  as a Cauchy hypersurface, i.e.  $\Sigma$  is achronal in  $\mathcal{M}$  and  $\mathcal{M}$  coincides with the domain of dependence of  $\Sigma$  in  $\mathcal{M}$ .<sup>2</sup>
- ◇  $(\mathcal{M}, \mathbf{g})$  is maximal for the inclusion among all the spacetimes satisfying the above conditions.

The spacetime  $(\mathcal{M}, \mathbf{g})$  is called the maximal globally hyperbolic development of  $(\Sigma, g, k)$ .

### Remarks on Theorem 1.2

1.2a In [Fou52], the local existence result of Theorem 1.2 is obtained by a Banach-Picard iteration using Kirchhoff-Sobolev parametrix for the Einstein equations in wave gauge (1.2).

1.2b Here and in the rest of this thesis, a *smooth* or  $\mathcal{C}^\infty$  manifold admits by definition an atlas of charts such that all coordinate changes are  $\mathcal{C}^\infty$  with respect to the standard  $\mathcal{C}^\infty$ -topology of  $\mathbb{R}^n$ . As all manifolds we consider will be smooth submanifolds of a fixed smooth 4-dimensional manifold  $\mathcal{M}$  and as all vector bundles we consider will be constructed upon  $\mathbf{T}\mathcal{M}$  and  $\mathbf{T}^*\mathcal{M}$ , we shall assume that such an atlas is fixed on  $\mathcal{M}$ , which then canonically determines the  $\mathcal{C}^k$ -topology for all tensors on all smooth submanifolds of  $\mathcal{M}$  in this thesis.

## 1.3 The characteristic Cauchy problem

An alternative to the classical Cauchy problem where initial data are prescribed on an initial spacelike hypersurface is to prescribe initial data on *characteristic* or *null* hypersurfaces.<sup>3</sup>

Formulations of the Cauchy problem for initial data posed on characteristic hypersurfaces is of particular interest in the case of Einstein equations since, contrary to the classical Cauchy problem of Theorem 1.2 where initial data are posed on a spacelike hypersurface and have to satisfy elliptic constraint equations (1.3), initial data can be *freely prescribed* on null hypersurfaces (see the seminal [Sac62]). The *characteristic Cauchy problem* (or *Goursat problem*) is therefore used in numerical general relativity (see [SF82]), as well as in the construction and control of solutions to Einstein equations (see the dynamical formation of black holes solutions in [Chr09, KR12, AL17], the impulsive gravitational waves solutions in [LR15, LR17] or the

<sup>2</sup>We refer to [Wal84, Chapter 8] for further definitions.

<sup>3</sup>The null hypersurfaces coincide with the characteristic hypersurfaces for the underlying wave operator. See [Ren90, Section 3] for a definition of characteristic hypersurfaces for general differential operators.

naked singularities of [RS19]).

The following theorem is a rough statement of the seminal result [Ren90] which established the local well-posedness for the characteristic Cauchy problem (we refer to [Ren90, CP12, Luk12] for precise versions and further discussion).

**Theorem 1.3** ([Ren90]). *Let  $\mathcal{C}$  and  $\underline{\mathcal{C}}$  be two intersecting smooth 3-dimensional manifolds. Assume that smooth characteristic initial data are given on  $\mathcal{C} \cup \underline{\mathcal{C}}$ , i.e.*

- ◊ *a foliation by Riemannian 2-surfaces on  $\mathcal{C}$  and  $\underline{\mathcal{C}}$  respectively, coinciding with  $\mathcal{C} \cap \underline{\mathcal{C}}$ ,*
- ◊ *a set of auxiliary quantities on  $\mathcal{C} \cap \underline{\mathcal{C}}$ .<sup>4</sup>*

*Then, there exists a smooth vacuum spacetime  $(\mathcal{M}, \mathbf{g})$  such that*

- ◊  *$\mathcal{C}$  and  $\underline{\mathcal{C}}$  are null hypersurfaces of  $(\mathcal{M}, \mathbf{g})$  and the foliations of  $\mathcal{C}$  and  $\underline{\mathcal{C}}$  are geodesic,<sup>5</sup>*
- ◊ *the induced metrics by  $\mathbf{g}$  on the foliations of  $\mathcal{C}$  and  $\underline{\mathcal{C}}$  are conformal to the given Riemannian metrics,*
- ◊ *on  $\mathcal{C} \cap \underline{\mathcal{C}}$ , the auxiliary quantities coincide with geometric quantities associated to  $(\mathcal{M}, \mathbf{g})$ .<sup>4</sup>*

### Remarks on Theorem 1.3

- 1.3a The proof of Theorem 1.3 in [Ren90] relies in an extension argument in the exterior of  $\mathcal{C} \cup \underline{\mathcal{C}}$  and an application of Theorem 1.2 for the classical Cauchy problem.
- 1.3b The characteristic initial data of Theorem 1.3 are *freely prescribed*. They do not have to satisfy constraint equations as in the classical Cauchy problem.

## 1.4 Some conjectures in general relativity

### 1.4.1 The weak cosmic censorship conjecture

One of the most natural question for nonlinear evolution PDE is the large-data global-in-time existence of solutions. In general relativity, this question is relevant for *isolated systems*, i.e. for Cauchy data  $(\Sigma, g, k)$  which are spatially asymptotically close to the Minkowski initial data  $(\mathbb{R}^3, \delta, 0)$ . These data are said to be *asymptotically flat* (we refer to [Bar86] for definitions).

Such solutions might collapse in finite time (see discussions in [Pen69]). The global-in-time existence conjecture rather states that causal geodesics emanating from the asymptotic region of  $\Sigma$  are complete, which more colloquially speaking corresponds to the existence of a *complete future null infinity* and less colloquially to the fact that far away observers live forever. See [Wal84, Chapter 12] for precisions. The conjecture was later modified to take into account the existence of the *naked singularity solutions* of [Chr94] for which the conjecture fails, but which were proved to be *non-generic* [Chr99] (see also [RS19]).

**Conjecture 1.4** (Weak cosmic censorship [Pen69]). *For generic asymptotically flat Cauchy data, solutions to Einstein equations admit a complete null infinity.*

In the seminal works [Chr87, Chr91, Chr93, Chr99], it is shown that the weak cosmic censorship conjecture holds true in the case of spherical symmetry for Einstein equations coupled with a scalar field.

**Remark 1.5.** *Due to the rigidity results of [BL23], there are no non-trivial spherically symmetric solutions to Einstein vacuum equations (1.1). Einstein equations coupled with a scalar field can be seen as one of the simplest set of dynamical equations involving Einstein equations in spherical symmetry.*

In the final article [Chr99] in which the conjecture is proved, *naked singularities* are considered – i.e. singularities for which Conjecture 1.4 would fail. It is shown that suitable perturbations of their initial data produce spacetimes with *trapped surfaces* surrounding the singularity, which thus verify Conjecture 1.4. It uses a *breakdown criterion* obtained in [Chr93], a *trapped surface formation mechanism* obtained in [Chr91]

<sup>4</sup>With notations introduced in Section 2, the auxiliary quantities will coincide with the null connection coefficients  $\text{tr}\chi, \text{tr}\underline{\chi}, \zeta$  associated to the geodesic foliation on  $\mathcal{C}$  and  $\underline{\mathcal{C}}$  as well as with the initial conformal factor on  $\mathcal{C} \cap \underline{\mathcal{C}}$ . See [CP12] for further discussion.

<sup>5</sup>We refer to Section 2 for a definition of geodesic foliations.

and a *local existence result* for rough initial data proved in [Chr93].

It is crucial that in [Chr93] the breakdown criterion is sufficiently precise and the local existence result allows for the existence of sufficiently rough solutions. In [Chr93], these results are obtained at the *low regularity level* of initial data with bounded variations (which is adapted to the (1+1)-setting of spherical symmetry).

Obtaining equivalent results outside of spherical symmetry would be a major breakthrough towards the proof of Conjecture 1.4 in general. As of today, the sharpest known such result has been obtained by the resolution of the *bounded  $L^2$  curvature conjecture* which we present in the next section.

### 1.4.2 The bounded $L^2$ curvature conjecture

In the case of Einstein vacuum equations (1.1) without symmetry, local existence results are naturally formulated in terms of  $L^2$ -based function spaces. In that context, the sharpest known local existence/breakdown result in terms of regularity of the initial data has been obtained by the resolution of the *bounded  $L^2$  curvature conjecture*. The following is a rough statement of the *bounded  $L^2$  curvature theorem* obtained in [KRS15] and the companion papers [Sze12a]–[Sze16].

**Theorem 1.6** (Bounded  $L^2$  curvature theorem [KRS15]). *Assume that  $(\Sigma, g, k)$  are Cauchy data to the Einstein equations (1.1) such that*

$$\|\text{Ric}(g)\|_{L^2(\Sigma)} + \|\nabla^{\leq 1} k\|_{L^2(\Sigma)} < \infty. \quad (1.4)$$

*Then, there exists a vacuum spacetime which admits  $(\Sigma, g, k)$  as Cauchy data. Moreover, if the initial data are more regular, this additional regularity is propagated to the spacetime.*

#### Remarks on Theorem 1.6

1.6a The  $L^2(\Sigma)$ -bounds (1.4) are at the (low regular) level of an  $H_{loc}^2(\Sigma) \times H_{loc}^1(\Sigma)$  bound for the initial data  $(\mathbf{g}_{\mu\nu}, \partial_{x^0} \mathbf{g}_{\mu\nu})$ .

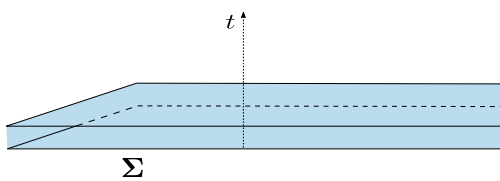
1.6b Theorem 1.6 is primarily to be understood as a *continuation result* or *breakdown criterion* for smooth solutions of the Einstein vacuum equations, see [KRS15, Remark 1.2].

The proof [Chr99] of the weak cosmic censorship conjecture in spherical symmetry crucially relies on the setting of the characteristic Cauchy problem, in particular for the trapped surface formation mechanism of [Chr91] and for the local existence result of [Chr93] (see also motivations to the characteristic Cauchy problem given in Section 1.3).

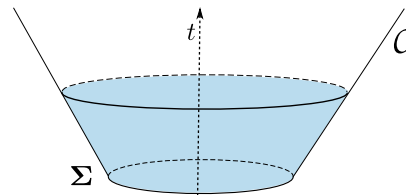
This raises the following question.

**Question 1.7.** *Can we obtain a generalisation of the bounded  $L^2$  curvature theorem to initial data posed on characteristic hypersurfaces?*

The first main result of this thesis provides a positive answer to the above question by establishing a *spacelike-characteristic bounded  $L^2$  curvature theorem*, which generalises Theorem 1.6 to the case of initial data posed on a null hypersurface. See Parts II and III of this thesis. See also Section 5 of this introduction for an overview of the result.



(a) The classical bounded  $L^2$  curvature theorem [KRS15].



(b) The spacelike-characteristic bounded  $L^2$  curvature theorem of this thesis.

### 1.4.3 The asymptotic stability of Minkowski space

Using the initial value formulation of Theorem 1.2, we have the following statement of the asymptotic stability conjecture for Minkowski space.

**Conjecture 1.8** (Asymptotic stability of Minkowski space). *For Cauchy data  $(\Sigma, g, k)$  close to the Minkowski initial data, the maximal globally hyperbolic development is geodesically complete and is asymptotic to Minkowski space  $(\mathbb{R}^4, \eta)$  in the limit along null and timelike geodesics.*

The breakthrough result which solved the asymptotic stability conjecture for Minkowski space is the following theorem.

**Theorem 1.9** (Stability of Minkowski space [CK93]). *For Cauchy data  $(\Sigma, g, k)$  such that*

- ◊  $\Sigma$  is diffeomorphic to  $\mathbb{R}^3$ ,
- ◊  $(\Sigma, g, k)$  is asymptotically flat (i.e. tends to Minkowski initial data  $(\mathbb{R}^3, \delta, 0)$  when  $r \rightarrow \infty$ )
- ◊  $(g, k)$  (and derivatives) are close to Minkowski initial data  $(\delta, 0)$  measured in an (weighted)  $L^2$ -sense,

*then its maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  is geodesically complete and admits global time and optical functions  $t$  and  $u$  such that, measured in these coordinates,  $\mathbf{g}$  is bounded and decays towards  $\eta$ .*

A localised version of Theorem 1.9 was proved for initial data posed on the exterior of a 3-disk.

**Theorem 1.10** (Exterior stability of Minkowski space [KN03]). *For Cauchy data  $(\Sigma, g, k)$  such that*

- ◊  $\Sigma$  is diffeomorphic to  $\mathbb{R}^3 \setminus \mathbb{D}$  where  $\mathbb{D}$  denotes the disk of  $\mathbb{R}^3$ ,
- ◊ the same asymptotic flatness and closeness to Minkowski space assumptions as in Theorem 1.9 hold,

*then, the maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  admits global optical functions  $u, \underline{u}$  such that, measured in these coordinates,  $\mathbf{g}$  is bounded and decays towards  $\eta$ .*

**Remark 1.11.** *In the proof of Theorem 1.9 in [CK93], the topology assumption  $\Sigma \simeq \mathbb{R}^3$  is crucially used to define a global time function such that its level sets are maximal hypersurfaces with prescribed asymptotic conditions when  $r \rightarrow \infty$ . The main novelty in the proof [KN03] of Theorem 1.10 is the definition of a double-null foliation by the level sets of two optical functions  $u, \underline{u}$ . It replaces the global time function and enables a localisation of the global nonlinear stability proof [CK93] to the exterior of a disk. See discussions in [KN03, Section 2].*

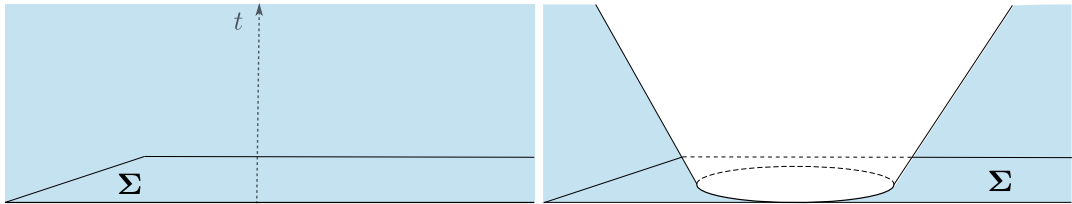


Figure 1.2: The stability of Minkowski space of Theorems 1.9, 1.10 proved in [CK93] and [KN03].

In view of Theorems 1.9 and 1.10, we have the following natural question.

**Question 1.12.** *Can we complete the result of Theorem 1.10 to re-obtain the result of Theorem 1.9? In other words, can we prove the global nonlinear stability of Minkowski space for initial data posed on a spacelike disk and an outgoing null hypersurface?*

The second main result of this thesis provides a positive answer to the above question. See Part IV of this thesis. See also Section 5 of this introduction for an overview of the result.

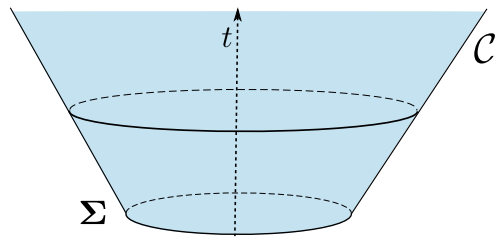


Figure 1.3: The stability of Minkowski space of this thesis.

## 2. Spacetime foliations

Einstein equations (1.1) form a well-determined system of equations, provided that additional *gauge conditions* are prescribed. In Section 1.1, we discussed the *wave coordinate* gauge, which is the prescription of four additional wave equations for the coordinates  $x^\mu$ . In many problems, other gauge choices are more adapted to the geometric situation, display better regularity properties, etc.

The gauge choices we use in this thesis are naturally formulated in terms of (natural) geometric constructions for the spacetime. They correspond to coordinate choices, such that their level sets foliate the spacetime by geometric hypersurfaces: *maximal* hypersurfaces (*i.e.* spacelike hypersurfaces which maximise their volume), *null* hypersurfaces, etc.

In this section, we review the formalisms associated to general spacetime foliations. Using these formalisms, we give definitions for the gauge conditions used in this thesis (as well as some other classical gauge choices). We also discuss the main features and motivations for these gauges. Most of the material of this section can be found in [CK93, Introduction] and [CK93, Chapter 7].

### 2.1 Foliations by spacelike hypersurfaces

In this section, we introduce the decompositions (of the spacetime metric and its derivatives, of Einstein vacuum equations) associated to foliations of the spacetime by spacelike hypersurfaces, which are level sets of a time function  $t$ . This formalism is called  $1 + 3$  (see [ADM62]). In that context, we formulate the *maximal gauge* condition and present the Einstein equations in that gauge. This gauge is used in Parts II and III of this thesis and we discuss motivations for that choice.

#### 2.1.1 The $1 + 3$ formalism

In this section as well as in the rest of this introduction, we shall assume that  $(\mathcal{M}, \mathbf{g})$  is an oriented and time-oriented vacuum spacetime.

We consider  $t$  a *time function* on  $\mathcal{M}$ , *i.e.* such that

$$\mathbf{g}(\mathbf{D}t, \mathbf{D}t) < 0.$$

We note  $\Sigma_t$  its level sets, which are *spacelike hypersurfaces* of  $(\mathcal{M}, \mathbf{g})$ . We define the *first fundamental form*  $g$  of  $\Sigma_t$  to be the induced Riemannian metric on  $\Sigma_t$ . We define the *second fundamental form*  $k$  of  $\Sigma_t$  to be the  $\Sigma_t$ -tangent tensor given by

$$k(Y, Z) := -\mathbf{g}(\mathbf{D}_Y T, Z),$$

where here and in the following  $T$  is the future-pointing unit normal to  $\Sigma_t$  and where  $Y, Z \in T\Sigma_t$ .

For simplicity, we shall from now on consider functions  $x^i$  which are *transported without shift* along the flow of  $t$ . Such a prescription can be written as

$$T(x^i) = 0, \quad i = 1, 2, 3. \quad (2.1)$$

We shall also assume that  $(t, x^i)$  forms a coordinate system on  $\mathcal{M}$ , and we note that the assumptions (2.1) consist in the 3 *gauge conditions*

$$\mathbf{g}_{0i} = 0.$$

We define the *time lapse*  $n$  of the foliation  $(\Sigma_t)$  by

$$n^{-2} := -\mathbf{g}(\mathbf{D}t, \mathbf{D}t).$$

With these definitions, the spacetime metric  $\mathbf{g}$  in the coordinates  $(t, x^i)$  decomposes as

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j.$$

Using the above definitions, the Einstein vacuum equations (1.1) rewrite in these coordinates as (see [CK93, Introduction])

$$\begin{aligned}\partial_t(g_{ij}) &= -\frac{1}{2}n k_{ij}, \\ \partial_t(k_{ij}) &= -\nabla_i \nabla_j n + n (\text{Ric}(g)_{ij} + (\text{tr}_g k)k_{ij} - 2k_{ia}k_j^a),\end{aligned}\tag{2.2a}$$

and

$$\mathbf{R}(g) = |k|^2 - (\text{tr}_g k)^2, \quad (\text{div } k)_i = \nabla_i(\text{tr}_g k).\tag{2.2b}$$

The system of equations (2.2a) is a well-determined system of evolution equations, up to a choice of time function  $t$ , which can be turned into a choice for  $n$  or  $\text{tr}_g k$ .

**Remark 2.1.** *The equations (2.2b) are the constraint equations. They hold by definition for initial Cauchy data to Einstein equations (1.1) (see Section 1). They are propagated by the evolution equation (2.2a). In this thesis, we always assume that we work on an a priori existing vacuum spacetime. Thus, we shall always consider that the constraint equations are satisfied, and do not investigate further the propagation of these equations.*

### 2.1.2 The maximal gauge

Additionally to the 3 gauge choices (2.1), we make the following additional (and last) *maximal gauge choice*

$$\text{tr}_g k = 0.\tag{2.3}$$

We remark that taking the trace in (2.2), this gauge choice implies the following equation for the lapse  $n$

$$\Delta_g n = n|k|_g^2.$$

**Remark 2.2.** *One can show that for a vacuum spacetime, any compact perturbation of each separate hypersurface  $\Sigma_t$  yield hypersurfaces with smaller volume. This justifies that the hypersurfaces  $\Sigma_t$  are called maximal hypersurfaces.*

To highlight the main features of equations (2.2) together with the maximal gauge choice (2.3), we first introduce the *electric-magnetic tensors*  $E$  and  $H$ , which are the  $\Sigma_t$ -tangent tensors defined by

$$E(X, Y) := \mathbf{R}(T, X, T, Y), \quad H(X, Y) := *\mathbf{R}(T, X, T, Y),\tag{2.4}$$

for all  $X, Y \in T\Sigma_t$  and where  $*\mathbf{R}$  is the Hodge dual of the spacetime curvature tensor  $\mathbf{R}$ .

Einstein vacuum equations (1.1) together with the additional gauge choices (2.1) and (2.3) are equivalent to the following *Einstein vacuum equations in maximal gauge*, which is the system of coupled quasilinear transport-elliptic-Maxwell equations (see [CK93, pp. 8-9 and p. 146]):

the *first variation* transport equation for  $g$

$$n^{-1}\partial_t g_{ij} = -2k_{ij},\tag{2.5a}$$

the *second variation* transport equation for  $k$

$$n^{-1}\partial_t k_{ij} = -n^{-1}\nabla_i \nabla_j n + E_{ij} - k_{il}k_j^l,\tag{2.5b}$$

the Hodge-type elliptic equations for  $k$

$$\begin{aligned}\text{tr}_g k &= 0, \\ \text{div}_g k_i &= 0, \\ \text{curl}_g k_{ij} &= H_{ij},\end{aligned}\tag{2.5c}$$

the Laplace equation for  $n$

$$\Delta_g n = n|k|_g^2,\tag{2.5d}$$



the Poisson-type elliptic equation for  $g^1$

$$\text{Ric}(g)_{ij} = E_{ij} + k_i^l k_{lj}, \quad (2.5e)$$

and the Maxwell-type equations for  $E$  and  $H$

$$\begin{aligned} \text{tr}_g E &= \text{tr}_g H = 0, \\ \text{div}_g E_i &= (k \wedge H)_i, \\ \text{div}_g H_i &= -(k \wedge E)_i, \\ n^{-1} \partial_t E_{ij} + \text{curl}_g H_{ij} &= -n^{-1} (\nabla n \wedge H)_{ij} + \frac{1}{2} (k \times E)_{ij} - \frac{2}{3} (k \cdot E) g_{ij}, \\ -n^{-1} \partial_t H_{ij} + \text{curl}_g E_{ij} &= -n^{-1} (\nabla n \wedge E)_{ij} - \frac{1}{2} (k \times H)_{ij} + \frac{2}{3} (k \cdot H) g_{ij}, \end{aligned} \quad (2.5f)$$

where  $\text{curl}_g$  is the tensorial rotational operator with respect to the metric  $g$  and  $\wedge$ ,  $\times$ ,  $\cdot$  are standard tensorial operations with respect to  $g$ . We refer to [CK93, Introduction] for definitions.

The 1 + 3 decomposition and the maximal gauge were used in a large number of work in general relativity from which we only cite [CK93], [KRS15], [Sze12a] – [Sze16], as well as Parts II and III and Part IV of this thesis.

The first motivation for that choice is that the system of equations (2.5) displays a clear hyperbolic structure due to the Maxwell-type equations (2.5f) for the electric-magnetic tensors  $E$  and  $H$ . In particular, *energy estimates* can be obtained for  $E, H$ .<sup>2</sup> The metric and connection coefficients  $g, k$  and  $n$  are then determined only by solving the transport (2.5a), (2.5b) or elliptic (2.5c), (2.5d), (2.5e) equations, for which the electric-magnetic tensors  $E$  and  $H$  are source terms.

**Remark 2.3.** *The time lapse  $n$  is only defined through the elliptic equation (2.5d) on each slice  $\Sigma_t$ , and is therefore well-determined up to a choice of a boundary condition at a finite/infinite boundary. Making such a choice is roughly equivalent to prescribing the boundary values for the maximal hypersurfaces  $\Sigma_t$ . In [CK93, KRS15] where  $\Sigma_t \simeq \mathbb{R}^3$ , the chosen condition is  $n \rightarrow 1$  at spatial infinity, which physically corresponds to considering a centre-of-mass frame for the system (see the discussion in the introduction of [CK93]). In Parts II, III and IV where  $\Sigma_t$  is a 3-disk, we shall see that one of the crucial step is to make an appropriate choice of (finite) boundaries for  $\Sigma_t$  (which is equivalent to a choice for  $n$ ).*

The second – and more specific – motivation, which is used in [CK93] and is crucial in [KRS15], [Sze12a] – [Sze16], in Parts II and III and in Part IV of this thesis is the *regularity* of the foliation: from the elliptic equations (2.5c), (2.5d) (2.5e) on  $\Sigma_t$ , the metric and connection coefficients  $g, k, n$  can be shown to have *optimal regularity* in the tangential directions to  $\Sigma_t$  with respect to the source terms  $E$  and  $H$ .

## 2.2 Foliations by 2-spheres

In this section, we consider foliations of the spacetime by spacelike 2-spheres  $S_{u, \underline{u}}$  which are intersections of the level sets of two functions  $u, \underline{u}$ . We present the *null decompositions* (of the spacetime metric and derivatives, of the Einstein equations) along orthogonal null directions to  $S_{u, \underline{u}}$ . This framework is particularly adapted to the definition of *null gauges*, which are used in Part IV and for which we give motivations.

### 2.2.1 The null decompositions

Let  $S_{u, \underline{u}}$  be a (local) foliation of  $\mathcal{M}$  by spacelike 2-spheres which are intersections of the level sets of two functions  $u, \underline{u}$ . We note  $\mathcal{g}$  the Riemannian metric induced by  $\mathbf{g}$  on  $S_{u, \underline{u}}$ , and we note  $r$  its area radius  $4\pi r^2 := |S|$ . A *null pair*  $(e_3, e_4)$  is a pair of vectorfields on  $\mathcal{M}$  orthogonal to the 2-spheres  $S_{u, \underline{u}}$  such that

$$\mathbf{g}(e_4, e_3) = -2, \quad \mathbf{g}(e_4, e_4) = \mathbf{g}(e_3, e_3) = 0.$$

**Remark 2.4.** *A null pair  $(e_3, e_4)$  decomposes the spacetime metric  $\mathbf{g}$  as follows*

$$\mathbf{g} = -\frac{1}{2} (e_3)^\flat \otimes (e_4)^\flat - \frac{1}{2} (e_4)^\flat \otimes (e_3)^\flat + \mathcal{g},$$

where  $(e_3)^\flat, (e_4)^\flat \in T^* \mathcal{M}$  are the 1-forms canonically associated to  $e_3, e_4$ .

<sup>1</sup>In the so-called *harmonic coordinates* for the Riemannian metric  $g$ , the Ricci tensor  $\text{Ric}(g)_{ij}$  can be rewritten as the Laplace-Beltrami operator of the metric component  $\Delta_g(g_{ij})$  up to lower order terms.

<sup>2</sup>The Maxwell equations (2.5f) are projections of the more general (spacetime) *Bianchi equations*. General energy estimates for these equations is the subject of Section 3.



We define the *null connection coefficients* relative to a null pair  $(e_3, e_4)$  to be the  $S$ -tangent tensors defined by

$$\begin{aligned}\chi(X, Y) &:= \mathbf{g}(\mathbf{D}_X e_4, Y), & \xi(X) &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, X), & \eta(X) &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, X), \\ \zeta(X) &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_X e_4, e_3), & \omega &:= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3),\end{aligned}$$

and

$$\begin{aligned}\underline{\chi}(X, Y) &:= \mathbf{g}(\mathbf{D}_X e_3, Y), & \underline{\xi}(X) &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, X), & \underline{\eta}(X) &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, X), \\ \zeta(X) &:= -\frac{1}{2}\mathbf{g}(\mathbf{D}_X e_3, e_4), & \underline{\omega} &:= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4),\end{aligned}$$

where  $X, Y \in TS$ . In the following, we often split up  $\chi$  and  $\underline{\chi}$  into their trace and tracefree parts,

$$\begin{aligned}\mathrm{tr}\chi &:= \mathcal{g}^{ab}\chi_{ab} & \hat{\chi}_{ab} &:= \chi_{ab} - \frac{1}{2}\mathrm{tr}\chi\mathcal{g}_{ab}, \\ \mathrm{tr}\underline{\chi} &:= \mathcal{g}^{ab}\underline{\chi}_{ab} & \hat{\underline{\chi}}_{ab} &:= \underline{\chi}_{ab} - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathcal{g}_{ab}.\end{aligned}$$

The coefficients  $\mathrm{tr}\chi, \mathrm{tr}\underline{\chi}$  are called the *expansions*,  $\chi, \underline{\chi}$  are called the (null) *second fundamental forms*, and  $\zeta$  is called the *torsion coefficient*.

We define the *null curvature components* relative to a null pair  $(e_3, e_4)$  to be the  $S$ -tangent tensors defined by

$$\begin{aligned}\alpha(X, Y) &:= \mathbf{R}(e_4, X, e_4, Y), & \beta(X) &:= \frac{1}{2}\mathbf{R}(X, e_4, e_3, e_4), & \rho &:= \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4), \\ \underline{\alpha}(X, Y) &:= \mathbf{R}(e_3, X, e_3, Y), & \underline{\beta}(X) &:= \frac{1}{2}\mathbf{R}(X, e_3, e_3, e_4), & \sigma &:= \frac{1}{4}*\mathbf{R}(e_3, e_4, e_3, e_4),\end{aligned}$$

where  $X, Y \in TS$  and where  $*\mathbf{R}$  denotes the Hodge dual of  $\mathbf{R}$ .

We have the following *null structure equations* relating the null connection coefficients and the null curvature components (see [CK93, pp. 168-170]).

We have the following *first variation* transport equations along  $e_3$  and  $e_4$

$$\mathcal{L}_{e_3}\mathcal{g} = 2\underline{\chi}, \quad \mathcal{L}_{e_4}\mathcal{g} = 2\underline{\chi},$$

and the following *second variation* equations

$$\begin{aligned}\nabla_3 \hat{\chi} + \mathrm{tr}\underline{\chi}\hat{\chi} &= \nabla \hat{\otimes} \underline{\xi} - 2\underline{\omega}\hat{\chi} + (\eta + \underline{\eta} - 2\underline{\zeta}) \hat{\otimes} \underline{\xi} - \underline{\alpha}, \\ \nabla_3 \mathrm{tr}\underline{\chi} + \frac{1}{2}(\mathrm{tr}\underline{\chi})^2 &= 2\mathrm{div}\underline{\xi} - 2\underline{\omega}\mathrm{tr}\underline{\chi} + 2\underline{\xi} \cdot (\eta + \underline{\eta} - 2\underline{\zeta}) - |\hat{\chi}|^2, \\ \nabla_3 \zeta &= -2\nabla \underline{\omega} - \underline{\chi} \cdot (\zeta + \eta) + 2\underline{\omega}(\zeta - \eta) + \underline{\chi} \cdot \underline{\xi} + 2\underline{\omega}\underline{\xi} - \underline{\beta}, \\ \nabla_3 \hat{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} &= \nabla \hat{\otimes} \eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} + \underline{\xi} \otimes \underline{\xi} + \eta \hat{\otimes} \eta, \\ \nabla_3 \mathrm{tr}\chi + \frac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi &= 2\mathrm{div}\eta + 2\underline{\omega}\mathrm{tr}\chi - \hat{\chi} \cdot \hat{\chi} + 2(\underline{\xi} \cdot \underline{\xi} + |\eta|^2) + 2\rho, \\ \nabla_3 \xi - \nabla_4 \eta &= 4\underline{\omega}\underline{\xi} + \underline{\chi} \cdot (\eta - \underline{\eta}) + \beta, \\ \nabla_3 \eta - \nabla_4 \underline{\xi} &= -4\underline{\omega}\underline{\xi} - \underline{\chi} \cdot (\eta - \underline{\eta}) + \underline{\beta}, \\ \nabla_3 \omega + \nabla_4 \underline{\omega} &= \underline{\xi} \cdot \underline{\xi} + \underline{\zeta} \cdot (\eta - \underline{\eta}) - \eta \cdot \underline{\eta} + 4\underline{\omega}\underline{\omega} + \rho, \\ \nabla_4 \hat{\chi} + \mathrm{tr}\underline{\chi}\hat{\chi} &= \nabla \hat{\otimes} \underline{\xi} - 2\underline{\omega}\hat{\chi} + (\eta + \underline{\eta} + 2\underline{\zeta}) \hat{\otimes} \underline{\xi} - \alpha, \\ \nabla_4 \mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 &= 2\mathrm{div}\xi - 2\underline{\omega}\mathrm{tr}\chi + 2\underline{\xi} \cdot (\eta + \underline{\eta} + 2\underline{\zeta}) - |\hat{\chi}|^2, \\ \nabla_4 \zeta &= 2\nabla \omega + \underline{\chi} \cdot (-\zeta + \eta) + 2\underline{\omega}(\zeta + \eta) - \underline{\chi} \cdot \underline{\xi} - 2\underline{\omega}\underline{\xi} - \beta, \\ \nabla_4 \hat{\chi} + \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} &= \nabla \hat{\otimes} \eta + 2\underline{\omega}\hat{\chi} - \frac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi} + \underline{\xi} \otimes \underline{\xi} + \eta \hat{\otimes} \eta, \\ \nabla_4 \mathrm{tr}\underline{\chi} + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} &= 2\mathrm{div}\underline{\eta} + 2\underline{\omega}\mathrm{tr}\underline{\chi} - \hat{\chi} \cdot \hat{\chi} + 2(\underline{\xi} \cdot \underline{\xi} + |\eta|^2) + 2\rho.\end{aligned}$$

We have the following elliptic equations on the 2-spheres

$$\begin{aligned}
 \text{curl } \underline{\eta} &= -\text{curl } \eta = \frac{1}{2} \hat{\chi} \wedge \underline{\hat{\chi}} - \xi \wedge \underline{\xi} - \sigma, \\
 \text{div } \underline{\hat{\chi}} &= \nabla \text{tr} \underline{\chi} + \underline{\chi} \cdot \zeta - \text{tr} \underline{\chi} \zeta + \underline{\beta}, \\
 \text{div } \hat{\chi} &= \nabla \text{tr} \chi - \chi \cdot \zeta + \text{tr} \chi \zeta - \beta, \\
 \text{curl } \xi &= \xi \wedge (\eta + \underline{\eta} + 2\zeta), \\
 \text{curl } \underline{\xi} &= \underline{\xi} \wedge (\eta + \underline{\eta} - 2\zeta), \\
 K &= -\frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} + \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} - \rho.
 \end{aligned}$$

We have the following *null Bianchi equations*<sup>3</sup> relating the null connection coefficients and the null curvature components (see [CK93, p. 161]).

$$\begin{aligned}
 \nabla_3 \alpha + \frac{1}{2} \text{tr} \underline{\chi} \alpha &= \nabla \hat{\otimes} \beta + 2\underline{\omega} \alpha - 3(\hat{\chi} \rho + * \hat{\chi} \sigma) + (\zeta + 4\underline{\eta}) \hat{\otimes} \beta, \\
 \nabla_4 \beta + 2 \text{tr} \chi \beta &= \text{div } \alpha - 2\underline{\omega} \beta + (2\zeta + \underline{\eta}) \cdot \alpha + 3(\xi \rho + * \xi \sigma), \\
 \nabla_3 \beta + \text{tr} \underline{\chi} \beta &= \nabla \rho + * \nabla \sigma + 2\underline{\omega} \beta + \underline{\xi} \cdot \alpha + 3(\eta \rho + * \eta \sigma), \\
 \nabla_4 \rho + \frac{3}{2} \text{tr} \chi \rho &= \text{div } \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2(\underline{\eta} \cdot \beta - \xi \cdot \underline{\beta}), \\
 \nabla_3 \rho + \frac{3}{2} \text{tr} \underline{\chi} \rho &= -\text{div } \underline{\beta} - \frac{1}{2} \hat{\chi} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} + 2(\underline{\xi} \cdot \underline{\beta} - \eta \cdot \underline{\beta}), \\
 \nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma &= -\text{curl } \beta + \frac{1}{2} \hat{\chi} \cdot * \alpha - \zeta \cdot * \beta - 2(\underline{\eta} \cdot * \beta + 2\underline{\xi} \cdot * \underline{\beta}), \\
 \nabla_3 \sigma + \frac{3}{2} \text{tr} \underline{\chi} \sigma &= -\text{curl } \underline{\beta} - \frac{1}{2} \hat{\chi} \cdot * \underline{\alpha} + \zeta \cdot * \underline{\beta} - 2(\underline{\eta} \cdot * \underline{\beta} + \eta \cdot * \underline{\beta}), \\
 \nabla_4 \underline{\beta} + \text{tr} \chi \underline{\beta} &= -\nabla \rho + * \nabla \sigma + 2\underline{\hat{\chi}} \cdot \beta + 2\underline{\omega} \underline{\beta} - \xi \cdot \underline{\alpha} - 3(\eta \rho - * \eta \sigma), \\
 \nabla_3 \underline{\beta} + 2 \text{tr} \underline{\chi} \underline{\beta} &= -\text{div } \underline{\alpha} - 2\underline{\omega} \underline{\beta} - (-2\zeta + \underline{\eta}) \cdot \underline{\alpha} + 3(-\underline{\xi} \rho + * \underline{\xi} \sigma), \\
 \nabla_4 \underline{\alpha} + \frac{1}{2} \text{tr} \chi \underline{\alpha} &= -\nabla \hat{\otimes} \underline{\beta} + 4\underline{\omega} \underline{\alpha} - 3(\underline{\hat{\chi}} \rho - * \underline{\hat{\chi}} \sigma) + (\zeta - 4\underline{\eta}) \hat{\otimes} \underline{\beta}.
 \end{aligned}$$

Here  $\mathcal{L}_{e_3}, \mathcal{L}_{e_4}$  and  $\nabla_3, \nabla_4$  are tangential projections of the Lie derivative and the covariant derivative along  $e_3, e_4$  respectively,  $\nabla$  is the induced covariant derivative on the 2-spheres  $S_{u, \underline{u}}$ , the operators  $\nabla \hat{\otimes}, \text{div}, \text{curl}$  are build upon  $\nabla$  by standard tensorial constructions and the symbols  $\hat{\otimes}, |\cdot|, \cdot, \wedge, *$  denote standard tensorial operations with respect to the metric  $g$ . We refer to [CK93, Chapter 2] for precise definitions.

### 2.2.2 Null gauges

The null decompositions are particularly adapted to the choice of *null gauges*. That is, when one or both of the functions  $u, \underline{u}$  are assumed to be *optical*, *i.e.*

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) = 0, \quad \text{or} \quad \mathbf{g}(\mathbf{D}\underline{u}, \mathbf{D}\underline{u}) = 0, \quad (2.8)$$

in which case their level sets are *null hypersurfaces* of  $\mathcal{M}$ . In that case, the null pair  $(e_3, e_4)$  is chosen to be orthogonal to the foliation  $S_{u, \underline{u}}$  and such that  $e_3$  (or  $e_4$ ) is colinear to the gradient  $\mathbf{D}\underline{u}$  (or  $\mathbf{D}u$  respectively). The optical condition (2.8) implies a series of relations which precise the null structure equations from Section 2.2.1. The resulting equations are combination of transport equations in the  $e_3, e_4$  direction or elliptic equations on the 2-spheres.

Null gauges are a powerful tool to capture propagation features of the Einstein equations (see [Chr09] or [LR15, LR17] for particularly clear instances of that statement). In the context of the global stability results [CK93, KN03, KS17] (and in many other works), it is crucially used to obtain precise decay rates. In the context of the low regularity results [KRS15] and [Sze12a] – [Sze16] it is crucially used to construct a parametrix and obtain precise bilinear estimates.

We review some of the main null gauge choices used in the literature.

<sup>3</sup>As the Maxwell equations of Section 2.1.2, the null Bianchi equations are projections of the spacetime Bianchi equations. These last equations are discussed in Section 3.

- ◇ The *double null gauge* is the assumption that  $u, \underline{u}$  are both optical. This choice is typically done together with a choice of two shift-free transported spherical coordinates  $\vartheta, \varphi$ . It is made in a large number of work in general relativity, from which we only cite the seminals [KN03] and [Chr09]. It is adapted to the characteristic Cauchy problem, where initial data are posed on two transversely intersecting null hypersurfaces (see discussions in Section 1.3).
- ◇ The *maximal-null gauge* is the assumption that  $u$  is optical, together with the assumption that

$$\underline{u} = 2t - u,$$

where  $t$  is a *maximal* time function. This choice is made in [CK93] and [KRS15] and [Sze12a] – [Sze16] and combines the above mentioned features of the null and maximal gauges.

- ◇ The *geodesic-null gauge* is the assumption that  $u$  is optical, together with the assumption that  $\underline{u}$  is a *geodesic affine parameter* of the null hypersurfaces level sets of  $u$ . We postpone that last definition to the next Section 2.3. This gauge choice is made in [KS17] and in Part IV, in which it is used to deal with specific geometric situations (see the introduction of Part IV).

## 2.3 The geodesic and canonical foliations of null hypersurfaces

In this section, we present the *geodesic* and *canonical* foliation by 2-spheres of a null hypersurface (similar definitions hold in the spacelike case). These (gauges/initialisation of gauges) choices are used in Parts II and III and in Part IV of this thesis.

Let  $\mathcal{H}$  be a null hypersurface of  $\mathcal{M}$ , which we assume is emanating from a 2-sphere  $S$  (similar definitions hold in the case of a null hypersurface emanating from a point). Let  $L$  be a null geodesic generator of  $\mathcal{H}$ , *i.e.* such that  $L$  is everywhere null and tangent to  $\mathcal{H}$  and satisfies the following *geodesic equation*

$$\mathbf{D}_L L = 0.$$

To a fixed null geodesic generator, we associate an *affine geodesic parameter*  $s$  on  $\mathcal{H}$ , by

$$Ls = 1, \quad s|_S = 1.$$

We call  $(S'_s)$  the *geodesic foliation* on  $\mathcal{H}$ , where  $S'_s$  are the level sets of  $s$ . Note that this foliation depends on the choice of null geodesic generator  $L$  on  $S$  and on the choice of  $S$ .

Let  $v$  be a general function on  $\mathcal{H}$  such that its level sets  $S_v$  define a (local) foliation of  $\mathcal{H}$  by 2-spheres. We define the *null lapse*  $\Omega$  of the foliation  $(S_v)$  by

$$\Omega := Lv.$$

To a general foliation  $(S_v)$ , we associate the following *null pair*  $(e_3, e_4)$  by

$$e_4 = L, \quad \mathbf{g}(e_3, e_4) = -2, \quad \mathbf{g}(e_3, e_3) = 0, \quad e_3 \text{ is orthogonal to } S_v.$$

To describe the geometry of the foliation  $(S_v)$ , we use the null connection coefficients from Section 2.2.1, which make sense in the context of a null pair (only) defined on  $\mathcal{H}$  (and not on  $\mathcal{M}$ ), that is we consider

$$\chi, \underline{\chi}, \zeta, \eta, \xi, \omega.$$

We first note that, for a general foliation of 2-spheres on  $\mathcal{H}$ , we have

$$\xi = 0, \quad \omega = 0.$$

Moreover, one can prove that

$$\zeta = \eta - \nabla \log \Omega. \tag{2.9}$$

With these definitions, the *geodesic foliation* choice, writes

$$\Omega = 1.$$

In several situations, one rather perform a *canonical foliation* choice,<sup>4</sup> which is the prescription of the following elliptic equation for  $\Omega$

$$\begin{aligned} \Delta (\log \Omega) &= -\text{div} \zeta - \rho + \bar{\rho} + \text{lower order terms}, \\ \int_{S_v} \log \Omega &= 0, \end{aligned} \tag{2.10}$$

where  $\bar{\rho}$  is the mean value on  $S_v$  of  $\rho$ .

The motivation for that choice is that it cancels the right-hand side of the null structure transport equation for the transverse expansion  $\text{tr}\underline{\chi}$  which reads in general

$$e_4(\text{tr}\underline{\chi}) + \frac{1}{2}\text{tr}\underline{\chi}\text{tr}\underline{\chi} = -2\text{div} \zeta - 2\Delta (\log \Omega) + 2\rho + \text{lower order terms}. \tag{2.11}$$

In the geodesic case, the above equation writes

$$e_4(\text{tr}\underline{\chi}) + \frac{1}{2}\text{tr}\underline{\chi}\text{tr}\underline{\chi} = -2\text{div} \zeta + 2\rho + \text{lower order terms}, \tag{2.12}$$

while in the canonical foliation case

$$e_4(\text{tr}\underline{\chi}) + \frac{1}{2}\text{tr}\underline{\chi}\text{tr}\underline{\chi} = 2\bar{\rho} + \text{lower order terms}. \tag{2.13}$$

We note that tangential derivatives of the right-hand side of (2.13) are only composed of lower order terms, which is not the case for (2.12). This allows for an improved control of the transverse expansion  $\text{tr}\underline{\chi}$  and subsequently the whole transverse second fundamental form  $\underline{\chi}$  on  $\mathcal{H}$  in the canonical foliation case.

In many situations, this improved regularity is necessary to obtain a sufficient control of hypersurfaces transversely emanating from the foliation  $S_v$ . See [CK93, KN03, Nic04, Sze12a] and Parts II, III and IV.

**Remark 2.5.** *Elliptic equation (2.10) together with the lapse definition  $\frac{dv}{ds} = Lv = \Omega$ , forms a system of transport-elliptic equation provided that a background geodesic foliation is given (in that case the terms in the right-hand side of the Poisson equation (2.10) are considered as source terms). For such a system, local existence holds by a Banach-Picard fixed point argument, and thus the canonical foliation choice can be realised.*

---

<sup>4</sup>The denomination canonical is used in [CK93, KN03] in the context of a last slice initialisation in a bootstrap argument. It usually denotes foliations prescribed by elliptic equations. The terminology can be misleading since these foliations still consist in a (gauge) choice.



### 3. Bianchi equations and energy estimates

Einstein vacuum equations (1.1) together with the once contracted Bianchi identities yield the following *Bianchi equations* for the spacetime curvature tensor  $\mathbf{R}$

$$\mathbf{D}^\alpha \mathbf{R}_{\alpha\beta\gamma\delta} = 0. \quad (3.1)$$

These equations are of Maxwell-type.<sup>1</sup> Using (3.1), one can in particular prove *energy estimates* for the spacetime curvature tensor  $\mathbf{R}$ .

In this section, we introduce the general setup to perform these estimates in the general case of the spacetime equations (3.1). Most of the material is taken from [CK93, Chapter 7]. We then detail an example of energy estimate which can be performed using this setup. Last, we discuss the notion of *conformal Killing vectorfield* and its application to energy estimates.

#### 3.1 Weyl fields and Bel-Robinson tensors

We follow the treatment of [CK93, Chapter 7]. In this section, as well as in the next sections, we shall assume that  $\mathcal{M}$  is orientable and time-orientable. We note  $\in$  an associated spacetime volume form.

We say that a 4-tensor  $\mathbf{W}$  is a *Weyl tensor* if it has the same symmetries as the Riemann curvature tensor and is  $\mathbf{g}$ -tracefree, that is,

$$\begin{aligned} \mathbf{W}_{\alpha\beta\gamma\delta} &= -\mathbf{W}_{\beta\alpha\gamma\delta} = -\mathbf{W}_{\alpha\beta\delta\gamma}, & \mathbf{W}_{\alpha\beta\gamma\delta} &= \mathbf{W}_{\gamma\delta\alpha\beta}, \\ \mathbf{W}_{\alpha\beta\gamma\delta} + \mathbf{W}_{\alpha\gamma\delta\beta} + \mathbf{W}_{\alpha\delta\beta\gamma} &= 0, & \mathbf{W}_{\beta\delta} &:= \mathbf{W}^\alpha_{\beta\alpha\delta} = 0. \end{aligned}$$

We note that the spacetime curvature tensor  $\mathbf{R}$  of a vacuum spacetime is the prime example of a Weyl tensor.

We define the *Bel-Robinson tensor* of a Weyl tensor  $\mathbf{W}$  to be

$$Q(\mathbf{W})_{\alpha\beta\gamma\delta} := \mathbf{W}_{\alpha\nu\gamma\mu} \mathbf{W}_{\beta\delta}^{\nu\mu} + {}^* \mathbf{W}_{\alpha\nu\gamma\mu} {}^* \mathbf{W}_{\beta\delta}^{\nu\mu},$$

where the left dual  ${}^* \mathbf{W}$  of a Weyl tensor  $\mathbf{W}$  is defined by

$${}^* \mathbf{W}_{\alpha\beta\gamma\delta} := \frac{1}{2} \in_{\alpha\beta\mu\nu} \mathbf{W}^{\mu\nu}_{\gamma\delta}.$$

We define the *current*  $J(\mathbf{W})$  of a Weyl field to be the 3-tensor

$$J(\mathbf{W})_{\beta\gamma\delta} := \mathbf{D}^\alpha \mathbf{W}_{\alpha\beta\gamma\delta},$$

and we note that the Bianchi equations (3.1) imply  $J(\mathbf{R}) = 0$ .

We have the following equation for the spacetime divergence of a Bel-Robinson tensor

$$\begin{aligned} \mathbf{D}^\alpha Q(\mathbf{W})_{\alpha\beta\gamma\delta} &= J(\mathbf{W})_{\mu\gamma\nu} \mathbf{W}_{\beta\delta}^{\mu\nu} + J(\mathbf{W})_{\mu\delta\nu} \mathbf{W}_{\beta\gamma}^{\mu\nu} \\ &+ J^*(\mathbf{W})_{\mu\gamma\nu} {}^* \mathbf{W}_{\beta\delta}^{\mu\nu} + J^*(\mathbf{W})_{\mu\delta\nu} {}^* \mathbf{W}_{\beta\gamma}^{\mu\nu}, \end{aligned} \quad (3.3)$$

where

$$J^*_{\beta\gamma\delta} = \frac{1}{2} J_{\beta\mu\nu} \in^{\mu\nu}_{\gamma\delta},$$

and we note that this in the case of  $\mathbf{W} = \mathbf{R}$  implies

$$\mathbf{D}^\alpha Q(\mathbf{R})_{\alpha\beta\gamma\delta} = 0. \quad (3.4)$$

The Bel-Robinson tensor satisfies the following *coercivity properties*:

<sup>1</sup>See for example the projected equations (2.5f) in the 1 + 3 context of Section 2.1.

◇ for a unit timelike vectorfield  $T$ , we have

$$Q(\mathbf{W})(T, T, T, T) \simeq |E(\mathbf{W})|^2 + |H(\mathbf{W})|^2,$$

where the electric-magnetic decomposition  $E, H$  of  $\mathbf{W}$  is defined as in Section 2.1.2,

◇ for a null pair  $(e_3, e_4)$  we have

$$\begin{aligned} Q(\mathbf{W})(e_3 + e_4, e_3 + e_4, e_3 + e_4, e_3) &\simeq |\underline{\alpha}(\mathbf{W})|^2 + |\underline{\beta}(\mathbf{W})|^2 + \rho(\mathbf{W})^2 + \sigma(\mathbf{W})^2 + |\beta(\mathbf{W})|^2, \\ Q(\mathbf{W})(e_3 + e_4, e_3 + e_4, e_3 + e_4, e_4) &\simeq |\underline{\beta}(\mathbf{W})|^2 + \rho(\mathbf{W})^2 + \sigma(\mathbf{W})^2 + |\beta(\mathbf{W})|^2 + |\alpha(\mathbf{W})|^2, \end{aligned}$$

where the null decomposition  $\underline{\alpha}, \dots, \alpha$  of  $\mathbf{W}$  is defined as in Section 2.2.1.

The above properties make the Bel-Robinson the analogue for a Weyl field of a standard energy-momentum tensor. From the expression of its spacetime divergence in terms of the sources of the Weyl field (which vanish in the case of the spacetime curvature tensor) we obtain (boundary and spacetime) integral identities *via* a Stokes formula. From the above coercivity properties, the produced boundary terms are energy fluxes controlling the field.

In the next section, we give a statement of Stokes theorem. We then give an example of energy estimates that can be obtained using the tools from these two sections.

## 3.2 Stokes theorem

For the purpose of this section, we rewrite the volume form  $\in$  of  $\mathcal{M}$  under the form

$$\text{dvol}_{\mathcal{M}} := \in.$$

For all  $k$ -form on  $\mathcal{M}$ , we denote by  $i$  the contraction, defined by

$$i_P \omega := \omega(P, \cdot, \cdot, \cdot),$$

where  $P \in \text{T}\mathcal{M}$ .

We have the following theorem.

**Theorem 3.1** (Stokes theorem). *Let  $\mathcal{D}$  be a subdomain of  $\mathcal{M}$ . Let  $\omega$  be a 3-form on  $\mathcal{D}$ . We have*

$$\int_{\mathcal{D}} \text{d}\omega = \int_{\partial\mathcal{D}} \omega,$$

where the boundary  $\partial\mathcal{D}$  is oriented such that the 3-form  $i_{N_{\partial\mathcal{D}}} \text{dvol}_{\mathcal{M}}$  is a positive volume form of  $\partial\mathcal{D}$ , where  $N_{\partial\mathcal{D}}$  is an outgoing vectorfield to  $\partial\mathcal{D}$ .

We have the following corollary.

**Corollary 3.2.** *For all vectorfield  $P \in \text{T}\mathcal{M}$ , we have*

$$\int_{\mathcal{D}} (\text{div} P) \text{dvol}_{\mathcal{M}} = \int_{\partial\mathcal{D}} i_P \text{dvol}_{\mathcal{M}},$$

where  $\text{div} P := \mathbf{D}^\alpha P_\alpha$ .

*Proof.* This follows from the formula (which is easily checked in local normal coordinates)

$$\text{d}(i_P \text{dvol}_{\mathcal{M}}) = (\text{div} P) \text{dvol}_{\mathcal{M}},$$

and Stokes theorem. □

### 3.3 An example of energy estimate

Let  $\Sigma_1$  and  $\Sigma_2$  be two spacelike hypersurfaces and  $\mathcal{H}_1, \mathcal{H}_2$  be two null hypersurfaces enclosing a spacetime domain  $\mathcal{D}$ , as shown in Figure 3.1.

Let  $T$  be a future-pointing timelike vectorfield in  $\mathcal{D}$ , which for simplicity we will assume to be unitary and normal to the spacelike hypersurfaces  $\Sigma_1$  and  $\Sigma_2$ .

We use  $T$  as *multiplier vectorfield* for the Bel-Robinson tensor of  $\mathbf{R}$ , and we form the following 1-tensor

$$P := Q(\mathbf{R})(T, T, T, \cdot).$$

Using formula (3.4) and the symmetries of the Bel-Robinson tensor (see [CK93, Chapter 7]) we have

$$\mathbf{div} P = \frac{3}{2} Q(\mathbf{R})_{\alpha\beta\gamma\delta} {}^{(T)}\hat{\pi}^{\alpha\beta} T^\gamma T^\delta,$$

where

$${}^{(T)}\hat{\pi}^{\alpha\beta} := \mathbf{D}^\alpha T^\beta + \mathbf{D}^\beta T^\alpha - \frac{1}{2} (\mathbf{D}^\alpha T_\alpha) \mathbf{g}^{\alpha\beta}.$$

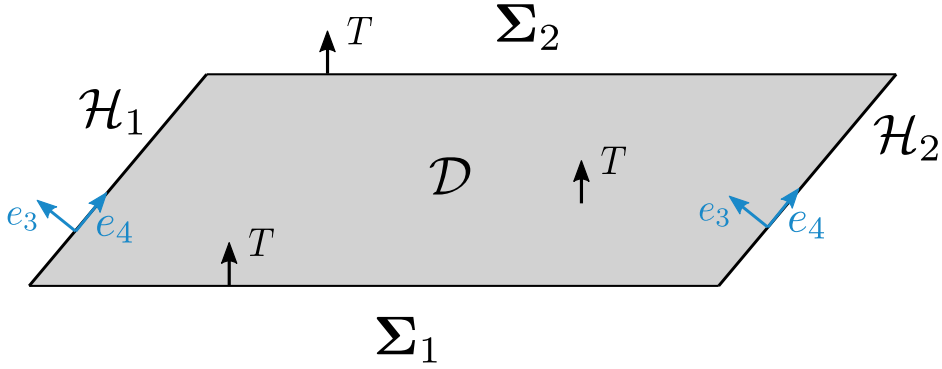


Figure 3.1: The spacetime domain  $\mathcal{D}$ .

From Corollary 3.2, we have

$$\frac{3}{2} \int_{\mathcal{D}} Q(\mathbf{R})_{\alpha\beta\gamma\delta} {}^{(T)}\hat{\pi}^{\alpha\beta} T^\gamma T^\delta \, \text{dvol}_{\mathcal{M}} = \int_{\partial\mathcal{D}} i_P \, \text{dvol}_{\mathcal{M}}, \quad (3.5)$$

where we recall that  $\partial\mathcal{D}$  is oriented such that  $i_N \, \text{dvol}_{\mathcal{M}}$  is a positive volume form of  $\partial\mathcal{D}$  for any outgoing vectorfield  $N$  to  $\partial\mathcal{D}$ .

Our goal in the rest of this section is to express the boundary terms of (3.5) in terms of contractions of the Bel-Robinson tensor and standard integrals on the hypersurfaces  $\Sigma$  and  $\mathcal{H}$ .

First, let denote by  $+$  the orientation of these hypersurfaces such that  $i_T \, \text{dvol}_{\mathcal{M}}$  is a positive volume form (*i.e.* the orientation given by the time-orientation of  $\mathcal{M}$ ), and by  $-$  the reverse orientation.

The boundary term of (3.5) write as (see Figure 3.1)

$$\int_{\partial\mathcal{D}} = \int_{\Sigma_1^-} + \int_{\Sigma_2^+} + \int_{\mathcal{H}_1^+} + \int_{\mathcal{H}_2^-}.$$

We define the canonical volume form of  $\Sigma_1, \Sigma_2$  by

$$\text{dvol}_{\Sigma} := i_T \, \text{dvol}_{\mathcal{M}},$$

which is positive for  $\Sigma_1^+, \Sigma_2^+$ .



Since we have  $P + \mathbf{g}(P, T)T \in T\Sigma$  and reversing the orientation in the case of  $\Sigma_1$ , we deduce on  $\Sigma_1$  and  $\Sigma_2$

$$\begin{aligned} \int_{\Sigma_1^-} i_P \mathrm{dvol}_{\mathcal{M}} &= \int_{\Sigma_1^+} \mathbf{g}(P, T) \mathrm{dvol}_{\Sigma_1}, \\ \int_{\Sigma_2^+} i_P \mathrm{dvol}_{\mathcal{M}} &= - \int_{\Sigma_2^+} \mathbf{g}(P, T) \mathrm{dvol}_{\Sigma_2}. \end{aligned}$$

Assume that a future-oriented null pair  $(e_3, e_4)$  is given on  $\mathcal{H}_1$  and  $\mathcal{H}_2$  (see the definition of Section 2.2.1). Assume moreover that it is adapted to  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , *i.e.* that  $e_3 \in T\mathcal{H}_1$  and  $e_4 \in T\mathcal{H}_2$ . For fixed null pairs, we define the volume forms  $\mathrm{dvol}_{\mathcal{H}_1}$  and  $\mathrm{dvol}_{\mathcal{H}_2}$  of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  to be

$$\mathrm{dvol}_{\mathcal{H}_1} := \frac{1}{2} i_{e_4} \mathrm{dvol}_{\mathcal{M}}, \quad \mathrm{dvol}_{\mathcal{H}_2} := \frac{1}{2} i_{e_3} \mathrm{dvol}_{\mathcal{M}}.$$

**Remark 3.3.** *For spacelike hypersurfaces and for timelike hypersurfaces there exists a canonical volume form – provided that a spacetime orientation and time-orientation is given. For null hypersurfaces, the definition of a volume form depends on a choice of a transverse null vectorfield (which can for example be chosen to be orthogonal to a foliation of the hypersurface).*

Since we have

$$\begin{aligned} P + \frac{1}{2} \mathbf{g}(P, e_3) e_4 &\in T\mathcal{H}_1, \\ P + \frac{1}{2} \mathbf{g}(P, e_4) e_3 &\in T\mathcal{H}_2, \end{aligned}$$

and reversing the orientation in the case of  $\mathcal{H}_2$ , we deduce that

$$\begin{aligned} \int_{\mathcal{H}_1^+} i_P \mathrm{dvol}_{\mathcal{M}} &= - \int_{\mathcal{H}_1^+} \mathbf{g}(P, e_3) \mathrm{dvol}_{\mathcal{H}_1}, \\ \int_{\mathcal{H}_2^-} i_P \mathrm{dvol}_{\mathcal{M}} &= + \int_{\mathcal{H}_2^+} \mathbf{g}(P, e_4) \mathrm{dvol}_{\mathcal{H}_2}. \end{aligned}$$

We now express the contractions of  $P$ :

$$\begin{aligned} \mathbf{g}(P, T) &= Q(\mathbf{R})(T, T, T, T), \\ \mathbf{g}(P, e_3) &= Q(\mathbf{R})(T, T, T, e_3), \\ \mathbf{g}(P, e_4) &= Q(\mathbf{R})(T, T, T, e_4). \end{aligned}$$

This finally gives the following identity

$$\begin{aligned} &\int_{\Sigma_2} Q(\mathbf{R})(T, T, T, T) \mathrm{dvol}_{\Sigma_2} + \int_{\mathcal{H}_1} Q(\mathbf{R})(T, T, T, e_3) \mathrm{dvol}_{\mathcal{H}_1} \\ &= \int_{\Sigma_1} Q(\mathbf{R})(T, T, T, T) \mathrm{dvol}_{\Sigma_1} + \int_{\mathcal{H}_2} Q(\mathbf{R})(T, T, T, e_4) \mathrm{dvol}_{\mathcal{H}_2} - \frac{3}{2} \int_{\mathcal{D}} Q(\mathbf{R})_{\alpha\beta\gamma\delta} {}^{(T)} \hat{\pi}^{\alpha\beta} T^\gamma T^\delta \mathrm{dvol}_{\mathcal{M}}. \end{aligned}$$

where all hypersurfaces are positively oriented.

Assuming that  $T \simeq \frac{1}{2}(e_3 + e_4)$ , and using the coercivity properties of the Bel-Robinson tensor from Section 3.1, this gives the following *energy estimate*

$$\begin{aligned} &\int_{\Sigma_2} (|E|^2 + |H|^2) \mathrm{dvol}_{\Sigma_2} + \int_{\mathcal{H}_1} (|\underline{\alpha}|^2 + |\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2 + |\beta|^2) \mathrm{dvol}_{\mathcal{H}_1} \\ &\simeq \int_{\Sigma_1} (|E|^2 + |H|^2) \mathrm{dvol}_{\Sigma_1} + \int_{\mathcal{H}_2} (|\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2 + |\beta|^2 + |\alpha|^2) \mathrm{dvol}_{\mathcal{H}_2} \\ &\quad - \int_{\mathcal{D}} Q(\mathbf{R})_{\alpha\beta\gamma\delta} {}^{(T)} \hat{\pi}^{\alpha\beta} T^\gamma T^\delta \mathrm{dvol}_{\mathcal{M}}. \end{aligned}$$

**Remark 3.4.** *All volume forms above can also be expressed using coordinates and standard integrals.*

In the next section, we introduce the *conformal Killing vectorfields* and discuss the treatment of the term

$$\int_{\mathcal{D}} Q(\mathbf{R})_{\alpha\beta\gamma\delta} {}^{(T)} \hat{\pi}^{\alpha\beta} T^\gamma T^\delta \mathrm{dvol}_{\mathcal{M}}.$$

We also discuss alternative multiplier/commuting vectorfields.

### 3.4 Conformal Killing vectorfields

In this section, we discuss how to treat the spacetime integral terms obtained in the energy estimates of the last section, introducing the notion of *conformal Killing vectorfield*. We further discuss generalisations of energy estimates to derivatives of the spacetime curvature tensor.

#### 3.4.1 Multiplier vectorfields

Using symmetries of the Bel-Robinson tensor (see [CK93, Chapter 7]), we have the following general formula

$$\begin{aligned} \operatorname{div}(Q(\mathbf{W})(X, Y, Z)) &= \mathbf{D}^\alpha Q(\mathbf{W})_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta + \frac{1}{2} Q(\mathbf{W})_{\alpha\beta\gamma\delta} {}^{(X)}\hat{\pi}^{\alpha\beta} Y^\gamma Z^\delta \\ &\quad + \frac{1}{2} Q(\mathbf{W})_{\alpha\beta\gamma\delta} {}^{(Y)}\hat{\pi}^{\alpha\beta} X^\gamma Z^\delta + \frac{1}{2} Q(\mathbf{W})_{\alpha\beta\gamma\delta} {}^{(Z)}\hat{\pi}^{\alpha\beta} X^\gamma Y^\delta, \end{aligned} \quad (3.6)$$

for  $X, Y, Z \in \mathcal{TM}$ , and where  ${}^{(U)}\hat{\pi}$  is the following *traceless part of the deformation tensor* of the 1-tensor  $U$

$${}^{(U)}\hat{\pi}_{\mu\nu} := \mathbf{D}_\mu U_\nu + \mathbf{D}_\nu U_\mu - \frac{1}{2} (\mathbf{D}^\alpha U_\alpha) \mathbf{g}_{\mu\nu}.$$

We say that  $U$  is a *conformal Killing vectorfield* if

$${}^{(U)}\hat{\pi} = 0.$$

If  $X, Y, Z$  are conformal Killing vectorfields, we have from formula (3.6)

$$\operatorname{div}(Q(\mathbf{W})(X, Y, Z)) = \mathbf{D}^\alpha Q(\mathbf{W})_{\alpha\beta\gamma\delta} X^\beta Y^\gamma Z^\delta.$$

In particular, in the case of  $\mathbf{W} = \mathbf{R}$ , we deduce from the above and (3.4) that

$$\operatorname{div}(Q(\mathbf{R})(X, Y, Z)) = 0.$$

In the case of the energy estimate of Section 3.3, if the timelike vectorfield  $T$  is conformal Killing, we thus have the following energy estimate

$$\begin{aligned} &\int_{\Sigma_2} (|E|^2 + |H|^2) \operatorname{dvol}_{\Sigma_2} + \int_{\mathcal{H}_1} (|\underline{\alpha}|^2 + |\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2 + |\beta|^2) \operatorname{dvol}_{\mathcal{H}_1} \\ &\simeq \int_{\Sigma_1} (|E|^2 + |H|^2) \operatorname{dvol}_{\Sigma_1} + \int_{\mathcal{H}_2} (|\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2 + |\beta|^2 + |\alpha|^2) \operatorname{dvol}_{\mathcal{H}_2}. \end{aligned}$$

Timelike conformal Killing vectorfields are thus generally used as multiplier to perform energy estimates. For *approximate* conformal Killing vectorfields, we obtain energy estimates with a spacetime integrated *error term*, corresponding to the fact that  $\hat{\pi} \simeq 0$  but does not vanish.

In the case of Minkowski space  $(\mathbb{R}^4, \boldsymbol{\eta})$ , we mention the following timelike Killing vectorfields which are used as multiplier in energy estimates

- ◊ the *time translation Killing vectorfield*  $T := \partial_t$ ,
- ◊ the *conformal Morawetz Killing vectorfield*  $K := (t^2 + r^2)\partial_t + 2tr\partial_r$ .

We refer the reader to [CK93, Bie10] for instances of energy estimates performed using (approximation of) these vectorfields as multipliers, and to [CK90] for the full set of conformal Killing vectorfields of Minkowski space. In Part II, we use approximate time translation Killing vectorfields as multipliers to perform local energy estimates for  $\mathbf{R}$ . In Part IV, we use both approximate time translation and Morawetz conformal Killing vectorfields as multipliers to perform global energy estimates.

**Remark 3.5.** *In the case of the Morawetz multiplier  $K$ , the weights in  $t, r$  produce the boundedness of weighted  $L^2$ -fluxes for the spacetime curvature tensor  $\mathbf{R}$ . This can be turned into (pointwise) decay for  $\mathbf{R}$  via Sobolev and Klainerman-Sobolev estimates.<sup>2</sup>*

<sup>2</sup>See [CK90] for an introduction to Klainerman-Sobolev estimates in that setting.

### 3.4.2 Commuting vectorfields

For a Weyl field  $\mathbf{W}$  and a vectorfield  $X$ , we define the *modified Lie derivative*, which takes Weyl tensors into Weyl tensors, by

$$\hat{\mathcal{L}}_X \mathbf{W} := \mathcal{L}_X \mathbf{W} - \frac{1}{2} {}^{(X)}[\mathbf{W}] + \frac{3}{8} \text{tr}_{\mathbf{g}} \pi \mathbf{W},$$

where  $\pi := \mathcal{L}_X \mathbf{g}$  and

$${}^{(X)}[\mathbf{W}]_{\alpha\beta\ \gamma\delta} := \pi^\mu{}_\alpha \mathbf{W}_{\mu\beta\gamma\delta} + \pi^\mu{}_\beta \mathbf{W}_{\alpha\mu\gamma\delta} + \pi^\mu{}_\gamma \mathbf{W}_{\alpha\beta\ \mu\delta} + \pi^\mu{}_\delta \mathbf{W}_{\alpha\beta\gamma\ \mu}.$$

We have the following formula (see [CK93, Proposition 7.1.2]) for the current of  $\hat{\mathcal{L}}_X \mathbf{W}$

$$\begin{aligned} J(\hat{\mathcal{L}}_X \mathbf{W})_{\beta\gamma\delta} &= \hat{\mathcal{L}}_X J_{\beta\gamma\delta} + \frac{1}{2} \hat{\pi}^{\mu\nu} \mathbf{D}_\nu \mathbf{W}_{\mu\beta\gamma\delta} + \frac{1}{2} \mathbf{D}^\alpha \hat{\pi}_{\alpha\lambda} \mathbf{W}_{\beta\gamma\delta}^\lambda \\ &\quad + \frac{1}{2} (\mathbf{D}_\beta \hat{\pi}_{\alpha\lambda} - \mathbf{D}_\lambda \hat{\pi}_{\alpha\beta}) \mathbf{W}_{\gamma\delta}^{\alpha\lambda} \\ &\quad + \frac{1}{2} (\mathbf{D}_\gamma \hat{\pi}_{\alpha\lambda} - \mathbf{D}_\lambda \hat{\pi}_{\alpha\gamma}) \mathbf{W}_{\beta\delta}^{\alpha\lambda} \\ &\quad + \frac{1}{2} (\mathbf{D}_\delta \hat{\pi}_{\alpha\lambda} - \mathbf{D}_\lambda \hat{\pi}_{\alpha\delta}) \mathbf{W}_{\beta\gamma}^{\alpha\lambda}, \end{aligned} \tag{3.7}$$

where  $\hat{\pi}$  denotes the traceless deformation tensor of  $X$ , and where

$$\hat{\mathcal{L}}_X J_{\beta\gamma\delta} := \mathcal{L}_X J_{\delta\gamma\delta} - \frac{1}{2} (\hat{\pi}_\delta^\mu J_{\mu\gamma\delta} + \hat{\pi}_\gamma^\mu J_{\beta\mu\delta} + \hat{\pi}_\delta^\mu J_{\beta\gamma\mu}) + \frac{1}{8} \text{tr}_{\mathbf{g}} \pi J_{\beta\gamma\delta}.$$

The above notion of derivation is used together with the Bel-Robinson tensor to commute Bianchi equations (3.1) and prove energy estimates for derivatives of the curvature tensor  $\mathbf{R}$ . This is done by considering the following *commuted and contracted* Bel-Robinson tensor

$$Q(\hat{\mathcal{L}}_X \mathbf{R})(Y_1, Y_2, Y_3, \cdot),$$

to which we apply Stokes formula from Corollary 3.2.

From formula (3.7), we deduce that if  $X, Y_1, Y_2, Y_3$  are conformal Killing vectorfields, we have

$$\text{div} \left( Q(\hat{\mathcal{L}}_X \mathbf{R})(Y_1, Y_2, Y_3, \cdot) \right) = 0,$$

and we can thus obtain energy estimates for the commuted spacetime curvature tensor  $\hat{\mathcal{L}}_X \mathbf{R}$ .

In Minkowski space  $(\mathbb{R}^4, \boldsymbol{\eta})$ , we mention the following conformal Killing vectorfields

- ◇ the four *spacetime translation Killing vectorfields*  $\partial_0, \partial_1, \partial_2, \partial_3$ ,
- ◇ the *scaling conformal Killing vectorfield*  $t\partial_t + r\partial_r$ ,
- ◇ the three *rotation Killing vectorfields*  $x^1\partial_2 - x^2\partial_1, x^2\partial_3 - x^3\partial_2$  and  $x^3\partial_1 - x^1\partial_3$ .

In Part II, we use approximate timelike translation  $T$  as commuting vectorfield to obtain energy estimates for derivatives of  $\mathbf{R}$ . In Part IV, we use approximate spacetime translations, scaling and rotation conformal Killing vectorfields, to control derivatives of  $\mathbf{R}$ .

# 4. The bounded $L^2$ curvature theorem and the stability of Minkowski space

The main results of this thesis are a bounded  $L^2$  curvature theorem (see Parts II and III) and a global nonlinear stability of Minkowski space theorem (see Part IV) for initial data posed on a characteristic hypersurface. Their final statements are adaptations to the characteristic setting of the statements of the classical bounded  $L^2$  curvature theorem [KRS15] and of the global nonlinear stability of Minkowski space theorem [CK93].

Our proofs in Parts II and III and in Part IV are based on new geometric constructions which are designed so that the results of [KRS15] and [CK93] can respectively either be directly applied *as a black box*, or re-obtained under some modifications.

In this section, we provide precised statements to the bounded  $L^2$  curvature theorem [KRS15] and to the global nonlinear stability of Minkowski space theorem [CK93]. Since the basic scheme of proof in both Parts II, III and Part IV follow the general features of the seminal proof in [CK93], we also give an overview of that proof. It uses the spacetime decompositions and energy estimates which are reviewed in Sections 2 and 3.

## 4.1 The bounded $L^2$ curvature theorem

We have the following precised version of the bounded  $L^2$  curvature theorem of [KRS15] and [Sze12a]–[Sze16].

**Theorem 4.1** (The bounded  $L^2$  curvature theorem [KRS15], precised version). *Let  $(\Sigma, g, k)$  be smooth Cauchy data such that*

- ◇  $\Sigma$  is a maximal hypersurface, diffeomorphic to  $\mathbb{R}^3$ ,
- ◇ the following  $L^2$  bounds for the curvature hold on  $\Sigma$

$$\|\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{\leq 1} k\|_{L^2(\Sigma)} \leq \varepsilon. \quad (4.1)$$

*Then, there exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$  the following holds for the smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  of  $(\Sigma, g, k)$ .*

- ◇ *There exists a smooth time function  $t$  ranging from 0 to 1, locally foliating  $\mathcal{M}$  by spacelike maximal hypersurface  $\Sigma_t$  such that  $\Sigma_0 = \Sigma$ .*
- ◇ *The following bounds hold on all hypersurfaces  $\Sigma_t$*

$$\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^{\leq 1} k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad (4.2)$$

$$\|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} \lesssim \varepsilon, \quad (4.3)$$

*where we recall that  $\text{Ric}$  and  $k$  denote the intrinsic Ricci tensor and second fundamental form of  $\Sigma_t \subset \mathcal{M}$  and where  $n$  denotes the time lapse of the foliation  $(\Sigma_t)$ .*

### Remarks on Theorem 4.1

4.1a Theorem 4.1 is a small-data time 1 result (this is ensured by the bound (4.3) on the time lapse) that can be turned into a large-data small-time result by a rescaling argument (see also the rough version of Theorem 1.6 and [KRS15, Theorem 2.2]).

4.1b The  $L^2$  bounds of (4.1) and (4.2) are equivalent to  $L^2$  bounds for the full spacetime curvature tensor  $\mathbf{R}$  of  $(\mathcal{M}, \mathbf{g})$ , or alternatively to  $L^2$  bounds for the electric-magnetic decomposition  $E, H$  of  $\mathbf{R}$ . This can be seen using the Einstein equations in maximal gauge from Section 2.1.2. These bounds correspond to bounds on the boundary fluxes naturally arising from energy estimates for Bianchi equations for  $\mathbf{R}$ , see Section 3.

4.1c The proof of Theorem 4.1 is based on bilinear and trilinear estimates which crucially rely on a plane wave representation formula for the wave equation on low regularity spacetimes developed in [Sze12a] – [Sze16]. This plane wave representation formula is constructed as a Fourier integral operator which necessitates the assumption  $\Sigma \simeq \mathbb{R}^3$ .

## 4.2 The global nonlinear stability of Minkowski space

We have the following precised version of the global nonlinear stability result of [CK93].

**Theorem 4.2** (Stability of Minkowski space [CK93], precised version). *Let  $(\Sigma, g, k)$  be Cauchy data such that:*

- ◇  $\Sigma$  is maximal, diffeomorphic to  $\mathbb{R}^3$ ,
- ◇  $\Sigma$  is asymptotically flat, i.e. there exists coordinates  $(x^1, x^2, x^3)$  in a neighbourhood of infinity such that

$$(r\partial)^{\leq 4} \left( g_{ij} - \left( 1 + \frac{2M}{r} \right) \delta_{ij} \right) = O(r^{-3/2}), \quad (4.4a)$$

when  $r \rightarrow \infty$  and where here  $r := \sqrt{\sum_{i=1}^3 (x^i)^2}$  and  $M \geq 0$ , and we have the following sup-norm bound for the curvature of  $(\Sigma, g)$

$$\|(1+d)^3 \text{Ric}\|_{L^\infty(\Sigma)} \leq \varepsilon, \quad (4.4b)$$

where  $\text{Ric}$  denotes the Ricci tensor of the metric  $g$  and  $d$  denotes the geodesic distance to a fixed point of  $\Sigma$ ,

- ◇ the following bounds hold for curvature  $L^2$ -fluxes through  $\Sigma$

$$\left\| (1+d) \left( (1+d)\nabla \right)^{\leq 3} k \right\|_{L^2(\Sigma)} + \left\| (1+d)^3 \left( (1+d)\nabla \right)^{\leq 1} \mathbf{B} \right\|_{L^2(\Sigma)} \leq \varepsilon, \quad (4.4c)$$

where  $\mathbf{B} := \text{curl}_g \left( \text{Ric} - \frac{1}{3} \mathbf{R}g \right)$ .

Then, there exists  $\varepsilon_0 > 0$ , such that if  $\varepsilon < \varepsilon_0$ , the following holds for the maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  of  $(\Sigma, g, k)$ .

- ◇  $(\mathcal{M}, \mathbf{g})$  is geodesically complete.
- ◇ There exists a global time function  $t$  on  $\mathcal{M}$  ranging from  $-\infty$  to  $+\infty$  which foliates  $\mathcal{M}$  by maximal spacelike hypersurfaces  $\Sigma_t$  such that  $\Sigma_0 = \Sigma$ .
- ◇ There exists a future exterior region<sup>1</sup>  $\mathcal{M}^{\text{ext}}$  foliated by outgoing null hypersurfaces  $\mathcal{C}_u$  level sets of a global optical function  $u$  ranging from  $-\infty$  to  $+\infty$  on  $\mathcal{M}^{\text{ext}}$ , and a past exterior region with symmetric constructions.
- ◇ We have the following decay in the interior region  $\mathcal{M}^{\text{int}} := \mathcal{M} \setminus \mathcal{M}^{\text{ext}}$  of the spacetime curvature tensor  $\mathbf{R}$  of  $\mathbf{g}$

$$|E(\mathbf{R})| + |H(\mathbf{R})| \lesssim \varepsilon t^{-7/2}. \quad (4.5a)$$

- ◇ We have the following differentiated decay in the exterior region  $\mathcal{M}^{\text{ext}}$  of the spacetime curvature tensor  $\mathbf{R}$  according to its null decomposition<sup>2</sup>

$$\begin{aligned} |\alpha(\mathbf{R})| &\lesssim \varepsilon r^{-7/2}, & |\beta(\mathbf{R})| &\lesssim \varepsilon r^{-7/2}, & |\rho(\mathbf{R})| &\lesssim \varepsilon r^{-3}, \\ |\underline{\alpha}(\mathbf{R})| &\lesssim \varepsilon r^{-1} u^{-5/2}, & |\underline{\beta}(\mathbf{R})| &\lesssim \varepsilon r^{-2} u^{-3/2}, & |\sigma(\mathbf{R})| &\lesssim \varepsilon r^{-3} u^{-1/2}, \end{aligned} \quad (4.5b)$$

where here  $r := t - u$ .

- ◇ The induced metric and connection coefficients adapted to the maximal foliation  $\Sigma_t$  and maximal-null foliation  $\Sigma_t$  and  $\mathcal{C}_u$  satisfy decay statements consistent with (4.5a) and (4.5b).
- ◇ The spacetime  $(\mathcal{M}, \mathbf{g})$  admits a past/future timelike, past/future null and spacelike infinities  $i^-, i^+$ ,  $\mathcal{I}^-, \mathcal{I}^+$  and  $i_0$  on which one can make sense of asymptotic quantities and their evolution equations.

<sup>1</sup>The future exterior region is of the type  $\{u \leq ct\}$ .

<sup>2</sup>The null pair is adapted to the 2-spheres intersections of the maximal-null foliation of  $\mathcal{M}^{\text{ext}}$  by  $\Sigma_t$  and  $\mathcal{C}_u$ .

### Remarks on Theorem 4.2

- 4.2a A first stability result for initial data with stronger decay assumptions was obtained in [Fri83]. A global stability result has been obtained in [Bie10, BZ09] using the same general techniques as in [CK93] but under relaxed assumptions for both the regularity and decay of the initial data. A global stability result for Minkowski space has also been obtained using wave coordinates, see [LR10].
- 4.2b The proof of Theorem 4.2 relies on the *vectorfield method* wrapped into an elaborate *bootstrap argument*. It also requires the constructions of appropriate spacetime decompositions. In the next section, we discuss these main features and give an overview of the proof of Theorem 4.2. Bootstrap arguments underlie all the proofs of the main theorems of this thesis. The vectorfield method is used in the global nonlinear stability result of Part IV. Constructing and controlling new geometric spacetime decompositions is one of the main achievements of Parts II and III and Part IV.

## 4.3 Overview of the proof of Theorem 4.2

The proof of Theorem 4.2 in [CK93] is an elaborate *bootstrap argument*, which is used to deal with the nonlinear character of Einstein equations (1.1). The main step of a bootstrap argument is to show that a set of estimates can be obtained in a *bootstrap region*, where *bootstrap assumptions* are assumed. Applying local existence and extension results, one obtains by a standard continuity argument that the bootstrap region covers the whole desired spacetime and that the estimates globally hold.

### 4.3.1 The *vectorfield method*

The estimates in the proof of Theorem 4.2 are obtained using the *vectorfield method*, which proceeds in the following two steps.

**Step 1: global energy estimates.** As a first step, *global energy estimates* for the Bianchi equations (3.1) are performed. These estimates are obtained following the general framework described in Section 3, applying Stokes formula to contracted and commuted Bel-Robinson tensors. See Section 3.3 for an example of how such an estimate is performed.

The Bel-Robinson tensors are contracted and commuted with a set of *multiplier* and *commuting* vectorfields. These vectorfields are chosen to be *approximations of the Minkowskian conformal Killing vectorfields*. In Minkowski space, contracting and commuting by conformal Killing vectorfields produces exact conservation laws, see the discussion in Section 3.4. From their approximations arise *nonlinear error terms*. The global energy estimates hold provided that these error terms are controlled.

The outcome of the energy estimates is a control of *energy boundary fluxes* through hypersurfaces  $\Sigma_t$  and  $\mathcal{C}_u$  by the boundary flux through the initial hypersurface  $\Sigma$ . Assumption (4.4c) of Theorem 4.2 guarantees that these initial boundary fluxes through  $\Sigma$  are controlled.

**Step 2: boundedness and decay estimates.** The control of the energy boundary fluxes of Step 1 can be turned into *boundedness of  $L^2$  norms* for (derivatives of) the spacetime curvature tensor  $\mathbf{R}$  on the boundary hypersurfaces. From the boundedness of the  $L^2$  norms, one obtains *decay estimates* for  $\mathbf{R}$  using *Klainerman-Sobolev embeddings*. As a consequence of the boundedness and decay for the spacetime curvature  $\mathbf{R}$ , one obtains consistent boundedness and decay for the metric and connection coefficients associated to the maximal-null foliation. This is done using the structure equations for the maximal and null decompositions, displayed in Sections 2.1 and 2.2. These equations schematically read

$$\mathbf{D}\Gamma = \mathbf{R} + \mathbf{D}\Gamma + \Gamma \cdot \Gamma.$$

Here  $\mathbf{D}$  are derivatives,  $\Gamma$  are connection coefficients, the terms  $\mathbf{R}, \mathbf{D}\Gamma$  on the right-hand side are treated as linear source terms and the terms  $\Gamma \cdot \Gamma$  as nonlinear error terms.

The crux of the proof of Theorem 4.2 in [CK93] is the control of the nonlinear error terms arising in the global energy estimates of Step 1. The approximate conformal Killing vectorfields are constructed upon the maximal-null decompositions of the spacetime. The nonlinear error terms of Step 1 can thus be expressed in terms of the spacetime curvature  $\mathbf{R}$  and the connection coefficients  $\Gamma$ . Their control thus crucially relies on the decay estimates obtained in Step 2.

### 4.3.2 The maximal-null foliation

The vectorfield method can be performed provided that the spacetime is decomposed along foliations by hypersurfaces according to the framework of Section 2. In [CK93], the spacetime is foliated by the level sets  $\Sigma_t$  of a time function  $t$  and by the level sets  $\mathcal{C}_u$  of an optical function  $u$ .

The global time function  $t$  is constructed by imposing that its level sets  $\Sigma_t$  are maximal hypersurfaces of  $\mathcal{M}$ , that  $\Sigma = \{t = 0\}$  and that  $n \rightarrow 1$  when  $r \rightarrow \infty$ . These last conditions are equivalent to the choice of boundary for  $\Sigma_t$  at infinity. See also discussions in Section 2.1.2. It physically corresponds to considering a *centre-of-mass frame* for the system (see the discussion in the introduction of [CK93]).

The global optical function  $u$  is constructed by an *initialisation on the last slice*, *i.e.* by imposing that the outgoing null hypersurfaces  $\mathcal{C}_u$  are backwards emanating from the 2-spheres of a *canonical foliation* on a last slice  $\Sigma^*$  corresponding to the future boundary of the bootstrap region.

That the construction of the optical function  $u$  is performed *within the bootstrap argument* from the last slice of the bootstrap region, is done so that the function  $u$  *asymptotically* matches the corresponding Minkowskian optical function. In that case, the null cones  $\mathcal{C}_u$  do approach the Minkowskian null cones at infinity and sufficient decay estimates for the induced metric and null connection coefficients can be obtained to control the crucial energy estimates nonlinear error terms discussed in the previous section.

# 5. Main results of the thesis

In Sections 5.1 and 5.2, we present the main results obtained in this thesis, which respectively answer Questions 1.7 and 1.12 raised in Section 1.

## 5.1 The spacelike-characteristic bounded $L^2$ curvature theorem

We consider the *spacelike-characteristic Cauchy problem* for Einstein equations (1.1), where initial data are posed on

- ◊ a maximal spacelike hypersurface  $\Sigma$  diffeomorphic to the unit disk of  $\mathbb{R}^3$ ,
- ◊ an outgoing null hypersurface  $\mathcal{H}$  emanating from  $\partial\Sigma$ .

**Remark 5.1.** *Initial data to the spacelike-characteristic Cauchy problem must satisfy constraint equations as in Sections 1.2 and 1.3 of this introduction, together with compatibility conditions at the intersection  $\Sigma \cap \mathcal{H}$ . We do not intend to give a characterisation of these initial prescriptions. In Parts II and III and in Part IV, we shall in fact assume that for smooth data a combination of the results of [Fou52, CG69] and [Ren90] yields the existence of a smooth maximal globally hyperbolic development of  $\Sigma \cup \mathcal{H}$ . The essence of our results is a characterisation of that development provided that a specific control holds on  $\Sigma \cup \mathcal{H}$ .*

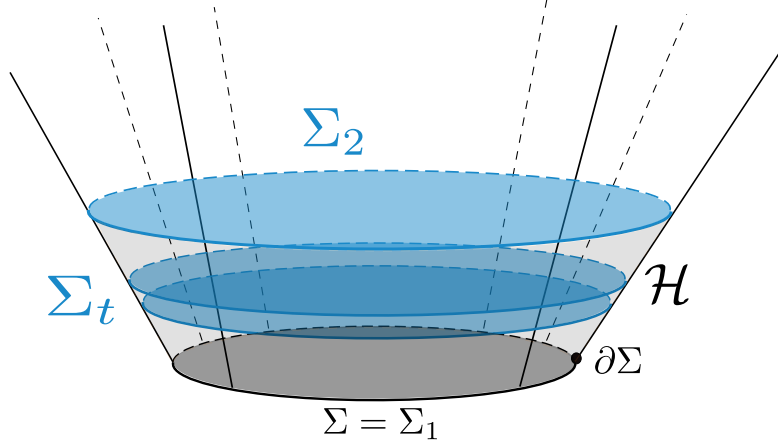


Figure 5.1: The spacelike-characteristic bounded  $L^2$  curvature theorem.

We obtain the following result, which is the subject of Part II (and III) of this manuscript.

**Theorem 5.2** (Spacelike-characteristic bounded  $L^2$  curvature theorem [CG19a, CG19b]). *Let smooth spacelike-characteristic initial data be given on a maximal spacelike 3-disk  $\Sigma$  and on a null hypersurface  $\mathcal{H}$  emanating from  $\partial\Sigma$ . Assume that*

- ◊ *the following bounds hold on  $\Sigma$*

$$\|\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{\leq 1} k\|_{L^2(\Sigma)} \leq \varepsilon, \quad (5.1)$$

*together with mild geometric assumptions for  $\Sigma$ ,*

- ◊ *the hypersurface  $\mathcal{H}$  is foliated by a smooth geodesic foliation  $(S'_s)$  ranging from  $s = 1$  to  $s = 5/2$ , such that  $S'_1 = \partial\Sigma$ , and such that for the null curvature components associated to the geodesic foliation, we have*

$$\|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\underline{\beta}\|_{L^2(\mathcal{H})} \leq \varepsilon, \quad (5.2)$$

- ◊ *low regularity bounds consistent with (5.1) and (5.2) hold on  $S'_1 = \Sigma \cap \mathcal{H}$  for the induced metric and null connection coefficients of the geodesic foliation.*



Then, there exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$  the following holds for the smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  of  $\Sigma \cup \mathcal{H}$ .

- ◇ There exists a smooth time function  $t$  ranging from  $t = 1$  to  $t = 2$ , locally foliating the future of  $\Sigma \cup \mathcal{H}$  in  $\mathcal{M}$  by spacelike maximal hypersurfaces  $\Sigma_t$ , such that  $\Sigma_1 = \Sigma$ .
- ◇ The following bounds hold on  $\Sigma_t$

$$\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^{\leq 1} k\|_{L_t^\infty L^2(\Sigma_t)} + \|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} \lesssim \varepsilon, \quad (5.3)$$

together with a mild geometric control of  $\Sigma_t$ .

### Remarks on Theorem 5.2

5.2a Theorem 5.2 assumes solely initial data bounds at the level of curvature in  $L^2$  and makes no symmetry assumptions. In contrast, in the available literature the Cauchy problem for the Einstein vacuum equations with initial data on null hypersurfaces outside of symmetry is studied under the assumption of

- ◇ higher regularity of the full initial data, see for example [Ren90, CCM11, CP12],
- ◇ higher regularity of specific components of the initial data, see for example [Luk12, LR15, LR17]. For instance, in [LR15] the null curvature component  $\alpha$  is only assumed to be a measure on  $\mathcal{H}$  while  $\beta, \rho, \sigma$  and  $\underline{\beta}$  are assumed to be controlled up to two angular derivatives in  $L^2$ .

5.2b The  $L^2$  bounds assumptions (5.1) and (5.2) on  $\Sigma$  and  $\mathcal{H}$  are equivalent to a control of the boundary fluxes naturally arising from energy estimates for the Bianchi equations for  $\mathbf{R}$  (see Section 3).

5.2c The proof of Theorem 5.2 features an *energy estimate* for the Bianchi equations for  $\mathbf{R}$  in  $\mathcal{M}$  (using the framework described in Section 3). The control of the spacetime error term is obtained invoking the *bounded  $L^2$  curvature theorem* [KRS15] as a black box, which requires to apply the *extension of initial data procedure* from [Czi18].

To apply these results, we obtain a sharp control on the metric and connection coefficients by considering *global elliptic boundary problems* on each separate slice  $\Sigma_t$ . The elliptic equations are the structure equations (2.5c), (2.5d), (2.5e) for  $g, k, n$  from the Einstein equations in maximal gauge. The boundary conditions are (implicit) mixed Dirichlet and Neumann conditions and are related to the intrinsic geometry of the foliation of boundaries  $(\partial\Sigma_t)_{1 \leq t \leq 2}$ .

5.2d The time function  $t$  and the maximal hypersurfaces of the foliation  $\Sigma_t$  are determined by the choice of the boundaries  $\partial\Sigma_t = \Sigma_t \cap \mathcal{H}$ . The most natural choice is to impose that they coincide with the 2-spheres  $S'_s$  of the geodesic foliation on  $\mathcal{H}$ , *i.e.*  $\partial\Sigma_t = S'_t$  for all  $1 \leq t \leq 2$ . It turns out that the regularity of the geodesic foliation on  $\mathcal{H}$  is not sufficient to control the maximal hypersurfaces  $\Sigma_t$ . In Part III we prove that, under the small  $L^2$ -bound assumption on  $\mathcal{H}$ , one can deform the geodesic foliation to the *canonical foliation* on  $\mathcal{H}$ , which provides the required regularity to control the hypersurfaces  $\Sigma_t$  (see Theorem 5.3 below).

In Part III, we obtain the following result which ensures the existence and control of the canonical foliation of  $\mathcal{H}$  needed in the proof of Theorem 5.2 (see Item 5.2d).

**Theorem 5.3** (The canonical foliation with bounded  $L^2$  curvature [CG19a]). *Under the assumptions of Theorem 5.2, there exists a smooth canonical foliation  $(S_v)$  on  $\mathcal{H}$  ranging from  $v = 1$  to  $v = 2$ , such that  $S_1 = S'_1 = \Sigma \cap \mathcal{H}$ , and such that the following bounds hold for its associated null connection coefficients*

$$\left\| \text{tr}\chi - \frac{2}{v}, \text{tr}\underline{\chi} + \frac{2}{v}, \hat{\chi}, \hat{\underline{\chi}}, \zeta, \Omega - 1, \nabla\Omega \right\|_{H^1(\mathcal{H})} \lesssim \varepsilon, \quad (5.4)$$

where  $\|F\|_{H^1(\mathcal{H})} := \|F\|_{L^2(\mathcal{H})} + \|\nabla F\|_{L^2(\mathcal{H})} + \|\nabla_L F\|_{L^2(\mathcal{H})}$ , and together with additional, refined estimates.

**Remarks on Theorem 5.3**

- 5.3a The crucial result of Theorem 5.3 is that the *tangential derivatives* of  $\text{tr}\underline{\chi}$  and  $\hat{\chi}$  are controlled at a similar level as the other connection coefficients, which is not the case for the geodesic foliation. This is obtained thanks to a simplified transport equation for  $\text{tr}\underline{\chi}$  in the case of the canonical foliation, and it is sufficient for the control of the maximal hypersurfaces of Theorem 5.2.
- 5.3b The proof of Theorem 5.3 features a *triangularisation* of the system of null structure equations for the canonical foliation on  $\mathcal{H}$ , the proof of *geometric trace norms* estimates at low regularity, which relies on a *comparison argument* to the geodesic foliation and the use of the *sharp bilinear estimates* of [KR06a].
- 5.3c The functional calculus tools are mostly taken from [Sha14], which is the latest version of the ideas from the groundbreaking [KR05, KR06a, KR06b] (see also [Wan09] and [Sze12a]–[Sze16]).

## 5.2 The global nonlinear stability of Minkowski space for characteristic initial data

In Part IV of this thesis, we obtain the following theorem.

**Theorem 5.4** (Global nonlinear stability of Minkowski space for characteristic initial data [Gra20]). *Consider smooth spacelike-characteristic initial data posed on a spacelike 3-disk  $\Sigma$  and on an outgoing null hypersurface  $\mathcal{C}$  emanating from  $\partial\Sigma$ . Assume that*

- ◇ *we have the following curvature fluxes bounds through  $\Sigma$*

$$\int_{\Sigma} |\mathbf{D}^{\leq 2} \mathbf{R}|^2 \leq \varepsilon^2, \quad (5.5a)$$

*together with consistent bounds for the metric and connection coefficients,*

- ◇ *the null hypersurface  $\mathcal{C}$  is future geodesically complete, foliated by the 2-spheres of a geodesic foliation  $(S'_s)_{1 \leq s < +\infty}$ , and for the associated null decompositions, we have the following fluxes bounds through  $\mathcal{C}$*

$$\int_{\mathcal{C}} \left( |\nabla^{\leq 2} \underline{\beta}|^2 + |s\nabla^{\leq 2} \rho|^2 + |s\nabla^{\leq 2} \sigma|^2 + |s^2\nabla^{\leq 2} \beta|^2 + |s^2\nabla^{\leq 2} \alpha|^2 \right) \leq \varepsilon^2 \quad (5.5b)$$

*where  $\nabla \in \{(s\nabla), (s\nabla)_4, \nabla_3\}$ , together with consistent bounds for the metric and connection coefficients.*

*There exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$ , the following holds for the smooth maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  of  $\Sigma \cup \mathcal{C}$ .*

- ◇ *The spacetime  $(\mathcal{M}, \mathbf{g})$  is future causally geodesically complete.*
- ◇ *The spacetime  $(\mathcal{M}, \mathbf{g})$  is covered by an interior and an exterior region  $\mathcal{M}^{\text{int}}$  and  $\mathcal{M}^{\text{ext}}$ , intersecting at a timelike transition hypersurface  $\mathcal{T} = \mathcal{M}^{\text{int}} \cap \mathcal{M}^{\text{ext}}$ .*
- ◇ *There exists a global time function  $t$  on  $\mathcal{M}^{\text{int}}$  ranging up to  $+\infty$  foliating  $\mathcal{M}^{\text{int}}$  by spacelike maximal hypersurface  $\Sigma_t$ .*
- ◇ *There exists a global optical function  $u$  on  $\mathcal{M}^{\text{ext}}$  ranging up to  $+\infty$  foliating  $\mathcal{M}^{\text{ext}}$  by outgoing null hypersurfaces  $\mathcal{C}_u$ . There exists a global function  $\underline{u}$  on  $\mathcal{M}^{\text{ext}}$  which is a geodesic affine parameter on  $\mathcal{C}_u$ , foliating  $\mathcal{C}_u$  by 2-spheres  $S_{u, \underline{u}}$ . Moreover, on the transition hypersurface  $\mathcal{T}$ , we have*

$$u = \tau \underline{u}, \quad t = \frac{1}{2}(u + \underline{u}),$$

*where  $0 < \tau < 1$  is a fixed parameter.*

- ◇ *We have the following decay bounds in  $\mathcal{M}^{\text{int}}$*

$$|E| + |H| \lesssim \varepsilon t^{-7/2},$$

*together with consistent bounds for the metric and connection coefficients.*

◇ We have the following decay bounds in  $\mathcal{M}^{\text{ext}1}$

$$\begin{aligned} |\alpha| &\lesssim \varepsilon \underline{u}^{-7/2}, & |\beta| &\lesssim \varepsilon \underline{u}^{-7/2}, & |\rho| &\lesssim \varepsilon \underline{u}^{-3} u^{-1/2}, \\ |\underline{\alpha}| &\lesssim \varepsilon \underline{u}^{-1} u^{-5/2}, & |\underline{\beta}| &\lesssim \varepsilon \underline{u}^{-2} u^{-3/2}, & |\sigma| &\lesssim \varepsilon \underline{u}^{-3} u^{-1/2}, \end{aligned}$$

together with consistent bounds for the metric and connection coefficients.

◇ The spacetime  $(\mathcal{M}, \mathbf{g})$  admits a future timelike and future null infinity  $i^+$  and  $\mathcal{I}^+$ . The future null infinity  $\mathcal{I}^+$  is future geodesically complete, admits well-defined notions of Bondi mass and angular momentum for which we obtain Bondi mass loss formula and angular momentum evolution equation along  $\mathcal{I}^+$ , and which tend to 0 at future timelike infinity  $i^+$ .

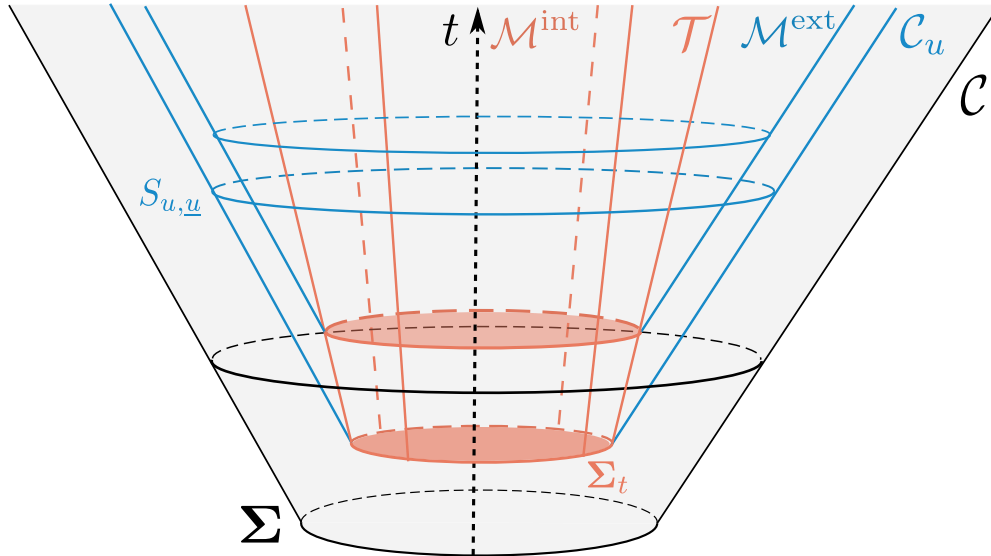


Figure 5.2: The global nonlinear stability of Minkowski space for characteristic data.

#### Remarks on Theorem 5.4

- 5.4a The closeness assumptions (5.5) match what can be obtained for an outgoing null hypersurface in [CK93] or [KN03]. Therefore, Theorem 5.4 provides a stability result for the complementary region to the exterior region considered in [KN03]. Together, they amount to a stability result for initial data posed on a spacelike hypersurface.
- 5.4b Theorem 5.4 was conjectured to hold true in [KN03, Tay17] and its conclusions were used in [Tay17].
- 5.4c The basic scheme of proof of Theorem 5.4 is a vectorfield method wrapped in a bootstrap argument as in [CK93]. See Section 4.3. The main novelty is the introduction and control of new geometric constructions, which provide suitable spacetime decompositions to run these arguments.
- 5.4d Our constructions display the following new crucial geometric features.
- ◇ They virtually emanate from the future infinity of a (timelike) central axis. This guarantees optimal decay rates. It replaces an asymptotically flat spacelike infinity which plays a similar crucial role in [CK93, KN03].
  - ◇ In the interior region, our constructions are build on *spacelike maximal hypersurfaces with prescribed boundaries* and *global harmonic coordinates*. This makes any reference to null decompositions and spherical foliations – which degenerate at the central axis – disappear in that region.
- 5.4e In the proof of Theorem 5.4, we match discontinuous gauge choices across the timelike interface  $\mathcal{T}$  without using the gluing procedure of [CK93, KS17]. Our matching features a *mean value argument* which compensates regularity losses at the timelike interface. We believe that this new treatment gains in concision and clarity.

<sup>1</sup>The null decomposition is taken with respect to a null pair adapted to the geodesic-null foliation of  $\mathcal{M}^{\text{ext}}$ .

5.4f In the appendices of Part [IV](#), we also provide new optimal estimates and control for harmonic coordinates on a Riemannian manifold, only based on elementary energy and Bochner estimates. We moreover give a full statement and proof for general limits of the metric and connection coefficients and their derivatives in all directions at the vertex of a (general family of) null cones.



## II The spacelike-characteristic bounded $L^2$ curvature theorem

joint work with Stefan Czimek



# 1. Introduction

## 1.1 Einstein vacuum equations and the Cauchy problem of general relativity

A Lorentzian 4-manifold  $(\mathcal{M}, \mathbf{g})$  is called a *vacuum spacetime* if it solves the Einstein vacuum equations

$$\mathbf{Ric} = 0, \tag{1.1}$$

where  $\mathbf{Ric}$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ . The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a non-linear geometric coupled system of partial differential equations of order 2 for  $\mathbf{g}$ . In suitable coordinates, for example so-called *wave coordinates*, it can be shown that (1.1) is hyperbolic and hence admits an initial value formulation.

One way of prescribing initial data for the Einstein vacuum equations is by specifying a triplet  $(\Sigma, g, k)$  where  $(\Sigma, g)$  is a Riemannian 3-manifold and  $k$  is a symmetric 2-tensor on  $\Sigma$  satisfying the *constraint equations*,

$$\begin{aligned} \mathbf{R}_{scal} &= |k|_g^2 - (\mathrm{tr}_g k)^2, \\ \mathrm{div} k &= \mathrm{d}(\mathrm{tr}_g k), \end{aligned} \tag{1.2}$$

where  $\mathbf{R}_{scal}$  denotes the scalar curvature of  $g$ ,  $\mathrm{d}$  the exterior derivative on  $(\Sigma, g)$  and for a symmetric 2-tensor  $F$  on  $\Sigma$ ,

$$|F|_g^2 := g^{ad}g^{bc}F_{ab}F_{cd}, \quad \mathrm{tr}_g F := g^{ij}F_{ij}, \quad (\mathrm{div} F)_i := \nabla^j F_{ij}.$$

Here  $\nabla$  denotes the covariant derivative on  $(\Sigma, g)$  and we use, as in the rest of this paper, the Einstein summation convention. In the future development  $(\mathcal{M}, \mathbf{g})$  of such initial data  $(\Sigma, g, k)$ ,  $\Sigma \subset \mathcal{M}$  is a spacelike hypersurface with induced metric  $g$  and second fundamental form  $k$ . Hence we say that such initial data is *posed on a spacelike hypersurface*.

For the purposes of this paper, it suffices to consider initial data posed on *maximal* spacelike hypersurfaces, that is, satisfying  $\mathrm{tr}_g k = 0$  in addition to (1.2); see also [Bar84]. In this case, we say that  $(\Sigma, g, k)$  is *maximal initial data*, and the constraint equations (1.2) reduce to

$$\begin{aligned} \mathbf{R}_{scal} &= |k|_g^2, \\ \mathrm{div} k &= 0, \\ \mathrm{tr}_g k &= 0. \end{aligned}$$

## 1.2 Weak cosmic censorship and the bounded $L^2$ curvature theorem

One of the main open questions in general relativity is the so-called *weak cosmic censorship conjecture* formulated by Penrose in 1969, see [Pen69].

**Conjecture 1.1** (Weak cosmic censorship conjecture). *Generically, all singularities forming in the context of gravitational collapse are covered by black holes.*

In the pioneering work [Chr99], Christodoulou proves the weak cosmic censorship conjecture for the Einstein vacuum-scalar field equations *in spherical symmetry*. In Christodoulou's proof, a low regularity control of the Einstein equations is essential for analysing the dynamical formation of black holes. This strongly suggests that a crucial step to prove the weak cosmic censorship in the absence of symmetry is to control the Einstein vacuum equations in very low regularity.

We remark that in the  $(1+1)$ -setting of spherical symmetry, Christodoulou bounds the regularity of initial data in a scale-invariant BV-norm. Outside of spherical symmetry, however, this BV-norm is not suitable anymore and regularity should be measured with respect to  $L^2$ -based spaces; we refer the reader to the introduction of [KRS15].



A breakthrough result in the low regularity control of the Einstein equations in absence of symmetry is the *bounded  $L^2$  curvature theorem* by Klainerman-Rodnianski-Szeftel [KRS15]. Before stating it, we define the volume radius of a Riemannian 3-manifold.

**Definition 1.2** (Volume radius). *Let  $(\Sigma, g)$  be a Riemannian 3-manifold, and let  $r > 0$  be a real number. The volume radius of  $\Sigma$  at scale  $r$  is defined by*

$$r_{vol}(\Sigma, r) := \inf_{p \in \Sigma} \inf_{0 < r' < r} \frac{\text{vol}_g(B_g(p, r'))}{r'^3},$$

where  $B_g(p, r')$  denotes the geodesic ball of radius  $r'$  centred at  $p \in \Sigma$ .

The following theorem is proved in [KRS15], see also the companion papers [Sze12a]-[Sze16]. We state a more technical version in Section 3.6, see Theorem 3.14.

**Theorem 1.3** (The bounded  $L^2$  curvature theorem, version 1). *Let  $(\Sigma, g, k)$  be asymptotically flat, maximal initial data for the Einstein vacuum equations such that  $\Sigma \simeq \mathbb{R}^3$ . Assume further that for some  $\varepsilon > 0$ ,*

$$\|\text{Ric}\|_{L^2(\Sigma)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon \text{ and } r_{vol}(\Sigma, 1) \geq \frac{1}{2},$$

where  $\text{Ric}$  denotes the Ricci tensor of  $(\Sigma, g)$ . Then:

1.  **$L^2$ -regularity.** *There is a universal constant  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then the maximal globally hyperbolic future development  $(\mathcal{M}, \mathbf{g})$  of the initial data  $(\Sigma, g, k)$  contains a foliation  $(\Sigma_t)_{0 \leq t \leq 1}$  of maximal spacelike hypersurfaces defined as level sets of a time function  $t$  such that  $\Sigma_0 = \Sigma$  and for  $0 \leq t \leq 1$ ,*

$$\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \inf_{0 \leq t \leq 1} r_{vol}(\Sigma_t, 1) \geq \frac{1}{4},$$

where  $\text{Ric}$  and  $k$  denote the intrinsic Ricci tensor and second fundamental form of  $\Sigma_t \subset \mathcal{M}$ , respectively.

2. **Propagation of smoothness.** *Smoothness of the initial data is propagated into the spacetime up to  $\Sigma_1 = \{t = 1\}$ .*

### Remarks on Theorem 1.3

1. By the finite speed of propagation for the Einstein vacuum equations (1.1), Theorem 1.3 is local in nature, and hence we do not specify here further the asymptotic flatness condition on  $(\Sigma, g, k)$ , see also Remark 2.3 in [KRS15].
2. Theorem 1.3 is primarily to be understood as a *continuation result for smooth solutions of the Einstein vacuum equations*, see Remark 1.2 in the introduction of [KRS15]. This holds similarly for the results of this paper.
3. The proof of Theorem 1.3 relies crucially on a plane wave representation formula for the wave equation on low regularity spacetimes developed in [Sze12a]-[Sze16]. This plane wave representation formula is constructed as a Fourier integral operator which necessitates the assumption  $\Sigma \simeq \mathbb{R}^3$ .

However, Christodoulou's work [Chr99] as well as related results on the formation of trapped surfaces [Chr09] [KR12] [KLR14] [AL17] and gravitational impulses [LR15] [LR17] consider initial data posed on null hypersurfaces rather than on a spacelike hypersurface as assumed in Theorem 1.3. This motivates the study of the Cauchy problem of general relativity in low regularity with initial data posed on null hypersurfaces.

## 1.3 The spacelike-characteristic Cauchy problem

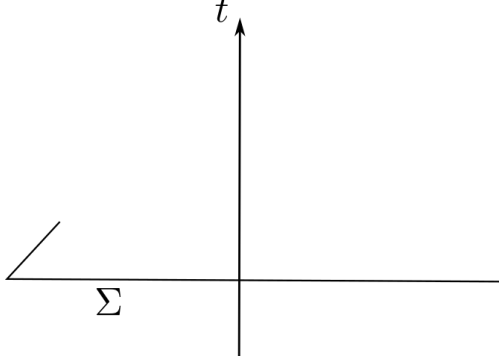
In this paper, we consider the *spacelike-characteristic Cauchy problem of general relativity*, where initial data is posed on

1. a maximal spacelike hypersurface with boundary  $\Sigma \simeq \overline{B_1} \subset \mathbb{R}^3$ ,
2. the outgoing null hypersurface  $\mathcal{H}$  emanating from  $\partial\Sigma$ .

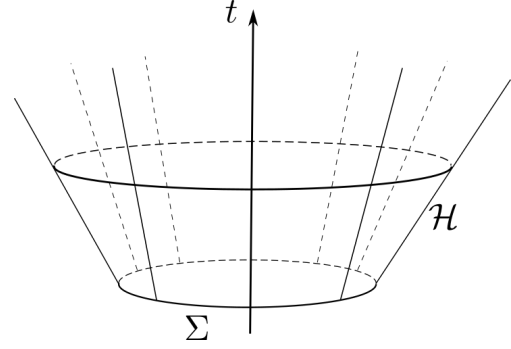
**Remark 1.4.** *Initial data for the Einstein vacuum equations posed on null hypersurfaces must satisfy constraint equations, namely the so-called null structure equations, see for example [CG19a]. We do not state them here as they do not play a role in this paper.*

**Remark 1.5.** *Initial data for the spacelike-characteristic Cauchy problem of general relativity must satisfy additional algebraic compatibility conditions on  $\partial\Sigma$ , see for example Section 7.6 in [CP12]. We do not state them here as they do not play a role in this paper.*

Local existence for the spacelike-characteristic Cauchy problem for smooth initial data follows from [Fou52] [Ren90], see Proposition 2.28.



(a) The spacelike Cauchy hypersurface of Theorem 1.3.



(b) The spacelike-characteristic Cauchy hypersurface of Theorem 1.6.

Before stating our main theorem, we give preliminar definitions. We consider

- ◊  $T$  the future-pointing unit normal to  $\Sigma$ ,
- ◊  $L$  the null geodesic generator of  $\mathcal{H}$  such that  $\mathbf{g}(L, T)|_{\partial\Sigma} = -1$ ,
- ◊  $(L, \underline{L})$  the null pair associated to a general foliation  $(S_v)$  of  $\mathcal{H}$  by 2-spheres, which is defined by

$$\mathbf{g}(L, \underline{L}) = -2, \quad \mathbf{g}(\underline{L}, \underline{L}) = 0, \quad \underline{L} \text{ is orthogonal to } S_v.$$

To a null pair, we associate the following *null connection coefficients*, which are the  $(S_v)$ -tangent tensors defined by

$$\begin{aligned} \chi(X, Y) &:= \mathbf{g}(\mathbf{D}_X L, Y), & \zeta(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_X L, e_3), \\ \underline{\chi}(X, Y) &:= \mathbf{g}(\mathbf{D}_X \underline{L}, Y), & \underline{\eta}(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_X \underline{L}, X), \end{aligned} \tag{1.3}$$

where  $X, Y \in \text{TS}_v$ . We moreover associate the following *null curvature component* to be the  $(S_v)$ -tangent tensors defined by

$$\begin{aligned} \alpha(X, Y) &:= \mathbf{R}(L, X, L, Y), & \beta(X) &:= \frac{1}{2} \mathbf{R}(X, L, \underline{L}, L), & \rho &:= \frac{1}{4} \mathbf{R}(\underline{L}, L, \underline{L}, L), \\ \underline{\alpha}(X, Y) &:= \mathbf{R}(\underline{L}, X, \underline{L}, Y), & \underline{\beta}(X) &:= \frac{1}{2} \mathbf{R}(X, \underline{L}, \underline{L}, L), & \sigma &:= \frac{1}{4} \mathbf{R}(\underline{L}, L, \underline{L}, L), \end{aligned} \tag{1.4}$$

where  $X, Y \in \text{TS}_v$  and where  $\mathbf{R}$  is the Hodge dual of  $\mathbf{R}$ .

The next theorem is a rough version of our main result, see Theorem 2.27 for a precise statement.

**Theorem 1.6** (Main result, version 1). *Consider initial data for the spacelike-characteristic Cauchy problem. Assume that for some real number  $\varepsilon > 0$ ,*

$$\|\text{Ric}\|_{L^2(\Sigma)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon, \quad r_{\text{vol}}(\Sigma, 1/2) \geq 1/4, \quad \text{vol}_g(\Sigma) \leq 8\pi. \tag{1.5}$$

Assume further that with respect to the so-called canonical foliation by spacelike 2-spheres  $(S_v)_{1 \leq v \leq 2}$  of  $\mathcal{H}$ , see Definition 2.7, it holds that

$$\begin{aligned} & \|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\underline{\beta}\|_{L^2(\mathcal{H})} \leq \varepsilon, \\ & \left\| \text{tr}\chi - \frac{2}{v} \right\|_{L^\infty H^{1/2}(S_v)} + \left\| \text{tr}\underline{\chi} + \frac{2}{v} \right\|_{L^\infty H^{1/2}(S_v)} + \|\zeta\|_{L^\infty H^{1/2}(S_v)} \leq \varepsilon. \end{aligned} \quad (1.6)$$

There is a universal constant  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then the following holds for the maximal globally hyperbolic future development  $(\mathcal{M}, \mathbf{g})$  of the initial data.

1.  **$L^2$ -regularity.**  $(\mathcal{M}, \mathbf{g})$  contains a foliation  $(\Sigma_t)_{1 \leq t \leq 2}$  of maximal spacelike hypersurfaces defined as level sets of a time function  $t$  with  $\Sigma_1 = \Sigma$  such that for  $1 \leq t \leq 2$ ,

$$\partial\Sigma_t = S_t$$

and

$$\begin{aligned} & \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \quad \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \\ & \inf_{1 \leq t \leq 2} r_{\text{vol}}(\Sigma_t, 1/2) \geq \frac{1}{8}, \quad \text{vol}_g(\Sigma_t) \leq 32\pi. \end{aligned}$$

2. **Propagation of smoothness.** Smoothness of the initial data is propagated into the spacetime up to  $\Sigma_2 = \{t = 2\}$ .

### Remarks on Theorem 1.6

1. Theorem 1.6 assumes solely initial data bounds at the level of curvature in  $L^2$  and makes no symmetry assumptions. In contrast, in the available literature the Cauchy problem for the Einstein vacuum equations with initial data on null hypersurfaces outside of symmetry is studied under the assumption of
  - ◇ higher regularity of the full initial data, see for example [Ren90] [CCM11] [CP12],
  - ◇ higher regularity of specific components of the initial data, see for example [Luk12] [LR15] [LR17].
For instance, in [LR15] the null curvature component  $\alpha$  is only assumed to be a measure on  $\mathcal{H}$  while  $\beta, \rho, \sigma$  and  $\underline{\beta}$  are assumed to be controlled up to two angular derivatives in  $L^2$ .
2. The assumed geometric control (1.6) of the foliation  $(S_v)_{v \geq 1}$  on  $\mathcal{H}$  is essential for the regularity of the spacetime. In the authors' companion paper [CG19a], it is shown that assuming small bounded  $L^2$  curvature flux on  $\mathcal{H}$  (with respect to the geodesic foliation) and further low regularity geometry bounds on the initial sphere  $S_1 = \Sigma \cap \mathcal{H}$ , the canonical foliation  $(S_v)$  exists for  $1 \leq v \leq 2$  and satisfies *stronger* regularity estimates than (1.6). For ease of presentation, in the rest of this paper we assume these stronger estimates for the canonical foliation.
3. The assumptions  $r_{\text{vol}}(\Sigma, 1/2) \geq 1/4$  and  $\text{vol}_g(\Sigma) \leq 8\pi$  on  $\Sigma$  are only used to invoke the Cheeger-Gromov theory developed in Section 7, see Theorem 4.1.
4. The proof of Theorem 1.6 uses as *black boxes* the *bounded  $L^2$  curvature theorem*, see Theorems 1.3 and 3.14, and the *extension procedure for the constraint equations*, see Theorem 3.12.
5. The methods developed in this paper and [Czi18] [Czi19a] [Czi19b] [CG19a] appear promising for a future study of the *characteristic Cauchy problem of general relativity* where initial data is posed on two transversally intersecting null hypersurfaces.

## 1.4 Overview of the proof of Theorem 1.6

The proof of Theorem 1.6 goes by a standard continuity argument. Let  $t^* \geq 1$  be the maximal time such that there exists a smooth family of smooth maximal spacelike hypersurfaces  $(\Sigma_t)_{1 \leq t \leq t^*}$  locally foliating the future of  $\Sigma$  in  $\mathcal{M}$  and such that the following *bootstrap assumptions* hold for all  $t \in [1, t^*]$ ,

$$\|\text{Ric}\|_{L^2(\Sigma_t)}^2 + \|\nabla^{\leq 1} k\|_{L^2(\Sigma_t)}^2 \leq (D\varepsilon)^2, \quad (1.7)$$

where  $D > 0$  is a fixed (large) constant.<sup>1</sup>

Our aim is to show that  $t^* \geq 2$ . Using classical local existence results, it can be shown that  $t^* > 1$  and that the solution can be extended as long as it remains smooth. In what follows, we shall therefore restrict to the improvement of the bootstrap assumption (1.7) which is the crucial step in the continuity argument.

In the next section, we perform the energy estimate for Bianchi equations, which is at the centre of the improvement of the bootstrap assumption (1.7).

### 1.4.1 The energy estimate

From Einstein vacuum equations (1.1) and the once contracted Bianchi identities, we have the following *Bianchi equations* for  $\mathbf{R}$

$$\mathbf{D}^\alpha \mathbf{R}_{\alpha\beta\gamma\delta} = 0. \quad (1.8)$$

Let define the *Bel-Robinson tensor*  $Q(\mathbf{R})$  associated to the spacetime curvature tensor  $\mathbf{R}$  by

$$Q(\mathbf{R})_{\alpha\beta\gamma\delta} := \mathbf{R}_{\alpha\nu\gamma\mu} \mathbf{R}_{\beta\delta}^{\nu\mu} + {}^* \mathbf{R}_{\alpha\nu\gamma\mu} {}^* \mathbf{R}_{\beta\delta}^{\nu\mu}.$$

We have the following consequence of the Bianchi equations (1.8)

$$\mathbf{D}^\alpha Q(\mathbf{R})_{\alpha\beta\gamma\delta} = 0.$$

Thus, applying Stokes formula to the contracted Bel-Robinson tensor

$$Q(\mathbf{R})(T, T, T, \cdot),$$

in the spacetime domain  $\mathcal{D}$  comprised between  $\Sigma$ ,  $\mathcal{H}$  and  $\Sigma_t$  yields the following *energy estimate*

$$\int_{\Sigma_t} Q(\mathbf{R})(T, T, T, T) \lesssim \int_{\Sigma} Q(\mathbf{R})(T, T, T, T) + \int_{\mathcal{H}} Q(\mathbf{R})(T, T, T, L) + \left| \int_{\mathcal{D}} Q(\mathbf{R})_{\alpha\beta\gamma\delta} \mathbf{D}^\alpha T^\beta T^\gamma T^\delta \right|, \quad (1.9)$$

where we recall that  $T$  is the future-pointing unit normal to  $\Sigma_t$  and  $L$  is the null geodesic generator of  $\mathcal{H}$ .

Let define the *electric-magnetic decomposition* of the spacetime curvature tensor  $\mathbf{R}$  to be the  $\Sigma_t$ -tangent tensors  $E, H$  such that

$$E(X, Y) := \mathbf{R}(T, X, T, Y), \quad H := {}^* \mathbf{R}(T, X, T, Y),$$

where  $X, Y \in T\Sigma_t$ .

From the definition of the Bel-Robinson tensor, the electric-magnetic tensors  $E, H$  and the null decomposition of  $\mathbf{R}$ , one has<sup>2</sup>

$$Q(\mathbf{R})(T, T, T, T) \simeq |E|^2 + |H|^2, \quad Q(\mathbf{R})(T, T, T, L) \simeq |\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2.$$

Moreover, we have the following *Gauss* and *Gauss-Codazzi* equations

$$E_{ij} = \text{Ric}_{ij} - k_{ia} k_j^a, \quad H_{ij} = \text{curl} k_{ij}. \quad (1.10)$$

Thus, using (1.10) and the bounds (1.5) on  $\Sigma$ , and from the bounds (1.6) on  $\mathcal{H}$ , we deduce

$$\int_{\Sigma} Q(\mathbf{R})(T, T, T, T) \lesssim \varepsilon^2, \quad \int_{\mathcal{H}} Q(\mathbf{R})(T, T, T, L) \lesssim \varepsilon^2,$$

which plugged in the energy estimate (1.9) gives

$$\int_{\Sigma_t} |E|^2 + |H|^2 \lesssim \varepsilon^2 + \left| \int_{\mathcal{D}} Q(\mathbf{R})_{\alpha\beta\gamma\delta} \mathbf{D}^\alpha T^\beta T^\gamma T^\delta \right|. \quad (1.11)$$

<sup>1</sup>For the simplicity of this introduction, we do only state part of the actual bootstrap assumptions. See Section 2.

<sup>2</sup>Up to easily controlled error terms in the null decomposition case, which are due to a non-trivial *slope*. See also Section 1.4.3 below.

The covariant derivatives of  $T$  rewrite in terms of  $k$  and  $\nabla n$  (see Section 2.3) as

$$\mathbf{D}_i T_j = -k_{ij}, \quad \mathbf{D}_T T_i = n^{-1} \nabla_i n, \quad (1.12)$$

and the term in the right-hand side of (1.11) is a *trilinear error term* which rewrites schematically as

$$\int_{\mathcal{D}} (\nabla n, k) \cdot \mathbf{R} \cdot \mathbf{R}.$$

Using the bootstrap assumptions (1.7), one wishes to obtain the following control of the error term

$$\left| \int_{\mathcal{D}} (\nabla n, k) \cdot \mathbf{R} \cdot \mathbf{R} \right| \lesssim (D\varepsilon)^3, \quad (1.13)$$

which plugged in the energy estimate (1.11) would give

$$\|E\|_{L^2(\Sigma_t)}^2 + \|H\|_{L^2(\Sigma_t)}^2 \lesssim \varepsilon^2 + (D\varepsilon)^3 \lesssim \varepsilon^2,$$

for  $\varepsilon$  sufficiently small. In view of (1.10), this bound should be sufficient to close the bootstrap argument (see Section 1.4.3).

### 1.4.2 The extension procedure

Obtaining the trilinear error term control (1.13) at our level of regularity is the heart of the proof of the classical bounded  $L^2$  curvature theorem.

In this paper, we circumvent this difficulty by applying the bounded  $L^2$  curvature Theorem 1.3 *from the slice  $\Sigma_t$  backwards* and by performing an energy estimate in the region  $\mathcal{D}$ .

To apply Theorem 1.3, the data  $(g, k)$  on  $\Sigma_t$  is *extended* to data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$  such that

$$\tilde{\Sigma} \simeq \mathbb{R}^3$$

and

$$\|\text{Ric}(\tilde{g})\|_{L^2(\tilde{\Sigma})}^2 + \|\tilde{\nabla} \tilde{k}\|_{L^2(\tilde{\Sigma})}^2 \leq \varepsilon',$$

where  $0 < \varepsilon' < \varepsilon_0$ , with  $\varepsilon_0 > 0$  the constant of Theorem 1.3.

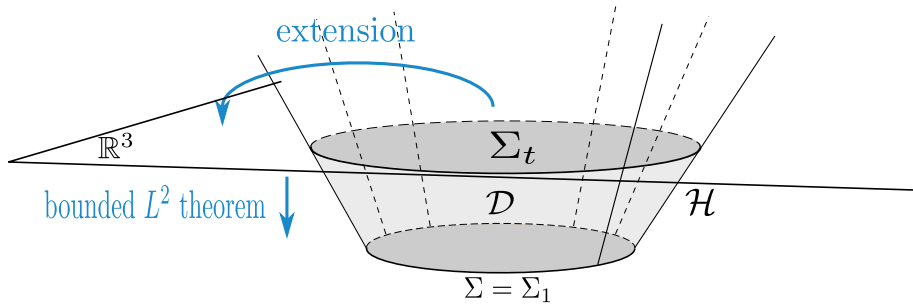


Figure 1.2: Extension procedure and backward application of the bounded  $L^2$  curvature theorem.

Using the trilinear estimates of the proof of Theorem 1.3 in [KRS15] as a black box, we then obtain the following control of the trilinear error term

$$\left| \int_{\mathcal{D}} (\nabla n, k) \cdot \mathbf{R} \cdot \mathbf{R} \right| \lesssim \varepsilon' (D\varepsilon)^2 \lesssim \varepsilon^2,$$

for  $\varepsilon' > 0$  sufficiently small, and from the energy estimate, we infer

$$\|E\|_{L^2(\Sigma_t)}^2 + \|H\|_{L^2(\Sigma_t)}^2 \lesssim \varepsilon^2. \quad (1.14)$$

**Remark 1.7.** *In this paper, we only have the implicit bounds  $\varepsilon'(\varepsilon)$  for the extended spacetime due to the contradiction argument used to obtain coordinates (see Section 7). We thus have to use the precise trilinear estimates of [KRS15] (see Section 4).*

The extension procedure was established in [Czi18]<sup>3</sup> and requires to obtain  $H^2$  and  $H^1$  estimates for respectively  $g$  and  $k$  on  $\Sigma_t$

$$\|g_{ij} - \delta_{ij}\|_{H^2(\Sigma_t)} \leq \varepsilon'', \quad (1.15)$$

$$\|k_{ij}\|_{H^1(\Sigma_t)} \leq \varepsilon'', \quad (1.16)$$

with  $\varepsilon''(\varepsilon') > 0$  sufficiently small.

The bound (1.15) is obtained by contradiction, assuming that for all  $\varepsilon > 0$ , there exists Riemannian manifolds  $(\Sigma_t, g)$  satisfying bootstrap bounds (with bound  $D\varepsilon$ ) for their Ricci curvature and fundamental forms of the boundary  $\partial\Sigma$ , such that for  $\varepsilon'' > 0$  fixed, no coordinate system satisfying (1.15) exists. Using Cheeger-Gromov convergence theory, we can extract a limit from these manifolds which must be isometric to the Euclidean unit disk and thus brings a contradiction.<sup>4</sup>

The bound (1.16) is obtained from the bootstrap assumptions 1.7.

### 1.4.3 Improvement of the bounds on Ric and $\nabla^{\leq 1}k$

From the Gauss equation (1.10), estimate (1.14) and the bootstrap assumptions (1.7) together with Sobolev estimates, we have

$$\|\text{Ric}\|_{L^2(\Sigma_t)} \lesssim \|E\|_{L^2(\Sigma_t)} + \|k\|_{L^4(\Sigma_t)}^2 \lesssim \varepsilon + (D\varepsilon)^2 \lesssim \varepsilon,$$

and the bounds on Ric of the bootstrap assumptions (1.7) is directly improved.

From (1.2), the maximality of  $\Sigma_t$  and equation (1.10), the second fundamental form  $k$  satisfies the following Hodge-type elliptic equations

$$\text{tr}k = 0, \quad \text{div}k = 0, \quad \text{curl}k = H. \quad (1.17)$$

The standard energy estimate for (1.17) reads schematically

$$\int_{\Sigma_t} |\nabla k|^2 \lesssim \int_{\Sigma_t} |H|^2 + \int_{\partial\Sigma_t} k \cdot \nabla k, \quad (1.18)$$

where  $\nabla$  denotes the tangential covariant derivative on  $\partial\Sigma_t$  and  $k \cdot \nabla k$  are contractions of  $k$  and tangential derivative of  $k$ . To explicit this boundary term, let first decompose the tensor  $k$  into its normal and tangential components on the boundary  $\partial\Sigma_t$ . We define  $N$  to be the outgoing unit normal to  $\partial\Sigma_t$  in  $\Sigma_t$  and the  $\partial\Sigma_t$ -tangent tensors  $\delta, \epsilon, \eta$  by

$$\delta := k_{NN}, \quad \epsilon_A := k_{NA}, \quad \eta_{AB} := k_{AB}, \quad (1.19)$$

where capital Latin indices range from 1 to 2 and denote the evaluation with respect to  $\partial\Sigma_t$ -tangent vectors. With these definitions, the boundary integral in (1.18) writes

$$\int_{\partial\Sigma_t} k \cdot \nabla k = \int_{\partial\Sigma_t} \epsilon \cdot \nabla \delta - \int_{\partial\Sigma_t} (\delta^2 + |\epsilon|^2 + |\eta|^2) + \text{trilinear error terms}, \quad (1.20)$$

where it should be noted that the second term appears with a favourable sign. Using this fact, the energy estimate (1.18) and the bound (1.14) on  $H$ , we can control the full  $H^1$ -norm of  $k$  on  $\Sigma_t$ , and we obtain

$$\begin{aligned} \|\nabla^{\leq 1}k\|_{L^2(\Sigma_t)}^2 &\lesssim \|H\|_{L^2(\Sigma_t)}^2 + \int_{\partial\Sigma_t} \epsilon \cdot \nabla \delta + \text{trilinear error terms} \\ &\lesssim \varepsilon^2 + \int_{\partial\Sigma_t} \epsilon \cdot \nabla \delta, \end{aligned} \quad (1.21)$$

<sup>3</sup>In establishing such an extension procedure, the main difficulty is that the constraint equations (1.2) have to be satisfied by the extended data  $(\tilde{\Sigma}, \tilde{g}, \tilde{k})$ . The result can not be obtained by a simple cut-off procedure.

<sup>4</sup>This proof by contradiction is typical from proofs of so-called ‘‘curvature pinching results’’ in Riemannian geometry. See [GPL97].

provided that the trilinear error terms can be controlled.<sup>5</sup> Obtaining the desired improvement of the bootstrap bound (1.7) for  $k$  thus requires to control the last boundary integral in (1.21). This can be achieved provided that one has an  $H^{1/2}$ -control of  $\delta$  on  $\partial\Sigma_t$ .

Obtaining the  $H^{1/2}$ -control of  $\delta$  on  $\partial\Sigma_t$  will depend on the choice of the (foliation of) prescribed boundaries  $\partial\Sigma_t$  on  $\mathcal{H}$ . Let assume that the boundaries of the maximal hypersurfaces  $\Sigma_t$  coincide with the 2-spheres of a foliation ( $S_v = \partial\Sigma_{t=v}$ ), as generally described in Section 1.3. There exists a *slope factor*  $\nu > 0$  such that the future-directed unit normal  $T$  to  $\Sigma_t$  is related to the null vector fields  $L, \underline{L}$  by<sup>6</sup>

$$T = \frac{1}{2}\nu L + \frac{1}{2}\nu^{-1}\underline{L}. \quad (1.22)$$

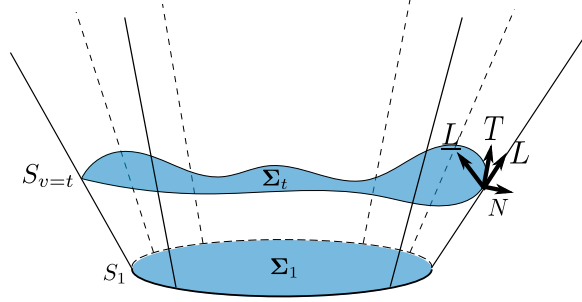


Figure 1.3: Null decomposition on  $S_v$ .

Using relations (1.3), (1.12), the maximal condition  $\text{tr}_g k = 0$ , and relation (1.22), one can obtain

$$\delta = \frac{1}{2}\nu \text{tr}\chi + \frac{1}{2}\nu^{-1}\text{tr}\underline{\chi}. \quad (1.23)$$

Plugging this relation into the boundary integral term in (1.21), we have

$$\begin{aligned} \int_{\partial\Sigma_t} \epsilon \cdot \nabla \delta &= \frac{1}{2} \int_{\partial\Sigma_t} \nu \epsilon \cdot \nabla \text{tr}\chi + \frac{1}{2} \int_{\partial\Sigma_t} \nu^{-1} \epsilon \cdot \nabla \text{tr}\underline{\chi} \\ &+ \frac{1}{2} \int_{\partial\Sigma_t} \nu \text{tr}\chi \epsilon \cdot \nabla \log \nu - \frac{1}{2} \int_{\partial\Sigma_t} \nu^{-1} \text{tr}\underline{\chi} \epsilon \cdot \nabla \log \nu. \end{aligned} \quad (1.24)$$

From this computation, we deduce two observations. First, that the required regularity on the foliation ( $S_v$ ) to estimate the boundary integral (1.24) is that the null connection coefficients  $\text{tr}\chi$  and  $\text{tr}\underline{\chi}$  must be controlled in  $L_v^\infty H^{1/2}(S_v)$ . Second, that from writing  $\delta$  in terms of the geometric quantities  $\text{tr}\chi$  and  $\text{tr}\underline{\chi}$ , one encounters an additional factor  $\nabla \log \nu$  in the boundary integral (1.24). We expect that the terms  $\text{tr}\chi$ ,  $\text{tr}\underline{\chi}$  and  $\nu$  are close to their value in Minkowski space, which is respectively  $\text{tr}\chi \simeq 2/v$ ,  $\text{tr}\underline{\chi} \simeq -2/v$  and  $\nu \simeq 1$ . This implies that for the two last boundary integrals in (1.24) we have

$$\frac{1}{2} \int_{\partial\Sigma_t} \nu \text{tr}\chi \epsilon \cdot \nabla \log \nu - \frac{1}{2} \int_{\partial\Sigma_t} \nu^{-1} \text{tr}\underline{\chi} \epsilon \cdot \nabla \log \nu \simeq \frac{2}{v} \int_{\partial\Sigma_t} \epsilon \cdot \nabla \log \nu. \quad (1.25)$$

At first sight, this seems to prevent us from closing the energy estimate for  $k$  (1.21) since  $\nu$  can only be estimated using *both* the control of  $\delta, \epsilon, \eta$  and  $\chi, \underline{\chi}, \zeta$ , but the  $k$ -components  $\delta, \epsilon, \eta$  are only determined *after* solving equation (1.17). However, using relations (1.3), (1.12), (1.19), and relation (1.22), one can obtain

$$\epsilon = -\nabla \log \nu + \zeta, \quad (1.26)$$

which, plugged into the boundary integral (1.25), gives

$$\frac{2}{v} \int_{\partial\Sigma_t} \epsilon \cdot \nabla \log \nu = -\frac{2}{v} \int_{\partial\Sigma_t} |\nabla \log \nu|^2 + \frac{2}{v} \int_{\partial\Sigma_t} \zeta \cdot \nabla \nu, \quad (1.27)$$

<sup>5</sup>Unlike in the case of the energy estimate (1.11) for Bianchi equations, the control of the nonlinear error terms in the energy estimate (1.18) for the elliptic equations (1.17) is easily obtained using standard Sobolev embeddings and the respective  $H^2$  and  $H^1$  control for  $g$  and  $k$ .

<sup>6</sup>The fact that  $\nu$  is not in general equal to 1 is related to a non-trivial *slope* between the maximal hypersurface  $\Sigma_t$  and the null hypersurface  $\mathcal{H}$ .

where it should be noted that the first term has a *favourable sign* and that the second term is controlled if the null connection coefficient  $\zeta$  of the foliation  $(S_v)$  is bounded in  $L_v^\infty H^{1/2}(S_v)$ . We therefore conclude that we can close the energy estimate for  $k$  and control the slope factor  $\nu$  if the null connection coefficients  $\text{tr}\chi$ ,  $\text{tr}\underline{\chi}$  and  $\zeta$ , which only depend on the geometry of the foliation  $(S_v)$ , are controlled in  $L_v^\infty H^{1/2}(S_v)$ .

These bounds coincide with the assumed bounds (1.6) of Theorem 1.6 for the canonical foliation. We refer the reader to our companion paper [CG19a] for the proof that these estimates hold for the canonical foliation on  $\mathcal{H}$  under assumptions at the level of bounded  $L^2$  curvature.<sup>7</sup>

## 1.5 Organisation of the paper

We outline the organisation and give a reading order suggestion of the paper.

- ◊ In Section 2, we collect the definitions and formulas used in this paper. We also state and prove the main theorem, assuming that a set of bootstrap assumptions can be improved, regularity/smoothness can be propagated, and a smooth local existence and continuation result holds.
- ◊ In Section 4, we obtain the improvement of the (low regular) bootstrap assumptions.
- ◊ In Section 5, we obtain a propagation of regularity/smoothness result.
- ◊ In Section 6, we obtain a smooth local existence and continuation result for the spacetime and geometric constructions of this paper.
- ◊ In Section 7, we obtain global coordinates *via* Cheeger-Gromov theory, which are used in Sections 4 and 5.
- ◊ In Section 3, we state functional estimates, as well as literature results which are applied in this paper.
- ◊ Appendices A – E are dedicated to the proof of functional estimates and auxiliary results.

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<sup>7</sup>We shall in fact have a stronger control than  $L_v^\infty H^{1/2}$  for  $\text{tr}\chi$ , which is required to control the geodesic/canonical foliation on  $\mathcal{H}$ . We shall also have in [CG19a] a stronger control for  $\text{tr}\underline{\chi}$  than  $L_v^\infty H^{1/2}$  – namely an  $L_v^\infty H^1$  control –, which is obtained by stronger assumptions – namely an  $H^1(S_1)$  control of  $\text{tr}\underline{\chi}$  – on the initial sphere  $S_1$ . Since – contrary to the case of  $\text{tr}\chi$  – this stronger control is not used to control the nonlinear structure of the canonical foliation, we believe that it is not necessary and that the initial assumptions on  $S_1$  in [CG19a] can be relaxed to obtain the assumptions of the present paper.





## 2. Geometric setup and main results

In this section, we introduce the notation and main equations of this paper, state the precise version of our main result (see Section 2.12) and give its proof (see Section 2.9).

**Notation.** For a real number  $r > 1$ , let  $B_r \subset \mathbb{R}^3$  denote the open ball of radius  $r$ . Lowercase Latin letters range over  $\{1, 2, 3\}$  and uppercase Latin letters over  $\{1, 2\}$ . Greek letters range over  $\{0, 1, 2, 3\}$ . We tacitly use the Einstein summation convention. In an inequality, a constant  $C_{\alpha_1, \dots, \alpha_k}$  depends on the quantities  $\alpha_1, \dots, \alpha_k$ .

### 2.1 Weyl tensors on vacuum spacetimes

In this section, we define Weyl tensors and the Bel-Robinson tensor of a Weyl tensor. We follow the presentation in the introduction and Sections 7 and 8 of [CK93]. The Bel-Robinson tensor is used in this paper to prove energy estimates for the curvature tensor, see Sections 3.4, 4 and 5.

**Definition 2.1** (Weyl tensor). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime. A 4-tensor  $\mathbf{W}$  is a Weyl tensor if it has the same symmetries as the Riemann curvature tensor and is  $\mathbf{g}$ -tracefree, that is,*

$$\begin{aligned} \mathbf{W}_{\alpha\beta\gamma\delta} &= -\mathbf{W}_{\beta\alpha\gamma\delta} = -\mathbf{W}_{\alpha\beta\delta\gamma}, & \mathbf{W}_{\alpha\beta\gamma\delta} &= \mathbf{W}_{\gamma\delta\alpha\beta}, \\ \mathbf{W}_{\alpha\beta\gamma\delta} + \mathbf{W}_{\alpha\gamma\delta\beta} + \mathbf{W}_{\alpha\delta\beta\gamma} &= 0, & \mathbf{W}_{\beta\delta} &= \mathbf{W}^{\alpha}_{\beta\alpha\delta} = 0. \end{aligned}$$

Further, let the left dual  ${}^*\mathbf{W}$  of a Weyl-tensor  $\mathbf{W}$  be

$${}^*\mathbf{W}_{\alpha\beta\gamma\delta} := \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} \mathbf{W}^{\mu\nu}_{\gamma\delta}$$

where  $\epsilon$  denotes the volume form on  $(\mathcal{M}, \mathbf{g})$ .

We note that the Riemann curvature tensor  $\mathbf{R}$  of a vacuum spacetime is the prime example of a Weyl tensor.

**Definition 2.2** (Bel-Robinson tensor). *Let  $\mathbf{W}$  be a Weyl tensor on a vacuum spacetime  $(\mathcal{M}, \mathbf{g})$ . The Bel-Robinson tensor of  $\mathbf{W}$  is defined by*

$$Q(\mathbf{W})_{\alpha\beta\gamma\delta} := \mathbf{W}_{\alpha\nu\gamma\mu} \mathbf{W}_{\beta\delta}^{\nu\mu} + {}^*\mathbf{W}_{\alpha\nu\gamma\mu} {}^*\mathbf{W}_{\beta\delta}^{\nu\mu}.$$

The following *modified Lie derivative* takes Weyl tensors into Weyl tensors, see Lemma 7.1.2 in [CK93]. Together with the Bel-Robinson tensor, it is used to derive higher regularity energy estimates for the Riemann curvature tensor in Section 5.

**Definition 2.3** (Modified Lie derivative). *Let  $\mathbf{W}$  be a Weyl field and  $X$  a vectorfield on a vacuum spacetime  $(\mathcal{M}, \mathbf{g})$ . Define the modified Lie derivative by*

$$\widehat{\mathcal{L}}_X \mathbf{W} := \mathcal{L}_X \mathbf{W} - \frac{1}{2} {}^{(X)}[\mathbf{W}] + \frac{3}{8} \text{tr}_{\mathbf{g}} \pi \mathbf{W},$$

where the deformation tensor  $\pi := \mathcal{L}_X \mathbf{g}$  and

$${}^{(X)}[\mathbf{W}]_{\alpha\beta\gamma\delta} := \pi^\mu_{\alpha} \mathbf{W}_{\mu\beta\gamma\delta} + \pi^\mu_{\beta} \mathbf{W}_{\alpha\mu\gamma\delta} + \pi^\mu_{\gamma} \mathbf{W}_{\alpha\beta\mu\delta} + \pi^\mu_{\delta} \mathbf{W}_{\alpha\beta\gamma\mu}.$$

### 2.2 Foliations of null hypersurfaces

Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime and let  $\mathcal{H}$  be an outgoing null hypersurface emanating from a spacelike 2-sphere  $(S_1, \mathcal{g})$ . Let moreover  $T$  be a given timelike vectorfield on  $S_1$ . In the following we introduce the geometric setup of foliations on  $\mathcal{H}$  following the notations and normalisations of [CG19a] and [KR05].

**Definition 2.4** (Geodesic foliation on  $\mathcal{H}$ ). *Let  $L$  be the unique  $\mathcal{H}$ -tangential null vectorfield on  $S_1$  with  $\mathbf{g}(L, T) = -1$ . Extend  $L$  as null geodesic vectorfield onto  $\mathcal{H}$ . Let  $s$  be the affine parameter of  $L$  on  $\mathcal{H}$  defined by*

$$Ls = 1 \text{ on } \mathcal{H}, \quad s|_{S_1} = 1.$$

Denote the level sets of  $s$  by  $S'_s$  and the geodesic foliation by  $(S'_s)$ .

**Definition 2.5** (General foliations on  $\mathcal{H}$ ). *Let  $v$  be a given scalar function on  $\mathcal{H}$ . We denote the level sets of  $v$  by  $S_{v_0} = \{v = v_0\}$  and the corresponding foliation by  $(S_v)$ . We define the null lapse  $\Omega$  of  $(S_v)$  on  $\mathcal{H}$  by*

$$\Omega := Lv. \quad (2.2)$$

**Definition 2.6** (Orthonormal null frame). *Let  $(S_v)$  be a foliation on  $\mathcal{H}$ . Let  $\underline{L}$  be the unique null vector field on  $\mathcal{H}$  orthogonal to each  $S_v$  and such that  $\mathbf{g}(L, \underline{L}) = -2$ . The pair  $(L, \underline{L})$  is called a null pair for the foliation  $(S_v)$ . Let  $(e_1, e_2)$  be an orthonormal frame tangential to each  $S_v$ . The frame  $(L, \underline{L}, e_1, e_2)$  is called an orthonormal null frame for the foliation  $(S_v)$ .*

Let  $(S_v)$  be a foliation on  $\mathcal{H}$  and let  $(L, \underline{L}, e_1, e_2)$  be an orthonormal null frame for  $(S_v)$ .

- ◇ Denote by  $\mathcal{g}$  and  $\nabla$  the induced metric and covariant derivative on  $S_v$ ,
- ◇ For a given  $S_v$ -tangential  $n$ -tensor  $W$ , define

$$\nabla_L W_{A_1 \dots A_n} := \Pi_{A_1}^{\beta_1} \dots \Pi_{A_n}^{\beta_n} \mathbf{D}_L W_{\beta_1 \dots \beta_n},$$

where  $\Pi$  denotes the projection operator onto the tangent space of  $S_v$  and  $\mathbf{D}$  is the covariant derivative on  $(\mathcal{M}, \mathbf{g})$ .

- ◇ Let the *null connection coefficients* be defined by

$$\chi_{AB} := \mathbf{g}(\mathbf{D}_A L, e_B), \quad \underline{\chi}_{AB} := \mathbf{g}(\mathbf{D}_A \underline{L}, e_B), \quad \zeta_A := \frac{1}{2} \mathbf{g}(\mathbf{D}_A L, \underline{L}).$$

Further decompose  $\chi$  and  $\underline{\chi}$  into their trace and tracefree parts,

$$\begin{aligned} \text{tr} \chi &:= \mathcal{g}^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2} \text{tr} \chi \mathcal{g}_{AB}, \\ \text{tr} \underline{\chi} &:= \mathcal{g}^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2} \text{tr} \underline{\chi} \mathcal{g}_{AB}. \end{aligned}$$

- ◇ For a given Weyl tensor  $\mathbf{W}$  on  $(\mathcal{M}, \mathbf{g})$ , define its null decomposition by

$$\begin{aligned} \underline{\alpha}_{AB}(\mathbf{W}) &:= \mathbf{W}_{A \underline{L} B \underline{L}}, & \underline{\beta}_A(\mathbf{W}) &:= \frac{1}{2} \mathbf{W}_{A \underline{L} \underline{L} \underline{L}}, & \rho(\mathbf{W}) &:= \frac{1}{4} \mathbf{W}_{\underline{L} \underline{L} \underline{L} \underline{L}}, \\ \sigma(\mathbf{W}) &:= \frac{1}{4} {}^* \mathbf{W}_{\underline{L} \underline{L} \underline{L} \underline{L}}, & \beta_A(\mathbf{W}) &:= \frac{1}{2} \mathbf{W}_{A \underline{L} \underline{L} \underline{L}}, & \alpha_{AB}(\mathbf{W}) &:= \mathbf{W}_{A \underline{L} B \underline{L}}. \end{aligned}$$

In particular, for the Riemann curvature tensor  $\mathbf{R}$  of a vacuum spacetime, we denote the null curvature components by

$$\begin{aligned} \underline{\alpha}_{AB} &:= \mathbf{R}_{A \underline{L} B \underline{L}}, & \underline{\beta}_A &:= \frac{1}{2} \mathbf{R}_{A \underline{L} \underline{L} \underline{L}}, & \rho &:= \frac{1}{4} \mathbf{R}_{\underline{L} \underline{L} \underline{L} \underline{L}}, \\ \sigma &:= \frac{1}{4} {}^* \mathbf{R}_{\underline{L} \underline{L} \underline{L} \underline{L}}, & \beta_A &:= \frac{1}{2} \mathbf{R}_{A \underline{L} \underline{L} \underline{L}}, & \alpha_{AB} &:= \mathbf{R}_{A \underline{L} B \underline{L}}. \end{aligned}$$

- ◇ For  $S_v$ -tangent vectorfields  $X$  define

$$d\mathcal{I}_v X := \nabla_A X^A, \quad \text{curl} X := \epsilon_{AB} \nabla^A X^B,$$

where  $\epsilon_{AB} := \epsilon_{A \underline{L} B \underline{L}}$ .

- ◇ Define on  $\mathcal{H}$  the positive-definite metric  $\mathbf{h}^v$  with respect to the foliation  $(S_v)$  by

$$\mathbf{h}^v_{\alpha \beta} := \mathbf{g}_{\alpha \beta} + \frac{1}{2} (L + \underline{L})_{\alpha} (L + \underline{L})_{\beta}.$$

For a given  $k$ -tensor  $\mathbf{W}$  on  $\mathcal{M}$ , let on  $\mathcal{H}$

$$|\mathbf{W}|_{\mathbf{h}^v}^2 := \mathbf{W}_{\alpha_1 \dots \alpha_k} \mathbf{W}_{\alpha'_1 \dots \alpha'_k} (\mathbf{h}^v)^{\alpha_1 \alpha'_1} \dots (\mathbf{h}^v)^{\alpha_k \alpha'_k}. \quad (2.3)$$

In a vacuum spacetime, the following *Ricci equations* hold, see [CK93],

$$\begin{aligned}
 \mathbf{D}_L L &= 0, & \mathbf{D}_L \underline{L} &= 2\underline{\eta}_A e_A, \\
 \mathbf{D}_A L &= \chi_{AB} e_B - \zeta_A L, & \mathbf{D}_A \underline{L} &= \underline{\chi}_{AB} e_B + \zeta_A \underline{L}, \\
 \mathbf{D}_L e_A &= \nabla_L e_A + \underline{\eta}_A L, & \mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2} \chi_{AB} \underline{L} + \frac{1}{2} \underline{\chi}_{AB} L.
 \end{aligned} \tag{2.4}$$

We turn to the definition of the *canonical foliation* on  $\mathcal{H}$ .

**Definition 2.7** (Canonical foliation on  $\mathcal{H}$ ). *Let  $(S_v)$  be a foliation on  $\mathcal{H}$ . We say that  $(S_v)$  is the canonical foliation on  $\mathcal{H}$  if  $v|_{S_1} = 1$  and*

$$\begin{aligned}
 \Delta \log \Omega &= -\text{div} \zeta + \left( \rho - \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) - \left( \bar{\rho} - \frac{1}{2} \overline{\hat{\chi} \cdot \hat{\chi}} \right), \\
 \overline{\log \Omega} &= 0,
 \end{aligned}$$

where for scalar functions  $f$  we denote by  $\bar{f}$  the average of  $f$  over the 2-sphere  $(S_v, g)$ .

In [CG19a] it is shown that the canonical foliation is well-defined under the assumption of small  $L^2$  curvature flux and small low regularity foliation geometry on the initial sphere  $S_1$ , see the introduction of [CG19a].

## 2.3 Foliations of vacuum spacetimes by spacelike maximal hypersurfaces

Let  $t$  be a scalar function on a vacuum spacetime  $(\mathcal{M}, \mathbf{g})$  whose level sets  $\Sigma_t$  constitute a foliation of spacelike maximal hypersurfaces.

- ◇ Let  $g$  denote the induced metric on  $\Sigma_t$  and  $\nabla$  its covariant derivative. Let  $\Delta$  denote the Laplace-Beltrami operator of  $g$ .
- ◇ Let  $e_0 := T$  denote the future-pointing timelike unit normal to  $\Sigma_t$ , and let  $(e_i)_{i=1,2,3}$  be an orthonormal frame tangent to  $\Sigma_t$ . Define the second fundamental form  $k$  of  $\Sigma_t$  by

$$k_{ij} := -\mathbf{g}(\mathbf{D}_i T, e_j).$$

Define the foliation lapse  $n$  by

$$n^{-2} := -\mathbf{g}(\mathbf{D}t, \mathbf{D}t),$$

satisfying in particular,

$$T = -n\mathbf{D}t, \mathbf{D}_T T = n^{-1} \nabla n, T(t) = n^{-1}. \tag{2.5}$$

We remark that the deformation tensor  $\pi := \mathcal{L}_T \mathbf{g}$  can be expressed as

$$\pi_{\alpha\beta} = -2k_{\alpha\beta} - n^{-1} (T_\alpha \nabla_\beta n + T_\beta \nabla_\alpha n) \text{ for } \alpha, \beta = 0, 1, 2, 3, \tag{2.6}$$

where, as in the rest of this paper,  $k$  is extended to a tensor on  $\mathcal{M}$  by  $k_{0\mu} = 0$  for  $\mu = 0, 1, 2, 3$ .

- ◇ For two symmetric  $g$ -tracefree 2-tensors  $V$  and  $W$  and a vectorfield  $X$  on  $\Sigma_t$ , define

$$\begin{aligned}
 \text{div} V_i &:= \nabla^j V_{ji}, \\
 \text{curl} V_{ij} &:= \frac{1}{2} \left( \epsilon_{ilm} \nabla^l V_j^m + \epsilon_{jlm} \nabla^l V_i^m \right), \\
 (V \times W)_{ij} &:= \epsilon_i^{ab} \epsilon_j^{cd} V_{ac} W_{bd} + \frac{1}{3} (V \cdot W) g_{ij}, \\
 (V \wedge W)_i &:= \epsilon_i^{mn} V_m^l W_{ln}, \\
 (X \wedge V)_{ij} &:= \epsilon_i^{mn} X_m V_{nj} + \epsilon_j^{mn} X_m V_{in},
 \end{aligned}$$

where  $\epsilon_{abc} := \epsilon_{abc} T$ .

- ◇ For a Weyl tensor  $\mathbf{W}$  on a vacuum spacetime  $(\mathcal{M}, \mathbf{g})$ , define its *electric-magnetic decomposition with respect to  $T$*  as follows,

$$E(\mathbf{W})_{ab} := \mathbf{W}_{aTbT}, \quad H(\mathbf{W})_{ab} := {}^*\mathbf{W}_{aTbT}.$$

In particular, for the Riemann curvature tensor  $\mathbf{R}$  of a vacuum spacetime, let

$$E_{ab} := \mathbf{R}_{aTbT}, \quad H_{ab} := {}^*\mathbf{R}_{aTbT}.$$

The 2-tensors  $E(\mathbf{W})$  and  $H(\mathbf{W})$  are  $\Sigma_t$ -tangent, symmetric and  $g$ -tracefree, see Section 7.2 in [CK93]. By definition of the modified Lie derivative, see Definition 2.3, it holds that (see page 188 in [CK93])

$$\begin{aligned} \hat{\mathcal{L}}_T E(\mathbf{W}) &= E(\hat{\mathcal{L}}_T \mathbf{W}) - k \times E(\mathbf{W}) + 2n^{-1} \nabla n \wedge H(\mathbf{W}), \\ \hat{\mathcal{L}}_T H(\mathbf{W}) &= H(\hat{\mathcal{L}}_T \mathbf{W}) - k \times H(\mathbf{W}) - 2n^{-1} \nabla n \wedge E(\mathbf{W}), \end{aligned} \quad (2.7)$$

where  $\hat{\mathcal{L}}_T H(\mathbf{W})$  and  $\hat{\mathcal{L}}_T E(\mathbf{W})$  are the  $g$ -tracefree parts of  $\mathcal{L}_T H(\mathbf{W})$  and  $\mathcal{L}_T E(\mathbf{W})$ , respectively. Moreover, by definition of the Bel-Robinson tensor, see Definition 2.2,

$$|E(\mathbf{W})|^2 + |H(\mathbf{W})|^2 = Q(\mathbf{W})_{TTTT}. \quad (2.8)$$

- ◇ Define the positive-definite metric  $\mathbf{h}^t$  on  $\mathcal{M}$  by

$$\mathbf{h}_{\alpha\beta}^t := \mathbf{g}_{\alpha\beta} + 2T_\alpha T_\beta, \quad (2.9)$$

and for  $n$ -tensors  $\mathbf{W}$  on  $\Sigma_t$ , let

$$|\mathbf{W}|_{\mathbf{h}^t}^2 := \mathbf{W}_{\alpha_1 \dots \alpha_n} \mathbf{W}_{\alpha'_1 \dots \alpha'_n} (\mathbf{h}^t)^{\alpha_1 \alpha'_1} \dots (\mathbf{h}^t)^{\alpha_n \alpha'_n}. \quad (2.10)$$

In particular, for Weyl tensors  $\mathbf{W}$  it holds by (7.2.1) in [CK93] and (2.8) that

$$|\mathbf{W}|_{\mathbf{h}^t}^2 \lesssim Q(\mathbf{W})_{TTTT} = |E(\mathbf{W})|^2 + |H(\mathbf{W})|^2 \lesssim |\mathbf{W}|_{\mathbf{h}^t}^2. \quad (2.11)$$

In particular,

$$|Q(\mathbf{W})_{\mu\nu\lambda T}| \lesssim Q(\mathbf{W})_{TTTT}, \quad (2.12)$$

where evaluation is made with respect to an orthonormal frame  $(e_\mu)_{\mu=0,1,2,3}$ .

The Einstein vacuum equations imply the following *structure equations of the maximal foliation*, see equations (1.0.11a)-(1.0.14d) in [CK93]. We have the *first variation equation*,

$$\mathcal{L}_T g_{ij} = -2k_{ij},$$

the *second variation equation*,

$$\mathbf{D}_T k_{ij} = E_{ij} - n^{-1} \nabla_i \nabla_j n - k_{il} k^l{}_j, \quad (2.13a)$$

the *Gauss-Codazzi equations*

$$\operatorname{div} k_i = 0, \quad (2.13b)$$

$$\operatorname{curl} k_{ij} = H_{ij}, \quad (2.13c)$$

the *maximality of  $\Sigma_t$* ,

$$\operatorname{tr}_g k = 0, \quad (2.13d)$$

the *lapse equation*,

$$\Delta n = n |k|_g^2, \quad (2.13e)$$

the *traced Gauss equation*,

$$\operatorname{Ric}_{ij} = E_{ij} + k_{ia} k_j^a, \quad (2.13f)$$

and the *twice-traced Gauss equation*,

$$\mathbf{R}_{scal} = |k|_g^2. \quad (2.13g)$$

With respect to a foliation  $(\Sigma_t)$  by maximal hypersurfaces, the Bianchi equations of  $(\mathcal{M}, \mathbf{g})$  can be written as follows, see Proposition 7.2.1 in [CK93].

**Proposition 2.8** (Maxwell's equations for  $E(\mathbf{W})$  and  $H(\mathbf{W})$ ). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime. Let  $E(\mathbf{W})$  and  $H(\mathbf{W})$  be the electric-magnetic decomposition of a Weyl tensor  $\mathbf{W}$  relative to a maximal spacelike foliation  $(\Sigma_t)$  on  $\mathcal{M}$ . Assume that  $\mathbf{W}$  satisfies the inhomogeneous Bianchi equations*

$$\mathbf{D}^\alpha \mathbf{W}_{\alpha\beta\ \gamma\delta} = J_{\beta\gamma\delta} \text{ for } \beta, \gamma, \delta = 0, 1, 2, 3.$$

Then, with  $J_{\beta\gamma\delta}^* := \frac{1}{2} J_{\beta\mu\nu} \epsilon^{\mu\nu}{}_{\gamma\delta}$ ,

$$\begin{aligned} \operatorname{div} E(\mathbf{W}) &= k \wedge H(\mathbf{W}) + J, \\ \operatorname{div} H(\mathbf{W}) &= -k \wedge E(\mathbf{W}) + J^*, \\ -\hat{\mathcal{L}}_T H(\mathbf{W}) + \operatorname{curl} E(\mathbf{W}) &= -n^{-1} \nabla n \wedge E(\mathbf{W}) - \frac{1}{2} k \times H(\mathbf{W}) - J^*, \\ \hat{\mathcal{L}}_T E(\mathbf{W}) + \operatorname{curl} H(\mathbf{W}) &= -n^{-1} \nabla n \wedge H(\mathbf{W}) + \frac{1}{2} k \times E(\mathbf{W}) - J. \end{aligned} \tag{2.14}$$

*Remarks.*

1. In Appendix A, we interpret (2.14) as 3-dimensional Hodge system for  $E(\mathbf{W})$  and  $H(\mathbf{W})$  and, interpreting  $\hat{\mathcal{L}}_T E(\mathbf{W})$  and  $\hat{\mathcal{L}}_T H(\mathbf{W})$  as given source terms, prove global elliptic estimates on  $\Sigma_t$ .
2. In particular, using that in a vacuum spacetime it holds that

$$\mathbf{D}^\alpha \mathbf{R}_{\alpha\beta\ \gamma\delta} = 0,$$

it follows by Proposition 2.8 that

$$\begin{aligned} \operatorname{div} E &= k \wedge H, \\ \operatorname{div} H &= -k \wedge E, \\ -\hat{\mathcal{L}}_T H + \operatorname{curl} E &= -n^{-1} \nabla n \wedge E - \frac{1}{2} k \times H, \\ \hat{\mathcal{L}}_T E + \operatorname{curl} H &= -n^{-1} \nabla n \wedge H + \frac{1}{2} k \times E. \end{aligned} \tag{2.15}$$

The following commutator identity allows us to derive elliptic estimates for  $T(n)$ , see (18.4) in Appendix E in [KRS15] for a proof.

**Lemma 2.9** (Commutator identity). *Let  $f$  be a scalar function on a vacuum spacetime  $(\mathcal{M}, \mathbf{g})$ . Let  $(\Sigma_t)$  be a foliation on  $\mathcal{M}$  by maximal spacelike hypersurfaces given as level sets of a time function  $t$ . Then it holds that*

$$[\Delta, T]f = 2k\nabla^2 f - 2n^{-1} \nabla n \nabla T(f) - |k|^2 T(f) + 2n^{-1} k \nabla n \nabla f.$$

## 2.4 Spherical coordinates on spacelike hypersurfaces

Let  $(\Sigma, g)$  be a maximal spacelike hypersurface in a vacuum spacetime  $(\mathcal{M}, \mathbf{g})$ . Assume there exists a real number  $1 \leq t \leq 2$  such that there is a global coordinate chart  $\phi: \overline{B}_t \rightarrow \Sigma$ . Using the chart  $\phi$ , define standard spherical coordinates  $(r, \theta^1, \theta^2)$  on  $\Sigma$  with  $r \in [0, t]$ . We denote the level sets of  $r$  by  $\mathcal{S}_r$ , and for two reals  $0 \leq r_1, r_2 \leq t$ , let  $A(r_1, r_2)$  denote the coordinate annulus

$$A(r_1, r_2) := \{p \in \Sigma : r_1 \leq r(p) \leq r_2\}. \tag{2.16}$$

Then:

- ◊ The metric  $g$  can be expressed in coordinates  $(r, \theta^1, \theta^2)$  for  $r > 0$  as

$$g = a^2 dr^2 + \mathcal{g}_{AB} (b^A dr + d\theta^A) (b^B dr + d\theta^B),$$

where

- $a$  is the *foliation lapse*,
- $\mathcal{g}$  is the *induced metric* on  $\mathcal{S}_r$ ,
- $b$  is the  $\mathcal{S}_r$ -*tangent shift vector*.

- ◇ Let  $N$  be the outward pointing unit normal to  $\mathcal{S}_r$  and let  $(e_1, e_2)$  denote an orthonormal frame tangent to  $\mathcal{S}_r$ . Define the second fundamental form of  $\mathcal{S}_r$  for  $r > 0$  by

$$\Theta_{AB} := g(\nabla_A N, e_B).$$

We split  $\Theta$  into its trace and tracefree part,

$$\text{tr}\Theta := g^{AB}\Theta_{AB}, \quad \hat{\Theta}_{AB} := \Theta_{AB} - \frac{1}{2}\text{tr}\Theta g_{AB},$$

Further, in coordinates  $(r, \theta^1, \theta^2)$  we can express for  $r > 0$ ,

$$N = \frac{1}{a}\partial_r - \frac{1}{a}b, \quad \Theta_{AB} = -\frac{1}{2a}\partial_r(g_{AB}) + \frac{1}{2a}(\mathcal{L}_b g)_{AB}, \quad (2.17)$$

where  $\mathcal{L}$  denotes the Lie derivative on  $\mathcal{S}_r$ .

- ◇ Let  $\nabla$  and  $\Delta$  denote the induced covariant derivative and Laplace-Beltrami operator on  $\mathcal{S}_r$ , respectively. We note the relations (see Chapter 3 in [CK93])

$$\nabla_N N = -a^{-1}\nabla a, \quad \nabla_A N = \Theta_{AB}e_B, \quad \text{div}N = \text{tr}\Theta.$$

- ◇ we decompose the second fundamental form  $k$  on  $\Sigma$  into  $\mathcal{S}_r$ -tangential tensors as follows,

$$\delta := k_{NN}, \quad \epsilon_A := k_{NA}, \quad \eta_{AB} := k_{AB}.$$

We note that  $\text{tr}\eta = -\delta$  because  $\text{tr}_g k = 0$  on  $\Sigma$  by maximality.

With this notation, we can decompose  $\nabla k$  as follows (see Sections 3.1 and 4.4 in [CK93]),

$$\begin{aligned} \nabla_A k_{BC} &= \nabla_A \eta_{BC} + \Theta_{AB}\epsilon_C + \Theta_{AC}\epsilon_B, & \nabla_N k_{NN} &= N(\delta) + 2a^{-1}\nabla a \cdot \epsilon, \\ \nabla_N k_{AB} &= \nabla_N \eta_{AB} - a^{-1}\nabla_A \epsilon_B - a^{-1}\nabla_B \epsilon_A, & \nabla_A k_{NN} &= \nabla_A \delta - 2\Theta_{AC}\epsilon^C, \\ \nabla_N k_{NA} &= \nabla_N \epsilon_A + a^{-1}\nabla_C a \eta^C_A - a^{-1}\nabla_A \delta, & \nabla_B k_{NA} &= \nabla_B \epsilon_A - \eta_A^C \Theta_{CB} + \delta \Theta_{AB}, \end{aligned}$$

where for an  $\mathcal{S}_r$ -tangential tensor  $F$ ,  $\nabla_N F$  is defined as the projection of  $\nabla_N F$  onto  $\mathcal{S}_r$ .

Further, the Gauss-Codazzi equations (2.13b) and (2.13c) imply the following relations (see Proposition 4.4.3 in [CK93]),

$$\begin{aligned} N(\delta) + d\text{iv}\epsilon &= -\delta \text{tr}\Theta + \eta \cdot \Theta - 2a^{-1}\nabla a \cdot \epsilon, \\ \nabla_N \epsilon_A - \nabla_A \delta &= {}^*H_{N^A} - 2\Theta_{AC}\epsilon^C - a^{-1}\nabla^C a \eta_{CA} + a^{-1}\nabla_A a \delta, \\ \nabla_N \hat{\eta}_{AB} + \frac{1}{2}\text{tr}\Theta \hat{\eta}_{AB} &= {}^*\hat{H}_{AB} + \frac{1}{2}(\nabla \hat{\otimes} \epsilon)_{AB} + \frac{3}{2}\delta \hat{\Theta}_{AB} - (a^{-1}\nabla a \hat{\otimes} \epsilon)_{AB}, \\ d\text{iv}\eta_A + \nabla_A \delta &= -{}^*H_{N^A} + \Theta_{AC}\epsilon^C - \text{tr}\Theta \epsilon_A, \\ \text{curl}\epsilon &= H_{NN} + \hat{\Theta} \wedge \hat{\eta}, \end{aligned} \quad (2.18)$$

where we decomposed the source term  $H$  into  $\mathcal{S}_r$ -tangential tensors

$$H_{NN}, \quad H_{N^A} := H_{NA}, \quad \hat{H}_{AB} := H_{AB}, \quad (2.19)$$

and for  $\mathcal{S}_r$ -tangential vectorfields  $X$  and  $Y$  and symmetric tracefree 2-tensors  $F$  and  $G$ , we denote

$$\begin{aligned} {}^*X_A &:= \epsilon_{AB} X^B, \quad (X \hat{\otimes} Y)_{AB} := X_A Y_B + X_B Y_A - (X \cdot Y)g_{AB}, \\ {}^*F_{AB} &:= \epsilon_{AC} F^C_B, \quad (F \wedge G)_{AB} := \epsilon^{AC} F_{AC} G^C_B. \end{aligned} \quad (2.20)$$

## 2.5 Relations between foliations on vacuum spacetimes and null hypersurfaces

Let  $(\mathcal{M}, g)$  be a vacuum spacetime. Let  $\Sigma \simeq \overline{B_1}$  be a spacelike maximal hypersurface and let  $\mathcal{H}$  denote the outgoing null hypersurface emanating from  $\partial\Sigma$ . Let

- ◊  $(S_v)_{1 \leq v \leq 2}$  be a foliation on  $\mathcal{H}$  by spacelike 2-spheres such that  $S_1 = \partial\Sigma$ . Let  $(L, \underline{L}, e_1, e_2)$  be an orthonormal null frame of  $(S_v)$ .
- ◊  $(\Sigma_t)_{1 \leq t \leq 2}$  be a foliation on  $\mathcal{M}$  by maximal spacelike hypersurfaces given as level sets of a time function  $t$  such that  $\Sigma_1 = \Sigma$  and such that for  $1 \leq t \leq 2$ ,

$$\partial\Sigma_t = S_t, \text{ i.e. } t = v \text{ on } \mathcal{H}. \quad (2.21)$$

Let  $T$  denote the unit normal to  $\Sigma_t$ .

**Definition 2.10.** *The slope  $\nu$  on  $\mathcal{H}$  is defined by*

$$\nu := -\mathbf{g}(\underline{L}, T). \quad (2.22)$$

The proof of the next lemma is left to the reader.

**Lemma 2.11** (Algebraic relations). *On  $\mathcal{H}$  it holds that*

$$\begin{aligned} T &= \frac{1}{2}\nu L + \frac{1}{2}\nu^{-1}\underline{L}, & N &= \frac{1}{2}\nu L - \frac{1}{2}\nu^{-1}\underline{L}, \\ L &= \nu^{-1}(T + N), & \underline{L} &= \nu(T - N), \end{aligned} \quad (2.23)$$

and further,

$$\begin{aligned} \Theta_{AB} &= \frac{1}{2}\nu\chi_{AB} - \frac{1}{2}\nu^{-1}\underline{\chi}_{AB}, \\ \eta_{AB} &= -\frac{1}{2}\nu\chi_{AB} - \frac{1}{2}\nu^{-1}\underline{\chi}_{AB}, \\ \delta &= -\text{tr}\eta = \frac{1}{2}\nu\text{tr}\chi + \frac{1}{2}\nu^{-1}\text{tr}\underline{\chi}. \end{aligned}$$

**Remark 2.12.** *By Lemma 2.11, the definition of  $\nu$  in (2.22) is equivalent to*

$$\nu^{-1} := -\mathbf{g}(L, T).$$

Thus, by Definition 2.4 it follows that we have the normalisation

$$\nu = 1 \text{ on } S_1.$$

**Lemma 2.13** (Slope equation). *On  $\mathcal{H}$  it holds that*

$$\nu^{-1}\nabla_A\nu = -\epsilon_A + \zeta_A. \quad (2.24)$$

*Proof.* Using (2.5), (2.22) and Lemma 2.11, we have

$$\begin{aligned} \nu^{-1}\nabla_A\nu &= -\nu^{-1}\nabla_A(\mathbf{g}(\underline{L}, T)) \\ &= -\nu^{-1}(\mathbf{g}(\mathbf{D}_A\underline{L}, T) + \mathbf{g}(\underline{L}, \mathbf{D}_AT)) \\ &= -\nu^{-1}(\mathbf{g}(\underline{\chi}_{AB}e^B + \zeta_A\underline{L}, T) - \mathbf{g}(\underline{L}, k_{Aj}e_j)) \\ &= \zeta_A + \nu^{-1}\epsilon_A\mathbf{g}(\underline{L}, N) \\ &= \zeta_A - \epsilon_A. \end{aligned}$$

This finishes the proof of (2.24). □

We further have the following transport equation for  $\nu$  on  $\mathcal{H}$ . It is used in Sections 4 and 5 to prove estimate for  $T(n)$ .

**Lemma 2.14** (Transport equation for  $\nu$  on  $\mathcal{H}$ ). *It holds on  $\mathcal{H}$  that*

$$L(\nu) = n^{-1}N(n) - \delta.$$



*Proof.* By (2.5), (2.22) and Lemma 2.11,

$$\begin{aligned}
 L(\nu) &= -L(\mathbf{g}(\underline{L}, T)) \\
 &= -\mathbf{g}(\mathbf{D}_L \underline{L}, T) - \mathbf{g}(\underline{L}, \mathbf{D}_L T) \\
 &= -\nu^{-1} \mathbf{g}(\underline{L}, \mathbf{D}_{T+N} T) \\
 &= -\mathbf{g}(T - N, \mathbf{D}_{T+N} T) \\
 &= n^{-1} N(n) - \delta,
 \end{aligned}$$

where we used that  $\mathbf{D}_T T = n^{-1} \nabla n$ , see (2.5).  $\square$

Moreover, the lapse  $n$  can be expressed on  $\partial\Sigma_t$  as follows.

**Lemma 2.15.** *It holds on  $\partial\Sigma_t$  that*

$$n = \nu^{-1} \Omega^{-1}. \quad (2.25)$$

*Proof.* Indeed, by (2.1) and Lemma 2.11,

$$\Omega = L(v) = L(t) = \nu^{-1} T(t) = \nu^{-1} n^{-1},$$

where we used (2.21) and that  $T(t) = n^{-1}$ , see (2.5).  $\square$

The next lemma follows from Lemma 7.3.1 in [CK93] and Lemma 2.11.

**Lemma 2.16.** *It holds that on  $\mathcal{H}$ ,*

$$Q(\mathbf{R})_{LTTT} = \frac{1}{4} \nu^3 |\alpha|^2 + \frac{3}{2} \nu |\beta|^2 + \frac{3}{2} \nu^{-1} (\rho^2 + \sigma^2) + \frac{1}{2} \nu^{-3} |\underline{\beta}|^2.$$

Moreover, for  $\|\nu - 1\|_{L^\infty(\mathcal{H})}$  sufficiently small,

$$|Q_{\mu\nu\lambda L}| \lesssim Q_{LTTT}. \quad (2.26)$$

## 2.6 Integration and norms

In this section, we define integration and norms.

**Definition 2.17** (Norms on  $S$ ). *Let  $(S, \mathfrak{g})$  be a Riemannian 2-sphere. Let  $F$  be an  $S$ -tangent tensor on  $S$ . For real numbers  $1 \leq p < \infty$ , define*

$$\|F\|_{L^p(S)}^p := \int_S |F|^p,$$

where the integrals over  $S$  are with respect to the metric  $\mathfrak{g}$ . Moreover, let

$$\|F\|_{L^\infty(S)} := \sup_S |F|.$$

**Definition 2.18** (Integration on  $\mathcal{H}$ ). *Let  $1 < v_0 < \infty$  be a real number. Let  $\mathcal{H} \subset \mathcal{M}$  be a null hypersurface foliated by spacelike 2-spheres  $(S_v)_{1 \leq v \leq v_0}$ . Let  $f$  be a scalar function on  $\mathcal{H}$ . Let*

$$\int_{\mathcal{H}} f := \int_1^{v_0} \left( \int_{S_v} \Omega^{-1} f \right) dv,$$

where the integral over  $S_v$  is with respect to the induced metric  $\mathfrak{g}$  and  $\Omega := L(v)$  is the null lapse of the foliation  $(S_v)$ .

**Definition 2.19** (Norms on  $\mathcal{H}$ ). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime and let  $\mathcal{H} \subset \mathcal{M}$  be a null hypersurface. For a real number  $1 < v_0 < \infty$ , let  $(S_v)_{1 \leq v \leq v_0}$  be a foliation of spacelike 2-spheres on  $\mathcal{H}$  and denote*

$\mathcal{H}_{v_0} := \mathcal{H} \cap (S_v)_{1 \leq v \leq v_0}$ . Let  $1 \leq p < \infty$  be a real number and let  $F$  be an  $S_v$ -tangent tensor on  $\mathcal{H}$ . For integers  $m \geq 0$ , define

$$\begin{aligned} \|F\|_{L^2(\mathcal{H}_{v_0})} &:= \left( \int_{\mathcal{H}_{v_0}} |F|^2 \right)^{1/2}, \\ \|F\|_{L_{[1, v_0]}^\infty L^p(S_v)} &:= \sup_{1 \leq v \leq v_0} \|F\|_{L^p(S_v)}, \\ \|F\|_{L_{[1, v_0]}^\infty L^\infty(S_v)} &:= \sup_{1 \leq v \leq v_0} \|F\|_{L^\infty(S_v)}, \\ \|F\|_{L_{[1, v_0]}^\infty H^{1/2}(S_v)} &:= \sup_{1 \leq v \leq v_0} \|F\|_{H^{1/2}(S_v)}, \end{aligned}$$

where the fractional Sobolev space  $H^{1/2}(S_v)$  is defined in Section 3.1. Further, for tensors  $\mathbf{W}$  on  $\mathcal{M}$ , define

$$\|\mathbf{W}\|_{L^\infty(\mathcal{H}_{v_0})} := \sup_{1 \leq v \leq v_0} \sup_{S_v} |\mathbf{W}|_{\mathbf{h}^v},$$

where  $\mathbf{h}^v$  denotes the positive-definite metric on  $\mathcal{H}$  associated to the foliation  $(S_v)$ , see (2.3).

**Notation.** For ease of presentation, we leave away the index  $v_0$  in this paper whenever it is clear what interval we consider. For example, we write  $\|\cdot\|_{L^2(\mathcal{H})}$  instead of  $\|\cdot\|_{L^2(\mathcal{H}_{v_0})}$  and  $\|\cdot\|_{L_{[1, v_0]}^\infty L^p(S_v)}$  instead of  $\|\cdot\|_{L_{[1, v_0]}^\infty L^p(S_v)}$ .

**Definition 2.20** (Norms on  $\mathcal{M}$ ). Let  $t_1$  and  $t_2$  be two real numbers. Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime foliated by spacelike hypersurfaces  $(\Sigma_t)_{t_1 \leq t \leq t_2}$  given as level sets of a time function  $t$  on  $\mathcal{M}$ . For  $\Sigma_t$ -tangential tensors  $F$  define

$$\|F\|_{L_{[t_1, t_2]}^\infty L^2(\Sigma_t)} := \sup_{t_1 \leq t \leq t_2} \left( \int_{\Sigma_t} |F|^2 \right)^{1/2},$$

where the integral over  $\Sigma_t$  is with respect to the induced metric  $g$ . Further, for tensors  $\mathbf{W}$  on  $\mathcal{M}$ , let

$$\begin{aligned} \|\mathbf{W}\|_{L_{[t_1, t_2]}^\infty L^2(\Sigma_t)} &:= \sup_{t_1 \leq t \leq t_2} \left( \int_{\Sigma_t} |\mathbf{W}|_{\mathbf{h}^t}^2 \right)^{1/2}, \\ \|\mathbf{W}\|_{L^\infty(\Sigma_t)} &:= \sup_{\Sigma_t} |\mathbf{W}|_{\mathbf{h}^t}, \end{aligned}$$

where  $\mathbf{h}^t$  denotes the positive-definite metric associated to the foliation  $(\Sigma_t)$ , see (2.10).

**Notation.** For ease of presentation, we leave away the index  $[t_1, t_2]$  in this paper whenever it is clear what interval of integration we consider. For example, we write  $\|\cdot\|_{L_t^\infty L^2(\Sigma_t)}$  instead of  $\|\cdot\|_{L_{[t_1, t_2]}^\infty L^2(\Sigma_t)}$ .

## 2.7 Initial data regularity and norms for the spacelike-characteristic Cauchy problem

In this section, we introduce the notions of regularity and the initial data norms used in our main result.

**Weakly regular 2-spheres.** First, we have the following definition of weak regularity of 2-spheres, see [CG19a] and [Sha14]. This level of regularity allows to apply the basic calculus tools on 2-spheres of Section 3.

**Definition 2.21** (Weakly regular 2-spheres). Let  $1 \leq N < \infty$  be an integer and  $c > 0$  be a real number. A Riemannian 2-sphere  $(S, g)$  is a weakly regular sphere with constants  $N, c$  if

- ◊ it can be covered by  $N$  coordinate patches,
- ◊ there is a partition of unity  $\eta$  adapted to the above coordinate patches,
- ◊ there are functions  $0 \leq \tilde{\eta} \leq 1$  which are compactly supported in the coordinate patches and equal to 1 on the support of  $\eta$ ,

◇ on each coordinate patch there exists an orthonormal frame  $(e_1, e_2)$

such that on each coordinate patch,

$$\begin{aligned} c^{-1} &\leq \sqrt{\det g} \leq c, \\ c^{-1}|\xi|^2 &\leq g_{AB}\xi^A\xi^B \leq c|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2, \\ |\partial\eta| + |\partial^2\eta| + |\partial\tilde{\eta}| &\leq c, \\ \|\nabla\partial_A\|_{L^2} + \|\nabla e_A\|_{L^4} &\leq c \text{ for } A = 1, 2, \end{aligned}$$

where here for  $\xi \in \mathbb{R}^2$ ,  $|\xi|^2 := (\xi^1)^2 + (\xi^2)^2$ .

**Remark 2.22.** In [CG19a] it is shown that for the canonical foliation, the spheres  $(S_v)$  are weakly regular 2-spheres with constants  $N, c$  uniformly controlled by the  $L^2$  curvature flux through  $\mathcal{H}$  and low regularity assumptions on the foliation geometry of the initial sphere  $S_1$ . Therefore, for ease of presentation, in this paper we do not explicitly indicate the dependence of estimates on  $N, c$ .

**Weakly regular 3-balls.** The following regularity property is related to the existence of global coordinates, see Theorem 4.1.

**Definition 2.23** (Weakly regular 3-balls). *Let  $r > 0$  and  $0 < C_{\text{ball}} < 1/2$  be two real numbers. A Riemannian 3-manifold with boundary  $(\Sigma, g)$  is a weakly regular ball of radius  $r$  with constant  $C_{\text{ball}}$  if there is a global coordinate chart  $\phi : \overline{B_r} \rightarrow \Sigma$  such that on  $\overline{B_r}$ ,*

$$\begin{aligned} (1 - C_{\text{ball}})|\xi|^2 &\leq g_{ij}\xi^i\xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2, \\ \|\partial g_{ij}\|_{L^2(B_r)} + \|\partial^2 g_{ij}\|_{L^2(B_r)} &\leq C_{\text{ball}}, \end{aligned}$$

and moreover, for integers  $m \geq 1$  the higher regularity of the metric components  $g_{ij}$  can be estimated by the Ricci tensor as follows,

$$\sum_{0 \leq m' \leq m+2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_r)} \leq C_r \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + C_{\text{ball}},$$

where  $e_{ij}$  denotes the standard Cartesian components of the Euclidean metric.

**Low regularity initial data norms.** Consider initial data for the spacelike-characteristic Cauchy problem of general relativity on the spacelike hypersurface  $\Sigma \simeq \overline{B_1}$  and the outgoing null hypersurface  $\mathcal{H}$  emanating from  $\partial\Sigma$ . Let further  $(S_v)_{1 \leq v \leq 2}$  denote the canonical foliation on  $\mathcal{H}$  with  $S_1 = \partial\Sigma$ . Define

$$\begin{aligned} \mathcal{O}_0^\Sigma &:= \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)}, \\ \mathcal{R}_0^\Sigma &:= \|\text{Ric}\|_{L^2(\Sigma)}, \\ \mathcal{O}_0^\mathcal{H} &:= \left\| \text{tr}\chi - \frac{2}{v} \right\|_{L_v^\infty L^\infty(S_v)} + \left\| \text{tr}\underline{\chi} + \frac{2}{v} \right\|_{L_v^\infty L^\infty(S_v)} + \|\nabla \text{tr}\chi\|_{L_v^\infty L^2(S_v)} + \|\nabla \text{tr}\underline{\chi}\|_{L_v^\infty L^2(S_v)} \\ &\quad + \|\hat{\chi}\|_{L_v^\infty L^4(S_v)} + \|\hat{\underline{\chi}}\|_{L_v^\infty L^4(S_v)} + \|\zeta\|_{L_v^\infty H^{1/2}(S_v)} \\ &\quad + \|\nabla\Omega\|_{L_v^\infty L^4(S_v)} + \|\nabla\Omega\|_{L_v^\infty H^{1/2}(S_v)} + \|\Omega - 1\|_{L_v^\infty L^\infty(S_v)}, \\ \mathcal{R}_0^\mathcal{H} &:= \|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\underline{\beta}\|_{L^2(\mathcal{H})}. \end{aligned}$$

Here  $H^{1/2}(S_v)$  is an  $L^2$ -based fractional Sobolev space on  $S_v$  bounding 1/2 derivatives, see Definition 3.2.

**Remark 2.24.** In [CG19a] it is shown that the norm  $\mathcal{O}_0^\mathcal{H}$  can be bounded by the  $L^2$  curvature flux  $\mathcal{R}_0^\mathcal{H}$  and low regularity bounds on the geometry of the initial sphere  $S_1 = \Sigma \cap \mathcal{H}$ , see the introduction of [CG19a].

**Higher regularity initial data norms.** For integers  $m \geq 1$ , let

$$\begin{aligned}\mathcal{O}_m^\Sigma &:= \sum_{0 \leq m' \leq m+1} \|\nabla^{m'} k\|_{L^\infty(\Sigma)}, \\ \mathcal{R}_m^\Sigma &:= \sum_{0 \leq m' \leq m} \|\mathbf{D}^{m'} \mathbf{R}\|_{L^\infty(\Sigma)} + \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^\infty(\Sigma)}, \\ \mathcal{O}_m^{\mathcal{H}} &:= \sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \left( \text{tr} \chi - \frac{2}{v} \right) \right\|_{L^\infty(\mathcal{H})} + \left\| \nabla^{m'} \left( \text{tr} \underline{\chi} + \frac{2}{v} \right) \right\|_{L^\infty(\mathcal{H})} \\ &\quad + \sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \hat{\chi} \right\|_{L^\infty(\mathcal{H})} + \left\| \nabla^{m'} \underline{\hat{\chi}} \right\|_{L^\infty(\mathcal{H})} + \|\nabla^{m'} \zeta\|_{L^\infty(\mathcal{H})} \\ &\quad + \sum_{0 \leq m'+m'' \leq m+2} \|\nabla^{m'} \Omega\|_{L^\infty(\mathcal{H})} + \|\nabla^{m'} L^{m''}(\Omega)\|_{L^\infty(\mathcal{H})}, \\ \mathcal{R}_m^{\mathcal{H}} &:= \sum_{0 \leq m' \leq m} \|\mathbf{D}^{m'} \mathbf{R}\|_{L^\infty(\mathcal{H})}.\end{aligned}$$

**Remark 2.25.** For ease of presentation of the proof of the higher regularity estimates, we give ourselves leeway and choose  $L^\infty$ -based norms instead of  $L^2$ -based norms for higher regularity initial data. As consequence, the higher regularity estimates of this paper are not sharp. However, they are sufficient for the proof of the main theorem, see Section 2.9.

**Remark 2.26.** In [CG19a] it is shown that for integers  $m \geq 1$  the norm  $\mathcal{O}_m^{\mathcal{H}}$  can be bounded by higher regularity curvature fluxes and bounds on the geometry of the initial sphere  $S_1 = \Sigma \cap \mathcal{H}$ , see the introduction of [CG19a].

## 2.8 Main result

The following is the main result of this paper.

**Theorem 2.27** (Main theorem, version 2). *Consider smooth initial data for the spacelike-characteristic Cauchy problem on  $\Sigma$  and  $\mathcal{H}$ , and let  $(S_v)_{1 \leq v \leq 2}$  denote the canonical foliation on  $\mathcal{H}$  with  $S_1 = \Sigma \cap \mathcal{H}$ . Assume that the 2-spheres  $S_v$  are uniformly weakly regular with constants  $N, c$  and that for some real number  $\varepsilon > 0$ ,*

$$\mathcal{O}_0^\Sigma + \mathcal{R}_0^\Sigma + \mathcal{O}_0^{\mathcal{H}} + \mathcal{R}_0^{\mathcal{H}} \leq \varepsilon, \quad 1/4 \leq r_{\text{vol}}(\Sigma, 1/2) \leq 8, \quad 2\pi \leq \text{vol}_g(\Sigma) \leq 8\pi. \quad (2.27)$$

Then the following holds.

1.  **$L^2$ -regularity.** *Let  $0 < C_{\text{ball}} < 1/2$  be a real number. There exists a universal constant  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then the maximal smooth globally hyperbolic future development  $(\mathcal{M}, \mathbf{g})$  contains a foliation  $(\Sigma_t)_{1 \leq t \leq 2}$  of maximal spacelike hypersurfaces defined as level sets of a time function  $t$  with  $\Sigma_1 = \Sigma$  and such that for  $1 \leq t \leq 2$ ,*

$$\begin{aligned}\partial \Sigma_t &= S_t, \\ \Sigma_t &\text{ is a weakly regular ball of radius } t \text{ with constant } C_{\text{ball}},\end{aligned}$$

and moreover,

$$\begin{aligned}\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ 1/8 \leq r_{\text{vol}}(\Sigma_t, 1/2) &\leq 16, \\ \pi/2 \leq \text{vol}_g(\Sigma_t) &\leq 32\pi.\end{aligned} \quad (2.28)$$

2. **Propagation of smoothness.** *Smoothness of the initial data is propagated into the spacetime up to  $\Sigma_2 = \{t = 2\}$ . Specifically, for  $1 \leq t \leq 2$  and integers  $m \geq 1$ ,*

$$\begin{aligned}\sum_{0 \leq m' \leq m} \|\mathbf{D}^{m'} \mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\leq C(\mathcal{O}_m^\Sigma, \mathcal{R}_m^\Sigma, \mathcal{O}_m^{\mathcal{H}}, \mathcal{R}_m^{\mathcal{H}}, m), \\ \sum_{0 \leq m' \leq m+1} \|\mathbf{D}^{m'} \pi\|_{L_t^\infty L^2(\Sigma_t)} &\leq C(\mathcal{O}_m^\Sigma, \mathcal{R}_m^\Sigma, \mathcal{O}_m^{\mathcal{H}}, \mathcal{R}_m^{\mathcal{H}}, m),\end{aligned} \quad (2.29)$$

where  $\pi := \mathcal{L}_T \mathbf{g}$  denotes the deformation tensor of  $T$ .

*Remarks on Theorem 2.27.*

1. In the proof of Theorem 2.27, we derive energy estimates for the curvature tensor  $\mathbf{R}$  by using the Bel-Robinson tensor  $Q(\mathbf{R})$ , see Proposition 3.10, which in turn requires a trilinear estimate for the corresponding error term. It is due to this trilinear estimate that we need to invoke the bounded  $L^2$  curvature theorem, see Theorem 3.14. We note that for the proof of the higher regularity estimates  $m \geq 2$  of Theorem 2.27, the corresponding error term can be bounded by a classical Grönwall argument.
2. In (1) of Theorem 2.27, each hypersurface  $\Sigma_t$  is a weakly regular ball of radius  $t$  with constant  $C_{\text{ball}}$  which means that there are global coordinates on  $\Sigma_t$  such that  $\partial^2 g_{ij} \in L^2(\Sigma_t)$ . However, because these global coordinates are constructed by Cheeger-Gromov theory on  $\Sigma_t$  (see Theorem 4.1), we have no control on the regularity of the coordinate components  $g_{ij}$  in the  $t$ -direction.
3. In [CG19a] it is shown that the weak regularity of the 2-spheres  $S_v$  and the norm  $\mathcal{O}_0^{\mathcal{H}}$  can be bounded by the  $L^2$  curvature flux  $\mathcal{R}_0^{\mathcal{H}}$  and low regularity bounds on the geometry of the initial sphere  $S_1 = \Sigma \cap \mathcal{H}$ . Analogously, the norms  $\mathcal{O}_m^{\mathcal{H}}$  can be bounded by higher regularity curvature fluxes and bounds on the geometry of the initial sphere  $S_1$ .
4. The regularity assumptions (2.27) on the canonical foliation (succesfully established in [CG19a]) are crucial for the proof of Theorem 2.27. However, the exact definition of the canonical foliation is not used in this paper. Thus, any other foliation with similar regularity bounds could be used to prove Theorem 2.27.
5. The higher regularity estimates proved in Theorem 2.27 are not sharp, see Remark 2.25. Nevertheless, they are sufficient for the proof of Theorem 2.27.

## 2.9 Proof of the main result

The proof of Theorem 2.27 goes by a bootstrapping argument. Let  $T \in [1, 2]$  be defined as

$$T := \sup_{t^* \in [1, 2]} \left\{ \begin{array}{l} \text{There is a time function } 1 \leq t \leq t^* \text{ as in Theorem 2.27} \\ \text{such that (2.28) and (2.29) hold} \end{array} \right\}.$$

In the following, we show that  $T = 2$  for  $\varepsilon > 0$  sufficiently small.

**Step 1.** *It holds that  $T > 1$ .* Indeed, this follows from the following local existence result. Its proof is given in Section 6.

**Proposition 2.28** (Classical local existence and continuation). *Consider smooth initial data for the spacelike-characteristic Cauchy problem on  $\Sigma$  and  $\mathcal{H}$ , and let  $(S_v)_{v \geq 1}$  denote a smooth foliation on  $\mathcal{H}$  by spacelike 2-spheres such that  $S_1 = \partial\Sigma$ . Then, for a small real number  $\tau > 0$ , the maximal globally hyperbolic future development  $(\mathcal{M}, \mathbf{g})$  contains a foliation by smooth spacelike maximal hypersurfaces  $(\Sigma_t)_{1 \leq t \leq 1 + \tau}$  given as level sets of a smooth time function  $t$  such that  $\Sigma_1 = \Sigma$  and for each  $1 \leq t \leq 1 + \tau$ ,*

$$\partial\Sigma_t = S_t.$$

*Moreover, the foliation  $(\Sigma_t)$  can be locally continued in a smooth fashion as long as the foliation  $(\Sigma_t)$  and the spacetime remain smooth.*

*Remarks on Proposition 2.28.*

- ◇ The existence of a local maximal foliation in Proposition 2.28 follows by a classical perturbation argument of Bruhat [Cho76], see Theorem 6.2.
- ◇ We could explicitly formulate Proposition 2.28 in terms of function spaces of finite regularity, but for ease of presentation we choose the smooth class, that is,  $C^k$  for every integer  $k \geq 0$ .

**Step 2.** *Assuming that a set of bootstrap assumptions holds up to  $1 < t_0^* < 2$ , we show that we can improve them for  $\varepsilon > 0$  sufficiently small.* Indeed, the following proposition is proved in Section 4.

**Proposition 2.29** (Improvement of bootstrap assumptions). *Let  $(\mathcal{M}, \mathbf{g})$  be the maximal globally hyperbolic future development of initial data for the spacelike-characteristic Cauchy problem posed on  $\Sigma$  and  $\mathcal{H}$ . Let  $(S_v)_{1 \leq v \leq 2}$  be the canonical foliation on  $\mathcal{H}$  with  $S_1 = \Sigma \cap \mathcal{H}$  and assume that the  $S_v$  are weakly regular*

2-spheres with constants  $N, c$ . Let  $1 < t_0^* \leq 2$  be a real number, and let  $t$  be a time function on  $\mathcal{M}$  such that its level sets  $(\Sigma_t)_{1 \leq t \leq t_0^*}$  are spacelike maximal hypersurfaces with  $\Sigma_1 = \Sigma$ , satisfying for each  $1 \leq t \leq t_0^*$ ,

$$\partial\Sigma_t = S_t.$$

Assume that for some small  $\varepsilon > 0$ ,

$$\mathcal{R}_0^\Sigma + \mathcal{O}_0^\Sigma + \mathcal{O}_0^\mathcal{H} + \mathcal{R}_0^\mathcal{H} \leq \varepsilon,$$

and for some fixed large real number  $D > 0$ , for  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\leq D\varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(S_t)} &\leq D\varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(S_t)} &\leq D\varepsilon, \\ 1/8 \leq r_{\text{vol}}(\Sigma_t, 1/2) &\leq 16, \\ \pi/2 \leq \text{vol}_g(\Sigma_t) &\leq 32\pi. \end{aligned}$$

Let further  $0 < C_{\text{ball}} < 1/2$  be a real number. There exists  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then for  $1 \leq t \leq t_0^*$ ,

$$\Sigma_t \text{ is a weakly regular ball of radius } t \text{ with constant } C_{\text{ball}},$$

and

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\leq D'\varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(S_t)} + \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} &\leq D'\varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(S_t)} &\leq D'\varepsilon, \\ \|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} &\leq D'\varepsilon, \\ 1/8 < r_{\text{vol}}(\Sigma_t, 1/2) &< 16, \\ \pi/2 < \text{vol}_g(\Sigma_t) &< 32\pi, \end{aligned}$$

for a constant  $0 < D' < D$ .

**Step 3.** The following *higher regularity estimates* are proved in Section 5.

**Proposition 2.30** (Higher regularity estimates). *Let  $(\mathcal{M}, \mathbf{g})$  be the maximal globally hyperbolic future development of initial data for the spacelike-characteristic Cauchy problem posed on  $\Sigma$  and  $\mathcal{H}$ . Let  $(S_v)_{1 \leq v \leq 2}$  be the canonical foliation on  $\mathcal{H}$  with  $S_1 = \Sigma \cap \mathcal{H}$  and assume that  $S_v$  are weakly regular 2-spheres with constants  $N, c$ . Let  $1 < t_0^* \leq 2$  be a real number and assume that there is a time function  $1 \leq t \leq t_0^*$  in  $\mathcal{M}$  such that its level sets  $\Sigma_t$  are maximal spacelike hypersurfaces with  $\Sigma_1 = \Sigma$  and such that for  $1 \leq t \leq t_0^*$  and a real number  $0 < C_{\text{ball}} < 1/2$ ,*

$$\partial\Sigma_t = S_t,$$

$$\Sigma_t \text{ is a weakly regular ball of radius } t \text{ with constant } C_{\text{ball}},$$

and assume moreover there is a real number  $\varepsilon > 0$  such that for  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(S_t)} + \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} + \|\nabla \nu\|_{L_t^\infty L^4(S_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(S_t)} &\lesssim \varepsilon, \\ \|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ 1/8 \leq r_{\text{vol}}(\Sigma_t, 1/2) &\leq 16, \\ \pi/2 \leq \text{vol}_g(\Sigma_t) &\leq 32\pi. \end{aligned}$$

For  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, it holds that for all integers  $m \geq 1$ , on  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned} \sum_{0 \leq m' \leq m} \|\mathbf{D}^{m'} \mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\leq C(\mathcal{O}_m^\Sigma, \mathcal{R}_m^\Sigma, \mathcal{O}_m^\mathcal{H}, \mathcal{R}_m^\mathcal{H}, m), \\ \sum_{0 \leq m' \leq m+1} \|\mathbf{D}^{m'} \pi\|_{L_t^\infty L^2(\Sigma_t)} &\leq C(\mathcal{O}_m^\Sigma, \mathcal{R}_m^\Sigma, \mathcal{O}_m^\mathcal{H}, \mathcal{R}_m^\mathcal{H}, m), \end{aligned}$$

where  $\pi := \mathcal{L}_T \mathbf{g}$  denotes the deformation tensor of  $T$ .

*Remarks on Proposition 2.30.*

1. The estimates for  $m = 1$  require a trilinear estimate based on the null structure of the Einstein vacuum equations, see Section 5. In Appendix E we show that this trilinear estimate follows from the  $(m = 1)$ -estimates of the proof of the bounded  $L^2$  curvature theorem [KRS15].
2. The estimates for  $m \geq 2$  are based on a classical Grönwall argument together with the estimates for  $m = 1$ ; see Section 5.2.
3. The smallness of  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  is only used in the proof of the case  $m = 1$  in context of the trilinear estimate.

**Step 4.** By the higher regularity estimates of Proposition 2.30, it follows that on each hypersurface  $\Sigma_t$ , the induced data is smooth. Thus by Proposition 2.28 and Proposition 2.29, we can *continue the spacetime such that (2.28) and (2.29) hold beyond  $T$  for each  $T < 2$* , which shows by the maximality of  $T$  that  $T = 2$ . This concludes the proof of Theorem 2.27.

## 2.10 Organisation of the paper

The paper is organised as follows.

- ◇ In Section 3, we recapitulate calculus inequalities and prerequisite results.
- ◇ In Section 4, we prove the low regularity estimates of Proposition 2.29.
- ◇ In Section 5, we prove the higher regularity estimates of Proposition 2.30.
- ◇ In Section 6, we prove classical local existence and continuation for the Einstein equations, see Proposition 2.28.
- ◇ In Section 7, we prove the existence of global coordinates by Cheeger-Gromov theory, see Theorem 4.1, which is used to establish that  $\Sigma_t$  are weakly regular 3-balls in Section 4.3.
- ◇ In Appendix A, we prove global elliptic estimates for 3-dimensional Hodge systems.
- ◇ In Appendix B, we prove the calculus Lemmas 3.6 and 3.8.
- ◇ In Appendix C, we prove a Riemannian rigidity result, see Lemma 7.4, used in the proof of existence of global coordinates in Section 7.
- ◇ In Appendix D, we prove comparison estimates between two maximal foliations on a vacuum spacetime used in Sections 4 and 5.
- ◇ In Appendix E, we prove the necessary trilinear estimate for the  $(m = 1)$ -higher regularity estimates, see Proposition 5.2.

## 3. Calculus inequalities and prerequisite results

### 3.1 Calculus on Riemannian 2-spheres

In this section, we recapitulate calculus prerequisites on weakly regular Riemannian 2-spheres  $(S, g)$ . The following lemma is proved in [Sha14].

**Lemma 3.1** (Sobolev inequalities). *Let  $(S, g)$  be a weakly regular Riemannian 2-sphere with constants  $N, c$ . Let  $1 \leq p < \infty$  be a real number. Then it holds that for each tensor  $F$  on  $S$ ,*

$$\begin{aligned} \|F\|_{L^p(S)} &\lesssim \|\nabla F\|_{L^2(S)} + \|F\|_{L^2(S)}, \\ \|F\|_{L^\infty(S)} &\lesssim \|\nabla F\|_{L^4(S)} + \|F\|_{L^2(S)}, \end{aligned}$$

where the constants depend on  $p, N, c$  and  $N, c$ , respectively.

We introduce the following fractional Sobolev spaces.

**Definition 3.2** (Fractional Sobolev spaces). *Let  $(S, g)$  be a Riemannian 2-sphere and let  $-\infty < s < \infty$  be a real number. For tensors  $F$  on  $S$ , define the norm*

$$\|F\|_{H^s(S)} := \|(1 - \Delta)^{s/2} F\|_{L^2(S)},$$

where the fractional Laplace-Beltrami operator is defined by standard spectral decomposition, see [Sha14].

The following properties of fractional Sobolev spaces are well-known, see for example Section 2 and Theorem 3.6 in [Sha14] and Section 3 and Appendix B in [CG19a].

**Lemma 3.3** (Properties of fractional Sobolev spaces). *Let  $(S, g)$  be a weakly regular Riemannian 2-sphere with constants  $N, c$ . Let  $F, F_1$  and  $F_2$  be tensors on  $S$ . Then it holds that*

$$\begin{aligned} \|F\|_{L^4(S)} &\lesssim \|F\|_{H^{1/2}(S)}, \\ \|\nabla F\|_{H^{-1/2}(S)} &\lesssim \|F\|_{H^{1/2}(S)}, \\ \|F\|_{H^{1/2}(S)} &\lesssim \|F\|_{H^1(S)}, \\ \|F_1 F_2\|_{H^{1/2}(S)} &\lesssim (\|F_1\|_{L^\infty(S)} + \|\nabla F_1\|_{L^2(S)}) \|F_2\|_{H^{1/2}(S)}, \end{aligned}$$

where the constants depend on  $N, c$ .

### 3.2 Calculus on Riemannian 3-manifolds with boundary

In this section, we recall low regularity calculus prerequisites on compact Riemannian 3-manifolds with boundary  $(\Sigma, g)$ . The following Sobolev inequalities are well-known, see for example page 44 in [Sze18].

**Lemma 3.4** (Sobolev inequalities on  $\Sigma$ ). *Let  $(\Sigma, g)$  be a compact Riemannian 3-manifold with boundary such that, in local charts,*

$$\frac{1}{4}|\xi| \leq g_{ij}\xi^i\xi^j \leq 2|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2. \quad (3.1)$$

Then for each tensor  $F$  on  $\Sigma$ ,

$$\begin{aligned} \|F\|_{L^\infty(\Sigma)} &\lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)} + \|\nabla^2 F\|_{L^2(\Sigma)}, \\ \|F\|_{L^6(\Sigma)} &\lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}. \end{aligned}$$

**Remark 3.5.** *Assumption (3.1) is in particular satisfied by weakly regular 3-balls with suitable constant  $0 < C_{\text{ball}} < 1/2$ .*

The following trace estimates are well-known, see for example [AF08] and [Sha14]. For completeness, a proof is provided in Appendix B.



**Lemma 3.6** (Trace estimates onto  $\partial\Sigma$ ). *Let  $1 \leq r \leq 2$  and  $0 < C_{\text{ball}} < 1/2$  be real numbers, and let  $(\Sigma, g)$  be a weakly regular ball of radius  $r$  with constant  $C_{\text{ball}}$ . Then for each tensor  $F$  on  $\Sigma$ , it holds that*

$$\begin{aligned} \|F\|_{L^4(\partial\Sigma)} &\lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}, \\ \|F\|_{H^{1/2}(\partial\Sigma)} &\lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}. \end{aligned} \quad (3.2)$$

Moreover, for all integers  $m \geq 1$ ,

$$\sum_{0 \leq m' \leq m} \|\nabla^{m'} F\|_{H^{1/2}(\partial\Sigma)} \lesssim \sum_{0 \leq m' \leq m+1} \|\nabla^{m'} F\|_{L^2(\Sigma)} + \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + C_m C_{\text{ball}}.$$

**Remark 3.7.** *The first of (3.2) more generally holds under the assumption that  $g_{ij} \in C^0$ .*

The following lemma allows to estimate the  $L^2$ -norm of a tensor by the  $L^2$ -norm of its covariant derivative and its boundary value; see Appendix B for a proof.

**Lemma 3.8.** *Let  $1 \leq r \leq 2$  and  $0 < C_{\text{ball}} < 1/2$  be two real numbers. Let  $(\Sigma, g)$  be a weakly regular ball of radius  $r$  with constant  $C_{\text{ball}}$ , and let  $F$  be a tensor on  $\Sigma$ . Then,*

$$\|F\|_{L^2(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)} + \|F\|_{L^2(\partial\Sigma)}.$$

For real numbers  $1 \leq r \leq 2$  and  $0 < C_{\text{ball}} < 1/2$ , let  $(\Sigma, g)$  be a weakly regular ball of radius  $r$  with constant  $C_{\text{ball}}$ . In Section 2.4, we defined spherical coordinates  $(r, \theta^1, \theta^2)$  on  $\Sigma$  and expressed the metric as follows,

$$g = a^2 dr^2 + g_{AB} (b^A dr + d\theta^A)(b^B dr + d\theta^B).$$

Moreover, we defined for real numbers  $1 \leq r_1 \leq r_2 \leq 2$  the annulus  $A(r_1, r_2)$  in (2.16). The following lemma shows that in spherical coordinates, the metric components are estimated by the constant  $C_{\text{ball}}$ ; see Lemma 2.22 in [Czi18] for a proof.

**Lemma 3.9** (Estimates for metric components in spherical coordinates). *Let  $1 \leq r \leq 2$  and  $0 < C_{\text{ball}} < 1/2$  be two real numbers. Let  $(\Sigma, g)$  be a weakly regular ball of radius  $r$  with constant  $C_{\text{ball}}$ . Then*

$$\begin{aligned} \|a - 1\|_{H^2(A(r/2, r))} + \|b\|_{H^2(A(r/2, r))} + \|g_{AB} - \gamma_{AB}\|_{H^2(A(r/2, r))} &\lesssim C_{\text{ball}}, \\ \left\| \text{tr}\Theta - \frac{2}{r'} \right\|_{H^1(A(r/2, r))} + \|\widehat{\Theta}\|_{H^1(A(r/2, r))} + \|\nabla a\|_{H^1(A(r/2, r))} &\lesssim C_{\text{ball}}, \end{aligned} \quad (3.3)$$

where  $\gamma_{AB}$  denotes the standard round metric on  $\mathcal{S}_r$ .

### 3.3 Global elliptic estimates for the Laplace-Beltrami operator on $\Sigma$

The following global elliptic estimates are applied in this paper to the maximal lapse equation to control the lapse  $n$ . The proof follows from a straight-forward generalisation of the estimates proved in Appendix B in [Czi19a].

**Proposition 3.10.** *Let  $1 \leq r \leq 2$  and  $0 < C_{\text{ball}} < 1/2$  be two real numbers. Let  $(\Sigma, g)$  be a weakly regular ball of radius  $r$  with constant  $C_{\text{ball}}$ . Then for any scalar function  $f$  on  $\Sigma$ ,*

$$\begin{aligned} \sum_{0 \leq m' \leq 2} \|\nabla^{m'} f\|_{L^2(\Sigma)} &\lesssim \|\Delta f\|_{L^2(\Sigma)} + \|\nabla f\|_{H^{1/2}(\partial\Sigma)} + \|f\|_{L^2(\partial\Sigma)}, \\ \sum_{0 \leq m' \leq 3} \|\nabla^{m'} f\|_{L^2(\Sigma)} &\lesssim \sum_{0 \leq m' \leq 1} \|\nabla^{m'} (\Delta f)\|_{L^2(\Sigma)} + \sum_{1 \leq m' \leq 2} \|\nabla^{m'} f\|_{H^{1/2}(\partial\Sigma)} + \|f\|_{L^2(\partial\Sigma)}. \end{aligned}$$

Furthermore, for integers  $m \geq 1$ ,

$$\begin{aligned} &\sum_{0 \leq m' \leq m+2} \|\nabla^{m'} f\|_{L^2(\Sigma)} \\ &\lesssim \sum_{0 \leq m' \leq m} \|\nabla^{m'} \Delta f\|_{L^2(\Sigma)} + \sum_{0 \leq m' \leq m+1} \|\nabla^{m'} f\|_{H^{1/2}(\partial\Sigma)} + \|f\|_{L^2(\partial\Sigma)} \\ &\quad + C_m \left( \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + C_{\text{ball}} \right) (\|\Delta f\|_{L^2(\Sigma)} + \|\nabla f\|_{H^{1/2}(\partial\Sigma)} + \|f\|_{L^2(\partial\Sigma)}). \end{aligned}$$

### 3.4 Energy estimates for Weyl tensors on vacuum spacetimes

The following classical energy estimate for Weyl tensors is proved in [CK93], see the introduction and Lemma 8.1.1 therein.

**Proposition 3.11** (Classical energy estimate for Weyl tensors). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime bounded by two disjoint maximal spacelike hypersurfaces  $\Sigma_1$  and  $\Sigma_2$  and an outgoing null hypersurface  $\mathcal{H}$ , and assume that  $\mathcal{M}$  is foliated by the spacelike level sets  $(\Sigma_t)_{1 \leq t \leq 2}$  of a time function  $t$  such that  $\{t = 1\} = \Sigma_1$  and  $\{t = 2\} = \Sigma_2$ . Let  $T$  denote the timelike unit normal to  $\Sigma_t$ . Let further  $\mathbf{W}$  be a Weyl tensor on  $\mathcal{M}$ . Then it holds that*

$$\begin{aligned} \int_{\Sigma_2} Q(\mathbf{W})_{TTTT} &= \int_{\Sigma_1} Q(\mathbf{W})_{TTTT} + \int_{\mathcal{H}} Q(\mathbf{W})_{LTTT} \\ &\quad - \int_{\mathcal{M}} \mathbf{D}^\mu Q(\mathbf{W})_{\mu TTT} - \int_{\mathcal{M}} \frac{3}{2} Q(\mathbf{W})_{\alpha\beta TT} \pi^{\alpha\beta}, \end{aligned} \quad (3.4)$$

where  $\pi := \mathcal{L}_T \mathbf{g}$  denotes the deformation tensor of  $T$ , the integral over  $\mathcal{H}$  is defined in Definition 2.18, and  $L$  is the fixed null generator of  $\mathcal{H}$  defined in Section 2.2.

### 3.5 An extension procedure for the constraint equations

The following result of [Czi18] is used as a *black box* in this paper.

**Theorem 3.12** (Extension procedure for the constraints, [Czi18]). *Let  $1 \leq r \leq 2$  be a real number. Let  $(\bar{g}, \bar{k})$  be maximal initial data for the Einstein equations on  $B_r \subset \mathbb{R}^3$ . There exists a universal constant  $\varepsilon > 0$  such that if*

$$\sum_{0 \leq m' \leq 2} \|\partial^{m'}(\bar{g}_{ij} - e_{ij})\|_{L^2(B_r)} + \sum_{0 \leq m' \leq 1} \|\partial^{m'} \bar{k}_{ij}\|_{L^2(B_r)} < \varepsilon,$$

where  $e_{ij}$  denotes the standard Euclidean metric in Cartesian coordinates, then  $(\bar{g}, \bar{k})$  can be smoothly extended to maximal initial data  $(g, k)$  on  $\mathbb{R}^3$  with

$$(g, k)|_{B_r} = (\bar{g}, \bar{k}),$$

such that

$$\|g_{ij} - e_{ij}\|_{H^2_{-1/2}(\mathbb{R}^3)} + \|k_{ij}\|_{H^1_{-3/2}(\mathbb{R}^3)} \lesssim \sum_{0 \leq m' \leq 2} \|\partial^{m'}(\bar{g}_{ij} - e_{ij})\|_{L^2(B_r)} + \sum_{0 \leq m' \leq 1} \|\partial^{m'} \bar{k}_{ij}\|_{L^2(B_r)},$$

where  $H^2_{-1/2}(\mathbb{R}^3)$  and  $H^1_{-3/2}(\mathbb{R}^3)$  are weighted Sobolev spaces bounding 2 and 1 coordinate derivatives, respectively, and measuring asymptotic flatness, see [Czi18]. In particular, the constructed maximal initial data  $(g, k)$  on  $\mathbb{R}^3$  satisfies

$$r_{vol}(\mathbb{R}^3, 1/2) > 1/4$$

and

$$\|\text{Ric}\|_{L^2(\mathbb{R}^3)} + \|k\|_{L^2(\mathbb{R}^3)} + \|\nabla k\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{0 \leq m' \leq 2} \|\partial^{m'}(\bar{g}_{ij} - e_{ij})\|_{L^2(B_r)} + \sum_{0 \leq m' \leq 1} \|\partial^{m'} \bar{k}_{ij}\|_{L^2(B_r)}.$$

More generally, in case of higher regularity, for integers  $m \geq 1$ ,

$$\begin{aligned} &\|g_{ij} - e_{ij}\|_{H^{m+2}_{-1/2}(\mathbb{R}^3)} + \|k_{ij}\|_{H^{m+1}_{-3/2}(\mathbb{R}^3)} \\ &\leq C_m \left( \sum_{0 \leq m' \leq m+2} \|\partial^{m'}(\bar{g}_{ij} - e_{ij})\|_{L^2(B_r)} + \sum_{0 \leq m' \leq m+1} \|\partial^{m'} \bar{k}_{ij}\|_{L^2(B_r)} \right), \end{aligned}$$

where  $H^{m+2}_{-1/2}(\mathbb{R}^3)$  and  $H^{m+1}_{-3/2}(\mathbb{R}^3)$  are weighted Sobolev spaces bounding  $m+2$  and  $m+1$  coordinate derivatives, respectively, and measuring asymptotic flatness, see [Czi18]. In particular, it holds that for integers  $m \geq 1$ ,

$$\begin{aligned} &\sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\mathbb{R}^3)} + \sum_{0 \leq m' \leq m-1} \|\nabla^{m'} k\|_{L^2(\mathbb{R}^3)} \\ &\leq C_m \left( \sum_{0 \leq m' \leq m+2} \|\partial^{m'}(\bar{g}_{ij} - e_{ij})\|_{L^2(B_r)} + \sum_{0 \leq m' \leq m+1} \|\partial^{m'} \bar{k}_{ij}\|_{L^2(B_r)} \right). \end{aligned}$$

**Remark 3.13.** In [Czi18], Theorem 3.12 is proved for maximal initial data  $(\bar{g}, \bar{k})$  given on the unit ball  $B_1$ . However, it is straight-forward to generalise that result to maximal initial data given on  $B_r$  for  $1 \leq r \leq 2$ , as stated in Theorem 3.12 above.

### 3.6 The bounded $L^2$ curvature theorem

The following theorem is a paraphrase of Theorems 2.4 and 2.5 in [Sze12a] and Theorem 2.18 in [Sze18], and used as a *black box* in this paper.

**Theorem 3.14** (The bounded  $L^2$  curvature theorem, version 2). *Let  $(\Sigma, g, k)$  be asymptotically flat maximal initial data for the Einstein vacuum equations such that  $\Sigma \simeq \mathbb{R}^3$ . Assume moreover that for some  $\varepsilon > 0$ ,*

$$\|\text{Ric}\|_{L^2(\Sigma)} \leq \varepsilon, \quad \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon, \quad r_{\text{vol}}(\Sigma, 1/2) > 1/4.$$

Then.

1.  **$L^2$ -regularity.** *There is a universal constant  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then the maximal globally hyperbolic future development  $(\mathcal{M}, \mathbf{g})$  of the initial data  $(\Sigma, g, k)$  contains a foliation of maximal spacelike hypersurfaces  $(\Sigma_t)_{0 \leq t \leq 1}$  with  $\Sigma_0 = \Sigma$  such that on each  $\Sigma_t$ ,*

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|n - 1\|_{L^\infty(\mathcal{M})} + \|\nabla n\|_{L^\infty(\mathcal{M})} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla T(n)\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ r_{\text{vol}}(\Sigma_t, 1/2) &\geq 1/8. \end{aligned}$$

Moreover, for each  $\omega \in \mathbb{S}^2$ , there is a foliation  $(\mathcal{H}_{\omega_u})_{\omega_u \in \mathbb{R}}$  of  $\mathcal{M}$  by weakly regular (see remarks below) null hyperplanes  $\mathcal{H}_{\omega_u}$  given as level sets of an optical function  $\omega u$  such that

$$\sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot L\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})} \lesssim \varepsilon,$$

where  $L$  denotes the  $\mathcal{H}_{\omega_u}$ -tangential null vectorfield with  $\mathbf{g}(T, L) = -1$ . Here  $\mathbf{R} \cdot L$  denotes contractions of the Riemann curvature tensor  $\mathbf{R}$  on  $\mathcal{M}$  with  $L$ . In addition, the following trilinear estimate holds,

$$\left| \int_{\mathcal{M}} Q(\mathbf{R})_{ijTT} k^{ij} \right| \lesssim \varepsilon \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)}^2 + \varepsilon \|\mathbf{R}\|_{L^2(\mathcal{M})} \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot L\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})}. \quad (3.5)$$

2. **Higher regularity.** *For integers  $m \geq 1$ , it holds that*

$$\begin{aligned} \sum_{0 \leq m' \leq m} \|\mathbf{D}^{m'} \mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{m'} \nabla k\|_{L^2(\Sigma)}, \\ \sum_{0 \leq m' \leq m+1} \|\mathbf{D}^{m'} \pi\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{m'} \nabla k\|_{L^2(\Sigma)}, \end{aligned}$$

where  $\pi := \mathcal{L}_T \mathbf{g}$  denotes the deformation tensor of  $T$ .

*Remarks.*

1. Theorem 1.3 is the *small data version* of the bounded  $L^2$  curvature theorem. A corresponding large data version is obtained in [KRS15] by a rescaling procedure.
2. We refer to Definition 5.3 in [KRS15] for a definition of weakly regular null hypersurfaces. For the purposes of this paper, it suffices to note that weak regularity is sufficient for an application of Stokes' theorem as in Proposition 3.11.
3. In Appendix E, we give more details about the wave parametrix formalism of [KRS15].

## 4. Low regularity estimates

In this section we prove Proposition 2.29. Let  $\varepsilon > 0$  and  $1 < t_0^* < 2$  be real numbers. Assume that

$$\mathcal{O}_0^\Sigma + \mathcal{R}_0^\Sigma + \mathcal{O}_0^{\mathcal{H}} + \mathcal{R}_0^{\mathcal{H}} \leq \varepsilon, \quad (4.1)$$

and that, for a large fixed constant  $D > 0$  and for  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\leq D\varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(S_t)} &\leq D\varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(S_t)} &\leq D\varepsilon, \\ 1/4 \leq r_{\text{vol}}(\Sigma_t, 1/2) &\leq 8, \\ \pi/2 \leq \text{vol}_g(\Sigma_t) &\leq 32\pi. \end{aligned} \quad (4.2)$$

In the following, we prove that for  $\varepsilon > 0$  sufficiently small, for  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\leq D'\varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(S_t)} &\leq D'\varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(S_t)} &\leq D'\varepsilon, \\ 1/4 < r_{\text{vol}}(\Sigma_t, 1/2) &< 8, \\ \pi/2 < \text{vol}_g(\Sigma_t) &< 32\pi. \end{aligned} \quad (4.3)$$

for a constant  $0 < D' < D$ , and in addition, we show that

$$\begin{aligned} \|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon. \end{aligned} \quad (4.4)$$

**Notation.** Pick  $1 \leq t^* \leq t_0^*$ . In the following, we prove (4.3) and (4.4) on  $\Sigma_{t^*}$ . As  $t^*$  was chosen arbitrarily, this implies (4.3) and (4.4) for  $1 \leq t \leq t_0^*$ .

### 4.1 Overview of the proof

In the following, we outline the main steps of the proof of Proposition 2.29. An important tool is the following theorem about the existence of global coordinates which is applied in the proof of Proposition 2.29 to each hypersurface  $\Sigma_t$ . Its proof is given in Section 7 and based on the Cheeger-Gromov theory of manifold convergence [Czi19a].

**Theorem 4.1** (Existence of global regular coordinates). *Let  $(M, g)$  be a compact Riemannian 3-manifold with boundary such that  $M \simeq \overline{B_1} \subset \mathbb{R}^3$ . Assume that for real numbers  $1 \leq t \leq 2$ ,  $\varepsilon > 0$  and  $0 < V < \infty$ ,*

$$\|\text{Ric}\|_{L^2(M)} \leq \varepsilon, \quad \left\| \text{tr}\Theta - \frac{2}{t} \right\|_{L^4(\partial M)} + \|\widehat{\Theta}\|_{L^4(\partial M)} \leq \varepsilon, \quad r_{\text{vol}}(M, 1/2) \geq 1/4, \quad \text{vol}_g(M) \leq V,$$

where  $\Theta$  denotes the second fundamental form of  $\partial M \subset M$ . Then for every real number  $0 < C_{\text{ball}} < 1/2$  there is an  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then

$$(M, g) \text{ is a weakly regular ball of radius } t \text{ with constant } C_{\text{ball}},$$

that is,

1.  **$H^2$ -regularity.** There is a global coordinate chart  $\phi: \overline{B_t} \rightarrow M$  such that

$$\begin{aligned} (1 - C_{\text{ball}})|\xi|^2 &\leq g_{ij}\xi^i\xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2, \\ \sum_{0 \leq m' \leq 2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_t)} &\lesssim C_{\text{ball}}. \end{aligned}$$

2. **Higher regularity.** For integers  $m \geq 1$  it holds that

$$\sum_{0 \leq m' \leq m+2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_t)} \lesssim C_V \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(M)} + C_{m,V},$$

where  $e$  denotes the standard Euclidean metric on  $B_t$ .

We are now in position to give an overview of the proof of Proposition 2.29. As noted above, it suffices to improve the bootstrap assumptions on  $\Sigma_{t^*}$  for a fixed real number

$$1 \leq t^* \leq t_0^*.$$

1. Let  $0 < C_{\text{ball}} < 1/2$  be a real number to be determined below. By the bootstrap assumptions (4.2) together with Theorem 4.1, we deduce that for  $\varepsilon > 0$  sufficiently small,  $\Sigma_{t^*}$  is a weakly regular ball of radius  $t^*$  with constant  $C_{\text{ball}}$ . For  $C_{\text{ball}} > 0$  sufficiently small, this directly improves the bootstrap assumptions on  $\text{vol}_g(\Sigma_{t^*})$  and  $r_{\text{vol}}(\Sigma_{t^*}, 1/2)$ , see Section 4.3.
2. For  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, the *extension procedure for the constraint equations* (Theorem 3.12) can be applied to  $\Sigma_{t^*}$ . This yields an extension of the maximal initial data  $(\Sigma_{t^*}, g, k)$  to an asymptotically flat maximal initial data set of size bounded by  $C_{\text{ball}}$ , see Section 4.5.
3. For  $C_{\text{ball}} > 0$  sufficiently small, we can subsequently apply *backwards* the bounded  $L^2$  curvature theorem (Theorem 3.14) to the above extended initial data set. This yields a foliation of the past of  $\Sigma_{t^*}$  in  $\mathcal{M}$  by maximal hypersurfaces  $(\tilde{\Sigma}_{\tilde{t}})_{0 \leq \tilde{t} \leq t^*}$  which satisfies in particular  $\tilde{\nabla} \tilde{n} \in L^\infty(\mathcal{M}_{t^*})$  and admits a trilinear estimate, see Section 4.5.
4. Using the  $\tilde{\Sigma}_{\tilde{t}}$ -foliation, we apply the Bel-Robinson energy estimate for  $\mathbf{R}$  (see Proposition 3.10) to estimate the curvature flux through  $\Sigma_{t^*}$  by the curvature fluxes through  $\mathcal{H}$  and  $\Sigma$ . As the latter are bounded by the initial data norms (see Remark 4.2 below), this improves the bootstrap assumption on the curvature flux on  $\Sigma_{t^*}$ . It is in this energy estimate that the control of  $\tilde{\nabla} \tilde{n} \in L^\infty(\mathcal{M}_{t^*})$  and the trilinear estimate for the  $(\tilde{\Sigma}_{\tilde{t}})_{0 \leq \tilde{t} \leq t^*}$ -foliation are essential, see Section 4.6.
5. The second fundamental form  $k$  on  $\Sigma_{t^*}$  satisfies a Hodge system. By applying global elliptic estimates (Corollary A.6), we improve the bootstrap assumptions on  $\nabla k \in L^2(\Sigma_{t^*})$ . Here we use that the source terms in the Hodge system are curvature terms which were improved in the previous step. Moreover, here it is crucial to analyse the boundary integrals appearing in the global elliptic estimates for  $k$ . Indeed, they admit a special structure which allows to split them up into two parts: One part which *has the right sign* to control the slope  $\nu$  between  $\mathcal{H}$  and  $\Sigma_{t^*}$  and the value of  $k$  on  $\partial\Sigma$ , and a second part which can be estimated by the initial data norm on  $\mathcal{H}$ . See Section 4.7 for details.
6. The bootstrap assumptions for  $\nu$  on  $\partial\Sigma_{t^*}$  are fully improved by using the slope equation (2.24) together with the previously improved bounds for  $k$  and the initial data norms on  $\mathcal{H}$ , see Section 4.10.
7. The foliation lapse  $n$  of the foliation  $\Sigma_t$  is improved by global elliptic estimates applied to the maximal lapse equation, using that  $k$  on  $\Sigma_{t^*}$  and the boundary value  $n = \Omega^{-1}\nu^{-1}$  on  $\partial\Sigma_{t^*}$  are improved in the previous steps, see (4.38). The estimate for  $\mathbf{D}_T k$  on  $\Sigma_{t^*}$  follows from the second variation equation (2.13a), see Corollary 4.11.

**Remark 4.2.** *To bound the curvature fluxes through  $\mathcal{H}$  and  $\Sigma$  by the initial data norms on  $\mathcal{H}$  and  $\Sigma$ , one needs a comparison argument between the two maximal foliations  $(\Sigma_t)_{1 \leq t \leq t^*}$  and  $(\tilde{\Sigma})_{0 \leq \tilde{t} \leq t^*}$ , see Lemma 4.6. This comparison requires the control of  $n \in L^\infty(\mathcal{M}_{t^*})$ , and hence in the proof of Proposition 2.29 we first bound  $n - 1$  of size  $D\varepsilon$  by using the smallness assumption (4.1) and the bootstrap assumptions (4.2), before improving the curvature estimates on  $\Sigma_{t^*}$ .*

## 4.2 First consequences of the bootstrap assumptions

We first remark that by (4.1) and (4.2),

$$\begin{aligned} \left\| \text{tr}\chi - \frac{2}{t} \right\|_{L_t^\infty L^\infty(S_t)} + \left\| \text{tr}\underline{\chi} + \frac{2}{t} \right\|_{L_t^\infty L^\infty(S_t)} &\lesssim \varepsilon, \\ \|\hat{\chi}\|_{L_t^\infty L^4(S_t)} + \|\hat{\underline{\chi}}\|_{L_t^\infty L^4(S_t)} &\lesssim \varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} &\lesssim D\varepsilon. \end{aligned} \tag{4.5}$$

Using that by Lemma 2.11

$$\begin{aligned} \text{tr}\Theta - \frac{2}{t} &= \frac{1}{2}\nu \text{tr}\chi - \frac{1}{2}\nu^{-1} \text{tr}\underline{\chi} - \frac{2}{t} \\ &= \frac{1}{2}(\nu - 1)\text{tr}\chi + \frac{1}{2} \left( \text{tr}\chi - \frac{2}{t} \right) - \frac{1}{2}(\nu^{-1} - 1)\text{tr}\underline{\chi} - \frac{1}{2} \left( \text{tr}\underline{\chi} + \frac{2}{t} \right), \end{aligned}$$

we therefore have by (4.5) for  $\varepsilon > 0$  sufficiently small that

$$\left\| \text{tr}\Theta - \frac{2}{t} \right\|_{L_t^\infty L^\infty(S_t)} \lesssim D\varepsilon. \quad (4.6)$$

Moreover, using that by Lemma 2.11

$$\widehat{\Theta} = \frac{1}{2}\nu\widehat{\chi} - \frac{1}{2}\nu^{-1}\widehat{\underline{\chi}},$$

we get by (4.5) for  $\varepsilon > 0$  sufficiently small that

$$\left\| \widehat{\Theta} \right\|_{L_t^\infty L^4(S_t)} \lesssim \varepsilon. \quad (4.7)$$

### 4.3 Weakly regular ball property of $\Sigma_{t^*}$

From (4.2), (4.6) and (4.7), we have for  $\varepsilon > 0$  sufficiently small that

$$\begin{aligned} \|\text{Ric}\|_{L^2(\Sigma_{t^*})} &\lesssim D\varepsilon, \\ \left\| \text{tr}\Theta - \frac{2}{t^*} \right\|_{L^4(S_{t^*})} + \left\| \widehat{\Theta} \right\|_{L^4(S_{t^*})} &\lesssim D\varepsilon, \\ r_{\text{vol}}(\Sigma_{t^*}, 1/2) &\geq 1/4, \\ \text{vol}_g(\Sigma_{t^*}) &\leq 32\pi. \end{aligned}$$

Let  $0 < C_{\text{ball}} < 1/2$  be a real number to be determined below. By Theorem 4.1, there exists  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then  $\Sigma_{t^*}$  is a *weakly regular ball of radius  $t^*$  with constant  $C_{\text{ball}}$* , that is, there is a global coordinate chart  $\phi : \overline{B_{t^*}} \rightarrow \Sigma_{t^*}$  such that

$$\begin{aligned} (1 - C_{\text{ball}})|\xi|^2 &\leq g_{ij}\xi^i\xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2, \\ \sum_{0 \leq m' \leq 2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_{t^*})} &\lesssim C_{\text{ball}}. \end{aligned} \quad (4.8)$$

For  $C_{\text{ball}} > 0$  sufficiently small, it follows from (4.8) that

$$1/4 < r_{\text{vol}}(\Sigma_{t^*}, 1/2) < 8, \quad \pi/2 < \text{vol}_g(\Sigma_{t^*}) < 32\pi,$$

which improves the bootstrap assumptions on  $r_{\text{vol}}(\Sigma_{t^*}, 1/2)$  and  $\text{vol}_g(\Sigma_{t^*})$  in (4.2).

**Remark 4.3.** *In this paper it generally holds that  $C_{\text{ball}} \gg \varepsilon$ , see Theorem 4.1. In particular, demanding  $C_{\text{ball}} > 0$  to be sufficiently small thus stipulates that  $\varepsilon > 0$  be sufficiently small. For ease of presentation, this is tacitly acknowledged in the following.*

**Remark 4.4.** *In particular, the above shows that the calculus results of Section 3.2 for weakly regular balls hold on  $\Sigma_t$ .*

### 4.4 Estimates for the lapse $n$ on $\Sigma_{t^*}$

The lapse function  $n$  is by (2.13e) and (2.25) a solution to the following elliptic boundary value problem,

$$\begin{aligned} \Delta n &= n|k|_g^2 && \text{on } \Sigma_{t^*}, \\ n &= \nu^{-1}\Omega^{-1} && \text{on } \partial\Sigma_{t^*}. \end{aligned} \quad (4.9)$$

In this section, we use (4.1), (4.2) and global elliptic estimates to prove that for  $\varepsilon > 0$  sufficiently small,

$$\|n - 1\|_{L^\infty(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 n\|_{L^2(\Sigma_{t^*})} \lesssim D\varepsilon. \quad (4.10)$$

**Remark 4.5.** *In accordance with the continuity argument of the proof of the main result, we do not have any bootstrap assumptions on  $n$  in (4.2).*

On the one hand, by (4.1), (4.2) and Lemma 3.3, the boundary value  $n = \nu^{-1}\Omega^{-1}$  satisfies for  $\varepsilon > 0$  sufficiently small

$$\begin{aligned} \|n - 1\|_{L^\infty(\partial\Sigma_{t^*})} &\lesssim \|\nu - 1\|_{L^\infty(\partial\Sigma_{t^*})} + \|\Omega - 1\|_{L^\infty(\partial\Sigma_{t^*})} \\ &\lesssim D\varepsilon, \\ \|\nabla n\|_{L^2(\partial\Sigma_{t^*})} &\lesssim \|\nabla \nu\|_{L^2(\partial\Sigma_{t^*})} + \|\nabla \Omega\|_{L^2(\partial\Sigma_{t^*})} \\ &\lesssim D\varepsilon, \end{aligned}$$

as well as

$$\begin{aligned} &\|\nabla n\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ &= \left\| \frac{1}{\nu^2\Omega} \nabla \nu + \frac{1}{\nu\Omega^2} \nabla \Omega \right\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ &\lesssim \left( \left\| \nabla \left( \frac{1}{\Omega\nu^2} \right) \right\|_{L^2(\partial\Sigma_{t^*})} + \left\| \frac{1}{\Omega\nu^2} \right\|_{L^\infty(\partial\Sigma_{t^*})} + \left\| \nabla \left( \frac{1}{\Omega^2\nu} \right) \right\|_{L^2(\partial\Sigma_{t^*})} + \left\| \frac{1}{\Omega^2\nu} \right\|_{L^\infty(\partial\Sigma_{t^*})} \right) \\ &\quad \cdot \left( \|\nabla \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla \Omega\|_{H^{1/2}(\partial\Sigma_{t^*})} \right) \\ &\lesssim \|\nabla \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla \Omega\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ &\lesssim D\varepsilon. \end{aligned}$$

On the other hand, by (4.2), (4.8), (4.9) and Lemma 3.4, we have that

$$\begin{aligned} \|\Delta n\|_{L^2(\Sigma_{t^*})} &= \|n|k|^2\|_{L^2(\Sigma_{t^*})} \\ &\lesssim \|n\|_{L^6(\Sigma_{t^*})} \|k\|_{L^6(\Sigma_{t^*})}^2 \\ &\lesssim (1 + \|n - 1\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})}) (\|k\|_{L^2(\Sigma_{t^*})} + \|\nabla k\|_{L^2(\Sigma_{t^*})})^2 \\ &\lesssim (1 + \|n - 1\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})}) (D\varepsilon)^2. \end{aligned}$$

By the elliptic estimates of Proposition 3.10, we hence get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} &\|n - 1\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 n\|_{L^2(\Sigma_{t^*})} \\ &\lesssim \|\Delta n\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|n\|_{L^2(\partial\Sigma_{t^*})} \\ &\lesssim (\|n - 1\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})}) (D\varepsilon)^2 + D\varepsilon. \end{aligned}$$

For  $\varepsilon > 0$  sufficiently small, we can absorb the first term on the right-hand side into the left-hand side to get that

$$\|n - 1\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 n\|_{L^2(\Sigma_{t^*})} \lesssim D\varepsilon.$$

The estimate (4.10) follows then by Lemma 3.4.

## 4.5 Construction of a background foliation of the past of $\Sigma_{t^*}$

Let  $\mathcal{M}_{t^*}$  denote the past of  $\Sigma_{t^*}$  in  $\mathcal{M}$ . In this section, we apply *backwards* the bounded  $L^2$  curvature theorem on  $\Sigma_{t^*}$  to construct a background foliation of  $\mathcal{M}_{t^*}$ .

First, on the one hand, we have by (4.8) that

$$\begin{aligned} (1 - C_{\text{ball}})|\xi|^2 &\leq g_{ij}\xi^i\xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2, \\ \sum_{0 \leq m' \leq 2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_{t^*})} &\lesssim C_{\text{ball}}. \end{aligned} \tag{4.11}$$

On the other hand, by (4.2),

$$\|k\|_{L^2(\Sigma_{t^*})} + \|\nabla k\|_{L^2(\Sigma_{t^*})} \leq D\varepsilon. \tag{4.12}$$

For  $\varepsilon > 0$  sufficiently small, (4.11) and (4.12) imply by standard product estimates that

$$\sum_{0 \leq m' \leq 1} \|\partial^{m'} k_{ij}\|_{L^2(B_{t^*})} \lesssim C_{\text{ball}}.$$

In particular, with (4.11), it holds that

$$\sum_{0 \leq m' \leq 2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_{t^*})} + \sum_{0 \leq m' \leq 1} \|\partial^{m'} k_{ij}\|_{L^2(B_{t^*})} \lesssim C_{\text{ball}}. \quad (4.13)$$

Second, using (4.13) and assuming  $C_{\text{ball}} > 0$  is sufficiently small, we can apply the extension procedure of Theorem 3.12 to extend  $(g, k)$  from  $\Sigma_{t^*} = \phi(\bar{B}_{t^*})$  to an asymptotically flat maximal initial data  $(g', k')$  on  $\mathbb{R}^3$  satisfying

$$\begin{aligned} r_{\text{vol}}(\mathbb{R}^3, 1/2) &> 1/4, \\ \|\text{Ric}'\|_{L^2(\mathbb{R}^3)} + \|k'\|_{L^2(\mathbb{R}^3)} + \|\nabla' k'\|_{L^2(\mathbb{R}^3)} &\lesssim C_{\text{ball}}. \end{aligned} \quad (4.14)$$

Here  $r_{\text{vol}}(\mathbb{R}^3, 1/2)$ ,  $\text{Ric}'$  and  $\nabla'$  denote the volume radius, the Ricci curvature and the covariant derivative with respect to  $g'$  on  $\mathbb{R}^3$ .

Third, by (4.14) the maximal initial data  $(\mathbb{R}^3, g', k')$  satisfies for  $C_{\text{ball}} > 0$  sufficiently small the assumptions of the bounded  $L^2$  curvature theorem (see Theorem 3.14). Thus, Theorem 3.14 yields the following.

1.  $\mathcal{M}_{t^*}$  is foliated by spacelike maximal hypersurfaces  $(\tilde{\Sigma}_{\tilde{t}})_0 \leq \tilde{t} \leq t^*$  given as level sets of a time function  $\tilde{t}$  with  $\Sigma_{t^*} = \tilde{\Sigma}_{t^*}$  and satisfying for  $0 \leq \tilde{t} \leq t^*$ ,

$$\begin{aligned} \|\widetilde{\text{Ric}}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, & \|\tilde{k}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\nabla} \tilde{k}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \\ \|\mathbf{R}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, & \|\tilde{n} - 1\|_{L_{\tilde{t}}^\infty L^\infty(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\nabla} \tilde{n}\|_{L_{\tilde{t}}^\infty L^\infty(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \end{aligned} \quad (4.15)$$

where  $\widetilde{\text{Ric}}$ ,  $\tilde{k}$  and  $\widetilde{\nabla}$  denote the induced Ricci curvature, the second fundamental form and the induced covariant derivative on  $\tilde{\Sigma}_{\tilde{t}}$ , respectively, and  $\tilde{n}$  denotes the lapse of the foliation  $(\tilde{\Sigma}_{\tilde{t}})_0 \leq \tilde{t} \leq t^*$ .

2. For each  $\omega \in \mathbb{S}^2$ , the spacetime portion  $\mathcal{M}_{t^*}$  is foliated by a family of null hyperplanes  $(\mathcal{H}^{\omega_u})_{\omega_u \in \mathbb{R}}$  given as level sets of an optical function  $\omega_u$  satisfying

$$\sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_{\omega_u}^\infty L^2(\mathcal{H}^{\omega_u})} \lesssim C_{\text{ball}},$$

where  $\tilde{L}$  is the unique  $\mathcal{H}^{\omega_u}$ -tangent null vectorfield with  $\mathbf{g}(\tilde{L}, \tilde{T}) = -1$  and  $\tilde{T}$  denotes the future-pointing time-like unit normal to  $\tilde{\Sigma}_{\tilde{t}}$ .

3. Define the *angle*  $\tilde{\nu}$  between  $T$  and  $\tilde{T}$  by

$$\tilde{\nu} := -\mathbf{g}(T, \tilde{T}). \quad (4.16)$$

The proof of the next lemma is provided in Appendix D.

**Lemma 4.6** (Comparison of maximal foliations on  $\mathcal{M}_{t^*}$ ). *For  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\tilde{\nu} - 1\|_{L_{\tilde{t}}^\infty L^\infty(\Sigma_{\tilde{t}})} \lesssim C_{\text{ball}}, \quad \|\tilde{k}\|_{L_{\tilde{t}}^\infty L^4(\Sigma_{\tilde{t}})} \lesssim C_{\text{ball}}, \quad (4.17)$$

where  $\tilde{k}$  denotes the second fundamental form of  $\tilde{\Sigma}_{\tilde{t}}$ .

## 4.6 Energy estimates for the curvature tensor on $\Sigma_{t^*}$

In this section, we prove that

$$\|\mathbf{R}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} \lesssim \varepsilon. \quad (4.18)$$

Using that by construction  $\Sigma_{t^*} = \tilde{\Sigma}_{t^*}$ , (4.18) implies in particular that

$$\|\mathbf{R}\|_{L^2(\Sigma_{t^*})} \lesssim \varepsilon. \quad (4.19)$$

We turn to the proof of (4.18). Recall from Section 4.5 that for each  $\omega \in \mathbb{S}^2$ ,  $\mathcal{M}_{t^*}$  is foliated by families of null hyperplanes  $(\mathcal{H}^{\omega_u})_{\omega_u \in \mathbb{R}}$ . Applying the classical Bel-Robinson energy estimate (3.4) for  $W = \mathbf{R}$  with  $\tilde{T}$



as multiplier field over the spacetime region bounded by  $\Sigma$ ,  $\mathcal{H}$ ,  $\Sigma_{t^*}$  and  $\mathcal{H}_{\omega_u}$  (for some  $\omega \in \mathbb{S}^2$  and  $\omega_u \in \mathbb{R}$ ) and subsequently taking the supremum over  $\omega \in \mathbb{S}^2$  and  $\omega_u \in \mathbb{R}$ , we get that

$$\begin{aligned} & \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\ & \lesssim \int_{\Sigma_1} Q(\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T} + \int_{\mathcal{H}} Q(\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} + \underbrace{\left| \int_{\mathcal{M}_{t^*}} Q(\mathbf{R})_{\alpha\beta} \tilde{T}\tilde{T}\tilde{\pi}^{\alpha\beta} \right|}_{=:\mathcal{E}}, \end{aligned} \quad (4.20)$$

where we used (2.11) and Lemma 2.16. The error term  $\mathcal{E}$  on the right-hand side of (4.20) is bounded as follows. By (2.6) the components of  $\tilde{\pi} := \hat{\mathcal{L}}_{\tilde{T}}\mathbf{g}$  are

$$\tilde{\pi}_{\tilde{T}\tilde{T}} = 0, \quad \tilde{\pi}_{\tilde{T}j} = \tilde{n}^{-1} \nabla_j \tilde{n}, \quad \tilde{\pi}_{ab} = -2\tilde{k}_{ab}.$$

Hence, by (4.15),

$$\begin{aligned} \mathcal{E} &= \left| \int_{\mathcal{M}_{t^*}} Q(\mathbf{R})_{\alpha\beta} \tilde{T}\tilde{T}\tilde{\pi}^{\alpha\beta} \right| \\ &\lesssim \left| \int_{\mathcal{M}_{t^*}} Q(\mathbf{R})_{ab} \tilde{T}\tilde{T}\tilde{k}^{ab} \right| + \left| \int_{\mathcal{M}_{t^*}} \tilde{n}^{-1} Q(\mathbf{R})_{\tilde{T}j} \tilde{T}\tilde{T}\tilde{\nabla}^j \tilde{n} \right| \\ &\lesssim \left| \int_{\mathcal{M}_{t^*}} Q(\mathbf{R})_{ab} \tilde{T}\tilde{T}\tilde{k}^{ab} \right| + \left(1 + \|\tilde{n} - 1\|_{L^\infty(\mathcal{M}_{t^*})}\right) \|\tilde{\nabla}\tilde{n}\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 \\ &\lesssim \left| \int_{\mathcal{M}_{t^*}} Q(\mathbf{R})_{ab} \tilde{T}\tilde{T}\tilde{k}^{ab} \right| + C_{\text{ball}} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2. \end{aligned} \quad (4.21)$$

The first term on the right-hand side of (4.21) is estimated by a localisation of the trilinear estimate (3.5) of Theorem 3.14. Indeed, a direct inspection of its proof on page 112 in [KRS15] yields that the following estimate holds on  $\mathcal{M}_{t^*}$ ,

$$\begin{aligned} \left| \int_{\mathcal{M}_{t^*}} Q(\mathbf{R})_{ab} \tilde{T}\tilde{T}\tilde{k}^{ab} \right| &\lesssim C_{\text{ball}} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + C_{\text{ball}} \|\mathbf{R}\|_{L^2(\mathcal{M}_{t^*})} \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})} \\ &\lesssim C_{\text{ball}} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + C_{\text{ball}} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)} \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})} \\ &\lesssim C_{\text{ball}} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + C_{\text{ball}} \left( \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})} \right)^2. \end{aligned}$$

Plugging (4.21) and the above into (4.20), we get

$$\begin{aligned} & \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\ & \lesssim \int_{\Sigma_1} Q(\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T} + \int_{\mathcal{H}} Q(\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} \\ & \quad + C_{\text{ball}} \|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + C_{\text{ball}} \left( \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})} \right)^2. \end{aligned}$$

For  $C_{\text{ball}} > 0$  sufficiently small we can absorb the third and fourth term on the right-hand side into the left-hand side, which yields

$$\|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})}^2 \lesssim \underbrace{\int_{\Sigma_1} Q(\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T}}_{:=\mathcal{I}_1} + \underbrace{\int_{\mathcal{H}} Q(\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L}}_{:=\mathcal{I}_2}, \quad (4.22)$$

It remains to control the boundary integrals  $\mathcal{I}_1$  and  $\mathcal{I}_2$  on the right-hand side of (4.22).

**Control of  $\mathcal{I}_1$ .** Let  $(e_i)_{i=1,2,3}$  be an orthonormal frame of  $\Sigma_1$ . Decompose  $\tilde{T}$  with respect to this frame into

$$\tilde{T} = \tilde{\nu}T + C^1 e_1 + C^2 e_2 + C^3 e_3, \quad (4.23)$$

and denote  $C^0 := \tilde{\nu} - 1$  and  $e_0 := T$ . By (4.17), we have that

$$\|C^0\|_{L^\infty(\Sigma_1)} \lesssim C_{\text{ball}}. \quad (4.24)$$

Using that  $T$  and  $\tilde{T}$  are unit timelike vectors, that  $(e_1, e_2, e_3)$  is an orthonormal frame tangent to  $\Sigma_1$  and (4.24), we have that

$$-1 = \mathbf{g}(\tilde{T}, \tilde{T}) = -\tilde{\nu}^2 + \sum_{i=1}^3 |C^i|^2,$$

so that on  $\Sigma_1$ ,

$$|C^i| \leq \sqrt{\nu^2 - 1} = \sqrt{C^0} \sqrt{\nu + 1} \lesssim \sqrt{C_{\text{ball}}} \quad \text{for } i = 1, 2, 3. \quad (4.25)$$

Using (2.11), (2.12), (4.1), (4.23), (4.24) and (4.25), we get that

$$\begin{aligned} \mathcal{I}_1 &:= \int_{\Sigma_1} Q_{\tilde{T}\tilde{T}\tilde{T}T} \\ &= \int_{\Sigma_1} Q_{TTTT} + \int_{\Sigma_1} C^\mu Q_{\mu TTT} + \int_{\Sigma_1} C^\mu C^\nu Q_{\mu\nu TT} + \int_{\Sigma_1} C^\mu C^\nu C^\lambda Q_{\mu\nu\lambda T} \\ &\lesssim \left(1 + \sqrt{C_{\text{ball}}} + \sqrt{C_{\text{ball}}^2} + \sqrt{C_{\text{ball}}^3}\right) \int_{\Sigma_1} Q_{TTTT} \\ &\lesssim \|\mathbf{R}\|_{L^2(\Sigma_1)}^2 \\ &\lesssim \varepsilon^2. \end{aligned}$$

**Control of  $\mathcal{I}_2$ .** Let  $(N, e_1, e_2)$  be a local frame on  $\mathcal{H}$  such that  $(e_1, e_2)$  is an orthonormal frame tangent to  $\partial\Sigma_t$  and  $N$  is tangent to  $\Sigma_t$  and normal to  $\partial\Sigma_t$ . Decompose  $\tilde{T}$  into

$$\tilde{T} = \tilde{\nu}T + C'^1 e_1 + C'^2 e_2 + C'^3 N, \quad (4.26)$$

and denote  $C'^0 := \tilde{\nu} - 1$  and  $e_0 = T$ . Using (4.17), we have that

$$\|C'^0\|_{L^\infty(\Sigma_1)} \lesssim C_{\text{ball}}. \quad (4.27)$$

Using that  $T$  and  $\tilde{T}$  are timelike unit, that  $(N, e_1, e_2)$  is an orthonormal frame and (4.27), it follows that

$$\|C'^i\|_{L^\infty(\mathcal{H})} \lesssim \sqrt{C_{\text{ball}}} \quad \text{for } i = 1, 2, 3. \quad (4.28)$$

By Lemma 2.16, (2.26), (4.1), (4.17), (4.26), (4.27) and (4.28), we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \mathcal{I}_2 &:= \int_{\mathcal{H}} Q_{\tilde{T}\tilde{T}\tilde{T}L} \\ &= \int_{\mathcal{H}} Q_{TTTL} + \int_{\mathcal{H}} C'^\mu Q_{\mu TTL} + \int_{\mathcal{H}} C'^\mu C'^\nu Q_{\mu\nu TL} + \int_{\mathcal{H}} C'^\mu C'^\nu C'^\lambda Q_{\mu\nu\lambda L} \\ &\lesssim \left(1 + \sqrt{C_{\text{ball}}} + \sqrt{C_{\text{ball}}^2} + \sqrt{C_{\text{ball}}^3}\right) \int_{\mathcal{H}} Q_{TTTL} \\ &\lesssim \|\tilde{\nu}\|_{L^\infty(\mathcal{H})}^3 \|\alpha\|_{L^2(\mathcal{H})}^2 + \|\tilde{\nu}\|_{L^\infty(\mathcal{H})} \|\beta\|_{L^2(\mathcal{H})}^2 \\ &\quad + \|\tilde{\nu}^{-1}\|_{L^\infty(\mathcal{H})} (\|\rho\|_{L^2(\mathcal{H})}^2 + \|\sigma\|_{L^2(\mathcal{H})}^2) + \|\tilde{\nu}^{-3}\|_{L^\infty(\mathcal{H})} \|\underline{\beta}\|_{L^2(\mathcal{H})}^2 \\ &\lesssim \|\alpha\|_{L^2(\mathcal{H})}^2 + \|\beta\|_{L^2(\mathcal{H})}^2 + \|\rho\|_{L^2(\mathcal{H})}^2 + \|\sigma\|_{L^2(\mathcal{H})}^2 + \|\underline{\beta}\|_{L^2(\mathcal{H})}^2 \\ &\lesssim \varepsilon^2. \end{aligned}$$

To summarise, plugging the above control of  $\mathcal{I}_1$  and  $\mathcal{I}_2$  into (4.22), we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\|\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \tilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})}^2 \lesssim \varepsilon^2.$$

In particular, this implies (4.18) and subsequently (4.19).

## 4.7 Elliptic estimates for the second fundamental form $k$ on $\Sigma_{t^*}$

In this section we prove the following proposition to improve the bootstrap assumption (4.2) for  $k$ .

**Proposition 4.7** (Global elliptic estimate for  $k$ ). *It holds that*

$$\|\nabla k\|_{L^2(\Sigma_{t^*})}^2 + \|k\|_{L^2(\Sigma_{t^*})}^2 + \|k\|_{L^4(\Sigma_{t^*})}^4 + \|k\|_{L^2(\partial\Sigma_{t^*})}^2 + \|\nabla\nu\|_{L^2(\partial\Sigma_{t^*})}^2 \lesssim (\sqrt{D}\varepsilon)^2.$$

*Proof.* Using (2.13b), (2.13c), (2.13d) and Corollary A.6, we have the following well-known classical global elliptic estimate for  $k$ ,

$$\int_{\Sigma_{t^*}} |\nabla k|^2 + \frac{1}{4}|k|^4 - \int_{\partial\Sigma_{t^*}} \nabla_a k_{bN} k^{ba} \lesssim \int_{\Sigma_{t^*}} |\mathbf{R}|_{\mathbf{h}^t}^2, \quad (4.29)$$

where  $N$  denotes the outward-pointing unit normal to  $\partial\Sigma_{t^*} \subset \Sigma_{t^*}$ .

By  $\operatorname{div} k = 0$ , see (2.13b), the boundary term on the left-hand side of (4.29) can be rewritten as

$$\begin{aligned} - \int_{\partial\Sigma_{t^*}} \nabla_a k_{bN} k^{ba} &= - \int_{\partial\Sigma_{t^*}} \left( \nabla_N k_{bN} k^{bN} + \nabla_C k_{bN} k^{bC} \right) \\ &= - \int_{\partial\Sigma_{t^*}} \left( -\nabla^C k_{bC} k^{bN} + \nabla_C k_{NN} k^{NC} + \nabla_C k_{AN} k^{AC} \right) \\ &= \int_{\partial\Sigma_{t^*}} \nabla^C k_{AC} k^{AN} + \nabla_C k_{NC} k^{NN} - \nabla_C k_{NN} k^{NC} - \nabla_C k_{AN} k^{AC}, \end{aligned} \quad (4.30)$$

where summation over  $A, C = 1, 2$  indicates taking the trace with respect to a local orthonormal frame  $(e_1, e_2)$  on  $\partial\Sigma_{t^*}$ . Using the boundary decomposition of Section 2.4 for  $\nabla k$  on  $\partial\Sigma_{t^*}$ , that is,

$$\begin{aligned} \nabla^C k_{AC} &= \mathfrak{d}\nabla\nu\eta_A + \Theta_{AB}\epsilon^B + \operatorname{tr}\Theta\epsilon_A, & \nabla^C k_{NC} &= \mathfrak{d}\nabla\nu\epsilon - \eta \cdot \Theta + \delta\operatorname{tr}\Theta, \\ \nabla_C k_{NN} &= \nabla_C\delta - 2\Theta_{CB}\epsilon^B, & \nabla_C k_{AN} &= \nabla_C\epsilon_A - \eta_A^B\Theta_{BC} + \delta\Theta_{AC}, \end{aligned}$$

where  $\delta = k_{NN}$ ,  $\epsilon_A = k_{NA}$  and  $\eta_{AB} = k_{AB}$ , and integrating by parts, we get that (4.30) becomes

$$\begin{aligned} - \int_{\partial\Sigma_{t^*}} \nabla_a k_{bN} k^{ba} &= \int_{\partial\Sigma_{t^*}} 2(\mathfrak{d}\nabla\nu)_A \epsilon^A - 2\epsilon \cdot \nabla\delta + 3\Theta_{AB}\epsilon^A\epsilon^B + \operatorname{tr}\Theta|\epsilon|^2 \\ &\quad + \int_{\partial\Sigma_{t^*}} \delta^2\operatorname{tr}\Theta - 2\delta\eta_{AB}\Theta^{AB} + \eta_{AB}\Theta_{BC}\eta^{CA} \\ &= \int_{\partial\Sigma_{t^*}} -4\epsilon \cdot \nabla\delta - 2{}^*H_{N^A} \epsilon^A + 5\Theta_{AB}\epsilon^A\epsilon^B - \operatorname{tr}\Theta|\epsilon|^2 \\ &\quad + \int_{\partial\Sigma_{t^*}} \delta^2\operatorname{tr}\Theta - 2\delta\eta_{AB}\Theta^{AB} + \eta_{AB}\Theta_{BC}\eta^{CA}, \end{aligned} \quad (4.31)$$

where we used the following Gauss-Codazzi equation in the last equality, see (2.18),

$$\mathfrak{d}\nabla\nu\eta_A = -\nabla_A\delta - {}^*H_{N^A} + \Theta \cdot \epsilon - \operatorname{tr}\Theta\epsilon.$$

We can re-arrange (4.31) as

$$\begin{aligned} - \int_{\partial\Sigma_{t^*}} \nabla_a k_{bN} k^{ba} &= - \int_{\partial\Sigma_{t^*}} 4\epsilon \cdot \nabla\delta + 2\epsilon^{AB} H_{BN}\epsilon_A \\ &\quad + \int_{\partial\Sigma_{t^*}} \left( \operatorname{tr}\Theta - \frac{2}{t^*} \right) \left( \frac{3}{2}|\epsilon|^2 + 2|\delta|^2 + \frac{1}{2}|\eta|^2 \right) \\ &\quad + \int_{\partial\Sigma_{t^*}} 5\widehat{\Theta}_{AB}\epsilon^A\epsilon^B - 2\delta\widehat{\Theta}_{AB}\eta^{AB} - \eta_{AC}\widehat{\Theta}^{CB}\eta_B^A \\ &\quad + \int_{\partial\Sigma_{t^*}} \frac{3}{t^*}|\epsilon|^2 + \frac{4}{t^*}|\delta|^2 + \frac{1}{t^*}|\eta|^2, \end{aligned} \quad (4.32)$$

where we used that by definition, see (2.19) and (2.20),

$${}^*H_N \epsilon^A = \epsilon^{AB} H_{BN} \epsilon_A.$$

Since  $\partial\Sigma_{t^*} \subset \mathcal{H}$ , we can use Lemmas 2.11 and 2.13, that is, the relations

$$\begin{aligned} \epsilon_A &= -\nu^{-1} \nabla_A \nu + \zeta_A, \\ \delta &= \frac{1}{2} \nu \operatorname{tr} \chi + \frac{1}{2} \nu^{-1} \operatorname{tr} \underline{\chi}, \\ \nabla \delta &= \nabla \left( \frac{1}{2} \nu \operatorname{tr} \chi + \frac{1}{2} \nu^{-1} \operatorname{tr} \underline{\chi} \right) \\ &= \underbrace{\left( \frac{1}{t^*} \left( 1 + \frac{1}{\nu^2} \right) + \frac{1}{2} \left( \operatorname{tr} \chi - \frac{2}{t^*} \right) - \frac{1}{2} \frac{1}{\nu^2} \left( \operatorname{tr} \underline{\chi} + \frac{2}{t^*} \right) \right)}_{=: F(\nu, \operatorname{tr} \chi, \operatorname{tr} \underline{\chi})} \nabla \nu \\ &\quad + \frac{1}{2} \nu \nabla \operatorname{tr} \chi + \frac{1}{2} \nu^{-1} \nabla \operatorname{tr} \underline{\chi}, \end{aligned} \tag{4.33}$$

to rewrite the first term on the right-hand side of (4.32) as follows,

$$\begin{aligned} - \int_{\partial\Sigma_{t^*}} 4\epsilon \cdot \nabla \delta &= - \int_{\partial\Sigma_{t^*}} 4\zeta \cdot \nabla \delta + \int_{\partial\Sigma_{t^*}} 4\nu^{-1} \nabla \nu \cdot \nabla \delta \\ &= - \int_{\partial\Sigma_{t^*}} 4\zeta \cdot \nabla \delta + \int_{\partial\Sigma_{t^*}} 4\nu^{-1} F(\nu, \operatorname{tr} \chi, \operatorname{tr} \underline{\chi}) |\nabla \nu|^2 \\ &\quad + \int_{\partial\Sigma_{t^*}} 2 (\nabla \nu \cdot \nabla \operatorname{tr} \chi + \nu^{-2} \nabla \nu \cdot \nabla \operatorname{tr} \underline{\chi}). \end{aligned} \tag{4.34}$$

By (4.1) and (4.2), it holds for  $\varepsilon > 0$  sufficiently small that on  $\partial\Sigma_{t^*}$ ,

$$\nu^{-1} F(\nu, \operatorname{tr} \chi, \operatorname{tr} \underline{\chi}) \geq \frac{1}{8}.$$

Hence, for  $\varepsilon > 0$  sufficiently small, (4.34) yields

$$- \int_{\partial\Sigma_{t^*}} 4\epsilon \cdot \nabla \delta \geq \int_{\partial\Sigma_{t^*}} \frac{1}{2} |\nabla \nu|^2 + \int_{\partial\Sigma_{t^*}} 2 (\nabla \nu \cdot \nabla \operatorname{tr} \chi + \nu^{-2} \nabla \nu \cdot \nabla \operatorname{tr} \underline{\chi}) - \int_{\partial\Sigma_{t^*}} 4\zeta \cdot \nabla \delta. \tag{4.35}$$

Plugging (4.32) and (4.35) into (4.29), we get that for  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} &\int_{\Sigma_{t^*}} |\nabla k|^2 + \frac{1}{4} |k|^4 + \int_{\partial\Sigma_{t^*}} |k|^2 + \int_{\partial\Sigma_{t^*}} |\nabla \nu|^2 \\ &\lesssim \underbrace{\int_{\Sigma_{t^*}} |\mathbf{R}|_{\mathbf{h}^t}^2}_{:=I_1} + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \epsilon^{AB} H_{AN} \epsilon_B \right|}_{:=I_2} + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \left( \operatorname{tr} \Theta - \frac{2}{t^*} \right) |k|^2 \right|}_{:=I_3} \\ &\quad + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \widehat{\Theta}_{AB} \epsilon^A \epsilon^B \right|}_{:=I_4} + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \delta \widehat{\Theta}_{AB} \eta^{AB} \right|}_{:=I_5} + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \eta_{AC} \widehat{\Theta}^{CB} \eta_B^A \right|}_{:=I_6} \\ &\quad + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \zeta \cdot \nabla \delta \right|}_{:=I_7} + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \nabla \nu \cdot \nabla \operatorname{tr} \chi \right|}_{:=I_8} + \underbrace{\left| \int_{\partial\Sigma_{t^*}} \nu^{-2} \nabla \nu \cdot \nabla \operatorname{tr} \underline{\chi} \right|}_{:=I_9}. \end{aligned} \tag{4.36}$$

In the following, we control the terms  $I_1$ - $I_9$  on the right-hand side of (4.36).

**Control of  $I_1$ .** From (4.19), we have

$$I_1 := \int_{\Sigma_{t^*}} |\mathbf{R}|_{\mathfrak{h}^t}^2 \lesssim \varepsilon^2. \quad (4.37a)$$

**Control of  $I_2$ .** In the following we use, based on the control (4.8), the spherical coordinates  $(r, \theta^1, \theta^2)$  on  $\overline{B_{t^*}} = \phi^{-1}(\Sigma_{t^*})$  as defined in Section 2.4. Let  $\psi : \overline{B_{t^*}} \rightarrow [0, 1]$  be a smooth radial cut-off function such that  $\psi(x) = 1$  for  $|x| \geq 3t^*/4$  and  $\psi(x) = 0$  for  $|x| \leq t^*/2$ , where  $|\cdot|$  denotes the Euclidean norm on  $\overline{B_{t^*}}$ . Then by the fundamental theorem of calculus, (4.2), Lemma 3.9 and the fact that due to (2.11) and (4.19),

$$\|E\|_{L^2(\Sigma_{t^*})} + \|H\|_{L^2(\Sigma_{t^*})} \lesssim \|\mathbf{R}\|_{L^2(\Sigma_{t^*})} \lesssim \varepsilon, \quad (4.37b)$$

we have that

$$\begin{aligned} & \int_{\partial\Sigma_{t^*}} \in^{AB} H_{AN} \epsilon_B \\ &= \int_{t^*/2}^{t^*} \partial_r \left( \psi \int_{\mathcal{S}_r} \in^{AB} H_{AN} \epsilon_B \right) dr \\ &\lesssim \int_{t^*/2}^{t^*} \psi \partial_r \left( \int_{\mathcal{S}_r} \in^{AB} H_{AN} \epsilon_B \right) dr + \|H\|_{L^2(\Sigma_{t^*})} \|k\|_{L^2(\Sigma_{t^*})} \\ &\lesssim \int_{t^*/2}^{t^*} \psi \left( \int_{\mathcal{S}_r} (\partial_r (\in^{AB} H_{AN} \epsilon_B) + \in^{AB} H_{AN} \epsilon_B a \operatorname{tr}\Theta) \right) dr + \varepsilon(D\varepsilon) \\ &\lesssim \int_{t^*/2}^{t^*} \psi \int_{\mathcal{S}_r} \partial_r (\in^{AB} H_{AN} \epsilon_B) dr + \varepsilon(D\varepsilon). \end{aligned} \quad (4.37c)$$

Using that  $\partial_r = aN + b$  and  $\in^{AB} = \in^{ABN}$ , we can express the integrand on the right-hand side of (4.37c) as

$$\begin{aligned} \partial_r (\in^{AB} H_{AN} \epsilon_B) &= \partial_r (\in^{ijN} H_{iN} k_{jN}) \\ &= a \nabla_N (\in^{ijN} H_{iN} k_{jN}) + b (\in^{AB} H_{AN} \epsilon_B) \\ &= a \nabla_N (\in^{ijN} H_{iN} k_{jN}) + b^c \nabla_c (\in^{AB} H_{AN} \epsilon_B), \end{aligned} \quad (4.37d)$$

Further, using that  $\nabla_N N = -a^{-1} \nabla a$  and (2.15), that is,

$$\nabla_N H_{AN} = -\mathfrak{d}\nabla H_A - \Theta_{AC} H_N^C - \operatorname{tr}\Theta H_{AN} - (k \wedge E)_A,$$

where  $\mathfrak{H}$  denotes the projection of  $H$  onto  $\mathcal{S}_r$ , the first term on the right-hand side of (4.37d) can be expressed as follows,

$$\begin{aligned} & a \nabla_N (\in^{ijN} H_{iN} k_{jN}) \\ &= a \in^{ijl} (-a^{-1} \nabla_l a) H_{iN} k_{jN} + a \in^{ijN} (\nabla_N H_{iN} - a^{-1} \nabla^C a H_{iC}) k_{jN} \\ &\quad + a \in^{ijN} H_{iN} (\nabla_N k_{jN} - a^{-1} \nabla^C a k_{jC}) \\ &= a \in^{ijl} (-a^{-1} \nabla_l a) H_{iN} k_{jN} + a \in^{AB} (\nabla_N H_{AN} - a^{-1} \nabla^C a H_{AC}) \epsilon_B \\ &\quad + a \in^{AB} H_{AN} (\nabla_N k_{BN} - a^{-1} \nabla^C a k_{BC}) \\ &= a \in^{ijl} (-a^{-1} \nabla_l a) H_{iN} k_{jN} \\ &\quad + a \in^{AB} (-\mathfrak{d}\nabla H_A - \Theta_{AC} H_N^C - \operatorname{tr}\Theta H_{NA} - (k \wedge E)_A - a^{-1} \nabla^C a H_{AC}) \epsilon_B \\ &\quad + a \in^{AB} H_{AN} (\nabla_N k_{BN} - a^{-1} \nabla^C a k_{BC}). \end{aligned}$$

Plugging (4.37d) and the above into (4.37c) and integrating by parts on  $\mathcal{S}_r$  the terms  $-a \in^{AB} \mathfrak{d}\nabla H_A \epsilon_B$  and  $b^c \nabla_c (\in^{AB} H_{AN} \epsilon_B)$ , we get that

$$\begin{aligned}
 & \int_{\partial\Sigma_{t^*}} \in^{AB} H_{AN} \epsilon_B \\
 & \lesssim \int_{t^*/2}^{t^*} \psi \left( \int_{\mathcal{S}_r} (\in^{ijkl} (-a^{-1} \nabla_l a) H_{iN} k_{jN} - \text{div} b (\in^{AB} H_{AN} \epsilon_B)) \right) dr \\
 & \quad + \int_{t^*/2}^{t^*} \psi \left( \int_{\mathcal{S}_r} \in^{AB} \nabla^C (a \epsilon_B) \mathbb{H}_{AC} \right) dr \\
 & \quad + \int_{t^*/2}^{t^*} \psi \left( \int_{\mathcal{S}_r} a \in^{AB} (-\Theta_{AC} H_N^C - \text{tr} \Theta H_{NA} - (k \wedge E)_A - a^{-1} \nabla^C a H_{AC}) \epsilon_B \right) dr \\
 & \quad + \int_{t^*/2}^{t^*} \psi \left( \int_{\mathcal{S}_r} a \in^{AB} H_{AN} (\nabla_N k_{BN} - a^{-1} \nabla^C a k_{BC}) \right) dr + \varepsilon D \varepsilon.
 \end{aligned}$$

By Lemmas 3.4 and 3.9, (4.2), (4.8) and (4.37b), the right-hand side can be estimated as

$$\begin{aligned}
 I_2 := \left| \int_{\partial\Sigma_{t^*}} \in^{AB} H_{AN} \epsilon_B \right| & \lesssim (1 + C_{\text{ball}}) \|H\|_{L^2(\Sigma_{t^*})} (\|\nabla k\|_{L^2(\Sigma_{t^*})} + \|k\|_{L^2(\Sigma_{t^*})}) \\
 & \lesssim (1 + C_{\text{ball}}) \varepsilon D \varepsilon \\
 & \lesssim (\sqrt{D} \varepsilon)^2.
 \end{aligned} \tag{4.37e}$$

**Control of  $I_3$ .** Using (4.2) and (4.6), we have

$$\begin{aligned}
 I_3 := \left| \int_{\partial\Sigma_{t^*}} \left( \text{tr} \Theta - \frac{2}{t^*} \right) |k|^2 \right| & \leq \left\| \text{tr} \Theta - \frac{2}{t^*} \right\|_{L^\infty(\partial\Sigma_{t^*})} \|k\|_{L^2(\partial\Sigma_{t^*})}^2 \\
 & \lesssim (D\varepsilon)^3.
 \end{aligned} \tag{4.37f}$$

**Control of  $I_4, I_5$  and  $I_6$ .** By Lemma 3.6 and (4.7), we have

$$\begin{aligned}
 I_4 := \left| \int_{\partial\Sigma_{t^*}} \widehat{\Theta}_{AB} \epsilon^A \epsilon^B \right| & \lesssim \|\widehat{\Theta}\|_{L^4(\partial\Sigma_{t^*})} \|\epsilon\|_{L^2(\partial\Sigma_{t^*})} \|\epsilon\|_{L^4(\partial\Sigma_{t^*})} \\
 & \lesssim \|\widehat{\Theta}\|_{L^4(\partial\Sigma_{t^*})} \left( \|\epsilon\|_{L^2(\Sigma_{t^*})} + \|\nabla \epsilon\|_{L^2(\Sigma_{t^*})} \right)^2 \\
 & \lesssim \|\widehat{\Theta}\|_{L^4(\partial\Sigma_{t^*})} (\|k\|_{L^2(\Sigma_{t^*})} + \|\nabla k\|_{L^2(\Sigma_{t^*})})^2 \\
 & \lesssim \varepsilon (D\varepsilon)^2.
 \end{aligned} \tag{4.37g}$$

The terms  $I_5$  and  $I_6$  are bounded similarly as

$$I_5 + I_6 \lesssim \varepsilon (D\varepsilon)^2. \tag{4.37h}$$

**Control of  $I_7, I_8$  and  $I_9$ .** By (4.1), (4.2) and (4.33), we have

$$\begin{aligned}
 I_7 &:= \left| \int_{\partial\Sigma_{t^*}} \zeta \cdot \nabla \delta \right| \\
 &\leq \left| \int_{\partial\Sigma_{t^*}} F(\nu, \text{tr } \chi, \text{tr } \underline{\chi}) \zeta \cdot \nabla \nu + \frac{1}{2} \zeta \cdot (\nu \nabla \text{tr } \chi + \nu^{-1} \nabla \text{tr } \underline{\chi}) \right| \\
 &\lesssim \left( 1 + \|\nu - 1\|_{L^\infty(\partial\Sigma_{t^*})} + \left\| \text{tr } \chi - \frac{2}{t^*} \right\|_{L^\infty(\partial\Sigma_{t^*})} + \left\| \text{tr } \underline{\chi} + \frac{2}{t^*} \right\|_{L^\infty(\partial\Sigma_{t^*})} \right) \\
 &\quad \cdot \|\zeta\|_{L^2(\partial\Sigma_{t^*})} \left( \|\nabla \nu\|_{L^2(\partial\Sigma_{t^*})} + \|\nabla \text{tr } \chi\|_{L^2(\partial\Sigma_{t^*})} + \|\nabla \text{tr } \underline{\chi}\|_{L^2(\partial\Sigma_{t^*})} \right) \\
 &\lesssim \varepsilon(D\varepsilon) + \varepsilon^2.
 \end{aligned} \tag{4.37i}$$

The terms  $I_8$  and  $I_9$  are bounded similarly by (4.1) and (4.2),

$$I_8 + I_9 \lesssim (\sqrt{D}\varepsilon)^2. \tag{4.37j}$$

Plugging (4.37a)-(4.37j) into (4.36), we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\int_{\Sigma_{t^*}} |\nabla k|^2 + \int_{\Sigma_{t^*}} |k|^4 + \int_{\partial\Sigma_{t^*}} |k|^2 + \int_{\partial\Sigma_{t^*}} |\nabla \nu|^2 \lesssim (\sqrt{D}\varepsilon)^2.$$

In particular, this implies by Lemma 3.8 that

$$\begin{aligned}
 \|k\|_{L^2(\Sigma_{t^*})} &\lesssim \|\nabla k\|_{L^2(\Sigma_{t^*})} + \|k\|_{L^2(\partial\Sigma_{t^*})} \\
 &\lesssim (\sqrt{D}\varepsilon)^2.
 \end{aligned}$$

Summarising the above, we have that

$$\int_{\Sigma_{t^*}} (|\nabla k|^2 + |k|^2 + |k|^4) + \int_{\partial\Sigma_{t^*}} (|k|^2 + |\nabla \nu|^2) \lesssim (\sqrt{D}\varepsilon)^2.$$

This finishes the proof of Proposition 4.7. □

Proposition 4.7 and Lemma 3.6 yield the following corollary.

**Corollary 4.8.** *It holds that*

$$\|k\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|k\|_{L^4(\partial\Sigma_{t^*})} \lesssim \sqrt{D}\varepsilon.$$

## 4.8 Estimates for the slope $\nu$ on $\partial\Sigma_{t^*}$

We first prove the next lemma.

**Lemma 4.9.** *It holds that*

$$\|\nu - 1\|_{L^2(\partial\Sigma_{t^*})} \lesssim \sqrt{D}\varepsilon.$$

*Proof.* Recall from Lemma 2.11 that on  $\partial\Sigma_{t^*}$ ,

$$\delta = \frac{1}{2} \nu \text{tr } \chi + \frac{1}{2} \nu^{-1} \text{tr } \underline{\chi}.$$

This can be rearranged as

$$\delta = \frac{1}{2} \nu \left( \text{tr } \chi - \frac{2}{t^*} \right) + \frac{1}{2} \nu^{-1} \left( \text{tr } \underline{\chi} + \frac{2}{t^*} \right) + \frac{1}{\nu t^*} (\nu + 1)(\nu - 1),$$

which leads to

$$\nu - 1 = \frac{\nu t^*}{\nu + 1} \left( \delta - \frac{1}{2} \nu \left( \text{tr} \chi - \frac{2}{t^*} \right) - \frac{1}{2} \nu^{-1} \left( \text{tr} \underline{\chi} + \frac{2}{t^*} \right) \right).$$

Consequently, using (4.1) and Proposition 4.7, we can estimate for  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \|\nu - 1\|_{L^2(\partial\Sigma_{t^*})} &\lesssim (1 + D\varepsilon) \left( \|\delta\|_{L^2(\partial\Sigma_{t^*})} + \left\| \text{tr} \chi - \frac{2}{t^*} \right\|_{L^2(\partial\Sigma_{t^*})} + \left\| \text{tr} \underline{\chi} + \frac{2}{t^*} \right\|_{L^2(\partial\Sigma_{t^*})} \right) \\ &\lesssim \sqrt{D}\varepsilon. \end{aligned}$$

This finishes the proof of Lemma 4.9.  $\square$

Moreover, we have the following.

**Lemma 4.10.** *It holds that*

$$\|\nabla \nu\|_{L^4(\partial\Sigma_{t^*})} + \|\nu - 1\|_{L^\infty(\partial\Sigma_{t^*})} + \|\nabla \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} \lesssim \sqrt{D}\varepsilon.$$

*Proof.* Indeed, by Lemma 2.13, (4.1) and Corollary 4.8, we have

$$\begin{aligned} \|\nabla \nu\|_{L^4(\partial\Sigma_{t^*})} &\lesssim \|\epsilon\|_{L^4(\partial\Sigma_{t^*})} + \|\zeta\|_{L^4(\partial\Sigma_{t^*})} \\ &\lesssim \sqrt{D}\varepsilon + \varepsilon. \end{aligned}$$

Consequently, by Lemmas 3.6 and 4.9, we have

$$\begin{aligned} \|\nu - 1\|_{L^\infty(\partial\Sigma_{t^*})} &\lesssim \|\nabla(\nu - 1)\|_{L^4(\partial\Sigma_{t^*})} + \|\nu - 1\|_{L^2(\partial\Sigma_{t^*})} \\ &\lesssim \sqrt{D}\varepsilon. \end{aligned}$$

By the above and Lemmas 2.13, 3.3 and 3.6, (4.1) and Proposition 4.7,

$$\begin{aligned} &\|\nabla \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ &= \|\nu (\nu^{-1} \nabla \nu)\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ &\lesssim (\|\nu\|_{L^\infty(\partial\Sigma_{t^*})} + \|\nabla \nu\|_{L^2(\partial\Sigma_{t^*})}) (\|\zeta\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\epsilon\|_{H^{1/2}(\partial\Sigma_{t^*})}) \\ &\lesssim (1 + \|\nu - 1\|_{L^\infty(\partial\Sigma_{t^*})} + \|\nabla \nu\|_{L^2(\partial\Sigma_{t^*})}) (\|\zeta\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\epsilon\|_{L^2(\Sigma_{t^*})} + \|\nabla \epsilon\|_{L^2(\Sigma_{t^*})}) \\ &\lesssim \sqrt{D}\varepsilon. \end{aligned}$$

This finishes the proof of Lemma 4.10.  $\square$

This finishes the improvement of the bootstrap assumptions (4.2).

At this point we can reapply the estimates of Section 4.4 for  $n$  to get

$$\|n - 1\|_{L^\infty(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 n\|_{L^2(\Sigma_{t^*})} \lesssim \sqrt{D}\varepsilon. \quad (4.38)$$

As a consequence of the above, we can prove the following additional bound.

**Lemma 4.11.** *It holds that*

$$\|\mathbf{D}_T k\|_{L^2(\Sigma_{t^*})} \lesssim \sqrt{D}\varepsilon.$$

*Proof.* Indeed, by the second variation equation (2.13a), that is,

$$\mathbf{D}_T k_{ij} = E_{ij} - n^{-1} \nabla_i \nabla_j n - k_{il} k^l{}_j,$$

we get that for  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \|\mathbf{D}_T k\|_{L^2(\Sigma_{t^*})} &\lesssim \|E\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 n\|_{L^2(\Sigma_{t^*})} + (\|\nabla k\|_{L^2(\Sigma_{t^*})} + \|k\|_{L^2(\Sigma_{t^*})})^2 \\ &\lesssim \sqrt{D}\varepsilon + (\sqrt{D}\varepsilon)^2. \end{aligned}$$

This finishes the proof of Lemma 4.11.  $\square$





## 5. Higher regularity estimates

In this section we prove Proposition 2.30. In Section 5.1 we prove the higher regularity estimates for  $m = 1$  and in Section 5.2 we prove the estimates for  $m \geq 2$ . As emphasised in Section 2.9, the case  $m = 1$  requires a trilinear estimate which necessitates an inspection of the wave parametrix formalism of [KRS15], see Proposition 5.2 and its proof in Appendix E. On the contrary, the cases  $m \geq 2$  are proved by a classical Grönwall argument together with straight-forward generalisations of the methods for  $m = 1$ .

### 5.1 Higher regularity estimates for $m = 1$

Assume that the spheres  $(S_v)_{1 \leq v \leq 2}$  of the canonical foliation on  $\mathcal{H}$  are weakly regular 2-spheres with constants  $N, c$ . Assume further that for real numbers  $1 < t_0^* \leq 2$ ,  $\varepsilon > 0$  and  $0 < C_{\text{ball}} < 1/2$ , it holds for  $1 \leq t \leq t_0^*$  that

$$\Sigma_t \text{ is a weakly regular ball of radius } t \text{ with constant } C_{\text{ball}} \quad (5.1)$$

and moreover,

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} + \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(\partial\Sigma_t)} + \|\mathbf{D}_T k\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \|\nu - 1\|_{L_t^\infty L^\infty(S_t)} + \|\nabla \nu\|_{L_t^\infty H^{1/2}(S_t)} &\lesssim \varepsilon, \\ \|n - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \end{aligned} \quad (5.2)$$

In the following we show that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, it holds that for  $1 \leq t \leq t_0^*$ ,

$$\|\nabla E\|_{L^2(\Sigma_t)} + \|\nabla H\|_{L^2(\Sigma_t)} + \|\nabla \text{Ric}\|_{L^2(\Sigma_t)} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}, \quad (5.3)$$

$$\|\nabla^2 k\|_{L^2(\Sigma_t)} + \|\nabla^2 \nu\|_{H^{1/2}(S_t)} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}, \quad (5.4)$$

$$\|\mathbf{DR}\|_{L^2(\Sigma_t)} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}, \quad (5.5)$$

$$\|\nabla^3 n\|_{L^2(\Sigma_t)} + \|\nabla^2 T(n)\|_{L^2(\Sigma_t)} + \|\nabla T^2(n)\|_{L^2(\Sigma_t)} \lesssim \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}, \quad (5.6)$$

$$\|\mathbf{D}_T \mathbf{D}_T k\|_{L^2(\Sigma_t)} + \|\nabla \mathbf{D}_T k\|_{L^2(\Sigma_t)} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}. \quad (5.7)$$

The estimates (5.3), (5.4), (5.5), (5.6) and (5.7) then prove the case  $m = 1$  of Proposition 2.30.

**Remark 5.1.** *The smallness of  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  is only used in the proof of the ( $m = 1$ )-estimates in context of the trilinear estimate. For the cases  $m \geq 2$  in the next section, no further smallness assumption is made.*

**Notation.** Pick  $1 \leq t^* \leq t_0^*$ . In the following, we prove (5.3), (5.4), (5.5) and (5.6) on  $\Sigma_{t^*}$ . As  $t^*$  was chosen arbitrarily, this implies (5.3), (5.4), (5.5) and (5.6) for  $1 \leq t \leq t_0^*$ .

First, similarly as in Section 4.5, we use the bounded  $L^2$  curvature theorem (Theorem 3.14) to construct a background foliation  $(\tilde{\Sigma}_{\tilde{t}})_{0 \leq \tilde{t} \leq t^*}$  of the past of  $\Sigma_{t^*}$ . For completeness, we recall the construction in the following.

By (5.1) and (5.2), it holds for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small that

$$\sum_{0 \leq m' \leq 2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(\Sigma_{t^*})} + \sum_{0 \leq m' \leq 1} \|\partial^{m'} k_{ij}\|_{L^2(\Sigma_{t^*})} \lesssim C_{\text{ball}}. \quad (5.8)$$

Thus for  $C_{\text{ball}} > 0$  sufficiently small, we can extend  $(\Sigma_{t^*}, g, k)$  by Theorem 3.12 to an asymptotically flat, maximal initial data set on  $\mathbb{R}^3$  which satisfies the assumptions of the bounded  $L^2$  curvature theorem (Theorem 3.14). Consequently, applying Theorem 3.14 *backwards* from  $\Sigma_{t^*}$ , we get:

1. The past of  $\Sigma_{t^*}$  in  $\mathcal{M}$ , denoted by  $\mathcal{M}_{t^*}$ , is foliated by maximal spacelike hypersurfaces  $(\tilde{\Sigma}_{\tilde{t}})_{0 \leq \tilde{t} \leq t^*}$  given as level sets of a time function  $\tilde{t}$  with  $\tilde{\Sigma}_{t^*} = \Sigma_{t^*}$  and satisfying

$$\begin{aligned} \|\widetilde{\text{Ric}}\|_{L_t^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\mathbf{R}}\|_{L_t^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \\ \|\widetilde{k}\|_{L_t^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\nabla k}\|_{L_t^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \\ \|\widetilde{n} - 1\|_{L_t^\infty L^\infty(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\nabla n}\|_{L_t^\infty L^\infty(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\nabla}^2 \widetilde{n}\|_{L_t^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\nabla T}(\widetilde{n})\|_{L_t^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \end{aligned} \quad (5.9)$$

where  $\widetilde{\text{Ric}}$  and  $\widetilde{\nabla}$  denote the Ricci curvature and the covariant derivative on  $\widetilde{\Sigma}_{\tilde{t}}$ . Let  $\tilde{e}_0 := \widetilde{T}$  denote the timelike unit normal to  $\widetilde{\Sigma}_{\tilde{t}}$ , and let  $\widetilde{E}$  and  $\widetilde{H}$  be the electric-magnetic decomposition of  $\mathbf{R}$  with respect to  $\widetilde{T}$ .

2. By the higher regularity estimates of Theorems 3.12 and 3.14, it holds that for  $0 \leq \tilde{t} \leq t^*$ ,

$$\|\mathbf{DR}\|_{L_t^\infty L^2(\widetilde{\Sigma}_{\tilde{t}})} + \|\mathbf{D}^2 \tilde{\pi}\|_{L_t^\infty L^2(\widetilde{\Sigma}_{\tilde{t}})} \lesssim \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}}.$$

3. For each  $\omega \in \mathbb{S}^2$ , the spacetime  $\mathcal{M}_{t^*}$  is foliated by a family of null hyperplanes  $(\mathcal{H}_{\omega_u})_{\omega_u \in \mathbb{R}}$  given as level sets of an optical function  $\omega u$  satisfying

$$\sup_{\omega \in \mathbb{S}^2} \|\mathbf{R} \cdot \widetilde{L}\|_{L_u^\infty L^2(\mathcal{H}_{\omega_u})} \lesssim C_{\text{ball}},$$

where  $\widetilde{L}$  is the  $\mathcal{H}_{\omega_u}$ -tangent null vectorfield with  $\mathbf{g}(\widetilde{L}, \widetilde{T}) = -1$ .

4. Recall from (4.16) that the angle  $\tilde{\nu}$  between  $T$  and  $\widetilde{T}$  is defined by

$$\tilde{\nu} := -\mathbf{g}(T, \widetilde{T}).$$

By Lemma 4.6, for  $\varepsilon > 0$  and  $C_{\text{ball}} > 0$  sufficiently small, the following comparison estimates hold along the foliation  $(\Sigma_t)_{1 \leq t \leq t^*}$ ,

$$\|\tilde{\nu} - 1\|_{L_t^\infty L^\infty(\Sigma_t)} \lesssim C_{\text{ball}}, \quad \|\tilde{k}\|_{L_t^\infty L^4(\Sigma_t)} \lesssim C_{\text{ball}}, \quad (5.10)$$

where  $\tilde{k}$  denotes the second fundamental form of  $\widetilde{\Sigma}_{\tilde{t}}$ .

In the rest of this section we proceed as follows.

◇ In Section 5.1.1 we prove by *elliptic estimates* that on  $\Sigma_{t^*}$ ,

$$\|\nabla E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} \lesssim \|\hat{\mathcal{L}}_{\widetilde{T}} \mathbf{R}\|_{L^2(\Sigma_{t^*})} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}. \quad (5.11)$$

◇ In Section 5.1.2 we use a Bel-Robinson energy estimate applied to  $W = \hat{\mathcal{L}}_{\widetilde{T}} \mathbf{R}$  to prove that for  $0 \leq \tilde{t} \leq t^*$ ,

$$\|\hat{\mathcal{L}}_{\widetilde{T}} \mathbf{R}\|_{L^2(\Sigma_{t^*})} \lesssim \mathcal{R}_1^\Sigma + \mathcal{R}_1^{\mathcal{H}} + \sqrt{C_{\text{ball}}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} \right) + C_{\text{ball}}. \quad (5.12)$$

This energy estimate requires a trilinear estimate for the error term.

◇ In Section 5.1.3, we prove by *elliptic estimates for  $k$*  that on  $\Sigma_{t^*}$ ,

$$\|\nabla^2 k\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla^2 \nu\|_{L^2(\partial \Sigma_{t^*})}^2 \lesssim \|\nabla H\|_{L^2(\Sigma_{t^*})}^2 + C_{\text{ball}}^2 + \|\text{l.o.t.}\|_{L^1(\partial \Sigma_{t^*})}, \quad (5.13)$$

where l.o.t. denotes lower order product terms.

◇ In Section 5.1.4, we combine (5.11), (5.12) and (5.13) to conclude the proof of (5.3) and (5.4), that is,

$$\begin{aligned} \|\nabla E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} + \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} &\lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}, \\ \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 \nu\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 \nu\|_{H^{1/2}(\Sigma_{t^*})} &\lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}. \end{aligned}$$

◇ In Section 5.1.5, we prove (5.5), that is,

$$\|\mathbf{DR}\|_{L^2(\Sigma_{t^*})} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}.$$

◇ In Section 5.1.6, we prove (5.6) and (5.7), that is,

$$\begin{aligned} \|\nabla^3 n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 T(n)\|_{L^2(\Sigma_{t^*})} + \|\nabla T^2(n)\|_{L^2(\Sigma_{t^*})} &\lesssim \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}, \\ \|\mathbf{D}_T \mathbf{D}_T k\|_{L^2(\Sigma_t)} + \|\nabla \mathbf{D}_T k\|_{L^2(\Sigma_t)} &\lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + C_{\text{ball}}. \end{aligned}$$

### 5.1.1 Elliptic estimates for curvature

In this section, we prove (5.11). First we note that by construction,  $E = \tilde{E}, H = \tilde{H}, \tilde{\nabla}\tilde{E} = \nabla E$  and  $\tilde{\nabla}\tilde{H} = \nabla H$  on  $\Sigma_{t^*}$ . Therefore it suffices to prove that

$$\|\tilde{\nabla}\tilde{E}\|_{L^2(\Sigma_{t^*})} + \|\tilde{\nabla}\tilde{H}\|_{L^2(\Sigma_{t^*})} \lesssim \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L^2(\tilde{\Sigma}_{\tilde{t}})} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}.$$

By (2.7) and (2.15),  $\tilde{E}$  and  $\tilde{H}$  satisfy the following Hodge system on  $\Sigma_{t^*}$ ,

$$\begin{aligned} \widetilde{\text{div}}\tilde{E} &= +\tilde{k} \wedge \tilde{H}, \\ \widetilde{\text{curl}}\tilde{E} &= +\tilde{H}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}) - \frac{3}{2}\tilde{k} \times \tilde{H} - 3\tilde{n}^{-1}\tilde{\nabla}\tilde{n} \wedge \tilde{E}, \\ \widetilde{\text{div}}\tilde{H} &= -\tilde{k} \wedge \tilde{E}, \\ \widetilde{\text{curl}}\tilde{H} &= -\tilde{E}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}) + \frac{3}{2}\tilde{k} \times \tilde{E} - 3\tilde{n}^{-1}\tilde{\nabla}\tilde{n} \wedge \tilde{H}, \end{aligned} \tag{5.14}$$

where  $\widetilde{\text{div}}$  and  $\widetilde{\text{curl}}$  denote the divergence and symmetrised curl operators on  $\Sigma_{t^*}$ , respectively, see (2.3). Applying classical global elliptic estimates to the Hodge system (5.14) (see Corollary A.7) and using (5.9), we get

$$\begin{aligned} \int_{\Sigma_{t^*}} |\tilde{\nabla}\tilde{E}|^2 + |\tilde{\nabla}\tilde{H}|^2 &\lesssim \int_{\Sigma_{t^*}} |\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}|^2 + \int_{\partial\Sigma_{t^*}} \tilde{\nabla}_b \tilde{E}_{aN} \tilde{E}^{ab} + \int_{\partial\Sigma_{t^*}} \tilde{\nabla}_b \tilde{H}_{aN} \tilde{H}^{ab} + C_{\text{ball}}^2 \\ &\lesssim \int_{\Sigma_{t^*}} |\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}|^2 + \int_{\partial\Sigma_{t^*}} \nabla_b E_{aN} E^{ab} + \int_{\partial\Sigma_{t^*}} \nabla_b H_{aN} H^{ab} + C_{\text{ball}}^2, \end{aligned} \tag{5.15}$$

where we used that  $\nabla = \tilde{\nabla}$  and  $E = \tilde{E}, H = \tilde{H}$  on  $\Sigma_{t^*}$ .

Using the spacetime relations

$$\begin{aligned} \nabla_a E_{bN} &= \mathbf{D}_a \mathbf{R}_{TbTN} - k_{ac} \mathbf{R}_{cbTN} - k_{ac} \mathbf{R}_{TbcN}, \\ \nabla_a H_{bN} &= \mathbf{D}_a^* \mathbf{R}_{TbTN} - k_{ac}^* \mathbf{R}_{cbTN} - k_{ac}^* \mathbf{R}_{TbcN}, \end{aligned}$$

and Lemma 3.6, we can estimate the boundary integrals on the right-hand side of (5.15) by

$$\begin{aligned} &\int_{\partial\Sigma_{t^*}} \nabla_b E_{aN} E^{ab} + \int_{\partial\Sigma_{t^*}} \nabla_b H_{aN} H^{ab} \\ &\lesssim \int_{\partial\Sigma_{t^*}} |\mathbf{D}\mathbf{R}|_{\mathbf{h}^t}^2 + |\mathbf{R}|_{\mathbf{h}^t}^2 + |k| |\mathbf{R}|_{\mathbf{h}^t}^2 \\ &\lesssim \|\mathbf{D}\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + \|k\|_{L^1(\partial\Sigma_{t^*})} \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 \\ &\lesssim \|\mathbf{D}\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + (\|\nabla k\|_{L^2(\Sigma_{t^*})} + \|k\|_{L^2(\Sigma_{t^*})}) \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 \\ &\lesssim \|\mathbf{D}\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + \varepsilon \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 \\ &\lesssim (\mathcal{R}_1^{\mathcal{H}})^2. \end{aligned}$$

To summarise, we get that for  $0 \leq \tilde{t} \leq t^*$ ,

$$\|\nabla E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} \lesssim \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L^2(\tilde{\Sigma}_{\tilde{t}^*})} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}.$$

This finishes the proof of (5.11).

### 5.1.2 Bel-Robinson energy estimate

In this section we prove (5.12), that is, we have for  $0 \leq \tilde{t} \leq t^*$ ,

$$\|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} \lesssim \mathcal{R}_1^\Sigma + \mathcal{R}_1^{\mathcal{H}} + \sqrt{C_{\text{ball}}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} \right) + C_{\text{ball}}.$$

Indeed, applying the classical energy estimate (3.4) to the Weyl tensor  $W := \hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}$  with multiplier field  $\tilde{T}$  yields (similarly as in Section 4.6)

$$\begin{aligned}
 & \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R} \cdot \tilde{L}\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\
 & \lesssim \int_{\Sigma_1} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T} + \int_{\mathcal{H}} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} \\
 & \quad - \underbrace{\int_{\mathcal{M}_{t^*}} \frac{3}{2} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\alpha\beta\tilde{T}\tilde{T}} \tilde{\pi}^{\alpha\beta}}_{:=\mathcal{E}_1} - \underbrace{\int_{\mathcal{M}_{t^*}} \mathbf{D}^\alpha Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\alpha\tilde{T}\tilde{T}\tilde{T}}}_{:=\mathcal{E}_2}.
 \end{aligned} \tag{5.16}$$

The terms  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are bounded by the following proposition proved in Appendix E.

**Proposition 5.2** (Trilinear estimate for  $m = 1$ ). *For  $\varepsilon > 0$  and  $C_{\text{ball}} > 0$  sufficiently small, it holds that*

$$\begin{aligned}
 |\mathcal{E}_1| + |\mathcal{E}_2| & \lesssim C_{\text{ball}} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R} \cdot \tilde{L}\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\
 & \quad + C_{\text{ball}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}} \right)^2 + C_{\text{ball}}^2.
 \end{aligned} \tag{5.17}$$

**Remark 5.3.** *In Appendix E, we prove Proposition 5.16 by reducing (5.17) to the estimates proved in Section 13 of [KRS15].*

Plugging (5.17) into (5.16), we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned}
 & \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R} \cdot \tilde{L}\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\
 & \lesssim \int_{\mathcal{H}} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} + \int_{\Sigma_1} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T} \\
 & \quad + C_{\text{ball}} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R} \cdot \tilde{L}\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\
 & \quad + C_{\text{ball}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}} \right)^2 + C_{\text{ball}}^2.
 \end{aligned}$$

For  $C_{\text{ball}} > 0$  sufficiently small, we can absorb the second and third term on the right-hand side into the left-hand side and get

$$\begin{aligned}
 & \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R} \cdot \tilde{L}\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\
 & \lesssim \underbrace{\int_{\mathcal{H}} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L}}_{:=\mathcal{I}_1} + \underbrace{\int_{\Sigma_1} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T}}_{:=\mathcal{I}_2} \\
 & \quad + C_{\text{ball}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} \right)^2 + C_{\text{ball}}^2.
 \end{aligned} \tag{5.18}$$

In the following, we control the terms  $\mathcal{I}_1$  and  $\mathcal{I}_2$  on the right-hand side of (5.18).

**Control of  $\mathcal{I}_1$ .** By definition of the Bel-Robinson tensor  $Q$  in Definition 2.2,

$$Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} = (\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\mu\tilde{T}\nu} (\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}}{}^\nu{}_\mu + \text{dual term}.$$

By Definition 2.3 and using that  $(\tilde{\Sigma}_t)$  is maximal,

$$(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\alpha\beta\gamma\delta} := (\mathcal{L}_{\tilde{T}}\mathbf{R})_{\alpha\beta\gamma\delta} - \frac{1}{2} \left( \tilde{\pi}^\mu{}_\alpha \mathbf{R}_{\mu\beta\gamma\delta} + \tilde{\pi}^\mu{}_\beta \mathbf{R}_{\alpha\mu\gamma\delta} + \tilde{\pi}^\mu{}_\gamma \mathbf{R}_{\alpha\beta\mu\delta} + \tilde{\pi}^\mu{}_\delta \mathbf{R}_{\alpha\beta\gamma\mu} \right),$$

which can be written schematically as

$$\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R} = \mathbf{D}\mathbf{R} + \tilde{\pi} \cdot \mathbf{R},$$

and hence,

$$Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} = \mathbf{DR} \cdot \mathbf{DR} + \tilde{\pi} \cdot \mathbf{R} \cdot \mathbf{DR} + \tilde{\pi} \cdot \tilde{\pi} \cdot \mathbf{R} \cdot \mathbf{R}.$$

Therefore, for  $\varepsilon > 0$  and  $C_{\text{ball}} > 0$  sufficiently small,

$$\begin{aligned} \mathcal{I}_1 &:= \int_{\mathcal{H}} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}L} \lesssim \int_{\mathcal{H}} |\mathbf{DR}|_{\mathbf{h}^{\tilde{t}}}^2 + |\tilde{\pi}|_{\mathbf{h}^{\tilde{t}}} |\mathbf{R}|_{\mathbf{h}^{\tilde{t}}} |\mathbf{DR}|_{\mathbf{h}^{\tilde{t}}} + |\tilde{\pi}|_{\mathbf{h}^{\tilde{t}}}^2 |\mathbf{R}|_{\mathbf{h}^{\tilde{t}}}^2 \\ &\lesssim \int_{\mathcal{H}} |\mathbf{DR}|_{\mathbf{h}^v}^2 + |\tilde{\pi}|_{\mathbf{h}^{\tilde{t}}} |\mathbf{R}|_{\mathbf{h}^v} |\mathbf{DR}|_{\mathbf{h}^v} + |\tilde{\pi}|_{\mathbf{h}^{\tilde{t}}}^2 |\mathbf{R}|_{\mathbf{h}^v}^2 \end{aligned} \quad (5.19)$$

where we used (5.2) and (5.10) to compare  $\mathbf{h}^{\tilde{t}}$  with  $\mathbf{h}^v$  on  $\mathcal{H}$ .

By Lemma 3.6, Remark 3.7 and (5.9),

$$\begin{aligned} \int_{\mathcal{H}} |\tilde{\pi}|_{\mathbf{h}^{\tilde{t}}} + \int_{\mathcal{H}} |\tilde{\pi}|_{\mathbf{h}^{\tilde{t}}}^2 &\lesssim \|\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\partial\tilde{\Sigma}_{\tilde{t}})} + \|\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\partial\tilde{\Sigma}_{\tilde{t}})}^2 \\ &\lesssim \left( \|\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\tilde{\nabla}\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} \right) + \left( \|\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\tilde{\nabla}\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} \right)^2 \\ &\lesssim C_{\text{ball}} + C_{\text{ball}}^2. \end{aligned}$$

Plugging this into (5.19) yields that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\mathcal{I}_1 \lesssim \|\mathbf{R}\|_{L^\infty(\mathcal{H})}^2 + \|\mathbf{DR}\|_{L^\infty(\mathcal{H})}^2 \lesssim (\mathcal{R}_1^{\mathcal{H}})^2. \quad (5.20)$$

**Control of  $\mathcal{I}_2$ .** First, by (5.10), (2.8) and (2.12), that is,

$$|Q_{\mu\nu\lambda\tilde{T}}| \lesssim Q_{\tilde{T}\tilde{T}\tilde{T}\tilde{T}} \text{ for } \mu, \nu, \lambda = 0, 1, 2, 3,$$

it follows that

$$\int_{\tilde{\Sigma}_1} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T} \lesssim \int_{\tilde{\Sigma}_1} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}\tilde{T}} \lesssim \int_{\tilde{\Sigma}_1} |\tilde{E}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})|^2 + |\tilde{H}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})|^2.$$

Second, by definition of  $\hat{\mathcal{L}}_{\tilde{T}}$ , see Definition 2.3, and using that  $(\tilde{\Sigma}_{\tilde{t}})$  is maximal, we have for an  $\tilde{\Sigma}_{\tilde{t}}$ -tangential frame  $(\tilde{e}_a)_{a=1,2,3}$ ,

$$\tilde{E}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{ab} = (\mathcal{L}_{\tilde{T}}\mathbf{R})_{\tilde{T}a\tilde{T}b} - \frac{1}{2} \left( \tilde{\pi}_{\tilde{T}}^c \tilde{\mathbf{R}}_{ca\tilde{T}b} + \tilde{\pi}_{\tilde{T}}^c \mathbf{R}_{\tilde{T}c\tilde{T}b} + \tilde{\pi}_{\tilde{T}}^c \mathbf{R}_{\tilde{T}acb} + \tilde{\pi}_{\tilde{T}}^c \mathbf{R}_{\tilde{T}a\tilde{T}c} \right),$$

The Lie derivative on the right-hand side can be expressed as

$$(\mathcal{L}_{\tilde{T}}\mathbf{R})_{\tilde{T}a\tilde{T}b} = \mathbf{D}_{\tilde{T}}\mathbf{R}_{\tilde{T}a\tilde{T}b} - \tilde{n}^{-1} \tilde{\nabla}_c \tilde{n} (\mathbf{R}_{ca\tilde{T}b} + \mathbf{R}_{\tilde{T}acb}) - \tilde{k}_{ac} \mathbf{R}_{\tilde{T}c\tilde{T}b} - \tilde{k}_{bc} \mathbf{R}_{\tilde{T}a\tilde{T}c}.$$

From the above two, we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \int_{\tilde{\Sigma}_1} |\tilde{E}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})|^2 &\lesssim \| |\mathbf{DR}|_{\mathbf{h}^{\tilde{t}}} \|_{L^2(\Sigma_1)}^2 + \| |\mathbf{R}|_{\mathbf{h}^{\tilde{t}}} \|_{L^2(\Sigma_1)}^2 \|\tilde{\nabla}\tilde{n}\|_{L^\infty(\tilde{\Sigma}_{\tilde{t}})}^2 + \| |\mathbf{R}|_{\mathbf{h}^{\tilde{t}}} \|_{L^\infty(\Sigma_1)}^2 \|\tilde{k}\|_{L^2(\Sigma_1)}^2 \\ &\lesssim \|\mathbf{DR}\|_{L^\infty(\Sigma_1)}^2 + \|\mathbf{R}\|_{L^\infty(\Sigma_1)}^2 \\ &\lesssim (\mathcal{R}_1^\Sigma)^2, \end{aligned}$$

where we used (5.9) and applied (5.10) to compare  $\mathbf{h}^{\tilde{t}}$  and  $\mathbf{h}^t$ . Similarly, it follows that

$$\int_{\tilde{\Sigma}_1} |\tilde{H}(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})|^2 \lesssim (\mathcal{R}_1^\Sigma)^2.$$

To summarise, we proved that

$$\mathcal{I}_2 := \int_{\Sigma_1} Q(\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R})_{\tilde{T}\tilde{T}\tilde{T}T} \lesssim (\mathcal{R}_1^\Sigma)^2. \quad (5.21)$$

Plugging (5.20) and (5.21) into (5.18) shows that

$$\|\hat{\mathcal{L}}_{\tilde{T}}\mathbf{R}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})}^2 \lesssim (\mathcal{R}_1^\Sigma)^2 + (\mathcal{R}_1^{\mathcal{H}})^2 + C_{\text{ball}} \left( \|\nabla\text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} \right)^2 + C_{\text{ball}}^2.$$

This finishes the proof of (5.12).

### 5.1.3 Elliptic estimates for $k$ on $\Sigma_{t^*}$ for $m = 1$

In this section we prove (5.13), that is,

$$\|\nabla^2 k\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla^2 \nu\|_{L^2(\partial\Sigma_{t^*})}^2 \lesssim \|\nabla H\|_{L^2(\Sigma_{t^*})}^2 + C_{\text{ball}}^2 + \|\text{l.o.t.}\|_{L^1(\partial\Sigma_{t^*})},$$

where l.o.t. denotes lower order product terms.

Analogously to Section 4.7, the idea is to use global elliptic estimates for  $k$ . We recall from (2.13b), (2.13c) and (2.13d) that  $k$  satisfies on  $\Sigma_{t^*}$  the Hodge system

$$\begin{aligned} \operatorname{div}_g k &= 0, \\ \operatorname{curl}_g k &= H, \\ \operatorname{tr}_g k &= 0. \end{aligned}$$

Moreover, it holds by (2.15) that

$$\operatorname{div} H = -k \wedge E. \quad (5.22)$$

In the following, we rely on the notations of Appendix A. In particular, the above Hodge system of  $k$  implies that

$$A(k)_{iab} = \epsilon_{ab}^m H_{im}, \quad D(k) = 0, \quad (5.23)$$

where  $A(k)$  and  $D(k)$  are given by Definition A.1.

We note that by (5.23) together with Lemma A.4, we can express the symmetrised derivative  $\bar{\nabla} k$  of  $k$  as

$$(\bar{\nabla} k)_{a_1 a_2 b} = \nabla_b k_{a_1 a_2} + \frac{1}{3} \epsilon_{ba_1}^m H_{a_2 m} + \frac{1}{3} \epsilon_{ba_2}^m H_{a_1 m}. \quad (5.24)$$

By Lemmas A.2, A.4 and A.5, (5.23), (5.24), and the bounds (5.2), we get the following global elliptic estimate,

$$\begin{aligned} & \int_{\Sigma_{t^*}} |\nabla^2 k|^2 + \int_{\partial\Sigma_{t^*}} (\bar{\nabla} k)^{a_1 a_2 N} D(\bar{\nabla} k)_{a_1 a_2} - \int_{\partial\Sigma_{t^*}} \nabla_b (\bar{\nabla} k)_{a_1 a_2 N} (\bar{\nabla} k)^{a_1 a_2 b} \\ & \lesssim \int_{\Sigma_{t^*}} |\nabla H|^2 + C_{\text{ball}}^2. \end{aligned} \quad (5.25)$$

By the definition of  $D(\bar{\nabla} k)$ , see Definition A.1, we can rewrite the boundary integrals on the left-hand side of (5.25) as

$$\begin{aligned} & \int_{\partial\Sigma_{t^*}} (\bar{\nabla} k)^{a_1 a_2 N} D(\bar{\nabla} k)_{a_1 a_2} - \int_{\partial\Sigma_{t^*}} \nabla_b (\bar{\nabla} k)_{a_1 a_2 N} (\bar{\nabla} k)^{a_1 a_2 b} \\ & = \int_{\partial\Sigma_{t^*}} (\bar{\nabla} k)^{a_1 a_2 N} \nabla^b (\bar{\nabla} k)_{a_1 a_2 b} - \int_{\partial\Sigma_{t^*}} \nabla_b (\bar{\nabla} k)_{a_1 a_2 N} (\bar{\nabla} k)^{a_1 a_2 b} \\ & = \int_{\partial\Sigma_{t^*}} (\bar{\nabla} k)^{a_1 a_2 N} \nabla^D (\bar{\nabla} k)_{a_1 a_2 D} - \int_{\partial\Sigma_{t^*}} \nabla_D (\bar{\nabla} k)_{a_1 a_2 N} (\bar{\nabla} k)^{a_1 a_2 D}, \end{aligned} \quad (5.26)$$

where we used that the integrals cancel each other for  $b = N$ .

**Notation.** To ease presentation, in the following we write l.o.t. for lower order product terms and norms thereof.

Consider the first term on the right-hand side of (5.26). We distinguish three cases,

1.  $a_1, a_2 \in \{1, 2\}$ ,
2.  $a_1 \in \{1, 2\}, a_2 = N$ ,

3.  $a_1 = a_2 = N$ .

**Control of Case 1:**  $a_1, a_2 \in \{1, 2\}$ . In this case denote  $A_1 := a_1$  and  $A_2 := a_2$ . By (5.2), (5.22), (5.23), Lemma 3.6 and integration by parts on  $\partial\Sigma_{t^*}$ ,

$$\begin{aligned}
 & \int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{A_1 A_2 N} \nabla^D (\bar{\nabla}k)_{A_1 A_2 D} \\
 &= \int_{\partial\Sigma_{t^*}} \left( \nabla^{A_1} k^{A_2 N} + \frac{1}{3} \in^{m N A_1} H_m^{A_2} + \frac{1}{3} \in^{m N A_2} H_m^{A_1} \right) \\
 & \quad \cdot \nabla^D \left( \nabla_{A_2} k_{A_1 D} + \frac{1}{3} \in_{m D A_2} H_{A_1}^m + \frac{1}{3} \in_{m D A_1} H_{A_2}^m \right) \\
 &= \int_{\partial\Sigma_{t^*}} \nabla_{A_2} k^{A_2 N} \nabla^{A_1} \nabla^D k_{A_1 D} + \text{l.o.t.} \\
 &= \int_{\partial\Sigma_{t^*}} d\check{\nu} \epsilon d\check{\nu} (d\check{\nu} \eta) + \text{l.o.t.} \\
 &= - \int_{\partial\Sigma_{t^*}} d\check{\nu} \epsilon d\check{\nu} \nabla \delta + \text{l.o.t.},
 \end{aligned} \tag{5.27}$$

where we used (2.18) in the last equation, that is,

$$d\check{\nu} \eta_A = - \nabla_A \delta - {}^* H_N / A + \Theta \cdot \epsilon - \text{tr} \Theta \epsilon.$$

**Remark 5.4.** The lower order boundary terms in (5.27) involving  $H$  are estimated by standard product estimates together with trace estimates (see Lemma 3.6) and bilinear trace estimates for  $H$  on  $\partial\Sigma_{t^*}$  based on (5.22) (see the derivation of the bilinear trace estimate (4.37e) at the level of  $m = 0$ ).

Using the slope equation (2.24) and (4.33) for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, that is,

$$\begin{aligned}
 \epsilon_A &= - \nu^{-1} \nabla_A \nu + \zeta_A, \\
 \nabla \delta &= \nabla \left( \frac{1}{2} \nu \text{tr} \chi + \frac{1}{2} \nu^{-1} \text{tr} \underline{\chi} \right) = \underbrace{F(\nu, \text{tr} \chi, \text{tr} \underline{\chi})}_{\geq 1/8} \nabla \nu + \frac{1}{2} \nu \nabla \text{tr} \chi + \frac{1}{2} \nu^{-1} \nabla \text{tr} \underline{\chi},
 \end{aligned}$$

we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned}
 \int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{A_1 A_2 N} \nabla^D (\bar{\nabla}k)_{A_1 A_2 D} &= \int_{\partial\Sigma_{t^*}} \nu^{-1} F(\nu, \text{tr} \chi, \text{tr} \underline{\chi}) |\underline{\Delta} \nu|^2 + \text{l.o.t.} \\
 &\gtrsim \int_{\partial\Sigma_{t^*}} |\underline{\Delta} \nu|^2 + \text{l.o.t.} \\
 &\gtrsim \int_{\partial\Sigma_{t^*}} |\nabla^2 \nu|^2 + \text{l.o.t.},
 \end{aligned}$$

where we applied standard elliptic estimates for  $\underline{\Delta}$  on  $\partial\Sigma_{t^*}$ , see Lemma 3.17 in [CG19a]. We remark that the lower order terms in  $\nabla \text{tr} \chi$  and  $\nabla \text{tr} \underline{\chi}$  are estimated by standard product estimates and the initial data norm  $\mathcal{O}_1^{\mathcal{H}}$ . This finishes the control of Case 1.

**Control of Case 2:**  $a_1 \in \{1, 2\}, a_2 = N$ . In this case let  $A_1 := a_1$ . We have by (5.2), (5.22), Lemma 3.6 and integration by parts on  $\partial\Sigma_{t^*}$ ,

$$\begin{aligned}
 \int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{A_1 N N} \nabla^D (\bar{\nabla}k)_{A_1 N D} &= \int_{\partial\Sigma_{t^*}} \nabla^{A_1} k^{N N} \nabla^D \nabla_{A_1} k_{N D} + \text{l.o.t.} \\
 &= \int_{\partial\Sigma_{t^*}} \nabla^{A_1} \delta \nabla_{A_1} d\check{\nu} \epsilon + \text{l.o.t.} \\
 &= - \int_{\partial\Sigma_{t^*}} d\check{\nu} \nabla \delta d\check{\nu} \epsilon + \text{l.o.t.},
 \end{aligned}$$



which thus reduced Case 2 to Case 1, see the right-hand side of (5.27). Hence we get that

$$\int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{A_1 N N} \nabla^D (\bar{\nabla}k)_{A_1 N D} \gtrsim \int_{\partial\Sigma_{t^*}} |\nabla^2 \nu|^2 + \text{l.o.t.}$$

This finishes the control of Case 2.

**Control of Case 3:**  $a_1 = a_2 = N$ . Using that  $\text{div}k = 0$ , we can decrease the numbers of  $N$ 's to reduce to (5.27) in Case 1 as follows,

$$\begin{aligned} \int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{N N N} \nabla^D (\bar{\nabla}k)_{N N D} &= \int_{\partial\Sigma_{t^*}} \nabla_N k_{N N} \nabla^D \nabla_D k_{N N} + \text{l.o.t.} \\ &= - \int_{\partial\Sigma_{t^*}} \nabla^A k_{N A} \nabla^D \nabla_D k_{N N} + \text{l.o.t.} \\ &= - \int_{\partial\Sigma_{t^*}} \text{div} \nabla \delta + \text{l.o.t.}, \end{aligned}$$

where we used (5.2), (5.22) and Lemma 3.6 to estimate the error terms. Hence we have that

$$\int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{N N N} \nabla^D (\bar{\nabla}k)_{N N D} \gtrsim \int_{\partial\Sigma_{t^*}} |\nabla^2 \nu|^2 + \text{l.o.t.}$$

This finishes the control of Case 3.

To summarise, we proved that the first integral on the right-hand side of (5.26) is bounded as follows,

$$\int_{\partial\Sigma_{t^*}} (\bar{\nabla}k)^{a_1 a_2 N} \nabla^D (\bar{\nabla}k)_{a_1 a_2 D} \gtrsim \int_{\partial\Sigma_{t^*}} |\nabla^2 \nu|^2 + \text{l.o.t.} \quad (5.28)$$

The second integral on the right-hand side of (5.26) is, after an integration by parts, equal to the first (up to lower order product terms) and is thus similarly estimated from below.

Therefore, plugging (5.28) into (5.25) yields

$$\|\nabla^2 k\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla^2 \nu\|_{L^2(\partial\Sigma_{t^*})}^2 \lesssim \|\nabla H\|_{L^2(\Sigma_{t^*})}^2 + C_{\text{ball}}^2 + \|\text{l.o.t.}\|_{L^1(\partial\Sigma_{t^*})}.$$

This finishes the proof of (5.13).

#### 5.1.4 Conclusion of the proof of (5.3) and (5.4)

In this section, we conclude the proof of (5.3) and (5.4), that is,

$$\begin{aligned} \|\nabla E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} + \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} &\lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}, \\ \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} &\lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}. \end{aligned}$$

First, we note that by (2.13f), that is,

$$\text{Ric}_{ij} = E_{ij} + k_{ia} k_j^a,$$

it holds that

$$\|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} \lesssim \|\nabla E\|_{L^2(\Sigma_{t^*})} + \|k \cdot \nabla k\|_{L^2(\Sigma_{t^*})}. \quad (5.29)$$

Combining (5.11), (5.12), (5.13) and (5.29), we get

$$\begin{aligned} &\|\nabla E\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla H\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla^2 \nu\|_{L^2(\partial\Sigma_{t^*})}^2 + \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})}^2 \\ &\lesssim (\mathcal{O}_1^{\mathcal{H}})^2 + (\mathcal{R}_1^{\mathcal{H}})^2 + (\mathcal{O}_1^{\Sigma})^2 + (\mathcal{R}_1^{\Sigma})^2 + C_{\text{ball}} \left( \|\nabla^2 k\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})}^2 \right) \\ &\quad + C_{\text{ball}}^2 + \|\text{l.o.t.}\|_{L^1(\partial\Sigma_{t^*})} + \|k \cdot \nabla k\|_{L^2(\Sigma_{t^*})}^2 \\ &\lesssim (\mathcal{O}_1^{\mathcal{H}})^2 + (\mathcal{R}_1^{\mathcal{H}})^2 + (\mathcal{O}_1^{\Sigma})^2 + C_{\text{ball}} \left( \|\nabla^2 k\|_{L^2(\Sigma_{t^*})}^2 + \|\nabla E\|_{L^2(\Sigma_{t^*})}^2 + \|k \cdot \nabla k\|_{L^2(\Sigma_{t^*})}^2 \right) \\ &\quad + C_{\text{ball}}^2 + \|\text{l.o.t.}\|_{L^1(\partial\Sigma_{t^*})} + \|k \cdot \nabla k\|_{L^2(\Sigma_{t^*})}^2. \end{aligned}$$

By standard product estimates applied to l.o.t. (see also Remark 5.4) and assuming that  $C_{\text{ball}} > 0$  is sufficiently small to absorb terms from the right-hand side into the left-hand side, we get that

$$\begin{aligned} & \|\nabla E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 \nu\|_{L^2(\partial\Sigma_{t^*})} + \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} \\ & \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}. \end{aligned}$$

This finishes the proof of (5.3) and (5.4).

It remains to estimate  $\|\nabla^2 \nu\|_{H^{1/2}(\partial\Sigma_{t^*})}$ . First, by Lemma 3.6 and (5.3),

$$\begin{aligned} \|\nabla \nu\|_{L^8(\partial\Sigma_{t^*})} & \lesssim \|\nabla^2 \nu\|_{L^2(\partial\Sigma_{t^*})} + \|\nabla \nu\|_{L^2(\partial\Sigma_{t^*})} \\ & \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}. \end{aligned} \quad (5.30)$$

Using the slope equation (2.24), that is,

$$\nu^{-1} \nabla_A \nu = -\epsilon_A + \zeta_A,$$

we have that

$$\begin{aligned} \nabla_A \nabla_B \nu & = \nu \nabla_A (\nu^{-1} \nabla_B \nu) + \nu^{-1} \nabla_A \nu \nabla_B \nu \\ & = \nu \nabla_A (-\epsilon + \zeta)_B + \nu^{-1} \nabla_A \nu \nabla_B \nu. \end{aligned} \quad (5.31)$$

Therefore, by Lemmas 3.3 and 3.6, (5.2) and (5.30),

$$\begin{aligned} \|\nabla \nu\|_{L^\infty(\partial\Sigma_{t^*})} & \lesssim \|\nabla^2 \nu\|_{L^4(\partial\Sigma_{t^*})} + \|\nabla \nu\|_{L^2(\partial\Sigma_{t^*})} \\ & \lesssim \|\nabla \zeta\|_{L^4(\partial\Sigma_{t^*})} + \|\nabla \epsilon\|_{L^4(\partial\Sigma_{t^*})} + \|\nabla \nu\|_{L^8(\partial\Sigma_{t^*})}^2 \\ & \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}. \end{aligned} \quad (5.32)$$

By Lemma 3.3 with (5.2), (5.30), (5.31) and (5.32), we get that for  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \|\nabla^2 \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} & \lesssim \|\nabla \zeta\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla \epsilon\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ & \quad + (\|\nabla \nu\|_{L^\infty(\partial\Sigma_{t^*})} + \|\nabla^2 \nu\|_{L^2(\partial\Sigma_{t^*})}) \|\nabla \nu\|_{H^{1/2}(\partial\Sigma_{t^*})} \\ & \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}. \end{aligned}$$

This finishes the proof of (5.3) and (5.4).

### 5.1.5 Proof of (5.5)

In this section we prove (5.5), that is,

$$\|\mathbf{DR}\|_{L^2(\Sigma_{t^*})} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}.$$

Consider first  $\mathbf{D}_T \mathbf{R}$  where  $T$  denotes the timelike unit normal to  $\Sigma_{t^*}$ . By decomposing  $\mathbf{D}_T \mathbf{R}$  into  $\mathbf{D}_T \mathbf{R}_{T.T}$  and  $\mathbf{D}_T^* \mathbf{R}_{T.T}$ . (see the electric-magnetic decomposition of Weyl tensors in Section 2.3), it suffices to prove that

$$\int_{\Sigma_{t^*}} |\mathbf{D}_T \mathbf{R}_{T.T}|_{\mathbf{h}}^2 + |\mathbf{D}_T^* \mathbf{R}_{T.T}|_{\mathbf{h}}^2 \lesssim (\mathcal{O}_1^{\mathcal{H}})^2 + (\mathcal{R}_1^{\mathcal{H}})^2 + (\mathcal{O}_1^{\Sigma})^2 + (\mathcal{R}_1^{\Sigma})^2 + C_{\text{ball}}^2. \quad (5.33)$$

On the one hand,

$$\begin{aligned} \mathbf{D}_T \mathbf{R}_{TaTb} & = \mathbf{D}_T E_{ab} + n^{-1} \nabla^c n (\mathbf{R}_{caTb} - \mathbf{R}_{Tab}), \\ \mathbf{D}_T^* \mathbf{R}_{TaTb} & = \mathbf{D}_T H_{ab} + n^{-1} \nabla^c n (*\mathbf{R}_{caTb} - *\mathbf{R}_{Tab}). \end{aligned}$$

On the other hand, by definition of the Lie derivative and the Bianchi equations (2.14),

$$\begin{aligned} \mathbf{D}_T E_{ab} & = \hat{\mathcal{L}}_T E_{ab} - (k_{ac} E_{cb} + k_{bc} E_{ca} - k \cdot E g_{ab}), \\ & = -\text{curl} H_{ab} - (n^{-1} \nabla n \wedge H)_{ab} + \frac{1}{2} (k \times E)_{ab} - (k_{ac} E_{cb} + k_{bc} E_{ca} - k \cdot E g_{ab}), \\ \mathbf{D}_T H_{ab} & = \hat{\mathcal{L}}_T H_{ab} - (k_{ac} H_{cb} + k_{bc} H_{ca} - k \cdot H g_{ab}) \\ & = \text{curl} E_{ab} + (n^{-1} \nabla n \wedge E)_{ab} + \frac{1}{2} (k \times H)_{ab} - (k_{ac} H_{cb} + k_{bc} H_{ca} - k \cdot H g_{ab}). \end{aligned}$$

Combining the two above and using (5.2), (5.3), (5.4) and (5.12), we get that

$$\begin{aligned}
 & \|\mathbf{D}_T \mathbf{R}_{TaTb}\|_{L^2(\Sigma_{t^*})} + \|\mathbf{D}_T^* \mathbf{R}_{TaTb}\|_{L^2(\Sigma_{t^*})} \\
 & \lesssim \|\nabla E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} \\
 & \quad + (\|\nabla^2 n\|_{L^2(\Sigma_{t^*})} + \|\nabla n\|_{L^2(\Sigma_{t^*})} + \|n-1\|_{L^2(\Sigma_{t^*})} + \|\nabla k\|_{L^2(\Sigma_{t^*})} + \|k\|_{L^2(\Sigma_{t^*})}) \\
 & \quad \cdot (\|\nabla E\|_{L^2(\Sigma_{t^*})} + \|E\|_{L^2(\Sigma_{t^*})} + \|\nabla H\|_{L^2(\Sigma_{t^*})} + \|H\|_{L^2(\Sigma_{t^*})}) \\
 & \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}.
 \end{aligned}$$

This finishes the proof of (5.33).

It remains to bound  $\mathbf{D}_c \mathbf{R}$  where  $(e_1, e_2, e_3)$  is an orthonormal frame tangent to  $\Sigma_{t^*}$ . By decomposing  $\mathbf{D}_c \mathbf{R}$  into  $\mathbf{D}_c \mathbf{R}_{T.T.}$  and  $\mathbf{D}_c^* \mathbf{R}_{T.T.}$  (see the electric-magnetic decomposition of Weyl tensors in Section 2.3), it suffices to prove that for  $c = 1, 2, 3$ ,

$$\int_{\Sigma_{t^*}} |\mathbf{D}_c \mathbf{R}_{T.T.}|_{\mathfrak{h}}^2 + |\mathbf{D}_c^* \mathbf{R}_{T.T.}|_{\mathfrak{h}}^2 \lesssim (\mathcal{O}_1^{\mathcal{H}})^2 + (\mathcal{R}_1^{\mathcal{H}})^2 + (\mathcal{O}_1^{\Sigma})^2 + (\mathcal{R}_1^{\Sigma})^2 + C_{\text{ball}}^2. \quad (5.34)$$

Using the relations

$$\begin{aligned}
 \mathbf{D}_c \mathbf{R}_{TaTb} &= \nabla_c E_{ab} + k_{cd} \mathbf{R}_{daTb} + k_{cd} \mathbf{R}_{TaTd}, \\
 \mathbf{D}_c^* \mathbf{R}_{TaTb} &= \nabla_c H_{ab} + k_{cd}^* \mathbf{R}_{daTb} + k_{cd}^* \mathbf{R}_{TaTd},
 \end{aligned}$$

we get with (5.2), (5.3) and (5.4) that

$$\|\mathbf{D}_c \mathbf{R}_{TaTb}\|_{L^2(\Sigma_{t^*})} + \|\mathbf{D}_c^* \mathbf{R}_{TaTb}\|_{L^2(\Sigma_{t^*})} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}.$$

This finishes the proof of (5.34) and hence (5.5).

### 5.1.6 Proof of (5.6) and (5.7)

In this section we prove (5.6) and (5.7), that is,

$$\|\nabla^3 n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 T(n)\|_{L^2(\Sigma_{t^*})} + \|\nabla T^2(n)\|_{L^2(\Sigma_{t^*})} \lesssim \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}},$$

and

$$\|\mathbf{D}_T \mathbf{D}_T k\|_{L^2(\Sigma_t)} + \|\nabla \mathbf{D}_T k\|_{L^2(\Sigma_t)} \lesssim \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + C_{\text{ball}}.$$

First, by applying standard elliptic estimates (see Proposition 3.10) to the boundary value problem for  $n$  in (4.9), that is,

$$\begin{aligned}
 \Delta n &= n|k|_g^2 && \text{on } \Sigma_{t^*}, \\
 n &= \nu^{-1} \Omega^{-1} && \text{on } \partial \Sigma_{t^*},
 \end{aligned} \quad (5.35)$$

and using (5.3) and (5.4), it follows that for  $C_{\text{ball}} > 0$  sufficiently small,

$$\begin{aligned}
 & \sum_{0 \leq m' \leq 3} \|\nabla^{m'} n\|_{L^2(\Sigma_{t^*})} \\
 & \lesssim \|\nabla \Delta n\|_{L^2(\Sigma_{t^*})} + \|\Delta n\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 n\|_{H^{1/2}(\partial \Sigma_{t^*})} + \|\nabla n\|_{H^{1/2}(\partial \Sigma_{t^*})} + \|n\|_{L^2(\partial \Sigma_{t^*})} \\
 & \lesssim \|\nabla(n|k|^2)\|_{L^2(\Sigma_{t^*})} + \|n|k|^2\|_{L^2(\Sigma_{t^*})} + \|\nabla^2(\Omega^{-1}\nu^{-1})\|_{H^{1/2}(\partial \Sigma_{t^*})} \\
 & \quad + \|\nabla(\Omega^{-1}\nu^{-1})\|_{H^{1/2}(\partial \Sigma_{t^*})} + \|\Omega^{-1}\nu^{-1}\|_{L^2(\partial \Sigma_{t^*})} \\
 & \lesssim \mathcal{O}_1^{\Sigma} + \mathcal{R}_1^{\Sigma} + \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}.
 \end{aligned}$$

Second, we turn to the control of  $T(n)$  and  $TT(n)$ . On the one hand, by Lemma 2.9 and (2.13a),  $T(n)$  satisfies on  $\Sigma_{t^*}$  the equation

$$\begin{aligned}
 \Delta(T(n)) &= T(\Delta n) + [\Delta, T]n \\
 &= T(n)|k|^2 + 2nk \cdot \mathbf{D}_T k + 2k \cdot \nabla^2 n - 2n^{-1} \nabla n \nabla T(n) - |k|^2 T(n) \\
 & \quad + 2n^{-1} k |\nabla n|^2 \\
 &= 2nk(E - n^{-1} \nabla^2 n + k \cdot k) + 2k \cdot \nabla^2 n - 2n^{-1} \nabla n \nabla T(n) + 2n^{-1} k |\nabla n|^2
 \end{aligned} \quad (5.36)$$

On the other hand, we have by Lemma 2.14 that on  $\partial\Sigma_t$ ,

$$\begin{aligned}
 T(n) &= \nu L(n) - N(n) \\
 &= \nu L(\nu^{-1}\Omega^{-1}) - N(n) \\
 &= -\frac{1}{\nu\Omega}L(\nu) - \frac{1}{\Omega^2}L(\Omega) - N(n) \\
 &= -\frac{1}{\nu\Omega}\left(n^{-1}N(n) - \delta\right) - \frac{1}{\Omega^2}L(\Omega) - N(n),
 \end{aligned} \tag{5.37}$$

which implies by Lemma 3.6, (5.2) and (5.4) that

$$\begin{aligned}
 \|T(n)\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla T(n)\|_{H^{1/2}(\partial\Sigma_{t^*})} &\lesssim \|N(n)\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla N(n)\|_{H^{1/2}(\partial\Sigma_{t^*})} \\
 &\quad + \|\delta\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla \delta\|_{H^{1/2}(\partial\Sigma_{t^*})} \\
 &\quad + \|L(\Omega)\|_{H^{1/2}(\partial\Sigma_{t^*})} + \|\nabla L(\Omega)\|_{H^{1/2}(\partial\Sigma_{t^*})} \\
 &\lesssim \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}.
 \end{aligned} \tag{5.38}$$

Applying Proposition 3.10 to (5.36) and using (5.38), (5.2), (5.3) and (5.4), we get that

$$\|\nabla^2 T(n)\|_{L^2(\Sigma_t)} + \|\nabla T(n)\|_{L^2(\Sigma_t)} \lesssim \mathcal{O}_1^\Sigma + \mathcal{R}_1^\Sigma + \mathcal{O}_1^{\mathcal{H}} + \mathcal{R}_1^{\mathcal{H}} + C_{\text{ball}}. \tag{5.39}$$

The proof of the control of  $\nabla T^2(n)$  follows by commuting (5.35) once more with  $T$ , applying standard elliptic estimates (Proposition 3.10) and using Lemmas 2.14 and 3.6 and (5.2) to control the boundary value of  $T^2(n)$ ; see also Appendix E in [KRS15]. This finishes the proof of (5.6).

The estimate (5.7) for  $\mathbf{D}_T \mathbf{D}_T k$  and  $\nabla \mathbf{D}_T k$  follows similarly by respectively applying  $\mathbf{D}_T$  and  $\nabla$  to the second variation equation (2.13a), that is,

$$\mathbf{D}_T k_{ij} = E_{ij} - n^{-1} \nabla_i \nabla_j n - k_{il} k^l{}_j,$$

and using the above estimates for  $\mathbf{R}$ ,  $k$  and  $n$ . This finishes the proof of (5.7) and the case  $m = 1$  of Proposition 2.30.

## 5.2 Higher regularity estimates for $m \geq 2$

In this section we outline the proof of the cases  $m \geq 2$  of Proposition 2.30. The proof is based on induction in  $m \geq 1$ . The base case  $m = 1$  is proved in the previous sections. In the following we discuss the induction step  $m \rightarrow m + 1$ .

By assumption in Proposition 2.30, the foliation  $(\Sigma_t)_{1 \leq t \leq t_0^*}$  satisfies for a real number  $\varepsilon > 0$ , on  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned}
 \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} + \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\
 \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(\partial\Sigma_t)} &\lesssim \varepsilon.
 \end{aligned}$$

For the proof by induction of Proposition 2.30, assume as *induction hypothesis* that for an integer  $m \geq 1$ , it holds that for  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned}
 &\sum_{0 \leq m' \leq m} \|\mathbf{D}^{m'} \mathbf{R}\|_{L^2(\Sigma_t)} + \sum_{0 \leq m' \leq m+1} \|\mathbf{D}^{m'} \pi\|_{L^2(\Sigma_t)} \\
 &+ \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma_t)} + \|\nabla^{m'} \nabla \nu\|_{H^{1/2}(\partial\Sigma_t)} \\
 &\lesssim C(\mathcal{O}_m^{\mathcal{H}}, \mathcal{R}_m^{\mathcal{H}}, \mathcal{O}_m^\Sigma, \mathcal{R}_m^\Sigma, m).
 \end{aligned} \tag{5.40}$$

In the following we prove the *induction step*, that is, we show that for  $0 \leq t \leq t_0^*$ ,

$$\begin{aligned}
 &\sum_{0 \leq m' \leq m+1} \|\mathbf{D}^{m'} \mathbf{R}\|_{L^2(\Sigma_t)} + \sum_{0 \leq m' \leq m+2} \|\mathbf{D}^{m'} \pi\|_{L^2(\Sigma_t)} \\
 &+ \sum_{0 \leq m' \leq m+1} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma_t)} + \|\nabla^{m'} \nabla \nu\|_{H^{1/2}(\partial\Sigma_t)} \\
 &\lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^\Sigma, \mathcal{R}_{m+1}^\Sigma, m+1).
 \end{aligned} \tag{5.41}$$

We proceed as follows.

1. In Section 5.2.1, we prove that for each  $0 \leq t \leq t_0^*$ ,

$$\begin{aligned} & \sum_{0 \leq m' + m'' \leq m+1} \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} E \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} H \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \end{aligned} \quad (5.42)$$

The proof of (5.42) is based on the fact that for each  $0 \leq m' \leq m$ ,  $E(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  and  $H(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  satisfy a 3-dimensional Hodge system on  $\Sigma_t$  by the Bianchi equations, see (2.7) and Proposition 2.8.

2. In Section 5.2.2, we prove that for  $0 \leq t \leq t_0^*$ ,

$$\left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \quad (5.43)$$

The proof of (5.43) is based on an energy estimate for the curvature using the Bel-Robinson tensor together with the classical Grönwall lemma.

**Remark 5.5.** *Contrary to the case  $m = 1$  where the error integral in the Bel-Robinson energy estimate is bounded by a trilinear estimate, in the case  $m \geq 2$  we can argue solely by the classical Grönwall lemma and the estimates for  $m = 1$ . In particular, we do not use the bounded  $L^2$  curvature theorem and its trilinear estimate in this case.*

3. In Section 5.2.3, we show that for  $1 \leq t \leq t_0^*$ ,

$$\begin{aligned} & \sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \nabla k \right\|_{L^2(\Sigma_t)} + \sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \nabla \nu \right\|_{L^2(\partial \Sigma_t)} \\ & \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \end{aligned} \quad (5.44)$$

The proof of (5.44) is based on standard higher regularity estimates for the Hodge system satisfied by  $k$ .

4. In Section 5.2.4, we conclude the proof of the induction step (5.41).

### 5.2.1 Elliptic curvature estimates on $\Sigma_t$

In this section we prove (5.42), that is,

$$\begin{aligned} & \sum_{0 \leq m' + m'' \leq m+1} \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} E \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} H \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \end{aligned}$$

By Proposition 2.8 with (2.7) for each  $0 \leq m' \leq m$ ,  $E(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  and  $H(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  satisfy the following 3-dimensional Hodge system on  $\Sigma_t$ ,

$$\begin{aligned} \operatorname{div} E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_a &= + \left( k \wedge H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right)_a + J \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{TaT}, \\ \operatorname{curl} E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{ab} &= + H \left( \hat{\mathcal{L}}_T^{m'+1} \mathbf{R} \right)_{ab} - 3 \left( n^{-1} \nabla n \wedge E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right)_{ab} \\ &\quad - \frac{3}{2} \left( k \times H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right)_{ab} - J^* \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{aTb}, \\ \operatorname{div} H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_a &= - \left( k \wedge E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right)_a + J^* \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{TaT}, \\ \operatorname{curl} H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{ab} &= - E \left( \hat{\mathcal{L}}_T^{m'+1} \mathbf{R} \right)_{ab} - 3 \left( n^{-1} \nabla n \wedge H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right)_{ab} \\ &\quad + \frac{3}{2} \left( k \times E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right)_{ab} - J \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{aTb}, \end{aligned} \quad (5.45)$$

where we denoted

$$J \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{\beta\gamma\delta} := \mathbf{D}^\alpha \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{\alpha\beta\gamma\delta}.$$

By standard higher regularity elliptic estimates (proved by combining Lemma A.2 with Lemmas A.4 and A.5) applied to (5.45) for  $0 \leq m' \leq m$ , it follows that

$$\begin{aligned} & \left\| \nabla \nabla^{m-m'} E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right\|_{L^2(\Sigma_t)} + \left\| \nabla \nabla^{m-m'} H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \nabla^{m-m'} \hat{\mathcal{L}}_T^{m'+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m-m'} J \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m-m'} J^* \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right\|_{L^2(\Sigma_t)} \\ & \quad + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1) + \|\text{l.o.t.}\|_{L^2(\Sigma_{t^*})}, \end{aligned} \quad (5.46)$$

where we applied (5.40), denoted lower order product terms by l.o.t. and controlled the boundary terms of the global elliptic estimate by initial data norms analogously as in Section 5.1.1.

Further, we have that  $\nabla^{m-m'} J(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  and  $\nabla^{m-m'} J^*(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  on the right-hand side of (5.46) are lower order product terms. Indeed, on the one hand, by definition (see also Proposition 7.1.2 in [CK93]), we have the recursive relation

$$\begin{aligned} J \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) & := \mathbf{D}^\alpha \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)_{\alpha\beta\gamma\delta} \\ & = \hat{\mathcal{L}}_T J \left( \hat{\mathcal{L}}_T^{m'-1} \mathbf{R} \right)_{\beta\gamma\delta} + \frac{1}{2} \pi^{\mu\nu} \mathbf{D}_\nu \left( \hat{\mathcal{L}}_T^{m'-1} \mathbf{R} \right)_{\mu\beta\gamma\delta} \\ & \quad + \frac{1}{2} \mathbf{D}^\alpha \pi_{\alpha\lambda} \left( \hat{\mathcal{L}}_T^{m'-1} \mathbf{R} \right)_{\beta\gamma\delta}^\lambda + \frac{1}{2} (\mathbf{D}_\beta \pi_{\alpha\lambda} - \mathbf{D}_\lambda \pi_{\alpha\beta}) \left( \hat{\mathcal{L}}_T^{m'-1} \mathbf{R} \right)_{\gamma\delta}^{\alpha\lambda} \\ & \quad + \frac{1}{2} (\mathbf{D}_\gamma \pi_{\alpha\lambda} - \mathbf{D}_\lambda \pi_{\alpha\gamma}) \left( \hat{\mathcal{L}}_T^{m'-1} \mathbf{R} \right)_{\beta\delta}^{\alpha\lambda} \\ & \quad + \frac{1}{2} (\mathbf{D}_\delta \pi_{\alpha\lambda} - \mathbf{D}_\lambda \pi_{\alpha\delta}) \left( \hat{\mathcal{L}}_T^{m'-1} \mathbf{R} \right)_{\beta\gamma}^{\alpha\lambda}, \end{aligned} \quad (5.47)$$

with

$$\begin{aligned} \hat{\mathcal{L}}_T J \left( \hat{\mathcal{L}}_T^{i-1} \mathbf{R} \right)_{\beta\gamma\delta} & := \mathcal{L}_T J \left( \hat{\mathcal{L}}_T^{i-1} \mathbf{R} \right)_{\beta\gamma\delta} - \frac{1}{2} \pi_\beta^\mu J \left( \hat{\mathcal{L}}_T^{i-1} \mathbf{R} \right)_{\mu\gamma\delta} \\ & \quad - \frac{1}{2} \pi_\gamma^\mu J \left( \hat{\mathcal{L}}_T^{i-1} \mathbf{R} \right)_{\beta\mu\delta} - \frac{1}{2} \pi_\delta^\mu J \left( \hat{\mathcal{L}}_T^{i-1} \mathbf{R} \right)_{\beta\gamma\mu}. \end{aligned}$$

On the other hand, it holds that by the Bianchi equations,

$$J(\mathbf{R}) = J^*(\mathbf{R}) = 0.$$

Combining the above two, it follows that  $J(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  and  $J^*(\hat{\mathcal{L}}_T^{m'} \mathbf{R})$  consist only of lower order product terms.

Therefore we can write (5.46) for  $0 \leq m' \leq m$  as follows,

$$\begin{aligned} & \left\| \nabla \nabla^{m-m'} E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right\|_{L^2(\Sigma_t)} + \left\| \nabla \nabla^{m-m'} H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right) \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \nabla^{m-m'} \hat{\mathcal{L}}_T^{m'+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1) + \|\text{l.o.t.}\|_{L^2(\Sigma_t)}. \end{aligned} \quad (5.48)$$

In particular, using that  $E \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)$  and  $H \left( \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right)$  are the electric-magnetic decomposition of  $\hat{\mathcal{L}}_T^{m'} \mathbf{R}$ , (5.48) implies the estimate

$$\begin{aligned} & \left\| \nabla \nabla^{m-m'} \hat{\mathcal{L}}_T^{m'} \mathbf{R} \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \nabla^{m-m'} \hat{\mathcal{L}}_T^{m'+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1) + \|\text{l.o.t.}\|_{L^2(\Sigma_t)} \end{aligned} \quad (5.49)$$

Consequently, recursive application of (5.49) for  $0 \leq m' \leq m$  shows that

$$\begin{aligned} & \sum_{0 \leq m' + m'' \leq m+1} \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} \mathbf{R} \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1) + \|\text{l.o.t.}\|_{L^2(\Sigma_t)}, \end{aligned} \quad (5.50)$$

which, using (5.40) and that  $\hat{\mathcal{L}}_T^0 \mathbf{R} = \mathbf{R}$ , implies

$$\begin{aligned} & \sum_{0 \leq m' + m'' \leq m+1} \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} E \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} H \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1) + \|\text{l.o.t.}\|_{L^2(\Sigma_t)}. \end{aligned}$$

The lower order product terms on the right-hand side can be estimated by standard product estimates together with the induction hypothesis (5.50), which yields

$$\begin{aligned} & \sum_{0 \leq m' + m'' \leq m+1} \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} E \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m'} \hat{\mathcal{L}}_T^{m''} H \right\|_{L^2(\Sigma_t)} \\ & \lesssim \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \end{aligned} \quad (5.51)$$

This finishes the proof of (5.42).

## 5.2.2 Energy estimate for the curvature

In this section we prove (5.43), that is, for  $0 \leq t \leq t^*$ ,

$$\left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)} \leq C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m).$$

Applying the classical Bel-Robinson energy estimate (3.4) to the Weyl tensor  $\hat{\mathcal{L}}_T^{m+1} \mathbf{R}$  with multiplier field  $T$  yields for  $0 \leq t \leq t_0^*$ ,

$$\begin{aligned} \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_t)}^2 & \lesssim \int_{\Sigma_1} Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{TTTT} + \int_{\mathcal{H}} Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{TTTL} \\ & \quad - \underbrace{\int_{\mathcal{M}_t} \frac{3}{2} Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{\alpha\beta TT} \pi^{\alpha\beta}}_{:=\mathcal{E}_1} - \underbrace{\int_{\mathcal{M}_t} \mathbf{D}^\alpha Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{\alpha TTT}}_{:=\mathcal{E}_2}, \end{aligned} \quad (5.52)$$

where  $\mathcal{M}_t$  denotes the past of  $\Sigma_t$  in  $\mathcal{M}$ . In the following, we first control the error terms  $\mathcal{E}_1$  and  $\mathcal{E}_2$ .

**Control of  $\mathcal{E}_1$ .** By Lemma 2.16,

$$\begin{aligned} \mathcal{E}_1 & := \int_{\mathcal{M}_t} \frac{3}{2} Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{\alpha\beta TT} \pi^{\alpha\beta} \\ & \lesssim \|\pi\|_{L^\infty(\mathcal{M}_t)} \int_{\mathcal{M}_t} |Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{\alpha\beta TT}| \\ & \lesssim \|\pi\|_{L^\infty(\mathcal{M}_t)} \left(1 + \|n-1\|_{L^\infty(\mathcal{M}_t)}\right) \int_0^t \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_{t'})}^2 dt' \\ & \lesssim C(\mathcal{O}_m^{\mathcal{H}}, \mathcal{R}_m^{\mathcal{H}}, \mathcal{O}_m^{\Sigma}, \mathcal{R}_m^{\Sigma}, m) \int_0^t \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_{t'})}^2 dt'. \end{aligned}$$

This finishes the control of  $\mathcal{E}_1$ .

**Control of  $\mathcal{E}_2$ .** Using that

$$\begin{aligned} \mathbf{D}^\alpha Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{\alpha TTT} & = 2 \left( \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right)_{T T}^{\mu \nu} J \left( \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right)_{\mu T \nu} \\ & \quad + 2^* \left( \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right)_{T T}^{\mu \nu} J^* \left( \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right)_{\mu T \nu}, \end{aligned}$$

we can estimate

$$\mathcal{E}_2 := \int_{\mathcal{M}_t} \mathbf{D}^\alpha Q(\hat{\mathcal{L}}_T^{m+1} \mathbf{R})_{\alpha TTT} \lesssim \int_0^t \left\| \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right\|_{L^2(\Sigma_{t'})} \left\| J \left( \hat{\mathcal{L}}_T^{m+1} \mathbf{R} \right) \right\|_{L^2(\Sigma_{t'})} dt'. \quad (5.53)$$

Using that  $J(\hat{\mathcal{L}}_T^{m+1}\mathbf{R})$  consists of lower order product terms (see the discussion in Section 5.2.1) together with the elliptic estimates (5.50) and (5.51), we have that

$$\left\| J(\hat{\mathcal{L}}_T^{m+1}\mathbf{R}) \right\|_{L^2(\Sigma_{t'})} \lesssim \left\| \hat{\mathcal{L}}_T^{m+1}\mathbf{R} \right\|_{L^2(\Sigma_{t'})} + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \quad (5.54)$$

Plugging (5.54) into (5.53), we get that

$$\mathcal{E}_2 \lesssim \int_0^t \left\| \hat{\mathcal{L}}_T^{m+1}\mathbf{R} \right\|_{L^2(\Sigma_{t'})}^2 dt' + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1).$$

Plugging the above estimates for  $\mathcal{E}_1$  and  $\mathcal{E}_2$  into (5.52), we get that for  $0 \leq t \leq t_0^*$ ,

$$\begin{aligned} \left\| \hat{\mathcal{L}}_T^{m+1}\mathbf{R} \right\|_{L^2(\Sigma_t)}^2 &\lesssim \int_{\Sigma_1} Q(\hat{\mathcal{L}}_T^{m+1}\mathbf{R})_{TTTT} + \int_{\mathcal{H}} Q(\hat{\mathcal{L}}_T^{m+1}\mathbf{R})_{TTTL} + \int_0^t \left\| \hat{\mathcal{L}}_T^{m+1}\mathbf{R} \right\|_{L^2(\Sigma_{t'})}^2 dt' \\ &\quad + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \end{aligned}$$

Therefore Grönwall's lemma yields that for  $0 \leq t \leq t_0^*$ ,

$$\begin{aligned} \left\| \hat{\mathcal{L}}_T^{m+1}\mathbf{R} \right\|_{L^2(\Sigma_t)}^2 &\lesssim \underbrace{\int_{\Sigma_1} Q(\hat{\mathcal{L}}_T^{m+1}\mathbf{R})_{TTTT}}_{:=\mathcal{I}_1} + \underbrace{\int_{\mathcal{H}} Q(\hat{\mathcal{L}}_T^{m+1}\mathbf{R})_{TTTL}}_{:=\mathcal{I}_2} \\ &\quad + C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m). \end{aligned}$$

The boundary integrals  $\mathcal{I}_1$  and  $\mathcal{I}_2$  by initial data norms as follows,

$$\mathcal{I}_1 + \mathcal{I}_2 \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1);$$

see also Sections 4.6 and 5.1.2. This finishes the proof of (5.43).

### 5.2.3 Elliptic estimate for $k$ on $\Sigma_{t^*}$

In this section, we prove (5.44), that is,

$$\sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \nabla k \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m'} \nu \right\|_{L^2(\partial\Sigma_t)} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1).$$

By standard higher regularity elliptic estimates (see Sections A.1 and A.2) applied to the Hodge system satisfied by  $k$ , see (2.13b), (2.13c) and (2.13d), and using the previous higher regularity estimates (5.40), (5.42) and (5.43) we have

$$\sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \nabla k \right\|_{L^2(\Sigma_{t^*})} + \left\| \nabla^{m'} \nabla \nu \right\|_{L^2(\partial\Sigma_{t^*})} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1).$$

Indeed, the boundary integrals of the elliptic estimate are shown to bound  $\left\| \nabla^{m'} \nabla \nu \right\|_{L^2(\partial\Sigma_{t^*})}^2$  in a similar way as in Section 5.1.3. The only difference in the analysis of the cases  $m \geq 2$  is that normal derivatives in the boundary integral are systematically reduced to  $\partial\Sigma_t$ -tangential derivatives by using  $\operatorname{div} k = 0$  and  $\operatorname{curl} k = H$  (see also (2.18)) an *even number of times*. This is due to the fact that the integrand of the boundary integral is a contraction of two tensors. As a consequence, *the sign is conserved* and the constant in the estimate is bounded from below by a positive constant independent of  $m$ . This finishes the proof of (5.44).

### 5.2.4 Conclusion of (5.41)

In this section, we conclude the proof of (5.41), that is,

$$\begin{aligned} &\sum_{0 \leq m' \leq m+1} \left\| \mathbf{D}^{m'} \mathbf{R} \right\|_{L^2(\Sigma_t)} + \sum_{0 \leq m' \leq m+2} \left\| \mathbf{D}^{m'} \pi \right\|_{L^2(\Sigma_t)} \\ &\quad + \sum_{0 \leq m' \leq m+1} \left\| \nabla^{m'} \operatorname{Ric} \right\|_{L^2(\Sigma_t)} + \left\| \nabla^{m'} \nabla \nu \right\|_{H^{1/2}(\partial\Sigma_t)} \\ &\lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1). \end{aligned}$$



Indeed, first, the estimate

$$\sum_{0 \leq m' \leq m+1} \|\mathbf{D}^{m'} \mathbf{R}\|_{L^2(\Sigma_t)} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1)$$

follows in a straight-forward way from the induction hypothesis estimates (5.40) and the previous estimates (5.42) and (5.43) for  $\nabla^{m'} \mathbf{R}$  and  $\hat{\mathcal{L}}_T^{m'} \mathbf{R}$  for  $0 \leq m' \leq m+1$ , see also Section 5.1.5.

Second, the estimate

$$\sum_{0 \leq m' \leq m+2} \|\mathbf{D}^{m'} \pi\|_{L^2(\Sigma_t)} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1)$$

follows by the induction hypothesis estimates (5.40) and previous estimates (5.42) and (5.44) for  $\nabla^{m'} \mathbf{R}$  and  $\nabla^{m'} k$ ,  $0 \leq m' = m+1$ , by using the relation between  $\pi$  and  $k$  and  $n$  in (2.6), the second variation equation (2.13a) and  $T$ -derivatives thereof (see Lemma 4.11), and applying higher regularity elliptic estimates to the maximal lapse equation for  $n$  and equations satisfied by  $T^{m'}(n)$ ,  $0 \leq m' \leq m+2$ , where the boundary value is controlled by generalisations of Lemma 2.14; see Section 5.1.6.

Third, the estimate

$$\sum_{0 \leq m' \leq m+1} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma_t)} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1)$$

follows by the previous estimates (5.42) and (5.44) for  $\nabla^{m'} \mathbf{R}$  and  $\nabla^{m'} k$ ,  $0 \leq m' \leq m+1$ , by using the traced Gauss equation (2.13f).

Fourth, the estimate

$$\|\nabla^{m'} \nabla \nu\|_{H^{1/2}(\partial\Sigma_t)} \lesssim C(\mathcal{O}_{m+1}^{\mathcal{H}}, \mathcal{R}_{m+1}^{\mathcal{H}}, \mathcal{O}_{m+1}^{\Sigma}, \mathcal{R}_{m+1}^{\Sigma}, m+1)$$

follows by using the slope equation (2.24), that is,

$$\nu^{-1} \nabla_{A\nu} \nu = -\epsilon_A + \zeta_A \text{ on } \partial\Sigma_t.$$

together with (5.42), (5.43), (5.44) and Lemmas 3.6, 3.3 and 3.6. This finishes our discussion of the proof of (5.41).

## 6. Classical local existence of vacuum space-time with maximal foliation

In this section, we prove Proposition 2.28. First, we have the following classical local existence result for the spacelike-characteristic Cauchy problem of general relativity.

**Theorem 6.1** (Classical local existence). *Let there be given smooth initial data for the spacelike-characteristic Cauchy problem on a maximal spacelike hypersurface with boundary  $\Sigma$  and the outgoing null hypersurface  $\mathcal{H}$  emanating from  $\partial\Sigma$ . Let  $(S_v)_{v \geq 1}$  be a smooth foliation on  $\mathcal{H}$  by spacelike 2-spheres such that  $S_1 = \partial\Sigma$ . Then there exists a real number  $\tau > 0$  such that the maximal smooth globally hyperbolic future development  $(\mathcal{M}, \mathbf{g})$  has past boundary  $\Sigma \cup (S_v)_{1 \leq v \leq 1+\tau}$ .*

*Proof.* The proof follows from [Fou52] and [Ren90], see also [Luk12]. First, by classical local existence for the spacelike Cauchy problem [Fou52] it follows that the maximal smooth globally hyperbolic future development of the initial data on  $\Sigma$ , denoted by  $\mathcal{D}(\Sigma)$ , has past boundary  $\Sigma$ . Given that  $\mathcal{D}(\Sigma) \subset \mathcal{M}$ , it follows that the past boundary of  $\mathcal{M}$  contains  $\Sigma$ .

Second, considering the induced data on  $\partial\mathcal{D}(\Sigma)$  together with the characteristic initial data on  $\mathcal{H}$ , it follows by classical local existence for the characteristic Cauchy problem [Ren90] that there is  $\tau > 0$  such that the past boundary of  $\mathcal{M}$  contains  $\Sigma \cup (S_v)_{1 \leq v \leq 1+\tau}$ . This finishes the proof of Theorem 6.1.  $\square$

We turn to the proof of Proposition 2.28. Let  $\Sigma$  be a compact maximal spacelike hypersurface with boundary and let  $\mathcal{H}$  be the outgoing null hypersurface emanating from  $\partial\Sigma$ . Let  $(S_v)_{v \geq 1}$  be a smooth foliation on  $\mathcal{H}$  by spacelike 2-spheres  $S_v$  such that  $S_1 = \partial\Sigma$ . Let  $(\mathcal{M}, \mathbf{g})$  denote the maximal globally hyperbolic future development of the spacelike-characteristic Cauchy problem.

In the following, we prove that there is a real number  $\tau > 0$  and a local time function  $t$  in the future of  $\Sigma$  such that

- $\diamond$   $t = 1$  on  $\Sigma$ ,
- $\diamond$   $t = v$  on  $\mathcal{H}$  for  $1 \leq t \leq 1 + \tau$ ,
- $\diamond$  the level sets  $\Sigma_t$  of  $t$  are maximal spacelike hypersurfaces for  $1 \leq t \leq 1 + \tau$ .

The main ingredient for this construction is the work [Cho76] of Bruhat which shows that in shift-free background coordinates the linearisation of the mean curvature functional is an isomorphism, see Theorem 6.2 below.

We split the construction of the time function into three steps.

1. Construction of local shift-free background coordinates  $(x^\mu)_{\mu=0,1,2,3}$ ,
2. Construction of a family of maximal spacelike hypersurfaces in  $\mathcal{M}$ ,
3. Proof that the above family of spacelike maximal hypersurfaces can be written as level sets of a smooth time function  $t$  satisfying  $t = 1$  on  $\Sigma$  and  $t = v$  on  $\mathcal{H}$  for  $1 \leq t \leq 1 + \tau$  for  $\tau > 0$  sufficiently small.

**Step 1. Construction of shift-free background coordinates.** First, define a scalar function  $x^0$  on  $\Sigma \cup \mathcal{H}$  by

$$x^0 = 1 \text{ on } \Sigma, \quad x^0 = v \text{ on } \mathcal{H}, \quad (6.1)$$

where  $v$  denotes the parameter of the given foliation  $(S_v)_{v \geq 1}$  on  $\mathcal{H}$ . By the Whitney extension theorem (see its similar application in [Ren90] and references therein) there exists a smooth extension of  $x^0$  into  $\mathcal{M}$  such that its level sets  $(\tilde{\Sigma}_{x^0})_{x^0 \geq 1}$  are spacelike and form a local foliation of the future of  $\Sigma$  in  $(\mathcal{M}, \mathbf{g})$ . We note that by (6.1),

$$\partial\tilde{\Sigma}_{x^0} = S_{x^0} \subset \mathcal{H}. \quad (6.2)$$

Let  $e_0$  be the future-pointing timelike unit normal to  $\tilde{\Sigma}_{x^0}$ .

Second, let  $(\bar{x}^i)_{i=1,2,3}$  be given coordinates on  $\Sigma$ . We extend them as local coordinates  $(x^i)_{i=1,2,3}$  onto  $\mathcal{M}$  as follows. First let

$$x^i := f_i \text{ on } \mathcal{H}, \quad (6.3)$$

where  $(f_i)_{i=1,2,3} \in C^\infty(\mathcal{H})$  are smooth, increasing functions chosen below. Then define  $(x^i)_{i=1,2,3}$  on  $\mathcal{M}$  as solution to

$$e_0(x^i) = 0 \text{ on } \mathcal{M}, \quad x^i = \bar{x}^i \text{ on } \Sigma_0, \quad x^i = f_i \text{ on } \mathcal{H}.$$

The smoothness of  $x^i$  in  $\mathcal{M}$  requires algebraic compatibility conditions on  $(f_i)_{i=1,2,3}$  and their derivatives at  $\partial\Sigma = \mathcal{H} \cap \Sigma$ . By the Whitney extension theorem (see its similar application in [Ren90] and references therein), there exists a choice  $(f_i)_{i=1,2,3}$  such that these compatibility conditions are satisfied.

By construction,  $(x^\mu)_{\mu=0,1,2,3}$  locally form a coordinate system in the future of  $\Sigma$ . Moreover, the coordinates are by construction shift-free, that is,  $e_0(x^i) = 0$  for  $i = 1, 2, 3$ .

**Step 2: Construction of a foliation of maximal spacelike hypersurfaces.** In the following, for  $\tau > 0$  sufficiently small, we perturb the level sets  $(\tilde{\Sigma}_{x^0})_{1 \leq x^0 \leq 1+\tau}$  to maximal hypersurfaces, using that  $\tilde{\Sigma}_1 = \Sigma$  by assumption maximal. Our main tool to do so is the following paraphrase of [Cho76].

**Theorem 6.2** (Construction of nearby maximal spacelike hypersurfaces by perturbation). *Let  $m \geq 0$  be an integer. Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime and let  $\Sigma \subset \mathcal{M}$  be a compact maximal spacelike hypersurface with boundary, that is, satisfying*

$$H_{\mathbf{g}}(\Sigma) = 0,$$

where  $H_{\mathbf{g}}(\Sigma)$  denotes the mean curvature of  $\Sigma$  with respect to  $\mathbf{g}$ . Let  $(x^\mu)_{\mu=0,1,2,3}$  be a shift-free coordinate system on  $\mathcal{M}$  such that  $\Sigma = \{x^0 = 1\}$ . Let  $\mathbf{g}'$  be another Lorentzian metric on  $\mathcal{M}$  such that for some  $\varepsilon > 0$ , with respect to the coordinate system  $(x^\mu)_{\mu=0,1,2,3}$ ,

$$\|\mathbf{g} - \mathbf{g}'\|_{C^{m'}(\mathcal{M})} < \varepsilon.$$

There are universal  $m'_0 > 0$  and  $\varepsilon_0 > 0$  such that if  $m' \geq m'_0$  and  $0 < \varepsilon < \varepsilon_0$ , then there is a  $C^{m'}(\Sigma)$ -function  $\varphi : \Sigma \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \varphi|_{\partial\Sigma} &= 0, \\ H_{\mathbf{g}'}(\text{graph}_\Sigma(\varphi)) &= 0, \end{aligned} \quad (6.4)$$

where

$$\text{graph}_\Sigma(\varphi) := \{x^0 = \varphi(x^1, x^2, x^3) + 1\} \subset \mathcal{M}.$$

Moreover, we have the bound

$$\|\varphi\|_{C^m(\Sigma)} \leq C\varepsilon,$$

where the constant  $C > 0$  depends on  $(\mathcal{M}, \mathbf{g})$ ,  $\Sigma$  and  $m$ .

*Remarks.* In Theorem 6.2, due to the first of (6.4), the boundary of  $\partial\Sigma$  is not perturbed.

*Proof of Theorem 6.2.* The proof is based on the implicit function theorem. It is shown in [Cho76] that the linearisation of the mean curvature functional under graphs in shift-free coordinates is an isomorphism. We refer the reader to [Cho76] for more details.  $\square$

In the following we use Theorem 6.2 to construct for sufficiently small  $\tau > 0$  a family of functions

$$(\varphi_{1+\tau'} : \tilde{\Sigma}_{1+\tau'} \rightarrow \mathbb{R})_{0 \leq \tau' \leq \tau} \quad (6.5)$$

such that

$$\begin{aligned} \varphi_{1+\tau'}|_{\partial\tilde{\Sigma}_{1+\tau'}} &= 0, \\ H_{\mathbf{g}}(\text{graph}_{\tilde{\Sigma}_{1+\tau'}}(\varphi_{1+\tau'})) &= 0, \end{aligned} \quad (6.6)$$

where

$$\text{graph}_{\tilde{\Sigma}_{1+\tau'}}(\varphi_{1+\tau'}) := \{x^0 = \varphi_{1+\tau'}(x^1, x^2, x^3) + (1 + \tau')\}.$$

We note in particular that (6.6) implies that

$$\partial \left( \text{graph}_{\tilde{\Sigma}_{1+\tau'}}(\varphi_{1+\tau'}) \right) = \partial \tilde{\Sigma}_{1+\tau'} = S_{1+\tau'}. \quad (6.7)$$

We turn to the construction of (6.5). By (6.3), the hypersurfaces  $\tilde{\Sigma}_{x^0}$  vary smoothly in  $x^0$  with respect to the coordinates  $(x^0, x^1, x^2, x^3)$ . Therefore, for  $\tau > 0$  sufficiently small, the hypersurfaces  $(\tilde{\Sigma}_{1+\tau'})_{0 \leq \tau' \leq \tau}$  are diffeomorphic to each other. That is, for  $\tau > 0$  sufficiently small, there exists a family of diffeomorphisms

$$(\Psi_{\tau'} : \mathbb{R}^4 \rightarrow \mathbb{R}^4)_{0 \leq \tau' \leq \tau}$$

such that

$$\Psi_{\tau'}(\tilde{\Sigma}_{1+\tau'}) = \tilde{\Sigma}_1,$$

and moreover, for any given integer  $m \geq 0$  and real number  $\varepsilon > 0$ , it holds that for  $\tau > 0$  sufficiently small, with respect to the background coordinates  $(x^\mu)_{\mu=0,1,2,3}$ ,

$$\|(\Psi_{\tau'})_* \mathbf{g} - \mathbf{g}\|_{C^m(V)} \leq \varepsilon, \quad (6.8)$$

for  $0 \leq \tau' \leq \tau$ , where  $(\Psi_{\tau'})_* \mathbf{g}$  denotes the push-forward of  $\mathbf{g}$  under  $\Psi_{\tau'}$  and  $V$  is a fixed open coordinate neighbourhood of  $\Sigma$ .

By (6.8) and the maximality of  $\Sigma_1$ , we can apply Theorem 6.2 to get that for  $\tau > 0$  sufficiently small, there is a family of functions

$$(\varphi'_{1+\tau'} : \Sigma_1 \rightarrow \mathbb{R})_{0 \leq \tau' \leq \tau}$$

such that

$$\begin{aligned} \varphi'_{1+\tau'}|_{\partial \Sigma_1} &= 0, \\ H_{\Psi_{\tau'}^* \mathbf{g}}(\text{graph}_{\Sigma_1}(\varphi'_{1+\tau'})) &= 0. \end{aligned}$$

In particular, using the diffeomorphism  $\Psi_{\tau'}$ , it follows that for  $0 \leq \tau' \leq \tau$ ,

$$\begin{aligned} \varphi'_{1+\tau'} \circ \Psi_{\tau'}|_{\partial \tilde{\Sigma}_{1+\tau'}} &= 0, \\ H_{\mathbf{g}}(\text{graph}_{\tilde{\Sigma}_{1+\tau'}}(\varphi'_{1+\tau'} \circ \Psi_{\tau'})) &= 0. \end{aligned}$$

To summarise, for  $\tau > 0$  sufficiently small, we can define the family of functions (6.5) by taking

$$\varphi_{1+\tau'} := \varphi'_{1+\tau'} \circ \Psi_{\tau'} : \tilde{\Sigma}_{1+\tau'} \rightarrow \mathbb{R}.$$

**Step 3: Analysis of the time function.** Consider the mapping

$$\Phi : (t, x^1, x^2, x^3) \mapsto (x^0 = \varphi_t(x^1, x^2, x^3) + t, x^1, x^2, x^3). \quad (6.9)$$

**Remark 6.3.** Denoting by  $\Sigma_t$  the level sets of  $t$ , it follows by (6.2), (6.7) and (6.9) that

$$\Sigma_t = \text{graph}_{\tilde{\Sigma}_t}(\varphi_t), \quad \partial \Sigma_t = \partial \tilde{\Sigma}_t = S_t.$$

To show that  $(t, x^1, x^2, x^3)$  is a smooth coordinate system, we show that  $\Phi$  is a smooth diffeomorphism from a neighbourhood of  $\{t = 1\}$  to a neighbourhood of  $\{x^0 = 1\}$ . By the inverse function theorem, it suffices to show that the Jacobian of  $\Phi$ , denoted by  $D\Phi$ , is invertible at  $t = 1$ . This follows in a standard way (we leave details to the reader) from

$$\partial_t \varphi_t|_{t=1} = 0. \quad (6.10)$$

In the following we prove (6.10). On the one hand,  $\varphi_{1+\tau'}$  satisfies for each  $0 \leq \tau' \leq \tau$  the maximal surface equation

$$H_{\mathbf{g}}(\text{graph}_{\tilde{\Sigma}_{1+\tau'}}(\varphi_{1+\tau'})) = 0. \quad (6.11)$$

On the other hand, using that

$$\varphi_{1+\tau'}|_{\tau'=0} = 0, \quad \varphi_{1+\tau'}|_{\partial\tilde{\Sigma}_{1+\tau'}} = 0 \text{ for } 0 \leq \tau' \leq \tau,$$

it follows that

$$\partial_{\tau'}\varphi_{1+\tau'}|_{\partial\tilde{\Sigma}_1} = 0. \quad (6.12)$$

Consequently, differentiating (6.11) in  $\tau'$  and using (6.12) shows that  $\partial_{\tau'}\varphi_{1+\tau'}|_{\tau'=0}$  lies in the kernel of the linearisation of the mean curvature functional at  $t = 0$ . By Theorem 6.2 (see also the given remarks on the proof) this kernel is trivial and therefore

$$\partial_{\tau'}\varphi_{1+\tau'}|_{\tau'=0} = 0. \quad (6.13)$$

This finishes the proof of (6.10) and hence shows that  $(t, x^1, x^2, x^3)$  is a smooth local coordinate system. In other words, the time function  $t$  is well-defined. This finishes the proof of Proposition 2.28.

## 7. Existence of global coordinates by Cheeger-Gromov theory

In this section we prove Theorem 4.1 by applying the Cheeger-Gromov theory developed in [Czi19a]. Theorem 4.1 is a low regularity *curvature pinching result* and its proof is based, like other curvature pinching results (see for example [GPL97]), on a convergence result and a rigidity result, see Lemmas 7.3 and 7.4 below, respectively.

In the following, we first introduce the necessary definitions and prerequisite results before turning to the proof of Theorem 4.1.

**Notation.** We denote diffeomorphism equivalence and isometry of manifolds by  $\simeq$  and  $\cong$ , respectively. Moreover, given a scalar function  $f$  on a subset  $U$  of Euclidean 3-space, let

$$\|f\|_{H^2(U)} := \sum_{0 \leq m' \leq 2} \|\partial^{m'} f\|_{L^2(U)}.$$

**Definition 7.1** ( $H^2$ -convergence of functions and tensors). *Let  $(M, g)$  be a compact Riemannian 3-manifold with boundary. Let  $(\varphi_i)$  be a finite number of fixed charts covering  $M$ . A sequence of functions  $(f_n)_{n \in \mathbb{N}}$  on  $M$  is said to converge in  $H^2$  as  $n \rightarrow \infty$ , if for each  $\varphi_i$ , the pullbacks  $(\varphi_i)^* f_n$  converge in  $H^2$  as  $n \rightarrow \infty$ . The convergence of a sequence of tensors on  $M$  in  $H^2$  is defined similarly.*

**Definition 7.2** ( $H^2$ -convergence of manifolds with boundary). *A sequence  $(M_n, g_n)$  of compact Riemannian 3-manifolds with boundary is said to converge to a Riemannian manifold with boundary  $(M, g)$  in the  $H^2$ -topology as  $n \rightarrow \infty$ , if for large  $n$  there exist diffeomorphisms  $\Psi_n : M \rightarrow M_n$  such that  $(\Psi_n)^* g_n \rightarrow g$  in the  $H^2$ -topology on  $M$ .*

The following convergence result is applied in the proof of Theorem 4.1.

**Lemma 7.3** ( $H^2$ -convergence). *Let  $(M_n, g_n)$  be a sequence of smooth compact Riemannian 3-manifolds with boundary such that  $M_n \simeq \overline{B_1} \subset \mathbb{R}^3$  and for real numbers  $1 \leq t \leq 2$  and  $0 < V < \infty$ ,*

$$\begin{aligned} & \|\text{Ric}_n\|_{L^2(M_n)} \rightarrow 0 \text{ as } n \rightarrow \infty, \\ & \left\| \text{tr} \Theta_n - \frac{2}{t} \right\|_{L^4(\partial M_n)} + \|\widehat{\Theta}_n\|_{L^4(\partial M_n)} \rightarrow 0 \text{ as } n \rightarrow \infty, \\ & r_{\text{vol}}(M_n, 1/2) \geq 1/4, \\ & \text{vol}_{g_n}(M_n) \leq V. \end{aligned} \tag{7.1}$$

*Then, there is a smooth compact Riemannian 3-manifold  $(M, g)$  with  $M \simeq \overline{B_1} \subset \mathbb{R}^3$  such that as  $n \rightarrow \infty$ ,*

$$(M_n, g_n) \rightarrow (M, g) \text{ in the } H^2\text{-topology,} \tag{7.2}$$

*that is, for large  $n$  there are global diffeomorphisms  $\Psi_n : M \rightarrow M_n$  such that, with respect to a fixed chart on  $M$ , for  $i, j = 1, 2, 3$ ,*

$$\sum_{0 \leq m' \leq 2} \|\partial^{m'} ((\Psi_n^* g_n)_{ij} - g_{ij})\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{7.3}$$

*Moreover, for integers  $m \geq 1$ ,*

$$\begin{aligned} & \sum_{0 \leq m' \leq m+2} \|\partial^{m'} ((\Psi_n^* g_n)_{ij} - g_{ij})\|_{L^2} \\ & \leq C_V \sum_{0 \leq m' \leq m} \|(n)\nabla^{m'} \text{Ric}_n\|_{L^2(M_n)} + C_{m,V} \sum_{i,j=1,2,3} \sum_{0 \leq m' \leq 2} \|\partial^{m'} ((\Psi_n^* g_n)_{ij} - g_{ij})\|_{L^2}, \end{aligned} \tag{7.4}$$

*where  $(n)\nabla$  and  $\text{Ric}_n$  denote the covariant derivative and Ricci curvature on  $(M_n, g_n)$ .*

*Proof of Lemma 7.3.* The  $H^2$ -convergence (7.2) of the sequence (7.1) follows directly by the low regularity pre-compactness result established in [Czi19a], see Theorem 4.1 and Corollary 3.11 therein, together with  $H^2$ -regularity elliptic estimates for boundary harmonic coordinates, see Section 4 in [Czi19a]. We note that the limit manifold is flat by (7.1).

The higher regularity estimates (7.4) follow by construction from the above  $H^2$ -convergence together with the methods of proof of the pre-compactness result in [Czi19a]. More precisely, an inspection of the so-called *center-of-mass construction* used to prove the *fundamental theorem of convergence theory* (see Theorem 3.6 in [Czi19a] as well as Theorem 11.3.6 and its proof in [GPL97]) leads, together with the established  $H^2$ -convergence, directly to (7.4). This finishes the proof of Lemma 7.3.  $\square$

In the proof of Theorem 4.1, we also use the following rigidity result. A proof is provided in Appendix C.

**Lemma 7.4** (Rigidity result). *Let  $(M, g)$  be a smooth compact Riemannian 3-manifold with boundary such that  $M \simeq \overline{B_1} \subset \mathbb{R}^3$  and for a real number  $1 \leq t \leq 2$ ,*

$$\text{Ric} = 0 \text{ on } M, \text{ tr } \Theta = \frac{2}{t}, \widehat{\Theta} = 0 \text{ on } \partial M.$$

Then,

$$(M, g) \cong (\overline{B_t}, e).$$

We are now in position to prove Theorem 4.1.

*Proof of Theorem 4.1.* The proof follows in a straight-forward way from Lemmas 7.3 and 7.4 by contradiction. For completeness, we provide full details. Let  $(M_n, g_n)$  be a sequence of smooth compact Riemannian 3-manifolds with boundary such that  $M_n \simeq \overline{B_1} \subset \mathbb{R}^3$  and for real numbers  $1 \leq t \leq 2$  and  $0 < V < \infty$ ,

$$\begin{aligned} \|\text{Ric}_n\|_{L^2(M_n)} &\leq \frac{1}{n}, \\ \left\| \text{tr } \Theta_n - \frac{2}{t} \right\|_{L^4(\partial M_n)} + \|\widehat{\Theta}_n\|_{L^4(\partial M_n)} &\leq \frac{1}{n}, \\ r_{\text{vol}}(M_n, 1/2) &\geq 1/4, \\ \text{vol}_{g_n}(M_n) &\leq V. \end{aligned} \tag{7.5}$$

Given  $0 < C_{\text{ball}} < 1/2$ , assume there do not exist an integer  $N \geq 1$ , real numbers  $C_V > 0$  and  $C_{m,V} > 0$  and a family of global charts

$$(\varphi_n : \overline{B_t} \rightarrow M_n)_{n \geq N}$$

such that on  $\overline{B_t}$ , for  $i, j = 1, 2, 3$ ,

$$\begin{aligned} \sum_{0 \leq m' \leq 2} \|\partial^{m'}((g_n)_{ij} - e_{ij})\|_{L^2(B_t)} &\lesssim C_{\text{ball}}, \\ (1 - C_{\text{ball}})|\xi|^2 &\leq (g_n)_{ij} \xi^i \xi^j \leq (1 + C_{\text{ball}})|\xi|^2 \text{ for all } \xi \in \mathbb{R}^2. \end{aligned} \tag{7.6}$$

and for integers  $m \geq 1$ ,

$$\sum_{0 \leq m' \leq m+2} \|\partial^{m'}((g_n)_{ij} - e_{ij})\|_{L^2(B_t)} \lesssim C_V \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}_n\|_{L^2(M_n)} + C_{m,V}, \tag{7.7}$$

where we abused notation by writing  $g_n$  instead of  $\Psi_n^* g_n$ .

By Lemma 7.3, there is a smooth limit manifold  $(M, g)$  such that

$$(M_n, g_n) \rightarrow (M, g) \text{ in the } H^2\text{-topology as } n \rightarrow \infty, \tag{7.8}$$

and the estimates (7.3) and (7.4) hold.

By (7.5) and (7.8), it follows that  $(M, g)$  satisfies

$$\text{Ric} = 0 \text{ in } M, \text{ tr } \Theta = \frac{2}{t}, \widehat{\Theta} = 0 \text{ on } \partial M,$$

so by Lemma 7.4,

$$(M, g) \cong (\overline{B}_t, e), \quad (7.9)$$

which trivially admits global smooth coordinates.

Therefore, on the one hand, by (7.3), (7.8) and (7.9), we get that for  $n$  large there are global diffeomorphisms  $\Psi_n : \overline{B}_1 \rightarrow M_n$  satisfying

$$\sum_{0 \leq m' \leq 2} \|\partial^{m'}((g_n)_{ij} - e_{ij})\|_{L^2(B_t)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (7.10)$$

On the other hand, by (7.4) and (7.9), for integers  $m \geq 1$ ,

$$\begin{aligned} & \sum_{0 \leq m' \leq m+2} \|\partial^{m'}((g_n)_{ij} - e_{ij})\|_{L^2(B_t)} \\ & \leq C_V \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}_n\|_{L^2(M_n)} + C_{m,V} \sum_{i,j=1,2,3} \sum_{0 \leq m' \leq 2} \|\partial^{m'}((g_n)_{ij} - e_{ij})\|_{L^2(B_t)} \\ & \lesssim C_V \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}_n\|_{L^2(M_n)} + C_{m,V}, \end{aligned}$$

where we used (7.9) and (7.10). This yields a contradiction to (7.6) and (7.7), and hence finishes the proof of Theorem 4.1.  $\square$





# A. Global elliptic estimates for Hodge systems

In this section we discuss global elliptic estimates for general Hodge systems on compact Riemannian 3-manifolds with boundary  $\Sigma$ . This is a slight generalisation of the elliptic estimates in [CK93] where non-compact manifolds without boundary are considered.

## A.1 General Hodge systems

In this section, we introduce tools and results to obtain elliptic estimates for general Hodge systems. They are applied in Sections A.2 and A.3 to the specific Hodge systems of this paper. We have the following notation.

**Definition A.1.** *Let  $m \geq 0$  be an integer. For a given totally symmetric  $(m+2)$ -tensor  $F$ , define*

$$\begin{aligned} A(F)_{a_1 \dots a_{m+1}bc} &:= \nabla_c F_{a_1 \dots a_{m+1}b} - \nabla_b F_{a_1 \dots a_{m+1}c}, \\ D(F)_{a_1 \dots a_{m+1}} &:= \nabla^c F_{a_1 \dots a_{m+1}c}. \end{aligned}$$

The following lemma is a straight-forward generalisation of Lemma 4.4.1 in [CK93] to manifolds with boundary. The proof is by integration by parts and left to the reader.

**Lemma A.2** (Fundamental integral identity for Hodge systems). *Let  $(\Sigma, g)$  be a compact Riemannian 3-manifold with boundary and let  $m \geq 0$  be an integer. Let  $F$  be a totally symmetric  $(m+2)$ -tensor on  $\Sigma$ . Then it holds that*

$$\begin{aligned} \int_{\Sigma} |\nabla F|^2 &= \int_{\Sigma} \frac{1}{2} |A(F)|^2 + |D(F)|^2 \\ &\quad - \int_{\Sigma} \sum_{i=1}^{m+1} \left( R^l{}_{a_i b c} F_{a_1 \dots l \dots a_{m+1} c} + \text{Ric}^l{}_b F_{a_1 \dots a_{m+1} l} \right) F^{a_1 \dots a_{m+1} b} \\ &\quad - \int_{\partial \Sigma} F^{a_1 \dots a_{m+1} N} D(F)_{a_1 \dots a_{m+1}} + \int_{\partial \Sigma} \nabla_b F_{a_1 \dots a_{m+1} N} F^{a_1 \dots a_{m+1} b}. \end{aligned}$$

In this paper we use Lemma A.2 to derive (higher regularity) elliptic estimates of Hodge systems. As Lemma A.2 applies only to symmetric tensors, one introduces the symmetrised derivative as follows.

**Definition A.3** (Symmetrised derivative). *Let  $m \geq 0$  be an integer. For a given totally symmetric  $(m+2)$ -tensor  $F$ , let*

$$(\overline{\nabla} F)_{a_1 \dots a_{m+2}c} := \frac{1}{m+3} \left( \nabla_c F_{a_1 \dots a_{m+2}} + \sum_{i=1}^{m+2} \nabla_{a_i} F_{a_1 \dots c \dots a_{m+2}} \right).$$

The symmetrised derivative is related to the standard derivative as follows.

**Lemma A.4.** *Let  $m \geq 0$  be an integer. Let  $F$  be a totally symmetric  $(m+2)$ -tensor. Then, schematically,*

$$\nabla F = \overline{\nabla} F + A(F).$$

*Proof of Lemma A.4.* By direct calculation,

$$\begin{aligned} \nabla_b F_{a_1 \dots a_{m+2}} &= (m+3)(\overline{\nabla} F)_{a_1 \dots a_{m+2}b} - \sum_{i=1}^{m+2} \nabla_{a_i} F_{a_1 \dots b \dots a_{m+2}} \\ &= (m+3)(\overline{\nabla} F)_{a_1 \dots a_{m+2}b} - \sum_{i=1}^{m+2} \nabla_{a_i} F_{a_1 \dots a_{i-1} a_{i+1} \dots a_{m+2} b} \\ &= (m+3)(\overline{\nabla} F)_{a_1 \dots a_{m+2}b} - \sum_{i=1}^{m+2} \left( \nabla_b F_{a_1 \dots a_{m+2}} + A(F)_{a_1 \dots a_{i-1} a_{i+1} \dots a_{m+2} b a_i} \right) \\ &= (m+3)(\overline{\nabla} F)_{a_1 \dots a_{m+2}b} - (m+2) \nabla_b F_{a_1 \dots a_{m+2}} - \sum_{i=1}^{m+2} A(F)_{a_1 \dots a_{i-1} a_{i+1} \dots a_{m+2} b a_i} \end{aligned}$$

which shows that

$$\nabla_b F_{a_1 \dots a_{m+2}} = (\bar{\nabla} F)_{a_1 \dots a_{m+2} b} - \frac{1}{m+3} \sum_{i=1}^{m+2} A(F)_{a_1 \dots a_{i-1} a_{i+1} \dots a_{m+2} b a_i}.$$

This finishes the proof of Lemma A.4.  $\square$

The Hodge systems satisfied by a tensor  $F$  and its symmetrised derivative  $\bar{\nabla} F$  are related as follows.

**Lemma A.5.** *Let  $F$  be a totally symmetric  $(m+2)$ -tensor. Then it holds that*

$$\begin{aligned} A(\bar{\nabla} F) &= \nabla A(F) + R \cdot F, \\ D(\bar{\nabla} F) &= \nabla A(F) + \nabla D(F) + R \cdot F, \end{aligned}$$

where  $R \cdot F$  denotes contractions between  $R$  and  $F$ .

*Proof.* First,

$$\begin{aligned} A(\bar{\nabla} F)_{a_1 \dots a_{m+2} b c} &= \nabla_c (\bar{\nabla} F)_{a_1 \dots a_{m+2} b} - \nabla_b (\bar{\nabla} F)_{a_1 \dots a_{m+2} c} \\ &= \frac{1}{m+3} \nabla_c \left( \nabla_b F_{a_1 \dots a_{m+2}} + \sum_{i=1}^{m+2} \nabla_{a_i} F_{a_1 \dots b \dots a_{m+2}} \right) \\ &\quad - \frac{1}{m+3} \nabla_b \left( \nabla_c F_{a_1 \dots a_{m+2}} + \sum_{i=1}^{m+2} \nabla_{a_i} F_{a_1 \dots c \dots a_{m+2}} \right) \\ &= \frac{1}{m+3} (\nabla_c \nabla_b - \nabla_b \nabla_c) F_{a_1 \dots a_{m+2}} \\ &\quad + \frac{1}{m+3} \sum_{i=1}^{m+2} ((\nabla_c \nabla_{a_i} - \nabla_{a_i} \nabla_c) F_{a_1 \dots b \dots m+2} - (\nabla_b \nabla_{a_i} - \nabla_{a_i} \nabla_b) F_{a_1 \dots c \dots m+2}) \\ &\quad + \frac{1}{m+3} \sum_{i=1}^{m+2} \nabla_{a_i} \underbrace{(\nabla_c F_{a_1 \dots b \dots a_{m+2}} - \nabla_b F_{a_1 \dots c \dots a_{m+2}})}_{=A(F)_{a_1 \dots a_{i-1} a_{i+1} \dots a_{m+2} b c}}, \end{aligned}$$

where we can further express

$$(\nabla_c \nabla_b - \nabla_b \nabla_c) F_{a_1 \dots a_{m+2}} = \sum_{i=1}^{m+2} R^d_{a_i b c} F_{a_1 \dots d \dots a_{m+2}}.$$

Second,

$$\begin{aligned} D(\bar{\nabla} F)_{a_1 \dots a_{m+2}} &= \nabla^c (\bar{\nabla} F)_{a_1 \dots a_{m+2} c} \\ &= \frac{1}{m+3} \nabla^c \left( \nabla_c F_{a_1 \dots a_{m+2}} + \sum_{i=1}^{m+2} \nabla_{a_i} F_{a_1 \dots c \dots a_{m+2}} \right) \\ &= \frac{1}{m+3} \nabla^c \nabla_c F_{a_1 \dots a_{m+2}} + \frac{1}{m+3} \sum_{i=1}^{m+2} \nabla^c \nabla_{a_i} F_{a_1 \dots c \dots a_{m+2}}, \end{aligned}$$

where the first term on the right-hand side equals

$$\nabla^c \nabla_c F_{a_1 \dots a_{m+2}} = \nabla^c A(F)_{a_1 \dots a_{m+2} c} + (\nabla^c \nabla_{a_{m+2}} - \nabla_{a_{m+2}} \nabla^c) F_{a_1 \dots a_{m+1} c} + \nabla_{a_{m+2}} D(F)_{a_1 \dots a_{m+1}},$$

and for the sum on the right-hand side,

$$\nabla^c \nabla_{a_i} F_{a_1 \dots c \dots a_{m+2}} = (\nabla^c \nabla_{a_i} - \nabla_{a_i} \nabla^c) F_{a_1 \dots c \dots a_{m+2}} + \nabla_{a_i} D(F)_{a_1 \dots a_{i-1} a_{i+1} \dots a_{m+2}}.$$

This finishes the proof of Lemma A.5.  $\square$

Higher regularity estimates for Hodge systems are proved by induction, using the basic integral identity of Lemma A.2, the relation between  $\bar{\nabla}$  and  $\nabla$  of Lemma A.4 and the recursive relations of Lemma A.5. In the next sections we discuss more specifically the Hodge systems which appear in this paper.

## A.2 Elliptic estimates for the second fundamental form $k$

Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime and let  $\Sigma \simeq \overline{B_1}$  be a compact spacelike maximal hypersurface in  $\mathcal{M}$ . By (2.13b), (2.13c) and (2.13d), the second fundamental form  $k$  of  $\Sigma$  satisfies the following Hodge system,

$$\begin{aligned}\operatorname{div}_g k &= 0, \\ \operatorname{curl}_g k &= H, \\ \operatorname{tr}_g k &= 0.\end{aligned}$$

In the notation of Definition A.1,  $k$  satisfies in particular

$$A(k)_{iab} = \epsilon_{ab}^m H_{im}, \quad D(k) = 0. \quad (\text{A.1})$$

Lemma A.2 together with (A.1) yields the following corollary (see Section 8.3 in [KR10] for the case of manifolds without boundary).

**Corollary A.6** (Fundamental global elliptic estimate for  $k$ ). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime and let  $\Sigma \simeq \overline{B_1}$  be a compact spacelike maximal hypersurface in  $\mathcal{M}$ . Then it holds that*

$$\int_{\Sigma} |\nabla k|^2 + \frac{1}{4} |k|^4 - \int_{\partial\Sigma} \nabla_a k_{bN} k^{ba} \lesssim \int_{\Sigma} |\mathbf{R}|_{\mathbf{h}^t}^2,$$

where  $N$  denotes the outward-pointing unit normal to  $\partial\Sigma \subset \Sigma$ ,  $T$  denotes the timelike unit normal to  $\Sigma$  and  $\mathbf{h}^t$  denotes the positive-definite norm on  $\Sigma$  defined with  $T$ , see (2.9).

*Proof.* By Lemma A.2 with  $F = k$ , we have

$$\int_{\Sigma} |\nabla k|^2 = \int_{\Sigma} \frac{1}{2} |H|^2 - \int_{\Sigma} \left( \mathbf{R}^l{}_{abc} k_{lc} + \operatorname{Ric}^l{}_b k_{al} \right) k^{ab} + \int_{\partial\Sigma} \nabla_b k_{aN} k^{ab}. \quad (\text{A.2})$$

In dimension  $n = 3$ , the full Riemann curvature tensor is determined by  $\operatorname{Ric}$ , yielding

$$\mathbf{R}^l{}_{abc} k_{lc} k^{ab} = 2 \operatorname{Ric}_{jl} k^{ji} k^l{}_i - \frac{1}{2} \mathbf{R}_{scal} |k|^2.$$

Plugging this into (A.2), we get

$$\int_{\Sigma} |\nabla k|^2 = \int_{\Sigma} \frac{1}{2} |H|^2 - \int_{\Sigma} \left( 3 \operatorname{Ric}^l{}_b k_{al} k^{ab} - \frac{1}{2} \mathbf{R}_{scal} |k|^2 \right) + \int_{\partial\Sigma} \nabla_b k_{aN} k^{ab}. \quad (\text{A.3})$$

On the one hand, by (2.13f) and (2.13g), we have

$$E_{ij} = \operatorname{Ric}_{ij} - k_{im} k^m{}_j, \quad \mathbf{R}_{scal}(g) = |k|_g^2.$$

On the other hand, in dimension  $n = 3$ , it holds for symmetric tracefree 2-tensors  $F$  that

$$3 \operatorname{tr}(F^4) \geq |F|^4.$$

Hence it follows from (A.3) that

$$\begin{aligned}\int_{\Sigma} \frac{1}{2} |H|^2 &= \int_{\Sigma} |\nabla k|^2 + 3 \operatorname{Ric}^s{}_a k_{bs} k^{ba} - \frac{1}{2} |k|^4 - \int_{\partial\Sigma} \nabla_a k_{bN} k^{ba} \\ &= \int_{\Sigma} |\nabla k|^2 + 3(E^s{}_a + k_m^s k^m{}_a) k_{bs} k^{ba} - \frac{1}{2} |k|^4 - \int_{\partial\Sigma} \nabla_a k_{bN} k^{ba} \\ &\geq \int_{\Sigma} |\nabla k|^2 + 3E^s{}_a k_{bs} k^{ba} + \frac{1}{2} |k|^4 - \int_{\partial\Sigma} \nabla_a k_{bN} k^{ba}.\end{aligned}$$

Using that  $|E|^2 + |H|^2 \lesssim |\mathbf{R}|_{\mathbf{h}^t}^2$  by (2.8) and (2.11), we obtain

$$\int_{\Sigma} |\nabla k|^2 + \frac{1}{4} |k|^4 - \int_{\partial\Sigma} \nabla_a k_{bN} k^{ba} \lesssim \int_{\Sigma} |\mathbf{R}|_{\mathbf{h}^t}^2.$$

This finishes the proof of Corollary A.6.  $\square$

### A.3 Elliptic estimates for the curvature tensor

Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime and let  $(\Sigma_t)$  be a maximal foliation on  $\mathcal{M}$ . Let  $T$  denote the timelike unit normal to  $\Sigma_t$ . We recall from Proposition 2.8 and (2.7) that for a Weyl tensors  $\mathbf{W}$  satisfying the inhomogeneous Bianchi equations

$$\mathbf{D}^\alpha \mathbf{W}_{\alpha\beta\ \gamma\delta} = J_{\beta\gamma\delta},$$

it holds that

$$\begin{aligned} \operatorname{div} E(\mathbf{W})_a &= + (k \wedge H(\mathbf{W}))_a + J_{TaT}, \\ \operatorname{curl} E(\mathbf{W})_{ab} &= + H \left( \hat{\mathcal{L}}_T \mathbf{W} \right)_{ab} - 3 (n^{-1} \nabla n \wedge E(\mathbf{W}))_{ab} \\ &\quad - \frac{3}{2} (k \times H(\mathbf{W}))_{ab} - J_{aTb}^*, \\ \operatorname{div} H(\mathbf{W})_a &= - (k \wedge E(\mathbf{W}))_a + J_{TaT}^*, \\ \operatorname{curl} H(\mathbf{W})_{ab} &= - E \left( \hat{\mathcal{L}}_T \mathbf{W} \right)_{ab} - 3 (n^{-1} \nabla n \wedge H(\mathbf{W}))_{ab} \\ &\quad + \frac{3}{2} (k \times E(\mathbf{W}))_{ab} - J_{aTb}. \end{aligned} \tag{A.4}$$

Interpreting (A.4) as coupled Hodge system for  $E(\mathbf{W})$  and  $H(\mathbf{W})$  with the right-hand side as source terms, Lemma A.2 yields the following global elliptic estimates on  $\Sigma$ .

**Corollary A.7** (Elliptic estimates for  $E(\mathbf{W})$  and  $H(\mathbf{W})$ ). *Let  $E(\mathbf{W})$  and  $H(\mathbf{W})$  be solutions to (A.4) on  $\Sigma_t$ . Then it holds that*

$$\begin{aligned} &\int_{\Sigma_t} |\nabla E(\mathbf{W})|^2 + |\nabla H(\mathbf{W})|^2 \\ &\lesssim \int_{\Sigma_t} |\hat{\mathcal{L}}_T \mathbf{W}|_{\mathfrak{h}^t}^2 + |J|^2 + \sum_{0 \leq m' \leq 2} \|\nabla^{m'}(n-1)\|_{L^2(\Sigma_t)}^2 + \sum_{0 \leq m' \leq 1} \|\nabla^{m'} k\|_{L^2(\Sigma_t)}^2 \\ &\quad + \int_{\partial\Sigma_t} \nabla_b E(\mathbf{W})_{aN} E(\mathbf{W})^{ab} + \int_{\partial\Sigma_t} \nabla_b H(\mathbf{W})_{aN} H(\mathbf{W})^{ab} \\ &\quad - \int_{\partial\Sigma_t} J_{TaT} E(\mathbf{W})^{aN} - \int_{\partial\Sigma_t} J_{TaT}^* H(\mathbf{W})^{aN}. \end{aligned}$$

## B. Proof of Lemmas 3.6 and 3.8

In this section we prove Lemmas 3.6 and 3.8.

### B.1 Proof of Lemma 3.6

We have to show that on weakly regular balls  $(\Sigma, g)$  of radius  $1 \leq r \leq 2$  with constant  $0 < C_{\text{ball}} < 1/2$ , it holds that

$$\begin{aligned} \|F\|_{L^2(\partial\Sigma)} &\lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}, \\ \|F\|_{L^4(\partial\Sigma)} &\lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}, \end{aligned} \tag{B.1}$$

and moreover,

$$\|F\|_{H^{1/2}(\partial\Sigma)} \lesssim \|F\|_{L^2(\Sigma)} + \|\nabla F\|_{L^2(\Sigma)}, \tag{B.2}$$

and for integers  $m \geq 1$ ,

$$\begin{aligned} \sum_{0 \leq m' \leq m} \|\nabla^{m'} F\|_{H^{1/2}(\partial\Sigma)} &\lesssim \sum_{0 \leq m' \leq m+1} \|\nabla^{m'} F\|_{L^2(\Sigma)} \\ &+ \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + C_m C_{\text{ball}}. \end{aligned} \tag{B.3}$$

First, the estimates (B.1) are straight-forward, see for example Lemma 3.26 and Corollary 3.27 in [Sze18] for a concise proof.

We turn to the proof of (B.2). On the one hand, by Sections 7.50 to 7.56 in [AF08], for each coordinate patch  $U \subset \partial\overline{B_r}$  and smooth open set  $V \subset \overline{B_r}$  with  $U \subset (V \cap \partial\overline{B_r})$ , it holds that

$$H^1(V) \hookrightarrow H^{1/2}(U),$$

where  $H^{1/2}(U)$  denotes a local coordinate-defined fractional Sobolev space on  $U$ .

On the other hand, if  $g_{ij} \in H^2(B_r)$  then in particular  $g_{AB} \in W^{1,4}(\partial\overline{B_r})$  in local coordinates. By Proposition 3.2 in [Sha14], this control suffices to compare the coordinate-defined spaces  $H^{1/2}(U)$  with the space  $H^{1/2}(\partial\overline{B_r})$  defined in Definition 3.2, see also Appendix B of [Sha14]. This finishes the proof of (B.2).

The higher regularity trace estimate (B.3) is a straight-forward generalisation of (B.2) using that on a weakly regular ball we have by definition, for  $i, j = 1, 2, 3$ ,

$$\sum_{0 \leq m' \leq m+2} \|\partial^{m'}(g_{ij} - e_{ij})\|_{L^2(B_r)} \lesssim \sum_{0 \leq m' \leq m} \|\nabla^{m'} \text{Ric}\|_{L^2(\Sigma)} + C_m C_{\text{ball}}.$$

This finishes the proof of Lemma 3.6.

### B.2 Proof of Lemma 3.8

Let  $1 \leq r_0 \leq 2$  and  $0 < C_{\text{ball}} < 1/2$  be two real numbers. Let  $(\Sigma, g)$  be a weakly regular ball of radius  $r_0$  with constant  $C_{\text{ball}}$ . Let  $F$  be a tensor on  $\Sigma$ . We have to show that

$$\|F\|_{L^2(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)} + \|F\|_{L^2(\partial\Sigma)}.$$

Define spherical coordinates  $(r, \theta^1, \theta^2)$  with  $r \in [0, r_0]$  and  $(\theta^1, \theta^2) \in \mathbb{S}^2$  on  $\Sigma \simeq \overline{B_{r_0}}$  as in Section 2.4. Let  $\gamma$  and  $d\mu_\gamma$  on  $\mathcal{S}_r$  denote the *standard round metric* of radius  $r > 0$  and its volume element, respectively. Denote by  $\overset{\circ}{\gamma}$  and  $d\mu_{\overset{\circ}{\gamma}}$  the *standard round metric on the unit sphere* and its volume element, respectively. Note that

$\gamma = r^2 \circlearrowleft$ . By the fundamental theorem of calculus and using that  $\partial\Sigma = \mathcal{S}_{r_0}$ , we have for  $0 < r \leq r_0$ ,

$$\begin{aligned} \int_{\mathcal{S}_r} |F|_g^2 d\mu_\gamma &= \int_{r_0}^r \partial_{r'} \left( \int_{\mathcal{S}_{r'}} |F|_g^2 d\mu_\gamma \right) dr' + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma \\ &= \int_{r_0}^r \left( \int_{\mathcal{S}_{r'}} \nabla_{\partial_{r'}} F \cdot F d\mu_\gamma \right) dr' + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma \\ &\leq \|\partial_r |g|^2\|_{L^\infty(\Sigma)} \left( \int_{r_0}^r \int_{\mathcal{S}_{r'}} |\nabla F|_g^2 d\mu_\gamma dr' \right)^{1/2} \left( \int_{r_0}^r \int_{\mathcal{S}_{r'}} |F|_g^2 d\mu_\gamma dr' \right)^{1/2} + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma. \end{aligned}$$

Using that by definition of the spherical coordinates on  $B_{r_0}$  and the weakly regular ball property of  $(\Sigma, g)$ ,

$$|\partial_r |g|^2 = g_{rr} = \frac{x^i x^j}{r} \frac{x^j}{r} g_{ij} \leq (1 + C_{\text{ball}}) \frac{x^i x^j}{r} \frac{x^j}{r} e_{ij} = 1 + C_{\text{ball}},$$

we can estimate the right-hand side as follows,

$$\begin{aligned} \int_{\mathcal{S}_r} |F|_g^2 d\mu_\gamma &\lesssim \left( \int_{r_0}^r \int_{\mathcal{S}_{r'}} |\nabla F|_g^2 d\mu_\gamma dr' \right)^{1/2} \left( \int_{r_0}^r \int_{\mathcal{S}_{r'}} |F|_g^2 d\mu_\gamma dr' \right)^{1/2} + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma \\ &\lesssim \frac{1}{r^2} \left( \int_{r_0}^r \int_{\mathcal{S}_{r'}} |\nabla F|_g^2 d\mu_\gamma dr' \right)^{1/2} \left( \int_{r_0}^r \int_{\mathcal{S}_{r'}} |F|_g^2 d\mu_\gamma dr' \right)^{1/2} + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma \\ &\lesssim \frac{1}{r^2} \left( \int_{B_{r_0}} |\nabla F|_g^2 d\mu_e \right)^{1/2} \left( \int_{B_{r_0}} |F|_g^2 d\mu_e \right)^{1/2} + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma \\ &\lesssim \frac{1}{r^2} \|\nabla F\|_{L^2(B_{r_0})} \|F\|_{L^2(B_{r_0})} + \int_{\partial\Sigma} |F|_g^2 d\mu_\gamma, \end{aligned}$$

where  $d\mu_e$  denotes the measure with respect to the standard Euclidean metric  $e$  in Cartesian coordinates, and we used in the last inequality that  $\Sigma$  is a weakly regular ball to compare the Euclidean integral with the  $g$ -dependent norm.

Multiplying the above by  $r^2$  and using that by Lemmas 3.6, 3.6 and 3.9,

$$\int_{\partial\Sigma} |F|_g^2 d\mu_{r_0^2 \circlearrowleft} \lesssim \|F\|_{L^2(\partial\Sigma)}^2,$$

we get that for  $0 < r \leq r_0$ ,

$$\begin{aligned} \int_{\mathcal{S}_r} |F|_g^2 d\mu_\gamma &\lesssim \|\nabla F\|_{L^2(\Sigma)} \|F\|_{L^2(\Sigma)} + \frac{r^2}{r_0^2} \int_{\partial\Sigma} |F|_g^2 d\mu_{r_0^2 \circlearrowleft} \\ &\lesssim \|\nabla F\|_{L^2(\Sigma)} \|F\|_{L^2(\Sigma)} + \frac{r^2}{r_0^2} \|F\|_{L^2(\partial\Sigma)}^2. \end{aligned} \tag{B.4}$$

By (B.4) and using that  $(\Sigma, g)$  is a weakly regular ball of radius  $r_0$ , we get that

$$\|F\|_{L^2(\Sigma)}^2 \lesssim \int_{B_{r_0}} |F|_g^2 d\mu_e \lesssim \int_0^{r_0} \left( \int_{\mathcal{S}_r} |F|_g^2 d\mu_\gamma \right) dr \lesssim \|\nabla F\|_{L^2(\Sigma)} \|F\|_{L^2(\Sigma)} + \|F\|_{L^2(\partial\Sigma)}^2,$$

which implies that

$$\|F\|_{L^2(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)} + \|F\|_{L^2(\partial\Sigma)}.$$

This finishes the proof of Lemma 3.8.

## C. Proof of Lemma 7.4

In this section we prove Lemma 7.4. Let  $(M, g)$  be a smooth compact Riemannian 3-manifold with boundary such that  $M \simeq \overline{B_1} \subset \mathbb{R}^3$  and for a real number  $1 \leq t \leq 2$ ,

$$\text{Ric} = 0 \text{ on } M, \quad \text{tr } \Theta = \frac{2}{t}, \quad \widehat{\Theta} = 0 \text{ on } \partial M. \quad (\text{C.1})$$

We have to prove that

$$(M, g) \cong (\overline{B_t}, e).$$

First, by the Gauss equation

$$2K = (\text{tr}\Theta)^2 - |\Theta|^2 + R_{\text{scal}} - 2\text{Ric}(N, N),$$

it follows that the Gauss curvature of  $(\partial M, g)$  is constant  $K = \frac{1}{t^2}$ . Hence by classical differential geometry, there exist coordinates  $(\theta^1, \theta^2)$  on  $\partial M$  such that

$$g = t^2 ((d\theta^1)^2 + \sin^2(\theta^1)(d\theta^2)^2). \quad (\text{C.2})$$

Second, by the smoothness of  $(M, g)$ , there exists in an open neighbourhood  $\mathcal{U} \subset M$  of  $\partial M \subset M$  a so-called *Gaussian coordinate system*  $(r, \theta^1, \theta^2)$ , see for example Section 3.3 in [Wal84], which coincide with (C.2) on  $\partial M$  and are such that for some small real number  $\delta > 0$ ,

$$\begin{aligned} \mathcal{U} &= \{r \in (t - \delta, t]\}, \\ \mathcal{U} \cap \partial M &= \{r = t\}, \\ \nabla_{\partial_r} \partial_r &= 0, \\ \partial_r|_{\partial M} &\text{ is normal to } \partial M, \\ g(\partial_r, \partial_r) &= 1. \end{aligned}$$

In such coordinates it holds that

$$g = dr^2 + g_{AB} d\theta^A d\theta^B.$$

Third, we claim that the Riemannian manifold  $(M, g)$  smoothly extends onto  $\mathbb{R}^3 \setminus \overline{B_t}$  when identifying  $\partial M = \partial \overline{B_t} \subset \mathbb{R}^3$ . It suffices to show that the induced metric

$$\tilde{g}_{AB}(r, \theta^1, \theta^2) := \begin{cases} \gamma_{AB}(r, \theta^1, \theta^2), & \text{if } r \geq t, \\ g_{AB}(r, \theta^1, \theta^2) & \text{if } t - \delta < r \leq t, \end{cases}$$

is smooth across  $\{r = t\}$ . Here  $\gamma_{AB}(r, \theta^1, \theta^2)$  is the standard round metric of radius  $r$ .

By (C.2), it follows that  $\tilde{g}$  is continuous across  $\{r = t\}$ . Further, on the one hand, on  $\partial \overline{B_t} \subset \mathbb{R}^3$ , it holds that

$$\partial_r \gamma_{AB}|_{r=t} = 2t \overset{\circ}{\gamma}_{AB}, \quad \partial_r^2 \gamma_{AB}|_{r=t} = 2 \overset{\circ}{\gamma}_{AB}, \quad \partial_r^{m'} \gamma_{AB}|_{r=t} = 0 \text{ for } m' \geq 3, \quad (\text{C.3})$$

where  $\overset{\circ}{\gamma}_{AB}$  denotes the metric components of the standard round metric on  $\mathbb{S}^2$ .

On the other hand, by construction of  $(r, \theta^1, \theta^2)$  it holds on  $\mathcal{U}$  that

$$\partial_r g_{AB} = 2g(\nabla_r \partial_A, \partial_B) = 2g(\nabla_A \partial_r, \partial_B). \quad (\text{C.4})$$

By (C.1), (C.2) and (C.4), on  $\partial M$  we thus have that

$$\partial_r g_{AB}|_{r=t} = 2\Theta_{AB} = 2t \overset{\circ}{\gamma}_{AB}, \quad (\text{C.5})$$

Differentiating (C.4) in  $r$  yields

$$\begin{aligned} \partial_r^2 g_{AB} &= 2g(\nabla_r \partial_A, \nabla_r \partial_B) + 2g(\partial_A, \nabla_r \nabla_r \partial_B) \\ &= 2g(\nabla_A \partial_r, \nabla_B \partial_r) + 2g(\partial_A, \nabla_r \nabla_B \partial_r) \\ &= 2g(\nabla_A \partial_r, \nabla_B \partial_r) + 2g(\partial_A, \text{Rm}(\partial_r, \partial_B) \partial_r) + 2g(\partial_A, \nabla_B \nabla_r \partial_r) \\ &= 2g(\nabla_A \partial_r, \nabla_B \partial_r), \end{aligned} \quad (\text{C.6})$$



## Appendix C. Proof of Lemma 7.4

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where we used that  $\text{Rm} = 0$  in  $(M, g)$  and  $\nabla_{\partial_r} \partial_r = 0$  in Gaussian coordinates in  $\mathcal{U}$ . From (C.5) and (C.6) and using that  $a = 1$ , it follows that

$$\partial_r^2 \mathcal{g}_{AB}|_{r=t} = 2t^{-2} \overset{\circ}{\gamma}{}^{CD} t \overset{\circ}{\gamma}{}_{AC} t \overset{\circ}{\gamma}{}_{BD} = 2 \overset{\circ}{\gamma}{}_{AB}. \quad (\text{C.7})$$

Differentiating (C.4) further in  $r$  shows that

$$\partial_r^{m'} \mathcal{g}_{AB}|_{r=t} = 0 \text{ for } m' \geq 3. \quad (\text{C.8})$$

Comparing (C.3) with (C.5), (C.7) and (C.8) shows that all  $r$ -derivatives of  $\mathcal{g}$  agree on  $r = t$ . Hence  $(M, g)$  smoothly extends as Riemannian manifold onto  $(\mathbb{R}^3 \setminus \overline{B_t}, e)$ .

The resulting smooth Riemannian 3-manifold is in particular flat, complete and has cubic volume growth of geodesic balls. By Proposition 4.4 in [Czi19a] it must therefore be isometric to  $(\mathbb{R}^3, e)$ . We conclude that by the above construction,

$$(M, g) \cong (\mathbb{R}^3 \setminus (\mathbb{R}^3 \setminus \overline{B_t}), e) \cong (\overline{B_t}, e).$$

This finishes the proof of Lemma 7.4.

## D. Comparison estimates between two maximal spacelike foliations

In this section we prove Lemma 4.6. By assumption, for real numbers  $D > 0$  and  $\varepsilon > 0$ ,  $\mathcal{M}_{t^*}$  is foliated by maximal spacelike hypersurface  $(\Sigma_t)_{1 \leq t \leq t^*}$  given as level sets of a time function  $t$  with  $\Sigma_1 = \Sigma$  and satisfying

$$\begin{aligned} \|\mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} + \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} &\leq D\varepsilon, \\ \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} + \|k\|_{L_t^\infty L^2(\Sigma_t)} &\leq D\varepsilon, \\ \|n - 1\|_{L^\infty(\mathcal{M}_{t^*})} + \|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim D\varepsilon. \end{aligned} \tag{D.1}$$

Let  $T$  denote the timelike unit normal to  $\Sigma_t$ .

Moreover, by assumption there is another foliation on  $\mathcal{M}_{t^*}$  (constructed by the bounded  $L^2$  curvature theorem) of maximal hypersurfaces  $(\tilde{\Sigma}_{\tilde{t}})_{0 \leq \tilde{t} \leq t^*}$  given as level sets of a time function  $\tilde{t}$  with  $\Sigma_{t^*} = \tilde{\Sigma}_{t^*}$  and satisfying

$$\begin{aligned} \|\mathbf{R}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\widetilde{\text{Ric}}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \\ \|\widetilde{\nabla} \tilde{k}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\tilde{k}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} + \|\mathbf{D}\tilde{\pi}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \\ \|\tilde{n} - 1\|_{L^\infty(\mathcal{M}_{t^*})} + \|\widetilde{\nabla} \tilde{n}\|_{L^\infty(\mathcal{M}_{t^*})} + \|\widetilde{\nabla}^2 \tilde{n}\|_{L_{\tilde{t}}^\infty L^2(\tilde{\Sigma}_{\tilde{t}})} &\lesssim C_{\text{ball}}, \end{aligned} \tag{D.2}$$

where  $\widetilde{\text{Ric}}$ ,  $\widetilde{\nabla}$  and  $\tilde{k}$  denote the Ricci curvature, covariant derivative and the second fundamental form on  $\tilde{\Sigma}_{\tilde{t}}$ , respectively. Let  $\tilde{T}$  denote the timelike unit normal to  $\tilde{\Sigma}_{\tilde{t}}$ .

We need to show that for  $\varepsilon > 0$  and  $C_{\text{ball}} > 0$  sufficiently small, for  $1 \leq t \leq t^*$ ,

$$\|\tilde{\nu} - 1\|_{L_{\tilde{t}}^\infty L^\infty(\Sigma_t)} \lesssim C_{\text{ball}}, \tag{D.3}$$

$$\|\tilde{k}\|_{L_{\tilde{t}}^\infty L^4(\Sigma_t)} \lesssim C_{\text{ball}}, \tag{D.4}$$

where the *angle*  $\tilde{\nu}$  between  $T$  and  $\tilde{T}$  is defined as

$$\tilde{\nu} := -\mathbf{g}(T, \tilde{T}). \tag{D.5}$$

The proof of (D.3) and (D.4) is based on a standard continuity argument going backwards in  $t$  and starting at  $\Sigma_{t^*} = \tilde{\Sigma}_{t^*}$  where by construction

$$\tilde{\nu} = 1, \quad k = \tilde{k}.$$

In the following, we only discuss the bootstrap assumption and its improvement.

**Bootstrap assumption.** Let  $1 \leq t_0^* < t^*$  be a real. Assume that for a large constant  $M > 0$ , for  $t_0^* \leq t \leq t^*$ ,

$$\|\tilde{\nu} - 1\|_{L_{\tilde{t}}^\infty L^\infty(\Sigma_t)} \leq MC_{\text{ball}}. \tag{D.6}$$

**First consequences of the bootstrap assumption.** Let  $(\tilde{e}_i)_{i=1,2,3}$  be an orthonormal frame on  $\tilde{\Sigma}_{\tilde{t}}$  and let  $\tilde{e}_0 = \tilde{T}$ . Expressing  $T$  as

$$T = \tilde{\nu} \tilde{T} + \mathbf{g}(T, \tilde{e}_i) \tilde{e}_i,$$

and using that  $\mathbf{g}(T, T) = -1$ , it follows that

$$-1 = -|\tilde{\nu}|^2 + \sum_{i=1,2,3} |\mathbf{g}(T, \tilde{e}_i)|^2.$$

This implies by the bootstrap assumption (D.6) for  $C_{\text{ball}} > 0$  sufficiently small that for  $t_0^* \leq t \leq t^*$ ,

$$\|\mathbf{g}(T, \tilde{e}_i)\|_{L_{\tilde{t}}^\infty L^\infty(\Sigma_t)} \lesssim \sqrt{MC_{\text{ball}}}. \tag{D.7}$$

Let  $(e_i)_{i=1,2,3}$  be the orthonormal frame tangent to  $\Sigma_t$  constructed by the Gram-Schmidt method applied to the  $\Sigma_t$ -tangential frame

$$(\tilde{e}_1 + \mathbf{g}(\tilde{e}_1, T)T, \tilde{e}_2 + \mathbf{g}(\tilde{e}_2, T)T, \tilde{e}_3 + \mathbf{g}(\tilde{e}_3, T)T),$$

and set moreover  $e_0 := T$ . By the Gram-Schmidt construction and (D.6) and (D.7), we get that for  $C_{\text{ball}} > 0$  sufficiently small, for  $t_0^* \leq t \leq t^*$ ,

$$\begin{aligned} & \sum_{i=1}^3 \left( \|\mathbf{g}(e_i, \tilde{e}_i) - 1\|_{L_t^\infty L^\infty(\Sigma_t)} + \sum_{j \neq i} \|\mathbf{g}(\tilde{e}_i, e_j)\|_{L_t^\infty L^\infty(\Sigma_t)} + \|\mathbf{g}(\tilde{T}, e_i)\|_{L_t^\infty L^\infty(\Sigma_t)} \right) \\ & \lesssim \sqrt{MC_{\text{ball}}}. \end{aligned} \quad (\text{D.8})$$

**Improvement of the bootstrap assumption.** First, by definition of  $\tilde{\nu}$ , see (D.5), and using that  $\mathbf{D}_T T = n^{-1} \nabla n$ ,

$$\begin{aligned} T(\tilde{\nu}) &= -\mathbf{g}(\mathbf{D}_T T, \tilde{T}) - \mathbf{g}(T, \mathbf{D}_T \tilde{T}) \\ &= -\left( \mathbf{g}(\tilde{T}, e_i) \mathbf{g}(\mathbf{D}_T T, e_i) + \mathbf{g}(T, \tilde{e}_\mu) \mathbf{g}(T, \tilde{e}_i) \mathbf{g}(\tilde{e}_i, \mathbf{D}_{\tilde{e}_\mu} \tilde{T}) \right) \\ &= -\left( \mathbf{g}(\tilde{T}, e_i) (n^{-1} \nabla_{e_i} n) - \mathbf{g}(T, \tilde{e}_j) \mathbf{g}(T, \tilde{e}_i) \tilde{k}_{ij} + \tilde{\nu} \mathbf{g}(T, \tilde{e}_i) (\tilde{n}^{-1} \tilde{\nabla}_{\tilde{e}_i} \tilde{n}) \right). \end{aligned} \quad (\text{D.9})$$

Define shift-free coordinates  $(t, x^1, x^2, x^3)$  on  $\mathcal{M}_{t^*}$  by transporting  $(x^1, x^2, x^3)$  from  $\Sigma_{t^*}$  backwards along  $T$ . Then it holds that  $\partial_t = n^{-1} T$ , and we get from (D.9) that

$$\partial_t \tilde{\nu} = -n^{-1} \left( \mathbf{g}(\tilde{T}, e_i) (n^{-1} \nabla_{e_i} n) - \mathbf{g}(T, \tilde{e}_j) \mathbf{g}(T, \tilde{e}_i) \tilde{k}_{ij} + \tilde{\nu} \mathbf{g}(T, \tilde{e}_i) (\tilde{n}^{-1} \tilde{\nabla}_{\tilde{e}_i} \tilde{n}) \right).$$

Integrating this equation in  $t$ , using that  $\tilde{\nu}|_{t=t^*} = 1$  and applying (D.1), (D.2), (D.6), (D.7), (D.8) and Lemma 3.4, we get that for  $\varepsilon > 0$  and  $C_{\text{ball}} > 0$  sufficiently small, for  $t_0^* \leq t \leq t^*$ , it holds that

$$\begin{aligned} & \|\tilde{\nu} - 1\|_{L_t^\infty L^4(\Sigma_t)} \\ & \lesssim \|\mathbf{g}(\tilde{T}, e_i)\|_{L^\infty(\mathcal{M}_{t^*})} \|\nabla_{e_i} n\|_{L^4(\mathcal{M}_{t^*})} \\ & \quad + \|\mathbf{g}(T, \tilde{e}_i)\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{g}(T, \tilde{e}_j)\|_{L^\infty(\mathcal{M}_{t^*})} \|\tilde{k}_{ij}\|_{L^4(\mathcal{M}_{t^*})} \\ & \quad + \|\tilde{\nu}\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{g}(T, \tilde{e}_i)\|_{L^\infty(\mathcal{M}_{t^*})} \|\tilde{\nabla}_{\tilde{e}_i} \tilde{n}\|_{L^4(\mathcal{M}_{t^*})} \\ & \lesssim \|\mathbf{g}(\tilde{T}, e_i)\|_{L^\infty(\mathcal{M}_{t^*})} (\|\nabla n\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla^2 n\|_{L_t^\infty L^2(\Sigma_t)}) \\ & \quad + \|\mathbf{g}(T, \tilde{e}_i)\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{g}(T, \tilde{e}_j)\|_{L^\infty(\mathcal{M}_{t^*})} \left( \|\tilde{k}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)} + \|\tilde{\nabla} \tilde{k}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)} \right) \\ & \quad + \|\tilde{\nu}\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{g}(T, \tilde{e}_i)\|_{L^\infty(\mathcal{M}_{t^*})} \left( \|\tilde{\nabla} \tilde{n}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)} + \|\tilde{\nabla}^2 \tilde{n}\|_{L_t^\infty L^2(\tilde{\Sigma}_i)} \right) \\ & \lesssim \sqrt{MC_{\text{ball}}}(D\varepsilon) + \sqrt{MC_{\text{ball}}} C_{\text{ball}}. \end{aligned} \quad (\text{D.10})$$

Second, by definition of  $\tilde{\nu}$  in (D.5), for  $i = 1, 2, 3$ ,

$$\begin{aligned} \nabla_{e_i} \tilde{\nu} &= -\mathbf{g}(\mathbf{D}_{e_i} T, \tilde{T}) - \mathbf{g}(T, \mathbf{D}_{e_i} \tilde{T}) \\ &= -\left( \mathbf{g}(\tilde{T}, e_j) k_{ij} + \mathbf{g}(T, \tilde{e}_j) \mathbf{g}(e_i, \tilde{e}_\mu) \mathbf{g}(\tilde{e}_j, \mathbf{D}_{\tilde{e}_\mu} \tilde{T}) \right) \\ &= -\left( \mathbf{g}(\tilde{T}, e_j) k_{ij} - \mathbf{g}(T, \tilde{e}_j) \mathbf{g}(e_i, \tilde{e}_l) \tilde{k}_{lj} + \mathbf{g}(T, \tilde{e}_j) \mathbf{g}(e_i, \tilde{T}) (\tilde{n}^{-1} \tilde{\nabla}_{\tilde{e}_j} \tilde{n}) \right) \end{aligned}$$

By (D.1), (D.2), (D.8) and Lemma 3.4, we have for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small,

$$\begin{aligned} \|\nabla_{e_i} \tilde{\nu}\|_{L_t^\infty L^4(\Sigma_t)} &\lesssim \|\mathbf{g}(\tilde{T}, e_j)\|_{L^\infty(\mathcal{M}_{t^*})} \|k_{ij}\|_{L_t^\infty L^4(\Sigma_t)} \\ &\quad + \|\mathbf{g}(T, \tilde{e}_j)\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{g}(e_i, \tilde{e}_l)\|_{L^\infty(\mathcal{M}_{t^*})} \|\tilde{k}_{lj}\|_{L_t^\infty L^4(\Sigma_t)} \\ &\quad + \|\mathbf{g}(T, \tilde{e}_j)\|_{L^\infty(\mathcal{M}_{t^*})} \|\mathbf{g}(e_i, \tilde{T})\|_{L^\infty(\mathcal{M}_{t^*})} \|\tilde{\nabla}_{\tilde{e}_j} \tilde{n}\|_{L_t^\infty L^\infty(\Sigma_t)} \\ &\lesssim \sqrt{MC_{\text{ball}}} D\varepsilon + \sqrt{MC_{\text{ball}}} \|k_{ij}\|_{L_t^\infty L^4(\Sigma_t)} + (MC_{\text{ball}}) C_{\text{ball}}. \end{aligned} \quad (\text{D.11})$$

By (D.10), (D.11) and Lemma 3.4, it follows that

$$\|\tilde{\nu} - 1\|_{L^\infty(\mathcal{M}_{t^*})} \lesssim \sqrt{MC_{\text{ball}}}(D\varepsilon + \sqrt{MC_{\text{ball}}} C_{\text{ball}}) + \sqrt{MC_{\text{ball}}} \|k_{ij}\|_{L_t^\infty L^4(\Sigma_t)}. \quad (\text{D.12})$$

To estimate the remaining term  $\|\tilde{k}_{lj}\|_{L_t^\infty L^4(\Sigma_t)}$  on the right-hand side of (D.12), we apply the following technical lemma whose proof is postponed to the end of this section.

**Lemma D.1** (Technical lemma). *Under the assumption of (D.1), (D.2), (D.6) for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, it holds for every scalar function  $f$  on  $\mathcal{M}_{t^*}$  that for  $t_0^* \leq t \leq t^*$ ,*

$$\|f\|_{L_t^\infty L^4(\Sigma_t)} \lesssim \|\mathbf{D}f\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^{1/4} \|f\|_{L_t^\infty L^6(\tilde{\Sigma}_t)}^{3/4} + \|f\|_{L^4(\Sigma_{t^*})}.$$

Applying Lemma D.1 to  $f = \tilde{k}_{ij}$ , we have for  $\varepsilon > 0$  and  $C_{\text{ball}} > 0$  sufficiently small, for  $t_0^* \leq t \leq t^*$ ,

$$\begin{aligned} \|\tilde{k}_{ij}\|_{L_t^\infty L^4(\Sigma_t)} &\lesssim \|\mathbf{D}(\tilde{k}_{ij})\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^{1/4} \|\tilde{k}_{ij}\|_{L_t^\infty L^6(\tilde{\Sigma}_t)}^{3/4} + \|\tilde{k}_{ij}\|_{L^4(\Sigma_{t^*})} \\ &\lesssim \left( \|\mathbf{D}\tilde{\pi}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)} + \|\tilde{\pi}\|_{L_t^\infty L^4(\tilde{\Sigma}_t)} \|\tilde{\mathbf{A}}\|_{L_t^\infty L^4(\tilde{\Sigma}_t)} \right)^{1/4} \|\tilde{k}\|_{L_t^\infty L^6(\tilde{\Sigma}_t)}^{3/4} \\ &\quad + \|k\|_{L^4(\Sigma_{t^*})} \\ &\lesssim C_{\text{ball}}^{1/4} C_{\text{ball}}^{3/4} + D\varepsilon, \end{aligned} \tag{D.13}$$

where  $\tilde{\mathbf{A}}$  denotes the connection 1-form defined by

$$(\tilde{\mathbf{A}}_\mu)_{\alpha\beta} := \mathbf{g}(\mathbf{D}_{\tilde{e}_\mu} \tilde{e}_\beta, \tilde{e}_\alpha), \quad \text{for } \mu, \alpha, \beta = 0, 1, 2, 3,$$

and we used that the foliation  $(\tilde{\Sigma}_{\tilde{t}})_{0 \leq \tilde{t} \leq t^*}$  constructed by the bounded  $L^2$  curvature theorem (Theorem 3.14, see also [KRS15]) satisfies in addition to (D.2) the following bound on  $0 \leq \tilde{t} \leq t^*$ ,

$$\|\tilde{\mathbf{A}}\|_{L_t^\infty L^4(\tilde{\Sigma}_t)} \lesssim C_{\text{ball}}.$$

Plugging (D.13) into (D.12), we get that for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, for  $t_0^* \leq t \leq t^*$ ,

$$\begin{aligned} \|\tilde{\nu} - 1\|_{L_t^\infty L^\infty(\Sigma_t)} &\lesssim \sqrt{MC_{\text{ball}}}(D\varepsilon + C_{\text{ball}} + \sqrt{MC_{\text{ball}}C_{\text{ball}}}) \\ &\leq M'C_{\text{ball}}, \end{aligned}$$

for a constant  $0 < M' < M$ . This improves the bootstrap assumption (D.6) and hence finishes the proof of (D.3). The estimate (D.4) follows directly from (D.13). This finishes the proof of Lemma 4.6.

It remains to prove Lemma D.1. Let  $(t, x^1, x^2, x^3)$  denote the shift-free coordinate system constructed above. Using that  $\text{tr}k = 0$  on  $\Sigma_t$  for  $1 \leq t \leq t^*$ , (D.1), (D.2) and  $\partial_t = n^{-1}T$ , we have for  $C_{\text{ball}} > 0$  and  $\varepsilon > 0$  sufficiently small, on  $t_0^* \leq t \leq t^*$ ,

$$\begin{aligned} \int_{\Sigma_t} f^4 d\mu_g &= \int_{t^*}^t \partial_{t'} \left( \int_{\tilde{\Sigma}_{t'}} f^4 d\mu_g \right) dt' + \int_{\Sigma_{t^*}} f^4 d\mu_g \\ &= 4 \int_{t^*}^t \left( \int_{\tilde{\Sigma}_{t'}} \partial_t f f^3 d\mu_g \right) dt' + \int_{\Sigma_{t^*}} f^4 d\mu_g \\ &\lesssim \int_{\mathcal{M}_{t^*}} |\mathbf{D}f|_{\mathbf{h}^t} |f|^3 + \int_{\Sigma_{t^*}} f^4 d\mu_g \\ &\lesssim \int_{\mathcal{M}_{t^*}} |\mathbf{D}f|_{\mathbf{h}^{\tilde{t}}} |f|^3 + \int_{\Sigma_{t^*}} f^4 d\mu_g \\ &\lesssim \|\mathbf{D}f\|_{L_t^\infty L^2(\tilde{\Sigma}_t)} \|f\|_{L_t^\infty L^6(\tilde{\Sigma}_t)}^3 + \int_{\Sigma_{t^*}} f^4 d\mu_g, \end{aligned}$$

where we used (D.6) to compare the positive-definite norms  $\mathbf{h}^{\tilde{t}}$  and  $\mathbf{h}^t$ . This finishes the proof of Lemma D.1.



## E. Proof of Proposition 5.2

In this section, we prove Proposition 5.2, that is, the claim that for

$$\mathcal{E}_1 := \int_{\mathcal{M}_{t^*}} \frac{3}{2} Q(\hat{\mathcal{L}}_{\tilde{T}} \mathbf{R})_{\alpha\beta\tilde{T}\tilde{T}} \tilde{\pi}^{\alpha\beta}, \quad \mathcal{E}_2 := \int_{\mathcal{M}_{t^*}} \mathbf{D}^\alpha Q(\hat{\mathcal{L}}_{\tilde{T}} \mathbf{R})_{\alpha\tilde{T}\tilde{T}\tilde{T}}, \quad (\text{E.1})$$

it holds that

$$\begin{aligned} |\mathcal{E}_1| + |\mathcal{E}_2| &\lesssim C_{\text{ball}} \|\hat{\mathcal{L}}_{\tilde{T}} \mathbf{R}\|_{L_t^\infty L^2(\tilde{\Sigma}_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_{\tilde{T}} \mathbf{R} \cdot \tilde{L}\|_{L_{\tilde{\omega}_u}^\infty L^2(\mathcal{H}_{\omega_u})}^2 \\ &\quad + C_{\text{ball}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}} \right)^2 + C_{\text{ball}}^2. \end{aligned} \quad (\text{E.2})$$

The trilinear estimate (E.2) follows from the trilinear estimates proved in Sections 11-13 of [KRS15]. As the latter are stated and proved in [KRS15] using a wave parametrix construction, in the next section we first recapitulate the notation and the wave parametrix construction of [KRS15]. Subsequently, in Sections E.2 and E.3 we prove the bound (E.2) for  $\mathcal{E}_1$  and  $\mathcal{E}_2$ .

### E.1 Recapitulation of the wave parametrix formalism of [KRS15]

In this section, we recapitulate the wave parametrix formalism and the trilinear estimates of [KRS15] relevant to our paper. We adopt the notation of [KRS15] and suppress the tilde notation in the following.

Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime foliated by maximal spacelike hypersurfaces  $(\Sigma_t)_{0 \leq t \leq 1}$ . Let  $e_0 := T$  denote the timelike unit normal to  $\Sigma_t$  and let  $(e_i)_{i=1,2,3}$  be an orthonormal frame on  $\Sigma_t$ . The connection 1-form  $\mathbf{A}$  is defined by

$$(\mathbf{A}_\mu)_{\alpha\beta} := \mathbf{g}(\mathbf{D}_{e_\mu} e_\beta, e_\alpha) \quad \text{for } \mu, \alpha, \beta = 0, 1, 2, 3.$$

The indices  $l, m$  of  $(\mathbf{A}_i)_{lm}$  are interpreted as *internal* and the 1-form  $\mathbf{A}_\mu dx^\mu$  as having values in the Lie algebra of  $so(3, 1)$ .

Denoting  $A_0 := \mathbf{A}_0$ ,  $A_j := \mathbf{A}_j$ ,  $j = 1, 2, 3$ , the second fundamental form  $k$  of  $\Sigma_t$  can be expressed as

$$k_{ij} = -(A_i)_{0j}, \quad (\text{E.3})$$

and the spacetime curvature tensor  $\mathbf{R}$  as

$$\mathbf{R}_{ijlm} = \partial_l (A_m)_{ij} - \partial_m (A_l)_{ij} - ([A_l, A_m])_{ij}, \quad (\text{E.4})$$

where  $i, j, l, m = 1, 2, 3$ .

In [KRS15] it is shown that  $A_i$  satisfies the following structural equations (see Lemmas 6.5 and 13.1 in [KRS15])

$$\begin{aligned} A_i &= (\text{curl} B)_i + E_i, \\ \partial A_i &= (\text{curl}(\partial B))_i + E'_i, \end{aligned} \quad (\text{E.5})$$

where the curl of a 1-form  $\omega$  is defined by

$$(\text{curl} \omega)_i = \epsilon_i^{jl} \partial_j (\omega_l),$$

and  $E$  and  $E'$  are error terms with better regularity.

In Sections 7.2 and 13.2 of [KRS15] it is shown that the 1-forms  $B_i$  and  $\partial B_i$  of (E.5) satisfy non-linear wave equations which exhibit a *null structure*. Exploiting the null structure, the following energy estimates for  $B$  and  $\partial B$  are derived in [KRS15].

**Proposition E.1** (Energy estimates for  $B$  and  $\partial B$ ). *Denoting the initial data norms on  $\Sigma = \Sigma_0$  by*

$$\begin{aligned}\mathcal{Q}_0(\Sigma) &:= \|\text{Ric}\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)}, \\ \mathcal{Q}_1(\Sigma) &:= \|\nabla \text{Ric}\|_{L^2(\Sigma)} + \|\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^2 k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)}.\end{aligned}$$

it holds that

$$\|\partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\omega \in \mathbb{S}^2} \|\nabla(\partial B)\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})} + \sup_{\omega \in \mathbb{S}^2} \|L(\partial B)\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})} \lesssim \mathcal{Q}_0, \quad (\text{E.6})$$

and

$$\|\partial \partial^2 B\|_{L_t^\infty L^2(\Sigma_t)} + \sup_{\omega \in \mathbb{S}^2} \|\nabla(\partial^2 B)\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})} + \sup_{\omega \in \mathbb{S}^2} \|L(\partial^2 B)\|_{L_{\omega_u}^\infty L^2(\mathcal{H}_{\omega_u})} \lesssim \mathcal{Q}_1, \quad (\text{E.7})$$

where  $\partial \in \{\partial_0, \partial_1, \partial_2, \partial_3\}$ . Here, for fixed  $\omega \in \mathbb{S}^2$ ,  $(\mathcal{H}_{\omega_u})_{\omega_u \in \mathbb{R}}$  is a foliation of the vacuum spacetime  $\mathcal{M}$  by null hyperplanes (see also Theorem 3.14).

*Proof.* The estimate (E.6) follows directly from Proposition 7.4 and Lemma 8.3 in [KRS15] (see also (9.13) and the bottom of page 186 therein). The estimate (E.7) follows directly from Proposition 13.2 in [KRS15].  $\square$

Further, in Section 10 of [KRS15], see in particular Theorem 10.8, a parametrix for the wave equation is constructed. As shown in Sections 11.2 and 13.3.2, applying the parametrix construction to  $B$  and  $\partial B$  and using the special curl-structure of (E.5) allows to prove general trilinear estimates for error terms appearing in Bel-Robinson energy estimates. We refer also to Section 11.1 of [KRS15] for instructive similar proofs of bilinear estimates.

In slightly more detail, the rough idea of proof for the trilinear estimates is as follows. First, consider an error term of the form

$$\int_{\mathcal{M}} \mathcal{C}(U, \partial\phi)$$

where  $U$  is a tensor,  $\phi$  is a solution to the wave equation (typically,  $\phi = B$  or  $\phi = \partial B$ ) and  $\mathcal{C}$  denotes a contraction between  $U$  and  $\partial\phi$ . By plugging in the wave parametrix for  $\phi$  and assuming energy estimates such as (E.6) for  $\phi$  (with  $\mathcal{Q}_0$  sufficiently small), it can be shown that

$$\left| \int_{\mathcal{M}} \mathcal{C}(U, \partial\phi) \right| \lesssim \mathcal{Q}_0 \sup_{\omega \in \mathbb{S}^2} \|\mathcal{C}(U, N)\|_{L_{\omega_u}^2 L^1(\mathcal{H}_{\omega_u})}, \quad (\text{E.8})$$

where  $\mathcal{C}(U, N)$  denotes a contraction between  $U$  and  $N$ , the unit normal to  $\mathcal{H}_{\omega_u} \cap \Sigma_t$  along  $\Sigma_t$ . In our cases of interest, the tensor  $U$  is composed of  $\mathbf{R}$  and derivatives of solutions to a scalar wave equation. Therefore if one can show that  $\mathcal{C}(U, N)$  only contains terms of the form

$$\mathbf{R} \cdot L, \nabla \psi \text{ and } L(\psi), \quad (\text{E.9})$$

where  $\psi$  is a solution to a wave equation, then one can apply the energy estimates for  $\mathbf{R}$  and  $\psi$  to bound the right-hand side of (E.8), see the explanatory Remark 11.1 and the discussion in Section 11.2 in [KRS15].

Coming back to this paper, to prove (E.2) it thus suffices to verify that all terms in  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are of the form  $\mathcal{C}(U, \partial\phi)$  (with  $\phi$  solving a wave equation) leading to contractions  $\mathcal{C}(U, N)$  which contain at least one of (E.9). This is exactly the content of the following Sections E.2 and E.3.

Note that in Section 5.1 we apply the bounded  $L^2$  curvature theorem to a maximal hypersurface  $\tilde{\Sigma}_{t^*}$  which satisfies by construction the bounds

$$\begin{aligned}\mathcal{Q}_0(\tilde{\Sigma}_{t^*}) &\lesssim C_{\text{ball}}, \\ \mathcal{Q}_1(\tilde{\Sigma}_{t^*}) &\lesssim \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}}.\end{aligned} \quad (\text{E.10})$$

The above bounds therefore precisely lead to the estimate (E.2).

## E.2 Control of $\mathcal{E}_1$

Recall from (E.1) that

$$\mathcal{E}_1 := \int_{\mathcal{M}_{t^*}} \frac{3}{2} Q(\hat{\mathcal{L}}_T \mathbf{R})_{\alpha\beta TT} \pi^{\alpha\beta}.$$

The Weyl tensor  $\hat{\mathcal{L}}_T \mathbf{R}$  has the same symmetries as  $\mathbf{R}$  and thus, an inspection of Section 11.2 in [KRS15] directly yields the trilinear estimate

$$\begin{aligned} \mathcal{E}_1 &\lesssim C_{\text{ball}} \|\hat{\mathcal{L}}_T \mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)}^2 + C_{\text{ball}} \|\hat{\mathcal{L}}_T \mathbf{R}\|_{L^2(\mathcal{M}_{t^*})} \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_T \mathbf{R} \cdot L\|_{L^\infty_u L^2(\mathcal{H}_{\omega_u})} \\ &\lesssim C_{\text{ball}} \|\hat{\mathcal{L}}_T \mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)}^2 + C_{\text{ball}} \sup_{\omega \in \mathbb{S}^2} \|\hat{\mathcal{L}}_T \mathbf{R} \cdot L\|_{L^\infty_u L^2(\mathcal{H}_{\omega_u})}^2. \end{aligned}$$

## E.3 Control of $\mathcal{E}_2$

Recall from (E.1) that

$$\mathcal{E}_2 := \int_{\mathcal{M}_{t^*}} \mathbf{D}^\alpha Q(\hat{\mathcal{L}}_T \mathbf{R})_{\alpha TTT}.$$

By explicit calculation (see Propositions 7.1.1 and 7.1.2 in [CK93]) and expanding in an orthonormal frame  $(T, e_i), i = 1, 2, 3$ , it holds that

$$\begin{aligned} &\mathbf{D}^\alpha Q(\hat{\mathcal{L}}_T \mathbf{R})_{\alpha TTT} \\ &= (\hat{\mathcal{L}}_T \mathbf{R})_{T^a T^b} (\pi^{\alpha\beta} \mathbf{D}_\alpha \mathbf{R}_{\beta a T b} + (\text{div} \pi)^\beta \mathbf{R}_{\beta a T b}) \\ &\quad + (\hat{\mathcal{L}}_T \mathbf{R})_{T^a T^b} \left( (\mathbf{D}_a \pi_{\alpha\beta} - \mathbf{D}_\beta \pi_{\alpha a}) \mathbf{R}^{\alpha\beta}_{T b} + (\mathbf{D}_T \pi_{\alpha\beta} - \mathbf{D}_\beta \pi_{\alpha T}) \mathbf{R}^{\alpha\beta}_{a b} \right) \\ &\quad + (\hat{\mathcal{L}}_T \mathbf{R})_{T^a T^b} \left( (\mathbf{D}_b \pi_{\alpha\beta} - \mathbf{D}_\beta \pi_{\alpha b}) \mathbf{R}^{\alpha\beta}_{a T} \right) + \text{dual terms}. \end{aligned} \tag{E.11}$$

Careful inspection of the right-hand side of (E.11) based on further decomposition with respect to  $(T, e_i), i = 1, 2, 3$  shows that each term is controlled by either

1. applying the wave parametrix construction of [KRS15] (to  $\phi = B$  or to  $\phi = \partial B$ , depending on the term) and using that the resulting contractions  $\mathcal{C}(U, N)$  lead to terms of the desired form (E.9). Once this is shown, the estimate (E.2) follows from the estimates of [KRS15], see Proposition E.1 and the discussion in Section E.1,
2. by cancelling each other out with another term from the right-hand side of (E.11),
3. by directly applying the precise estimates of the bounded  $L^2$  curvature theorem.

In the following, for explicitness, we discuss in correspondence to the above the control of three representative terms of the right-hand side of (E.11). Namely, we prove the control of

$$\begin{aligned} \mathcal{T}_1 &:= (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{D}_i \mathbf{R}_{j A T B} k^{ij}, \\ \mathcal{T}_2 &:= (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{R}_{C T N A} \mathbf{R}^C_{T B}, \\ \mathcal{T}_3 &:= (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{D}_i \pi_{j T} \mathbf{R}^i_{A B}, \end{aligned}$$

where  $A, B, C = 1, 2$  denotes frame elements tangential to  $\mathcal{H}_{\omega_u} \cap \Sigma_t$ .

**Control of  $\mathcal{T}_1$ .** Consider the integral

$$\int_{\mathcal{M}_{t^*}} \mathcal{T}_1 = \int_{\mathcal{M}_{t^*}} (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{D}_i \mathbf{R}_{j A T B} k^{ij}.$$

Using (E.3) and (E.5), that is,

$$k_{ij} = -(A_i)_{0j}, \quad A_i = (\text{curl} B)_i + E_i, \tag{E.12}$$



we can write

$$\int_{\mathcal{M}_{t^*}} \mathcal{T}_1 = \int_{\mathcal{M}_{t^*}} \mathcal{C}(U, \partial B) + \int_{\mathcal{M}_{t^*}} (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \mathbf{DR} \cdot E, \quad (\text{E.13})$$

where  $B$  solves the wave equation,  $U$  is a tensor involving  $\hat{\mathcal{L}}_T \mathbf{R}$  and  $\mathbf{DR}$ ,  $\mathcal{C}(U, \partial B)$  is a contraction thereof and  $E$  is an error term with better regularity. Using the wave parametrix formalism and energy estimates for  $B$  (see Proposition E.1 and (E.8)) and the improved regularity of  $E$  (see [KRS15] for details), we get that

$$\int_{\mathcal{M}_{t^*}} \mathcal{T}_1 \lesssim \mathcal{Q}_0 \sup_{\omega \in \mathbb{S}^2} \|\mathcal{C}(U, N)\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} + \mathcal{Q}_0 \|\hat{\mathcal{L}}_T \mathbf{R} \cdot \mathbf{DR}\|_{L_t^\infty L^1(\Sigma_t)}. \quad (\text{E.14})$$

It remains to show that  $\mathcal{C}(U, N)$  contains terms of the form (E.9). From (E.12) and (E.13), we have that

$$\begin{aligned} \mathcal{C}(U, \partial B) &= (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \mathbf{D}^i \mathbf{R} (\text{curl} B)_i \\ &= (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \mathbf{D}^i \mathbf{R} \in_i^{j \cdot} \partial_j B, \end{aligned}$$

where we suppressed the internal indices of  $B$ . Therefore

$$\begin{aligned} \mathcal{C}(U, N) &= (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \mathbf{D}^i \mathbf{R} \cdot \in_i^{j \cdot} N_j \\ &= (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \mathbf{D}^i \mathbf{R} \cdot \in_i^{N \cdot} \\ &= (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \mathbf{D}^A \mathbf{R} \\ &= (\hat{\mathcal{L}}_T \mathbf{R}) \cdot \nabla^A \mathbf{R}, \end{aligned} \quad (\text{E.15})$$

where  $A$  is tangential to  $\mathcal{H}_{\omega_u}$ , we used that  $\in_i^{N \cdot} = 0$  for  $i = N$ , and we suppressed lower order error terms.

Plugging (E.15) into (E.14) and using (E.4) and (E.5) to write

$$\mathbf{R}_{ijlm} = \partial_l (A_m)_{ij} - \partial_m (A_l)_{ij} - ([A_l, A_m])_{ij} = \partial^2 B$$

and applying Proposition E.1 and (E.10), we get that

$$\begin{aligned} \int_{\mathcal{M}_{t^*}} \mathcal{T}_1 &\lesssim \mathcal{Q}_0 \sup_{\omega \in \mathbb{S}^2} \left\| \hat{\mathcal{L}}_T \mathbf{R} \cdot \nabla \mathbf{R} \right\|_{L^2_{\omega_u} L^1(\mathcal{H}_{\omega_u})} + \mathcal{Q}_0 \|\hat{\mathcal{L}}_T \mathbf{R} \cdot \mathbf{DR}\|_{L_t^\infty L^1(\Sigma_t)} + \mathcal{Q}_0^2 \\ &\lesssim \mathcal{Q}_0 \left( \left\| \hat{\mathcal{L}}_T \mathbf{R} \right\|_{L^2(\mathcal{M}_{t^*})}^2 + \sup_{\omega \in \mathbb{S}^2} \left\| \nabla (\partial^2 B) \right\|_{L^\infty_{\omega_u} L^2(\mathcal{H}_{\omega_u})}^2 \right) \\ &\quad + \mathcal{Q}_0 \left( \left\| \hat{\mathcal{L}}_T \mathbf{R} \right\|_{L_t^\infty L^2(\Sigma_t)}^2 + \left\| \partial \partial^2 B \right\|_{L_t^\infty L^2(\Sigma_t)}^2 \right) + \mathcal{Q}_0^2 \\ &\lesssim C_{\text{ball}} \left\| \hat{\mathcal{L}}_T \mathbf{R} \right\|_{L_t^\infty L^2(\Sigma_t)}^2 + C_{\text{ball}} \left( \|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}} \right)^2 + C_{\text{ball}}^2. \end{aligned}$$

This finishes the control  $\mathcal{T}_1$ .

**Control of  $\mathcal{T}_2$ .** By the (anti)-symmetries of the curvature tensor, we have that

$$\begin{aligned} \mathcal{T}_2 &:= (\hat{\mathcal{L}}_T \mathbf{R})_T^A \mathbf{R}_{CTNA}^B \mathbf{R}_{TB}^{CN} \\ &= (\hat{\mathcal{L}}_T \mathbf{R})_T^1 T^2 (\mathbf{R}_{CTN1} \mathbf{R}_{T2}^{CN} + \mathbf{R}_{CTN2} \mathbf{R}_{T1}^{CN}) + (\hat{\mathcal{L}}_T \mathbf{R})_T^1 T^1 (\mathbf{R}_{CTN1} \mathbf{R}_{T1}^{CN} - \mathbf{R}_{CTN2} \mathbf{R}_{T2}^{CN}) \\ &= (\hat{\mathcal{L}}_T \mathbf{R})_T^1 T^2 (\mathbf{R}_{1TN1} \mathbf{R}_{T2}^{1N} + \mathbf{R}_{1TN2} \mathbf{R}_{T1}^{1N} + \mathbf{R}_{2TN1} \mathbf{R}_{T2}^{2N} + \mathbf{R}_{2TN2} \mathbf{R}_{T1}^{2N}) \\ &\quad + (\hat{\mathcal{L}}_T \mathbf{R})_T^1 T^1 (\mathbf{R}_{1TN1} \mathbf{R}_{T1}^{1N} - \mathbf{R}_{1TN2} \mathbf{R}_{T2}^{1N} + \mathbf{R}_{2TN1} \mathbf{R}_{T1}^{2N} - \mathbf{R}_{2TN2} \mathbf{R}_{T2}^{2N}). \end{aligned}$$

Given that  $\mathbf{R}_{\mu\nu} = 0$ , it holds that

$$\begin{aligned} &\mathbf{R}_{1TN1} \mathbf{R}_{T2}^{1N} + \mathbf{R}_{1TN2} \mathbf{R}_{T1}^{1N} + \mathbf{R}_{2TN1} \mathbf{R}_{T2}^{2N} + \mathbf{R}_{2TN2} \mathbf{R}_{T1}^{2N} \\ &= \mathbf{R}_{1NT1} (\mathbf{R}_{1NT2} + \mathbf{R}_{1TN2}) + \mathbf{R}_{2TN2} (\mathbf{R}_{2TN1} + \mathbf{R}_{2NT1}) \\ &= \mathbf{R}_{1TN1} (\mathbf{R}_{2TN1} + \mathbf{R}_{1TN2}) - \mathbf{R}_{1TN1} (\mathbf{R}_{2TN1} + \mathbf{R}_{2NT1}) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} & \mathbf{R}_{1TN1}\mathbf{R}^{1N}_{T1} - \mathbf{R}_{1TN2}\mathbf{R}^{1N}_{T2} + \mathbf{R}_{2TN1}\mathbf{R}^{2N}_{T1} - \mathbf{R}_{2TN2}\mathbf{R}^{2N}_{T2} \\ &= (-\mathbf{R}_{2TN2})(-\mathbf{R}_{2TN2}) - \mathbf{R}_{2TN1}\mathbf{R}_{2NT1} + \mathbf{R}_{1TN2}\mathbf{R}_{1NT2} - \mathbf{R}_{2TN2}\mathbf{R}_{2TN2} \\ &= 0. \end{aligned}$$

Therefore, the term turns out to vanish, that is,

$$\mathcal{T}_2 := (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{R}_{CTNA} \mathbf{R}^{CN}_{TB} = 0.$$

This finishes our discussion  $\mathcal{T}_2$ .

**Control of  $\mathcal{T}_3$ .** Recall that

$$\mathcal{T}_3 := (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{D}_i \pi_{jT} \mathbf{R}^i{}_A{}^j{}_B.$$

Using that

$$\pi_{iT} = n^{-1} \nabla_i n, \quad (A_0)_{0i} = -n^{-1} \nabla_i n,$$

together with estimate (5.6) of [KRS15], that is,

$$\|\partial A_0\|_{L_t^\infty L^3(\Sigma_t)} \lesssim \mathcal{Q}_0,$$

we get by (E.10) and the estimates of the bounded  $L^2$ -curvature theorem that

$$\begin{aligned} \int_{\mathcal{M}_{t^*}} (\hat{\mathcal{L}}_T \mathbf{R})_{T^A T^B} \mathbf{D}_i \pi_{jT} \mathbf{R}^j{}_A{}^i{}_B &\lesssim \|\hat{\mathcal{L}}_T \mathbf{R}\|_{L_t^\infty L^2(\Sigma_t)} \|\partial A_0\|_{L_t^\infty L^3(\Sigma_t)} \|\mathbf{R}\|_{L_t^\infty L^6(\Sigma_t)} \\ &\lesssim \mathcal{Q}_0 (\mathcal{Q}_1)^2 \\ &\lesssim C_{\text{ball}} (\|\nabla \text{Ric}\|_{L^2(\Sigma_{t^*})} + \|\nabla^2 k\|_{L^2(\Sigma_{t^*})} + C_{\text{ball}})^2. \end{aligned}$$

This finishes the control of  $\mathcal{T}_3$ .



# III The canonical foliation with bounded $L^2$ curvature

joint work with Stefan Czimek



# 1. Introduction

## 1.1 Einstein vacuum equations

A Lorentzian 4-manifold  $(\mathcal{M}, \mathbf{g})$  is called a *vacuum spacetime* if it solves the Einstein vacuum equations

$$\mathbf{Ric} = 0, \tag{1.1}$$

where  $\mathbf{Ric}$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ . Expressed in general coordinates, (1.1) is a non-linear coupled system of partial differential equations of order 2 for  $\mathbf{g}$ . In so-called *wave coordinates*, it can be shown that (1.1) is a system of nonlinear wave equations. It therefore admits an initial value formulation. Moreover, the characteristic hypersurfaces of these equations are the *null* hypersurfaces of the spacetime  $(\mathcal{M}, \mathbf{g})$ .

## 1.2 The *weak cosmic censorship conjecture* and the *bounded $L^2$ curvature theorem*

The global behaviour of solutions to (1.1) is subject to the celebrated conjecture of *weak cosmic censorship* formulated by Penrose [Pen69].

**Conjecture 1.1** (Weak cosmic censorship conjecture, [Pen69]). *For generic initial data, all singularities forming in gravitational collapse are hidden in a black hole region.*

In the seminal work [Chr99], it is shown that the conjecture holds true in the case of spherical symmetry for Einstein equations coupled with a scalar field. The result relies on the sharp breakdown criterion and local existence result proved in [Chr93] at the level of data with bounded variation, which is adapted to the  $(1+1)$ -setting of spherical symmetry.

In the case of Einstein vacuum equations (1.1) without symmetry, local existence results are naturally formulated in terms of  $L^2$ -based function spaces (see the discussion in the introduction of [KRS15]). In this context, the sharpest known local existence result in terms of regularity of the initial data is the celebrated *bounded  $L^2$  curvature theorem* (see [KRS15] and the companion papers [Sze12a]–[Sze16]). The following is a rough statement of that result.

**Theorem 1.2** (Bounded  $L^2$  curvature theorem, [KRS15]). *For initial data to the Einstein equations (1.1) on a spacelike hypersurface  $\Sigma$  such that the spacetime curvature tensor  $\mathbf{R}$  is bounded in  $L^2(\Sigma)$ , there exists a local Cauchy development that satisfies Einstein equations (1.1).*

In the proof [Chr99] of the weak cosmic censorship conjecture in spherical symmetry, it is crucial that the local existence result in [Chr93] is formulated on null hypersurfaces, in order to highlight a trapped surface formation mechanism (see [Chr91], [Chr99] and also [LL18] for further discussion).

The aim of the present paper is to initiate the proof of a local existence result for initial data on null hypersurfaces with no symmetry assumption, assuming only finite  $L^2$  curvature. Together with the companion paper [CG19b], this will amount to a proof of a spacelike-characteristic bounded  $L^2$  curvature theorem that generalises the bounded  $L^2$  curvature Theorem 1.2 to the case of initial data posed on a characteristic hypersurface instead of a spacelike hypersurface (see Section 1.6 for further discussion).

## 1.3 Null hypersurfaces, foliations and geometry

In various problems, foliating vacuum spacetimes by null hypersurfaces is a powerful tool to capture the propagation features of the Einstein equations. We refer the reader for example to [CK93], [KRS15], [Sze12a]–[Sze16], where the spacetimes are foliated by a mixed spacelike-null foliation (one family of null hypersurfaces and one family of spacelike hypersurfaces), and for example to [Chr09], [KN03], [LR15] where the spacetimes are foliated by a double null foliation (two families of transversely intersecting null hypersurfaces).

When using mixed spacelike-null foliations, the family of null hypersurfaces is typically determined by prescribing the corresponding induced foliation on an initial spacelike hypersurface. This is equivalent to prescribing the values of an optical function  $u$ , whose level sets are the null hypersurfaces, on the initial

spacelike hypersurface. In particular, the regularity of the induced foliation on the initial spacelike hypersurface determines the regularity of the corresponding foliation by null hypersurfaces and hence needs to be carefully picked depending on the situation (see the different constructions of the optical functions in [CK93] and in [Sze12a] for example).

The family of spacelike hypersurfaces is itself typically determined by defining them to be maximal hypersurfaces and fixing their asymptotics towards spacelike infinity or prescribing their finite boundary. In case of spacelike-null foliations these boundaries can be naturally prescribed by choosing the induced foliation on an initial null hypersurface, see [Bar84]. Similarly, in double null foliations, the two families of null hypersurfaces are entirely determined by the foliation they induce on two transversely intersecting initial null hypersurfaces.

A standard choice of foliation on initial null hypersurfaces is the *geodesic foliation* (see below for a definition and [Chr09], [Luk12], [LR15] for examples where this foliation is used as an initial foliation). In more specific situations, other foliations have to be considered: the so-called *canonical foliation* on null hypersurfaces in [KN03] and [Nic04] (see also Definition 1.4) for its additional regularity features, the so-called *constant expansion* and *constant mass aspect function* foliations in [Sau08] to obtain monotonicity properties for the Hawking mass.

In this paper, we consider foliations on an outgoing truncated null hypersurface  $\mathcal{H}$  emanating from a spacelike 2-sphere  $S$ , given by the level sets  $S_v$  of a scalar function  $v \in [1, 2]$  and we assume that the first leaf of this foliation coincides with  $S$ , *i.e.*  $S = S_{v=1}$ . Given a null geodesic generator  $L$  of  $\mathcal{H}$ , we define the *null lapse*  $\Omega$  of the foliation ( $S_v$ ) to be

$$\Omega := Lv.$$

The *geodesic foliation* corresponds to  $\Omega = 1$  and we call its parameter  $s$ . Note that it depends on the choice of  $L$ , which we assume to be fixed, here and in the rest of the paper.

We denote by  $\underline{L}$  the null vector field orthogonal to  $S_v$  and transverse to  $\mathcal{H}$  such that  $\mathbf{g}(L, \underline{L}) = -2$ . The geometry of the foliation ( $S_v$ ) on  $\mathcal{H}$  is described by the induced metric  $\mathcal{g}$  on the 2-spheres  $S_v$  by  $\mathbf{g}$ , the *intrinsic null second fundamental form*  $\chi$ , the *torsion*  $\zeta$  and the *extrinsic null second fundamental form*  $\underline{\chi}$ , respectively defined by

$$\chi(X, Y) := \mathbf{g}(\mathbf{D}_X L, Y), \quad \zeta(X) := \frac{1}{2} \mathbf{g}(\mathbf{D}_X L, \underline{L}), \quad \underline{\chi}(X, Y) := \mathbf{g}(D_X \underline{L}, Y),$$

where  $X, Y$  are  $S_v$ -tangent vectors and  $\mathbf{D}$  denotes the covariant derivative on  $(\mathcal{M}, \mathbf{g})$ . The quantities  $\chi, \zeta$  and  $\underline{\chi}$  are called the *null connection coefficients*. In the following, we often split up  $\chi$  and  $\underline{\chi}$  into their trace and tracefree parts,

$$\begin{aligned} \text{tr}\chi &:= \mathcal{g}^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2} \text{tr}\chi \mathcal{g}_{AB}, \\ \text{tr}\underline{\chi} &:= \mathcal{g}^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2} \text{tr}\underline{\chi} \mathcal{g}_{AB}. \end{aligned}$$

Geometrically, the intrinsic second fundamental form  $\chi$  measures how the spheres  $S_v$  and their first fundamental form  $\mathcal{g}$  change along  $\mathcal{H}$  and the extrinsic second fundamental form  $\underline{\chi}$  measures how the 2-spheres  $S_v$  change in the null direction transverse to  $\mathcal{H}$  given by  $\underline{L}$ . In particular, we see that the geometry of hypersurfaces emanating from the 2-spheres  $S_v$  transversely to  $\mathcal{H}$  must critically depend on the regularity of  $\underline{\chi}$  on  $S_v$ .

We also have the following decomposition of the spacetime curvature tensor  $\mathbf{R}$  into the *null curvature components* relative to  $L$  and  $\underline{L}$ .

$$\begin{aligned} \alpha(X, Y) &:= \mathbf{R}(X, L, Y, L), & \beta(X) &:= \frac{1}{2} \mathbf{R}(X, L, \underline{L}, L), & \rho &:= \frac{1}{4} \mathbf{R}(L, \underline{L}, L, \underline{L}), \\ \sigma &:= \frac{1}{4} {}^* \mathbf{R}(\underline{L}, L, \underline{L}, L), & \underline{\beta}(X) &:= \frac{1}{2} \mathbf{R}(X, \underline{L}, \underline{L}, L), & \underline{\alpha}(X, Y) &:= \mathbf{R}(\underline{L}, X, \underline{L}, Y), \end{aligned}$$

where  $X$  and  $Y$  are  $S_v$ -tangent vectors. We define the  $L^2$  *curvature flux* through  $\mathcal{H}$  by

$$\mathcal{R}_{\mathcal{H}} := \left( \|\alpha\|_{L^2(\mathcal{H})}^2 + \|\beta\|_{L^2(\mathcal{H})}^2 + \|\rho\|_{L^2(\mathcal{H})}^2 + \|\sigma\|_{L^2(\mathcal{H})}^2 + \|\underline{\beta}\|_{L^2(\mathcal{H})}^2 \right)^{1/2}. \quad (1.2)$$

## 1.4 Regularity of the foliation on $\mathcal{H}$

Our goal in this paper is to provide a local initial foliation  $(S_v)$  on  $\mathcal{H}$  such that the geometry of a family of hypersurfaces transverse to  $\mathcal{H}$  emanating from the 2-spheres  $S_v$  can be locally controlled under the assumption of finite  $L^2$  curvature flux. One needs to control the intrinsic and extrinsic geometry of the foliation  $(S_v)$ , that is, to provide bounds on  $\mathcal{H}$  for the null lapse  $\Omega$ , the induced metric  $\mathcal{g}$  and the null connection coefficients  $\chi, \zeta$  and  $\underline{\chi}$  of the foliation  $(S_v)$  assuming only a control on the  $L^2$  curvature flux.

Here and in the rest of the paper, all quantities specific to the geodesic foliation will be noted with a prime. In [KR05], the following groundbreaking result is proved for the geodesic foliation.

**Theorem 1.3** (Control of the geodesic foliation, [KR05]). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime. Let  $\mathcal{H}$  be an outgoing null hypersurface emanating from a spacelike 2-sphere  $(S, \mathcal{g})$  foliated by the geodesic foliation associated to the affine parameter  $s$  going from  $s|_S = 1$  to  $s = 2$ . Let  $\mathcal{I}'_S$  denote low regularity norms on  $\chi', \zeta'$  and  $\underline{\chi}'$  on  $S$  (see Section 2.11 for a definition). Assume that*

$$\mathcal{I}'_S + \mathcal{R}'_{\mathcal{H}} \leq \varepsilon.$$

Then, there exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$ , the following bounds hold

$$\begin{aligned} \left\| \operatorname{tr}\chi' - \frac{2}{s}, \hat{\chi}', \zeta' \right\|_{H^1(\mathcal{H})} &\lesssim \varepsilon, \\ \left\| \operatorname{tr}\underline{\chi}' + \frac{2}{s}, \hat{\underline{\chi}}' \right\|_{L^2(\mathcal{H})} &\lesssim \varepsilon, \end{aligned}$$

where  $\|F\|_{H^1(\mathcal{H})} := \|F\|_{L^2(\mathcal{H})} + \|\nabla F\|_{L^2(\mathcal{H})} + \|\nabla_L F\|_{L^2(\mathcal{H})}$  (see Definition 2.5). This holds together with additional, more specific estimates for  $\chi', \zeta'$  and  $\underline{\chi}'$  (see Section 2.11 for further estimates which are used in this paper, and see [KR05] for the full estimates).

### Remarks.

1. The smallness assumption on  $\mathcal{I}'_S$  implies that the 2-sphere  $S$  is close to the Euclidean 2-sphere of radius 1 in a weak sense, and the smallness assumption on  $\mathcal{R}'_{\mathcal{H}}$  implies that the null curvature components are close to their (trivial) value in Minkowski spacetime in  $L^2$  on  $\mathcal{H}$ .
2. The proof of Theorem 1.3 is obtained by analysing the so-called *null structure equations* and *null Bianchi equations* which are consequences of geometric constraints and the Einstein equations (1.1) and relate  $\mathcal{g}', \chi', \zeta', \underline{\chi}'$  to  $\alpha', \beta', \rho', \sigma', \underline{\beta}'$ , see Sections 2.3 and 2.4 and [CK93] and [KR05] for details.
3. Sharp bilinear and trace estimates and geometric Littlewood-Paley theory are used in the proof of Theorem 1.3, see also [KR06a] [KR06b] and the subsequent [AS16] [AS14] [Sha14] [Sze12a]–[Sze16] [Wan09].
4. This theorem gives a local control of the geodesic foliation in terms of the  $L^2$  curvature flux. In [AS14, AS16], a global control on the geodesic foliation was obtained provided that the (weighted)  $L^2$  curvature flux is sufficiently close to Schwarzschild data. In [Wan09], a local control result was obtained when  $\mathcal{H}$  is the null cone emanating from a point.
5. The control of the extrinsic coefficients  $\operatorname{tr}\underline{\chi}'$  and  $\hat{\underline{\chi}}'$  in Theorem 1.3 is significantly weaker than the control of  $\operatorname{tr}\chi'$  and  $\hat{\chi}'$  on  $\mathcal{H}$ .

One needs bounds for  $\operatorname{tr}\underline{\chi}$  and  $\hat{\underline{\chi}}$  comparable to the ones for  $\operatorname{tr}\chi$  and  $\hat{\chi}$  in order to control transversely emanating hypersurfaces. As outlined above, one does not obtain such bounds using the techniques from [KR05, KR06a, KR06b] in the case of the geodesic foliation.

In the next section we turn to the study of the *canonical foliation* on  $\mathcal{H}$ , which was defined in [KN03] and [Nic04] for its improved regularity features for  $\operatorname{tr}\underline{\chi}$  and  $\hat{\underline{\chi}}$  (see the discussion in [KN99]). Relying on the same ideas, we can obtain a similar regularity improvement for  $\operatorname{tr}\underline{\chi}$  and  $\hat{\underline{\chi}}$  which is sufficient for the proof of the spacelike-characteristic bounded  $L^2$  curvature theorem in our companion paper [CG19b] (see also Section 1.6 for more details).



## 1.5 The canonical foliation and a first version of the main result

In this section, we introduce the canonical foliation on  $\mathcal{H}$ . Notation and more precise definitions are given in Section 2.

**Definition 1.4** (Canonical foliation). *A foliation  $(S_v)$  on  $\mathcal{H}$  is called canonical foliation, if the null lapse  $\Omega$  satisfies*

$$\begin{aligned} \Delta (\log \Omega) &= -\text{div} \zeta + \rho - \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}} - \bar{\rho} + \frac{1}{2} \overline{\hat{\chi} \cdot \underline{\hat{\chi}}}, \\ \int_{S_v} \log \Omega &= 0. \end{aligned} \tag{1.3}$$

where  $\Delta$  denotes the induced Laplace-Beltrami operator on  $S_v$ , and  $\text{div}$  the divergence operator acting on  $S_v$ -tangent vector fields,  $\bar{\rho}$  and  $\overline{\hat{\chi} \cdot \underline{\hat{\chi}}}$  denote the average of  $\rho$  and  $\hat{\chi} \cdot \underline{\hat{\chi}}$  on  $S_v$  respectively.

**Remark 1.5.** *In this paper, we use the definitions for the connection coefficients from [KR05], for this choice makes the null lapse  $\Omega$  disappear from the null structure equations (see Section 2.3). This accounts for the apparent discrepancy with the original canonical foliation definition (see Definition 3.3.2 in [KN03]).*

The following is a first version of the main result of this paper, see Theorem 2.36 for the precise version.

**Theorem 1.6** (Existence and control of the canonical foliation, version 1). *Let  $(\mathcal{M}, \mathbf{g})$  be a vacuum spacetime. Let  $\mathcal{H}$  be an outgoing null hypersurface emanating from a spacelike 2-sphere  $(S, \mathbf{g})$  and foliated by a smooth geodesic foliation associated to the affine parameter  $s$  taking values between  $s|_S = 1$  and  $s = 5/2$ . Assume that we have the following bounds*

$$\mathcal{I}_S + \mathcal{R}'_{\mathcal{H}} \leq \varepsilon, \tag{1.4}$$

for the initial data norm  $\mathcal{I}_S$  at  $S_1$  and the  $L^2$  curvature flux  $\mathcal{R}'_{\mathcal{H}}$  (with respect to the geodesic foliation). There exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$ , the following holds.

1.  **$L^2$ -regularity.** *The canonical foliation  $(S_v)$  on  $\mathcal{H}$  is well-defined from  $v = 1$  to 2 and*

$$\left\| \text{tr} \underline{\chi} - \frac{2}{v}, \text{tr} \underline{\chi} + \frac{2}{v}, \hat{\chi}, \underline{\hat{\chi}}, \zeta, \Omega - 1, \nabla \Omega \right\|_{H^1(\mathcal{H})} \lesssim \varepsilon, \tag{1.5}$$

*Additional, more specific estimates hold for  $\chi$ ,  $\zeta$ ,  $\Omega$  and  $\underline{\chi}$  (see Theorem 2.36).*

2. **Higher regularity.** *The smoothness of the geodesic foliation implies smoothness of the canonical foliation.*

### Remarks.

1. In Theorem 1.6, the regularity of  $\text{tr} \underline{\chi}$  and  $\hat{\chi}$  is improved compared to Theorem 1.3. In particular, the regularity of  $\underline{\chi}$  is sufficient for the spacelike-characteristic bounded  $L^2$  curvature theorem in the companion paper [CG19b] (see also Section 1.6).
2. The canonical foliation displays better regularity features for  $\underline{\chi}$  than the geodesic foliation because of a simplified transport equation for  $\text{tr} \underline{\chi}$ . More precisely, while in the geodesic foliation it holds that

$$L(\text{tr} \underline{\chi}') + \frac{1}{2} \text{tr} \underline{\chi}' \text{tr} \underline{\chi}' = -2 \text{div} \zeta' + 2 \left( \rho' - \frac{1}{2} \hat{\chi}' \cdot \underline{\hat{\chi}}' \right) + 2 |\zeta'|^2$$

where a low regularity curvature term is present on the right-hand side, in the canonical foliation we have

$$L(\text{tr} \underline{\chi}) + \frac{1}{2} \text{tr} \underline{\chi} \text{tr} \underline{\chi} = 2 \bar{\rho} - \overline{\hat{\chi} \cdot \underline{\hat{\chi}}} + 2 |\nabla \Omega - \zeta|^2,$$

where the right-hand side has improved tangential regularity (see Lemma 2.22). This allows for an improved control of  $\text{tr} \underline{\chi}$  and subsequently  $\underline{\hat{\chi}}$  on  $\mathcal{H}$ .

3. The methods in the proof of Theorem 1.6 are reminiscent of [KR05, KR06a, KR06b] and the subsequent [AS14, AS16], [Sha14] and [Wan09] where the geodesic foliation is studied (see also [Sze12a]–[Sze16]). A new difficulty that arises in our analysis is that, in contrast to the geodesic foliation where  $\Omega \equiv 1$ , the null lapse  $\Omega$  has only low regularity and hence must be treated with care (see Sections 2, 4, 5 and 6).

4. The functional calculus tools are listed in Section 3 and are mostly taken from [Sha14], which is the latest version of the ideas from the groundbreaking [KR05, KR06a, KR06b] (see also [Wan09] and [Sze12a]–[Sze16]).
5. The quantity  $\mathcal{I}_S$  contains the same norms as  $\mathcal{I}'_S$  in Theorem 1.3 together with additional norms on  $\text{tr}\underline{\chi}'$  and  $\underline{\chi}'$  in order for the new bounds to hold. These additional norms are at the same level of regularity as the required norms for  $\text{tr}\chi'$  and  $\hat{\chi}'$ .
6. We use that the geodesic connection coefficients are controlled by Theorem 1.3 and that a small change of foliation leaves the null curvature components, the second fundamental form  $\chi$  and some geometric norms essentially invariant (see Sections 2, 4, 5 and 6).
7. The proof of Theorem 1.6 implies in particular that if a given canonical foliation on  $\mathcal{H}$  has small initial norm  $\mathcal{I}_S$  at  $S$  and small  $L^2$  curvature flux  $\mathcal{R}_{\mathcal{H}}$  with respect to the canonical foliation on  $\mathcal{H}$ , then the foliation geometry on  $\mathcal{H}$  is controlled as stated in Theorem 1.6 with  $\mathcal{I}_S$  and  $\mathcal{R}_{\mathcal{H}}$  on the right-hand side. Since such a formulation would require that the canonical foliation a priori exists, we preferred to state the smallness assumptions with respect to the geodesic foliation.

## 1.6 The spacelike-characteristic Cauchy problem of general relativity in low regularity

Our motivation for the main Theorem 1.6 in this paper is its application to the authors' spacelike-characteristic bounded  $L^2$  curvature theorem [CG19b]. First we define the volume radius of a Riemannian 3-manifold.

**Definition 1.7** (Volume radius). *Let  $(\Sigma, g)$  be a Riemannian 3-manifold with boundary, and let  $r > 0$  be a real number. The volume radius of  $\Sigma$  at scale  $r$  is defined by*

$$r_{\text{vol}}(\Sigma, r) := \inf_{p \in \Sigma} \inf_{0 < r' < r} \frac{\text{vol}_g(B_g(p, r'))}{(r')^3},$$

where  $B_g(p, r')$  denotes the geodesic ball of radius  $r'$  centred at  $p \in \Sigma$ .

**Theorem 1.8** (The spacelike-characteristic bounded  $L^2$  curvature theorem, [CG19b]). *Consider smooth initial data for the Einstein vacuum equations posed on a maximal spacelike hypersurface  $\Sigma \simeq \overline{B} \subset \mathbb{R}^3$  and the outgoing null hypersurface  $\mathcal{H}$  emanating from  $S := \partial\Sigma$ . Assume that for some  $\varepsilon > 0$ ,*

$$\begin{aligned} \mathcal{I}_S + \mathcal{R}'_{\mathcal{H}} &\leq \varepsilon, & \|\text{Ric}\|_{L^2(\Sigma)} &\leq \varepsilon, & \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} &\leq \varepsilon, \\ r_{\text{vol}}(\Sigma, 1/2) &\geq 1/4, & \text{vol}_g(\Sigma) &< \infty, \end{aligned}$$

where the initial foliation geometry  $\mathcal{I}_S$  and the  $L^2$  curvature flux  $\mathcal{R}'_{\mathcal{H}}$  are the same as in Theorem 1.6, and  $\text{Ric}$  and  $k$  denote the Ricci tensor and second fundamental form of  $\Sigma \subset \mathcal{M}$ . Then:

1.  **$L^2$ -regularity.** *There is a universal constant  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then the maximal globally hyperbolic development of  $(\mathcal{M}, \mathbf{g})$  contains a future region of  $\Sigma \cup \mathcal{H}$  which is foliated by maximal spacelike hypersurfaces  $\Sigma_t$  given as level sets of a time function  $t$  such that  $\Sigma_1 = \Sigma$  and*

$$\partial\Sigma_t = S_t \text{ on } \mathcal{H},$$

where  $(S_t)_{1 \leq t \leq 2}$  is the canonical foliation on  $\mathcal{H}$ , and the following control holds for  $1 \leq t \leq 2$ ,

$$\begin{aligned} \|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, & \|k\|_{L_t^\infty L^2(\Sigma_t)} + \|\nabla k\|_{L_t^\infty L^2(\Sigma_t)} &\lesssim \varepsilon, \\ \inf_{1 \leq t \leq 2} r_{\text{vol}}(\Sigma_t, 1/2) &\geq \frac{1}{8}, & \text{vol}_g(\Sigma_t) &< \infty. \end{aligned}$$

2. **Higher regularity.** *Smoothness is propagated from initial data into  $\mathcal{M}$  up to  $t = 2$ .*

**Remarks.**

1. In the proof of Theorem 1.8, the boundary regularity of the hypersurfaces  $\Sigma_t$  is directly related to the regularity of the canonical foliation  $(S_v)$  on  $\mathcal{H}$ . More specifically, for the control of  $k$  on  $\partial\Sigma_t$ , it is necessary to have a control for

$$\left\| \text{tr}\chi - \frac{2}{v}, \text{tr}\underline{\chi} + \frac{2}{v}, \zeta \right\|_{L_v^\infty H^{1/2}(S_v)}.$$

These estimates are obtained by  $H^1(\mathcal{H})$  to  $H^{1/2}(S_v)$  trace estimates and the  $H^1(\mathcal{H})$  estimates (1.5) of Theorem 1.6.

2. In addition to the estimates of this paper, the proof of Theorem 1.8 relies on the *bounded  $L^2$  curvature theorem* [KRS15], the *extension procedure for the constraint equations* [Czi18], Cheeger-Gromov convergence theory on manifolds (with boundary) in low regularity [Czi19a, Czi19b], and global estimates for maximal spacelike hypersurfaces.

## 1.7 Overview of the proof of Theorem 1.6

### 1.7.1 The null structure equations

General properties of the Riemann curvature tensor together with Einstein vacuum equations (1.1) imply that the induced metric  $\mathcal{g}$ , the null lapse  $\Omega$  and the null connection coefficients  $\chi$ ,  $\zeta$  and  $\underline{\chi}$  for a general foliation of  $\mathcal{H}$  by 2-spheres  $S_v$  satisfy a system of *null structure equations* on  $\mathcal{H}$ , which is the following system of coupled quasilinear transport and elliptic equations (see [CK93, pp. 168–170]):

the *first variation* transport equation for  $\mathcal{g}$

$$\mathcal{L}_L \mathcal{g} = 2\chi, \quad (1.6a)$$

the *second variation* transport equations for  $\chi, \underline{\chi}, \zeta$

$$\nabla_L \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 = -|\hat{\chi}|^2, \quad (1.6b)$$

$$\nabla_L \hat{\chi} + \text{tr} \chi \hat{\chi} = -\alpha, \quad (1.6c)$$

$$\nabla_L \text{tr} \underline{\chi} + \frac{1}{2} \text{tr} \chi \text{tr} \underline{\chi} = -2\text{d}\hat{v}\zeta - 2\Delta(\log \Omega) + 2\left(\rho - \frac{1}{2}\hat{\chi} \cdot \hat{\chi}\right) + 2|\zeta + \nabla \log \Omega|^2, \quad (1.6d)$$

$$\nabla_L \hat{\chi} + \frac{1}{2} \text{tr} \chi \hat{\chi} = -\nabla \hat{\otimes} \zeta - (\nabla \hat{\otimes} \nabla) \log \Omega - \frac{1}{2} \text{tr} \chi \hat{\chi} + \text{n.l.t.}, \quad (1.6e)$$

$$\nabla_L \zeta + \text{tr} \chi \zeta = -\frac{1}{2} \text{tr} \chi \nabla \log \Omega - \beta + \text{n.l.t.}, \quad (1.6f)$$

the Hodge-type elliptic equations for  $\zeta$

$$\begin{aligned} \text{d}\hat{v}\zeta &= -\rho - \mu + \text{n.l.t.}, \\ \text{curl} \zeta &= \sigma + \text{n.l.t.}, \end{aligned} \quad (1.6g)$$

the transport equation for the *mass aspect function*  $\mu$

$$\nabla_L \mu + \frac{3}{2} \text{tr} \chi \mu = \frac{1}{2} \text{tr} \chi \Delta \log \Omega + \text{n.l.t.}, \quad (1.6h)$$

the Hodge-type elliptic *Codazzi* equations for  $\hat{\chi}$  and  $\underline{\hat{\chi}}$

$$\text{d}\hat{v}\hat{\chi} = \frac{1}{2} \nabla \text{tr} \chi - \zeta \cdot \hat{\chi} + \frac{1}{2} \zeta \text{tr} \chi - \beta, \quad (1.6i)$$

$$\text{d}\hat{v}\underline{\hat{\chi}} = \frac{1}{2} \nabla \text{tr} \underline{\chi} + \zeta \cdot \underline{\hat{\chi}} - \frac{1}{2} \zeta \text{tr} \underline{\chi} + \underline{\beta}, \quad (1.6j)$$

and the *Gauss equation* for the *Gauss curvature*  $K$  of the 2-spheres  $S_v$

$$K = -\frac{1}{4} \text{tr} \underline{\chi} \text{tr} \chi - \rho + \frac{1}{2} \hat{\chi} \cdot \underline{\hat{\chi}}, \quad (1.6k)$$

where n.l.t. denotes (additional) nonlinear error terms.

The system of null structure equations has the null curvature components  $\alpha, \beta, \rho, \sigma, \underline{\beta}$  as source terms. Using the initial small  $L^2$ -bound (1.4), and provided that the foliation  $(S_v)$  is close to the geodesic foliation  $(S'_s)$  of  $\mathcal{H}$ , one can obtain the following  $L^2$ -control of the null curvature components

$$\|\alpha\|_{L^2(\mathcal{H})}^2 + \|\beta\|_{L^2(\mathcal{H})}^2 + \|\sigma\|_{L^2(\mathcal{H})}^2 + \|\rho\|_{L^2(\mathcal{H})}^2 + \|\underline{\beta}\|_{L^2(\mathcal{H})}^2 \lesssim \varepsilon^2. \quad (1.7)$$

Our goal is to estimate the induced metric  $\mathcal{g}$  and the null connection coefficients  $\chi, \underline{\chi}, \zeta, \nabla^{\leq 1} \Omega$  using the null structure equations (1.6), the bound (1.7) on the null curvature source terms, and bounds from the regularity assumptions at the sphere  $S_1 = \Sigma \cap \mathcal{H}$ .<sup>1</sup>

<sup>1</sup>Bounds for  $\mathcal{g}$  and the null connection coefficients on the first sphere  $S_1$  are needed when integrating the transport equation in the  $L$ -direction.

### 1.7.2 Linear estimates and the geodesic foliation case

Estimates can only be obtained when (an equation for) the null lapse  $\Omega$  is fixed *-i.e.* once the foliation has been chosen.<sup>2</sup> Let first assume that the geodesic foliation choice  $\Omega = 1$  has been made and try to obtain the desired  $H^1(\mathcal{H})$  estimates for  $\zeta$ ,  $\text{tr}\chi$  and  $\text{tr}\underline{\chi}$  *at the linear level*. Using equation (1.6f) for  $\nabla_L \zeta$ , taking the  $L^2(\mathcal{H})$ -norm and using the bound (1.7), one obtains

$$\begin{aligned} \|\nabla_L \zeta\|_{L^2(\mathcal{H})} &\lesssim \|\beta\|_{L^2(\mathcal{H})} + \text{l.o.t.} \\ &\lesssim \varepsilon, \end{aligned} \tag{1.8}$$

provided that the lower order terms are controlled. Using the Hodge-type elliptic equation (1.6g) for  $\zeta$ , the bound (1.7) and an appropriate elliptic energy estimate, one obtains

$$\begin{aligned} \|\nabla \zeta\|_{L^2(\mathcal{H})} + \|\zeta\|_{L^2(\mathcal{H})} &\lesssim \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\mu\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + \|\mu\|_{L^2(\mathcal{H})}. \end{aligned} \tag{1.9}$$

Using the transport equation (1.6h) for the mass aspect function  $\mu$  at the linear level and integrating in the  $L$ -direction one deduces that

$$\begin{aligned} \|\mu\|_{L^2(\mathcal{H})} &\lesssim \|\mu\|_{L_v^\infty L^2(S_v)} \\ &\lesssim \|\mu\|_{L^2(S_1)} + \|\Delta \log \Omega\|_{L_v^1 L^2(S_v)} \\ &\lesssim \varepsilon, \end{aligned} \tag{1.10}$$

since  $\Omega = 1$  and where we used the initial control (1.4) on the sphere  $S_1$ . Using the bounds (1.8), (1.9) and (1.10), we deduce that

$$\|\zeta\|_{H^1(\mathcal{H})} \lesssim \varepsilon,$$

which gives the desired estimate for  $\zeta$ .

We turn to obtaining the same estimate for  $\text{tr}\chi$ . One does not have an elliptic equation of the type (1.6g) by which the tangential derivatives of  $\text{tr}\chi$  would be controlled in  $L^2(\mathcal{H})$ . Fortunately, there are no curvature source terms in the transport equation (1.6b) for  $\text{tr}\chi$ . Commuting this transport equation with a tangential derivative  $\nabla$ , one thus obtains that  $\nabla_L \nabla \text{tr}\chi$  are only lower order terms. Integrating this equation in the  $L$ -direction we therefore have

$$\|\nabla \text{tr}\chi\|_{L_v^\infty L^2(S_v)} \lesssim \|\nabla \text{tr}\chi\|_{L^2(S_1)} + \text{l.o.t.} \lesssim \varepsilon, \tag{1.11}$$

where we used the initial control (1.4) for  $\nabla \text{tr}\chi$  on  $S_1$  and provided that the lower order terms are controlled. Estimating  $\nabla_L \text{tr}\chi$  is done by a direct inspection of equation (1.6b) and we deduce the desired  $H^1(\mathcal{H})$  (and  $L_v^\infty H^1(S_v)$ ) control for  $\text{tr}\chi$ .

For the null component  $\text{tr}\underline{\chi}$ , one does not have an elliptic equation of the type (1.6g). One can only rely on the transport equation (1.6d). Unlike in the transport equation (1.6b) for  $\text{tr}\chi$ , there are curvature and high order source terms to equation (1.6d), which namely reads at the linear level and when  $\Omega = 1$

$$\nabla_L \text{tr}\underline{\chi} = 2\rho - 2\text{d}\not{v}\zeta + \text{l.o.t.}$$

To obtain an  $H^1(\mathcal{H})$  control of  $\text{tr}\underline{\chi}$ , one would need to control the null curvature term  $\rho$  and the high order term  $\text{d}\not{v}\zeta$  at an  $L_v^1 H^1(S_v)$  level.<sup>3</sup> Such a control cannot be obtained with the assumed  $L^2(\mathcal{H})$  regularity (1.7) for the curvature. The geodesic foliation choice thus fails – at the linear level – to provide the required regularity for the study of the spacelike-characteristic bounded  $L^2$  curvature theorem.

The *canonical foliation* choice (1.3) is designed so that the transport equation (1.6d) for  $\text{tr}\underline{\chi}$  writes

$$\nabla_L \text{tr}\underline{\chi} + \frac{1}{2} \text{tr}\chi \text{tr}\underline{\chi} = 2\bar{\rho} - \overline{\hat{\chi} \cdot \hat{\chi}} + 2|\nabla \log \Omega + \zeta|^2, \tag{1.12}$$

where it should be noted that the higher order terms on the right-hand side are now *constant* in the tangential direction. By commuting the transport equation (1.12) with tangential derivatives, we therefore deduce that at the linear level  $\nabla_L \nabla \text{tr}\underline{\chi}$  is only composed of lower order terms. Arguing as for the null connection coefficient  $\text{tr}\chi$ , one can then obtain the desired  $H^1(\mathcal{H})$  estimate for  $\text{tr}\underline{\chi}$ .

<sup>2</sup>As for Einstein vacuum equations, the system of null structure equations (1.6) is determined only up to a gauge choice, which in this case geometrically corresponds to a choice of foliation on  $\mathcal{H}$ .

<sup>3</sup>The (weaker)  $L_v^\infty H^{1/2}(S_v)$  control required in the proof of Theorem 1.8 would also require higher regularity than bounded  $L^2$  curvature.

### 1.7.3 Construction and control of the canonical foliation

In this section, we give a sketch of proof for the existence and control from  $v = 1$  to  $v = 2$  of the canonical foliation  $(S_v)$  and – motivated by the study of the linear case in Section 1.7.2 – for the obtention of the desired  $H^1(\mathcal{H})$  estimates for its null connection coefficients.

Using the smooth background geodesic foliation, this reduces to proving that solutions to the quasilinear system of transport and elliptic null structure equations (1.6) together with the additional elliptic equation (1.3) for the null lapse  $\Omega$  exist and remain controlled from  $v = 1$  to  $v = 2$ . This time 1 existence result has to be obtained using only low regularity smallness assumptions on the initial sphere  $S_1$  and the  $L^2$ -smallness assumption (1.7) on the null curvature source terms.

The proof of Theorem 1.6 thus goes by a standard continuity argument relying on bootstrap assumptions for the estimates (1.5), on propagation of regularity, and on an higher regularity local existence and continuation result. In the rest of this section, we shall review the key elements for the improvement of the bootstrap assumptions.

To improve the set of bootstrap assumptions for estimates (1.5), we have to show that we can estimate the  $H^1(\mathcal{H})$  norms of the null connection coefficients one-by-one in a suitable order by the  $L^2(\mathcal{H})$  norm of the null curvature components. This virtually amounts to a *triangularisation* of the system of null structure equations (1.6) and (1.3). It has to take into account the presence of a non-trivial null lapse  $\Omega$  and differs from the geodesic foliation case studied in [KR05] because of the intertwined equations for  $\zeta$ ,  $\mu$  and  $\log \Omega$  (1.6f), (1.6g), (1.6h) and (1.3).

Provided that this can be done, we obtain the desired  $H^1(\mathcal{H})$  control (1.5) for the null connection coefficients arguing as in the linear case of Section 1.7.2, using standard elliptic energy estimates, (deriving) and integrating the transport equations. Here the improved form of the transport equation for  $\text{tr}\underline{\chi}$  (1.12), which is a consequence of the canonical foliation choice, is crucial to establish the desired  $H^1$  estimate for  $\text{tr}\underline{\chi}$  (and subsequently the desired  $H^1$  control of  $\hat{\chi}$ ).

The main difficulty is to control the nonlinear error terms arising in the null structure transport equations (1.6b), (1.6h) and (1.12) using only the low regularity smallness assumptions on the initial sphere  $S_1$  and the  $L^2$ -smallness assumption (1.7). Integrating these equations in the  $L$ -direction and taking the  $L^2$ -norm in the tangential direction requires to deal with error terms of the form

$$\left\| \int_1^2 A \cdot R \, dv \right\|_{L^2(S)} \quad (1.13)$$

where  $A$  denotes the null connection coefficients  $\text{tr}\chi - \frac{2}{v}$ ,  $\hat{\chi}$ ,  $\zeta$  and  $\nabla\Omega$  and where  $R$  is only bounded in  $L^2(\mathcal{H})$  (such as the null curvature components). Using Hölder estimates, the control of (1.13) is achieved provided that the following crucial *geometric trace norms* estimates for  $\hat{\chi}$ ,  $\zeta$  and  $\nabla\Omega$

$$\sup_{\omega \in S} \int_1^2 |\hat{\chi}(v, \omega)|^2 \, dv + \sup_{\omega \in S} \int_1^2 |\zeta(v, \omega)|^2 \, dv + \sup_{\omega \in S} \int_1^2 |\nabla\Omega(v, \omega)|^2 \, dv \lesssim \varepsilon^2, \quad (1.14)$$

and the following *uniform* bound for  $\text{tr}\chi$

$$\left\| \text{tr}\chi - \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} \lesssim \varepsilon, \quad (1.15)$$

can be obtained.<sup>4</sup>

In the case of the geodesic foliation, the control of the geometric trace norms for  $\hat{\chi}$  and  $\zeta$  was obtained in the seminal series of papers [KR05] [KR06a] [KR06b]. This required to prove sharp bilinear estimates for transport equations, using Besov spaces and Littlewood-Paley calculus, and therefore to make sense to a Littlewood-Paley theory for tensors on the 2-spheres  $S'_s$  relying only on low regularity geometric estimates (see [KR06a], [KR06b] and also [Sha14]). In this paper, we obtain the bounds for the corresponding connection coefficients  $\hat{\chi}$  and  $\zeta$  in the canonical foliation using a *comparison argument* with the background geodesic foliation, taking advantage of the estimates proved for the geodesic foliation in [KR05]. This uses

<sup>4</sup>Note that estimate (1.15) follows from estimate (1.14) for  $\hat{\chi}$  by integrating the transport equation (1.6b).

that the null curvature fluxes, the geometric norms and the null connection coefficients  $\chi$  and  $\zeta$  are essentially invariant if the two foliations are close in an appropriate sense.

Obtaining the last geometric trace norm estimate

$$\sup_{\omega \in \mathcal{S}} \int_1^2 |\nabla \Omega(v, \omega)|^2 dv \lesssim \varepsilon^2$$

is the most delicate point of our analysis.<sup>5</sup> To this end, we highlight that equation (1.3) displays the appropriate structure to apply the sharp bilinear estimate theorem of [KR06b]. Applying this theorem requires to use the geometric Littlewood-Paley theory and geometric Besov spaces developed in [Sha14].

## 1.8 Organisation of the paper

We outline the organisation and give a reading order suggestion.

- ◇ In Section 2, we collect definitions and formulas which are used in this paper. We also state and prove the main theorem, provided that bootstrap assumptions can be improved, higher regularity/smoothness of the canonical foliation can be propagated on  $\mathcal{H}$  and local existence for the canonical foliation can be obtained.
- ◇ In Section 3, we state the key functional results in low regularity which are taken from [Sha14].
- ◇ In Section 4, we obtain the improvement of (low regularity) bootstrap assumptions on  $\mathcal{H}$ .
- ◇ In Section 5, we show that higher regularity/smoothness of the geodesic foliation implies higher regularity/smoothness of the canonical foliation.
- ◇ In Section 6, we prove a local existence and continuation result for the canonical foliation.
- ◇ In Appendices A and B, we prove auxiliary formulas and functional estimates.

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<sup>5</sup>This estimate is also the key to close the aforementioned comparison argument.



## 2. Geometric setup and main results

In this section, we introduce the geometric setup of this paper, give a precise statement of our main result (see Section 2.12) as well as a proof, assuming that the results from Sections 3, 4, 5, 6 can be obtained.

### 2.1 Foliations on null hypersurfaces

In this section, we set up foliations on null hypersurfaces following the notations (and normalisations) of [KR05]. Let  $(\mathcal{M}, \mathbf{g})$  be a Lorentzian 4-manifold and let  $S \subset \mathcal{M}$  be a spacelike 2-sphere. Let  $\mathcal{H}$  denote the outgoing null hypersurface emanating from  $S$ .

**Definition 2.1** (Geodesic foliation on  $\mathcal{H}$ ). *Let  $L$  be an  $\mathcal{H}$ -tangential null vector field on  $S$  orthogonal to  $S$ . Extend  $L$  as null geodesic vector field onto  $\mathcal{H}$ . Define the affine parameter  $s$  of  $L$  on  $\mathcal{H}$  by*

$$Ls = 1 \text{ on } \mathcal{H}, \quad s|_S = 1.$$

Denote the level sets of  $s$  by

$$S'_{s_0} = \{s = s_0\}$$

and denote the geodesic foliation by  $(S'_s)$ .

**Definition 2.2** (General foliations on  $\mathcal{H}$ ). *Let  $v$  be a given scalar function on  $\mathcal{H}$ . We denote the level sets of  $v$  by*

$$S_{v_0} = \{v = v_0\},$$

and the foliation by  $(S_v)$ . We define the null lapse  $\Omega$  of  $(S_v)$  on  $\mathcal{H}$  by

$$\Omega := Lv. \tag{2.1}$$

**Definition 2.3** (Orthonormal null frame). *Let  $(S_v)$  be a foliation on  $\mathcal{H}$ . Let  $\underline{L}$  be the unique null vector field on  $\mathcal{H}$  orthogonal to the 2-spheres  $S_v$  and such that  $\mathbf{g}(\underline{L}, L) = -2$ . The pair  $(L, \underline{L})$  is called a null pair for the foliation  $(S_v)$ . Let  $(e_1, e_2)$  be an orthonormal frame tangential to the 2-spheres  $S_v$ . The frame  $(L, \underline{L}, e_1, e_2)$  is called an orthonormal null frame for the foliation  $(S_v)$ .*

**Remark 2.4.** *Let  $(S_v)$  be a foliation on  $\mathcal{H}$ ,  $(L, \underline{L})$  an associated null pair and  $X$  an  $\mathcal{H}$ -tangent vector field. Decomposing  $X$  onto an orthonormal null frame for the foliation  $(S_v)$  and using (2.1) we have*

$$Xv = -\frac{1}{2}\Omega\mathbf{g}(\underline{L}, X). \tag{2.2}$$

**Definition 2.5.** *Let  $(S_v)$  be a foliation on  $\mathcal{H}$ . We denote by  $\mathcal{g}$  and  $\nabla$  the induced Riemannian metric and covariant derivative on the 2-spheres  $S_v$  and define for any  $S_v$ -tangential  $k$ -tensor  $T$  the derivative  $\nabla_L T$  by*

$$\nabla_L T_{A_1 \dots A_k} := \Pi_{A_1}^{\beta_1} \dots \Pi_{A_k}^{\beta_k} \mathbf{D}_L T_{\beta_1 \dots \beta_k},$$

where  $\Pi$  denotes the projection operator onto the tangent space of  $S$ ,  $\mathbf{D}$  is the covariant derivative on  $(\mathcal{M}, \mathbf{g})$  and we tacitly use, as in the rest of this paper, the Einstein summation convention.

Here and in the following, indices  $A, B, C, D, E \in \{1, 2\}$  denote evaluation of  $S_v$ -tangent tensors on the components  $(e_1, e_2)$  of an orthonormal frame  $(L, \underline{L}, e_1, e_2)$  for the foliation  $(S_v)$ .

**Definition 2.6** (Null connection coefficients). *We define the null connection coefficients, to be the  $S_v$ -tangent tensors such that*

$$\begin{aligned} \chi_{AB} &:= \mathbf{g}(\mathbf{D}_A L, e_B), & \underline{\chi}_{AB} &:= \mathbf{g}(\mathbf{D}_A \underline{L}, e_B), \\ \zeta_A &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_A L, \underline{L}), & \underline{\eta}_A &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_A), \end{aligned} \tag{2.3}$$

**Lemma 2.7.** *The connection coefficients  $\underline{\eta}$  and  $\zeta$  and the null lapse  $\Omega$  verify*

$$\underline{\eta} = -\zeta - \nabla(\log \Omega). \tag{2.4}$$



*Proof.* Using equations (2.1) and (2.2), we have

$$\begin{aligned}
 \eta_A &= -\frac{1}{2}\mathbf{g}(\underline{L}, \mathbf{D}_L e_A) \\
 &= -\frac{1}{2}\mathbf{g}(\underline{L}, \mathbf{D}_A L) - \frac{1}{2}\mathbf{g}(\underline{L}, [L, e_A]) \\
 &= -\zeta_A + \Omega^{-1}[L, e_A](v) \\
 &= -\zeta_A - \nabla_A(\log \Omega),
 \end{aligned}$$

as desired.  $\square$

We have the following relations between covariant derivatives and null connection coefficients (see [CK93]),

$$\begin{aligned}
 \mathbf{D}_L L &= 0, & \mathbf{D}_L \underline{L} &= 2\underline{\eta}_A e_A, \\
 \mathbf{D}_A L &= \chi_{AB} e_B - \zeta_A L, & \mathbf{D}_A \underline{L} &= \underline{\chi}_{AB} e_B + \zeta_A \underline{L} \\
 \mathbf{D}_L e_A &= \nabla_L e_A + \underline{\eta}_A L, & \mathbf{D}_A e_B &= \nabla_A e_B + \frac{1}{2}\chi_{AB} \underline{L} + \frac{1}{2}\underline{\chi}_{AB} L.
 \end{aligned} \tag{2.5}$$

If the orthonormal null frame is such that  $\nabla_L e_A = 0$ , we call it *Fermi propagated*.

We have the following decomposition of  $\chi$  and  $\underline{\chi}$  into their trace and tracefree parts

$$\begin{aligned}
 \text{tr}\chi &:= \mathcal{J}^{AB} \chi_{AB}, & \widehat{\chi}_{AB} &:= \chi_{AB} - \frac{1}{2}\text{tr}\chi \mathcal{J}_{AB}, \\
 \text{tr}\underline{\chi} &:= \mathcal{J}^{AB} \underline{\chi}_{AB}, & \widehat{\underline{\chi}}_{AB} &:= \underline{\chi}_{AB} - \frac{1}{2}\text{tr}\underline{\chi} \mathcal{J}_{AB}.
 \end{aligned}$$

**Definition 2.8** (Null curvature components). *We define the null curvature components to be the  $S_v$ -tangent tensors such that*

$$\begin{aligned}
 \alpha_{AB} &:= \mathbf{R}(L, e_A, L, e_B), & \beta_A &:= \frac{1}{2}\mathbf{R}(e_A, L, \underline{L}, L), \\
 \rho &:= \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), & \sigma &:= \frac{1}{4}{}^*\mathbf{R}(\underline{L}, L, \underline{L}, L), \\
 \underline{\beta}_A &:= \frac{1}{2}\mathbf{R}(e_A, \underline{L}, \underline{L}, L), & \underline{\alpha}_{AB} &:= \mathbf{R}(\underline{L}, e_A, \underline{L}, e_B),
 \end{aligned}$$

where  ${}^*\mathbf{R}$  denotes the Hodge dual of  $\mathbf{R}$ , given by  ${}^*\mathbf{R}_{\alpha\beta\gamma\delta} = \frac{1}{2}\in_{\alpha\beta\mu\nu}\mathbf{R}^{\mu\nu}_{\gamma\delta}$ , with  $\in$  the volume form associated to the metric  $\mathbf{g}$ .

## 2.2 Tensor calculus on 2-spheres

We introduce the following notation.

**Definition 2.9** (Hodge duals). *For a  $S_v$ -tangent 1-tensor  $\phi$ , we define its left Hodge dual by*

$${}^*\phi_A := \not\epsilon_{AB}\phi_B,$$

where  $\not\epsilon_{AB} := \in_{AB}\underline{L}\underline{L}$ . Similarly, for a  $S_v$ -tangent symmetric 2-tensor  $\phi$ , let

$${}^*\phi_{AB} := \not\epsilon_{AC}\phi_{CB}.$$

**Definition 2.10** ( $S_v$ -tangent tensor calculus). *For  $S_v$ -tangent  $r$ -tensors  $\phi$ ,  $\phi^{(1)}$  and  $\phi^{(2)}$ , we define*

$$\phi^{(1)} \cdot \phi^{(2)} := \mathcal{J}^{A_1 B_1} \dots \mathcal{J}^{A_r B_r} \phi_{A_1 \dots A_r}^{(1)} \phi_{B_1 \dots B_r}^{(2)}, \quad |\phi|^2 = \phi \cdot \phi,$$

and

$$\text{div}\phi_{A_2 \dots A_r} := \mathcal{J}^{AB} \nabla_A \phi_{BA_2 \dots A_r}, \quad \text{curl}\phi_{A_2 \dots A_r} := \not\epsilon^{AB} \nabla_A \phi_{BA_2 \dots A_r}.$$

For a 1-form  $\phi$ , we define

$$(\nabla \widehat{\otimes} \phi)_{AB} := \nabla_A \phi_B + \nabla_B \phi_A - \text{div}\mathcal{J}_{AB}.$$

For 1-forms  $\phi^{(1)}$  and  $\phi^{(2)}$  we define

$$\begin{aligned} (\phi^{(1)} \widehat{\otimes} \phi^{(2)})_{AB} &:= \phi_A^{(1)} \phi_B^{(2)} + \phi_B^{(1)} \phi_A^{(2)} - g_{AB} \phi^{(1)} \cdot \phi^{(2)}, \\ \phi^{(1)} \wedge \phi^{(2)} &:= \epsilon^{AB} \phi_A^{(1)} \phi_B^{(2)}. \end{aligned}$$

For symmetric 2-tensors  $\phi^{(1)}$  and  $\phi^{(2)}$  we define the wedge product,

$$(\phi^{(1)} \wedge \phi^{(2)}) := \epsilon^{AB} g^{CD} \phi_{AC}^{(1)} \phi_{BD}^{(2)}.$$

## 2.3 Null structure equations on $\mathcal{H}$

The Einstein vacuum equations (1.1) induce the following *null structure equations* on  $\mathcal{H}$ , see ([CK93], pp. 168-170). We have the *first variation equation*,

$$\mathcal{L}_L g = 2\chi,$$

the *null transport equations*,

$$\begin{aligned} \nabla_L \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 &= -|\widehat{\chi}|^2, \\ \nabla_L \widehat{\chi} + \text{tr} \chi \widehat{\chi} &= -\alpha, \\ \nabla_L \text{tr} \underline{\chi} + \frac{1}{2} \text{tr} \chi \text{tr} \underline{\chi} &= 2 \text{div} \underline{\eta} + 2\rho - \widehat{\chi} \cdot \widehat{\chi} + 2|\underline{\eta}|^2, \\ \nabla_L \widehat{\chi} + \frac{1}{2} \text{tr} \chi \widehat{\chi} &= (\nabla \widehat{\otimes} \underline{\eta}) - \frac{1}{2} \text{tr} \chi \widehat{\chi} + (\underline{\eta} \widehat{\otimes} \underline{\eta}), \\ \nabla_L \zeta + \frac{1}{2} \text{tr} \chi \zeta &= \frac{1}{2} \text{tr} \chi \underline{\eta} - \widehat{\chi} \cdot (\zeta - \underline{\eta}) - \beta, \end{aligned}$$

the *torsion equation*,

$$\text{curl} \underline{\eta} = -\text{curl} \zeta = -\sigma + \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi},$$

and the *Gauss-Codazzi equations*,

$$\begin{aligned} K &= -\frac{1}{4} \text{tr} \chi \text{tr} \chi - \rho + \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi}, \\ \text{div} \widehat{\chi} - \frac{1}{2} \nabla \text{tr} \chi &= -\zeta \cdot \widehat{\chi} + \frac{1}{2} \zeta \text{tr} \chi - \beta, \\ \text{div} \widehat{\chi} - \frac{1}{2} \nabla \text{tr} \chi &= \zeta \cdot \widehat{\chi} - \frac{1}{2} \zeta \text{tr} \chi + \underline{\beta}, \end{aligned}$$

where  $K$  denotes the Gauss curvature of  $S_v$ .

**Remark 2.11.** Only the trace and the symmetrised traceless part of the transport equation for  $\underline{\chi}$  are stated in [CK93]. By rederiving the equation, or simply using the null transport equation for the traced and the symmetrised traceless tensor together with the torsion equation, one has more generally the following transport equation for the full tensor  $\underline{\chi}$

$$\nabla_L \underline{\chi}_{AB} + \chi_{AC} \underline{\chi}_{CB} = 2 \nabla_A \underline{\eta}_B + 2 \underline{\eta}_A \underline{\eta}_B + \rho g_{AB} + \sigma \epsilon_{AB}.$$

**Remark 2.12.** Similarly, only the divergence part of Codazzi equations are stated in [CK93]. By rederiving the equation, or simply using the Gauss-Codazzi equation for  $\text{div} \widehat{\chi}$ , more generally it holds that

$$\text{curl} \chi = -\zeta \cdot \widehat{\chi} + \frac{1}{2} \text{tr} \chi \zeta - \beta.$$

## 2.4 Null Bianchi identities on $\mathcal{H}$

The Einstein vacuum equations (1.1) further yield the following *null Bianchi identities* on  $\mathcal{H}$ , see ([CK93], p. 161).

$$\nabla_L \underline{\alpha} + \frac{1}{2} \text{tr} \chi \underline{\alpha} = -(\nabla \widehat{\otimes} \underline{\beta}) - 3 \widehat{\chi} \rho + 3 \widehat{\chi} \sigma + ((\zeta - 4\underline{\eta}) \widehat{\otimes} \underline{\beta}), \quad (2.7a)$$

$$\nabla_L \underline{\beta} + \text{tr} \chi \underline{\beta} = -\nabla \rho + \widehat{\nabla} \sigma + 2 \widehat{\chi} \cdot \beta - 3 \underline{\eta} \rho + 3 \widehat{\nabla} \sigma, \quad (2.7b)$$

$$\nabla_L \rho + \frac{3}{2} \text{tr} \chi \rho = \text{div} \beta - \frac{1}{2} \widehat{\chi} \cdot \alpha + \zeta \cdot \beta + 2 \underline{\eta} \cdot \beta, \quad (2.7c)$$

$$\nabla_L \sigma + \frac{3}{2} \text{tr} \chi \sigma = -\text{curl} \beta + \frac{1}{2} \widehat{\chi} \wedge \alpha - \zeta \wedge \beta - 2 \underline{\eta} \wedge \beta, \quad (2.7d)$$

$$\nabla_L \beta + 2 \text{tr} \chi \beta = \text{div} \alpha + (2\zeta + \underline{\eta}) \cdot \alpha. \quad (2.7e)$$

**Definition 2.13** (Renormalised null curvature components). *Let the renormalised curvature components  $\check{\rho}$ ,  $\check{\sigma}$ ,  $\check{\beta}$  be*

$$\check{\rho} := \rho - \frac{1}{2} \widehat{\chi} \cdot \widehat{\chi}, \quad \check{\sigma} := \sigma - \frac{1}{2} \widehat{\chi} \wedge \widehat{\chi}, \quad \check{\beta} := \beta + 2 \widehat{\chi} \cdot \zeta. \quad (2.8)$$

We have the following transport equations for  $\check{\rho}$ ,  $\check{\sigma}$  and  $\check{\beta}$ .

**Lemma 2.14.** *The renormalised null curvature component  $\check{\rho}$ ,  $\check{\sigma}$  and  $\check{\beta}$  satisfy*

$$\nabla_L \check{\rho} + \frac{3}{2} \text{tr} \chi \check{\rho} = \text{div} \beta + \zeta \cdot \beta + 2 \underline{\eta} \cdot \beta - \frac{1}{2} (\nabla \widehat{\otimes} \underline{\eta}) \cdot \widehat{\chi} + \frac{1}{4} \text{tr} \chi |\widehat{\chi}|^2 - \frac{1}{2} (\underline{\eta} \otimes \underline{\eta}) \cdot \widehat{\chi}, \quad (2.9a)$$

$$\nabla_L \check{\sigma} + \frac{3}{2} \text{tr} \chi \check{\sigma} = -\text{curl} \beta - \zeta \wedge \beta - 2 \underline{\eta} \wedge \beta - \frac{1}{2} \widehat{\chi} \wedge (\nabla \widehat{\otimes} \underline{\eta}) - \frac{1}{2} \widehat{\chi} \wedge (\underline{\eta} \otimes \underline{\eta}) \quad (2.9b)$$

$$\begin{aligned} \nabla_L \check{\beta} + \text{tr} \chi \check{\beta} &= -\nabla \rho + \widehat{\nabla} \sigma + 2(\nabla \widehat{\otimes} \underline{\eta}) \cdot \zeta - 3 \underline{\eta} \rho + 3 \widehat{\nabla} \sigma \\ &\quad - \text{tr} \chi \zeta \cdot \widehat{\chi} + \text{tr} \chi \underline{\eta} \cdot \widehat{\chi} + 2\zeta \cdot (\underline{\eta} \widehat{\otimes} \underline{\eta}) - 2 \widehat{\chi} \cdot \widehat{\chi} \cdot (\zeta - \underline{\eta}). \end{aligned} \quad (2.9c)$$

*Proof.* From p. 14 in [KR05], we have the first two equations. Using Bianchi equation (2.7b) and the null structure equation for  $\widehat{\chi}$  and  $\zeta$  from Section 2.3, we have

$$\begin{aligned} \nabla_L \check{\beta} + \text{tr} \chi \check{\beta} &= \nabla_L \underline{\beta} + \text{tr} \chi \underline{\beta} + 2 \left( \nabla_L \widehat{\chi} + \frac{1}{2} \text{tr} \chi \widehat{\chi} \right) \cdot \zeta + 2 \widehat{\chi} \cdot \left( \nabla_L \zeta + \frac{1}{2} \text{tr} \chi \zeta \right) \\ &= -\nabla \rho + \widehat{\nabla} \sigma + 2 \widehat{\chi} \cdot \beta - 3 \underline{\eta} \rho + 3 \widehat{\nabla} \sigma \\ &\quad + 2\zeta \cdot \left( \nabla \widehat{\otimes} \underline{\eta} - \frac{1}{2} \text{tr} \chi \widehat{\chi} + \underline{\eta} \widehat{\otimes} \underline{\eta} \right) \\ &\quad + 2 \widehat{\chi} \cdot \left( \frac{1}{2} \text{tr} \chi \underline{\eta} - \widehat{\chi} \cdot (\zeta - \underline{\eta}) - \beta \right) \\ &= -\nabla \rho + \widehat{\nabla} \sigma + 2(\nabla \widehat{\otimes} \underline{\eta}) \cdot \zeta - 3 \underline{\eta} \rho + 3 \widehat{\nabla} \sigma \\ &\quad - \text{tr} \chi \zeta \cdot \widehat{\chi} + \text{tr} \chi \underline{\eta} \cdot \widehat{\chi} + 2\zeta \cdot (\underline{\eta} \widehat{\otimes} \underline{\eta}) - 2 \widehat{\chi} \cdot \widehat{\chi} \cdot (\zeta - \underline{\eta}), \end{aligned}$$

as desired.  $\square$

## 2.5 Commutation formulas on $\mathcal{H}$

The next proposition follows from p. 159 in [CK93].

**Proposition 2.15** (Commutation formulas). *For an  $S_v$ -tangent  $r$ -tensor  $\phi$ , it holds that*

$$\begin{aligned} [\nabla_L, \nabla_B] \phi_{A_1 \dots A_r} &= -\frac{1}{2} \text{tr} \chi \nabla_B \phi_{A_1 \dots A_r} - \widehat{\chi}_{BC} \nabla_C \phi_{A_1 \dots A_r} + (\underline{\eta}_B + \zeta_B) \nabla_L \phi_{A_1 \dots A_r} \\ &\quad + \sum_{i=1}^r (\chi_{A_i B} \underline{\eta}_C - \chi_{BC} \underline{\eta}_{A_i} + \not\chi_{A_i C} \widehat{\nabla} \beta_B) \phi_{A_1 \dots C \dots A_r}. \end{aligned} \quad (2.10)$$

For an  $S_v$ -tangent 1-form  $\phi$ , it holds that

$$[\nabla_L, \mathfrak{d}\mathfrak{v}] \phi = -\frac{1}{2} \text{tr} \chi \mathfrak{d}\mathfrak{v} \phi - \hat{\chi} \cdot \nabla \phi + (\underline{\eta} + \zeta) \cdot \nabla_L \phi + \text{tr} \chi \underline{\eta} \cdot \phi - \underline{\eta}_A \phi_C \chi_{AC} + \beta \cdot \phi, \quad (2.11)$$

$$[\nabla_A, \mathfrak{d}\mathfrak{v}] \phi_B = -3K \nabla_A \phi_B + 2\mathfrak{g}_{AB} K \mathfrak{d}\mathfrak{v} \phi + 2\mathfrak{e}_{AB} K \text{curl} \phi + \mathfrak{g}_{AB} \phi \cdot K - \phi_A \nabla_B K. \quad (2.12)$$

where  $K$  denotes the Gauss curvature of  $S_v$ . For a scalar function  $\phi$ , it holds that

$$[\nabla_L, \nabla_B] \phi = -\frac{1}{2} \text{tr} \chi \nabla_B \phi - \hat{\chi}_{BC} \nabla_C \phi + (\underline{\eta}_B + \zeta_B) L \phi, \quad (2.13)$$

$$[\nabla_L, \mathfrak{d}\mathfrak{v}] \phi = -\text{tr} \chi \mathfrak{d}\mathfrak{v} \phi - 2\chi \cdot \nabla^2 \phi + (\underline{\eta} + \zeta) \cdot (\nabla \nabla_L + \nabla_L \nabla) \phi + (\text{tr} \chi \underline{\eta} - \mathfrak{d}\mathfrak{v} \chi) \cdot \nabla \phi - \underline{\eta}_A \chi_{AB} \nabla_B \phi + (\mathfrak{d}\mathfrak{v} \underline{\eta} + \mathfrak{d}\mathfrak{v} \zeta) \nabla_L \phi + \beta \cdot \nabla \phi, \quad (2.14)$$

$$[\nabla_A, \mathfrak{d}\mathfrak{v}] \phi = -K \nabla_A \phi. \quad (2.15)$$

**Proposition 2.16.** For any scalar function  $f$  on  $\mathcal{H}$ , it holds that

$$\Omega^{-1} L(\bar{f}) = \overline{\Omega^{-1} L f} + \overline{\Omega^{-1} \text{tr} \chi f} - \overline{\Omega^{-1} \text{tr} \chi} \cdot \bar{f},$$

where  $\bar{f}$  denotes the mean value of  $f$  on  $S_v$ .

*Proof.* Using the null structure equations from Section 2.3, we have

$$\begin{aligned} \Omega^{-1} L \left( \int_{S_v} f \right) &= \frac{d}{dv} \left( \int_{S_v} f \right) \\ &= \int_{S_v} \Omega^{-1} (L f + \text{tr} \chi f). \end{aligned}$$

We therefore deduce

$$\begin{aligned} \Omega^{-1} L(\bar{f}) &= -\Omega^{-1} L(\log(|S_v|)) \bar{f} + \overline{\Omega^{-1} (L f + \text{tr} \chi f)} \\ &= -\overline{\Omega^{-1} \text{tr} \chi} \cdot \bar{f} + \overline{\Omega^{-1} (L f + \text{tr} \chi f)}, \end{aligned}$$

where  $|S_v|$  denotes the area of  $S_v$ . This proves the desired result.  $\square$

## 2.6 The mass aspect function on $\mathcal{H}$

**Definition 2.17.** Let the mass aspect function  $\mu$  on  $\mathcal{H}$  be defined by

$$\mu := -\check{\rho} - \mathfrak{d}\mathfrak{v} \zeta. \quad (2.16)$$

We have the following transport equation for  $\mu$ .

**Lemma 2.18.** The mass aspect function  $\mu$  verifies

$$\begin{aligned} L(\mu) + \text{tr} \chi \mu &= \frac{1}{2} \text{tr} \chi \check{\rho} - \frac{1}{2} \text{tr} \chi \mathfrak{d}\mathfrak{v} \underline{\eta} - 2\zeta \cdot \beta + (\zeta - \underline{\eta}) \cdot \nabla \text{tr} \chi + \hat{\chi} \cdot \nabla \zeta \\ &\quad + \frac{1}{2} \hat{\chi} \cdot \nabla \underline{\eta} + \text{tr} \chi \left( |\zeta|^2 - \zeta \cdot \underline{\eta} - \frac{1}{2} |\underline{\eta}|^2 \right) - \frac{1}{4} \text{tr} \chi |\hat{\chi}|^2 \\ &\quad + 2\hat{\chi} \cdot \zeta \cdot \underline{\eta} - \frac{1}{2} \hat{\chi} \cdot \underline{\eta} \cdot \underline{\eta}. \end{aligned}$$

*Proof.* Using the transport equation for  $\zeta$  from the null structure equations from Section 2.3 and commutation formula (2.11), we have

$$\begin{aligned} L(\mathfrak{d}\mathfrak{v} \zeta) &= \mathfrak{d}\mathfrak{v} \nabla_L \zeta + [\nabla_L, \mathfrak{d}\mathfrak{v}] \zeta \\ &= \mathfrak{d}\mathfrak{v} \left( -\frac{1}{2} \text{tr} \chi \zeta + \frac{1}{2} \text{tr} \chi \underline{\eta} - \hat{\chi} \cdot (\zeta - \underline{\eta}) - \beta \right) \\ &\quad - \frac{1}{2} \text{tr} \chi \mathfrak{d}\mathfrak{v} \zeta - \hat{\chi} \cdot \nabla \zeta + (\underline{\eta} + \zeta) \cdot \nabla_L \zeta + \text{tr} \chi \underline{\eta} \cdot \zeta - \underline{\eta} \cdot \zeta \cdot \chi + \beta \cdot \zeta \\ &= -\text{tr} \chi \mathfrak{d}\mathfrak{v} \zeta + \frac{1}{2} \text{tr} \chi \mathfrak{d}\mathfrak{v} \underline{\eta} - \mathfrak{d}\mathfrak{v} \beta + F_1, \end{aligned}$$

where

$$\begin{aligned}
 F_1 := & -\frac{1}{2}\zeta \cdot \nabla \operatorname{tr}\chi + \frac{1}{2}\underline{\eta} \cdot \nabla \operatorname{tr}\chi - \operatorname{d}\not\!v\hat{\chi} \cdot (\zeta - \underline{\eta}) - \hat{\chi} \cdot (\nabla \zeta - \nabla \underline{\eta}) \\
 & - \hat{\chi} \cdot \nabla \zeta + (\underline{\eta} + \zeta) \cdot \left( -\frac{1}{2}\operatorname{tr}\chi\zeta + \frac{1}{2}\operatorname{tr}\chi\underline{\eta} - \hat{\chi} \cdot (\zeta - \underline{\eta}) - \beta \right) \\
 & + \operatorname{tr}\chi\underline{\eta} \cdot \zeta - \underline{\eta} \cdot \zeta \cdot \chi + \beta \cdot \zeta.
 \end{aligned}$$

From the above and transport equation (2.9a) for  $\check{\rho}$ , we have

$$\begin{aligned}
 L(\mu) + \operatorname{tr}\chi\mu &= -L(\check{\rho}) - \operatorname{tr}\chi\check{\rho} - L(\operatorname{d}\not\!v\zeta) - \operatorname{tr}\chi\operatorname{d}\not\!v\zeta \\
 &= \frac{3}{2}\operatorname{tr}\chi\check{\rho} - \operatorname{d}\not\!v\beta + F_2 - \operatorname{tr}\chi\check{\rho} \\
 &\quad - \frac{1}{2}\operatorname{tr}\chi\operatorname{d}\not\!v\underline{\eta} + \operatorname{d}\not\!v\beta - F_1 \\
 &= \frac{1}{2}\operatorname{tr}\chi\check{\rho} - \frac{1}{2}\operatorname{tr}\chi\operatorname{d}\not\!v\underline{\eta} + F_2 - F_1,
 \end{aligned}$$

where the nonlinear term  $F_2$  is given by

$$F_2 := -\zeta \cdot \beta - 2\underline{\eta} \cdot \beta + \frac{1}{2}(\nabla \underline{\eta}) \cdot \hat{\chi} - \frac{1}{4}\operatorname{tr}\chi|\hat{\chi}|^2 + \frac{1}{2}\underline{\eta} \cdot \underline{\eta} \cdot \hat{\chi}.$$

Rearranging the nonlinear term  $F_2 - F_1$  then gives the desired result.  $\square$

## 2.7 The canonical foliation on $\mathcal{H}$

**Definition 2.19** (Canonical foliation). *Let  $\mathcal{H}$  be an outgoing null hypersurface emanating from a spacelike 2-sphere  $S$ . A foliation  $(S_v)$  of  $\mathcal{H}$  is called a canonical foliation if the null lapse  $\Omega$  satisfies the following elliptic equation on the leaves  $S_v$*

$$\begin{aligned}
 \Delta(\log \Omega) &= -\operatorname{d}\not\!v\zeta + \check{\rho} - \bar{\rho}, \\
 \int_{S_v} \log \Omega &= 0,
 \end{aligned} \tag{2.17}$$

where  $\bar{\rho}$  is the average of  $\check{\rho}$  on the 2-sphere  $S_v$ .

In the following, we will moreover consider canonical foliations with  $v|_S = 1$ .

**Remark 2.20.** Using (2.16), the elliptic equation (2.17) can also be rewritten

$$\begin{aligned}
 \Delta(\log \Omega) &= -2\operatorname{d}\not\!v\zeta - \mu + \bar{\mu}, \\
 &= 2(\check{\rho} - \bar{\rho}) + \mu - \bar{\mu}.
 \end{aligned} \tag{2.18}$$

**Notation.** From now on, primed quantities on  $\mathcal{H}$  will correspond to the geodesic foliation of  $\mathcal{H}$ , while unprimed quantities correspond to the canonical foliation. Moreover, we call  $S_1 = S = S'_1$ .

As a first consequence of Definition 2.19, we note that in a canonical foliation, the quantities  $\underline{\eta}$  and  $\operatorname{tr}\underline{\chi}$  satisfy the following equations.

**Lemma 2.21.** *In a canonical foliation,  $\underline{\eta}$  satisfies the following equation*

$$\operatorname{d}\not\!v\underline{\eta} = -\check{\rho} + \bar{\rho}. \tag{2.19}$$

*Proof.* Using relation (2.4) and the elliptic equation (2.17), we have

$$\begin{aligned}
 \operatorname{d}\not\!v\underline{\eta} &= -\operatorname{d}\not\!v\zeta - \Delta(\log \Omega) \\
 &= -\check{\rho} + \bar{\rho},
 \end{aligned}$$

as desired.  $\square$

**Lemma 2.22.** *In a canonical foliation,  $\operatorname{tr}\underline{\chi}$  satisfies the next null transport equation*

$$L(\operatorname{tr}\underline{\chi}) + \frac{1}{2}\operatorname{tr}\chi\operatorname{tr}\underline{\chi} = 2\bar{\rho} + 2|\underline{\eta}|^2. \tag{2.20}$$

*Proof.* Using the transport equation for  $\underline{\chi}$  from Section 2.3 and relation from Lemma 2.21, we have

$$\begin{aligned} L(\underline{\text{tr}}\underline{\chi}) + \frac{1}{2}\underline{\text{tr}}\underline{\chi}\underline{\text{tr}}\underline{\chi} &= 2\underline{\text{div}}\underline{\eta} + 2\bar{\rho} + 2|\underline{\eta}|^2 \\ &= 2\bar{\rho} + 2|\underline{\eta}|^2, \end{aligned}$$

as desired.  $\square$

**Lemma 2.23.** *In a canonical foliation, the transport equation for the mass aspect function can be written in the following form*

$$\begin{aligned} L(\mu) + \underline{\text{tr}}\underline{\chi}\mu &= \underline{\text{tr}}\underline{\chi}\bar{\rho} - \frac{1}{2}\underline{\text{tr}}\underline{\chi}\bar{\rho} - 2\zeta \cdot \beta + (\zeta - \underline{\eta}) \cdot \nabla \underline{\text{tr}}\underline{\chi} + \hat{\chi} \cdot \nabla \zeta \\ &\quad + \frac{1}{2}\hat{\chi} \cdot \nabla \underline{\eta} + \underline{\text{tr}}\underline{\chi} \left( |\zeta|^2 - \zeta \cdot \underline{\eta} - \frac{1}{2}|\underline{\eta}|^2 \right) - \frac{1}{4}\underline{\text{tr}}\underline{\chi}|\hat{\chi}|^2 \\ &\quad + 2\hat{\chi} \cdot \zeta \cdot \underline{\eta} - \frac{1}{2}\hat{\chi} \cdot \underline{\eta} \cdot \underline{\eta}. \end{aligned} \quad (2.21)$$

For convenience, we summarise the full *null structure equations in a canonical foliation*

$$\not\llcorner_L \not\mathcal{G} = 2\underline{\chi}, \quad (2.22a)$$

$$\nabla_L \underline{\text{tr}}\underline{\chi} + \frac{1}{2}(\underline{\text{tr}}\underline{\chi})^2 = -|\hat{\chi}|^2, \quad (2.22b)$$

$$\nabla_L \hat{\chi} + \underline{\text{tr}}\underline{\chi}\hat{\chi} = -\alpha, \quad (2.22c)$$

$$\nabla_L \underline{\chi} + \underline{\chi} \cdot \underline{\chi} = 2\nabla \underline{\eta} + 2\underline{\eta}\underline{\eta} + \rho \not\mathcal{G} + \sigma \not\llcorner, \quad (2.22d)$$

$$\nabla_L \underline{\text{tr}}\underline{\chi} + \frac{1}{2}\underline{\text{tr}}\underline{\chi}\underline{\text{tr}}\underline{\chi} = 2\bar{\rho} + 2|\underline{\eta}|^2, \quad (2.22e)$$

$$\nabla_L \hat{\chi} + \frac{1}{2}\underline{\text{tr}}\underline{\chi}\hat{\chi} = (\nabla \hat{\otimes} \underline{\eta}) - \frac{1}{2}\underline{\text{tr}}\underline{\chi}\hat{\chi} + (\underline{\eta} \hat{\otimes} \underline{\eta}), \quad (2.22f)$$

$$\nabla_L \zeta + \frac{1}{2}\underline{\text{tr}}\underline{\chi}\zeta = \frac{1}{2}\underline{\text{tr}}\underline{\chi}\underline{\eta} + \hat{\chi} \cdot (\underline{\eta} - \zeta) - \beta, \quad (2.22g)$$

$$\text{curl}\underline{\eta} = -\text{curl}\zeta = -\check{\sigma}, \quad (2.22h)$$

$$K = -\frac{1}{4}\underline{\text{tr}}\underline{\chi}\underline{\text{tr}}\underline{\chi} - \bar{\rho}, \quad (2.22i)$$

$$\underline{\text{div}}\hat{\chi} - \frac{1}{2}\nabla \underline{\text{tr}}\underline{\chi} = -\zeta \cdot \hat{\chi} + \frac{1}{2}\zeta \underline{\text{tr}}\underline{\chi} - \beta, \quad (2.22j)$$

$$\underline{\text{div}}\underline{\chi} - \frac{1}{2}\nabla \underline{\text{tr}}\underline{\chi} = \zeta \cdot \underline{\chi} - \frac{1}{2}\zeta \underline{\text{tr}}\underline{\chi} + \underline{\beta}, \quad (2.22k)$$

$$\underline{\text{div}}\underline{\eta} = -\bar{\rho} + \bar{\rho}, \quad (2.22l)$$

$$\text{curl}\underline{\chi} = -\zeta \cdot \hat{\chi} + \frac{1}{2}\underline{\text{tr}}\underline{\chi} \text{curl}\zeta - \text{curl}\beta, \quad (2.22m)$$

$$\Delta(\log \Omega) = -\underline{\text{div}}\zeta + \bar{\rho} - \bar{\rho}. \quad (2.22n)$$

## 2.8 Comparison of foliations

In this section, we derive equations that are used to compare a geodesic foliation and a canonical foliation starting from a common sphere  $S$ .

We first introduce the derivative of the geodesic parameter  $s$  in the canonical foliation.

**Definition 2.24.** *We define the  $S_v$ -tangent 1-form  $\Upsilon$  to be*

$$\Upsilon := \nabla s.$$

We have the following proposition.

**Proposition 2.25** (Null frame comparison). *Let  $(e'_A)_{A=1,2}$  and  $(e_A)_{A=1,2}$  be Fermi propagated null frames respectively for the geodesic and the canonical foliation, such that  $e'_A = e_A$  on  $S$ . For  $A = 1, 2$ , it holds that*

$$e'_A = e_A - \Upsilon_A L, \quad (2.23)$$

and

$$\underline{L}' = \underline{L} - 2\Upsilon_A e_A + |\Upsilon|^2 L. \quad (2.24)$$

*Proof.* It is straight-forward to verify that the vectors  $e_A - \Upsilon_A L$ ,  $A = 1, 2$ , are  $S'_s$ -tangent vector fields that coincide with  $e_{A'}$  on  $S$ . Moreover, for any  $X \in TS'_s$ , we have

$$\begin{aligned} \mathbf{g}(X, \mathbf{D}_L(e_A - \Upsilon_A L)) &= \mathbf{g}(X, \underline{\eta}_A L - L(\Upsilon_A)L) \\ &= 0. \end{aligned}$$

Thus, the vectors  $e_A - \Upsilon_A L$  are Fermi propagated with respect to the geodesic foliation and we deduce that they coincide with  $e_{A'}$  on  $\mathcal{H}$ .

One then directly checks that the vector field

$$Z := \underline{L} - 2\Upsilon_A e_A + |\Upsilon|^2 L$$

satisfies  $g(Z, e_{A'}) = g(Z, Z) = 0$  and  $g(Z, L) = -2$ .  $\square$

We have the following definition of the projection of tensors from one foliation to another, see also Section 2.2 in [AS16].

**Definition 2.26.** Let  $\phi'$  be a  $S'_s$ -tangent  $r$ -tensor. We define the projection  $(\phi')^\dagger$  to be the  $S_v$ -tangent  $r$ -tensor defined by

$$(\phi')^\dagger_{A_1 \dots A_r} = (\phi')^\dagger(e_{A_1}, \dots, e_{A_r}) := \phi'(e'_{A_1}, \dots, e'_{A_r}) = \phi'_{A_1 \dots A_r}. \quad (2.25)$$

Reciprocally, for  $\phi$  a  $S_v$ -tangent  $r$ -tensor, we define the projection  $(\phi)^\ddagger$  to be the  $S'_s$ -tangent  $r$ -tensor defined by

$$(\phi)^\ddagger_{A_1 \dots A_r} = (\phi)^\ddagger(e'_{A_1}, \dots, e'_{A_r}) := \phi(e_{A_1}, \dots, e_{A_r}) = \phi_{A_1 \dots A_r}. \quad (2.26)$$

We introduce the following projection of  $\Upsilon$ .

**Definition 2.27.** We define the  $S'_s$ -tangent 1-form  $\Upsilon'$  to be

$$\Upsilon' := -(\Upsilon)^\ddagger.$$

We have the following relation between  $\Upsilon'$  and the derivative of  $v$  in the geodesic foliation.

**Lemma 2.28.** We have

$$\Upsilon' = \Omega^{-1} \nabla' v.$$

*Proof.* We have

$$\nabla'_A v = (e_A - \Upsilon_A L)v = -\Omega \Upsilon_A = \Omega \Upsilon'_A,$$

as desired  $\square$

We have the following correspondences for  $S_v$ -tangential derivatives of projected tensors.

**Proposition 2.29** (Projection calculus). Let  $r \geq 1$  be an integer. Let  $\phi'$  be an  $S'_s$ -tangent  $r$ -tensor. Then it holds that

$$\begin{aligned} \nabla'_L (\phi')^\dagger_{A_1 \dots A_r} &= (\nabla'_L \phi')^\dagger_{A_1 \dots A_r}, \\ \nabla'_A (\phi')^\dagger_{A_1 \dots A_r} &= (\nabla'_A \phi')^\dagger_{A_1 \dots A_r} + \Upsilon_A (\nabla'_L \phi')^\dagger_{A_1 \dots A_r} \\ &\quad + \chi_{AA_i} \Upsilon_B (\phi')^\dagger_{A_1 \dots B \dots A_r} - \chi_{AB} \Upsilon_{A_i} (\phi')^\dagger_{A_1 \dots B \dots A_r} \end{aligned} \quad (2.27)$$

Similarly, for a given  $S_v$ -tangent  $r$ -tensor  $\phi$ , it holds that

$$\begin{aligned} \nabla'_L \phi^\ddagger_{A_1 \dots A_r} &= (\nabla'_L \phi)^\ddagger_{A_1 \dots A_r}, \\ \nabla'_A \phi^\ddagger_{A_1 \dots A_r} &= (\nabla'_A \phi)^\ddagger_{A_1 \dots A_r} + \Upsilon'_A (\nabla'_L \phi)^\ddagger_{A_1 \dots A_r} \\ &\quad + \chi'_{AA_i} \Upsilon'_B (\phi)^\ddagger_{A_1 \dots B \dots A_r} - \chi'_{AB} \Upsilon'_{A_i} (\phi)^\ddagger_{A_1 \dots B \dots A_r}. \end{aligned} \quad (2.28)$$

*Proof.* First, using that both frames  $(e_1, e_2)$  and  $(e'_1, e'_2)$  are Fermi propagated, we have

$$\begin{aligned}\nabla_L(\phi')^\dagger_{A_1 \dots A_r} &= L((\phi')^\dagger(e_{A_1}, \dots, e_{A_r})) \\ &= L(\phi'(e'_{A_1}, \dots, e'_{A_r})) \\ &= \nabla'_L \phi'(e'_{A_1}, \dots, e'_{A_r}) \\ &= (\nabla'_L \phi')^\dagger(e_{A_1}, \dots, e_{A_r}) \\ &= (\nabla'_L \phi')^\dagger_{A_1 \dots A_r}.\end{aligned}$$

Second,

$$\nabla_A(\phi')^\dagger_{A_1 \dots A_r} = e_A(\phi'_{A_1 \dots A_r}) - (\phi')^\dagger(e_{A_1}, \dots, \nabla_A e_{A_i}, \dots, e_{A_r}),$$

and

$$\begin{aligned}e_A(\phi'_{A_1 \dots A_r}) &= e'_A(\phi'_{A_1 \dots A_r}) + \Upsilon_A L(\phi'_{A_1 \dots A_r}) \\ &= \nabla'_A \phi'_{A_1 \dots A_r} + \phi'(e'_{A_1}, \dots, \nabla'_A e'_{A_i}, \dots, e'_{A_r}) + \Upsilon_A \nabla'_L \phi'_{A_1 \dots A_r}.\end{aligned}$$

Therefore, we deduce

$$\begin{aligned}\nabla_A(\phi')^\dagger_{A_1 \dots A_r} &= (\nabla'_L \phi')^\dagger_{A_1 \dots A_r} + \Upsilon_A (\nabla'_L \phi')^\dagger_{A_1 \dots A_r} \\ &\quad + \phi'(e'_{A_1}, \dots, \nabla'_A e'_{A_i} - g(\nabla_A e_{A_i}, e'_B) e'_B, \dots, e'_{A_r}).\end{aligned}\tag{2.29}$$

To compute the third term on the right-hand side of (2.29), write

$$\begin{aligned}g(\nabla_A e_{A_i}, e'_B) &= g\left(\mathbf{D}_A e_{A_i} - \frac{1}{2} \chi_{AA_i} \underline{L} - \frac{1}{2} \underline{\chi}_{AA_i} L, e'_B\right) \\ &= g(\mathbf{D}_A e_{A_i}, e'_B) - \chi_{AA_i} \Upsilon_B \\ &= g(\mathbf{D}_{e'_A - \Upsilon'_A L}(e'_{A_i} + \Upsilon_{A_i} L), e'_B) - \chi_{AA_i} \Upsilon_B \\ &= g(\mathbf{D}_{e'_A} e'_{A_i}, e'_B) + \Upsilon_{A_i} g(\mathbf{D}_{e'_A} L, e'_B) - \chi_{AA_i} \Upsilon_B \\ &= g(\mathbf{D}_{e'_A} e'_{A_i}, e'_B) + \Upsilon_{A_i} \chi'_{AB} - \chi_{AA_i} \Upsilon_B.\end{aligned}$$

where we used (2.5), (2.24) and the fact that both frames are Fermi propagated. Moreover, it follows from (2.23) that  $\chi'_{AA_i} = \chi_{AA_i}$ , and therefore

$$\begin{aligned}\phi'(e'_{A_1}, \dots, \nabla'_A e'_{A_i} - g(\nabla_A e_{A_i}, e'_B) e'_B, \dots, e'_{A_r}) \\ = -\Upsilon_{A_i} \chi_{AB} (\phi')^\dagger_{A_1 \dots B \dots A_r} + \chi_{AA_i} \Upsilon_B (\phi')^\dagger_{A_1 \dots A_r}.\end{aligned}\tag{2.30}$$

Plugging (2.30) into (2.29) concludes the proof of (2.27). In view of (2.23) and Lemma 2.28, the proof of (2.28) follows by replacing  $\Upsilon_A$  by  $\Upsilon'_A$ . This finishes the proof of Proposition 2.29.  $\square$

We have the following transport equation for  $\Upsilon$ .

**Lemma 2.30.** *We have*

$$\nabla_L \Upsilon = -\nabla(\log \Omega) - \chi \cdot \Upsilon.\tag{2.31}$$

*Proof.* Using commutation formula (2.10) and  $Ls = 1$ , we have

$$\begin{aligned}\nabla_L \Upsilon_A &= \nabla_L \nabla s \\ &= \nabla L(s) + [\nabla_L, \nabla]_A s \\ &= -\nabla_A(\log \Omega) - \chi_{AB} \nabla_B s,\end{aligned}$$

as desired.  $\square$

We have the following comparison between null curvature components and connection coefficients and projected null curvature components and connections coefficients. The proofs are postponed to Appendix A.



**Proposition 2.31** (Null curvature component comparison). *The following relations hold.*

$$\begin{aligned}
 \alpha_{AB} &= (\alpha')^\dagger_{AB}, \\
 \beta_A &= (\beta')^\dagger_A + \Upsilon_B (\alpha')^\dagger_{AB}, \\
 \rho &= \rho' + \Upsilon \cdot (\beta')^\dagger + \Upsilon \cdot \Upsilon \cdot (\alpha')^\dagger, \\
 \sigma &= \sigma' - \Upsilon \cdot (*\beta')^\dagger - \Upsilon \cdot \Upsilon' \cdot (*\alpha')^\dagger, \\
 \underline{\beta}_A &= (\underline{\beta}')^\dagger_A - 3\rho' \Upsilon_A + 3\sigma' * \Upsilon_A - 2(\Upsilon \cdot (*\beta')^\dagger) * \Upsilon_A \\
 &\quad + |\Upsilon|^2 \beta'_A - 2(\Upsilon \cdot \Upsilon \cdot (\alpha')^\dagger) \Upsilon_A + |\Upsilon|^2 \Upsilon \cdot (\alpha')^\dagger_A.
 \end{aligned} \tag{2.32}$$

**Proposition 2.32** (Null connection coefficients comparison). *The following relations hold.*

$$\begin{aligned}
 \chi_{AB} &= (\chi')^\dagger_{AB}, \\
 \zeta_A &= (\zeta')^\dagger_A + (\chi')^\dagger_{AB} \Upsilon_B, \\
 \underline{\eta}_A &= (\underline{\eta}')^\dagger_A + \nabla_L \Upsilon_A, \\
 \underline{\chi}_{AB} &= (\underline{\chi}')^\dagger_{AB} + 2\Upsilon_A (\underline{\eta}')^\dagger_B - 2\Upsilon_B (\zeta')^\dagger_A + 2\nabla_A \Upsilon_B - |\Upsilon|^2 \chi'_{AB}.
 \end{aligned} \tag{2.33}$$

## 2.9 Norms on $\mathcal{H}$

In this section, we define norms on  $\mathcal{H}$ . Throughout this section, we denote by  $(\tilde{S}_{\tilde{v}})_{1 \leq \tilde{v} \leq v^*}$  either the geodesic foliation  $(S'_s)$  or the canonical foliation  $(S_v)$ .

**Definition 2.33** ( $\tilde{S}_{\tilde{v}}$ -mixed norms). *Let  $v^* \geq 1$ . Let  $F$  be an  $\tilde{S}_{\tilde{v}}$ -tangent tensor. We define the mixed norms on  $\mathcal{H}$  with respect to the foliation  $(\tilde{S}_{\tilde{v}})_{\tilde{v} \in [1, v^*]}$ ,*

$$\begin{aligned}
 \|F\|_{L^p_{\tilde{v}}([1, v^*])L^q} &:= \left( \int_1^{v^*} \|F\|_{L^q(\tilde{S}_{\tilde{v}})}^p d\tilde{v} \right)^{\frac{1}{p}}, \\
 \|F\|_{L^q L^p_{\tilde{v}}([1, v^*])} &:= \left\| \left( \int_1^{v^*} |F|^p d\tilde{v} \right)^{\frac{1}{p}} \right\|_{L^q(S_1)}.
 \end{aligned}$$

**Definition 2.34.** *Let  $v^* \geq 1$  and let  $F$  be an  $\tilde{S}_{\tilde{v}}$ -tangent tensor. Define*

$$\mathcal{N}_1^{\tilde{v}, [1, v^*]}(F) := \|F\|_{H^{1/2}(S_1)} + \|F\|_{L^2_{\tilde{v}}([1, v^*])L^2} + \|\tilde{\nabla} F\|_{L^2_{\tilde{v}}([1, v^*])L^2} + \|\tilde{\nabla}_L F\|_{L^2_{\tilde{v}}([1, v^*])L^2},$$

where  $\tilde{\nabla}$  and  $\tilde{\nabla}_L$  denote the induced covariant derivatives on  $\tilde{S}_{\tilde{v}}$ . Moreover, for  $m \geq 1$ , define

$$\mathcal{N}_m^{\tilde{v}, [1, v^*]}(F) := \sum_{k \leq m-1} \|\tilde{\nabla}^k F\|_{H^{1/2}(S_1)} + \sum_{k \leq m} \|\tilde{\nabla}^k F\|_{L^2_{\tilde{v}}([1, v^*])L^2},$$

where  $\tilde{\nabla} \in \{\tilde{\nabla}, \tilde{\nabla}_L\}$ . We refer to Section 3.2 for a precise definition of the space of tensors  $H^{1/2}(S_1)$ .

## 2.10 Weak regularity of 2-spheres

In this section, we define the weak regularity assumption on  $S_1$ , see [Sha14] Section 2.4.

**Definition 2.35.** *Let  $N \geq 1$  be an integer and  $C > 0$  a real number. A Riemannian 2-sphere  $(S, \mathcal{g})$  is weakly regular with constants  $N, c$  if it can be covered by  $N$  coordinate patches  $(x^1, x^2)$  with a partition of unity  $\eta$  adapted to the coordinate patches and with functions  $0 \leq \tilde{\eta} \leq 1$  that are compactly supported in the patches and equal to 1 on the support of  $\eta$ , and if on each patch there exists an orthonormal frame  $(e_1, e_2)$  such that for  $a, b = 1, 2$  and  $A = 1, 2$ ,*

$$c^{-1} \leq \sqrt{\det \mathcal{g}} \leq c, \tag{2.34}$$

$$c^{-1}((\xi^1)^2 + (\xi^2)^2) \leq \mathcal{g}_{ab} \xi^a \xi^b \leq c((\xi^1)^2 + (\xi^2)^2), \quad \forall (\xi^1, \xi^2) \in \mathbb{R}^2, \tag{2.35}$$

$$|\partial_{x^a} \eta| + |\partial_{x^a} \partial_{x^b} \eta| + |\partial_{x^a} \tilde{\eta}| \leq c, \tag{2.36}$$

$$\|\tilde{\nabla} \partial_{x^a}\|_{L^2(S)} + \|\tilde{\nabla} e_A\|_{L^4(S)} \leq c. \tag{2.37}$$

## 2.11 Norms for the geodesic and canonical foliation geometry

In this section, we introduce norms to measure the geometry of the geodesic foliation and the canonical foliation on  $\mathcal{H}$  at the level of bounded  $L^2$  curvature. The definitions of the Besov spaces  $B^0(S)$  and fractional Sobolev space  $H^{1/2}(S)$  are postponed to Section 3.2.

**Norms for null connection coefficients of the geodesic foliation on  $S_1$ .**

$$\begin{aligned} \mathcal{I}'_{S_1} := & \|\mathrm{tr}\chi' - 2\|_{L^\infty(S_1)} + \|\nabla\mathrm{tr}\chi'\|_{B^0(S_1)} + \|\mathrm{tr}\underline{\chi}' + 2\|_{L^\infty(S_1)} + \|\nabla\mathrm{tr}\underline{\chi}'\|_{L^2(S_1)} \\ & + \|\mu'\|_{B^0(S_1)} + \|\zeta'\|_{H^{1/2}(S_1)} + \|\hat{\chi}'\|_{H^{1/2}(S_1)} + \|\hat{\underline{\chi}}'\|_{H^{1/2}(S_1)}. \end{aligned}$$

**Norms for null connection coefficients of the canonical foliation on  $S_1$ .**

$$\begin{aligned} \mathcal{I}_{S_1} := & \|\mathrm{tr}\chi - 2\|_{L^\infty(S_1)} + \|\mathrm{tr}\underline{\chi} + 2\|_{L^\infty(S_1)} + \|\nabla\mathrm{tr}\chi\|_{B^0(S_1)} + \|\nabla\mathrm{tr}\underline{\chi}\|_{L^2(S_1)} \\ & + \|\mu\|_{B^0(S_1)} + \|\zeta\|_{H^{1/2}(S_1)} + \|\hat{\chi}\|_{H^{1/2}(S_1)} + \|\hat{\underline{\chi}}\|_{H^{1/2}(S_1)} \\ & + \|\nabla\log\Omega\|_{H^{1/2}(S_1)} + \|\underline{\eta}\|_{H^{1/2}(S_1)} + \|\log\Omega\|_{L^2(S_1)} + \|\Omega - 1\|_{L^\infty(S_1)} + \|\mu\|_{L^2(S_1)}. \end{aligned}$$

**Norms for null connection coefficients of the geodesic foliation on  $\mathcal{H}$ .**

$$\begin{aligned} \mathcal{O}'_{[1,s^*]} := & \left\| \mathrm{tr}\chi' - \frac{2}{s} \right\|_{L^\infty_{s^*}([1,s^*])L^\infty} + \|\hat{\chi}'\|_{L^\infty L^2_{s^*}([1,s^*])} + \|\zeta'\|_{L^\infty L^2_{s^*}([1,s^*])} \\ & + \mathcal{N}_1^{s,[1,s^*]} \left( \mathrm{tr}\chi' - \frac{2}{s} \right) + \mathcal{N}_1^{s,[1,s^*]}(\hat{\chi}') + \mathcal{N}_1^{s,[1,s^*]}(\zeta'). \end{aligned}$$

**Norms for null connection coefficients of the canonical foliation on  $\mathcal{H}$ .**

$$\begin{aligned} \mathcal{O}_{[1,v^*]} := & \mathcal{N}_1^{v,[1,v^*]} \left( \mathrm{tr}\chi - \frac{2}{v} \right) + \mathcal{N}_1^{v,[1,v^*]}(\hat{\chi}) + \mathcal{N}_1^{v,[1,v^*]}(\zeta) + \mathcal{N}_1^{v,[1,v^*]}(\underline{\eta}) \\ & + \mathcal{N}_1^{v,[1,v^*]} \left( \mathrm{tr}\underline{\chi} + \frac{2}{v} \right) + \mathcal{N}_1^{v,[1,v^*]}(\hat{\underline{\chi}}) \\ & + \|\Omega - 1\|_{L^\infty_{v^*}([1,v^*])L^\infty} + \|L(\log\Omega)\|_{L^2_{v^*}([1,v^*])L^4} + \mathcal{N}_1^{v,[1,v^*]}(\nabla\log\Omega) \\ & + \left\| \mathrm{tr}\chi - \frac{2}{v} \right\|_{L^\infty_{v^*}([1,v^*])L^\infty} + \|\hat{\chi}\|_{L^\infty L^2_{v^*}([1,v^*])} + \|\zeta\|_{L^\infty L^2_{v^*}([1,v^*])} \\ & + \|\underline{\eta}\|_{L^\infty L^2_{v^*}([1,v^*])} + \left\| \mathrm{tr}\underline{\chi} + \frac{2}{v} \right\|_{L^\infty_{v^*}([1,v^*])L^\infty} \\ & + \|\nabla\mathrm{tr}\underline{\chi}\|_{L^2 L^\infty_{v^*}([1,v^*])} + \|\mu\|_{L^2 L^\infty_{v^*}([1,v^*])} + \|\nabla\mathrm{tr}\chi\|_{L^2 L^\infty_{v^*}([1,v^*])}. \end{aligned}$$

**Norms for null curvature components of the geodesic foliation on  $\mathcal{H}$ .**

$$\begin{aligned} \mathcal{R}'_{[1,s^*]} := & \|\alpha'\|_{L^2_{s^*}([1,s^*])L^2} + \|\beta'\|_{L^2_{s^*}([1,s^*])L^2} + \|\rho'\|_{L^2_{s^*}([1,s^*])L^2} \\ & + \|\sigma'\|_{L^2_{s^*}([1,s^*])L^2} + \|\underline{\beta}'\|_{L^2_{s^*}([1,s^*])L^2}. \end{aligned}$$

**Norms for null curvature components of the canonical foliation on  $\mathcal{H}$ .**

$$\begin{aligned} \mathcal{R}_{[1,v^*]} := & \|\alpha\|_{L^2_{v^*}([1,v^*])L^2} + \|\beta\|_{L^2_{v^*}([1,v^*])L^2} + \|\rho\|_{L^2_{v^*}([1,v^*])L^2} \\ & + \|\sigma\|_{L^2_{v^*}([1,v^*])L^2} + \|\underline{\beta}\|_{L^2_{v^*}([1,v^*])L^2}. \end{aligned}$$

## 2.12 Main results

The following is the main result of this paper.

**Theorem 2.36** (Existence and control of the canonical foliation, version 2). *Let  $(\mathcal{M}, \mathbf{g})$  be a smooth spacetime and  $\mathcal{H}$  be a smooth null hypersurface emanating from a spacelike 2-sphere  $S$ . Assume that the geodesic foliation  $(S'_s)$  starting at  $S$  with  $s = 1$  is well-defined and smooth up to  $s = 5/2$ . Let  $N \geq 1$  be an integer and  $c > 0$  be a real number and assume that  $S$  is weakly regular with constants  $N, c$ . Assume moreover that for some  $\varepsilon > 0$ ,*

$$\mathcal{I}'_{S_1} + \mathcal{R}'_{[1,5/2]} + \mathcal{O}'_{[1,5/2]} \leq \varepsilon. \quad (2.38)$$

Then, there is a universal constant  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , the following holds.

1. **Existence of the canonical foliation.** *The canonical foliation (see Definition 2.17) is well-defined from  $v = 1$  to  $v = 2$ , and we have the following comparison estimate with respect to the geodesic foliation,*

$$\|\Omega - 1\|_{L^\infty_{[1,2]}L^\infty} \lesssim \varepsilon, \quad \|\Upsilon\|_{L^\infty_{[1,2]}L^\infty} \lesssim \varepsilon. \quad (2.39)$$

2.  **$L^2$ -regularity.** *There is a constant  $C = C(N, c) > 0$  such that the canonical foliation is uniformly weakly regular with constants  $N, C$  for  $v = 1$  to  $v = 2$ , and moreover,*

$$\mathcal{I}_{S_1} + \mathcal{R}_{[1,2]} + \mathcal{O}_{[1,2]} \lesssim \varepsilon. \quad (2.40)$$

3. **Smoothness.** *The canonical foliation is smooth up to  $v = 2$ .*

**Remarks.**

1. The comparison estimate (2.39) implies in particular that  $|s - v| \lesssim \varepsilon$ , so that the foliations remain close to each other.
2. Using the conclusions of [KR05], we have that under the assumption  $\mathcal{I}'_{S_1} + \mathcal{R}'_{[1,5/2]} \leq \varepsilon$ , control of the geodesic connection coefficient norm  $\mathcal{O}'_{[1,5/2]} \lesssim \varepsilon$  can be obtained. The smallness hypothesis (2.38) can therefore be replaced by  $\mathcal{I}'_{S_1} + \mathcal{R}'_{[1,5/2]} \leq \varepsilon$  and involves only  $L^2$ -norms of curvature components on  $\mathcal{H}$  and low regularity connection coefficients bounds on  $S_1$ .
3. Using the a priori estimates from the previous remark, together with a topological assumption on the hypersurface  $\mathcal{H}$  could lead to existence and non-degeneracy for the geodesic foliation from  $s = 1$  to  $s = 5/2$ . For simplicity, we rather make the existence and smoothness of the geodesic foliation from  $s = 1$  to  $s = 5/2$  an assumption.
4. The null curvature components are essentially invariant by a change of foliation (see Proposition 2.31) and so is the smallness assumption  $\mathcal{R} \leq \varepsilon$ . Regarding the previous remarks, it is consistent however to assume it for the geodesic foliation, since we rely on its existence and on the control on the connection coefficient norm  $\mathcal{O}'_{[1,5/2]} \lesssim \varepsilon$  to obtain existence for the canonical foliation together with bounds for the corresponding canonical foliation connection coefficients.
5. The equations for the canonical foliation reduce to a system of coupled quasilinear elliptic and transport equations on  $\mathcal{H}$  (see equations (2.22b)-(2.22n)), having curvature components as source terms, which are essentially invariant under a change of foliation. Thus, Theorem 2.36 can be seen as a small data time 1 existence result for which the smallness is measured in terms of  $L^2(\mathcal{H})$ -norms of the null curvature components.
6. The desired bound for  $\text{tr}\chi$  is obtained as part of the estimates (2.40), which are all needed to obtain existence and control of the canonical foliation on the interval  $1 \leq v \leq 2$ .
7. Here and in the rest of the paper, *smooth* means  $C^\infty$  with respect to the  $C^\infty$ -topology of the manifolds  $\mathcal{M}, \mathcal{H}$ , etc.
8. The smoothness of the canonical foliation is a consequence of the smoothness of the geodesic foliation and is obtained by higher regularity comparison estimates (see Step 3 in Section 2.13 and Section 5). Since we only work with smooth foliations, we did not seek any sharpness in these higher regularity estimates. For example in the proof of Proposition 2.40, we assume  $C^k$ -regularity with  $k$  arbitrarily large of the geodesic foliation to prove  $C^{k'}$ -regularity with  $k' \ll k$  of the canonical foliation.

## 2.13 Proof of Theorem 2.36

The proof of Theorem 2.36 relies on a bootstrap argument which we set up and prove in this section, assuming for the moment the local existence result and estimates that will be proved in Sections 4, 5 and 6.

Let  $D > 0$  be a fixed (large) constant and let  $v^* \in [1, 2]$ . We say that a foliation  $(S_v)_{1 \leq v \leq v^*}$  satisfies the *bootstrap assumptions*  $BA_{D\varepsilon, [1, v^]}$ , if

$$\|\Omega - 1\|_{L_v^\infty([1, v^*])L^\infty} + \|\Upsilon\|_{L_v^\infty([1, v^*])L^\infty} + \mathcal{O}_{[1, v^*]} \leq D\varepsilon.$$

We say that a foliation  $(S_v)_{1 \leq v \leq v^*}$  is *regular* if  $v$  is a  $C^1$ -function and if

$$\sum_{l \leq 5} \left( \|\nabla^l s\|_{L_v^\infty([1, v^*])L^2} + \|\nabla^l(\Omega - 1)\|_{L_v^\infty([1, v^*])L^2} \right) < \infty.$$

We define  $V \in [1, 2]$  as

$$V := \sup_{v^* \in [1, 2]} \left\{ \begin{array}{l} \text{There exists a regular function } v \text{ on } \mathcal{H} \text{ taking values} \\ \text{from } 1 \text{ to } v^* \text{ and such that the assumptions } BA_{D\varepsilon, [1, v^*]} \text{ are satisfied} \end{array} \right\},$$

and we show in the rest of this section that  $V = 2$ .

**Step 1** *It holds that  $V > 1$ .* Indeed, this follows by the next local existence result.

**Theorem 2.37** (Local existence and continuation for the canonical foliation). *Let  $(\mathcal{M}, \mathbf{g})$  be a smooth vacuum spacetime and  $\mathcal{H} \subset \mathcal{M}$  a smooth null hypersurface foliated by a well-defined and smooth geodesic foliation  $(S'_s)_{1 \leq s \leq s^*}$ . Let  $v^* \in [1, 2]$  be a real number. Assume there exists a  $C^1$ -function  $v$  on  $\mathcal{H}$  taking values from 1 to  $v^*$  and defining a canonical foliation. Assume moreover that*

$$\|s\|_{H^5(S_{v^*})} < \infty, \quad \|\Omega - 1\|_{L^\infty(S_{v^*})} < \frac{1}{100}$$

*and that  $S_{v^*}$  is close to the Euclidean 2-sphere in a weak sense (see Definition 3.14). There exists  $\delta > 0$  and a  $C^1$ -function  $v$  taking values from 1 to  $v^* + \delta$ , coinciding with  $v$  on  $\{1 \leq v \leq v^*\}$  and such that  $(S_v)_{1 \leq v \leq v^* + \delta}$  is a canonical foliation.*

### Remarks on Theorem 2.37

1. In Theorem 2.37, the time of existence  $\delta$  depends on  $\sum_{l \leq 5} \|\nabla^l(s - v^*)\|_{L^2(S_{v^*})}$  and  $\|g'\|_{C^5(\mathcal{H})}$  (see Section 6).
2. The proof of Theorem 2.37 is made by a fixed-point argument for a more general system of coupled quasilinear elliptic and transport equations and is detailed in Section 6. The required assumption  $|\Omega - 1| < 1/100$  at the initial sphere  $v = s = 1$  and its weak sphericity are consequences of the low regularity bounds (2.42) proved in Proposition 2.38 (see also Section 4.1 and Remark 4.1).
3. A local existence result for canonical foliations is proved in [Nic04] but under the stronger smallness assumption

$$\mathcal{R}'_2 \leq \varepsilon,$$

where  $\mathcal{R}'_2$  contain  $L^\infty$ -norms of curvature components on  $\mathcal{H}$ . Similarly, a local existence result was proved in [Sau08] for general foliations under  $L^\infty(\mathcal{H})$ -smallness assumptions on the curvature. As such smallness conditions can not be assumed in our low regularity setting, we give a new proof of a stronger local existence result.

**Step 2** For  $v^* \in [1, 2]$  we can improve  $BA_{D\varepsilon, [1, v^*]}$  to  $BA_{D'\varepsilon, [1, v^*]}$  for a real number  $D' < D$ . We first show that the assumptions of Theorem 2.36 imply that the canonical connection coefficients are controlled on  $S_1$ .

**Proposition 2.38** (Connection coefficients bounds on  $S_1$ ). *Assume that for some real  $\varepsilon > 0$ ,*

$$\mathcal{I}'_{S_1} \leq \varepsilon. \quad (2.41)$$

There exists  $\varepsilon_0 > 0$  small such that if  $0 < \varepsilon < \varepsilon_0$ , then we have

$$\mathcal{I}_{S_1} \lesssim \varepsilon. \quad (2.42)$$

The proof of Proposition 2.38 is carried out in Section 4.1 and goes by direct comparison between the geodesic and the canonical connection coefficients. Namely most of the coefficients are identical since the first 2-spheres of the two foliations coincide,  $S'_{s=1} = S = S_{v=1}$ .

Then the next proposition shows that we can improve the bootstrap assumptions.

**Proposition 2.39** (Low regularity estimates). *Assume that for some real number  $\varepsilon > 0$  and that*

$$\mathcal{R}'_{[1, 5/2]} + \mathcal{O}'_{[1, 5/2]} \leq \varepsilon, \quad \mathcal{I}_{S_1} \leq \varepsilon.$$

Let  $1 < v^* \leq 2$  and assume that the canonical foliation  $(S_v)_{1 \leq v \leq v^*}$  is regular and satisfies the bootstrap assumptions  $BA_{D\varepsilon, [1, v^*]}$  with  $D > 0$  a fixed constant. There exists  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$ , then the canonical foliation satisfies the bootstrap assumptions  $BA_{D'\varepsilon, [1, v^*]}$  for a real number  $D' < D$ .

Proposition 2.39 is proved in Section 4. The first step is to show that under the bootstrap assumptions  $BA_{D\varepsilon, [1, v^*]}$ , the null curvature components in the canonical foliation are comparable to the geodesic null curvature components

$$\mathcal{R}_{[1, v^*]} \lesssim \mathcal{R}'_{[1, 5/2]}. \quad (2.43)$$

Then, under the bounds (2.43) and weak regularity of  $S_1$  with constants  $N, c$ , we can show that the foliation  $(S_v)$  is uniformly weakly regular and spherical with constants depending only on  $N, c, \varepsilon$  (see Definitions 3.1 and 3.14). At this level of regularity, calculus inequalities can be derived on  $\mathcal{H}$  with constants depending only on  $N, c, \varepsilon$ . Using these inequalities together with the null structure equations (2.22b)-(2.22n), the bounds obtained on  $S_1$  (2.42) and the obtained bounds for the null curvature components (2.43), it follows that there exists  $\varepsilon_0 > 0$  small enough such that if  $0 < \varepsilon < \varepsilon_0$ , then the bootstrap assumptions  $BA_{D\varepsilon, [1, v^*]}$  can be improved to  $BA_{D'\varepsilon, [1, v^*]}$ .

**Step 3** The canonical foliation is regular on  $[1, V]$ . Indeed, we show more generally the following proposition.

**Proposition 2.40** (Higher regularity comparison estimates). *Assume that the geodesic foliation  $(S'_s)$  is smooth and well-defined from  $1 \leq s \leq 5/2$ , and assume that for  $1 < v^* < 2$  and for some real number  $\varepsilon > 0$  the canonical foliation  $(S_v)_{1 \leq v \leq v^*}$  is regular and satisfies the bootstrap assumptions  $BA_{\varepsilon, [1, v^*]}$ . There exists  $\varepsilon_0 > 0$  such that if  $0 < \varepsilon < \varepsilon_0$  then, we have for all integers  $m \geq 0$*

$$\sum_{l+k \leq m} \|\nabla^l_L \nabla^k(s-v)\|_{L^\infty_{[1, v^*]} L^2} \lesssim C \left( \|\mathcal{G}'\|_{C^{m+2}(\mathcal{H})}, m \right).$$

The proof of Proposition 2.40 goes by standard Grönwall argument and is carried out in Section 5. In particular, for  $m = 6$ , this gives the desired regularity result in our continuity argument.

By continuity, we therefore deduce that the canonical foliation is regular on the full interval  $[1, V]$ . Additionally, using these higher regularity estimates, one can deduce the smoothness of the canonical foliation.

**Step 4** The foliation can be continued past  $V$  for  $V < 2$ . Using the estimates from Step 2 and 3, the assumptions of the local existence and continuation Theorem 2.37 are satisfied and therefore we deduce that the canonical foliation can be extended past  $V$  which therefore implies that  $V \geq 2$ .

## 2.14 Organisation of the paper

The rest of the paper is organised as follows.

- ◇ In Section 3, we state the calculus prerequisites that hold under weak regularity conditions for a foliation on  $\mathcal{H}$ .
- ◇ Section 4 is dedicated to the proof of the low regularity bounds for the canonical connection coefficients on the sphere  $S_1$  and the improvement of the bootstrap assumptions.
- ◇ Section 5 is dedicated to the proof of higher regularity bounds for the canonical foliation.
- ◇ Section 6 is dedicated to the proof of the local existence Theorem 2.37.



### 3. Calculus prerequisites

In this section, we state the necessary calculus prerequisites for Sections 4, 5 and 6. The results are based on the pioneering works [KR05], [KR06a] and [KR06b] (see also [Wan09]), with further improvements and simplifications taken from [Sha14], whose presentation we shall follow and whose calculus results we shall use as a black box.

#### 3.1 Uniform weak regularity of foliations

We now state the definition of uniform weak regularity for a foliation which allows to develop uniform calculus on the leaves of the foliation (see [Sha14] Sections 3.3 and 4.3).

**Definition 3.1.** *Let  $N \geq 1$  be an integer and  $C > 0$  a real number. Let  $v^* > 1$  be a real number. We say that a foliation  $(S_v)_{1 \leq v \leq v^*}$  on  $\mathcal{H}$  is uniformly weakly regular with constants  $N, C$ , if the 2-sphere  $S_1$  is weakly regular with constants  $N, C$  in the sense of Definition 2.35, the following bounds are satisfied*

$$\begin{aligned} \|\Omega - 1\|_{L_v^\infty L^\infty} &\leq 1/10, \\ \|\mathrm{tr}\chi\|_{L_v^\infty L^\infty} &\leq C, \\ \|\hat{\chi}\|_{L^\infty L_v^2} &\leq C, \\ \mathcal{N}_1(\chi) + \mathcal{N}_1(\Omega) &\leq C, \end{aligned} \tag{3.1}$$

and there exists a  $S_v$ -tangent 3-tensor  $\Psi$  satisfying

$$\nabla_L \Psi_{ABC} = \Omega \nabla_A (\Omega^{-1} \chi_{BC}) - \Omega \nabla_C (\Omega^{-1} \chi_{BA}), \tag{3.2}$$

such that

$$\|\Psi\|_{L^4 L_v^\infty} \leq C. \tag{3.3}$$

**Remarks.**

1. For simplicity these assumptions are stronger than necessary and imply in particular assumptions **(F2)** of Section 4.5 in [Sha14] with constants  $N, C$  and  $B \equiv C$ , since the tensor  $k$  in [Sha14] reads in the present paper  $k \equiv \Omega^{-1} \chi$ .
2. Under the assumptions (3.1) (3.2) (3.3), one can deduce that each 2-sphere  $S_v$  is weakly regular in the sense of Definition 2.35 with uniform constants  $N, C'$ , where  $C'(N, C) > 0$  (see Proposition 4.13 in [Sha14]).
3. The assumption (3.3) is designed so that using the local frames  $(e_A)_{A=1,2}$  defined on the first 2-sphere  $S$ , the regularity of their associated Fermi propagated frames on  $\mathcal{H}$  is transported, *i.e.*  $\|\nabla e_A\|_{L^4 L_v^\infty} \leq C$ . This regularity is then sufficient in [Sha14] for running a scalarisation procedure for tensorial estimates and then comparing geometric Besov norms for scalars to coordinate-based Besov norms (see Sections 4,5 and Appendix A in that paper).
4. From the weak regularity of each 2-sphere  $S_v$ , we deduce in particular that

$$C^{-1} \lesssim \sqrt{\det(\not{g})} \lesssim C,$$

uniformly on  $\mathcal{H}$ . As a consequence, for all  $S_v$ -tangent tensor  $F$  and for all  $1 \leq p \leq q \leq \infty$ , we have

$$\|F\|_{L^q L_v^p} \lesssim \|F\|_{L_v^p L^q},$$

where the constant depends only on  $N, C$ .

**Notations.** Here and in the rest of this section, we take out any reference to  $v^*$  in the  $L^p L^q$ -norms for ease of notation and we moreover denote  $\mathcal{N}_m := \mathcal{N}_m^v$ .



### 3.2 Littlewood-Paley theory and Besov spaces

In this section, we define Littlewood-Paley projections and Besov spaces on Riemannian 2-spheres  $(S, g)$ .

Let  $\Delta$  denote the Laplacian on  $(S, g)$ . Interpreting  $-\Delta$  as a positive self-adjoint unbounded operator acting on tensors in  $L^2(S)$ , we have the spectral decomposition (see [Sha14] for details)

$$-\Delta = \int_0^\infty \lambda dE_\lambda.$$

We define the corresponding Littlewood-Paley operators as follows.

- ◇ Let  $\phi \in C^\infty(\mathbb{R})$  be a function such that  $\text{supp } \phi \subset \{1/2 \leq |\xi| \leq 2\}$  and

$$\sum_{k \in \mathbb{Z}} \phi(2^{-2k} \xi) = 1 \text{ for all } \xi \in \mathbb{R} \setminus \{0\}.$$

- ◇ For each  $k \in \mathbb{Z}$ , define the Littlewood-Paley operator acting on tensors in  $L^2(S)$  by

$$P_k = \phi(-2^{-2k} \Delta), \quad P_- = \delta_{\{0\}}(-\Delta),$$

where  $\delta_{\{0\}}(-\Delta)$  denotes the  $L^2$ -projection onto the kernel of  $-\Delta$ .

- ◇ For  $k \in \mathbb{Z}$ , define the aggregated operators

$$P_{<k} = P_- + \sum_{l < k} P_l,$$

where the summation is in the strong operator topology. In particular,

$$P_{<0} + \sum_{k \geq 0} P_k = 1. \tag{3.4}$$

Using the Littlewood-Paley operators, we next define Besov spaces.

**Definition 3.2** (Geometric tensorial Besov space). *For a  $S$ -tangent tensor  $F$  we define the norms*

$$\|F\|_{B^0(S)} := \sum_{k \geq 0} \|P_k F\|_{L^2(S)} + \|P_{<0} F\|_{L^2(S)},$$

We have moreover the following  $v$ -integrated Besov spaces.

**Definition 3.3** (Geometric tensorial  $v$ -integrated Besov spaces). *Define for a  $S_v$ -tangent tensor  $F$  on  $\mathcal{H}$ ,*

$$\begin{aligned} \|F\|_{\mathcal{P}_v^0} &:= \sum_{k \geq 0} \|P_k F\|_{L_v^2 L^2} + \|P_{<0} F\|_{L_v^2 L^2}, \\ \|F\|_{\mathcal{Q}_v^{1/2}} &:= \left( \sum_{k \geq 0} 2^k \|P_k F\|_{L_v^\infty L^2}^2 + \|P_{<0} F\|_{L_v^\infty L^2}^2 \right)^{1/2}. \end{aligned}$$

**Remark 3.4.** *The space  $B^0(S)$  corresponds to the  $L^2(S)$ -based Besov space on  $S$  with parameters  $s = 0$  and  $a = 1$  in [Sha14].*

*The  $v$ -integrated spaces  $\mathcal{P}_v^0$  and  $\mathcal{Q}_v^{1/2}$  correspond to the  $L^2(S)$ -based  $v$ -integrated Besov space with parameters respectively  $s = 0$ ,  $a = 1$  and  $p = 2$  and  $s = 1/2$ ,  $a = 2$  and  $p = \infty$  in [Sha14].*

Finally, set for real numbers  $s \in \mathbb{R}$  and  $S$ -tangent tensors  $F$ ,

$$\|F\|_{H^s(S)} := \|(I - \Delta)^{s/2} F\|_{L^2(S)},$$

where the fractional Laplace operator is defined as in [Sha14]. With this definition, we have

$$\|F\|_{H^1(S)} \simeq \|\nabla F\|_{L^2(S)} + \|F\|_{L^2(S)}.$$

The next lemma is proved in Appendix B.

**Lemma 3.5.** *For an  $S$ -tangent tensor  $F$ , we have*

$$\|\nabla F\|_{H^{-1/2}(S)} \lesssim \|F\|_{H^{1/2}(S)}.$$

### 3.3 Sobolev inequalities on 2-spheres

The next lemma is proved in Section 2.5 in [Sha14].

**Lemma 3.6** (Classical Sobolev inequalities on  $S$ ). *Let  $(S, \mathfrak{g})$  be weakly regular 2-sphere with constants  $N, C$ . Then for a scalar function  $f$  and for an  $S$ -tangent tensor  $F$ , we have*

$$\begin{aligned} \|F\|_{L^4(S)} &\lesssim \|\nabla F\|_{L^2(S)}^{1/2} \|F\|_{L^2(S)}^{1/2} + \|F\|_{L^2(S)}, \\ \|F\|_{L^\infty(S)} &\lesssim \|\nabla^2 F\|_{L^2(S)}^{1/2} \|F\|_{L^2(S)}^{1/2} + \|F\|_{L^2(S)}, \\ \|F\|_{L^\infty(S)} &\lesssim \|\nabla F\|_{L^4(S)} + \|F\|_{L^4(S)}, \end{aligned}$$

where the constants depend only on  $N, C$ .

The next lemma follows from Proposition 3.3 in [Sha14].

**Lemma 3.7** (Besov-Sobolev inequalities on  $S$ ). *Let  $(S, \mathfrak{g})$  be a weakly regular 2-sphere with constants  $N, C$ . Then for an  $S$ -tangent tensor  $F$ , we have*

$$\|F\|_{L^4(S)} \lesssim \|F\|_{H^{1/2}(S)},$$

where the constant depends only on  $N, C$ .

### 3.4 Sobolev inequalities on $\mathcal{H}$

Let  $\mathcal{H}$  be a null hypersurface.

**Lemma 3.8** (Classical Sobolev inequalities on  $\mathcal{H}$ ). *Let  $(S_v)$  be a uniformly weakly regular foliation on  $\mathcal{H}$  with constants  $N, C$ . Then for an  $S_v$ -tangent tensor  $F$  on  $\mathcal{H}$ ,*

$$\begin{aligned} \|F\|_{L^4 L_v^\infty} &\lesssim \mathcal{N}_1(F), \\ \|F\|_{L_v^6 L^6} &\lesssim \mathcal{N}_1(F), \\ \|F\|_{L_v^\infty L^\infty} &\lesssim \mathcal{N}_1(\nabla F) + \mathcal{N}_1(F), \\ \|F\|_{L_v^2 L^4} &\lesssim \|\nabla F\|_{L_v^2 L^2}^{1/2} \|F\|_{L_v^2 L^2}^{1/2} + \|F\|_{L_v^2 L^2}. \end{aligned}$$

All constants in these estimates depend only on  $N, C$ .

We have the following Besov-Sobolev estimate, see Proposition 5.3 in [Sha14].

**Lemma 3.9** (Besov-Sobolev inequalities on  $\mathcal{H}$ ). *Let  $(S_v)$  be a uniformly weakly regular foliation on  $\mathcal{H}$  with constants  $N, C$ . Let  $F$  be a  $S_v$ -tangent tensor. We have*

$$\|F\|_{\mathcal{Q}_v^{1/2}} \lesssim \mathcal{N}_1(F),$$

where the constant depends only on  $N, C$ .

We have the following product estimate in Besov spaces, see Theorem 3.6 in [Sha14].

**Lemma 3.10** (Besov product estimates). *Let  $(S_v)$  be a uniformly weakly regular foliation on  $\mathcal{H}$  with constants  $N, C$ . Let  $F$  and  $G$  be two  $S_v$ -tangent tensors. We have*

$$\|FG\|_{\mathcal{P}_v^0} \lesssim \mathcal{N}_1(F)(\|G\|_{L_v^2 L^2} + \|\nabla G\|_{L_v^2 L^2}),$$

where the constant depends only on  $N, C$ .

### 3.5 Null transport equations on $\mathcal{H}$

We have the following  $L^p L_v^\infty$ -estimates for solutions of null transport equations. .

**Lemma 3.11** ( $L^p L_v^\infty$ -estimates for transport equations). *Let  $(S_v)$  be a uniformly weakly regular foliation of  $\mathcal{H}$  with constants  $N, C$ . Let  $\kappa$  be a real number. For  $F$  an  $S_v$ -tangent tensor satisfying on  $\mathcal{H}$*

$$\nabla_L F + \kappa \text{tr} \chi F = W,$$

and for all  $1 \leq p \leq \infty$ , we have

$$\|F\|_{L^p L_v^\infty} \lesssim \|F\|_{L^p(S_1)} + \|W\|_{L^p L_v^1},$$

where the constant depends only on  $N, C, p, \kappa$ .

*Proof.* The proof follows by applying Proposition 4.6 in [Sha14] to the transport equation obtained for the renormalised quantity  $\exp\left(\int_1^v \Omega^{-1} \text{tr} \chi \, dv'\right) F$  and using the uniform weak regularity bounds (3.1) for  $\Omega$  and  $\text{tr} \chi$ . Details are left to the reader.  $\square$

**Remark 3.12.** Proposition 4.6 in [Sha14] does not require assumption (3.3). This will be used when proving that assumption (3.3) holds.

### 3.6 $L^\infty L_v^2$ trace estimate

By Theorem 5.7 in [Sha14], we have the next trace estimate.

**Lemma 3.13** (Trace estimate). *Let  $(S_v)$  be a uniformly weakly regular foliation with constants  $N, C$  and let  $F$  be a  $S_v$ -tangent tensor such that*

$$\nabla F = \nabla_L P + E.$$

Then

$$\|F\|_{L^\infty L_v^2} \lesssim \mathcal{N}_1(P) + \|E\|_{\mathcal{P}_0} + \mathcal{N}_1(F),$$

where the constant only depends on  $N, C$ .

*Proof.* This is Theorem 5.7 in [Sha14] together with the bounds (3.1) of Definition 3.1 and Sobolev Lemma 3.8.  $\square$

### 3.7 Uniform weak sphericity

In order to have uniform estimates for Hodge systems on 2-spheres  $S_v$ , we introduce the following definition of uniform weak sphericity (see Section 6.1 in [Sha14]).

**Definition 3.14.** *Let  $N \geq 1$  be an integer and  $C, D_{sph} > 0$  be reals. We say that a 2-sphere  $S$  is weakly spherical with constants  $N, C, D_{sph}$  and radius  $R$  if it is weakly regular in the sense of Definition 2.35 with constants  $N, C$  and if the Gauss curvature  $K$  of  $S$  can be written as*

$$K - \frac{1}{R^2} = d\mathfrak{I}_v \Psi + \Theta,$$

where

$$\|\Psi\|_{H^{1/2}(S)} + \|\Theta\|_{L^2(S)} \leq D_{sph}.$$

We say that a foliation  $(S_v)$  of  $\mathcal{H}$  is uniformly weakly spherical with constants  $N, C, D_{sph}$  if it is uniformly weakly regular in the sense of 3.1 with constants  $N, C$  and such that the Gauss curvature  $K$  of  $S_v$  can be written as

$$K - \frac{1}{v^2} = d\mathfrak{I}_v \Psi + \Theta,$$

where

$$\|\Psi\|_{\mathcal{Q}_v^{1/2}} + \|\Theta\|_{L^\infty L^2} \leq D_{sph}.$$

**Remark 3.15.** From the Definition 3.3 of the Besov space  $\mathcal{Q}_v^{1/2}$ , it is clear that every 2-sphere  $S_v$  of a uniformly weakly spherical foliation is weakly spherical with radius  $v$  and uniform constants.

### 3.8 Bochner identities on 2-spheres and consequences

We first recall the Bochner identity on spheres (see [KR05], p. 483 and p. 488).

**Lemma 3.16** (Bochner identities). *Let  $(S, g)$  be a Riemannian 2-sphere. For scalar functions  $f$  on  $S$ , we have*

$$\int_S |\nabla^2 f|^2 = \int_S |\Delta f|^2 - \int_S K |\nabla f|^2,$$

where  $K$  denotes the Gauss curvature of  $S$ . For an  $S$ -tangent 1-form  $F$ , we have

$$\int_S |\nabla^2 F|^2 = \int_S |\Delta F|^2 - 2 \int_S K |\nabla F|^2 + \int_S K |d\mathfrak{I}_v F|^2 + \int_S K^2 |F|^2.$$

The Bochner identities of Lemma 3.16 imply the next estimates, see Section 6 in [Sha14].

**Lemma 3.17** (Bochner estimates). *For a weakly spherical 2-sphere  $S$  of radius 1 with constants  $N, C, D_{sph}$ , there exists a universal constant  $D_0 > 0$  such that if  $D_{sph} < D_0$ , then the following holds.*

1. For scalar function  $f$  on  $S$ , we have

$$\|\nabla^2 f\|_{L^2(S)} + \|\nabla f\|_{L^2(S)} \lesssim \|\Delta f\|_{L^2(S)}.$$

2. For an  $S$ -tangent 1-form  $F$ , we have

$$\|\nabla^2 F\|_{L^2(S)} \lesssim \|\Delta F\|_{L^2(S)} + \|\nabla F\|_{L^2(S)} + \|F\|_{L^2(S)}.$$

Moreover, for a uniformly weakly spherical foliation  $(S_v)$  of  $\mathcal{H}$  with constants  $N, C, D$ , with  $D_{sph} < D_0$ , the following holds.

1. For scalar functions  $f$  on  $\mathcal{H}$ , we have

$$\|\nabla^2 f\|_{L_v^2 L^2} + \|\nabla f\|_{L_v^2 L^2} \lesssim \|\Delta f\|_{L_v^2 L^2}.$$

2. For a  $S_v$ -tangent 1-form  $F$ , we have

$$\|\nabla^2 F\|_{L_v^2 L^2} \lesssim \|\Delta F\|_{L_v^2 L^2} + \|\nabla F\|_{L_v^2 L^2} + \|F\|_{L_v^2 L^2}.$$

### 3.9 Hodge systems on 2-spheres

In this section, we recall standard Hodge theory on Riemannian 2-spheres, see for example [CK93].

**Definition 3.18.** *Let  $(S, g)$  be a Riemannian 2-sphere. We define the Hodge operators  $\mathcal{D}_1$  and  $\mathcal{D}_2$  that act respectively on  $S$ -tangent 1-forms  $\phi$  and on  $S$ -tangent traceless symmetric 2-tensors  $\psi$  by*

$$\begin{aligned} \mathcal{D}_1 \phi &:= (\operatorname{div} \phi, \operatorname{curl} \phi), \\ (\mathcal{D}_2 \psi)_A &:= \operatorname{div} \psi_A. \end{aligned}$$

We denote by  ${}^* \mathcal{D}_1$  and  ${}^* \mathcal{D}_2$  their  $L^2$ -adjoint. For scalar functions  $f, h$  on  $S$  and for a  $S$ -tangent 1-form  $\phi$ , we have

$$\begin{aligned} {}^* \mathcal{D}_1(f, h) &= -\nabla_A f + {}^* \nabla_A h, \\ {}^* \mathcal{D}_2 \phi &= -\frac{1}{2} \nabla \widehat{\otimes} \phi. \end{aligned}$$

**Remarks.**

1. The following identities hold (see [CK93]),

$$\begin{aligned} \mathcal{D}_1 {}^* \mathcal{D}_1 &= -\Delta, & \mathcal{D}_1 {}^* \mathcal{D}_1 &= -\Delta + K, \\ \mathcal{D}_2 {}^* \mathcal{D}_2 &= -\frac{1}{2} \Delta - \frac{1}{2} K, & {}^* \mathcal{D}_2 \mathcal{D}_2 &= -\frac{1}{2} \Delta + K. \end{aligned} \tag{3.5}$$

2. The operator  $\mathcal{D}_1$  is a bijection between the space of vector fields and the space of pairs of functions with vanishing mean.
3. The operator  $\mathcal{D}_2$  is a bijection between the space of symmetric tracefree 2-tensors and the orthogonal complement of the space of conformal Killing vector fields.
4. We denote by  $\mathcal{D}_1^{-1}$  and  $\mathcal{D}_2^{-1}$  the inverses of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  composed with the projections onto their respective domain.

We have the following  $L^2$ -estimates for Hodge systems, see Proposition 6.5 in [Sha14].

**Lemma 3.19** (Estimate for Hodge systems). *For a weakly spherical 2-sphere of radius 1  $S$  with constants  $N, C, D_{sph}$ , there exists  $D_0 > 0$  such that if  $D_{sph} < D_0$ , the following holds. For an  $S$ -tangent tensor of appropriate type  $F$ , we have*

$$\begin{aligned} \|\nabla \mathcal{P}^{-1} F\|_{L^2(S)} + \|\mathcal{P}^{-1} F\|_{L^2(S)} &\lesssim \|F\|_{L^2(S)}, \\ \|\nabla (*\mathcal{P}_1)^{-1} F\|_{L^2(S)} + \|(*\mathcal{P}_1)^{-1} F\|_{L^2(S)} &\lesssim \|F\|_{L^2(S)}, \end{aligned}$$

where  $\mathcal{P}^{-1} \in \{\mathcal{P}_1^{-1}, \mathcal{P}_2^{-1}\}$  and the constants depend only on  $N, C$ . For a uniformly weakly spherical foliation  $(S_v)$  of  $\mathcal{H}$  with constants  $N, C, D_{sph}$ , there exists  $D_0 > 0$  such that if  $D_{sph} < D_0$ , the following holds. For an  $S_v$ -tangent tensor of appropriate type  $F$ , we have

$$\begin{aligned} \|\nabla \mathcal{P}^{-1} F\|_{L_v^2 L^2} + \|\mathcal{P}^{-1} F\|_{L_v^2 L^2} &\lesssim \|F\|_{L_v^2 L^2}, \\ \|\nabla (*\mathcal{P}_1)^{-1} F\|_{L_v^2 L^2} + \|(*\mathcal{P}_1)^{-1} F\|_{L_v^2 L^2} &\lesssim \|F\|_{L_v^2 L^2}, \end{aligned}$$

where  $\mathcal{P}^{-1} \in \{\mathcal{P}_1^{-1}, \mathcal{P}_2^{-1}\}$  and the constants depend only on  $N, C$ .

We have the following elliptic estimates. The proof is postponed to Appendix B.

**Lemma 3.20.** *For a weakly spherical 2-sphere  $S$  of radius 1 with constants  $N, C, D_{sph}$ , there exists a universal constant  $D_0 > 0$  such that if  $D_{sph} < D_0$ , then the following holds. Assume that  $f$  satisfies the equation*

$$\Delta f = \text{div} P + h, \tag{3.6}$$

then

$$\|\nabla f\|_{L^2(S)} + \|f - \bar{f}\|_{L^2(S)} \lesssim \|P\|_{L^2(S)} + \|h\|_{L^{4/3}(S)}.$$

Moreover, for a uniformly weakly spherical foliation  $(S_v)$  of  $\mathcal{H}$  with constants  $N, C, D$ , with  $D_{sph} < D_0$ , the following holds. Assume that  $f$  satisfies (3.6) then,

$$\|\nabla f\|_{L_v^2 L^2} + \|f - \bar{f}\|_{L_v^2 L^2} \lesssim \|P\|_{L_v^2 L^2} + \|h\|_{L_v^2 L^{4/3}},$$

where the constants only depend on  $N, C$ .

The next lemma follows from Proposition 6.10 and Theorem 6.8 in [Sha14].

**Lemma 3.21** (Elliptic estimates in fractional Sobolev spaces). *For a weakly spherical 2-sphere  $S$  with constants  $N, C, D_{sph}$ , there exists a constant  $D_0 > 0$  such that if  $D_{sph} < D_0$ , then the following holds.*

1. Let  $f$  be a scalar function on  $S$  and  $X$  a  $S$ -tangent 1-form satisfying

$$\Delta f = \text{div} X, \quad \int_S f = 0.$$

Then,

$$\|\nabla f\|_{H^{1/2}(S)} + \|f\|_{L^2(S)} \lesssim \|X\|_{H^{1/2}(S)}.$$

2. For an  $S$ -tangent 1-form  $F$ , we have

$$\|*\mathcal{P}_1^{-1} F\|_{H^{1/2}(S)} \lesssim \|F\|_{H^{-1/2}(S)}.$$

## 4. Low regularity estimates

This section is dedicated to the proofs of Propositions 2.38 and 2.39 (see Step 2 in Section 2.13). Let  $(S'_s)_{1 \leq s \leq 5/2}$  denote the geodesic foliation on  $\mathcal{H}$ , and assume that for some  $\varepsilon > 0$ ,

$$\mathcal{I}'_{S_1} + \mathcal{R}'_{[1,5/2]} + \mathcal{O}'_{[1,5/2]} \leq \varepsilon. \quad (4.1)$$

◇ In Section 4.1, we show that, for  $\varepsilon$  small enough, we have  $\mathcal{I}_{S_1}$ . This proves Proposition 2.38.

◇ From Section 4.2 on, we assume that  $1 < v^* < 2$  is a real number and that the canonical foliation  $(S_v)_{1 \leq v \leq v^*}$  is regular. We suppose further that for a fixed large constant  $D$ , for  $1 \leq v \leq v^*$ ,

$$\|\Omega - 1\|_{L^\infty_{\mathbb{V}}([1,v^*])L^\infty} + \|\Upsilon\|_{L^\infty_{\mathbb{V}}([1,v^*])L^\infty} + \mathcal{O}_{[1,v^*]} \leq D\varepsilon. \quad (4.2)$$

◇ We prove in Sections 4.2- 4.7 that for  $\varepsilon > 0$  sufficiently small, we can improve (4.2), *i.e.* we show that,

$$\|\Omega - 1\|_{L^\infty_{\mathbb{V}}([1,v^*])L^\infty} + \|\Upsilon\|_{L^\infty_{\mathbb{V}}([1,v^*])L^\infty} + \mathcal{O}_{[1,v^*]} + \mathcal{R}_{[1,v^*]} \leq D'\varepsilon$$

for a constant  $0 < D' < D$ . This proves Proposition 2.39.

**Notation.** To ease the notations, we suppress all references to  $v^*$  in the following and we denote by  $\mathcal{N}'_m := \mathcal{N}'_m$  and by  $\mathcal{N}'_m := \mathcal{N}'_m$ .

### 4.1 Bounds for the connection coefficients on $S_1$

This section is dedicated to the proof of Proposition 2.38.

By Proposition 2.32 and the fact that  $\Upsilon = 0$  on  $S_1$  since  $s = 1$ , we have

$$\chi = (\chi')^\dagger, \quad \underline{\chi} = (\underline{\chi}')^\dagger, \quad \zeta = (\zeta')^\dagger,$$

which by using (2.16) also gives  $\mu = \mu'$ .

Using the elliptic equation for  $\log \Omega$  (2.18), using the hypotheses estimates (4.1) and applying Lemma 3.21, we have

$$\begin{aligned} \|\nabla \log \Omega\|_{H^{1/2}(S_1)} + \|\log \Omega\|_{L^2(S_1)} &\lesssim \|\zeta\|_{H^{1/2}(S_1)} + \|\mu - \bar{\mu}\|_{L^2(S_1)} \\ &\lesssim \varepsilon. \end{aligned}$$

From this and relation (2.4), we deduce

$$\|\underline{\eta}\|_{H^{1/2}(S_1)} \lesssim \|\zeta\|_{H^{1/2}(S_1)} + \|\nabla \log \Omega\|_{H^{1/2}(S_1)} \lesssim \varepsilon.$$

Moreover, using Sobolev Lemmas 3.6 and 3.7, together with the previous estimates, we have

$$\begin{aligned} \|\log \Omega\|_{L^\infty(S_1)} &\lesssim \|\log \Omega\|_{H^{1/2}(S_1)} + \|\log \Omega\|_{L^2(S_1)} \\ &\lesssim \varepsilon, \end{aligned}$$

which implies

$$\|\Omega - 1\|_{L^\infty(S_1)} \lesssim \varepsilon.$$

This finishes the proof of the bound (2.42) and of Proposition 2.38.

**Remark 4.1.** From the Gauss equation (2.34t) and the definition of the mass aspect function (2.16), we have on  $S_1$

$$K - 1 = \Theta + \text{div} \zeta,$$

where

$$\Theta := \frac{1}{2} (\text{tr} \chi - 2) - \frac{1}{2} (\text{tr} \underline{\chi} + 2) - \frac{1}{4} (\text{tr} \chi - 2) (\text{tr} \underline{\chi} + 2) + \mu.$$

Using estimate (2.42), this implies that the 2-sphere  $S_1$  is weakly spherical with radius 1 and constants  $N, C, C'\varepsilon$  in the sense of Definition 3.14, where  $C' > 0$  is a universal constant. This assumption is required to apply Theorem 2.37 in Step 1 of Section 2.13.

## 4.2 Uniform weak regularity and sphericity of the canonical foliation

In this section, we show that under the bootstrap assumptions (4.2), the regularity of  $S_1$  is propagated to the canonical foliation. More specifically, we show that the canonical foliation is uniformly weakly regular in Lemma 4.2 and uniformly weakly spherical in Lemma 4.3.

**Lemma 4.2** (Uniform weak regularity). *Let  $S_1$  be a weakly regular 2-sphere with constants  $N, c$ . There exists  $\varepsilon_0 > 0$  and  $C(N, c) > 0$  such that for  $\varepsilon < \varepsilon_0$  the canonical foliation is uniformly weakly regular with constants  $N, C$ .*

*Proof.* The bounds (3.1) directly follow from the bootstrap assumptions (4.2). Using Gauss-Codazzi equation (2.22m) for  $\text{curl}\chi$ , we deduce that for  $\Psi$  a 3-tensor verifying equation (3.2), we have

$$\begin{aligned} \nabla_L \Psi_{ABC} &= \Omega \nabla_A (\Omega^{-1} \chi_{BC}) - \Omega \nabla_C (\Omega^{-1} \chi_{BA}) \\ &= -\Omega^{-1} (\nabla_A \Omega \chi_{BC} - \nabla_C \Omega \chi_{BA}) \\ &\quad + \not\chi_{AC} \text{curl} \chi_B \\ &= -\Omega^{-1} (\nabla_A \Omega \chi_{BC} - \nabla_C \Omega \chi_{BA}) \\ &\quad + \not\chi_{AC} \left( -\zeta_D \hat{\chi}_{DB} + \frac{1}{2} \text{tr} \chi \zeta_B - \beta_B \right). \end{aligned}$$

Using the transport equation (2.22g) for  $\zeta$ , we have

$$*\beta_B = \frac{1}{2} \text{tr} \chi (-*\zeta_B + *\eta_B) + *\hat{\chi}_{BD} (\eta_D - \zeta_D) - \nabla_L *\zeta_B.$$

Therefore, we have

$$\nabla_L (\Psi_{ABC} - \not\chi_{AC} *\zeta_B) = E_{ABC},$$

with

$$\begin{aligned} E_{ABC} &:= -\Omega^{-1} (\nabla_A \Omega \chi_{BC} - \nabla_C \Omega \chi_{BA}) \\ &\quad + \not\chi_{AC} \left( \text{tr} \chi *\zeta_B - \frac{1}{2} \text{tr} \chi *\eta_B - *\hat{\chi}_{BD} \eta_D \right). \end{aligned}$$

Using the transport Lemma 3.11 (see also Remark 3.12), the bootstrap assumptions (4.2) and choosing  $\Psi_{ABC} := \not\chi_{AC} *\zeta_B$  on  $S_1$ , we have

$$\begin{aligned} \|\Psi - \not\chi *\zeta\|_{L^4 L^\infty} &\lesssim \|E\|_{L^4 L^1} \\ &\lesssim \|\Omega^{-1}\|_{L^\infty L^\infty} \|\chi\|_{L^\infty L^2} \|\nabla \Omega\|_{L^4 L^2} \\ &\quad + \|\text{tr} \chi\|_{L^\infty L^\infty} \left( \|\zeta\|_{L^4 L^1} + \|\eta\|_{L^4 L^1} \right) \\ &\quad + \|\hat{\chi}\|_{l^\infty L^2} \|\eta\|_{L^4 L^2} \\ &\lesssim D\varepsilon + (D\varepsilon)^2. \end{aligned}$$

Using the bootstrap assumptions (4.2), we deduce

$$\begin{aligned} \|\Psi\|_{L^4 L^\infty} &\lesssim \|\zeta\|_{L^4 L^\infty} + (D\varepsilon) + (D\varepsilon)^2 \\ &\lesssim D\varepsilon + (D\varepsilon)^2 \\ &\leq C. \end{aligned}$$

This finishes the proof of Lemma 4.2.  $\square$

**Lemma 4.3** (Uniform weak sphericity). *There exists  $\varepsilon_0 > 0$ , such that for  $\varepsilon < \varepsilon_0$ , the canonical foliation is uniformly weakly spherical with constants  $N, C, D_{sph}$  and we have  $D_{sph} < D_0$ , where  $D_0$  is the constant from Lemmas 3.17, 3.19, 3.20, and 3.21.*

*Proof.* We have by the Gauss equation (2.34t) and the definition of the mass aspect function (2.16)

$$K - \frac{1}{v^2} = \Theta + \mathfrak{d}v\zeta,$$

where

$$\Theta := \frac{1}{2}v^{-1} \left( \operatorname{tr}\chi - \frac{2}{v} \right) - \frac{1}{2}v^{-1} \left( \operatorname{tr}\underline{\chi} + \frac{2}{v} \right) - \frac{1}{4} \left( \operatorname{tr}\chi - \frac{2}{v} \right) \left( \operatorname{tr}\underline{\chi} + \frac{2}{v} \right) + \mu.$$

Using bootstrap assumptions (4.2), we have

$$\begin{aligned} \|\Theta\|_{L_v^\infty L^2} &\lesssim \left\| \operatorname{tr}\chi - \frac{2}{v} \right\|_{L_v^\infty L^\infty} + \left\| \operatorname{tr}\underline{\chi} + \frac{2}{v} \right\|_{L_v^\infty L^\infty} + \left\| \operatorname{tr}\chi - \frac{2}{v} \right\|_{L_v^\infty L^\infty} \left\| \operatorname{tr}\underline{\chi} + \frac{2}{v} \right\|_{L_v^\infty L^\infty} \\ &\quad + \|\mu\|_{L_v^\infty L^2} \\ &\lesssim D\varepsilon + (D\varepsilon)^2. \end{aligned}$$

And moreover, using bootstrap assumptions (4.2) and Lemma 3.9, we have

$$\|\zeta\|_{\mathcal{Q}_v^{1/2}} \lesssim \mathcal{N}_1(\zeta) \lesssim D\varepsilon.$$

Therefore

$$\|\Theta\|_{L_v^\infty L^2} + \|\zeta\|_{\mathcal{Q}_v^{1/2}} \lesssim (D\varepsilon) + (D\varepsilon)^2 \leq D_0,$$

for  $\varepsilon > 0$  small enough. This finishes the proof of Lemma 4.3.  $\square$

The next lemma will allow us to compare norms for the geodesic and the canonical foliation components.

**Lemma 4.4** (Integral comparison). *For all  $1 \leq p \leq \infty$ , and for all  $S_v$ -tangent tensor  $F$ , we have*

$$\|F\|_{L_v^p([1, v^*])L^p} \lesssim \|F^\ddagger\|_{L_s^p([1, 5/2])L^p}.$$

*To stress this foliation independence, we shall replace all  $L_v^p L^p$  and  $L_s^p L^p$  norms by  $L^p(\mathcal{H})$  in the rest of the paper.*

*Proof.* Let  $(x^1, x^2)$  be local coordinates on  $S_1$ . Extend  $(x^1, x^2)$  on  $\mathcal{H}$  by

$$L(x^a) = 0 \text{ on } \mathcal{H} \text{ for } a = 1, 2.$$

The triplets  $(s, x^1, x^2)$  and  $(v, x^1, x^2)$  are local coordinates on  $\mathcal{H}$ . Let  $(\partial_s, \partial'_{x^1}, \partial'_{x^2})$  and  $(\partial_v, \partial_{x^1}, \partial_{x^2})$  be the respective corresponding coordinate vector fields. Then the following relations hold,

$$\begin{aligned} \partial_v &= \Omega^{-1} \partial_s, \\ \partial_{x^a} &= \partial'_{x^a} - \Omega^{-1} (\partial'_{x^a} v) \partial_s \text{ for } a = 1, 2. \end{aligned}$$

In particular,

$$\mathfrak{g}'_{ab} = \mathfrak{g}(\partial'_{x^a}, \partial'_{x^b}) = \mathfrak{g}(\partial_{x^a}, \partial_{x^b}) = \mathfrak{g}_{ab} \text{ for } a, b = 1, 2,$$

and

$$\gamma := \sqrt{\det(\mathfrak{g}'_{ab})} = \sqrt{\det(\mathfrak{g}_{ab})},$$

where the indices  $a, b$  with  $a, b = 1, 2$  correspond to evaluation with respect to the coordinate vector fields  $\partial_{x^a}, \partial_{x^b}$ . Performing a change of variable in the integrals, using the previous relations, and the bootstrap assumption (4.2) on  $\Omega$ , we therefore deduce

$$\begin{aligned} \|F\|_{L_v^p([1, v^*])L^p}^p &= \int_v \int_{S_v} |F|^p \, dv \\ &= \int_s \int_{S'_s} |F^\ddagger|^p \Omega \, ds \\ &\simeq \int_s \int_{S'_s} |F^\ddagger|^p \, ds \\ &\lesssim \|F^\ddagger\|_{L_s^p([1, 5/2])L^p}^p, \end{aligned}$$

as desired.  $\square$



### 4.3 Bounds for the null curvature components of the canonical foliation on $\mathcal{H}$

In this section, we estimate the canonical curvature components  $(\alpha, \beta, \rho, \sigma, \underline{\beta})$  and  $(\check{\rho}, \check{\sigma}, \check{\underline{\beta}})$  on  $\mathcal{H}$  by comparing them to the geodesic components.

**Lemma 4.5.** *For  $\varepsilon > 0$ , sufficiently small, it holds that*

$$\|\alpha\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} + \|\rho\|_{L^2(\mathcal{H})} + \|\sigma\|_{L^2(\mathcal{H})} + \|\underline{\beta}\|_{L^2(\mathcal{H})} \lesssim \varepsilon. \quad (4.3)$$

*Proof.* By Proposition 2.31, it holds schematically that

$$R = R' + (\Upsilon')R' + (\Upsilon')^2R' + (\Upsilon')^3R',$$

where

$$R \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}\} \text{ and } R' \in \{\alpha', \beta', \rho', \sigma', \underline{\beta}'\}.$$

Using the above and the bootstrap assumptions (4.1), we have

$$\begin{aligned} \|R\|_{L^2(\mathcal{H})} &\lesssim \|R'\|_{L^2(\mathcal{H})} + \|\Upsilon'\|_{L^\infty(\mathcal{H})} \|R'\|_{L^2(\mathcal{H})} + \|\Upsilon'\|_{L^\infty(\mathcal{H})}^2 \|R'\|_{L^2(\mathcal{H})} \\ &\quad + \|\Upsilon'\|_{L^\infty(\mathcal{H})}^3 \|R'\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + (D\varepsilon)\varepsilon + (D\varepsilon)^2\varepsilon + (D\varepsilon)^3\varepsilon \\ &\lesssim \varepsilon. \end{aligned}$$

This finishes the proof of Lemma 4.5. □

Moreover, we have the following estimates for the renormalised canonical curvature components.

**Lemma 4.6.** *We have,*

$$\|\check{\rho}\|_{L^2(\mathcal{H})} + \|\check{\sigma}\|_{L^2(\mathcal{H})} + \|\check{\underline{\beta}}\|_{L^2(\mathcal{H})} \lesssim \varepsilon. \quad (4.4)$$

*Proof.* Using equation (2.8), the bootstrap assumptions (4.2) and the previous estimate (4.3), we have

$$\begin{aligned} \|\check{\rho}\|_{L^2(\mathcal{H})} &\lesssim \|\rho\|_{L^2(\mathcal{H})} + \|\hat{\chi}\|_{L_v^\infty L^4} \|\underline{\hat{\chi}}\|_{L_v^\infty L^4} \\ &\lesssim \|\rho\|_{L^2(\mathcal{H})} + \mathcal{N}_1(\hat{\chi})\mathcal{N}_1(\underline{\hat{\chi}}) \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

The estimates for  $\check{\sigma}$  and  $\check{\underline{\beta}}$  follow analogously and are left to the reader. □

### 4.4 Schematic notation for null connection coefficients and null curvature components

For ease of presentation, in the following sections, we employ the next schematic notation for null connection coefficients and null curvature components.

**Notation.** Let

$$\begin{aligned} A &\in \left\{ \text{tr}\chi - \frac{2}{v}, \hat{\chi}, \zeta, \underline{\eta}, \nabla \log \Omega, \text{tr}\underline{\chi} + \frac{2}{v} \right\} \cup \{\Omega - 1, \log \Omega, \Omega^{-1} - 1\}, \\ \underline{A} &\in A \cup \{\underline{\hat{\chi}}\}, \\ R &\in \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \check{\rho}, \check{\sigma}, \check{\underline{\beta}}\}. \end{aligned}$$

Let moreover

$$\nabla \in \{\nabla, \nabla_L\}.$$

Using the above notation, the bootstrap assumptions (4.2), the bounds on  $S_1$  (2.42), the improved curvature bounds (4.3) and (4.4), can be written as follows.

**Lemma 4.7.** *It holds that*

$$\begin{aligned}\|\underline{A}\|_{H^{1/2}(S_1)} &\lesssim \varepsilon, \\ \|R\|_{L^2(\mathcal{H})} &\lesssim \varepsilon, \\ \|A\|_{L^\infty L_v^2} + \mathcal{N}_1(\underline{A}) + \|\underline{A}\|_{L^4 L_v^\infty} &\lesssim D\varepsilon.\end{aligned}$$

**Remark 4.8.** *To improve the bootstrap assumption for  $\mathcal{N}_1(\underline{A})$ , it is enough to improve only  $\|\nabla \underline{A}\|_{L^2(\mathcal{H})} + \|\nabla \underline{A}\|_{L^2(\mathcal{H})} + \|\underline{A}\|_{L^2(\mathcal{H})}$ , since the  $H^{1/2}(S_1)$  norms are already controlled by Lemma 2.38.*

## 4.5 Improvement of the estimates for $\Omega$ and $\underline{A}$

In this section, we improve the bounds for  $\Omega$  and  $\underline{A}$  using the previously improved bounds for the null curvature components.

**Lemma 4.9.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\mu\|_{L^2 L_v^\infty} \lesssim \varepsilon. \quad (4.5)$$

*Proof.* We rewrite schematically the transport equation (2.16) under the form

$$L(\mu) + \text{tr}\chi\mu = \check{\rho} - \frac{1}{2}\bar{\rho} + AR + A\nabla A + A^2 + A^3.$$

Using Lemma 3.11, the bootstrap assumptions (4.2), the bounds on  $S_1$  (2.42), and the renormalised curvature bounds (4.3) and (4.4), we have

$$\begin{aligned}\|\mu\|_{L^2 L_v^\infty} &\lesssim \|\mu\|_{L^2(S_1)} + \|\check{\rho}\|_{L^2 L_v^1} \\ &\quad + \|AR\|_{L^2 L_v^1} + \|A\nabla A\|_{L^2 L_v^1} + \|A^2\|_{L^2 L_v^1} + \|A^3\|_{L^2 L_v^1} \\ &\lesssim \|\mu\|_{L^2(S_1)} + \|\check{\rho}\|_{L^2(\mathcal{H})} \\ &\quad + \|A\|_{L^\infty L_v^2} (\|R\|_{L^2(\mathcal{H})} + \|\nabla A\|_{L^2(\mathcal{H})} + \|A\|_{L^2(\mathcal{H})} + \|A\|_{L^4(\mathcal{H})}^2) \\ &\lesssim \varepsilon + (D\varepsilon)^2 + (D\varepsilon)^3 \\ &\lesssim \varepsilon,\end{aligned}$$

as desired.  $\square$

**Remark 4.10.** *From now on, we include  $\mu$  in the schematic notation  $R$ .*

**Lemma 4.11.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\nabla^2 \log \Omega\|_{L^2(\mathcal{H})} + \|\nabla \log \Omega\|_{L^2(\mathcal{H})} + \|\log \Omega\|_{L^2(\mathcal{H})} \lesssim \varepsilon. \quad (4.6)$$

*Proof.* Using Lemma 3.20 on the elliptic equation (2.18) and the improved estimates (4.5) and (4.4), we get that

$$\begin{aligned}\|\nabla \log \Omega\|_{L^2(\mathcal{H})} + \|\log \Omega\|_{L^2(\mathcal{H})} &\lesssim \|\mu\|_{L_v^2 L^{4/3}} + \|\check{\rho}\|_{L_v^2 L^{4/3}} \\ &\lesssim \varepsilon + \|\mu\|_{L^2 L_v^\infty} + \|\check{\rho}\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon.\end{aligned}$$

Using Lemma 3.17 on the same elliptic equation (2.18) gives

$$\begin{aligned}\|\nabla^2 \log \Omega\|_{L^2(\mathcal{H})} + \|\nabla \log \Omega\|_{L^2(\mathcal{H})} &\lesssim \|\check{\rho}\|_{L^2(\mathcal{H})} + \|\mu\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon,\end{aligned}$$

as desired.  $\square$

**Lemma 4.12.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\nabla L(\log \Omega)\|_{L^2(\mathcal{H})} + \|L(\log \Omega)\|_{L^2(\mathcal{H})} \lesssim \varepsilon, \quad (4.7)$$

$$\|\nabla L \nabla \log \Omega\|_{L^2(\mathcal{H})} \lesssim \varepsilon. \quad (4.8)$$

*Proof.* Consider first (4.7). We want to derive an equation for  $L(\log \Omega)$  and apply Lemma 3.20. Commuting equation (2.18) with  $L$  gives the following elliptic equation for  $L(\log \Omega)$

$$\Delta (L(\log \Omega)) = F$$

with the source term  $F$

$$F := L(\mu) + 2L(\check{\rho}) + [\Delta, L] \log \Omega - 2L(\check{\rho}) - L(\bar{\mu}).$$

Using commutation formula (2.14), we have

$$\begin{aligned} [\Delta, L] \log \Omega &= -\operatorname{tr} \chi \Delta (\log \Omega) - 2\hat{\chi} \cdot \nabla^2 \log \Omega \\ &\quad + (\zeta + \underline{\eta}) (\nabla (L \log \Omega) + \nabla_L \nabla \log \Omega) \\ &\quad + (\operatorname{tr} \chi \underline{\eta} - \operatorname{div} \hat{\chi}) \cdot \nabla \log \Omega - \underline{\eta} \cdot \chi \cdot \nabla \log \Omega \\ &\quad + (\operatorname{div} \zeta + \operatorname{div} \underline{\eta}) L(\log \Omega) + \beta \cdot \nabla \log \Omega \\ &= -2v^{-1} \Delta \log \Omega \\ &\quad - \left( \operatorname{tr} \chi - \frac{2}{v} \right) \Delta \log \Omega - 2\hat{\chi} \cdot \nabla^2 \log \Omega \\ &\quad + (\zeta + \underline{\eta}) (\nabla (L \log \Omega) + \nabla_L \nabla \log \Omega) \\ &\quad + (\operatorname{tr} \chi \underline{\eta} - \operatorname{div} \hat{\chi}) \cdot \nabla \log \Omega - \underline{\eta} \cdot \chi \cdot \nabla \log \Omega \\ &\quad + (\operatorname{div} \zeta + \operatorname{div} \underline{\eta}) L(\log \Omega) + \beta \cdot \nabla \log \Omega. \end{aligned} \tag{4.9}$$

Using Bianchi equation (2.9a) for  $\check{\rho}$ , and the transport equation (2.21) for  $\mu$ , we obtain the formula

$$\begin{aligned} L(\mu) + 2L(\check{\rho}) &= -2v^{-1} \mu - 4v^{-1} \check{\rho} - v^{-1} \bar{\rho} + 2\operatorname{div} \beta \\ &\quad - \left( \operatorname{tr} \chi - \frac{2}{v} \right) \mu - 2 \left( \operatorname{tr} \chi - \frac{2}{v} \right) \check{\rho} \\ &\quad + 4\underline{\eta} \cdot \beta + (\zeta - \underline{\eta}) \cdot \nabla \operatorname{tr} \chi + \hat{\chi} \cdot \nabla \zeta \\ &\quad - \hat{\chi} \cdot \nabla \underline{\eta} + \operatorname{tr} \chi \left( |\zeta|^2 - \zeta \cdot \underline{\eta} - \frac{1}{2} |\underline{\eta}|^2 \right) \\ &\quad + \frac{1}{4} \operatorname{tr} \chi |\hat{\chi}|^2 + 2\hat{\chi} \cdot \zeta \cdot \underline{\eta} - \frac{3}{2} \hat{\chi} \cdot \underline{\eta} \cdot \underline{\eta}. \end{aligned} \tag{4.10}$$

Using Proposition 2.16 and Remark 4.10, we can moreover schematically write

$$L(\bar{\mu}) + 2L(\bar{\rho}) = \overline{L(\mu) + 2L(\check{\rho})} + AR. \tag{4.11}$$

Using the three equations (4.9), (4.10) and (4.11) we deduce that  $F$  can be rewritten in the following schematic form

$$F = F^L + F^{NL},$$

with the linear source terms  $F^L$

$$F^L = -2v^{-1} \Delta \log \Omega - 2v^{-1} (\mu - \bar{\mu}) - 4v^{-1} (\check{\rho} - \bar{\rho}) + 2\operatorname{div} \beta,$$

and the non-linear source terms  $F^{NL}$  being of the form

$$F^{NL} = \nabla (AL \log \Omega) + (A + A^2)L(\log \Omega) + A \nabla L \log \Omega + AR + A \nabla A + A^2 + A^3.$$

On the one hand, it holds that

$$F^L = \operatorname{div} P^L + W^L,$$

with

$$\begin{aligned} P^L &= 2\beta - 2v^{-1} \nabla \log \Omega, \\ W^L &= -4v^{-1} (\check{\rho} - \bar{\rho}) - 2v^{-1} (\mu - \bar{\mu}), \end{aligned}$$

and using the already improved bounds (4.3), (4.4), (4.5) and (4.6), we get

$$\begin{aligned} \|P^L\|_{L^2(\mathcal{H})} + \|W^L\|_{L_v^2 L^{4/3}} &\lesssim \|\beta\|_{L^2(\mathcal{H})} + \|\nabla \log \Omega\|_{L^2(\mathcal{H})} \\ &\quad + \|\check{\rho}\|_{L^2(\mathcal{H})} + \|\mu\|_{L^2 L_v^\infty} \\ &\lesssim \varepsilon. \end{aligned} \tag{4.12}$$

On the other hand, we have

$$F^{NL} = \nabla P^{NL} + W^{NL},$$

with

$$\begin{aligned} P^{NL} &= AL(\log \Omega) \\ W^{NL} &= (A + A^2)L(\log \Omega) + A\nabla L \log \Omega + AR + A\nabla A + A^2 + A^3, \end{aligned}$$

and using the bootstrap assumptions (4.2) and the estimates (4.3), we get

$$\begin{aligned} \|P^{NL}\|_{L^2(\mathcal{H})} + \|W^{NL}\|_{L_v^2 L^{4/3}} &\lesssim \|A\|_{L_v^\infty L^4} \|L(\log \Omega)\|_{L_v^2 L^4} \\ &\quad + \left( \|A\|_{L_v^\infty L^2} + \|A\|_{L_v^\infty L^4}^2 \right) \|L(\log \Omega)\|_{L_v^2 L^4} \\ &\quad + \|A\|_{L_v^\infty L^4} \left( \|\nabla L(\log \Omega)\|_{L^2(\mathcal{H})} + \|R\|_{L^2(\mathcal{H})} \right. \\ &\quad \left. + \|\nabla A\|_{L^2(\mathcal{H})} + \|A\|_{L^2(\mathcal{H})} + \|A\|_{L^4(\mathcal{H})}^2 \right) \\ &\lesssim (D\varepsilon)^2 + (D\varepsilon)^3 \\ &\lesssim \varepsilon. \end{aligned} \tag{4.13}$$

Applying Lemma 3.20, using the bounds (4.12) and (4.13), we have

$$\begin{aligned} &\|\nabla L(\log \Omega)\|_{L^2(\mathcal{H})} + \left\| L(\log \Omega) - \overline{L(\log \Omega)} \right\|_{L^2(\mathcal{H})} \\ &\lesssim \|P^L\|_{L^2(\mathcal{H})} + \|W^L\|_{L_v^2 L^{4/3}} + \|P^{NL}\|_{L^2(\mathcal{H})} + \|W^{NL}\|_{L_v^2 L^{4/3}} \\ &\lesssim \varepsilon. \end{aligned} \tag{4.14}$$

Using Proposition 2.16 and the equation (2.17), we have

$$\begin{aligned} \overline{L(\log \Omega)} &= \overline{\Omega^{-1}L(\log \Omega)} + \overline{(1 - \Omega^{-1})L(\log \Omega)} \\ &= \Omega^{-1}L(\log \Omega) - \overline{\Omega^{-1}\text{tr}\chi \log \Omega} + \overline{\Omega^{-1}\text{tr}\chi \cdot \log \Omega} + \overline{(1 - \Omega^{-1})L(\log \Omega)} \\ &= -\overline{\Omega^{-1}\text{tr}\chi \log \Omega} + \overline{(1 - \Omega^{-1})L(\log \Omega)}. \end{aligned}$$

Using the improved bound (4.6) and the bootstrap assumptions (4.2) we therefore deduce

$$\begin{aligned} \left\| \overline{L(\log \Omega)} \right\|_{L^2(\mathcal{H})} &\lesssim \|\log \Omega\|_{L^2(\mathcal{H})} \|\Omega^{-1}\|_{L^\infty(\mathcal{H})} \|\text{tr}\chi\|_{L^\infty(\mathcal{H})} \\ &\quad + \|\Omega - 1\|_{L^\infty(\mathcal{H})} \|\Omega^{-1}\|_{L^\infty(\mathcal{H})} \|L(\log \Omega)\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

Using (4.14), we finally get

$$\|\nabla L(\log \Omega)\|_{L^2(\mathcal{H})} + \|L(\log \Omega)\|_{L^2(\mathcal{H})} \lesssim \varepsilon.$$

By the above bounds, commutation formula (2.10) and the bootstrap assumptions (4.2) we also get that

$$\begin{aligned} \|\nabla_L \nabla \log \Omega\|_{L^2(\mathcal{H})} &\lesssim \|\nabla L(\log \Omega)\|_{L^2(\mathcal{H})} + \|\text{tr}\chi\|_{L^\infty(\mathcal{H})} \|\nabla \log \Omega\|_{L^2(\mathcal{H})} \\ &\quad + \|\hat{\chi}\|_{L_v^\infty L^4} \|\nabla \log \Omega\|_{L_v^\infty L^4} \\ &\quad + \left( \|\zeta\|_{L_v^\infty L^4} + \|\underline{\eta}\|_{L_v^\infty L^4} \right) \|L(\log \Omega)\|_{L_v^2 L^4} \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

This finishes the proof of Lemma 4.12.  $\square$

**Lemma 4.13.** *For  $\varepsilon > 0$  sufficiently small, we have*

$$\|\log \Omega\|_{L^\infty(\mathcal{H})} + \|\Omega - 1\|_{L^\infty(\mathcal{H})} + \|L(\log \Omega)\|_{L^2_v L^4} \lesssim \varepsilon. \quad (4.15)$$

*Proof.* This is a consequence of estimates (4.6), (4.7), (4.8) and Sobolev embeddings from Lemma 3.8.  $\square$

**Lemma 4.14.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\mathcal{N}_1(\zeta) + \mathcal{N}_1(\eta) \lesssim \varepsilon. \quad (4.16)$$

*Proof.* By equations (2.16) and (2.22h),  $\zeta$  satisfies the Hodge system

$$\mathcal{D}_1(\zeta) = (-\mu - \check{\rho}, \check{\sigma}).$$

Using Lemma 3.19, the improved bounds (4.4) and (4.5), we get that

$$\begin{aligned} \|\zeta\|_{L^2(\mathcal{H})} + \|\nabla \zeta\|_{L^2(\mathcal{H})} &\lesssim \|\mu\|_{L^2(\mathcal{H})} + \|\check{\rho}\|_{L^2(\mathcal{H})} + \|\check{\sigma}\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon. \end{aligned}$$

By relation (2.4), the improved bounds (4.6) and the above improvement, we directly deduce

$$\|\eta\|_{L^2(\mathcal{H})} + \|\nabla \eta\|_{L^2(\mathcal{H})} \lesssim \varepsilon.$$

Using equation (2.22g), the bootstrap assumptions (4.2), the curvature bounds (4.3) and the just obtained improved estimate for  $\zeta$  and  $\eta$ , we have

$$\begin{aligned} \|\nabla_L \zeta\|_{L^2(\mathcal{H})} &\lesssim \|\text{tr}\chi\|_{L^\infty} (\|\zeta\|_{L^2(\mathcal{H})} + \|\eta\|_{L^2(\mathcal{H})}) \\ &\quad + \|\hat{\chi}\|_{L^\infty_v L^4} (\|\zeta\|_{L^\infty_v L^4} + \|\eta\|_{L^\infty_v L^4}) + \|\beta\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

By relation (2.4) and the improved bounds (4.8), we therefore also deduce

$$\|\nabla_L \eta\|_{L^2(\mathcal{H})} \lesssim \varepsilon,$$

and this finishes the proof of Lemma 4.14.  $\square$

**Lemma 4.15.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\nabla \text{tr}\chi\|_{L^2 L^\infty_v} \lesssim \varepsilon, \quad (4.17)$$

$$\left\| \text{tr}\chi - \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} \lesssim \varepsilon, \quad (4.18)$$

$$\mathcal{N}_1\left(\text{tr}\chi - \frac{2}{v}\right) + \mathcal{N}_1(\hat{\chi}) \lesssim \varepsilon. \quad (4.19)$$

*Proof.* Consider first (4.17). Commuting equation (2.22b) with  $\nabla$ , we get

$$\nabla_L(\nabla \text{tr}\chi) + \frac{3}{2}\text{tr}\chi \nabla \text{tr}\chi = G,$$

with

$$G = -2\hat{\chi} \cdot \nabla \hat{\chi} - \hat{\chi} \cdot \nabla \text{tr}\chi + (\zeta + \eta) \left( \frac{1}{2}(\text{tr}\chi)^2 + |\hat{\chi}|^2 \right).$$

By the improved bounds (4.16) for  $\zeta$  and  $\eta$  and the bootstrap assumptions (4.1), we have

$$\begin{aligned} \|G\|_{L^2 L^1_v} &\lesssim \|\hat{\chi}\|_{L^\infty L^2_v} \|\nabla \hat{\chi}\|_{L^2(\mathcal{H})} + \|\hat{\chi}\|_{L^\infty L^2_v} \|\nabla \text{tr}\chi\|_{L^2(\mathcal{H})} \\ &\quad + (\|\zeta\|_{L^2(\mathcal{H})} + \|\eta\|_{L^2(\mathcal{H})}) \|\text{tr}\chi\|_{L^\infty(\mathcal{H})}^2 \\ &\quad + (\|\zeta\|_{L^\infty L^2_v} + \|\eta\|_{L^\infty L^2_v}) \|\hat{\chi}\|_{L^4(\mathcal{H})}^2 \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

Therefore, we deduce from Lemma 3.11 with (4.1) that (4.17) holds.

Next, we consider (4.18). The transport equation for  $\text{tr}\chi$  (2.22b) can be rewritten

$$L\left(\text{tr}\chi - \frac{2}{v}\right) + \text{tr}\chi\left(\text{tr}\chi - \frac{2}{v}\right) = 2v^{-2}(\Omega - 1) - |\hat{\chi}|^2 + \frac{1}{2}\left(\text{tr}\chi - \frac{2}{v}\right)^2. \quad (4.20)$$

Using Lemma 3.11, the bootstrap assumptions (4.2), the bounds on  $S_1$  (2.42), and the improved bound (4.15), we have

$$\begin{aligned} \left\|\text{tr}\chi - \frac{2}{v}\right\|_{L^\infty(\mathcal{H})} &\lesssim \|\text{tr}\chi - 2\|_{L^\infty(S_1)} + \|\Omega - 1\|_{L^\infty(\mathcal{H})} + \|\hat{\chi}\|_{L^\infty L_v^2}^2 + \left\|\text{tr}\chi - \frac{2}{v}\right\|_{L^\infty(\mathcal{H})}^2 \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon, \end{aligned}$$

which proves (4.18).

It remains to prove (4.19). Using transport equation (4.20) for  $\text{tr}\chi$ , we deduce that

$$\left\|L\left(\text{tr}\chi - \frac{2}{v}\right)\right\|_{L^2(\mathcal{H})} \lesssim \varepsilon,$$

and therefore

$$\mathcal{N}_1\left(\text{tr}\chi - \frac{2}{v}\right) \lesssim \varepsilon.$$

Applying Hodge Lemma 3.19 to the Codazzi equation on  $\hat{\chi}$  (2.34q), with the curvature bounds (4.3), the improved bound (4.16), and the bound just proven for  $\nabla \text{tr}\chi$  gives

$$\begin{aligned} \|\nabla \hat{\chi}\|_{L^2(\mathcal{H})} + \|\hat{\chi}\|_{L^2(\mathcal{H})} &\lesssim \|\nabla \text{tr}\chi\|_{L^2 L_v^\infty} + \|\zeta\|_{L^4 L_v^\infty} \|\hat{\chi}\|_{L^4 L_v^\infty} \\ &\quad + \|\text{tr}\chi\|_{L^\infty(\mathcal{H})} \|\zeta\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

Taking directly the  $L^2(\mathcal{H})$ -norm in the transport equation for  $\hat{\chi}$  (2.22c), we finally obtain

$$\|\nabla L\hat{\chi}\|_{L^2(\mathcal{H})} \lesssim \varepsilon$$

and this finishes the proof of Lemma 4.15.  $\square$

**Lemma 4.16.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\nabla \text{tr}\underline{\chi}\|_{L^2 L_v^\infty} \lesssim \varepsilon, \quad (4.21)$$

$$\left\|\text{tr}\underline{\chi} + \frac{2}{v}\right\|_{L^\infty(\mathcal{H})} \lesssim \varepsilon, \quad (4.22)$$

$$\mathcal{N}_1\left(\text{tr}\underline{\chi} + \frac{2}{v}\right) + \mathcal{N}_1(\hat{\chi}) \lesssim \varepsilon. \quad (4.23)$$

*Proof.* Consider (4.21). Commuting the transport equation for  $\text{tr}\underline{\chi}$  (2.22e) with  $\nabla$ , we get

$$\nabla L \nabla \text{tr}\underline{\chi} + \text{tr}\underline{\chi} \nabla \text{tr}\underline{\chi} = G,$$

with

$$G = 4\underline{\eta} \cdot \nabla \underline{\eta} - \hat{\chi} \cdot \nabla \text{tr}\underline{\chi} + (\zeta + \underline{\eta}) \left( -\frac{1}{2} \text{tr}\underline{\chi} \text{tr}\underline{\chi} + 4\bar{\rho} + 4|\underline{\eta}|^2 \right),$$

which can be rewritten in the schematic form

$$G = 2v^{-2}(\zeta + \underline{\eta}) + A(\nabla A + A + A^2 + R).$$

Using Lemma 3.11, the bootstrap assumptions (4.2), the initial bounds (2.42) and the improved bounds (4.7) (4.16), we have

$$\begin{aligned}
 \|\nabla \operatorname{tr} \underline{\chi}\|_{L^2 L_v^\infty} &\lesssim \|\nabla \operatorname{tr} \underline{\chi}\|_{L^2(S_1)} + \|G\|_{L^2 L_v^1} \\
 &\lesssim \varepsilon + \|\zeta\|_{L^2(\mathcal{H})} + \|\underline{\eta}\|_{L^2(\mathcal{H})} \\
 &\quad + \|A\|_{L^\infty L_v^2} (\|\nabla A\|_{L^2(\mathcal{H})} + \|A\|_{L^2(\mathcal{H})} + \|A\|_{L^4(\mathcal{H})}^2) + \|R\|_{L^2(\mathcal{H})} \\
 &\lesssim \varepsilon + (D\varepsilon)^2 + (D\varepsilon)^3 \\
 &\lesssim \varepsilon.
 \end{aligned}$$

We turn to estimate (4.22). The transport equation for  $\operatorname{tr} \underline{\chi}$  (2.22e) can be rewritten in the following form

$$\begin{aligned}
 &L\left(\operatorname{tr} \underline{\chi} + \frac{2}{v}\right) + \frac{1}{2} \operatorname{tr} \underline{\chi} \left(\operatorname{tr} \underline{\chi} + \frac{2}{v}\right) \\
 &= -2v^{-2}(\Omega - 1) + v^{-1} \left(\operatorname{tr} \underline{\chi} - \frac{2}{v}\right) + 2\bar{\rho} + 2|\underline{\eta}|^2.
 \end{aligned}$$

Using Lemma 3.11, the bootstrap assumptions (4.2), the bounds on  $S_1$  (2.42) and the improved bounds (4.18) (4.15), we have

$$\begin{aligned}
 \left\| \operatorname{tr} \underline{\chi} + \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} &\lesssim \left\| \operatorname{tr} \underline{\chi} + 2 \right\|_{L^\infty(S_1)} + \|\Omega - 1\|_{L^\infty(\mathcal{H})} \\
 &\quad + \left\| \operatorname{tr} \underline{\chi} - \frac{2}{v} \right\|_{L^\infty(\mathcal{H})} + \|\bar{\rho}\|_{L^\infty L_v^1} + \|\underline{\eta}\|_{L^\infty L_v^2}^2 \\
 &\lesssim \varepsilon + (D\varepsilon)^2 \\
 &\lesssim \varepsilon.
 \end{aligned}$$

To prove estimate (4.23), we apply Hodge Lemma 3.19 to the Codazzi equation for  $\hat{\chi}$  and since  $\nabla \operatorname{tr} \underline{\chi}$  and  $\operatorname{tr} \underline{\chi}$  have already been estimated the  $\nabla$ -control of  $\operatorname{tr} \underline{\chi} + \frac{2}{v}$  and  $\hat{\chi}$  follows. The estimates for  $L(\operatorname{tr} \underline{\chi} + \frac{2}{v})$  and  $\nabla_L \hat{\chi}$  are obtained by taking directly the  $L^2(\mathcal{H})$  norm in the transport equations for  $\operatorname{tr} \underline{\chi}$  and  $\hat{\chi}$  (2.22e) and (2.22f) since all linear source terms have already been estimated. This concludes the proof of Lemma 4.16.  $\square$

## 4.6 Improvement of $\Upsilon$

In this section, we improve the estimate for  $\|\Upsilon\|_{L^\infty(\mathcal{H})}$  which is the key quantity to compare the geodesic and canonical foliations. Using the estimates proved in the previous sections, we can first improve the  $L^\infty L_v^2$  estimate for  $\underline{\eta}$ .

**Lemma 4.17.** *For  $\varepsilon > 0$  sufficiently small, it holds that*

$$\|\underline{\eta}\|_{L^\infty L_v^2} \lesssim \varepsilon. \tag{4.24}$$

*Proof.* Our goal is to apply the trace estimate of Lemma 3.13. By the improved estimates (4.16) for  $\underline{\eta}$ , it suffices to prove that there exist  $P$  and  $E$  such that

$$\nabla \underline{\eta} = \nabla_L P + E,$$

with

$$\mathcal{N}_1(P) \lesssim \varepsilon, \quad \|E\|_{\mathcal{P}_0} \lesssim \varepsilon.$$

From the transport equation for  $\underline{\chi}$  (2.22d), we have

$$\begin{aligned}
 \nabla^A \underline{\eta}_B &= \frac{1}{2} \nabla_L \underline{\chi}_{AB} - \frac{1}{2} \rho \underline{g}_{AB} - \frac{1}{2} \sigma \underline{\not{e}}_{AB} + \frac{1}{2} \chi_{AC} \underline{\chi}_{CB} \\
 &= \frac{1}{2} \nabla_L \left( \underline{\chi}_{AB} + \frac{2}{v} \underline{g}_{AB} \right) - \frac{1}{2} \rho \underline{g}_{AB} - \frac{1}{2} \sigma \underline{\not{e}}_{AB} + E_{AB},
 \end{aligned}$$

where

$$\begin{aligned} E_{AB} := & \left( \frac{1}{2} \text{tr} \underline{\chi} \left( \text{tr} \chi - \frac{2}{v} \right) + 2v^{-2}(\Omega - 1) + \frac{1}{2} \text{tr} \chi \left( \text{tr} \underline{\chi} + \frac{2}{v} \right) \right. \\ & + \frac{1}{2} \left( \text{tr} \chi - \frac{2}{v} \right) \left( \text{tr} \underline{\chi} + \frac{2}{v} \right) \underline{g}_{AB} \\ & \left. + \frac{1}{4} \text{tr} \underline{\chi} \hat{\chi}_{AB} + \frac{1}{4} \text{tr} \chi \hat{\chi}_{AB} + \frac{1}{2} \hat{\chi}_{AC} \hat{\chi}_{CB} \right). \end{aligned}$$

First, using the results of Lemma 4.16, we have

$$\mathcal{N}_1 \left( \underline{\chi} + \frac{2}{v} \underline{g} \right) \lesssim \varepsilon.$$

Second, using Lemma 3.10 and the improved bounds for  $\Omega$ ,  $\chi$  and  $\underline{\chi}$  (4.15), (4.19), (4.23), we have

$$\begin{aligned} \|E\|_{\mathcal{P}_v^0} & \lesssim \left( \mathcal{N}_1(\Omega - 1) + \mathcal{N}_1 \left( \chi - \frac{2}{v} \underline{g} \right) + \mathcal{N}_1 \left( \underline{\chi} + \frac{2}{v} \underline{g} \right) \right) \\ & \quad \times \left( 1 + \mathcal{N}_1(\Omega - 1) + \mathcal{N}_1 \left( \chi - \frac{2}{v} \underline{g} \right) + \mathcal{N}_1 \left( \underline{\chi} + \frac{2}{v} \underline{g} \right) \right) \\ & \lesssim \varepsilon. \end{aligned}$$

Third, we define  $(\phi, \psi)$  to be the solution of the transport equation

$$\begin{aligned} L\phi & = \rho, \\ L\psi & = \sigma, \\ (\phi, \psi)|_{S_1} & = {}^* \mathcal{D}_1^{-1} \check{\beta}. \end{aligned} \tag{4.25}$$

Using the curvature bounds (4.3), we have directly

$$\|L\phi\|_{L^2(\mathcal{H})} + \|L\psi\|_{L^2(\mathcal{H})} \lesssim \varepsilon \tag{4.26}$$

Using the definition of  $\check{\beta}$  (2.8) and the Codazzi equation for  $\hat{\chi}$  (2.22k), we have schematically

$$\check{\beta} = \text{div} \hat{\chi} - \frac{1}{2} \nabla \text{tr} \chi + A + \underline{A} \underline{A}.$$

Therefore, using Lemma 3.5 and the bounds on  $S_1$  (2.42), we have

$$\begin{aligned} \|\check{\beta}\|_{H^{-1/2}(S_1)} & \lesssim \|\hat{\chi}\|_{H^{1/2}(S_1)} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{H^{1/2}(S_1)} + \|\underline{A} \underline{A}\|_{L^2(S_1)} \\ & \lesssim \|\hat{\chi}\|_{H^{1/2}(S_1)} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{H^{1/2}(S_1)} + \|A\|_{L^4(S_1)} \|\underline{A}\|_{L^4(S_1)} \\ & \lesssim \|\hat{\chi}\|_{H^{1/2}(S_1)} + \left\| \text{tr} \chi + \frac{2}{v} \right\|_{H^{1/2}(S_1)} + \|A\|_{H^{1/2}(S_1)} \|\underline{A}\|_{H^{1/2}(S_1)} \\ & \lesssim \varepsilon. \end{aligned}$$

Thus, using Lemma 3.21, we have

$$\|(\phi, \psi)\|_{H^{1/2}(S_1)} \lesssim \left\| {}^* \mathcal{D}_1^{-1} \check{\beta} \right\|_{H^{1/2}(S_1)} \lesssim \|\check{\beta}\|_{H^{-1/2}(S_1)} \lesssim \varepsilon. \tag{4.27}$$

Using the transport Lemma 3.11 with these bounds, we deduce

$$\|(\phi, \psi)\|_{L^2 L^\infty} \lesssim \|L(\phi, \psi)\|_{L^2(\mathcal{H})} + \|(\phi, \psi)\|_{H^{1/2}(S_1)} \lesssim \varepsilon. \tag{4.28}$$

Commuting the transport equation (4.25) by  ${}^* \mathcal{D}_1$ , using Bianchi equation (2.9c) for  $\check{\beta}$  and commutation formula (2.13) gives

$$\begin{aligned} \nabla_L {}^* \mathcal{D}_1(\phi, \psi) & = {}^* \mathcal{D}_1(\rho, \sigma) + [\nabla_L, {}^* \mathcal{D}_1](\phi, \psi) \\ & = \nabla_L \check{\beta} + \text{tr} \chi \check{\beta} + A(R + \nabla \underline{A} + \underline{A} + \underline{A}^2) + A \nabla(\phi, \psi). \end{aligned}$$



Using Lemma 3.11, the bootstrap assumptions (4.2), the curvature bounds (4.4) and the condition (4.25) on  $S_1$  for  $(\phi, \psi)$ , we have

$$\begin{aligned} \|\mathcal{D}_1^*(\phi, \psi) - \check{\beta}\|_{L_v^\infty L^2} &\lesssim \|\mathcal{D}_1^*(\phi, \psi) - \check{\beta}\|_{L^2(S_1)} + \|\text{tr}\chi\|_{L^\infty(\mathcal{H})} \|\check{\beta}\|_{L^2(\mathcal{H})} \\ &\quad + \|A\|_{L^\infty L_v^2} (\|R\|_{L^2(\mathcal{H})} + \|\nabla A\|_{L^2(\mathcal{H})} + \|A\|_{L^2(\mathcal{H})} + \|A\|_{L^4(\mathcal{H})}^2) \\ &\quad + \|A\|_{L^\infty L_v^2} \|\nabla(\phi, \psi)\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + (D\varepsilon)^2 + (D\varepsilon)\mathcal{N}_1(\phi, \psi). \end{aligned}$$

Using Hodge Lemma 3.19 and the curvature bound (4.4), we deduce from the above that

$$\begin{aligned} \|\nabla(\phi, \psi)\|_{L^2(\mathcal{H})} &\lesssim \|\mathcal{D}_1^*(\phi, \psi)\|_{L^2(\mathcal{H})} \\ &\lesssim \|\check{\beta}\|_{L^2(\mathcal{H})} + \|\mathcal{D}_1^*(\phi, \psi) - \check{\beta}\|_{L^\infty L_v^2} \\ &\lesssim \varepsilon + D\varepsilon\mathcal{N}_1(\phi, \psi). \end{aligned} \tag{4.29}$$

For  $\varepsilon > 0$  sufficiently small, we therefore have by (4.26), (4.27), (4.28) and (4.29), that

$$\mathcal{N}_1(\phi, \psi) \lesssim \varepsilon.$$

This finishes the proof of the lemma.  $\square$

**Lemma 4.18.** *We have the improved bound*

$$\|\Upsilon'\|_{L^\infty(\mathcal{H})} + \|\Upsilon\|_{L^\infty(\mathcal{H})} \lesssim \varepsilon. \tag{4.30}$$

*Proof.* From Proposition 2.32, we have

$$\begin{aligned} \nabla'_L \Upsilon'_A &= \underline{\eta}'_A - (\underline{\eta})_A^\dagger \\ &= \underline{\eta}'_A - \underline{\eta}(e_A). \end{aligned}$$

Integrating in  $s$  and since  $\Upsilon' = 0$  on  $S_1$ , we deduce

$$\begin{aligned} \Upsilon'_A &= \int_1^s L(\Upsilon'_A) ds' \\ &= \int_1^s (\underline{\eta}'_A - \underline{\eta}(e_A)) ds' \\ &= \int_1^s \underline{\eta}'_A ds' - \int_{v'=1}^{v(s)} \underline{\eta}_A \Omega^{-1} dv'. \end{aligned}$$

Therefore, using the assumed bound (4.1) on the geodesic connection coefficient  $\underline{\eta}' = -\zeta'$  and the improved bound for  $\underline{\eta}$  (4.24), we obtain

$$\begin{aligned} \|\Upsilon'\|_{L^\infty(\mathcal{H})} &\lesssim \|\underline{\eta}'\|_{L^\infty L_v^2} + \|\Omega^{-1}\|_{L^\infty(\mathcal{H})} \|\underline{\eta}\|_{L^\infty L_v^2} \\ &\lesssim \varepsilon, \end{aligned}$$

which, together with Definition 2.27 proves (4.30).  $\square$

## 4.7 Improvement of $L^\infty L_v^2$ estimates for $A$

In section 4.6, we proved  $L^\infty L_v^2$  estimate for  $\underline{\eta}$ . In this section, we prove the remaining  $L^\infty L_v^2$  estimates for  $\hat{\chi}, \zeta, \nabla \log \Omega$  by comparing the canonical foliation to the geodesic foliation on  $\mathcal{H}$ . This concludes the improvement of the bootstrap assumptions (4.2), thus finishes the proof of Proposition 2.39.

**Lemma 4.19.** *For  $\varepsilon > 0$  sufficiently small, we have*

$$\|\hat{\chi}\|_{L^\infty L_v^2} \lesssim \varepsilon, \tag{4.31}$$

$$\|\zeta\|_{L^\infty L_v^2} \lesssim \varepsilon, \tag{4.32}$$

$$\|\nabla \log \Omega\|_{L^\infty L_v^2} \lesssim \varepsilon. \tag{4.33}$$

*Proof.* From Proposition 2.32, we have

$$\begin{aligned}\chi_{AB} &= \chi'_{AB} \\ \zeta_A &= \zeta'_A - 2\Upsilon'_B \chi'_{AB},\end{aligned}$$

Thus, the estimate (4.31) is a direct consequence of the bounds (4.1) and (4.30).

Further, by the improved bound on  $\Upsilon'$  (4.30) and the assumed bound (2.38) for the geodesic connection coefficients, we get

$$\|\zeta\|_{L^\infty L^2_v} \lesssim \|\zeta'\|_{L^\infty L^2_s} + \|\Upsilon'\|_{L^\infty(\mathcal{H})} \|\chi'\|_{L^\infty L^2_s} \lesssim \varepsilon.$$

Estimate (4.33) then follows directly using relation (2.4). This finishes the proof of Lemma 4.19.  $\square$

## 4.8 Additional bounds for $\Upsilon$

In Section 5, we will use the following additional estimates.

**Lemma 4.20.** *For  $\varepsilon > 0$  sufficiently small, we have*

$$\|\nabla_L \Upsilon\|_{L^2(\mathcal{H})} + \|\nabla \Upsilon\|_{L^2 L^\infty_v} \lesssim \varepsilon. \quad (4.34)$$

*Proof.* Taking the  $L^2(\mathcal{H})$ -norm in the transport equation (2.31) for  $\Upsilon$ , using the improved bound (4.6) for  $\Omega$  and the improved bound on  $\Upsilon$  (4.30), we have

$$\begin{aligned}\|\nabla_L \Upsilon\|_{L^2(\mathcal{H})} &\lesssim \|\mathrm{tr}\chi\|_{L^\infty(\mathcal{H})} \|\Upsilon\|_{L^\infty(\mathcal{H})} + \|\hat{\chi}\|_{L^2 L^\infty_v} \|\Upsilon\|_{L^\infty(\mathcal{H})} + \|\nabla \log \Omega\|_{L^2(\mathcal{H})} \\ &\lesssim \varepsilon + \varepsilon^2 \\ &\lesssim \varepsilon.\end{aligned}$$

To obtain the other bound, we make the additional bootstrap assumption  $\|\nabla \Upsilon\|_{L^2 L^\infty_v} \leq D\varepsilon$ . We commute the transport equation (2.31) by  $\nabla$

$$\begin{aligned}\nabla_L \nabla_A \Upsilon_B &= -\frac{1}{2} \mathrm{tr}\chi \nabla_A \Upsilon_B - \frac{1}{2} \nabla_A \mathrm{tr}\chi \Upsilon_B + \nabla_A \nabla_B \log \Omega \\ &\quad - \nabla_A \hat{\chi}_{BC} \Upsilon_C - \hat{\chi}_{BC} \nabla_A \Upsilon_C + [\nabla_L, \nabla]_A \Upsilon_B,\end{aligned} \quad (4.35)$$

where by using formula (2.10) and (2.31) we have

$$[\nabla_L, \nabla]_A \Upsilon_B = -\frac{1}{2} \mathrm{tr}\chi \nabla_A \Upsilon_B - \hat{\chi}_{AC} \nabla_C \Upsilon_B - |\nabla(\log \Omega)|^2 + \chi \cdot \nabla \log \Omega \cdot \Upsilon - {}^* \beta_A {}^* \Upsilon_B.$$

Therefore, applying Lemma 3.11, using that  $\Upsilon = 0$  on  $S_1$ , the improved bounds and the additional bootstrap assumption, we obtain

$$\begin{aligned}\|\nabla \Upsilon\|_{L^2 L^\infty_v} &\lesssim \|\nabla^2 \log \Omega\|_{L^2(\mathcal{H})} + \|\nabla \log \Omega\|_{L^2_v L^4}^2 \\ &\quad + \|\Upsilon\|_{L^\infty(\mathcal{H})} (\|\nabla \chi\|_{L^2(\mathcal{H})} + \|\chi\|_{L^\infty L^2_v} \|\nabla \log \Omega\|_{L^2(\mathcal{H})} + \|\beta\|_{L^2(\mathcal{H})}) \\ &\quad + \|\nabla \Upsilon\|_{L^2 L^\infty_v} \|\hat{\chi}\|_{L^\infty L^2_v} \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon,\end{aligned}$$

which improves the additional bootstrap assumption and hence finishes the proof of Lemma 4.20.  $\square$



## 5. Higher regularity estimates

This section is dedicated to the proof of Proposition 2.40 and completes Step 3 in Section 2.13. We assume that  $(\mathcal{M}, \mathbf{g})$  is a smooth spacetime and  $\mathcal{H}$  a smooth null hypersurface foliated by a smooth geodesic foliation. We assume moreover that the following bounds hold on  $[1, v^*]$ ,

$$\begin{aligned} \|\Omega - 1\|_{L_v^\infty([1, v^*])L^\infty} + \mathcal{N}_1^{v, v^*}(\nabla \log \Omega) &\lesssim \varepsilon, \\ \|\Upsilon\|_{L_v^\infty([1, v^*])L^\infty} + \|\nabla \Upsilon\|_{L^2 L_v^\infty([1, v^*])} &\lesssim \varepsilon. \end{aligned} \quad (5.1)$$

For all  $m \geq 0$ , we will prove the following estimates

$$\sum_{l \leq m} \left( \|\nabla^l (\Omega - 1)\|_{L_v^\infty([1, v^*])L^2} + \|\nabla^l \Upsilon\|_{L_v^\infty([1, v^*])L^2} \right) \leq C \left( \|\mathcal{G}'\|_{C^{m+2}(\mathcal{H})}, m \right). \quad (5.2)$$

Moreover, we will also have the following estimates on the  $L$ -derivatives

$$\sum_{l \leq m} \|\nabla^l L^k(\Omega)\|_{L_v^\infty([1, v^*])L^2} \leq C \left( \|\mathcal{G}'\|_{C^{m+k+2}(\mathcal{H})}, m+k \right), \quad (5.3)$$

for all  $m \geq 0$  and all  $k \geq 0$ . This will complete the proof of Proposition 2.40.

Before turning to the proof of (5.2) and (5.3), we prove the following lemma that is a rewriting of equations (2.31) and (2.17). This will also be used in the proof of the local existence Theorem 2.37.

**Lemma 5.1.** *We have*

$$\nabla_L \Upsilon = -(\chi')^\dagger \cdot \Upsilon - \nabla \log \Omega, \quad (5.4)$$

$$\log \Omega = \mathbb{A}^{-1} (F'_1 + (F'_2)^\dagger \cdot \Upsilon + (F'_3)^\dagger \cdot \Upsilon \cdot \Upsilon + (F'_4)^\dagger \cdot \nabla \Upsilon), \quad (5.5)$$

where  $F'_1, F'_2, F'_3, F'_4$  are (contractions of) geodesic quantities, and  $u := \mathbb{A}^{-1} f$  denotes the solution of the following elliptic equation

$$\begin{aligned} \mathbb{A} u &= f - \int_{S_v} f, \\ \int_{S_v} u &= 0. \end{aligned}$$

**Remark 5.2.** *As far as higher regularity is concerned, we are not interested in proving sharp estimates. Thus, the specific structure of the terms  $(F'_i)^\dagger$  is not needed.*

*Proof.* Equation (5.4) is a rewriting of (2.31). Equation (5.5) is a rewriting of (2.22n). Namely, we have using the relations from Proposition 2.32 and the derivatives relation from Proposition 2.29

$$\begin{aligned} d\dot{v}\zeta &= d\dot{v}((\zeta')^\dagger + (\chi')^\dagger \cdot \Upsilon) \\ &= d\dot{v}'\zeta' + (\nabla'_L \zeta')^\dagger \cdot \Upsilon + (\text{tr}\chi')^\dagger (\zeta')^\dagger \cdot \Upsilon \\ &\quad - (\chi')^\dagger \cdot (\zeta')^\dagger \cdot \Upsilon + (\chi')^\dagger \cdot \nabla \Upsilon + (d\dot{v}'\chi')^\dagger \cdot \Upsilon \\ &\quad + \nabla'_L \chi' \cdot \Upsilon \cdot \Upsilon + (\text{tr}\chi')^\dagger (\chi')^\dagger \cdot \Upsilon \cdot \Upsilon - 2(\chi')^\dagger \cdot (\chi')^\dagger \cdot \Upsilon \cdot \Upsilon. \end{aligned}$$

Using the relations from Proposition 2.31, we have

$$\rho = \rho' + (\beta')^\dagger \cdot \Upsilon + (\alpha')^\dagger \cdot \Upsilon \cdot \Upsilon.$$

Using the relations from Proposition 2.32, we have

$$-\frac{1}{2} \hat{\chi} \cdot \hat{\chi} = -\frac{1}{2} (\hat{\chi}')^\dagger \cdot ((\hat{\chi}')^\dagger - 4(\zeta')^\dagger \cdot \Upsilon + 2\nabla \Upsilon - |\Upsilon|^2 (\chi')^\dagger).$$

Therefore, using the definition of  $\check{\rho}$  (2.8) and defining

$$\begin{aligned} F'_1 &:= -\text{d}^{\check{v}'}\zeta' + \rho' - \frac{1}{2}\hat{\chi}' \cdot \underline{\hat{\chi}}', \\ F'_2 &:= -\nabla'_{L'}\zeta' - \text{tr}\chi'\zeta' + \chi' \cdot \zeta' - (\text{d}^{\check{v}'}\chi')^\dagger + \beta' + 2\hat{\chi}' \cdot \zeta', \\ F'_3 &:= +\alpha' + \frac{1}{2}\hat{\chi}' \cdot \chi' - \frac{1}{2}|\hat{\chi}'|^2 - (\text{tr}\chi')^\dagger(\chi')^\dagger + 2(\chi')^\dagger \cdot (\chi')^\dagger, \\ F'_4 &:= -\text{tr}\chi'\check{g}' - 2\hat{\chi}', \end{aligned}$$

we have

$$\log \Omega = \mathbb{A}^{-1} (F'_1 + (F'_2)^\dagger \cdot \Upsilon + (F'_3)^\dagger \cdot \Upsilon \cdot \Upsilon + (F'_4)^\dagger \cdot \nabla' \Upsilon).$$

This finishes the proof of Lemma 5.1.  $\square$

*Proof of (5.2) and (5.3).* The proof of (5.2) goes by induction on  $m$ . The cases  $m = 0$  and  $m = 1$  were already obtained in Section 4. We prove the case  $m = 2$  and the cases  $m \geq 3$  are proved similarly and are left to the reader. In what follows, we use that the quantities  $F'_1, F'_2, F'_3, F'_4$  appearing in Lemma 5.1 are smooth in the geodesic foliation. More precisely, we are going to obtain bounds in terms of

$$\sum_{k \leq 2} \left( \|(\nabla')^k F'_1\|_{L^\infty(\mathcal{H})} + \|(\nabla')^k F'_2\|_{L^\infty(\mathcal{H})} + \|(\nabla')^k F'_3\|_{L^\infty(\mathcal{H})} + \|(\nabla')^k F'_4\|_{L^\infty(\mathcal{H})} \right).$$

For simplicity, we do not write the exact bound and this quantity shall be always implicitly included in the constants  $C$  appearing in the following.

First, applying Lemma 3.17 to elliptic equation (5.5), we have

$$\begin{aligned} \|\nabla'^2 \log \Omega\|_{L^\infty_v L^2} &\lesssim \|F'_1\|_{L^\infty(\mathcal{H})} + \|F'_2\|_{L^\infty(\mathcal{H})} \|\Upsilon\|_{L^\infty(\mathcal{H})} \\ &\quad + \|F'_3\|_{L^\infty(\mathcal{H})} \|\Upsilon\|_{L^\infty(\mathcal{H})}^2 + \|F'_4\|_{L^\infty(\mathcal{H})} \|\nabla' \Upsilon\|_{L^\infty_v L^2} \\ &\lesssim C(\varepsilon). \end{aligned}$$

Second, commuting equation (5.4) by  $\nabla'^2$ , we have schematically

$$\nabla'_L \nabla'^2 \Upsilon = -(\chi')^\dagger \cdot \nabla'^2 \Upsilon + G'(\Upsilon) \cdot (\nabla' \Upsilon + \nabla' \log \Omega) + \nabla'^3(\log \Omega) + [\nabla'_L, \nabla'^2] \Upsilon,$$

where  $G'(\Upsilon)$  denotes an arbitrary number of contractions of geodesic quantities with  $\Upsilon$ . Using commutation formula (2.10), the commutator can be schematically rewritten

$$\begin{aligned} [\nabla'_L, \nabla'^2] \Upsilon &= \nabla' [\nabla'_L, \nabla'] \Upsilon + [\nabla'_L, \nabla'] \nabla' \Upsilon \\ &= G'(\Upsilon) \cdot (\nabla' \Upsilon + \nabla' \log \Omega) \\ &\quad + (\chi')^\dagger \cdot \nabla'^2 \Upsilon + \nabla' \log \Omega \cdot \nabla'^2 \log \Omega. \end{aligned}$$

We therefore obtain the following schematic formula

$$\begin{aligned} \nabla'_L \nabla'^2 \Upsilon &= (\chi')^\dagger \cdot \nabla'^2 \Upsilon + \nabla'^3(\log \Omega) + \nabla' \log \Omega \cdot \nabla'^2 \log \Omega \\ &\quad + G'(\Upsilon) \cdot (\nabla' \Upsilon + \nabla' \log \Omega). \end{aligned}$$

Using Lemma 3.11, the assumptions (5.1), and the above formulas, we therefore get

$$\begin{aligned} \|\nabla'^2 \Upsilon\|_{L^\infty_v L^2} &\lesssim \|\chi'\|_{L^\infty(\mathcal{H})} \|\nabla'^2 \Upsilon\|_{L^1_v L^2} + \|\nabla'^3 \log \Omega\|_{L^1_v L^2} \\ &\quad + \|\nabla' \log \Omega\|_{L^\infty L^2_v} \|\nabla'^2 \log \Omega\|_{L^2(\mathcal{H})} \\ &\quad + \|G'(\Upsilon)\|_{L^\infty(\mathcal{H})} \left( \|\nabla' \Upsilon\|_{L^\infty_v L^2} + \|\nabla' \log \Omega\|_{L^\infty_v L^2} \right) \\ &\lesssim C + C \int_v \left( \|\nabla'^2 \Upsilon\|_{L^2(S_v)} + \|\nabla'^3 \log \Omega\|_{L^2(S_v)} \right) dv. \end{aligned} \tag{5.6}$$

On the other hand, commuting elliptic equation (5.5), with  $\nabla$ , we obtain schematically

$$\Delta \nabla \log \Omega = G'(\Upsilon) \cdot \nabla \Upsilon + (F_4')^\dagger \cdot \nabla^2 \Upsilon + [\Delta, \nabla] \log \Omega,$$

where using formula (2.15) and Propositions 2.31 and 2.32, the commutator can be rewritten

$$\begin{aligned} [\Delta, \nabla] \log \Omega &= -K \nabla \log \Omega \\ &= - \left( -\frac{1}{4} \text{tr} \chi \text{tr} \underline{\chi} + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \rho \right) \nabla \log \Omega \\ &= (G'(\Upsilon) + G'(\Upsilon) \cdot \nabla \Upsilon) \nabla \log \Omega. \end{aligned}$$

We therefore obtain the following formula

$$\Delta \nabla \log \Omega = G'(\Upsilon) \cdot (\nabla \Upsilon + \nabla \log \Omega) + G'(\Upsilon) \cdot \nabla \Upsilon \cdot \nabla \log \Omega + (F_4')^\dagger \cdot \nabla^2 \Upsilon.$$

Therefore, using Lemma 3.17, assumptions (5.1) and Sobolev Lemma 3.8, we have

$$\begin{aligned} \|\nabla^3 \log \Omega\|_{L^2(S_v)} &\lesssim \|G'(\Upsilon)\|_{L^\infty(\mathcal{H})} \left( \|\nabla \Upsilon\|_{L_v^\infty L^2} + \|\nabla \log \Omega\|_{L_v^\infty L^2} \right) \\ &\quad + \|G'(\Upsilon)\|_{L^\infty(\mathcal{H})} \|\nabla \log \Omega\|_{L_v^\infty L^4} \|\nabla \Upsilon\|_{L^4(S_v)} \\ &\quad + \|F_4'\|_{L^\infty(\mathcal{H})} \|\nabla^2 \Upsilon\|_{L^2(S_v)} \\ &\lesssim C \left( 1 + \|\nabla^2 \Upsilon\|_{L^2(S_v)} \right). \end{aligned}$$

Plugging this estimate into (5.6), we obtain

$$\|\nabla^2 \Upsilon\|_{L_v^\infty L^2} \lesssim C + C \int_v \|\nabla^2 \Upsilon\|_{L^2(S_v)} \, dv.$$

By a Grönwall argument, we deduce

$$\|\nabla^2 \Upsilon\|_{L_v^\infty L^2} \lesssim C.$$

This finishes the proof of the Lemma in the case  $m = 2$ .

To prove estimates (5.3), we commute elliptic equation (5.5) for  $\log \Omega$  with  $L$  and using the formula (5.4), the right-hand side can be expressed in terms of lower order derivatives of  $\log \Omega$ . Details are left to the reader.  $\square$



## 6. Proof of local existence

In this section, we prove Theorem 2.37 by showing a more general local existence theorem for equations of the type (5.4)-(5.5), where the unknown is the function

$$(v, \omega) \in [1, 2] \times \mathbb{S}^2 \mapsto s(v, \omega).$$

This strategy is similar to writing a foliation by geometric flows as family of graphs and was already used in [Nic04] and [Sau08].

### 6.1 Geometric setup and theorem

Let  $v_0 \in [1, 2)$ , and define

$$\mathcal{C} := \{(v, \omega) \in [v_0, 2] \times \mathbb{S}^2\}, \quad S_v := \{v\} \times \mathbb{S}^2 \subset \mathcal{C}.$$

Similarly, let

$$\mathcal{C}' := \{(s, \omega) \in [1, 5/2] \times \mathbb{S}^2\}, \quad S'_s := \{s\} \times \mathbb{S}^2 \subset \mathcal{C}'.$$

Let  $g$  be a smooth degenerate metric on  $\mathcal{C}'$  such that the induced metric on  $S'_s$  is Riemannian. Let  $F'_1, F'_2, F'_3, F'_4$  be respectively a fixed scalar field, a fixed 1, 2 and 2  $S'_s$ -tangent tensor.

For a function  $s : \mathcal{C} \rightarrow [1, 5/2]$ , we define

$$\begin{aligned} \Phi(s) : \mathcal{C} &\rightarrow \mathcal{C}' \\ (v, \omega) &\mapsto (s(v, \omega), \omega). \end{aligned}$$

For a function  $s : \mathcal{C} \rightarrow [1, 5/2]$ , we define  $\mathcal{g}(s)$  to be the induced Riemannian metric on  $S_v$  by  $\Phi(s)^*g$ , and  $F_i(s) := (\Phi(s)^*F'_i)^\dagger$ , where  $\dagger$  denotes the projection of  $\mathcal{C}$ -tangent tensors to  $S_v$ -tangent tensors defined in Definition 2.26.

**Remark 6.1.** *By the degeneracy of the metric  $g$  and the definition of  $\Phi$ , the metric  $\mathcal{g}(s)$  depends only on  $s$  and not on derivatives of  $s$ . Similarly, since the tensors  $F'_i$  are  $S'_s$ -tangent and by the definition of  $\Phi$ , the tensors  $F_i(s)$  only depend on  $s$ .*

In what follows, our goal is to prove local existence for the system of quasilinear elliptic transport equations in  $\mathcal{C}$

$$\begin{aligned} \log \Omega &= \Delta^{-1} (F_1(s) + F_2(s) \cdot \nabla s + F_3(s) \cdot \nabla s \cdot \nabla s + F_4(s) \cdot \nabla^2 s), \\ \partial_v s &= \Omega^{-1}, \end{aligned} \tag{6.1}$$

where  $\nabla$  and  $\Delta$  are respectively the covariant derivative and the Laplacian associated to  $\mathcal{g}(s)$  and where for a Riemannian 2-sphere  $(S, \mathcal{g})$ ,  $u := \Delta^{-1}f$  is the solution of

$$\begin{aligned} \Delta u &= f - \int_S f, \\ \int_S u &= 0, \end{aligned}$$

with integrals taken with respect to the metric  $\mathcal{g}$ .

For ease of notation, we shall define  $F(s, \nabla s, \nabla^2 s)$  to denote schematically

$$F(s, \nabla s, \nabla^2 s) := F_1(s) + F_2(s) \cdot \nabla s + F_3(s) \cdot \nabla s \cdot \nabla s + F_4(s) \cdot \nabla^2 s.$$

Let  $s_0$  be a function  $S_{v_0} \rightarrow [1, 5/2]$  and extend it to  $\mathcal{C}$  by requiring that  $\partial_v s_0 = 0$ . Define  ${}^{(0)}\nabla$  and  ${}^{(0)}\Delta$  to be respectively the covariant derivative and Laplacian on all spheres  $S_v$  associated to  $\mathcal{g}(s_0)$ . Define  $\log \Omega_0$  on all spheres  $S_v$  by

$$\log \Omega_0 := {}^{(0)}\Delta^{-1} \left( F(s_0, {}^{(0)}\nabla s_0, {}^{(0)}\nabla^2 s_0) \right).$$

We have the following result.



**Theorem 6.2.** *Assume that  $s_0 \in H^5(S_{v_0})$  and that  $|\log \Omega_0| \leq 1/100$ . Assume moreover that  $(S_{v_0}, \mathcal{g}(s_0))$  is a weakly spherical 2-sphere of radius  $v_0$  (see Definition 3.14) with constants such that the Bochner and Hodge estimates from Lemmas 3.17, 3.19, 3.20 hold true. There exists*

$$\delta \left( \|(s_0 - 5/2)^{-1}\|_{L^\infty(S_{v_0})}, \|s_0\|_{H^5(S_{v_0})}, \|F'_i\|_{C^3} \right) > 0,$$

and

$$s \in C^0([v_0, v_0 + \delta], H^5) \cap C^1([v_0, v_0 + \delta], H^4),$$

such that on  $\mathcal{C}$  the system of equations (6.1) is satisfied for  $s$ , with the initial condition  $s|_{S_{v_0}} = s_0$ . Moreover, we have

$$|\log \Omega| < 1/10. \quad (6.2)$$

## 6.2 Proof of Theorem 6.2

The proof goes by a classical Banach-Picard fixed-point theorem.

**Definition of the iteration** As defined previously, we have  $s_0(v, \omega) := s_0(\omega)$  and  $\Omega_0(v, \omega) := \Omega_0(\omega)$ . For all  $n \geq 0$ , we define  $s_{n+1}$  and  $\log \Omega_{n+1}$  on  $\mathcal{C}$  by

$$s_{n+1}(v, \omega) := s_0(\omega) + \int_{v^*}^v \Omega_n^{-1}(v', \omega) dv', \quad (6.3)$$

$$\log(\Omega_{n+1}) := {}^{(n+1)}\Delta^{-1} \left( F \left( s_{n+1}, {}^{(n+1)}\nabla s_{n+1}, {}^{(n+1)}\nabla^2 s_{n+1} \right) \right). \quad (6.4)$$

We define

$$M_0 := \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l s_0 \right\|_{L^2(S_{v_0})},$$

and

$$M_n^\delta := \sup_{v_0 \leq v \leq v_0 + \delta} \left( \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (s_n(v) - s_0) \right\|_{L^2(S_{v_0})} + \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l \log \Omega_n(v) \right\|_{L^2(S_{v_0})} \right).$$

**Boundedness of the iteration** In this section, we show that if

$$\delta < \delta \left( \|(s_0 - 5/2)^{-1}\|_{L^\infty(S_{v_0})}, M_0, \|F\|_{C^3} \right),$$

then, defining  $M := 2M_0$ , we have for all  $n \geq 0$

$$M_n^\delta \leq M, \quad (6.5)$$

$$\sup_{v_0 \leq v \leq v_0 + \delta} s_n(v) < 5/2, \quad (6.6)$$

$$\sup_{v_0 \leq v \leq v_0 + \delta} \|\log \Omega_n\|_{L^\infty(S_v)} \leq 1/10. \quad (6.7)$$

We argue by induction and assume that these assumptions hold for an arbitrary  $n \in \mathbb{N}$ . First, using the transport equation (6.3) and estimate (6.7), if  $\delta$  is small enough depending on  $\|(s_0 - 5/2)^{-1}\|_{L^\infty(S_{v_0})}$ , we have

$$\sup_{v_0 \leq v \leq v_0 + \delta} s_{n+1}(v) < 5/2,$$

and therefore (6.6) is proved for  $n + 1$ .

Second, using estimate (6.5) and (6.7) at step  $n$ , we obtain that

$$\sup_{v_0 \leq v \leq v_0 + \delta} \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (\Omega_n^{-1}) \right\|_{L^2(S_{v_0})} \leq C(M).$$

Therefore, deriving and estimating equation (6.3), we obtain

$$\sup_{v_0 \leq v \leq v_0 + \delta} \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (s_{n+1}(v) - s_0) \right\|_{L^2(S_{v_0})} \leq \delta C(M). \quad (6.8)$$

Third, we can rewrite equation (6.4)

$${}^{(0)}\Delta (\log \Omega_{n+1} - \log \Omega_0) = \left( {}^{(0)}\Delta - {}^{(n+1)}\Delta \right) (\log \Omega_{n+1} - \log \Omega_0) + E_{n+1}, \quad (6.9)$$

with

$$\begin{aligned} E_{n+1} &:= \left( {}^{(0)}\Delta - {}^{(n+1)}\Delta \right) (\log \Omega_0) \\ &\quad + F(s_{n+1}, {}^{(n+1)}\nabla s_{n+1}, {}^{(n+1)}\nabla^2 s_{n+1}) - F(s_0, {}^{(0)}\nabla s_0, {}^{(0)}\nabla^2 s_0). \end{aligned}$$

From the weak sphericity assumption from Theorem 6.2, we can apply the elliptic estimate from Lemma 3.20 and we therefore deduce that

$$\begin{aligned} &\left\| {}^{(0)}\nabla (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} + \left\| \log \Omega_{n+1} - \log \Omega_0 - \overline{\log \Omega_{n+1}} \right\|_{L^2(S_v)} \\ &\lesssim \left\| \left( {}^{(0)}\Delta - {}^{(n+1)}\Delta \right) (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^{4/3}(S_v)} + \|E_{n+1}\|_{L^{4/3}(S_v)}. \end{aligned} \quad (6.10)$$

Using Remark 6.1, we have

$$\begin{aligned} \left| \int_{(S_v, \mathcal{g}(s_0))} \log \Omega_{n+1} \right| &\lesssim \left| \int_{(S_v, \mathcal{g}(s_0))} \log \Omega_{n+1} - \int_{(S_v, \mathcal{g}(s_{n+1}))} \log \Omega_{n+1} \right| \\ &\lesssim C \|s_{n+1} - s_0\|_{L^\infty(S_v)} \|\log \Omega_{n+1}\|_{L^2(S_v)} \\ &\lesssim \delta C(M) \|\log \Omega_{n+1}\|_{L^2(S_v)} \\ &\lesssim \delta C(M) \|\log \Omega_{n+1} - \log \Omega_0\|_{L^2(S_v)} + \delta C(M). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\left\| \left( {}^{(0)}\Delta - {}^{(n+1)}\Delta \right) (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^{4/3}(S_v)} \\ &\lesssim C \left( \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l s_0 \right\|_{L^2(S_v)}, \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l s_{n+1} \right\|_{L^2(S_v)} \right) \\ &\quad \times \left( \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (s_{n+1} - s_0) \right\|_{L^2(S_v)} \right) \left( \sum_{l \leq 2} \left\| {}^{(0)}\nabla^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \right) \\ &\lesssim \delta C(M) \left( \sum_{l \leq 2} \left\| {}^{(0)}\nabla^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \right), \end{aligned}$$

and similarly

$$\|E_{n+1}\|_{L^{4/3}(S_v)} \lesssim \delta C(M).$$

Therefore, for  $\delta C(M)$  small enough we can perform a standard absorption argument in the elliptic estimate (6.10) and we finally deduce

$$\left\| {}^{(0)}\nabla (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} + \|\log \Omega_{n+1} - \log \Omega_0\|_{L^2(S_v)} \lesssim \delta C(M).$$

We now prove that we have the following bounds for the remaining higher order derivatives

$$\sum_{l=2}^5 \left\| {}^{(0)}\nabla^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \lesssim \delta C(M).$$

The proof goes by induction on  $l$  for  $l = 2$  to  $l = 5$ . We only do the case  $l = 5$  assuming that the bounds for  $l \leq 4$  have been obtained, since it will be clear that the proof for the other cases is almost identical. Commuting equation (6.9) with  $(0)\nabla^3$  gives

$$\begin{aligned} (0)\nabla^3 (0)\nabla^3 (\log \Omega_{n+1} - \log \Omega_0) &= [(0)\nabla^3, (0)\nabla^3] (\log \Omega_{n+1} - \log \Omega_0) \\ &\quad + (0)\nabla^3 \left( \left( (0)\nabla^3 - (n+1)\nabla^3 \right) (\log \Omega_{n+1} - \log \Omega_0) \right) \\ &\quad + (0)\nabla^3 (E_{n+1}). \end{aligned} \quad (6.11)$$

We have the following three estimates, which proofs are left to the reader

$$\begin{aligned} \left\| [(0)\nabla^3, (0)\nabla^3] (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} &\lesssim \delta C(M), \\ \left\| (0)\nabla^3 E_{n+1} \right\|_{L^2(S_v)} &\lesssim \delta C(M), \end{aligned}$$

and

$$\begin{aligned} &\left\| (0)\nabla^3 \left( \left( (0)\nabla^3 - (n+1)\nabla^3 \right) (\log \Omega_{n+1} - \log \Omega_0) \right) \right\|_{L^2(S_v)} \\ &\lesssim \delta C(M) \left( \sum_{l \leq 5} \left\| \nabla^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \right). \end{aligned}$$

Using these together with a Bochner estimate similar to Lemma 3.17 for tensors of arbitrary type applied to equation (6.11), we deduce that

$$\begin{aligned} \left\| (0)\nabla^5 (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} &\lesssim \left\| [(0)\nabla^3, (0)\nabla^3] (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \\ &\quad + \left\| (0)\nabla^3 \left( \left( (0)\nabla^3 - (n+1)\nabla^3 \right) (\log \Omega_{n+1} - \log \Omega_0) \right) \right\|_{L^2(S_v)} \\ &\quad + \left\| (0)\nabla^3 E_{n+1} \right\|_{L^2(S_v)} \\ &\lesssim \delta C(M) + \delta C(M) \left\| \nabla^5 (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)}, \end{aligned}$$

which, performing a standard absorption argument, gives the desired bound.

We therefore have proved that

$$M_{n+1}^\delta \leq M_0 + \delta C(M) \leq M,$$

provided that  $\delta$  has been chosen small enough.

Moreover, by Sobolev embedding, we deduce that

$$\begin{aligned} \|\log \Omega_{n+1} - \log \Omega_0\|_{L^\infty(S_v)} &\lesssim \sum_{l \leq 5} \left\| (0)\nabla^l (\log \Omega_{n+1} - \log \Omega_0) \right\|_{L^2(S_v)} \\ &\leq \delta C(M). \end{aligned}$$

Therefore, for  $\delta$  such that  $\delta C(M) < 1/100$ , and using the assumption  $|\log \Omega_0| \leq 1/100$ , this proves the bound (6.7) for  $n + 1$ . This finishes the proof of the boundedness of the sequence  $s_{n+1}$ .

**Contraction of the iteration** We define

$$\Delta_{n+1}^\delta := \sup_{v_0 \leq v \leq v_0 + \delta} \left( \sum_{l \leq 5} \left\| (0)\nabla^l (s_{n+1} - s_n) \right\|_{L^2(S_v)} + \sum_{l \leq 5} \left\| (0)\nabla^l (\log \Omega_{n+1} - \log \Omega_n) \right\| \right),$$

and we show, provided that  $\delta$  has been chose small enough, that we have

$$\Delta_{n+1}^\delta \leq \kappa \Delta_n^\delta,$$

with  $\kappa < 1$ . The proof follows the lines of the proof of the boundedness. First, we have

$$\sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (\Omega_n^{-1} - \Omega_{n-1}^{-1}) \right\|_{L^2(S_v)} \leq C(M) \left( \sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (\log \Omega_n - \log \Omega_{n-1}) \right\|_{L^2(S_v)} \right).$$

Therefore we deduce using equation (6.3) that

$$\sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (s_{n+1} - s_n) \right\|_{L^2(S_v)} \leq \delta C(M) \Delta_n^\delta.$$

Performing a similar elliptic estimate as in the proof of the boundedness of  $s_n$ , we therefore deduce that

$$\sum_{l \leq 5} \left\| {}^{(0)}\nabla^l (\log \Omega_{n+1} - \log \Omega_n) \right\|_{L^2(S_v)} \leq \delta C(M) \Delta_n^\delta.$$

Thus, for  $\delta$  such that  $\delta C(M) < 1$ , we deduce the result and this finishes the proof of the contraction and of Theorem 6.2.

### 6.3 Proof of Theorem 2.37

In this section, we show how Theorem 2.37 follows from Theorem 6.2.

We define  $v_0 := v^*$  and  $s_0 := s^* = s|_{v^*}$ , and  $F'_1, F'_2, F'_3, F'_4$  to be the tensors defined in Lemma 5.1. By assumptions and since the  $F'_i$  have been defined as in Lemma 5.1, the quantity  $\log \Omega_0$  defined in Section 6.1 coincides with  $\log \Omega|_{S_{v_0}}$ . By assumption of Theorem 2.37, we have  $s_0 \in H^5(S_{v_0})$ ,  $|\log \Omega_0| \leq 1/100$  and that  $(S_{v_0}, \mathcal{g}(s_0))$  is a weakly spherical 2-sphere of radius  $v_0$ .

Applying Theorem 6.2, there exists  $\delta > 0$  and a function  $s \in C^1([v_0, v_0 + \delta], \mathbb{S}^2)$ , satisfying the system of equations (6.1). Since by estimate (6.2) we have  $|\partial_v s - 1| < 1/5$ , the map  $\Phi(s) : [v_0, v_0 + \delta] \times \mathbb{S}^2 \rightarrow [1, 5/2] \times \mathbb{S}^2$  admits a  $C^1$ -inverse by the global inverse theorem. This defines a  $C^1$ -function  $v$  in geodesic coordinates, and therefore on  $\mathcal{H}$ , taking values from  $v^* = v_0$  to  $v^* + \delta$ , which, using the conclusion of Theorem 6.2, is regular. Since equations (6.1) are satisfied and since the  $F'_i$  have been defined as in Lemma 5.1, we deduce that  $(S_v)_{v^* \leq v \leq v^* + \delta}$  is a canonical foliation. In case  $(S_v)_{1 \leq v \leq v^*}$  was a regular canonical foliation, using equations (6.1), one deduces that  $(S_v)_{v^* \leq v \leq v^* + \delta}$  is a regular extension thereof. This finishes the proof of Theorem 2.37.



## A. Proof of Propositions 2.31 and 2.32

In this section we prove the formulas from Proposition 2.31 and 2.32. The following computations are standard and can be found in various forms in [AS16], [Nic04], [Sau08] for instance. In what follows, we use the formulas from [CK93] pp. 149-150. We have

$$\alpha_{AB} = \mathbf{R}(L, e_A, L, e_B) = \mathbf{R}(L, e'_A + \Upsilon_A L, L, e'_B + \Upsilon_B L) = \mathbf{R}(L, e'_A, L, e'_B),$$

and

$$\begin{aligned} \beta_A &= \frac{1}{2} \mathbf{R}(e_A, L, \underline{L}, L) \\ &= \frac{1}{2} \mathbf{R}(e'_A + \Upsilon_A L, L, \underline{L}' + 2\Upsilon_A e'_A + |\Upsilon|^2 L, L) \\ &= \frac{1}{2} \mathbf{R}(e'_A, L, \underline{L}', L) + \Upsilon_A \mathbf{R}(e'_A, L, e'_B, L) \\ &= \beta'_A + \Upsilon_B \alpha_{AB}, \end{aligned}$$

and

$$\begin{aligned} \rho &= \frac{1}{4} \mathbf{R}(\underline{L}, L, \underline{L}, L) \\ &= \frac{1}{4} \mathbf{R}(\underline{L}' + 2\Upsilon_A e'_A + |\Upsilon| L, L, \underline{L}' + 2\Upsilon_B e'_B + |\Upsilon|^2 L, L) \\ &= \frac{1}{4} \mathbf{R}(\underline{L}', L, \underline{L}', L) + \frac{1}{2} \Upsilon_A \mathbf{R}(e'_A, L, \underline{L}', L) + \frac{1}{2} \Upsilon_B \mathbf{R}(\underline{L}', L, e'_B, L) + \Upsilon_A \Upsilon_B \mathbf{R}(e'_A, L, e'_B, L) \\ &= \rho' + (\beta')^\dagger \cdot \Upsilon + (\alpha')^\dagger \cdot \Upsilon \cdot \Upsilon. \end{aligned}$$

Following the previous computation for  $\rho$  we also obtain

$$\begin{aligned} \sigma &= \frac{1}{4} {}^* \mathbf{R}(\underline{L}, L, \underline{L}, L) \\ &= \sigma' + \Upsilon_A {}^* \mathbf{R}(e_A, L, \underline{L}', L) + \Upsilon_A \Upsilon_B {}^* \mathbf{R}(e_A, L, e_B, L) \\ &= \sigma' - ({}^* \beta')^\dagger \cdot (\Upsilon) - ({}^* \alpha')^\dagger \cdot \Upsilon \cdot \Upsilon. \end{aligned}$$

We have

$$\begin{aligned} \underline{\beta}_A &= \frac{1}{2} \mathbf{R}(e_A, \underline{L}, \underline{L}, L) \\ &= \frac{1}{2} \mathbf{R}(e'_A + \Upsilon_A L, \underline{L}' + 2\Upsilon_B e'_B + |\Upsilon|^2 L, \underline{L}' + 2\Upsilon_C e'_C + |\Upsilon|^2 L, L) \\ &= \frac{1}{2} \mathbf{R}(e'_A, \underline{L}', \underline{L}', L) + \frac{1}{2} \Upsilon_A \mathbf{R}(L, \underline{L}', \underline{L}', L) + \Upsilon_B \mathbf{R}(e'_A, e'_B, \underline{L}', L) + \Upsilon_C \mathbf{R}(e'_A, \underline{L}', e'_C, L) \\ &\quad + \Upsilon_A \Upsilon_B \mathbf{R}(L, e'_B, \underline{L}', L) + \Upsilon_A \Upsilon_C \mathbf{R}(L, \underline{L}', e'_C, L) + 2\Upsilon_B \Upsilon_C \mathbf{R}(e'_A, e'_B, e'_C, L) \\ &\quad + \frac{1}{2} |\Upsilon|^2 \mathbf{R}(e'_A, L, \underline{L}', L) + 2\Upsilon_A \Upsilon_B \Upsilon_C \mathbf{R}(L, e'_B, e'_C, L) + |\Upsilon|^2 \Upsilon_C \mathbf{R}(e'_A, L, e'_C, L) \\ &= \underline{\beta}'_A - 2\Upsilon_A \rho' + 2 {}^* \Upsilon_A \sigma' + (-\Upsilon_A \rho' + {}^* \Upsilon_A \sigma') - 2\Upsilon_A \Upsilon \cdot (\underline{\beta}')^\dagger + 2\Upsilon_A \Upsilon \cdot (\underline{\beta}')^\dagger \\ &\quad - 2 {}^* \Upsilon_A \Upsilon \cdot ({}^* \beta')^\dagger + |\Upsilon|^2 \beta'_A - 2\Upsilon_A \Upsilon \cdot \Upsilon \cdot (\alpha')^\dagger + |\Upsilon|^2 \Upsilon \cdot (\alpha')^\dagger_A \\ &= \underline{\beta}'_A - 3\Upsilon_A \rho' + 3 {}^* \Upsilon_A \sigma' - 2 {}^* \Upsilon_A \Upsilon \cdot ({}^* \beta')^\dagger + |\Upsilon|^2 \beta'_A - 2\Upsilon_A \Upsilon \cdot \Upsilon \cdot (\alpha')^\dagger + |\Upsilon|^2 \Upsilon \cdot (\alpha')^\dagger_A. \end{aligned}$$

This finishes the proof of Proposition 2.31. We turn to the connection coefficients. We have immediately  $\chi_{AB} = \chi'_{AB}$ . We also have

$$\begin{aligned} \zeta_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_A L, \underline{L}) \\ &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e'_A + \Upsilon_A L} L, \underline{L}' + 2\Upsilon_B e'_B + |\Upsilon|^2 L) \\ &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e'_A} L, \underline{L}') + \Upsilon_B \mathbf{g}(\mathbf{D}_{e'_A} L, e'_B) \\ &= \zeta'_A + \Upsilon \cdot \chi_A, \end{aligned}$$

and

$$\begin{aligned}
 \underline{\eta}_A &= \frac{1}{2} \mathbf{g}(\mathbf{D}_L \underline{L}, e_A) \\
 &= \frac{1}{2} \mathbf{g}(\mathbf{D}_L(\underline{L}' + 2\Upsilon_B e'_B + |\Upsilon|^2 L, e'_A + \Upsilon_A L)) \\
 &= \frac{1}{2} \mathbf{g}(\mathbf{D}_L \underline{L}', e'_A) + \mathbf{g}(\mathbf{D}_L(\Upsilon_B e'_B), e'_A) \\
 &= \underline{\eta}'_A + \nabla_L \Upsilon_A.
 \end{aligned}$$

Finally, we have

$$\begin{aligned}
 \underline{\chi}_{AB} &= \mathbf{g}(\mathbf{D}_A \underline{L}, e_B) \\
 &= \mathbf{g}(\mathbf{D}_{e_A}(\underline{L}' + 2\Upsilon_C e_C - |\Upsilon|^2 L), e_B) \\
 &= 2\nabla_A \Upsilon_B + \mathbf{g}(\mathbf{D}_{e_A}(\underline{L}' - |\Upsilon|^2 L), e_B) \\
 &= 2\nabla_A \Upsilon_B + \mathbf{g}(\mathbf{D}_{e'_A + \Upsilon_A L}(\underline{L}' - |\Upsilon|^2 L), e'_B + \Upsilon_B L) \\
 &= 2\nabla_A \Upsilon_B + \mathbf{g}(\mathbf{D}_{e'_A} \underline{L}', e'_B) + \Upsilon_A \mathbf{g}(\mathbf{D}_L \underline{L}', e'_B) + \Upsilon_B \mathbf{g}(\mathbf{D}_{e'_A} \underline{L}', L) - |\Upsilon|^2 \mathbf{g}(\mathbf{D}_{e'_A} L, e'_B) \\
 &= 2\nabla_A \Upsilon_B + \underline{\chi}'_{AB} + 2\Upsilon_A \underline{\eta}'_B - 2\Upsilon_B \zeta'_A - |\Upsilon|^2 \chi'_{AB}.
 \end{aligned}$$

This finishes the proof of Proposition 2.32.

## B. Proof of Lemmas 3.5 and 3.20

### B.1 Proof of Lemma 3.5

This section is dedicated to the proof of Lemma 3.5.

In fact, we prove the following more general estimate

$$\|\nabla F\|_{H^s(S)} \lesssim \|F\|_{H^{s+1}(S)}, \quad (\text{B.1})$$

for  $-1 < s < 0$ .

**Remark B.1.** *As it will be clear from what follows, the proof below does not work for other ranges of exponents  $s$ , and would require additional regularity assumptions on the 2-sphere  $S$ .*

From Proposition 2.3 in [Sha14], we have the following characterisation of  $H^s(S)$  using the Littlewood-Paley projectors defined in Section 3.2

$$\|F\|_{H^s(S)}^2 \simeq \sum_{k \geq 0} 2^{2sk} \|P_k F\|_{L^2(S)}^2 + \|P_{<0} F\|_{L^2(S)}^2. \quad (\text{B.2})$$

From Section 2.2 and Proposition 2.1 in [Sha14], we recall the following properties of the Littlewood-Paley projection operators defined in Section 3.2. For all  $k \in \mathbb{Z}$ , we have

$$P_k = P_k P_{k-1} + P_k P_k + P_k P_{k+1}, \quad (\text{B.3})$$

and for  $F$  an  $S$ -tangent tensor and for all  $k \in \mathbb{Z}$ , we have

$$\|P_k F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}, \quad \|P_{<0} F\|_{L^2(S)} \lesssim \|F\|_{L^2(S)}, \quad (\text{B.4})$$

and,

$$\begin{aligned} \|P_k \nabla F\|_{L^2(S)} &\lesssim 2^k \|F\|_{L^2(S)}, & \|\nabla P_k F\|_{L^2(S)} &\lesssim 2^k \|F\|_{L^2(S)}, \\ \|P_{<0} \nabla F\|_{L^2(S)} &\lesssim \|F\|_{L^2(S)}, & \|\nabla P_{<0} F\|_{L^2(S)} &\lesssim \|F\|_{L^2(S)}. \end{aligned} \quad (\text{B.5})$$

We turn to the proof of estimate (B.1). Using (3.4), (B.2) and (B.5), we have

$$\begin{aligned} \|\nabla F\|_{H^s(S)}^2 &\lesssim \sum_{k \geq 0} 2^{2sk} \|P_k \nabla F\|_{L^2(S)}^2 + \|P_{<0} \nabla F\|_{L^2(S)}^2 \\ &\lesssim \sum_{k \geq 0} 2^{2sk} \|P_k \nabla P_{>k} F\|_{L^2}^2 + \sum_{k \geq 0} 2^{2sk} \|P_k \nabla P_{\leq k} F\|_{L^2(S)}^2 + \|F\|_{L^2(S)}^2. \end{aligned} \quad (\text{B.6})$$

The first term in the right-hand side of (B.6) can be estimated using (B.3), (B.5) and that  $-1 < s < 0$

$$\begin{aligned} \sum_{k \geq 0} 2^{2sk} \|P_k \nabla P_{>k} F\|_{L^2(S)}^2 &\lesssim \sum_{k \geq 0} 2^{2(s+1)k} \|P_{>k} F\|_{L^2(S)}^2 \\ &\lesssim \sum_{k \geq 0} \sum_{l > k} 2^{2(s+1)k} \|P_l F\|_{L^2(S)}^2 \\ &\lesssim \sum_{l > 0} 2^{2(s+1)l} \|P_l F\|_{L^2(S)}^2 \sum_{k=0}^{l-1} 2^{2(s+1)(k-l)} \\ &\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \|P_l F\|_{L^2(S)}^2. \end{aligned}$$

For the second term in the right-hand side of (B.6), using (B.4), we first write the following decomposition

$$\begin{aligned} \|\nabla P_{\leq k} F\|_{L^2(S)}^2 &= \sum_{0 \leq l \leq k} \sum_{0 \leq l' \leq k} \int_S \nabla P_l F \cdot \nabla P_{l'} F + 2 \sum_{0 \leq l \leq k} \int_S \nabla P_l F \cdot \nabla P_{<0} F \\ &\quad + \int_S \nabla P_{<0} F \cdot \nabla P_{<0} F. \end{aligned}$$



The first term can be estimated, using that  $\Delta$  preserves the support of the projectors  $P_k$  (see also Section 2.2 in [Sha14]) and (3.4),

$$\begin{aligned}
 \sum_{0 \leq l \leq k} \sum_{0 \leq l' \leq k} \int_S \nabla P_l F \cdot \nabla P_{l'} F &= - \sum_{0 \leq l \leq k} \sum_{0 \leq l' \leq k} \int_S P_l F \Delta P_{l'} F \\
 &= - \sum_{0 \leq l \leq k} \sum_{l' = l-1}^{l+1} \int_S P_l F \Delta P_{l'} F \\
 &\lesssim \sum_{0 \leq l \leq k} \sum_{l' = l-1}^{l+1} 2^{2l} \|P_l F\|_{L^2(S)} \|P_{l'} F\|_{L^2(S)} \\
 &\lesssim \sum_{0 \leq l \leq k+1} 2^{2l} \|P_l F\|_{L^2(S)}^2,
 \end{aligned}$$

and similarly, we deduce for the last two terms, using (B.4) and (B.5)

$$2 \sum_{0 \leq l \leq k} \int_S \nabla P_l F \cdot \nabla P_{<0} F + \int_S \nabla P_{<0} F \cdot \nabla P_{<0} F \lesssim \|F\|_{L^2(S)}^2.$$

Using this, (B.4) and that  $-1 < s < 0$ , we therefore deduce that for the second term of (B.6) we have

$$\begin{aligned}
 \sum_{k \geq 0} 2^{2sk} \|P_k \nabla P_{\leq k} F\|_{L^2(S)}^2 &\lesssim \sum_{k \geq 0} 2^{2sk} \|\nabla P_{\leq k} F\|_{L^2(S)}^2 \\
 &\lesssim \sum_{k \geq 0} \sum_{0 \leq l \leq k+1} 2^{2sk} 2^{2l} \|P_l F\|_{L^2(S)}^2 + \sum_{k \geq 0} 2^{2sk} \|F\|_{L^2(S)}^2 \\
 &\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \|P_l F\|_{L^2(S)}^2 \left( \sum_{k \geq l-1} 2^{2s(k-l)} \right) + \|F\|_{L^2(S)}^2 \\
 &\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \|P_l F\|_{L^2(S)}^2 + \|F\|_{L^2(S)}^2.
 \end{aligned}$$

Finally, plugging the above estimates into (B.6) and using (B.2), we obtain

$$\begin{aligned}
 \|\nabla F\|_{H^s(S)}^2 &\lesssim \sum_{l \geq 0} 2^{2(s+1)l} \|P_l F\|_{L^2(S)}^2 + \|F\|_{L^2(S)}^2 \\
 &\lesssim \|F\|_{H^{s+1}(S)}^2.
 \end{aligned}$$

This finishes the proof of Lemma 3.5.

## B.2 Proof of Lemma 3.20

This section is dedicated to the proof of Lemma 3.20. We assume that  $f$  is a scalar function satisfying the elliptic equation (3.6)

$$\Delta f = \text{div} P + h.$$

Multiplying equation (3.6) by  $f - \bar{f}$  and integrating by part, we have

$$\|\nabla f\|_{L^2(S)}^2 \leq \|P\|_{L^2(S)} \|\nabla f\|_{L^2(S)} + \|h\|_{L^{4/3}(S)} \|f - \bar{f}\|_{L^4(S)}.$$

Using Lemma 3.19, we have the following Poincaré inequality

$$\|f - \bar{f}\|_{L^2(S)} = \|(f - \bar{f}, 0)\|_{L^2(S)} = \|({}^* \mathcal{D}_1)^{-1}(\nabla f)\|_{L^2(S)} \lesssim \|\nabla f\|_{L^2(S)}.$$

Therefore, using Sobolev Lemma 3.8, we have

$$\begin{aligned}
 \|\nabla f\|_{L^2(S)}^2 + \|f - \bar{f}\|_{L^2(S)}^2 &\lesssim \|P\|_{L^2(S)} \|\nabla f\|_{L^2(S)} \\
 &\quad + \|h\|_{L^{4/3}(S)} (\|\nabla f\|_{L^2(S)} + \|f - \bar{f}\|_{L^2(S)}),
 \end{aligned}$$

and the bound holds by a standard absorption argument. The bound on  $\mathcal{H}$  follows by integration in  $v$ . This finishes the proof of Lemma 3.20.

# **IV** The global nonlinear stability of Minkowski space for characteristic initial data



# 1. Introduction

## 1.1 Einstein equations and the stability of Minkowski space

A Lorentzian 4-dimensional manifold  $(\mathcal{M}, \mathbf{g})$  is called a *vacuum spacetime* if it solves the Einstein vacuum equations

$$\text{Ric}(\mathbf{g}) = 0, \tag{1.1}$$

where  $\text{Ric}(\mathbf{g})$  denotes the Ricci tensor of the Lorentzian metric  $\mathbf{g}$ . For the metric components  $\mathbf{g}_{\mu\nu}$  in general coordinates, equation (1.1) writes as a system of non-linear coupled partial differential equations of order 2 for  $\mathbf{g}_{\mu\nu}$ . In so-called *wave coordinates*, it can be shown that (1.1) is a system of nonlinear wave equations. It therefore admits a well-posed initial value formulation (or *Cauchy problem*).

Cauchy data for equations (1.1) are classically described by a triplet  $(\Sigma, g, k)$  such that

- ◊  $(\Sigma, g)$  is a 3-dimensional Riemannian manifold,
- ◊  $k$  is a symmetric covariant 2-tensor on  $\Sigma$ ,
- ◊  $(g, k)$  satisfy so-called constraint equations on  $\Sigma$ .<sup>1</sup>

The seminal well-posedness results for the Cauchy problem obtained in [Fou52, CG69] ensure that for any smooth Cauchy data, there exists a unique smooth *maximal globally hyperbolic development*  $(\mathcal{M}, \mathbf{g})$  solution of Einstein equations (1.1) such that  $\Sigma \subset \mathcal{M}$  and  $g, k$  are respectively the first and second fundamental forms of  $\Sigma$  in  $\mathcal{M}$ . We refer to [Wal84, Chapter 10] for definitions and further discussions on the Cauchy problem in general relativity.

**Remark 1.1.** *Here and in the rest of this paper, a smooth or  $\mathcal{C}^\infty$  manifold admits by definition an atlas of charts such that all coordinates changes are  $\mathcal{C}^\infty$  with respect to the standard  $\mathcal{C}^\infty$ -topology of  $\mathbb{R}^n$ . As all manifolds we consider will be smooth submanifolds of a fixed smooth 4-dimensional manifold  $\mathcal{M}$  and as all vector bundles we consider will be constructed upon  $\text{T}\mathcal{M}$  and  $\text{T}^*\mathcal{M}$ , we shall assume that such an atlas is fixed on  $\mathcal{M}$ , which then canonically determines the  $\mathcal{C}^k$ -topology for all tensors on all smooth submanifolds of  $\mathcal{M}$  in this paper.*

The prime example of a vacuum spacetime is *Minkowski spacetime*

$$\mathcal{M} = \mathbb{R}^4, \quad \mathbf{g} = -(\text{d}x^0)^2 + (\text{d}x^1)^2 + (\text{d}x^2)^2 + (\text{d}x^3)^2 =: \boldsymbol{\eta},$$

for which Cauchy data are given by

$$\Sigma = \mathbb{R}^3, \quad g = (\text{d}x^1)^2 + (\text{d}x^2)^2 + (\text{d}x^3)^2 =: \delta, \quad k = 0.$$

We have the following global stability conjecture for Minkowski space.

**Conjecture** (Asymptotic stability of Minkowski space). *For Cauchy data  $(\Sigma, g, k)$  close to the Minkowski initial data  $(\mathbb{R}^3, \delta, 0)$ , the maximal globally hyperbolic development is geodesically complete and is asymptotic to Minkowski space  $(\mathbb{R}^4, \boldsymbol{\eta})$  in the limit along null and timelike geodesics.*

The breakthrough result which solved the asymptotic stability conjecture for Minkowski space is the following theorem. See Theorem 1.7 for a precised version.

**Theorem 1.2** (Stability of Minkowski space [CK93], version 1). *For Cauchy data  $(\Sigma, g, k)$  such that*

- ◊  $\Sigma$  is diffeomorphic to  $\mathbb{R}^3$ ,
- ◊  $(\Sigma, g, k)$  is asymptotically flat (i.e. tends to Minkowski initial data  $(\mathbb{R}^3, \delta, 0)$  when  $r \rightarrow \infty$ )
- ◊  $(g, k)$  are close to Minkowski initial data  $(\delta, 0)$  measured in an (weighted)  $L^2$ -sense,

*then its maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  is geodesically complete and admits global time and optical functions  $t$  and  $u$  such that, measured in these coordinates,  $\mathbf{g}$  is bounded and decays towards  $\boldsymbol{\eta}$ .*

<sup>1</sup>See [Wal84, equations (10.2.28), (10.2.30)].

A localised version of Theorem 1.2 was proved for initial data posed on the exterior of a 3-disk.

**Theorem 1.3** (Exterior stability of Minkowski space [KN03], rough version). *For Cauchy data  $(\Sigma, g, k)$  such that*

◇  $\Sigma$  is diffeomorphic to  $\mathbb{R}^3 \setminus \mathbb{D}$  where  $\mathbb{D}$  denotes the disk of  $\mathbb{R}^3$ ,

◇ the same asymptotic flatness and closeness to Minkowski space assumptions as in Theorem 1.2 hold,

then, the maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  admits global optical functions  $u, \underline{u}$  such that, measured in these coordinates,  $\mathbf{g}$  is bounded and decays towards  $\eta$ .

**Remark 1.4.** *In the proof of Theorem 1.2 in [CK93], the topology assumption  $\Sigma \simeq \mathbb{R}^3$  is crucially used to define a global time function such that its level sets are maximal hypersurfaces with prescribed asymptotic conditions when  $r \rightarrow \infty$ . The main novelty in the proof [KN03] of Theorem 1.3 is the definition of a double-null foliation by the level sets of two optical functions  $u, \underline{u}$ . It replaces the global time function and enables a localisation of the global nonlinear stability proof [CK93] to the exterior of a disk. See definitions and discussions in Section 1.2 and see also discussions in [KN03, Section 2].*

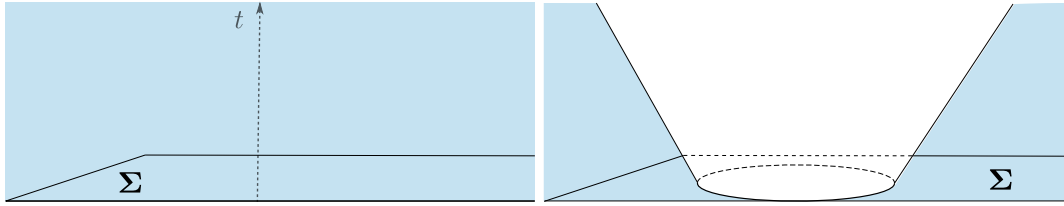


Figure 1: The stability of Minkowski space of Theorems 1.2, 1.3 proved in [CK93] and [KN03].

In view of Theorems 1.2 and 1.3, we have the following natural question.

**Question.** *Can we complete the result of Theorem 1.3 to re-obtain the result of Theorem 1.2? In other words, can we prove the global nonlinear stability of Minkowski space for initial data posed on a spacelike disk and an outgoing null hypersurface?*

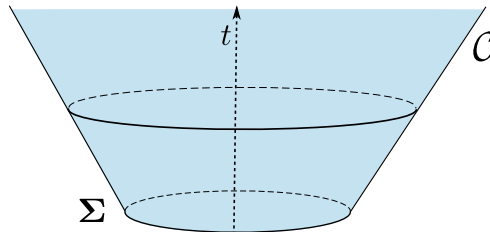


Figure 2: The stability of Minkowski space of Theorem 1.5 proved in the present paper.

This paper is dedicated to the proof of the following theorem, which provides a positive answer to this question. See Theorem 1.8 for a more precise version, and Theorem 4.2 for the exact result proved in this paper.

**Theorem 1.5** (Main theorem, rough version). *Let initial data for Einstein equations (1.1) be given on*

◇ an initial spacelike hypersurface  $\Sigma$  diffeomorphic to  $\mathbb{D}$  the unit disk of  $\mathbb{R}^3$ ,

◇ an initial outgoing null hypersurface  $\mathcal{C}$  emanating from  $\partial\Sigma$ .

Assume that

◇  $\mathcal{C}$  is future geodesically complete,

◇ the initial data are close to the corresponding Minkowski data on  $\Sigma$  and  $\mathcal{C}$  consistently with the boundedness and decay assumptions and results of [CK93] and [KN03].

Then the maximal globally hyperbolic development  $(\mathcal{M}, g)$  is future causally geodesically complete and admits global time and optical functions  $t$  and  $u$  such that, measured with respect to these coordinates,  $\mathbf{g}$  is bounded and decays towards  $\eta$  when  $t \rightarrow +\infty$ .

**Remark 1.6.** *The so-called spacelike-characteristic initial data of Theorem 1.5 have to satisfy constraints. In this paper, we do not discuss the prescription of such initial data. See [CP12] or [CN05] for discussions.*

## 1.2 The *global nonlinear stability of Minkowski space*

In this section, we give a more detailed statement of Theorem 1.2 and highlight key features of its proof in [CK93]. We first start with preliminary definitions.

For a fixed spacetime  $(\mathcal{M}, \mathbf{g})$ , we consider foliations of  $\mathcal{M}$  by the level sets  $\Sigma_t$  a time function  $t$ . We write  $T$  the future-pointing unit normal to  $\Sigma_t$ , and we recall that the second fundamental form  $k$  and the *time lapse*  $n$  of  $\Sigma_t \subset \mathcal{M}$  are defined by

$$k(X, Y) := -\mathbf{g}(\mathbf{D}_X T, Y), \quad n^{-2} := -\mathbf{g}(\mathbf{D}t, \mathbf{D}t),$$

where  $X, Y \in T\Sigma_t$  and  $\mathbf{D}$  denotes the spacetime covariant derivative. In this paper, we will consider *maximal hypersurfaces*  $\Sigma_t$ , i.e. such that

$$\mathrm{tr}_g k = 0,$$

where  $\mathrm{tr}_g$  is the trace with respect to the induced metric  $g$  on  $\Sigma$ .

We also consider foliations of  $\mathcal{M}$  by 2-spheres  $S_{u, \underline{u}}$ , intersections of the level sets two functions  $u, \underline{u}$ . A *null pair*  $(e_3, e_4)$  adapted to  $S_{u, \underline{u}}$  is a pair of vectorfields orthogonal to the 2-spheres  $S_{u, \underline{u}}$  which satisfy

$$\mathbf{g}(e_3, e_3) = \mathbf{g}(e_4, e_4) = 0, \quad \mathbf{g}(e_3, e_4) = -2.$$

To a null pair, we associate *null connection coefficients*, which are the  $S_{u, \underline{u}}$ -tangent tensors such that

$$\begin{aligned} \chi(X, Y) &:= \mathbf{g}(\mathbf{D}_X e_4, Y), & \xi(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, X), & \eta(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, X), \\ \zeta(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_X e_4, e_3), & \omega &:= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \\ \underline{\chi}(X, Y) &:= \mathbf{g}(\mathbf{D}_X e_3, Y), & \underline{\xi}(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, X), & \underline{\eta}(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, X), \\ \underline{\zeta}(X) &:= -\frac{1}{2} \mathbf{g}(\mathbf{D}_X e_3, e_4), & \underline{\omega} &:= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), \end{aligned} \tag{1.2}$$

where  $X, Y \in TS_{u, \underline{u}}$ , and the *null curvature components*, which are the  $S_{u, \underline{u}}$ -tangent tensors such that

$$\begin{aligned} \alpha(X, Y) &:= \mathbf{R}(e_4, X, e_4, Y), & \beta(X) &:= \frac{1}{2} \mathbf{R}(X, e_4, e_3, e_4), & \rho &:= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4), \\ \underline{\alpha}(X, Y) &:= \mathbf{R}(e_3, X, e_3, Y), & \underline{\beta}(X) &:= \frac{1}{2} \mathbf{R}(X, e_3, e_3, e_4), & \sigma &:= \frac{1}{4} {}^* \mathbf{R}(e_3, e_4, e_3, e_4), \end{aligned} \tag{1.3}$$

where  $X, Y \in TS_{u, \underline{u}}$  and where  $\mathbf{R}$  is the spacetime curvature tensor and  ${}^* \mathbf{R}$  denotes its Hodge dual.

We have the following precised version of the stability result of Theorem 1.2.

**Theorem 1.7** (Stability of Minkowski space [CK93], version 2). *Let  $(\Sigma, g, k)$  be Cauchy data such that:*

- ◇  $\Sigma$  is maximal, diffeomorphic to  $\mathbb{R}^3$ ,
- ◇  $\Sigma$  is asymptotically flat, i.e. there exists coordinates  $(x^1, x^2, x^3)$  in a neighbourhood of infinity such that

$$(r\partial)^{\leq 4} \left( g_{ij} - \left( 1 + \frac{2M}{r} \right) \delta_{ij} \right) = O(r^{-3/2}), \tag{1.4a}$$

when  $r \rightarrow \infty$  and where here  $r := \sqrt{\sum_{i=1}^3 (x^i)^2}$  and  $M \geq 0$ , and we have the following sup-norm bound for the curvature of  $(\Sigma, g)$

$$\| (1 + d)^3 \mathrm{Ric} \|_{L^\infty(\Sigma)} \leq \varepsilon, \tag{1.4b}$$

where  $\mathrm{Ric}$  denotes the Ricci tensor of the metric  $g$  and  $d$  denotes the geodesic distance to a fixed point of  $\Sigma$ ,

◇ the following bounds hold for curvature  $L^2$ -fluxes through  $\Sigma$

$$\left\| (1+d) \left( (1+d) \nabla \right)^{\leq 3} k \right\|_{L^2(\Sigma)} + \left\| (1+d)^3 \left( (1+d) \nabla \right)^{\leq 1} B \right\|_{L^2(\Sigma)} \leq \varepsilon, \quad (1.4c)$$

where  $B := \text{curl} \left( \text{Ric} - \frac{1}{3} \text{R}g \right)$ .

Then, there exists  $\varepsilon_0 > 0$ , such that if  $\varepsilon < \varepsilon_0$ , the following holds for the maximal globally hyperbolic development  $(\mathcal{M}, \mathbf{g})$  of  $(\Sigma, g, k)$ .

- ◇  $(\mathcal{M}, \mathbf{g})$  is geodesically complete.
- ◇ There exists a global time function  $t$  on  $\mathcal{M}$  ranging from  $-\infty$  to  $+\infty$  which foliates  $\mathcal{M}$  by maximal spacelike hypersurfaces  $\Sigma_t$  such that  $\Sigma_0 = \Sigma$ .
- ◇ There exists a future exterior region<sup>2</sup>  $\mathcal{M}^{\text{ext}}$  foliated by outgoing null hypersurfaces  $\mathcal{C}_u$  level sets of a global optical function  $u$  ranging from  $-\infty$  to  $+\infty$  on  $\mathcal{M}^{\text{ext}}$ , and a past exterior region with symmetric constructions.
- ◇ We have the following decay in the interior region  $\mathcal{M}^{\text{int}} := \mathcal{M} \setminus \mathcal{M}^{\text{ext}}$  of the spacetime curvature tensor  $\mathbf{R}$  of  $\mathbf{g}$ <sup>3</sup>

$$|\mathbf{R}| \lesssim \varepsilon t^{-7/2}. \quad (1.5a)$$

- ◇ We have the following differentiated decay in the exterior region  $\mathcal{M}^{\text{ext}}$  of the spacetime curvature tensor  $\mathbf{R}$  according to its null decomposition<sup>4</sup>

$$\begin{aligned} |\alpha(\mathbf{R})| &\lesssim \varepsilon r^{-7/2}, & |\beta(\mathbf{R})| &\lesssim \varepsilon r^{-7/2}, & |\rho(\mathbf{R})| &\lesssim \varepsilon r^{-3}, \\ |\underline{\alpha}(\mathbf{R})| &\lesssim \varepsilon r^{-1} u^{-5/2}, & |\underline{\beta}(\mathbf{R})| &\lesssim \varepsilon r^{-2} u^{-3/2}, & |\sigma(\mathbf{R})| &\lesssim \varepsilon r^{-3} u^{-1/2}, \end{aligned} \quad (1.5b)$$

where here  $r := t - u$ .

- ◇ The induced metric and connection coefficients adapted to the maximal foliation  $\Sigma_t$  and maximal-null foliation  $\Sigma_t$  and  $\mathcal{C}_u$  satisfy decay statements consistent with (1.5a) and (1.5b).
- ◇ The spacetime  $(\mathcal{M}, \mathbf{g})$  admits a past/future timelike, past/future null and spacelike infinities  $i^-, i^+$ ,  $\mathcal{I}^-, \mathcal{I}^+$  and  $i_0$  on which one can make sense of asymptotic quantities and their evolution equations.

### Remarks on Theorem 1.7

1.7a The proof of Theorem 1.7 in [CK93] is based on the *vectorfield method*, which proceeds in the following two steps:

Step 1 Einstein equations (1.1) induce the following *Bianchi equations*

$$\mathbf{D}^\alpha \mathbf{R}_{\alpha\beta\gamma\delta} = 0. \quad (1.6)$$

Multiplying and commuting (1.6) with a set of approximate conformal Killing vectorfields, wave-type energy estimates are obtained. These estimates hold provided that the nonlinear error terms produced by the conformal Killing approximations are controlled.

Step 2 From the boundedness of  $L^2$  fluxes for (derivatives of)  $\mathbf{R}$  through the hypersurfaces  $\Sigma_t$  and  $\mathcal{C}_u$  resulting from the energy estimates of Step 1, one deduces decay estimates for  $\mathbf{R}$  using Klainerman-Sobolev embeddings, as well as boundedness and decay estimates for the induced metric and connection coefficients associated to the maximal-null foliation. This is done using *structure equations* which schematically read

$$\nabla \Gamma = \mathbf{R} + \nabla \Gamma + \Gamma \cdot \Gamma.$$

Here  $\nabla$  are derivatives in the  $e_3, e_4$  and tangential directions,  $\Gamma$  are connection coefficients as defined in (1.2), the terms  $\mathbf{R}, \nabla \Gamma$  on the right-hand side are treated as linear source terms and the terms  $\Gamma \cdot \Gamma$  as nonlinear error terms.

<sup>2</sup>The future exterior region is of the type  $\{u \leq ct\}$  with  $c < 1$ .

<sup>3</sup>Here the norm of  $\mathbf{R}$  is taken with respect to an orthonormal frame adapted to the maximal hypersurfaces  $\Sigma_t$ . See precise definitions in Section 3.

<sup>4</sup>Here the null pair is adapted to the 2-spheres intersections of the maximal-null foliation of  $\mathcal{M}^{\text{ext}}$  by  $\Sigma_t$  and  $\mathcal{C}_u$ . See [CK93, Introduction]

The crux of the proof of Theorem 1.7 in [CK93] is the control of the nonlinear error terms of Step 1. Since the approximate conformal Killing vectorfields are constructed upon the geometric time and optical functions, these nonlinear error terms can be expressed in terms of the connection coefficients  $\Gamma$  for the foliations  $\Sigma_t$  and  $\mathcal{C}_u$ , and their control thus crucially relies on the decay estimates obtained in Step 2.

- 1.7b Assumption (1.4c), together with the maximal assumption on  $\Sigma$  guarantee that the boundary fluxes on  $\Sigma$  arising when performing energy estimates for Bianchi equations commuted with a set of approximate Killing vectorfields are controlled by  $\varepsilon$ .<sup>5</sup>
- 1.7c The global time function  $t$  is constructed by imposing that its level sets  $\Sigma_t$  are maximal hypersurfaces of  $\mathcal{M}$ , that  $\Sigma = \{t = 0\}$  and that  $n \rightarrow 1$  when  $r \rightarrow \infty$ . These last conditions are equivalent to the choice of boundary for  $\Sigma_t$  at infinity. It physically corresponds to considering a *centre-of-mass frame* for the system (see the discussion in the introduction of [CK93]).
- 1.7d The vectorfield method of [CK93] is wrapped in an elaborate *bootstrap argument*. One of the main challenge is to define the geometric constructions within this bootstrap argument to obtain sufficient decay rates for the associated metric and connection coefficients (see Item 1.7a). In [CK93] the global optical function  $u$  is constructed by an *initialisation on the last slice*, *i.e.* by imposing that the outgoing null hypersurfaces  $\mathcal{C}_u$  are backwards emanating from the 2-spheres level sets of a *canonical foliation* on a last slice  $\Sigma^*$  corresponding to the future boundary of the bootstrap region.
- 1.7e A first stability result for initial data with stronger decay assumptions was obtained in [Fri83]. A global stability result has been obtained in [Bie10, BZ09] using the same general techniques as in [CK93] but under relaxed assumptions for both the regularity and decay of the initial data. A global stability result for Minkowski space has also been obtained using wave coordinates, see [LR10]. See also [HV20] for an alternative proof.

### 1.3 Main theorem

This section is dedicated to the following precised version of Theorem 1.5 which is the main result of this paper. We also refer the reader to Theorem 4.2 for the detailed assumptions and conclusions.

**Theorem 1.8** (Main theorem, more precise version). *Let  $(\tilde{\Sigma}_1, \tilde{\mathcal{C}}_0)$  be smooth spacelike-characteristic initial data, such that*

- ◇ *we have the following curvature fluxes bounds through  $\tilde{\Sigma}_1$*

$$\int_{\tilde{\Sigma}_1} |\mathbf{D}^{\leq 2} \mathbf{R}|^2 \leq \varepsilon^2, \tag{1.7a}$$

*together with consistent bounds for a Cartesian coordinates system  $(x^i)$  on  $\tilde{\Sigma}_1$ ,*

- ◇ *the null hypersurface  $\tilde{\mathcal{C}}_0$  is future geodesically complete, foliated by the 2-spheres of a geodesic foliation  $(S'_s)_{1 \leq s < +\infty}$ , and for the associated geodesic null pair, we have the following curvature fluxes bounds*

$$\int_1^\infty \int_{S'_s} \left( |\nabla^{\leq 2} \underline{\beta}|^2 + |s \nabla^{\leq 2} \rho|^2 + |s \nabla^{\leq 2} \sigma|^2 + |s^2 \nabla^{\leq 2} \beta|^2 + |s^2 \nabla^{\leq 2} \alpha|^2 \right) ds \leq \varepsilon^2 \tag{1.7b}$$

*where  $\nabla \in \{(s \nabla), (s \nabla_4), \nabla_3\}$ , together with consistent bounds for the metric and connection coefficients.*

*There exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$ , the following holds for the future maximal globally hyperbolic development  $(\tilde{\mathcal{M}}, \mathbf{g})$  of  $(\tilde{\Sigma}_1, \tilde{\mathcal{C}}_0)$ .*

- ◇ *The spacetime  $(\tilde{\mathcal{M}}, \mathbf{g})$  is future geodesically complete.*
- ◇ *The spacetime  $(\tilde{\mathcal{M}}, \mathbf{g})$  is covered by an interior and an exterior region  $\mathcal{M}^{\text{int}}$  and  $\mathcal{M}^{\text{ext}}$ , intersecting at a timelike transition hypersurface  $\mathcal{T} = \mathcal{M}^{\text{int}} \cap \mathcal{M}^{\text{ext}}$ .*

---

<sup>5</sup>Estimates (1.4c) do not bound the boundary flux for the uncommuted Bianchi equations, which would in fact cause the ADM mass of  $\Sigma$  to vanish. The control of the ADM mass is obtained from (1.4a) and (1.4b).



- ◇ There exists a global time function  $t$  on  $\mathcal{M}^{\text{int}}$  ranging up to  $+\infty$  foliating  $\mathcal{M}^{\text{int}}$  by spacelike maximal hypersurface  $\Sigma_t$ .
- ◇ There exists a global optical function  $u$  on  $\mathcal{M}^{\text{ext}}$  ranging up to  $+\infty$  foliating  $\mathcal{M}^{\text{ext}}$  by outgoing null hypersurfaces  $\mathcal{C}_u$ . There exists a global function  $\underline{u}$  on  $\mathcal{M}^{\text{ext}}$  which is a geodesic affine parameter on  $\mathcal{C}_u$ , foliating  $\mathcal{C}_u$  by 2-spheres  $S_{u,\underline{u}}$ . Moreover, on the transition hypersurface  $\mathcal{T}$ , we have

$$u = \tau \underline{u}, \quad t = \frac{1}{2}(u + \underline{u}),$$

where  $0 < \tau < 1$  is a fixed parameter.

- ◇ We have the following decay bounds in  $\mathcal{M}^{\text{int}}$

$$|\mathbf{R}| \lesssim \varepsilon t^{-7/2},$$

together with consistent bounds for the metric and connection coefficients.

- ◇ We have the following decay bounds in  $\mathcal{M}^{\text{ext}}$

$$\begin{aligned} |\alpha| &\lesssim \varepsilon \underline{u}^{-7/2}, & |\beta| &\lesssim \varepsilon \underline{u}^{-7/2}, & |\rho| &\lesssim \varepsilon \underline{u}^{-3} u^{-1/2}, \\ |\underline{\alpha}| &\lesssim \varepsilon \underline{u}^{-1} u^{-5/2}, & |\underline{\beta}| &\lesssim \varepsilon \underline{u}^{-2} u^{-3/2}, & |\underline{\sigma}| &\lesssim \varepsilon \underline{u}^{-3} u^{-1/2}, \end{aligned}$$

together with consistent bounds for the metric and connection coefficients.

- ◇ The spacetime  $(\tilde{\mathcal{M}}, \mathbf{g})$  admits a future timelike and future null infinity  $i^+$  and  $\mathcal{I}^+$ . The future null infinity  $\mathcal{I}^+$  is future geodesically complete, admits well-defined notions of Bondi mass and angular momentum for which we obtain Bondi mass loss formula and angular momentum evolution equation along  $\mathcal{I}^+$ , and which tend to 0 at future timelike infinity  $i^+$ .

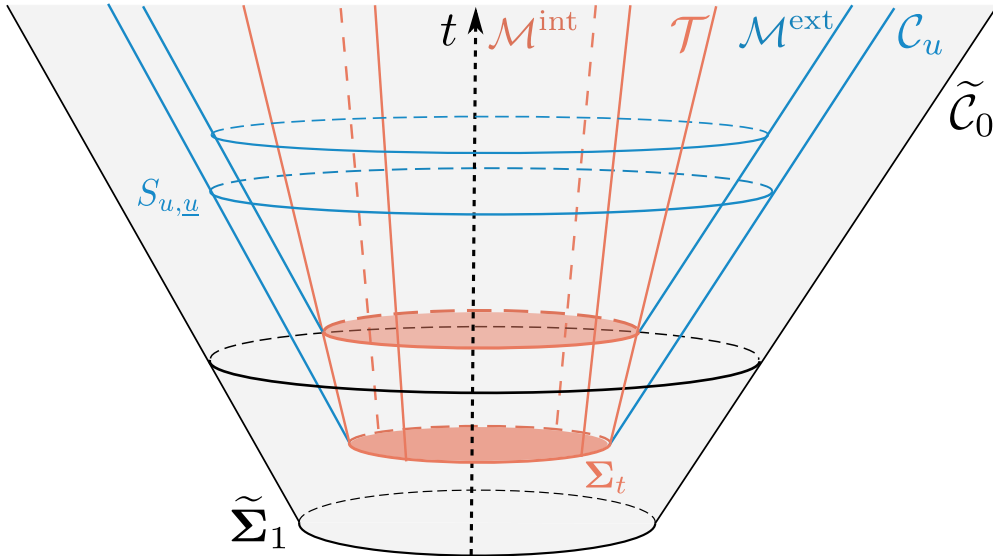


Figure 3: The global nonlinear stability of Minkowski space for characteristic data.

### Remarks on Theorem 1.8

- 1.8a The initial data assumptions (1.7) match what can be obtained for an outgoing null hypersurface in [CK93] or [KN03]. Therefore, Theorem 1.8 provides a stability result for the complementary region to the exterior region considered in [KN03]. Together, they amount to a stability result for initial data posed on a spacelike hypersurface.
- 1.8b Theorem 1.8 was conjectured to hold true in [KN03, Tay17] and its conclusions were used in [Tay17].

1.8c The basic scheme of proof of Theorem 1.8 is a vectorfield method wrapped in a bootstrap argument as in [CK93]. The main novelty is the introduction and control of new geometric constructions, which provide suitable spacetime decompositions to run these arguments.

1.8d Our constructions display the following new crucial geometric features.

- ◊ They virtually emanate from the future infinity of a (timelike) central axis. This guarantees optimal decay rates. It replaces an asymptotically flat spacelike infinity which plays a similar crucial role in [CK93, KN03].
- ◊ In the interior region, our constructions are build on *spacelike maximal hypersurfaces with prescribed boundaries* and *global harmonic coordinates*. This makes any reference to null decompositions and spherical foliations – which degenerate at the central axis – disappear in that region.

1.8e In the proof of Theorem 1.8, we match discontinuous gauge choices across the timelike interface  $\mathcal{T}$  without using the gluing procedure of [CK93, KS17]. Our matching features a *mean value argument* which compensates regularity losses at the timelike interface. We believe that this new treatment gains in concision and clarity.

1.8f In the appendix to this paper, we also provide new optimal estimates and control for harmonic coordinates on a 3-dimensional Riemannian manifold, only based on elementary energy and Bochner estimates (see Theorem 4.3 and Appendix A). We moreover give a full statement and proof for general limits of the metric and connection coefficients and their derivatives in all directions at the vertex of a (general family of) null cones (see Theorem 4.4 and Appendix B).

In the next section, we give an overview of the proof of Theorem 1.8. We postpone discussions/comparisons with other results to Section 1.5.

## 1.4 Overview of the proof of Theorem 1.8

The proof of the global nonlinear stability result of Theorem 1.8 goes by a standard continuity argument on the maximal parameter  $\underline{u}^*$  such that the smooth maximal globally hyperbolic development  $(\widetilde{\mathcal{M}}, \mathbf{g})$  admits a subregion  $\mathcal{M}_{\underline{u}^*}$  (of size  $\underline{u}^*$ ) which we geometrically describe next in Section 1.4.1. In this overview, we will focus on the geometric setup of  $\mathcal{M}_{\underline{u}^*}$  and the obtention of bounds for the curvature and the geometric structures of  $\mathcal{M}_{\underline{u}^*}$ . We refer the reader to Section 4 for the full setup and conclusion of the bootstrap argument and for its consequences in the limit  $\underline{u}^* \rightarrow +\infty$ , from which the conclusions of Theorem 1.8 follow.

### 1.4.1 Geometric setup of the bootstrap region $\mathcal{M}_{\underline{u}^*}$

Let  $O = \{x^i = 0\}$  be the centre of the initial spacelike hypersurface  $\widetilde{\Sigma}_1$  for the Cartesian coordinates  $x^i$  given as assumption on  $\widetilde{\Sigma}_1$ . We define the *central axis*  $\phi \subset \widetilde{\mathcal{M}}$  to be the timelike geodesic parametrised by arc-length such that  $\phi(1) = O$  and  $\dot{\phi}(1)$  is future-pointing and normal to  $\widetilde{\Sigma}_1$  at  $O$ .

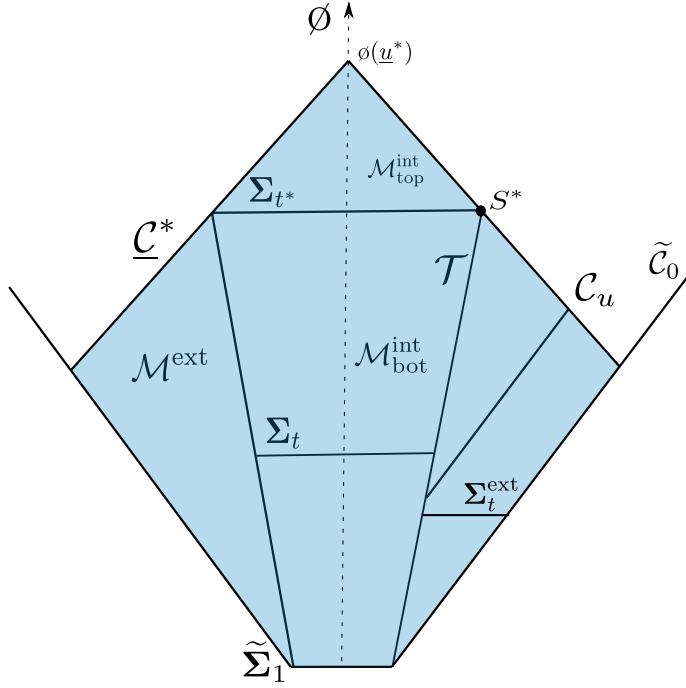
For a fixed parameter  $\underline{u}^* \geq 1$ , we define the *last cone*  $\mathcal{C}^*$  to be the ingoing null cone backwards emanating from the point  $\phi(\underline{u}^*)$ . The cone  $\mathcal{C}^*$  is foliated by the 2-spheres  $S_{u, \underline{u}^*}$  of a *canonical foliation* with parameter  $u$  which ranges from  $u|_{\phi(\underline{u}^*)} = \underline{u}^*$  to 0.

We define  $\mathcal{C}_u$  to be the outgoing null hypersurfaces backwards emanating from the 2-spheres  $S_{u, \underline{u}^*}$ , and we denote by  $u$  the associated optical function. We foliate the hypersurfaces  $\mathcal{C}_u$  by the 2-spheres  $S_{u, \underline{u}}$  of the *geodesic foliation* with parameter  $\underline{u}$  ranging from  $\underline{u}|_{\mathcal{C}^*} = \underline{u}^*$  to  $\tau^{-1}u$ , where  $0 < \tau < 1$  is a (suitably chosen) parameter. We define the following *exterior region* (see Figure 4 for a graphic representation)

$$\mathcal{M}^{\text{ext}} := \bigcup_{\substack{0 \leq u \leq \tau \underline{u}^* \\ \tau^{-1}u \leq \underline{u} \leq \underline{u}^*}} S_{u, \underline{u}},$$

the following (timelike) *transition hypersurface*

$$\mathcal{T} := \bigcup_{\substack{0 \leq u \leq \tau \underline{u}^* \\ \underline{u} = \tau^{-1}u}} S_{u, \underline{u}},$$


 Figure 4: The bootstrap region  $\mathcal{M}_{\underline{u}^*}$ 

and the following *last sphere*

$$S^* := S_{\tau \underline{u}^*, \underline{u}^*} = \underline{C}^* \cap \mathcal{T}.$$

Let  $\Sigma_t$  be the *maximal hypersurfaces* coinciding on the transition hypersurface with the 2-spheres  $S_{u, \underline{u}}$ , *i.e.* such that the associated maximal time function  $t$  satisfies

$$t|_{\mathcal{T}} := \frac{1}{2}(u + \underline{u}).$$

We define the following *bottom interior region*

$$\mathcal{M}_{\text{bot}}^{\text{int}} := \bigcup_{1 \leq t \leq t^*} \Sigma_t$$

where  $t^* := \frac{1}{2}(1 + \tau)\underline{u}^*$ , and we call  $\Sigma_{t^*}$  the *last maximal hypersurface* (note that the boundary of the last maximal hypersurface is the last sphere, *i.e.*  $\partial\Sigma_{t^*} = S^*$ ). We further define the *top interior region*  $\mathcal{M}_{\text{top}}^{\text{int}}$  to be the domain of dependence of  $\Sigma_{t^*}$ . We are now able to define the full spacetime bootstrap region  $\mathcal{M}_{\underline{u}^*}$

$$\mathcal{M}_{\underline{u}^*} := \mathcal{M}^{\text{ext}} \cup \mathcal{M}^{\text{int}},$$

where

$$\mathcal{M}^{\text{int}} := \mathcal{M}_{\text{bot}}^{\text{int}} \cup \mathcal{M}_{\text{top}}^{\text{int}}.$$

As it will play a key role in the control of the interior region, we also define *global harmonic Cartesian coordinates*  $(x^i)$  on the last slice  $\Sigma_{t^*}$  to be functions such that on  $\Sigma_{t^*}$

$$\Delta x^i = 0,$$

and such that Dirichlet boundary conditions for  $x^i$  at  $\partial\Sigma_{t^*} = S^*$  are fixed *via* (a suitable class of) conformal isomorphism of  $S^*$  to the Euclidean 2-sphere.

**Remarks**

- ◇ The central axis and its last point in the future direction is the starting point of our *initialisation from timelike infinity* procedure.
- ◇ The last cone  $\underline{\mathcal{C}}^*$  and the last maximal hypersurface  $\Sigma_{t^*}$  play a similar role as the last maximal hypersurface in [CK93] or the last cone in [KN03].
- ◇ The canonical foliation on  $\underline{\mathcal{C}}^*$  must provide sufficient regularity for its transverse geometric quantities (*i.e.*  $\text{tr}\chi$  and  $\hat{\chi}$ ) since this determines the regularity of the transversely emanating foliation of null cones  $\mathcal{C}_u$ . The canonical foliations on the last slices in [CK93, KN03] are built to satisfy the same transverse regularity features. See also the discussions in [CG19a, CG19b].
- ◇ Considering the geodesic foliation on the outgoing cones  $\tilde{\mathcal{C}}_u$  is geometrically simpler than constructing a double null foliation which requires a second transverse hypersurface to be initialised. The counterpart is additional difficulty in the analysis of the null structure equations to avoid regularity loss. See a similar foliation choice and difficulties in [KS17].
- ◇ Since they are not required to emanate from one point, but are defined from 2-spheres on  $\underline{\mathcal{C}}^*$ , the outgoing null hypersurfaces  $\mathcal{C}_u$  may degenerate before reaching their potential vertex  $\underline{u} \rightarrow u$ . We thus restrict their definition to the exterior region  $\tau^{-1}u \leq \underline{u}$ , with  $0 < \tau < 1$ , on which they remain regular.
- ◇ In this overview, we shall assume that the initial hypersurfaces  $\tilde{\Sigma}_1$  and  $\tilde{\mathcal{C}}_0$  match the geometric constructions described above, *i.e.*  $\tilde{\Sigma}_1 = \Sigma_1$  and  $\tilde{\mathcal{C}}_0 = \mathcal{C}_0$ . In general this does not hold. See Section 12 for the *initial layer* existence and comparisons arguments. In particular, in Sections 2–12, we will only assume that  $u$  ranges up to 1 (and not to 0) and  $t$  ranges up to  $(1 + \tau^{-1})/2$  (and not to 1).
- ◇ That the harmonic functions  $x^i$  on the last slice  $\Sigma_{t^*}$  form a global coordinate system on  $\Sigma_{t^*}$  is a result, obtained as a consequence of estimates for the functions  $x^i$ . These estimates are obtained using energy and Bochner estimates for the above defined Dirichlet problem on  $\Sigma_{t^*}$ . This (only) involves intrinsic quantities (the Ricci curvature and fundamental forms of the boundary) and basic functional estimates on  $\Sigma_{t^*}$ . See Theorem 4.3.

**1.4.2 Global energy estimates**

The final goal is to obtain bounds on the spacetime metric evaluated on the above defined geometric structures of  $\mathcal{M}_{\underline{u}^*}$  and decay estimates to the corresponding Minkowskian quantities, in terms of the initial data on  $\Sigma_1 \cup \mathcal{C}_0$ . To that end, the spacetime curvature  $\mathbf{R}$  is the key dynamical object since it satisfies Bianchi equations (1.6) for which wave-type energy estimates can be obtained. We use the *vectorfield method* (see Item 1.7a) for these equations in the region  $\mathcal{M}_{\underline{u}^*}$ .

We first recall tools developed in [CK93] to perform energy estimates for Bianchi equations.

First, commuting Bianchi equation (1.6) with a vectorfield  $\mathbf{X}$  gives schematically

$$\mathbf{D}^\alpha \left( \hat{\mathcal{L}}_{\mathbf{X}} \mathbf{R} \right)_{\alpha\beta\gamma\delta} = \mathbf{D}^{\leq 1} \left( {}^{(\mathbf{X})} \hat{\pi} \right) \cdot \mathbf{D}^{\leq 1} \mathbf{R}, \quad (1.8)$$

where  $\hat{\mathcal{L}}$  is a normalised derivative in the  $\mathbf{X}$  direction, and where  ${}^{(\mathbf{X})} \hat{\pi}$  is the following traceless part of the *deformation tensor* of  $\mathbf{X}$

$${}^{(\mathbf{X})} \hat{\pi}_{\mu\nu} := \mathbf{D}_\mu \mathbf{X}_\nu + \mathbf{D}_\nu \mathbf{X}_\mu - \frac{1}{2} (\mathbf{D}^\alpha \mathbf{X}_\alpha) \mathbf{g}_{\mu\nu}.$$

**Remark 1.9.** For a conformal Killing vectorfield  $\mathbf{X}$ , we have  ${}^{(\mathbf{X})} \hat{\pi} = 0$ . Thus from (1.8) we obtain that commuting Bianchi equation by  $\mathbf{X}$  gives

$$\mathbf{D}^\alpha \left( \hat{\mathcal{L}}_{\mathbf{X}} \mathbf{R} \right)_{\alpha\beta\gamma\delta} = 0.$$

For approximate conformal Killing vectorfields, the source terms of the commuted Bianchi equations are nonlinear error terms.

Second, we introduce the *Bel-Robinson tensor*  $Q(\mathbf{W})$  associated to a 4-tensor  $W$  by

$$Q(\mathbf{W})_{\alpha\beta\gamma\delta} := \mathbf{W}_{\alpha\nu\gamma\mu} \mathbf{W}_{\beta\delta}^{\nu\mu} + {}^* \mathbf{W}_{\alpha\nu\gamma\mu} {}^* \mathbf{W}_{\beta\delta}^{\nu\mu},$$

where  $*$  denotes the Hodge dual of  $\mathbf{W}$ . When  $\mathbf{W} = \mathbf{R}$ , we have the following consequence of the Bianchi equations (1.6)

$$\mathbf{D}^\alpha Q(\mathbf{R})_{\alpha\beta\gamma\delta} = 0.$$

Multiplying and commuting the Bel-Robinson tensor by vectorfields  $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ , we similarly obtain from (1.8) the following schematic formula

$$\mathbf{div} \left( Q \left( \hat{\mathcal{L}}_{\mathbf{X}_1} \hat{\mathcal{L}}_{\mathbf{X}_2} \mathbf{R} \right) (\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3) \right) = \mathbf{D}^{\leq 2} \hat{\pi} \cdot \mathbf{D}^{\leq 2} \mathbf{R}, \quad (1.9)$$

where  $\mathbf{div}$  is the spacetime divergence for 1-tensor and where  $\hat{\pi}$  denotes the traceless deformation tensors of  $\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ .

**Remark 1.10.** For exact conformal Killing vectorfields, formula (1.9) produces a spacetime divergence-free vectorfield

$$Q \left( \hat{\mathcal{L}}_{\mathbf{X}_1} \hat{\mathcal{L}}_{\mathbf{X}_2} \mathbf{R} \right) (\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3).$$

An application of Stokes theorem yields an exact identity of boundary fluxes for the above vectorfield. Provided that the multiplier vectorfields  $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$  are suitably chosen, this yields an energy estimate in  $\mathcal{M}$ .

**Remark 1.11.** For approximate conformal Killing vectorfields, the same procedure using formula (1.9) produces an energy estimate with a spacetime integral of nonlinear error terms.

Using the above tools, we can now perform *global energy estimates* in  $\mathcal{M}^{\text{ext}} \cup \mathcal{M}_{\text{bot}}^{\text{int}}$ . These estimates are obtained using the following set of contracted and commuted Bel-Robinson tensors

$$\begin{aligned} & Q \left( \hat{\mathcal{L}}_{\mathbf{T}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{K}), \quad Q \left( \hat{\mathcal{L}}_{\mathbf{O}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{T}), \\ & Q \left( \hat{\mathcal{L}}_{\mathbf{O}} \hat{\mathcal{L}}_{\mathbf{O}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{T}), \quad Q \left( \hat{\mathcal{L}}_{\mathbf{S}} \hat{\mathcal{L}}_{\mathbf{T}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{K}), \quad Q \left( \hat{\mathcal{L}}_{\mathbf{O}} \hat{\mathcal{L}}_{\mathbf{T}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{T}), \end{aligned} \quad (1.10)$$

where

- ◇  $\mathbf{T}$  is an approximation for the *time translation Killing vectorfield*  $\partial_t$  of Minkowski space,
- ◇  $\mathbf{S}$  is an approximation for the *scaling conformal Killing vectorfield*  $t\partial_t + r\partial_r$  of Minkowski space,
- ◇  $\mathbf{K}$  is an approximation for the *Morawetz conformal Killing vectorfield*  $(t^2 + r^2)\partial_t + 2tr\partial_r$  of Minkowski space,
- ◇  $\mathbf{O}$  are approximations for the three *rotation Killing vectorfields*  $x^1\partial_2 - x^2\partial_1$ ,  $x^2\partial_3 - x^3\partial_2$  and  $x^3\partial_1 - x^1\partial_3$  of Minkowski space.

We assume for the moment that these vectorfields are given and postpone their respective definitions in the bottom interior and exterior region to Section 1.4.4.

Applying Stokes theorem and formula (1.9) to the set of vectorfields (1.10) simultaneously in the bottom interior region  $\mathcal{M}_{\text{bot}}^{\text{int}}$  and the exterior region  $\mathcal{M}^{\text{ext}}$ , we obtain the following energy estimates (see Figure 4 for a graphic representation of these hypersurfaces)

$$\int_{\Sigma_t} + \int_{\mathcal{C}_u} + \int_{\Sigma_t^{\text{ext}}} + \int_{\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}} \lesssim \int_{\Sigma_1} + \int_{\mathcal{C}_0} + \mathcal{E}^{\mathcal{T}} + \mathcal{E}^{\text{int}} + \mathcal{E}^{\text{ext}},$$

where

- ◇ the integrands are the contracted and commuted Bel-Robinson tensors (1.10),
- ◇ for all  $1 \leq t \leq t^*$  and all  $0 \leq u \leq \tau u^*$  (we recall that)
  - $\Sigma_t$  are the maximal hypersurfaces of  $\mathcal{M}^{\text{int}}$ ,

- $\mathcal{C}_u$  are the outgoing null hypersurface of  $\mathcal{M}^{\text{ext}}$ ,
  - $\Sigma_t^{\text{ext}}$  are the level sets of the time function  $\frac{1}{2}(u + \underline{u})$  in  $\mathcal{M}^{\text{ext}}$ ,
  - $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  is the exterior part of the last cone  $\underline{\mathcal{C}}^*$ ,
  - $\Sigma_1$  and  $\mathcal{C}_0$  are the initial hypersurfaces for which we have bounds for the curvature fluxes (see the assumptions of Theorem 1.8),
- ◇ the nonlinear error term  $\mathcal{E}^{\mathcal{T}}$  is the difference of boundary fluxes on the timelike transition hypersurface  $\mathcal{T}$  for Stokes formula applied in  $\mathcal{M}_{\text{bot}}^{\text{int}}$  and  $\mathcal{M}^{\text{ext}}$ ,
  - ◇ the nonlinear error terms  $\mathcal{E}^{\text{int}}$  and  $\mathcal{E}^{\text{ext}}$  are spacetime integrals over  $\mathcal{M}_{\text{bot}}^{\text{int}}$  and  $\mathcal{M}^{\text{ext}}$  respectively, involving (two derivatives of) the spacetime curvature tensor  $\mathbf{R}$  and (two derivatives of) the deformations tensors  $\hat{\pi}$  for the approximate conformal Killing vectorfields.

The control of the error terms  $\mathcal{E}^{\text{int}}$  and  $\mathcal{E}^{\text{ext}}$  is the crux of the analysis and is obtained provided that sufficient *decay* can be obtained for (null decompositions of)  $\mathbf{R}$  and (null decompositions of)  $\hat{\pi}$ , as well as sufficient *regularity* can be obtained for  $\hat{\pi}$ .

The control of the interface error term  $\mathcal{E}^{\mathcal{T}}$  is obtained provided that the difference of the corresponding interior/exterior approximate conformal Killing vectorfields at the interface  $\mathcal{T}$  can be controlled with sufficient decay and regularity. It also requires that the spacetime curvature tensor  $\mathbf{R}$  has optimal regularity on  $\mathcal{T}$ , *i.e.* that  $\mathbf{D}^2\mathbf{R} \in L^2(\mathcal{T})$ . Since the hypersurface  $\mathcal{T}$  is timelike, this cannot be obtained from bounds on energy estimates boundary fluxes on  $\mathcal{T}$ . We thus rely on a *mean value argument* which selects a suitable transition parameter  $\tau$ /transition hypersurface  $\mathcal{T}$  on which such a control holds.

From these controls, we obtain

$$\int_{\Sigma_t} + \int_{\mathcal{C}_u} + \int_{\Sigma_t^{\text{ext}}} + \int_{\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}} \lesssim \varepsilon^2 + (D\varepsilon)^3 \lesssim \varepsilon^2.$$

### Remarks

- ◇ The contracted and commuted Bel-Robinson tensors (1.10) are identical to the ones used in [CK93, KN03].
- ◇ The decay rates obtained in this paper are similar, but slightly different to the decay rates of [CK93, KN03] due to the difference of geometric constructions.
- ◇ In Section 5, we use systematically that the tensors  $\mathbf{R}$  and  $\hat{\pi}$  are respectively controlled with the following regularity in  $\mathcal{M}^{\text{ext}}$

$$\begin{aligned} \mathbf{R} &\in L^\infty(\mathcal{M}^{\text{ext}}), & \mathbf{DR} &\in L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}}), & \mathbf{D}^2\mathbf{R} &\in L^2(\mathcal{M}^{\text{ext}}), \\ \hat{\pi} &\in L^\infty(\mathcal{M}^{\text{ext}}), & \mathbf{D}\hat{\pi} &\in L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}}), & \mathbf{D}^2\hat{\pi} &\in L^2(\mathcal{M}^{\text{ext}}), \end{aligned}$$

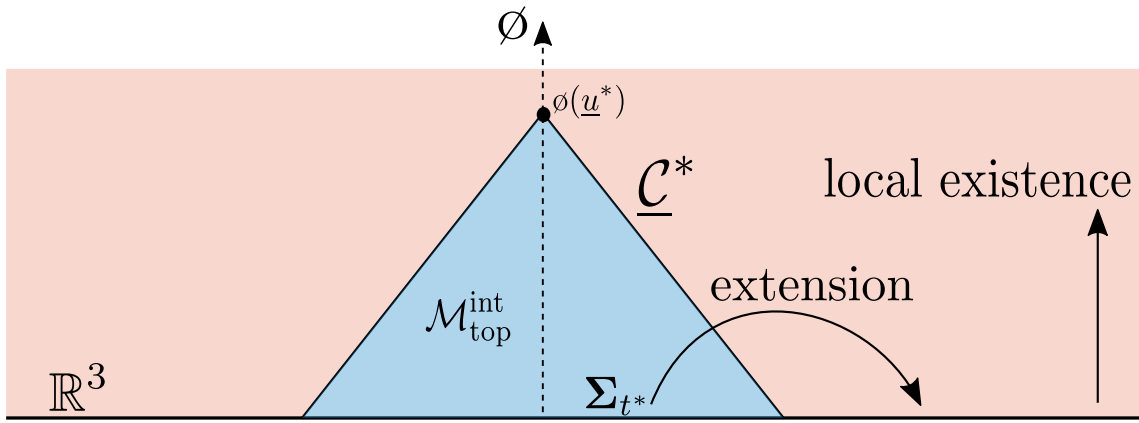
and with the following regularity in  $\mathcal{M}_{\text{bot}}^{\text{int}}$

$$\begin{aligned} \mathbf{R} &\in L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}}), & \mathbf{DR} &\in L_t^\infty L^6(\Sigma_t), & \mathbf{D}^2\mathbf{R} &\in L^2(\mathcal{M}_{\text{bot}}^{\text{int}}), \\ \hat{\pi} &\in L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}}), & \mathbf{D}\hat{\pi} &\in L_t^\infty L^6(\Sigma_t), & \mathbf{D}^2\hat{\pi} &\in L^2(\mathcal{M}_{\text{bot}}^{\text{int}}), \end{aligned}$$

which is in the spirit of the (bulk) Morawetz estimates of the  $r^p$ -method (see for example [Hol10]) and simplifies the analysis of [CK93, KN03].

- ◇ The main new feature of the global energy estimates in the region  $\mathcal{M}^{\text{ext}} \cup \mathcal{M}^{\text{int}}$  is to avoid the gluing procedure of [CK93, KS17] for functions/vectorfields across the interface  $\mathcal{T}$ . This is replaced by a treatment of transition nonlinear error terms  $\mathcal{E}^{\mathcal{T}}$  which arise from the discontinuity of gauge choices. We believe that this new procedure gains in clarity.

To obtain a control for the geometry of the cone  $\underline{\mathcal{C}}^*$ , we also need a control of the boundary fluxes through  $\underline{\mathcal{C}}^* \cap \mathcal{M}_{\text{top}}^{\text{int}}$ . To this end, we perform a  *$t^*$ -rescaling* of the spacetime region  $\mathcal{M}_{\text{top}}^{\text{int}}$ . The metric, connection and curvature components scale homogeneously in that region, which thus reduces to a size-1 region. Then, we *extend* the data on the last slice  $\Sigma_{t^*}$  to  $\mathbb{R}^3$  using the result of [Czi18] and apply a *local (time-1) existence*


 Figure 5: Energy estimates in (the rescaled)  $\mathcal{M}_{\text{top}}^{\text{int}}$ 

result for the extended (small) data. Then, we can perform energy estimates in  $\mathcal{M}_{\text{top}}^{\text{int}}$  – which we recall is the future domain of dependence of  $\Sigma_{t^*}$  –, using *Cartesian* approximate Killing vectorfields (*i.e.* the approximate Minkowskian time and space translations  $\partial_\mu$ ). The error terms are controlled using the local existence result. From a comparison argument on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ , we obtain control for the (null decomposition of the) curvature on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ , that is, an estimate of the type

$$\int_{\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}} \lesssim \int_{\Sigma_{t^*}} + \text{nonlinear error terms} \lesssim \varepsilon^2 + (D\varepsilon)^3 \lesssim \varepsilon^2,$$

where we used that the energy boundary fluxes through the last slice  $\Sigma_{t^*}$  were controlled by the previous energy estimates in  $\mathcal{M}_{\text{bot}}^{\text{int}} \cap \mathcal{M}^{\text{ext}}$ .

### Remarks

- ◇ Spacetime regions as  $\mathcal{M}_{\text{top}}^{\text{int}}$  with conical degeneracies and outside of spherical symmetry are not treated as such in other works. Our procedure is in the spirit of [CG19a, CG19b].
- ◇ Energy estimates using the set of (spherical) vectorfields  $\mathbf{T}, \mathbf{S}, \mathbf{K}, \mathbf{O}$  do not provide optimal bounds and is the cause of degeneracies – even for the wave equation in Minkowski space – in the region  $\mathcal{M}_{\text{top}}^{\text{int}}$ .<sup>6</sup>
- ◇ Classical Sobolev estimates degenerate at the vertex of the cone, which without the extension to a larger spacetime (artificially) complicates the control of the nonlinear terms at the vertex.
- ◇ The extension result for  $\Sigma_{t^*}$  uses an *optimal control* for the fundamental forms of  $\Sigma_{t^*}$  which can be obtained using global harmonic Cartesian coordinates and the new optimal control for these coordinates established in this paper (see Theorem 4.3 and Appendix A).

### 1.4.3 Curvature, connection and metric control

From the boundary fluxes controls obtained in Section 1.4.2, control for the spacetime curvature follows similarly as in [CK93, KN03].

From the control of the spacetime curvature and using the structure equations, one deduces a control for the connection and metric by

- ◇ integration of transport equations from the vertex  $\phi(\underline{u}^*)$  of  $\underline{\mathcal{C}}^*$  to  $\underline{\mathcal{C}}^*$  and elliptic estimates on the 2-spheres of the canonical foliation on  $\underline{\mathcal{C}}^*$ ,
- ◇ integration of transport equations from  $\underline{\mathcal{C}}^*$  to  $\mathcal{M}^{\text{ext}}$  and  $\mathcal{T}$ , and elliptic estimates on the 2-spheres of the geodesic foliation in  $\mathcal{M}^{\text{ext}}$ ,
- ◇ elliptic estimates on the maximal slices  $\Sigma_t$  using mixed (implicit) Dirichlet-Neumann boundary values on  $\partial\Sigma_t \subset \mathcal{T}$ .

<sup>6</sup>At the vertex,  $|\mathbf{O}| \rightarrow 0$  and  $\mathbf{T}, \mathbf{S}, \mathbf{K}$  become colinear.

**Remarks**

- ◇ We use that the spacetime is assumed to be *smooth* and in consequence, we establish general limits for (all derivatives of) the metric and connection coefficients at the vertex of a cone.<sup>7</sup> These limits are used as initial data to integrate the transport equations on  $\mathcal{C}^*$ .
- ◇ The main challenge in the succession of geometric constructions of  $\mathcal{M}_{\underline{u}^*}$  which are build upon one another, is to avoid the addition of *loss of regularity* for the connection coefficients. The regularity must eventually be sufficient for the control of the nonlinear error terms in the energy estimates of Section 1.4.2.

**1.4.4 Approximate conformal Killing vectorfields**

In the exterior region  $\mathcal{M}^{\text{ext}}$ , the functions  $u, \underline{u}$  are associated to a null pair  $(e_3, e_4)$ , such that

$$e_4 = -(\mathbf{D}u)^\sharp, \quad \mathbf{g}(e_3, e_3) = 0, \quad \mathbf{g}(e_4, e_3) = -2,$$

and such that  $e_3, e_4$  are orthogonal to the 2-spheres  $S_{u, \underline{u}}$  of the null-geodesic foliation. Here  $(\mathbf{D}u)^\sharp$  is the spacetime gradient of  $u$ . These vectorfields serve as approximations for the Minkowskian  $\partial_t - \partial_r$  and  $\partial_t + \partial_r$ . Upon these we thus define the following approximate exterior conformal Killing vectorfields

$$\mathbf{T}^{\text{ext}} := \frac{1}{2}(e_3 + e_4), \quad \mathbf{S}^{\text{ext}} := \frac{1}{2}(ue_3 + \underline{u}e_4), \quad \mathbf{K}^{\text{ext}} := \frac{1}{2}(u^2e_3 + \underline{u}^2e_4),$$

and we postpone the definition of the exterior rotation vectorfields  $\mathbf{O}^{\text{ext}}$  to the end of this section.

In the bottom interior region  $\mathcal{M}_{\text{bot}}^{\text{int}}$ , we define  $\mathbf{T}^{\text{int}}$  to be the future-pointing unit normal to the maximal hypersurfaces  $\Sigma_t$ . To obtain further definitions for  $\mathbf{S}^{\text{int}}$  and  $\mathbf{K}^{\text{int}}$  we need an approximation for the Minkowskian vectorfield  $r\partial_r$ . This can be obtained by defining the vectorfield  $\mathbf{X}^{\text{int}}$  on the last slice  $\Sigma_{t^*}$  using the global harmonic Cartesian coordinates

$$\mathbf{X}^{\text{int}} := \sum_{i=1}^3 x^i \nabla x^i,$$

and by extending it on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by parallel transport along the flow of  $t$ . Using  $\mathbf{X}^{\text{int}}$ , we have the following definitions for  $\mathbf{S}^{\text{int}}$  and  $\mathbf{K}^{\text{int}}$

$$\mathbf{S}^{\text{int}} := t\mathbf{T}^{\text{int}} + \mathbf{X}^{\text{int}}, \quad \mathbf{K}^{\text{int}} := (t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) \mathbf{T}^{\text{int}} + 2t\mathbf{X}^{\text{int}}.$$

Similarly, we can define the interior rotation vectorfields on the last slice  $\Sigma_{t^*}$  by

$$(1) \mathbf{O}^{\text{int}} := x^2 \nabla x^3 - x^3 \nabla x^2, \quad (2) \mathbf{O}^{\text{int}} := x^3 \nabla x^1 - x^1 \nabla x^3, \quad (3) \mathbf{O}^{\text{int}} := x^1 \nabla x^2 - x^2 \nabla x^1,$$

and extend this definition on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by parallel transport along the flow of  $t$ .

To match with the definitions of the interior rotations, the definition of the exterior rotation vectorfields is initialised at  $S^* = \partial\Sigma_{t^*}$  using the harmonic coordinates of  $\Sigma_{t^*}$  by

$$(1) \mathbf{O}^{\text{ext}} := x^2 \nabla x^3 - x^3 \nabla x^2, \quad (2) \mathbf{O}^{\text{ext}} := x^3 \nabla x^1 - x^1 \nabla x^3, \quad (3) \mathbf{O}^{\text{ext}} := x^1 \nabla x^2 - x^2 \nabla x^1.$$

We then define them by Lie transport along  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$

$$\left[ {}^{(\ell)} \mathbf{O}^{\text{ext}}, e_3 \right] = 0,$$

and in  $\mathcal{M}^{\text{ext}}$

$$\left[ {}^{(\ell)} \mathbf{O}^{\text{ext}}, e_4 \right] = 0,$$

for all  $\ell = 1, 2, 3$ .

The control of the deformation tensors of

<sup>7</sup>In fact, we even establish limits in the more general setting of a foliation of cones emanating from a central axis.



- ◇  $\mathbf{T}^{\text{ext}}, \mathbf{S}^{\text{ext}}, \mathbf{K}^{\text{ext}}$  is obtained from the control of the null connection coefficients associated to the null pair  $(e_3, e_4)$  in  $\mathcal{M}^{\text{ext}}$ ,
- ◇  $\mathbf{O}^{\text{ext}}$  is obtained by estimates for the harmonic coordinates on  $S^*$  and integration of transport equations from  $S^*$  to  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  and to  $\mathcal{M}^{\text{ext}}$ ,
- ◇  $\mathbf{T}^{\text{int}}$  is obtained from the control of the maximal connection coefficients in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ ,
- ◇  $\mathbf{X}^{\text{int}}, \mathbf{S}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{O}^{\text{int}}$  is obtained from estimates on  $\Sigma_{t^*}$ , using the bounds on the harmonic coordinates  $(x^i)$ , and by integration in  $t$ .

The control at the interface  $\mathcal{T}$  of the difference of vectorfields

- ◇  $\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}}$  is obtained by a control of the *slope* between the maximal hypersurfaces and the boundary  $\mathcal{T}$  (see a similar result in [CG19b]),
- ◇  $\mathbf{X}^{\text{ext}} - \mathbf{X}^{\text{int}}$ <sup>8</sup> (and subsequently  $\mathbf{S}^{\text{ext}} - \mathbf{S}^{\text{int}}$  and  $\mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}$ ) is obtained by a control on  $S^*$  using the harmonic coordinates and the control of the slope, and by integration along  $\mathcal{T}$ , using that  $\mathbf{D}^2\mathbf{X} \simeq 0$ ,
- ◇  $\mathbf{O}^{\text{ext}} - \mathbf{O}^{\text{int}}$  is obtained by a control on  $S^*$  using the harmonic coordinates control, and by integration along  $\mathcal{T}$ , using that  $\mathbf{D}^2\mathbf{O} \simeq 0$ .

### Remarks

- ◇ The rotation vectorfields in the interior region are only defined to extend the rotation vectorfields of the exterior region. They are not used to estimate the curvature in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ .
- ◇ The rotation vectorfields in the exterior region are used to estimate the tangential derivatives of the curvature. Thus, they have to be tangent to the 2-spheres of the canonical and geodesic foliation on  $\underline{\mathcal{C}}^*$  and  $\mathcal{M}^{\text{ext}}$  respectively. This motivates their definition by Lie transport.

## 1.5 Comparison to previous works

In this section, we discuss the strategy and techniques of proof of Theorem 1.8 and compare them to other works.

1. In the literature, the characteristic Cauchy problem outside of spherical symmetry is rather studied under a local (*e.g.* [Ren90, CCM11, CG19b]) or semi-global perspective – *i.e.* in a size 1 region from the initial null hypersurface – as in [Luk12, LZ18] or [CN05, CN10].
2. To obtain the *full* global result of Theorem 1.8, the spacetime geometric constructions are *initialised at timelike infinity*, which corresponds in this paper to starting our construction at the last point  $\phi(\underline{u}^*)$  on the central axis  $\phi$ . This differs from:
  - ◇ the semi-global existence results of Item 1 where the geometric constructions are initialised from the null initial data hypersurface,
  - ◇ the exterior region stability result of [KN03], where the double null foliation is constructed by imposing that:
    - the ingoing null hypersurfaces  $\underline{\mathcal{C}}_u$  emanate from the 2-spheres of a canonical foliation of the initial spacelike hypersurface  $\Sigma$ ,
    - the outgoing null hypersurfaces  $\mathcal{C}_u$  are backwards emanating from the 2-spheres of a canonical foliation on the last ingoing null hypersurface  $\underline{\mathcal{C}}^*$ ,
and which thus rather uses *spacelike infinity* to initialise the constructions,
  - ◇ the original stability result of [CK93] where a global maximal time function is given, based on the existence of an asymptotically flat spacelike infinity,
  - ◇ the stability of Schwarzschild spacetime established in [KS17] (see also constructions for Kerr spacetime in [KS19a, KS19b]), where the initialisation is performed from a *last sphere* corresponding to timelike infinity, but which tracks *the* central axis of Schwarzschild/Kerr spacetime. This is done by the construction of the last sphere as *the intrinsic* GCM sphere (see [KS19b, Section 7] or [KS17]) and uses that the mass of the spacetime is *non vanishing*. In the case of the

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<sup>8</sup>where  $\mathbf{X}^{\text{ext}} := \frac{1}{4}(\underline{u} - u)(e_4 - e_3)$ .

perturbations of Minkowski space of Theorem 1.8, there is no such intrinsic choice since there is no canonical central axis (the mass is not non-vanishing). We rather *prescribe* a central axis, at the future timelike infinity of which we perform our initialisation.

3. We treat an *interior region* enclosing the central axis, where constructions related to spherical coordinates *degenerate* (*i.e.* null pairs, null decompositions, etc.). This again differs from the semi-global results of Item 1, as well as from the exterior region global stability result of [KN03] and from the global stability of Schwarzschild spacetime<sup>9</sup> from [KS17], which do not treat such an interior region and rely on null projected equations. In particular, the  $r^p$ -methods developed in [Hol10] for Bianchi equations projected on null pairs (see also the seminal  $r^p$ -method of [DR10]) are unsuited for such a region.

In [CK93], the interior region is treated using the global maximal time function constructed from spacelike infinity and an interior optical function initialised at the central axis.

In the case of Theorem 1.8, no such global time function is available, and we rely instead on a construction of maximal hypersurfaces in the interior region, by prescribing their boundaries on the transition hypersurface, which is the timelike boundary between the interior and the exterior region. This is close in spirit to the proof of the spacelike-characteristic bounded  $L^2$  curvature result from [CG19b].

We moreover get rid of the interior optical function of [CK93], and rather use (transported) *Cartesian harmonic coordinates*, which virtually makes any reference to the central axis disappear in the analysis of that region. See Section 1.4.

4. Matching the (Cartesian) setting of the interior region to the (spherical) setting of the exterior region, is *not performed* following functions or frames gluing procedures as in [CK93, KS17], but is rather done by integration by parts, allowing discontinuities for the gauge choices at the interfaces. This procedure features a *mean value argument* to avoid regularity losses at the timelike interface. We believe that this treatment – although not fundamentally different – gains in clarity with respect to previous works.
5. At the core of the proof of Theorem 1.8 are the global energy estimates obtained by performing simultaneous energy estimates in the interior and exterior regions. Because of the different Cartesian/spherical setting used in each region, we choose to rely on the fully geometric framework (the spacetime Bel-Robinson tensors, the interior/exterior approximate conformal Killing vectorfields) of [CK93] to match these estimates across the timelike transition hypersurface.

## 1.6 Organisation of the paper

We outline the structure of this paper.

- ◇ The geometric set up, definitions and formulas are collected in Section 2.
- ◇ Section 3 is dedicated to collecting the definitions of the norms and bootstrap assumptions used in Sections 5 – 12.
- ◇ Section 4 is dedicated to the statement of the main theorem as well as auxiliary theorems (global existence of harmonic coordinates, vertex/axis limits, etc.). We also set up and prove the bootstrap argument from which the conclusions of the main theorem follow. This relies on the improvement of the bootstrap assumptions, which is the core of this paper and is obtained in Sections 5 – 12.
- ◇ In Section 5, we perform the global energy estimates which are split into simultaneous energy estimates in the interior and exterior region. That is, we analyse and control the nonlinear error terms at the timelike transition interface, in the interior and in the exterior region.
- ◇ In Sections 6 and 7, we deduce from the bounds for the boundary energy fluxes of Section 5, bounds for the curvature in the exterior and bottom interior region.
- ◇ In Section 8, we perform local energy estimates to obtain the remaining curvature bounds. In particular in the interior top region, we use an extension and local existence argument to obtain curvature bounds on the last cone.

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<sup>9</sup>We recall that in the case of Schwarzschild spacetime, the axis  $r = 0$  is the Schwarzschild (spacelike) singularity and lies inside the Schwarzschild black hole.

- ◇ In Sections 9 and 10, we obtain bounds for the null connection and rotation coefficients on the last cone and in the exterior region.
- ◇ In Section 11, we obtain bounds for the interior connection coefficients and approximate interior Killing vectorfields. We also obtain the bounds for the difference of Killing fields at the interface.
- ◇ Section 12 is dedicated to the initial layer comparisons arguments.
- ◇ Appendices A and B are respectively dedicated to the obtention of global harmonic coordinates and vertex/axis limits, which are the main, independent and general results of the auxiliary theorems stated in Section 4.
- ◇ Appendices C and D are dedicated to the obtention of auxiliary local existence and functional results.

## 2. Definitions & formulas

Let  $(\tilde{\mathcal{M}}, \mathbf{g})$  be a smooth vacuum spacetime. Let  $\tilde{\Sigma}_1 \subset \tilde{\mathcal{M}}$  be a smooth spacelike hypersurface diffeomorphic to the unit disk of  $\mathbb{R}^3$  and  $\tilde{\mathcal{C}}_0 \subset \tilde{\mathcal{M}}$  the outgoing null hypersurface emanating from  $\partial\tilde{\Sigma}_1$ . We call  $(\tilde{\Sigma}_1, \tilde{\mathcal{C}}_0)$  *spacelike-characteristic initial data* and we assume that  $\tilde{\mathcal{M}}$  coincides with the future maximal globally hyperbolic development of  $\tilde{\Sigma}_1 \cup \tilde{\mathcal{C}}_0$ .

Let  $O$  be a fixed point of  $\tilde{\Sigma}_1$ , which we call the *centre of  $\tilde{\Sigma}_1$* .<sup>1</sup> We note  $\emptyset \subset \tilde{\mathcal{M}}$  the timelike geodesic emanating from  $O$  orthogonally to  $\tilde{\Sigma}_1$  parametrised by arc-length and such that  $\emptyset(1) = O$ . We call  $\emptyset$  the *central axis* of the spacetime  $\tilde{\mathcal{M}}$ .

### 2.1 The bootstrap region $\mathcal{M}_{\underline{u}^*}$

In this section, we define the spacetime region  $\mathcal{M}_{\underline{u}^*} \subset \tilde{\mathcal{M}}$  involved in our bootstrap argument, as well as its foliation by geometric hypersurfaces and its decomposition into subdomains.<sup>2</sup>

The constructions of this section are graphically summarised in Figure 1.

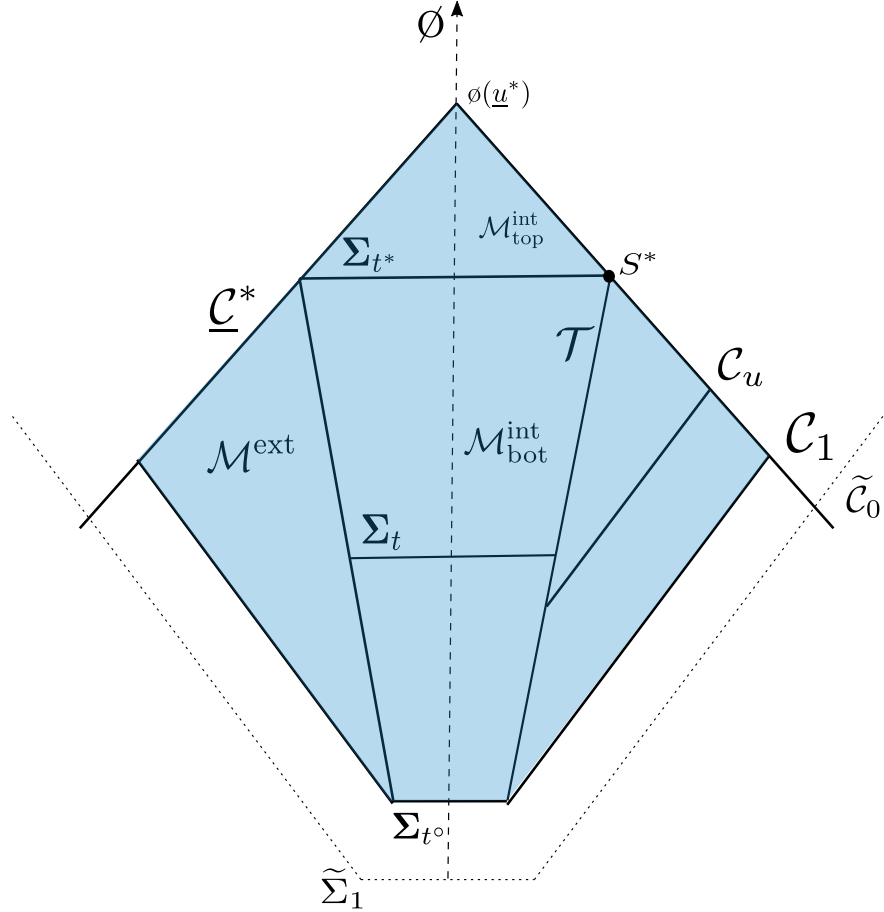


Figure 1: The spacetime region  $\mathcal{M} = \mathcal{M}_{\underline{u}^*}$ .

**The last cone  $\underline{\mathcal{C}}^*$  and the optical function  $u$ .** Let  $u^* \in [1, \infty)$ . We define  $\underline{\mathcal{C}}^*$  to be the incoming null cone with vertex  $\emptyset(u^*)$ . Let  $u$  be a scalar function on  $\underline{\mathcal{C}}^*$ . We assume that the level sets of  $u$  define a foliation

<sup>1</sup>The centre  $O$  will be defined in Section 4 to be the point  $x^i = 0$ , where  $(x^i)$  are coordinates on  $\tilde{\Sigma}_1$  such that  $\tilde{\Sigma}_1$  is close to the unit Euclidean disk of  $\mathbb{R}^3$ .

<sup>2</sup>The existence and regularity of the spacetime region  $\mathcal{M}_{\underline{u}^*} \subset \tilde{\mathcal{M}}$  will be an assumption of the bootstrap argument of Section 4.

of  $\underline{\mathcal{C}}^*$  by regular 2-spheres  $S_{u,\underline{u}^*}$ , which will further be defined to be the 2-spheres of the so-called *canonical foliation* (see Definition 2.1). We consider  $\mathcal{C}_u$  the outgoing null hypersurface backward emanating from the 2-sphere  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$ . We define  $u$  to be the associated optical function, and  $L$  to be the (normalised) spacetime gradient of  $u$ , *i.e.*  $L := -(\mathbf{D}u)^\sharp$ .

**The geodesic parameter  $\underline{u}$ .** We define  $\underline{u}$  to be the following null geodesic affine parameter of the hypersurfaces  $\mathcal{C}_u$

$$\begin{aligned} L(\underline{u}) &= 2, \\ \underline{u}|_{\underline{\mathcal{C}}^*} &= \underline{u}^*. \end{aligned}$$

We assume that its level sets define a regular foliation of hypersurfaces transverse to the null hypersurfaces  $\mathcal{C}_u$  and that their intersections  $S_{u,\underline{u}}$  define spacelike regular 2-spheres.

Let  $\underline{Y} := -(\mathbf{D}\underline{u})^\sharp$  be the (normalised) spacetime gradient of  $\underline{u}$ . By definition of  $\underline{u}$ , we have

$$\mathbf{g}(\underline{Y}, L) = -2.$$

We define the *optical defect* of the foliation  $S_{u,\underline{u}}$  to be the scalar function  $\eta$  given by

$$\mathbf{g}(\underline{Y}, \underline{Y}) =: -2\eta.$$

We define the *null pair*  $(e_3, e_4)$  associated to the foliation  $S_{u,\underline{u}}$  to be the pair of null vectorfields orthogonal to the 2-spheres  $S_{u,\underline{u}}$  such that

$$e_4 = L, \quad \text{and} \quad \mathbf{g}(e_3, e_4) = -2.$$

From the above definition, we have the following relations

$$\begin{aligned} e_3(\underline{u}) &= \eta, & e_3(u) &= 2, \\ e_4(\underline{u}) &= 2, & e_4(u) &= 0, \end{aligned} \tag{2.1}$$

and

$$\underline{Y} = e_3 + \frac{1}{2}\eta e_4. \tag{2.2}$$

**The canonical foliation on  $\underline{\mathcal{C}}^*$ .** We define  $\chi, \xi, \eta, \zeta, \omega$  and  $\underline{\chi}, \underline{\xi}, \underline{\eta}, \underline{\zeta}, \underline{\omega}$  to be the null decomposition of the connection coefficients associated to the null pair  $(e_3, e_4)$ , and  $\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}$  to be the null decomposition of the spacetime curvature tensor with respect to the null pair  $(e_3, e_4)$  (see Section 2.7 for definitions).

With respect to these null decompositions, we have the following definition, which determines the function  $u$  on  $\underline{\mathcal{C}}^*$ .

**Definition 2.1** (Canonical foliation on  $\underline{\mathcal{C}}^*$ ). *The scalar function  $u$  is said to be canonical on  $\underline{\mathcal{C}}^*$  if on each 2-sphere  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$ , the following condition holds*

$$\begin{aligned} \mathring{d}\eta + \rho &= \bar{\rho}, \\ \bar{\omega} &= 0, \end{aligned} \tag{2.3}$$

where we refer to Section 2.7 for definitions, and if at the vertex  $\mathring{o}(\underline{u}^*)$ , the function  $u$  is normalised by

$$u|_{\mathring{o}(\underline{u}^*)} = \underline{u}^*, \quad \mathbf{g}(e_3, \mathring{o})|_{\mathring{o}(\underline{u}^*)} = -1.$$

**Remark 2.2.** *The null pair  $(e_3, e_4)$  is constructed to be adapted to the canonical foliation on  $\underline{\mathcal{C}}^*$  (e.g.  $e_3(u) = 2$ ). It can be related to a geodesic null pair via a null lapse. This will only be introduced in the proof of the existence of the canonical foliation in Appendix C. Other arguments do not require the existence or control of a background geodesic foliation.*

**Remark 2.3.** *The canonical foliation of Definition 2.1 does not coincide with the canonical foliations of [KN03, Nic04, CG19a], since we replaced for the conciseness of the argument the condition on the mean value of the null lapse in that papers by the condition  $\bar{\omega} = 0$ . This is purely to ease the notations and does not change anything to the motivations, its local or global existence on  $\underline{\mathcal{C}}^*$ . See Theorem 4.5.*

**The exterior region  $\mathcal{M}_{\underline{u}^*}^{\text{ext}}$ .** Let  $0 < \tau_0 < 1$  be a constant sufficiently close to 1, which we call the *transition constant*. Its value is determined in Section 7.3 (see also Section 3.3.1 for a recapitulation of the constants of this paper and their dependencies).

Let  $\tau$  be a *transition parameter*, such that

$$\tau_0 \leq \tau \leq \frac{1}{2}(1 + \tau_0).$$

For a fixed transition parameter, we define the *exterior region*  ${}^{(\tau)}\mathcal{M}_{\underline{u}^*}^{\text{ext}}$  to be

$${}^{(\tau)}\mathcal{M}_{\underline{u}^*}^{\text{ext}} := \{1 \leq u \leq \tau \underline{u}\},$$

and we define the *timelike transition hypersurface*  ${}^{(\tau)}\mathcal{T}$  to be

$${}^{(\tau)}\mathcal{T} := \{u = \tau \underline{u}\}.$$

**Remark 2.4.** *The freedom on the transition parameter  $\tau$  is used to perform a mean value argument (see Section 5.1.1). This mean value argument provides optimal regularity for the spacetime curvature tensor on a well chosen timelike interface  ${}^{(\tau)}\mathcal{T}$ . This enables a control of the nonlinear transition error terms. See Remark 5.6 and Section 5.1.2.*

**The interior region  $\mathcal{M}_{\underline{u}^*}^{\text{int}}$ .** Let

$${}^{(\tau)}t^\circ := (1 + \tau^{-1})/2, \quad {}^{(\tau)}t^* := (1 + \tau) \underline{u}^*/2.$$

For all  $t^\circ \leq t \leq t^*$ , we define  ${}^{(\tau)}\Sigma_t$  to be the *maximal hypersurfaces*, i.e. such that

$$\text{tr}_g k = 0,$$

where  $g$  and  $k$  are the first and second fundamental form of  ${}^{(\tau)}\Sigma_t$  (see Section 2.2.1 for definitions), with prescribed boundary

$$\partial^{(\tau)}\Sigma_t = S_{u=2t\tau/(1+\tau), \underline{u}=2t/(1+\tau)} \subset {}^{(\tau)}\mathcal{T},$$

i.e.

$${}^{(\tau)}t|_{{}^{(\tau)}\mathcal{T}} = \frac{1}{2}(u + \underline{u}). \quad (2.4)$$

We define the *bottom interior region*  ${}^{(\tau)}\mathcal{M}_{\text{bot}, \underline{u}^*}^{\text{int}}$  to be

$${}^{(\tau)}\mathcal{M}_{\text{bot}, \underline{u}^*}^{\text{int}} := \left\{ {}^{(\tau)}t^\circ \leq {}^{(\tau)}t \leq {}^{(\tau)}t^* \right\}.$$

We call  ${}^{(\tau)}\Sigma_{t^*}$  the *last maximal slice* and we note its boundary  ${}^{(\tau)}S^* := \partial^{(\tau)}\Sigma_{t^*} = S_{\tau \underline{u}^*, \underline{u}^*} \subset \underline{\mathcal{C}}^*$ . We define the *top interior region*  ${}^{(\tau)}\mathcal{M}_{\text{top}, \underline{u}^*}^{\text{int}}$  to be the domain of dependence of the last maximal hypersurface  ${}^{(\tau)}\Sigma_{t^*}$ .

**Remark 2.5.** *Since  $\partial^{(\tau)}\Sigma_{t^*} \subset \underline{\mathcal{C}}^*$ , the future boundary of  ${}^{(\tau)}\mathcal{M}_{\text{top}, \underline{u}^*}^{\text{int}}$  is contained in the last cone  $\underline{\mathcal{C}}^*$ .*

We define the *interior region*  ${}^{(\tau)}\mathcal{M}_{\underline{u}^*}^{\text{int}}$  and the spacetime region  ${}^{(\tau)}\mathcal{M}_{\underline{u}^*}$  by

$$\begin{aligned} {}^{(\tau)}\mathcal{M}_{\underline{u}^*}^{\text{int}} &:= {}^{(\tau)}\mathcal{M}_{\text{bot}, \underline{u}^*}^{\text{int}} \cup {}^{(\tau)}\mathcal{M}_{\text{top}, \underline{u}^*}^{\text{int}}, \\ {}^{(\tau)}\mathcal{M}_{\underline{u}^*} &:= {}^{(\tau)}\mathcal{M}_{\underline{u}^*}^{\text{ext}} \cup {}^{(\tau)}\mathcal{M}_{\underline{u}^*}^{\text{int}}. \end{aligned}$$

From now on, we shall drop the labels  $\underline{u}^*, \tau$  in the above geometric constructions, unless that precision is relevant.

## 2.2 Maximal hypersurfaces $\Sigma_t$ of $\mathcal{M}_{\text{bot}}^{\text{int}}$

### 2.2.1 Fundamental forms

The *first fundamental form* is defined to be the induced metric  $g$  by  $\mathbf{g}$  on  $\Sigma_t$ .

We define the *time lapse*  $n$  of the maximal hypersurfaces  $\Sigma_t$  by

$$n^{-2} := -\mathbf{g}(\mathbf{D}t, \mathbf{D}t). \quad (2.5)$$

We define  $\bar{T}$  to be the future-oriented unit normal to  $\Sigma_t$ . It satisfies the following relations

$$\bar{T} = -n(\mathbf{D}t)^\sharp, \quad \mathbf{D}_{\bar{T}}\bar{T} = n^{-1}\nabla n, \quad \bar{T}(t) = n^{-1}, \quad (2.6)$$

where  $\nabla$  denotes the induced covariant derivative on  $\Sigma_t$ .

We define the *second fundamental form*  $k$  of  $\Sigma_t$  to be

$$k(X, Y) := -\mathbf{g}(\mathbf{D}_X\bar{T}, Y), \quad (2.7)$$

for all  $X, Y$  in  $T\Sigma_t$ . From the definitions of Section 2.1, we recall that we have

$$\text{tr}_g k = 0. \quad (2.8)$$

### 2.2.2 Electric-magnetic decompositions

The *electric-magnetic* decomposition of the curvature tensor  $\mathbf{R}$  consists of the  $\Sigma_t$  tangent tensors  $E$  and  $H$  defined by

$$E(X, Y) := \mathbf{R}(\bar{T}, X, \bar{T}, Y), \quad H(X, Y) := *\mathbf{R}(\bar{T}, X, \bar{T}, Y), \quad (2.9)$$

for all  $X, Y$  in  $T\Sigma_t$ .

Using the maximal condition (2.8), we have the following Gauss equation for the Ricci curvature tensor of  $\Sigma_t$

$$\text{Ric}_{ij} = E_{ij} + k_{ia}k_j^a. \quad (2.10)$$

Using the maximal condition (2.8), we have the following Hodge-type system for  $k$

$$\text{div}k = 0, \quad \text{curl}k = H, \quad \text{tr}k = 0, \quad (2.11)$$

and Laplace-type equation for  $n$

$$\Delta n = n|k|^2, \quad (2.12)$$

see [CK93, p. 6].

One also has the following equation for  $\hat{\mathcal{L}}_{\bar{T}}k$  (see [CK93, p. 6])

$$n\hat{\mathcal{L}}_{\bar{T}}k = -\nabla^2 n + n(E - k \cdot k). \quad (2.13)$$

The tensors  $E$  and  $H$  satisfy the following Maxwell-type system of equations (see [CK93, pp. 144-146])

$$\begin{aligned} \text{div}E &= k \wedge H, \\ \text{div}H &= -k \wedge E, \\ -\hat{\mathcal{L}}_{\bar{T}}H + \text{curl}E &= -n^{-1}\nabla n \wedge E - \frac{1}{2}k \times H, \\ \hat{\mathcal{L}}_{\bar{T}}E + \text{curl}H &= -n^{-1}\nabla n \wedge H + \frac{1}{2}k \times E, \end{aligned} \quad (2.14)$$

where for a  $\Sigma_t$ -tangent symmetric traceless 2-tensor  $U$ ,  $\hat{\mathcal{L}}_{\bar{T}}U_{ij} := \mathcal{L}_{\bar{T}}U_{ij} + \frac{2}{3}(k \cdot U)g_{ij}$ .

## 2.3 Transition relations on the boundary $\mathcal{T}$

In this section, we derive boundary relations between the maximal hypersurface decompositions of Sections 2.2.1 and 2.2.2 and the null decompositions of Section 2.7.

There exists  $\nu > 0$  such that on  $\mathcal{T}$ , we have

$$\bar{T} := \frac{1}{2} (\nu^{-1} e_3 + \nu e_4), \quad (2.15)$$

and the outward-pointing unit normal to the boundary  $\partial\Sigma_t$  writes

$$\bar{N} := \frac{1}{2} (-\nu^{-1} e_3 + \nu e_4). \quad (2.16)$$

We define the boundary decompositions of  $k$  by

$$\delta := k_{\bar{N}\bar{N}}, \quad \epsilon_a := k_{\bar{N}a}, \quad \kappa_{ab} := k_{ab}. \quad (2.17)$$

**Lemma 2.6.** *We have*

$$\begin{aligned} \kappa_{ab} &= -\frac{1}{2}\nu^{-1}\chi_{ab} - \frac{1}{2}\nu\chi_{ab}, \\ \epsilon_a &= -\nabla_a(\log \nu) + \zeta_a, \\ \delta &= \frac{1}{2}\nu^{-1}\mathbf{D}_3(\log \nu) - \frac{1}{2}\nu\mathbf{D}_4(\log \nu) + \nu^{-1}\underline{\omega} + \nu\omega. \end{aligned} \quad (2.18)$$

*Proof.* The first relations for  $\kappa$  immediately follow from (2.33) and (2.7). Using (2.16), we have

$$\begin{aligned} k_{\bar{N}a} &= -\frac{1}{4}\mathbf{g}(\mathbf{D}_a(\nu^{-1}e_3 + \nu e_4), -\nu^{-1}e_3 + \nu e_4) \\ &= -\nabla_a(\log \nu) + \zeta_a, \end{aligned}$$

and

$$\begin{aligned} k_{\bar{N}\bar{N}} &= -\frac{1}{8}\mathbf{g}(\mathbf{D}_{-\nu^{-1}e_3 + \nu e_4}(\nu^{-1}e_3 + \nu e_4), -\nu^{-1}e_3 + \nu e_4) \\ &= \frac{1}{2}\nu^{-1}\mathbf{D}_3(\log \nu) - \frac{1}{2}\nu\mathbf{D}_4(\log \nu) \\ &\quad + \frac{1}{8}\nu^{-1}\mathbf{g}(\mathbf{D}_{e_3}(e_3 + e_4), -e_3 + e_4) \\ &\quad - \frac{1}{8}\nu\mathbf{g}(\mathbf{D}_{e_4}(e_3 + e_4), -e_3 + e_4) \\ &= \frac{1}{2}\nu^{-1}\mathbf{D}_3(\log \nu) - \frac{1}{2}\nu\mathbf{D}_4(\log \nu) + \nu^{-1}\underline{\omega} + \nu\omega. \end{aligned}$$

This finishes the proof of the lemma.  $\square$

**Lemma 2.7.** *We have at  $\mathcal{T}$*

$$n = \frac{\tau}{\tau+1} \left( \nu + \left( \tau^{-1} - \frac{1}{2}\eta \right) \nu^{-1} \right). \quad (2.19)$$

*Proof.* Using relations (2.1), the future-pointing unit normal  $Z$  to  $S_{u,\underline{u}}$  in  $\mathcal{T}$  writes

$$Z = \frac{1}{2} \left( a^{-1/2} e_3 + a^{1/2} e_4 \right),$$

with

$$a := \tau^{-1} - \frac{1}{2}\eta.$$

Using that we have on  $\mathcal{T}$

$$Z(t) = \frac{1}{2} Z(u + \underline{u}),$$



we therefore deduce at  $\mathcal{T}$  that

$$\begin{aligned} a^{-1/2}e_3(t) + a^{1/2}e_4(t) &= \frac{1}{2}(a^{-1/2}e_3 + a^{1/2}e_4)(u + \underline{u}) \\ &= a^{-1/2} + a^{1/2} + \frac{1}{2}a^{-1/2}\eta. \end{aligned} \tag{2.20}$$

Moreover from the relations (2.6) and definitions (2.15), (2.16), we have at  $\mathcal{T}$

$$\begin{aligned} \frac{1}{2}\nu^{-1}e_3(t) + \frac{1}{2}\nu e_4(t) &= n^{-1}, \\ -\frac{1}{2}\nu^{-1}e_3(t) + \frac{1}{2}\nu e_4(t) &= 0, \end{aligned}$$

which gives

$$e_3(t) = \nu n^{-1}, \quad e_4(t) = \nu^{-1}n^{-1}. \tag{2.21}$$

Plugging this into (2.20) gives

$$n = \frac{a^{-1/2}\nu + a^{1/2}\nu^{-1}}{a^{-1/2} + a^{1/2} + \frac{1}{2}a^{-1/2}\eta},$$

which, after rewriting, using the expression for  $a$ , gives the desired formula.  $\square$

## 2.4 Uniformisation of the sphere $S^*$ and harmonic Cartesian coordinates on $\Sigma_{t^*}$

We recall the following definition from Section 2.1

$$S^* := \partial\Sigma_{t^*} = S_{u=\tau\underline{u}^*, \underline{u}^*},$$

and we note  $(r^*)^2 := \frac{1}{4\pi}|S^*|$  its area radius.

A *conformal isomorphism* between  $S^*$  and the Euclidean unit 2-sphere  $\mathbb{S}$  is a diffeomorphism  $\Phi : S^* \rightarrow \mathbb{S}$  such that there exists a *conformal factor*  $\phi > 0$  on  $\mathbb{S}$  satisfying

$$\Phi_{\#}\mathcal{g}_{S^*} = (r^*)^2\phi^2\mathcal{g}_{\mathbb{S}},$$

where  $\Phi_{\#}\mathcal{g}_{S^*}$  is the push-forward of the metric  $\mathcal{g}$  by  $\Phi$ .

To a fixed conformal isomorphism of  $S^*$ , we associate the (normalised) *Cartesian coordinates*  $\left(\frac{x^i}{r^*}\right)_{i=1\dots 3}$  on  $S^*$  to be the pull-back by  $\Phi$  of the standard Cartesian coordinates on  $\mathbb{S}$ .

We say that the conformal isomorphism  $\Phi$  is *centred* if the functions  $x^i$  satisfy the following conditions on  $S^*$

$$\int_{S^*} x^i = 0, \quad i = 1, \dots, 3.$$

Using these coordinates, we further define the associated *harmonic Cartesian coordinates* on  $\Sigma_{t^*}$  to be the solution of the following Dirichlet problem on  $\Sigma_{t^*}$

$$\begin{aligned} \Delta_g x^i &= 0, \\ x^i|_{S^*=\partial\Sigma_{t^*}} &= x^i, \end{aligned} \tag{2.22}$$

for all  $i = 1, 2, 3$ .

**Remark 2.8.** *Further constructions and bounds in this paper will hold for all centred conformal isomorphisms. In this paper, we will use the existence, uniqueness and control of all the centred conformal isomorphism established in [KS19b, Theorem 3.1].*

## 2.5 Approximate interior Killing fields $\mathbf{T}^{\text{int}}$ , $\mathbf{S}^{\text{int}}$ , $\mathbf{K}^{\text{int}}$ and $\mathbf{O}^{\text{int}}$ in $\mathcal{M}_{\text{bot}}^{\text{int}}$

We first define the *approximate Killing time translation vectorfield*  $\mathbf{T}^{\text{int}}$  on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by

$$\mathbf{T}^{\text{int}} := \bar{T}.$$

We define  $\mathbf{X}^{\text{int}}$  on  $\Sigma_{t^*}$  to be the  $\Sigma_{t^*}$ -tangent vectorfield given by

$$\mathbf{X}^{\text{int}} := \sum_{i=1}^3 x^i \nabla x^i, \quad (2.23)$$

where  $\nabla$  is the induced gradient on  $\Sigma_{t^*}$  and  $x^i$  are the harmonic Cartesian coordinates defined in Section 2.4.

We define  $\mathbf{X}^{\text{int}}$  on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by parallel transport, *i.e.*

$$\mathbf{D}_{\bar{T}} \mathbf{X}^{\text{int}} = 0, \quad (2.24)$$

We define the *approximate conformal Killing scaling vectorfield*  $\mathbf{S}^{\text{int}}$  on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by

$$\mathbf{S}^{\text{int}} := t \mathbf{T}^{\text{int}} + \mathbf{X}^{\text{int}}. \quad (2.25)$$

We define the *approximate conformal Killing Morawetz vectorfield*  $\mathbf{K}^{\text{int}}$  on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by

$$\mathbf{K}^{\text{int}} := (t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) \mathbf{T}^{\text{int}} + 2t \mathbf{X}^{\text{int}}. \quad (2.26)$$

We define the *approximate Killing rotation vectorfields*  $\mathbf{O}^{\text{int}}$  on  $\Sigma_{t^*}$  by

$${}^{(\ell)} \mathbf{O}^{\text{int}} := \epsilon_{\ell ij} x^i \nabla x^j, \quad (2.27)$$

for  $\ell = 1, 2, 3$ , and we extend them on  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by parallel transport along  $\bar{T}$ , *i.e.*

$$\mathbf{D}_{\bar{T}} {}^{(\ell)} \mathbf{O}^{\text{int}} = 0. \quad (2.28)$$

We recall that the spacetime *deformation tensor* of a spacetime vectorfield  $X$  is given by

$${}^{(X)} \pi_{\mu\nu} := \mathbf{D}_\mu X_\nu + \mathbf{D}_\nu X_\mu,$$

and that we note  $\hat{\pi}$  its traceless part, *i.e.*  $\hat{\pi} := \pi - \frac{1}{4} \text{tr} \pi \mathbf{g}$ .

Using the maximal condition (2.8), we have

$$\text{tr} \left( {}^{(\mathbf{T}^{\text{int}})} \pi \right) = 0, \quad (2.29)$$

and using relations (2.6), we have

$${}^{(\mathbf{T}^{\text{int}})} \pi_{\bar{T}\bar{T}} = 0, \quad {}^{(\mathbf{T}^{\text{int}})} \pi_{\bar{T}i} = n^{-1} \nabla_i n, \quad {}^{(\mathbf{T}^{\text{int}})} \pi_{ij} = -2k_{ij}. \quad (2.30)$$

## 2.6 Exterior hypersurfaces $\Sigma_t^{\text{ext}}$ of $\mathcal{M}^{\text{ext}}$

For all  $t^\circ \leq t \leq t^*$ , we define the exterior spacelike hypersurfaces  $\Sigma_t^{\text{ext}}$  by

$$\Sigma_t^{\text{ext}} := \{u + \underline{u} = 2t\} \cap \mathcal{M}^{\text{ext}}.$$

From (2.1), one has the following relations for the future-pointing unit normal  $\bar{T}^{\text{ext}}$  to  $\Sigma_t^{\text{ext}}$  and the outward-pointing unit normal  $\bar{N}^{\text{ext}}$  to the 2-spheres  $S_{u, \underline{u}} \subset \Sigma_t^{\text{ext}}$  in  $\Sigma_t^{\text{ext}}$

$$\begin{aligned} \bar{T}^{\text{ext}} &= \frac{1}{2} \left(1 + \frac{1}{2} \mathfrak{h}\right)^{-1/2} e_3 + \frac{1}{2} \left(1 + \frac{1}{2} \mathfrak{h}\right)^{1/2} e_4, \\ \bar{N}^{\text{ext}} &= -\frac{1}{2} \left(1 + \frac{1}{2} \mathfrak{h}\right)^{-1/2} e_3 + \frac{1}{2} \left(1 + \frac{1}{2} \mathfrak{h}\right)^{1/2} e_4. \end{aligned} \quad (2.31)$$

Moreover, we have the following definitions and expression of the time lapse  $n^{\text{ext}}$

$$\begin{aligned} n^{\text{ext}} &:= \left( -\mathbf{g} \left( \mathbf{D} \left( \frac{1}{2} (u + \underline{u}) \right), \mathbf{D} \left( \frac{1}{2} (u + \underline{u}) \right) \right) \right)^{-1/2} \\ &= \left( 1 + \frac{1}{2} \mathfrak{h} \right)^{-1/2}. \end{aligned} \quad (2.32)$$

## 2.7 Null decompositions

In this section, we recall the general null decomposition of the connection and curvature as defined in [CK93].

Let  $S_{u,\underline{u}}$  be a (local) foliation of  $\mathcal{M}$  by spacelike 2-spheres. We note  $\mathcal{g}$  the Riemannian metric induced by  $\mathbf{g}$  on  $S_{u,\underline{u}}$ , and we note  $r$  its area radius  $4\pi r^2 := |S|$ . A *null pair*  $(e_4, e_3)$  is a pair of vectorfields on  $\mathcal{M}$  orthogonal to the 2-spheres  $S_{u,\underline{u}}$  such that

$$\mathbf{g}(e_4, e_3) = -2, \quad \mathbf{g}(e_4, e_4) = \mathbf{g}(e_3, e_3) = 0.$$

We define the connection coefficients relative to a null pair  $(e_4, e_3)$  to be the  $S$ -tangent tensors defined by

$$\begin{aligned} \chi(X, Y) &:= \mathbf{g}(\mathbf{D}_X e_4, Y), & \xi(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_4, X), & \eta(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_4, X), \\ \zeta(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_X e_4, e_3), & \omega &:= \frac{1}{4} \mathbf{g}(\mathbf{D}_4 e_4, e_3), \end{aligned}$$

and

$$\begin{aligned} \underline{\chi}(X, Y) &:= \mathbf{g}(\mathbf{D}_X e_3, Y), & \underline{\xi}(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, X), & \underline{\eta}(X) &:= \frac{1}{2} \mathbf{g}(\mathbf{D}_4 e_3, X), \\ \zeta(X) &:= -\frac{1}{2} \mathbf{g}(\mathbf{D}_X e_3, e_4), & \underline{\omega} &:= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4), \end{aligned}$$

where  $X, Y \in TS$ .

**Remark 2.9.** *These correspond to the same definitions as in ([CK93, p. 147]) with*

$$\chi = H, \quad \xi = Y, \quad \eta = Z, \quad \zeta = V, \quad \omega = \Omega,$$

and

$$\underline{\chi} = \underline{H}, \quad \underline{\xi} = \underline{Y}, \quad \underline{\eta} = \underline{Z}, \quad \zeta = V, \quad \underline{\omega} = \underline{\Omega}.$$

For a (local) orthonormal frame  $(e_a)_{a=1,2}$  on  $TS$ , we have the following relations for the covariant derivatives of the orthonormal null frame  $(e_4, e_3, e_a)$  (see [CK93, p. 147])

$$\begin{aligned} \mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \zeta_a e_3, & \mathbf{D}_a e_4 &= \chi_{ab} e_b - \zeta_a e_4, \\ \mathbf{D}_3 e_3 &= 2\underline{\xi}_a e_a - 2\underline{\omega} e_3, & \mathbf{D}_3 e_4 &= 2\underline{\eta}_a e_a + 2\underline{\omega} e_4, \\ \mathbf{D}_4 e_3 &= 2\underline{\eta}_a e_a + 2\underline{\omega} e_3, & \mathbf{D}_4 e_4 &= 2\underline{\xi}_a e_a - 2\underline{\omega} e_4, \\ \mathbf{D}_3 e_a &= \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4, & \mathbf{D}_4 e_a &= \nabla_4 e_a + \xi_a e_3 + \underline{\eta}_a e_4, \\ \mathbf{D}_a e_b &= \nabla_a e_b + \frac{1}{2} \chi_{ab} e_3 + \frac{1}{2} \underline{\chi}_{ab} e_4. \end{aligned} \tag{2.33}$$

We define the curvature components relative to a null pair  $(e_4, e_3)$  to be the  $S$ -tangent tensors defined by

$$\begin{aligned} \alpha(X, Y) &:= \mathbf{R}(e_4, X, e_4, Y), & \beta(X) &:= \frac{1}{2} \mathbf{R}(X, e_4, e_3, e_4), & \rho &:= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4), \\ \underline{\alpha}(X, Y) &:= \mathbf{R}(e_3, X, e_3, Y), & \underline{\beta}(X) &:= \frac{1}{2} \mathbf{R}(X, e_3, e_3, e_4), & \sigma &:= \frac{1}{4} {}^* \mathbf{R}(e_3, e_4, e_3, e_4), \end{aligned}$$

where  $X, Y \in TS$  and where  ${}^* \mathbf{R}$  denotes the Hodge dual of  $\mathbf{R}$ .

We have the following null structure relating the null connection coefficients and the null curvature components (see [CK93, pp. 168-170]).

We have the following transport equations along  $e_3$  and  $e_4$

$$\mathcal{L}_{e_3} \mathcal{g} = 2\underline{\chi}, \quad \mathcal{L}_{e_4} \mathcal{g} = 2\underline{\chi}, \tag{2.34a}$$

and

$$\nabla_3 \hat{\chi} + \text{tr}\chi \hat{\chi} = \nabla \hat{\otimes} \underline{\xi} - 2\underline{\omega} \hat{\chi} + (\eta + \underline{\eta} - 2\underline{\zeta}) \hat{\otimes} \underline{\xi} - \underline{\alpha}, \quad (2.34b)$$

$$\nabla_3 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 = 2\text{d}\hat{\nu} \underline{\xi} - 2\underline{\omega} \text{tr}\chi + 2\underline{\xi} \cdot (\eta + \underline{\eta} - 2\underline{\zeta}) - |\hat{\chi}|^2, \quad (2.34c)$$

$$\nabla_3 \underline{\zeta} = -2\nabla \underline{\omega} - \underline{\chi} \cdot (\zeta + \eta) + 2\underline{\omega}(\zeta - \eta) + \chi \cdot \underline{\xi} + 2\underline{\omega} \underline{\xi} - \underline{\beta}, \quad (2.34d)$$

$$\nabla_3 \hat{\chi} + \frac{1}{2} \text{tr}\chi \hat{\chi} = \nabla \hat{\otimes} \eta + 2\underline{\omega} \hat{\chi} - \frac{1}{2} \text{tr}\chi \hat{\chi} + \underline{\xi} \otimes \underline{\xi} + \eta \hat{\otimes} \eta, \quad (2.34e)$$

$$\nabla_3 \text{tr}\chi + \frac{1}{2} \text{tr}\chi \text{tr}\chi = 2\text{d}\hat{\nu} \eta + 2\underline{\omega} \text{tr}\chi - \hat{\chi} \cdot \hat{\chi} + 2(\underline{\xi} \cdot \underline{\xi} + |\eta|^2) + 2\rho, \quad (2.34f)$$

$$\nabla_3 \underline{\xi} - \nabla_4 \eta = 4\underline{\omega} \underline{\xi} + \chi \cdot (\eta - \underline{\eta}) + \beta, \quad (2.34g)$$

$$\nabla_3 \underline{\eta} - \nabla_4 \underline{\xi} = -4\underline{\omega} \underline{\xi} - \underline{\chi} \cdot (\eta - \underline{\eta}) + \underline{\beta}, \quad (2.34h)$$

$$\nabla_3 \omega + \nabla_4 \underline{\omega} = \underline{\xi} \cdot \underline{\xi} + \zeta \cdot (\eta - \underline{\eta}) - \eta \cdot \underline{\eta} + 4\underline{\omega} \omega + \rho. \quad (2.34i)$$

$$\nabla_4 \hat{\chi} + \text{tr}\chi \hat{\chi} = \nabla \hat{\otimes} \underline{\xi} - 2\underline{\omega} \hat{\chi} + (\eta + \underline{\eta} + 2\underline{\zeta}) \hat{\otimes} \underline{\xi} - \alpha, \quad (2.34j)$$

$$\nabla_4 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 = 2\text{d}\hat{\nu} \underline{\xi} - 2\underline{\omega} \text{tr}\chi + 2\underline{\xi} \cdot (\eta + \underline{\eta} + 2\underline{\zeta}) - |\hat{\chi}|^2, \quad (2.34k)$$

$$\nabla_4 \underline{\zeta} = 2\nabla \omega + \chi \cdot (-\zeta + \eta) + 2\underline{\omega}(\zeta + \eta) - \underline{\chi} \cdot \underline{\xi} - 2\underline{\omega} \underline{\xi} - \beta, \quad (2.34l)$$

$$\nabla_4 \hat{\chi} + \frac{1}{2} \text{tr}\chi \hat{\chi} = \nabla \hat{\otimes} \eta + 2\underline{\omega} \hat{\chi} - \frac{1}{2} \text{tr}\chi \hat{\chi} + \underline{\xi} \otimes \underline{\xi} + \eta \hat{\otimes} \eta, \quad (2.34m)$$

$$\nabla_4 \text{tr}\chi + \frac{1}{2} \text{tr}\chi \text{tr}\chi = 2\text{d}\hat{\nu} \eta + 2\underline{\omega} \text{tr}\chi - \hat{\chi} \cdot \hat{\chi} + 2(\underline{\xi} \cdot \underline{\xi} + |\eta|^2) + 2\rho. \quad (2.34n)$$

We have the following elliptic equations on the 2-spheres

$$\text{curl} \underline{\eta} = -\text{curl} \eta = \frac{1}{2} \hat{\chi} \wedge \hat{\chi} - \underline{\xi} \wedge \underline{\xi} - \sigma, \quad (2.34o)$$

$$\text{d}\hat{\nu} \hat{\chi} = \nabla \text{tr}\chi + \underline{\chi} \cdot \zeta - \text{tr}\chi \zeta + \beta, \quad (2.34p)$$

$$\text{d}\hat{\nu} \hat{\chi} = \nabla \text{tr}\chi - \chi \cdot \zeta + \text{tr}\chi \zeta - \beta, \quad (2.34q)$$

$$\text{curl} \underline{\xi} = \underline{\xi} \wedge (\eta + \underline{\eta} + 2\underline{\zeta}), \quad (2.34r)$$

$$\text{curl} \underline{\xi} = \underline{\xi} \wedge (\eta + \underline{\eta} - 2\underline{\zeta}), \quad (2.34s)$$

$$K = -\frac{1}{4} \text{tr}\chi \text{tr}\chi + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} - \rho. \quad (2.34t)$$

We have the following Bianchi equations relating the null connection coefficients and the null curvature components (see [CK93, p. 161]).

$$\nabla_3 \alpha + \frac{1}{2} \text{tr}\chi \alpha = \nabla \hat{\otimes} \beta + 2\underline{\omega} \alpha - 3(\hat{\chi} \rho + {}^* \hat{\chi} \sigma) + (\zeta + 4\underline{\eta}) \hat{\otimes} \beta, \quad (2.35a)$$

$$\nabla_4 \beta + 2 \text{tr}\chi \beta = \text{d}\hat{\nu} \alpha - 2\underline{\omega} \beta + (2\underline{\zeta} + \underline{\eta}) \cdot \alpha + 3(\underline{\xi} \rho + {}^* \underline{\xi} \sigma), \quad (2.35b)$$

$$\nabla_3 \beta + \text{tr}\chi \beta = \nabla \rho + {}^* \nabla \sigma + 2\underline{\omega} \beta + \underline{\xi} \cdot \alpha + 3(\eta \rho + {}^* \eta \sigma), \quad (2.35c)$$

$$\nabla_4 \rho + \frac{3}{2} \text{tr}\chi \rho = \text{d}\hat{\nu} \beta - \frac{1}{2} \hat{\chi} \cdot \alpha + \zeta \cdot \beta + 2(\underline{\eta} \cdot \beta - \underline{\xi} \cdot \underline{\beta}), \quad (2.35d)$$

$$\nabla_3 \rho + \frac{3}{2} \text{tr}\chi \rho = -\text{d}\hat{\nu} \underline{\beta} - \frac{1}{2} \hat{\chi} \cdot \underline{\alpha} + \zeta \cdot \underline{\beta} + 2(\underline{\xi} \cdot \underline{\beta} - \eta \cdot \underline{\beta}), \quad (2.35e)$$

$$\nabla_4 \sigma + \frac{3}{2} \text{tr}\chi \sigma = -\text{curl} \beta + \frac{1}{2} \hat{\chi} \cdot {}^* \alpha - \zeta \cdot {}^* \beta - 2(\underline{\eta} \cdot {}^* \beta + 2\underline{\xi} \cdot {}^* \underline{\beta}), \quad (2.35f)$$

$$\nabla_3 \sigma + \frac{3}{2} \text{tr}\chi \sigma = -\text{curl} \underline{\beta} - \frac{1}{2} \hat{\chi} \cdot {}^* \underline{\alpha} + \zeta \cdot {}^* \underline{\beta} - 2(\underline{\eta} \cdot {}^* \underline{\beta} + \eta \cdot {}^* \underline{\beta}), \quad (2.35g)$$

$$\nabla_4 \underline{\beta} + \text{tr}\chi \underline{\beta} = -\nabla \rho + {}^* \nabla \sigma + 2\underline{\hat{\chi}} \cdot \beta + 2\underline{\omega} \underline{\beta} - \underline{\xi} \cdot \underline{\alpha} - 3(\eta \rho - {}^* \eta \sigma), \quad (2.35h)$$

$$\nabla_3 \underline{\beta} + 2 \text{tr}\chi \underline{\beta} = -\text{d}\hat{\nu} \alpha - 2\underline{\omega} \underline{\beta} - (-2\underline{\zeta} + \underline{\eta}) \cdot \alpha + 3(-\underline{\xi} \rho + {}^* \underline{\xi} \sigma), \quad (2.35i)$$

$$\nabla_4 \alpha + \frac{1}{2} \text{tr}\chi \alpha = -\nabla \hat{\otimes} \underline{\beta} + 4\underline{\omega} \alpha - 3(\hat{\chi} \rho - {}^* \hat{\chi} \sigma) + (\zeta - 4\underline{\eta}) \hat{\otimes} \underline{\beta}. \quad (2.35j)$$

We have the following commutations formulas for the covariant derivatives  $\nabla, \nabla_4, \nabla_3$  (see [CK93, p. 159])

$$[\nabla_4, \nabla]F = -\frac{1}{2}\text{tr}\chi\nabla F - \hat{\chi} \cdot \nabla F + \xi\nabla_3 F + (\underline{\eta} + \zeta)\nabla_4 F + \text{E}(\nabla_4, \nabla) \cdot F, \quad (2.36a)$$

$$[\nabla_3, \nabla]F = -\frac{1}{2}\text{tr}\underline{\chi}\nabla F - \hat{\underline{\chi}} \cdot \nabla F + \underline{\xi}\nabla_4 F + (\eta - \zeta)\nabla_3 F + \text{E}(\nabla_3, \nabla) \cdot F, \quad (2.36b)$$

$$[\nabla_3, \nabla_4]F = 2\omega\nabla_3 F + 2\underline{\omega}\nabla_4 F + (\eta - \underline{\eta}) \cdot \nabla F + \text{E}(\nabla_3, \nabla_4) \cdot F, \quad (2.36c)$$

where  $F$  is a  $S$ -tangent  $k$ -tensor and where the tensors  $\text{E}$  are given by

$$\begin{aligned} (\text{E}(\nabla_4, \nabla) \cdot F)_{aa_1 \dots a_k} &:= \sum_{i=1}^k \left( \underline{\chi}_{a_i a} \xi_b - \underline{\chi}_{ab} \xi_{a_i} + \chi_{a_i a} \underline{\eta}_b - \chi_{ab} \underline{\eta}_{a_i} + \epsilon_{a_i b} \beta_a \right) F_{a_1 \dots b \dots a_k}, \\ (\text{E}(\nabla_3, \nabla) \cdot F)_{aa_1 \dots a_k} &:= \sum_{i=1}^k \left( \chi_{a_i a} \underline{\xi}_b - \chi_{ab} \underline{\xi}_{a_i} + \underline{\chi}_{a_i a} \eta_b - \underline{\chi}_{ab} \eta_{a_i} - \epsilon_{a_i b} \beta_a \right) F_{a_1 \dots b \dots a_k}, \\ (\text{E}(\nabla_3, \nabla_4) \cdot F)_{aa_1 \dots a_k} &:= 2 \sum_{i=1}^k \left( \xi_{a_i} \underline{\xi}_b - \xi_b \underline{\xi}_{a_i} + \underline{\eta}_{a_i} \eta_b - \underline{\eta}_b \eta_{a_i} + \epsilon_{a_i b} \sigma \right) F_{a_1 \dots b \dots a_k}. \end{aligned}$$

## 2.8 Commutation relations for integrals and averages on $S_{u, \underline{u}}$

We have the following commutation relation between  $e_3, e_4$ -derivatives and the integral on a 2-sphere  $S_{u, \underline{u}}$

**Lemma 2.10.** *For all scalar function  $\phi$ , we have*

$$\begin{aligned} e_3 \left( \int_S \phi \right) &= \int_S (e_3(\phi) + \text{tr}\chi\phi) \quad \text{on } \underline{\mathcal{C}}^*, \\ e_4 \left( \int_S \phi \right) &= \int_S (e_4(\phi) + \text{tr}\chi\phi) \quad \text{on } \mathcal{M}^{\text{ext}}. \end{aligned} \quad (2.37)$$

*Proof.* Using (2.1) and defining appropriate transported spherical coordinates, we have  $e_4 = 2\partial_{\underline{u}}$  in  $\mathcal{M}^{\text{ext}}$ , which thus commutes with  $\int_{S_{u, \underline{u}}}$ . Using (2.34a), the desired formula in  $\mathcal{M}^{\text{ext}}$  follows.

Since  $\underline{\mathcal{C}}^*$  is null,  $\underline{Y}$  is null and  $\underline{\eta} = 0$ . Using (2.1), and defining appropriate transported spherical coordinates, we have  $e_3 = 2\partial_u$  on  $\underline{\mathcal{C}}^*$ . Using (2.34a), the desired formula follows on  $\underline{\mathcal{C}}^*$ . This finishes the proof of the lemma.  $\square$

Using (2.37), we also have the following commutation relation between  $e_3, e_4$ -derivatives and the mean value on a 2-sphere  $S_{u, \underline{u}}$

$$\begin{aligned} e_3(\bar{\phi}) &= \overline{e_3(\phi)} + \overline{(\text{tr}\chi - \text{tr}\underline{\chi})\phi} \quad \text{on } \underline{\mathcal{C}}^*, \\ e_4(\bar{\phi}) &= \overline{e_4(\phi)} + \overline{(\text{tr}\chi - \text{tr}\underline{\chi})\phi} \quad \text{on } \mathcal{M}^{\text{ext}}, \end{aligned} \quad (2.38)$$

where  $\phi$  is a scalar function and  $\bar{\phi} := (4\pi r^2)^{-1} \int_S \phi$ .

Using (2.37), we also have

$$e_3(r) = \frac{1}{2} r \overline{\text{tr}\chi} \quad \text{on } \underline{\mathcal{C}}^*, \quad (2.39)$$

$$e_4(r) = \frac{1}{2} r \overline{\text{tr}\underline{\chi}}, \quad \text{on } \mathcal{M}^{\text{ext}}. \quad (2.40)$$

## 2.9 Null decomposition of the geodesic-null foliation in $\mathcal{M}^{\text{ext}}$

In this section, we derive additional equations to the general null decompositions defined in Section 2.7, in the case of the null pair  $(e_3, e_4)$  of the geodesic-null foliation defined in Section 2.1.

We have the following relations.

**Lemma 2.11.** *In  $\mathcal{M}^{\text{ext}}$  the following relations hold*

$$\xi = 0, \quad (2.41)$$

$$\omega = 0, \quad (2.42)$$

$$0 = \eta - \zeta, \quad (2.43)$$

$$0 = \underline{\eta} + \zeta, \quad (2.44)$$

$$\nabla(\mathfrak{h}) = -2\underline{\xi}, \quad (2.45)$$

$$\nabla_4(\mathfrak{h}) = -4\underline{\omega}. \quad (2.46)$$

*Proof.* Identities (2.41) and (2.42) are a consequence of the geodesic equation  $\mathbf{D}_L L = 0$  and the definition  $e_4 = L$ .

From the definition of  $\eta$  and  $\zeta$ , we have

$$\eta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_a) = -\frac{1}{2}\mathbf{g}(e_4, [e_3, e_a]) - \frac{1}{2}\mathbf{g}(e_4, \mathbf{D}_a e_3) = -\frac{1}{2}\mathbf{g}(e_4, [e_3, e_a]) + \zeta_a.$$

From the definition of  $e_4$  and the relations (2.1), we have

$$-\frac{1}{2}\mathbf{g}(e_4, [e_3, e_a]) = \frac{1}{2}[e_3, e_a](u) = -e_a(1) = 0,$$

and we deduce the first identity of (2.44).

From the definition of  $\underline{\eta}$  and  $\zeta$ , we have

$$\underline{\eta}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_a) = -\frac{1}{2}\mathbf{g}(e_3, [e_4, e_a]) - \zeta_a.$$

From the definition of  $e_3$  and the relations (2.2), (2.1), we have

$$\begin{aligned} \mathbf{g}(e_3, [e_4, e_a]) &= \mathbf{g}(\underline{Y}, [e_4, e_a]) - \frac{1}{2}\mathfrak{h}\mathbf{g}(e_4, [e_4, e_a]), \\ &= -[e_4, e_a](\underline{u}) + \frac{1}{2}\mathfrak{h}[e_4, e_a](u) \\ &= 0, \end{aligned}$$

and we deduce the second identity of (2.44).

From the definition of  $\underline{\xi}$ , we have

$$\underline{\xi}_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_a) = -\frac{1}{2}\mathbf{g}(e_3, [e_3, e_a]).$$

Using the definition of  $e_3$  and  $\underline{Y}$ , we infer

$$\begin{aligned} \underline{\xi}_a &= -\frac{1}{2}\mathbf{g}\left(\underline{Y} - \frac{1}{2}\mathfrak{h}e_4, [e_3, e_a]\right) \\ &= \frac{1}{2}[e_3, e_a](\underline{u}) - \frac{1}{4}\mathfrak{h}[e_3, e_a](u) \\ &= -\frac{1}{2}e_a(\mathfrak{h}), \end{aligned}$$

and we deduce the identity (2.45).

From the definition of  $\underline{\omega}$ , the definition of  $e_4$  and the relations (2.1), we have

$$\begin{aligned}\underline{\omega} &= \frac{1}{4} \mathbf{g}(\mathbf{D}_3 e_3, e_4) \\ &= -\frac{1}{4} \mathbf{g}(e_3, [e_3, e_4]) \\ &= -\frac{1}{4} \mathbf{g}\left(\underline{Y} - \frac{1}{2} \eta e_4, [e_3, e_4]\right) \\ &= \frac{1}{4} [e_3, e_4](\underline{u}) - \frac{1}{8} \eta [e_3, e_4](u) \\ &= -\frac{1}{4} e_4(\eta),\end{aligned}$$

and we deduce the identity (2.46). This finishes the proof of the lemma.  $\square$

### 2.9.1 Averages and renormalisations

**Lemma 2.12** (Average of  $\rho$  and  $\sigma$ ). *The following relations hold in  $\mathcal{M}^{\text{ext}}$*

$$\nabla_4 \bar{\rho} + \frac{3}{2} \overline{\text{tr}\chi} \bar{\rho} = \text{Err}(\nabla_4, \bar{\rho}), \quad (2.47)$$

$$\bar{\sigma} = \frac{1}{2} \overline{\hat{\chi} \wedge \hat{\chi}}, \quad (2.48)$$

where

$$\text{Err}(\nabla_4, \bar{\rho}) := -\frac{1}{2} \overline{\hat{\chi} \cdot \underline{\alpha}} - \overline{\zeta \cdot \beta} - \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\rho - \bar{\rho})}.$$

*Proof.* Equation (2.47) follows from taking the average in equation (2.35d) and using commutation formula (2.38). Equation (2.48) follows from taking the average in equation (2.34o).  $\square$

**Lemma 2.13** (Average and renormalisation of  $\text{tr}\chi$ ). *We have the following transport equations in the  $e_4$  direction*

$$\nabla_4 (\text{tr}\chi - \overline{\text{tr}\chi}) + \overline{\text{tr}\chi} (\text{tr}\chi - \overline{\text{tr}\chi}) = \text{Err}(\nabla_4, \text{tr}\chi - \overline{\text{tr}\chi}), \quad (2.49)$$

$$\nabla_4 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) + \frac{1}{2} \overline{\text{tr}\chi} \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) = \text{Err} \left( \nabla_4, \overline{\text{tr}\chi} - \frac{2}{r} \right), \quad (2.50)$$

where

$$\begin{aligned}\text{Err}(\nabla_4, \text{tr}\chi - \overline{\text{tr}\chi}) &:= -|\hat{\chi}|^2 - \frac{1}{2} (\text{tr}\chi - \overline{\text{tr}\chi})^2 + \overline{|\hat{\chi}|^2} - \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})^2}, \\ \text{Err} \left( \nabla_4, \overline{\text{tr}\chi} - \frac{2}{r} \right) &:= -\overline{|\hat{\chi}|^2} + \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})^2}.\end{aligned}$$

*Proof.* Rewriting (2.34k), we have

$$\nabla_4 \text{tr}\chi + \frac{1}{2} (\text{tr}\chi)^2 = -|\hat{\chi}|^2. \quad (2.51)$$

Taking the mean value in (2.51), using commutation formula (2.38), we obtain

$$\begin{aligned}\nabla_4 \overline{\text{tr}\chi} + \frac{1}{2} \overline{\text{tr}\chi}^2 &= -\overline{|\hat{\chi}|^2} + \frac{1}{2} \left( \overline{\text{tr}\chi}^2 - \overline{(\text{tr}\chi)^2} \right) + \overline{(\text{tr}\chi - \overline{\text{tr}\chi}) \text{tr}\chi} \\ &= -\overline{|\hat{\chi}|^2} + \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})^2}.\end{aligned} \quad (2.52)$$

Combining equations (2.51) and (2.52) we obtain (2.49). Using equation (2.52) and relation (2.39) we obtain (2.50) and it finishes the proof of the lemma.  $\square$

**Lemma 2.14** (Average and renormalisation of  $\text{tr}\chi$ ). *We have the following equations in the  $e_4$  direction*

$$\begin{aligned}\nabla_4 (\text{tr}\chi - \overline{\text{tr}\chi}) + \frac{1}{2} \overline{\text{tr}\chi} (\text{tr}\chi - \overline{\text{tr}\chi}) &= -\frac{1}{2} \text{tr}\chi (\text{tr}\chi - \overline{\text{tr}\chi}) + 2\text{div}\zeta + 2(\rho - \bar{\rho}) \\ &\quad + \text{Err}(\nabla_4, \text{tr}\chi - \overline{\text{tr}\chi})\end{aligned} \quad (2.53)$$

$$\nabla_4 \left( \overline{\text{tr}\chi} + \frac{2}{r} \right) + \frac{1}{2} \overline{\text{tr}\chi} \left( \overline{\text{tr}\chi} + \frac{2}{r} \right) = 2\bar{\rho} + \text{Err} \left( \nabla_4, \overline{\text{tr}\chi} + \frac{2}{r} \right), \quad (2.54)$$

where

$$\begin{aligned} \text{Err}(\nabla_4, \text{tr}\underline{\chi} - \overline{\text{tr}\chi}) &:= -\hat{\chi} \cdot \hat{\underline{\chi}} + 2|\zeta|^2 + \overline{\hat{\chi} \cdot \hat{\underline{\chi}} - 2|\zeta|^2} - \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\text{tr}\underline{\chi} - \overline{\text{tr}\chi})}, \\ \text{Err}\left(\nabla_4, \overline{\text{tr}\chi} + \frac{2}{r}\right) &:= -\overline{\hat{\chi} \cdot \hat{\underline{\chi}}} + 2|\zeta|^2 + \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\text{tr}\underline{\chi} - \overline{\text{tr}\chi})}. \end{aligned}$$

*Proof.* Equation (2.34n) together with the relations of Lemma 2.11 rewrites

$$\nabla_4 \text{tr}\underline{\chi} + \frac{1}{2} \text{tr}\chi \text{tr}\underline{\chi} = 2\text{d}\not{v}\zeta - \hat{\chi} \cdot \hat{\underline{\chi}} + 2|\zeta|^2 + 2\rho. \quad (2.55)$$

Taking the average in (2.55), using commutation formula (2.38) gives

$$\nabla_4 \overline{\text{tr}\chi} + \frac{1}{2} \overline{\text{tr}\chi} \overline{\text{tr}\chi} = -\overline{\hat{\chi} \cdot \hat{\underline{\chi}}} + 2|\zeta|^2 + 2\bar{\rho} + \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\text{tr}\underline{\chi} - \overline{\text{tr}\chi})}. \quad (2.56)$$

Combining (2.55) and (2.56) directly gives (2.53). Combining (2.56) and formula (2.39) gives (2.54) and this finishes the proof of the lemma.  $\square$

**Lemma 2.15** (Average and renormalisation of  $\underline{\omega}$ ). *We have the following equations in the  $e_4$  direction*

$$\nabla_4(\underline{\omega} - \overline{\omega}) = \rho - \bar{\rho} + \text{Err}(\nabla_4, \underline{\omega} - \overline{\omega}), \quad (2.57)$$

$$\nabla_4 \overline{\omega} = \bar{\rho} + \text{Err}(\nabla_4, \overline{\omega}), \quad (2.58)$$

where

$$\begin{aligned} \text{Err}(\nabla_4, \underline{\omega} - \overline{\omega}) &:= 3|\zeta|^2 - 3\overline{|\zeta|^2} - \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\underline{\omega} - \overline{\omega})}, \\ \text{Err}(\nabla_4, \overline{\omega}) &:= 3\overline{|\zeta|^2} + \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\underline{\omega} - \overline{\omega})}. \end{aligned}$$

*Proof.* Using the result of Lemma 2.11, equation (2.34i) rewrites

$$\nabla_4 \underline{\omega} = 3|\zeta|^2 + \rho.$$

The result of the lemma then follows from taking the average in the above equation. The details are left to the reader.  $\square$

## 2.9.2 The null coefficient $\zeta$ and renormalisations

**The mass aspect function  $\mu$**  Equation (2.34l) together with the relations of Lemma 2.11 rewrites

$$\nabla_4 \zeta + \text{tr}\chi \zeta = -2\hat{\chi} \cdot \zeta - \beta. \quad (2.59)$$

We define the mass aspect function  $\mu$  by

$$\mu := \text{d}\not{v}\zeta + \rho \quad (2.60)$$

Using the relations of Lemma 2.11 and (2.60) and (2.34o), we have

$$\begin{aligned} \mathcal{P}_1(\zeta) &= (-\rho + \mu, \check{\sigma}), \\ &= (-\rho + \bar{\rho} + \mu - \bar{\mu}, \check{\sigma} - \bar{\sigma}). \end{aligned} \quad (2.61)$$

where for a  $S$ -tangent 1-tensor  $U$ , we have  $\mathcal{P}_1 U := (\text{d}\not{v}U, \text{curl}U)$ , and where

$$\check{\sigma} := \sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\underline{\chi}}.$$

We have the following transport equation in the  $e_4$  direction for  $\mu$ .

**Lemma 2.16.** *We have*

$$\nabla_4 \mu + \frac{3}{2} \text{tr}\chi \mu = \text{Err}(\nabla_4, \mu), \quad (2.62)$$

where

$$\text{Err}(\nabla_4, \mu) := -2\nabla\hat{\chi} \cdot \zeta - 3\hat{\chi} \cdot \nabla\zeta - \chi \cdot \zeta \cdot \zeta + \text{tr}\chi|\zeta|^2 - 2\zeta \cdot \beta - \nabla\text{tr}\chi \cdot \zeta - \frac{1}{2} \hat{\underline{\chi}} \cdot \alpha.$$



*Proof.* Commuting the transport equation (2.59) with  $d\dot{v}$ , using commutation formula (2.36a), we have

$$\nabla_4 d\dot{v}\zeta + \frac{3}{2}\text{tr}\chi d\dot{v}\zeta = -d\dot{v}\beta + \text{Err}_1, \quad (2.63)$$

where

$$\begin{aligned} \text{Err}_1 &:= \left( [\nabla_4, d\dot{v}]\zeta + \frac{1}{2}\text{tr}\chi d\dot{v}\zeta \right) - \nabla \text{tr}\chi \cdot \zeta - 2\nabla \hat{\chi} \cdot \zeta - 2\hat{\chi} \cdot \nabla \zeta \\ &= (-\hat{\chi} \cdot \nabla \zeta - \chi \cdot \zeta \cdot \zeta + \text{tr}\chi |\zeta|^2 - \zeta \cdot \beta) - \nabla \text{tr}\chi \cdot \zeta - 2\nabla \hat{\chi} \cdot \zeta - 2\hat{\chi} \cdot \nabla \zeta. \end{aligned}$$

From Bianchi identity (2.35d), we have

$$\nabla_4 \rho + \frac{3}{2}\rho = d\dot{v}\beta + \text{Err}_2, \quad (2.64)$$

where

$$\text{Err}_2 := -\frac{1}{2}\hat{\chi} \cdot \alpha - \zeta\beta,$$

and summing (2.63) and (2.64) gives the desired formula (2.62).  $\square$

We have the following transport equation for  $\mu - \bar{\mu}$ .

**Lemma 2.17** (Average and renormalisation of  $\mu$ ). *We have*

$$\bar{\mu} = \bar{\rho}, \quad (2.65)$$

and

$$\nabla_4(\mu - \bar{\mu}) + \frac{3}{2}\overline{\text{tr}\chi}(\mu - \bar{\mu}) = \text{Err}(\nabla_4, \mu - \bar{\mu}), \quad (2.66)$$

where

$$\text{Err}(\nabla_4, \mu - \bar{\mu}) := \text{Err}(\nabla_4, \mu) - \overline{\text{Err}(\nabla_4, \mu)} + \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})(\mu - \bar{\mu}) + 3/2(\text{tr}\chi - \overline{\text{tr}\chi})\mu.$$

*Proof.* Equation (2.65) follows from taking the average in the definition (2.60) of  $\mu$ .

Taking the average in equation (2.62) using formula (2.38), we have

$$\nabla_4 \bar{\mu} + \frac{3}{2}\overline{\text{tr}\chi}\bar{\mu} = -\frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})(\mu - \bar{\mu}) + \overline{\text{Err}(\nabla_4, \mu)}. \quad (2.67)$$

Combining (2.62) and (2.67) then gives the desired (2.66).  $\square$

**Renormalisation of  $\nabla_3\zeta$**  We define the auxiliary coefficients  $\underline{\omega}_\rho$  and  $\underline{\omega}_\sigma$  to be the solution of the following transport equations in the  $e_4$  direction

$$\begin{aligned} \nabla_4(r\underline{\omega}_\rho) &= r(\rho - \bar{\rho}), & \nabla_4(r\underline{\omega}_\sigma) &= r(\sigma - \bar{\sigma}), \\ \underline{\omega}_\rho|_{\underline{\mathcal{C}}^*} &= 0, & \underline{\omega}_\sigma|_{\underline{\mathcal{C}}^*} &= 0. \end{aligned} \quad (2.68)$$

We define  $\iota$  to be the following  $S$ -tangent tensor

$$\iota := \nabla \underline{\omega}_\rho - {}^*\nabla \underline{\omega}_\sigma + \underline{\beta}. \quad (2.69)$$

The tensor  $\iota$  satisfies the following transport equation.

**Lemma 2.18.** *We have*

$$\nabla_4(r^2\iota) = \text{Err}(\nabla_4, \iota), \quad (2.70)$$

where

$$\begin{aligned} \text{Err}(\nabla_4, \iota) &:= -\frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})r\nabla(r\underline{\omega}_\rho) - \hat{\chi} \cdot (r\nabla)(r\underline{\omega}_\rho) \\ &\quad + \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})(r^*\nabla)(r\underline{\omega}_\sigma) - \hat{\chi} \cdot (r^*\nabla)(r\underline{\omega}_\sigma) \\ &\quad - r^2(\text{tr}\chi - \overline{\text{tr}\chi})\underline{\beta} - 2r^2\hat{\chi} \cdot \underline{\beta} - 3r^2(\zeta\rho - {}^*\zeta\sigma). \end{aligned}$$

*Proof.* Commuting the transport equations (2.68) by respectively  $r\nabla$  and  $r^*\nabla$ , using commutation formula (2.36a), we have

$$\begin{aligned}
 \nabla_4(r^2\nabla\underline{\omega}_\rho) &= r^2\nabla\rho + [\nabla_4, r\nabla](r\underline{\omega}_\rho) \\
 &= r^2\nabla\rho - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})(r\nabla)(r\underline{\omega}_\rho) - \hat{\chi} \cdot (r\nabla)(r\underline{\omega}_\rho), \\
 \nabla_4(r^*\nabla\underline{\omega}_\sigma) &= r^{2*}\nabla\sigma + *([\nabla_4, r\nabla](r\underline{\omega})) \\
 &= r^{2*}\nabla\sigma - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})r^*\nabla(r\underline{\omega}_\sigma) - *(\hat{\chi} \cdot (r\nabla)(r\underline{\omega}_\sigma)) \\
 &= r^{2*}\nabla\sigma - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi})r^*\nabla(r\underline{\omega}_\sigma) + \hat{\chi} \cdot (r^*\nabla)(r\underline{\omega}_\sigma),
 \end{aligned} \tag{2.71}$$

where we used standard Hodge dual computations. From Bianchi identity (2.35h) and (2.39) we have

$$\begin{aligned}
 -r^2\nabla\rho + r^{2*}\nabla\sigma &= r^2\nabla_4\underline{\beta} + r^2\text{tr}\chi\underline{\beta} + 2r^2\hat{\chi} \cdot \underline{\beta} + 3r^2(\zeta\rho - *\zeta\sigma) \\
 &= \nabla_4(r^2\underline{\beta}) + r^2(\text{tr}\chi - \overline{\text{tr}\chi})\underline{\beta} + 2r^2\hat{\chi} \cdot \underline{\beta} + 3r^2(\zeta\rho - *\zeta\sigma).
 \end{aligned} \tag{2.72}$$

Combining (2.71) and (2.72), then directly gives the desired formula (2.70).  $\square$

We define  $\varsigma$  to be the following renormalisation of  $\nabla_3\zeta$

$$\varsigma := \nabla_3\zeta + \nabla\underline{\omega}_\rho + \nabla\underline{\omega}_\sigma. \tag{2.73}$$

We have the following transport equation for  $\varsigma$  in the  $e_4$  direction.

**Lemma 2.19.** *We have*

$$\nabla_4(r^2\varsigma) = r^2\text{tr}\chi\underline{\beta} + \frac{1}{2}r^2\text{tr}\chi\text{tr}\chi\underline{\zeta} + \text{Err}(\nabla_4, \varsigma), \tag{2.74}$$

where

$$\begin{aligned}
 \text{Err}(\nabla_4, \varsigma) &:= (r^2\overline{\text{tr}\chi} - r^2\text{tr}\chi)\nabla_3\zeta \\
 &\quad - 2r^2\underline{\omega}\beta - r^2\underline{\xi} \cdot \alpha - 3r^2(\zeta\rho + *\zeta\sigma) \\
 &\quad - 2r^2(\text{div}\zeta)\zeta - 2r^2\underline{\omega}\text{tr}\chi\zeta + r^2\hat{\chi} \cdot \underline{\chi}\zeta - 2r^2|\zeta|^2\zeta - 2r^2\rho\zeta \\
 &\quad - 2r^2\nabla_3(\hat{\chi} \cdot \zeta) - 2r^2\underline{\omega}\nabla_4\zeta - 2r^2\zeta \cdot \nabla\zeta - 2r^{2*}\zeta\sigma \\
 &\quad + \frac{1}{2}(\overline{\text{tr}\chi} - \text{tr}\chi)\nabla(r\underline{\omega}_\rho) - \hat{\chi} \cdot \nabla(r\underline{\omega}_\rho) \\
 &\quad + \frac{1}{2}(\overline{\text{tr}\chi} - \text{tr}\chi)^*\nabla(r\underline{\omega}_\sigma) + \hat{\chi} \cdot *\nabla(r\underline{\omega}_\sigma).
 \end{aligned}$$

*Proof.* Using formula (2.39), commuting the transport equation (2.59) with  $\nabla_3$ , using commutation formula (2.36c), Bianchi equation (2.35c) for  $\nabla_3\beta$  and the equation (2.34f) for  $\nabla_3\text{tr}\chi$  together with the relations from Lemma 2.11 gives

$$\begin{aligned}
 \nabla_4(r^2\nabla_3\zeta) &= r^2(\nabla_4\nabla_3\zeta + \text{tr}\chi\nabla_3\zeta) + \text{Err}_1 \\
 &= -r^2\nabla_3\beta - r^2\nabla_3(\text{tr}\chi)\zeta - 2r^2\nabla_3(\hat{\chi} \cdot \zeta) + r^2[\nabla_4, \nabla_3]\zeta + \text{Err}_1 \\
 &= r^2\text{tr}\chi\underline{\beta} - r^2\nabla\rho - r^{2*}\nabla\sigma + \frac{1}{2}r^2\text{tr}\chi\text{tr}\chi\underline{\zeta} + \text{Err}_1 + \text{Err}_2,
 \end{aligned} \tag{2.75}$$

where

$$\text{Err}_1 := (r^2\overline{\text{tr}\chi} - r^2\text{tr}\chi)\nabla_3\zeta,$$

$$\begin{aligned}
 \text{Err}_2 &:= -2r^2\underline{\omega}\beta - r^2\underline{\xi} \cdot \alpha - 3r^2(\zeta\rho + *\zeta\sigma) \\
 &\quad - 2r^2(\text{div}\zeta)\zeta - 2r^2\underline{\omega}\text{tr}\chi\zeta + r^2\hat{\chi} \cdot \underline{\chi}\zeta - 2r^2|\zeta|^2\zeta - 2r^2\rho\zeta \\
 &\quad - 2r^2\nabla_3(\hat{\chi} \cdot \zeta) - 2r^2\underline{\omega}\nabla_4\zeta - 2r^2\zeta \cdot \nabla\zeta - 2r^{2*}\zeta\sigma.
 \end{aligned}$$

From the transport equation (2.68) for the auxiliary coefficients  $\underline{\omega}_\rho$  and  $\underline{\omega}_\sigma$ , using commutation formula (2.36a), we have

$$\begin{aligned} -r^2 \nabla \rho - r^{2*} \nabla \sigma &= -r \nabla \nabla_4 (r \underline{\omega}_\rho) - r^* \nabla \nabla_4 (r \underline{\omega}_\sigma) \\ &= -\nabla_4 (r^2 \nabla \underline{\omega}_\rho) - \nabla_4 (r^2 \nabla \underline{\omega}_\sigma) + \text{Err}_3, \end{aligned} \quad (2.76)$$

where

$$\begin{aligned} \text{Err}_3 &:= -[r \nabla, \nabla_4](r \underline{\omega}_\rho) - [r^* \nabla, \nabla_4] \underline{\omega}_\sigma \\ &= \frac{1}{2}(\overline{\text{tr}\chi} - \text{tr}\chi) \nabla (r \underline{\omega}_\rho) - \hat{\chi} \cdot \nabla (r \underline{\omega}_\rho) \\ &\quad + \frac{1}{2}(\overline{\text{tr}\chi} - \text{tr}\chi)^* \nabla (r \underline{\omega}_\sigma) + \hat{\chi} \cdot {}^* \nabla (r \underline{\omega}_\sigma). \end{aligned}$$

Combining (2.75) and (2.76) gives the desired formula.  $\square$

### 2.9.3 The null coefficient $\underline{\omega}$ and renormalisations

Using equation (2.34i) and the relations from Lemma 2.11, we have

$$\nabla_4 \underline{\omega} = 3|\zeta|^2 + \rho. \quad (2.77)$$

Let  ${}^* \underline{\omega}$  be the *ad hoc* dual of  $\underline{\omega}$ , which we define as the solution of the following transport equation

$$\begin{aligned} \nabla_4 {}^* \underline{\omega} &:= \sigma, \\ {}^* \underline{\omega}|_{\mathcal{C}^*} &:= 0. \end{aligned} \quad (2.78)$$

We define  $\underline{\iota}$  to be the  $S$ -tangent tensor given by

$$\underline{\iota} := \nabla \underline{\omega} - {}^* \nabla {}^* \underline{\omega} + \underline{\beta}. \quad (2.79)$$

The tensor  $\underline{\iota}$  satisfies the following transport equation.

**Lemma 2.20.** *We have*

$$\nabla_4 (r \underline{\iota}) = -\frac{1}{2} r \overline{\text{tr}\chi} \underline{\beta} + \text{Err}(\nabla_4, \underline{\iota}), \quad (2.80)$$

where

$$\begin{aligned} \text{Err}(\nabla_4, \underline{\iota}) &:= 6\zeta \cdot (r \nabla) \zeta \\ &\quad - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi}) r \nabla \underline{\omega} - \hat{\chi} \cdot (r \nabla) \underline{\omega} \\ &\quad + \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi}) (r^* \nabla) {}^* \underline{\omega} - \hat{\chi} \cdot (r^* \nabla) {}^* \underline{\omega} \\ &\quad - r(\text{tr}\chi - \overline{\text{tr}\chi}) \underline{\beta} - 2r \hat{\chi} \cdot \underline{\beta} - 3r(\zeta \rho - {}^* \zeta \sigma). \end{aligned}$$

*Proof.* Commuting the transport equations (2.77) and (2.78) by respectively  $r \nabla$  and  $r^* \nabla$ , using commutation formula (2.36a), we have

$$\begin{aligned} \nabla_4 (r \nabla \underline{\omega}) &= 6\zeta \cdot (r \nabla) \zeta + (r \nabla) \rho + [\nabla_4, r \nabla] \underline{\omega} \\ &= 6\zeta \cdot (r \nabla) \zeta + (r \nabla) \rho \\ &\quad - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi}) (r \nabla) \underline{\omega} - \hat{\chi} \cdot (r \nabla) \underline{\omega}, \\ \nabla_4 (r^* \nabla {}^* \underline{\omega}) &= (r^* \nabla) \sigma + ([\nabla_4, r^* \nabla] {}^* \underline{\omega}) \\ &= (r^* \nabla) \sigma - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi}) r^* \nabla {}^* \underline{\omega} - {}^* (\hat{\chi} \cdot (r \nabla) {}^* \underline{\omega}) \\ &= (r^* \nabla) \sigma - \frac{1}{2}(\text{tr}\chi - \overline{\text{tr}\chi}) r^* \nabla {}^* \underline{\omega} + \hat{\chi} \cdot (r^* \nabla) {}^* \underline{\omega}, \end{aligned} \quad (2.81)$$

where we used standard Hodge dual computations. From Bianchi identity (2.35h) and (2.39) we have

$$\begin{aligned} -(r \nabla) \rho + (r^* \nabla) \sigma &= r \nabla_4 \underline{\beta} + r \text{tr}\chi \underline{\beta} + 2r \hat{\chi} \cdot \underline{\beta} + 3r(\zeta \rho - {}^* \zeta \sigma) \\ &= \nabla_4 (r \underline{\beta}) + \frac{1}{2} r \overline{\text{tr}\chi} \underline{\beta} + r(\text{tr}\chi - \overline{\text{tr}\chi}) \underline{\beta} + 2r \hat{\chi} \cdot \underline{\beta} + 3r(\zeta \rho - {}^* \zeta \sigma). \end{aligned} \quad (2.82)$$

Combining (2.81) and (2.82), then directly gives the desired formula (2.80).  $\square$

## 2.10 Null decomposition of the canonical foliation in $\underline{\mathcal{C}}^*$

Additionally to the relations of Section 2.9, we have the following relations on  $\underline{\mathcal{C}}^*$ .

**Lemma 2.21.** *On  $\underline{\mathcal{C}}^*$ , the following relations hold*

$$\eta = 0, \quad (2.83)$$

$$\xi = 0, \quad (2.84)$$

*Proof.* Since  $\underline{\mathcal{C}}^*$  is null, the gradient  $\underline{Y}$  of  $\underline{u}$  is null and  $\eta = 0$  on  $\underline{\mathcal{C}}^*$ . This proves (2.83).

We also deduce on  $\underline{\mathcal{C}}^*$  that

$$\xi_a = \frac{1}{2} \mathbf{g}(\mathbf{D}_3 e_3, e_a) = \frac{1}{2} \mathbf{g}(e_3, [e_a, e_3]) = -\frac{1}{2} \mathbf{g}(\mathbf{D}\underline{u}, [e_a, e_3]) = -\frac{1}{2} [e_a, e_3](\underline{u}) = 0.$$

This finishes the proof of the lemma.  $\square$

### 2.10.1 Averages and renormalisations

**Lemma 2.22** (Average of  $\rho$  and  $\sigma$ ). *The following relations hold on  $\underline{\mathcal{C}}^*$*

$$\nabla_3 \bar{\rho} + \frac{3}{2} \overline{\text{tr}\chi} \bar{\rho} = \text{Err}(\nabla_3, \bar{\rho}), \quad (2.85)$$

$$\bar{\sigma} = \frac{1}{2} \overline{\hat{\chi} \wedge \hat{\chi}}, \quad (2.86)$$

where

$$\text{Err}(\nabla_3, \bar{\rho}) := -\frac{1}{2} \overline{\hat{\chi} \cdot \underline{\alpha} - \zeta \cdot \underline{\beta}} - \frac{1}{2} \overline{(\text{tr}\chi - \overline{\text{tr}\chi})(\rho - \bar{\rho})}.$$

*Proof.* Equation (2.85) follows from taking the average in Bianchi equation (2.35e), using the relations of Lemmas 2.11 and 2.21, and using formula (2.38).  $\square$

**Lemma 2.23** (Average and renormalisation of  $\text{tr}\chi$ ). *We have the following transport equation along  $\underline{\mathcal{C}}^*$*

$$\nabla_3 (\text{tr}\chi - \overline{\text{tr}\chi}) + \overline{\text{tr}\chi} (\text{tr}\chi - \overline{\text{tr}\chi}) = -2(\underline{\omega} - \overline{\omega}) \text{tr}\chi + \text{Err}(\nabla_3, \text{tr}\chi - \overline{\text{tr}\chi}), \quad (2.87)$$

$$\nabla_3 \left( \overline{\text{tr}\chi} + \frac{2}{r} \right) + \frac{1}{2} \overline{\text{tr}\chi} \left( \overline{\text{tr}\chi} + \frac{2}{r} \right) = \text{Err} \left( \nabla_3, \overline{\text{tr}\chi} + \frac{2}{r} \right), \quad (2.88)$$

where<sup>3</sup>

$$\text{Err}(\nabla_3, \text{tr}\chi - \overline{\text{tr}\chi}) := -|\hat{\chi}|^2 + |\overline{\hat{\chi}}|^2 + 2(\underline{\omega} - \overline{\omega})(\text{tr}\chi - \overline{\text{tr}\chi}) - \frac{1}{2} (\text{tr}\chi - \overline{\text{tr}\chi})^2 - \frac{1}{2} (\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})^2,$$

$$\text{Err} \left( \nabla_3, \overline{\text{tr}\chi} + \frac{2}{r} \right) := -2(\underline{\omega} - \overline{\omega})(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}}) - |\overline{\hat{\chi}}|^2 + \frac{1}{2} (\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})^2.$$

*Proof.* Using the relations from Lemmas 2.11 and 2.21, equation (2.34c) rewrites

$$\nabla_3 \text{tr}\chi + \frac{1}{2} (\text{tr}\chi)^2 = -2(\underline{\omega} - \overline{\omega}) \text{tr}\chi - |\hat{\chi}|^2, \quad (2.89)$$

where we used that  $\overline{\omega} = 0$  for the canonical foliation on  $\underline{\mathcal{C}}^*$  (see Definition 2.1). Taking the average in (2.89), using formula (2.38), we have

$$\nabla_3 \overline{\text{tr}\chi} + \frac{1}{2} (\overline{\text{tr}\chi})^2 = -2(\underline{\omega} - \overline{\omega})(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}}) - |\overline{\hat{\chi}}|^2 + \frac{1}{2} (\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})^2. \quad (2.90)$$

Combining (2.89) and (2.90), we obtain (2.87). Combining (2.90) and formula (2.39) gives (2.88).  $\square$

<sup>3</sup>On  $\underline{\mathcal{C}}^*$  we have by definition of the canonical foliation  $\overline{\omega} = 0$ . For completeness, we keep track of the factor  $\overline{\omega}$  in the formulas of this section.

**Lemma 2.24** (Average and renormalisation of  $\text{tr}\chi$ ). *We have the following transport equation along  $\underline{\mathcal{C}}^*$*

$$\nabla_3 (\text{tr}\chi - \overline{\text{tr}\chi}) + \frac{1}{2} \overline{\text{tr}\chi} (\text{tr}\chi - \overline{\text{tr}\chi}) = 2(\underline{\omega} - \overline{\omega}) \text{tr}\chi + \text{Err}(\nabla_3, \text{tr}\chi - \overline{\text{tr}\chi}), \quad (2.91)$$

$$\nabla_3 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) + \frac{1}{2} \overline{\text{tr}\chi} \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) = 2\overline{\rho} + \text{Err} \left( \nabla_3, \overline{\text{tr}\chi} - \frac{2}{r} \right), \quad (2.92)$$

where

$$\begin{aligned} \text{Err}(\nabla_3, \text{tr}\chi - \overline{\text{tr}\chi}) &:= -\hat{\chi} \cdot \hat{\chi} + 2|\zeta|^2 + \overline{\hat{\chi} \cdot \hat{\chi}} - 2\overline{|\zeta|^2} - 2\overline{(\underline{\omega} - \overline{\omega})(\text{tr}\chi - \overline{\text{tr}\chi})} \\ &\quad - \frac{1}{2}(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})(\text{tr}\chi - \overline{\text{tr}\chi}) - \frac{1}{2}(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}}), \\ \text{Err} \left( \nabla_3, \overline{\text{tr}\chi} - \frac{2}{r} \right) &:= 2\overline{(\underline{\omega} - \overline{\omega})(\text{tr}\chi - \overline{\text{tr}\chi})} - \overline{\hat{\chi} \cdot \hat{\chi}} + 2\overline{|\zeta|^2} + \frac{1}{2}(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}}). \end{aligned}$$

*Proof.* Rewriting equation (2.34f), using the relations of Lemmas 2.11 and 2.21, and the Definition 2.1 of the canonical foliation on  $\underline{\mathcal{C}}^*$ , we have

$$\nabla_3 \text{tr}\chi + \frac{1}{2} \overline{\text{tr}\chi} \text{tr}\chi = 2\overline{\rho} + 2(\underline{\omega} - \overline{\omega}) \text{tr}\chi - \hat{\chi} \cdot \hat{\chi} + 2|\zeta|^2. \quad (2.93)$$

Taking the average in (2.93) gives

$$\nabla_3 \overline{\text{tr}\chi} + \frac{1}{2} \overline{\text{tr}\chi} \overline{\text{tr}\chi} = 2\overline{\rho} + 2\overline{(\underline{\omega} - \overline{\omega})(\text{tr}\chi - \overline{\text{tr}\chi})} - \overline{\hat{\chi} \cdot \hat{\chi}} + 2\overline{|\zeta|^2} + \frac{1}{2}(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}})(\overline{\text{tr}\chi} - \overline{\overline{\text{tr}\chi}}). \quad (2.94)$$

Combining (2.93) and (2.94) gives (2.91). Combining (2.94) and formula (2.39) gives (2.92).  $\square$

## 2.11 Approximate exterior Killing vectorfields $\mathbf{T}^{\text{ext}}, \mathbf{S}^{\text{ext}}, \mathbf{K}^{\text{ext}}$

Associated to the foliation  $S_{u, \underline{u}}$  and the null pair  $(e_4, e_3)$ , we define the following *approximate conformal Killing fields* in the exterior region  $\mathcal{M}^{\text{ext}}$

$$\begin{aligned} \mathbf{T}^{\text{ext}} &:= \frac{1}{2}(e_4 + e_3), & \mathbf{S}^{\text{ext}} &:= \frac{1}{2}(\underline{u}e_4 + ue_3), \\ \mathbf{K}^{\text{ext}} &:= \frac{1}{2}(\underline{u}^2e_4 + u^2e_3). \end{aligned} \quad (2.95)$$

We remark that we have

$$\mathbf{K}^{\text{ext}} = (u + \underline{u})\mathbf{S}^{\text{ext}} - \frac{1}{4}(\underline{u}u)\mathbf{T}^{\text{ext}}. \quad (2.96)$$

## 2.12 Approximate exterior Killing rotations $\mathbf{O}^{\text{ext}}$

### 2.12.1 Approximate exterior Killing rotations on $\underline{\mathcal{C}}^*$

For a fixed centred conformal isomorphism  $\Phi$  of  $S^*$  and for the associated Cartesian functions  $x^i$  (see the definitions of Section 2.4), we define the *approximate Killing exterior rotations* on  $S^*$  by

$${}^{(1)}\mathbf{O}^{\text{ext}} := x^2 \nabla x^3 - x^3 \nabla x^2, \quad {}^{(2)}\mathbf{O}^{\text{ext}} := x^3 \nabla x^1 - x^1 \nabla x^3, \quad {}^{(3)}\mathbf{O}^{\text{ext}} := x^1 \nabla x^2 - x^2 \nabla x^1, \quad (2.97)$$

where  $x^i$  are the Cartesian functions on  $S^*$  defined in Section 2.4.

We extend the rotations  $\mathbf{O}^{\text{ext}}$  by Lie transport along  $\underline{\mathcal{C}}^*$ , *i.e.*

$$\left[ e_3, {}^{(\ell)}\mathbf{O}^{\text{ext}} \right] = 0, \quad (2.98)$$

for all  $\ell = 1, 2, 3$ .

**Lemma 2.25.** *The vectorfields  ${}^{(\ell)}\mathbf{O}^{\text{ext}}$  are tangent to the 2-spheres  $S_{u, \underline{u}}$  of the canonical foliation on  $\underline{\mathcal{C}}^*$ .*

*Proof.* From the definition (2.98) and relations (2.1), we have

$$e_3 \left( {}^{(\ell)}\mathbf{O}^{\text{ext}}(\underline{u}) \right) = {}^{(\ell)}\mathbf{O}^{\text{ext}}(e_3(\underline{u})) = 0, \quad \text{and} \quad e_3 \left( {}^{(\ell)}\mathbf{O}^{\text{ext}}(u) \right) = {}^{(\ell)}\mathbf{O}^{\text{ext}}(e_3(u)) = 0.$$

From the definition of  ${}^{(\ell)}\mathbf{O}^{\text{ext}}$  on  $S^*$ , we have  ${}^{(\ell)}\mathbf{O}^{\text{ext}}(u)|_{S^*} = {}^{(\ell)}\mathbf{O}^{\text{ext}}(\underline{u})|_{S^*} = 0$ , and the result follows.  $\square$

**Remark 2.26.** *The exterior rotation vectorfields have to be exactly tangent to the 2-spheres of the foliation  $S_{u,\underline{y}}$  since they are used to estimate the tangential derivatives of the curvature. This is the reason for the definition of  $\mathbf{O}^{\text{ext}}$  by Lie transport. In contrast, the interior rotation  $\mathbf{O}^{\text{int}}$  are not used to estimate the curvature and thus can simply be defined by parallel transport. See Section 2.5.*

In the rest of this section, we simply call  $\mathbf{O}$  a rotation vectorfield  $(\ell)\mathbf{O}^{\text{ext}}$ .

Projecting equation (2.98) on  $S$  provides in particular

$$\nabla_3 \mathbf{O}_a = \underline{\chi}_{ab} \mathbf{O}^b. \quad (2.99)$$

Commuting equation (2.99) with  $\nabla$  using commutation formula (2.36b), we further have the following transport equations,

$$\begin{aligned} \nabla_3 \nabla_a \mathbf{O}_b &= \frac{1}{2} \nabla_a (\text{tr} \underline{\chi}) \mathbf{O}_b + \nabla_a \hat{\chi}_{bc} \mathbf{O}^c + \hat{\chi}_{bc} \nabla_a \mathbf{O}^c - (\hat{\chi} \cdot \nabla)_a \mathbf{O}_b \\ &\quad + (\zeta \cdot \mathbf{O}) \underline{\chi}_{ab} - (\underline{\chi} \cdot \mathbf{O})_a \zeta_b - {}^* \underline{\beta}_a {}^* \mathbf{O}_b, \end{aligned} \quad (2.100)$$

and the symmetrised version

$$\begin{aligned} \nabla_3 H_{ab} &= (\nabla (\text{tr} \underline{\chi}) \otimes \mathbf{O})_{ab} + ((\nabla \otimes \hat{\chi}) \cdot \mathbf{O})_{ab} + \hat{\chi}_{bc} \nabla_a \mathbf{O}^c \\ &\quad + \hat{\chi}_{ac} \nabla_b \mathbf{O}^c - ((\hat{\chi} \cdot \nabla) \otimes \mathbf{O})_{ab} + 2(\zeta \cdot \mathbf{O}) \underline{\chi}_{ab} \\ &\quad - (\underline{\chi} \cdot \mathbf{O} \otimes \zeta)_{ab} - ({}^* \underline{\beta} \otimes {}^* \mathbf{O})_{ab}, \end{aligned} \quad (2.101)$$

where

$$H_{ab} := \nabla_a \mathbf{O}_b + \nabla_b \mathbf{O}_a. \quad (2.102)$$

We also define the following  $S$ -tangent 3-tensor  $\Psi^4$

$$\Psi_{abc} := \nabla_{a,b}^2 \mathbf{O}_c - r^{-2} \mathbf{O}_b \not\partial_{ac} + r^{-2} \mathbf{O}_c \not\partial_{ab}. \quad (2.103)$$

**Lemma 2.27.** *We have the following transport equation for  $r\Psi$*

$$\begin{aligned} \nabla_3 (r\Psi)_{abc} &= \frac{1}{2} (r\nabla)_a \nabla_b (\text{tr} \underline{\chi}) \mathbf{O}_c + (r\nabla)_a \nabla_b \hat{\chi}_{cd} \mathbf{O}_d + (r\nabla)_a \hat{\chi}_{cd} \nabla_b \mathbf{O}_d - (r\nabla)_a \hat{\chi}_{bd} \nabla_d \mathbf{O}_c \\ &\quad + (r\nabla)_a \zeta_d \mathbf{O}_d \underline{\chi}_{bc} - (r\nabla)_a \underline{\chi}_{bd} \mathbf{O}_d \zeta_c - (r\nabla)_a {}^* \underline{\beta}_b {}^* \mathbf{O}_c + \frac{1}{2} \nabla_b (\text{tr} \underline{\chi}) (r\nabla)_a \mathbf{O}_c \\ &\quad + \nabla_b \hat{\chi}_{cd} (r\nabla)_a \mathbf{O}_d + \hat{\chi}_{cd} (r\nabla)_a \nabla_b \mathbf{O}_d - \hat{\chi}_{bd} (r\nabla)_a \nabla_d \mathbf{O}_c + \zeta_d (r\nabla)_a \mathbf{O}_d \underline{\chi}_{bc} \\ &\quad - \underline{\chi}_{bd} (r\nabla)_a \mathbf{O}_d \zeta_b - {}^* \underline{\beta}_b (r\nabla)_a {}^* \mathbf{O}_c + \zeta_d \mathbf{O}_d (r\nabla)_a \underline{\chi}_{bc} - \underline{\chi}_{bd} \mathbf{O}_d (r\nabla)_a \zeta_b \\ &\quad + \frac{1}{2} (\overline{\text{tr} \underline{\chi}} - \text{tr} \underline{\chi}) (r\nabla)_a \nabla_b \mathbf{O}_c - r \hat{\chi}_{ad} \nabla_d \nabla_b \mathbf{O}_c \\ &\quad + \left( \underline{\chi}_{ba} \zeta_d - \underline{\chi}_{da} \zeta_b - \not\partial_{bd} {}^* \underline{\beta}_a \right) \nabla_d \mathbf{O}_c \\ &\quad + \left( \underline{\chi}_{ca} \zeta_d - \underline{\chi}_{da} \zeta_c - \not\partial_{cd} {}^* \underline{\beta}_a \right) \nabla_b \mathbf{O}_d \\ &\quad - \frac{1}{2} r^{-1} (\text{tr} \underline{\chi} - \overline{\text{tr} \underline{\chi}}) \mathbf{O}_b \not\partial_{ac} - r^{-1} \hat{\chi}_{bd} \mathbf{O}_d \not\partial_{ac} \\ &\quad + \frac{1}{2} r^{-1} (\text{tr} \underline{\chi} - \overline{\text{tr} \underline{\chi}}) \mathbf{O}_c \not\partial_{ab} + r^{-1} \hat{\chi}_{cd} \mathbf{O}_d \not\partial_{ab}. \end{aligned} \quad (2.104)$$

*Proof.* Using commutation formula (2.36b) and the relations of Lemmas 2.11 and 2.21, we have

$$\begin{aligned} \nabla_3 r \nabla_a \nabla_b \mathbf{O}_c &= (r\nabla)_a \nabla_3 \nabla_b \mathbf{O}_c + \frac{1}{2} (\overline{\text{tr} \underline{\chi}} - \text{tr} \underline{\chi}) (r\nabla)_a \nabla_b \mathbf{O}_c - r \hat{\chi}_{ad} \nabla_d \nabla_b \mathbf{O}_c \\ &\quad + \left( \underline{\chi}_{ba} \zeta_d - \underline{\chi}_{da} \zeta_b - \not\partial_{bd} {}^* \underline{\beta}_a \right) \nabla_d \mathbf{O}_c + \left( \underline{\chi}_{ca} \zeta_d - \underline{\chi}_{da} \zeta_c - \not\partial_{cd} {}^* \underline{\beta}_a \right) \nabla_b \mathbf{O}_d. \end{aligned} \quad (2.105)$$

<sup>4</sup>The definition of  $\Psi$  is motivated by the fact that  $\Psi = 0$  in the Minkowskian case. See Lemma 2.30.

Using equation (2.100), we have

$$\begin{aligned}
 & (r\nabla)_a \nabla_3 \nabla_b \mathbf{O}_c \\
 &= \frac{1}{2} (r\nabla)_a \nabla_b (\text{tr}\underline{\chi}) \mathbf{O}_c + (r\nabla)_a \nabla_b \hat{\chi}_{cd} \mathbf{O}_d + (r\nabla)_a \hat{\chi}_{cd} \nabla_b \mathbf{O}_d - (r\nabla)_a \hat{\chi}_{bd} \nabla_d \mathbf{O}_c \\
 & \quad + (r\nabla)_a \zeta_d \mathbf{O}_d \underline{\chi}_{bc} - (r\nabla)_a \underline{\chi}_{bd} \mathbf{O}_d \zeta_c - (r\nabla)_a {}^* \underline{\beta}_b {}^* \mathbf{O}_c \\
 & \quad + \frac{1}{2} \nabla_b (\text{tr}\underline{\chi}) (r\nabla)_a \mathbf{O}_c + \nabla_b \hat{\chi}_{cd} (r\nabla)_a \mathbf{O}_d + \hat{\chi}_{cd} (r\nabla)_a \nabla_b \mathbf{O}_d - \hat{\chi}_{bd} (r\nabla)_a \nabla_d \mathbf{O}_c \\
 & \quad + \zeta_d (r\nabla)_a \mathbf{O}_d \underline{\chi}_{bc} - \underline{\chi}_{bd} (r\nabla)_a \mathbf{O}_d \zeta_b - {}^* \underline{\beta}_b (r\nabla)_a {}^* \mathbf{O}_c \\
 & \quad + \zeta_d \mathbf{O}_d (r\nabla)_a \underline{\chi}_{bc} - \underline{\chi}_{bd} \mathbf{O}_d (r\nabla)_a \zeta_b.
 \end{aligned} \tag{2.106}$$

Using formula (2.39), we have

$$\begin{aligned}
 -\nabla_3 r^{-1} \mathbf{O}_b \not\!{d}_{ac} &= -\frac{1}{2} r^{-1} (\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}) \mathbf{O}_b \not\!{d}_{ac} - r^{-1} \hat{\chi}_{bd} \mathbf{O}_d \not\!{d}_{ac}, \\
 \nabla_3 r^{-1} \mathbf{O}_c \not\!{d}_{ab} &= \frac{1}{2} r^{-1} (\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}) \mathbf{O}_c \not\!{d}_{ab} + r^{-1} \hat{\chi}_{cd} \mathbf{O}_d \not\!{d}_{ab}.
 \end{aligned} \tag{2.107}$$

Combining (2.105), (2.106) and (2.107) gives the desired formula and finishes the proof of the lemma.  $\square$

### 2.12.2 Approximate exterior Killing rotations in $\mathcal{M}^{\text{ext}}$

We define the vectorfields  $\mathbf{O}^{\text{ext}}$  in  $\mathcal{M}^{\text{ext}}$  by Lie transport from  $\underline{\mathcal{C}}^*$  along  $e_4$ , i.e.

$$\left[ e_4, {}^{(\ell)} \mathbf{O}^{\text{ext}} \right] = 0, \tag{2.108}$$

Arguing as in Lemma 2.25, the vectorfields  ${}^{(\ell)} \mathbf{O}^{\text{ext}}$  are tangent to the 2-spheres  $S_{u, \underline{u}}$ .

In the rest of this section, we simply call  $\mathbf{O}$  a rotation vectorfield  ${}^{(\ell)} \mathbf{O}^{\text{ext}}$ .

From (2.108), Lemma 2.25 and the relations (2.33), we have the following relations

$$\begin{aligned}
 \mathbf{g}(\mathbf{D}_4 \mathbf{O}, e_4) &= 0, \\
 \mathbf{g}(\mathbf{D}_4 \mathbf{O}, e_3) &= 2\underline{\zeta}_a \mathbf{O}^a, \\
 \mathbf{g}(\mathbf{D}_4 \mathbf{O}, e_a) &= \underline{\chi}_{ab} \mathbf{O}^b, \\
 \mathbf{g}(\mathbf{D}_3 \mathbf{O}, e_3) &= -2\underline{\xi}_a \mathbf{O}^a, \\
 \mathbf{g}(\mathbf{D}_3 \mathbf{O}, e_4) &= -2\underline{\zeta}_a \mathbf{O}^a, \\
 \mathbf{g}(\mathbf{D}_a \mathbf{O}, e_4) &= -\underline{\chi}_{ab} \mathbf{O}^b, \\
 \mathbf{g}(\mathbf{D}_a \mathbf{O}, e_3) &= -\underline{\chi}_{ab} \mathbf{O}^b, \\
 \mathbf{g}(\mathbf{D}_3 \mathbf{O}, e_a) &= \underline{\chi}_{ab} \mathbf{O}^b + Y_a,
 \end{aligned} \tag{2.109}$$

where

$$Y_a := \mathbf{g}(\mathbf{D}_3 \mathbf{O}, e_a) - \underline{\chi}_{ab} \mathbf{O}^b. \tag{2.110}$$

**Remark 2.28.** From the definition of  $\mathbf{O}^{\text{ext}}$  on  $\underline{\mathcal{C}}^*$  and the definition (2.110) of  $Y$ , we have

$$Y|_{\underline{\mathcal{C}}^*} = 0. \tag{2.111}$$

We deduce the following expression for the deformation tensor  $\pi$  of  $\mathbf{O}$

$$\begin{aligned}
 {}^{(\mathbf{O})} \pi_{44} &= 0, \\
 {}^{(\mathbf{O})} \pi_{34} &= 0, \\
 {}^{(\mathbf{O})} \pi_{33} &= -4\underline{\xi}_a \mathbf{O}^a, \\
 {}^{(\mathbf{O})} \pi_{4a} &= 0, \\
 {}^{(\mathbf{O})} \pi_{3a} &= Y_a, \\
 {}^{(\mathbf{O})} \pi_{ab} &=: H_{ab},
 \end{aligned} \tag{2.112}$$

and

$$\text{tr}^{(\mathbf{O})}\pi = \text{tr}H.$$

From equation (2.108), we have in particular the following transport equation for  $\mathbf{O}$  in the  $e_4$  direction

$$\nabla_4 \mathbf{O}_a = \chi_{ab} \mathbf{O}^b. \quad (2.113)$$

We further have the following transport equations for  $\nabla \mathbf{O}$ ,  $Y$ ,  $H$  and  $\Psi$  in the  $e_4$  direction.

**Lemma 2.29.** *We have*

$$\begin{aligned} \nabla_4 \nabla_a \mathbf{O}_b &= \frac{1}{2} \nabla_a (\text{tr}\chi) \mathbf{O}_b + \nabla_a \hat{\chi}_{bc} \mathbf{O}^c + \hat{\chi}_{bc} \nabla_a \mathbf{O}^c - (\hat{\chi} \cdot \nabla)_a \mathbf{O}_b \\ &\quad - (\zeta \cdot \mathbf{O}) \chi_{ab} + (\chi \cdot \mathbf{O})_a \zeta_b + {}^* \beta_a {}^* \mathbf{O}_b. \end{aligned} \quad (2.114)$$

We have

$$\begin{aligned} \nabla_4 Y - \frac{1}{2} \text{tr}\chi Y &= 2(\text{div}\zeta) \mathbf{O} + (2\nabla \hat{\chi} \zeta + \hat{\chi} \cdot \hat{\chi} - \hat{\chi} \cdot \hat{\chi}) \cdot \mathbf{O} \\ &\quad + \hat{\chi} \cdot Y - 4\zeta \cdot \nabla \mathbf{O} + 2\sigma {}^* \mathbf{O} + 2(\underline{\xi} \cdot \mathbf{O}) \zeta, \end{aligned} \quad (2.115)$$

We have

$$\begin{aligned} \nabla_4 H_{ab} &= (\nabla (\text{tr}\chi) \otimes \mathbf{O})_{ab} + ((\nabla \otimes \hat{\chi}) \cdot \mathbf{O})_{ab} + \hat{\chi}_{bc} \nabla_a \mathbf{O}^c \\ &\quad + \hat{\chi}_{ac} \nabla_b \mathbf{O}^c - ((\hat{\chi} \cdot \nabla) \otimes \mathbf{O})_{ab} - 2(\zeta \cdot \mathbf{O}) \chi_{ab} \\ &\quad + (\chi \cdot \mathbf{O} \otimes \zeta)_{ab} + ({}^* \beta \otimes {}^* \mathbf{O})_{ab}. \end{aligned} \quad (2.116)$$

We have

$$\begin{aligned} \nabla_4 (r\Psi)_{abc} &= \frac{1}{2} (r\nabla)_a \nabla_b (\text{tr}\chi) \mathbf{O}_c + (r\nabla)_a \nabla_b \hat{\chi}_{cd} \mathbf{O}_d + (r\nabla)_a \hat{\chi}_{cd} \nabla_b \mathbf{O}_d - (r\nabla)_a \hat{\chi}_{bd} \nabla_c \mathbf{O}_c \\ &\quad - (r\nabla)_a \zeta_d \mathbf{O}_d \chi_{bc} + (r\nabla)_a \chi_{bd} \mathbf{O}_d \zeta_c + (r\nabla)_a {}^* \beta_b {}^* \mathbf{O}_c + \frac{1}{2} \nabla_b (\text{tr}\chi) (r\nabla)_a \mathbf{O}_c \\ &\quad + \nabla_b \hat{\chi}_{cd} (r\nabla)_a \mathbf{O}_d + \hat{\chi}_{cd} (r\nabla)_a \nabla_b \mathbf{O}_d - \hat{\chi}_{bd} (r\nabla)_a \nabla_c \mathbf{O}_c - \zeta_d (r\nabla)_a \mathbf{O}_d \chi_{bc} \\ &\quad + \chi_{bd} (r\nabla)_a \mathbf{O}_d \zeta_b + {}^* \beta_b (r\nabla)_a {}^* \mathbf{O}_c - \zeta_d \mathbf{O}_d (r\nabla)_a \chi_{bc} + \chi_{bd} \mathbf{O}_d (r\nabla)_a \zeta_b \\ &\quad + \frac{1}{2} (\overline{\text{tr}\chi} - \text{tr}\chi) (r\nabla)_a \nabla_b \mathbf{O}_c - r \hat{\chi}_{ad} \nabla_d \nabla_b \mathbf{O}_c \\ &\quad + (-\chi_{ba} \zeta_d + \chi_{da} \zeta_b + \not\epsilon_{bd} {}^* \beta_a) \nabla_d \mathbf{O}_c \\ &\quad + (-\chi_{ca} \zeta_d + \chi_{da} \zeta_c + \not\epsilon_{cd} {}^* \beta_a) \nabla_b \mathbf{O}_d \\ &\quad - \frac{1}{2} r^{-1} (\text{tr}\chi - \overline{\text{tr}\chi}) \mathbf{O}_b \not\partial_{ac} - r^{-1} \hat{\chi}_{bd} \mathbf{O}_d \not\partial_{ac} \\ &\quad + \frac{1}{2} r^{-1} (\text{tr}\chi - \overline{\text{tr}\chi}) \mathbf{O}_c \not\partial_{ab} + r^{-1} \hat{\chi}_{cd} \mathbf{O}_d \not\partial_{ab}. \end{aligned} \quad (2.117)$$

*Proof.* Equations (2.114), (2.116) and (2.117) follow by duality from respectively (2.100), (2.101) and (2.104).

Using relations (2.109), we have

$$\mathbf{D}_3 \mathbf{O} = (\underline{\xi} \cdot \mathbf{O}) e_4 + (\zeta \cdot \mathbf{O}) e_3 + (\underline{\chi} \cdot \mathbf{O}) + Y.$$

Using equation (2.108), we have

$$\begin{aligned} \mathbf{D}_4 (\mathbf{D}_3 \mathbf{O}) &= \mathbf{D}_3 (\mathbf{D}_4 \mathbf{O}) + \mathbf{D}_{[e_4, e_3]} \mathbf{O} + \mathbf{R}(e_4, e_3, e_b, e_a) \mathbf{O}^a e_b \\ &= \mathbf{D}_3 (\chi \cdot \mathbf{O}) + \mathbf{D}_{[e_4, e_3]} \mathbf{O} + \mathbf{R}(e_4, e_3, e_b, e_a) \mathbf{O}^a e_b. \end{aligned}$$

Using the above two equations and projecting on  $\text{TS}$ , this gives

$$\nabla_4 Y = \nabla_3 (\chi \cdot \mathbf{O}) + \nabla_{[e_4, e_3]} \mathbf{O} - 2\sigma {}^* \mathbf{O} + 2(\underline{\xi} \cdot \mathbf{O}) \zeta - \nabla_4 (\underline{\chi} \cdot \mathbf{O}).$$

Using relations (2.33), we have

$$\begin{aligned} \nabla_{[e_4, e_3]} \mathbf{O} &= \nabla_{-4\zeta_a e_a - 2\omega e_4} \mathbf{O} \\ &= -4\zeta \cdot \nabla \mathbf{O} - 2\underline{\omega} \nabla_4 \mathbf{O} \\ &= -4\zeta \cdot \nabla \mathbf{O} - 2\underline{\omega} (\chi \cdot \mathbf{O}), \end{aligned}$$



and, using (2.109) again, we obtain

$$\begin{aligned} \nabla_4 Y - \frac{1}{2} \text{tr} \chi Y &= (\nabla_3 \underline{\chi} - \nabla_4 \underline{\chi}) \cdot \mathbf{O} + (\chi \cdot \underline{\chi} - \underline{\chi} \cdot \chi) \cdot \mathbf{O} + \hat{\chi} \cdot Y \\ &\quad - 4\zeta \cdot \nabla \mathbf{O} - 2\underline{\omega} \chi \cdot \mathbf{O} - 2\sigma^* \mathbf{O} + 2(\underline{\xi} \cdot \mathbf{O}) \zeta. \end{aligned} \quad (2.118)$$

Using equations (2.34f) and (2.34n) together with the relations of Lemma 2.11, we have

$$\frac{1}{2} (\nabla_3 \text{tr} \chi - \nabla_4 \text{tr} \underline{\chi}) = 2d\text{iv} \zeta + \underline{\omega} \text{tr} \chi.$$

Using equations (2.34e) and (2.34m) together with the relations of Lemma 2.11, we have

$$\nabla_3 \hat{\chi} - \nabla_4 \hat{\chi} = 2\nabla \hat{\otimes} \zeta + \underline{\omega} \hat{\chi}.$$

Additionally, we also check that

$$\begin{aligned} (\chi \cdot \underline{\chi} - \underline{\chi} \cdot \chi)_{ab} &= \chi_{ac} \underline{\chi}_{cb} - \underline{\chi}_{ac} \chi_{cb} \\ &= \hat{\chi}_{ac} \hat{\chi}_{cb} - \hat{\chi}_{ac} \hat{\chi}_{cb} \\ &= (\hat{\chi} \cdot \hat{\chi} - \hat{\chi} \cdot \hat{\chi})_{ab}. \end{aligned}$$

Using the above formulas, equation (2.118) directly rewrites as (2.115) as desired.  $\square$

We record the following lemma, which motivates the definition of  $\Psi$ .<sup>5</sup>

**Lemma 2.30.** *Let  $X$  be an  $S$ -tangent vectorfield. We have the following formula*

$$\begin{aligned} \mathbf{D}_{a,b}^2 X &= \nabla_{a,b}^2 X - r^{-2} X_b e_a + \frac{1}{2} r^{-1} g_{ab} (\mathbf{D}_4 - \mathbf{D}_3) X \\ &\quad + \frac{1}{2} r^{-1} (\nabla_a X_b + \nabla_b X_a) (e_3 - e_4) + (\mathbf{E}(\mathbf{D}^2, \nabla^2) \cdot X)_{ab}, \end{aligned} \quad (2.119)$$

where

$$\begin{aligned} (\mathbf{E}(\mathbf{D}^2, \nabla^2) \cdot X)_{ab} &:= \frac{1}{2} X_c \nabla_a (\chi - r^{-1} g)_{bc} e_3 + \frac{1}{2} X_c \nabla_a (\underline{\chi} + r^{-1} g)_{bc} e_4 \\ &\quad + \frac{1}{2} \nabla_b X_c (\chi_{ca} - r^{-1} g_{ca}) e_3 + \frac{1}{2} \nabla_b X_c (\underline{\chi}_{ca} + r^{-1} g_{ca}) e_4 \\ &\quad + \frac{1}{2} r^{-1} X_b \left( (\underline{\chi}_{ac} + r^{-1} g_{ac}) e_c - (\chi_{ac} - r^{-1} g_{ac}) e_c + \zeta_a e_3 + \zeta_a e_4 \right) \\ &\quad - \frac{1}{2} (\chi_{ab} - r^{-1} g_{ab}) \mathbf{D}_3 X - \frac{1}{2} (\underline{\chi}_{ab} + r^{-1} g_{ab}) \mathbf{D}_4 X. \end{aligned}$$

*Proof.* We first start by recording the following formula, which follows from the relations (2.33)

$$\begin{aligned} \mathbf{D}_b X &= \nabla_b X + \frac{1}{2} X_c \chi_{cb} e_3 + \frac{1}{2} X_c \underline{\chi}_{cb} e_4 \\ &= \nabla_b X + \frac{1}{2} r^{-1} X_b (e_3 - e_4) + (\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_b, \end{aligned} \quad (2.120)$$

where

$$(\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_b := \frac{1}{2} X_c (\chi_{cb} - r^{-1} g_{cb}) e_3 + \frac{1}{2} X_c (\underline{\chi}_{cb} + r^{-1} g_{cb}) e_4.$$

Using formula (2.120) and the relations (2.33), we have

$$\begin{aligned} \mathbf{D}_{a,b}^2 X &= \mathbf{D}_a (\mathbf{D}_b X) - \mathbf{D}_{\mathbf{D}_{e_a} e_b} X \\ &= \mathbf{D}_a \left( \nabla_b X + \frac{1}{2} r^{-1} X_b (e_3 - e_4) + (\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_b \right) \\ &\quad - \mathbf{D}_{\nabla_{e_a} e_b} X - \frac{1}{2} \chi_{ab} \mathbf{D}_3 X - \frac{1}{2} \underline{\chi}_{ab} \mathbf{D}_4 X. \end{aligned} \quad (2.121)$$

<sup>5</sup>Using that  $\mathbf{D}^2 \mathbf{O} = 0$  in the Euclidean space, one can deduce from the formula of Lemma 2.30 that  $\Psi = 0$  in the Euclidean case (see Lemma 9.11). Reciprocally, from the control of the rotation coefficients  $\Psi, H, Y \simeq 0$  defined in this section, one can deduce from this formula a control of  $\mathbf{D}^2 \mathbf{O}^{\text{ext}} \simeq 0$  (see Lemma 11.20).

Using formula (2.120), we have the following equations

$$\begin{aligned} \mathbf{D}_a(\nabla_b X) &= \nabla_a(\nabla_b X) + \frac{1}{2}r^{-1}\nabla_b X_a(e_3 - e_4) + (\mathbf{E}(\mathbf{D}, \nabla) \cdot \nabla_b X)_a \\ &= \nabla_{a,b}^2 X + \nabla_{\nabla_a e_b} X + \frac{1}{2}r^{-1}\nabla_b X_a(e_3 - e_4) + (\mathbf{E}(\mathbf{D}, \nabla) \cdot \nabla_b X)_a, \end{aligned} \quad (2.122)$$

and

$$\begin{aligned} \mathbf{D}_a\left(\frac{1}{2}r^{-1}X_b(e_3 - e_4)\right) &= \frac{1}{2}r^{-1}\nabla_a X_b(e_3 - e_4) + \frac{1}{2}r^{-1}X_c(\nabla_a e_b)^c(e_3 - e_4) \\ &\quad + \frac{1}{2}r^{-1}X_b\mathbf{D}_a(e_3 - e_4). \end{aligned} \quad (2.123)$$

Combining the terms of (2.121), (2.122) and (2.123) containing  $\nabla_{e_a e_b}$ , we obtain

$$\begin{aligned} &\nabla_{\nabla_a e_b} X + \frac{1}{2}r^{-1}X_c(\nabla_a e_b)^c(e_3 - e_4) - \mathbf{D}_{\nabla_a e_b} X \\ &= \nabla_{\nabla_a e_b} X + \frac{1}{2}r^{-1}X_c(\nabla_a e_b)^c(e_3 - e_4) \\ &\quad - \left( \nabla_{\nabla_a e_b} X + \frac{1}{2}r^{-1}X_c(\nabla_a e_b)^c(e_3 - e_4) + (\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_c(\nabla_a e_b)^c \right) \\ &= -(\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_c(\nabla_a e_b)^c. \end{aligned} \quad (2.124)$$

Thus, we rewrite (2.121) using (2.122), (2.123) and (2.124) as

$$\begin{aligned} \mathbf{D}_{a,b}^2 X &= \nabla_{a,b}^2 X + \frac{1}{2}r^{-1}\nabla_b X_a(e_3 - e_4) + \frac{1}{2}r^{-1}\nabla_a X_b(e_3 - e_4) \\ &\quad + \frac{1}{2}r^{-1}X_b\mathbf{D}_a(e_3 - e_4) - \frac{1}{2}\chi_{ab}\mathbf{D}_3 X - \frac{1}{2}\underline{\chi}_{ab}\mathbf{D}_4 X + \mathbf{E}^1, \end{aligned} \quad (2.125)$$

where

$$\mathbf{E}^1 := \mathbf{D}_a((\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_b) + (\mathbf{E}(\mathbf{D}, \nabla) \cdot \nabla_b X)_a - (\mathbf{E}(\mathbf{D}, \nabla) \cdot X)_c(\nabla_a e_b)^c.$$

We rewrite

$$\frac{1}{2}r^{-1}X_b\mathbf{D}_a(e_3 - e_4) = -r^{-2}X_b e_a + \mathbf{E}^2, \quad (2.126)$$

and

$$-\frac{1}{2}\chi_{ab}\mathbf{D}_3 X - \frac{1}{2}\underline{\chi}_{ab}\mathbf{D}_4 X = \frac{1}{2}r^{-1}\not{g}_{ab}(\mathbf{D}_4 - \mathbf{D}_3)X + \mathbf{E}^3, \quad (2.127)$$

where

$$\begin{aligned} \mathbf{E}^2 &:= \frac{1}{2}r^{-1}X_b(\mathbf{D}_a e_3 + r^{-1}e_a) - \frac{1}{2}r^{-1}X_b(\mathbf{D}_a e_4 - r^{-1}e_a), \\ \mathbf{E}^3 &:= -\frac{1}{2}(\chi_{ab} - r^{-1}\not{g}_{ab})\mathbf{D}_3 X - \frac{1}{2}(\underline{\chi}_{ab} + r^{-1}\not{g}_{ab})\mathbf{D}_4 X. \end{aligned}$$

Thus, we obtain the following rewriting of (2.125) using (2.126) and (2.127)

$$\begin{aligned} \mathbf{D}^2 X_{a,b} &= \nabla_{a,b}^2 X + \frac{1}{2}r^{-1}(\nabla_b X_a + \nabla_a X_b)(e_3 - e_4) - r^{-2}X_b e_a \\ &\quad + \frac{1}{2}r^{-1}\not{g}_{ab}(\mathbf{D}_4 - \mathbf{D}_3)X + \mathbf{E}^4, \end{aligned} \quad (2.128)$$

with

$$\mathbf{E}^4 := \mathbf{E}^1 + \mathbf{E}^2 + \mathbf{E}^3.$$

Using formula (2.120) and (2.33), we simplify the terms composing  $E^4$  as

$$\begin{aligned} E^1 &= \frac{1}{2}X_c \nabla_a (\chi - r^{-1}g)_{bc} e_3 + \frac{1}{2}X_c \nabla_a (\underline{\chi} + r^{-1}g)_{bc} e_4 \\ &\quad + \frac{1}{2}X_c (\chi_{cb} - r^{-1}g_{cb}) (\underline{\chi}_{ad} e_d + \zeta_a e_3) + \frac{1}{2}X_c (\underline{\chi}_{cb} + r^{-1}g_{cb}) (\chi_{ad} e_d \zeta_a e_4) \\ &\quad + \frac{1}{2}\nabla_b X_c (\chi_{ca} - r^{-1}g_{ca}) e_3 + \frac{1}{2}\nabla_b X_c (\underline{\chi}_{ca} + r^{-1}g_{ca}) e_4, \end{aligned}$$

and

$$E^2 = \frac{1}{2}r^{-1}X_b \left( (\underline{\chi}_{ac} + r^{-1}g_{ac})e_c - (\chi_{ac} - r^{-1}g_{ac})e_c + \zeta_a e_3 + \zeta_a e_4 \right),$$

and the formula (2.119) follows. This finishes the proof of the lemma.  $\square$

## 2.13 The last cones geodesic foliation

For all  $11/4 \leq \underline{u}' \leq \underline{u}^*$ , we define  $\underline{\mathcal{C}}_{\underline{u}'}$  to be the incoming null cones with vertex  $\phi(\underline{u}')$  and we note  $\underline{u}'$  its associated optical function. We define  $e_3'$  by  $e_3' := -(\mathbf{D}\underline{u}')^\sharp$ . We define  $u'$  to be the associated normalised affine parameter on the cones  $\underline{\mathcal{C}}_{\underline{u}'}$ , *i.e.*

$$\begin{aligned} e_3'(u') &= 2, \\ u'|_{\phi(\underline{u}')} &= \underline{u}'. \end{aligned}$$

This yields a foliation by 2-spheres that we note

$$S' := \left( S'_{\underline{u}', \underline{u}'} \right)_{11/4 \leq \underline{u}' \leq \underline{u}^*, 5/4 \leq u' \leq \underline{u}^*}$$

and that we call the *last cones geodesic foliation*. See Figure 2 for a graphic representation of the domain

$$\mathcal{M}' := \bigcup_{11/4 \leq \underline{u}' \leq \underline{u}^*, 5/4 \leq u' \leq \underline{u}^*} S'_{\underline{u}', \underline{u}'}$$

covered by  $S'$ .

We define  $e_4'$  such that  $(e_3', e_4')$  forms a null pair orthogonal to  $S'$ . With respect to this null pair, we let  $\chi', \xi', \eta', \zeta', \omega'$  and  $\underline{\chi}', \underline{\xi}', \underline{\eta}', \underline{\zeta}', \underline{\omega}'$  denote the associated null connection coefficients and  $\alpha', \beta', \rho', \sigma', \underline{\beta}', \underline{\alpha}'$  the associated null curvature components (see Section 2.7).

We denote  $Y' := -(\mathbf{D}u')^\sharp$ , and  $\eta'$  the optical defect

$$\mathbf{g}(Y', Y') =: -2\eta'.$$

From the above definitions, we have

$$\begin{aligned} e_3'(u') &= 2, & e_4'(u') &= \eta', \\ e_3'(\underline{u}') &= 0, & e_4'(\underline{u}') &= 2. \end{aligned} \tag{2.129}$$

and

$$Y' = e_4' + \frac{1}{2}\eta' e_3'. \tag{2.130}$$

From the geodesic equation  $\mathbf{D}_{e_3'} e_3'$  and relations (2.33), we have

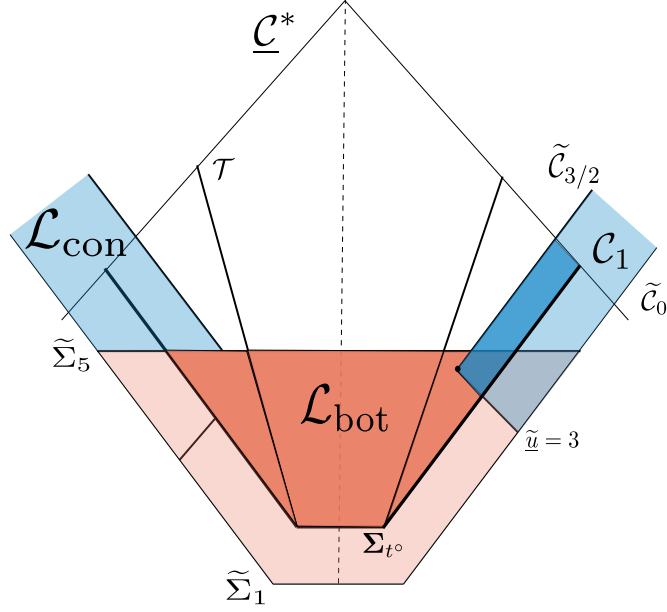
$$\underline{\xi}' = 0, \quad \underline{\omega}' = 0. \tag{2.131}$$

From relations (2.129) and (2.130) we have (see the analogous derivations of the relations of Lemma 2.11)

$$\zeta' = \eta' = -\underline{\eta}'. \tag{2.132}$$

From relations (2.129) and (2.130) we have (see the analogous derivations of relations of Lemma 2.21)

$$e_3'(\eta') = -4\omega', \quad \nabla' \eta' = -2\xi'. \tag{2.133}$$


 Figure 2: The initial layers  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$ .

## 2.14 The initial layers $\mathcal{L}_{\text{bot}}$ and $\mathcal{L}_{\text{con}}$

In this section, we define spacetime regions  $\mathcal{L}_{\text{bot}}, \mathcal{L}_{\text{con}} \subset \tilde{\mathcal{M}}$  which we respectively call the *bottom initial layer* and the *conical initial layer*. The construction and its overlay with the spacetime region  $\mathcal{M}_{\underline{u}^*}$  defined in Section 2.1 is graphically summarised in Figure 2.

**The bottom initial layer  $\mathcal{L}_{\text{bot}}$**  We define a bottom initial layer  $\mathcal{L}_{\text{bot}} \subset \tilde{\mathcal{M}}$  to be a spacetime region covered by a coordinates system, which we call *bottom initial layer coordinates*  $(x^\mu)$  such that in these coordinates  $\mathcal{L}_{\text{bot}} \simeq \cup_{x^0 \in [1, 5]} B((x^0, 0, 0, 0), x^0)$ , where  $B$  denote the coordinate balls. The region  $\mathcal{L}_{\text{bot}}$  shall be a future neighbourhood of  $\tilde{\Sigma}_1 = \{x^0 = 1\}$ , with boundary locally coinciding with the null hypersurface  $\tilde{\mathcal{C}}_0$ , *i.e.*  $\cup_{x^0 \in [1, 5]} \partial B((x^0, 0, 0, 0), x^0) \subset \tilde{\mathcal{C}}_0$ . We denote by  $\tilde{\Sigma}_{x^0}$  the level sets of the time coordinate  $x^0$ . We denote by  $\bar{T}^{\text{bot}}$  the timelike future-pointing unit normal to  $\tilde{\Sigma}$ . We denote by  $\bar{N}^{\text{bot}}$  the outward-pointing unit normal to the 2-spheres level sets of  $x^0$  and  $\sum_{i=1}^3 (x^i)^2$  which is orthogonal to  $\bar{T}^{\text{bot}}$ .

**The conical initial layer  $\mathcal{L}_{\text{con}}$**  We define a conical initial layer  $\mathcal{L}_{\text{con}} \subset \tilde{\mathcal{M}}$  to be a spacetime region covered by two optical functions  $\tilde{u}, \underline{\tilde{u}}$  such that  $\mathcal{L}_{\text{con}} \simeq [0, 3/2]_{\tilde{u}} \times [3, +\infty)_{\underline{\tilde{u}}} \times \tilde{S}_{\tilde{u}, \underline{\tilde{u}}}$ . Moreover, we require that  $\tilde{\mathcal{C}}_0 = \{\tilde{u} = 0\}$ .

We note  $\tilde{r}$  the area radius of the 2-spheres  $\tilde{S}_{\tilde{u}, \underline{\tilde{u}}}$ . We define its null lapse  $2(\tilde{\Omega})^{-2} := -\mathbf{g}(\mathbf{D}\tilde{u}, \mathbf{D}\underline{\tilde{u}})$  and the associated null pair  $(\tilde{e}_3, \tilde{e}_4)$  by  $\tilde{e}_4 := -\tilde{\Omega}(\mathbf{D}\tilde{u})^\sharp$  and  $\tilde{e}_3 := -\tilde{\Omega}(\mathbf{D}\underline{\tilde{u}})^\sharp$ .

We write  $\tilde{\chi}, \tilde{\xi}, \tilde{\eta}, \tilde{\zeta}, \tilde{\omega}$  and  $\underline{\tilde{\chi}}, \underline{\tilde{\xi}}, \underline{\tilde{\eta}}, \underline{\tilde{\zeta}}, \underline{\tilde{\omega}}$  the associated null connection coefficients and  $\tilde{\alpha}, \tilde{\beta}, \tilde{\rho}, \tilde{\sigma}, \tilde{\beta}, \tilde{\alpha}$  the associated null curvature components (see Section 2.7). Note that since the functions  $\tilde{u}, \underline{\tilde{u}}$  are assumed to be optical, we have  $\tilde{\xi} = \underline{\tilde{\xi}} = 0$ .

We moreover define the following intersections of domains

$$\begin{aligned} \mathcal{L}_{\text{bot}}^{\text{int}} &:= \mathcal{M}^{\text{int}} \cap \mathcal{L}_{\text{bot}}, \\ \mathcal{L}_{\text{bot}}^{\text{ext}} &:= \mathcal{M}^{\text{ext}} \cap \mathcal{L}_{\text{bot}}, \\ \mathcal{L}_{\text{con}}^{\text{ext}} &:= \mathcal{M}^{\text{ext}} \cap \mathcal{L}_{\text{con}}. \end{aligned}$$

## 2.15 General change of null frames

Let  $S = (S_{u, \underline{u}})$  and  $S' = (S'_{u', \underline{u}'})$  be two (local) foliations by spacelike 2-spheres. Let  $(e_4, e_3)$  and  $(e_4', e_3')$  be two null pairs associated to the foliations  $S$  and  $S'$  respectively. We have the following lemma (see [KS19a,

Lemma 3.1] for a proof).

**Lemma 2.31.** *There exists two  $S$ -tangent vectorfields  $f$  and  $\underline{f}$  and a scalar function  $\lambda$  such that*

$$\begin{aligned} e_4' &= \lambda \left( e_4 + f + \frac{1}{4}|f|^2 e_3 \right), \\ e_3' &= \lambda^{-1} \left( \left( 1 + \frac{1}{2}f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2 \right) e_3 + \underline{f} + \frac{1}{4}|\underline{f}|^2 f + \frac{1}{4}|\underline{f}|^2 e_4 \right). \end{aligned}$$

Moreover, if  $(e_a)_{a=1,2}$  is a (local) orthonormal null frame of  $TS$ , the pair of vectorfields  $(e_a')_{a=1,2}$  defined for  $a = 1, 2$  by

$$e_a' := e_a + \frac{1}{2}\underline{f}_a f + \frac{1}{2}\underline{f}_a e_4 + \left( \frac{1}{2}f_a + \frac{1}{8}|f|^2 \underline{f}_a \right) e_3, \quad (2.134)$$

is a (local) orthonormal frame of  $TS'$ . The triplet  $(\lambda, f, \underline{f})$  is called the transition coefficients of the change of frame.

We have the following definition for projections of  $S$ -tangent tensors to  $S'$ -tangent tensors.

**Definition 2.32.** *Let  $(e_a)_{a=1,2}$  be an orthonormal frame of  $TS$ . For  $\phi'$  an  $S'$ -tangent  $r$ -tensor, we define its projection  $(\phi')^\dagger$  to be the  $S$ -tangent  $r$ -tensor defined by*

$$(\phi')^\dagger_{a_1 \dots a_r} = (\phi')^\dagger(e_{a_1}, \dots, e_{a_r}) := \phi'(e_{a_1}', \dots, e_{a_r}') = \phi'_{a_1 \dots a_r},$$

where the frame  $(e_a')_{a=1,2}$  is associated to  $(e_a)_{a=1,2}$  via formula (2.134).

Reciprocally, for  $\phi$  an  $S$ -tangent  $r$ -tensor, we define its projection  $(\phi)^\ddagger$  to be the  $S'$ -tangent  $r$ -tensor defined by

$$\phi^\ddagger_{a_1 \dots a_r} = \phi^\ddagger(e_{a_1}', \dots, e_{a_r}') := \phi(e_{a_1}, \dots, e_{a_r}) = \phi_{a_1 \dots a_r}.$$

**Remark 2.33.** *In other terms, we have*

$$(\phi')^\dagger := (\phi')^a e_a, \quad \phi^\ddagger := \phi^a e_a',$$

where the frames  $e_a'$  and  $e_a$  are associated via formula (2.134). It can be seen from formula (2.134) that this definition does not depend on the choice of frame on  $S$

We have the following transition formulas for projected covariant derivatives.

**Proposition 2.34.** *Let  $\phi'$  be an  $S'$ -tangent  $r$ -tensor. We have*

$$\begin{aligned} \nabla_4(\phi')^\dagger &= \lambda^{-1}(\nabla'_4 \phi')^\dagger - f \cdot \nabla(\phi')^\dagger - \frac{1}{4}|f|^2 \nabla_3(\phi')^\dagger + (\phi')^\dagger \cdot \text{Err}(\nabla_4, \nabla'_4), \\ \nabla(\phi')^\dagger &= (\nabla' \phi')^\dagger - \frac{1}{2}\underline{f} f \cdot \nabla(\phi')^\dagger - \frac{1}{2}\underline{f} \nabla_4(\phi')^\dagger - \left( \frac{1}{2}f + \frac{1}{8}|f|^2 \underline{f} \right) \nabla_3(\phi')^\dagger + (\phi')^\dagger \cdot \text{Err}(\nabla, \nabla'), \\ \nabla_3(\phi')^\dagger &= \lambda(\nabla'_3 \phi')^\dagger - \left( \frac{1}{2}f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2 \right) \nabla_3(\phi')^\dagger - \left( \underline{f} + \frac{1}{4}|\underline{f}|^2 f \right) \cdot \nabla(\phi')^\dagger \\ &\quad - \frac{1}{4}|\underline{f}|^2 \nabla_4(\phi')^\dagger + (\phi')^\dagger \cdot \text{Err}(\nabla_3, \nabla'_3). \end{aligned}$$

where the  $S$ -tangent tensors  $\text{Err}(\nabla_4, \nabla'_4)$ ,  $\text{Err}(\nabla, \nabla')$  and  $\text{Err}(\nabla_3, \nabla'_3)$  are bilinear (or higher nonlinear) error terms composed of  $f$ ,  $\underline{f}$ , their first order derivatives and null connection coefficients for the  $S$  and  $S'$  foliations.

Reciprocally, using that for an  $S$ -tangent  $r$ -tensor  $\phi$  we have  $(\phi)^\ddagger = \phi$ , we obtain symmetric formulas for  $\nabla'_4 \phi^\ddagger$ ,  $\nabla' \phi^\ddagger$  and  $\nabla'_3 \phi^\ddagger$ .

*Proof.* We compute the tensorial formulas at a point where we choose a normal frame  $(e_a)_{a=1,2}$ , i.e.  $\mathbf{g}(e_a, e_b) = \delta_{ab}$  and  $\mathbf{D}e_a = 0$ . We denote by  $(e_a')_{a=1,2}$  the frame given by the frame transformation (2.134).<sup>6</sup>

<sup>6</sup>The frame  $(e_a')_{a=1,2}$  is not geodesic normal and the error terms  $\text{Err}$  will correspond to the projection of its connection coefficients  $\mathbf{D}e_a'$ .

We start with the first formula. We shall do the computations for  $\phi'$  an  $S'$ -tangent 1-form, and the results of Proposition 2.34 will follow by simple generalisation. With respect to the frames  $(e_a)_{a=1,2}$  and  $(e'_a)_{a=1,2}$ , we have

$$\begin{aligned}\nabla_4(\phi')^\dagger_a &= e_4(\phi'_a) \\ &= \left( \lambda^{-1}e'_4 - f - \frac{1}{4}|f|^2e_3 \right) (\phi'_a) \\ &= \lambda^{-1}\nabla'_4\phi'_a + \lambda^{-1}\phi'_b\mathbf{g}(\nabla'_4e'_a, e'_b) - f \cdot \nabla(\phi')^\dagger_a - \frac{1}{4}|f|^2\nabla_3(\phi')^\dagger_a.\end{aligned}$$

We deduce that the  $S$ -tangent tensor  $\text{Err}(\nabla_4, \nabla'_4)$  can be expressed as

$$\text{Err}(\nabla_4, \nabla'_4)_{ba} = \lambda^{-1}\mathbf{g}(\nabla'_4e'_a, e'_b) = \mathbf{g}(\lambda^{-1}\mathbf{D}_{e'_4}e'_a, e'_b).$$

Using the transition formulas from Lemma 2.31, we have

$$\begin{aligned}\mathbf{g}(\lambda^{-1}\mathbf{D}_{e'_4}e'_a, e'_b) &= \mathbf{g}\left(\mathbf{D}_{e_4+f}\left(e_a + \frac{1}{2}\underline{f}_a e_4 + \frac{1}{2}f_a e_3\right), e'_b\right) + l.o.t. \\ &= \mathbf{g}\left(\frac{1}{2}e_4(\underline{f}_a)e_4 + \frac{1}{2}e_4(f_a)e_3, e'_b\right) + l.o.t. \\ &= l.o.t.,\end{aligned}$$

where  $l.o.t.$  denotes bilinear (or higher) error terms.

For the second formula, we have

$$\begin{aligned}\nabla_b(\phi')^\dagger_a &= e_b(\phi'_a) \\ &= \left( e'_b - \frac{1}{2}\underline{f}_b f - \frac{1}{2}\underline{f}_b e_4 - \left( \frac{1}{2}\underline{f}_b + \frac{1}{8}|f|^2\underline{f}_b \right) e_3 \right) (\phi'_a) \\ &= (\nabla'_b\phi')^\dagger_a + \phi'_c\mathbf{g}(\mathbf{D}_{e'_b}e'_a, e'_c) - \frac{1}{2}\underline{f}_b f \cdot \nabla(\phi')^\dagger_a - \frac{1}{2}\underline{f}_b \nabla_4(\phi')^\dagger_a - \left( \frac{1}{2}\underline{f}_b + \frac{1}{8}|f|^2\underline{f}_b \right) \nabla_3(\phi')^\dagger_a.\end{aligned}$$

We therefore have  $\text{Err}(\nabla, \nabla')_{cba} = \mathbf{g}(\mathbf{D}_{e'_b}e'_a, e'_c)$  and we check similarly as before that  $\text{Err}(\nabla, \nabla') = l.o.t.$ .

For the third and last formula, we have

$$\begin{aligned}\nabla_3(\phi')^\dagger_a &= e_3(\phi'_a) \\ &= \left( \lambda e'_3 - \left( \frac{1}{2}f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2 \right) e_3 - \underline{f} - \frac{1}{4}|\underline{f}|^2 f - \frac{1}{4}|\underline{f}|^2 e_4 \right) \\ &= \lambda(\nabla'_3\phi')^\dagger_a + \lambda\phi'_b\mathbf{g}(\nabla'_3e'_a, e'_b) - \left( \frac{1}{2}f \cdot \underline{f} + \frac{1}{16}|f|^2|\underline{f}|^2 \right) \nabla_3(\phi')^\dagger_a \\ &\quad - \left( \underline{f} + \frac{1}{4}|\underline{f}|^2 f \right) \cdot \nabla(\phi')^\dagger_a - \frac{1}{4}|\underline{f}|^2 \nabla_4(\phi')^\dagger_a.\end{aligned}$$

And thus  $\text{Err}(\nabla_3, \nabla'_3)_{ba} = \mathbf{g}(\mathbf{D}_{e'_3}e'_a, e'_b) = l.o.t.$  as desired.  $\square$

We have the following transformation formulas for the null connection coefficients (see [KS19a, Proposition 3.3 and Appendix A]).

**Proposition 2.35.** *Under the transitions formulas of Lemma 2.31, we have*

$$\lambda^{-2}\xi' = \xi^\ddagger + \frac{1}{2}\lambda^{-1}\nabla'_4 f^\ddagger + \frac{1}{4}\text{tr}\chi f^\ddagger + \text{Err}(\xi, \xi'), \quad (2.135a)$$

$$\lambda^2\underline{\xi}' = \underline{\xi}^\ddagger + \frac{1}{2}\lambda\nabla'_3 \underline{f}^\ddagger + \frac{1}{4}\text{tr}\underline{\chi} \underline{f}^\ddagger + \text{Err}(\underline{\xi}', \underline{\xi}), \quad (2.135b)$$

$$\lambda^{-1}\chi' = \chi^\ddagger + \nabla' f^\ddagger + \text{Err}(\chi, \chi'), \quad (2.135c)$$

$$\lambda\underline{\chi}' = \underline{\chi}^\ddagger + \nabla' \underline{f}^\ddagger + \text{Err}(\underline{\chi}, \underline{\chi}'), \quad (2.135d)$$

$$\zeta' = \zeta^\ddagger - \nabla'(\log \lambda) - \frac{1}{4}\text{tr}\underline{\chi} f^\ddagger + \frac{1}{4}\text{tr}\chi \underline{f}^\ddagger + \text{Err}(\zeta, \zeta'), \quad (2.135e)$$

$$\eta' = \eta^\ddagger + \frac{1}{2}\lambda\nabla'_3 f^\ddagger + \frac{1}{4}\text{tr}\chi \underline{f}^\ddagger + \text{Err}(\eta, \eta'), \quad (2.135f)$$

$$\underline{\eta}' = \underline{\eta}^\ddagger + \frac{1}{2}\lambda^{-1}\nabla'_4 \underline{f}^\ddagger + \frac{1}{4}\text{tr}\underline{\chi} \underline{f}^\ddagger + \text{Err}(\underline{\eta}, \underline{\eta}'), \quad (2.135g)$$

$$\lambda^{-1}\omega' = \omega - \frac{1}{2}\lambda^{-1}e_4'(\log \lambda) + \text{Err}(\omega, \omega'), \quad (2.135h)$$

$$\lambda\underline{\omega}' = \underline{\omega} + \frac{1}{2}\lambda e_3'(\log \lambda) + \text{Err}(\underline{\omega}, \underline{\omega}'), \quad (2.135i)$$

where

$$\text{Err}(\xi, \xi') := \omega f^\ddagger + \frac{1}{2}f \cdot \hat{\chi} + l.o.t.,$$

$$\text{Err}(\underline{\xi}, \underline{\xi}') := \underline{\omega} \underline{f} + \frac{1}{2}\underline{f} \cdot \hat{\chi} + l.o.t.,$$

$$\text{Err}(\chi, \chi') := f^\ddagger \otimes \eta^\ddagger + f^\ddagger \otimes \zeta^\ddagger + \underline{f}^\ddagger \otimes \xi^\ddagger + \frac{1}{2}(f \cdot f)\chi^\ddagger - \frac{1}{2}(f \cdot \underline{f})\chi' + \frac{1}{2}f^\ddagger \cdot \chi' \otimes \underline{f}^\ddagger - \frac{1}{4}|f^2|_\chi + l.o.t.,$$

$$\text{Err}(\underline{\chi}, \underline{\chi}') := \underline{f}^\ddagger \otimes \underline{\eta}^\ddagger - \underline{f}^\ddagger \otimes \underline{\zeta}^\ddagger + \underline{f}^\ddagger \otimes \underline{\xi}^\ddagger + \frac{1}{2}(\underline{f} \cdot \underline{f})\underline{\chi}^\ddagger - \frac{1}{2}(\underline{f} \cdot \underline{f})\underline{\chi}' + \frac{1}{2}\underline{f}^\ddagger \cdot \underline{\chi}' \otimes \underline{f}^\ddagger - \frac{1}{4}|\underline{f}^2|_{\underline{\chi}} + l.o.t.,$$

$$\text{Err}(\zeta, \zeta') := \omega \underline{f} - \underline{\omega} f - \frac{1}{2}\hat{\chi} \cdot f + \frac{1}{4}\lambda^{-1}\text{tr}\chi' \underline{f}^\ddagger + \frac{1}{2}\lambda^{-1}\underline{f} \cdot \hat{\chi}' + l.o.t.,$$

$$\text{Err}(\eta, \eta') := -\underline{\omega} f + \frac{1}{2}\underline{f} \cdot \hat{\chi}' + l.o.t.,$$

$$\text{Err}(\underline{\eta}, \underline{\eta}') := \frac{1}{2}f \cdot \hat{\chi}' + l.o.t.,$$

$$\text{Err}(\omega, \omega') := \frac{1}{2}f \cdot (\zeta - \eta) - \frac{1}{8}\text{tr}\chi |f|^2 + \frac{1}{2}\lambda^{-2}\underline{f} \cdot \xi' + l.o.t.,$$

$$\text{Err}(\underline{\omega}, \underline{\omega}') := -\frac{1}{2}\underline{f} \cdot \zeta - \frac{1}{2}\underline{f}^\ddagger \cdot \eta' + \frac{1}{2}f \cdot \underline{\xi} + \frac{1}{8}(f \cdot \underline{f})\text{tr}\underline{\chi} + l.o.t.,$$

where *l.o.t.* denotes trilinear (or higher nonlinear) error terms composed of a null connection coefficient and transition coefficients.

Using the change of derivatives formulas from Proposition 2.34, the same formulas as (2.135) hold for derivatives taken with respect to the frame  $(e_3, e_4, e_a)$ , up to additional nonlinear error terms.

We have the following transformation formulas for the null curvature components (see [KS19a, Proposition 3.3 and Appendix A]).

**Proposition 2.36.** *Under the transitions formulas of Lemma 2.31, we have*

$$\begin{aligned} \lambda^{-2}\alpha' &= \alpha^\ddagger + \text{Err}(\alpha, \alpha'), & \lambda^{-1}\beta' &= \beta^\ddagger + \text{Err}(\beta, \beta'), & \rho' &= \rho + \text{Err}(\rho, \rho'), \\ \lambda^2\underline{\alpha}' &= \underline{\alpha}^\ddagger + \text{Err}(\underline{\alpha}, \underline{\alpha}'), & \lambda\underline{\beta}' &= \underline{\beta}^\ddagger + \text{Err}(\underline{\beta}, \underline{\beta}'), & \sigma' &= \sigma + \text{Err}(\sigma, \sigma'), \end{aligned}$$

where

$$\begin{aligned}
 \text{Err}(\alpha, \alpha') &:= (f^\dagger \widehat{\otimes} \beta^\dagger - *f^\dagger \widehat{\otimes} * \beta^\dagger) + l.o.t., \\
 \text{Err}(\beta, \beta') &:= \frac{3}{2}(f^\dagger \rho + *f^\dagger \sigma) + \frac{1}{2} \alpha^\dagger \cdot \underline{f}^\dagger + l.o.t., \\
 \text{Err}(\rho, \rho') &:= \underline{f} \cdot \beta - f \cdot \underline{\beta} + l.o.t., \\
 \text{Err}(\sigma, \sigma') &:= -\underline{f} \cdot * \beta - f \cdot * \underline{\beta} + l.o.t., \\
 \text{Err}(\underline{\beta}, \underline{\beta}') &:= -\frac{3}{2}(\underline{f}^\dagger \rho + *\underline{f}^\dagger \sigma) - \frac{1}{2} \underline{\alpha}^\dagger \cdot f^\dagger + l.o.t., \\
 \text{Err}(\underline{\alpha}, \underline{\alpha}') &:= -(\underline{f}^\dagger \widehat{\otimes} \underline{\beta}^\dagger - *\underline{f}^\dagger \widehat{\otimes} * \underline{\beta}^\dagger) + l.o.t.,
 \end{aligned}$$

where *l.o.t.* denotes trilinear error terms composed of a null curvature component and transition coefficients.

**Remark 2.37.** Although we did not present the full explicit formulas for the nonlinear error terms appearing in Propositions 2.34, 2.35, 2.36, we claim that they satisfy a conservation of signature principle,<sup>7</sup> which roughly states that the transition coefficients  $f, \lambda$  and  $\underline{f}$  are respectively paired with higher, same and lower signature null curvature components, null connection coefficients, derivatives, etc. Since a drop of signature corresponds at most to a drop of decay by  $r^{-1}$ , the control of the error terms in Section 12 can be obtained without decay assumptions on  $\lambda, \underline{f}$ , provided that the coefficient  $f$  decays as  $r^{-1}$ . See also [KS17, Remark 4.1.4].

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<sup>7</sup>See [CK93] for further discussion on signature.





# 3. Norms, bootstrap assumptions and consequences

## 3.1 Preliminary definitions

We define the following local frame norms.

**Definition 3.1** (Frame norms). *For an orthonormal frame  $(e_\mu)_{\mu=0..3}$ , we define the following associated frame norm*

$$|F|^2 := \sum_{i_1, \dots, i_l=0..3} |F_{i_1 \dots i_l}|^2,$$

for all spacetime tensor  $F$  and where  $F_{i_1 \dots i_l}$  denotes the evaluation of  $F$  on the  $l$ -uplet  $(e_{i_1}, \dots, e_{i_l})$ . In this paper, most frame norms are equivalents. We shall precise with respect to which frame the norms are taken only when it is relevant.

**Remark 3.2.** *For an orthonormal frame  $(e_\mu)_{\mu=0..3}$ , the frame norm from the above definition does not depend on the choice of spacelike orthonormal vectors  $e_i$ , and we have*

$$|F|^2 = 2 |\mathbf{g}(e_0, F)|^2 + \mathbf{g}(F, F).$$

Using Definition 3.1, we define integrals,  $L^p$  and  $L^p L^q$  norms on the various submanifolds of this paper using the respective (intrinsic) induced metrics. In the case of the null hypersurfaces  $\mathcal{C}$ , the integral is defined consistently with the coarea formulas from Lemma 3.34. In the case of the null hypersurface  $\underline{\mathcal{C}}^*$ , the integral is defined consistently with the foliation by the 2-spheres of the canonical foliation.

**Definition 3.3** ( $\tilde{H}^{1/2}$  norm). *Let  $(S, \mathfrak{g})$  be a Riemannian 2-sphere with area radius  $r$ . Let  $F$  be an  $S$ -tangent tensor. We define the (scaling homogeneous)  $\tilde{H}^{1/2}$  norm of  $F$  on  $(S, \mathfrak{g})$  to be*

$$\|F\|_{\tilde{H}^{1/2}(S)} := r^{1/2} \|F\|_{H^{1/2}(S, r^{-2}\mathfrak{g})},$$

where for a Riemannian 2-sphere  $(S, \gamma)$ ,  $H^{1/2}(S, \gamma)$  denotes the standard fractional Sobolev space on  $(S, \gamma)$ , as defined in [Sha14].<sup>1</sup>

**Remark 3.4.** *The  $\tilde{H}^{1/2}$  norm are schematically of the form*

$$\|F\|_{\tilde{H}^{1/2}(S)} \sim r^{-1/2} \left\| (r\nabla)^{\leq \frac{1}{2}} F \right\|_{L^2(S)}.$$

**Remark 3.5.** *Throughout this paper we will extensively use  $L^\infty \tilde{H}^{1/2}(S)$  norms, which should be thought of as an upgraded version of  $L^\infty L^4(S)$  norms. Namely:*

- ◊ *They are at the same scaling level, similar Sobolev embeddings, elliptic and transport estimates hold (see Lemmas 3.36, 3.38, 10.2),*
- ◊  *$\tilde{H}^{1/2}(S)$  embeds in  $L^4(S)$  (see Lemma 3.36),*
- ◊  *$\tilde{H}^{1/2}(S)$  is a natural trace space for spacelike 3D Dirichlet/Neumann problems with boundary  $S$ .*

For these reasons, most of the Bootstrap Assumptions (see Section 3.3) are formulated using  $L^\infty \tilde{H}^{1/2}(S)$  norms. For conciseness, we did not write the  $L^\infty$  and  $L^\infty L^4(S)$  norms obtained via Sobolev embeddings (see Lemma 3.36) and shall implicitly assume that these norms are controlled as well.

**Remark 3.6.** *Using the  $\tilde{H}^{1/2}$  spaces of Definition 3.3 (see also Lemma 3.35) forces us to reprove classical (intrinsically obtained) estimates (see the Klainerman-Sobolev  $\tilde{H}^{1/2}$  embeddings of Lemmas 6.7, 6.6 and their proof in Appendix D, and the transport estimates of Lemma 10.2) relying on coordinate comparison with the Euclidean case.*

<sup>1</sup>Under the regularity assumptions of this paper (see the Bootstrap Assumptions 3.25 and 3.26), these intrinsic fractional norms are also equivalent to coordinate defined norms (see Lemma 3.35).

## 3.2 Norms

In this section, we define the norms of the curvature, connection coefficients and coordinates, upon which the bootstrap argument is constructed (see the Bootstrap Assumptions in Section 3.3).

### 3.2.1 Norms for the curvature in $\mathcal{M}^{\text{ext}}$ and $\underline{\mathcal{C}}^*$

**Norms on  $\underline{\mathcal{C}}^*$ .** We define

$$q := \min(r, u).$$

**Remark 3.7.** *In the interior of the cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ , we have*

$$q \simeq r.$$

*In the exterior of the cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ , we have*

$$q \simeq u.$$

We define

$$\mathcal{R}_{\leq 2}^*[R] := \left\| \nabla^{\leq 2} R \right\|_{L^2(\underline{\mathcal{C}}^*)},$$

where  $\nabla \in \{(r\nabla), (r\nabla_4), (q\nabla_3)\}$ . Using this notation, we have the following definition

$$\begin{aligned} \mathcal{R}_{\leq 2}^* &:= \mathcal{R}_{\leq 2}^*[u^2\alpha] + \mathcal{R}_{\leq 2}^*[uu\beta] + \mathcal{R}_{\leq 2}^*[u^2(\rho - \bar{\rho})] + \mathcal{R}_{\leq 2}^*[u^2(\sigma - \bar{\sigma})] \\ &\quad + \left( \mathcal{R}_{\leq 2}^*[u^2\beta] + \left\| u^2 \nabla^{\leq 1} (r\nabla_3)\beta \right\|_{L^2(\underline{\mathcal{C}}^*)} \right) \\ &\quad + \left( \mathcal{R}_{\leq 2}^*[u^2\alpha] + \left\| u^2 \nabla^{\leq 1} (r\nabla_3)\alpha \right\|_{L^2(\underline{\mathcal{C}}^*)} + \left\| u^2 (r\nabla_3)^2 \alpha \right\|_{L^2(\underline{\mathcal{C}}^*)} - \left\| (r\nabla_4)^2 (u^2\alpha) \right\|_{L^2(\underline{\mathcal{C}}^*)} \right). \end{aligned}$$

We define

$$\mathfrak{R}_{\leq 1}^* := \left\| r^{1/2} q^{1/2} \nabla^{\leq 1} R \right\|_{L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})},$$

and

$$\begin{aligned} \mathfrak{R}_{\leq 1}^* &:= \mathfrak{R}_{\leq 1}^*[u^2\alpha] + \mathfrak{R}_{\leq 1}^*[uu\beta] + \mathfrak{R}_{\leq 1}^*[u^2(\rho - \bar{\rho})] \\ &\quad + \mathfrak{R}_{\leq 1}^*[u^2(\sigma - \bar{\sigma})] + \mathfrak{R}_{\leq 1}^*[q^{-1/2} u^{5/2} \beta] + \mathfrak{R}_{\leq 1}^*[q^{-1/2} u^{5/2} \alpha]. \end{aligned}$$

We define

$$\begin{aligned} \bar{\mathfrak{R}}_{\leq 2}^*[\bar{\rho}] &:= \left\| u^3 q u \nabla^{\leq 2} \bar{\rho} \right\|_{L^\infty(\underline{\mathcal{C}}^*)}, \\ \bar{\mathfrak{R}}_{\leq 2}^*[\bar{\sigma}] &:= \left\| u^3 q u \nabla^{\leq 2} \bar{\sigma} \right\|_{L^\infty(\underline{\mathcal{C}}^*)}, \end{aligned}$$

and

$$\bar{\mathfrak{R}}_{\leq 2}^* := \bar{\mathfrak{R}}_{\leq 2}^*[\bar{\rho}] + \bar{\mathfrak{R}}_{\leq 2}^*[\bar{\sigma}].$$

**Remark 3.8.** *Here, as in the rest of this section, we implicitly assume that the  $L^\infty \tilde{H}^{1/2}$  bootstrap bounds for a tensor  $F$  come together with the corresponding  $L^\infty L^4$  bootstrap bounds for  $F$  (see the Sobolev embeddings from Lemma 3.36), and that the  $L^\infty \tilde{H}^{1/2}$  bootstrap bounds for  $(r\nabla)^{\leq 1} F$  come together with the corresponding  $L^\infty L^\infty$  bound for  $F$ .*

**Remark 3.9.** *In the exterior of the cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  these norms provide the expected optimal decay rates for the spacetime curvature. In the interior of the cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ , these norms provide a suboptimal control in terms of  $r$ , which is due to the degeneracy of the null decomposition when  $r \rightarrow 0$ . See also Remark 8.2. These norms are only used to estimate the null metric and connection coefficients on  $\underline{\mathcal{C}}^*$ . The (optimal) curvature norms used in the treatment of the interior region are presented in Section 3.2.2.*

**Norms in  $\mathcal{M}^{\text{ext}}$ .** We define

$$\begin{aligned} \mathcal{R}_{\leq 2, \gamma}^{\text{ext}} := & \left\| u^{-1/2-\gamma} \underline{u}^2 \nabla^{\leq 2} \alpha \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| u^{-1/2-\gamma} \underline{u}^2 \nabla^{\leq 2} \beta \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ & + \left\| u^{-1/2-\gamma} \underline{u}^2 \nabla^{\leq 2} (\rho - \bar{\rho}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| u^{-1/2-\gamma} \underline{u}^2 \nabla^{\leq 2} (\sigma - \bar{\sigma}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ & + \left\| u^{-1/2-\gamma} \underline{u} u \nabla^{\leq 2} \underline{\beta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| u^{-1/2-\gamma} \underline{u}^2 \nabla^{\leq 2} \underline{\alpha} \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \end{aligned}$$

for all  $\gamma > 0$  and where here  $\nabla \in \{(r\nabla), (\underline{u}\nabla_4), (u\nabla_3)\}$ , since  $q \simeq u$  in  $\mathcal{M}^{\text{ext}}$ . We define

$$\begin{aligned} \mathfrak{R}_{\leq 1}^{\text{ext}} := & \left\| \underline{u}^3 \nabla^{\leq 1} \alpha \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} + \left\| \underline{u}^3 \nabla^{\leq 1} \beta \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} \\ & + \left\| \underline{u}^{5/2} u^{1/2} \nabla^{\leq 1} (\rho - \bar{\rho}) \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} + \left\| \underline{u}^{5/2} u^{1/2} \nabla^{\leq 1} (\sigma - \bar{\sigma}) \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} \\ & + \left\| \underline{u}^{3/2} u^{3/2} \nabla^{\leq 1} \underline{\beta} \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{5/2} \nabla^{\leq 1} \underline{\alpha} \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})}. \end{aligned}$$

We define

$$\bar{\mathfrak{R}}_{\leq 2}^{\text{ext}} := \left\| \underline{u}^3 u^2 \nabla^{\leq 2} \bar{\rho} \right\|_{L^\infty(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^3 u^2 \nabla^{\leq 2} \bar{\sigma} \right\|_{L^\infty(\mathcal{M}^{\text{ext}})}.$$

**Remark 3.10.** *The mean values  $\bar{\rho}, \bar{\sigma}$  of the null curvature components  $\rho, \sigma$  have stronger decay rates than the other curvature components. This is a consequence of the average Bianchi equations (2.47), (2.85) and the vertex limit  $r^3 \bar{\rho} \rightarrow 0$  when  $r \rightarrow 0$ , and of the structure equation (2.34o) for  $\sigma$ . See Sections 9 and 10.*

### 3.2.2 Norms for the curvature in $\mathcal{M}^{\text{int}}$

We have the following definitions of the curvature boundedness norms in the interior region

$$\mathcal{R}_{\leq 2}^{\text{int}} := \sup_{t^0 \leq t \leq t^*} \|t^2 |(t\mathbf{D})^{\leq 2} \mathbf{R}|\|_{L^2(\Sigma_t)},$$

where we take the norms in the maximal frame (*i.e.*  $e_0 = \bar{T}$ ).

We have the following definitions of the curvature decay norms in the interior region

$$\mathfrak{R}_{\leq 1}^{\text{int}} := \sup_{t^0 \leq t \leq t^*} \left( \|t^{7/2} |\mathbf{R}|\|_{L^\infty(\Sigma_t)} + \|t^4 |\mathbf{DR}|\|_{L^6(\Sigma_t)} \right).$$

**Remark 3.11.** *These norms only cover the bottom interior region  $\mathcal{M}_{\text{bot}}^{\text{int}}$ . This is not an issue for the global energy estimates in which these norms are used, since these estimates are performed in that same region  $\mathcal{M}_{\text{bot}}^{\text{int}}$ . The top interior region  $\mathcal{M}_{\text{top}}^{\text{int}}$  is contained in the domain of dependence of the last maximal hypersurface  $\Sigma_{t^*}$  and is treated via a local existence argument (see Section 8.1).*

### 3.2.3 Norms for the null connection coefficients on the cone $\underline{\mathcal{C}}^*$

For an  $S$ -tangent tensor  $\Gamma$ , we define the following  $L^2(\underline{\mathcal{C}}^*)$  based norms

$$\begin{aligned} \mathcal{O}_{\leq \ell}^{*, \mathfrak{g}}[\Gamma] &:= \left\| r^{-1} \underline{u}^2 \nabla^{\leq \ell} \Gamma \right\|_{L^2(\underline{\mathcal{C}}^*)}, \\ \mathcal{O}_{\leq \ell}^{*, \mathfrak{b}}[\Gamma] &:= \left\| r^{-1} \underline{u} u \nabla^{\leq \ell} \Gamma \right\|_{L^2(\underline{\mathcal{C}}^*)}, \end{aligned}$$

where  $\nabla \in \{(r\nabla), (q\nabla_3)\}$  and where  $\ell \geq 0$ . We also define the following norm for  $\underline{\omega} - \bar{\omega}$

$$\mathcal{O}_{\leq 2+}^{*, \mathfrak{b}}[\underline{\omega} - \bar{\omega}] := \mathcal{O}_{\leq 3}^{*, \mathfrak{b}}[\underline{\omega} - \bar{\omega}] - \left\| r^{-1} \underline{u} u (q\nabla_3)^3 (\underline{\omega} - \bar{\omega}) \right\|_{L^2(\underline{\mathcal{C}}^*)}.$$

We define

$$\begin{aligned} \mathcal{O}_{\leq 3}^* &:= \mathcal{O}_{\leq 3}^{*, \mathfrak{g}}[\text{tr}\underline{\chi} - \overline{\text{tr}\chi}] + \mathcal{O}_{\leq 3}^{*, \mathfrak{g}}[\zeta] + \mathcal{O}_{\leq 3}^{*, \mathfrak{g}}[\text{tr}\chi - \overline{\text{tr}\chi}] + \mathcal{O}_{\leq 3}^{*, \mathfrak{g}}[\hat{\chi}] \\ &\quad + \mathcal{O}_{\leq 3}^{*, \mathfrak{b}}[\hat{\chi}] + \mathcal{O}_{\leq 2+}^{*, \mathfrak{b}}[\underline{\omega} - \bar{\omega}]. \end{aligned}$$

We define the following norms for the mean values of  $\text{tr}\chi$  and  $\text{tr}\underline{\chi}$

$$\begin{aligned}\overline{\mathfrak{D}}_{\leq 2}^* [\overline{\text{tr}\chi}] &:= \left\| \underline{u} \underline{u}^3 (q \nabla_3)^{\leq 2} \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(\underline{\mathcal{C}}^*)}, \\ \overline{\mathfrak{D}}_{\leq 2}^* [\underline{\text{tr}\chi}] &:= \left\| u^2 \underline{u}^2 (q \nabla_3)^{\leq 2} \left( \underline{\text{tr}\chi} + \frac{2}{r} \right) \right\|_{L^\infty(\underline{\mathcal{C}}^*)},\end{aligned}$$

and

$$\overline{\mathfrak{D}}_{\leq 2}^* := \overline{\mathfrak{D}}_{\leq 2}^* [\overline{\text{tr}\chi}] + \overline{\mathfrak{D}}_{\leq 2}^* [\underline{\text{tr}\chi}].$$

We define the following  $L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})$  based norms

$$\begin{aligned}\mathfrak{D}_{\leq 2}^{*,g} [\Gamma] &:= \left\| r^{-1/2} q^{1/2} \underline{u}^2 \nabla^{\leq 2} \Gamma \right\|_{L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})}, \\ \mathfrak{D}_{\leq 2}^{*,b} [\Gamma] &:= \left\| r^{-1/2} q^{1/2} \underline{u} u \nabla^{\leq 2} \Gamma \right\|_{L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})}, \\ \mathfrak{D}_{\leq 1+}^{*,b} [\underline{\omega} - \overline{\omega}] &:= \mathfrak{D}_{\leq 2}^{*,b} [\underline{\omega} - \overline{\omega}] - \left\| r^{-1/2} q^{1/2} \underline{u} u (q \nabla_3)^2 (\underline{\omega} - \overline{\omega}) \right\|_{L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})}.\end{aligned}$$

**Remark 3.12.** Together with the relations between the null connection coefficients from Lemmas 2.11 and 2.21, these norms provide a control for all the null connection coefficients.

**Remark 3.13.** These norms give optimal decay rates in the exterior of the cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ . They give a suboptimal control in  $r$  in the region  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ . This is a consequence of the suboptimal control for the curvature. See Remark 8.2. Except to establish their own control in Section 9, these norms are only used in the exterior region where they are optimal.

We also have the following norms for  $\mathbf{O}^{\text{ext}}$

$$\mathcal{O}_{\leq 3}^{*,\mathbf{O}} := \mathcal{O}_{\leq 2}^{*,g} [r^{-1} H] + \mathcal{O}_{\leq 1}^{*,g} [\Psi],$$

where in that case the norms are restricted to the exterior region  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  where the exterior rotations are defined. See the definitions from Section 2.12.

### 3.2.4 Norms for the null connection coefficients in $\mathcal{M}^{\text{ext}}$

We recall that in the exterior region  $\mathcal{M}^{\text{ext}}$ , we have  $r \simeq \underline{u}$ . For an  $S$ -tangent tensor  $\Gamma$ , we define the following  $\tilde{H}^{1/2}$  norms

$$\begin{aligned}\mathfrak{D}_{\leq 1}^{\text{ext},g} [\Gamma] &:= \left\| r^{3/2} u^{1/2} (\nabla) \leq 1 \Gamma \right\|_{L_{u, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})}, \\ \mathfrak{D}_{\leq 1}^{\text{ext},b} [\Gamma] &:= \left\| r^{1/2} u^{3/2} (\nabla) \leq 1 \Gamma \right\|_{L_{u, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})},\end{aligned}$$

where  $\nabla \in \{(r \nabla), (u \nabla_3), (\underline{u} \nabla_4)\}$ .

For an  $S$ -tangent tensor  $\Gamma$ , we define the following  $L^2(\mathcal{M}^{\text{ext}})$  norms

$$\begin{aligned}\mathcal{O}_{\leq 2, \gamma}^{\text{ext},g} [\Gamma] &:= \left\| \underline{u}^{-1/2-\gamma} \underline{u} (\nabla) \leq 2 \Gamma \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \\ \mathcal{O}_{\leq 2, \gamma}^{\text{ext},b} [\Gamma] &:= \left\| \underline{u}^{-1/2-\gamma} u (\nabla) \leq 2 \Gamma \right\|_{L^2(\mathcal{M}^{\text{ext}})},\end{aligned}$$

for all  $\gamma > 0$  and where  $\nabla \in \{(r \nabla), (u \nabla_3), (\underline{u} \nabla_4)\}$ .

We define

$$\begin{aligned}\mathfrak{D}_{\leq 1}^{\text{ext}} &:= \mathfrak{D}_{\leq 1}^{\text{ext},g} [\text{tr}\chi - \overline{\text{tr}\chi}] + \mathfrak{D}_{\leq 1}^{\text{ext},g} [\hat{\chi}] + \mathfrak{D}_{\leq 1}^{\text{ext},g} [\text{tr}\chi - \overline{\text{tr}\chi}] + \mathfrak{D}_{\leq 1}^{\text{ext},g} [\zeta] \\ &\quad + \mathfrak{D}_{\leq 1}^{\text{ext},b} [\hat{\chi}] + \mathfrak{D}_{\leq 1}^{\text{ext},b} [\underline{\omega} - \overline{\omega}, {}^* \underline{\omega}] + \mathfrak{D}_{\leq 1}^{\text{ext},b} [\underline{\xi}], \\ \mathfrak{D}_{\leq 2}^{\text{ext}} &:= \mathfrak{D}_{\leq 1}^{\text{ext}} + \mathfrak{D}_{\leq 1}^{\text{ext},g} [(r \nabla)(\text{tr}\chi - \overline{\text{tr}\chi})] + \mathfrak{D}_{\leq 1}^{\text{ext},g} [(r \nabla)\hat{\chi}] + \mathfrak{D}_{\leq 1}^{\text{ext},g} [(r \nabla)\zeta] \\ &\quad + \mathfrak{D}_{\leq 1}^{\text{ext},g} [(u \nabla_3)\zeta] + \mathfrak{D}_{\leq 1}^{\text{ext},b} [(r \nabla)\underline{\omega}_\rho, (r \nabla)\underline{\omega}_\sigma] + \mathfrak{D}_{\leq 1}^{\text{ext},b} [(r \nabla)\underline{\omega}, (r \nabla) {}^* \underline{\omega}],\end{aligned}$$

where we recall that  $^*\underline{\omega}, \underline{\omega}_\rho, \underline{\omega}_\sigma$  are defined in Section 2.9.

We define

$$\begin{aligned} \mathcal{O}_{\leq 2, \gamma}^{\text{ext}} &:= \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [\text{tr}\chi - \overline{\text{tr}\chi}] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [\hat{\chi}] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [\zeta] \\ &\quad + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}} [\hat{\chi}] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}} [\underline{\omega} - \overline{\underline{\omega}}, ^*\underline{\omega}] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}} [\underline{\xi}], \\ \mathcal{O}_{\leq 3, \gamma}^{\text{ext}} &:= \mathcal{O}_{\leq 2, \gamma}^{\text{ext}} + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [(r\nabla)(\text{tr}\chi - \overline{\text{tr}\chi})] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [(r\nabla)\hat{\chi}] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [(r\nabla)\zeta] \\ &\quad + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [(u\nabla_3)\zeta] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}} [(r\nabla)\underline{\omega}_\rho, (r\nabla)\underline{\omega}_\sigma] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}} [(r\nabla)\underline{\omega}, (r\nabla)^*\underline{\omega}]. \end{aligned}$$

We define

$$\begin{aligned} \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\underline{\omega}] &:= \left\| \underline{u}^2 u^2 \nabla^{\leq 1} \underline{\omega} \right\|_{L^\infty(\mathcal{M}^{\text{ext}})}, \\ \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\overline{\text{tr}\chi}] &:= \left\| \underline{u}^3 u^1 \nabla^{\leq 1} \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(\mathcal{M}^{\text{ext}})}, \\ \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\overline{\text{tr}\underline{\chi}}] &:= \left\| \underline{u}^2 u^2 \nabla^{\leq 1} \left( \overline{\text{tr}\underline{\chi}} + \frac{2}{r} \right) \right\|_{L^\infty(\mathcal{M}^{\text{ext}})}, \end{aligned}$$

where  $\nabla \in \{(r\nabla), (u\nabla_3), (\underline{u}\nabla_4)\}$ , and we note

$$\overline{\mathfrak{D}}_{\leq 1}^{\text{ext}} := \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\underline{\omega}] + \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\overline{\text{tr}\chi}] + \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\overline{\text{tr}\underline{\chi}}].$$

We define

$$\begin{aligned} \overline{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}}[\underline{\omega}] &:= \left\| \underline{u}^{-1/2-\gamma} u^{3/2} \underline{u} \nabla^{\leq 2} \underline{\omega} \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \\ \overline{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}}[\overline{\text{tr}\chi}] &:= \left\| \underline{u}^{-1/2-\gamma} u^{1/2} \underline{u}^2 \nabla^{\leq 2} \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \\ \overline{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}}[\overline{\text{tr}\underline{\chi}}] &:= \left\| \underline{u}^{-1/2-\gamma} u^{3/2} \underline{u} \nabla^{\leq 2} \left( \overline{\text{tr}\underline{\chi}} + \frac{2}{r} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \end{aligned}$$

for all  $\gamma > 0$  and where  $\nabla \in \{(r\nabla), (u\nabla_3), (\underline{u}\nabla_4)\}$ .

**Remark 3.14.** *The absence of control for the higher derivatives of the coefficients  $\underline{\chi}$  and  $\underline{\xi}$  is due to the classical loss of regularity for the geodesic foliation. See Section 10.*

We have the following definitions for the norms of the defect  $\eta$  in  $\mathcal{M}^{\text{ext}}$  and on  $\mathcal{T}$

$$\begin{aligned} \mathfrak{D}_{\leq 1}^{\text{ext}}[\eta] &:= \left\| \underline{u} u^2 \nabla^{\leq 1} \eta \right\|_{L^\infty(\mathcal{M}^{\text{ext}})} + \left\| r^{-1/2} u^{3/2} \nabla^{\leq 1} (r\nabla)\eta \right\|_{L_{\underline{u}, \underline{u}}^\infty \tilde{H}^{1/2}}, \\ \mathcal{O}_{\leq 2, \gamma}^{\text{ext}}[\eta] &:= \left\| \underline{u}^{-1/2-\gamma} u^{3/2} \nabla^{\leq 2} \eta \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma} \underline{u}^{-1} u \nabla^{\leq 2} (r\nabla)\eta \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \\ \mathcal{O}_{\leq 2, \gamma}^{\mathcal{T}}[\eta] &:= \left\| t^{3/2-\gamma} \nabla^{\leq 2} \eta \right\|_{L^2(\mathcal{T})} + \left\| t^{-\gamma} \nabla^{\leq 2} (r\nabla)\eta \right\|_{L^2(\mathcal{T})}, \end{aligned}$$

for all  $\gamma > 0$  and where  $\nabla \in \{(r\nabla), (u\nabla_3), (\underline{u}\nabla_4)\}$ .

We define the following norms for  $\mathbf{O}^{\text{ext}}$  in  $\mathcal{M}^{\text{ext}}$

$$\begin{aligned} \mathfrak{D}_{\leq 2}^{\text{ext}, \mathbf{O}} &:= \mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}} [r^{-1}H] + \mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}} [r^{-1}Y] + \mathfrak{D}_{\leq 0}^{\text{ext}, \mathfrak{g}} [\Psi], \\ \mathcal{O}_{\leq 3, \gamma}^{\text{ext}, \mathbf{O}} &:= \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [r^{-1}H] + \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [r^{-1}Y] + \mathcal{O}_{\leq 1, \gamma}^{\text{ext}, \mathfrak{g}} [\Psi], \end{aligned}$$

for all  $\gamma > 0$ .

### 3.2.5 Norms for the maximal connection coefficients in $\mathcal{M}_{\text{bot}}^{\text{int}}$

We have the following definitions for the norms for the time lapse in  $\mathcal{M}_{\text{bot}}^{\text{int}}$

$$\begin{aligned} \mathcal{O}_{\leq 3, \gamma}^{\text{int}}[n] &:= \left\| (t\nabla)^{\leq 3}(n-1) \right\|_{L_t^\infty L^2(\Sigma_t)} + \left\| t(t\nabla)^{\leq 2} \bar{T}(n-1) \right\|_{L_t^\infty L^2(\Sigma_t)} \\ &\quad + \left\| t^{-1/2-\gamma} t^2 (t\nabla)^{\leq 1} \bar{T}^2(n-1) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})}, \end{aligned}$$

for all  $\gamma > 0$ .

We have the following definitions for the boundedness norms for the plane second fundamental form in  $\mathcal{M}^{\text{int}}$

$$\mathcal{O}_{\leq 2}^{\text{int}}[k] := \left\| t \left( t\nabla, t\hat{\mathcal{L}}_{\bar{T}} \right)^{\leq 2} k \right\|_{L_t^\infty L^2(\Sigma_t)}.$$

Additionally, we have the following definition for the transition factors  $\nu$  at  $\mathcal{T}$

$$\mathfrak{D}_{\leq 2}^{\mathcal{T}}[\nu] := \left\| t(t\nabla, tZ)^{\leq 2}(\nu-1) \right\|_{L_t^\infty \tilde{H}^{1/2}(\partial\Sigma_t)},$$

where  $Z$  is the future-pointing unit normal to  $\partial\Sigma_t$  in  $\mathcal{T}$ .

Moreover, we control the following more regular norms on  $\Sigma_{t^*}$

$$\mathcal{O}_{\leq 3}^{\Sigma_{t^*}}[k] := \left\| t(t\nabla)^3 k \right\|_{L^2(\Sigma_{t^*})}.$$

### 3.2.6 Norms for the approximate Killing fields in $\mathcal{M}_{\text{bot}}^{\text{int}}$

We have the following definitions for the control of the derivatives and deformation tensors of  $\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}}, \mathbf{S}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{O}^{\text{int}}$

$$\begin{aligned} \mathcal{O}_{\leq 3, \gamma}^{\text{int}}[\mathbf{T}^{\text{int}}] &:= \left\| t^{5/2} \mathbf{D}\mathbf{T}^{\text{int}} \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \\ &\quad + \left\| t^2 (t\mathbf{D})^{\leq 1} \mathbf{D}\mathbf{T}^{\text{int}} \right\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \left\| t^{-1/2-\gamma} t (t\mathbf{D})^{\leq 2} \mathbf{D}\mathbf{T}^{\text{int}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})}, \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{\leq 3, \gamma}^{\text{int}}[\mathbf{X}^{\text{int}}] &:= \left\| t^{3/2} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \\ &\quad + \left\| t (t\mathbf{D})^{\leq 1} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \left\| t^{-1/2-\gamma} (t\mathbf{D})^{\leq 2} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{\leq 3, \gamma}^{\text{int}}[\mathbf{S}^{\text{int}}] &:= \left\| t^{3/2} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \\ &\quad + \left\| t (t\mathbf{D})^{\leq 1} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \left\| t^{-1/2-\gamma} (t\mathbf{D})^{\leq 2} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \end{aligned}$$

$$\begin{aligned} \mathcal{O}_{\leq 3, \gamma}^{\text{int}}[\mathbf{K}^{\text{int}}] &:= \left\| t^{1/2} \left( (\mathbf{K}^{\text{int}})_\pi - 4t\mathbf{g} \right) \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \\ &\quad + \left\| (t\mathbf{D})^{\leq 1} \left( (\mathbf{K}^{\text{int}})_\pi - 4t\mathbf{g} \right) \right\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \left\| t^{-1/2-\gamma} t^{-1} (t\mathbf{D})^{\leq 2} \left( (\mathbf{K}^{\text{int}})_\pi - 4t\mathbf{g} \right) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{O}_{\leq 3, \gamma}^{\text{int}}[\mathbf{O}^{\text{int}}] &:= \left\| t^{3/2} (\mathbf{O}^{\text{int}})_\pi \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \\ &\quad + \left\| t^2 \mathbf{D}^2 \mathbf{O}^{\text{int}} \right\|_{L_t^\infty L^6(\Sigma_t)} \\ &\quad + \left\| t^{-1/2-\gamma} t (t\mathbf{D})^{\leq 1} \mathbf{D}^2 \mathbf{O}^{\text{int}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})}, \end{aligned}$$

for all  $\gamma > 0$ .

### 3.3 The Bootstrap Assumptions

In this section, we collect the *Bootstrap Assumptions* which are used throughout this paper. See Section 4.4 for the description of the associated bootstrap argument.

#### 3.3.1 The constants used in this paper

In this section, we recapitulate the constants used in this paper, and in particular the constants used in the following bootstrap assumptions. All these constants are *independent of the smallness parameter  $\varepsilon$* .

- ◇ The constant  $\gamma_0$  is used in the Bootstrap Assumptions of Sections 3.3.3 and is a fixed constant such that  $0 < \gamma_0 < 1/4$ . The bootstrap assumptions involving  $\gamma_0$  are improved for all  $\gamma > 0$ , and in particular for  $\gamma = \gamma_0$ .
- ◇ The constant  $C > 0$  is used in the mild Bootstrap Assumptions of Section 3.3.2 and is a fixed large constant.
- ◇ The constant  $D > 0$  is used in the strong Bootstrap Assumptions of Section 3.3.3 and is a fixed large constant.
- ◇ The constant  $0 < \wp < 1$  is used in the elliptic estimates of Lemma 7.5 is a small constant, which depends on  $C$ .
- ◇ The transition constant  $0 < \tau_0 < 1$  is used to determine the timelike transition hypersurface. It is a small constant, which depends on  $\wp$  (see Section 7.3). The transition parameter  $\tau$  is allowed to range in  $[\tau_0, (1 + \tau_0)/2]$ . This freedom is used in the mean value argument of Section 5.1.1.

**Remark 3.15.** *The bootstrap assumptions of Sections 3.3.2 and 3.3.3 will be assumed to hold uniformly for all transition parameter  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$ . See Section 4.4.*

#### 3.3.2 Mild bootstrap assumptions

**Bootstrap Assumption 3.16** (Mild bootstrap assumptions for the rotation vectorfield in  $\mathcal{M}^{\text{ext}}$ ). *Let  $C > 0$  be a (large) numerical constant. We assume that*

$$\|r^{-1}\mathbf{O}^{\text{ext}}\|_{L^\infty(\mathcal{M}^{\text{ext}})} + \|\nabla\mathbf{O}^{\text{ext}}\|_{L^\infty(\mathcal{M}^{\text{ext}})} \leq C.$$

*Moreover, we assume that for all  $S$ -tangent scalar  $f$  and for all 1-tensor or symmetric traceless 2-tensor  $F$ , the following bound holds*

$$\begin{aligned} \int_{S_{u,\underline{u}}} |(r\nabla)f|^2 &\leq C \sum_{\ell=1}^3 \int_{S_{u,\underline{u}}} \left| {}^{(\ell)}\mathbf{O}^{\text{ext}}(f) \right|^2, \\ \int_{S_{u,\underline{u}}} |(r\nabla)^{\leq 1}F|^2 &\leq C \sum_{\ell=1}^3 \int_{S_{u,\underline{u}}} \left| \hat{\mathcal{L}}^{(\ell)}\mathbf{O}^{\text{ext}}F \right|^2, \end{aligned}$$

*on all 2-sphere  $S_{u,\underline{u}} \subset \mathcal{M}^{\text{ext}}$ .*

**Bootstrap Assumption 3.17** (Mild bootstrap assumptions for the maximal hypersurfaces  $\Sigma_t \subset \mathcal{M}_{\text{bot}}^{\text{int}}$ ). *Let  $C > 0$  be a (large) numerical constant. We assume that on each separate maximal hypersurface  $\Sigma_t$ , for  $t^\circ \leq t \leq t^*$ , there exists (harmonic) global coordinates  $(x^i)$  such that*

$$\begin{aligned} \left\{ (x^1)^2 + (x^2)^2 + (x^3)^2 = \left( \frac{1-\tau}{1+\tau} \right)^2 t^2 \right\} &= \partial\Sigma_t, \\ \left\{ (x^1)^2 + (x^2)^2 + (x^3)^2 \leq \left( \frac{1-\tau}{1+\tau} \right)^2 t^2 \right\} &= \Sigma_t, \end{aligned}$$

*and such that we have the following uniform bounds for the metric  $g$  in these coordinates*

$$\begin{aligned} |g_{ij} - \delta_{ij}| &< 1/4, \\ |\partial_k g_{ij}| &\leq C, \end{aligned}$$

*on each separate maximal hypersurface  $\Sigma_t$ .*



**Bootstrap Assumption 3.18** (Mild bootstrap assumptions for the Killing fields in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ ). *Let  $C > 0$  be a (large) numerical constant. We assume that the norm of the vectorfields  $\mathbf{T}^{\text{int}}, \mathbf{S}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{O}^{\text{int}}$  satisfy the following mild bounds in  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$|\mathbf{X}^{\text{int}}| \leq Ct, \quad |\mathbf{S}^{\text{int}}| \leq Ct, \quad |\mathbf{K}^{\text{int}}| \leq Ct^2, \quad |\mathbf{O}^{\text{int}}| \leq Ct, \quad |\mathbf{DO}^{\text{int}}| \leq C, \quad (3.1)$$

where the norms are taken with respect to the maximal frame. We also assume that  $\mathbf{K}^{\text{int}}$  is a future-pointing timelike vectorfield in  $\mathcal{M}_{\text{bot}}^{\text{int}}$  and that we have the following mild bounds

$$\mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \leq -C^{-1}t^2, \quad |\mathbf{K}^{\text{int}} + \mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \mathbf{T}^{\text{int}}| \leq (1 - C^{-1}) |\mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}})|. \quad (3.2)$$

### 3.3.3 Strong bootstrap assumptions

Let  $0 < \gamma_0 < 1/4$  be a fixed numerical constant.<sup>2</sup> We have the following strong bootstrap assumptions in  $\mathcal{M}$ .

**Bootstrap Assumption 3.19** (Spacetime curvature in  $\underline{\mathcal{C}}^*$ ). *We assume that on  $\underline{\mathcal{C}}^*$*

$$\mathcal{R}_{\leq 2}^* + \mathfrak{R}_{\leq 1}^* + \overline{\mathfrak{R}}_{\leq 2}^* \leq D\varepsilon,$$

and we refer to Section 3.2.1 for definitions.

**Bootstrap Assumption 3.20** (Spacetime curvature in  $\mathcal{M}^{\text{ext}}$ ). *We assume that on  $\mathcal{M}^{\text{ext}}$*

$$\mathcal{R}_{\leq 2, \gamma_0}^{\text{ext}} + \mathfrak{R}_{\leq 1}^{\text{ext}} + \overline{\mathfrak{R}}_{\leq 2}^{\text{ext}} \leq D\varepsilon,$$

and we refer to Section 3.2.1 for definitions.

**Bootstrap Assumption 3.21** (Spacetime curvature in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ ). *We assume that on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$\mathcal{R}_{\leq 2}^{\text{int}} + \mathfrak{R}_{\leq 1}^{\text{int}} \leq D\varepsilon,$$

and we refer to Section 3.2.2 for definitions.

**Bootstrap Assumption 3.22** (Null connection in  $\underline{\mathcal{C}}^*$ ). *We assume that on  $\underline{\mathcal{C}}^*$ , we have*

$$\mathcal{O}_{\leq 3}^* + \mathfrak{D}_{\leq 2}^* + \overline{\mathfrak{D}}_{\leq 2} + \mathcal{O}_{\leq 3}^{*, \mathbf{O}} \leq D\varepsilon,$$

and we refer to Section 3.2.3 for definitions. Moreover, we assume that we have the following bootstrap bound for  $r$  on  $\underline{\mathcal{C}}^*$

$$\left| r - \frac{1}{2}(\underline{u}^* - u) \right| \leq D\varepsilon \underline{u}^{*-3} r^2 u^{-1}. \quad (3.3)$$

**Bootstrap Assumption 3.23** (Null connection in  $\mathcal{M}^{\text{ext}}$ ). *We assume that on  $\mathcal{M}^{\text{ext}}$ , we have*

$$\begin{aligned} \mathfrak{D}_{\leq 2}^{\text{ext}} + \mathcal{O}_{\leq 3, \gamma_0}^{\text{ext}} + \overline{\mathcal{O}}_{2, \gamma_0}^{\text{ext}} &\leq D\varepsilon, \\ \mathfrak{D}_{\leq 1}^{\text{ext}}[\eta] + \mathcal{O}_{2, \gamma_0}^{\text{ext}}[\eta] + \mathcal{O}_{\leq 2, \gamma_0}^{\mathcal{T}}[\eta] &\leq D\varepsilon, \\ \mathfrak{D}_{\leq 2}^{\text{ext}, \mathbf{O}} + \mathcal{O}_{\leq 3, \gamma_0}^{\text{ext}, \mathbf{O}} &\leq D\varepsilon, \end{aligned}$$

where we refer to Section 3.2.4 for definitions. Moreover, we assume that we have the following bootstrap bound for  $r$  on  $\mathcal{M}^{\text{ext}}$

$$\left| r - \frac{1}{2}(\underline{u} - u) \right| \leq D\varepsilon \underline{u}^{-1} u^{-1}. \quad (3.4)$$

**Bootstrap Assumption 3.24** (Maximal connection in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ ). *We assume that on  $\mathcal{M}_{\text{bot}}^{\text{int}}$ , we have*

$$\mathcal{O}_{\leq 3, \gamma_0}^{\text{int}}[n] + \mathcal{O}_{\leq 2}^{\text{int}}[k] + \mathfrak{D}_{\leq 2}^{\mathcal{T}}[\nu] + \mathcal{O}_{\leq 3}^{\Sigma_t^*}[k] \leq D\varepsilon,$$

and we refer to Section 3.2.5 for definitions.

<sup>2</sup>Fixing the constant  $\gamma_0$  breaks the scaling in the norms below. However, these norms are only used in the control of the nonlinear error terms of Section 5, which do not require a sharp control. Imposing  $\gamma_0 < 1/4$  is sufficient for this analysis. See Section 5.

**Bootstrap Assumption 3.25** (Spherical coordinates in  $\underline{\mathcal{C}}^*$ ). *We assume that there exists (two) spherical coordinate systems  $(u, \vartheta, \varphi)$  covering  $\underline{\mathcal{C}}^* \setminus \phi(\underline{u}^*)$  and ranging into*

$$[1, \underline{u}^*]_u \times [\pi/8, 7\pi/8]_{\vartheta} \times [0, 2\pi]_{\varphi},$$

where  $u$  is the canonical parameter on  $\underline{\mathcal{C}}^*$  as defined in Section 2.3. Moreover, we assume that for the induced metric on the 2-spheres  $S_{u, \underline{u}^*}$  in coordinates  $\vartheta, \varphi$ , we have the following bounds

$$|\partial_{\underline{a}}^{\leq 1} (r^{-2} \mathcal{g}_{bc} - (\mathcal{g}_{\mathbb{S}})_{bc})| \leq D\varepsilon \underline{u}^{*-1} u^{-1} q^{1/2},$$

where  $a, b, c \in \{\vartheta, \varphi\}$  and

$$\mathcal{g}_{\mathbb{S}} := d\vartheta^2 + \sin^2 \vartheta d\varphi^2.$$

**Bootstrap Assumption 3.26** (Spherical coordinates in  $\mathcal{M}^{\text{ext}}$ ). *We assume that there exists (two) spherical coordinate systems  $(u, \underline{u}, \vartheta, \varphi)$  covering  $\mathcal{M}^{\text{ext}}$  and ranging into*

$$[1, \tau \underline{u}^*]_u \times [\tau^{-1}, \underline{u}^*]_{\underline{u}} \times [\pi/8, 7\pi/8]_{\vartheta} \times [0, 2\pi]_{\varphi},$$

where  $u$  and  $\underline{u}$  are respectively the optical and affine parameter functions defined in Section 2, and which coincide with the spherical coordinates system  $(u, \vartheta, \varphi)$  on  $\underline{\mathcal{C}}^*$  from the Bootstrap Assumptions 3.25.<sup>3</sup> Moreover, we assume that for the induced metric on the 2-spheres  $S_{u, \underline{u}}$  in coordinates  $\vartheta, \varphi$ , we have the following bounds

$$|\partial_{\underline{a}}^{\leq 1} (r^{-2} \mathcal{g}_{bc} - (\mathcal{g}_{\mathbb{S}})_{bc})| \leq D\varepsilon \underline{u}^{-1} u^{-1/2},$$

where  $a, b, c \in \{\vartheta, \varphi\}$  and

$$\mathcal{g}_{\mathbb{S}} := d\vartheta^2 + \sin^2 \vartheta d\varphi^2.$$

**Bootstrap Assumption 3.27** (Harmonic coordinates in  $\Sigma_{t^*}$ ). *We assume that on  $\Sigma_{t^*}$ , we have*

$$\begin{aligned} \sum_{i,j=1}^3 \left\| t^{3/2} (g(\nabla x^i, \nabla x^j) - \delta_{ij}) \right\|_{L^\infty(\Sigma_{t^*})} + \sum_{i=1}^3 \left\| t(t\nabla)^{\leq 3} \nabla^2 x^i \right\|_{L^2(\Sigma_{t^*})} &\leq D\varepsilon, \\ \sum_{i=1}^3 \left\| t^{1/2} (r^* \bar{N}(x^i) - x^i) \right\|_{L^\infty(S^*)} &\leq D\varepsilon, \end{aligned}$$

where we refer to Section 2.4 for the definition of the harmonic coordinates  $x^i$ .

**Bootstrap Assumption 3.28** (Interior Killing vectorfields in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ ). *We assume that*

$$\mathcal{O}_{\leq 3, \gamma_0}^{\text{int}} [\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}}, \mathbf{S}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{O}^{\text{int}}] \leq D\varepsilon, \quad (3.5)$$

and we refer to Section 3.2.6 for definitions. We also assume that

$$\sup_{t^0 \leq t \leq t^*} t^{3/2} \left| \sup_{\Sigma_t} (t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) - \left( \frac{1-\tau}{1+\tau} \right)^2 \right| \leq D\varepsilon, \quad (3.6)$$

and

$$\sup_{t^0 \leq t \leq t^*} t^{3/2} |(t\mathbf{D})^{\leq 1} \mathbf{g}(t^{-1} \mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}})| \leq (D\varepsilon). \quad (3.7)$$

**Bootstrap Assumption 3.29** (Killing fields at  $\mathcal{T}$ ). *We assume that at the interface  $\mathcal{T}$  the following bootstrap bounds hold*

$$\begin{aligned} |\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}}| &\leq D\varepsilon t^{-3/2}, \\ |\mathbf{S}^{\text{ext}} - \mathbf{S}^{\text{int}}| &\leq D\varepsilon t^{-1/2}, \\ |\mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}| &\leq D\varepsilon t^{1/2}, \\ |\mathbf{O}^{\text{ext}} - \mathbf{O}^{\text{int}}| &\leq D\varepsilon t^{-1/2}. \end{aligned} \quad (3.8)$$

<sup>3</sup> The (two) spherical coordinates patch  $\vartheta, \varphi$  correspond to (two) different axis on the sphere. The choice of ranges for  $\vartheta, \varphi$  ensures that two such patches cover the full 2-spheres  $S_{u, \underline{u}}$ .

Moreover, we assume that for the first order derivatives, we have

$$\begin{aligned} \|t^{-\gamma_0} t^2 (t\mathbf{D})^{\leq 1} (\mathbf{DT}^{\text{ext}} - \mathbf{DT}^{\text{int}})\|_{L_t^\infty L^4(\partial\Sigma_t)} &\lesssim \varepsilon, \\ \|t^{-\gamma_0} t (t\mathbf{D})^{\leq 1} (t\mathbf{D})^{\leq 1} (\mathbf{DS}^{\text{ext}} - \mathbf{DS}^{\text{int}})\|_{L_t^\infty L^4(\partial\Sigma_t)} &\lesssim \varepsilon, \\ \|t^{-\gamma_0} t (t\mathbf{D})^{\leq 1} (\mathbf{DO}^{\text{ext}} - \mathbf{DO}^{\text{int}})\|_{L_t^\infty L^4(\partial\Sigma_t)} &\lesssim \varepsilon. \end{aligned} \quad (3.9)$$

**Bootstrap Assumption 3.30** (Last cones geodesic foliation). *We assume that  $\mathcal{M}' \subset \mathcal{M}$  and that in the exterior region  $\mathcal{M}' \cap \mathcal{M}^{\text{ext}}$ , the following bounds hold*

$$|u - u'| \leq D\varepsilon u'^{-1/2}, \quad |\underline{u} - \underline{u}'| \leq D\varepsilon \underline{u}',$$

and

$$|f'| \leq D\varepsilon \underline{u}'^{-3/2}, \quad |\underline{f}', \log \lambda'| \leq D\varepsilon u'^{-3/2},$$

where  $f', \underline{f}', \lambda'$  denote the transition coefficients between the null pairs  $(e_3, e_4)$  and  $(e_3', e_4')$ , i.e. the  $S_{u, \underline{u}}$ -tangent tensors such that<sup>4</sup>

$$\begin{aligned} e_4' &= \lambda' \left( e_4 + f' + \frac{1}{4} |f'|^2 e_3 \right), \\ e_3' &= (\lambda')^{-1} \left( \left( 1 + \frac{1}{2} f' \cdot \underline{f}' + \frac{1}{16} |f'|^2 |\underline{f}'|^2 \right) e_3 + \underline{f}' + \frac{1}{4} |f'|^2 f' + \frac{1}{4} |\underline{f}'|^2 e_4 \right). \end{aligned}$$

Moreover, we assume that in the interior region  $\mathcal{M}' \cap \mathcal{M}_{\text{bot}}^{\text{int}}$ , the following bounds hold

$$\left| t - \frac{1}{2}(u' + \underline{u}') \right| \leq D\varepsilon \underline{u}'^{-1/2},$$

and

$$\left| \mathbf{g} \left( \bar{T}, \frac{1}{2}(e_3' + e_4') \right) + 1 \right| \leq D\varepsilon \underline{u}'^{-3/2}.$$

**Bootstrap Assumption 3.31** (Bottom initial layer). *We assume that the spacetimes regions  $\mathcal{L}_{\text{bot}}^{\text{ext}}$  and  $\mathcal{L}_{\text{bot}}^{\text{int}}$  are not empty, and that in  $\mathcal{L}_{\text{bot}}^{\text{ext}}$  and  $\mathcal{L}_{\text{bot}}^{\text{int}}$ , the frames of the respective regions are comparable, i.e.*

$$\left| \mathbf{g} \left( \frac{1}{2}(e_3 + e_4), \bar{T}^{\text{bot}} \right) + 1 \right| \leq D\varepsilon, \quad \left| \mathbf{g}(\bar{T}, \bar{T}^{\text{bot}}) + 1 \right| \leq D\varepsilon. \quad (3.10)$$

We moreover assume that in the respective regions  $\mathcal{L}_{\text{bot}}^{\text{ext}}$  and  $\mathcal{L}_{\text{bot}}^{\text{int}}$  the following comparisons between time functions hold

$$\left| x^0 - \frac{1}{2}(u + \underline{u}) \right| \leq D\varepsilon, \quad |x^0 - t| \leq D\varepsilon. \quad (3.11)$$

**Bootstrap Assumption 3.32** (Conical initial layer). *We assume that the spacetime region  $\mathcal{L}_{\text{con}}^{\text{ext}}$  is not empty, and that in  $\mathcal{L}_{\text{con}}^{\text{ext}}$ , we have*

$$\begin{aligned} |u - \tilde{u}| &\leq D\varepsilon, \\ |\underline{u} - \underline{\tilde{u}}| &\leq D\varepsilon \underline{u}. \end{aligned} \quad (3.12)$$

Let  $(\tilde{\lambda}, \tilde{f}, \tilde{\underline{f}})$  be the transition coefficients associated to the null pairs  $(\tilde{e}_3, \tilde{e}_4)$  and  $(e_3, e_4)$ , i.e. the  $S_{u, \underline{u}}$ -tangent tensors such that

$$\begin{aligned} \tilde{e}_4 &= \tilde{\lambda} \left( e_4 + \tilde{f} + \frac{1}{4} |\tilde{f}|^2 \tilde{e}_3 \right), \\ \tilde{e}_3 &= (\tilde{\lambda})^{-1} \left( \left( 1 + \frac{1}{2} \tilde{f} \cdot \tilde{\underline{f}} + \frac{1}{16} |\tilde{f}|^2 |\tilde{\underline{f}}|^2 \right) e_3 + \tilde{\underline{f}} + \frac{1}{4} |\tilde{f}|^2 \tilde{f} + \frac{1}{4} |\tilde{\underline{f}}|^2 e_4 \right). \end{aligned}$$

We assume that in the region  $\mathcal{L}_{\text{con}}^{\text{ext}}$ , the following bounds hold

$$|\tilde{f}| \leq D\varepsilon \underline{u}^{-1}, \quad |\log \tilde{\lambda}| \leq D\varepsilon, \quad |\tilde{\underline{f}}| \leq D\varepsilon. \quad (3.13)$$

<sup>4</sup>To avoid confusion between the transition coefficients for the different change of frames in this paper, we renamed  $(f', \underline{f}', \lambda')$  the transition coefficients  $(f, \underline{f}, \lambda)$  of Section 2.15.

### 3.3.4 First consequences of the Bootstrap Assumptions

In this section, we collect lemmas which follow from the mild and strong Bootstrap Assumptions and which are used throughout Sections 5 – 12.

**Remark 3.33.** *Here and in the following, we write*

$$f_1 \lesssim f_2,$$

if there exists a constant  $C' > 0$ , independent of  $\varepsilon$ , such that

$$f_1 \leq C' f_2.$$

We write

$$f_1 \simeq f_2$$

if

$$f_1 \lesssim f_2 \quad \text{and} \quad f_2 \lesssim f_1.$$

**Lemma 3.34** (Coarea formulas). *In this paper, we use the following coarea formulas*

$$\int_{\mathcal{M}_{\text{bot}}^{\text{int}}} f \simeq \int_{t^\circ}^{t^*} \int_{\Sigma_t} f \, dt, \quad (3.14)$$

and

$$\int_{\mathcal{M}^{\text{ext}}} f \simeq \int_{t^\circ}^{t^*} \int_{\Sigma_t^{\text{ext}}} f \, dt, \quad \int_{\Sigma_t^{\text{ext}}} f \simeq \int_{2t/(1+\tau^{-1})}^{2t-\underline{u}^*} \int_{S_{u,\underline{u}=2t-u}} f \, du, \quad (3.15)$$

and

$$\int_{\mathcal{M}^{\text{ext}}} f \simeq \int_{\tau^{-1}}^{\underline{u}^*} \int_1^{\tau \underline{u}} \int_{S_{u,\underline{u}}} f \, d\underline{u} du = \int_1^{\tau \underline{u}^*} \int_{\tau^{-1}u}^{\underline{u}^*} \int_{S_{u,\underline{u}}} f \, d\underline{u} du =: \int_1^{\tau \underline{u}^*} \int_{\mathcal{C}_u} f \, du, \quad (3.16)$$

for all scalar function  $f$ .<sup>5</sup>

*Proof.* Formula (3.14) follows from the expression of  $\int_{\mathcal{M}_{\text{bot}}^{\text{int}}}$  in coordinates

$$\int_{\mathcal{M}_{\text{bot}}^{\text{int}}} f = \int_{t^\circ}^{t^*} \int_{\Sigma_t} f n \, dt$$

and the bound  $|n - 1| \leq D\varepsilon$  from the Bootstrap Assumptions 3.24. Formulas (3.15) follow similarly from the following control of the exterior time lapse  $n^{\text{ext}}$

$$|n^{\text{ext}} - 1| \lesssim |\eta| \lesssim D\varepsilon,$$

and from

$$\left| \overline{N}^{\text{ext}}(u) + 1 \right| \lesssim |\eta| \lesssim D\varepsilon$$

where we used relations (2.32) and (2.31) and the Bootstrap Assumptions 3.23 for  $\eta$ .

For the spherical coordinates from the Bootstrap Assumptions 3.26, we have the following expression of the coordinate vectorfields  $\partial_u, \partial_{\underline{u}}$

$$\partial_u = \frac{1}{2}e_3 - \frac{1}{4}\eta e_4 + b^a e_a, \quad \partial_{\underline{u}} = \frac{1}{2}e_4, \quad (3.17)$$

which is a consequence of relations (2.1). Thus, the spacetime metric  $\mathbf{g}$  writes in these coordinates

$$\mathbf{g} = -\frac{1}{2}dud\underline{u} - \frac{1}{2}d\underline{u}du + \left( \frac{1}{2}\eta + |b|^2 \right) du^2 + b^b du + du b^b + g$$

<sup>5</sup>The constants in the above estimates do not depend on  $f$ .

as an element of  $T^*\mathcal{M} \otimes T^*\mathcal{M}$ , and where  $\flat$  is the canonical musical isomorphism.

In  $\mathcal{M}^{\text{ext}}$ , using equation (3.17) and relations (2.33) we have the following formula

$$\nabla_4 b_a = b^c \chi_{ca} + 2\zeta_a.$$

From a straight-forward Grönwall argument, using that  $b = 0$  on  $\underline{\mathcal{C}}^*$  and the Bootstrap Assumptions 3.23 for  $\chi$  and  $\zeta$  we obtain

$$|b| \lesssim D\varepsilon \underline{u}^{-1} u^{-1/2} \quad (3.18)$$

in  $\mathcal{M}^{\text{ext}}$ .

Expressing  $\int_{\mathcal{M}^{\text{ext}}}$  in coordinates, using the Bootstrap Assumptions 3.23 for  $\eta$  and estimate (3.18) for  $b$  then yields the desired formulas (3.16).  $\square$

We have the following lemma, which follows from the Bootstrap Assumptions 3.25 and 3.26 and the results of [Sha14].<sup>6</sup>

**Lemma 3.35** (Coordinates fractional Sobolev spaces). *For all 2-spheres  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$  and all 2-spheres  $S_{u,\underline{u}} \subset \mathcal{M}^{\text{ext}}$ , and for all  $S$ -tangent  $k$ -tensor  $F$ , we have*

$$r^{-1/2} \|F\|_{\tilde{H}^{1/2}(S_{u,\underline{u}})} \simeq \sum_{(\vartheta,\varphi)} \sum_{a_1 \dots a_k \in \{\vartheta,\varphi\}} \|r^{-k} F_{a_1 \dots a_k}\|_{H_{\vartheta,\varphi}^{1/2}},$$

where the sum is taken over the (two) spherical coordinates systems covering  $S_{u,\underline{u}}$  given by the Bootstrap Assumptions 3.25 and 3.26, and where  $H_{\vartheta,\varphi}^{1/2}$  denotes the standard fractional Sobolev space on  $[\pi/8, 7\pi/8]_{\vartheta} \times [0, 2\pi]_{\varphi}$ .

We have the following Sobolev estimates.

**Lemma 3.36** (Sobolev estimates on 2-spheres). *For all 2-spheres  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$  and all 2-spheres  $S_{u,\underline{u}} \subset \mathcal{M}^{\text{ext}}$ , we have*

$$\begin{aligned} \|F\|_{L^4(S_{u,\underline{u}})} &\lesssim \|F\|_{\tilde{H}^{1/2}(S_{u,\underline{u}})}, \\ \|F\|_{L^\infty(S_{u,\underline{u}})} &\lesssim r^{-1/2} \|(r\nabla)^{\leq 1} F\|_{\tilde{H}^{1/2}(S_{u,\underline{u}})}, \end{aligned}$$

for all  $S$ -tangent tensor  $F$ . Moreover, we have

$$r^{-1/2} \|F\|_{L^2(S_{u,\underline{u}})} \lesssim \|F\|_{\tilde{H}^{1/2}(S_{u,\underline{u}})} \lesssim r^{-1/2} \|(r\nabla)^{\leq 1} F\|_{L^2(S_{u,\underline{u}})}.$$

*Proof.* From the Bootstrap Assumptions 3.25 and 3.26, the 2-spheres  $S_{u,\underline{u}}$  admit weakly regular coordinates systems in the sense of [Sha14]. The proof of the lemma then follows from the results of [Sha14].<sup>7</sup>  $\square$

We have the following product estimates for  $\tilde{H}^{1/2}$  norms (see [Sha14, Corollary 3.4]).

**Lemma 3.37** (Product estimates). *For all 2-spheres  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$  and all 2-spheres  $S_{u,\underline{u}} \subset \mathcal{M}^{\text{ext}}$ , we have*

$$\begin{aligned} \|FG\|_{\tilde{H}^{1/2}(S)} &\lesssim \left( \|F\|_{L^\infty(S)} + r^{-1} \|(r\nabla)F\|_{L^2(S)} \right) \|G\|_{\tilde{H}^{1/2}(S)}, \\ \|FG\|_{L^2(S)} &\lesssim \|F\|_{\tilde{H}^{1/2}(S)} \|G\|_{\tilde{H}^{1/2}(S)}, \end{aligned}$$

for all  $S$ -tangent tensors  $F, G$ .

We have the following elliptic estimates on 2-spheres.

<sup>6</sup>The strong sup-norm bounds from the Bootstrap Assumptions 3.25 and 3.26 imply that the assumptions of [Sha14] are satisfied and that we can build the orthonormal frame of [Sha14] upon the coordinates vectorfields.

<sup>7</sup>It can also be obtained directly from Lemma 3.35, consistently with [Sha14].

**Lemma 3.38** (Elliptic estimates on 2-spheres). *For all 2-spheres  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$  and all 2-spheres  $S_{u,\underline{u}} \subset \mathcal{M}^{\text{ext}}$  and for all  $S$ -tangent tensor  $U$  of appropriate type, we have*

$$\begin{aligned} \|r^{-1}(r\nabla)^{\leq 1}U\| &\lesssim \|\mathcal{D}_1U\|, \\ \|r^{-1}(r\nabla)^{\leq 1}(U - \bar{U})\| &\lesssim \|\mathcal{D}_1^*U\|, \\ \|r^{-1}(r\nabla)^{\leq 1}U\| &\lesssim \|\mathcal{D}_2U\|, \end{aligned} \tag{3.19}$$

where the norms are either  $L^2(S)$  or  $\tilde{H}^{1/2}(S)$  and where  $\mathcal{D}$  correspond to the classical Hodge-type elliptic operators (see [CK93]).<sup>8</sup>

Moreover, for all  $S$ -tangent 1-tensor  $U$  satisfying<sup>9</sup>

$$\mathcal{D}_1U = (\text{div} X, \text{curl} Y),$$

we have

$$\|U\|_{L^2(S)} \lesssim \|X\|_{L^2(S)} + \|Y\|_{L^2(S)}. \tag{3.20}$$

*Proof.* From the Bootstrap Assumptions 3.19, 3.20, 3.22, 3.23 and Gauss equation (2.34t), we have

$$\left|K - \frac{1}{r^2}\right| \lesssim D\varepsilon \underline{u}^{*-2} r^{-1} q^{-1/2}, \quad \left|K - \frac{1}{r^2}\right| \lesssim D\varepsilon \underline{u}^{-3} u^{-1/2}, \tag{3.21}$$

in  $\underline{\mathcal{C}}^*$  and  $\mathcal{M}^{\text{ext}}$  respectively. Since  $K(r^{-2}\mathcal{g}) = r^2K(\mathcal{g})$ , we deduce from (3.21) that  $(S, r^{-2}\mathcal{g})$  is weakly spherical in the sense of [Sha14]. Thus, the elliptic estimates (3.19) for  $L^2(S)$  and  $\tilde{H}^{1/2}(S)$  norms follow from [Sha14] and rescaling in  $r$ . The proof of the additional estimate (3.20) follows from writing  $U = \mathcal{D}_1^*(f, g)$  with  $\bar{f} = \bar{g} = 0$ , and standard energy estimates for Poisson equation.  $\square$

<sup>8</sup>Poincaré inequality is a consequence of the estimate for the elliptic operator  $\mathcal{D}_1^*$ .

<sup>9</sup>Such system of equations are consequences of the canonical foliation choice. See Section 9.



## 4. Main results

### 4.1 Initial layers $\varepsilon$ -close to Minkowski space

Let  $(\widetilde{\mathcal{M}}, \mathbf{g})$  be a smooth vacuum spacetime, and let  $(\widetilde{\Sigma}_1, \widetilde{\mathcal{C}}_0)$  be spacelike-characteristic initial data for  $\widetilde{\mathcal{M}}$ . We say that  $\widetilde{\mathcal{M}}$  contains a *bottom initial layer*  $\mathcal{L}_{\text{bot}}$  and a *conical initial layer*  $\mathcal{L}_{\text{con}}$  which are  $\varepsilon$ -close to Minkowski space, if it admits spacetime subregions  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$  as defined in Section 2.14 and if the following smallness assumptions holds on  $\mathcal{L}_{\text{bot}}, \mathcal{L}_{\text{con}}$ .

**The bottom initial layer  $\mathcal{L}_{\text{bot}}$**  With respect to the bottom initial layer coordinates  $(x^\mu)$  on  $\mathcal{L}_{\text{bot}}$  defined in Section 2.14, we have the following spacetime closeness requirement to the Minkowski metric  $\boldsymbol{\eta}$

$$\begin{aligned} \sum_{\mu, \nu=0}^3 \|\mathbf{g}_{\mu\nu} - \boldsymbol{\eta}_{\mu\nu}\|_{L^\infty(\mathcal{L}_{\text{bot}})} &\leq \varepsilon, \\ \sum_{\mu, \nu=0}^3 \|\partial^{(\leq 3)}(\mathbf{g}_{\mu\nu} - \boldsymbol{\eta}_{\mu\nu})\|_{L^2(\mathcal{L}_{\text{bot}})} &\leq \varepsilon, \end{aligned} \tag{4.1a}$$

and for all  $1 \leq x^0 \leq 5$ ,

$$\int_{\widetilde{\Sigma}_{x^0}} |\mathbf{D}^{\leq 2} \mathbf{R}|^2 \leq \varepsilon^2, \tag{4.1b}$$

where in the last estimate the norm is taken with respect to the frame associated to the bottom initial layer coordinates.<sup>1</sup>

**Remark 4.1.** The centre  $\varphi(1)$  of  $\widetilde{\Sigma}_1$  is chosen with respect to bottom initial layer coordinates, i.e. such that

$$x^i(\varphi(1)) = 0,$$

for all  $1 \leq i \leq 3$ .

The closeness to Minkowski assumptions (4.1) imply that

$$|x^i(\varphi(t))| \lesssim \varepsilon, \quad |x^0(\varphi(t)) - t| \lesssim \varepsilon,$$

for all  $1 \leq i \leq 3$ , and

$$|\dot{\varphi}(t) - \partial_{x^0}| \lesssim \varepsilon,$$

for all  $t$  such that  $\varphi(t) \in \mathcal{L}_{\text{bot}}$ .

**The conical initial layer  $\mathcal{L}_{\text{con}}$**  With respect to the initial layer optical functions on  $\mathcal{L}_{\text{con}}$  and the associated null pair (see Section 2.14), we have the following closeness requirement to Minkowski spacetime

- ◊ We have the following bounds on the area radius and the null lapse in  $\mathcal{L}_{\text{con}}$

$$\begin{aligned} \left| \widetilde{r}(\underline{u}, \underline{u}) - \frac{1}{2}(\underline{u} - \underline{u}) \right| &\leq \varepsilon \widetilde{r}(\underline{u}, \underline{u}), \\ |\widetilde{\Omega} - 1| &\leq \varepsilon \underline{u}^{-1}, \end{aligned} \tag{4.2a}$$

- ◊ We have the following sup-norm bounds on the null connection coefficients in  $\mathcal{L}_{\text{con}}$

$$\begin{aligned} \left| \text{tr} \widetilde{\chi} - \frac{2}{\widetilde{r}} \right| &\leq \varepsilon \underline{u}^{-2}, \quad \left| \widehat{\chi} \right| \leq \varepsilon \underline{u}^{-2}, \quad |\widetilde{\eta}| \leq \varepsilon \underline{u}^{-2}, \quad |\widetilde{\omega}| \leq \varepsilon \underline{u}^{-2}, \quad |\underline{\omega}| \leq \varepsilon \underline{u}^{-1}, \\ \left| \text{tr} \widetilde{\underline{\chi}} + \frac{2}{\widetilde{r}} \right| &\leq \varepsilon \underline{u}^{-2}, \quad \left| \widehat{\underline{\chi}} \right| \leq \varepsilon \underline{u}^{-1}, \quad \left| \widetilde{\zeta} \right| \leq \varepsilon \underline{u}^{-2}, \quad |\widetilde{\eta}| \leq \varepsilon \underline{u}^{-2}, \end{aligned} \tag{4.2b}$$

together with  $\widetilde{\xi} = \widetilde{\xi} = 0$ .

<sup>1</sup>With the notations of Definition 3.1, this frame is defined such that  $e_0 = \overline{T}^{\text{bot}}$ .



◇ We have the following  $L^2(\mathcal{L}_{\text{con}})$  bounds for (derivatives of) the null connection coefficients

$$\begin{aligned}
 & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \left( \text{tr} \widetilde{\chi} - \frac{2}{\underline{\tilde{r}}} \right) \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \widehat{\chi} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, \\
 & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \left( \text{tr} \widetilde{\chi} + \frac{2}{\underline{\tilde{r}}} \right) \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \widehat{\chi} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, \\
 & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \underline{\tilde{\omega}} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \underline{\tilde{\omega}} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, \\
 & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \underline{\tilde{\eta}} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \underline{\tilde{\eta}} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon, \\
 & \left\| \underline{\tilde{u}}^{-1/2-\gamma} \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \underline{\tilde{\zeta}} \right\|_{L^2(\mathcal{L}_{\text{con}})} \leq \gamma \varepsilon,
 \end{aligned} \tag{4.2c}$$

where  $\widetilde{\nabla} \in \left\{ \tilde{r} \widetilde{\nabla}, \tilde{r} \widetilde{\nabla}_4, \widetilde{\nabla}_3 \right\}$  and for all  $\gamma > 0$ .

◇ We have the following curvature flux bounds on the cones  $\widetilde{\mathcal{C}}_{\underline{\tilde{u}}}$ , for all  $0 \leq \underline{\tilde{u}} \leq 3/2$

$$\begin{aligned}
 & \int_3^\infty \int_{\widetilde{S}_{\underline{\tilde{u}}, \underline{\tilde{u}}}} \left( \left| \widetilde{\nabla}^{\leq 2} \underline{\tilde{\beta}} \right|^2 + \left| \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} (\underline{\tilde{\rho}} - \bar{\underline{\rho}}) \right|^2 + \left| \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} (\underline{\tilde{\sigma}} - \bar{\underline{\sigma}}) \right|^2 \right. \\
 & \quad \left. + \left| \underline{\tilde{u}}^2 \widetilde{\nabla}^{\leq 2} \underline{\tilde{\beta}} \right|^2 + \left| \underline{\tilde{u}}^2 \widetilde{\nabla}^{\leq 2} \underline{\tilde{\alpha}} \right|^2 \right) d\underline{\tilde{u}} \leq \varepsilon^2,
 \end{aligned} \tag{4.2d}$$

the following curvature flux bounds for all  $3 \leq \underline{\tilde{u}} < +\infty$

$$\begin{aligned}
 & \int_0^{3/2} \int_{\widetilde{S}_{\underline{\tilde{u}}, \underline{\tilde{u}}}} \left( \left| \widetilde{\nabla}^{\leq 2} \underline{\tilde{\alpha}} \right|^2 + \left| \underline{\tilde{u}} \widetilde{\nabla}^{\leq 2} \underline{\tilde{\beta}} \right|^2 + \left| \underline{\tilde{u}}^2 \widetilde{\nabla}^{\leq 2} (\underline{\tilde{\rho}} - \bar{\underline{\rho}}) \right|^2 \right. \\
 & \quad \left. + \left| \underline{\tilde{u}}^2 \widetilde{\nabla}^{\leq 2} (\underline{\tilde{\sigma}} - \bar{\underline{\sigma}}) \right|^2 + \left| \underline{\tilde{u}}^2 \widetilde{\nabla}^{\leq 2} \underline{\tilde{\beta}} \right|^2 \right) d\underline{\tilde{u}} \leq \varepsilon^2,
 \end{aligned} \tag{4.2e}$$

and the following sup-norm estimates in  $\mathcal{L}_{\text{con}}$  for the averages  $\bar{\underline{\rho}}$  and  $\bar{\underline{\sigma}}$

$$\left| \widetilde{\nabla}^{\leq 2} \bar{\underline{\rho}} \right| \leq \varepsilon \underline{\tilde{u}}^{-3}, \quad \left| \widetilde{\nabla}^{\leq 2} \bar{\underline{\sigma}} \right| \leq \varepsilon \underline{\tilde{u}}^{-3}, \tag{4.2f}$$

where  $\widetilde{\nabla} \in \left\{ \tilde{r} \widetilde{\nabla}, \tilde{r} \widetilde{\nabla}_4, \widetilde{\nabla}_3 \right\}$ .

**The bottom and conical initial layer intersection**  $\mathcal{L}_{\text{bot}} \cap \mathcal{L}_{\text{con}}$  We assume that on  $\mathcal{L}_{\text{bot}} \cap \mathcal{L}_{\text{con}}$  we have the following frame comparison

$$\begin{aligned}
 & \left| \mathbf{g} \left( \overline{T}^{\text{bot}}, \frac{1}{2}(\tilde{e}_3 + \tilde{e}_4) \right) + 1 \right| \leq \varepsilon, \\
 & \left| \mathbf{g} \left( \overline{N}^{\text{bot}}, \frac{1}{2}(\tilde{e}_4 - \tilde{e}_3) \right) - 1 \right| \lesssim \varepsilon,
 \end{aligned} \tag{4.3a}$$

and the following coordinates comparisons

$$\left| x^0 - \frac{1}{2}(\underline{\tilde{u}} + \tilde{u}) \right| \leq \varepsilon, \quad \left| \sqrt{\sum_{i=1}^3 (x^i)^2} - \frac{1}{2}(\underline{\tilde{u}} - \tilde{u}) \right| \leq \varepsilon. \tag{4.3b}$$

## 4.2 Main theorem

The following theorem is the main result of this paper.

**Theorem 4.2** (Main theorem, version 2). *Let  $(\widetilde{\mathcal{M}}, \mathbf{g})$  be a smooth vacuum spacetime, and let  $(\widetilde{\Sigma}_1, \widetilde{\mathcal{C}}_0)$  be smooth spacelike-characteristic initial data for  $\widetilde{\mathcal{M}}$ . Assume that  $\widetilde{\mathcal{M}}$  contains a bottom and conical initial layer  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$  adapted to  $(\widetilde{\Sigma}_1, \widetilde{\mathcal{C}}_0)$  which are  $\varepsilon$ -close to Minkowski space. There exists  $\varepsilon_0 > 0$  such that if  $\varepsilon < \varepsilon_0$ , the following holds.*

- $\diamond$  The spacetime  $(\widetilde{\mathcal{M}}, \mathbf{g})$  is future geodesically complete.  
 $\diamond$  There exists two spacetime regions  $\mathcal{M}_\infty^{\text{int}}, \mathcal{M}_\infty^{\text{ext}} \subset \widetilde{\mathcal{M}}$  such that

$$\widetilde{\mathcal{M}} = \mathcal{M}_\infty^{\text{int}} \cup \mathcal{M}_\infty^{\text{ext}} \cup \mathcal{L}_{\text{bot}} \cup \mathcal{L}_{\text{con}},$$

and such that

- $\mathcal{M}_\infty^{\text{int}}$  is foliated by spacelike maximal hypersurfaces  $\Sigma_t$  which are the level sets of a global time function  $t$  on  $\mathcal{M}_\infty^{\text{int}}$  ranging from  $t^\circ = (1 + \tau^{-1})/2$  to  $+\infty$ , where  $0 < \tau < 1$  is a fixed parameter, sufficiently close to 1.
- $\mathcal{M}_\infty^{\text{ext}}$  is foliated by outgoing null hypersurfaces  $\mathcal{C}_u$  which are the level sets of a global optical function  $u$  on  $\mathcal{M}_\infty^{\text{ext}}$  ranging from 1 to  $+\infty$ .
- There exists a global affine parameter  $\underline{u}$  on  $\mathcal{M}_\infty^{\text{ext}}$  foliating the null hypersurfaces  $\mathcal{C}_u$  ranging from  $\tau^{-1}$  to  $+\infty$ .
- The transition hypersurface  $\mathcal{T}$  satisfies

$$\mathcal{T} := \mathcal{M}_\infty^{\text{int}} \cap \mathcal{M}_\infty^{\text{ext}} = \{u = \tau \underline{u}\},$$

and on  $\mathcal{T}$ , we have

$$t = \frac{1}{2}(u + \underline{u}).$$

- $\diamond$  In  $\mathcal{M}_\infty^{\text{int}}$ , the following curvature decay holds<sup>2</sup>

$$|\mathbf{R}| \lesssim \varepsilon t^{-7/2}, \quad (4.4a)$$

we have the following control of the time function  $t$

$$|\mathbf{g}(\mathbf{D}t, \mathbf{D}t) + 1| \lesssim \varepsilon t^{-3/2}, \quad |\mathbf{D}^2 t| \lesssim \varepsilon t^{-5/2}, \quad (4.4b)$$

and the maximal hypersurfaces  $\Sigma_t$  approach the Euclidean disks in the following (intrinsic) sense<sup>3</sup>

$$|\text{Ric}| \lesssim \varepsilon t^{-7/2}, \quad \left| \text{tr}\theta - \frac{2}{t} \left( \frac{1+\tau}{1-\tau} \right) \right| \lesssim \varepsilon t^{-5/2}, \quad |\hat{\theta}| \lesssim \varepsilon t^{-5/2}, \quad (4.4c)$$

where  $\text{Ric}$  is the Ricci curvature tensor of  $\Sigma_t$  and  $\theta$  is the second fundamental form of the boundaries  $\partial\Sigma_t \subset \mathcal{T}$ .

- $\diamond$  In  $\mathcal{M}_\infty^{\text{ext}}$ , the following curvature decay holds

$$\begin{aligned} |\alpha| &\lesssim \varepsilon \underline{u}^{-7/2}, & |\beta| &\lesssim \varepsilon \underline{u}^{-7/2}, & |\rho - \bar{\rho}| &\lesssim \varepsilon \underline{u}^{-3} \underline{u}^{-1/2}, \\ |\underline{\alpha}| &\lesssim \varepsilon \underline{u}^{-1} \underline{u}^{-5/2}, & |\underline{\beta}| &\lesssim \varepsilon \underline{u}^{-2} \underline{u}^{-5/2}, & |\sigma - \bar{\sigma}| &\lesssim \varepsilon \underline{u}^{-3} \underline{u}^{-1/2}, \end{aligned} \quad (4.4d)$$

as well as

$$|\bar{\rho}| \lesssim \varepsilon^2 \underline{u}^{-3} \underline{u}^{-2}, \quad |\bar{\sigma}| \lesssim \varepsilon^2 \underline{u}^{-3} \underline{u}^{-2}, \quad (4.4e)$$

and we have the following control of the optical function and affine parameter  $u$  and  $\underline{u}$

$$\mathbf{g}(\mathbf{D}u, \mathbf{D}u) = 0, \quad \mathbf{g}(\mathbf{D}\underline{u}, \mathbf{D}\underline{u}) = -2, \quad |\mathbf{g}(\mathbf{D}u, \mathbf{D}\underline{u})| \lesssim \varepsilon \underline{u}^{-3/2}, \quad (4.4f)$$

and

$$\begin{aligned} \left| \text{tr}\chi - \frac{2}{r} \right| &\lesssim \varepsilon \underline{u}^{-2} \underline{u}^{-1/2}, & |\hat{\chi}| &\lesssim \varepsilon \underline{u}^{-2} \underline{u}^{-1/2}, & |\zeta| &\lesssim \varepsilon \underline{u}^{-2} \underline{u}^{-1/2}, & |\underline{\omega}| &\lesssim \varepsilon \underline{u}^{-1} \underline{u}^{-3/2}, \\ \left| \text{tr}\underline{\chi} + \frac{2}{r} \right| &\lesssim \varepsilon \underline{u}^{-2} \underline{u}^{-1/2}, & |\hat{\underline{\chi}}| &\lesssim \varepsilon \underline{u}^{-1} \underline{u}^{-3/2}, & |\underline{\xi}| &\lesssim \varepsilon \underline{u}^{-1} \underline{u}^{-3/2}, \end{aligned} \quad (4.4g)$$

<sup>2</sup> The norms are taken with respect to the frame associated to the spacelike hypersurfaces  $\Sigma_t$ .

<sup>3</sup> Using global harmonic coordinates (see Theorem 4.3), this can be alternatively formulated using coordinates.

together with  $\omega = |\xi| = 0$  and  $\zeta = \eta = -\underline{\eta}$ .

Moreover, the 2-spheres  $S_{u, \underline{u}}$  level sets of  $u, \underline{u}$  approach the Euclidean 2-spheres in the following (intrinsic) sense

$$\left| K - \frac{1}{r^2} \right| \lesssim \varepsilon \underline{u}^{-3} u^{-1/2}, \quad \left| r - \frac{1}{2}(\underline{u} - u) \right| \lesssim \varepsilon \underline{u}^{-1} u^{-1}, \quad (4.4h)$$

where  $r$  denotes the area radius of  $S_{u, \underline{u}}$ .

- ◇ The spacetime  $\widetilde{\mathcal{M}}$  admits a future timelike and future null infinity  $i^+$  and  $\mathcal{I}^+$ , and the future null infinity  $\mathcal{I}^+$  is future geodesically complete.<sup>4</sup> Moreover,  $\widetilde{\mathcal{M}}$  admits well-defined Bondi mass and angular momentum at null infinity, which satisfy respectively a Bondi mass loss formula and an angular momentum evolution equation on  $\mathcal{I}^+$ , and which tend to 0 at timelike infinity  $i^+$ .<sup>5</sup>

### Remarks on Theorem 4.2

- 4.2a Alternatively, the spacetime region  $\mathcal{M}_\infty = \mathcal{M}_\infty^{\text{int}} \cup \mathcal{M}_\infty^{\text{ext}}$  can also be foliated by the 2-spheres  $S'_{u', \underline{u}'}$  of the geodesic foliation on the incoming null cones backward emanating from the central axis  $\emptyset$  (see the definitions of the optical function  $\underline{u}'$  and the geodesic parameter  $u'$  of the so-called last cones geodesic foliation in Section 2.13). Analogous decay estimates to (4.4) can be obtained in interior and exterior regions with respect to the null frame adapted to  $u'$  and  $\underline{u}'$ .  
Since the classical definition of future null infinity and associated asymptotic quantities involve taking limits along the outgoing null cones, and also since the proof of Theorem 4.2 foremost relies on the outgoing null cones level sets of  $u$ , we preferred to state Theorem 4.2 using the time, optical and geodesic affine parameter functions  $t, u$  and  $\underline{u}$ .
- 4.2b More specific  $L^p$ -based decay statements, or boundedness statements for  $L^2$ -fluxes can be obtained for derivatives of the curvature and connection coefficients.
- 4.2c The decay rates of (4.4) match the decay rates obtained in [CK93, KN03]. A notable exception to that statement is the strong decay rate (4.4e) for the mean value  $\bar{\rho}$ . This is due to the two different spacetime regions studied in this paper and in [CK93, KN03]. In [CK93] and [KN03], the mean value  $\bar{\rho}$  satisfies a weak decay rate of the type  $|\bar{\rho}| \lesssim \varepsilon \underline{u}^{-3}$ . Namely,  $\bar{\rho}$  is not controlled by energy estimates and is only determined by integrating Bianchi equation along  $e_3$  from the initial spacelike hypersurface into the spacetime. Since on the initial spacelike hypersurface  $\bar{\rho}$  is related to the ADM mass  $M$  via  $\bar{\rho} \sim -2M/r^3$ , the decay rate for  $\bar{\rho}$  obtained by integration is at most  $\bar{\rho} \sim \underline{u}^{-3}$ . In the present paper, we determine  $\bar{\rho}$  by integrating Bianchi equation from the central axis  $\emptyset$  backwards in  $\mathcal{M}$ . Since  $r^3 \bar{\rho} \rightarrow 0$  when  $r \rightarrow 0$ , the initial value of  $\bar{\rho}$  on  $\emptyset$  is virtually 0, thus its decay rate in  $\mathcal{M}$  is only dictated by the nonlinear terms in Bianchi equation (2.85), from which we deduce the strong decay rate  $\bar{\rho} \sim \underline{u}^{-3} u^{-2}$ . This corresponds to obtaining bounds for the Bondi mass in  $\mathcal{I}^+$  integrating Bondi mass loss formula from timelike infinity  $i^+$  backwards on  $\mathcal{I}^+$ , provided that it is known that the final Bondi mass vanishes at  $i^+$ .<sup>6</sup>

## 4.3 Auxiliary theorems

In this section, we state auxiliary results to the main theorem (Theorems 4.3, 4.4 and 4.5), which are independent and of more general interest. Their respective proofs are given in Appendix A, B and C. We also state an existence of initial layers theorem (see Theorem 4.6), which we claim can be obtained from previous literature results.

### 4.3.1 Global harmonic coordinates

**Theorem 4.3** (Global harmonic coordinates on  $\Sigma$ ). *Let  $\Sigma$  be a Riemannian manifold diffeomorphic to the unit coordinate disk  $\mathbb{D}$  of  $\mathbb{R}^3$ . Let  $\varepsilon > 0$  and assume that on  $\Sigma$  the following  $L^2$  bounds for the Ricci curvature*

<sup>4</sup>See [Wal84] and also [CK93], [KN03] for definitions and discussions.

<sup>5</sup>See Section 4.4.3 for precise definitions.

<sup>6</sup>All the statements for  $\bar{\rho}$  can also be formulated in terms of the Hawking mass/average of mass aspect function.

tensor of  $\Sigma$  and the second fundamental form  $\theta$  of the boundary  $\partial\Sigma$  hold<sup>7</sup>

$$\begin{aligned} \|\text{Ric}\|_{L^2(\Sigma)} &\leq \varepsilon, \\ \|\text{tr}\theta - 2\|_{H^{1/2}(\partial\Sigma)} + \|\hat{\theta}\|_{H^{1/2}(\partial\Sigma)} &\leq \varepsilon. \end{aligned} \quad (4.5)$$

Assume moreover that a set of Sobolev embeddings, Poincaré inequality and trace estimate hold with constant  $C > 0$  on  $\Sigma$  (see the exact functional assumptions in Section A.1). Then, there exists  $\varepsilon_0(C) > 0$  such that if  $\varepsilon < \varepsilon_0$ , there exists global harmonic coordinates  $(x^i)$  from  $\Sigma$  onto  $\mathbb{D}$  with the following bounds

$$\sum_{i,j=1}^3 \|\mathbf{g}(\nabla x^i, \nabla x^j) - \delta_{ij}\|_{L^\infty(\Sigma)} + \sum_{i=1}^3 \|\nabla^{\leq 1} \nabla^2 x^i\|_{L^2(\Sigma)} + \sum_{i=1}^3 \|N(x^i) - x^i\|_{L^\infty(\partial\Sigma)} \lesssim \varepsilon, \quad (4.6)$$

where  $N$  denotes the outward-pointing unit normal to  $\partial\Sigma$ . Moreover, for all  $k \geq 0$ , we have the following higher regularity estimates

$$\sum_{i=1}^3 \|\nabla^k \nabla^3 x^i\|_{L^2(\Sigma)} \leq C_k \left( \|\nabla^{\leq k} \text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{\leq k}(\theta - \mathcal{J})\|_{H^{1/2}(\partial\Sigma)} + \varepsilon \right). \quad (4.7)$$

### Remarks on Theorem 4.3

4.3a For the metric components  $g_{ij}$  in the harmonic coordinate  $(x^i)$ , we deduce from (4.6) and (4.7) the following respective  $L^2$  estimates<sup>8</sup>

$$\begin{aligned} \|\partial^{\leq 2}(g_{ij} - \delta_{ij})\|_{L^2(\Sigma)} &\lesssim \varepsilon, \\ \|\partial^{\leq k+2}(g_{ij} - \delta_{ij})\|_{L^2(\Sigma)} &\lesssim C_k \left( \|\nabla^{\leq k} \text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{\leq k}(\theta - \mathcal{J})\|_{H^{1/2}(\partial\Sigma)} + \varepsilon \right). \end{aligned}$$

4.3b Theorem 4.3 can be cast as an existence and control of solutions to the *Dirichlet problem for harmonic maps* result with the Euclidean unit disk  $\mathbb{D}$  of  $\mathbb{R}^3$  as target manifold.

4.3c Theorem 4.3 improves on the results of [CG19b, Section 7] since it only uses elementary (energy and Bochner) estimates and provides optimal quantitative bounds for the metric components.

4.3d In the context of the present paper, we use estimate (4.6) and estimate (4.7) with  $k = 2$ .

4.3e The functional hypothesis which are collected in Section A.1 are at a weak regularity level, and are in particular satisfied if there exists weakly regular coordinates on  $\Sigma$ . These coordinates do not need to be harmonic. In the present paper, the induced coordinates obtained when applying the existence of maximal hypersurfaces result of [Cho76] are such coordinates.

4.3f Using the harmonic coordinates of Theorem 4.3 and standard analysis results, one can directly improve on the constants in the assumed functional estimates of Section A.1 for  $\varepsilon > 0$  sufficiently small. This suggests that the (already weak) functional assumptions of Theorem 4.3 could be removed.

4.3g The crux of the proof of Theorem 4.3 is the obtention of a Bochner identity for the Dirichlet problem on  $\Sigma$  with coercive boundary terms directly controlling Neumann data. See equation (A.12).

### 4.3.2 Axis limits

The following theorem provides limits for (all derivatives of) the metric, connection and curvature at the central axis  $\emptyset$  for the incoming null cones emanating from  $\emptyset$  foliated by geodesic affine parameter. Its proof is given in Appendix B.

<sup>7</sup>The second fundamental form  $\theta$  of  $\partial\Sigma \subset \Sigma$  is defined by

$$\theta(X, Y) := g(\nabla_X N, Y),$$

for  $X, Y \in T\partial\Sigma$ , where  $N$  is the outward-pointing unit normal to  $\partial\Sigma$  in  $\Sigma$ .

<sup>8</sup> $L^p$  and sup-norm estimates can be deduced by Sobolev embeddings.

**Theorem 4.4** (Axis limits). *Let  $(\mathcal{M}, \mathbf{g})$  be a smooth Lorentzian manifold. Let  $\phi$  be a timelike geodesic. There exists coordinates  $x^\mu$  in a neighbourhood of  $\phi$ , smooth in  $\mathcal{M} \setminus \phi$ , such that the level sets of*

$$\underline{u} := x^0 + \sqrt{\sum_{i=1}^3 (x^i)^2}$$

are the incoming null cones  $\underline{C}_{\underline{u}}$  emanating from the axis  $\phi$ , and such that

$$u := x^0 - \sqrt{\sum_{i=1}^3 (x^i)^2}$$

is the null geodesic affine parameter on the cones  $\underline{C}_{\underline{u}}$ .<sup>9</sup> We call the coordinates  $x^\mu$  (Cartesian) optical normal coordinates.

Moreover, there exists (transported along  $\underline{C}_{\underline{u}}$ ) spherical coordinates  $\vartheta, \varphi$  on the 2-spheres  $S_{u, \underline{u}}$  such that for the induced metric  $\mathcal{g}$ , we have

$$(\partial_u^k, (\partial_u + \partial_{\underline{u}})^l, \partial_\omega^m) \left( \mathcal{g} - \left( \frac{u - \underline{u}}{2} \right)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right) = O(|x|^{4-k}), \quad (4.8a)$$

when  $|x| \rightarrow 0$ , for  $k, l, m \geq 0$ , and where  $(\partial_u^k, (\partial_u + \partial_{\underline{u}})^l, \partial_\omega^m)$  denotes all combinations of partial derivatives containing respectively  $k, l, m$ -derivatives of  $u, u + \underline{u}, \omega = \vartheta, \varphi$ . In particular, for the area radius  $r$  of  $S_{u, \underline{u}}$ , we have

$$(\partial_u^k, (\partial_u + \partial_{\underline{u}})^l) \left( r(u, \underline{u}) - \frac{u - \underline{u}}{2} \right) = O(|x|^{3-k}). \quad (4.8b)$$

Furthermore, the following limits hold when  $|x| \rightarrow 0$  and for  $k, l, m \geq 0$

◇ for the optical defect  $\eta := -\frac{1}{2}\mathbf{g}(\mathbf{D}u, \mathbf{D}u)$

$$|((r\nabla_3)^k, (\nabla_3 + \nabla_4)^l, (r\nabla)^m) \eta| = O(|x|^2), \quad (4.8c)$$

◇ for the null connection coefficients associated to the null pair  $(e_3, e_4)$  (defined such that  $e_3 = -\mathbf{D}\underline{u}$ )

$$\left| ((r\nabla_3)^k, (\nabla_3 + \nabla_4)^l, (r\nabla)^m) \left( \chi - \frac{1}{r}\mathcal{g}, \underline{\chi} + \frac{1}{r}\mathcal{g}, \eta, \zeta, \underline{\eta}, \omega, \underline{\omega}, \xi, \underline{\xi} \right) \right| = O(|x|), \quad (4.8d)$$

◇ for the null curvature components

$$|((r\nabla_3)^k, (\nabla_3 + \nabla_4)^l, (r\nabla)^m) (\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha})| = O(1), \quad (4.8e)$$

#### Remarks on Theorem 4.4

4.4a The vertex limits (4.8) are consequences of the fact that in a classical normal coordinates system, the metric and its first derivatives are trivial at the point  $O$ . They are obtained by a change of coordinates from classical to Cartesian and subsequently spherical optical normal coordinates and by expressing the null coefficients in terms of the spherical optical normal coordinates. See Sections B.2 and B.3.

4.4b The vanishing of all  $\nabla_3 + \nabla_4$  derivatives is a consequence of the translation invariance along the axis.

4.4c The vertex limits (4.8c)–(4.8e) are sharp in terms of the asymptotic behaviour of the  $\nabla_3 + \nabla_4$  and  $r\nabla$  derivatives. For the  $\nabla_3$  derivatives, these limits allow for a blow-up of the  $k$  derivatives of the type  $\nabla_3^k F \sim r^{-k}$ , which we believe is far from being optimal in most cases.

4.4d Better limits can hold for a vacuum spacetime. For example, using that  $\text{tr}\underline{\alpha} = 0$  in that case and integrating the Raychaudhuri equation (2.34c), one can obtain better bounds for  $\text{tr}\underline{\chi}$ .

4.4e The coordinates  $u, \underline{u}$  from Theorem 4.4 correspond in the present paper to the coordinates  $u', \underline{u}'$  of the so-called last cones geodesic foliation. See Section 2.13.

<sup>9</sup>See also the definition of  $u'$  in Section 2.13.

### 4.3.3 Well-posedness of the canonical foliation

The following theorem ensures that the canonical foliation on  $\underline{\mathcal{C}}^*$  defined in Section 2.1 is locally well defined and provides vertex limits for its associated metric and null connection coefficients.

**Theorem 4.5** (The canonical foliation on  $\underline{\mathcal{C}}^*$ ). *Let  $(\mathcal{M}, \mathbf{g})$  be a smooth Lorentzian manifold. Let  $\underline{\mathcal{C}}^* \subset \mathcal{M}$  be a smooth null cone emanating from a point  $\phi(\underline{u}^*)$ , where we also assume that a unit timelike vector  $\dot{\phi}(\underline{u}^*)$  is given. There exists a function  $u$  in a neighbourhood of  $\phi(\underline{u}^*)$  in  $\underline{\mathcal{C}}^*$ , smooth on  $\underline{\mathcal{C}}^* \setminus \phi(\underline{u}^*)$ , such that its level sets  $S_u \subset \underline{\mathcal{C}}^*$  define a canonical foliation of  $\underline{\mathcal{C}}^*$  in the following sense (see also the definitions of Section 2.1):*

- ◇ For the null connection coefficients associated to the null pair  $(e_3, e_4)$  orthogonal to  $S_u$  (defined such that  $e_3(u) = 2$ ), the following elliptic condition is satisfied

$$\begin{aligned} \text{div} \eta + \rho &= \bar{\rho}, \\ \bar{\omega} &= 0, \end{aligned} \tag{4.9}$$

on each 2-sphere  $S_u$ .

- ◇ The following limits hold at the vertex  $\phi(\underline{u}^*)$

$$u|_{\phi(\underline{u}^*)} = \underline{u}^*, \quad \text{and} \quad \mathbf{g}(e_3, \dot{\phi}(\underline{u}^*))|_{\phi(\underline{u}^*)} = -1. \tag{4.10}$$

Moreover, there exists (transported along  $\underline{\mathcal{C}}^*$ ) spherical coordinates  $\vartheta, \varphi$  such that the induced metric  $\mathcal{g}$  on  $S_u$  in these coordinates satisfy the same limits at the vertex  $\phi(\underline{u}^*)$  as in Theorem 4.4. The limits for the area radius, null connection coefficients associated to the canonical foliation are also identical to the ones of Theorem 4.4.

#### Remarks on Theorem 4.5

- 4.5a Expressed using geodesic affine parameter, the system (4.9) rewrites as a coupled system of elliptic and transport equation, with initial value given by (4.10). The proof of Theorem 4.5 then relies on a standard Banach-Picard iteration similar to the one performed in [CG19a, Section 6]. See Appendix C.
- 4.5b As a byproduct of the Banach-Picard iteration – using an implicit function theorem –, one obtains that the solutions to the above mentioned system of elliptic-transport equations are also unique and stable under a perturbation of the background spacetime metric. This justifies the continuity argument of Section 4.4.2.
- 4.5c The vertex limits for the metric and null connection coefficients are used in Section 9 as initial data to obtain global estimates on  $\underline{\mathcal{C}}^*$ .

### 4.3.4 Existence and control of initial layers

We have the following theorem, which enables to relate the result of the main Theorem 4.2 proved in this paper, to the most general Theorem 1.8 stated in the introduction.

**Theorem 4.6** (Existence and control of initial layers [CG19b], [LZ18]). *Let  $\varepsilon > 0$ . There exists  $\varepsilon' > 0$  such that if  $(\widetilde{\mathcal{M}}, \mathbf{g})$  is a smooth vacuum spacetime admitting spacelike-characteristic initial data  $(\widetilde{\Sigma}_1, \widetilde{\mathcal{C}}_0)$  which are  $\varepsilon'$ -close to Minkowski space, then  $(\widetilde{\mathcal{M}}, \mathbf{g})$  contains a bottom initial layer  $\mathcal{L}_{\text{bot}}$  and a conical initial layer  $\mathcal{L}_{\text{con}}$ , which are  $\varepsilon$ -close to Minkowski space, as defined in Section 4.1.*

#### Remarks on Theorem 4.6

- 4.6a Combining the existence and control of initial layer Theorem 4.6 and the main Theorem 4.2, we obtain the general Theorem 1.8.
- 4.6b The existence and control of the bottom initial layer  $\mathcal{L}_{\text{bot}}$  follows from a small data local existence result for the spacelike-characteristic Cauchy problem. Such a result has been obtained in [CG19a, CG19b] under regularity assumptions weaker than in the present paper, and applies in particular in the present case.<sup>10</sup>

<sup>10</sup>It can also easily be reproved at the present regularity using the elementary techniques of this paper.

- 4.6c The conical initial layer decay rates for the metric, connection and curvature from estimates (4.2) correspond to the decay rates of [CK93, KN03]. The required regularity is also provided by [CK93, KN03]. In particular, a last layer along an outgoing null cone  $\tilde{\mathcal{C}}_0$  in [KN03] would be an admissible conical initial layer in the present paper.
- 4.6d The existence of the conical initial layer  $\mathcal{L}_{\text{con}}$  can be obtained from the existence result of [LZ18] for initial data posed on characteristic hypersurfaces.
- 4.6e The comparisons in the intersection of the initial layers  $\mathcal{L}_{\text{bot}} \cap \mathcal{L}_{\text{con}}$  can also be obtained from [LZ18]. It could also be obtained in the case of a last outgoing conical initial layer from [KN03].

## 4.4 Proof of the main theorem

The main part of the proof of Theorem 4.2 is a bootstrap argument. We define  $\underline{U}^*$  to be the supremum of all  $\underline{u}^*$  such that  $\tilde{\mathcal{M}}$  admits subregions  $(\tau)\mathcal{M}_{\underline{u}^*}$  – for all  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$  – satisfying a set of mild and strong bootstrap assumptions which are collected in Sections 3.3.2 and 3.3.3, and which we further call *the Bootstrap Assumptions*.<sup>11</sup> We assume by contradiction that  $\underline{U}^* < \infty$ . From the closedness of the Bootstrap Assumptions and by propagation of regularity,  $\tilde{\mathcal{M}}$  admits smooth subregions  $(\tau)\tilde{\mathcal{M}}_{\underline{U}^*}$  satisfying the Bootstrap Assumptions.<sup>11</sup>

**Remark 4.7.** *The fact that  $\tilde{\mathcal{M}}$  admits first subregions, say  $(\tau)\mathcal{M}_{\underline{u}^*=4}$ , satisfying the Bootstrap Assumptions follows from local constructions in the (domain of dependence of the) initial layer  $\mathcal{L}_{\text{bot}}$ . See also the arguments of Step 2 in Section 4.4.2.*

**Remark 4.8.** *We do not state nor prove a precise propagation of regularity/smoothness result in this paper, which is used to guarantee the existence of the above smooth spacetime subregion  $\mathcal{M}_{\underline{U}^*}$  and to run the extraction argument of Section 4.4.3. We claim that such a result can be obtained repeating the global energy estimates and subsequent arguments of Sections 5–12. The smooth character of the spacetime is crucially used so that the limits of Theorem 4.4 for (derivatives of) the null connection coefficients at the vertex  $\phi(\underline{u}^*)$  of the last cone  $\underline{\mathcal{C}}^*$  hold.<sup>12</sup> It is also used for convenience in the local existence results applied in the extension procedure of Section 4.4.2.*

In Sections 5–12, we show that under the Bootstrap Assumptions in the subregions  $(\tau)\mathcal{M}_{\underline{u}^*}$  and the  $\varepsilon$ -closeness to Minkowski of the initial layers  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$ , all the Bootstrap Assumptions from Sections 3.3.2 and 3.3.3 can be improved. We give an overview of the results obtained in these Sections in Section 4.4.1.

In Section 4.4.2, we show that provided that the Bootstrap Assumptions with improved constants hold in  $(\tau)\mathcal{M}_{\underline{u}^*}$ , the spacetime region  $(\tau)\mathcal{M}_{\underline{u}^*} \cup \mathcal{L}_{\text{con}}$  can be extended to a smooth spacetime  $(\tau)\mathcal{M}_{\underline{u}^*+\delta} \cup \mathcal{L}_{\text{con}}$  for  $\delta > 0$  such that the Bootstrap Assumptions hold. This contradicts the finiteness of  $\underline{U}^*$ .

In Section 4.4.3, we deduce from  $\underline{U}^* = +\infty$  the main conclusions of Theorem 4.2. This finishes the proof of Theorem 4.2.

### 4.4.1 Improvement of the Bootstrap Assumptions

In this section, we give an overview of the improvement of the Bootstrap Assumptions which is performed in Sections 5–12.

- ◇ In Section 12.1, we prove that under the Bootstrap Assumptions and the  $\varepsilon$ -closeness assumption of the initial layers to Minkowski space (see Section 4.1), improved bounds for the energy fluxes for the curvature through the hypersurfaces  $\Sigma_{t^\circ}$  and  $\mathcal{C}_1$  hold.
- ◇ In Section 5, we show by performing global energy estimates in  $\mathcal{M}^{\text{ext}} \cup \mathcal{M}_{\text{bot}}^{\text{int}}$ , using the improved initial energy fluxes, that improved energy bounds for the curvature hold on all hypersurfaces  $\Sigma_t$ ,  $\mathcal{C}_u$ ,  $\Sigma_t^{\text{ext}}$  and  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ . The global energy estimates of Section 5 are performed for *one* transition parameter  $\tau$  chosen by a mean value argument.

<sup>11</sup>We assume that the Bootstrap Assumptions are *uniformly satisfied* for all transition parameter  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$ .

<sup>12</sup>The limits of Theorem 4.4 which are actually used in this paper require to obtain limits for up to three derivatives of the connection coefficients. This requires to control the metric in at least a  $C^4$ -sense at the vertex. Given the optimal possible regularity  $\partial^{\leq 4} g_{\mu\nu} \in L_t^\infty L^2$  for the metric component, consistently with the curvature control  $\mathbf{D}^{\leq 2} \mathbf{R} \in L_t^\infty L^2$ , such a control can only be obtained under an higher regularity assumption. This is the reason for the smoothness assumption in this paper.



- ◇ In Section 6, we show that using the improved energy bounds on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  and  $\mathcal{C}_u, \Sigma_t^{\text{ext}}$ , one obtains improved control of the spacetime curvature tensor in  $\mathcal{M}^{\text{ext}}$ .
- ◇ In Section 7, we show that using the improved energy bounds on  $\Sigma_t$  and the improved trace bounds for the curvature on the interface  $\mathcal{T} = \mathcal{M}^{\text{int}} \cap \mathcal{M}^{\text{ext}}$  from Section 6, one obtains improved curvature bounds in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ .
- ◇ In Section 8.1, we show that using the curvature control of Section 7 on  $\Sigma_{t^*}$ , applying a rescaling, extension and local existence results and performing a local energy estimate in  $\mathcal{M}_{\text{top}}^{\text{int}}$ , one obtains improved curvature bounds on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ . In Section 8.2, we show by rescaling and local energy estimates that curvature estimates are improved for all the constructions related to *all* transition parameters  $\tau$ .
- ◇ In Section 9, we show that using the vertex limits of Theorems 4.4 and 4.5, the Bootstrap Assumptions and the improved curvature bounds on  $\underline{\mathcal{C}}^*$ , we obtain improved bounds for the null connection coefficients of the canonical foliation. We moreover show that under these improved bounds, the conformal factor and exterior rotation vectorfields satisfy improved bounds on  $S^*$ . Using the improved bounds for the exterior rotation on  $S^*$ , we improve their bounds on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  in Section 9.14.
- ◇ In Section 10, we show that using the improved bounds for the connection and rotation coefficients on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ , we obtain improved bounds for the connection and rotation coefficients on  $\mathcal{M}^{\text{ext}}$ .
- ◇ In Section 11, we show using the improved curvature estimates in  $\mathcal{M}_{\text{bot}}^{\text{int}}$  and the improved trace estimates for the null connection coefficients on  $\mathcal{T}$  that the connection coefficients related to the maximal foliation are improved, and thus the control of the time translation approximate Killing field  $\mathbf{T}^{\text{int}}$  is improved. From an application of Theorem 4.3, we obtain bounds for the harmonic coordinates of the last slice  $\Sigma_{t^*}$ . We further show using these bounds that the remaining interior Killing fields are controlled in  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by integration from  $\Sigma_{t^*}$ . Last, we show that at the interface  $\mathcal{T}$ , the differences of interior and exterior Killing fields is controlled.
- ◇ In Section 12, we show using the previous improvement for all connection and curvature coefficients that the comparison of the constructions of  $\mathcal{M}$  to the constructions of the initial layers  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$  are improved. This finishes the improvement of the Bootstrap Assumptions.

#### 4.4.2 Extension of $\mathcal{M}_{\underline{u}^*}$

In this section, we assume that  $\widetilde{\mathcal{M}}$  admits a smooth spacetime subregion  $(\tau)\mathcal{M}_{\underline{u}^*}$  satisfying the Bootstrap Assumptions of Sections 3.3.2 and 3.3.3 with improved constants. We show that  $\widetilde{\mathcal{M}}$  admits a smooth spacetime region  $(\tau)\widetilde{\mathcal{M}}_{\underline{u}^*+\delta}$  for  $\delta > 0$  satisfying the Bootstrap Assumptions.

**Remark 4.9.** *Here and in the rest of this section, the transition parameter  $\tau$  is fixed. We thus omit the label  $\tau$  in the arguments below.*

**Step 1: Spacetime extension.** We first show that there exists a smooth spacetime  $\mathcal{N}$  containing  $\mathcal{M}_{\underline{u}^*} \cup \mathcal{L}_{\text{con}}$  such that  $\mathcal{M}_{\underline{u}^*}$  is strictly included in  $\mathcal{N}$ .<sup>13</sup>

By a comparison argument, the null hypersurface  $\underline{\mathcal{C}}^* \cap \{1 \leq u \leq (1 + \tau)\underline{u}^*/2\}$  can be foliated by a smooth non-degenerate geodesic foliation starting from the 2-sphere  $\underline{\mathcal{C}}^* \cap \widetilde{\mathcal{C}}_{3/2}$ . Similarly, the initial layer null hypersurface  $\widetilde{\mathcal{C}}_{3/2}$  can be foliated by a smooth geodesic foliation starting from  $\underline{\mathcal{C}}^* \cap \widetilde{\mathcal{C}}_{3/2}$ . Applying the local existence result of [Luk12] to the induced characteristic initial data on the hypersurfaces

$$(\underline{\mathcal{C}}^* \cap \{u \leq (1 + \tau)\underline{u}^*/2\}) \cup \widetilde{\mathcal{C}}_{3/2},$$

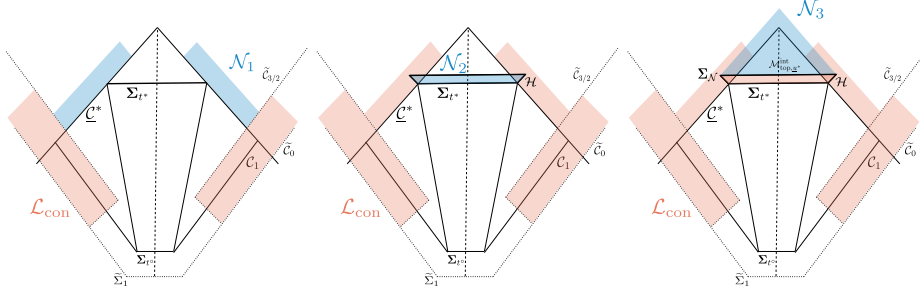
we deduce that there exists a smooth spacetime  $\mathcal{N}_1$  which strictly contains  $\underline{\mathcal{C}}^* \cap \{u \leq (1 + \tau)\underline{u}^*/2\}$  (see also Figure 1).

We consider the smooth null hypersurface  $\mathcal{H}$  emanating from  $S^*$  in  $\mathcal{N}_1$ . Applying a spacelike-characteristic local existence result to  $\Sigma_{t^*} \cup \mathcal{H}$  (see [CG19b, Section 6]), we deduce that there exists a spacetime  $\mathcal{N}_2$  which contains  $\Sigma_{t^*} \cup \mathcal{H}$  in which the hypersurface  $\Sigma_{t^*}$  is strictly included.

<sup>13</sup>Here and in the following, strict inclusion means that  $\mathcal{M}_{\underline{u}^*}$  is composed of interior points of the manifold  $\mathcal{N}$ .



In particular,  $\mathcal{N}_2$  contains a maximal spacelike hypersurface  $\Sigma_{\mathcal{N}}$  strictly in the future of  $\Sigma_{t^*}$  with boundary included in  $\mathcal{H}$ . We define  $\mathcal{N}_3$  to be its smooth maximal globally hyperbolic development of  $\Sigma_{\mathcal{N}}$ . By continuity, one can assume that on  $\Sigma_{\mathcal{N}}$ , the harmonic coordinates and second fundamental form bounds from the Bootstrap Assumptions 3.24 and 3.27 hold. From the  $\varepsilon$ -smallness of the initial data on  $\Sigma_{\mathcal{N}}$ , a rescaling to a time-1 situation and a local existence result (see [Fou52]), one deduces that  $\mathcal{N}_3$  is diffeomorphic and  $\varepsilon$ -close to a subregion of Minkowski space. Thus  $\mathcal{N}_3$  strictly contains  $\mathcal{M}_{\text{top}, \underline{u}^*}^{\text{int}} \cap \mathcal{N}_3$ . Patching together  $\mathcal{L}_{\text{con}}$ ,  $\mathcal{M}_{\underline{u}^*}$ ,  $\mathcal{N}_1$ ,  $\mathcal{N}_2$  and  $\mathcal{N}_3$ , we obtain a spacetime  $\mathcal{N}$  strictly extending  $\mathcal{M}_{\underline{u}^*}$ , as desired.


 Figure 1: The spacetime  $\mathcal{N}$ .

**Step 2: Constructions of optical, affine parameters and time functions.** Since in particular  $\phi(\underline{u}^*) \in \mathcal{M}_{\underline{u}^*} \subset \mathcal{N}$  is an interior point of  $\mathcal{N}$ , the timelike geodesic  $\phi$  can be continued past  $\phi(\underline{u}^*)$  and there exists  $\delta > 0$  sufficiently small such that  $\phi(\underline{u}^* + \delta) \in \mathcal{N}$ .

Let denote by  $\underline{\mathcal{C}}^{*,\delta}$  the incoming null cone backward emanating from  $\phi(\underline{u}^* + \delta)$ , which local existence, smoothness and local foliation by geodesic affine parameter is guaranteed by the (null) geodesic equation and Cauchy-Lipschitz theorem. Since the cone  $\underline{\mathcal{C}}^*$  globally exists and can be foliated by a geodesic foliation going from  $s = \underline{u}^*$  to  $s = 1/2$ , applying Cauchy-Lipschitz again provided that  $\delta > 0$  is sufficiently small, the cone  $\underline{\mathcal{C}}^{*,\delta}$  can be foliated by geodesic affine parameter going from  $s = \underline{u}^* + \delta$  to  $s = 2/3$  and the associated null connection coefficients are close to the null connection coefficients associated to  $\underline{\mathcal{C}}^*$ .

Applying the well-posedness Theorem 4.5 for the canonical foliation on  $\underline{\mathcal{C}}^{*,\delta}$  (see in particular Item 4.5b), one deduces that for  $\delta > 0$  sufficiently small the canonical foliation exists from  $u = \underline{u}^* + \delta$  to  $u = 1$ .

We denote by  $\mathcal{C}_u^\delta$  the outgoing null cones backward emanating from the 2-spheres of the canonical foliation on  $\underline{\mathcal{C}}^{*,\delta}$ . From Cauchy-Lipschitz, since the 2-spheres on  $\underline{\mathcal{C}}^*$  and  $\underline{\mathcal{C}}^{*,\delta}$  are close, the cones globally exist with geodesic affine parameter  $\underline{u}$  ranging from  $\underline{u}^*$  to  $\tau^{-1}u$ , and the associated null connection coefficients satisfy the Bootstrap Assumptions.

Since the 2-spheres of the interfaces  $\mathcal{T}^\delta$  and  $\mathcal{T}$  are close, one can apply the implicit function theorem of [Cho76] (see also a similar application in [CG19b, Section 6]) and we deduce that there exists a foliation by maximal hypersurfaces  $(\Sigma_t^\delta)_{t^\circ \leq t \leq (1+\tau)(\underline{u}^* + \delta)/2}$  with boundaries  $\partial \Sigma_t^\delta = S_{u, \underline{u}}^\delta \subset \mathcal{T}^\delta$ , which is close to the analogous maximal foliation of  $\mathcal{M}_{\text{bot}, \underline{u}^*}^{\text{int}}$ .

The existence, uniqueness and control of all centred conformal isomorphism from the new sphere  $S^{*,\delta}$  can be obtained from the Uniformisation Theorem [KS19b, Theorem 3.1]. From induced coordinates on the new last maximal hypersurface  $\Sigma_{(1+\tau)(\underline{u}^* + \delta)/2}^\delta$ , the hypothesis of Theorem 4.3 are satisfied and there exists global harmonic coordinates on  $\Sigma_{(1+\tau)(\underline{u}^* + \delta)/2}^\delta$  adapted to any centred conformal isomorphism on  $S^{*,\delta}$ . From the closeness to the constructions of the previous spacetime region, we deduce that the Bootstrap Assumptions are satisfied and we have obtained the desired spacetime region  $\mathcal{M}_{\underline{u}^* + \delta} \subset \mathcal{N} \subset \mathcal{M}$ .

### 4.4.3 Conclusions

In this section, we deduce from  $\underline{U}^* = +\infty$  the main conclusions of Theorem 4.2.<sup>14</sup>

<sup>14</sup>In this section, the transition parameter  $\tau$  is fixed in  $[\tau_0, (1 + \tau_0)/2]$ .

**Global time, optical and affine parameter functions** First, we infer from  $\underline{U}^* = +\infty$  that the central axis  $\phi(t)$  exists for  $t = 1$  to  $+\infty$  and there exists global functions  $u', \underline{u}'$  such that its level sets is the last cones geodesic foliation defined in Section 2.13.

From the Bootstrap Assumptions and propagation of regularity, the functions  $(\underline{u}^*)\underline{u}, (\underline{u}^*)u$  and  $(\underline{u}^*)t$  and their derivatives are bounded and equicontinuous uniformly in  $\underline{u}^*$  on each compact region  $\{\underline{u}' \leq C\}$ . Thus, applying Arzelà-Ascoli theorem, one can deduce that there exists a sequence  $\underline{u}^*_n \rightarrow +\infty$  when  $n \rightarrow \infty$  and functions  $\underline{u}, u, t$  such that

$$(\underline{u}^*_n)\underline{u} \rightarrow \underline{u}, \quad (\underline{u}^*_n)u \rightarrow u, \quad (\underline{u}^*_n)t \rightarrow t,$$

in  $\mathcal{C}^k$  topology in each compact region  $\{\underline{u}' \leq C\}$  and for all  $k \geq 0$ .

We define the subregions

$$\begin{aligned} \mathcal{M}_\infty^{\text{ext}} &:= \{\tau^{-1} \leq \underline{u} < +\infty, 1 \leq u \leq \tau \underline{u}\}, \\ \mathcal{M}_\infty^{\text{int}} &:= \{t^\circ \leq t < +\infty\}, \\ \mathcal{T} &:= \{1 \leq u < +\infty, u = \tau \underline{u}\}. \end{aligned}$$

Passing to the limits the definitions of  $\underline{u}, u$  and  $t$ , one deduces that they are respectively affine parameter, optical functions and maximal time function, and that on the interface  $\mathcal{T} := \{u = \tau \underline{u}\}$ , one has  $t = \frac{1}{2}(u + \underline{u})$ . Moreover, passing to the limit the sup-norm estimates from the Bootstrap Assumptions, one deduces the bounds (4.4) of Theorem 4.2.

Using the bounds (4.4) and taking limits along the null cones  $\mathcal{C}_u$ , a proper notion of *future null infinity*  $\mathcal{I}^+$  can be obtained (see [KN03, Section 8]). From the fact that

$$|\mathbf{D}_{e_3} e_3| \lesssim |\underline{\omega}| + |\underline{\xi}| \lesssim \varepsilon \underline{u}^{-1} u^{-3/2} \rightarrow 0 \quad \text{when } \underline{u} \rightarrow 0,$$

that  $e_3(u) = 2$  and that  $u$  ranges from 1 to  $+\infty$ , one can deduce that  $\mathcal{I}^+$  is *future geodesically complete*.

**Hawking and Bondi mass** We define the *Hawking mass*  $m$  in  $\mathcal{M}_{\underline{u}^*}^{\text{ext}}$  by<sup>15</sup>

$$m(u, \underline{u}) := \frac{r}{8\pi} \int_{S_{u, \underline{u}}} \left( -\rho + \frac{1}{2} \hat{\chi} \cdot \hat{\chi} \right) = \frac{r}{2} + \frac{r}{32\pi} \int_{S_{u, \underline{u}}} \text{tr} \chi \text{tr} \underline{\chi}. \quad (4.11)$$

From the (improved) Bootstrap Assumptions 3.20 and 3.23, we have

$$|m(u, \underline{u})| \lesssim \varepsilon^2 u^{-2}. \quad (4.12)$$

Using the null structure equations (2.34) and the Bootstrap Assumptions, one can prove that in  $\mathcal{M}_{\underline{u}^*}^{\text{ext}}$ , the following bound holds

$$|e_4(m)| \lesssim \varepsilon \underline{u}^{-3} u^{-3}.$$

Passing to the limit in  $\underline{u}^*$ , the estimate still holds in  $\mathcal{M}_\infty^{\text{ext}}$  and we deduce that  $e_4(m)$  is integrable along  $e_4$ , and  $m(u, \underline{u})$  admits a limit  $M(u)$  when  $\underline{u} \rightarrow +\infty$ . We call  $M$  the *Bondi mass*. From (4.12), one deduces

$$|M(u)| \lesssim \varepsilon^2 u^{-2},$$

and in particular

$$M(u) \rightarrow 0 \quad \text{when } u \rightarrow +\infty,$$

that is, the *final Bondi mass* is 0.

<sup>15</sup>The identity is given by Gauss equation (2.34t) and Gauss-Bonnet formula.

**Bondi mass loss formula** Using equation (2.85) (2.34e) and (2.34b) together with the Bootstrap Assumptions, we have

$$\begin{aligned} e_3(r^3\bar{\rho}) &= -\frac{1}{2}r^3\overline{\hat{\chi} \cdot \underline{\alpha}} + O(\underline{u}^{-1}), \\ \nabla_3(r\hat{\chi}) &= -\frac{1}{2}r\text{tr}\chi\hat{\chi} + O(\underline{u}^{-2}), \\ \nabla_3(r^2\hat{\chi}) &= -r^2\underline{\alpha} + O(1), \end{aligned}$$

when  $\underline{u} \rightarrow +\infty$ . Thus,

$$\begin{aligned} e_3(m) &= -\frac{1}{2}e_3(r^3\bar{\rho}) + \frac{1}{4}e_3\left(\overline{(r\hat{\chi}) \cdot (r^2\hat{\chi})}\right) \\ &= -\frac{r^3}{8}\overline{\text{tr}\chi|\hat{\chi}|^2} + O(\underline{u}^{-1}), \end{aligned}$$

and passing to the limit when  $\underline{u}^* \rightarrow +\infty$  and taking the limit when  $\underline{u} \rightarrow +\infty$ , we infer the following *Bondi mass loss formula*

$$\frac{d}{du}M(u) = \lim_{\underline{u} \rightarrow +\infty} \left( -\frac{r}{64\pi} \int_{S_{u,\underline{u}}} \text{tr}\chi|\hat{\chi}|^2 \right).$$

**Angular momentum** According to [Riz98], we define the following local angular momentum

$${}^{(\ell)}L(u, \underline{u}) := \frac{1}{8\pi r} \int_{S_{u,\underline{u}}} \zeta \cdot {}^{(\ell)}\mathbf{O}^{\text{ext}}, \quad (4.13)$$

for  $\ell = 1, 2, 3$ .<sup>16</sup> Using the Bootstrap Assumptions, we directly obtain

$$\left| {}^{(\ell)}L(u, \underline{u}) \right| \lesssim \varepsilon u^{-1/2}. \quad (4.14)$$

Using equations (2.34g) and (2.113), we have

$$\begin{aligned} r^2\zeta &= O(1), & r^{-1}\mathbf{O}^{\text{ext}} &= O(1), \\ \nabla_4(r^2\zeta) &= O(\underline{u}^{-3/2}), & \nabla_4(r^{-1}\mathbf{O}^{\text{ext}}) &= O(\underline{u}^{-2}), \end{aligned}$$

when  $\underline{u} \rightarrow +\infty$ . Thus, deriving (4.13) by  $e_4$  gives

$$e_4(L) = \frac{1}{2}e_4\left(\overline{(r^2\zeta) \cdot r^{-1}\mathbf{O}^{\text{ext}}}\right) = O(\underline{u}^{-3/2}),$$

and the bound still holds when passing to the limit  $\underline{u}^* \rightarrow +\infty$ . We infer that  $e_4(L)$  is integrable along  $e_4$  and we define the *angular momentum at null infinity*  ${}^{(\ell)}L(u)$  to be the limit of  ${}^{(\ell)}L(u, \underline{u})$  when  $\underline{u} \rightarrow \infty$ . From (4.14), we directly obtain

$$\left| {}^{(\ell)}L(u) \right| \lesssim \varepsilon u^{-1/2},$$

and in particular

$${}^{(\ell)}L(u) \rightarrow 0, \quad \text{when } u \rightarrow +\infty,$$

that is, *the final angular momentum is 0*.

Using equations (2.34d) and (2.109), we have

$$\nabla_3(r^2\zeta) = -2r^2\nabla\omega - r^2\underline{\beta} + O(\underline{u}^{-1}), \quad \nabla_3(r^{-1}\mathbf{O}^{\text{ext}}) = O(\underline{u}^{-1}),$$

when  $\underline{u} \rightarrow +\infty$ . Arguing as previously, we infer the following *angular momentum evolution equation along null infinity*

$$\frac{d}{du}{}^{(\ell)}L(u) = \lim_{\underline{u} \rightarrow \infty} \left( \frac{1}{16\pi r} \int_{S_{u,\underline{u}}} (-2\nabla\omega - \underline{\beta}) \cdot {}^{(\ell)}\mathbf{O}^{\text{ext}} \right),$$

for  $\ell = 1, 2, 3$ . This finishes the proof of the conclusions of Theorem 4.2.

<sup>16</sup>Arguing similarly as before, one can deduce the existence of exterior rotation vectorfields in the limit  $\underline{u}^* \rightarrow +\infty$ .

## 5. Global energy estimates in $\mathcal{M}$

In this section, we perform the global energy estimates in  $\mathcal{M}$ , which are used to estimate the curvature in the exterior and interior bottom regions respectively in Sections 6 and 7. We prove the following proposition.

**Proposition 5.1.** *Recall that from the initial layer results of Section 12, we have<sup>1</sup>*

$$\int_{(\tau)\Sigma_{t^0}} P \cdot \bar{T} + \int_{(\tau)\mathcal{C}_1} P \cdot e_4 \lesssim \varepsilon^2, \quad (5.1)$$

where  $P$  are the following Bel-Robinson tensors defined respectively in the interior and exterior region  $(\tau)\mathcal{M}_{\text{bot}}^{\text{int}}$  and  $(\tau)\mathcal{M}^{\text{ext}}$  by

$$\begin{aligned} & Q(\hat{\mathcal{L}}_{\mathbf{T}}\mathbf{R})(\mathbf{K}, \mathbf{K}, \mathbf{K}), \quad Q(\hat{\mathcal{L}}_{\mathbf{O}}\mathbf{R})(\mathbf{K}, \mathbf{K}, \mathbf{T}), \\ & Q(\hat{\mathcal{L}}_{\mathbf{O}}\hat{\mathcal{L}}_{\mathbf{O}}\mathbf{R})(\mathbf{K}, \mathbf{K}, \mathbf{T}), \quad Q(\hat{\mathcal{L}}_{\mathbf{S}}\hat{\mathcal{L}}_{\mathbf{T}}\mathbf{R})(\mathbf{K}, \mathbf{K}, \mathbf{K}), \quad Q(\hat{\mathcal{L}}_{\mathbf{O}}\hat{\mathcal{L}}_{\mathbf{T}}\mathbf{R})(\mathbf{K}, \mathbf{K}, \mathbf{T}), \end{aligned} \quad (5.2)$$

where  $\mathbf{T}, \mathbf{S}, \mathbf{K}, \mathbf{O}$  correspond to the approximate conformal Killing vectorfields defined respectively in the interior region in Section 2.5 and in the exterior region in Section 2.11.<sup>2</sup>

Under the Bootstrap Assumptions and for  $\varepsilon > 0$  sufficiently small, there exists a transition parameter  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$  such that we have

$$\int_{(\tau)\Sigma_t} P \cdot \bar{T} + \int_{\mathcal{C}_u \cap (\tau)\mathcal{M}^{\text{ext}}} P \cdot e_4 + \int_{(\tau)\Sigma_t^{\text{ext}}} P \cdot \bar{T}^{\text{ext}} + \int_{\underline{\mathcal{C}}^* \cap (\tau)\mathcal{M}^{\text{ext}}} P \cdot e_3 \lesssim \varepsilon^2, \quad (5.3)$$

where  $P$  denotes the Bel-Robinson tensor from (5.2).

**Remark 5.2.** *The transition parameter  $\tau$  is determined by a mean value argument in Section 5.1.1. The control for the curvature on the hypersurfaces associated to all the transition parameters  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$  is obtained in Section 8.2.*

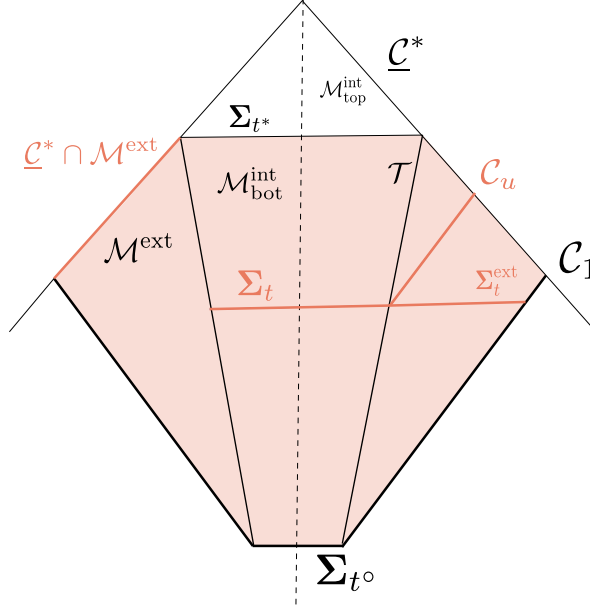


Figure 1: Global energy estimates in  $\mathcal{M}$ .

The proof of Proposition 5.1 follows from the following energy estimate

$$\int_{\Sigma_t} P \cdot \bar{T} + \int_{\mathcal{C}_u \cap \mathcal{M}^{\text{ext}}} P \cdot e_4 + \int_{\Sigma_t^{\text{ext}}} P \cdot \bar{T}^{\text{ext}} + \int_{\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}} P \cdot e_3 \lesssim \int_{\Sigma_{t^0}} P \cdot \bar{T} + \int_{\mathcal{C}_1} P \cdot e_4 + \mathcal{E}^{\mathcal{T}} + \mathcal{E}^{\text{int}} + \mathcal{E}^{\text{ext}},$$

<sup>1</sup>These results are postponed to the initial layer Section 12 for consistency. They only use the Bootstrap Assumptions and the  $\varepsilon$ -closeness to Minkowski space assumptions from Section 4.1.

<sup>2</sup>The bound (5.1) holds for all transition parameters  $\tau$ .

which holds for all  $t^\circ \leq t \leq t^*$  and for all  $1 \leq u \leq \tau u^*$ , for all contracted and commuted Bel-Robinson tensors  $P$  of (5.2), and where the nonlinear error terms  $\mathcal{E}$  are defined below.

The nonlinear error terms are decomposed as follows

$$\mathcal{E}^{\mathcal{T}} = \mathcal{E}_1^{\mathcal{T}} + \mathcal{E}_2^{\mathcal{T}}, \quad \mathcal{E}^{\text{int}} = \mathcal{E}_1^{\text{int}} + \mathcal{E}_2^{\text{int}}, \quad \mathcal{E}^{\text{ext}} = \mathcal{E}_1^{\text{ext}} + \mathcal{E}_2^{\text{ext}},$$

where we have the following definitions for each respective factor.

◇ We have

$$\mathcal{E}_1^{\mathcal{T}} := \int_{\mathcal{T}} (P^{\text{ext}} \cdot N^{\mathcal{T}} - P^{\text{int}} \cdot N^{\mathcal{T}}),$$

where  $N^{\mathcal{T}}$  denotes the inward-pointing unit normal to the timelike hypersurface  $\mathcal{T}$  and where

$$P \in \left\{ Q \left( \hat{\mathcal{L}}_{\mathbf{T}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{K}), Q \left( \hat{\mathcal{L}}_{\mathbf{O}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{T}) \right\}.$$

◇ We have

$$\mathcal{E}_2^{\mathcal{T}} := \int_{\mathcal{T}} (P^{\text{ext}} \cdot N^{\mathcal{T}} - P^{\text{int}} \cdot N^{\mathcal{T}}),$$

where

$$P \in \left\{ Q \left( \hat{\mathcal{L}}_{\mathbf{O}} \hat{\mathcal{L}}_{\mathbf{O}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{T}), Q \left( \hat{\mathcal{L}}_{\mathbf{S}} \hat{\mathcal{L}}_{\mathbf{T}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{K}), Q \left( \hat{\mathcal{L}}_{\mathbf{O}} \hat{\mathcal{L}}_{\mathbf{T}} \mathbf{R} \right) (\mathbf{K}, \mathbf{K}, \mathbf{T}) \right\}.$$

◇ We have

$$\mathcal{E}_1^{\text{int}} := \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \text{div}(P),$$

where

$$P \in \left\{ Q \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}), Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \right\}.$$

◇ We have

$$\mathcal{E}_2^{\text{int}} := \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \text{div}(P),$$

where

$$P \in \left\{ Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}), Q \left( \hat{\mathcal{L}}_{\mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}), \right. \\ \left. Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \right\}.$$

◇ We have

$$\mathcal{E}_1^{\text{ext}} := \int_{\mathcal{M}^{\text{ext}}} \text{div}(P),$$

where

$$P \in \left\{ Q \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}), Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \right\}.$$

◇ We have

$$\mathcal{E}_2^{\text{ext}} := \int_{\mathcal{M}^{\text{ext}}} \text{div}(P),$$

where

$$P \in \left\{ Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}), Q \left( \hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}), \right. \\ \left. Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \right\}.$$

This section is dedicated to proving that, under the Bootstrap Assumptions, there exists a transition parameter  $\tau$  such that we have

$$\mathcal{E}_{\leq 2}^{\mathcal{T}} + \mathcal{E}_{\leq 2}^{\text{int}} + \mathcal{E}_{\leq 2}^{\text{ext}} \lesssim (D\varepsilon)^3. \quad (5.4)$$

The result of Proposition 5.1 then directly follows, provided that  $\varepsilon > 0$  is sufficiently small.

To control the interior and exterior error terms in Sections 5.2 and 5.3, we make the following additional bootstrap assumption. In view of the above, it will directly be improved by the energy estimate once the control (5.4) of the error terms has been obtained.

**Bootstrap Assumption 5.3.** *Assume that the above fluxes through the hypersurfaces  $\Sigma_t, \mathcal{C}_u \cap \mathcal{M}^{\text{ext}}$  and  $\Sigma_t^{\text{ext}}$  satisfy the following additional bootstrap bounds*

$$\int_{\Sigma_t} P \cdot \bar{T} + \int_{\mathcal{C}_u \cap \mathcal{M}^{\text{ext}}} P \cdot e_4 + \int_{\Sigma_t^{\text{ext}}} P \cdot \bar{T}^{\text{ext}} \leq (D\varepsilon)^2,$$

where  $P$  denotes the contracted Bel-Robinson tensors (5.2).

**Remark 5.4** (Decay in the exterior region). *From the point of view of the decay, the most difficult error term to treat is the exterior term  $\mathcal{E}_{\leq 2}^{\text{ext}}$ , whose estimate is the crux of [CK93] and [KN03]. To this end, one has to exhibit a null structure which pairs curvature/connection coefficients (decomposed in the null directions) with compensating decay rates. In this paper, we choose the same commuting and multiplying vectorfields as in [CK93] and [KN03], and the structure of the error terms in the exterior region is formally identical to the one in these books. Since our vectorfields are constructed upon a geodesic-null foliation which is different from the maximal-null and the double-null foliations of respectively [CK93] and [KN03], the deformation tensors of our approximate conformal Killing fields satisfy slightly different decay rates.<sup>3</sup> We analyse in detail these differences in Sections 5.3.3, 5.3.5, 5.3.7 and show that the error terms are still integrable in the spacetime region  $\mathcal{M}^{\text{ext}}$ .*

## 5.1 Estimates for the interface error terms $\mathcal{E}^{\mathcal{T}}$

### 5.1.1 The mean value argument

In this section, we obtain  $L^2(\mathcal{T})$  bounds for the spacetime curvature tensor and its first and second derivatives. This is done by a *mean value argument* which selects a particular transition parameter  $\tau$ .

We consider the spacetime region

$$\mathcal{D} := \{(u, \underline{u}) : \tau_0 \underline{u} \leq u \leq (1 + \tau_0) \underline{u}/2\} \cap J^+(\tilde{\Sigma}_3).$$

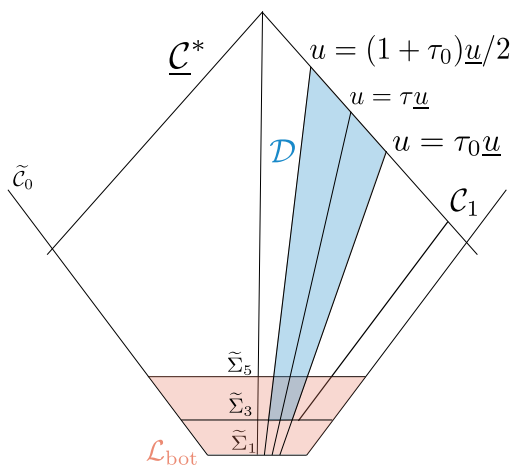


Figure 2: The mean value argument.

<sup>3</sup>This is due to the presence of  $\xi$ , which vanishes in the double-null setting of [KN03].

From the Bootstrap Assumptions 3.20 when the transition parameter is set to  $\tau = (1 + \tau_0)/2$ , we have the following  $L^2(\mathcal{D})$  control for the spacetime curvature tensor (we recall that in the region  $\mathcal{D}$ , we have  $u \simeq \underline{u} \simeq r \simeq t$ )

$$\int_{\mathcal{D}} \underline{u}^{-1-1/2} \underline{u}^4 |(\underline{u}\mathbf{D})^{\leq 2} \mathbf{R}|^2 \lesssim (\mathcal{R}_{\leq 2, \tau_0}^{\text{ext}})^2 \lesssim (D\varepsilon)^2, \quad (5.5)$$

where the norm  $|\cdot|$  is taken with respect to the orthonormal frame associated to the null pair  $(e_3, e_4)$ .<sup>4</sup>

The region  $\mathcal{D}$  can be foliated as follows

$$\mathcal{D} = \cup_{\tau_0 \leq \tau \leq (1+\tau_0)/2} \{(u, \underline{u}) : u = \tau \underline{u}\} \cap J^+(\tilde{\Sigma}_3).$$

With respect to this foliation, we have the following coarea formula

$$\begin{aligned} & \int_{\tau_0}^{(1+\tau_0)/2} \left( \int_{\{u=\tau \underline{u}\} \cap J^+(\tilde{\Sigma}_3)} (\underline{u}^{-1-1/2} \underline{u}^4 |(\underline{u}\mathbf{D})^{\leq 2} \mathbf{R}|^2) \underline{u} \right) d\tau \\ & \lesssim \int_{\mathcal{D}} \underline{u}^{-1-1/2} \underline{u}^4 |(\underline{u}\mathbf{D})^{\leq 2} \mathbf{R}|^2 \\ & \lesssim (D\varepsilon)^2. \end{aligned}$$

Using the mean value theorem, we deduce that there exists a transition parameter  $\tau \in [\tau_0, (1 + \tau_0)/2]$  such that

$$\int_{\{u=\tau \underline{u}\} \cap J^+(\tilde{\Sigma}_3)} \underline{u}^{-1/2} \underline{u}^4 |(\underline{u}\mathbf{D})^{\leq 2} \mathbf{R}|^2 \lesssim (D\varepsilon)^2.$$

This rewrites

$$\int_{(\tau)\mathcal{T}} \underline{u}^{-1/2} \underline{u}^4 |(\underline{u}\mathbf{D})^{\leq 2} \mathbf{R}|^2 \lesssim (D\varepsilon)^2. \quad (5.6)$$

**Remark 5.5.** *The integral bound (5.6) on  $\mathcal{T} \cap J^-(\tilde{\Sigma}_3) \subset \mathcal{L}_{\text{bot}}$  can be obtained using the bottom initial layer assumptions and comparison arguments. Details are left to the reader.*

**Remark 5.6.** *One cannot obtain a control on a timelike boundary such as  $\mathcal{T}$  for the curvature components, without either loosing derivatives or decay. For the error terms of  $\mathcal{E}^{\mathcal{T}}$ , we need to control the highest order terms  $\mathbf{D}^2 \mathbf{R}$  in  $L^2(\mathcal{T})$  and are bound to use the above mean value argument which loses  $\underline{u}^{1/2}$  decay. However, this loss of decay is (more than) compensated by the decay rates for the difference between interior and exterior Killing fields, see Section 5.1.2.*

In the rest of this section, the transition parameter  $\tau$  is fixed by the above mean value argument and we drop its label in the notations.

### 5.1.2 Control of $\mathcal{E}_1^{\mathcal{T}}$ and $\mathcal{E}_2^{\mathcal{T}}$

We have

$$\begin{aligned} \mathcal{E}_1^{\mathcal{T}} & := \int_{\mathcal{T}} \left( Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) - Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \right) \\ & + \int_{\mathcal{T}} \left( Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) - Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \right). \end{aligned}$$

We write schematically

$$\begin{aligned} & Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) - Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \\ & = \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \cdot \left( \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} - \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \right) \mathbf{R} \right) \cdot (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \\ & + \left( \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} - \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \right) \mathbf{R} \right) \cdot \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \cdot (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \\ & + Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \\ & + Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{ext}}) \\ & + Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}). \end{aligned} \quad (5.7)$$

<sup>4</sup>With the notations of Definition 3.1, we define  $e_0 := \frac{1}{2}(e_3 + e_4)$  and the  $e_i$  to be any (local) vectorfields such that  $(e_\mu)$  forms an orthonormal frame.

We start with the estimates for the last three terms of (5.7). From the Bootstrap Assumptions 3.29 at  $\mathcal{T}$ , we recall that we have

$$\begin{aligned} |\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}}| &\lesssim D\varepsilon t^{-3/2}, \\ |\mathbf{S}^{\text{ext}} - \mathbf{S}^{\text{int}}| &\lesssim D\varepsilon t^{-1/2}, \\ |\mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}| &\lesssim D\varepsilon t^{1/2}. \end{aligned} \quad (5.8)$$

Using that from the mild Bootstrap Assumptions 3.18 we have  $|\mathbf{K}^{\text{int}}| \lesssim t^2$  at  $\mathcal{T}$  and using the  $L^2(\mathcal{T})$  bounds (5.6) – where we recall that  $t \simeq \underline{u}$  –, we infer

$$\begin{aligned} \int_{\mathcal{T}} \left| Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}) \right| &\lesssim \int_{\mathcal{T}} \left| Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R}) \right| (|\mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}| |\mathbf{K}^{\text{int}}|^2) \\ &\lesssim D\varepsilon \int_{\mathcal{T}} \left| Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R}) \right| t^{9/2} \\ &\lesssim D\varepsilon \int_{\mathcal{T}} t^{-1} t^{4-1/2} |(t\mathbf{D})^{\leq 1} \mathbf{R}|^2 \\ &\lesssim (D\varepsilon)^3. \end{aligned}$$

For the first two terms of (5.7), we have schematically

$$\left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} - \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \right) \mathbf{R} = \mathbf{D}_{\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}}} \mathbf{R} + (\mathbf{D}\mathbf{T}^{\text{ext}} - \mathbf{D}\mathbf{T}^{\text{int}}) \cdot \mathbf{R}.$$

Using (5.8) and that by the Bootstrap Assumptions 3.29 and 3.20, we respectively have

$$\begin{aligned} \|\mathbf{D}\mathbf{T}^{\text{ext}} - \mathbf{D}\mathbf{T}^{\text{int}}\|_{L^4(\partial\Sigma_t)} &\lesssim D\varepsilon t^{-2+\gamma_0}, \\ \|\mathbf{R}\|_{L^4(\partial\Sigma_t)} &\lesssim D\varepsilon t^{-3}, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_{\partial\Sigma_t} \left| \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \cdot \left( \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} - \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \right) \mathbf{R} \right) \cdot (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \right| \\ &\lesssim \|\mathbf{D}\mathbf{R}\|_{L^2(\partial\Sigma_t)}^2 \|\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}}\|_{L^\infty(\partial\Sigma_t)} \|\mathbf{K}^{\text{ext}}\|_{L^\infty(\partial\Sigma_t)}^3 \\ &\quad + \|\mathbf{D}\mathbf{R}\|_{L^2(\partial\Sigma_t)} \|\mathbf{R}\|_{L^4(\partial\Sigma_t)} \|\mathbf{D}\mathbf{T}^{\text{ext}} - \mathbf{D}\mathbf{T}^{\text{int}}\|_{L^4(\partial\Sigma_t)} \|\mathbf{K}^{\text{ext}}\|_{L^\infty(\partial\Sigma_t)}^3 \\ &\lesssim (D\varepsilon) t^{6-3/2} \|\mathbf{D}\mathbf{R}\|_{L^2(\partial\Sigma_t)}^2 + (D\varepsilon)^2 t^{6-5+\gamma_0} \|\mathbf{D}\mathbf{R}\|_{L^2(\partial\Sigma_t)}. \end{aligned}$$

Integrating on  $\mathcal{T}$  using a coarea formula, Cauchy-Schwartz, and the estimate (5.6), one obtains

$$\begin{aligned} &\int_{\mathcal{T}} \left| \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \cdot \left( \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} - \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \right) \mathbf{R} \right) \cdot (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \right| \\ &\lesssim (D\varepsilon) \int_{t^0}^{t^*} t^{-1} t^{4-1/2} \|(t\mathbf{D})^{\leq 1} \mathbf{R}\|_{L^2(\partial\Sigma_t)}^2 dt + (D\varepsilon)^2 \int_{t^0}^{t^*} t^{\gamma_0} \|(t\mathbf{D})^{\leq 1} \mathbf{R}\|_{L^2(\partial\Sigma_t)} dt \\ &\lesssim (D\varepsilon)^3 + (D\varepsilon)^2 \left( \int_{t^0}^{t^*} t^2 \|(t\mathbf{D})^{\leq 1} \mathbf{R}\|_{L^2(\partial\Sigma_t)}^2 dt \right)^{1/2} \\ &\lesssim (D\varepsilon)^3, \end{aligned}$$

where from the second to third line, we used that  $\gamma_0 < 1/4$ .

Thus, we have obtained

$$\left| \int_{\mathcal{T}} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) - Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \right| \lesssim (D\varepsilon)^3.$$

The estimates for the second term of  $\mathcal{E}_1^{\mathcal{T}}$  follow similarly, using that on  $\mathcal{T}$  by the Bootstrap Assumptions 3.29, we have

$$\|\mathbf{D}\mathbf{O}^{\text{ext}} - \mathbf{D}\mathbf{O}^{\text{int}}\|_{L^4(\partial\Sigma_t)} \lesssim D\varepsilon t^{-1+\gamma_0},$$



and this finishes the estimates for  $\mathcal{E}_1^{\mathcal{T}}$ .

The estimates for the highest order error term  $\mathcal{E}_2^{\mathcal{T}}$  follow along the same estimates as above, using that by the Bootstrap Assumptions 3.29, the following bounds hold on  $\mathcal{T}$

$$\|\mathbf{DS}^{\text{ext}} - \mathbf{DS}^{\text{int}}\|_{L^4(\partial\Sigma_t)} \lesssim D\varepsilon t^{-1+\gamma_0},$$

and

$$\begin{aligned} \|(t\mathbf{D})^{\leq 1}(\mathbf{DT}^{\text{ext}} - \mathbf{DT}^{\text{int}})\|_{L^4(\partial\Sigma_t)} &\lesssim D\varepsilon t^{-2+\gamma_0}, \\ \|(t\mathbf{D})^{\leq 1}(\mathbf{DO}^{\text{ext}} - \mathbf{DO}^{\text{int}})\|_{L^4(\partial\Sigma_t)} &\lesssim D\varepsilon t^{-1+\gamma_0}. \end{aligned}$$

This finishes the proof of the desired estimates for  $\mathcal{E}_2^{\mathcal{T}}$ .

## 5.2 Estimates for the interior error terms $\mathcal{E}^{\text{int}}$

### 5.2.1 Estimates for $\mathcal{E}_1^{\text{int}}$

We have

$$\begin{aligned} \mathcal{E}_1^{\text{int}} &:= \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \mathbf{div} \left( Q \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \\ &\quad + \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \mathbf{div} \left( Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \\ &\quad + \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} Q \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) \cdot (\mathbf{K}^{\text{int}})_{\hat{\pi}} \cdot (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \\ &\quad + \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \cdot (\mathbf{K}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{K}^{\text{int}} \cdot \mathbf{T}^{\text{int}} \\ &\quad + \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \cdot (\mathbf{T}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{K}^{\text{int}} \cdot \mathbf{K}^{\text{int}}. \end{aligned}$$

To treat the last three error terms, we use that from the mild Bootstrap Assumptions 3.18 and the Bootstrap Assumptions 3.28, we have respectively

$$\begin{aligned} |\mathbf{T}^{\text{int}}| &\lesssim 1, & |\mathbf{K}^{\text{int}}| &\lesssim t^2, \\ |(\mathbf{T}^{\text{int}})_{\hat{\pi}}| &\lesssim D\varepsilon t^{-5/2}, & |(\mathbf{K}^{\text{int}})_{\hat{\pi}}| &\lesssim D\varepsilon t^{-1/2}, \end{aligned}$$

where the norms are taken with respect to the maximal frame (*i.e.*  $e_0 = \bar{T}$ ).

Using the Bootstrap Assumptions 5.3 on the flux of the contracted Bel-Robinson tensor, its positivity properties (see [CK93, Chapter 7]), and a coarea formula, we obtain

$$\begin{aligned} &\int_{\mathcal{M}_{\text{bot}}^{\text{int}}} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R}) \cdot (\mathbf{T}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{K}^{\text{int}} \cdot \mathbf{K}^{\text{int}} \\ &\lesssim \int_{t^\circ}^{t^*} \int_{\Sigma_t} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R}) (\mathbf{T}^{\text{int}}, \mathbf{T}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \left| (\mathbf{T}^{\text{int}})_{\hat{\pi}} \right| dt \\ &\lesssim (D\varepsilon)^3 \int_{t^\circ}^{t^*} t^{-5/2} dt \\ &\lesssim (D\varepsilon)^3, \end{aligned}$$

and the other terms follow similarly.

For the first term, we have using Bianchi equations and the formula from [CK93, p. 141]

$$\mathbf{div} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) = (\mathbf{T}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{DR} + \mathbf{D}(\mathbf{T}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{R}.$$

Therefore, using [CK93, p. 137], we have

$$\begin{aligned} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R}) &= \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \cdot \left( \operatorname{div} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) \right) \\ &= \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \cdot \mathbf{DR} \cdot (\mathbf{T}^{\text{int}})_{\hat{\pi}} + \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{D}(\mathbf{T}^{\text{int}})_{\hat{\pi}}. \end{aligned}$$

The first term is handled as previously. For the second term, we use that, by the Bootstrap Assumptions 5.3 on the flux of the contracted Bel-Robinson tensors, the positivity properties of the Bel-Robinson tensor, and a coarea formula, one has

$$\begin{aligned} \left\| t^{-1/2-\gamma} t^3 \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim \left( \int_{t^0}^{t^*} \int_{\Sigma_t} t^{-1-2\gamma} t^6 \left| \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right|^2 dt \right)^{1/2} \\ &\lesssim \left( \int_{t^0}^{t^*} \int_{\Sigma_t} t^{-1-2\gamma} Q \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right) (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) dt \right)^{1/2} \\ &\lesssim (D\varepsilon) \left( \int_{t^0}^{t^*} t^{-1-2\gamma} dt \right)^{1/2} \\ &\lesssim_{\gamma} (D\varepsilon), \end{aligned}$$

for all  $\gamma > 0$ . We also use that by the Bootstrap Assumptions 3.21 and the Bootstrap Assumptions 3.28, we have respectively

$$\begin{aligned} \left\| t^{7/2} \mathbf{R} \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim D\varepsilon, \\ \left\| t^{-1/2-\gamma_0} t^2 \mathbf{D}(\mathbf{T}^{\text{int}})_{\hat{\pi}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim D\varepsilon. \end{aligned}$$

This gives

$$\begin{aligned} &\left| \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \cdot \mathbf{R} \cdot \mathbf{D}(\mathbf{T}^{\text{int}})_{\hat{\pi}} \cdot (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}) \right| \\ &\lesssim (D\varepsilon) \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} t^{5/2} \left| \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right| \left| \mathbf{D}(\mathbf{T}^{\text{int}})_{\hat{\pi}} \right| \\ &\lesssim (D\varepsilon) \left\| t^{-1/2-\gamma} t^3 \hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \left\| t^{-1/2-\gamma_0} t^2 \mathbf{D}(\mathbf{T}^{\text{int}})_{\hat{\pi}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \\ &\lesssim (D\varepsilon)^3, \end{aligned}$$

provided that  $\gamma$  is such that  $\gamma < 3/2 - \gamma_0$ .

The remaining term of  $\mathcal{E}_1^{\text{int}}$  follows along the same lines, using the Bootstrap Assumptions 3.28 for  $\mathbf{D}^{\leq 1}(\mathbf{O}^{\text{int}})_{\hat{\pi}}$ . This finishes the control of  $\mathcal{E}_1^{\text{int}}$ .

### 5.2.2 Estimates for $\mathcal{E}_2^{\text{int}}$

We treat the following error term

$$\int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \operatorname{div} \left( Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \right) \cdot (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}).$$

Using [CK93, p. 141] and that schematically  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} = \mathbf{D}_{\mathbf{O}^{\text{int}}} + \mathbf{DO}^{\text{int}}$ , we have

$$\begin{aligned} \operatorname{div} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) &= \mathbf{D}^2(\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{R} \cdot \mathbf{O}^{\text{int}} + \mathbf{D}(\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{DR} \cdot \mathbf{O}^{\text{int}} + (\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{D}^2 \mathbf{R} \cdot \mathbf{O}^{\text{int}} \\ &\quad + \mathbf{D}(\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{R} \cdot \mathbf{DO}^{\text{int}} + \mathbf{D}(\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{R} \cdot \mathbf{DO}^{\text{int}} + (\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{DR} \cdot \mathbf{DO}^{\text{int}} \\ &\quad + (\mathbf{O}^{\text{int}})_{\hat{\pi}} \cdot \mathbf{R} \cdot \mathbf{D}^2 \mathbf{O}^{\text{int}}. \end{aligned}$$

Using the  $L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})$  and  $L^2(\mathcal{M}_{\text{bot}}^{\text{int}})$  of the Bootstrap Assumptions 3.21 and 3.28 for  $\mathbf{R}$  and  $\mathbf{O}^{\text{int}}$  respectively, we check that

$$\left\| t^{4-\gamma_0} \operatorname{div} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim (D\varepsilon)^2. \quad (5.9)$$

**Remark 5.7.** For the second term, we use the  $L_t^\infty L^6(\Sigma_t)$  estimates of the Bootstrap Assumptions 3.21 and 3.28 and a coarea formula as follows

$$\begin{aligned}
 \left\| t^{4-\gamma_0} \mathbf{D}^{(\mathbf{O}^{\text{int}})} \hat{\pi} \cdot \mathbf{DR} \cdot \mathbf{O}^{\text{int}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim \left( \int_{t^0}^{t^*} \left\| t^{5-\gamma_0} \mathbf{D}^{(\mathbf{O}^{\text{int}})} \hat{\pi} \cdot \mathbf{DR} \right\|_{L^2(\Sigma_t)}^2 dt \right)^{1/2} \\
 &\lesssim \left( \int_{t^0}^{t^*} \|t^{-1-\gamma_0}\|_{L^6(\Sigma_t)}^2 \left\| t^2 \mathbf{D}^{(\mathbf{O}^{\text{int}})} \hat{\pi} \right\|_{L^6(\Sigma_t)}^2 \|t^4 \mathbf{DR}\|_{L^6(\Sigma_t)}^2 dt \right)^{1/2} \\
 &\lesssim (D\varepsilon)^2 \left( \int_{(1+\tau^{-1})/2}^{t^*} t^{-1-2\gamma_0} dt \right)^{1/2} \\
 &\lesssim (D\varepsilon)^2.
 \end{aligned}$$

Using (5.9) and  $L^2(\mathcal{M}_{\text{bot}}^{\text{int}})$  estimates obtained from the Bootstrap Assumptions 5.3 on the flux of the contracted Bel-Robinson tensors, we have

$$\begin{aligned}
 &\left| \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} \operatorname{div} \left( Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \right) \cdot (\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \right| \\
 &\lesssim \int_{\mathcal{M}_{\text{bot}}^{\text{int}}} t^4 \left| \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right| \left| \operatorname{div} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \right| \\
 &\lesssim \left\| t^{-1/2-\gamma} t^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \left\| t^{4-\gamma_0} \operatorname{div} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{int}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \\
 &\lesssim (D\varepsilon)^3,
 \end{aligned}$$

provided that  $\gamma \leq 3/2 - \gamma_0$ .

The estimates for the other error terms of  $\mathcal{E}_2^{\text{int}}$  are obtained either arguing as in Section 5.2.1 or as above, using the estimates for  $\mathbf{DT}^{\text{int}}, \mathbf{DS}^{\text{int}}$  from the Bootstrap Assumptions 3.28. This finishes the control of  $\mathcal{E}_2^{\text{int}}$ .

### 5.3 Estimates for the exterior error terms $\mathcal{E}^{\text{ext}}$

We denote by  $\mathcal{E}_{1,1}^{\text{ext}}, \mathcal{E}_{1,2}^{\text{ext}}, \mathcal{E}_{2,1}^{\text{ext}}$  and  $\mathcal{E}_{2,2}^{\text{ext}}$  the exterior error terms produced in the energy estimates, which are defined as follows

$$\begin{aligned}
 \mathcal{E}_{1,1}^{\text{ext}} &:= \int_{\mathcal{M}^{\text{ext}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi} + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}}) \hat{\pi} \right. \\
 &\quad \left. + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi} \right), \\
 \mathcal{E}_{1,2}^{\text{ext}} &:= \int_{\mathcal{M}^{\text{ext}}} \left( \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) + \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \right), \\
 \mathcal{E}_{2,1}^{\text{ext}} &:= \int_{\mathcal{M}^{\text{ext}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi} + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi} \right. \\
 &\quad \left. + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}}) \hat{\pi} + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi} \right), \\
 \mathcal{E}_{2,2}^{\text{ext}} &:= \int_{\mathcal{M}^{\text{ext}}} \left( \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) + \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \right. \\
 &\quad \left. + \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \right).
 \end{aligned}$$

#### 5.3.1 Preliminary definitions and computational results

**Null decompositions of  $\hat{\pi}$ .** We define the following null decompositions of the deformation tensor  $\hat{\pi}$

$$\begin{aligned}
 {}^{(X)}\mathbf{i}_{ab} &= {}^{(X)}\hat{\pi}_{ab}, & {}^{(X)}\mathbf{j} &= {}^{(X)}\hat{\pi}_{34}, \\
 {}^{(X)}\mathbf{m}_a &= {}^{(X)}\hat{\pi}_{4a}, & {}^{(X)}\mathbf{\underline{m}}_a &= {}^{(X)}\hat{\pi}_{3a}, \\
 {}^{(X)}\mathbf{n} &= {}^{(X)}\hat{\pi}_{44}, & {}^{(X)}\mathbf{\underline{n}} &= {}^{(X)}\hat{\pi}_{33}.
 \end{aligned}$$

We have for the time translation  $\mathbf{T}^{\text{ext}} = \frac{1}{2}(e_4 + e_3)$

$$\operatorname{tr}^{(\mathbf{T}^{\text{ext}})} \pi = -2\underline{\omega} + \operatorname{tr} \chi + \operatorname{tr} \underline{\chi},$$

and

$$\begin{aligned} (\mathbf{T}^{\text{ext}})\mathbf{n} &= 0, & (\mathbf{T}^{\text{ext}})\underline{\mathbf{n}} &= -4\underline{\omega}, \\ (\mathbf{T}^{\text{ext}})\mathbf{m}_a &= -2\underline{\zeta}_a, & (\mathbf{T}^{\text{ext}})\underline{\mathbf{m}}_a &= \underline{\xi}_a + 2\underline{\zeta}_a, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{T}^{\text{ext}})\mathbf{j} &= \underline{\omega} + \frac{1}{2}(\text{tr}\chi + \text{tr}\underline{\chi}), \\ (\mathbf{T}^{\text{ext}})\mathbf{i}_{ab} &= \frac{1}{2}\underline{\omega}\underline{g}_{ab} + \frac{1}{4}(\text{tr}\chi + \text{tr}\underline{\chi})\underline{g}_{ab} + \hat{\chi}_{ab} + \hat{\underline{\chi}}_{ab}. \end{aligned}$$

We have for the scaling vectorfield  $\mathbf{S}^{\text{ext}} = \frac{1}{2}(ue_3 + \underline{u}e_4)$

$$\text{tr}^{(\mathbf{S}^{\text{ext}})}\pi = 4 - 2u\underline{\omega} + \underline{u}\text{tr}\chi + \text{tr}\underline{\chi},$$

and

$$\begin{aligned} (\mathbf{S}^{\text{ext}})\mathbf{n} &= 0, & (\mathbf{S}^{\text{ext}})\underline{\mathbf{n}} &= -2\underline{\eta} - 4\underline{u}\underline{\omega}, \\ (\mathbf{S}^{\text{ext}})\mathbf{m}_a &= -2u\underline{\zeta}_a, & (\mathbf{S}^{\text{ext}})\underline{\mathbf{m}}_a &= u\underline{\xi}_a + 2\underline{u}\underline{\zeta}_a, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{S}^{\text{ext}})\mathbf{j} &= -2 + u\underline{\omega} + \frac{1}{2}(\underline{u}\text{tr}\chi + \text{tr}\underline{\chi}), \\ (\mathbf{S}^{\text{ext}})\mathbf{i}_{ab} &= -\underline{g}_{ab} + \frac{1}{2}u\underline{\omega}\underline{g}_{ab} + \frac{1}{4}(\underline{u}\text{tr}\chi + \text{tr}\underline{\chi})\underline{g}_{ab} + \underline{u}\hat{\chi}_{ab} + u\hat{\underline{\chi}}_{ab}. \end{aligned}$$

We have for the conformal Morawetz vectorfield  $\mathbf{K}^{\text{ext}} = \frac{1}{2}(u^2e_3 + \underline{u}^2e_4)$

$$\text{tr}^{(\mathbf{K}^{\text{ext}})}\pi = 4(u + \underline{u}) - 2u^2\underline{\omega} + \underline{u}^2\text{tr}\chi + u^2\text{tr}\underline{\chi},$$

and

$$\begin{aligned} (\mathbf{K}^{\text{ext}})\mathbf{n} &= 0, & (\mathbf{K}^{\text{ext}})\underline{\mathbf{n}} &= -4u\underline{\eta} - 4\underline{u}^2\underline{\omega}, \\ (\mathbf{K}^{\text{ext}})\mathbf{m}_a &= -2u^2\underline{\zeta}_a, & (\mathbf{K}^{\text{ext}})\underline{\mathbf{m}}_a &= u^2\underline{\xi}_a + 2\underline{u}^2\underline{\zeta}_a, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{K}^{\text{ext}})\mathbf{j} &= -2(u + \underline{u}) + u^2\underline{\omega} + \frac{1}{2}(u^2\text{tr}\chi + \underline{u}^2\text{tr}\underline{\chi}), \\ (\mathbf{K}^{\text{ext}})\mathbf{i}_{ab} &= -(u + \underline{u})\underline{g}_{ab} + \frac{1}{2}u^2\underline{\omega}\underline{g}_{ab} + \frac{1}{4}(u^2\text{tr}\chi + \underline{u}^2\text{tr}\underline{\chi})\underline{g}_{ab} + \underline{u}^2\hat{\chi}_{ab} + u^2\hat{\underline{\chi}}_{ab}. \end{aligned}$$

We have for the rotations  $\mathbf{O}^{\text{ext}}$

$$\text{tr}^{(\mathbf{O}^{\text{ext}})}\pi = 2\underline{\zeta} \cdot \mathbf{O}^{\text{ext}} + \text{tr}^{(i)}H,$$

and

$$\begin{aligned} (\mathbf{O}^{\text{ext}})\mathbf{n} &= 0, & (\mathbf{O}^{\text{ext}})\underline{\mathbf{n}} &= -4\underline{\xi} \cdot \mathbf{O}^{\text{ext}}, \\ (\mathbf{O}^{\text{ext}})\mathbf{m}_a &= 0, & (\mathbf{O}^{\text{ext}})\underline{\mathbf{m}}_a &= {}^{(i)}Y_a, \end{aligned}$$

and

$$\begin{aligned} (\mathbf{O}^{\text{ext}})\mathbf{j} &= -\underline{\zeta} \cdot \mathbf{O}^{\text{ext}} + \frac{1}{2}\text{tr}^{(i)}H, \\ (\mathbf{O}^{\text{ext}})\mathbf{i}_{ab} &= {}^{(i)}H_{ab} - \frac{1}{4}\left(2\underline{\zeta} \cdot \mathbf{O}^{\text{ext}} + \text{tr}^{(i)}H\right)\underline{g}_{ab}. \end{aligned}$$

**Null decomposition of  $\mathbf{D}\hat{\pi}$**  We have the following definitions for (contractions of)  $\mathbf{D}\hat{\pi}$

$$\begin{aligned} {}^{(X)}p_\mu &:= \text{div}^{(X)}\hat{\pi}_\mu, \\ {}^{(X)}q_{\mu\nu\lambda} &:= \mathbf{D}_\nu^{(X)}\hat{\pi}_{\lambda\mu} - \mathbf{D}_\lambda^{(X)}\hat{\pi}_{\nu\mu} - \frac{1}{3}\left({}^{(X)}p_\lambda\mathbf{g}_{\mu\nu} - {}^{(X)}p_\nu\mathbf{g}_{\mu\lambda}\right). \end{aligned}$$

We note  $p_3, p_4$  and  $\not{p}$  the null decomposition of the spacetime vectorfield  $p$ .

We note  $\Lambda(q), K(q), \Xi(q), I(q), \Theta(q)$  and  $\underline{\Lambda}(q), \underline{K}(q), \underline{\Xi}(q), \underline{I}(q), \underline{\Theta}(q)$  the null decomposition of the spacetime 3-covariant tensor  $q$ , and we refer the reader to [CK93, p. 212] for its definition.

**The currents  $J$**  We define the *current*  $J$  of a Weyl tensor  $W$  to be

$$J(W)_{\nu\lambda\gamma} := \mathbf{D}^\mu W_{\mu\nu\lambda\gamma}.$$

We note  $\Lambda(J), K(J), \Xi(J), I(J), \Theta(J)$  and  $\underline{\Lambda}(J), \underline{K}(J), \underline{\Xi}(J), \underline{I}(J), \underline{\Theta}(J)$  the null decomposition of  $J$  (see [CK93]).

For a vectorfield  $X$ , and a Weyl tensor  $W$ , we note  $J(X, W)$  the current of  $\hat{\mathcal{L}}_X W$ .

The divergence of the Bel-Robinson tensors is related to the null decomposition of the associated Weyl field and the null decomposition of its current. When the Weyl field is a modified Lie derivative  $\hat{\mathcal{L}}_X W$ , its current  $J(X, W)$  can be expressed in terms of the null decomposition of  ${}^{(X)}\hat{\pi}$ , the null decomposition of  $\mathbf{D}^{(X)}\hat{\pi}$  and null decompositions of  $W$  and  $\mathbf{D}W$ . See [CK93, pp. 214–218] for the computations. We use these formulas in the following Sections 5.3.3–5.3.8.

### 5.3.2 Preliminary sup-norm estimates for the deformation tensors $\pi$

We define the following decay norms for the null decompositions of the deformation tensors of  $\mathbf{T}^{\text{ext}}, \mathbf{S}^{\text{ext}}, \mathbf{K}^{\text{ext}}$  and  $\mathbf{O}^{\text{ext}}$ .

$$\begin{aligned} \mathfrak{D}_0^{\text{ext}}[\mathbf{T}^{\text{ext}}] &:= \left\| u^{3/2} \underline{u}^{(\mathbf{T}^{\text{ext}})} \mathbf{i} \right\|_{L^\infty} + \left\| u^{3/2} \underline{u}^{(\mathbf{T}^{\text{ext}})} \mathbf{j} \right\|_{L^\infty} + \left\| \underline{u}^2 u^{1/2(\mathbf{T}^{\text{ext}})} \mathbf{m} \right\|_{L^\infty} + \left\| \underline{u} u^{3/2(\mathbf{T}^{\text{ext}})} \underline{\mathbf{m}} \right\|_{L^\infty} \\ &\quad + \left\| \underline{u} u^{3/2(\mathbf{T}^{\text{ext}})} \underline{\mathbf{n}} \right\|_{L^\infty} \end{aligned}$$

$$\begin{aligned} \mathfrak{D}_0^{\text{ext}}[\mathbf{S}^{\text{ext}}] &:= \left\| \underline{u} u^{1/2(\mathbf{S}^{\text{ext}})} \mathbf{i} \right\|_{L^\infty} + \left\| \underline{u} u^{1/2(\mathbf{S}^{\text{ext}})} \mathbf{j} \right\|_{L^\infty} + \left\| \underline{u}^2 u^{1/2(\mathbf{S}^{\text{ext}})} \mathbf{m} \right\|_{L^\infty} + \left\| \underline{u} u^{1/2(\mathbf{S}^{\text{ext}})} \underline{\mathbf{m}} \right\|_{L^\infty} \\ &\quad + \left\| u^{3/2(\mathbf{S}^{\text{ext}})} \underline{\mathbf{n}} \right\|_{L^\infty}, \end{aligned}$$

$$\begin{aligned} \mathfrak{D}_0^{\text{ext}}[\mathbf{K}^{\text{ext}}] &:= \left\| u^{1/2(\mathbf{K}^{\text{ext}})} \mathbf{i} \right\|_{L^\infty} + \left\| u^{1/2(\mathbf{K}^{\text{ext}})} \mathbf{j} \right\|_{L^\infty} + \left\| \underline{u}^2 u^{-3/2(\mathbf{K}^{\text{ext}})} \mathbf{m} \right\|_{L^\infty} + \left\| u^{1/2(\mathbf{K}^{\text{ext}})} \underline{\mathbf{m}} \right\|_{L^\infty} \\ &\quad + \left\| u^{3/2} \underline{u}^{-1(\mathbf{K}^{\text{ext}})} \underline{\mathbf{n}} \right\|_{L^\infty}, \end{aligned}$$

$$\mathfrak{D}_0^{\text{ext}}[\mathbf{O}^{\text{ext}}] := \left\| \underline{u} u^{1/2(\mathbf{O}^{\text{ext}})} \mathbf{i} \right\|_{L^\infty} + \left\| \underline{u} u^{1/2(\mathbf{O}^{\text{ext}})} \mathbf{j} \right\|_{L^\infty} + \left\| \underline{u} u^{1/2(\mathbf{O}^{\text{ext}})} \underline{\mathbf{m}} \right\|_{L^\infty} + \left\| u^{3/2(\mathbf{O}^{\text{ext}})} \underline{\mathbf{n}} \right\|_{L^\infty},$$

together with

$$\begin{aligned} {}^{(\mathbf{T}^{\text{ext}})} \underline{\mathbf{n}} = {}^{(\mathbf{S}^{\text{ext}})} \underline{\mathbf{n}} = {}^{(\mathbf{K}^{\text{ext}})} \underline{\mathbf{n}} = {}^{(\mathbf{O}^{\text{ext}})} \underline{\mathbf{n}} &= 0, \\ {}^{(\mathbf{O}^{\text{ext}})} \mathbf{m} &= 0. \end{aligned}$$

From the Bootstrap Assumptions 3.23 and the formulas from Section 5.3.1 relating the null decompositions of the deformation tensors of  $\mathbf{T}^{\text{ext}}, \mathbf{S}^{\text{ext}}, \mathbf{K}^{\text{ext}}$  and  $\mathbf{O}^{\text{ext}}$ , the null connection coefficients and the rotation coefficients  ${}^{(i)}Y$  and  ${}^{(i)}H$ , we have

$$\mathfrak{D}_0^{\text{ext}}[\mathbf{T}^{\text{ext}}, \mathbf{S}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{O}^{\text{ext}}] \lesssim D\varepsilon. \quad (5.10)$$

**Remark 5.8.** We detail the estimate for  ${}^{(\mathbf{S}^{\text{ext}})} \mathbf{j}$ . We have

$$\begin{aligned} {}^{(\mathbf{S}^{\text{ext}})} \mathbf{j} &= -2 + \frac{1}{2}(\underline{u} \text{tr} \chi + u \text{tr} \underline{\chi}) + u \underline{\omega} \\ &= -2 + \frac{u-u}{r} + \frac{1}{2} \underline{u} \left( \text{tr} \chi - \frac{2}{r} \right) + \frac{1}{2} u \left( \text{tr} \underline{\chi} + \frac{2}{r} \right) + u \underline{\omega} \\ &= \frac{u-u-2r}{r} + \frac{1}{2} \underline{u} \left( \text{tr} \chi - \frac{2}{r} \right) + \frac{1}{2} u \left( \text{tr} \underline{\chi} + \frac{2}{r} \right) + u \underline{\omega}, \end{aligned}$$

and therefore

$$\left\| \underline{u} u^{1/2(\mathbf{S}^{\text{ext}})} \mathbf{j} \right\|_{L^\infty(\mathcal{M}^{\text{ext}})} \lesssim \left\| u^{1/2} \left( r - \frac{1}{2}(\underline{u} - u) \right) \right\|_{L^\infty(\mathcal{M}^{\text{ext}})} + \mathfrak{D}_{\leq 1}^{\text{ext}} \lesssim D\varepsilon,$$

where we used the bootstrap bound (3.4) for  $r$  and where we refer to Section 3.2.4 for the definition of  $\mathfrak{D}_{\leq 1}^{\text{ext}}$ .

**Remark 5.9.** The decay rates (5.10) are different from the ones obtained in [CK93, p. 223]. They are stronger to the ones obtained in [KN03, pp. 251-254] – in particular due to the stronger bounds (3.4) for the area radius  $r$  that we derived in this paper –, except for  $(\mathbf{T}^{\text{ext}})\underline{\mathbf{m}}$  which lose  $\underline{u}u^{-1}$  and for  $(\mathbf{O}^{\text{ext}})\underline{\mathbf{m}}$  which is non zero in our case. Both these differences are due to the presence of  $\underline{\xi}$  which is vanishing in the double null foliation of [KN03]. In the next sections we prove in particular that the control of the error terms is still valid despite these differences.

### 5.3.3 Estimates for $\mathcal{E}_{1,1}^{\text{ext}}, \mathcal{E}_{2,1}^{\text{ext}}$

The estimates for the error terms  $\mathcal{E}_{1,1}^{\text{ext}}$  and  $\mathcal{E}_{2,1}^{\text{ext}}$  are obtained exactly in the same manner and each split into obtaining a control of the spacetime integrals in  $\mathcal{M}^{\text{ext}}$  of the following three integrands:

$$Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi}, \quad Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}}) \hat{\pi}, \quad Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) \cdot (\mathbf{K}^{\text{ext}}) \hat{\pi}, \quad (5.11)$$

where  $W$  is a Weyl tensor such that energy estimates are performed for the following contracted Bel-Robinson tensors respectively

$$Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}), \quad Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}), \quad Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}).$$

Since we have same or stronger decay rates for the sup-norm of  $(\mathbf{K}^{\text{ext}})\hat{\pi}$  as in [KN03], we only treat the second term of (5.11) which involves  $(\mathbf{T}^{\text{ext}})\hat{\pi}$ , *i.e.*

$$\int_{\mathcal{M}^{\text{ext}}} Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}}) \hat{\pi}.$$

Arguing as in Section 5.2.1, one can obtain from the Bootstrap Assumptions 5.3 on the boundedness of the energy fluxes through the hypersurfaces  $\mathcal{C}_u$  and  $\Sigma_t^{\text{ext}}$  the following  $L^2(\mathcal{M}^{\text{ext}})$  bounds for the null decomposition of  $W$

$$\begin{aligned} \left\| u^{-1/2-\gamma} \underline{u}^2 \alpha(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| u^{-1/2-\gamma} \underline{u}^2 \beta(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \rho(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \sigma(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u} \underline{u} \beta(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \underline{\alpha}(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \end{aligned} \quad (5.12)$$

for all  $\gamma > 0$ .

We now decompose  $Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}})\hat{\pi}$  in terms of the null decompositions of  $W$  and  $(\mathbf{T}^{\text{ext}})\hat{\pi}$ . We have

$$Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}})\hat{\pi} = \underline{u}^4 Q(W)_{44}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} + \underline{u}^2 \underline{u}^2 Q(W)_{34}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} + \underline{u}^4 Q(W)_{33}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu}.$$

Rewriting schematically the formulas (6.2.45) (6.2.46) and (6.2.47) of [KN03, p. 275] (see also [CK93, pp. 248–250]), we have

$$\begin{aligned} &\underline{u}^4 Q(W)_{44}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} \\ &= \underline{u}^4 \left( |\alpha|^2 \underline{\mathbf{n}} + (\rho^2 + \sigma^2) \underline{\mathbf{n}} + |\beta|^2 \underline{\mathbf{j}} + \alpha \beta \underline{\mathbf{m}} + \rho \beta \underline{\mathbf{m}} + \sigma \beta \underline{\mathbf{m}} + |\beta|^2 \underline{\mathbf{i}} + \rho \alpha \underline{\mathbf{i}} + \sigma \alpha \underline{\mathbf{i}} \right), \\ &\underline{u}^2 \underline{u}^2 Q(W)_{34}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} \\ &= \underline{u}^2 \underline{u}^2 \left( |\beta|^2 \underline{\mathbf{n}} + |\underline{\beta}|^2 \underline{\mathbf{n}} + (\rho^2 + \sigma^2) \underline{\mathbf{j}} + \rho \beta \underline{\mathbf{m}} + \sigma \beta \underline{\mathbf{m}} + \rho \underline{\beta} \underline{\mathbf{m}} + \sigma \underline{\beta} \underline{\mathbf{m}} + (\rho^2 + \sigma^2) \underline{\mathbf{i}} + \beta \underline{\beta} \underline{\mathbf{i}} \right), \\ &\underline{u}^4 Q(W)_{33}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} \\ &= \underline{u}^4 \left( (\rho^2 + \sigma^2) \underline{\mathbf{n}} + |\alpha|^2 \underline{\mathbf{n}} + |\beta|^2 \underline{\mathbf{j}} + \underline{\alpha} \underline{\beta} \underline{\mathbf{m}} + \rho \underline{\beta} \underline{\mathbf{m}} + \sigma \underline{\beta} \underline{\mathbf{m}} + |\beta|^2 \underline{\mathbf{i}} + \rho \underline{\alpha} \underline{\mathbf{i}} + \sigma \underline{\alpha} \underline{\mathbf{i}} \right), \end{aligned}$$

where  $(\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}) = \text{null}(W)$  and  $(\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{m}}, \underline{\underline{\mathbf{m}}}, \underline{\mathbf{n}}, \underline{\underline{\mathbf{n}}}) = \left( (\mathbf{T}^{\text{ext}})\underline{\mathbf{i}}, (\mathbf{T}^{\text{ext}})\underline{\mathbf{j}}, (\mathbf{T}^{\text{ext}})\underline{\mathbf{m}}, (\mathbf{T}^{\text{ext}})\underline{\underline{\mathbf{m}}}, (\mathbf{T}^{\text{ext}})\underline{\mathbf{n}}, (\mathbf{T}^{\text{ext}})\underline{\underline{\mathbf{n}}} \right)$ .

Using the sup norm estimates (5.10) for the null decomposition  $(\mathbf{i}, \mathbf{j}, \mathbf{m}, \underline{\mathbf{m}}, \mathbf{n}, \underline{\mathbf{n}})$  of the deformation tensor  $(\mathbf{T}^{\text{ext}})_{\hat{\pi}}$ , we obtain that

$$\begin{aligned} & \left| \underline{u}^4 Q(W)_{44}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} \right| \\ & \lesssim D\varepsilon \left( \underline{u}^3 u^{-3/2} |\alpha|^2 + \underline{u}^3 u^{-3/2} |\beta|^2 + \underline{u}^3 u^{-3/2} |\alpha||\beta| \right. \\ & \quad + \underline{u}^2 u^{-1/2} |\rho||\beta| + \underline{u}^2 u^{-1/2} |\sigma||\beta| + \underline{u}^3 u^{-3/2} |\beta|^2 + \underline{u}^3 u^{-3/2} |\rho||\alpha| \\ & \quad \left. + \underline{u}^3 u^{-3/2} |\sigma||\alpha| \right), \end{aligned} \tag{5.13}$$

$$\begin{aligned} & \left| \underline{u}^2 u^2 Q(W)_{34}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} \right| \\ & \lesssim D\varepsilon \left( \underline{u} u^{1/2} |\beta|^2 + \underline{u} u^{1/2} (\rho^2 + \sigma^2) + \underline{u} u^{1/2} |\rho||\beta| \right. \\ & \quad + \underline{u} u^{1/2} |\sigma||\beta| + u^{3/2} |\rho||\beta| + u^{3/2} |\sigma||\beta| + \underline{u} u^{1/2} (\rho^2 + \sigma^2) \\ & \quad \left. + \underline{u} u^{1/2} |\beta||\beta| \right), \end{aligned} \tag{5.14}$$

$$\begin{aligned} & \left| u^4 Q(W)_{33}^{\mu\nu(\mathbf{T}^{\text{ext}})} \hat{\pi}_{\mu\nu} \right| \\ & \lesssim D\varepsilon \left( \underline{u}^{-1} u^{5/2} (\rho^2 + \sigma^2) + \underline{u}^{-1} u^{5/2} |\beta| + \underline{u}^{-2} u^{7/2} |\alpha||\beta| \right. \\ & \quad + \underline{u}^{-1} u^{5/2} |\rho||\beta| + \underline{u}^{-1} u^{5/2} |\sigma||\beta| + \underline{u}^{-1} u^{5/2} |\beta|^2 + \underline{u}^{-1} u^{5/2} |\rho||\alpha| \\ & \quad \left. + \underline{u}^{-1} u^{5/2} |\sigma||\alpha| \right). \end{aligned} \tag{5.15}$$

Applying Cauchy-Schwartz, one checks that the spacetime integral in  $\mathcal{M}^{\text{ext}}$  of all the terms from (5.13) can be controlled by the  $L^2(\mathcal{M}^{\text{ext}})$  norms of (5.12), which thus gives

$$\left| \int_{\mathcal{M}^{\text{ext}}} Q(W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) \cdot (\mathbf{T}^{\text{ext}})_{\hat{\pi}} \right| \lesssim (D\varepsilon)^3.$$

This concludes the estimate of the error terms  $\mathcal{E}_{1,1}^{\text{ext}}$  and  $\mathcal{E}_{2,1}^{\text{ext}}$ .

### 5.3.4 Preliminary $L^\infty L^4(S)$ estimates for $\mathbf{D}\pi$

We have the following definition of the decay norms for the null decompositions of the tensors  $p$  and  $q$

$$\begin{aligned} \mathfrak{D}_{1,p}^{\text{ext}}[\mathbf{T}^{\text{ext}}] & := \left\| \underline{u}^{1/2} u^{5/2} (\mathbf{T}^{\text{ext}}) p_3 \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} + \left\| \underline{u}^{3/2} u^{3/2} (\mathbf{T}^{\text{ext}}) p_4 \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} \\ & \quad + \left\| \underline{u}^{3/2} u^{3/2} (\mathbf{T}^{\text{ext}}) \hat{p} \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})}, \\ \mathfrak{D}_{1,q}^{\text{ext}}[\mathbf{T}^{\text{ext}}] & := \left\| \underline{u}^{3/2} u^{3/2} \Lambda(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} + \left\| \underline{u}^{5/2} u^{1/2} K(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} \\ & \quad + \left\| \underline{u}^{5/2} u^{1/2} \Xi(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} + \left\| \underline{u}^{3/2} u^{3/2} I(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} \\ & \quad + \left\| \underline{u}^{3/2} u^{3/2} \Theta(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} + \left\| \underline{u}^{1/2} u^{5/2} \underline{\Lambda}(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} \\ & \quad + \left\| \underline{u}^{3/2} u^{3/2} \underline{K}(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} + \left\| \underline{u}^{1/2} u^{5/2} \underline{\Xi}(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} \\ & \quad + \left\| \underline{u}^{3/2} u^{3/2} \underline{I}(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})} + \left\| \underline{u}^{1/2} u^{5/2} \underline{\Theta}(\mathbf{T}^{\text{ext}}) q \right\|_{L_{u,\underline{u}}^\infty L^4(S_{u,\underline{u}})}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{D}_{1,p}^{\text{ext}}[\mathbf{S}^{\text{ext}}] &:= \left\| \underline{u}^{1/2} u^{3/2} (\mathbf{S}^{\text{ext}}) p_3 \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{3/2} u^{1/2} (\mathbf{S}^{\text{ext}}) p_4 \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} (\mathbf{S}^{\text{ext}}) \not{p} \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})}, \\ \mathfrak{D}_{1,q}^{\text{ext}}[\mathbf{S}^{\text{ext}}] &:= \left\| \underline{u}^{3/2} u^{1/2} \Lambda((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{5/2} u^{-1/2} K((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{5/2} u^{-1/2} \Xi((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{3/2} u^{1/2} I((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} \Theta((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{3/2} \underline{\Lambda}((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} \underline{K}((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{3/2} \underline{\Xi}((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} \underline{I}((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{3/2} \underline{\Theta}((\mathbf{S}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})}, \end{aligned}$$

and

$$\begin{aligned} \mathfrak{D}_{1,p}^{\text{ext}}[\mathbf{O}^{\text{ext}}] &:= \left\| \underline{u}^{1/2} u^{3/2} (\mathbf{O}^{\text{ext}}) p_3 \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{3/2} u^{1/2} (\mathbf{O}^{\text{ext}}) p_4 \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})}, \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} (\mathbf{O}^{\text{ext}}) \not{p} \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})}, \\ \mathfrak{D}_{1,q}^{\text{ext}}[\mathbf{O}^{\text{ext}}] &:= \left\| \underline{u}^{3/2} u^{1/2} \Lambda((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{5/2} u K((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} I((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} \Theta((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{3/2} \underline{\Lambda}((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} \underline{K}((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{3/2} \underline{\Xi}((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} \\ &\quad + \left\| \underline{u}^{3/2} u^{1/2} \underline{I}((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})} + \left\| \underline{u}^{1/2} u^{3/2} \underline{\Theta}((\mathbf{O}^{\text{ext}})q) \right\|_{L_{\underline{u}, \underline{u}}^\infty L^4(S_{u, \underline{u}})}, \end{aligned}$$

together with  $\Xi((\mathbf{O}^{\text{ext}})q) = 0$ .

Using formulas from [CK93, pp. 231–232], the Bootstrap Assumptions 3.23 for  $\tilde{H}^{1/2}$  norms of (one derivative of) the null connection coefficients in  $\mathcal{M}^{\text{ext}}$ , the product estimate from Lemma 3.37, the Sobolev embeddings from Lemma 3.36, we have

$$\mathfrak{D}_{1,p}^{\text{ext}} + \mathfrak{D}_{1,q}^{\text{ext}} \lesssim \mathfrak{D}_{\leq 1}^{\text{ext}} \lesssim D\varepsilon. \quad (5.18)$$

**Remark 5.10.** *The decay rate of each term is easily checked using that  $r\nabla$ ,  $\underline{u}\nabla_4$  and  $u\nabla_3$  preserve the decay rates, and that the decay of the  $L^4(S)$  norm loses a weight  $r^{1/2}$  with respect to the decay of the  $L^\infty(S)$  norm.*

**Remark 5.11.** *The decay rates in the norms  $\mathfrak{D}_1^{\text{ext}}[\mathbf{T}^{\text{ext}}]$ ,  $\mathfrak{D}_1^{\text{ext}}[\mathbf{S}^{\text{ext}}]$  and  $\mathfrak{D}_1^{\text{ext}}[\mathbf{O}^{\text{ext}}]$  are the same or better as the ones obtained in [KN03, pp. 250–258]<sup>5</sup> except for  $\underline{\Xi}(\mathbf{T}^{\text{ext}}q)$ , which loses a  $\underline{u}^{-1}$  factor due to the non vanishing of  $\underline{\xi}$  in the present paper.*

### 5.3.5 Estimates for $\mathcal{E}_{1,2}^{\text{ext}}$

We start with the estimate for

$$\int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}).$$

<sup>5</sup>By a scaling consideration, we do believe that there is a typo in the estimates (6.1.53) which should be multiplied by  $r^{-1}$ . Moreover, we do also believe that there is a typo in the estimate for  $\underline{\Xi}(\mathbf{S}^{\text{ext}}q)$  in [KN03] which should be multiplied by  $r^{-1}$ .



Arguing as in Section 5.2.1 using the Bootstrap Assumption 5.3, we first record the following  $L^2(\mathcal{M}^{\text{ext}})$  bounds for the null decomposition of  $\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R}$

$$\begin{aligned} \left\| \underline{u}^{-1/2-\gamma} \underline{u}^3 \alpha \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^3 \beta \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 u \rho \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 u \sigma \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u} u^2 \underline{\beta} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} u^3 \underline{\alpha} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \end{aligned} \quad (5.19)$$

for all  $\gamma > 0$ .

We have

$$\begin{aligned} \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) &= \underline{u}^6 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{444} + \underline{u}^4 u^2 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{443} \\ &\quad + \underline{u}^2 u^4 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{433} + u^6 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{333}. \end{aligned} \quad (5.20)$$

From [CK93, p. 213], we have

$$\begin{aligned} \underline{u}^6 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{444} &= \underline{u}^6 \alpha \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \Theta(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \\ &\quad + \underline{u}^6 \beta \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \Xi(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})), \\ \underline{u}^4 u^2 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{443} &= \underline{u}^4 u^2 \rho \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \Lambda(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \\ &\quad + \underline{u}^4 u^2 \sigma \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) K(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \\ &\quad + \underline{u}^4 u^2 \beta \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) I(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})), \\ \underline{u}^2 u^4 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{433} &= \underline{u}^2 u^4 \rho \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \underline{\Lambda}(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \\ &\quad + \underline{u}^2 u^4 \sigma \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \underline{K}(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \\ &\quad + \underline{u}^2 u^4 \beta \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \underline{I}(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})), \\ u^6 \text{div}Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})_{333} &= u^6 \underline{\alpha} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \underline{\Theta}(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \\ &\quad + u^6 \underline{\beta} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \underline{\Xi}(J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})), \end{aligned} \quad (5.21)$$

We only examine the term

$$\int_{\mathcal{M}^{\text{ext}}} \underline{u}^6 \beta(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R}) \cdot \Xi(J). \quad (5.22)$$

of (5.21) which is the hardest to treat due to the pairing of high weights in  $\underline{u}$  with slower decaying null components of  $\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R}$ .

Using the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (5.19), our goal in what follows is to obtain the following  $L^2(\mathcal{M}^{\text{ext}})$  estimates for  $\Xi(J)$

$$\left\| \underline{u}^{3+1/2+\gamma} \Xi(J) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim (D\varepsilon)^2. \quad (5.23)$$

Using the formulas from [CK93, p. 215], the estimates (5.10) for the sup-norm and the estimates (5.18) for

the  $L^\infty L^4$  norm of the null decompositions of  $\pi$  and  $\mathbf{D}\pi$  respectively, we have

$$\begin{aligned} \left\| \Xi(J^1(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-1}u^{-3/2} \|\nabla\alpha(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \right. \\ &\quad + \underline{u}^{-2}u^{-1/2} \|\nabla_3\alpha(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-1}u^{-3/2} \|\nabla_4\alpha(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-2}u^{-1/2} \|\nabla\beta(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-1}u^{-3/2} \|\nabla_4\beta(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-2}u^{-3/2} \|\alpha(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-2}u^{-3/2} \|\beta(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad \left. + \underline{u}^{-3}u^{-1/2} \|(\rho(\mathbf{R}), \sigma(\mathbf{R}))\|_{L^2(S_{u,\underline{u}})} \right), \end{aligned} \quad (5.24)$$

and

$$\begin{aligned} \left\| \Xi(J^2(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-3/2}u^{-3/2} \|\alpha(\mathbf{R})\|_{L^4(S_{u,\underline{u}})} \right. \\ &\quad \left. + \underline{u}^{-3/2}u^{-3/2} \|\beta(\mathbf{R})\|_{L^4(S_{u,\underline{u}})} \right), \\ \left\| \Xi(J^3(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-3/2}u^{-3/2} \|\alpha(\mathbf{R})\|_{L^4(S_{u,\underline{u}})} \right. \\ &\quad + \underline{u}^{-3/2}u^{-3/2} \|\beta(\mathbf{R})\|_{L^4(S_{u,\underline{u}})} \\ &\quad \left. + \underline{u}^{-5/2}u^{-1/2} \|(\rho(\mathbf{R}), \sigma(\mathbf{R}))\|_{L^4(S_{u,\underline{u}})} \right). \end{aligned} \quad (5.25)$$

Using the Sobolev estimates on the spheres  $S_{u,\underline{u}}$  from Lemma 3.36, we further deduce from (5.25)

$$\begin{aligned} \left\| \Xi(J^2(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-2}u^{-3/2} \|(r\nabla)^{\leq 1}\alpha(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \right. \\ &\quad \left. + \underline{u}^{-2}u^{-3/2} \|(r\nabla)^{\leq 1}\beta(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \right), \\ \left\| \Xi(J^3(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-2}u^{-3/2} \|(r\nabla)^{\leq 1}\alpha(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \right. \\ &\quad + \underline{u}^{-2}u^{-3/2} \|(r\nabla)^{\leq 1}\beta(\mathbf{R})\|_{L^2(S_{u,\underline{u}})} \\ &\quad \left. + \underline{u}^{-3}u^{-1/2} \|(r\nabla)^{\leq 1}(\rho(\mathbf{R}), \sigma(\mathbf{R}))\|_{L^2(S_{u,\underline{u}})} \right). \end{aligned} \quad (5.26)$$

From (5.24) and (5.26), we obtain

$$\begin{aligned} \left\| \underline{u}^{3+1/2+\gamma} \Xi \left( J \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R} \right) \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim (D\varepsilon) \left( \left\| \underline{u}^{1+1/2+\gamma} u^{-3/2} (r\nabla)^{\leq 1} \alpha(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \right. \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-3/2} (u\nabla_3)\alpha(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-3/2} (\underline{u}\nabla_4)\alpha(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-1/2} (r\nabla)^{\leq 1} \beta(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-3/2} (\underline{u}\nabla_4)\beta(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad \left. + \left\| \underline{u}^{1/2+\gamma} u^{-1/2} (r\nabla)^{\leq 1} (\rho(\mathbf{R}), \sigma(\mathbf{R})) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \right). \end{aligned} \quad (5.27)$$

We check that for  $\gamma < 1/4$ , all the above  $L^2(\mathcal{M}^{\text{ext}})$  norms can be bounded by  $\mathcal{R}_{\leq 2, \gamma_0}^{\text{ext}}$ , where we recall that  $\gamma_0 < 1/4$ . Thus, from the Bootstrap Assumptions 3.20 on the  $L^2(\mathcal{M}^{\text{ext}})$  norms of the null curvature

components, we have

$$\left\| \underline{u}^{3+1/2+\gamma} \Xi \left( J \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim (D\varepsilon) \mathcal{R}_{\leq 2, \gamma_0}^{\text{ext}} \lesssim (D\varepsilon)^2,$$

as desired. This finishes the control of (5.22).

We now repeat the procedure to estimate

$$\int_{\mathcal{M}^{\text{ext}}} \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}),$$

which is the second error term of  $\mathcal{E}_{1,2}^{\text{ext}}$ .

We first record the following  $L^2(\mathcal{M}^{\text{ext}})$  bounds for the null decomposition of  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}$

$$\begin{aligned} \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \rho \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \sigma \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u} \underline{\beta} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \underline{\alpha} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \end{aligned} \quad (5.28)$$

for all  $\gamma > 0$ .

We have

$$\begin{aligned} \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) &\lesssim \underline{u}^4 \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})_{444} + \underline{u}^4 \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})_{443} \\ &\quad + \underline{u}^2 \underline{u}^2 \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})_{433} + \underline{u}^4 \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})_{333}. \end{aligned} \quad (5.29)$$

The most critical terms of (5.29) are  $\underline{u}^4 \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})_{443}$  since it has the highest  $\underline{u}$  weight and the lowest signature.

We have

$$\begin{aligned} \underline{u}^4 \mathbf{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})_{443} &= \underline{u}^4 \rho \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \Lambda \left( J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right) \\ &\quad + \underline{u}^4 \sigma \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) K \left( J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right) \\ &\quad + \underline{u}^4 \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \cdot I \left( J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right). \end{aligned} \quad (5.30)$$

We only check the first error term of (5.30), the second will follow by duality and the third is easier to treat because of the stronger decay for  $\beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right)$ .

In view of the bounds (5.28), our goal is to prove

$$\left\| \underline{u}^{2+1/2+\gamma} \Lambda \left( J \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim (D\varepsilon)^2. \quad (5.31)$$

Using the formulas from [CK93, p. 216], the estimates (5.10) for the sup-norm and the estimates (5.18) for the  $L^\infty L^4$  norm of the null decompositions of  $\pi$  and  $\mathbf{D}\pi$  respectively, we have

$$\begin{aligned} \left\| \Lambda \left( J^1 \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) \right) \right\|_{L^2(S_{u, \underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-1} \underline{u}^{-1/2} \|\nabla \beta(\mathbf{R})\|_{L^2(S_{u, \underline{u}})} \right. \\ &\quad + \underline{u}^{-1} \underline{u}^{-1/2} \|\nabla_4 \beta(\mathbf{R})\|_{L^2(S_{u, \underline{u}})} \\ &\quad + \underline{u}^{-1} \underline{u}^{-1/2} \|(\rho_4(\mathbf{R}), \sigma_4(\mathbf{R}))\|_{L^2(S_{u, \underline{u}})} \\ &\quad + \underline{u}^{-2} \underline{u}^{-1/2} \|\alpha(\mathbf{R})\|_{L^2(S_{u, \underline{u}})} \\ &\quad + \underline{u}^{-2} \underline{u}^{-1/2} \|\beta(\mathbf{R})\|_{L^2(S_{u, \underline{u}})} \\ &\quad \left. + \underline{u}^{-2} \underline{u}^{-1/2} \|(\rho(\mathbf{R}), \sigma(\mathbf{R}))\|_{L^2(S_{u, \underline{u}})} \right), \end{aligned} \quad (5.32)$$

and

$$\begin{aligned}
 \left\| \Lambda(J^2(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-3/2} u^{-1/2} \|\beta(\mathbf{R})\|_{L^4(S_{u,\underline{u}})} \right. \\
 &\quad \left. + \underline{u}^{-3/2} u^{-1/2} \|(\rho(\mathbf{R}), \sigma(\mathbf{R}))\|_{L^4(S_{u,\underline{u}})} \right), \\
 \left\| \Lambda(J^3(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-1/2} u^{-3/2} \|\alpha(\mathbf{R})\|_{L^4(S_{u,\underline{u}})} \right. \\
 &\quad \left. + \underline{u}^{-3/2} u^{-1/2} \|(\rho(\mathbf{R}), \sigma(\mathbf{R}))\|_{L^4(S_{u,\underline{u}})} \right).
 \end{aligned} \tag{5.33}$$

Arguing as previously, using Sobolev estimates on the 2-spheres  $S_{u,\underline{u}}$ , we deduce from (5.32) and (5.33)

$$\begin{aligned}
 \left\| \underline{u}^{2+1/2+\gamma} \Lambda \left( J \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R} \right) \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim (D\varepsilon) \left( \left\| \underline{u}^{1/2+\gamma} u^{-1/2} (r\nabla) \beta(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \right. \\
 &\quad + \left\| \underline{u}^{1/2+\gamma} u^{-1/2} (\underline{u}\nabla_4) \beta(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{1/2+\gamma} u^{-1/2} (\underline{u}\rho_4(\mathbf{R}), \underline{u}\sigma_4(\mathbf{R})) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{3/2+\gamma} u^{-3/2} (r\nabla)^{\leq 1} \alpha(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{1/2+\gamma} u^{-1/2} (r\nabla)^{\leq 1} \beta(\mathbf{R}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad \left. + \left\| \underline{u}^{1/2+\gamma} u^{-3/2} (r\nabla)^{\leq 1} (\rho(\mathbf{R}), \sigma(\mathbf{R})) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \right).
 \end{aligned} \tag{5.34}$$

We verify that from (5.34), one has

$$\left\| \underline{u}^{2+1/2+\gamma} \Lambda \left( J \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R} \right) \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim (D\varepsilon) \mathcal{R}_{\leq 2, \gamma_0}^{\text{ext}} \lesssim (D\varepsilon)^2,$$

for  $0 < \gamma < 1/4$ , and (5.31) is proved. This finishes the control of the error term  $\mathcal{E}_{1,2}^{\text{ext}}$ .

### 5.3.6 Preliminary $L^2(\mathcal{M}^{\text{ext}})$ estimates for $\mathbf{D}^2\pi$

We have the following definitions for the  $L^2(\mathcal{M}^{\text{ext}})$  norms for derivatives of  $(\mathbf{T}^{\text{ext}})_p, (\mathbf{T}^{\text{ext}})_q$  and  $(\mathbf{O}^{\text{ext}})_p, (\mathbf{O}^{\text{ext}})_q$

$$\begin{aligned}
 \mathfrak{D}_{2,p,\gamma_0}^{\text{ext}}[\mathbf{T}^{\text{ext}}] &:= \left\| \underline{u}^{-1/2-\gamma_0} u^2 \hat{\mathcal{L}}_X(\mathbf{T}^{\text{ext}}) p_3 \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X(\mathbf{T}^{\text{ext}}) p_4 \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X(\mathbf{T}^{\text{ext}}) p_a \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 \mathfrak{D}_{2,q,\gamma_0}^{\text{ext}}[\mathbf{T}^{\text{ext}}] &:= \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X \Lambda(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} u^2 \hat{\mathcal{L}}_X K(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{-1/2-\gamma_0} u^2 \hat{\mathcal{L}}_X \Xi(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X I(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X \Theta(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} u^2 \hat{\mathcal{L}}_X \underline{\Lambda}(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X \underline{K}(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} u^2 \hat{\mathcal{L}}_X \underline{\Xi}(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} u \hat{\mathcal{L}}_X \underline{I}(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} u^2 \hat{\mathcal{L}}_X \underline{\Theta}(\mathbf{T}^{\text{ext}}) q \right\|_{L^2(\mathcal{M}^{\text{ext}})},
 \end{aligned}$$

where  $X \in \{\mathbf{O}^{\text{ext}}, \mathbf{S}^{\text{ext}}\}$ , and

$$\begin{aligned} \mathfrak{D}_{2,p,\gamma_0}^{\text{ext}}[\mathbf{O}^{\text{ext}}] &:= \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}(\mathbf{O}^{\text{ext}}) p_3 \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}(\mathbf{O}^{\text{ext}}) p_4 \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \\ &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}(\mathbf{O}^{\text{ext}}) p_a \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \\ \mathfrak{D}_{2,q,\gamma_0}^{\text{ext}}[\mathbf{O}^{\text{ext}}] &:= \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \Lambda((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u}^2 u^{1/2} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} K((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} I((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \Theta((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\Delta}((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{K}((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\Xi}((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{I}((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| \underline{u}^{-1/2-\gamma_0} \underline{u} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\Theta}((\mathbf{O}^{\text{ext}})q) \right\|_{L^2(\mathcal{M}^{\text{ext}})}, \end{aligned}$$

together with  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\Xi}((\mathbf{O}^{\text{ext}})q) = 0$ .

Differentiating the formulas for  $p, q$  from [CK93, pp. 231–232] and the formulas for the null decompositions of  $(\mathbf{T}^{\text{ext}})\hat{\pi}$  and  $(\mathbf{O}^{\text{ext}})\hat{\pi}$  from Sections 5.3.1, using the Bootstrap Assumptions 3.23 for the  $L^2(\mathcal{M}^{\text{ext}})$  norms of (two derivatives) of the null connection coefficients, we obtain

$$\mathfrak{D}_{2,\gamma_0}^{\text{ext}} \lesssim \mathcal{O}_{\leq 2,\gamma_0}^{\text{ext}} \lesssim D\varepsilon, \quad (5.35)$$

for all  $\gamma > 0$ .

**Remark 5.12.** *The decay rates are easily checked, using that deriving with respect to  $\mathbf{O}^{\text{ext}}$  or  $\mathbf{S}^{\text{ext}}$  does not change the asymptotic behaviour of the components.*

### 5.3.7 Estimates for $\mathcal{E}_{2,2}^{\text{ext}}$

We start with the estimate for

$$\int_{\mathcal{M}^{\text{ext}}} \text{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}). \quad (5.36)$$

We first record the following  $L^2(\mathcal{M}^{\text{ext}})$  bounds for the null decomposition of  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R}$  which are consequences of the Bootstrap Assumptions 5.3

$$\begin{aligned} \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \rho \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \sigma \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u} u \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 \underline{\alpha} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \end{aligned} \quad (5.37)$$

for all  $\gamma > 0$ .

Similarly to Section 5.3.5, the decompositions on the null directions  $(e_3, e_4)$  gives

$$\begin{aligned} \text{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}) &\lesssim \underline{u}^4 \text{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{444} + \underline{u}^4 \text{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{443} \\ &\quad + \underline{u}^2 u^2 \text{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{433} + u^4 \text{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{333}. \end{aligned} \quad (5.38)$$

and

$$\begin{aligned}
 \underline{u}^4 \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{444} &= \underline{u}^4 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \Theta(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})) \\
 &\quad + \underline{u}^4 \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \Xi(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})), \\
 \underline{u}^4 \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{443} &= \underline{u}^4 \rho \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \Lambda(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})) \\
 &\quad + \underline{u}^4 \sigma \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) K(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})) \\
 &\quad + \underline{u}^4 \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) I(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})), \\
 \underline{u}^2 u^2 \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{433} &= \underline{u}^2 u^2 \rho \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \underline{\Lambda}(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})) \\
 &\quad + \underline{u}^2 u^2 \sigma \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \underline{K}(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})) \\
 &\quad + \underline{u}^2 u^2 \beta \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \underline{I}(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})), \\
 u^4 \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})_{333} &= u^4 \underline{\alpha} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \underline{\Theta}(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})) \\
 &\quad + u^4 \underline{\beta} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \underline{\Xi}(J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R})).
 \end{aligned} \tag{5.39}$$

We have from the formulas [CK93, p. 206]

$$J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R}) = \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) + J^i(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}),$$

where the terms  $J^i(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})$  are the same as the terms  $J^i(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})$  from Section 5.3.5 with  $\mathbf{R}$  replaced by  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}$ . Therefore, the estimates of that section carry over and we only need to treat the term

$$\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} J(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}).$$

This term again has the same structure as the terms treated in Section 5.3.5, although differentiated by  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}$ . This does not change the decay of the components, and the difference with Section 5.3.5 is that one does not have a control for the  $L_{u, \underline{u}}^\infty L^4(S_{u, \underline{u}})$  norm of the components of the type  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{D}\hat{\pi}$ .

That case is actually (more) simply handled using the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (5.37) for the null decomposition of  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R}$ , the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (5.35) for  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{D}\hat{\pi}$ , the Bootstrap Assumptions 3.20 for the sup-norm of the curvature  $\mathbf{R}$  and Cauchy-Schwartz. As an example, we treat the first term of (5.39) containing a term of the type  $\mathbf{D}^2 \hat{\pi}$  which is

$$\underline{u}^4 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^{(\mathbf{O}^{\text{ext}})} p_3 \cdot \alpha(\mathbf{R}).$$

We have

$$\begin{aligned}
 &\left| \int_{\mathcal{M}^{\text{ext}}} \underline{u}^4 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^{(\mathbf{O}^{\text{ext}})} p_3 \cdot \alpha(\mathbf{R}) \right| \\
 &\lesssim (D\varepsilon) \int_{\mathcal{M}^{\text{ext}}} \underline{u}^{1/2} \left| \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right| \left| \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^{(\mathbf{O}^{\text{ext}})} p_3 \right| \\
 &\lesssim (D\varepsilon) \left\| u^{-1/2-\gamma} \underline{u}^2 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \left\| u^{1/2+\gamma} \underline{u}^{-3/2} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^{(\mathbf{O}^{\text{ext}})} p_3 \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\lesssim (D\varepsilon) \left\| u^{-1/2-\gamma} \underline{u}^2 \alpha \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2 \mathbf{R} \right) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \left\| \underline{u}^{-1/2-\gamma_0} u \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^{(\mathbf{O}^{\text{ext}})} p_3 \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\
 &\lesssim (D\varepsilon)^3,
 \end{aligned}$$

provided that  $0 < \gamma < 1/2$ .

All the other terms of (5.36) follow similarly, using the analysis of the decay rates already performed in Section 5.3.5.

To handle the last error terms of  $\mathcal{E}_{2,2}^{\text{ext}}$ , we have the following lemma.

**Lemma 5.13.** *Let  $W$  be a current-free Weyl field such that the following  $L^2(\mathcal{M}^{\text{ext}})$  bounds hold*

$$\begin{aligned} \left\| \underline{u}^{-1/2-\gamma} \underline{u}^3 \nabla^{\leq 1} \alpha(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^3 \nabla^{\leq 1} \beta(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 u \nabla^{\leq 1} \rho(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} \underline{u}^2 u \nabla^{\leq 1} \sigma(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \\ \left\| \underline{u}^{-1/2-\gamma} \underline{u} u^2 \nabla^{\leq 1} \underline{\beta}(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, & \left\| \underline{u}^{-1/2-\gamma} u^3 \nabla^{\leq 1} \underline{\alpha}(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim_{\gamma} D\varepsilon, \end{aligned} \quad (5.40)$$

for all  $\gamma > 0$  and where  $\nabla \in \{r\nabla, u\nabla_3, \underline{u}\nabla_4\}$ . Then, we have

$$\begin{aligned} \int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) &\lesssim (D\varepsilon)^3, \\ \int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}) &\lesssim (D\varepsilon)^3. \end{aligned}$$

The proof of Lemma 5.13 is similar to the control of the error terms from Section 5.3.5 and consists in checking that the integrands have the appropriate  $u, \underline{u}$  decay and are integrable on  $\mathcal{M}^{\text{ext}}$ . It is postponed to Section 5.3.8.

The estimates for the terms

$$\int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}})$$

and

$$\int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}})$$

now follow from Lemma 5.13, arguing as above, and using the decay rates analysis for

$$\int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}})$$

performed in Section 5.3.5 to handle the terms from the non-vanishing current  $\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})$  and  $\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} J(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})$ . This finishes the control of the error terms  $\mathcal{E}^{\text{ext}}$ .

### 5.3.8 Proof of Lemma 5.13

Since from an inspection of (5.10) and (5.18), the deformation tensors of  $\mathbf{O}^{\text{ext}}$  satisfy better decay estimates than the deformation tensor of  $\mathbf{S}^{\text{ext}}$ , it is enough to control

$$\int_{\mathcal{M}^{\text{ext}}} \operatorname{div} Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W)(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}).$$

Decomposing the divergence on the null directions  $(e_3, e_4)$ , the most difficult terms to treat are

$$\int_{\mathcal{M}^{\text{ext}}} \underline{u}^6 \beta(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W) \cdot \Xi(J(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W)),$$

since it has the highest  $\underline{u}$  weight and the  $\beta$  component satisfies a weaker control than  $\alpha$ .

Using estimates (5.10) and (5.18) for the null decomposition of  $(\mathbf{S}^{\text{ext}})^{\hat{\pi}}$ , the formulas from [CK93, p. 215–217],

and Sobolev estimates on the 2-sphere  $S_{u,\underline{u}}$  we have

$$\begin{aligned} \left\| \Xi(J^1(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W)) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-1} u^{-1/2} \|\nabla \alpha(W)\|_{L^2(S_{u,\underline{u}})} \right. \\ &\quad + \underline{u}^{-2} u^{1/2} \|\nabla_3 \alpha(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-1} u^{-1/2} \|\nabla_4 \alpha(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-2} u^{1/2} \|\nabla \beta(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-1} u^{-1/2} \|\nabla_4 \beta(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-2} u^{-1/2} \|\alpha(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad + \underline{u}^{-2} u^{-1/2} \|\beta(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad \left. + \underline{u}^{-3} u^{1/2} \|(\rho(W), \sigma(W))\|_{L^2(S_{u,\underline{u}})} \right), \end{aligned}$$

and

$$\begin{aligned} \left\| \Xi(J^2(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-2} u^{-1/2} \|(r\nabla)^{\leq 1} \alpha(W)\|_{L^2(S_{u,\underline{u}})} \right. \\ &\quad \left. + \underline{u}^{-2} u^{-1/2} \|(r\nabla)^{\leq 1} \beta(W)\|_{L^2(S_{u,\underline{u}})} \right), \\ \left\| \Xi(J^3(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \mathbf{R})) \right\|_{L^2(S_{u,\underline{u}})} &\lesssim (D\varepsilon) \left( \underline{u}^{-2} u^{-1/2} \|(r\nabla)^{\leq 1} \alpha(W)\|_{L^2(S_{u,\underline{u}})} \right. \\ &\quad + \underline{u}^{-2} u^{-1/2} \|(r\nabla)^{\leq 1} \beta(W)\|_{L^2(S_{u,\underline{u}})} \\ &\quad \left. + \underline{u}^{-3} u^{1/2} \|(r\nabla)^{\leq 1} (\rho(W), \sigma(W))\|_{L^2(S_{u,\underline{u}})} \right). \end{aligned}$$

Thus,

$$\begin{aligned} \left\| \underline{u}^{3+1/2+\gamma} \Xi(J(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W)) \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim (D\varepsilon) \left( \left\| \underline{u}^{1+1/2+\gamma} u^{-1/2} (r\nabla)^{\leq 1} \alpha(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \right. \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-1/2} (u\nabla_3) \alpha(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-1/2} (\underline{u}\nabla_4) \alpha(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-1/2} (r\nabla)^{\leq 1} \beta(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad + \left\| \underline{u}^{1+1/2+\gamma} u^{-1/2} (\underline{u}\nabla_4) \beta(W) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\quad \left. + \left\| \underline{u}^{1/2+\gamma} u^{1/2} (r\nabla)^{\leq 1} (\rho(W), \sigma(W)) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \right), \end{aligned}$$

and using the  $L^2(\mathcal{M}^{\text{ext}})$  bounds (5.40), we have

$$\left\| \underline{u}^{3+1/2+\gamma} \Xi(J(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} W)) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim (D\varepsilon)^2,$$

provided that  $0 < \gamma < 1$ . Arguing as previously, using Cauchy-Schwartz and the  $L^2(\mathcal{M}^{\text{ext}})$  bounds (5.40) gives the desired control. This finishes the proof of Lemma 5.13.





## 6. Null curvature estimates in $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ and $\mathcal{M}^{\text{ext}}$

In this section, we prove the following proposition.

**Proposition 6.1.** *Recall that from Proposition 5.1, we have*

$$\int_{(\tau)\Sigma_t^{\text{ext}}} P \cdot \bar{T}^{\text{ext}} + \int_{\mathcal{C}_u \cap (\tau)\mathcal{M}^{\text{ext}}} P \cdot e_4 + \int_{\underline{\mathcal{C}}^* \cap (\tau)\mathcal{M}^{\text{ext}}} P \cdot e_3 \lesssim \varepsilon^2, \quad (6.1)$$

for a fixed transition parameter  $\tau$ , for all  $1 \leq u \leq \tau u^*$  and for all  $t^0 \leq t \leq t^*$ , and where  $P$  denote the following contracted and commuted Bel-Robinson tensors

$$\begin{aligned} & Q\left(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R}\right)\left(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}\right), \quad Q\left(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R}\right)\left(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}\right), \\ & Q\left(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R}\right)\left(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}\right), \quad Q\left(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R}\right)\left(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}\right), \\ & Q\left(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R}\right)\left(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}\right). \end{aligned}$$

Under the Bootstrap Assumptions, the energy estimates (6.1), and for  $\varepsilon > 0$  sufficiently small, we have

◇ the following  $L^2$  bounds on the exterior cone  $\underline{\mathcal{C}}^* \cap (\tau)\mathcal{M}^{\text{ext}}$

$$\mathcal{R}_{\leq 2}^* \lesssim \varepsilon, \quad (6.2a)$$

◇ the following  $L^2$  bounds on  $(\tau)\mathcal{M}^{\text{ext}}$

$$\mathcal{R}_{\leq 2, \gamma}^{\text{ext}} \lesssim \gamma \varepsilon, \quad (6.2b)$$

for all  $\gamma > 0$ ,<sup>1</sup>

◇ the following  $L_u^\infty \tilde{H}^{1/2}$  bounds on  $\underline{\mathcal{C}}^* \cap (\tau)\mathcal{M}^{\text{ext}}$

$$\mathfrak{N}_{\leq 1}^* \lesssim \varepsilon, \quad (6.2c)$$

where the norms are restricted to  $\underline{\mathcal{C}}^* \cap (\tau)\mathcal{M}^{\text{ext}}$ ,

◇ the following  $L_{u, \underline{u}}^\infty \tilde{H}^{1/2}$  bounds in  $(\tau)\mathcal{M}^{\text{ext}}$

$$\mathfrak{N}_{\leq 1}^{\text{ext}} \lesssim \varepsilon. \quad (6.2d)$$

We refer the reader to the norm definitions of Section 3.2.1.

**Remark 6.2.** *Proposition 6.1 does not provide bounds for the mean value  $\bar{\rho}, \bar{\sigma}$ . These are obtained in Sections 9 and 10. See also Remark 6.4.*

The proof of the estimates (6.2) relies on the following localised control on the 2-spheres  $S_{u, \underline{u}}$  for the null curvature tensors and their derivatives in terms of the (contracted) Bel-Robinson tensors used in Section 5. The proof of Proposition 6.3 is provided in Section 6.3.

**Proposition 6.3.** *On each 2-sphere  $S_{u, \underline{u}}$  of the exterior region  $\mathcal{M}^{\text{ext}}$ , the following control holds*

$$\left(\underline{\mathcal{R}}_{\leq 1}(u, \underline{u})\right)^2 \lesssim \underline{\mathcal{Q}}_1(u, \underline{u}) + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2, \quad (6.3a)$$

$$\left(\mathcal{R}_{\leq 1}(u, \underline{u})\right)^2 \lesssim \mathcal{Q}_1(u, \underline{u}) + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2, \quad (6.3b)$$

$$\left(\underline{\mathcal{R}}_{\leq 2}(u, \underline{u})\right)^2 \lesssim \underline{\mathcal{Q}}_{\leq 2}(u, \underline{u}) + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2, \quad (6.3c)$$

$$\left(\mathcal{R}_{\leq 2}(u, \underline{u})\right)^2 \lesssim \mathcal{Q}_{\leq 2}(u, \underline{u}) + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2, \quad (6.3d)$$

<sup>1</sup>In particular, the bound holds with  $\gamma = \gamma_0$  where  $0 < \gamma_0 < 1/4$  is the fixed numerical constant of Section 3.3.3. This improves the bound from the Bootstrap Assumption 3.20.



and

$$\begin{aligned}
\mathcal{R}_1[\underline{\alpha}](u, \underline{u}) &:= \|u^2(q\nabla)\underline{\alpha}\|_{L^2(S_{u, \underline{u}})} + \|u^2(r\nabla_4)\underline{\alpha}\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_1^{add}[\underline{\alpha}](u, \underline{u}) &:= \|u^2(q\nabla_3)\underline{\alpha}\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_1[\underline{\beta}](u, \underline{u}) &:= \|u^2(r\nabla)\underline{\beta}\|_{L^2(S_{u, \underline{u}})} + \|u^2(r\nabla_4)\underline{\beta}\|_{L^2(S_{u, \underline{u}})} + \|u^2(q\nabla_3)\underline{\beta}\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_1[\rho](u, \underline{u}) &:= \|\underline{u}u(r\nabla)\rho\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}ur\nabla_4(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}uq\nabla_3(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_1[\sigma](u, \underline{u}) &:= \|\underline{u}u(r\nabla)\sigma\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}ur\nabla_4(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}uq\nabla_3(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_1[\beta](u, \underline{u}) &:= \|\underline{u}^2(r\nabla)\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(r\nabla_4)\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(q\nabla_3)\beta\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_1[\alpha](u, \underline{u}) &:= \|\underline{u}^2(r\nabla)\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(r\nabla_3)\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(r\nabla_4)\alpha\|_{L^2(S_{u, \underline{u}})},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}_2[\underline{\alpha}](u, \underline{u}) &:= \|u^2qr\nabla^2\underline{\alpha}\|_{L^2(S_{u, \underline{u}})} + \|u^2r^2\nabla\nabla_4\underline{\alpha}\|_{L^2(S_{u, \underline{u}})} + \|u^2r^2\nabla_4\nabla_4\underline{\alpha}\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|u^2q^2\nabla\nabla_3\underline{\alpha}\|_{L^2(S_{u, \underline{u}})} + \|u^2(q\nabla_3)(r\nabla_4)\underline{\alpha}\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_2[\underline{\beta}](u, \underline{u}) &:= \|u^2r^2\nabla^2\underline{\beta}\|_{L^2(S_{u, \underline{u}})} + \|u^2r^2\nabla\nabla_4\underline{\beta}\|_{L^2(S_{u, \underline{u}})} + \|u^2r^2\nabla_4\nabla_4\underline{\beta}\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|u^2rq\nabla\nabla_3\underline{\beta}\|_{L^2(S_{u, \underline{u}})} + \|u^2q^2\nabla_3\nabla_3\underline{\beta}\|_{L^2(S_{u, \underline{u}})} + \|u^2(r\nabla_3)(r\nabla_4)\underline{\beta}\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_2[\rho](u, \underline{u}) &:= \|\underline{u}ur^2\nabla^2\rho\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}ur^2\nabla\nabla_4(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}ur^2\nabla_4\nabla_4(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|\underline{u}urq\nabla\nabla_3(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}uq^2\nabla_3\nabla_3(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|\underline{u}u(r\nabla_3)r\nabla_4(\rho - \bar{\rho})\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_2[\sigma](u, \underline{u}) &:= \|\underline{u}ur^2\nabla^2\sigma\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}ur^2\nabla\nabla_4(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}ur^2\nabla_4\nabla_4(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|\underline{u}urq\nabla\nabla_3(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}uq^2\nabla_3\nabla_3(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|\underline{u}u(r\nabla_3)r\nabla_4(\sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_2[\beta](u, \underline{u}) &:= \|\underline{u}^2r^2\nabla^2\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2r^2\nabla\nabla_4\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2r^2\nabla_4\nabla_4\beta\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|\underline{u}^2rq\nabla\nabla_3\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2q^2\nabla_3\nabla_3\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(r\nabla_3)(r\nabla_4)\beta\|_{L^2(S_{u, \underline{u}})}, \\
\mathcal{R}_2[\alpha](u, \underline{u}) &:= \|\underline{u}^2r^2\nabla^2\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2r^2\nabla\nabla_4\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2r^2\nabla\nabla_3\alpha\|_{L^2(S_{u, \underline{u}})} \\
&\quad + \|\underline{u}^2r^2\nabla_3\nabla_3\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(r\nabla_3)(r\nabla_4)\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(r\nabla_4)^2\alpha\|_{L^2(S_{u, \underline{u}})}.
\end{aligned}$$

We define

$$\begin{aligned}
\mathcal{R}_{\leq 1}(u, \underline{u}) &:= \mathcal{R}_0(u, \underline{u}) + \mathcal{R}_0^{add}(u, \underline{u}) + \mathcal{R}_1(u, \underline{u}), & \mathcal{R}_{\leq 2}(u, \underline{u}) &:= \mathcal{R}_2(u, \underline{u}) + \mathcal{R}_1^{add}(u, \underline{u}) + \mathcal{R}_{\leq 1}(u, \underline{u}), \\
\mathcal{R}_{\leq 1}(u, \underline{u}) &:= \mathcal{R}_0(u, \underline{u}) + \mathcal{R}_0^{add}(u, \underline{u}) + \mathcal{R}_1(u, \underline{u}), & \mathcal{R}_{\leq 2}(u, \underline{u}) &:= \mathcal{R}_2(u, \underline{u}) + \mathcal{R}_1^{add}(u, \underline{u}) + \mathcal{R}_{\leq 1}(u, \underline{u}).
\end{aligned}$$

We have the following definitions for integrals of the Bel-Robinson tensors

$$\begin{aligned}\underline{\mathcal{Q}}_1(u, \underline{u}) &:= \int_{S_{u, \underline{u}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}, e_3) \right), \\ \mathcal{Q}_1(u, \underline{u}) &:= \int_{S_{u, \underline{u}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_4) + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}, e_4) \right), \\ \underline{\mathcal{Q}}_2(u, \underline{u}) &:= \int_{S_{u, \underline{u}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}, e_3) + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) \right. \\ &\quad \left. + Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) \right), \\ \mathcal{Q}_2(u, \underline{u}) &:= \int_{S_{u, \underline{u}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}^2\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}, e_3) + Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) \right. \\ &\quad \left. + Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}}\mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) \right),\end{aligned}$$

and

$$\underline{\mathcal{Q}}_{\leq 2}(u, \underline{u}) := \underline{\mathcal{Q}}_1(u, \underline{u}) + \underline{\mathcal{Q}}_2(u, \underline{u}), \quad \mathcal{Q}_{\leq 2}(u, \underline{u}) := \mathcal{Q}_1(u, \underline{u}) + \mathcal{Q}_2(u, \underline{u}).$$

**Remark 6.4.** The terms  $\bar{\rho}, \bar{\sigma}$  in the right-hand side of (6.3) are not controlled in terms of the Bel-Robinson tensors.<sup>2</sup> However, as it will be shown in the next section, the terms  $\|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}$  and  $\|u(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}$  have sufficient decay to be integrated in  $u$  or  $\underline{u}$ .

## 6.1 Proof of the $L^2$ bounds (6.2a) and (6.2b) on $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ and $\mathcal{M}^{\text{ext}}$

In this section, we use the control of Proposition 6.3 and the energy estimates of Section 5 to derive the estimates (6.2a) and (6.2b).

We first have the following control of the curvature fluxes through the hypersurfaces  $\mathcal{C}_u, \Sigma_t^{\text{ext}}$  and  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ .

**Lemma 6.5.** We have

◇ for all  $1 \leq u \leq \tau \underline{u}^*$

$$\int_{\tau^{-1}u}^{\underline{u}^*} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 d\underline{u} \lesssim \varepsilon^2, \quad (6.4a)$$

◇ for all  $t^\circ \leq t \leq t^*$

$$\int_{2t/(1+\tau^{-1})}^{2t-\underline{u}^*} (\mathcal{R}_{\leq 2}(u, \underline{u} = 2t - u))^2 + (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u} = 2t - u))^2 du \lesssim \varepsilon^2, \quad (6.4b)$$

◇ on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$

$$\int_1^{\tau \underline{u}^*} (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}^*))^2 du \lesssim \varepsilon^2. \quad (6.4c)$$

*Proof.* From the bounds (6.1) through the hypersurface  $\mathcal{C}_u$ , the Bootstrap Assumptions 3.20 on  $\bar{\rho}$  and  $\bar{\sigma}$ , and the results of Proposition 6.3, we have for all  $1 \leq u \leq \tau \underline{u}^*$

$$\begin{aligned}\int_{\tau^{-1}u}^{\underline{u}^*} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 d\underline{u} &\lesssim \int_{\tau^{-1}u}^{\underline{u}^*} \mathcal{Q}_{\leq 2}(u, \underline{u}) d\underline{u} + \int_{\tau^{-1}u}^{\underline{u}^*} (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2 d\underline{u} \\ &\lesssim \varepsilon^2 + (D\varepsilon)^4 \int_{\tau^{-1}u}^{\underline{u}^*} \underline{u}^4 \underline{u}^{-6} u^{-4} d\underline{u} \\ &\lesssim \varepsilon^2 + (D\varepsilon)^4 \\ &\lesssim \varepsilon^2,\end{aligned}$$

which proves (6.4a). Estimates (6.4b) and (6.4c) follow similarly from the bounds (6.1) on the Bel-Robinson tensors through respectively  $\Sigma_t^{\text{ext}}$  and  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ . Details are left to the reader.  $\square$

<sup>2</sup>They are controlled by integrating the associated Bianchi equation (2.35e) and by using the null structure (2.34o) respectively. See Sections 9 and 10.

From the result of Lemma 6.5, and an inspection of the definitions of Section 3.2.1, we have on  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$

$$\mathcal{R}_{\leq 2}^* \simeq \left( \int_1^{\tau \underline{u}^*} (\mathcal{R}_{\leq 2}(u, \underline{u}^*))^2 du \right)^{1/2} \lesssim \varepsilon,$$

which proves the desired bound (6.2a).

From an inspection of the norms defined in Section 3.2.1 and the results of Lemma 6.5, we deduce

$$\begin{aligned} (\mathcal{R}_{\leq 2, \gamma}^{\text{ext}})^2 &\lesssim \int_1^{\tau \underline{u}^*} \int_{\tau^{-1} \underline{u}}^{\underline{u}^*} \left( u^{-1-2\gamma} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 + \underline{u}^{-1-2\gamma} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 \right) d\underline{u} du \\ &\lesssim \int_1^{\tau \underline{u}^*} u^{-1-2\gamma} \int_{\tau^{-1} \underline{u}}^{\underline{u}^*} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 d\underline{u} du \\ &\quad + \int_{t^0}^{t^*} t^{-1-2\gamma} \int_{2t/(1+\tau^{-1})}^{2t-\underline{u}^*} (\mathcal{R}_{\leq 2}(u, \underline{u} = 2t - u))^2 + (\mathcal{R}_{\leq 2}(u, \underline{u} = 2t - u))^2 dudt \\ &\lesssim \varepsilon^2 \int_1^{\tau \underline{u}^*} u^{-1-2\gamma} du + \varepsilon^2 \int_{t^0}^{t^*} t^{-1-2\gamma} dt \\ &\lesssim_{\gamma} \varepsilon^2, \end{aligned}$$

for all  $\gamma > 0$ , and where we used the coarea formulas from Lemma 3.34 and  $t \simeq \underline{u}$  in  $\mathcal{M}^{\text{ext}}$ . This finishes the proof of estimate (6.2b).

## 6.2 Proof of the $L^\infty \tilde{H}^{1/2}$ estimates (6.2c) and (6.2d)

The proof of the estimates (6.2c) and (6.2d) boils down to the following two Klainerman-Sobolev estimates on  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$  and  $\Sigma_t^{\text{ext}}$  respectively. Their proof are postponed to Appendix D.

**Lemma 6.6** (Klainerman-Sobolev estimates on  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$ ). *For all  $S$ -tangent tensor  $F$  on  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$ , we have the following  $L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})$  estimates in  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$*

$$\|rF\|_{L_{1 \leq u \leq \tau \underline{u}^*}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})} \lesssim \|F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|r\nabla F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|r\nabla_3 F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})},$$

and

$$\left\| r^{1/2} q^{1/2} F \right\|_{L_{1 \leq u \leq \tau \underline{u}^*}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})} \lesssim \|F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|r\nabla F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|q\nabla_3 F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})}.$$

**Lemma 6.7** (Klainerman-Sobolev estimates on  $\Sigma_t^{\text{ext}}$ ). *For all  $S$ -tangent tensor  $F$  we have the following  $L_{u, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})$  estimates in  $\mathcal{M}^{\text{ext}}$*

$$\begin{aligned} \|\underline{u}F\|_{L_{u, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \|\underline{u}^*F\|_{L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})} + \|F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})} \\ &\quad + \|\underline{u}\nabla F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})} + \|\underline{u}\nabla_3 F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})} + \|\underline{u}\nabla_4 F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})}, \end{aligned}$$

and

$$\begin{aligned} \left\| \underline{u}^{1/2} u^{1/2} F \right\|_{L_{u, \underline{u}}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \left\| \underline{u}^{*1/2} u^{1/2} F \right\|_{L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})} + \|F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})} \\ &\quad + \|\underline{u}\nabla F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})} + \|\underline{u}\nabla_3 F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})} + \|\underline{u}\nabla_4 F\|_{L_t^\infty L^2(\Sigma_t^{\text{ext}})}. \end{aligned}$$

First, we apply the degenerate version of the Klainerman-Sobolev estimates of Lemma 6.6, with  $F$  the following respective tensors

$$u^2 \nabla^{\leq 1} \underline{\alpha}, \quad u \underline{u} \nabla^{\leq 1} \underline{\beta}, \quad \underline{u}^2 \nabla^{\leq 1} (\rho - \bar{\rho}), \quad \underline{u}^2 \nabla^{\leq 1} (\sigma - \bar{\sigma}),$$

and the non-degenerate version with  $F$  respectively

$$\underline{u}^2 \nabla^{\leq 1} \beta, \quad \underline{u}^2 (r\nabla)^{\leq 1} \alpha, \quad \underline{u}^2 (r\nabla_3) \alpha,$$

where  $\nabla \in \{(r\nabla), (q\nabla)_3\}$ . From an inspection of the definitions of Section 3.2.1, we deduce from the bounds (6.4c) that

$$\mathfrak{R}_{\leq 1}^* \lesssim \mathcal{R}_{\leq 2}^* \lesssim \varepsilon,$$

where the norms are restricted to  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ . This proves (6.2c).

From an inspection of the definitions of Section 3.2.1, the Klainerman-Sobolev estimates from Lemma 6.7, the coarea formulas from Lemma 3.34, the above bound for  $\mathfrak{R}_{\leq 1}^*$  and the bounds from Lemma 6.5, we have

$$\begin{aligned} \mathfrak{R}_{\leq 1}^{\text{ext}} &\lesssim \mathfrak{R}_{\leq 1}^* + \sup_{t^0 \leq t \leq t^*} \left( \int_{2t/(1+\tau^{-1})}^{2t-u^*} (\mathcal{R}_{\leq 2}(u, \underline{u} = 2t - u))^2 + (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u} = 2t - u))^2 du \right) \\ &\lesssim \varepsilon^2. \end{aligned}$$

This finishes the proof of (6.2d).

## 6.3 Proof of Proposition 6.3

### 6.3.1 Control of $\underline{\mathcal{R}}_{\leq 1}(u, \underline{u})$ and $\mathcal{R}_{\leq 1}(u, \underline{u})$

From the definition of  $\mathbf{T}^{\text{ext}}, \mathbf{K}^{\text{ext}}$ , and the decomposition [CK93, p. 150] of the Bel-Robinson tensors in the null frame  $(e_3, e_4)$ , we have

$$\begin{aligned} &\left| u^2 \underline{\alpha}(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| uu \underline{\beta}(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| \underline{u}^2 \rho(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| \underline{u}^2 \sigma(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| \underline{u}^2 \beta(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 \\ &\lesssim Q(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}, e_3), \end{aligned}$$

where we denote by  $\mathbf{O}^{\text{ext}}$  any exterior rotation vectorfield  ${}^{(\ell)}\mathbf{O}^{\text{ext}}$  for  $\ell = 1, 2, 3$ .

Moreover, from [CK93, pp. 152-153], we have

$$\begin{aligned} &\left| u^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\alpha} \right|^2 + \left| uu \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\beta} \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \rho \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \sigma \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \beta \right|^2 \\ &\lesssim \left| u^2 \underline{\alpha}(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| uu \underline{\beta}(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| \underline{u}^2 \rho(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| \underline{u}^2 \sigma(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 + \left| \underline{u}^2 \beta(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}) \right|^2 \\ &\quad + |\text{Err}_1|^2, \end{aligned}$$

where  $\text{Err}_1$  is an error term, which is treated in Section (6.3.2), and for which we have

$$\int_{S_{u, \underline{u}}} |\text{Err}_1|^2 \lesssim (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 1}(u, \underline{u}))^2 + (D\varepsilon)^2 \int_{S_{u, \underline{u}}} |\underline{u}(\bar{\rho}, \bar{\sigma})|^2.$$

Thus, combining these estimates and integrating on  $S_{u, \underline{u}}$ , using the definition of  $\underline{\mathcal{Q}}_1$ , we obtain

$$\int_{S_{u, \underline{u}}} \left| u^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\alpha} \right|^2 + \left| uu \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\beta} \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \rho \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \sigma \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \beta \right|^2 \lesssim \mathcal{F}(u, \underline{u}), \quad (6.5)$$

where

$$\mathcal{F}(u, \underline{u}) := \underline{\mathcal{Q}}_1(u, \underline{u}) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 1}(u, \underline{u}))^2 + (D\varepsilon)^2 \int_{S_{u, \underline{u}}} |\underline{u}(\bar{\rho}, \bar{\sigma})|^2.$$

Using the estimates of the mild Bootstrap Assumptions 3.16 and the Poincaré estimates from Lemma 3.38 we deduce from (6.5) the following estimate

$$\begin{aligned} &\int_{S_{u, \underline{u}}} \left( \left| u^2 (r\nabla)^{\leq 1} \underline{\alpha} \right|^2 + \left| uu (r\nabla)^{\leq 1} \underline{\beta} \right|^2 + \left| \underline{u}^2 (r\nabla)^{\leq 1} (\rho - \bar{\rho}) \right|^2 \right. \\ &\quad \left. + \left| \underline{u}^2 (r\nabla)^{\leq 1} (\sigma - \bar{\sigma}) \right|^2 + \left| \underline{u}^2 (r\nabla)^{\leq 1} \beta \right|^2 \right) \\ &\quad \int_{S_{u, \underline{u}}} \left| u^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\alpha} \right|^2 + \left| uu \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \underline{\beta} \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \rho \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \sigma \right|^2 + \left| \underline{u}^2 \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \beta \right|^2 \\ &\lesssim \mathcal{F}(u, \underline{u}). \end{aligned} \quad (6.6)$$

Using the Bianchi identities (2.35), we deduce from (6.6) the following control for the  $\nabla_3, \nabla_4$  derivatives

$$\begin{aligned}
 & \int_{S_{u, \underline{u}}} \left( |u^2(r\nabla_4)\underline{\alpha}|^2 + |uu(u\nabla_3)\underline{\beta}|^2 + |uu(r\nabla_4)\underline{\beta}|^2 \right. \\
 & + \left| \underline{u}^2 u \left( \nabla_3 \rho + \frac{3}{2} \text{tr} \chi \rho \right) \right|^2 + \left| \underline{u}^2 u \left( \nabla_3 \sigma + \frac{3}{2} \text{tr} \chi \sigma \right) \right|^2 \\
 & + \left| \underline{u}^2 \underline{u} \left( \nabla_4 \rho + \frac{3}{2} \text{tr} \chi \rho \right) \right|^2 + \left| \underline{u}^2 \underline{u} \left( \nabla_4 \sigma + \frac{3}{2} \text{tr} \chi \sigma \right) \right|^2 + |\underline{u}^2 \underline{u} \nabla_3 \beta|^2 \right) \\
 & \lesssim \mathcal{F}(u, \underline{u}) + \int_{S_{u, \underline{u}}} |\text{Err}_2|^2 \\
 & \lesssim \mathcal{F}(u, \underline{u}),
 \end{aligned} \tag{6.7}$$

where the control for the error term  $\text{Err}_2$  by  $\mathcal{F}$  is postponed to Section 6.3.2. From (6.7) and the control (6.6) for  $\rho - \bar{\rho}, \sigma - \bar{\sigma}$  and the mild bounds  $|r\text{tr}\chi| + |\text{tr}\chi| \lesssim 1$ , we deduce

$$\begin{aligned}
 & \int_{S_{u, \underline{u}}} \left( |u^2(r\nabla_4)\underline{\alpha}|^2 + |uu(u\nabla_3)\underline{\beta}|^2 + |uu(r\nabla_4)\underline{\beta}|^2 \right. \\
 & + |\underline{u}^2 u \nabla_3(\rho - \bar{\rho})|^2 + |\underline{u}^2 u \nabla_3(\sigma - \bar{\sigma})|^2 \\
 & \left. + |\underline{u}^2 \underline{u} \nabla_4(\rho - \bar{\rho})|^2 + |\underline{u}^2 \underline{u} \nabla_4(\sigma - \bar{\sigma})|^2 + |\underline{u}^2 \underline{u} \nabla_3 \beta|^2 \right) \\
 & \lesssim \mathcal{F}(u, \underline{u}).
 \end{aligned} \tag{6.8}$$

Using the other contracted Bel-Robinson tensor of  $\underline{\mathcal{Q}}_1$ , we have

$$\begin{aligned}
 \int_{S_{u, \underline{u}}} |u^2 u \nabla_{\mathbf{T}^{\text{ext}}} \underline{\alpha}|^2 + |\underline{u}^2 \underline{u} \nabla_{\mathbf{T}^{\text{ext}}} \beta|^2 & \lesssim \int_{S_{u, \underline{u}}} \left( Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R})(\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) \right) + \int_{S_{u, \underline{u}}} |\text{Err}_3|^2 \\
 & \lesssim \mathcal{F}(u, \underline{u}),
 \end{aligned}$$

where the control of the error term  $\text{Err}_3$  by  $\mathcal{F}$  is postponed to Section 6.3.2. Combining with (6.8), we obtain

$$\int_{S_{u, \underline{u}}} |u^2 u \nabla_3 \underline{\alpha}|^2 + \int_{S_{u, \underline{u}}} |\underline{u}^3 \nabla_4 \beta|^2 \lesssim \mathcal{F}(u, \underline{u}). \tag{6.9}$$

Using Bianchi equation (2.35b), the estimates (6.6) and (6.9) respectively for  $\beta$  and  $\nabla_4 \beta$ , and the elliptic estimates on 2-spheres from Lemma 3.38, we obtain

$$\begin{aligned}
 \int_{S_{u, \underline{u}}} |\underline{u}^2 (r\nabla_4)^{\leq 1} \alpha|^2 & \lesssim \int_{S_{u, \underline{u}}} \underline{u}^4 |(r\nabla_4)^{\leq 1} \beta|^2 + \int_{S_{u, \underline{u}}} |\text{Err}_4|^2 \\
 & \lesssim \mathcal{F}(u, \underline{u}),
 \end{aligned}$$

where the control of the error term is obtained in Section 6.3.2. Using Bianchi equation (2.35a), one further obtains

$$\int_{S_{u, \underline{u}}} |\underline{u}^2 r \nabla_3 \alpha|^2 \lesssim \mathcal{F}(u, \underline{u}) + \int_{S_{u, \underline{u}}} |\text{Err}_5|^2 \lesssim \mathcal{F}(u, \underline{u}).$$

Summarising the bounds obtained in this section, we have proved

$$(\underline{\mathcal{R}}_{\leq 1}(u, \underline{u}))^2 \lesssim \mathcal{F}(u, \underline{u}) = \underline{\mathcal{Q}}_1(u, \underline{u}) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 1}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2,$$

which after an absorption argument, concludes the desired control of  $\underline{\mathcal{R}}_{\leq 1}(u, \underline{u})$ . The control of  $\mathcal{R}_{\leq 1}(u, \underline{u})$  is obtained along the same lines and is left to the reader.

### 6.3.2 Error term estimates from Section 6.3.1

In this section, we show that, under the Bootstrap Assumptions 3.23 on the null connection and rotation coefficients in  $\mathcal{M}^{\text{ext}}$ , we have for the error terms  $\text{Err}_1, \dots, \text{Err}_5$  from Section 6.3.1

$$\|\text{Err}\|_{L^2(S_{u, \underline{u}})} \lesssim (D\varepsilon) \underline{\mathcal{R}}_{\leq 1}(u, \underline{u}) + (D\varepsilon) \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}.$$

The error terms for the dual estimates for  $\mathcal{R}$  are controlled similarly, and their estimate is left to the reader.



**Estimates for  $\text{Err}_1$  and  $\text{Err}_3$**  We first recall from [CK93, pp. 152–153], relations (2.33) and (2.109) that for the vectorfields  $\mathbf{T}^{\text{ext}}$  and  $\mathbf{O}^{\text{ext}}$ , we have

$$\begin{aligned} [\mathbf{T}^{\text{ext}}, e_3] &= -2\zeta_a e_a - \underline{\omega} e_4, \\ [\mathbf{T}^{\text{ext}}, e_4] &= 2\zeta_a e_a + \underline{\omega} e_4, \\ [\mathbf{T}^{\text{ext}}, e_a] &= \Pi([T, e_a]) + \frac{1}{2}\underline{\xi}_a e_4, \end{aligned}$$

and

$$\begin{aligned} [\mathbf{O}^{\text{ext}}, e_3] &= -(\zeta \cdot \mathbf{O}^{\text{ext}})e_3 - (\underline{\xi} \cdot \mathbf{O}^{\text{ext}})e_4 - Y, \\ [\mathbf{O}^{\text{ext}}, e_4] &= 0, \\ [\mathbf{O}^{\text{ext}}, e_a] &= \Pi([\mathbf{O}^{\text{ext}}, e_a]), \end{aligned}$$

where  $\Pi([T, e_a])$  and  $\Pi([\mathbf{O}^{\text{ext}}, e_a])$  are the projection on  $S$  of  $[T, e_a]$  and  $[\mathbf{O}^{\text{ext}}, e_a]$  respectively, and play no role in the argument below.

With the notations of [CK93, pp. 152–153], we thus have

$$\begin{aligned} (\mathbf{T}^{\text{ext}})\underline{P} &= -2\zeta, & (\mathbf{T}^{\text{ext}})P &= 2\zeta, \\ (\mathbf{T}^{\text{ext}})\underline{M} &= 0, & (\mathbf{T}^{\text{ext}})M &= \underline{\omega}, \\ (\mathbf{T}^{\text{ext}})\underline{N} &= -\underline{\omega}, & (\mathbf{T}^{\text{ext}})N &= 0, \\ (\mathbf{T}^{\text{ext}})\underline{Q} &= \underline{\xi}, & (\mathbf{T}^{\text{ext}})Q &= 0, \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} (\mathbf{O}^{\text{ext}})\underline{P} &= -Y, & (\mathbf{O}^{\text{ext}})P &= 0, \\ (\mathbf{O}^{\text{ext}})\underline{M} &= -\zeta \cdot \mathbf{O}^{\text{ext}}, & (\mathbf{O}^{\text{ext}})M &= 0, \\ (\mathbf{O}^{\text{ext}})\underline{N} &= -\underline{\xi} \cdot \mathbf{O}^{\text{ext}}, & (\mathbf{O}^{\text{ext}})N &= 0, \\ (\mathbf{O}^{\text{ext}})\underline{Q} &= 0, & (\mathbf{O}^{\text{ext}})Q &= 0, \end{aligned} \tag{6.11}$$

From direct computation or from relations (2.112), we also have

$$\text{tr}^{(\mathbf{T}^{\text{ext}})}\pi = -2\underline{\omega} + (\text{tr}\underline{\chi} + \text{tr}\underline{\chi}), \tag{6.12}$$

and

$$\text{tr}^{(\mathbf{O}^{\text{ext}})}\pi = 2\zeta \cdot \mathbf{O}^{\text{ext}} + \text{tr}H \tag{6.13}$$

and that

$${}^{(T)}\hat{\pi}_{ab} = \frac{1}{2}\underline{\omega} + \frac{1}{4}(\text{tr}\underline{\chi} + \text{tr}\underline{\chi})\mathcal{G}_{ab} + \hat{\chi}_{ab} + \hat{\underline{\chi}}_{ab}, \tag{6.14}$$

$${}^{(\mathbf{O}^{\text{ext}})}\hat{\pi}_{ab} = H_{ab} - \frac{1}{4}(2\zeta \cdot \mathbf{O}^{\text{ext}} + \text{tr}H)\mathcal{G}_{ab}. \tag{6.15}$$

From the formulas of [CK93, pp. 152–153], we have

$$\text{null}\left(\hat{\mathcal{L}}_X \mathbf{R}\right) - \hat{\mathcal{L}}_X \text{null}(\mathbf{R}) = A \cdot R, \tag{6.16}$$

where  $A$  are Lie coefficients  $P, \underline{P}, Q, \underline{Q}, M, \underline{M}$ ,  $\text{tr}\pi$ ,  $\hat{\pi}_{ab}$  from (6.10), (6.12) and (6.14) for  $X = \mathbf{T}^{\text{ext}}$  and (6.11), (6.13) and (6.15) for  $X = \mathbf{O}^{\text{ext}}$  and where  $R \in \{\underline{\alpha}, \dots, \alpha\}$  with signature  $s(R) = s(\text{null}) \pm 1$ .

We first estimate

$$\left\| u^p \underline{u}^q \left( \text{null}\left(\hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R}\right) - \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \text{null}(\mathbf{R}) \right) \right\|_{L^2(S_{u, \underline{u}})} \tag{6.17}$$

where  $u^p, \underline{u}^q$  are the appropriate powers of  $u$  and  $\underline{u}$  given in Section 3.2.1.

From the expression of the Lie coefficients  $A$  of  $\mathbf{O}^{\text{ext}}$  given in (6.11), (6.13) and (6.15) and the decay estimates of the Bootstrap Assumptions 3.23, we have

$$\|\underline{u}A\|_{L^\infty(\mathcal{M}^{\text{ext}})} \lesssim D\varepsilon.$$

From formula (6.16), we have

$$\left\| u^p \underline{u}^q \left( \text{null} \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) - \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \text{null}(\mathbf{R}) \right) \right\|_{L^2(S_{u, \underline{u}})} \lesssim \|\underline{u}A\|_{L^\infty} \|u^p \underline{u}^{q-1} R\|_{L^2(S_{u, \underline{u}})},$$

where  $R \in \{\underline{\alpha}, \dots, \alpha\}$  with  $s(R) = s(\text{null}) \pm 1$ .

We have

$$\|u^p \underline{u}^{q-1} R\|_{L^2(S_{u, \underline{u}})} \lesssim \mathcal{R}_{\leq 1}(u, \underline{u}) + \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})},$$

and the control of  $\text{Err}_1$  follows.

From the expression of the Lie coefficients  $A$  of  $\mathbf{T}^{\text{ext}}$  given in (6.10), (6.12) and (6.14) and the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \|\underline{u}uA\|_{L^\infty(\mathcal{M}^{\text{ext}})} &\lesssim D\varepsilon, \\ \|\underline{u}^2(P, Q)\|_{L^\infty(\mathcal{M}^{\text{ext}})} &\lesssim D\varepsilon. \end{aligned}$$

and we therefore deduce from an inspection of [CK93, pp. 152–153]

$$\left\| u^p \underline{u}^q \left( \text{null} \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) - \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \text{null}(\mathbf{R}) \right) \right\|_{L^2(S_{u, \underline{u}})} \lesssim (D\varepsilon) \left( \mathcal{R}_{\leq 1}(u, \underline{u}) + \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \right).$$

Moreover, we have schematically for  $R \in \{\underline{\alpha}, \underline{\beta}, \beta\}$

$$\hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} R - \nabla_{\mathbf{T}^{\text{ext}}} R = (\chi + \underline{\chi}) \cdot R,$$

and we therefore deduce, using the Bootstrap Assumptions 3.23, that

$$\begin{aligned} \left\| u^p \underline{u}^q \left( \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} R - \nabla_{\mathbf{T}^{\text{ext}}} R \right) \right\|_{L^2(S_{u, \underline{u}})} &\lesssim \|\underline{u}(\chi + \underline{\chi})\|_{L^\infty} \|u^p \underline{u}^{q-1} R\|_{L^2(S_{u, \underline{u}})} \\ &\lesssim (D\varepsilon) \mathcal{R}_{\leq 1}(u, \underline{u}), \end{aligned}$$

which concludes the control of  $\text{Err}_3$ .

**Estimates for  $\text{Err}_2, \text{Err}_4$  and  $\text{Err}_5$**  We shall only treat the term  $\text{Err}_5$ , for the other error terms will follow similarly.

From Bianchi equation (2.35a) and the relations of Lemma 2.11, we have

$$\text{Err}_5 = \underline{u}^3 (2\underline{\omega}\alpha - 3(\hat{\chi}\rho + {}^*\hat{\chi}\sigma) + 5\zeta \otimes \beta).$$

Thus, using the Bootstrap Assumptions 3.23

$$\begin{aligned} &\|\text{Err}_5\|_{L^2(S_{u, \underline{u}})} \\ &\lesssim (D\varepsilon) \left( \|\underline{u}^2\alpha\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2\beta\|_{L^2(S_{u, \underline{u}})} + \|\underline{u}^2(\rho - \bar{\rho}, \sigma - \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \right) \\ &\quad + (D\varepsilon) \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \\ &\lesssim (D\varepsilon) \left( \mathcal{R}_{\leq 1}(u, \underline{u}) + \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \right), \end{aligned}$$

as desired. This finishes the control of the error terms  $\text{Err}_2, \text{Err}_4$  and  $\text{Err}_5$ .

### 6.3.3 Control of $\underline{\mathcal{R}}_{\leq 2}(u, \underline{u})$ and $\mathcal{R}_{\leq 2}(u, \underline{u})$

In this section, we argue as in Section 6.3.1 to obtain the control of  $\underline{\mathcal{R}}_{\leq 2}(u, \underline{u})$  and  $\mathcal{R}_{\leq 2}(u, \underline{u})$ . The error terms are dealt with arguing as in Section 6.3.2, using  $L^\infty(\mathcal{M}^{\text{ext}})$  for the connection and rotation coefficients and  $L^\infty_{u, \underline{u}} L^4(S_{u, \underline{u}})$  estimates for derivatives of the connection and rotation coefficients. This treatment of the error terms is left to the reader.

**Remark 6.8.** From Bianchi equations (2.35e), (2.35g), (2.35d) and (2.35f) for  $\nabla_3, \nabla_4$  derivatives of  $\rho, \sigma$ , and the Bootstrap Assumptions 3.23 for the null connection coefficients, we have

$$\|\underline{u}^2(\nabla_3, \nabla_4)(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})} \lesssim (D\varepsilon)\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}) + \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}.$$

Thus, derivatives of  $\bar{\rho}, \bar{\sigma}$  shall in the sequel directly be replaced using the above estimate.

**Remark 6.9.** Using the mild Bootstrap Assumptions 3.16 and the Bootstrap Assumptions 3.23 for the rotation vectorfields  $\mathbf{O}^{\text{ext}}$ , one can obtain the following higher order tangential derivatives estimates

$$\int_{S_{u, \underline{u}}} |(r\nabla)^2 F|^2 \lesssim \sum_{\ell, \ell'=1}^3 \int_{S_{u, \underline{u}}} \left| \hat{\mathcal{L}}^{(\ell)} \mathbf{O}^{\text{ext}} \hat{\mathcal{L}}^{(\ell')} \mathbf{O}^{\text{ext}} F \right|^2 + \int_{S_{u, \underline{u}}} |(r\nabla)^{\leq 1} F|^2, \quad (6.18)$$

for all  $S$ -tangent tensor  $F$  and for all 2-sphere  $S_{u, \underline{u}} \subset \mathcal{M}^{\text{ext}}$ .

Arguing as in Sections 6.3.1 and 6.3.2, using (6.18) we have

$$\begin{aligned} & \int_{S_{u, \underline{u}}} \left( |u^2(r\nabla)^{\leq 2} \underline{\alpha}|^2 + |u\underline{u}(r\nabla)^{\leq 2} \underline{\beta}|^2 + |\underline{u}^2(r\nabla)^{\leq 2}(\rho - \bar{\rho})|^2 + |\underline{u}^2(r\nabla)^{\leq 2}(\sigma - \bar{\sigma})|^2 + |\underline{u}^2(r\nabla)^{\leq 2} \beta|^2 \right) \\ & \lesssim \int_{S_{u, \underline{u}}} Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{T}^{\text{ext}}, e_3) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2, \end{aligned}$$

and

$$\begin{aligned} & \int_{S_{u, \underline{u}}} \left( |u^3(r\nabla) \nabla_{\mathbf{T}^{\text{ext}}} \underline{\alpha}|^2 + |\underline{u}^3(r\nabla) \nabla_{\mathbf{T}^{\text{ext}}} \beta|^2 \right) \\ & \lesssim \int_{S_{u, \underline{u}}} Q \left( \hat{\mathcal{L}}_{\mathbf{O}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2. \end{aligned}$$

Using these two estimates with Bianchi equations (2.35) commuted with  $(r\nabla)$  as in Section 6.3.1, we deduce

$$\begin{aligned} & \int_{S_{u, \underline{u}}} \left( |u^2(r\nabla)(u\nabla_3)\underline{\alpha}|^2 + |u^2(r\nabla)(r\nabla_4)\underline{\alpha}|^2 + |u\underline{u}(r\nabla)(u\nabla_3)\beta|^2 + |u\underline{u}(r\nabla)(r\nabla_4)\beta|^2 \right. \\ & \quad \left. + |\underline{u}^2 u(r\nabla) \nabla_3(\rho - \bar{\rho})|^2 + |\underline{u}^2 u(r\nabla) \nabla_3(\sigma - \bar{\sigma})|^2 + |\underline{u}^2 \underline{u}(r\nabla) \nabla_4(\rho - \bar{\rho})|^2 \right. \\ & \quad \left. + |\underline{u}^2 \underline{u}(r\nabla) \nabla_4(\sigma - \bar{\sigma})|^2 + |\underline{u}^2 \underline{u}(r\nabla) \nabla_3 \beta|^2 + |\underline{u}^2(r\nabla)(u\nabla_4)\beta|^2 \right) \\ & \lesssim \underline{\mathcal{Q}}_{\leq 2}(u, \underline{u}) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2. \end{aligned}$$

Taking  $\nabla_3, \nabla_4$  derivative in Bianchi equations (2.35), we deduce from the above estimates

$$\begin{aligned} & (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}))^2 - \int_{S_{u, \underline{u}}} |u^2(u\nabla_3)^2 \underline{\alpha}|^2 - \int_{S_{u, \underline{u}}} |\underline{u}^2(u\nabla_4)^2 \beta|^2 - \int_{S_{u, \underline{u}}} |\underline{u}^4(\nabla, \nabla_3)(\nabla, \nabla_3, \nabla_4)\alpha|^2 \\ & \lesssim \underline{\mathcal{Q}}_{\leq 2}(u, \underline{u}) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2. \end{aligned}$$

Using the last Bel-Robinson tensor of  $\underline{\mathcal{Q}}_{\leq 2}$  and the above estimate, we have

$$\begin{aligned} & \int_{S_{u, \underline{u}}} \left( |u^4 \nabla_3^2 \underline{\alpha}|^2 + |\underline{u}^4 \nabla_4^2 \beta|^2 \right) \\ & \lesssim \int_{S_{u, \underline{u}}} Q \left( \hat{\mathcal{L}}_{\mathbf{S}^{\text{ext}}} \hat{\mathcal{L}}_{\mathbf{T}^{\text{ext}}} \mathbf{R} \right) (\mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, \mathbf{K}^{\text{ext}}, e_3) + (D\varepsilon)^2 (\underline{\mathcal{R}}_{\leq 2}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u, \underline{u}})}^2. \end{aligned}$$

Using (commuted) Bianchi equations (2.35a) (2.35b) for  $\nabla_3\alpha$  and  $d\check{v}\alpha$ , the elliptic estimates of Lemma 3.38 and the above estimates for (derivatives of)  $\beta$  we deduce

$$\begin{aligned} \int_{S_{u,\underline{u}}} |\underline{u}^4(\nabla, \nabla_3)(\nabla, \nabla_3, \nabla_4)\alpha|^2 &\lesssim \int_{S_{u,\underline{u}}} |\underline{u}^2(\underline{u}\nabla_4, \underline{u}\nabla)^{\leq 1}(\underline{u}\nabla, \underline{u}\nabla_3, \underline{u}\nabla_4)^{\leq 1}\beta|^2 \\ &\lesssim \underline{\mathcal{Q}}_{\leq 2}(u, \underline{u}) + (D\varepsilon)^2 (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 + (D\varepsilon)^2 \|\underline{u}(\bar{\rho}, \bar{\sigma})\|_{L^2(S_{u,\underline{u}})}^2. \end{aligned}$$

Using an absorption argument, this concludes the desired control of  $\underline{\mathcal{R}}_{\leq 2}(u, \underline{u})$ . The control of  $\mathcal{R}_{\leq 2}(u, \underline{u})$  follows along the same lines and this finishes the proof of Proposition 6.3.



## 7. Maximal curvature estimates in $\mathcal{M}_{\text{bot}}^{\text{int}}$

In this section, we prove the following proposition.

**Proposition 7.1.** *Recall that from Proposition 5.1, we have in  $(\tau)\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$\int_{(\tau)\Sigma_t} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}}\mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \bar{T}) \lesssim \varepsilon^2, \quad (7.1a)$$

$$\int_{(\tau)\Sigma_t} Q(\hat{\mathcal{L}}_{\mathbf{S}^{\text{int}}}\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}}\mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \bar{T}) \lesssim \varepsilon^2, \quad (7.1b)$$

for a fixed transition parameter  $\tau$  and for all  $(\tau)t^\circ \leq t \leq (\tau)t^*$ . Recall that from Proposition 6.1, we have on  $(\tau)\mathcal{T}$

$$\left\| t^3 \nabla^{\leq 1} R \right\|_{L_t^\infty \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon, \quad (7.2)$$

for  $\nabla \in \{(t\nabla, t\nabla_3, t\nabla_4)\}$  and  $R \in \{\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}\}$ .

Under the Bootstrap Assumptions, and the estimates (7.1) and (7.2), we have the following bound in  $(\tau)\mathcal{M}_{\text{bot}}^{\text{int}}$  for  $\varepsilon > 0$  sufficiently small

$$\mathcal{R}_{\leq 2}^{\text{int}} \lesssim \varepsilon, \quad (7.3)$$

where we refer the reader to Section 3.2.2 for definitions.

**Remark 7.2.** *Using Sobolev estimates (see Lemma 7.3), one directly deduces from (7.3) that*

$$\mathfrak{R}_{\leq 1}^{\text{int}} \lesssim \varepsilon$$

in  $(\tau)\mathcal{M}_{\text{bot}}^{\text{int}}$ . See Section 3.2.2 for definitions.

### 7.1 Preliminary results

We have the following Sobolev embeddings on the maximal hypersurfaces  $\Sigma_t$  of  $\mathcal{M}_{\text{bot}}^{\text{int}}$ .

**Lemma 7.3** (Sobolev estimates on  $\Sigma_t$ ). *Under the mild Bootstrap Assumptions 3.17, we have for all  $t^\circ \leq t \leq t^*$  and for all  $\Sigma_t$ -tangent tensor  $F$*

$$\begin{aligned} \|F\|_{L^6(\Sigma_t)} &\lesssim \|\nabla F\|_{L^2(\Sigma_t)}^{1/2} \|F\|_{L^2(\Sigma_t)}^{1/2} + t^{-1} \|F\|_{L^2(\Sigma_t)}, \\ \|F\|_{L^\infty(\Sigma_t)} &\lesssim \|\nabla^2 F\|_{L^2(\Sigma_t)}^{3/4} \|F\|_{L^2(\Sigma_t)}^{1/4} + t^{-3/2} \|F\|_{L^2(\Sigma_t)}. \end{aligned}$$

*Proof.* A rescaling in  $t$  of the estimates of [CG19b, Section 3] gives the result.  $\square$

We have the following elliptic estimates for div-curl systems on  $\Sigma_t$ .

**Lemma 7.4** (Elliptic estimates for div-curl systems on  $\Sigma_t$ ). *Assume that the mild Bootstrap Assumptions 3.17 are satisfied and that the following bounds hold*

$$\begin{aligned} \left\| t^{5/2} \left( \text{tr}\theta - \frac{2}{r} \right) \right\|_{L^\infty(\mathcal{T})} &\lesssim D\varepsilon, \\ \left\| t^{5/2} \hat{\theta} \right\|_{L^\infty(\mathcal{T})} &\lesssim D\varepsilon. \end{aligned}$$

Then, for  $\varepsilon > 0$  sufficiently small, we have for all symmetric traceless  $\Sigma_t$ -tangent 2-tensor  $F$

$$\int_{\Sigma_t} |(t\nabla)^{\leq 1} F|^2 \lesssim \int_{\Sigma_t} (t^2 |\text{div} F|^2 + t^2 |\text{curl} F|^2) + t^2 \left| \int_{\partial\Sigma_t} \mathcal{H}_{\bar{N}} \cdot \nabla (F_{\bar{N}\bar{N}}) \right|,$$

where  $\mathcal{H}_{\bar{N}}$  is the  $\partial\Sigma_t$ -tangent 1-tensor defined by  $\mathcal{H}_{\bar{N}a} := F_{\bar{N}a}$ .

*Proof.* The result of the lemma is a straight-forward generalisation of the estimates obtained for  $k$  in [CG19b, Section 4.7] and a  $t$ -rescaling (see in particular estimate (4.32) in that paper). Details are left to the reader.  $\square$

We have the following variation of Lemma 7.4, which will be used in Section 7.3 to control  $\hat{\mathcal{L}}_{\bar{T}}E$  and  $\hat{\mathcal{L}}_{\bar{T}}H$  (see also [CK93, pp. 104-105] for an analogous result in the case without boundary).

**Lemma 7.5** (Elliptic estimates for modified div-curl systems on  $\Sigma_t$ ). *Assume that the hypothesis of Lemma 7.4 hold. There exists a sufficiently small constant  $\varphi > 0$ , such that if  $X$  is a  $\Sigma_t$ -tangent vectorfield with  $|X| \leq \varphi$  and  $|(t\nabla)X| \lesssim 1$  on  $\Sigma_t$ , then, for all symmetric traceless  $\Sigma_t$ -tangent 2-tensors  $F, G$ , we have*

$$\begin{aligned} \int_{\Sigma_t} |(t\nabla)F|^2 + |(t\nabla)G|^2 &\lesssim \int_{\Sigma_t} (t^2|\text{div}F|^2 + t^2|\text{div}G|^2) \\ &\quad + \int_{\Sigma_t} t^2 \left( |\text{curl}F + \hat{\mathcal{L}}_X G|^2 + |\text{curl}G - \hat{\mathcal{L}}_X F|^2 \right) \\ &\quad + \int_{\Sigma_t} (|F|^2 + |G|^2) \\ &\quad + t^2 \left| \int_{\partial\Sigma_t} \mathcal{H}_N \cdot \nabla (F_{NN}) \right| + t^2 \left| \int_{\partial\Sigma_t} \mathcal{G}_N \cdot \nabla (G_{NN}) \right| \end{aligned}$$

*Proof.* The result follows from a rescaling of  $\Sigma_t$  to a disk of radius 1, a generalisation of the estimates from [CG19b] and an absorption argument, provided that  $|X| \leq \varphi$  is sufficiently small. Details are left to the reader.  $\square$

We have the following higher order elliptic estimates for div-curl systems, which is used to control  $\nabla^2 E$  and  $\nabla^2 H$  in Section 7.3.

**Lemma 7.6** (Higher order elliptic estimates on  $\Sigma_t$ ). *Under the Bootstrap Assumptions 3.17, we have for all traceless symmetric  $\Sigma_t$ -tangent 2-tensor  $F$*

$$\begin{aligned} \int_{\Sigma_t} |(t\nabla)^2 F|^2 &\lesssim \int_{\Sigma_t} t^4 |\nabla \text{div}F|^2 + \int_{\Sigma_t} t^4 |\nabla \text{curl}F|^2 + \int_{\Sigma_t} |(t\nabla)^{\leq 1} F|^2 \\ &\quad + t^4 \|\nabla F\|_{\dot{H}^{1/2}(\partial\Sigma_t)}^2 + t^4 \|\nabla \mathcal{H}_{\bar{N}}\|_{\dot{H}^{1/2}(\partial\Sigma_t)}^2 + t^4 \|\nabla (F_{\bar{N}\bar{N}})\|_{\dot{H}^{1/2}(\partial\Sigma_t)}^2, \end{aligned}$$

where  $\mathcal{H}, \mathcal{H}_{\bar{N}}$  are respectively the projections of  $F$  and  $F_{\bar{N}}$  as  $\partial\Sigma_t$ -tangent tensors.

*Proof.* The proof follows from rescaling in  $t$  and the results of [CG19b, Appendix A].  $\square$

## 7.2 Control of $\mathcal{R}_{\leq 1}^{\text{int}}$

**Control of  $\hat{\mathcal{L}}_{\bar{T}}E, \hat{\mathcal{L}}_{\bar{T}}H$**  Using (7.1a), the fact that  $\mathbf{T}^{\text{int}} = \bar{T}$ , and that – by the mild Bootstrap Assumptions 3.18 –,  $\mathbf{K}^{\text{int}}$  is a future-pointing timelike vectorfield satisfying comparison bounds (3.2) with  $\bar{T}$ , we have

$$\int_{\Sigma_t} t^6 \left| \hat{\mathcal{L}}_{\bar{T}} \mathbf{R} \right|^2 \lesssim \int_{\Sigma_t} Q(\hat{\mathcal{L}}_{\mathbf{T}^{\text{int}}} \mathbf{R})(\mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{K}^{\text{int}}, \bar{T}) \lesssim \varepsilon^2.$$

One has schematically

$$\hat{\mathcal{L}}_{\bar{T}}E, \hat{\mathcal{L}}_{\bar{T}}H = \hat{\mathcal{L}}_{\bar{T}}\mathbf{R} + \mathbf{R} \cdot \mathbf{D}\bar{T},$$

from which, using the Bootstrap Assumptions 3.21 and 3.28 on the curvature and on  $\mathbf{D}\bar{T}$  respectively, one deduces that

$$\left\| t^3 \hat{\mathcal{L}}_{\bar{T}}E \right\|_{L^2(\Sigma_t)} + \left\| t^3 \hat{\mathcal{L}}_{\bar{T}}H \right\|_{L^2(\Sigma_t)} \lesssim \varepsilon. \quad (7.4)$$

**Control of  $\nabla^{\leq 1}(E, H)$**  From Maxwell equations (2.14), the traceless symmetric  $\Sigma_t$ -tangent 2-tensor  $E$  satisfies schematically the following div-curl system

$$\begin{aligned} \text{tr}E &= 0, \\ \text{div}E &= \text{Err}(\text{div}, E), \\ \text{curl}E &= \hat{\mathcal{L}}_{\bar{T}}H + \text{Err}(\text{curl}, E), \end{aligned} \tag{7.5}$$

where the error terms  $\text{Err}(\text{div}, E)$ ,  $\text{Err}(\text{curl}, E)$  are of the following form

$$\text{Err} = (\nabla n, k) \cdot (E, H).$$

Using (7.5) and the elliptic estimate from Lemma 7.4, we have

$$\begin{aligned} \int_{\Sigma_t} t^4 |(t\nabla)^{\leq 1}E|^2 &\lesssim \int_{\Sigma_t} t^6 |\hat{\mathcal{L}}_{\bar{T}}H|^2 + t^6 \left| \int_{\partial\Sigma_t} \mathcal{H}_{\bar{N}} \cdot \nabla(E_{\bar{N}\bar{N}}) \right| + \int_{\Sigma_t} t^6 |\text{Err}|^2 \\ &\lesssim \varepsilon^2 + t^6 \left| \int_{\partial\Sigma_t} \mathcal{H}_{\bar{N}} \cdot \nabla(E_{\bar{N}\bar{N}}) \right|, \end{aligned}$$

where the error terms are estimated using the Bootstrap Assumptions 3.21 and 3.28.

From relations (2.15) and (2.16) between  $\bar{T}, \bar{N}$  and  $e_3, e_4$ , one has

$$\begin{aligned} E_{\bar{N}\bar{N}} &= \mathbf{R}(\bar{T}, \bar{N}, \bar{T}, \bar{N}) \\ &= \frac{1}{4} \mathbf{R}(e_3, e_4, e_3, e_4) \\ &= \rho. \end{aligned}$$

Using an  $H^{1/2} \times H^{-1/2}$  estimate on  $\partial\Sigma_t$ ,<sup>1</sup> we have

$$t^6 \left| \int_{\partial\Sigma_t} \mathcal{H}_{\bar{N}} \cdot \nabla(E_{\bar{N}\bar{N}}) \right| \lesssim t^6 \|\mathcal{H}_{\bar{N}}\|_{\dot{H}^{1/2}(\partial\Sigma_t)} \|\rho - \bar{\rho}\|_{\dot{H}^{1/2}(\partial\Sigma_t)}.$$

Using the trace estimates (7.2) for  $\rho - \bar{\rho}$  and an absorption argument for  $\mathcal{H}_{\bar{N}}$ , we therefore conclude

$$\int_{\Sigma_t} t^4 |(t\nabla)^{\leq 1}E|^2 \lesssim \varepsilon^2,$$

as desired. The estimates for  $\nabla^{\leq 1}H$  follow similarly. This concludes the control of  $\mathcal{R}_{\leq 1}^{\text{int}}$ .

### 7.3 Control of $\mathcal{R}_2^{\text{int}}$

**Control of  $\hat{\mathcal{L}}_{\mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}}(E, H)$**  Using (7.1b) and the mild Bootstrap Assumptions 3.18, one has

$$\int_{\Sigma_t} \left( t^6 \left| \hat{\mathcal{L}}_{\mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}}E \right|^2 + t^6 \left| \hat{\mathcal{L}}_{\mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}}H \right|^2 \right) \lesssim \varepsilon^2. \tag{7.6}$$

**Control of  $\nabla \hat{\mathcal{L}}_{\bar{T}}(E, H)$  and  $\hat{\mathcal{L}}_{\bar{T}}^2(E, H)$**  Commuting Maxwell equations (2.14) with  $\hat{\mathcal{L}}_{\bar{T}}$ , one has the following div-curl system for the symmetric traceless  $\Sigma_t$ -tangent 2-tensor  $\hat{\mathcal{L}}_{\bar{T}}E$  and  $\hat{\mathcal{L}}_{\bar{T}}H$

$$\begin{aligned} \text{tr} \hat{\mathcal{L}}_{\bar{T}}E &= 0, \\ \text{div} \hat{\mathcal{L}}_{\bar{T}}E &= \text{Err}(\text{div}, \hat{\mathcal{L}}_{\bar{T}}E), \\ \text{curl} \hat{\mathcal{L}}_{\bar{T}}E &= \hat{\mathcal{L}}_{\bar{T}} \hat{\mathcal{L}}_{\bar{T}}H + \text{Err}(\text{curl}, \hat{\mathcal{L}}_{\bar{T}}E), \end{aligned} \tag{7.7}$$

and

$$\begin{aligned} \text{tr} \hat{\mathcal{L}}_{\bar{T}}H &= 0, \\ \text{div} \hat{\mathcal{L}}_{\bar{T}}H &= \text{Err}(\text{div}, \hat{\mathcal{L}}_{\bar{T}}H), \\ \text{curl} \hat{\mathcal{L}}_{\bar{T}}H &= -\hat{\mathcal{L}}_{\bar{T}} \hat{\mathcal{L}}_{\bar{T}}E + \text{Err}(\text{curl}, \hat{\mathcal{L}}_{\bar{T}}H), \end{aligned} \tag{7.8}$$

<sup>1</sup>Such an estimate can be obtained from the definitions [Sha14, pp. 834–836] and [CG19a, Appendix B].



where the error terms are of the form

$$\begin{aligned} \text{Err} &= \left( \nabla^{\leq 2}(n-1) + \nabla^{\leq 1} \hat{\mathcal{L}}_{\bar{T}}^{\leq 1}(n-1) + \left( \nabla, \hat{\mathcal{L}}_{\bar{T}} \right)^{\leq 1} k \right) \cdot (E, H) \\ &\quad + \left( \nabla^{\leq 1}(n-1) + k \right) \cdot \left( \left( \nabla, \hat{\mathcal{L}}_{\bar{T}} \right)^{\leq 1}(E, H) \right). \end{aligned}$$

From the definition of  $\mathbf{S}^{\text{int}}$  and  $\mathbf{X}^{\text{int}}$  in Section 2.5, we have

$$\hat{\mathcal{L}}_{\bar{T}} \hat{\mathcal{L}}_{\bar{T}}(E, H) = \hat{\mathcal{L}}_{t^{-1} \mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}}(E, H) - \hat{\mathcal{L}}_{t^{-1} \mathbf{X}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}}(E, H). \quad (7.9)$$

Using (7.4), (7.6) and the Bootstrap Assumptions 3.24 to control the derivatives of  $t$ , we have

$$\int_{\Sigma_t} t^8 \left| \hat{\mathcal{L}}_{t^{-1} \mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}}(E, H) \right|^2 \lesssim \varepsilon^2. \quad (7.10)$$

We define  $X$  to be the projection of  $t^{-1} \mathbf{X}^{\text{int}}$  on  $\Sigma_t$ , *i.e.*

$$t^{-1} \mathbf{X}^{\text{int}} =: -\mathbf{g}(t^{-1} \mathbf{X}^{\text{int}}, \bar{T}) \bar{T} + X.$$

From the bootstrap bound (3.7), we have

$$\left| (t\mathbf{D})^{\leq 1} \mathbf{g}(t^{-1} \mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}}) \right| \lesssim (D\varepsilon) t^{-3/2},$$

and the  $\bar{T}$  component of  $\mathbf{X}^{\text{int}}$  in (7.9) can be treated as an error term in the following rewriting of the div-curl systems (7.7) and (7.8)

$$\begin{aligned} \text{tr} \hat{\mathcal{L}}_{\bar{T}} E &= 0, \\ \text{div} \hat{\mathcal{L}}_{\bar{T}} E &= \text{Err}, \\ \text{curl} \hat{\mathcal{L}}_{\bar{T}} E + \hat{\mathcal{L}}_X \hat{\mathcal{L}}_{\bar{T}} H &= \hat{\mathcal{L}}_{t^{-1} \mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}} H + \text{Err}, \end{aligned} \quad (7.11)$$

and

$$\begin{aligned} \text{tr} \hat{\mathcal{L}}_{\bar{T}} H &= 0, \\ \text{div} \hat{\mathcal{L}}_{\bar{T}} H &= \text{Err}, \\ \text{curl} \hat{\mathcal{L}}_{\bar{T}} H - \hat{\mathcal{L}}_X \hat{\mathcal{L}}_{\bar{T}} E &= -\hat{\mathcal{L}}_{t^{-1} \mathbf{S}^{\text{int}}} \hat{\mathcal{L}}_{\bar{T}} E + \text{Err}. \end{aligned} \quad (7.12)$$

From the bootstrap bound (3.6), we have

$$\left| \sup_{\Sigma_t} |X| - \frac{1-\tau}{1+\tau} \right| \lesssim \varepsilon t^{-3/2}.$$

Therefore, there exists a numerical constant  $0 < \tau_0 < 1$ , such that for all transition parameters  $\tau$ , *i.e.*  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$ , and for  $\varepsilon > 0$  sufficiently small, we have

$$|X| \leq \varrho,$$

in  $\mathcal{M}_{\text{bot}}^{\text{int}}$ , where  $\varrho$  is the constant from Lemma 7.5. Moreover, from the Bootstrap Assumptions 3.28 on  $\mathbf{D}\mathbf{X}^{\text{int}}$ , we have

$$|(t\nabla)X| \lesssim 1.$$

One can therefore apply the result of Lemma 7.5 to the modified div-curl systems (7.11) and (7.12) for  $\hat{\mathcal{L}}_{\bar{T}} E$  and  $\hat{\mathcal{L}}_{\bar{T}} H$ , using estimate (7.10) and the estimates for the lower order terms obtained in Section 7.2 and the Bootstrap Assumptions 3.21 and 3.24 to control the nonlinear error terms, and we have

$$\int_{\Sigma_t} t^6 \left| (t\nabla) \hat{\mathcal{L}}_{\bar{T}}(E, H) \right|^2 \lesssim \varepsilon^2 + \left| \int_{\partial\Sigma_t} \hat{\mathcal{L}}_{\bar{T}} E_{\bar{N}} \cdot \nabla \left( \hat{\mathcal{L}}_{\bar{T}} E_{NN} \right) \right| + \left| \int_{\partial\Sigma_t} \hat{\mathcal{L}}_{\bar{T}} H_{\bar{N}} \cdot \nabla \left( \hat{\mathcal{L}}_{\bar{T}} H_{NN} \right) \right|.$$

We have on  $\partial\Sigma_t$

$$\hat{\mathcal{L}}_{\bar{T}} E_{NN} = e_3(\rho) + e_4(\rho) + \text{Err}.$$

Therefore using the trace estimates (7.2) and  $H^{-1/2} \times H^{1/2}$  estimates as previously gives the control of the integral boundary term for  $\hat{\mathcal{L}}_{\bar{T}}E$ . Arguing similarly gives the control of the integral boundary term for  $\hat{\mathcal{L}}_{\bar{T}}H$ , and we finally obtain

$$\int_{\Sigma_t} t^6 \left| (t\nabla)^{\leq 1} \hat{\mathcal{L}}_{\bar{T}}(E, H) \right|^2 \lesssim \varepsilon^2. \quad (7.13)$$

Using (7.9) and the above mentioned controls for  $\mathbf{X}^{\text{int}}$ , we also deduce from (7.10) and (7.13) that

$$\int_{\Sigma_t} t^8 \left| \hat{\mathcal{L}}_{\bar{T}}^2(E, H) \right|^2 \lesssim \varepsilon^2.$$

**Control of  $\nabla^2(E, H)$**  The control of  $\nabla^2 E$  follows from an application of the higher order elliptic estimates of Lemma 7.6 to the div-curl systems (7.5), using the estimate (7.13) for  $\nabla \hat{\mathcal{L}}_{\bar{T}}H$  and the trace estimates (7.2). The estimate for  $\nabla^2 H$  follow similarly.

**Control of  $\mathbf{D}^{\leq 2}\mathbf{R}$**  Using the definition (2.9) of  $E, H$ , the control of  $\bar{T}, \mathbf{D}\bar{T}, \mathbf{D}^2\bar{T}$  from the Bootstrap Assumptions 3.28, and the Bootstrap Assumptions 3.21, one deduces from the bounds obtained for  $E, H$  that

$$\left\| t^2 (t\mathbf{D})^{\leq 2} \mathbf{R} \right\|_{L^2(\Sigma_t)} \lesssim \varepsilon,$$

for all  $t^\circ \leq t \leq t^*$ , and where the norms are taken with respect to the maximal frame. This finishes the proof of (7.3).



## 8. Remaining curvature estimates

In this section, we prove that curvature estimates hold in the interior of the cone  $\underline{\mathcal{C}}^* \cap^{(\tau)} \mathcal{M}^{\text{int}}$ . We also prove that the curvature estimates hold on all hypersurfaces associated to all transition parameter  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$ . Both arguments go by rescaling and local energy estimates.

### 8.1 Null curvature estimates on $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$

In this section, we prove the following proposition.

**Proposition 8.1.** *Recall that from Proposition 7.1, we have on  ${}^{(\tau)}\Sigma_{t^*}$*

$$\left\| t^{*2} (t^* \mathbf{D})^{\leq 2} \mathbf{R} \right\|_{L^2({}^{(\tau)}\Sigma_{t^*})} \lesssim \varepsilon, \quad (8.1)$$

for a fixed transition parameter  $\tau$ .

Under the Bootstrap Assumptions and (8.1) and for  $\varepsilon > 0$  sufficiently small, we have the following bounds for the null curvature components on the interior of the last cone  $\underline{\mathcal{C}}^* \cap {}^{(\tau)}\mathcal{M}^{\text{int}}$

$$\mathcal{R}_{\leq 2}^* \lesssim \varepsilon. \quad (8.2)$$

Moreover, we also have the following  $L^\infty \tilde{H}^{1/2}$  estimates in  $\underline{\mathcal{C}}^* \cap {}^{(\tau)}\mathcal{M}^{\text{int}}$

$$\mathfrak{R}_{\leq 1}^* \lesssim \varepsilon. \quad (8.3)$$

We refer the reader to Section 3.2.1 for definitions.

**Remark 8.2.** *The  $L^2$ -norms of (8.2) (see the definitions of Section 3.2.1) allow for a degeneracy of the type  $R \sim r^{-3/2}$  when  $r \rightarrow 0$  (see the Sobolev embeddings of Lemma 6.6). This is suboptimal by far and could be easily improved using that we obtain independently the uniform boundedness of the spacetime curvature tensor  $|\mathbf{R}| \lesssim \varepsilon$  in the extended spacetime of Section 8.1.1. In this section, we could improve the  $L^2$ -bounds (8.2) by a more careful analysis of relations between Cartesian and spherical derivatives, see Remark 8.3. However – since the null decompositions of  $\mathbf{R}$  are multi-valued at the vertex –, there is no hope to obtain  $L^2$  norms on  $\underline{\mathcal{C}}^*$  sufficiently regular to get boundedness of  $R$  when  $r \rightarrow 0$  via Sobolev embeddings, since continuity would follow as well. As a consequence, we do not seek for rates in  $r$  in the  $L^2$  norms better than the (easily obtained) ones of (8.2). Fortunately, we can still obtain a control of the null connection coefficients consistent with the potential  $r^{-3/2}$  singularity of the null curvature components on the cone  $\underline{\mathcal{C}}^*$  (see Remark 9.3 and [Wan09]).*

#### 8.1.1 $r^*$ -rescaling, extension and local existence from $\Sigma_{t^*}$

We perform a  $r^*$ -rescaling of the last maximal hypersurface  $\Sigma_{t^*}$  to a maximal hypersurface  $\bar{\Sigma}_1$  of size 1, *i.e.* such that  $\partial \bar{\Sigma}_1$  has area radius 1.<sup>1</sup> The bounds (8.1) rewrite as

$$\left\| \mathbf{D}^{\leq 2} \mathbf{R} \right\|_{L^2(\bar{\Sigma}_1)} \lesssim \varepsilon_1, \quad (8.4)$$

where

$$\varepsilon_1 := \varepsilon (t^*)^{-3/2},$$

and the bounds from the Bootstrap Assumptions 3.27 and 3.24 as<sup>2</sup>

$$\begin{aligned} \left\| \partial^{\leq 4} (g_{ij} - \delta_{ij}) \right\|_{L^2(\bar{\Sigma}_1)} &\lesssim D \varepsilon_1, \\ \left\| \nabla^{\leq 3} k \right\|_{L^2(\bar{\Sigma}_1)} &\lesssim D \varepsilon_1. \end{aligned} \quad (8.5)$$

Using the extension theorem [Czi18, Theorem 3.1], we obtain an extension  $\tilde{g}_{ij}, \tilde{k}_{ij}$  of  $(g_{ij}, k_{ij})$  defined on  $\mathbb{R}^3$ , *i.e.* such that  $\tilde{g}_{ij} = g_{ij}$  and  $\tilde{k}_{ij} = k_{ij}$  on  $\bar{\Sigma}_1 \simeq \mathbb{D} \subset \mathbb{R}^3$ , that satisfy the maximal constraint equations, and

<sup>1</sup>Recall that  $r^*$  is defined to be the area radius of  $\partial \Sigma_{t^*} = S^*$  and that  $r^* \simeq t^*$ .

<sup>2</sup>Bounds for harmonic coordinates are equivalent to bounds for the metric in these coordinates. See also Theorem 4.3.

such that

$$\begin{aligned} \|\partial^{\leq 4}(\tilde{g}_{ij} - \delta_{ij})\|_{L^2(\mathbb{R}^3)} &\lesssim D\varepsilon_1, \\ \|\partial^{\leq 3}\tilde{k}_{ij}\|_{L^2(\mathbb{R}^3)} &\lesssim D\varepsilon_1, \end{aligned}$$

with suitable fall-off rate at infinity (see [Czi18] for precisions).

Using the local existence result for such an initial data set satisfying the maximal constraint equations on  $\mathbb{R}^3$  (see [CK93, Theorem 10.2.2]), and provided that  $D\varepsilon_1$  is sufficiently small, there exists a spacetime  $D\varepsilon_1$ -close to  $[0, 3/2] \times \mathbb{R}^3 \subset \mathbb{R}^{1+3}$  (see [CK93] for the precise definitions). This spacetime admits a maximal time function  $\tilde{t}$  such that  $\tilde{t} = 0$  on  $\bar{\Sigma}_1$  and we have the following control for the norms of the spacetime curvature tensor in the maximal frame norm

$$\begin{aligned} \|\mathbf{R}\|_{L^\infty} &\lesssim D\varepsilon_1, \\ \|\mathbf{D}\mathbf{R}\|_{L_t^\infty L^6(\Sigma_{\tilde{t}})} &\lesssim D\varepsilon_1, \\ \|\mathbf{D}^2\mathbf{R}\|_{L_t^\infty L^2(\Sigma_{\tilde{t}})} &\lesssim D\varepsilon_1, \end{aligned} \tag{8.6}$$

locally in the domain of dependence of  $\bar{\Sigma}_1$ . Moreover, there exists approximate Cartesian Killing vectorfields  $\partial_\mu$  for  $\mu = 0, 1, 2, 3$ , such that

$$\begin{aligned} \|\mathbf{D}\partial_\mu\|_{L^\infty} &\lesssim D\varepsilon_1, \\ \|\mathbf{D}^2\partial_\mu\|_{L_t^\infty L^2(\Sigma_{\tilde{t}})} &\lesssim D\varepsilon_1, \\ \|\mathbf{D}^3\partial_\mu\|_{L_t^\infty L^2(\Sigma_{\tilde{t}})} &\lesssim D\varepsilon_1, \end{aligned} \tag{8.7}$$

and where  $\partial_0$  is defined as

$$\partial_0 := \frac{\mathbf{D}\tilde{t}}{(-\mathbf{g}(\mathbf{D}\tilde{t}, \mathbf{D}\tilde{t}))^{1/2}}.$$

### 8.1.2 Energy estimates in $\mathcal{M}_{\text{top}}^{\text{int}}$

We define the spacetime region  $\mathcal{D}_1$  to be the future domain of dependence of  $\bar{\Sigma}_1$ .<sup>3</sup> In this section, we apply energy estimates in the region  $\mathcal{D}_1$  to obtain the desired curvature control on the  $r^*$ -rescaling of the cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ .

More precisely, we integrate the following contracted Bel-Robinson tensors in  $\mathcal{D}_1$

$$Q(\mathbf{R})(\partial_0, \partial_0, \partial_0), \quad Q(\hat{\mathcal{L}}_{\partial_\mu}\mathbf{R})(\partial_0, \partial_0, \partial_0), \quad Q(\hat{\mathcal{L}}_{\partial_\mu}\hat{\mathcal{L}}_{\partial_\nu}\mathbf{R})(\partial_0, \partial_0, \partial_0),$$

where the vectorfields  $\partial_\mu$  are the approximate Cartesian Killing fields from Section 8.1.1.

For each estimates, the spacetime error term in  $\mathcal{D}_1$  is controlled using the estimates (8.6) and (8.7) (see also the treatment of the error terms in  $\mathcal{M}_{\text{bot}}^{\text{int}}$  and  $\mathcal{M}^{\text{ext}}$  in Section 5). We therefore obtain the following estimates on the future boundary  $\bar{\mathcal{C}}_1$  of  $\mathcal{D}_1$

$$\begin{aligned} \int_{\bar{\mathcal{C}}_1} Q(\mathbf{R})(\partial_0, \partial_0, \partial_0, e_3) &\lesssim \int_{\bar{\Sigma}_1} Q(\mathbf{R})(\partial_0, \partial_0, \partial_0, \bar{T}) + (D\varepsilon_1)^3, \\ \int_{\bar{\mathcal{C}}_1} Q(\hat{\mathcal{L}}_{\partial_\mu}\mathbf{R})(\partial_0, \partial_0, \partial_0, e_3) &\lesssim \int_{\bar{\Sigma}_1} Q(\hat{\mathcal{L}}_{\partial_\mu}\mathbf{R})(\partial_0, \partial_0, \partial_0, \bar{T}) + (D\varepsilon_1)^3, \\ \int_{\bar{\mathcal{C}}_1} Q(\hat{\mathcal{L}}_{\partial_\mu}\hat{\mathcal{L}}_{\partial_\nu}\mathbf{R})(\partial_0, \partial_0, \partial_0, e_3) &\lesssim \int_{\bar{\Sigma}_1} Q(\hat{\mathcal{L}}_{\partial_\mu}\hat{\mathcal{L}}_{\partial_\nu}\mathbf{R})(\partial_0, \partial_0, \partial_0, \bar{T}) + (D\varepsilon_1)^3, \end{aligned}$$

where  $\bar{T}$  is the future-pointing unit normal to  $\bar{\Sigma}_1$ .

<sup>3</sup> $\mathcal{D}_1$  corresponds to the  $r^*$  rescaling of  $\mathcal{M}_{\text{top}}^{\text{int}}$ .

Using the improved curvature estimate (8.4) and the control (8.7) of the vectorfields  $\partial_\mu$  on  $\bar{\Sigma}_1$ , this yields

$$\begin{aligned} \int_{\bar{\mathcal{C}}_1} Q(\mathbf{R})(\partial_0, \partial_0, \partial_0, e_3) &\lesssim \varepsilon_1^2, \\ \int_{\bar{\mathcal{C}}_1} Q(\hat{\mathcal{L}}_{\partial_\mu} \mathbf{R})(\partial_0, \partial_0, \partial_0, e_3) &\lesssim \varepsilon_1^2, \\ \int_{\bar{\mathcal{C}}_1} Q(\hat{\mathcal{L}}_{\partial_\mu} \hat{\mathcal{L}}_{\partial_\nu} \mathbf{R})(\partial_0, \partial_0, \partial_0, e_3) &\lesssim \varepsilon_1^2, \end{aligned} \quad (8.8)$$

for  $\varepsilon_1$  sufficiently small.

We now compare the coordinate vectorfields  $\partial_0$  to  $T := \frac{1}{2}(e_3 + e_4)$  on  $\bar{\mathcal{C}}_1$ . Since  $\partial_0 = \bar{T}$  on  $\bar{\Sigma}_1$ , we have on  $\partial\bar{\Sigma}_1$

$$\mathbf{g}(\partial_0, T) = \mathbf{g}(\bar{T}, T) = -\frac{1}{2}(\nu^{-1} + \nu).$$

From the (rescaled) bootstrap bound on  $\nu$  from the Bootstrap Assumptions 3.24, this gives on  $\partial\bar{\Sigma}_1$

$$|\mathbf{g}(\partial_0, T) + 1| \lesssim D\varepsilon_1.$$

Using the relations (2.33), we further have

$$\begin{aligned} |\mathbf{D}_{e_3}(\mathbf{g}(\partial_0, T))| &\lesssim |\mathbf{D}\partial_0||e_3| + |\mathbf{D}_{e_3}e_3| + |\mathbf{D}_{e_3}e_4| \\ &\lesssim |\mathbf{D}\partial_0||e_3| + |\eta||e_a| + |\underline{\omega}|(|e_3| + |e_4|), \end{aligned}$$

where the norms are taken with respect to the maximal frame (*i.e.*  $e_0 = \partial_0$  in the notations of Definition 3.1).

Using the Bootstrap Assumptions 3.22 on the null connection coefficients on  $\underline{\mathcal{C}}^*$  and (8.7), we obtain

$$|\mathbf{D}_{e_3}(\mathbf{g}(\partial_0, T))| \lesssim r^{-1/2}D\varepsilon_1 (|e_3| + |e_4| + |e_a|).$$

From Remark 3.2, using that  $e_3, e_4$  are future-pointing, we have

$$\begin{aligned} |e_3| + |e_4| &= -\sqrt{2}\mathbf{g}(e_3, \partial_0) - \sqrt{2}\mathbf{g}(e_4, \partial_0) = 2\sqrt{2}\mathbf{g}(T, \partial_0), \\ \sum_{a=1,2} |e_a|^2 &= 2 + 2 \sum_{a=1,2} |\mathbf{g}(\partial_0, e_a)|^2 \leq 2 + (|\partial_0|_T^2) \leq 1 + 2|\mathbf{g}(\partial_0, T)|^2, \end{aligned}$$

where  $|\cdot|_T$  is the frame norm with respect to  $T$ .

Thus, we have

$$|\mathbf{D}_{e_3}(\mathbf{g}(\partial_0, T))| \lesssim r^{-1/2}D\varepsilon_1 (1 + |\mathbf{g}(\partial_0, T)|),$$

and from a Grönwall argument, integrating from  $r = 1$  to  $r = 0$ , we obtain

$$|\mathbf{g}(\partial_0, T) + 1| \lesssim D\varepsilon_1 \quad (8.9)$$

on  $\bar{\mathcal{C}}_1$ , *i.e.* the frames adapted to  $\partial_0$  and  $T$  are comparable.

Using (8.8) and (8.9) we obtain

$$\begin{aligned} \int_{\bar{\mathcal{C}}_1} |\text{null}(\mathbf{R})|^2 &\lesssim \varepsilon_1^2, \\ \int_{\bar{\mathcal{C}}_1} |\text{null}(\hat{\mathcal{L}}_{\partial_\mu} \mathbf{R})|^2 &\lesssim \varepsilon_1^2, \\ \int_{\bar{\mathcal{C}}_1} |\text{null}(\hat{\mathcal{L}}_{\partial_\mu} \hat{\mathcal{L}}_{\partial_\nu} \mathbf{R})|^2 &\lesssim \varepsilon_1^2, \end{aligned} \quad (8.10)$$

where in that case  $\text{null} \in \{\alpha, \dots, \beta\}$ . This provides in particular the first desired estimates for the  $L^2$  norms of the curvature.

Using (8.7) and (8.4), we have

$$\begin{aligned} \int_{\bar{\mathcal{C}}_1} |\text{null}(\mathbf{D}_{\partial_\mu} \mathbf{R})|^2 &\lesssim \int_{\bar{\mathcal{C}}_1} |\text{null}(\hat{\mathcal{L}}_{\partial_\mu} \mathbf{R})|^2 + (D\varepsilon_1)^3, \\ \int_{\bar{\mathcal{C}}_1} |\text{null}(\mathbf{D}_{\partial_\mu, \partial_\nu}^2 \mathbf{R})|^2 &\lesssim \int_{\bar{\mathcal{C}}_1} |\text{null}(\hat{\mathcal{L}}_{\partial_\mu} \hat{\mathcal{L}}_{\partial_\nu} \mathbf{R})|^2 + (D\varepsilon_1)^3, \end{aligned}$$

which, using (8.9) and the resulting comparison between  $\partial_\mu$  and  $e_3, e_4, e_a$  further gives

$$\begin{aligned} \int_{\bar{\mathcal{C}}_1} |\text{null}(\mathbf{D}_X \mathbf{R})|^2 &\lesssim \varepsilon_1^2, \\ \int_{\bar{\mathcal{C}}_1} |\text{null}(\mathbf{D}_{X,Y}^2 \mathbf{R})|^2 &\lesssim \varepsilon_1^2, \end{aligned}$$

where  $X, Y \in \{e_3, e_4, e_a\}$ .

We have schematically

$$\nabla_X \text{null}(\mathbf{R}) = \text{null}(\mathbf{D}_X \mathbf{R}) + \mathbf{R} \cdot \Gamma,$$

where

$$\Gamma \in \{\text{tr}\chi, \text{tr}\underline{\chi}, \hat{\chi}, \hat{\underline{\chi}}, \eta, \zeta, \underline{\eta}, \underline{\omega}\}.$$

Multiplying by  $r$ , and since by the Bootstrap Assumptions 3.22,

$$|r\Gamma| \lesssim 1, \tag{8.11}$$

we therefore infer that for  $X \in \{e_3, e_4, e_a\}$  and since  $r \lesssim 1$ ,

$$\begin{aligned} \int_{\bar{\mathcal{C}}_1} |r\nabla_X \text{null}(\mathbf{R})|^2 &\lesssim \int_{\bar{\mathcal{C}}_1} r^2 |\text{null}(\mathbf{D}_X \mathbf{R})|^2 + \int_{\bar{\mathcal{C}}_1} |r\Gamma|^2 |\mathbf{R}|^2 \\ &\lesssim \varepsilon_1^2. \end{aligned}$$

Similarly, since we have schematically

$$\nabla_X \nabla_Y \text{null}(\mathbf{R}) = \text{null}(\mathbf{D}_{X,Y} \mathbf{R}) + \mathbf{D}\mathbf{R} \cdot \Gamma + \mathbf{R} \cdot \Gamma \cdot \Gamma,$$

for  $X, Y \in \{e_3, e_4, e_a\}$ , multiplying by  $r^2$  and taking the  $L^2$  norm on  $\bar{\mathcal{C}}_1$ , we conclude

$$\int_{\bar{\mathcal{C}}_1} |r^2 \nabla_{X,Y}^2 \text{null}(\mathbf{R})|^2 \lesssim \varepsilon_1^2.$$

Rescaling back these estimates on the original cone  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$  gives the desired bounds for the curvature, and finishes the proof of (8.2).

**Remark 8.3.** Estimate (8.11) is suboptimal in terms of  $r$ -weight for most of the connection coefficients. More precise bounds together with Hardy estimates (see Lemma 9.4) would lead to improve bounds. See Remark 8.2.

### 8.1.3 Proof of the decay estimate (8.3)

The proof of (8.3) boils down to the following Klainerman-Sobolev embeddings.

**Lemma 8.4** (Klainerman-Sobolev estimates on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ ). *For all  $S$ -tangent tensor  $F$  with vertex limit*

$$r^{3/2} |(r\nabla)^{\leq 1} F| \rightarrow 0 \quad \text{when } r \rightarrow 0,$$

*we have the following  $L_u^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})$  estimates in  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$*

$$\|rF\|_{L_{u \geq \tau \underline{u}^*}^\infty \tilde{H}^{1/2}(S_{u, \underline{u}^*})} \lesssim \|F\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} + \|r\nabla F\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} + \|r\nabla^3 F\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})}.$$

The proof of Lemma 8.4 is postponed to Appendix D. We apply the Klainerman-Sobolev estimates of Lemma 8.4, with  $F$  the following respective tensors

$$u^2 \nabla^{\leq 1} \underline{\alpha}, \quad u \underline{u} \nabla^{\leq 1} \underline{\beta}, \quad \underline{u}^2 \nabla^{\leq 1} (\rho - \bar{\rho}), \quad \underline{u}^2 \nabla^{\leq 1} (\sigma - \bar{\sigma}), \quad \underline{u}^2 \nabla^{\leq 1} \beta, \quad \underline{u}^2 (r \nabla)^{\leq 1} \alpha, \quad \underline{u}^2 (r \nabla_3) \alpha,$$

where  $\nabla \in \{(r \nabla), (q \nabla_3)\}$ . From Theorems 4.4 and 4.5, these tensors have the asymptotic behaviour  $r^{3/2} (r \nabla)^{\leq 1} R = O(r^{3/2})$  when  $r \rightarrow 0$  and satisfy in particular the required limits of Lemma 8.4. From an inspection of the definitions of Section 3.2.1, and using that in  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$  we have  $r \simeq q$  and  $u \simeq \underline{u}$ , we deduce from the bounds (8.2) that

$$\mathfrak{R}_{\leq 1}^* \lesssim \mathcal{R}_{\leq 2}^* \lesssim \varepsilon,$$

where the norms are restricted to  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}}$ . This proves (8.3).

## 8.2 Curvature estimates for all transition parameters

In this section, we call  $\tau_1 := \tau$  the fixed transition parameter given by the mean value argument of Section 5.1.1. We prove the following proposition.

**Proposition 8.5.** *Recall that from Propositions 7.1, we have*

$${}^{(\tau_1)} \mathcal{R}_{\leq 2}^{\text{int}} \lesssim \varepsilon, \quad (8.12)$$

in  ${}^{(\tau_1)} \mathcal{M}_{\text{bot}}^{\text{int}}$ . Recall that from Lemma 6.5, we have

$$\int_{\tau_1^{-1} u}^{\tau_0^{-1} u} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 \, d\underline{u} \lesssim \varepsilon^2, \quad (8.13)$$

for all  $1 \leq u \leq \tau_1 \underline{u}^*$ .

Under the Bootstrap Assumptions and estimates (8.12) and (8.13), and for  $\varepsilon > 0$  sufficiently small, we have the following bounds for all transition parameters  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$

$${}^{(\tau)} \mathcal{R}_{\leq 2, \gamma}^{\text{ext}} \lesssim \gamma \varepsilon, \quad (8.14)$$

$${}^{(\tau)} \mathfrak{R}_{\leq 1}^{\text{ext}} \lesssim \varepsilon, \quad (8.15)$$

for all  $\gamma > 0$  and

$${}^{(\tau)} \mathcal{R}_{\leq 2}^{\text{int}} \lesssim \varepsilon. \quad (8.16)$$

We refer to Sections 3.2.1 and 3.2.2 for the definitions of these norms. Here we indicate by  ${}^{(\tau)}$  the dependency on either just the covered domain (in estimates (8.14), (8.15)) or the covered domain and the nature of the hypersurfaces (in estimate (8.16)).

**Remark 8.6.** *As a result of Proposition 8.5, the curvature estimates of Propositions 6.1 and 7.1 obtained for the fixed transition parameter  $\tau_1$  given by the mean value argument of Section 5.1.1, hold for all transition parameter  $\tau_0 \leq \tau \leq (1 + \tau_0)/2$ . In the remaining Sections 9 – 12, we shall thus assume that  $\tau$  is any parameter in  $[\tau_0, (1 + \tau_0)/2]$ . This will improve the mild and strong Bootstrap Assumptions (see Sections 3.3.2 and 3.3.3) for all transition parameter  $\tau$ .*

The remaining  $L^2(\mathcal{M}^{\text{ext}})$  estimates of (8.14) not already obtained in Proposition 6.1 are directly obtained from (8.12) and a comparison of frame argument. The decay estimates (8.15) are obtained *via* the Klainerman-Sobolev embeddings of Lemma 6.7 and  $L^2$  bounds on  ${}^{(\tau)} \Sigma_t^{\text{ext}}$  (see the proof of the analogous estimates in Section 6.2). Thus Proposition 7.1 boils down to the following three lemmas.

**Lemma 8.7.** *For all  $\tau_1 \leq \tau \leq (1 + \tau_0)/2$ , we have*

$$\int_{{}^{(\tau)} \Sigma_t} t^4 |(t\mathbf{D})^{\leq 2} \mathbf{R}|^2 + \int_{\tau^{-1} u}^{\tau_1^{-1} u} (\mathcal{R}_{\leq 2}(u, \underline{u}))^2 \, d\underline{u} \lesssim \varepsilon^2, \quad (8.17)$$

for all  ${}^{(\tau)} t^\circ \leq t \leq {}^{(\tau)} t^*$  and for all  $1 \leq u \leq \tau \underline{u}^*$  and where the frame norm is adapted to  ${}^{(\tau)} \bar{T}$ .



**Lemma 8.8.** For all  $\tau_0 \leq \tau \leq \tau_1$ , we have

$$\int_{({}^\tau)\Sigma_t} t^4 |(t\mathbf{D})^{\leq 2} \mathbf{R}|^2 \lesssim \varepsilon^2, \quad (8.18)$$

for all  $({}^\tau)t^\circ \leq t \leq ({}^\tau)t^*$  and where the frame norm is adapted to  $({}^\tau)\bar{T}$ .

**Lemma 8.9.** For all  $\tau_1 \leq \tau \leq (1 + \tau_0)/2$ , we have

$$\int_{({}^\tau)\Sigma_t^{\text{ext}} \cap ({}^{\tau_1})\mathcal{M}^{\text{int}}} t^4 |(t\mathbf{D})^{\leq 2} \mathbf{R}|^2 \lesssim \varepsilon^2, \quad (8.19)$$

for all  $({}^\tau)t^\circ \leq t \leq ({}^\tau)t^*$  and where the frame norm is adapted to  $\bar{T}^{\text{ext}}$ .

### 8.2.1 Proof of Lemma 8.7

In this section, we assume that  $\tau_1 \leq \tau \leq (1 + \tau_0)/2$ .

Let  $({}^\tau)t^\circ \leq t \leq ({}^\tau)t^*$ . Let

$$u := \frac{2}{1 + \tau^{-1}} t, \quad t_1 := \frac{1 + \tau_1^{-1}}{2} u.$$

**Remark 8.10.** In the case where  $t_1 \geq ({}^{\tau_1})t^*$ , i.e. when  $({}^\tau)\Sigma_t$  is in the domain of dependence of the last slice  $({}^{\tau_1})\Sigma_{({}^{\tau_1})t^*}$  the bound (8.17) follows from the arguments of Section 8.1 and a comparison of frame. In the following, we restrict to the case  $t_1 \leq ({}^{\tau_1})t^*$ .

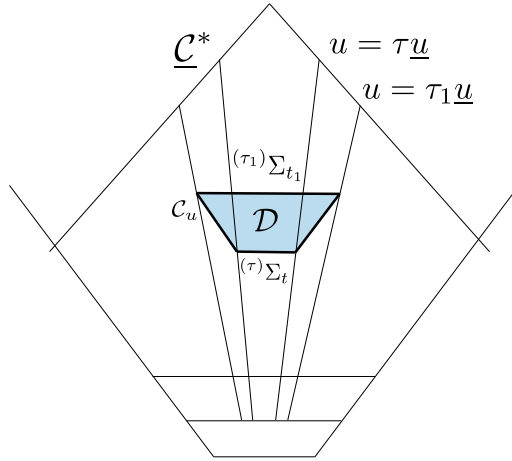


Figure 1: Local energy estimates for  $\tau \geq \tau_1$ .

We consider the maximal hypersurface  $({}^{\tau_1})\Sigma_{t_1}$ . We perform a  $t_1$ -rescaling of the region  $\mathcal{D}$  enclosed by  $({}^\tau)\Sigma_t \cup \mathcal{C}_u \cup ({}^{\tau_1})\Sigma_{t_1}$ , which rescales to a size 1 region (see Figure 1).<sup>4</sup> We call  $\varepsilon_1$  the rescaled smallness parameter  $\varepsilon$  (see the similar definition in Section 8.1.1).

Using that by the Bootstrap Assumptions 3.21, 3.23, 3.24, we have

$$\begin{aligned} \|(\nabla)^{\leq 1} \text{Ric}\|_{L^2({}^{\tau_1})\Sigma_{t_1}} &\lesssim D\varepsilon_1, \\ \|\nabla^{\leq 1}(\theta - \not{g})\|_{H^{1/2}(\partial({}^{\tau_1})\Sigma_{t_1})} &\lesssim D\varepsilon_1, \end{aligned}$$

and applying the results of Theorem 4.3, there exists harmonic coordinates  $x^i$  on  $({}^{\tau_1})\Sigma_{t_1}$  (see also the definition of Section 2.4) such that the following bounds hold on  $({}^{\tau_1})\Sigma_{t_1}$

$$\|\nabla^{\leq 2} \nabla^2 x^i\|_{L^2({}^{\tau_1})\Sigma_{t_1}} \lesssim D\varepsilon_1, \quad (8.20)$$

<sup>4</sup>From its definition, we have  $t_1 \simeq t$ .

for  $i = 1 \dots 3$ . Let define the Cartesian vectorfields  $X_i$  by parallel transport of  $\nabla x^i$ , *i.e.*

$$\begin{aligned} \mathbf{D}_{(\tau_1)\overline{T}}(X_i) &= 0, \\ X_i|_{(\tau_1)\Sigma_{t_1}} &= \nabla x^i, \end{aligned} \quad (8.21)$$

for  $i = 1 \dots 3$ . Commuting equation (8.21) by  $\mathbf{D}^{\leq 3}$  and integrating in  $(\tau_1)\overline{T}$ , using the Bootstrap Assumptions 3.21 and 3.24 in  $(\tau_1)\mathcal{M}_{\text{bot}}^{\text{int}}$ , estimate (8.20) on  $(\tau_1)\Sigma_{t_1}$  and the Sobolev estimates of Lemma 7.3, we have the following bounds in (the rescaled) region  $\mathcal{D}$

$$\|\mathbf{D}X_i\|_{L^\infty(\mathcal{D})} \lesssim D\varepsilon_1, \quad \|\mathbf{D}^2X_i\|_{L^\infty L^6} \lesssim D\varepsilon_1, \quad \|\mathbf{D}^3X_i\|_{L^2(\mathcal{D})} \lesssim D\varepsilon_1. \quad (8.22)$$

Thus, one can apply energy estimates (see Section 5.2 or Section 8.1) in the rescaled domain  $\mathcal{D}$  for the following contracted and commuted Bel-Robinson tensors

$$Q(\hat{\mathcal{L}}_{X_\mu}\mathbf{R})(X_0, X_0, X_0), \quad Q(\hat{\mathcal{L}}_{X_\mu}\hat{\mathcal{L}}_{X_\nu}\mathbf{R})(X_0, X_0, X_0)$$

for  $\mu, \nu = 0 \dots 3$  and where  $X_0 := (\tau_1)\overline{T}$ , and we obtain

$$\begin{aligned} & \int_{(\tau)S_{it}} Q(\hat{\mathcal{L}}_{X_\mu}^{\leq 1}\hat{\mathcal{L}}_{X_\nu}\mathbf{R})(X_0, X_0, X_0, (\tau)\overline{T}) + \int_{\mathcal{C}_u \cap \mathcal{D}} Q(\hat{\mathcal{L}}_{X_\mu}^{\leq 1}\hat{\mathcal{L}}_{X_\nu}\mathbf{R})(X_0, X_0, X_0, e_4) \\ & \lesssim \int_{(\tau_1)\Sigma_{t_1}} Q(\hat{\mathcal{L}}_{X_\mu}^{\leq 1}\hat{\mathcal{L}}_{X_\nu}\mathbf{R})(X_0, X_0, X_0, (\tau_1)\overline{T}) + (D\varepsilon_1)^3 \\ & \lesssim \varepsilon_1, \end{aligned} \quad (8.23)$$

where we used the curvature bounds (8.12) and (8.22).

Performing a frame comparison argument as in Section 8.1 and scaling back in  $t_1$ , we deduce from (8.23) that estimate (8.17) holds. This finishes the proof of Lemma 8.7.

### 8.2.2 Proof of Lemma 8.8

In this section, we assume that  $\tau_0 \leq \tau \leq \tau_1$ .

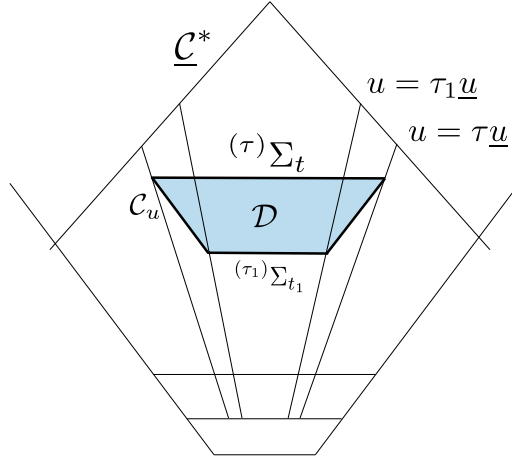


Figure 2: Local energy estimates for  $\tau \leq \tau_1$ .

Let  $(\tau)t^\circ \leq t \leq (\tau)t^*$ . Let

$$u := \frac{2}{1 + \tau^{-1}}t, \quad t_1 := \frac{1 + \tau_1^{-1}}{2}u.$$

**Remark 8.11.** *In the case  $t \leq 3\frac{1+\tau_0^{-1}}{1+\tau_1^{-1}}$ , the proof of estimate (8.18) follows from local arguments and is left to the reader. We shall thus consider that  $t_1 \geq 3$  (that is, the hypersurface  $(\tau_1)\Sigma_{t_1}$  is well defined).*

We consider the maximal hypersurfaces  $(\tau_1)\Sigma_{t_1}$  and  $(\tau)\Sigma_t$  (see Figure 2). Exchanging their roles in the proof of Lemma 8.7 in Section 8.2.1, defining harmonic coordinates on the hypersurface  $(\tau)\Sigma_t$  and running the same procedure yields estimate (8.18), where we use the curvature bounds (8.12) through  $(\tau_1)\Sigma_{t_1}$  and the curvature bounds (8.13) through  $\mathcal{C}_u$ . This finishes the proof of Lemma 8.8.

### 8.2.3 Proof of Lemma 8.9

In this section, we assume that  $\tau_1 \leq \tau \leq (1 + \tau_0)/2$ .

Let  $(\tau)t^\circ \leq t \leq (\tau)t^*$ . Let

$$u := \frac{2}{1 + \tau}t, \quad u_1 := \frac{2}{1 + \tau_1}t, \quad t_0 := \frac{1 + \tau_0}{2}u_1.$$

Let denote by  $\mathcal{D}$  the domain enclosed by  $(\tau)\Sigma_t^{\text{ext}}$ ,  $\mathcal{C}_{u_1}$ ,  $(\tau_0)\Sigma_{t_0}$  and  $\mathcal{C}_u$  (see Figure 3).

**Remark 8.12.** If  $t_0 \geq (\tau_0)t^*$ , i.e.  $(\tau)\Sigma_t^{\text{ext}}$  is in the domain of dependence of the last slice  $(\tau_0)\Sigma_{(\tau_0)t^*}$ , estimates follow along the arguments of Section 8.1. In the following we assume that  $t_0 \leq (\tau_0)t^*$ .

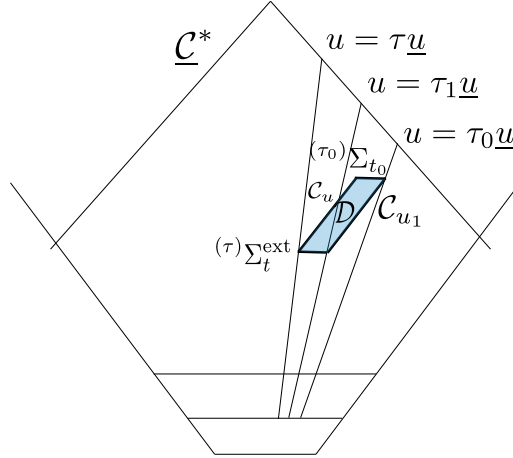


Figure 3: Local energy estimates for  $(\tau)\Sigma_t^{\text{ext}}$ .

Defining harmonic coordinates on the full slice  $(\tau_0)\Sigma_{t_0}$ , and arguing as in Section 8.2.1, one can perform energy estimates in the (rescaled) region  $\mathcal{D}$ , using the commutator vectorfields given by parallel transport of the harmonic vectorfields on  $(\tau_0)\Sigma_{t_0}$ . Estimate (8.19) thus follows from the energy bound (8.18) through  $(\tau_0)\Sigma_{t_0}$  from Lemma 8.8 and the energy bound (8.13) and (8.17) on  $\mathcal{C}_u$ , and a comparison argument. This finishes the proof of Lemma 8.9 and of Proposition 8.5.

## 9. Null connection estimates on $\underline{\mathcal{C}}^*$

In this section, we prove the following proposition.

**Proposition 9.1.** *Recall that from Propositions 6.1 and 8.1, the following estimates hold for the curvature boundedness norms on  $\underline{\mathcal{C}}^*$  (see the definitions of Section 3.2.1)*

$$\mathcal{R}_{\leq 2}^* + \mathfrak{R}_{\leq 1}^* \lesssim \varepsilon. \quad (9.1)$$

Under the Bootstrap Assumptions, estimates (9.1), the limits for the null connection coefficients at the vertex of the cone  $\underline{\mathcal{C}}^*$  from Theorems 4.4 and 4.5, we have for  $\varepsilon > 0$  sufficiently small (see the definitions of Sections 3.2.1 and 3.2.3)

$$\overline{\mathfrak{R}}_{\leq 2}^* + \mathcal{O}_{\leq 3}^* + \overline{\mathfrak{D}}_{\leq 2}^* + \mathcal{O}_{\leq 3}^{*\mathbf{O}} \lesssim \varepsilon, \quad (9.2)$$

and

$$\mathfrak{D}_{\leq 2}^* \lesssim \varepsilon. \quad (9.3)$$

Moreover, the Bootstrap Assumptions 3.25 for the spherical coordinates on  $\underline{\mathcal{C}}^*$  (see Section 9.16), the bootstrap bound (3.3) on the area radius (see Section 9.15), and the mild Bootstrap Assumptions 3.16 for the rotation vectorfields  $\mathbf{O}^{\text{ext}}$  on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  (see Section 9.14.1) are improved.<sup>1</sup>

**Remark 9.2.** The bound (9.3) will follow from (9.2) and the Klainerman-Sobolev estimates of Lemma 9.6.

**Remark 9.3.** The norms  $\mathfrak{D}_{\leq 2}^*$  (see the definitions of Section 3.2.3) allow for a degeneracy of the null connection coefficients of the type  $\check{\Gamma} \sim r^{-1/2}$  when  $r \rightarrow 0$  on the cone  $\underline{\mathcal{C}}^*$ , where  $\check{\Gamma} \in \{\text{tr}\chi - \frac{2}{r}, \text{tr}\underline{\chi} + \frac{2}{r}, \hat{\chi}, \hat{\underline{\chi}}, \zeta, \underline{\omega}\}$ . This is inherited from the degeneracy in  $r$  of the null curvature components (see Remark 8.2). These degeneracies in  $r$  still allow to close the estimates for the null connection coefficients. This is consistent with the low regularity control for the geodesic foliation on a null cone obtained in [Wan09]. In that article, it is namely proven that one can close (low regularity) estimates for the null connection coefficients of the geodesic foliation, only assuming an  $L^2$  control of the null curvature components on the cone. In the present paper, such an  $L^2$  control holds and is consistent with a potential  $r^{-3/2}$  singularity for the curvature at the axis, and we therefore expect that the  $r$  degeneracy is not an obstruction to close our estimates.<sup>2</sup>

### 9.1 Hardy estimates

We first state the following preliminary general Hardy estimates, which hold on the whole cone  $\underline{\mathcal{C}}^*$ . We recall that  $q := \min(r, u)$ .

**Lemma 9.4** (Hardy estimates on  $\underline{\mathcal{C}}^*$ ). *For all  $\kappa \in \mathbb{R}$ , the following holds. Assume that  $U$  is an  $S$ -tangent tensor satisfying*

$$\nabla_3 U + \frac{\kappa}{2} \text{tr}\underline{\chi} U = F,$$

with vertex limit

$$r^\kappa |U| \rightarrow 0, \quad \text{when } r \rightarrow 0.$$

We have the following Hardy inequality on  $\underline{\mathcal{C}}^*$

$$\|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \|r^\lambda q F\|_{L^2(\underline{\mathcal{C}}^*)}. \quad (9.4)$$

for all  $\lambda < \kappa - 3/2$ .

**Remark 9.5.** In the following, we will use the Hardy inequality, with  $\lambda = -1$  for the null connection coefficients  $\text{tr}\chi, \text{tr}\underline{\chi}$ , which satisfy transport equations with  $\kappa \geq 1$ . See the null structure equations (2.34).<sup>3</sup>

<sup>1</sup>This amounts to an improvement of the strong Bootstrap Assumptions 3.22.

<sup>2</sup>In view of the nonlinearities in the null structure equations, the critical degeneracy at which one cannot close (degenerate) estimates for the null connection coefficients is given by the (scaling-critical) blow-up rate  $R \sim r^{-2}$ .

<sup>3</sup>The fact that the null connection coefficients intrinsic to the geometry of the cones  $\chi, \underline{\chi}, \zeta$  satisfy transport equations with  $\kappa \geq 1$  enables the control of the intrinsic geometry of null cones at the  $L^2$  level in [Wan09].

*Proof.* We first prove the following two classical Hardy estimates corresponding respectively to the interior and exterior region

$$\int_{\tau \underline{u}^*}^{\underline{u}^*} \left( \frac{1}{(\underline{u}^* - u)^\alpha} \int_u^{\underline{u}^*} f \, du' \right)^2 (\underline{u}^* - u)^2 \, du \lesssim \int_{\tau \underline{u}^*}^{\underline{u}^*} (\underline{u}^* - u)^{2-2\alpha} |f|^2 (\underline{u}^* - u)^2 \, du, \quad (9.5)$$

$$\int_1^{u_0} \left( \int_u^{u_0} f \, du' \right)^2 \, du \lesssim \int_1^{u_0} u^2 |f|^2 \, du, \quad (9.6)$$

where  $f$  is a scalar function of the variable  $u$ , and where  $\alpha > 3/2$ . Here  $u_0 > 0$  is a parameter which will be determined in the sequel.

To prove (9.5), using Cauchy-Schwartz we write

$$\begin{aligned} & \int_{\tau \underline{u}^*}^{\underline{u}^*} \left( \int_u^{\underline{u}^*} f \, du' \right)^2 (\underline{u}^* - u)^{2-2\alpha} \, du \\ & \leq \int_{\tau \underline{u}^*}^{\underline{u}^*} \left( \int_u^{\underline{u}^*} (\underline{u}^* - u')^{2\gamma} |f|^2 \, du' \right) \left( \int_u^{\underline{u}^*} (\underline{u}^* - u')^{-2\gamma} \, du' \right) (\underline{u}^* - u)^{2-2\alpha} \, du, \end{aligned}$$

where  $\gamma \in \mathbb{R}$ . Provided that  $-2\gamma + 1 > 0$ , we have

$$\int_u^{\underline{u}^*} (\underline{u}^* - u')^{-2\gamma} \, du' \lesssim (\underline{u}^* - u)^{-2\gamma+1}.$$

Using Fubini theorem, and provided that  $-2\gamma + 4 - 2\alpha < 0$ , we obtain

$$\begin{aligned} & \int_{\tau \underline{u}^*}^{\underline{u}^*} \left( \int_u^{\underline{u}^*} f \, du' \right)^2 (\underline{u}^* - u)^{2-2\alpha} \, du \\ & \lesssim \int_{\tau \underline{u}^*}^{\underline{u}^*} (\underline{u}^* - u')^{2\gamma} |f|^2 \left( \int_{\tau \underline{u}^*}^{u'} (\underline{u}^* - u)^{-2\gamma+1+2-2\alpha} \, du \right) \, du' \\ & \lesssim \int_{\tau \underline{u}^*}^{\underline{u}^*} (\underline{u}^* - u')^{2\gamma} |f|^2 (\underline{u}^* - u')^{-2\gamma+4-2\alpha} \, du' \\ & = \int_{\tau \underline{u}^*}^{\underline{u}^*} (\underline{u}^* - u')^{2-2\alpha} |f|^2 (\underline{u}^* - u')^2 \, du'. \end{aligned}$$

The conditions  $-2\gamma + 1 > 0$  and  $-2\gamma + 4 - 2\alpha < 0$  can both be satisfied if and only if  $\alpha > 3/2$ , which finishes the proof of (9.5).

To prove (9.6), we write using Cauchy-Schwartz

$$\begin{aligned} & \int_1^{u_0} \left( \int_u^{u_0} f \, du' \right)^2 \, du \\ & \leq \int_1^{u_0} \left( \int_u^{u_0} (u')^{2\gamma} |f|^2 \, du' \right) \left( \int_u^{u_0} (u')^{-2\gamma} \, du' \right) \, du \\ & \lesssim \int_1^{u_0} \left( \int_u^{u_0} (u')^{2\gamma} |f|^2 \, du' \right) u^{-2\gamma+1} \, du, \end{aligned}$$

provided that  $-2\gamma + 1 < 0$ . Using Fubini theorem, we thus have

$$\begin{aligned} & \int_1^{u_0} \left( \int_u^{u_0} f \, du' \right)^2 \, du \\ & \lesssim \int_1^{u_0} (u')^{2\gamma} |f|^2 \left( \int_1^{u'} u^{-2\gamma+1} \, du \right) \, du' \\ & \lesssim \int_1^{u_0} (u')^{2\gamma} |f|^2 (u')^{-2\gamma+2} \, du', \end{aligned}$$

provided that  $-2\gamma + 2 > 0$ . Thus, choosing  $\gamma = 3/4$ , we obtained (9.6).

Using formula (2.39), we have

$$\nabla_3(r^\kappa U) = r^\kappa F + \frac{\kappa}{2} (\operatorname{tr}\chi - \overline{\operatorname{tr}\chi}) r^\kappa U.$$

Using the vertex limit  $r^\kappa|U| \rightarrow 0$ , the Bootstrap Assumptions 3.22 on  $\operatorname{tr}\chi$  and a Grönwall argument, we deduce

$$\|r^{-1}r^\kappa|U|\|_{L^2(S_{u,\underline{u}^*})} \lesssim \int_u^{\underline{u}^*} \|r^{-1}r^\kappa|F|\|_{L^2(S_{u',\underline{u}^*})} du'. \quad (9.7)$$

Applying the classical Hardy estimate (9.5) with  $f(u) = \|r^{-1}r^\kappa F\|_{L^2(S_{u,\underline{u}^*})}$ , using the Bootstrap Assumptions 3.25 on the metric in spherical coordinates and the bootstrap bounds (3.3) on the area radius, we deduce from (9.7) in the interior region

$$\|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} = \|r^{-\alpha} r^\kappa U\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} \lesssim \|r^{1-\alpha} r^\kappa F\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} = \|r^{\lambda+1} F\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})},$$

where  $\lambda := \kappa - \alpha$  and  $\alpha > 3/2$ . Rewriting this estimate, using that  $r \simeq q$  in the interior region, we thus have

$$\|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} \lesssim \|r^{\lambda+1} F\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} \lesssim \|r^\lambda q F\|_{L^2(\underline{\mathcal{C}}^*)}, \quad (9.8)$$

for all  $\lambda < \kappa - 3/2$ .

From a mean value argument, there exists  $\tau \underline{u}^* \leq u_0 \leq (1 + \tau) \underline{u}^*/2$  such that

$$\|r^\lambda U\|_{L^2(S_{u_0,\underline{u}^*})} \lesssim \underline{u}^{*-1/2} \|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^* \cap \{\tau \underline{u}^* \leq u_0 \leq (1+\tau)\underline{u}^*/2\})} \lesssim \underline{u}^{*-1/2} \|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})}. \quad (9.9)$$

Using (9.7), we have for  $\lambda < \kappa - 3/2$  and for  $u \leq u_0$

$$\|r^{-1}r^\lambda U\|_{L^2(S_{u,\underline{u}^*})} \lesssim \|r^{-1}(r^\lambda U)\|_{L^2(S_{u_0,\underline{u}^*})} + \int_u^{u_0} \|r^{-1}r^\lambda F\|_{L^2(S_{u',\underline{u}^*})} du'.$$

Thus,

$$\|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \lesssim \|(r^\lambda U)|_{u=u_0}\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} + \left\| \int_u^{u_0} \|r^{-1}r^\lambda F\|_{L^2(S_{u',\underline{u}^*})} du' \right\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})}.$$

Using (9.9) and (9.8), we have for the first term

$$\begin{aligned} \|(r^\lambda U)|_{u=u_0}\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} &\lesssim \underline{u}^{*1/2} \|r^\lambda U\|_{L^2(S_{u_0,\underline{u}^*})} \\ &\lesssim \|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{int}})} \\ &\lesssim \|r^\lambda q F\|_{L^2(\underline{\mathcal{C}}^*)}. \end{aligned}$$

For the second term, applying the classical Hardy estimate (9.6) to  $f(u) = \|r^{-1}r^\lambda|F|\|_{L^2(S_{u,\underline{u}^*})}$ , using the Bootstrap Assumption 3.25 on the metric in spherical coordinates and the bootstrap bounds (3.3) on the area radius, gives

$$\left\| \int_u^{u_0} \|r^{-1}r^\lambda F\|_{L^2(S_{u',\underline{u}^*})} du' \right\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \lesssim \|r^\lambda q F\|_{L^2(\underline{\mathcal{C}}^*)},$$

where we used that  $q \simeq u$  in the exterior region and  $q \simeq r$  in the interior region. This finishes the proof of the lemma.  $\square$

## 9.2 Klainerman-Sobolev estimates

We have the following Klainerman-Sobolev estimates, which are used to obtain estimates (9.3). Its proof is postponed to Appendix D.

**Lemma 9.6** (Klainerman-Sobolev estimates on  $\underline{\mathcal{C}}^*$ ). *For all  $S$ -tangent tensor  $F$  on  $\underline{\mathcal{C}}^*$  with vertex limit*

$$r^{3/2}|(r\nabla)\leq^1 F| \rightarrow 0 \quad \text{when } r \rightarrow 0,$$

we have the following  $L_u^\infty \tilde{H}^{1/2}(S_{u,\underline{u}^*})$  estimates in  $\underline{\mathcal{C}}^*$

$$\|rF\|_{L_u^\infty \tilde{H}^{1/2}(S_{u,\underline{u}^*})} \lesssim \|F\|_{L^2(\underline{\mathcal{C}}^*)} + \|r\nabla F\|_{L^2(\underline{\mathcal{C}}^*)} + \|r\nabla_3 F\|_{L^2(\underline{\mathcal{C}}^*)},$$

and

$$\left\| r^{1/2} q^{1/2} F \right\|_{L_u^\infty \tilde{H}^{1/2}(S_{u,\underline{u}^*})} \lesssim \|F\|_{L^2(\underline{\mathcal{C}}^*)} + \|r\nabla F\|_{L^2(\underline{\mathcal{C}}^*)} + \|q\nabla_3 F\|_{L^2(\underline{\mathcal{C}}^*)}.$$

## 9.3 Control of $\bar{\rho}$ and $\bar{\sigma}$

From the vertex limits of Theorems 4.4 and 4.5, we have  $r^3\bar{\rho} \rightarrow 0$  when  $r \rightarrow 0$ . Integrating equation (2.85) thus gives

$$r^3|\bar{\rho}| \lesssim \int_u^{\underline{u}^*} r^3 |\text{Err}(\nabla_3, \bar{\rho})| du'.$$

From the Bootstrap Assumptions 3.22 and (9.1), we have

$$\begin{aligned} |\text{Err}(\nabla_3, \bar{\rho})| &\lesssim |\hat{\chi}| |\underline{\alpha}| + |\zeta| |\underline{\beta}| + |\text{tr}\chi - \overline{\text{tr}\chi}| |\rho - \bar{\rho}| \\ &\lesssim (D\varepsilon)q^{-1/2}\underline{u}^{-2} \cdot \varepsilon r^{-1}q^{-1/2}u^{-2}. \end{aligned}$$

Thus,

$$r^3|\bar{\rho}| \lesssim D\varepsilon^2 \underline{u}^{*-2} \int_u^{\underline{u}^*} r(u', \underline{u}^*)^2 q(u', \underline{u}^*)^{-1} (u')^{-2} du'.$$

In the interior region  $u \geq \tau \underline{u}^*$ , this yields

$$r^3|\bar{\rho}| \lesssim D\varepsilon^2 \underline{u}^{*-4} \int_u^{\underline{u}^*} r(u', \underline{u}^*) du' \lesssim D\varepsilon^2 \underline{u}^{*-4} r^2.$$

In the exterior region  $u \leq \tau \underline{u}^*$ , this yields

$$\begin{aligned} r^3|\bar{\rho}| &\lesssim \underline{u}^{*3} |\bar{\rho}|_{S^*} + D\varepsilon^2 \int_u^{\underline{u}^*} (u')^{-3} du' \\ &\lesssim D\varepsilon^2 u^{-2}. \end{aligned}$$

Thus, we summarise the bounds on  $\bar{\rho}$  as

$$\left\| uq\underline{u}^{*3}\bar{\rho} \right\|_{L^\infty(\underline{\mathcal{C}}^*)} \lesssim \varepsilon.$$

Estimating directly the  $(q\nabla_3)$  derivatives using equation (2.85), we further obtain

$$\bar{\mathfrak{R}}_{\leq 2}^*[\bar{\rho}] \lesssim \varepsilon. \quad (9.10)$$

From equation (2.48), and the Bootstrap Assumptions 3.22, one directly deduces

$$\bar{\mathfrak{R}}_{\leq 2}^*[\bar{\sigma}] \lesssim \varepsilon. \quad (9.11)$$

We thus have proved the desired estimate

$$\bar{\mathfrak{R}}_{\leq 2}^* \lesssim \varepsilon.$$

## 9.4 Control of $\overline{\text{tr}\chi} + \frac{2}{r}$

Integrating the transport equation (2.88) for  $\overline{\text{tr}\chi} + \frac{2}{r}$ , using that from Theorems 4.4 and 4.5, one has the limit  $r(\overline{\text{tr}\chi} + \frac{2}{r}) \rightarrow 0$  when  $r \rightarrow 0$ , we have

$$\begin{aligned} \left| r \left( \overline{\text{tr}\chi} + \frac{2}{r} \right) \right| &\lesssim \int_u^{u^*} r \text{Err} \left( \nabla_3, \overline{\text{tr}\chi} + \frac{2}{r} \right) du' \\ &\lesssim (D\varepsilon)^2 \int_u^{u^*} r \underline{u}^{*-2} u^{-2} q^{-1} du', \end{aligned}$$

from which we deduce, arguing as in the previous section

$$\left\| \underline{u}^{*2} u^2 \left( \overline{\text{tr}\chi} + \frac{2}{r} \right) \right\|_{L^\infty} \lesssim \varepsilon.$$

Estimating directly equation (2.88) for  $\nabla_3 \left( \overline{\text{tr}\chi} + \frac{2}{r} \right)$ , one further infers

$$\overline{\mathfrak{D}}_{\leq 2}^* [\overline{\text{tr}\chi}] \lesssim \varepsilon. \quad (9.12)$$

## 9.5 Control of $\overline{\text{tr}\chi} - \frac{2}{r}$

Integrating the transport equation (2.92) for  $\overline{\text{tr}\chi} - \frac{2}{r}$ , using that from Theorems 4.4 and 4.5, one has the limit  $r(\overline{\text{tr}\chi} - \frac{2}{r}) \rightarrow 0$  when  $r \rightarrow 0$ , one has

$$\left| r \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right| \lesssim \int_u^{u^*} r \left( |\bar{\rho}| + |\text{Err}(\nabla_3, \overline{\text{tr}\chi} - \frac{2}{r})| \right) du.$$

Using the bound (9.10) obtained for  $\bar{\rho}$  and arguing as in the previous sections, one obtains

$$\left\| \underline{u} \underline{u}^{*3} \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(\mathcal{C}^*)} \lesssim \varepsilon.$$

Estimating directly equation (2.92) for  $\nabla_3 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right)$ , one further infers

$$\overline{\mathfrak{D}}_{\leq 2}^* [\overline{\text{tr}\chi}] \lesssim \varepsilon. \quad (9.13)$$

This finishes the improvement of

$$\overline{\mathfrak{D}}_{\leq 2}^* \lesssim \varepsilon.$$

## 9.6 Control of $\zeta$

Using the definition (2.3) of the canonical foliation on  $\mathcal{C}^*$ ,  $\zeta$  satisfies the following elliptic equation

$$\mathcal{P}_1 \zeta = \left( -\rho + \bar{\rho}, \sigma - \frac{1}{2} \hat{\chi} \wedge \hat{\underline{\chi}} \right). \quad (9.14)$$

Using the elliptic estimates of Lemma 3.38, the estimates (9.1) for the curvature and the Bootstrap Assumptions 3.22, one has

$$\begin{aligned} \|r^{-1} (r\nabla)^{\leq 3} \zeta\|_{L^2(\mathcal{C}^*)} &\lesssim \|(r\nabla)^{\leq 2} (\rho - \bar{\rho})\|_{L^2(\mathcal{C}^*)} + \|(r\nabla)^{\leq 2} (\sigma - \bar{\sigma})\|_{L^2(\mathcal{C}^*)} + \|(r\nabla)^{\leq 2} \hat{\chi} \wedge \hat{\underline{\chi}}\|_{L^2(\mathcal{C}^*)} \\ &\lesssim \varepsilon \underline{u}^{*-2} + (D\varepsilon)^2 \underline{u}^{*-2} \\ &\lesssim \varepsilon \underline{u}^{*-2}. \end{aligned}$$

Multiplying equation (9.14) with  $r$  and commuting with  $(q\nabla_3)$  gives

$$r\mathcal{P}_1((q\nabla_3)\zeta) = \left( (q\nabla_3)(-r\rho + r\bar{\rho}), (q\nabla_3)(r\sigma - \frac{1}{2}r\hat{\chi} \wedge \hat{\underline{\chi}}) \right) + \text{Err}, \quad (9.15)$$

where

$$\text{Err} := [(q\nabla_3), (r\mathcal{P}_1)] \zeta.$$



Applying the elliptic estimates of Lemma 3.38 gives

$$\|r^{-1}(r\nabla)^{\leq 2}(q\nabla_3)\zeta\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \varepsilon \underline{u}^{*-2}.$$

Commuting (9.15) further with  $(q\nabla_3)$  and arguing similarly gives

$$\|r^{-1}(r\nabla)^{\leq 1}(q\nabla_3)^2\zeta\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \varepsilon \underline{u}^{*-2}.$$

Commuting (9.15) with  $(q\nabla_3)^2$  gives, using the Bianchi equations (2.35e), (2.35g) for  $\nabla_3\rho$  and  $\nabla_3\sigma$  respectively

$$\begin{aligned} (r\mathcal{P}_1)(q\nabla_3)^3\zeta &= [r(q\nabla_3)^3(-\rho + \bar{\rho}), r(q\nabla_3)^3\sigma] + \text{l.o.t.} \\ &= [\text{div}(-rq(q\nabla_3)^2\beta), \text{curl}(rq(q\nabla_3)^2\beta)] + \text{l.o.t.} \end{aligned}$$

where the lower order terms can be estimated in  $L^2(\underline{\mathcal{C}}^*)$ . Using the elliptic estimate (3.20) of Lemma 3.38 with  $X = -rq(q\nabla_3)^2\beta$  and  $Y = rq(q\nabla_3)^2\beta$ , we have

$$\begin{aligned} \|r^{-1}(q\nabla_3)^3\zeta\|_{L^2(\underline{\mathcal{C}}^*)} &\lesssim \|r^{-1}q(q\nabla_3)^2\beta\|_{L^2(\underline{\mathcal{C}}^*)} + \|\text{l.o.t.}\|_{L^2(\underline{\mathcal{C}}^*)} \\ &\lesssim \varepsilon \underline{u}^{*-2}. \end{aligned}$$

We thus have proved

$$\mathcal{O}_{\leq 3}^{*,g}[\zeta] \lesssim \varepsilon. \quad (9.16)$$

## 9.7 Control of $\underline{\omega} - \overline{\omega}$

Using the results of Lemmas 2.11 and 2.21, equation (2.34d) rewrites

$$2\nabla\omega = -\nabla_3\zeta - 2\underline{\chi} \cdot \zeta - \underline{\beta}.$$

From the estimates (9.16) for  $\zeta$  and the estimates (9.1) for  $\underline{\beta}$ , one directly deduces

$$\begin{aligned} \|r^{-1}(r\nabla)^{\leq 2}q\nabla\omega\|_{L^2(\underline{\mathcal{C}}^*)} &\lesssim \varepsilon \underline{u}^{*-2}, \\ \|r^{-1}(r\nabla)^{\leq 1}(q\nabla_3)q\nabla\omega\|_{L^2(\underline{\mathcal{C}}^*)} &\lesssim \varepsilon \underline{u}^{*-2}, \\ \|r^{-1}(r\nabla)^{\leq 1}(q\nabla_3)^2q\nabla\omega\|_{L^2(\underline{\mathcal{C}}^*)} &\lesssim \varepsilon \underline{u}^{*-2}, \end{aligned}$$

which, using Poincaré estimates (see Lemma 3.38) further gives

$$\mathcal{O}_{\leq 2+}^{*,b}[\underline{\omega} - \overline{\omega}] \lesssim \varepsilon. \quad (9.17)$$

**Remark 9.7.** *It is expected for the canonical foliation that we do not control all  $\nabla_3$  derivatives of the coefficient  $\underline{\omega}$ , since the foliation is only defined by an elliptic equation on 2-spheres. See also the estimates for the lapse  $\Omega$  in [CG19a].*

## 9.8 Control of $\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}$

Applying the Hardy estimate of Lemma 9.4 with  $\kappa = 2, \lambda = -1$  to the transport equation (2.87) for  $\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}$ , using that from Theorems 4.4 and 4.5 one has the vertex limit  $r^2(\text{tr}\underline{\chi} + \frac{2}{r}) \rightarrow 0$  when  $r \rightarrow 0$ , we have

$$\|r^{-1}(\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}})\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \|r^{-1}q\text{tr}\underline{\chi}(\underline{\omega} - \overline{\omega})\|_{L^2(\underline{\mathcal{C}}^*)} + \|r^{-1}q\text{Err}(\nabla_3, \text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}})\|_{L^2(\underline{\mathcal{C}}^*)}.$$

Using the improved bound (9.17) for  $\underline{\omega}$  and the Bootstrap Assumptions 3.22, we deduce

$$\left\| r^{-1} \left( \text{tr}\underline{\chi} + \frac{2}{r} \right) \right\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \varepsilon \underline{u}^{*-2}.$$

Commuting equation (2.87) with  $(r\nabla)^{\leq 3}$  or directly estimating its  $(q\nabla_3)^{\leq 2}$  derivatives and arguing similarly, we further infer

$$\mathcal{O}_{\leq 3}^{*,g}[\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}] \lesssim \varepsilon. \quad (9.18)$$

## 9.9 Control of $\hat{\chi}$

Using the elliptic equation (2.34p), the elliptic estimates from Lemma 3.38, estimates (9.16), (9.18), the curvature estimates (9.1) and the Bootstrap Assumptions 3.22, we have

$$\begin{aligned} \|r^{-1}u(r\nabla)^{\leq 3}\hat{\chi}\|_{L^2(\underline{\mathcal{C}}_{\underline{u}})} &\lesssim \|u(r\nabla)^{\leq 2}\underline{\beta}\|_{L^2(\underline{\mathcal{C}}_{\underline{u}})} + \|u(r\nabla)^{\leq 2}\nabla\text{tr}\chi\|_{L^2(\underline{\mathcal{C}}_{\underline{u}})} \\ &\quad + \|u(r\nabla)^{\leq 2}(\text{tr}\chi\zeta)\|_{L^2(\underline{\mathcal{C}}_{\underline{u}})} + \text{Err} \\ &\lesssim \varepsilon\underline{u}^{-1}. \end{aligned}$$

Using the just obtained estimate for  $\hat{\chi}$  and directly estimating equation (2.34b) for  $\nabla_3\hat{\chi}$ , one further has

$$\mathcal{O}_{\leq 3}^{*,b}[\hat{\chi}] \lesssim \varepsilon \quad (9.19)$$

## 9.10 Control of $\text{tr}\chi - \overline{\text{tr}\chi}$

Applying the Hardy estimate of Lemma 9.4 with  $\kappa = 1$  and  $\lambda = -1$ , to the transport equation (2.91) for  $\text{tr}\chi - \overline{\text{tr}\chi}$ , using that from Theorems 4.4 and 4.5, one has the limit  $r(\text{tr}\chi - \overline{\text{tr}\chi}) \rightarrow 0$  when  $r \rightarrow 0$ , we have

$$\|r^{-1}(\text{tr}\chi - \overline{\text{tr}\chi})\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \|r^{-1}q\text{tr}\chi(\underline{\omega} - \overline{\omega})\|_{L^2(\underline{\mathcal{C}}^*)} + \|r^{-1}q\text{Err}(\nabla_3, \text{tr}\chi - \overline{\text{tr}\chi})\|_{L^2(\underline{\mathcal{C}}^*)}.$$

Using estimate (9.17) for  $\underline{\omega} - \overline{\omega}$  and the Bootstrap Assumptions 3.22, one deduces

$$\left\| r^{-1} \left( \text{tr}\chi - \frac{2}{r} \right) \right\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \varepsilon\underline{u}^{*-2}.$$

Commuting the transport equation (2.91) further with  $(r\nabla)^{\leq 3}$  or directly estimating its  $(q\nabla_3)^{\leq 2}$  derivatives, we further infer

$$\mathcal{O}_{\leq 3}^{*,g}[\text{tr}\chi - \overline{\text{tr}\chi}] \lesssim \varepsilon \quad (9.20)$$

## 9.11 Control of $\hat{\chi}$

Using elliptic equation (2.34q) for  $\hat{\chi}$ , the elliptic estimate from Lemma 3.38, and the bounds (9.1), (9.16) and (9.20), one has

$$\begin{aligned} \|r^{-1}(r\nabla)^{\leq 3}\hat{\chi}\|_{L^2(\underline{\mathcal{C}}^*)} &\lesssim \|(r\nabla)^{\leq 2}\beta\|_{L^2(\underline{\mathcal{C}}^*)} + \|(r\nabla)^{\leq 2}\nabla\text{tr}\chi\|_{L^2(\underline{\mathcal{C}}^*)} + \text{l.o.t.} \\ &\lesssim \varepsilon\underline{u}^{*-2}. \end{aligned}$$

Using equation (2.34e) for  $\nabla_3\hat{\chi}$  and estimating directly, one further infers

$$\mathcal{O}_{\leq 3}^{*,g}[\hat{\chi}] \lesssim \varepsilon. \quad (9.21)$$

This finishes the improvement of

$$\mathcal{O}_{\leq 3}^* \lesssim \varepsilon,$$

from which, using the Klainerman-Sobolev estimates of Lemma 6.6 and the vertex limits of Theorems 4.4 and 4.5, one also deduces

$$\mathfrak{D}_{\leq 2}^* \lesssim \varepsilon.$$

**Remark 9.8.** *Due to the choice of the canonical foliation, we have obtained the optimal tangential regularity for  $\chi$ , namely an  $L^2(\underline{\mathcal{C}})$  control of  $\nabla^3\chi$ . In view of the transport equation (2.34f), this would not have been the case for other foliations, such as the geodesic foliation. The need of this optimal regularity in the extension argument<sup>4</sup> of Section 8.1 motivates the choice of the canonical foliation. See also its use in [CG19a, CG19b, KN03] for similar tangential regularity reasons.*

<sup>4</sup>We recall that optimal regularity of the last slice  $\Sigma_{t^*}$  can be obtained provided that optimal regularity holds for its boundary  $\partial\Sigma_{t^*} = S^* = S_{T\underline{u}^*, \underline{u}^*} \subset \underline{\mathcal{C}}^*$ , which is a 2-sphere of the canonical foliation on  $\underline{\mathcal{C}}^*$ .

## 9.12 Uniformisation of $S^*$

We have the following lemma, which is a consequence of the Uniformisation Theorem [KS19b, Theorem 3.1].

**Lemma 9.9** (Uniformisation of  $S^*$ ). *There exists a unique – up to isomorphisms of  $\mathbb{S}$  – centred conformal isomorphism  $\Phi : S^* \rightarrow \mathbb{S}$  (see the definitions in Section 2.4). The associated conformal factor  $\phi$  on  $S^*$  satisfies*

$$\|t(t\nabla)^{\leq 3}(\phi - 1)\|_{\tilde{H}^{1/2}(S^*)} \lesssim \varepsilon. \quad (9.22)$$

*Proof.* Using the improved estimate for  $\rho$  from Sections 6 and estimate (9.10), and using the improved estimates (9.12), (9.13), (9.18), (9.20) together with the  $\tilde{H}^{1/2}$  Klainerman-Sobolev embeddings of Lemma 6.6, we have on  $S^*$

$$\begin{aligned} \|t^3(t\nabla)^{\leq 1}\rho\|_{\tilde{H}^{1/2}(S^*)} &\lesssim \varepsilon, \\ \|t^2(t\nabla)^{\leq 2}\left(\mathrm{tr}\chi - \frac{2}{r^*}\right)\|_{\tilde{H}^{1/2}(S^*)} &\lesssim \varepsilon, \\ \|t^2(t\nabla)^{\leq 2}\left(\mathrm{tr}\underline{\chi} + \frac{2}{r^*}\right)\|_{\tilde{H}^{1/2}(S^*)} &\lesssim \varepsilon. \end{aligned}$$

Using the above estimates, Bootstrap Assumptions 3.22 and Gauss equation (2.34t), we have

$$\|t^3(t\nabla)^{\leq 1}\left(K - \frac{1}{r^{*2}}\right)\|_{\tilde{H}^{1/2}(S^*)} \lesssim \varepsilon,$$

from which, applying the Uniformisation Theorem [KS19b, Theorem 3.1] one deduces the existence, uniqueness up to isometry of a centred conformal isomorphism  $\Phi$ , and the estimate (9.22) for  $\phi$ .  $\square$

## 9.13 Control of the rotation vectorfields $\mathbf{O}^{\mathrm{ext}}$ on $S^*$

**Lemma 9.10** (Mild bounds for  $\mathbf{O}^{\mathrm{ext}}$  on  $S^*$ ). *We have the following mild bounds on  $S^*$*

$$\begin{aligned} \|r^{-1}\mathbf{O}^{\mathrm{ext}}\|_{L^\infty(S^*)} &\lesssim 1, \\ \|r^{-2}(r\nabla)^{\leq 3}\mathbf{O}^{\mathrm{ext}}\|_{L^2(S^*)} &\lesssim 1. \end{aligned} \quad (9.23)$$

Moreover, for all  $S$ -tangent scalar  $f$  and for all 1-tensor or symmetric traceless 2-tensor  $F$ , we have

$$\begin{aligned} \|(r\nabla)f\|_{L^2(S^*)} &\lesssim \sum_{\ell=1}^3 \left\| \hat{\mathcal{L}}^{(\ell)}\mathbf{O}^{\mathrm{ext}}(f) \right\|_{L^2(S^*)}, \\ \|(r\nabla)^{\leq 1}F\|_{L^2(S^*)} &\lesssim \sum_{\ell=1}^3 \left\| \hat{\mathcal{L}}^{(\ell)}\mathbf{O}^{\mathrm{ext}}F \right\|_{L^2(S^*)}. \end{aligned} \quad (9.24)$$

*Proof.* By rescaling, we shall assume that  $r^* = 1$ . By definition (see Section 2.4), we have the following uniform bounds on  $S^*$  for the Cartesian coordinates  $x^i$  and their derivatives with respect to the Euclidean metric  $\mathcal{g}_\mathbb{S}$

$$\begin{aligned} |x^i| &\lesssim 1, \\ \left\| \left(\mathcal{g}_\mathbb{S}\nabla\right)^k x^i \right\|_{\mathcal{g}_\mathbb{S}} &\lesssim_k 1, \end{aligned} \quad (9.25)$$

for all  $k \geq 1$  and all  $i = 1, 2, 3$ .

We recall the definition of  $(3)\mathbf{O}^{\mathrm{ext}}$

$$(3)\mathbf{O}^{\mathrm{ext}} = x^1\mathcal{g}_\mathbb{S}\nabla x^2 - x^2\mathcal{g}_\mathbb{S}\nabla x^1 = x^1\mathcal{g}_\mathbb{S}\nabla x^2 - x^2\mathcal{g}_\mathbb{S}\nabla x^1.$$

Thus, by the definition of the conformal factor  $\phi$  from Section 2.4, we deduce the following uniform bound on  $S^*$

$$\left\| (3)\mathbf{O}^{\mathrm{ext}} \right\|_{\mathcal{g}} = \phi^{-1} \left\| (3)\mathbf{O}^{\mathrm{ext}} \right\|_{\mathcal{g}_\mathbb{S}} \lesssim 1,$$

where we used that from the bounds (9.22) and the Sobolev estimates of Lemma 3.36, one has

$$|\phi| + |\phi^{-1}| \lesssim 1. \quad (9.26)$$

We further have

$$\begin{aligned} \not\partial \nabla^{(3)} \mathbf{O}^{\text{ext}} &= \not\partial \nabla x^1 \not\partial \nabla x^2 - \not\partial \nabla x^2 \not\partial \nabla x^1 \\ &\quad + x^1 \not\partial \nabla \not\partial \nabla x^2 - x^2 \not\partial \nabla \not\partial \nabla x^1. \end{aligned} \quad (9.27)$$

From a standard computation of the Christoffel symbols for conformal metrics, we have for all scalar function  $f$

$$\begin{aligned} \not\partial \nabla_i \not\partial \nabla_j f &= \not\partial \nabla_i \not\partial \nabla_j f + \not\partial \nabla_i (\log \phi) \not\partial \nabla_j f + \not\partial \nabla_j (\log \phi) \not\partial \nabla_i f \\ &\quad - \left( \not\partial \nabla (\log \phi) \cdot \not\partial \nabla f \right) (\not\partial)_{ij}. \end{aligned} \quad (9.28)$$

From the bounds (9.22) for  $\phi$  and arguing as previously, we have

$$\left\| \not\partial \nabla (\log \phi) \right\|_{L^2(S^*)} \lesssim \varepsilon \lesssim 1. \quad (9.29)$$

Thus, applying formula (9.28) to equation (9.27) using (9.25), (9.26) and (9.29), we have

$$\left\| \not\partial \nabla^{(3)} \mathbf{O}^{\text{ext}} \right\|_{L^2(S^*)} \lesssim 1.$$

Differentiating equation (9.27) further by  $\nabla^{\leq 2}$ , generalising formula (9.28), using that by (9.22), one has

$$\left\| \left( \not\partial \nabla \right)^{\leq 3} (\log \phi) \right\|_{L^2(S^*)} \lesssim \varepsilon \lesssim 1, \quad (9.30)$$

and arguing as previously, we obtain

$$\left\| \not\partial \nabla^{\leq 3(3)} \mathbf{O}^{\text{ext}} \right\|_{L^2(S^*)} \lesssim 1,$$

as desired.

We turn to the proof of (9.24). From an exact computation in the Euclidean case, we have

$$\begin{aligned} |r \nabla f|_{\not\partial \nabla}^2 &\lesssim \sum_{\ell=1}^3 \left| \mathcal{L}^{(\ell)} \mathbf{O}^{\text{ext}}(f) \right|_{\not\partial \nabla}^2, \\ \left| (r \not\partial \nabla) F \right|_{\not\partial \nabla}^2 + |F|_{\not\partial \nabla}^2 &\lesssim \sum_{\ell=1}^3 \left| \hat{\mathcal{L}}^{(\ell)} \mathbf{O}^{\text{ext}} F \right|_{\not\partial \nabla}^2, \end{aligned}$$

for all  $S$ -tangent scalar  $f$  and for all 1-tensor or symmetric traceless 2-tensor  $F$ . From the first bound and (9.26), we directly deduce (9.24) for scalar functions.

From the second bounds, noting that  $\hat{\mathcal{L}}^{(\ell)} \mathbf{O}^{\text{ext}}$  is invariant under conformal change and arguing as earlier, using the control (9.29) for  $\nabla \log \phi$ , one has

$$\begin{aligned} \left| (r \not\partial \nabla) F \right|_{\not\partial \nabla}^2 + |F|_{\not\partial \nabla}^2 &\lesssim \left| r \not\partial \nabla F \right|_{\not\partial \nabla}^2 + (1 + |\nabla \log \phi|) |F|_{\not\partial \nabla}^2 \\ &\lesssim \left| r \not\partial \nabla F \right|_{\not\partial \nabla}^2 + |F|_{\not\partial \nabla}^2 \\ &\lesssim \sum_{\ell=1}^3 \left| \hat{\mathcal{L}}^{(\ell)} \mathbf{O}^{\text{ext}} F \right|_{\not\partial \nabla}^2 \\ &\lesssim \sum_{\ell=1}^3 \left| \hat{\mathcal{L}}^{(\ell)} \mathbf{O}^{\text{ext}} F \right|_{\not\partial \nabla}^2. \end{aligned}$$

This finishes the proof of the lemma.  $\square$

**Lemma 9.11** (Precise bounds for  $\mathbf{O}^{\text{ext}}$  on  $S^*$ ). *We have the following bounds*

$$\begin{aligned} \|r^{-1}(r\mathcal{V})^{\leq 2}H\|_{L^2(S^*)} &\lesssim \varepsilon, \\ \|(r\mathcal{V})^{\leq 1}\Psi\|_{L^2(S^*)} &\lesssim \varepsilon. \end{aligned} \quad (9.31)$$

*Proof.* By a rescaling argument, we may assume that  $r^* = 1$ . Arguing as in the proof of Lemma 9.10, using that from (9.22), one has for the conformal factor  $\phi$ ,  $|\phi| + |\phi^{-1}| \lesssim 1$  and  $\|\mathcal{V}^{\leq 3} \log \phi\|_{L^2(S^*)} \lesssim \varepsilon$ , it is enough to prove that in the Euclidean case  $H = 0$  and  $\Psi = 0$ .

From a direct computation, one has in the Euclidean case

$$\mathbf{D}^{2(\ell)}\mathbf{O} = \mathbf{D}^2(\in_{\ell ij} x^i \partial_{x^j}) = 0. \quad (9.32)$$

Using formula (2.119) in the Euclidean case where

$$\mathbf{D}_4\mathbf{O} = \mathbf{O}, \quad \mathbf{D}_3\mathbf{O} = -\mathbf{O}, \quad \mathbf{E}(\mathbf{D}^2, \mathcal{V}^2) = 0,$$

and the definitions (2.102) and (2.103) of  $H$  and  $\Psi$ , we deduce that

$$\begin{aligned} 0 &= \mathbf{D}_{a,b}^2\mathbf{O} \\ &= \mathcal{V}_{a,b}^2\mathbf{O} - \mathbf{O}_b e_a + \mathcal{J}_{ab}\mathbf{O} + \frac{1}{2}(\mathcal{V}_a\mathbf{O}_b + \mathcal{V}_b\mathbf{O}_a)(e_3 - e_4) \\ &= \Psi_{abc}e_c + \frac{1}{2}H_{ab}(e_3 - e_4), \end{aligned}$$

from which we deduce that  $\Psi = 0$  and  $H = 0$  in the Euclidean case. This finishes the proof of the lemma.  $\square$

## 9.14 Control of $\mathbf{O}^{\text{ext}}$ on $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$

### 9.14.1 Mild control of $\mathbf{O}^{\text{ext}}$

For convenience, we write  $\mathbf{O}$  any rotation vectorfield  $(\ell)\mathbf{O}^{\text{ext}}$ . We rewrite equation (2.99) under the following form

$$\mathcal{V}_3(r^{-1}\mathbf{O}) = r^{-1}\hat{\chi} \cdot \mathbf{O} + \frac{1}{2}r^{-1}(\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}})\mathbf{O}. \quad (9.33)$$

Integrating equation (9.33) from  $u = \tau u^* - i.e.$  from the sphere  $S^{*-}$ , using that from the result of Lemma 9.10, one has an improved mild bound  $\|r^{-1}\mathbf{O}\|_{L^\infty(S^*)} \lesssim 1$ , we have

$$|r^{-1}\mathbf{O}| \lesssim 1 + \int_u^{\tau u^*} (|r^{-1}\hat{\chi} \cdot \mathbf{O}| + |r^{-1}(\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}})\mathbf{O}|) du',$$

and from a Grönwall argument, we infer

$$\|r^{-1}\mathbf{O}\|_{L^\infty(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \lesssim 1.$$

Arguing similarly, using the commuted equation (2.100), and the improved mild bounds of Lemma 9.10, we obtain

$$\|r^{-1}(r\mathcal{V})^{\leq 1}\mathbf{O}\|_{L^\infty(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \lesssim 1. \quad (9.34)$$

Let  $S_{u,\underline{u}^*} \subset \underline{\mathcal{C}}^*$ . Let  $f, F$  be respectively an  $S_{u,\underline{u}^*}$ -tangent scalar function and 1-tensor or symmetric traceless 2-tensor. We want to improve the bootstrap bounds

$$\begin{aligned} \int_{S_{u,\underline{u}^*}} |(r\mathcal{V})f|^2 &\lesssim \sum_{\ell=1}^3 \int_{S_{u,\underline{u}^*}} \left| (\ell)\mathbf{O}^{\text{ext}}(f) \right|^2, \\ \int_{S_{u,\underline{u}^*}} |(r\mathcal{V})^{\leq 1}F|^2 &\lesssim \sum_{\ell=1}^3 \int_{S_{u,\underline{u}^*}} \left| \hat{\mathcal{L}}^{(\ell)}\mathbf{O}^{\text{ext}}F \right|^2. \end{aligned}$$

Let extend  $f, F$  on  $\underline{\mathcal{C}}^*$  as  $S$ -tangent tensors by parallel transport, *i.e.*

$$e_3(f) = 0, \quad \nabla_3 F = 0.$$

From the definition of  $(\ell)\mathbf{O}^{\text{ext}}$  on  $S^*$ , we obtained in Lemma 9.10 that there exists  $c > 0$  such that

$$\begin{aligned} \int_{S^*} |(r\nabla)f|^2 &\leq c \sum_{\ell=1}^3 \int_{S^*} |(\ell)\mathbf{O}^{\text{ext}}(f)|^2, \\ \int_{S^*} |(r\nabla)^{\leq 1} F|^2 &\leq c \sum_{\ell=1}^3 \int_{S^*} |\hat{\mathcal{L}}^{(\ell)}\mathbf{O}^{\text{ext}}F|^2. \end{aligned}$$

Let define

$$\mathcal{F}_f(u) := \int_{S_{u, \underline{u}^*}} |(r\nabla)f|^2 - c \sum_{\ell=1}^3 \int_{S_{u, \underline{u}^*}} |(\ell)\mathbf{O}^{\text{ext}}(f)|^2.$$

Using formula (2.37) and (2.36b), we have

$$\begin{aligned} e_3 \left( \int_{S_{u, \underline{u}^*}} |(r\nabla)f|^2 \right) &= \int_{S_{u, \underline{u}^*}} \text{tr}\underline{\chi} |r\nabla f|^2 + \int_{S_{u, \underline{u}^*}} 2(r\nabla f) \cdot \nabla_3(r\nabla)f \\ &= \int_{S_{u, \underline{u}^*}} \text{tr}\underline{\chi} |r\nabla f|^2 + \text{Err}[(r\nabla)f], \end{aligned}$$

where

$$\text{Err}[(r\nabla)f] := \int_{S_{u, \underline{u}^*}} 2(r\nabla f) \cdot ((\overline{\text{tr}\underline{\chi}} - \text{tr}\underline{\chi})(r\nabla)f + \hat{\chi} \cdot (r\nabla f)).$$

Using formula (2.37) and the Lie transport of  $(\ell)\mathbf{O}^{\text{ext}}$  (2.98), we have

$$e_3 \left( \int_{S_{u, \underline{u}^*}} |(\ell)\mathbf{O}^{\text{ext}}(f)|^2 \right) = \int_{S_{u, \underline{u}^*}} \text{tr}\underline{\chi} |(\ell)\mathbf{O}^{\text{ext}}(f)|^2.$$

Thus, we deduce that

$$e_3 \mathcal{F}_f = \overline{\text{tr}\underline{\chi}} \mathcal{F}_f + \text{Err}[(r\nabla)f]. \quad (9.35)$$

Integrating (9.35) along  $u$ , using that  $\mathcal{F}_f(\tau \underline{u}^*) \leq 0$  and the bounds (9.3), we have

$$\begin{aligned} \mathcal{F}_f(u) &\lesssim \left| r^2 \int_u^{\tau \underline{u}^*} r^{-2} \left( \varepsilon r^{-1} (u')^{-3/2} \right) \int_{S_{u', \underline{u}^*}} |(r\nabla)f|^2 du' \right| \\ &\lesssim \varepsilon \sup_{u \leq u' \leq \tau \underline{u}^*} \int_{S_{u', \underline{u}^*}} |(r\nabla)f|^2 \\ &\lesssim \varepsilon \int_{S_{u, \underline{u}^*}} |(r\nabla)f|^2, \end{aligned}$$

where one obtains the last estimate by using the transport of  $f$  along  $e_3$ . Using the definition of  $\mathcal{F}_f$ , we deduce that

$$\int_{S_{u, \underline{u}^*}} |(r\nabla)f|^2 \leq c \sum_{\ell=1}^3 \int_{S_{u, \underline{u}^*}} |(\ell)\mathbf{O}^{\text{ext}}(f)|^2 + \varepsilon \int_{S_{u, \underline{u}^*}} |(r\nabla)f|^2,$$

and the desired bound follows by absorption. The bounds when  $F$  is a 1-tensor or symmetric traceless 2-tensor are obtained similarly and left to the reader. This finishes the improvement of the mild Bootstrap Assumptions 3.16 on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ .

### 9.14.2 Control of $H$

Applying similar Hardy estimates to the ones of Lemma 9.4 with  $\kappa = 0, \lambda = -2$  – but integrating from  $S^*$  – to the transport equation (2.101) for  $H$ , using that from Lemma 9.11 we have  $\left\| \underline{u}^{*1/2} H \right\|_{L^2(S^*)} \lesssim \varepsilon$ , we have

$$\begin{aligned} \|r^{-2} H\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} &\lesssim \underline{u}^{*1/2} \|r^{-2} H\|_{L^2(S^*)} + \|r^{-2} q \nabla_3 H\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \\ &\lesssim \varepsilon \underline{u}^{*-2} + \|r^{-2} u \nabla_3 H\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})}. \end{aligned}$$

From an inspection of the terms composing  $\nabla_3 H$ , we only treat the terms  $\nabla \hat{\chi} \cdot \mathbf{O}$  and  $\hat{\chi} \cdot \nabla \mathbf{O}$  and the remaining terms will follow similarly. We have using estimate (9.19) and the mild estimate (9.34) for  $\mathbf{O}$

$$\begin{aligned} \|r^{-2} q \nabla \hat{\chi} \cdot \mathbf{O}\|_{L^2(\underline{\mathcal{C}}^*)} &\lesssim \|r^{-1} q \nabla \hat{\chi}\|_{L^2(\underline{\mathcal{C}}^*)} \|r^{-1} \mathbf{O}\|_{L^\infty(\underline{\mathcal{C}}^*)} \\ &\lesssim \underline{u}^{*-2} \|r^{-1} \underline{u}^* u(r \nabla) \hat{\chi}\|_{L^2(\underline{\mathcal{C}}^*)} \\ &\lesssim \varepsilon \underline{u}^{*-2}, \end{aligned}$$

and

$$\begin{aligned} \|r^{-2} q \hat{\chi} \cdot \nabla \mathbf{O}\|_{L^2(\underline{\mathcal{C}}^*)}^2 &= \int_1^{\underline{u}^*} \|r^{-2} q \hat{\chi} \cdot \nabla \mathbf{O}\|_{L^2(S_{u, \underline{u}^*})}^2 du \\ &\lesssim \int_1^{\underline{u}^*} \|r^{-1} q \hat{\chi}\|_{L^\infty(S_{u, \underline{u}^*})}^2 \|r^{-2} (r \nabla) \mathbf{O}\|_{L^\infty L^2(S_{u, \underline{u}^*})}^2 du \\ &\lesssim \int_1^{\underline{u}^*} \left( r^{-1} \|r^{-1} q(r \nabla) \hat{\chi}\|_{L^2(S_{u, \underline{u}^*})} \right)^2 du \\ &\lesssim \|r^{-1} \underline{u}^{*-1} u(r \nabla) \hat{\chi}\|_{L^2(\underline{\mathcal{C}}^*)}^2 \\ &\lesssim \varepsilon^2 \underline{u}^{*-4}, \end{aligned}$$

where we used the Sobolev embeddings from Lemma 3.36 in the third line. Thus,

$$\|r^{-2} H\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \lesssim \varepsilon \underline{u}^{*-2}.$$

Estimating directly equation (2.101), one also obtains

$$\|r^{-2} q \nabla_3 H\|_{L^2(\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}})} \lesssim \varepsilon \underline{u}^{*-2}.$$

From the definitions (2.102) and (2.103) of  $H$  and  $\Psi$ , the higher derivative of  $H$  are controlled by  $\Psi$  and from the results of this section we have

$$\mathcal{O}_{\leq 2}^{*,g} [r^{-1} H] \lesssim \varepsilon + \mathcal{O}_{\leq 1}^{*,g} [\Psi]. \quad (9.36)$$

We thus refer to the control of  $\Psi$  in the next section to obtain the full control of  $H$ .

### 9.14.3 Control of $\Psi$

The control for  $\Psi$  follows from applying the same Hardy estimate as in the previous section to the (commuted with  $r \nabla$ ) transport equation (2.104) for  $r \Psi$ , using the initial estimates from Lemma 9.11 on  $S^*$ . Details are left to the reader and we have

$$\mathcal{O}_{\leq 1}^{*,g} [\Psi] \lesssim \varepsilon. \quad (9.37)$$

We moreover deduce from (9.36) and (9.37)

$$\mathcal{O}_{\leq 2}^{*,g} [r^{-1} H] \lesssim \varepsilon,$$

and this finishes the improvement of (9.2).

## 9.15 Control of the area radius

**Lemma 9.12.** *Under the estimates obtained in the previous sections, we have on  $\underline{\mathcal{C}}^*$*

$$\left| r(u, \underline{u}^*) - \frac{1}{2}(\underline{u}^* - u) \right| \lesssim \varepsilon \underline{u}^{*-3} r^2 u^{-1}. \quad (9.38)$$

*Proof.* Using relations (2.1) and (2.39), we have on  $\underline{\mathcal{C}}^*$

$$e_3 \left( r(u, \underline{u}^*) - \frac{1}{2}(\underline{u}^* - u) \right) = \frac{1}{2} r \overline{\text{tr}\chi} + 1.$$

Integrating in  $u$ , using the improved estimate (9.12) for  $\overline{\text{tr}\chi} + \frac{2}{r}$  and the limit  $r(u, \underline{u}^*) \rightarrow 0$  when  $u \rightarrow \underline{u}^*$  (see Theorems 4.4 and 4.5), we have

$$\begin{aligned} \left| r(u, \underline{u}^*) - \frac{1}{2}(\underline{u}^* - u) \right| &\lesssim \lim_{u \rightarrow \underline{u}^*} \left| r(u, \underline{u}^*) - \frac{1}{2}(\underline{u}^* - u) \right| + \int_u^{\underline{u}^*} r \left| \overline{\text{tr}\chi} + \frac{2}{r} \right| du' \\ &\lesssim \varepsilon \int_u^{\underline{u}^*} r \underline{u}^{*-2} (u')^{-2} du' \\ &\lesssim \varepsilon \underline{u}^{*-3} r^2 u^{-1}, \end{aligned}$$

as desired.  $\square$

## 9.16 Control of spherical coordinates on $\underline{\mathcal{C}}^*$

For a spherical coordinate system as described in the Bootstrap Assumption 3.25, we have using (2.34a) the following transport equation in the  $e_3$  direction

$$e_3(\mathcal{J}_{ab}) = 2\underline{\chi}_{ab} = \overline{\text{tr}\chi} \mathcal{J}_{ab} + (\text{tr}\chi - \overline{\text{tr}\chi}) \mathcal{J}_{ab} + 2\hat{\chi}_{ab},$$

where  $a, b \in \{\vartheta, \varphi\}$ , which rewrites using (2.39) and the notation of the Bootstrap Assumption 3.25

$$e_3(r^{-2} \mathcal{J}_{ab} - (\mathcal{J}_{\mathbb{S}})_{ab}) = (\text{tr}\chi - \overline{\text{tr}\chi}) r^{-2} \mathcal{J}_{ab} + 2r^{-2} \hat{\chi}_{ab}. \quad (9.39)$$

Integrating along  $e_3$  from  $\varnothing(\underline{u}^*)$ , this yields

$$\begin{aligned} |r^{-2} \mathcal{J}_{ab} - (\mathcal{J}_{\mathbb{S}})_{ab}|(u, \vartheta, \varphi) &\lesssim \lim_{u \rightarrow \underline{u}^*} (|r^{-2} \mathcal{J}_{ab} - (\mathcal{J}_{\mathbb{S}})_{ab}|(u, \vartheta, \varphi)) \\ &\quad + \int_u^{\underline{u}^*} \left( |(\text{tr}\chi - \overline{\text{tr}\chi}) r^{-2} \mathcal{J}_{ab}| + 2r^{-2} |\hat{\chi}_{ab}| \right) du' \\ &\lesssim \varepsilon \int_u^{\underline{u}^*} (\underline{u}^*)^{-1} (u')^{-1} q^{-1/2} du' \\ &\lesssim \varepsilon \underline{u}^{*-1} u^{-1} q^{1/2}, \end{aligned}$$

where we used the vertex limits from Theorems 4.4 and 4.5, the improved bounds (9.18), (9.19) for  $\text{tr}\chi$  and  $\hat{\chi}$  and the Bootstrap Assumptions 3.25 for the coordinate component of  $\mathcal{J}$ .

Commuting (9.39) with  $\partial$  and integrating using the sup-norm bounds for  $(r\nabla)(\text{tr}\chi - \overline{\text{tr}\chi})$  and  $(r\nabla)\hat{\chi}$ , we further have

$$|\partial^{\leq 1} (r^{-2} \mathcal{J}_{ab} - (\mathcal{J}_{\mathbb{S}})_{ab})| \lesssim \varepsilon \underline{u}^{*-1} u^{-1} q^{1/2}.$$

This finishes the improvement of the Bootstrap Assumptions 3.25.





## 10. Null connection estimates in $\mathcal{M}^{\text{ext}}$

In this section, we prove the following proposition.

**Proposition 10.1.** *Recall that from Proposition 6.1, the following estimates hold for the curvature norms in  $\mathcal{M}^{\text{ext}}$  (see the definitions of Section 3.2.1)*

$$\mathfrak{R}_{\leq 1}^{\text{ext}} + \mathcal{R}_{\leq 2, \gamma}^{\text{ext}} \lesssim \varepsilon, \quad (10.1)$$

for all  $\gamma > 0$ . Recall that from Proposition 9.1 the following estimates hold for the null connection coefficient norms on  $\underline{\mathcal{C}}^*$  (see the definitions of Sections 3.2.1 and 3.2.3)

$$\overline{\mathfrak{R}}_{\leq 2}^* + \mathcal{O}_{\leq 3}^* + \overline{\mathfrak{D}}_{\leq 2}^* + \mathfrak{D}_{\leq 2}^* + \mathcal{O}_{\leq 3}^{*, \mathbf{O}} \lesssim \varepsilon, \quad (10.2)$$

and that the Bootstrap Assumptions 3.16, 3.22, 3.25 are improved on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$ .

Under the Bootstrap Assumptions, the estimates (10.1) and (10.2), and for  $\varepsilon > 0$  sufficiently small, we have the following bounds for the connection coefficients (see the definitions of Section 3.2.4)

$$\begin{aligned} \overline{\mathfrak{R}}_{\leq 2}^{\text{ext}} + \mathfrak{D}_{\leq 2}^{\text{ext}} + \mathcal{O}_{\leq 3, \gamma}^{\text{ext}} + \overline{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}} &\lesssim_{\gamma} \varepsilon, \\ \mathfrak{D}_{\leq 1}^{\text{ext}}[\eta] + \mathcal{O}_{2, \gamma}^{\text{ext}}[\eta] + \mathcal{O}_{\leq 2, \gamma}^{\mathcal{T}}[\eta] &\lesssim_{\gamma} \varepsilon, \end{aligned} \quad (10.3)$$

$$\mathfrak{D}_{\leq 2}^{\text{ext}, \mathbf{O}} + \mathcal{O}_{\leq 3, \gamma}^{\text{ext}, \mathbf{O}} \lesssim_{\gamma} \varepsilon,$$

for all  $\gamma > 0$ . Moreover, the Bootstrap Assumptions 3.26 for the spherical coordinates (see Section 10.15), the bootstrap bound (3.4) on the area radius (see Section 10.14), and the mild Bootstrap Assumptions 3.16 for the rotation vectorfields  $\mathbf{O}^{\text{ext}}$  (see Section 10.13.1) are improved.<sup>1</sup>

### 10.1 Evolution estimates

Let us first recall that in the region  $\mathcal{M}^{\text{ext}}$ , we have

$$r \simeq \underline{u}.$$

The following lemma provides estimates for solutions to transport equations in the  $e_4$  direction.

**Lemma 10.2** (Transport estimates in  $\mathcal{M}^{\text{ext}}$ ). *For all  $\kappa \in \mathbb{R}$ , the following holds. Assume that  $U$  is an  $S$ -tangent tensor satisfying*

$$\nabla_4 U + \frac{\kappa}{2} \text{tr} \chi U = F.$$

◇ We have the following  $L^\infty L^\infty$  estimates

$$\|r^\lambda U\|_{L^\infty(S_{\underline{u}, \underline{u}})} \lesssim \|r^\lambda U\|_{L^\infty(S_{\underline{u}, \underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \|r^\lambda F\|_{L^\infty(S_{\underline{u}', \underline{u}})} \, d\underline{u}', \quad (10.4)$$

for all  $S_{\underline{u}, \underline{u}} \subset \mathcal{M}^{\text{ext}}$  and for all  $\lambda \geq \kappa$ .

◇ We have the following  $L^\infty H^{1/2}$  estimate

$$\|r^\lambda U\|_{\tilde{H}^{1/2}(S_{\underline{u}, \underline{u}})} \lesssim \|r^\lambda U\|_{\tilde{H}^{1/2}(S_{\underline{u}, \underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \|r^\lambda F\|_{\tilde{H}^{1/2}(S_{\underline{u}', \underline{u}})} \, d\underline{u}', \quad (10.5)$$

for all  $S_{\underline{u}, \underline{u}} \subset \mathcal{M}^{\text{ext}}$  and for all  $\lambda \geq \kappa - 1/2$ .

<sup>1</sup>These last bounds together with (10.3) also amount to an improvement the strong Bootstrap Assumptions 3.23.

◇ We have the following  $L_{\underline{u}}^\infty L_u^2 L^2(S_{u, \underline{u}})$  estimate

$$\|r^\lambda U\|_{L_u^2 L^2(S_{u, \underline{u}})} \lesssim \|r^\lambda U\|_{L^2(\underline{\mathcal{C}}^*)} + \int_{\underline{u}}^{\underline{u}^*} \|r^\lambda F\|_{L_u^2 L^2(S_{u, \underline{u}'})} d\underline{u}', \quad (10.6)$$

for all  $\lambda \geq \kappa - 1$ .

*Proof.* We only perform the proof for the  $\tilde{H}^{1/2}$  estimate and for a  $S$ -tangent 1-tensor  $U$ . Rewriting the transport equation satisfied by  $U$ , using (2.39), we have

$$\nabla_4(r^\kappa U) = r^\kappa F + \frac{\kappa}{2}(\overline{\text{tr}\chi} - \text{tr}\chi)r^\kappa U. \quad (10.7)$$

Using the spherical coordinates in  $\mathcal{M}^{\text{ext}}$  from the Bootstrap Assumptions 3.26, we have

$$\nabla_4(r^\kappa U)_a = 2\partial_{\underline{u}}(r^\kappa U_a) - 2r^\kappa U^b \chi_{ab},$$

where  $a, b \in \{\vartheta, \varphi\}$ . Thus, using (2.39) and equation (10.7), we infer

$$\begin{aligned} 2\partial_{\underline{u}}(r^{-1+\kappa} U_a) &= r^{-1} \nabla_4(r^\kappa U)_a + r^{-1+\kappa} U_a (\text{tr}\chi - \overline{\text{tr}\chi}) + 2r^{-1+\kappa} U_b \hat{\chi}_a^b \\ &= r^{-1+\kappa} F_a + \left(\frac{\kappa}{2} - 1\right)(\overline{\text{tr}\chi} - \text{tr}\chi)r^{-1+\kappa} U_a + 2r^{-1+\kappa} U_b \hat{\chi}_a^b. \end{aligned}$$

Integrating the above equation, we obtain

$$\begin{aligned} r^{-1+\kappa} U_a(\underline{u}, u, \vartheta, \varphi) &= r^{-1+\kappa} U_a(\underline{u}^*, u, \vartheta, \varphi) \\ &+ \frac{1}{2} \int_{\underline{u}}^{\underline{u}^*} \left( r^{-1+\kappa} F_a + \left(\frac{\kappa}{2} - 1\right)(\overline{\text{tr}\chi} - \text{tr}\chi)r^{-1+\kappa} U_a + 2r^{-1+\kappa} U_b \hat{\chi}_a^b \right) (\underline{u}', u, \vartheta, \varphi) d\underline{u}'. \end{aligned}$$

Taking the coordinate  $H_{\vartheta, \varphi}^{1/2}$  norm in the above, we obtain

$$\begin{aligned} \|r^{-1+\kappa} U_a(\underline{u}, u)\|_{H_{\vartheta, \varphi}^{1/2}} &\lesssim \|r^{-1+\kappa} U_a(\underline{u}^*, u)\|_{H_{\vartheta, \varphi}^{1/2}} \\ &+ \int_{\underline{u}}^{\underline{u}^*} \left\| \left( r^{-1+\kappa} F_a + \left(\frac{\kappa}{2} - 1\right)(\overline{\text{tr}\chi} - \text{tr}\chi)r^{-1+\kappa} U_a + 2r^{-1+\kappa} U_b \hat{\chi}_a^b \right) (\underline{u}', u) \right\|_{H_{\vartheta, \varphi}^{1/2}} d\underline{u}'. \end{aligned}$$

Using the fractional Sobolev space comparison Lemma 3.35, we thus infer

$$\begin{aligned} \|r^{-1/2+\kappa} U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \|r^{-1/2+\kappa} U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \|r^{-1/2+\kappa} F\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}' \\ &+ \int_{\underline{u}}^{\underline{u}^*} \left( \|r^{-1/2+\kappa} (\text{tr}\chi - \overline{\text{tr}\chi}) U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} + \|r^{-1/2+\kappa} \hat{\chi} \cdot U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} \right) d\underline{u}'. \end{aligned}$$

Using the  $\tilde{H}^{1/2}$  product estimates from Lemma 3.37, and the Bootstrap Assumptions 3.23 for the  $L^\infty$  norms of  $\chi$  and  $\nabla\chi$ , we obtain

$$\begin{aligned} \|r^{-1/2+\kappa} U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \|r^{-1/2+\kappa} U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} + \int_{\underline{u}}^{\underline{u}^*} \|F\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}' \\ &+ D\varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-2} u^{-1/2} \|U\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}', \end{aligned}$$

and the conclusion follows from a standard Grönwall argument. The result for  $\lambda \geq \kappa - 1/2$  follows since  $r(u, \underline{u}) \lesssim r(u, \underline{u}')$  for  $\underline{u} \leq \underline{u}' \leq \underline{u}^*$ . The remaining estimates are obtained similarly. This finishes the proof of the lemma.  $\square$

**Remark 10.3.** The conclusions of Lemma 10.2 also hold with  $\text{tr}\chi$  replaced by  $\overline{\text{tr}\chi}$  or  $\frac{2}{r}$ .

## 10.2 Control of $\bar{\rho}$ and $\bar{\sigma}$

Applying the  $L^\infty$  estimate of Lemma 10.2, with  $\kappa = 3, \lambda = 3$  to the transport equation (2.47), we have

$$\|r^3 \bar{\rho}\|_{L^\infty(S_{u, \underline{u}})} \lesssim \|r^3 \bar{\rho}\|_{L^\infty(S_{u, \underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \|r^3 \text{Err}(\nabla_4, \bar{\rho})\|_{L^\infty(S_{u, \underline{u}'})} d\underline{u}'.$$

Using the bounds (10.2) for  $\bar{\rho}$  on  $\mathcal{C}^*$ , the curvature bounds (10.1) and the Bootstrap Assumptions 3.23, with the expression of  $\text{Err}(\nabla_4, \bar{\rho})$  from (2.47), we have

$$\begin{aligned} \|r^3 \bar{\rho}\|_{L^\infty(S_{u, \underline{u}^*})} &\lesssim \varepsilon u^{-2}, \\ \int_{\underline{u}}^{\underline{u}^*} \|r^3 \text{Err}(\nabla_4, \bar{\rho})\|_{L^\infty(S_{u, \underline{u}'})} d\underline{u}' &\lesssim (D\varepsilon)^2 \int_{\underline{u}}^{\underline{u}^*} r^3 (\underline{u}')^{-1} (\underline{u}')^{-7/2} u^{-3/2} d\underline{u}' \\ &\lesssim \varepsilon u^{-2}. \end{aligned}$$

Thus,

$$\|r^3 u^2 \bar{\rho}\|_{L^\infty(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon. \quad (10.8)$$

Estimating directly equation (2.47) for  $\nabla_4 \bar{\rho}$ , using (10.8), (10.1) and the Bootstrap Assumptions 3.23, we have

$$\|r^3 u^2 (\underline{u} \nabla_4) \bar{\rho}\|_{L^\infty} \lesssim \varepsilon.$$

Taking the average in equation (2.35e) for  $\nabla_3 \rho$ , using the bound (10.8) for  $\bar{\rho}$ , estimates (10.1) and the Bootstrap Assumptions 3.23, one checks that

$$\|r^3 u^2 (u \nabla_3) \bar{\rho}\|_{L^\infty} \lesssim \varepsilon.$$

Taking one more derivative in the above equations, we further have

$$\bar{\mathfrak{R}}_{\leq 2}^{\text{ext}}[\bar{\rho}] \lesssim \varepsilon.$$

Estimating directly equation (2.48) for  $\bar{\sigma}$ , using the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \|r^3 u^2 \bar{\sigma}\|_{L^\infty} &\lesssim \|r^3 u^2 \hat{\chi} \wedge \hat{\chi}\|_{L^\infty} \\ &\lesssim (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

Taking the average and directly estimating equations (2.35g) and (2.35f) for  $\nabla_3 \sigma$  and  $\nabla_4 \sigma$  respectively, one further has

$$\bar{\mathfrak{R}}_{\leq 2}^{\text{ext}}[\bar{\sigma}] \lesssim \varepsilon,$$

and we have thus proved

$$\bar{\mathfrak{R}}_{\leq 2}^{\text{ext}} \lesssim \varepsilon. \quad (10.9)$$

## 10.3 Control of $\bar{\omega}$

Applying the  $L^\infty$  estimate of Lemma 10.2, with  $\kappa = 0, \lambda = 1$ , to the transport equation (2.58) for  $\bar{\omega}$ , using the bounds (10.9) obtained for  $\bar{\rho}$  and the Bootstrap Assumptions 3.23, and using that  $\bar{\omega} = 0$  on  $\mathcal{C}^*$ , we have

$$\begin{aligned} \|r \bar{\omega}\|_{L^\infty(S_{u, \underline{u}})} &\lesssim \int_{\underline{u}}^{\underline{u}^*} (r |\bar{\rho}| + \text{Err}(\nabla_4, \bar{\omega})) d\underline{u}' \\ &\lesssim \varepsilon \underline{u}^{-1} u^{-2} \end{aligned}$$

Commuting (2.58) with  $\nabla^{\leq 1}$  and arguing similarly, we further have

$$\bar{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\bar{\omega}] \lesssim \varepsilon. \quad (10.10)$$

Applying the  $L_{\underline{u}}^\infty L_{\underline{u}}^2 L^2(S_{u, \underline{u}})$  estimate of Lemma 10.2, with  $\kappa = 0, \lambda = 1 - \gamma$ , to the transport equation (2.58) for  $\bar{\omega}$  multiplied by  $u^{3/2-\gamma}$ , using the estimates (10.9) for  $\bar{\rho}$ , from which one deduces  $\|u^{3/2-\gamma}\bar{\rho}\|_{L_{\underline{u}}^2 L^2(S_{u, \underline{u}})} \lesssim_\gamma \varepsilon \underline{u}^{-2}$  for all  $\gamma > 0$ , and the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \left\| u^{3/2-\gamma} \underline{u}^{-\gamma} \underline{u} \bar{\omega} \right\|_{L_{\underline{u}}^2 L^2(S_{u, \underline{u}})} &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left( \left\| u^{3/2-\gamma}(\underline{u}')^{1-\gamma} \bar{\rho} \right\|_{L_{\underline{u}}^2 L^2(S_{u, \underline{u}'})} + \left\| u^{-3/2-\gamma}(\underline{u}')^{1-\gamma} \text{Err}(\nabla_4, \bar{\omega}) \right\|_{L_{\underline{u}}^2 L^2(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\lesssim \varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-1-\gamma} d\underline{u}' \\ &\lesssim_\gamma \varepsilon. \end{aligned}$$

Thus, arguing as in Section 5.2.1, using that  $u \lesssim \underline{u}$  in  $\mathcal{M}^{\text{ext}}$

$$\left\| u^{3/2} \underline{u}^{-1/2-\gamma} \underline{u} \bar{\omega} \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim_\gamma \varepsilon,$$

for all  $\gamma > 0$ . Commuting equation (2.58) with  $\nabla^{\leq 2}$  and arguing similarly gives

$$\bar{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}}[\bar{\omega}] \lesssim \varepsilon, \quad (10.11)$$

for all  $\gamma > 0$ .

#### 10.4 Control of $\overline{\text{tr}\chi} - \frac{2}{r}$

Applying the  $L^\infty$  estimate of Lemma 10.2, with  $\kappa = 1, \lambda = 2$ , to the transport equation (2.50) for  $\overline{\text{tr}\chi} - \frac{2}{r}$ , we have

$$\left\| r^2 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(S_{u, \underline{u}})} \lesssim \left\| r^2 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(S_{u, \underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \left\| r^2 \text{Err} \left( \nabla_4, \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(S_{u, \underline{u}'})} d\underline{u}'.$$

From the bound (10.2) on  $\mathcal{C}^*$ , and the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \left\| r^2 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(S_{u, \underline{u}^*})} &\lesssim \varepsilon \underline{u}^{*-1} u^{-1}, \\ \int_{\underline{u}}^{\underline{u}^*} \left\| r^2 \text{Err} \left( \nabla_4, \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty(S_{u, \underline{u}'})} d\underline{u}' &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left( r^2 \|\hat{\chi}\|_{L^\infty}^2 + r^2 \|\overline{\text{tr}\chi} - \frac{2}{r}\|_{L^\infty}^2 \right) d\underline{u}' \\ &\lesssim (D\varepsilon)^2 \int_{\underline{u}}^{\underline{u}^*} r^2 (\underline{u}')^{-4} u^{-1} d\underline{u}' \\ &\lesssim \varepsilon \underline{u}^{-1} u^{-1}, \end{aligned}$$

and we therefore deduce

$$\left\| \underline{u} \underline{u}^3 \left( \overline{\text{tr}\chi} - \frac{2}{r} \right) \right\|_{L^\infty} \lesssim \varepsilon. \quad (10.12)$$

Estimating directly equation (2.50) or commuting equation (2.50) by  $u\nabla_3$ , and applying analogously the  $L^\infty$  estimates of Lemma 10.2, one obtains

$$\bar{\mathcal{O}}_{\leq 1}^{\text{ext}}[\overline{\text{tr}\chi}] \lesssim \varepsilon. \quad (10.13)$$

Commuting with  $\nabla^{\leq 2}$  derivatives and applying the  $L_{\underline{u}}^\infty L^2$  estimates of Lemma 10.2 as in the previous section further gives

$$\bar{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}}[\overline{\text{tr}\chi}] \lesssim \varepsilon, \quad (10.14)$$

for all  $\gamma > 0$ .

## 10.5 Control of $\overline{\text{tr}\chi} + \frac{2}{r}$

Applying the  $L^\infty$  estimate of Lemma 10.2, with  $\kappa = 1, \lambda = 2$ , to the transport equation (2.54) for  $\overline{\text{tr}\chi} + \frac{2}{r}$ , using the bounds (10.9) obtained for  $\bar{\rho}$  and the Bootstrap Assumptions 3.23, we have

$$\overline{\mathfrak{D}}_{\leq 1}^{\text{ext}}[\overline{\text{tr}\chi}] \lesssim \varepsilon. \quad (10.15)$$

Applying the  $L_u^\infty L^2$  estimate of Lemma 10.2 as in the previous sections gives

$$\overline{\mathcal{O}}_{\leq 2, \gamma}^{\text{ext}}[\overline{\text{tr}\chi}] \lesssim \varepsilon, \quad (10.16)$$

for all  $\gamma > 0$ .

## 10.6 Control of $\text{tr}\chi - \overline{\text{tr}\chi}$

Applying the  $L^\infty \tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 2, \lambda = 3/2$  to the transport equation (2.49), we have

$$\begin{aligned} \left\| r^{3/2}(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \left\| r^{3/2}(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}^*})} \\ &+ \int_{\underline{u}}^{\underline{u}^*} \left\| r^{3/2} \text{Err}(\nabla_4, \text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}'. \end{aligned}$$

Using the bounds (10.2) on  $\underline{\mathcal{C}}^*$ , the  $H$  product estimates of Lemma 3.37 and the Bootstrap Assumptions 3.23, we have

$$\left\| r^{3/2}(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}^*})} \lesssim \varepsilon u^{-1/2},$$

and

$$\begin{aligned} &\int_{\underline{u}}^{\underline{u}^*} \left\| r^{3/2} \text{Err}(\nabla_4, \text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}' \\ &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left( \left\| r^{3/2} \hat{\chi} \cdot \hat{\chi} \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} + \left\| r^{3/2}(\text{tr}\chi - \overline{\text{tr}\chi})^2 \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left( \left( \left\| \hat{\chi} \right\|_{L^\infty(S_{u, \underline{u}'})} + r^{-1} \left\| r \nabla \hat{\chi} \right\|_{L^2(S_{u, \underline{u}'})} \right) \left\| r^{3/2} \hat{\chi} \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} \right. \\ &\quad \left. + \left( \left\| \text{tr}\chi - \overline{\text{tr}\chi} \right\|_{L^\infty(S_{u, \underline{u}'})} + r^{-1} \left\| r \nabla(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{L^2(S_{u, \underline{u}'})} \right) \left\| r^{3/2}(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\lesssim (D\varepsilon)^2 \int_{\underline{u}}^{\underline{u}^*} \left( r^{-2} u^{-1/2} \right) u^{-1/2} d\underline{u}' \\ &\lesssim \varepsilon u^{-1/2}. \end{aligned}$$

Thus,

$$\left\| r^{3/2} u^{1/2} (\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{L_{u, \underline{u}}^\infty \tilde{H}^{1/2}} \lesssim \varepsilon. \quad (10.17)$$

Arguing similarly, commuting by  $(r\nabla)$ ,  $(u\nabla_3)$  or directly estimating equation (2.49), we obtain

$$\overline{\mathfrak{D}}_{\leq 1}^{\text{ext}, \mathfrak{g}}[\text{tr}\chi - \overline{\text{tr}\chi}] \lesssim \varepsilon. \quad (10.18)$$

Arguing similarly, commuting first by  $(r\nabla)$ , we obtain

$$\overline{\mathfrak{D}}_{\leq 1}^{\text{ext}, \mathfrak{g}}[(r\nabla)\text{tr}\chi] \lesssim \varepsilon. \quad (10.19)$$

Applying the  $L_u^\infty L^2$  estimates of Lemma 10.2 with  $\kappa = 2, \lambda = 1$  to the transport equation (2.49), we obtain

$$\begin{aligned} \left\| r(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{L_u^\infty L^2(S_{u, \underline{u}})} &\lesssim \left\| r(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{L^2(\underline{\mathcal{C}}^*)} \\ &+ \int_{\underline{u}}^{\underline{u}^*} \left\| r \text{Err}(\nabla_4, \text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{L_u^\infty L^2(S_{u, \underline{u}'})} d\underline{u}'. \end{aligned}$$

Using the estimate (10.2) on  $\mathcal{C}^*$ , we have

$$\|r(\text{tr}\chi - \overline{\text{tr}\chi})\|_{L^2(\mathcal{C}^*)} \lesssim \varepsilon.$$

We only treat the error terms involving  $\hat{\chi}$  and the other terms follow (more easily). Using the Bootstrap Assumptions 3.24 and Cauchy-Schwartz, gives

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \|\underline{u}'|\hat{\chi}|^2\|_{L_u^2 L^2(S_{u,\underline{u}'})} d\underline{u}' &\lesssim \int_{\underline{u}}^{\underline{u}^*} (D\varepsilon)(\underline{u}')^{-1} \|u^{-1/2}\hat{\chi}\|_{L_u^2 L^2(S_{u,\underline{u}'})} d\underline{u}' \\ &\lesssim (D\varepsilon) \|u^{-1/2}\underline{u}^{-\gamma}\hat{\chi}\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim (D\varepsilon) \|\underline{u}^{-1/2-\gamma}\underline{u}\hat{\chi}\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim (D\varepsilon)^2. \end{aligned}$$

Thus,

$$\|r(\text{tr}\chi - \overline{\text{tr}\chi})\|_{L_u^\infty L_u^2 L^2(S_{u,\underline{u}})} \lesssim \varepsilon,$$

and we infer

$$\|\underline{u}^{-1/2-\gamma}\underline{u}(\text{tr}\chi - \overline{\text{tr}\chi})\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon$$

for all  $\gamma > 0$ .

Commuting equation (2.49) by  $(r\nabla)^{\leq 1}$  and then by  $\nabla^{\leq 2}$  and arguing similarly then gives

$$\mathcal{O}_{\leq 2,\gamma}^{\text{ext},\mathfrak{g}}[\text{tr}\chi - \overline{\text{tr}\chi}] \lesssim \varepsilon, \quad \mathcal{O}_{\leq 2,\gamma}^{\text{ext},\mathfrak{g}}[(r\nabla)\text{tr}\chi] \lesssim \varepsilon, \quad (10.20)$$

for all  $\gamma > 0$ .

## 10.7 Control of $\zeta$

### 10.7.1 Control of $\mu - \bar{\mu}$ and $\nabla^{\leq 1}\zeta$

Arguing as in the control for  $\text{tr}\chi - \overline{\text{tr}\chi}$  in Section 10.6, using that  $\mu - \bar{\mu} = 0$  on  $\mathcal{C}^*$ , and using the control obtained for  $\text{tr}\chi - \overline{\text{tr}\chi}$ , we obtain using the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 3, \lambda = 5/2$

$$\mathfrak{D}_{\leq 1}^{\text{ext},\mathfrak{g}}[r(\mu - \bar{\mu})] \lesssim \varepsilon, \quad (10.21)$$

and, using the  $L_u^\infty L_u^2 L^2(S_{u,\underline{u}})$  estimates of Lemma 10.2 with  $\kappa = 3, \lambda = 5/2$  and arguing a previously,

$$\mathcal{O}_{\leq 2,\gamma}^{\text{ext},\mathfrak{g}}[r(\mu - \bar{\mu})] \lesssim \varepsilon, \quad (10.22)$$

for all  $\gamma > 0$ .

Using the elliptic equation (2.61) and the elliptic estimates from Lemma 3.38, one obtains from the  $\tilde{H}^{1/2}$  estimates (10.21) for  $\mu - \bar{\mu}$ ,

$$\mathfrak{D}_{\leq 1}^{\text{ext},\mathfrak{g}}[\zeta] \lesssim \varepsilon, \quad \mathfrak{D}_{\leq 1}^{\text{ext},\mathfrak{g}}[(r\nabla)\zeta] \lesssim \varepsilon, \quad (10.23)$$

and from the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (10.22) for  $\mu - \bar{\mu}$

$$\mathcal{O}_{\leq 2,\gamma}^{\text{ext},\mathfrak{g}}[\zeta] \lesssim \varepsilon, \quad \mathcal{O}_{\leq 2,\gamma}^{\text{ext},\mathfrak{g}}[(r\nabla)\zeta] \lesssim \varepsilon, \quad (10.24)$$

for all  $\gamma > 0$ .

### 10.7.2 Control of $\iota, \underline{\omega}_\rho, \underline{\omega}_\sigma$

Applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = -1/2$  to the transport equation (2.70), we have

$$\|r^{-1/2}(r^2\iota)\|_{\tilde{H}^{1/2}(S_{u,\underline{u}})} \lesssim \|r^{-1/2}(r^2\iota)\|_{\tilde{H}^{1/2}(S_{u,\underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \|r^{-1/2}\text{Err}(\nabla_4, \iota)\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} d\underline{u}'.$$

Using that  $\iota = \underline{\beta}$  on  $\underline{\mathcal{C}}^*$  and the estimates for the curvature (10.1), we have

$$\left\| r^{-1/2}(r^2\iota) \right\|_{\tilde{H}^{1/2}(S_{u,\underline{u}^*})} \lesssim \left\| r^{3/2}\underline{\beta} \right\|_{\tilde{H}^{1/2}(S_{u,\underline{u}^*})} \lesssim \varepsilon u^{-3/2}.$$

From the error term  $\text{Err}(\nabla_4, \iota)$ , we only treat the term  $r^2(\text{tr}\chi - \overline{\text{tr}\chi})$  and the estimates for the other terms will follow similarly. Using the product estimates from Lemma 3.37, estimates (10.1) and the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} & \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1/2}r^2(\text{tr}\chi - \overline{\text{tr}\chi})\underline{\beta} \right\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} \, d\underline{u}' \\ & \lesssim \int_{\underline{u}}^{\underline{u}^*} \left( \|\text{tr}\chi - \overline{\text{tr}\chi}\|_{L^\infty(S_{u,\underline{u}'})} + r^{-1} \|(r\nabla)\text{tr}\chi\|_{L^2(S_{u,\underline{u}'})} \right) \left\| r^{-1/2}r^2\underline{\beta} \right\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} \, d\underline{u}' \\ & \lesssim \int_{\underline{u}}^{\underline{u}^*} (D\varepsilon)\underline{u}^{-2}u^{-1/2}(D\varepsilon)u^{-3/2} \, d\underline{u}' \\ & \lesssim (D\varepsilon)^2 u^{-3/2}, \end{aligned}$$

Thus,

$$\left\| r^{3/2}u^{3/2}\iota \right\|_{L_{\underline{u},\underline{u}}^\infty \tilde{H}^{1/2}} \lesssim \varepsilon. \quad (10.25)$$

Commuting equation (2.70) by  $(r\nabla)$ ,  $(u\nabla)_3$  and  $\underline{u}\nabla_4$ , one further obtains

$$\mathfrak{D}_{\leq 1}^{\text{ext},\text{b}}[r\iota] \lesssim \varepsilon. \quad (10.26)$$

Using the  $L_{\underline{u}}^\infty L_u^2 L^2(S_{u,\underline{u}})$  estimates of Lemma 10.2, we have

$$\left\| (\underline{u}^{-1}u)(r^2\iota) \right\|_{L_u^2 L^2(S_{u,\underline{u}})} \lesssim \left\| (\underline{u}^{-1}u)(r^2\iota) \right\|_{L^2(\underline{\mathcal{C}}^*)} + \int_{\underline{u}}^{\underline{u}^*} \left\| ((\underline{u}')^{-1}u)\text{Err}(\nabla_4, \iota) \right\|_{L_u^2 L^2(S_{u,\underline{u}'})} \, d\underline{u}'.$$

From (10.2), we have

$$\left\| (\underline{u}^{-1}u)(r^2\iota) \right\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \varepsilon.$$

From estimates (10.1) and the Bootstrap Assumptions 3.23 and using Cauchy-Schwartz, we have

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \left\| ((\underline{u}')^{-1}u)r^2(\text{tr}\chi - \overline{\text{tr}\chi})\underline{\beta} \right\|_{L_u^2 L^2(S_{u,\underline{u}'})} \, d\underline{u}' & \lesssim \int_{\underline{u}}^{\underline{u}^*} \left\| (\underline{u}')^{-1}ur^2\underline{\beta}(\text{tr}\chi - \overline{\text{tr}\chi}) \right\|_{L_u^2 L^2(S_{u,\underline{u}'})} \, d\underline{u}' \\ & \lesssim \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-3/2+\gamma} \left\| (\underline{u}')^{-1/2-\gamma}\underline{u}'u^{1/2}\underline{\beta} \right\|_{L_u^2 L^2(S_{u,\underline{u}'})} \, d\underline{u}' \\ & \lesssim \left\| \underline{u}^{-1/2-\gamma}\underline{u}u^{1/2}\underline{\beta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ & \lesssim \left\| \underline{u}^{-1/2-\gamma}\underline{u}u\underline{\beta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ & \lesssim (D\varepsilon)^2. \end{aligned}$$

Thus,

$$\left\| (\underline{u}^{-1}u)(r^2\iota) \right\|_{L_{\underline{u}}^\infty L_u^2 L^2(S_{u,\underline{u}})} \lesssim \varepsilon.$$

Commuting with  $\nabla^{\leq 2}$  and arguing as previously, we infer

$$\mathcal{O}_{\leq 2,\gamma}^{\text{ext},\text{b}}[r\iota] \lesssim \varepsilon, \quad (10.27)$$

for all  $\gamma > 0$ .

Using the elliptic equation (2.69), the elliptic estimates of Lemma 3.38, the estimates for the curvature (10.1), we have using the  $\tilde{H}^{1/2}$  estimates (10.26)

$$\mathfrak{D}_{\leq 1}^{\text{ext},\text{b}}[(r\nabla)(\underline{\omega}_\rho, \underline{\omega}_\sigma)] \lesssim \varepsilon, \quad (10.28)$$

and using the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (10.27)

$$\mathcal{O}_{\leq 2,\gamma}^{\text{ext},\text{b}}[(r\nabla)(\underline{\omega}_\rho, \underline{\omega}_\sigma)] \lesssim \varepsilon, \quad (10.29)$$

for all  $\gamma > 0$ .



### 10.7.3 Control of $\varsigma$ and $\nabla_3 \zeta$

Applying the estimates from Lemma 10.2 with  $\kappa = 0, \lambda = -1/2$  to the transport equation (2.74) for  $\varsigma$ , we have

$$\begin{aligned} \left\| r^{-1/2} r^2 \varsigma \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \left\| r^{-1/2} r^2 \varsigma \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}^*})} \\ &\quad + \int_{\underline{u}}^{\underline{u}^*} \left( \left\| r^{-1/2} r^2 \text{tr} \underline{\chi} \beta \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} + \left\| r^{-1/2} r^2 \text{tr} \chi \text{tr} \underline{\chi} \zeta \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\quad + \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1/2} \text{Err}(\nabla_4, \varsigma) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}'. \end{aligned}$$

From (10.2), we have

$$\left\| r^{-1/2} r^2 \varsigma \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}^*})} = \left\| r^{-1/2} r^2 \nabla_3 \zeta \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}^*})} \lesssim \varepsilon u^{-3/2}.$$

From the curvature estimates (10.1), the Bootstrap Assumptions 3.23 for  $\text{tr} \chi$ ,  $\text{tr} \underline{\chi}$  and the previously obtained estimates (10.23) for  $\zeta$ , we have

$$\begin{aligned} &\int_{\underline{u}}^{\underline{u}^*} \left( \left\| r^{-1/2} r^2 \text{tr} \underline{\chi} \beta \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} + \left\| r^{-1/2} r^2 \text{tr} \chi \text{tr} \underline{\chi} \zeta \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\lesssim \int_{\underline{u}}^{\underline{u}^*} \varepsilon (\underline{u}')^{-5/2} + \varepsilon (\underline{u}')^{-2} u^{-1/2} d\underline{u}' \\ &\lesssim \varepsilon u^{-3/2}. \end{aligned}$$

From the product estimates from Lemma 3.37, estimates (10.1) and the Bootstrap Assumptions 3.23 and an inspection of the nonlinear terms composing  $\text{Err}(\nabla_4, \varsigma)$ , we have

$$\begin{aligned} &\int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1/2} \text{Err}(\nabla_4, \varsigma) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}' \\ &\lesssim \int_{\underline{u}}^{\underline{u}^*} (D\varepsilon)^2 (\underline{u}')^{-2} u^{-3/2} d\underline{u}' \\ &\lesssim (D\varepsilon)^2 u^{-3/2}. \end{aligned}$$

Thus,

$$\left\| r^{-1/2} u^{3/2} r^2 \varsigma \right\|_{L_{u, \underline{u}}^\infty \tilde{H}^{1/2}} \lesssim \varepsilon. \quad (10.30)$$

Commuting by  $(r\nabla)$ ,  $(u\nabla_3)$  and arguing similarly, or directly estimating  $\nabla_4 \varsigma$ , we have

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}}[u\varsigma] \lesssim \varepsilon. \quad (10.31)$$

Applying the  $L_u^\infty L^2$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = -1$  to the transport equation (2.74) multiplied with  $u^{1-\gamma}$  gives

$$\begin{aligned} &\left\| r^{-1} u^{1-\gamma} r^2 \varsigma \right\|_{L_u^2 L^2(S_{u, \underline{u}})} \\ &\lesssim \left\| r^{-1} u^{1-\gamma} r^2 \varsigma \right\|_{L^2(\mathcal{C}^*)} \\ &\quad + \int_{\underline{u}}^{\underline{u}^*} \left( \left\| r^{-1} u^{1-\gamma} r^2 (\text{tr} \underline{\chi} \beta) \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} + \left\| r^{-1} u^{1-\gamma} r^2 (\text{tr} \chi \text{tr} \underline{\chi} \zeta) \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\quad + \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1} u^{1-\gamma} \text{Err}(\nabla_4, \varsigma) \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} d\underline{u}'. \end{aligned}$$

From the estimates (10.2) for  $\nabla_3 \zeta$  on  $\mathcal{C}^*$ , we have

$$\left\| r^{-1} u^{1-\gamma} r^2 \varsigma \right\|_{L^2(\mathcal{C}^*)} = \left\| u^{1-\gamma} r \nabla_3 \zeta \right\|_{L^2(\mathcal{C}^*)} \lesssim \varepsilon.$$

Using Cauchy-Schwartz, the estimates (10.24) for the  $L^2(\mathcal{M}^{\text{ext}})$  norm of  $\zeta$ , and the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \|r^{-1}u^{1-\gamma}r^2\text{tr}\chi\text{tr}\underline{\chi}\zeta\|_{L^2_{\underline{u}}L^2(S_{u,\underline{u}'})} d\underline{u}' &\lesssim \int_{\underline{u}}^{\underline{u}^*} \|r^{-1}u^{1-\gamma}\zeta\|_{L^2_{\underline{u}}L^2(S_{u,\underline{u}'})} d\underline{u}' \\ &\lesssim \int_{\underline{u}}^{\underline{u}^*} \underline{u}^{-1/2-\gamma/2} \|\underline{u}^{-1/2-\gamma/2}\underline{u}^2r^{-1}\zeta\|_{L^2_{\underline{u}}L^2(S_{u,\underline{u}'})} d\underline{u}' \\ &\lesssim \|\underline{u}^{-1/2-\gamma/2}\underline{u}^2r^{-1}\zeta\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim \varepsilon. \end{aligned}$$

The linear source term  $\beta$  and the error term  $\text{Err}(\nabla_4\zeta)$  are estimated similarly, using the curvature estimate (10.1) and the Bootstrap Assumptions 3.23, and we have

$$\int_{\underline{u}}^{\underline{u}^*} \|r^{-1}u^{1-\gamma}r^2\text{tr}\chi\beta\|_{L^2_{\underline{u}}L^2(S_{u,\underline{u}'})} d\underline{u}' \lesssim \varepsilon.$$

Thus,

$$\|r^{-1}u^{1-\gamma}r^2\zeta\|_{L^\infty_{\underline{u}}L^2_{\underline{u}}L^2(S_{u,\underline{u}'})} \lesssim \varepsilon,$$

for all  $\gamma > 0$ , from which we also deduce

$$\|\underline{u}^{-1/2-\gamma}\underline{u}(u\zeta)\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon,$$

for all  $\gamma > 0$ . Commuting by  $\nabla^{\leq 2}$  and arguing similarly further gives

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}}[u\zeta] \lesssim \varepsilon. \quad (10.32)$$

Using equation (2.73) for  $\nabla_3\zeta$ , the  $\tilde{H}^{1/2}$  estimates (10.28) for  $\nabla\omega_\rho, \nabla\omega_\sigma$ , the  $\tilde{H}^{1/2}$  estimates (10.30), (10.31), for  $\zeta$ , we deduce

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}}[(u\nabla_3)\zeta] \lesssim \varepsilon. \quad (10.33)$$

From the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (10.29) for  $\nabla\omega_\rho, \nabla\omega_\sigma$ , and the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (10.32) for  $\zeta$ , we have

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}}[(u\nabla_3)\zeta] \lesssim \varepsilon, \quad (10.34)$$

for all  $\gamma > 0$ .

## 10.8 Control of $\hat{\chi}$

Using the elliptic estimates of Lemma 3.38 with the elliptic equation (2.34q) for  $\hat{\chi}$ , together with the improved estimates (10.18), (10.19), (10.20) for  $\nabla\text{tr}\chi$ , the improved estimates (10.23), (10.33), for  $\zeta$ , the Bootstrap Assumptions 3.23 and the curvature estimates (10.1), we have

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}}[\hat{\chi}] \lesssim \varepsilon, \quad \mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}}[(r\nabla)\hat{\chi}] \lesssim \varepsilon, \quad (10.35)$$

and using the corresponding  $L^2(\mathcal{M}^{\text{ext}})$  estimates,

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}}[\hat{\chi}] \lesssim \varepsilon, \quad \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}}[(r\nabla)\hat{\chi}] \lesssim \varepsilon, \quad (10.36)$$

for all  $\gamma > 0$ .

## 10.9 Control of $\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}$ and $\hat{\chi}$

Applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 1, \lambda = 1/2$  to the (commuted by  $\nabla^{\leq 1}$ ) transport equation (2.53) for  $\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}$ , using the improved estimates (10.23) for  $\nabla\zeta$  and the estimates (10.1) for  $\rho - \bar{\rho}$ , we have

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}}[\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}] \lesssim \varepsilon, \quad (10.37)$$

Using the corresponding  $L^2(\mathcal{M}^{\text{ext}})$  estimates, we also have

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}}[\text{tr}\underline{\chi} - \overline{\text{tr}\underline{\chi}}] \lesssim \varepsilon, \quad (10.38)$$

for all  $\gamma > 0$ .

Arguing similarly, using the transport equation (2.34m) for  $\hat{\chi}$ , we have

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{b}}[\hat{\chi}] \lesssim \varepsilon, \quad (10.39)$$

and

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}}[\hat{\chi}] \lesssim \varepsilon, \quad (10.40)$$

for all  $\gamma > 0$ .

## 10.10 Control of $\underline{\omega} - \overline{\underline{\omega}}$

### 10.10.1 Control of $\underline{\omega} - \overline{\underline{\omega}}$

Applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = -1/2$  to the transport equation (2.57) for  $\underline{\omega} - \overline{\underline{\omega}}$ , we obtain

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{b}}[\underline{\omega} - \overline{\underline{\omega}}] \lesssim \varepsilon, \quad (10.41)$$

and, applying the  $L^2(\mathcal{M}^{\text{ext}})$  estimates

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}}[\underline{\omega} - \overline{\underline{\omega}}] \lesssim \varepsilon, \quad (10.42)$$

for all  $\gamma > 0$ .

### 10.10.2 Control of $\underline{\iota}$ and $\nabla(\underline{\omega}, * \underline{\omega})$

Applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = -1/2$  to the transport equation (2.80) for  $\underline{\iota}$ ,

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{b}}[r\underline{\iota}] \lesssim \varepsilon, \quad (10.43)$$

and applying the  $L^2(\mathcal{M}^{\text{ext}})$  estimates of Lemma 10.2, we have

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}}[r\underline{\iota}] \lesssim \varepsilon, \quad (10.44)$$

for all  $\gamma > 0$ .

From the elliptic equation (2.69), the elliptic estimates of Lemma 3.38, we deduce from (10.43) and (10.44) and the estimates (10.1) for  $\underline{\beta}$

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{b}}[(r\nabla)\underline{\omega}] \lesssim \varepsilon, \quad \mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{b}}[(r\nabla)^* \underline{\omega}] \lesssim \varepsilon, \quad (10.45)$$

and

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}}[(r\nabla)\underline{\omega}] \lesssim \varepsilon, \quad \mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}}[(r\nabla)^* \underline{\omega}] \lesssim \varepsilon, \quad (10.46)$$

for all  $\gamma > 0$ .

## 10.11 Control of $\underline{\xi}$

We rewrite equation (2.34d) together with the results of Lemma 2.11 as

$$\frac{1}{2} \operatorname{tr} \chi \underline{\xi} = \nabla_3 \zeta + \underline{\beta} + 2 \nabla \omega + 2 \underline{\chi} \cdot \zeta - \hat{\chi} \cdot \underline{\xi}. \quad (10.47)$$

Estimating directly equation (10.47) together with the  $\tilde{H}^{1/2}$  estimates (10.33) for  $\nabla_3 \zeta$ , (10.45) for  $\nabla \omega$ , and the estimates (10.1) for  $\underline{\beta}$ , we have

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{b}} [\underline{\xi}] \lesssim \varepsilon. \quad (10.48)$$

Using the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (10.34), (10.46) and (10.1), we further have

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{b}} [\underline{\xi}] \lesssim \varepsilon, \quad (10.49)$$

for all  $\gamma > 0$ .

## 10.12 Control of $\eta$

### 10.12.1 Control of $\bar{\eta}$

Applying the  $L^\infty$  estimate of Lemma 10.2 with  $\kappa = 0, \lambda = 0$ , to the transport equation

$$\nabla_4 \bar{\eta} = -4 \bar{\omega} + \overline{(\operatorname{tr} \chi - \overline{\operatorname{tr} \chi})} (\eta - \bar{\eta}), \quad (10.50)$$

obtained by taking the average in (2.46) and using formula (2.38), we have

$$\|\bar{\eta}\|_{L^\infty(S_{u, \underline{u}})} \lesssim \int_{\underline{u}}^{\underline{u}^*} \left( \|\bar{\omega}\|_{L^\infty(S_{u, \underline{u}'})} + \|(\operatorname{tr} \chi - \overline{\operatorname{tr} \chi}) (\eta - \bar{\eta})\|_{L^\infty(S_{u, \underline{u}'})} \right) d\underline{u}',$$

where we used that  $\eta = 0$  on  $\underline{\mathcal{C}}^*$ . From the bounds (10.10) for  $\bar{\omega}$  and from the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \|\bar{\omega}\|_{L^\infty(S_{u, \underline{u}'})} d\underline{u}' &\lesssim \varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-2} u^{-2} d\underline{u}' \lesssim \varepsilon \underline{u}^{-1} u^{-2}, \\ \int_{\underline{u}}^{\underline{u}^*} \|(\operatorname{tr} \chi - \overline{\operatorname{tr} \chi}) (\eta - \bar{\eta})\|_{L^\infty(S_{u, \underline{u}'})} d\underline{u}' &\lesssim (D\varepsilon)^2 \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-2} u^{-1/2} u^{-3/2} d\underline{u}' \lesssim \varepsilon \underline{u}^{-1} u^{-2}. \end{aligned}$$

Thus,

$$\|\underline{u} u^2 \bar{\eta}\|_{L^\infty(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon.$$

Commuting equation (10.50) by  $\nabla \in \{r \nabla, u \nabla_3, \underline{u} \nabla_4\}$  and arguing similarly, we obtain

$$\|\underline{u} u^2 \nabla \bar{\eta}\|_{L^\infty(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon. \quad (10.51)$$

Applying the  $L_u^\infty L_u^2 L^2(S_{u, \underline{u}})$  estimate of Lemma 10.2 with  $\kappa = 0, \lambda = -\gamma$  to equation (10.50) multiplied by  $u^{3/2}$ , using Cauchy-Schwartz, the  $L^2(\mathcal{M}^{\text{ext}})$  estimates (10.11) for  $\bar{\omega}$  and the Bootstrap Assumptions 3.23, we have

$$\begin{aligned} \left\| u^{3/2} \underline{u}^{-\gamma} \bar{\eta} \right\|_{L_u^2 L^2(S_{u, \underline{u}})} &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left( \left\| u^{3/2} (\underline{u}')^{-\gamma} \bar{\omega} \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} + \left\| u^{3/2} (\underline{u}')^{-\gamma} (\operatorname{tr} \chi - \overline{\operatorname{tr} \chi}) (\eta - \bar{\eta}) \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} \right) d\underline{u}' \\ &\lesssim \left\| u^{3/2} \underline{u}^{-\gamma/2} u^{1/2} \bar{\omega} \right\|_{L^2(\mathcal{M}^{\text{ext}})} + (D\varepsilon) \left\| \underline{u} u^{-3/2-\gamma/2} (\eta - \bar{\eta}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim_\gamma \varepsilon. \end{aligned}$$

Thus,

$$\left\| \underline{u}^{-1/2-\gamma} u^{3/2} \bar{\eta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon,$$

for all  $\gamma > 0$ . Commuting with  $\nabla^{\leq 2}$  and arguing similarly, we obtain

$$\left\| \underline{u}^{-1/2-\gamma} u^{3/2} \nabla^{\leq 2} \bar{\eta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon. \quad (10.52)$$

To prove the estimates on  $\mathcal{T}$ , we write the  $L^2(\mathcal{T})$  norm of  $\bar{\eta}$  of Section 3.2.4 as

$$\begin{aligned} \int_1^{\tau u^*} u^{3-2\gamma} \bar{\eta}^2 u^2 du &\lesssim \int_1^{\tau u^*} u^{3-2\gamma} \left( \int_{\tau^{-1}u}^{u^*} |\nabla_4 \bar{\eta}| d\underline{u}' \right)^2 u^2 du \\ &\lesssim \int_1^{\tau u^*} \int_{\tau^{-1}u}^{u^*} u^{3-2\gamma} (\underline{u}')^{-2\alpha} |\nabla_4 \bar{\eta}|^2 u^{2\alpha+1} u^2 d\underline{u}' du, \\ &\lesssim \left\| u^{3-\gamma+\alpha} \underline{u}^{-1-\alpha} \nabla_4 \bar{\eta} \right\|_{L^2(\mathcal{M}^{\text{ext}})}^2 \\ &\lesssim \left\| u^{3/2} \underline{u}^{1/2-\gamma} \nabla_4 \bar{\eta} \right\|_{L^2(\mathcal{M}^{\text{ext}})}^2, \end{aligned}$$

where we used that  $\eta = 0$  on  $\mathcal{C}^*$  and where in the second line we chose  $\alpha$  such that  $2\alpha + 1 < 0$  and in the third line we chose  $\alpha$  such that  $\alpha + \frac{3}{2} - \gamma > 0$ .

Using equation (10.50) for  $\nabla_4 \bar{\eta}$ , the  $L^2(\mathcal{M}^{\text{ext}})$  bounds (10.11) for  $\bar{\omega}$  and the Bootstrap Assumptions, we have

$$\begin{aligned} \left\| u^{3/2} \underline{u}^{1/2-\gamma} \nabla_4 \bar{\eta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} &\lesssim \left\| u^{3/2} \underline{u}^{1/2-\gamma} \bar{\omega} \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| u^{3/2} \underline{u}^{1/2-\gamma} (\text{tr}\chi - \overline{\text{tr}\chi})(\eta - \bar{\eta}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim \varepsilon + (D\varepsilon) \left\| u \underline{u}^{-3/2-\gamma} (\eta - \bar{\eta}) \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim \varepsilon + (D\varepsilon)^2 \\ &\lesssim \varepsilon. \end{aligned}$$

Thus,

$$\left\| u^{3/2-\gamma} \bar{\eta} \right\|_{L^2(\mathcal{T})} \lesssim \varepsilon,$$

for all  $\gamma > 0$ . Commuting by  $\nabla^{\leq 2}$  and arguing similarly further gives the desired estimate on  $\mathcal{T}$

$$\left\| u^{3/2-\gamma} \nabla^{\leq 2} \bar{\eta} \right\|_{L^2(\mathcal{T})} \lesssim \varepsilon. \quad (10.53)$$

### 10.12.2 Control of $\nabla \eta$

From formula (2.45), we have  $\nabla \eta = -2\underline{\xi}$ . Compiling the bounds for  $\bar{\eta}$  obtained in Section 10.12.1 and the bounds (10.48) for  $\xi$ , one directly deduces

$$\mathcal{O}_{\leq 1}^{\text{ext}}[\eta] \lesssim \varepsilon, \quad \mathcal{O}_{\leq 2, \gamma}^{\text{ext}}[\eta] \lesssim \varepsilon. \quad (10.54)$$

Arguing as in Section 10.12.1, we have

$$\left\| u^{1-\gamma} \underline{\xi} \right\|_{L^2(\mathcal{T})} \lesssim \left\| u \underline{u}^{1/2-\gamma} \nabla_4 \underline{\xi} \right\|_{L^2(\mathcal{M}^{\text{ext}})},$$

where we used that  $\underline{\xi} = 0$  on  $\mathcal{C}^*$ . Rewriting equation (2.34h), we have the following equation for  $\nabla_4 \underline{\xi}$ .

$$\nabla_4 \underline{\xi} = -\nabla_3 \zeta - \underline{\beta} - 2\underline{\chi} \cdot \zeta. \quad (10.55)$$

Thus,

$$\left\| u^{1-\gamma} \underline{\xi} \right\|_{L^2(\mathcal{T})} \lesssim \left\| u \underline{u}^{1/2-\gamma} \nabla_3 \zeta \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| u \underline{u}^{1/2-\gamma} \underline{\beta} \right\|_{L^2(\mathcal{M}^{\text{ext}})} + \left\| u \underline{u}^{-1/2-\gamma} \underline{\chi} \cdot \zeta \right\|_{L^2(\mathcal{M}^{\text{ext}})}.$$

Using the  $L^2(\mathcal{M}^{\text{ext}})$  bounds (10.34) for  $\nabla_3 \zeta$ , the curvature bounds (10.1) for  $\underline{\beta}$ , the bounds (10.24) for  $\zeta$  and the Bootstrap Assumptions 3.23, one infers

$$\left\| u^{1-\gamma} \underline{\xi} \right\|_{L^2(\mathcal{T})} \lesssim \varepsilon.$$

Commuting equation (10.55) by  $\nabla^{\leq 2}$  and arguing similarly, we finally obtain the desired  $L^2(\mathcal{T})$  bounds for (derivatives of)  $\underline{\xi} = -\frac{1}{2} \nabla \eta$ , and we have

$$\mathcal{O}_{\leq 2, \gamma}^{\mathcal{T}}[\eta] \lesssim \varepsilon, \quad (10.56)$$

for all  $\gamma > 0$ , as desired.

## 10.13 Control of $\mathbf{O}^{\text{ext}}$

### 10.13.1 Mild control of $\mathbf{O}^{\text{ext}}$

We rewrite equation (2.113) under the following form

$$\nabla_4 (r^{-1}\mathbf{O}) = r^{-1}\hat{\chi} \cdot \mathbf{O} + \frac{1}{2}r^{-1}(\text{tr}\chi - \overline{\text{tr}\chi})\mathbf{O}. \quad (10.57)$$

Applying the  $L^\infty$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = 0$ , to the transport equation (10.57), using the improvement of the mild Bootstrap Assumptions for  $\mathbf{O}^{\text{ext}}$  on  $\underline{\mathcal{C}}^* \cap \mathcal{M}^{\text{ext}}$  obtained in Section 9.14.1, we obtain the following improvement of the mild Bootstrap Assumptions 3.16 on the full domain  $\mathcal{M}^{\text{ext}}$

$$\|r^{-1}(r\nabla) \leq 1 \mathbf{O}^{\text{ext}}\|_{L^\infty(\mathcal{M}^{\text{ext}})} \lesssim 1. \quad (10.58)$$

Arguing similarly as in Section 9.14.1, one also obtains the last estimates of the Bootstrap Assumptions 3.16. This finishes the improvement of 3.16 on  $\mathcal{M}^{\text{ext}}$ .

### 10.13.2 Control of $H$ and $\Psi$

Applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = -1/2$  to the transport equation (2.116) for  $H$ , we have

$$\|r^{-1/2}H\|_{\tilde{H}^{1/2}(S_{u,\underline{u}})} \lesssim \|r^{-1/2}H\|_{\tilde{H}^{1/2}(S_{u,\underline{u}^*})} + \int_{\underline{u}}^{\underline{u}^*} \|r^{-1/2}\nabla_3 H\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} d\underline{u}'.$$

From estimates (10.2), we have

$$\|r^{-1/2}H\|_{\tilde{H}^{1/2}(S_{u,\underline{u}^*})} \lesssim \varepsilon.$$

Using the product estimates of Lemma 3.37 and the estimates (10.35), we check that

$$\int_{\underline{u}}^{\underline{u}^*} \|r^{-1/2}\nabla\hat{\chi} \cdot \mathbf{O}\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} d\underline{u}' \lesssim \int_{\underline{u}}^{\underline{u}^*} \|r^{-1/2}(r\nabla)\hat{\chi}\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} d\underline{u}' \lesssim \varepsilon u^{-1/2} \underline{u}^{-1},$$

and

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \|r^{-1/2}\hat{\chi} \cdot \nabla \mathbf{O}\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} d\underline{u}' &\lesssim \varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-2} u^{-1/2} \|r^{-1/2}\nabla \mathbf{O}\|_{\tilde{H}^{1/2}(S_{u,\underline{u}'})} d\underline{u}' \\ &\lesssim \varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-2} u^{-1/2} \left( \|r^{-1}\mathbf{O}\|_{L^\infty(S_{u,\underline{u}'})} + \|\nabla \mathbf{O}\|_{L^\infty(S_{u,\underline{u}'})} + \|\Psi\|_{L^2(S_{u,\underline{u}'})} \right) d\underline{u}' \\ &\lesssim \varepsilon u^{-1/2} \underline{u}^{-1}, \end{aligned}$$

and the control of the other terms composing  $\nabla_3 H$  follows similarly. Thus, we have

$$\|r^{-1/2}u^{1/2}\underline{u}^2 r^{-1}H\|_{L_{u,\underline{u}}^\infty \tilde{H}^{1/2}} \lesssim \varepsilon.$$

Commuting equation (2.116) with  $(\nabla) \leq 2$ , where  $\nabla \in \{(r\nabla), u\nabla_3\}$  and applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2, or directly estimating the  $\nabla_4$  derivatives from equation (2.116) gives similar bounds and we have

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}} [r^{-1}H] \lesssim \varepsilon. \quad (10.59)$$

Applying the  $L_{\underline{u}}^\infty L_u^2 L^2(S_{u,\underline{u}})$  estimates of Lemma 10.2 with  $\kappa = 0, \lambda = -\gamma$  to the transport equation (2.116) for  $H$  gives

$$\|\underline{u}^{-\gamma}H\|_{L_{\underline{u}}^\infty L_u^2 L^2(S_{u,\underline{u}})} \lesssim \|\underline{u}^{-\gamma}H\|_{L^2(\underline{\mathcal{C}}^*)} + \int_{\underline{u}}^{\underline{u}^*} \left( \|\underline{u}^{-\gamma}\nabla_4 H\|_{L_u^2 L^2(S_{u,\underline{u}'})} \right) d\underline{u}'.$$

From estimate (10.2) on  $\underline{\mathcal{C}}^*$ , we have

$$\|\underline{u}^{-\gamma}H\|_{L^2(\underline{\mathcal{C}}^*)} \lesssim \varepsilon.$$

From the  $L^2(\mathcal{M}^{\text{ext}})$  estimate (10.36) for  $\hat{\chi}$  and Cauchy-Schwartz, we have

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \|(\underline{u}')^{-\gamma} \nabla \hat{\chi} \cdot \mathbf{O}\|_{L_u^2 L^2(S_{u, \underline{u}'})} d\underline{u}' &\lesssim \int_{\underline{u}}^{\underline{u}^*} \|(\underline{u}')^{-\gamma} (r \nabla) \hat{\chi}\|_{L_u^2 L^2(S_{u, \underline{u}'})} d\underline{u}' \\ &\lesssim \left\| \underline{u}^{1/2} \underline{u}^{-\gamma/2} (r \nabla) \hat{\chi} \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim \varepsilon. \end{aligned}$$

The other terms follow similarly. Thus, we deduce that

$$\left\| \underline{u}^{-1/2-\gamma} H \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon,$$

for all  $\gamma > 0$ . Commuting by  $(r \nabla)^{\leq 3}$  and arguing similarly, we obtain

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [r^{-1} H] \lesssim \varepsilon, \quad (10.60)$$

for all  $\gamma > 0$ .

The control of  $\Psi$  follows from similar estimates as the ones performed for  $H$ , using the transport equation (2.117) for  $r\Psi$ . Thus, we have

$$\mathfrak{D}_{\leq 0}^{\text{ext}, \mathfrak{g}} [\Psi] \lesssim \varepsilon, \quad \mathcal{O}_{\leq 1, \gamma}^{\text{ext}, \mathfrak{g}} [\Psi] \lesssim \varepsilon, \quad (10.61)$$

for all  $\gamma > 0$ .

### 10.13.3 Control of $Y$

Applying the  $\tilde{H}^{1/2}$  estimates of Lemma 10.2 with  $\kappa = -1, \lambda = -3/2$  to the transport equation (2.115) for  $Y$ , using that  $Y = 0$  on  $\underline{\mathcal{C}}^*$ , we have

$$\left\| r^{-3/2} Y \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} \lesssim \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-3/2} \left( \nabla_4 Y - \frac{1}{2} \text{tr} \chi Y \right) \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}'.$$

From an inspection of the terms in the right-hand side of equation (2.115) the term with slowest decay is  $2(d\check{v}\zeta)\mathbf{O}$ , and we have

$$\begin{aligned} \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-3/2} (d\check{v}\zeta)\mathbf{O} \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}' &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1/2} \nabla \zeta \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}'})} d\underline{u}' \\ &\lesssim \varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-3} \underline{u}^{-1/2} d\underline{u}' \\ &\lesssim \varepsilon \underline{u}^{-2} \underline{u}^{-1/2}, \end{aligned}$$

where we used the product estimates of Lemma 3.37, the estimates (10.23) for  $\zeta$  and (10.58) for  $\mathbf{O}$ . Commuting with  $(r \nabla), (q \nabla_3)$  or estimating directly equation (2.115) and arguing similarly, we obtain

$$\mathfrak{D}_{\leq 1}^{\text{ext}, \mathfrak{g}} [r^{-1} Y] \lesssim \varepsilon.$$

Applying the  $L_u^\infty L_u^2 L^2(S_{u, \underline{u}})$  estimates of Lemma 10.2 with  $\kappa = -1, \lambda = -\gamma$  to the transport equation (2.115), using that  $Y = 0$  on  $\underline{\mathcal{C}}^*$ , we have

$$\begin{aligned} \left\| r^{-1-\gamma} \underline{u} Y \right\|_{L_u^2 L^2(S_{u, \underline{u}})} &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1-\gamma} \underline{u}' d\check{v}\zeta \mathbf{O} \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} d\underline{u}' \\ &\lesssim \int_{\underline{u}}^{\underline{u}^*} \left\| r^{-1-\gamma} \underline{u}' (r \nabla) \zeta \right\|_{L_u^2 L^2(S_{u, \underline{u}'})} d\underline{u}' \\ &\lesssim \left\| r^{-1/2-\gamma/2} \underline{u} (r \nabla) \zeta \right\|_{L^2(\mathcal{M}^{\text{ext}})} \\ &\lesssim \varepsilon, \end{aligned}$$

where we only considered the slowest decaying term  $2(\operatorname{div}\zeta)\mathbf{O}$  of the right-hand side of (2.115). Therefore, we obtain

$$\left\| \underline{u}^{-1/2-\gamma} \underline{u}Y \right\|_{L^2(\mathcal{M}^{\text{ext}})} \lesssim \varepsilon,$$

for all  $\gamma > 0$ . Commuting with  $\nabla^{\leq 2}$  and arguing similarly, we have

$$\mathcal{O}_{\leq 2, \gamma}^{\text{ext}, \mathfrak{g}} [r^{-1}Y] \lesssim \varepsilon,$$

for all  $\gamma > 0$ . This finishes the proof of (10.3).

## 10.14 Control of the area radius

**Lemma 10.4.** *Under the improved estimates obtained in this section, we have in  $\mathcal{M}^{\text{ext}}$*

$$\left| r(u, \underline{u}) - \frac{1}{2}(\underline{u} - u) \right| \lesssim \varepsilon \underline{u}^{-1} u^{-1}.$$

*Proof.* Using relations (2.1) and (2.39), we have

$$e_4 \left( r(u, \underline{u}) - \frac{1}{2}(\underline{u} - u) \right) = \frac{1}{2} r \overline{\operatorname{tr}\chi} - 1.$$

Integrating in  $\underline{u}$ , using the improved estimate for  $r$  on  $\underline{\mathcal{C}}^*$  from Section 9.15 and using the improved estimate (10.12) for  $\overline{\operatorname{tr}\chi}$ , we have

$$\begin{aligned} \left| r(u, \underline{u}) - \frac{1}{2}(\underline{u} - u) \right| &\lesssim \left| r(u, \underline{u}^*) - \frac{1}{2}(\underline{u}^* - u) \right| + \int_{\underline{u}}^{\underline{u}^*} r \left| \overline{\operatorname{tr}\chi} - \frac{2}{r} \right| d\underline{u}' \\ &\lesssim \varepsilon \underline{u}^{*-1} u^{-1} + \varepsilon \int_{\underline{u}}^{\underline{u}^*} r(\underline{u}')^{-3} u^{-1} d\underline{u}' \\ &\lesssim \varepsilon \underline{u}^{-1} u^{-1}, \end{aligned}$$

as desired.  $\square$

## 10.15 Control of the spherical coordinates in $\mathcal{M}^{\text{ext}}$

For a spherical coordinate system as described in the Bootstrap Assumption 3.26, we have using (2.34a), the following transport equation in the  $e_4$  direction

$$e_4(\mathfrak{g}_{ab}) = 2\chi_{ab} = \overline{\operatorname{tr}\chi} \mathfrak{g}_{ab} + (\operatorname{tr}\chi - \overline{\operatorname{tr}\chi}) \mathfrak{g}_{ab} + 2\hat{\chi}_{ab},$$

where  $a, b \in \{\vartheta, \varphi\}$ , which rewrites using (2.39) and the notation of the Bootstrap Assumption 3.26

$$e_4(r^{-2} \mathfrak{g}_{ab} - (\mathfrak{g}_{\mathbb{S}})_{ab}) = (\operatorname{tr}\chi - \overline{\operatorname{tr}\chi}) r^{-2} \mathfrak{g}_{ab} + 2r^{-2} \hat{\chi}_{ab}. \quad (10.62)$$

Integrating along  $e_4$ , this yields

$$\begin{aligned} |r^{-2} \mathfrak{g}_{ab} - (\mathfrak{g}_{\mathbb{S}})_{ab}|(u, \underline{u}, \vartheta, \varphi) &\lesssim |r^{-2} \mathfrak{g}_{ab} - (\mathfrak{g}_{\mathbb{S}})_{ab}|(u, \underline{u}^*, \vartheta, \varphi) + \int_{\underline{u}}^{\underline{u}^*} (|(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi}) r^{-2} \mathfrak{g}_{ab}| + 2r^{-2} |\hat{\chi}_{ab}|) d\underline{u}' \\ &\lesssim \varepsilon + \varepsilon \int_{\underline{u}}^{\underline{u}^*} (\underline{u}')^{-2} u^{-1/2} d\underline{u}' \\ &\lesssim \varepsilon \underline{u}^{-1} u^{-1/2}, \end{aligned}$$

where we used the improved bounds for the metric in coordinates on  $\underline{\mathcal{C}}^*$  from Section 9.16, the improved bounds (10.18), (10.35) for  $\operatorname{tr}\chi$  and  $\hat{\chi}$  and the Bootstrap Assumptions 3.26 for the coordinate component of  $\mathfrak{g}$ .

Commuting (10.62) with  $\partial$  and integrating using the sup-norm bounds for  $(r\nabla)(\operatorname{tr}\chi - \overline{\operatorname{tr}\chi})$  and  $(r\nabla)\hat{\chi}$ , we further have

$$|\partial^{\leq 1} (r^{-2} \mathfrak{g}_{ab} - (\mathfrak{g}_{\mathbb{S}})_{ab})| \lesssim \varepsilon \underline{u}^{-1} u^{-1/2}.$$

This finishes the improvement of the Bootstrap Assumptions 3.26.





# 11. Maximal connection estimates in $\mathcal{M}_{\text{bot}}^{\text{int}}$

In this section, we prove the following proposition.

**Proposition 11.1.** *Recall that from Proposition 7.1 we have the following curvature control on the maximal hypersurfaces  $\Sigma_t$  (see the definitions of Section 3.2.2)*

$$\mathcal{R}_{\leq 2}^{\text{int}} + \mathfrak{R}_{\leq 1}^{\text{int}} \lesssim \varepsilon. \quad (11.1)$$

Recall that from Proposition 10.1 we have the following control of the null connection coefficients in  $\mathcal{M}^{\text{ext}}$  – and in particular on the interface  $\mathcal{T}$  –

$$\mathfrak{D}_{\leq 1}^{\text{ext}} + \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}} + \mathfrak{D}_{\leq 1}^{\text{ext}}[\mathfrak{h}] + \mathcal{O}_{2,\gamma}^{\mathcal{T}}[\mathfrak{h}] \lesssim \varepsilon, \quad (11.2)$$

for all  $\gamma > 0$  (see the definitions from Section 3.2.4). Recall that from Proposition 9.1, we have on  $\underline{\mathcal{C}}^*$  (see the definitions of Section 3.2.3)

$$\mathfrak{D}_{\leq 2}^* \lesssim \varepsilon. \quad (11.3)$$

Recall that from Proposition 10.1 we have the following control of the exterior rotation vectorfields in  $\mathcal{M}^{\text{ext}}$  – and in particular on the interface  $\mathcal{T}$  – (see the definitions from Section 3.2.4)

$$\mathfrak{D}_{\leq 2}^{\text{ext}, \mathbf{O}} \lesssim \varepsilon. \quad (11.4)$$

Under the Bootstrap Assumptions and estimates (11.1), (11.2), we have for  $\varepsilon > 0$  sufficiently small the following control on the connection coefficients of the maximal hypersurfaces (see Section 3.2.5 for definitions)

$$\mathcal{O}_{\leq 3,\gamma}^{\text{int}}[n] + \mathcal{O}_{\leq 2}^{\text{int}}[k] + \mathfrak{D}_{\leq 2}^{\mathcal{T}}[\nu] \lesssim_{\gamma} \varepsilon, \quad (11.5)$$

for all  $\gamma > 0$ . Moreover, under the same hypothesis and (11.3), we have the following harmonic coordinates control on the last maximal slice  $\Sigma_{t^*}$

$$\begin{aligned} \sum_{i,j=1}^3 \left\| t^{3/2} (g(\nabla x^i, \nabla x^j) - \delta_{ij}) \right\|_{L^\infty(\Sigma_{t^*})} + \sum_{i=1}^3 \left\| (t\nabla)^{\leq 3} \nabla^2 x^i \right\|_{L^2(\Sigma_{t^*})} &\lesssim \varepsilon, \\ \sum_{i=1}^3 \left\| t (r^* \overline{N}(x^i) - x^i) \right\|_{L^\infty(S^*)} &\lesssim \varepsilon. \end{aligned} \quad (11.6)$$

Under the same hypothesis, the respectively mild and strong Bootstrap Assumptions 3.18 and 3.28 for the interior approximate conformal Killing fields  $\mathbf{T}^{\text{int}}, \mathbf{S}^{\text{int}}, \mathbf{K}^{\text{int}}, \mathbf{O}^{\text{int}}$  are improved. Under the same hypothesis and using (11.4), the Bootstrap Assumptions 3.29 on the difference of the interior and exterior approximate Killing fields on  $\mathcal{T}$  are improved.

**Remark 11.2.** *We do not control all 3-derivatives of  $n$ , which is a far-reaching consequence of the lack of regularity for  $\underline{\xi}$ . However, we do control all 2-derivatives of  $\nabla n$ , which is enough to control  $\mathbf{D}^{(\mathbf{T}^{\text{int}})}\pi$  and  $\mathbf{D}^{2\mathbf{T}^{\text{int}}}\pi$  in the error term estimates of Section 5 and in the curvature estimates of Section 7, where it is used.*

## 11.1 Elliptic estimates

In Section 11.2, we will need the following elliptic estimates for the Laplace equation on  $\Sigma_t$ .

**Lemma 11.3** (Elliptic estimate for Laplace equation on  $\Sigma_t$ ). *Under the Bootstrap Assumptions 3.17, we have for all  $1 \leq t \leq t^*$  and for all scalar function  $f$*

$$\|(t\nabla)f\|_{L^2(\Sigma_t)} + \|f\|_{L^2(\Sigma_t)} \lesssim \|t^2 \Delta f\|_{L^2(\Sigma_t)} + \|tf\|_{\tilde{H}^{1/2}(\partial\Sigma_t)},$$

and for all  $k \geq 2$

$$\|(t\nabla)^{\leq k} f\|_{L^2(\Sigma_t)} \lesssim \|t^2 (t\nabla)^{k-2} (\Delta f)\|_{L^2(\Sigma_t)} + \|t(t\nabla)^{\leq k-1} f\|_{\tilde{H}^{1/2}(\partial\Sigma_t)}.$$

*Proof.* The proof follows from a rescaling in  $t$  and the results from [Czi19a] (see also [CG19b]).  $\square$

## 11.2 Control of the second fundamental form $k$ and lapse $n$

### 11.2.1 Control of $k, \nabla k$ on $\Sigma_t$

In this section, we use energy estimates for the elliptic div-curl systems (2.11) satisfied by  $k$  on  $\Sigma_t$ . The source term for this elliptic equation is  $H$  and the boundary conditions on  $\partial\Sigma_t$  are given by mixed implicit Dirichlet and Neumann conditions for  $k$  (see Lemma 2.6). This estimate has already been done in [CG19b, Sections 4.7 and 4.8], where, rescaling in  $t$  these estimates, it holds that

$$\begin{aligned} & \|t(t\nabla)k\|_{L^2(\Sigma_t)} + \|tk\|_{L^2(\Sigma_t)} + t^{3/2}\|k\|_{L^2(\partial\Sigma_t)} \\ & \lesssim \|t^2H\|_{L^2(\Sigma_t)} + \left\| t^2 \left( \text{tr}\chi - \frac{2}{r} \right) \right\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \left\| t^2 \left( \text{tr}\underline{\chi} + \frac{2}{r} \right) \right\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \|t^2\zeta\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \text{error terms.} \end{aligned}$$

Using the  $L^2(\Sigma_t)$  bounds (11.1) for  $H$  and the  $\tilde{H}^{1/2}(\partial\Sigma_t)$  bounds (11.2) for  $\text{tr}\chi - \frac{2}{r}, \text{tr}\underline{\chi} + \frac{2}{r}, \zeta$ , we thus obtain

$$\|t(t\nabla)^{\leq 1}k\|_{L^2(\Sigma_t)} \lesssim \varepsilon.$$

As a byproduct of these estimates, it is shown in [CG19b, Section 4] that we have the following control for  $\nu$

$$\|t(t\nabla)^{\leq 1}(\nu - 1)\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon.$$

**Remark 11.4.** *The  $\tilde{H}^{1/2}(\partial\Sigma_t)$  control of  $\nu$  uses the first two relations of (2.18). See [CG19b, Lemma 4.9] for the full argument.*

### 11.2.2 Control of $\nabla^2 k$ on $\Sigma_t$

To estimate  $\nabla^2 k$  on  $\Sigma_t$ , one cannot use the higher regularity elliptic estimates from Lemma 7.6 since the boundary terms for  $k$  are only implicitly related to the null connection coefficients on  $\partial\Sigma_t$ . We therefore have to show that commuting the elliptic equations (2.11) by a derivative does not change the coercive structure of the boundary terms which appeared in the original energy estimate. This analysis has already been carried out in [CG19b, Sections 5.1.3 and 5.2.3], and we refer the reader to that paper.

We therefore shall assume that we have the following elliptic estimate on  $\Sigma_t$

$$\begin{aligned} & \|t^3\nabla^2 k\|_{L^2(\Sigma_t)} + \|t^{5/2}\nabla k\|_{L^2(\partial\Sigma_t)} \\ & \lesssim \|t^3\nabla H\|_{L^2(\Sigma_t)} + \|t^2\nabla k\|_{L^2(\Sigma_t)} + \|tk\|_{L^2(\Sigma_t)} + \|t^3\nabla \text{tr}\chi\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \|t^3\nabla \text{tr}\underline{\chi}\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \\ & \quad + \|t^3\nabla \zeta\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \left\| t^2 \left( \text{tr}\chi - \frac{2}{r} \right) \right\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \left\| t^2 \left( \text{tr}\underline{\chi} + \frac{2}{r} \right) \right\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \|t^2\zeta\|_{\tilde{H}^{1/2}(\partial\Sigma_t)}. \end{aligned}$$

Using the  $L^2(\Sigma_t)$  bounds (11.1) for  $H$  and the  $\tilde{H}^{1/2}(\partial\Sigma_t)$  bounds (11.2) for  $\text{tr}\chi - \frac{2}{r}, \text{tr}\underline{\chi} + \frac{2}{r}, \zeta$ , we thus obtain

$$\|t^3\nabla^2 k\|_{L^2(\Sigma_t)} \lesssim \varepsilon. \quad (11.7)$$

As a byproduct of these estimates, one further has the following improved estimates for the tangential derivatives of  $\nu$

$$\|t(t\nabla)^{\leq 2}(\nu - 1)\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon. \quad (11.8)$$

### 11.2.3 Optimal control for $\nabla^3 k$ on the last hypersurface $\Sigma_{t^*}$

On the boundary of the last hypersurface  $\partial\Sigma_{t^*} = S^* \subset \mathcal{C}^*$ , we have from  $\mathfrak{D}_{\leq 2}^* \lesssim \varepsilon$  (see the definitions in Section 3.2.3 and the improvement in Section 9) the following additional bounds

$$\left\| t^2(t\nabla)^2 \left( \text{tr}\chi - \frac{2}{r} \right) \right\|_{\tilde{H}^{1/2}(S^*)} + \left\| t^2(t\nabla)^2 \left( \text{tr}\underline{\chi} + \frac{2}{r} \right) \right\|_{\tilde{H}^{1/2}(S^*)} + \|t^2(t\nabla)^2\zeta\|_{\tilde{H}^{1/2}(S^*)} \lesssim \varepsilon. \quad (11.9)$$

From the elliptic estimates of [CG19b, Sections 5.1.3 and 5.2.3], we have on the last slice  $\Sigma_{t^*}$

$$\begin{aligned} & \left\| t^4 \nabla^3 k \right\|_{L^2(\Sigma_{t^*})} + \left\| t^{7/2} \nabla^2 k \right\|_{L^2(\partial\Sigma_{t^*})} \\ & \lesssim \left\| t^4 \nabla^2 H \right\|_{L^2(\Sigma_{t^*})} + \left\| t^3 \nabla^2 k \right\|_{L^2(\Sigma_{t^*})} + \left\| t^2 \nabla k \right\|_{L^2(\Sigma_{t^*})} + \|tk\|_{L^2(\Sigma_{t^*})} \\ & \quad + \left\| t^4 \nabla^2 \text{tr}\chi \right\|_{\dot{H}^{1/2}(S^*)} + \left\| t^4 \nabla^2 \text{tr}\underline{\chi} \right\|_{\dot{H}^{1/2}(S^*)} + \left\| t^4 \nabla^2 \zeta \right\|_{\dot{H}^{1/2}(S^*)} \\ & \quad + \left\| t^3 \nabla \text{tr}\chi \right\|_{\dot{H}^{1/2}(S^*)} + \left\| t^3 \nabla \text{tr}\underline{\chi} \right\|_{\dot{H}^{1/2}(S^*)} + \left\| t^3 \nabla \zeta \right\|_{\dot{H}^{1/2}(S^*)} \\ & \quad + \left\| t^2 \left( \text{tr}\chi - \frac{2}{r} \right) \right\|_{\dot{H}^{1/2}(S^*)} + \left\| t^2 \left( \text{tr}\underline{\chi} + \frac{2}{r} \right) \right\|_{\dot{H}^{1/2}(S^*)} + \left\| t^2 \zeta \right\|_{\dot{H}^{1/2}(S^*)}. \end{aligned}$$

Thus, using the curvature estimates (11.1) and the additional bounds (11.9), we obtain

$$\left\| t^4 \nabla^3 k \right\|_{L^2(\Sigma_{t^*})} \lesssim \varepsilon.$$

**Remark 11.5.** *The optimal control in regularity for  $k$ ,  $\nabla^3 k \in L^2$ , can only be achieved provided that  $\text{tr}\chi$ ,  $\text{tr}\underline{\chi}$  and  $\zeta$  have optimal regularity on the boundary. The optimal regularity for these coefficients is only obtained on the last cone  $\underline{\mathcal{C}}^*$  and is a consequence of the choice of the canonical foliation (see Section 9). The optimal regularity for  $\underline{\chi}$  in  $\mathcal{M}^{\text{ext}} \setminus \underline{\mathcal{C}}^*$  cannot be obtained due to the classical loss of regularity for the geodesic foliation on the cones  $\mathcal{C}_u$ . In the present paper, this is not an issue since the optimal bounds for  $k$  are only needed on  $\Sigma_{t^*}$  to carry out the extension procedure of  $\Sigma_{t^*}$  in Section 8.1.*

#### 11.2.4 Control of $\nabla^{\leq 3} n$ on $\Sigma_t$

From relation (2.19), we have

$$n - 1 = \frac{\tau}{\tau + 1} \left( (\nu - 1) + \tau^{-1}(\nu^{-1} - 1) - \frac{1}{2} \eta \nu^{-1} \right). \quad (11.10)$$

From the assumption (11.2) (see the definitions of Section 3.2.4 and the improvements of Section 10), and from the just obtained (11.8), we have

$$\begin{aligned} & \left\| t(t\nabla)^{\leq 2} \eta \right\|_{\dot{H}^{1/2}(\partial\Sigma_t)} \lesssim \mathfrak{D}_{\leq 1}^{\text{ext}}[\eta] \lesssim \varepsilon, \\ & \left\| t(t\nabla)^{\leq 2} (\nu - 1) \right\|_{\dot{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon. \end{aligned}$$

Thus, using relation (11.10) and the product estimates of Lemma 3.37, we have

$$\left\| t(t\nabla)^{\leq 2} (n - 1) \right\|_{\dot{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon. \quad (11.11)$$

Applying the elliptic estimates of Lemma 11.3, using Laplace equation (2.12) for  $n$ , the Bootstrap Assumptions 3.24 and the boundary estimate (11.11), we therefore deduce

$$\left\| (t\nabla)^{\leq 3} (n - 1) \right\|_{L^2(\Sigma_t)} \lesssim \varepsilon, \quad (11.12)$$

as desired.

#### 11.2.5 Control of $\hat{\mathcal{L}}_{\bar{T}} k$ , $\nabla \hat{\mathcal{L}}_{\bar{T}} k$ on $\Sigma_t$

The improved estimates for  $\hat{\mathcal{L}}_{\bar{T}} k$ ,  $\nabla \hat{\mathcal{L}}_{\bar{T}} k$  on  $\Sigma_t$  are directly obtained using equation (2.13) for  $\hat{\mathcal{L}}_{\bar{T}} k$  together with the assumed estimates (11.1) for the curvature and the just improved estimates (11.12) for  $\nabla^{\leq 3} n$  from Section 11.2.4, and we have

$$\left\| t^2 (t\nabla)^{\leq 1} \hat{\mathcal{L}}_{\bar{T}} k \right\|_{L^2(\Sigma_t)} \lesssim \varepsilon. \quad (11.13)$$

#### 11.2.6 Control of $\nabla^{\leq 2} (\bar{T}(n))$ on $\Sigma_t$

On the boundary  $\partial\Sigma_t$ , we have

$$Z = \left( a^{1/2} \nu^{-1} + a^{-1/2} \nu \right) \bar{T} + \left( a^{1/2} \nu^{-1} - a^{-1/2} \nu \right) \bar{N},$$

where  $Z$  is the future-pointing unit normal to  $\partial\Sigma_t$  in  $\mathcal{T}$ , defined in the proof of Lemma 2.7, where  $a = \tau^{-1} - \frac{1}{2}\eta$  and where we recall that  $\bar{N}$  is the outward-pointing unit normal to  $\partial\Sigma_t$  in  $\Sigma_t$ .

We therefore have

$$\left(a^{1/2}\nu^{-1} + a^{-1/2}\nu\right)\bar{T}(n) = Z(n) - \left(a^{1/2}\nu^{-1} - a^{-1/2}\nu\right)\bar{N}(n). \quad (11.14)$$

From a trace estimate, using the improved estimates from Section 11.2.4, the second term and its tangent derivative are controlled in  $H^{1/2}$ .

Using relation (2.19) on  $\mathcal{T}$ , we have

$$Z(n) = Z\left(\frac{\tau}{\tau+1}\left(\nu + \left(\tau^{-1} - \frac{1}{2}\eta\right)\nu^{-1}\right)\right),$$

and  $\nabla^{\leq 1}Z(n)$  is schematically composed of

$$\nabla^{\leq 1}(\nabla_3, \nabla_4)\eta, \quad \nabla^{\leq 1}Z(\nu).$$

From the assumption (11.2) (see the definitions of Section 3.2.4 and the improvements of Section 10), we have

$$\|t(\nabla^{\leq 1}(\nabla_3)\eta)\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} + \|t(\nabla^{\leq 1}(\nabla_4)\eta)\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \mathfrak{D}_{\leq 1}^{\text{ext}}[\eta] \lesssim \varepsilon. \quad (11.15)$$

Using relation (2.18) for  $\text{tr}\kappa$ , we have on  $\mathcal{T}$

$$\nu^{-1} - \nu = -r\delta + r\left(\text{tr}\underline{\chi} + \frac{2}{r}\right) + r\left(\text{tr}\chi - \frac{2}{r}\right) + \text{Err},$$

which derived in  $Z$  gives

$$Z(\nu) = \frac{1}{2}Z(r\delta) - \frac{1}{2}Z\left(r\left(\text{tr}\underline{\chi} + \frac{2}{r}\right)\right) - \frac{1}{2}Z\left(r\left(\text{tr}\chi - \frac{2}{r}\right)\right) + \text{Err}.$$

Using the improved estimates (11.7), (11.13) for respectively  $\nabla^{\leq 2}k$  and  $\nabla^{\leq 1}\hat{\mathcal{L}}_{\bar{T}}k$  and trace estimates, one deduces from the above formula

$$\|t^2Z(\nu)\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon. \quad (11.16)$$

Using relations (2.18) for  $\epsilon$ , and the improved estimates (11.7), (11.13) for  $k$  and trace estimates, we have

$$\|t^2(\nabla^{\leq 1}Z(\nu))\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon. \quad (11.17)$$

Thus, combining (11.15), (11.16), (11.17) and equation (11.14), we obtain

$$\|t(\nabla^{\leq 1}\bar{T}(n))\|_{\tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon.$$

Applying the elliptic estimate of Lemma 11.3, commuting Laplace equation (2.12) with  $\bar{T}$  and using the above boundary estimate, we obtain

$$\|t(\nabla^{\leq 2}\bar{T}(n))\|_{L^2(\Sigma_t)} \lesssim \varepsilon, \quad (11.18)$$

as desired.

### 11.2.7 Control of $\hat{\mathcal{L}}_{\bar{T}}^2k$ on $\Sigma_t$

The improved estimates for  $\hat{\mathcal{L}}_{\bar{T}}^2k$  are obtained directly using equation (2.13), the estimates just obtained in Section 11.2.6 for  $\nabla^{\leq 2}\bar{T}(n)$  and the improved estimates for the curvature, and we have

$$\|t^3\hat{\mathcal{L}}_{\bar{T}}^2k\|_{L^2(\Sigma_t)} \lesssim \varepsilon. \quad (11.19)$$

### 11.2.8 Control of $\nabla^{\leq 1} \bar{T}^2(n)$ on $\Sigma_t$

Arguing as in Section 11.2.6, our goal is to obtain boundary estimates for  $\bar{T}^2(n)$  on  $\partial\Sigma_t$ . As in (11.14), one can obtain schematically obtain a formula

$$\bar{T}^2(n) = Z^2(n) + \bar{N}^2(n) + \bar{NT}(n) + \text{lower order terms.} \quad (11.20)$$

The term  $Z^2(n)$  is schematically composed of

$$Z^{\leq 2}(\eta) \qquad Z^{\leq 2}(\nu).$$

Using the embeddings of Lemma 3.36, we have the following  $L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)$  bounds for  $Z^2(\eta)$

$$\left\| t^{1/2-\gamma} t^2 Z^2(\eta) \right\|_{L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \left\| t^{-\gamma} t^2 (t\nabla)^{\leq 1} Z^2(\eta) \right\|_{L_t^2 L^2(\partial\Sigma_t)} \lesssim \mathcal{O}_{\leq 2, \gamma}^{\mathcal{T}}[\eta] \lesssim \varepsilon,$$

for all  $\gamma > 0$ .

As in Section 11.2.6,  $\tilde{H}^{1/2}$  estimates for  $\nu$  can be obtained from trace estimates and the improved estimates (11.7), (11.13), (11.19) for  $k$ , and we have

$$\left\| t^3 Z^2(\nu) \right\|_{L_t^\infty \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon.$$

Using the above estimate, we deduce the following  $L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)$  control

$$\left\| t^{1/2-\gamma} t^2 Z^2(\nu) \right\|_{L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \left\| t^3 Z^2(\nu) \right\|_{L_t^\infty \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon,$$

for all  $\gamma > 0$ .

For the lower order terms and the terms  $\bar{N}^2(n)$  and  $\bar{NT}(n)$ , one can obtain  $L_t^\infty \tilde{H}^{1/2}(\partial\Sigma_t)$  bounds, using trace estimates and the previous bounds (11.12), (11.18), and these bounds can similarly be turned into weaker  $L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)$  bounds. Combining these estimates, we obtain the following boundary control

$$\left\| t^{1/2-\gamma} t^2 \bar{T}^2(n) \right\|_{L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon. \quad (11.21)$$

Applying the elliptic estimates of Lemma 11.3, using that by (2.12)  $\Delta(n)$  is composed of lower order terms, we obtain on each separate slice  $\Sigma_t$

$$\left\| t^{-1/2-\gamma} t^2 (t\nabla)^{\leq 1} \bar{T}^2(n) \right\|_{L^2(\Sigma_t)} \lesssim \left\| t^{1/2-\gamma} t^2 \bar{T}^2(n) \right\|_{\tilde{H}^{1/2}(\partial\Sigma_t)}.$$

Taking the  $L_t^2$  norm in the above estimate and using the boundary bound (11.21), we obtain

$$\left\| t^{-1/2-\gamma} t^2 (t\nabla)^{\leq 1} \bar{T}^2(n) \right\|_{L_t^2 L^2(\Sigma_t)} \lesssim \left\| t^{1/2-\gamma} t^2 \bar{T}^2(n) \right\|_{L_t^2 \tilde{H}^{1/2}(\partial\Sigma_t)} \lesssim \varepsilon.$$

This finishes the proof of (11.5).

### 11.2.9 Control of $\mathbf{DT}^{\text{int}}$

From (11.5), the formula (2.30) and the Sobolev embeddings of Lemma 7.3, we directly deduce the following control for  $\mathbf{DT}^{\text{int}}$  and in particular for the deformation tensor  $(\mathbf{T}^{\text{int}})^{\hat{\pi}}$  of the interior approximate Killing field  $\mathbf{T}^{\text{int}}$

$$\begin{aligned} & \left\| t^{5/2} \mathbf{DT}^{\text{int}} \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon, \\ & \left\| t^2 (t\mathbf{D})^{\leq 1} \mathbf{DT}^{\text{int}} \right\|_{L_t^\infty L^6(\Sigma_t)} + \left\| t^2 (t\mathbf{D})^{\leq 1} \mathbf{DT}^{\text{int}} \right\|_{L_t^\infty L^4(\partial\Sigma_t)} \lesssim \varepsilon, \\ & \left\| t^{-1/2-\gamma} t (t\mathbf{D})^{\leq 2} \mathbf{DT}^{\text{int}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon, \end{aligned} \quad (11.22)$$

for all  $\gamma > 0$ .

### 11.3 Control of the harmonic Cartesian coordinates on $\Sigma_{t^*}$

From Gauss equation (2.10) on  $\Sigma_{t^*}$ , the curvature bounds (11.1) and the Bootstrap Assumptions 3.24, we have

$$\|t^2(t\nabla)^{\leq 2}\text{Ric}\|_{L^2(\Sigma_{t^*})} \lesssim \varepsilon. \quad (11.23)$$

From relation (2.16) for the unit normal  $\bar{N}$  to  $\partial\Sigma_{t^*}$ , the assumed bounds (11.3) for  $\chi$  and  $\underline{\chi}$  on  $S^* \subset \mathcal{C}^*$ , the improved bounds (11.8) for  $\nu$  on  $\partial\Sigma_{t^*}$ , we have

$$\left\| t^2(t\nabla)^{\leq 2} \left( \text{tr}\theta - \frac{2}{r^*} \right) \right\|_{\tilde{H}^{1/2}(S^*)} + \left\| t^2(t\nabla)^{\leq 2}\theta \right\|_{\tilde{H}^{1/2}(S^*)} \lesssim \varepsilon. \quad (11.24)$$

Thus, applying the (rescaled) results of Theorem 4.3, we deduce that (for all the centred conformal isomorphisms of  $S^*$ ), estimate (11.6) holds for the harmonic Cartesian coordinates of  $\Sigma_{t^*}$ .

### 11.4 Control of the interior Killing fields $\mathbf{T}^{\text{int}}$ , $\mathbf{S}^{\text{int}}$ , $\mathbf{K}^{\text{int}}$ and $\mathbf{O}^{\text{int}}$ in $\mathcal{M}_{\text{bot}}^{\text{int}}$

In Section 11.4.3, we improve the mild Bootstrap Assumptions 3.18 on the interior approximate Killing vectorfields  $\mathbf{T}^{\text{int}}$ ,  $\mathbf{S}^{\text{int}}$ ,  $\mathbf{K}^{\text{int}}$  and on the vectorfield  $\mathbf{X}^{\text{int}}$ . In Section 11.4.2, we improve the control (3.6) for  $\mathbf{X}^{\text{int}}$  from the Bootstrap Assumptions 3.28. In Section 11.4.1, we improve the bounds (3.5) and (3.7) on (derivatives of) the deformation tensors, which finishes the improvement of the Bootstrap Assumptions 3.28.

#### 11.4.1 Control of $\mathbf{DX}^{\text{int}}$ , $\mathbf{DS}^{\text{int}}$ , $\mathbf{DK}^{\text{int}}$ and $\mathbf{DO}^{\text{int}}$

**Lemma 11.6.** *The following bounds hold on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$\begin{aligned} \left\| t^{3/2} (\mathbf{DX}^{\text{int}} - g) \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim \varepsilon, \\ \left\| t(\mathbf{D})^{\leq 1} (\mathbf{DX}^{\text{int}} - g) \right\|_{L_t^\infty L^6(\Sigma_t)} &\lesssim \varepsilon, \\ \left\| t^{-1/2-\gamma} t(\mathbf{D})^{\leq 2} (\mathbf{DX}^{\text{int}} - g) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim \varepsilon, \end{aligned} \quad (11.25)$$

for all  $\gamma > 0$ . We also have the following bounds on  $\partial\Sigma_t$

$$\left\| t^{-\gamma} t(\mathbf{D})^{\leq 1} (\mathbf{DX}^{\text{int}} - g) \right\|_{L_t^\infty L^4(\partial\Sigma_t)}, \quad (11.26)$$

for all  $\gamma > 0$ . Moreover, we have the following control on  $\mathcal{M}_{\text{bot}}^{\text{int}}$

$$|(t\mathbf{D})^{\leq 1} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}})| \lesssim \varepsilon t^{-1/2}. \quad (11.27)$$

*Proof.* From the definition (2.23) on  $\Sigma_{t^*}$ ,  $\mathbf{X}^{\text{int}}$  is an  $\Sigma_{t^*}$  tangent vectorfield, such that

$$\begin{aligned} \nabla_k \mathbf{X}^{\text{int}}_l &= \sum_{i=1}^3 \nabla_k x^i \nabla_l x^i + \sum_{i=1}^3 x^i \nabla_{k,l}^2 x^i \\ &= g_{kl} + (\delta_{kl} - g_{kl}) + \sum_{i=1}^3 x^i \nabla_{k,l}^2 x^i. \end{aligned}$$

and from the bound (11.6), we deduce

$$\begin{aligned} \left\| (t\nabla)^{\leq 2} (\nabla \mathbf{X}^{\text{int}} - g) \right\|_{L^2(\Sigma_{t^*})} &\lesssim \left\| (t\nabla)^{\leq 2} (\delta_{kl} - g_{kl}) \right\|_{L^2(\Sigma_{t^*})} + \left\| t(t\nabla)^{\leq 2} \nabla^2 x^i \right\|_{L^2(\mathbf{S}^{\text{int}}_{t^*})} \\ &\lesssim \varepsilon. \end{aligned} \quad (11.28)$$

From the definition (2.24), we have schematically

$$\mathbf{DX}^{\text{int}} = \nabla \mathbf{X}^{\text{int}} + \mathbf{X}^{\text{int}} \cdot k,$$

thus, we deduce from (11.28), the bounds (11.5) for  $k$  and the mild Bootstrap Assumptions 3.18 and 3.28 for  $\mathbf{X}^{\text{int}}$  that

$$\left\| (t\mathbf{D})^{\leq 2} (\mathbf{DX}^{\text{int}} - g) \right\|_{L^2(\Sigma_{t^*})} \lesssim \varepsilon. \quad (11.29)$$

Commuting the transport equation (2.24) for  $\mathbf{X}^{\text{int}}$ , we have

$$\mathbf{D}_{\bar{T}} \mathbf{D}_{\mu} \mathbf{X}^{\text{int}}{}_{\nu} = -\mathbf{D}_{\mathbf{D}_{\mu} \bar{T}} \mathbf{X}^{\text{int}}{}_{\nu} + \mathbf{X}^{\text{int} \lambda} \bar{T}^{\lambda'} \mathbf{R}_{\lambda \nu \lambda' \mu}.$$

Moreover, we have

$$\mathbf{D}_{\bar{T}} g = \mathbf{D}_{\bar{T}}(\mathbf{g} + \bar{T} \otimes \bar{T}) = \mathbf{D}_{\bar{T}} \bar{T} \otimes \bar{T} + \bar{T} \otimes \mathbf{D}_{\bar{T}} \bar{T},$$

and therefore  $\mathbf{D}\mathbf{X}^{\text{int}} - 2g$  satisfy the following transport equation in  $\mathcal{M}_{\text{bot}}^{\text{int}}$

$$\mathbf{D}_{\bar{T}}(\mathbf{D}_{\mu} \mathbf{X}^{\text{int}}{}_{\nu} - g_{\mu\nu}) = -\mathbf{D}_{\mathbf{D}_{\mu} \bar{T}} \mathbf{X}^{\text{int}}{}_{\nu} + \mathbf{X}^{\text{int} \lambda} \bar{T}^{\lambda'} \mathbf{R}_{\lambda \nu \lambda' \mu} - \mathbf{D}_{\bar{T}} \bar{T}_{\mu} \bar{T}_{\nu} + \mathbf{D}_{\bar{T}} \bar{T}_{\nu} \bar{T}_{\mu}. \quad (11.30)$$

Commuting the transport equation (11.30) with  $(t\mathbf{D})^{\leq 2}$ , using the estimates (11.1) for the curvature, the mild Bootstrap Assumptions 3.18 for  $\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}}$ , the bootstrap bounds (3.5) for  $\mathbf{D}\mathbf{X}^{\text{int}}$ , and the bounds (11.22) for  $\mathbf{D}\mathbf{T}^{\text{int}}$ , we deduce from (11.30) that

$$\left\| t^{-1/2-\gamma} t \mathbf{D}_{\bar{T}} (t\mathbf{D})^{\leq 2} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon. \quad (11.31)$$

Integrating (11.31), using the bound (11.29) on  $\Sigma_{t^*}$ , we obtain

$$\left\| t^{-\gamma} t (t\mathbf{D})^{\leq 2} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L_t^{\infty} L^2(\Sigma_t)} \lesssim \varepsilon,$$

for all  $\gamma > 0$ , from which we also deduce the desired bounds

$$\left\| t^{-1/2-\gamma} t (t\mathbf{D})^{\leq 2} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon,$$

and, using a trace estimate on  $\partial\Sigma_t$ ,

$$\left\| t^{-\gamma} t (t\mathbf{D})^{\leq 1} (\mathbf{D}\mathbf{X}^{\text{int}} - g) \right\|_{L_t^{\infty} L^4(\partial\Sigma_t)},$$

for all  $\gamma > 0$ . The  $L^{\infty}(\mathcal{M}_{\text{bot}}^{\text{int}})$  and  $L_t^{\infty} L^6(\Sigma_t)$  estimates of (11.25) follow similarly and are left to the reader.

To obtain (11.27), we use the definition (2.24) of  $\mathbf{X}^{\text{int}}$ , from which we have

$$\begin{aligned} \bar{T}(\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}})) &= {}^{(\mathbf{T}^{\text{int}})} \pi(\bar{T}, \mathbf{X}^{\text{int}}), \\ \mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}})|_{\Sigma_{t^*}} &= 0. \end{aligned} \quad (11.32)$$

Thus, integrating (11.32), using the mild Bootstrap Assumptions 3.18 on  $\mathbf{X}^{\text{int}}$  and the bounds (11.22) on  $\mathbf{D}\mathbf{T}^{\text{int}}$ , we have

$$|\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}})| \lesssim \varepsilon t^{-1/2}. \quad (11.33)$$

Using the bounds (11.22) on  $\mathbf{D}\mathbf{T}^{\text{int}}$ , the bounds (11.25) on  $\mathbf{D}\mathbf{X}^{\text{int}}$  and the mild Bootstrap Assumptions 3.18 on  $\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}}$ , we further have

$$\begin{aligned} |t\mathbf{D}\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}})| &\lesssim |t\mathbf{g}(\mathbf{D}\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}})| + |t\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{D}\mathbf{X}^{\text{int}})| \\ &\lesssim \varepsilon t^{-1/2}, \end{aligned}$$

as desired. This finishes the proof of the lemma.  $\square$

**Lemma 11.7.** *The following bounds hold on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$\begin{aligned} \left\| t^{3/2} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L^{\infty}(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim \varepsilon, \\ \left\| t (t\mathbf{D})^{\leq 1} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L_t^{\infty} L^6(\Sigma_t)} &\lesssim \varepsilon, \\ \left\| t^{-1/2-\gamma} t (t\mathbf{D})^{\leq 2} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} &\lesssim \varepsilon, \\ \left\| t^{-\gamma} t (t\mathbf{D})^{\leq 1} (\mathbf{D}\mathbf{S}^{\text{int}} - \mathbf{g}) \right\|_{L_t^{\infty} L^4(\partial\Sigma_t)} &\lesssim \varepsilon, \end{aligned} \quad (11.34)$$



and

$$\begin{aligned}
 & \left\| t^{1/2} \left( (\mathbf{K}^{\text{int}}) \pi - 4t\mathbf{g} \right) \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon, \\
 & \left\| (t\mathbf{D})^{\leq 1} \left( (\mathbf{K}^{\text{int}}) \pi - 4t\mathbf{g} \right) \right\|_{L_t^\infty L^6(\Sigma_t)} \lesssim \varepsilon, \\
 & \left\| t^{-1/2-\gamma} (t\mathbf{D})^{\leq 2} \left( (\mathbf{K}^{\text{int}}) \pi - 4t\mathbf{g} \right) \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon.
 \end{aligned} \tag{11.35}$$

*Proof.* The estimates (11.34) are a direct consequence of the definition (2.25) of  $\mathbf{S}^{\text{int}}$  and the bounds (11.5) for the maximal connection coefficients, the bounds (11.22) for  $\mathbf{D}\mathbf{T}^{\text{int}}$  and the bounds (11.25) for  $\mathbf{X}^{\text{int}}$ .

From the definition (2.26) of  $\mathbf{K}^{\text{int}}$ , we have

$$\begin{aligned}
 \mathbf{D}\mathbf{K}^{\text{int}} &= 2t\mathbf{D}t \otimes \mathbf{T}^{\text{int}} + 2\mathbf{g}(\mathbf{D}\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) \otimes \mathbf{T}^{\text{int}} + (t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) \mathbf{D}\mathbf{T}^{\text{int}} + 2\mathbf{D}t \otimes \mathbf{X}^{\text{int}} + 2t\mathbf{D}\mathbf{X}^{\text{int}} \\
 &= -2n^{-1}t\mathbf{T}^{\text{int}} \otimes \mathbf{T}^{\text{int}} + 2\mathbf{X}^{\text{int}} \otimes \mathbf{T}^{\text{int}} + 2\mathbf{g}(\mathbf{D}\mathbf{X}^{\text{int}} - g, \mathbf{X}^{\text{int}}) \otimes \mathbf{T}^{\text{int}} + (t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) \mathbf{D}\mathbf{T}^{\text{int}} \\
 &\quad - 2n^{-1}\mathbf{T}^{\text{int}} \otimes \mathbf{X}^{\text{int}} + 2tg + 2t(\mathbf{D}\mathbf{X}^{\text{int}} - g) \\
 &= 2t\mathbf{g} + 2\mathbf{X}^{\text{int}} \otimes \mathbf{T}^{\text{int}} - 2\mathbf{T}^{\text{int}} \otimes \mathbf{X}^{\text{int}} + \mathbf{E},
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{E} &:= -2t(n^{-1} - 1)\mathbf{T}^{\text{int}} \otimes \mathbf{T}^{\text{int}} + 2\mathbf{g}(\mathbf{D}\mathbf{X}^{\text{int}} - g, \mathbf{X}^{\text{int}})\mathbf{T}^{\text{int}} + (t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) \mathbf{D}\mathbf{T}^{\text{int}} \\
 &\quad - 2(n^{-1} - 1)\mathbf{T}^{\text{int}} \otimes \mathbf{X}^{\text{int}} + 2t(\mathbf{D}\mathbf{X}^{\text{int}} - g).
 \end{aligned}$$

From the bounds (11.5) for the maximal connection coefficients, the bounds (11.22) for  $\mathbf{D}\mathbf{T}^{\text{int}}$  and the bounds (11.25) for  $\mathbf{X}^{\text{int}}$ , and the mild Bootstrap Assumptions 3.18, we have

$$\begin{aligned}
 & \left\| t^{1/2}\mathbf{E} \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon, \\
 & \left\| (t\mathbf{D})^{\leq 1}\mathbf{E} \right\|_{L_t^\infty L^6(\Sigma_t)} \lesssim \varepsilon, \\
 & \left\| t^{-1/2-\gamma} (t\mathbf{D})^{\leq 2}\mathbf{E} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon,
 \end{aligned}$$

and the desired bounds (11.35) follow. This finishes the proof of the lemma.  $\square$

**Lemma 11.8.** *The following bounds hold on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$\left\| \mathbf{D}\mathbf{O}^{\text{int}} \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim 1, \tag{11.36}$$

$$\left\| t^{3/2}(\mathbf{O}^{\text{int}})\pi \right\|_{L^\infty(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon, \tag{11.37}$$

$$\left\| t^2\mathbf{D}^2\mathbf{O}^{\text{int}} \right\|_{L_t^\infty L^6(\Sigma_t)} \lesssim \varepsilon, \tag{11.38}$$

$$\left\| t^{-1/2-\gamma} t^2 (t\mathbf{D})^{\leq 1} \mathbf{D}^2 \mathbf{O}^{\text{int}} \right\|_{L^2(\mathcal{M}_{\text{bot}}^{\text{int}})} \lesssim \varepsilon, \tag{11.39}$$

$$\left\| t^{-\gamma} t^2 \mathbf{D}^2 \mathbf{O}^{\text{int}} \right\|_{L_t^\infty L^4(\partial\Sigma_t)} \lesssim \varepsilon \tag{11.40}$$

for all  $\gamma > 0$ .

*Proof.* Commuting definition (2.28) of  $\mathbf{O}^{\text{int}}$  in  $\mathcal{M}_{\text{bot}}^{\text{int}}$  by  $\mathbf{D}$  and arguing as in the proof of Lemma 11.6, we have schematically

$$\mathbf{D}_{\bar{T}}\mathbf{D}\mathbf{O}^{\text{int}} = \mathbf{D}\bar{T} \cdot \mathbf{D}\mathbf{O}^{\text{int}} + \mathbf{R} \cdot \mathbf{O}^{\text{int}} \cdot \bar{T}. \tag{11.41}$$

Using the definition (2.27) of  $\mathbf{O}^{\text{int}}$  on  $\Sigma_{t^*}$ ,  $\mathbf{O}^{\text{int}}$  is a  $\Sigma_{t^*}$ -tangent vectorfield and we have

$$\begin{aligned}
 \nabla_l^{(\ell)} \mathbf{O}^{\text{int}}_m &= \epsilon_{lij} \nabla_l x^i \nabla_m x^j + \epsilon_{lij} x^i \nabla_{l,m}^2 x^j, \\
 \nabla_l^{(\ell)} \mathbf{O}^{\text{int}}_m + \nabla_m^{(\ell)} \mathbf{O}^{\text{int}}_l &= 2 \epsilon_{lij} x^i \nabla_{l,m}^2 x^j.
 \end{aligned} \tag{11.42}$$

Moreover, using (2.28), we have on  $\Sigma_{t^*}$

$$\begin{aligned} \mathbf{D}_{\bar{T}}^{(\ell)} \mathbf{O}^{\text{int}}_i &= 0, \\ \mathbf{D}_i^{(\ell)} \mathbf{O}^{\text{int}}_{\bar{T}} &= -\mathbf{g}^{(\ell)}(\mathbf{O}^{\text{int}}, \mathbf{D}_i \bar{T}) = k_{ij}^{(\ell)} \mathbf{O}^{\text{int}}_j, \\ \mathbf{D}_{\bar{T}}^{(\ell)} \mathbf{O}^{\text{int}}_{\bar{T}} &= 0. \end{aligned} \tag{11.43}$$

Thus, we deduce from (11.42) and (11.43) and the bounds (11.6) and (11.22) that

$$\begin{aligned} \|\mathbf{D}\mathbf{O}^{\text{int}}\|_{L^\infty(\Sigma_{t^*})} &\lesssim 1, \\ \left\| t^{3/2}(\mathbf{O}^{\text{int}})_\pi \right\|_{L^\infty(\Sigma_{t^*})} &\lesssim \varepsilon. \end{aligned}$$

Integrating (11.41), we obtain the desired bounds (11.36) and (11.37).

Differentiating formula (11.42), we obtain schematically on  $\Sigma_{t^*}$

$$\nabla^2 \mathbf{O}^{\text{int}} = \nabla x \nabla^2 x + x \nabla^3 x,$$

from which, arguing as previously, using the bounds (11.6) and (11.22), one deduces that

$$\begin{aligned} \|t^2 \mathbf{D}^2 \mathbf{O}^{\text{int}}\|_{L^6(\Sigma_{t^*})} &\lesssim \varepsilon, \\ \|t(\mathbf{tD})^{\leq 1} \mathbf{D}^2 \mathbf{O}^{\text{int}}\|_{L^2(\Sigma_{t^*})} &\lesssim \varepsilon. \end{aligned}$$

Arguing as in the proof of Lemma 11.6, integrating the transport equation (11.41) and applying a trace estimate, estimates (11.38), (11.39) and (11.40) follow. This finishes the proof of the lemma.  $\square$

### 11.4.2 Control of $\mathbf{X}^{\text{int}}$

This section is dedicated to the improvement of the bound (3.6) from the Bootstrap Assumptions 3.28.

**Proposition 11.9.** *We have for all  $t^\circ \leq t \leq t^*$*

$$\left| \sup_{\Sigma_t} (t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) - \left( \frac{1-\tau}{1+\tau} \right)^2 \right| \lesssim \varepsilon t^{-3/2}.$$

Before turning to the proof of Proposition 11.9, we have the following three lemmas.

**Lemma 11.10.** *The following bound holds on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$|\mathbf{D}_{\mathbf{X}^{\text{int}}} \mathbf{X}^{\text{int}} - \mathbf{X}^{\text{int}}| \lesssim \varepsilon t^{-1/2},$$

where the norm is taken with respect to the maximal frame.

*Proof.* The proof is a straight-forward adaptation of the estimates of Lemma 11.6.  $\square$

**Lemma 11.11.** *For all  $t^\circ \leq t \leq t^*$ , we have*

$$\sup_{\Sigma_t} t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) = \sup_{\partial \Sigma_t} t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}). \tag{11.44}$$

*Proof.* Let  $t^\circ \leq t \leq t^*$  and assume that the supremum on  $\Sigma_t$  is reached for  $p \in \Sigma_t \setminus \partial \Sigma_t$ . Since  $p$  is not on the boundary, we have in particular, for the derivative in the (projected on  $\Sigma_t$ )  $\mathbf{X}^{\text{int}}$ -direction

$$0 = \nabla_{\mathbf{X}^{\text{int}} + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}}) \mathbf{T}^{\text{int}}} (t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})).$$

Using (2.24) and relations (2.6), this gives

$$\begin{aligned} 0 &= t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) + t^{-2} \mathbf{g}(\mathbf{D}\mathbf{X}^{\text{int}} - \mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) \\ &\quad + \mathbf{D}_{\mathbf{X}^{\text{int}}} (t^{-2}) \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}}) \mathbf{D}_{\mathbf{T}^{\text{int}}} (t^{-2}) \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) \\ &= t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) + t^{-2} \mathbf{g}(\mathbf{D}\mathbf{X}^{\text{int}} - \mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) \\ &\quad + 2t^{-3} \mathbf{g}(\mathbf{X}^{\text{int}}, n^{-1} \mathbf{T}^{\text{int}}) \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) + 2t^{-3} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}}) \mathbf{g}(\mathbf{T}^{\text{int}}, n^{-1} \mathbf{T}^{\text{int}}) \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) \\ &= t^{-2} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) + t^{-2} \mathbf{g}(\mathbf{D}\mathbf{X}^{\text{int}} - \mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}). \end{aligned}$$

Using the bound from Lemma 11.10, we thus deduce

$$|t^{-2}\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})| \lesssim \varepsilon t^{-3/2} (|t^{-2}\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})|)^{1/2},$$

which we rewrite as

$$|t^{-2}\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})| \lesssim \varepsilon^2 t^{-3}.$$

For  $\varepsilon > 0$  sufficiently small, this contradicts the bootstrap bound (3.6), and (11.44) follows.  $\square$

**Remark 11.12.** From the result of Lemma 11.11 and the bootstrap bound (3.6), we also deduce the following mild control for  $\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})$  on  $\mathcal{T}$

$$t^2 \lesssim \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}). \quad (11.45)$$

**Lemma 11.13.** We have the following control on  $\mathcal{T}$

$$\left| \tilde{Z} - \left( \bar{T} + \frac{1-\tau}{1+\tau} \bar{X} \right) \right| \lesssim \varepsilon t^{-3/2}, \quad (11.46)$$

where  $\bar{X} := \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})^{-1/2} \mathbf{X}^{\text{int}}$  and  $\tilde{Z}$  is the  $\mathcal{T}$ -tangent vectorfield normal to  $\partial\Sigma_t$  in  $\mathcal{T}$  and such that  $\tilde{Z}(t) = 1$ .

*Proof.* Let define

$$\tilde{Z}^{\text{Err}} := \tilde{Z} - \left( \bar{T} + \frac{1-\tau}{1+\tau} \bar{X} \right).$$

From its definition, we have the following expression for  $\tilde{Z}$

$$\tilde{Z} = \left( \frac{\tau}{1+\tau} \right) e_3 + \left( \frac{2-\tau\eta}{2+2\tau} \right) e_4. \quad (11.47)$$

Using the Bootstrap Assumptions 3.23 and 3.24 on the exterior and interior connection coefficients, one can therefore deduce the following mild control

$$|\tilde{Z}| \lesssim 1, \quad (11.48)$$

and using additionally relations (2.33),

$$|\mathbf{D}_{\tilde{Z}} \tilde{Z}| \lesssim \varepsilon t^{-5/2}, \quad (11.49)$$

where norms are taken in the maximal frame.

Using (11.22) and (11.48), we have

$$|\mathbf{D}_{\tilde{Z}} \bar{T}| \lesssim \varepsilon t^{-5/2}. \quad (11.50)$$

We have

$$\mathbf{D}_{\tilde{Z}} \bar{X} = \mathbf{D}_{\bar{T} + \frac{1-\tau}{1+\tau} \bar{X}} \bar{X} + \tilde{Z}^{\text{Err}} \cdot \mathbf{D} \bar{X}. \quad (11.51)$$

We have

$$\begin{aligned} \mathbf{D}_{\bar{X}} \bar{X} &= \bar{X} + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})^{1/2} \bar{X} \left( \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})^{-1/2} \right) \bar{X} \\ &= \left( 1 - \frac{\mathbf{g}(\mathbf{D}_{\mathbf{X}^{\text{int}}} \mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})}{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})^{3/2}} \right) \bar{X}. \end{aligned}$$

Using the result of Lemma 11.10 and the mild control  $t \lesssim |\mathbf{X}^{\text{int}}| \lesssim t$  from the Bootstrap Assumptions 3.18 and (11.45) on  $\mathcal{T}$ , we therefore deduce

$$|\mathbf{D}_{\bar{X}} \bar{X}| \lesssim \varepsilon t^{-5/2},$$

which using that  $\mathbf{D}_{\bar{T}}\mathbf{X}^{\text{int}} = 0$  and (11.51) further gives on  $\mathcal{T}$

$$|\mathbf{D}_{\tilde{Z}}\bar{X}| \lesssim \varepsilon t^{-5/2} + \left| \tilde{Z}^{\text{Err}} \right| |\mathbf{D}\bar{X}|. \quad (11.52)$$

We deduce from the definition of  $\tilde{Z}^{\text{Err}}$  and estimates (11.49), (11.50) and (11.52) the following estimate on  $\mathcal{T}$

$$\left| \mathbf{D}_{\tilde{Z}}\tilde{Z}^{\text{Err}} \right| \lesssim \varepsilon t^{-5/2} + \left| \tilde{Z}^{\text{Err}} \right| |\mathbf{D}\bar{X}|. \quad (11.53)$$

Using the expression (11.47), that  $\mathfrak{v} = 0$  on  $\mathcal{C}^*$  and the relations (2.15), (2.16), we have on  $S^*$

$$\begin{aligned} \tilde{Z}^{\text{Err}} &= \frac{\tau}{1+\tau} e_3 + \frac{1}{1+\tau} e_4 - \bar{T} - \frac{1-\tau}{1+\tau} \bar{X} \\ &= \frac{\tau}{1+\tau} (1-\nu^{-1}) e_3 + \frac{1}{1+\tau} (1-\nu) e_4 - \frac{1-\tau}{1+\tau} (\bar{X} - \bar{N}). \end{aligned} \quad (11.54)$$

From the bound (11.6) on the harmonic coordinates on  $\Sigma_{t^*}$ , we have

$$|\nabla x^i - \partial_{x^i}| \lesssim \varepsilon t^{*-3/2}.$$

Thus, from the definition (2.23) of  $\mathbf{X}^{\text{int}}$  on  $\Sigma_{t^*}$ , we have

$$|\mathbf{X}^{\text{int}} - x^i \partial_{x^i}| \lesssim \varepsilon t^{*-1/2}.$$

From the bounds (11.6) on  $S^*$ , we have

$$|r^* \bar{N}(x^i) - x^i| \lesssim \varepsilon t^{*-1/2}$$

from which we deduce on  $S^*$

$$|\bar{X} - \bar{N}| \lesssim t^{*-1} |\mathbf{X}^{\text{int}} - r^* \bar{N}| \lesssim t^{*-1} |x^i \partial_{x^i} - r^* \bar{N}| + \varepsilon t^{*-3/2} \lesssim \varepsilon t^{*-3/2}.$$

Using this bound together with the estimates for  $\nu$  from (11.5) in equation (11.54), we deduce that on  $S^*$

$$\left| \tilde{Z}^{\text{Err}} \right| \lesssim \varepsilon t^{-3/2}. \quad (11.55)$$

Integrating the estimate (11.53) for  $\tilde{Z}^{\text{Err}}$  along  $\tilde{Z}$ , using the mild control  $|\mathbf{D}\bar{X}| \lesssim 1$ , the estimate (11.55) for  $\tilde{Z}^{\text{Err}}$  on  $S^*$  and a Grönwall argument, the result of the lemma follows.  $\square$

*Proof of Proposition 11.9.* We have

$$\tilde{Z}(\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) = \frac{1-\tau}{1+\tau} \bar{X}(\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) + \tilde{Z}^{\text{Err}}(\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})). \quad (11.56)$$

From Lemma 11.10 and the mild control  $|\mathbf{X}^{\text{int}}| \lesssim t$ , we have

$$\left| \bar{X}(\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})) - 2\sqrt{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})} \right| \lesssim \varepsilon t^{-1/2}.$$

Plugging this in (11.56) and using estimate (11.46) and the mild control  $t \lesssim |\mathbf{X}^{\text{int}}| \lesssim t$  and  $|\mathbf{D}\mathbf{X}^{\text{int}}| \lesssim 1$ , we have

$$\left| \frac{\tilde{Z}(\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}))}{2\sqrt{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})}} - \frac{1-\tau}{1+\tau} \right| \lesssim \varepsilon t^{-3/2}. \quad (11.57)$$

Integrating (11.57) from  $t^*$  to  $t$ , we obtain

$$\left| \left[ \sqrt{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})} \right]_t^{t^*} - \frac{1-\tau}{1+\tau} (t^* - t) \right| \lesssim \varepsilon t^{-1/2}. \quad (11.58)$$

From the definition of  $\mathbf{X}^{\text{int}}$  on  $\Sigma_{t^*}$ , and the bounds (11.6) on the harmonic coordinates, we have on  $S^*$

$$|\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) - (r^*)^2| = \left| \sum_{i,j=1}^3 x^i x^j g^{ij} - (r^*)^2 \right| = \left| \sum_{i,j=1}^3 x^i x^j (g^{ij} - \delta^{ij}) \right| \lesssim \varepsilon t^{*1/2},$$

from which we deduce

$$\left| \sqrt{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})}|_{S^*} - r^* \right| \lesssim \varepsilon t^{*-1/2}.$$

From the definition of  $t^*$  and the area radius estimate (9.38), we have

$$\left| r^* - \frac{1-\tau}{1+\tau} t^* \right| = \left| r^* - \frac{1}{2}(1-\tau)\underline{u}^* \right| \lesssim \varepsilon t^{*-2}.$$

Thus,

$$\left| \sqrt{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})}|_{S^*} - \frac{1-\tau}{1+\tau} t^* \right| \lesssim \varepsilon t^{*-1/2}.$$

Plugging this in (11.58), we infer

$$\left| \sqrt{\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})}|_{\partial\Sigma_t} - \frac{1-\tau}{1+\tau} t \right| \lesssim \varepsilon t^{-1/2}.$$

Using the result of Lemma 11.11, the result of Proposition 11.9 follows.  $\square$

### 11.4.3 Mild control of $\mathbf{X}^{\text{int}}$ , $\mathbf{S}^{\text{int}}$ , $\mathbf{K}^{\text{int}}$ and $\mathbf{O}^{\text{int}}$

The following lemmas improve the mild Bootstrap Assumptions 3.18.

**Lemma 11.14.** *We have the following mild (improved) control in  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$|\mathbf{X}^{\text{int}}| \lesssim t, \quad |\mathbf{S}^{\text{int}}| \lesssim t, \quad |\mathbf{K}^{\text{int}}| \lesssim t^2, \quad (11.59)$$

where the norm is taken with respect to the maximal frame.

*Proof.* The norm of  $\mathbf{X}^{\text{int}}$  in the maximal frame can be expressed as

$$|\mathbf{X}^{\text{int}}|^2 = 2|\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{X}^{\text{int}})|^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}). \quad (11.60)$$

Using the bootstrap bounds (3.6) and (3.7) from the Bootstrap Assumptions 3.28, we have

$$\begin{aligned} \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{T}^{\text{int}}) &\lesssim D\varepsilon t^{-3/2}, \\ \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) &\lesssim t^2 + (D\varepsilon)t^{1/2} \lesssim t^2, \end{aligned} \quad (11.61)$$

for  $\varepsilon > 0$  sufficiently small, and the first bound of (11.59) is improved, *i.e.*

$$|\mathbf{X}^{\text{int}}| \lesssim t. \quad (11.62)$$

Using the estimate (11.62) obtained for  $\mathbf{X}^{\text{int}}$ , and the definitions (2.25) and (2.26) for  $\mathbf{S}^{\text{int}}$  and  $\mathbf{K}^{\text{int}}$ , we have

$$t^{-1}|\mathbf{S}^{\text{int}}| \lesssim |\mathbf{T}^{\text{int}}| + t^{-1}|\mathbf{X}^{\text{int}}| \lesssim 1,$$

and

$$t^{-2}|\mathbf{K}^{\text{int}}| \lesssim t^{-2}|t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})| + t^{-1}|\mathbf{X}^{\text{int}}| \lesssim 1,$$

as desired.  $\square$

The following estimates are used in Section 7 to show that  $\mathbf{K}^{\text{int}}$  is an appropriate multiplying vectorfield.

**Lemma 11.15.** *The following (improved) mild control holds on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$\mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) \leq -\frac{1}{2}t^2, \quad |\mathbf{K}^{\text{int}} + \mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}})\mathbf{T}^{\text{int}}| \leq \frac{1}{2}|\mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}})|,$$

for  $1 - \tau > 0$  and  $\varepsilon > 0$  sufficiently small.

*Proof.* From the definition (2.26) of  $\mathbf{K}^{\text{int}}$  and the estimate (11.33), we have

$$|\mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) + t^2 + \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}})| \lesssim D\varepsilon t^{1/2}.$$

Using that  $\mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) \geq 0$  (this is a consequence of the definition (2.24)), we have

$$\begin{aligned} \mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}}) &\lesssim -t^2 - \mathbf{g}(\mathbf{X}^{\text{int}}, \mathbf{X}^{\text{int}}) + (D\varepsilon)t^{1/2} \\ &\lesssim -t^2 + (D\varepsilon)t^{1/2} \\ &\leq -\frac{1}{2}t^2, \end{aligned}$$

provided that  $\varepsilon > 0$  is sufficiently small.

Moreover, from the definition of  $\mathbf{K}^{\text{int}}$  and the bound (3.6), we have

$$\begin{aligned} |\mathbf{K}^{\text{int}} + \mathbf{g}(\mathbf{K}^{\text{int}}, \mathbf{T}^{\text{int}})\mathbf{T}^{\text{int}}| &\leq 2t|\mathbf{X}^{\text{int}}| + (D\varepsilon)t^{1/2} \\ &\leq 2\left(\frac{1-\tau}{1+\tau}\right)^2 t^2 + (D\varepsilon)t^{1/2} \\ &\leq \frac{1}{4}t^2, \end{aligned}$$

provided that  $1 - \tau > 0$  and  $\varepsilon > 0$  are sufficiently small. This finishes the proof of the lemma.  $\square$

**Lemma 11.16.** *The following (improved) mild control holds on  $\mathcal{M}_{\text{bot}}^{\text{int}}$*

$$|\mathbf{O}^{\text{int}}| \lesssim t. \quad (11.63)$$

*Proof.* Using the definition (2.28), we have

$$\begin{aligned} \bar{T}(\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{O}^{\text{int}})) &= {}^{(\mathbf{T}^{\text{int}})}\hat{\pi}(\bar{T}, \mathbf{O}^{\text{int}}), \\ \mathbf{g}(\bar{T}, \mathbf{O}^{\text{int}})|_{\Sigma_{t^*}} &= 0. \end{aligned} \quad (11.64)$$

Integrating (11.64), using the control (11.22) and the mild Bootstrap Assumptions 3.18, we have

$$|\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{O}^{\text{int}})| \lesssim (D\varepsilon)t^{-1/2}. \quad (11.65)$$

Using the definitions (2.28), (2.27), and estimates (11.6) for the global harmonic coordinates, we have

$$\sup_{\Sigma_t} \mathbf{g}(\mathbf{O}^{\text{int}}, \mathbf{O}^{\text{int}}) \leq \sup_{\Sigma_{t^*}} \mathbf{g}(\mathbf{O}^{\text{int}}, \mathbf{O}^{\text{int}}) \lesssim t^2. \quad (11.66)$$

Thus, combining (11.65) and (11.66), we have

$$|\mathbf{O}^{\text{int}}|^2 = 2|\mathbf{g}(\mathbf{T}^{\text{int}}, \mathbf{O}^{\text{int}})|^2 + \mathbf{g}(\mathbf{O}^{\text{int}}, \mathbf{O}^{\text{int}}) \lesssim t^2,$$

which improves (11.63) as desired.  $\square$

## 11.5 Control of the Killing fields at the interface $\mathcal{T}$

This section is dedicated to the improvement of the Bootstrap Assumptions 3.29.

**Lemma 11.17.** *The following bounds hold on  $\mathcal{T}$*

$$|\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}}| \lesssim \varepsilon t^{-3/2}, \quad (11.67)$$

$$|\mathbf{S}^{\text{ext}} - \mathbf{S}^{\text{int}}| \lesssim \varepsilon t^{-1/2}, \quad (11.68)$$

$$|\mathbf{K}^{\text{ext}} - \mathbf{K}^{\text{int}}| \lesssim \varepsilon t^{1/2}. \quad (11.69)$$

*Proof.* From the transition relations (2.15), we have

$$\mathbf{T}^{\text{ext}} - \mathbf{T}^{\text{int}} = \frac{1}{2}(1 - \nu^{-1})e_3 + \frac{1}{2}(1 - \nu)e_4,$$

and (11.67) follows from the bounds (11.5) for  $\nu$ .

From the definition (2.95) for  $\mathbf{S}^{\text{ext}}$ , and from the definitions of Section 2.1, we have

$$\begin{aligned} \mathbf{DS}^{\text{ext}} &= \frac{1}{2} (\mathbf{D}u \otimes e_3 + u\mathbf{D}e_3 + \mathbf{D}\underline{u} \otimes e_4 + \underline{u}\mathbf{D}e_4) \\ &= -\frac{1}{2}e_4 \otimes e_3 - \frac{1}{2}e_3 \otimes e_4 - \frac{1}{4}\eta e_4 \otimes e_4 + \frac{1}{2}u\mathbf{D}e_3 + \frac{1}{2}\underline{u}\mathbf{D}e_4 \\ &= -\frac{1}{2}e_4 \otimes e_3 - \frac{1}{2}e_3 \otimes e_4 + \mathcal{J} + \mathbf{E}, \end{aligned}$$

where

$$\mathbf{E} := \left( -\frac{1}{2}\frac{u}{r}\mathcal{J} + \frac{1}{2}\frac{\underline{u}}{r}\mathcal{J} - \mathcal{J} \right) - \frac{1}{4}\eta e_4 \otimes e_4 + \frac{1}{2}u \left( \mathbf{D}e_3 + \frac{1}{r}\mathcal{J} \right) + \frac{1}{2}\underline{u} \left( \mathbf{D}e_4 - \frac{1}{r}\mathcal{J} \right).$$

Using the relations (2.33) and the bounds (11.2) for the null connection coefficients in  $\mathcal{M}^{\text{ext}}$ , we have

$$\left\| t^{3/2}\mathbf{E} \right\|_{L^\infty(\mathcal{T})} \lesssim \varepsilon.$$

Thus, combining this bound together with the estimate (11.34) for  $\mathbf{DS}^{\text{int}}$ , we have

$$\begin{aligned} \left\| t^{3/2}\mathbf{D}(\mathbf{S}^{\text{ext}} - \mathbf{S}^{\text{int}}) \right\|_{L^\infty(\mathcal{T})} &\lesssim \left\| t^{3/2}(\mathbf{DS}^{\text{ext}} - \mathbf{g}) \right\|_{L^\infty(\mathcal{T})} + \left\| t^{3/2}(\mathbf{DS}^{\text{int}} - \mathbf{g}) \right\|_{L^\infty(\mathcal{T})} \\ &\lesssim \varepsilon. \end{aligned} \quad (11.70)$$

Arguing as in the proof of Lemma 11.13, we have on  $S^*$

$$|\mathbf{X}^{\text{int}} - r^*\bar{N}| \lesssim \varepsilon t^{*-1/2}.$$

Thus, from the definition (2.25) of  $\mathbf{S}^{\text{int}}$ , we have

$$|\mathbf{S}^{\text{int}} - t^*\bar{T} - r^*\bar{N}| \lesssim \varepsilon t^{*-1/2}, \quad (11.71)$$

on  $S^*$ . From the definition (2.95), relations (2.15) and (2.16) and the bounds (11.5) for  $\nu$ , we have

$$|\mathbf{S}^{\text{ext}} - t^*\bar{T} - r^*\bar{N}| \lesssim t^*|\nu - 1| \lesssim \varepsilon t^{*-1/2}. \quad (11.72)$$

Estimate (11.68) now follows from combining (11.71), (11.72) and integrating (11.70) along  $\mathcal{T}$ .

Estimate (11.69) follows from estimates (11.67) and (11.68) and the approximate formulas

$$|\mathbf{K} - 2t\mathbf{S} + (t^2 - r^2)\mathbf{T}| \lesssim \varepsilon t^{1/2},$$

which we leave to the reader. This finishes the proof of the lemma.  $\square$

**Lemma 11.18.** *The following bounds hold on  $\mathcal{T}$*

$$\left| {}^{(\ell)}\mathbf{O}^{\text{ext}} - {}^{(\ell)}\mathbf{O}^{\text{int}} \right| \lesssim \varepsilon t^{-1/2}, \quad (11.73)$$

for all  $\ell = 1, 2, 3$ .

*Proof.* We start by proving the following estimate for the covariant derivative of the interior vectorfields  $\mathbf{O}^{\text{int}}$

$$\left| \mathbf{D}_{\mathbf{X}^{\text{int}}} {}^{(\ell)}\mathbf{O}^{\text{int}} - {}^{(\ell)}\mathbf{O}^{\text{int}} \right| \lesssim \varepsilon t^{-1/2}, \quad (11.74)$$

for all  $\ell = 1, 2, 3$ .

By differentiating in  $\bar{T}$ , using the transport equations (2.24) and (2.28) for  $\mathbf{X}^{\text{int}}$  and  $\mathbf{O}^{\text{int}}$  and the sup-norm bounds (11.22) for  $\mathbf{D}\bar{T}$  and sup-norm bounds (11.1) for the curvature, it is enough to prove that (11.74) holds on the last slice  $\Sigma_{t^*}$ .

From the definitions (2.23) and (2.27) of  $\mathbf{X}^{\text{int}}$  and  $\mathbf{O}^{\text{int}}$  on  $\Sigma_{t^*}$ , we have

$$\begin{aligned}\nabla_{\mathbf{X}^{\text{int}}}^{(\ell)} \mathbf{O}^{\text{int}} &= x^k \nabla_{\nabla_{x^k}} (\in_{\ell ij} x^i \nabla x^j) \\ &= \in_{\ell ij} x^k \nabla_{x^k} x^i \nabla x^j + \in_{\ell ij} x^k x^i \nabla_{\nabla_{x^k}} \nabla x^j \\ &= \in_{\ell ij} x^i \nabla x^j + \mathbf{E},\end{aligned}$$

where

$$\mathbf{E} = \in_{\ell ij} x^k (\nabla_{x^k} x^i - \delta_{ki}) \nabla x^j + \in_{\ell ij} x^k x^i \nabla_{\nabla_{x^k}} \nabla x^j.$$

From the bounds (11.6), we have

$$|\mathbf{E}| \lesssim \varepsilon t^{-1/2}.$$

Using additionally the control (11.22) of  $\mathbf{DT}^{\text{int}}$ , we deduce that the estimate (11.74) holds on  $\Sigma_{t^*}$ , and thus (11.74) is proved.

Using that  $\mathbf{X}^{\text{int}} = -t\mathbf{T}^{\text{int}} + 2\mathbf{S}^{\text{int}}$ , and the bounds (11.68) obtained for  $\mathbf{S}^{\text{ext}} - \mathbf{S}^{\text{int}}$ , the relations (2.15), (2.16) and the bounds for  $\nu$ , we have

$$|\mathbf{X}^{\text{int}} - r\bar{N}| \lesssim \varepsilon t^{-1/2}, \quad (11.75)$$

on  $\mathcal{T}$ . Thus, we deduce from (11.74) that at  $\mathcal{T}$

$$|\mathbf{D}_{\bar{T}} \mathbf{O}^{\text{int}}| \lesssim \varepsilon t^{-3/2}, \quad \left| \mathbf{D}_{\bar{N}} \mathbf{O}^{\text{int}} - \frac{1}{r} \mathbf{O}^{\text{int}} \right| \lesssim \varepsilon t^{-3/2}. \quad (11.76)$$

Using the definitions from Section 2.12, the bounds (10.3) on  ${}^{(i)}Y$  and on  $\chi$  and  $\underline{\chi}$ , we have for the exterior rotation vectorfields at  $\mathcal{T}$

$$\left| \mathbf{D}_3 \mathbf{O}^{\text{ext}} + \frac{1}{r} \mathbf{O}^{\text{ext}} \right| + \left| \mathbf{D}_4 \mathbf{O}^{\text{ext}} - \frac{1}{r} \mathbf{O}^{\text{ext}} \right| \lesssim \varepsilon t^{-3/2},$$

from which, using the relations (2.15) and (2.16) and the bounds on  $\nu$ , we deduce

$$|\mathbf{D}_{\bar{T}} \mathbf{O}^{\text{ext}}| \lesssim \varepsilon t^{-3/2}, \quad \left| \mathbf{D}_{\bar{N}} \mathbf{O}^{\text{ext}} - \frac{1}{r} \mathbf{O}^{\text{ext}} \right| \lesssim \varepsilon t^{-3/2}. \quad (11.77)$$

Combining (11.76) and (11.77) and estimate (11.46) for  $\tilde{Z}$  we obtain

$$\left| \mathbf{D}_{\tilde{Z}} (\mathbf{O}^{\text{ext}} - \mathbf{O}^{\text{int}}) - \frac{1}{r} \left( \frac{1-\tau}{1+\tau} \right) (\mathbf{O}^{\text{ext}} - \mathbf{O}^{\text{int}}) \right| \lesssim \varepsilon t^{-3/2}, \quad (11.78)$$

where we recall that  $\tilde{Z}$  was defined in Section 11.4.2 to be the  $\mathcal{T}$ -tangent vectorfield normal to  $\partial\Sigma_t$  in  $\mathcal{T}$  and such that  $\tilde{Z}(t) = 1$ .

From the definitions (2.27) of  $\mathbf{O}^{\text{int}}$  on  $\Sigma_{t^*}$  and in particular on  $S^*$ , we have

$$\begin{aligned}{}^{(\ell)} \mathbf{O}^{\text{int}} &= \in_{\ell ij} x^i \nabla x^j \\ &= \in_{\ell ij} x^i \nabla x^j + \in_{\ell ij} x^i \bar{N}(x^j) \bar{N} \\ &= \in_{\ell ij} x^i \nabla x^j + \in_{\ell ij} x^i x^j \bar{N} + \in_{\ell ij} x^i (\bar{N}(x^j) - x^j) \bar{N} \\ &= {}^{(\ell)} \mathbf{O}^{\text{ext}} + \in_{\ell ij} x^i (\bar{N}(x^j) - x^j) \bar{N}.\end{aligned} \quad (11.79)$$

Thus, using the bounds (11.6), we have on  $S^*$

$$\left| {}^{(\ell)} \mathbf{O}^{\text{ext}} - {}^{(\ell)} \mathbf{O}^{\text{int}} \right| \lesssim \left| \in_{\ell ij} x^i (\bar{N}(x^j) - x^j) \right| \lesssim \varepsilon t^{*-1/2}.$$

Integrating (11.78) along  $\mathcal{T}$  then yields (11.73) and finishes the proof of the lemma.  $\square$



**Lemma 11.19.** *The following bounds hold on  $\mathcal{T}$*

$$\begin{aligned} \|t^{-\gamma}t^2(\mathbf{D}\mathbf{T})^{\leq 1}(\mathbf{D}\mathbf{T}^{\text{ext}} - \mathbf{D}\mathbf{T}^{\text{int}})\|_{L_t^\infty L^4(\partial\Sigma_t)} &\lesssim_\gamma \varepsilon, \\ \|t^{-\gamma}t(\mathbf{D}\mathbf{T})^{\leq 1}(\mathbf{D}\mathbf{S}^{\text{ext}} - \mathbf{D}\mathbf{S}^{\text{int}})\|_{L_t^\infty L^4(\partial\Sigma_t)} &\lesssim_\gamma \varepsilon, \end{aligned} \quad (11.80)$$

for all  $\gamma > 0$ .

*Proof.* The proof of the lemma follows directly from the precise control of each separate tensor already obtained in the previous sections and is left to the reader.  $\square$

**Lemma 11.20.** *Under the same assumptions as in the previous lemma, we have*

$$\|t^{-\gamma}t(\mathbf{D}\mathbf{O})^{\leq 1}(\mathbf{D}\mathbf{O}^{\text{ext}} - \mathbf{D}\mathbf{O}^{\text{int}})\|_{L_t^\infty L^4(\partial\Sigma_t)} \lesssim_\gamma \varepsilon, \quad (11.81)$$

for all  $\gamma > 0$ .

*Proof.* The main part of the proof is to obtain the following two estimates

$$\|t^{-\gamma}t^2\mathbf{D}^2\mathbf{O}^{\text{int}}\|_{L_t^\infty L^4(\partial\Sigma_t)} \lesssim \varepsilon, \quad (11.82)$$

$$\|t^{-\gamma}t^2\mathbf{D}^2\mathbf{O}^{\text{ext}}\|_{L_t^\infty L^4(\partial\Sigma_t)} \lesssim \varepsilon. \quad (11.83)$$

We first verify that we can obtain (11.81) from (11.82) and (11.83). Using these last estimates and integration along  $\tilde{Z}$  as in the proof of Lemma 11.18, it is enough to additionally show that

$$\|t(\mathbf{D}\mathbf{O}^{\text{ext}} - \mathbf{D}\mathbf{O}^{\text{int}})\|_{L^4(S^*)} \lesssim \varepsilon. \quad (11.84)$$

Using formula (11.79) and the bounds (11.6) on the harmonic coordinates, we have

$$\|t(\nabla\mathbf{O}^{\text{ext}} - \nabla\mathbf{O}^{\text{int}})\|_{L^4(S^*)} \lesssim \varepsilon.$$

From this formula and the bounds (11.73) on  $\mathbf{O}^{\text{ext}} - \mathbf{O}^{\text{int}}$  obtained previously, we deduce

$$\|t\mathbf{D}_\alpha(\mathbf{O}^{\text{ext}} - \mathbf{O}^{\text{int}})\|_{L^4(S^*)} \lesssim \varepsilon. \quad (11.85)$$

Using the estimates (11.76) and (11.77) already obtained in the proof of Lemma 11.18 for respectively  $\mathbf{O}^{\text{int}}$  and  $\mathbf{O}^{\text{ext}}$  on  $\mathcal{T}$ , we have in particular on  $S^*$  and for the  $L^4$  norm

$$\begin{aligned} \|t\mathbf{D}_{\bar{T}}\mathbf{O}^{\text{int}}\|_{L^4(S^*)} + \left\| t \left( \mathbf{D}_{\bar{N}}\mathbf{O}^{\text{int}} - \frac{1}{r}\mathbf{O}^{\text{int}} \right) \right\|_{L^4(S^*)} &\lesssim \varepsilon, \\ \|t\mathbf{D}_{\bar{T}}\mathbf{O}^{\text{ext}}\|_{L^4(S^*)} + \left\| t \left( \mathbf{D}_{\bar{N}}\mathbf{O}^{\text{ext}} - \frac{1}{r}\mathbf{O}^{\text{ext}} \right) \right\|_{L^4(S^*)} &\lesssim \varepsilon. \end{aligned}$$

These last estimates combined with (11.85) yield (11.84) as desired.

We now turn to the proof of (11.82) and (11.83), where we note that estimate (11.82) was already obtained in Lemma 11.8. To prove (11.83), we compute that for each component of  $\mathbf{D}_{\mu,\nu}^2\mathbf{O}^{\text{ext}}$ , we have

$$\mathbf{D}_{\mu,\nu}^2\mathbf{O}^{\text{ext}} = \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0, \quad (11.86)$$

where

$$\begin{aligned} \mathbf{E}^1 &\text{ is a linear combination of } && (\mathbf{D}^{\leq 1}\check{\Gamma}, \mathbf{D}^{\leq 1}Y, \mathbf{D}^{\leq 1}H, \Psi, \mathbf{R}), \\ \mathbf{E}^0 &\text{ is a (nonlinear) combination of } && (\check{\Gamma}, Y, H), \\ \mathbf{F}^0 &\text{ is a (nonlinear) combination of } && (1, \mathbf{D}\mathbf{O}^{\text{ext}}), \end{aligned}$$

with

$$\check{\Gamma} \in \left\{ \text{tr}\chi - \frac{2}{r}, \text{tr}\underline{\chi} + \frac{2}{r}, \hat{\chi}, \hat{\underline{\chi}}, \zeta, \underline{\omega}, \underline{\xi} \right\}.$$

Note that we do not precise the weights in  $u, \underline{u}, r, t$  in (11.86), which are recovered by a simple scaling consideration on  $\mathcal{T}$  where  $u \simeq \underline{u} \simeq r \simeq t$ . Using formula (11.86), the  $L_t^\infty \tilde{H}^{1/2}(\partial\Sigma_t)$  estimates of (11.2) for  $\check{\Gamma}, Y, H, \Psi, \mathbf{R}$  and  $\check{\nabla} \mathbf{O}^{\text{ext}}$ , the  $\tilde{H}^{1/2}$  product estimates of Lemma 3.37 and the Sobolev embeddings of Lemma 3.36, we obtain the desired estimate (11.83).

The rest of the proof is therefore dedicated to obtaining the formulas (11.86). For simplicity, we call  $\mathbf{O}$  any exterior rotation vectorfield  $(\ell)\mathbf{O}^{\text{ext}}$ . From (2.108) and relations (2.33), we have

$$\mathbf{D}_4 \mathbf{O} = \mathbf{D}_{\mathbf{O}} e_4 = \chi_{ab} \mathbf{O}^b e_a - \zeta_b \mathbf{O}^b e_4 = r^{-1} \mathbf{O} + \mathbf{E}^0, \quad (11.87)$$

from which we also deduce

$$\mathbf{D}_4 (r^{-1} \mathbf{O}) = \mathbf{E}^0. \quad (11.88)$$

Similarly, using the relations (2.109), we have

$$\begin{aligned} \mathbf{D}_3 \mathbf{O} &= -r^{-1} \mathbf{O} + \mathbf{E}^0, \\ \mathbf{D}_3 (r^{-1} \mathbf{O}) &= \mathbf{E}^0. \end{aligned} \quad (11.89)$$

Deriving equation (11.87) by  $\mathbf{D}_4$  and using (11.88) and relations (2.33), we obtain

$$\mathbf{D}_{4,4}^2 \mathbf{O} = \mathbf{D}_4 (\mathbf{D}_4 \mathbf{O}) + \mathbf{E}^1 \cdot \mathbf{F}^0 = \mathbf{D}_4 (r^{-1} \mathbf{O}) + \mathbf{D}_4 (\mathbf{E}^0) + \mathbf{E}^1 \cdot \mathbf{F}^0 = \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0.$$

Arguing similarly using  $\mathbf{D}_3, \mathbf{D}_4$  derivatives, we have

$$\mathbf{D}_{4,4}^2 \mathbf{O}, \mathbf{D}_{3,4}^2 \mathbf{O}, \mathbf{D}_{4,3}^2 \mathbf{O}, \mathbf{D}_{3,3}^2 \mathbf{O} = \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0.$$

For  $\mathbf{D}_{a,4}^2$ , using equation (11.87), we have

$$\begin{aligned} \mathbf{D}_{a,4}^2 \mathbf{O} &= \mathbf{D}_a (\mathbf{D}_4 \mathbf{O}) - (\mathbf{D}_a e_4)^\mu \mathbf{D}_\mu \mathbf{O} \\ &= \mathbf{D}_a (\mathbf{D}_4 \mathbf{O}) - r^{-1} \mathbf{D}_a \mathbf{O} + \mathbf{E}^1 \cdot \mathbf{F}^0 \\ &= \mathbf{D}_a (r^{-1} \mathbf{O}) - r^{-1} \mathbf{D}_a \mathbf{O} + \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0 \\ &= \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0. \end{aligned}$$

Using the above, we also obtain for  $\mathbf{D}_{4,a}^2$

$$\mathbf{D}_{4,a}^2 \mathbf{O} = \mathbf{D}_{a,4}^2 \mathbf{O} + \mathbf{R} = \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0.$$

Arguing similarly for the  $e_3$  derivatives, we also obtain

$$\mathbf{D}_{3,a}^2 \mathbf{O}, \mathbf{D}_{a,3}^2 \mathbf{O} = \mathbf{E}^1 \cdot \mathbf{F}^0 + \mathbf{E}^1 \cdot \mathbf{E}^0.$$

We now turn to  $\mathbf{D}_{a,b}^2 \mathbf{O}$ . From the result of Lemma 2.30, using (11.87) and (11.89), and the definitions of  $H, Y$  and  $\Psi$  from Section 2.12, we have

$$\begin{aligned} \mathbf{D}_{a,b}^2 \mathbf{O} &= \check{\nabla}_{a,b}^2 \mathbf{O} - r^{-2} \mathbf{O}_b e_a + \frac{1}{2} r^{-1} \check{g}_{ab} (\mathbf{D}_4 - \mathbf{D}_3) \mathbf{O} \\ &\quad + \frac{1}{2} r^{-1} (\check{\nabla}_a \mathbf{O}_b + \check{\nabla}_b \mathbf{O}_a) (e_3 - e_4) + (\mathbf{E} (\mathbf{D}^2, \check{\nabla}^2) \cdot \mathbf{O})_{ab}, \\ &= \check{\nabla}_{a,b}^2 \mathbf{O} - r^{-2} \mathbf{O}_b e_a + r^{-2} \check{g}_{ab} \mathbf{O} + \frac{1}{2} r^{-1} H_{ab} (e_3 - e_4) + \mathbf{E}^1 \cdot \mathbf{F}^0 \\ &= \Psi_{abc} e_c + \frac{1}{2} r^{-1} H_{ab} (e_3 - e_4) + \mathbf{E}^1 \cdot \mathbf{F}^0 \\ &= \mathbf{E}^1 \cdot \mathbf{F}^0, \end{aligned}$$

where we used that from the formulas of Lemma 2.30

$$\mathbf{E} (\mathbf{D}^2, \check{\nabla}^2) \cdot \mathbf{O} = \mathbf{E}^1 \cdot \mathbf{F}^0.$$

This finishes the proof of (11.86) and concludes the proof of the lemma.  $\square$



## 12. The initial layers

The goal of this section is twofold.

In Section 12.1 we obtain the initial bounds (5.1) for the energy fluxes of (contracted and commuted) Bel-Robinson tensors through  $\Sigma_{t^\circ}$  and  $\mathcal{C}_1$ , which are used in Section 5 to obtain improved energy fluxes bounds in  $\mathcal{M}$ .

In Sections 12.2 and 12.3, we use the improved estimates of Sections 5–11 to improve the Bootstrap Assumptions 3.31 and 3.32 on the comparison between the geometric constructions in  $\mathcal{M}$  and in  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$ , which were used in Section 12.1. This is done in two steps: in Section 12.2 we control the intermediate foliation by the last cones used to define the region  $\mathcal{M}$  (see its definition in Section 2.13). In Section 12.3 we use this intermediate control to obtain the desired comparisons in  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$ .

### 12.1 Initial bounds for energy fluxes through $\mathcal{C}_1$ and $\Sigma_{t^\circ}$

In this section, we assume that the Bootstrap Assumptions hold and that the initial layers  $\mathcal{L}_{\text{bot}}$  and  $\mathcal{L}_{\text{con}}$  are  $\varepsilon$ -close to Minkowski (see the definitions of Section 4.1), and we show that

$$\int_{\Sigma_{t^\circ}} |\mathbf{D}^{\leq 2} \mathbf{R}|^2 \lesssim \varepsilon^2, \quad (12.1)$$

and that

$$\int_{\mathcal{C}_1} \left( |\nabla^{\leq 2} \underline{\beta}|^2 + |\underline{u} \nabla^{\leq 2} (\rho - \bar{\rho})|^2 + |\underline{u} \nabla^{\leq 2} (\sigma - \bar{\sigma})|^2 + |\underline{u}^2 \nabla^{\leq 2} \beta|^2 + |\underline{u}^2 \nabla^{\leq 2} \alpha|^2 \right) \lesssim \varepsilon^2 \quad (12.2)$$

where  $\nabla \in \{(r\nabla), (\underline{u}\nabla_4), (u\nabla_3)\}$ .

**Remark 12.1.** *Together with the null connection and rotation coefficients Bootstrap Assumptions 3.23 on  $\mathcal{C}_1$ , this proves the desired bounds for the Bel-Robinson tensors (5.1) of Section 5. Details are left to the reader.*

#### 12.1.1 Energy fluxes through $\Sigma_{t^\circ}$

Under the Bootstrap Assumptions 3.31, the hypersurface  $\Sigma_{t^\circ}$  is included in  $\mathcal{L}_{\text{bot}}$ . One can therefore perform energy estimates in the past region of  $\Sigma_{t^\circ}$  in  $\mathcal{L}_{\text{bot}}$  for the following contracted and commuted Bel-Robinson tensors

$$Q \left( \hat{\mathcal{L}}_{\partial_\mu}^{\leq 2} \mathbf{R} \right) \left( \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}} \right),$$

for  $\mu = 0, 1, 2, 3$ . Arguing as in Section 5.2 to estimate the error terms using the bottom initial layer estimates (4.1), one obtains

$$\int_{\Sigma_{t^\circ}} Q \left( \hat{\mathcal{L}}_{\partial_\mu}^{\leq 2} \mathbf{R} \right) \left( \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T} \right) \lesssim \int_{\tilde{\Sigma}_1} Q \left( \hat{\mathcal{L}}_{\partial_\mu}^{\leq 2} \mathbf{R} \right) \left( \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}} \right) + \varepsilon^3.$$

Using the curvature flux bound (4.1b) on  $\tilde{\Sigma}_1$  and the bounds (4.1), one has

$$\int_{\tilde{\Sigma}_1} Q \left( \hat{\mathcal{L}}_{\partial_\mu}^{\leq 2} \mathbf{R} \right) \left( \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}} \right) \lesssim \varepsilon^2,$$

and thus

$$\int_{\Sigma_{t^\circ}} Q \left( \hat{\mathcal{L}}_{\partial_\mu}^{\leq 2} \mathbf{R} \right) \left( \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T} \right) \lesssim \varepsilon^2. \quad (12.3)$$

Using the Bootstrap Assumptions 3.31 for the comparisons of the frames, we deduce from (12.3)

$$\int_{\Sigma_{t^\circ}} \left| \hat{\mathcal{L}}_{\partial_\mu}^2 \mathbf{R} \right|^2 \lesssim \varepsilon^2, \quad (12.4)$$

where the norm is taken with respect to the frame with respect to either  $\bar{T}^{\text{bot}}$  or  $\bar{T}$ .

Using the bounds (4.1), one can obtain by integration along  $x^0$

$$\|\mathbf{D}^{\leq 1} \mathbf{D} \partial_\mu\|_{L^2(\Sigma_{t^0})} \lesssim \|\partial^{\leq 2}(\mathbf{g}_{\alpha\beta} - \boldsymbol{\eta}_{\alpha\beta})\|_{L^2(\Sigma_{t^0})} \lesssim \|\partial^{\leq 3}(\mathbf{g}_{\alpha\beta} - \boldsymbol{\eta}_{\alpha\beta})\|_{L^2(\mathcal{L}_{\text{bot}})} \lesssim \varepsilon. \quad (12.5)$$

Using that we schematically have

$$\mathbf{D}_{\partial_\mu}^2 \mathbf{R} = \hat{\mathcal{L}}_{\partial_\mu}^2 \mathbf{R} + \mathbf{D}\mathbf{R} \cdot \mathbf{D}\partial_\mu + \mathbf{R} \cdot \mathbf{D}^2\partial_\mu,$$

we deduce from (12.4) and (12.5)

$$\int_{\Sigma_{t^0}} |\mathbf{D}_{\partial_\mu}^{\leq 2} \mathbf{R}|^2 \lesssim \varepsilon^2,$$

for  $\mu = 0, 1, 2, 3$ , from which using (4.1) again, we deduce

$$\int_{\Sigma_{t^0}} |\mathbf{D}^{\leq 2} \mathbf{R}|^2 \lesssim \varepsilon^2,$$

in either the frame with respect to  $\bar{T}^{\text{bot}}$  or  $\bar{T}$ . This finishes the proof of (12.1).

### 12.1.2 Energy fluxes through $\mathcal{C}_1$

Under the Bootstrap Assumptions 3.31 and 3.32 (see also Figure 1), we have

$$\mathcal{C}_1 = (\mathcal{C}_1 \cap \mathcal{L}_{\text{bot}}) \cup (\mathcal{C}_1 \cap \mathcal{L}_{\text{con}}).$$

We first obtain bounds on the bottom initial layer part  $\mathcal{C}_1 \cap \mathcal{L}_{\text{bot}}$ , using the same contracted and commuted Bel-Robinson tensors as in Section 12.1.1 and the bottom initial layer assumptions (4.1) to control the error terms, and we have

$$\int_{\mathcal{C}_1 \cap \mathcal{L}_{\text{bot}}} Q(\hat{\mathcal{L}}_{\partial_\mu}^{\leq 2} \mathbf{R})(\bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, \bar{T}^{\text{bot}}, e_4) \lesssim \varepsilon^2. \quad (12.6)$$

Using (4.1), the Bootstrap Assumptions 3.31 in  $\mathcal{L}_{\text{bot}}^{\text{ext}}$  and arguing as in Section (12.1.1), we deduce from (12.6) that the desired bound (12.2) holds in  $\mathcal{C}_1 \cap \mathcal{L}_{\text{bot}}$ .

Performing a mean value argument between  $\mathcal{C}_1 \cap \mathcal{L}_{\text{con}}$  and  $\mathcal{C}_{4/3} \cap \mathcal{L}_{\text{con}}$  using the (integrated) curvature flux bound (4.2e) (see for example Section 5.1.1), one can obtain

$$\int_{\mathcal{C}_c \cap \mathcal{L}_{\text{con}}} \tilde{u}^{-1-\gamma} \left| \widetilde{\nabla}^{\leq 2} \tilde{\alpha} \right|^2 \lesssim_\gamma \varepsilon^2, \quad (12.7)$$

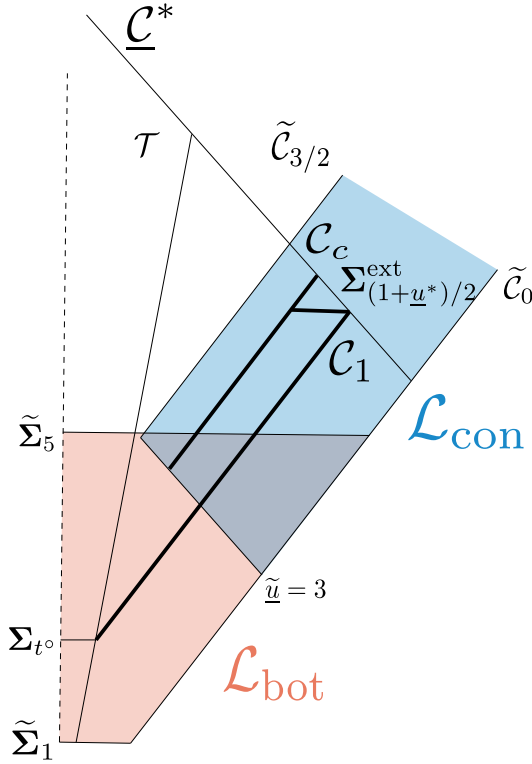
where  $\widetilde{\nabla} \in \left\{ \tilde{u} \widetilde{\nabla}, \tilde{u} \widetilde{\nabla}_4, \widetilde{\nabla}_3 \right\}$ , for  $\gamma > 0$  and where  $1 < c < 4/3$ .

Using energy estimates in the spacetime region between  $\tilde{\mathcal{C}}_0 \cup \{\underline{u}' = 3\}$  and  $(\mathcal{C}_c \cap \mathcal{L}_{\text{con}}) \cup (\underline{\mathcal{C}}^* \cap \mathcal{L}_{\text{con}})$ ,<sup>1</sup> for the null Bianchi equations satisfied by the null curvature components with respect to the null pair of the double null foliation of the conical initial layer, commuting with  $\tilde{u} \widetilde{\nabla}$  and  $\widetilde{\nabla}_3, \widetilde{\nabla}_4$ , we claim that one can obtain the following curvature flux control on  $\mathcal{C}_c$  (this follows from an adaptation of [LZ18])

$$\begin{aligned} & \int_{\mathcal{C}_c \cap \mathcal{L}_{\text{con}}} \left( \left| \widetilde{\nabla}^{\leq 2} \tilde{\beta} \right|^2 + \left| \tilde{u} \widetilde{\nabla}^{\leq 2} (\tilde{\rho} - \bar{\rho}) \right|^2 + \left| \tilde{u} \widetilde{\nabla}^{\leq 2} (\tilde{\sigma} - \bar{\sigma}) \right|^2 + \left| \tilde{u}^2 \widetilde{\nabla}^{\leq 2} \tilde{\beta} \right|^2 + \left| \tilde{u}^2 \widetilde{\nabla}^{\leq 2} \tilde{\alpha} \right|^2 \right) \\ & \lesssim \varepsilon^2 + \int_{\mathcal{C}_c \cap \mathcal{L}_{\text{con}}} (D\varepsilon)^2 \tilde{u}^{-2} \left( \left| \widetilde{\nabla}^{\leq 2} \tilde{\alpha} \right|^2 + \left| \tilde{u} \widetilde{\nabla}^{\leq 2} \tilde{\beta} \right|^2 + \left| \tilde{u}^2 \widetilde{\nabla}^{\leq 2} (\tilde{\rho} - \bar{\rho}) \right|^2 \right. \\ & \quad \left. + \left| \tilde{u}^2 \widetilde{\nabla}^{\leq 2} (\tilde{\sigma} - \bar{\sigma}) \right|^2 + \left| \tilde{u}^2 \widetilde{\nabla}^{\leq 2} \tilde{\beta} \right|^2 \right) \end{aligned} \quad (12.8)$$

where  $\widetilde{\nabla} \in \left\{ \tilde{u} \widetilde{\nabla}, \tilde{u} \widetilde{\nabla}_4, \widetilde{\nabla}_3 \right\}$ .

<sup>1</sup>To obtain no uncontrolled error terms on  $\underline{\mathcal{C}}^*$ , we would rather perform an energy estimate replacing  $\underline{\mathcal{C}}^* \cap \mathcal{L}_{\text{con}}$  by a spacelike boundary hypersurface which can be obtained by extension above  $\underline{\mathcal{C}}^*$  in  $\mathcal{L}_{\text{con}}$ .


 Figure 1: Energy flux through  $\mathcal{C}_1$ 

**Remark 12.2.** The  $(D\varepsilon)^2 \underline{u}^{-2}$  term comes from the fact that using the relation from Lemma 2.31, the normal  $e_4$  to  $\mathcal{C}_c$  can be expressed (at first orders) as  $e_4 = \lambda^{-1} \tilde{e}_4 - \lambda^{-1} \tilde{f}^a \tilde{e}_a - \frac{1}{4} |\tilde{f}|^2 \tilde{e}_3$ , and thus the difference between  $e_4$  and  $e_4'$  yields the presence of  $|\tilde{f}|^2 |\tilde{\alpha}|^2$ , which justifies the claim.

Thus, we deduce from (12.7) and (12.8) that

$$\int_{\mathcal{C}_c \cap \mathcal{L}_{\text{con}}} \left| \widetilde{\nabla}^{\leq 2} \tilde{\beta} \right|^2 + \left| \underline{u} \widetilde{\nabla}^{\leq 2} (\tilde{\rho} - \bar{\rho}) \right|^2 + \left| \underline{u} \widetilde{\nabla}^{\leq 2} (\tilde{\sigma} - \bar{\sigma}) \right|^2 + \left| \underline{u}^2 \widetilde{\nabla}^{\leq 2} \tilde{\beta} \right|^2 + \left| \underline{u}^2 \widetilde{\nabla}^{\leq 2} \tilde{\alpha} \right|^2 \lesssim \varepsilon^2. \quad (12.9)$$

Using the transition relations from Propositions 2.34, 2.35 and 2.36 and the Bootstrap Assumptions 3.32 on the transition coefficients  $\tilde{f}, \tilde{f}, \tilde{\lambda}$ , one can deduce from (12.9)

$$\int_{\mathcal{C}_c \cap \mathcal{L}_{\text{con}}} \left| \nabla^{\leq 2} \underline{\beta} \right|^2 + \left| \underline{u} \nabla^{\leq 2} (\rho - \bar{\rho}) \right|^2 + \left| \underline{u} \nabla^{\leq 2} (\sigma - \bar{\sigma}) \right|^2 + \left| \underline{u}^2 \nabla^{\leq 2} \beta \right|^2 + \left| \underline{u}^2 \nabla^{\leq 2} \alpha \right|^2 \lesssim \varepsilon^2 \quad (12.10)$$

where  $\nabla \in \{(r\nabla), (\underline{u}\nabla_4), \nabla_3\}$ .

**Remark 12.3.** From an inspection of the relations from Propositions 2.35 and 2.36, the bootstrap bounds  $|\tilde{f}| \lesssim D\varepsilon \underline{u}^{-1}$ ,  $|\tilde{f}|, |\log \tilde{\lambda}| \lesssim D\varepsilon$  are sufficient due to a conservation of signature principle, i.e.  $\tilde{\lambda}, \tilde{f}$  are only paired with lower signature null curvature component hence are not required to decay, while the transition coefficient  $\tilde{f}$  which is paired with higher signature components satisfies the sufficient decay  $\underline{u}^{-1}$ . See also [KS17, Remark 4.1.4] for further discussion.

**Remark 12.4.** We use the relations of Proposition 2.35 to obtain the control of all derivatives of  $\tilde{f}, \tilde{f}, \tilde{\lambda}$ , from the Bootstrap Assumptions 3.23 on the null connection coefficients of  $\mathcal{M}^{\text{ext}}$ , the Bootstrap Assumptions 3.32 on  $\tilde{f}, \tilde{f}, \tilde{\lambda}$  and the conical initial layer assumptions (4.2) on the connection coefficients.<sup>2</sup>

To obtain the desired bound on  $\mathcal{C}_1 \cap \mathcal{L}_{\text{con}}$  from the bound (12.10) on  $\mathcal{C}_c \cap \mathcal{L}_{\text{con}}$ , we first obtain an energy bound on the spacelike hypersurface  $\Sigma_{(1+\underline{u}^*)/2}^{\text{ext}} \cap \mathcal{L}_{\text{con}}$ , arguing similarly as for (12.10), except that no mean value argument is required to control the error terms. Then performing an energy estimate as in Section 5 in the spacetime region comprised between  $(\mathcal{C}_1 \cap \mathcal{L}_{\text{con}}) \cup \{\underline{u}' = 3\}$  and  $(\mathcal{C}_c \cap \mathcal{L}_{\text{con}}) \cup \left( \Sigma_{(1+\underline{u}^*)/2}^{\text{ext}} \cap \mathcal{L}_{\text{con}} \right)$ , one deduces the desired estimate (12.2) in  $\mathcal{C}_1 \cap \mathcal{L}_{\text{con}}$ , which combined with the bound obtained in  $\mathcal{C}_1 \cap \mathcal{L}_{\text{bot}}$  finishes the proof of (12.2).

<sup>2</sup> $L^2(\mathcal{C}_c)$  bounds for (one derivative of) the initial conical layer null connection coefficients can be obtained by integration from  $\mathcal{L}_{\text{con}}$  to  $\mathcal{C}_c$  of the  $L^2(\mathcal{L}_{\text{con}})$  estimates of (4.2), as in the bottom initial layer case. See Section 12.1.1.

## 12.2 Control of the last cones geodesic foliation

This section is dedicated to the independent control of the last cones geodesic foliation. The final goal of these results is the improvement of the Bootstrap Assumptions 3.30, which will be obtained in Section 12.3.

We have the following bounds for the null curvature components of the last cones geodesic foliation.

**Lemma 12.5.** *For all  $11/4 \leq \underline{u}' \leq \underline{u}^*$ ,  $5/4 \leq u' \leq \underline{u}^*$ , the following bounds hold*

$$\begin{aligned} |\alpha'| &\lesssim \varepsilon \underline{u}'^{-7/2}, & |\beta'| &\lesssim \varepsilon \underline{u}'^{-7/2}, \\ |\rho'| &\lesssim \varepsilon \underline{u}'^{-3} u'^{-1/2}, & |\sigma'| &\lesssim \varepsilon \underline{u}'^{-3} u'^{-1/2}, \\ |\underline{\beta}'| &\lesssim \varepsilon \underline{u}'^{-2} u'^{-3/2}, & |\underline{\alpha}'| &\lesssim \varepsilon \underline{u}'^{-1} u'^{-5/2}. \end{aligned} \quad (12.11)$$

*Proof.* In the exterior region, these bounds follow from the improved curvature bounds  $\mathfrak{R}_{\leq 1}^{\text{ext}} \lesssim \varepsilon$ , the Bootstrap Assumptions 3.30 and the transition formulas from Proposition 2.36. In the bottom interior region, these bounds follow from the the improved bounds in the maximal frame  $\mathfrak{R}_{\leq 1}^{\text{int}} \lesssim \varepsilon$  and the Bootstrap Assumptions 3.30. In the top interior region, these bounds follow from the following pointwise bound on the spacetime curvature tensor (where the norm is taken with respect to the frame  $(e_3', e_4')$ )

$$|\mathbf{R}| \lesssim \varepsilon t^{*-7/2}.$$

This bound holds by a  $t^*$ -rescaling, extension and local existence as in Section 6, and comparison between the frames given by the local existence and the frame  $(e_3', e_4')$ . See Section 8.1.1 and in particular estimate (8.6). This finishes the proof of the lemma.  $\square$

We have the following control for the null connection coefficients of the last cones geodesic foliation.

**Lemma 12.6.** *For all  $11/4 \leq \underline{u}' \leq \underline{u}^*$ ,  $5/4 \leq u' \leq \underline{u}^*$ , the following bounds hold*

$$\begin{aligned} \left| \text{tr} \underline{\chi}' + \frac{2}{r'} \right| &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, & |\hat{\chi}'| &\lesssim \varepsilon \underline{u}'^{-1} u'^{-3/2}, \\ |\zeta'| &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, & |\omega'| &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, \end{aligned} \quad (12.12)$$

and

$$\begin{aligned} \left\| r'^{-1/2} (r \nabla')^{\leq 1} \left( \text{tr} \underline{\chi}' + \frac{2}{r'} \right) \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, & \left\| r'^{-1/2} (r \nabla')^{\leq 1} \hat{\chi}' \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-1} u'^{-3/2}, \\ \left\| r'^{-1/2} (r \nabla')^{\leq 1} \zeta' \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, & \left\| r'^{-1/2} (r \nabla')^{\leq 1} \omega' \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, \end{aligned} \quad (12.13)$$

and

$$\begin{aligned} \left\| r'^{-1/2} \left( \text{tr} \chi' - \frac{2}{r'} \right) \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, & \left\| r'^{-1/2} \hat{\chi}' \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-2} u'^{-1/2}, \\ \left\| r'^{-1/2} \xi' \right\|_{L^\infty L^4(S')} &\lesssim \varepsilon \underline{u}'^{-5/2}, \end{aligned} \quad (12.14)$$

together with the additional relations (see (2.131) and (2.132))

$$\underline{\xi}' = 0, \quad \underline{\omega}' = 0, \quad \eta' = \zeta' = -\underline{\eta}', \quad (12.15)$$

and where we denoted by  $r'$  the area radius of the 2-spheres  $S'$ .

Moreover, we have the following control for the area radius  $r'$

$$\left| r' - \frac{1}{2}(\underline{u}' - u') \right| \lesssim \varepsilon r' \underline{u}'^{-2} u'^{-1/2}, \quad (12.16)$$

and for the optical defect  $\eta'$

$$|\eta'| \lesssim \varepsilon \underline{u}'^{-3/2}. \quad (12.17)$$

*Proof.* We first note that relations (12.15) are consequences of the definitions of Section 2.13.

The proof of the sup-norm estimates (12.12) are obtained by integration of the – rewritten using relations (12.15) – respective null transport equations (2.34c), (2.34b), (2.34d), (2.34i), using the sup-norm curvature bounds from Lemma 12.5 and the axis limits when  $r \rightarrow 0$  from Theorem 4.4 (the last cones geodesic foliation coincides with the foliation of Theorem 4.4).

The proof of the  $L^\infty L^4$  estimates (12.13) are obtained by

- ◊ Commuting by  $(r\nabla')$  and integrating equation (2.34c) for  $\nabla'_3 \text{tr}\underline{\chi}'$ ,
- ◊  $L^4(S)$  elliptic estimates<sup>3</sup> for equation (2.34p) for  $d\text{iv}'\underline{\hat{\chi}}'$ ,
- ◊ Commuting by  $d\text{iv}'$  and integrating equation (2.34d) for  $\nabla'_3 \zeta'$  renormalised using a mass aspect function  $\mu'$ , and using  $L^4(S)$  elliptic estimates,
- ◊ Commuting by  $(r\nabla')$  and integrating equation (2.34i) for  $\nabla'_3 \omega'$  renormalised using an auxiliary tensor  $\iota'$  (see the definition (2.69) of a similar tensor).

The proof of the  $L^\infty L^4$  estimates (12.14) then follow using (12.13) and integrating respectively equations (2.34f) and (2.34e) for  $\text{tr}\chi'$  and  $\hat{\chi}'$ , and equation (2.34g) that we rewrite using (2.34l) as

$$\nabla'_3 \xi' + \frac{1}{2} \text{tr}\underline{\chi}' \xi' = 2\nabla' \omega' - \underline{\hat{\chi}}' \cdot \xi' + 4\omega' \xi'.$$

The proof of the area radius estimate (12.16) is obtained from integrating the bounds (12.12) on  $\text{tr}\underline{\chi}'$  from the axis  $r = 0$ .

The proof of the optical defect estimate (12.17) follows from equations (2.133) and the limit  $\eta' \rightarrow 0$  when  $r' \rightarrow 0$  (see Theorem 4.4). This finishes the proof of the lemma.  $\square$

## 12.3 Control of transition coefficients and comparison of foliations

In this section, we recall that the Bootstrap Assumptions hold, that we have the following improved bounds for the null connection coefficients (see the definitions of Section 3.2.4)

$$\mathfrak{D}_{\leq 1}^{\text{ext}} + \overline{\mathfrak{D}}_{\leq 1}^{\text{ext}} \lesssim \varepsilon, \tag{12.18}$$

and for the interior connection coefficients (see the definitions of Section 3.2.5)

$$\mathcal{O}_{\leq 3, \gamma}^{\text{int}}[n] + \mathcal{O}_{\leq 2}^{\text{int}}[k] + \mathfrak{D}_{\leq 2}^{\mathcal{T}}[\nu] \lesssim \varepsilon, \tag{12.19}$$

that the results of Lemma 12.6 hold and that the bottom and conical initial layer are  $\varepsilon$ -close to Minkowski (see the definitions of Section 4.1).

We prove in Section 12.3.1 that under these assumptions, the following bounds hold on the last cones geodesic foliation in  $\mathcal{M}' \cap \mathcal{M}^{\text{ext}}$

$$|f'| \lesssim \varepsilon \underline{u}^{-3/2}, \quad |\underline{f}', \log \lambda'| \lesssim \varepsilon u'^{-3/2}, \tag{12.20}$$

and

$$|u - u'| \lesssim \varepsilon u'^{-1/2}, \quad |\underline{u} - \underline{u}'| \lesssim \varepsilon \underline{u}', \tag{12.21}$$

and in  $\mathcal{M}' \cap \mathcal{M}_{\text{bot}}^{\text{int}}$

$$\left| \mathbf{g} \left( \overline{T}, \frac{1}{2}(e_3' + e_4') \right) + 1 \right| \lesssim \varepsilon \underline{u}'^{-3/2}, \tag{12.22}$$

<sup>3</sup>We refer to [CK93, pp. 43–47] for statement and proof of  $L^4(S)$  elliptic estimates using Calderon-Zygmund theory.



and

$$\left| t - \frac{1}{2}(\underline{u}' + u') \right| \lesssim \varepsilon \underline{u}'^{-1/2}. \quad (12.23)$$

We also prove that the following bounds hold in the conical initial layer  $\mathcal{L}_{\text{con}}^{\text{ext}}$

$$\left| \tilde{f} \right| \lesssim \varepsilon \underline{u}^{-1}, \quad \left| \tilde{f}, \log \tilde{\lambda} \right| \lesssim \varepsilon, \quad (12.24)$$

and

$$|u - \tilde{u}| \lesssim \varepsilon, \quad |\underline{u} - \tilde{\underline{u}}| \lesssim \varepsilon \underline{u}, \quad (12.25)$$

and that the following bounds hold in the bottom initial layer:

$$\left| \mathbf{g} \left( \frac{1}{2}(e_3 + e_4), \bar{T}^{\text{bot}} \right) + 1 \right| \lesssim \varepsilon, \quad (12.26)$$

and

$$\left| x^0 - \frac{1}{2}(\underline{u} + u) \right| \lesssim \varepsilon, \quad (12.27)$$

in  $\mathcal{L}_{\text{bot}}^{\text{ext}}$ , and

$$\left| \mathbf{g}(\bar{T}, \bar{T}^{\text{bot}}) + 1 \right| \lesssim \varepsilon, \quad (12.28)$$

and

$$|t - x^0| \lesssim \varepsilon \quad (12.29)$$

in  $\mathcal{L}_{\text{bot}}^{\text{int}}$ .

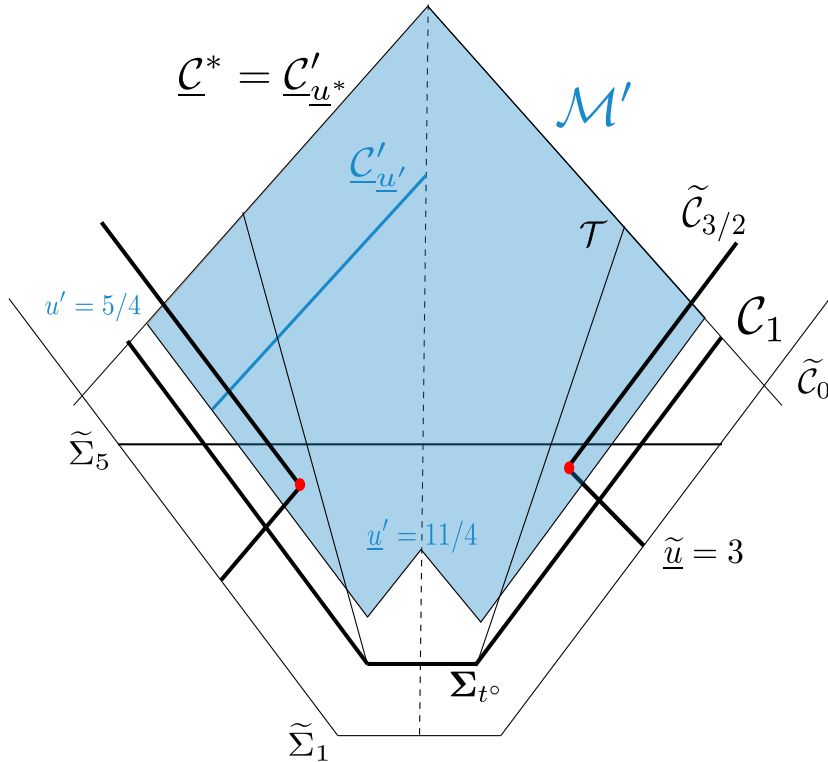


Figure 2: The last cones geodesic foliation and the initial layers.<sup>4</sup>

**Remark 12.7.** The bounds (12.20)–(12.23) improve the Bootstrap Assumptions 3.30 for the comparison to the last cones geodesic foliation. The bounds (12.24) and (12.25) improve the Bootstrap Assumptions 3.32 for the comparison to the double null conical initial layer foliation. The bounds (12.26)–(12.29) improve the Bootstrap Assumptions 3.31 for the comparison to the bottom initial layer coordinates.

<sup>4</sup>In red the 2-sphere  $\{\tilde{u} = 3/2, \underline{\tilde{u}} = 3\}$  used in Sections 12.3.2 and 12.3.3.

### 12.3.1 Last cones geodesic foliation comparisons

From the definition of the last cone geodesic foliation, we have at the tip of the axis  $\phi(\underline{u}^*)$

$$|f'| = |\underline{f}'| = \log \lambda' = 0. \quad (12.30)$$

Moreover, since  $\underline{\mathcal{C}}^* = \underline{\mathcal{C}}'_{\underline{u}'=\underline{u}^*}$ , we have (see the relations of Lemma 2.31)

$$\underline{f}' = 0 \quad (12.31)$$

on  $\underline{\mathcal{C}}^*$ .

Integrating equation (2.135f) on  $\underline{\mathcal{C}}^*$ , using the limits (12.30) and (12.31), the sup-norm estimates (12.18) for the null connection coefficients and the estimates from Lemma 12.6, we have

$$\left\| r'^{-1/2} f' \right\|_{L^\infty_{\underline{u}', \underline{u}^*} L^4(S')} \lesssim \varepsilon \underline{u}'^{-3/2}. \quad (12.32)$$

Integrating equation (2.135i), we also obtain on  $\underline{\mathcal{C}}^*$

$$\left\| r'^{-1/2} \log \lambda' \right\|_{L^\infty_{\underline{u}', \underline{u}^*} L^4(S')} \lesssim \varepsilon \underline{u}'^{-1} u'^{-1/2}. \quad (12.33)$$

We now integrate equation (2.135a) in the  $e_4'$  direction along the last cones geodesic foliation from  $\underline{\mathcal{C}}^*$  to  $\mathcal{T}^5$ , using the estimates (12.32) for  $f'$  on  $\underline{\mathcal{C}}^*$ , the estimate  $\xi' \sim \varepsilon \underline{u}'^{-5/2}$  from Lemma 12.6, and we have

$$\left\| r'^{-1/2} f' \right\|_{L^\infty L^4(S')} \lesssim \varepsilon \underline{u}'^{-3/2}. \quad (12.34)$$

Integrating (2.135g), using (12.34) and (12.31), we have

$$\left\| r'^{-1/2} \underline{f}' \right\|_{L^\infty L^4(S')} \lesssim \varepsilon \underline{u}'^{-1} u'^{-1/2}. \quad (12.35)$$

Integrating (2.135h), using (12.33), we have

$$\left\| r'^{-1/2} \log \lambda' \right\|_{L^\infty L^4(S')} \lesssim \varepsilon \underline{u}'^{-1} u'^{-1/2}. \quad (12.36)$$

Using moreover the equations (2.135c), (2.135d) and (2.135e) to obtain  $L^\infty L^4(S')$  control for tangential derivatives of  $f'$ ,  $\underline{f}'$ ,  $\log \lambda'$ , together with Sobolev estimates, we further deduce that in  $\mathcal{M}' \cap \mathcal{M}^{\text{ext}}$ , the following sup-norm bounds hold<sup>6</sup>

$$|f'| \lesssim \varepsilon \underline{u}'^{-3/2}, \quad |\underline{f}'| \lesssim \varepsilon \underline{u}'^{-3/2}, \quad |\log \lambda'| \lesssim \varepsilon \underline{u}'^{-3/2}. \quad (12.37)$$

Let define

$$\phi := \mathbf{g} \left( \bar{T}, \frac{1}{2}(e_3' + e_4') \right).$$

Using the bounds (12.37) on  $\mathcal{T}$ , relations (2.15) and estimates (12.19) for  $\nu$ , we obtain on  $\mathcal{T}$

$$|\phi + 1| \lesssim \varepsilon \underline{u}'^{-3/2}. \quad (12.38)$$

Differentiating  $\phi$  by  $e_3'$  and integrating up to the central axis, using the bounds (12.19) and the estimates from Lemma 12.6, one obtains  $L^\infty L^4(S')$  estimates in  $\mathcal{M}'_{\text{bot}}^{\text{int}}$  for  $\phi$ . Differentiating by  $\nabla'$ , one further obtains  $L^\infty L^4(S')$  estimates for  $\nabla' \phi$ , which together with Sobolev embeddings on the 2-spheres  $S'$  gives

$$|\phi + 1| \lesssim \varepsilon \underline{u}'^{-3/2} \quad (12.39)$$

<sup>5</sup>In order to obtain the desired  $L^4(S')$  estimates for  $f, \underline{f}, \lambda$  in a region covering  $\mathcal{T}$ , one actually integrates from  $\underline{\mathcal{C}}^*$  to the last sphere of  $S'$  touching  $\mathcal{T}$ . This requires to extend the double null foliation in a neighbourhood interior to  $\mathcal{T}$ , which can be easily achieved.

<sup>6</sup>Note that since  $\nabla' \underline{f}$  is related to  $\underline{\chi}$  which only decays as  $\underline{u}'^{-1} u'^{-3/2}$ , we lose the decay weight  $\underline{u}'^{-1}$  from (12.35). A similar loss occurs for  $\log \lambda'$ .

in  $\mathcal{M}' \cap \mathcal{M}_{\text{bot}}^{\text{int}}$ .

We now turn to the improvement of (12.21). At the tip of the central axis  $\phi(\underline{u}^*)$ , we have

$$\underline{u} = \underline{u}' = \underline{u}' = \underline{u}. \quad (12.40)$$

On  $\underline{\mathcal{C}}^* = \underline{\mathcal{C}}'_{\underline{u}'=\underline{u}^*}$ , we have

$$\underline{u}' = \underline{u}. \quad (12.41)$$

Using relations (2.1) and (2.129), that  $\underline{f}' = 0$  on  $\underline{\mathcal{C}}^*$ , and the definitions of the transitions coefficients from Lemma 2.31, we have

$$e_3'(u' - u) = -(e_3' - e_3)(u) = 2 \left(1 - \lambda'^{-1}\right). \quad (12.42)$$

Integrating the last equation, using (12.40) and the bounds (12.37) for  $\lambda'$ , we obtain on  $\underline{\mathcal{C}}^*$

$$|u' - u| \lesssim \varepsilon u'^{-1/2}. \quad (12.43)$$

From relations (2.1), (2.129) and the relations of Lemma 2.31, we have

$$e_4'(u' - u) = \eta' - e_4'(u) = \eta' - \frac{1}{2}\lambda|f'|^2.$$

Integrating the above equation, using that  $|\eta'| \lesssim \varepsilon \underline{u}'^{-3/2}$  and  $|f'|^2 \lesssim \varepsilon^2 \underline{u}'^{-3}$ , and estimate (12.43) on  $\underline{\mathcal{C}}^*$ , we deduce that in  $\mathcal{M}' \cap \mathcal{M}^{\text{ext}}$

$$|u' - u| \lesssim \varepsilon u'^{-1/2}. \quad (12.44)$$

Let  $Z$  be defined on  $\mathcal{T}$  by

$$Z := \frac{1}{2}\tau e_3 + \frac{1}{2} \left(1 - \frac{1}{2}\tau\eta\right) e_4,$$

such that  $Z(\underline{u}) = 1$ . Using the relations of Lemma 2.31 and the bounds  $|\eta| \lesssim \varepsilon \underline{u}^{-3/2}$ ,  $|\eta'| \lesssim \varepsilon \underline{u}^{-3/2}$ ,  $|f'|, |\underline{f}'|, |\log \lambda'| \lesssim \varepsilon \underline{u}^{-3/2}$  on  $\mathcal{M}' \cap \mathcal{T}$ , we have

$$|Z(\underline{u}' - \underline{u})| \lesssim \varepsilon \underline{u}^{-3/2},$$

on  $\mathcal{M}' \cap \mathcal{T}$ , and we infer by integration from  $S^*$  along  $Z$

$$|\underline{u}' - \underline{u}| \lesssim \varepsilon \underline{u}^{-1/2}, \quad (12.45)$$

on  $\mathcal{M}' \cap \mathcal{T}$ . Using (2.1), (2.129) and the relations of Lemma 2.31, we have

$$e_4'(\underline{u}' - \underline{u}) = 2(1 - \lambda') - \frac{1}{4}\lambda'|f'|^2\eta.$$

Integrating the above equation from  $\mathcal{M}' \cap \mathcal{T}$  to  $\underline{\mathcal{C}}^*$ , using the bounds (12.37) for  $\lambda'$  and  $f'$ , the bounds (12.18) for  $\eta$ , and the bounds (12.45) on  $\mathcal{T}$  we infer

$$|\underline{u}' - \underline{u}| \lesssim \varepsilon \underline{u}' \quad (12.46)$$

on  $\mathcal{M}' \cap \mathcal{M}^{\text{ext}}$ .

We now turn to the proof of (12.23). From the definition of  $t$  on  $\mathcal{T}$  and the bounds obtained for  $u' - u$  and  $\underline{u}' - \underline{u}$  in  $\mathcal{M}' \cap \mathcal{M}^{\text{ext}}$  and on  $\mathcal{M}' \cap \mathcal{T}$ , we have on  $\mathcal{M}' \cap \mathcal{T}$

$$\left| t - \frac{1}{2}(\underline{u}' + u') \right| \lesssim \varepsilon \underline{u}'^{-1/2}, \quad (12.47)$$

On  $\mathcal{M}_{\text{bot}}^{\text{int}}$ , using relations (2.6) and (2.130), we have

$$-\mathbf{D} \left( t - \frac{1}{2}(\underline{u}' + u') \right) = n^{-1}\bar{T} - \frac{1}{2}(e_3' + Y') = n^{-1}\bar{T} - \frac{1}{2}(e_3' + e_4') - \frac{1}{4}\eta'e_3'.$$

Using the bounds  $|\eta'| \lesssim \varepsilon \underline{u}'^{-3/2}$ , (12.18), and the bounds (12.39) obtained for  $\phi$ , we deduce

$$\left| \mathbf{D} \left( t - \frac{1}{2}(\underline{u}' + u') \right) \right| \lesssim \varepsilon \underline{u}'^{-3/2},$$

which combined with (12.47) on  $\mathcal{T}$  gives (12.23) as desired.

### 12.3.2 Conical initial layer comparisons

We first have the following lemma, which is a consequence of the  $\varepsilon$ -closeness of the bottom initial layer to Minkowski (see the definitions from Section 4.1) and the (local) definition of the last cones geodesic foliation.

**Lemma 12.8.** *In  $\mathcal{M}' \cap \mathcal{L}_{\text{bot}}$ , the following comparison bounds hold between the last cones geodesic foliation and the bottom initial layer spacetime coordinates*

$$\begin{aligned} \left| x^0 - \frac{1}{2}(\underline{u}' + u') \right| &\lesssim \varepsilon, \\ \left| \sqrt{\sum_{i=1}^3 (x^i)^2} - \frac{1}{2}(\underline{u}' - u') \right| &\lesssim \varepsilon. \end{aligned} \quad (12.48)$$

Moreover, we have in  $\mathcal{M}' \cap \mathcal{L}_{\text{bot}}$

$$\left| \mathbf{g} \left( \overline{T}^{\text{bot}}, \frac{1}{2}(e_3' + e_4') \right) + 1 \right| \lesssim \varepsilon, \quad (12.49)$$

and in  $\mathcal{M}' \cap \mathcal{L}_{\text{bot}}^{\text{ext}}$

$$\left| \mathbf{g} \left( \overline{N}^{\text{bot}}, \frac{1}{2}(e_4' - e_3') \right) - 1 \right| \lesssim \varepsilon. \quad (12.50)$$

*Proof.* The centre  $\phi(1)$  of  $\widetilde{\Sigma}_1$  is chosen such that  $x^i(\phi(1)) = 0$  (see also Remark 4.1). The proof then follows from a local control of the null geodesic equation using sup-norm estimates in  $\mathcal{L}_{\text{bot}}$ . Details are left to the reader.  $\square$

From the estimates (12.21) for  $u', \underline{u}'$ , one deduces that for  $\varepsilon$  sufficiently small, we have the following inclusion (see also Figure 2)

$$\{\tilde{u} = 3/2, \tilde{\underline{u}} = 3\} \subset \mathcal{L}_{\text{bot}} \cap \mathcal{L}_{\text{con}} \cap \mathcal{M}^{\text{ext}} \cap \mathcal{M}'.$$

From the comparison bounds (12.37) for the null frames  $(e_3', e_4')$  and  $(e_3, e_4)$ , and from the initial bounds (12.49) and (12.50), we deduce

$$\begin{aligned} \left| \mathbf{g} \left( \overline{T}^{\text{bot}}, \frac{1}{2}(e_3 + e_4) \right) + 1 \right| &\lesssim \varepsilon, \\ \left| \mathbf{g} \left( \overline{N}^{\text{bot}}, \frac{1}{2}(e_4 - e_3) \right) - 1 \right| &\lesssim \varepsilon, \end{aligned} \quad (12.51)$$

on  $\{\tilde{u} = 3/2, \tilde{\underline{u}} = 3\}$ . Combining (12.51) and the compatibility assumptions (4.3a) between the bottom initial layer and the conical initial layer frames, we deduce the following bounds for the transition coefficients  $(\tilde{f}, \tilde{\underline{f}}, \tilde{\lambda})$  between the null frames  $(\tilde{e}_3, \tilde{e}_4)$  and  $(e_3, e_4)$  on  $\{\tilde{u} = 3/2, \tilde{\underline{u}} = 3\}$

$$\left| \tilde{f}, \tilde{\underline{f}}, \log \tilde{\lambda} \right| \lesssim \varepsilon. \quad (12.52)$$

Integrating successively equations (2.135b), (2.135f), (2.135i) along  $\tilde{e}_3$  from  $\{\tilde{u} = 3/2, \tilde{\underline{u}} = 3\}$  to  $\mathcal{C}_1 \cap \{\tilde{\underline{u}} = 3\}$ , using the bounds (12.18) and (4.2b) for the null connection coefficients of each foliation, and the bounds (12.52) on  $\{\tilde{u} = 3/2, \tilde{\underline{u}} = 3\}$ , we infer that the bound

$$\left| \tilde{f}, \tilde{\underline{f}}, \log \tilde{\lambda} \right| \lesssim \varepsilon \quad (12.53)$$

holds on  $\mathcal{M}^{\text{ext}} \cap \{\tilde{\underline{u}} = 3\}$ . We rewrite (2.135a) schematically as

$$\nabla_4 \tilde{f} + \frac{1}{2} \text{tr} \chi \tilde{f} = \text{Err},$$

which integrated along  $e_4$ , from  $\{\tilde{\underline{u}} = 3\} \cap \mathcal{M}^{\text{ext}}$  into the region  $\mathcal{L}_{\text{con}}^{\text{ext}}$ , using the Bootstrap Assumptions 3.32, the bounds (12.18) and (4.2b) for the null connection coefficients in  $\mathcal{L}_{\text{con}}^{\text{ext}}$  and the bounds (12.53) on  $\{\tilde{\underline{u}} = 3\}$  gives

$$\underline{u} |\tilde{f}| \lesssim \varepsilon, \quad (12.54)$$

on  $\mathcal{L}_{\text{con}}^{\text{ext}}$ . Integrating equations (2.135g) and (2.135h) from  $\{\tilde{u} = 3\} \cap \mathcal{M}^{\text{ext}}$  into  $\mathcal{L}_{\text{con}}^{\text{ext}}$  yields

$$\left| \underline{\tilde{f}}, \log \tilde{\lambda} \right| \lesssim \varepsilon, \quad (12.55)$$

as desired. This finishes the proof of (12.24).

We turn to the proof of (12.25). We deduce from the bounds (12.21), (12.48) and the compatibility assumption (4.3b) between the bottom and conical initial layers that

$$|\underline{u} - \underline{\tilde{u}}| \lesssim \varepsilon, \quad |u - \tilde{u}| \lesssim \varepsilon,$$

at the 2-sphere  $\{\tilde{u} = 3/2, \underline{\tilde{u}} = 3\}$ . Differentiating by  $\tilde{e}_3$  and integrating from  $\{\tilde{u} = 3/2, \underline{\tilde{u}} = 3\}$  to  $\mathcal{C}_1 \cap \{\underline{\tilde{u}} = 3\}$ , we further obtain

$$|\underline{u} - \underline{\tilde{u}}| \lesssim \varepsilon, \quad |u - \tilde{u}| \lesssim \varepsilon, \quad (12.56)$$

on  $\{\underline{\tilde{u}} = 3\} \cap \mathcal{M}^{\text{ext}}$ . Using relations (2.1) and the definitions from Section 2.14 we have the following equations

$$\begin{aligned} e_4(u - \tilde{u}) &= -\frac{1}{4}|\tilde{f}|^2 \tilde{e}_3(\tilde{u}) = -\frac{1}{2}|\tilde{f}|^2 \tilde{\Omega}^{-1}, \\ e_4(\underline{u} - \underline{\tilde{u}}) &= 2(1 - \tilde{\lambda}). \end{aligned}$$

Integrating the above equations along  $e_4$  from  $\{\underline{\tilde{u}} = 3\} \cap \mathcal{M}^{\text{ext}}$  into  $\mathcal{L}_{\text{con}}^{\text{ext}}$ , using the bounds (12.56) on  $\{\underline{\tilde{u}} = 3\}$  and (12.54), (12.55), (4.2b) we deduce

$$|u - \tilde{u}| \lesssim \varepsilon, \quad |\underline{u} - \underline{\tilde{u}}| \lesssim \varepsilon \underline{u},$$

as desired. This finishes the proof of (12.25).

### 12.3.3 Bottom initial layer comparisons

In this section, we prove the comparison estimates (12.26), (12.27), (12.28), (12.29) for the bottom initial layer. Following similar arguments as previously, one has

$$\left| \mathbf{D} \left( \mathbf{g} \left( \bar{T}^{\text{bot}}, \frac{1}{2}(e_3 + e_4) \right) \right) \right| \lesssim \varepsilon$$

in  $\mathcal{L}_{\text{bot}} \cap \mathcal{M}^{\text{ext}}$ , and we thus deduce from the bound (12.51) on the 2-sphere  $\{\tilde{u} = 3/2, \underline{\tilde{u}} = 3\} \subset \mathcal{L}_{\text{bot}}^{\text{ext}}$  by integration in  $\mathcal{L}_{\text{bot}}^{\text{ext}}$  that (12.26) holds in  $\mathcal{L}_{\text{bot}}^{\text{ext}}$ . Further integration in  $\mathcal{L}_{\text{bot}}^{\text{int}}$  also gives (12.28). The bounds (12.27) and (12.29) are then also deduced by integration, using the just obtained bounds (12.51) and (12.28) and the bounds (12.21), (12.48) on  $\{\tilde{u} = 3/2, \underline{\tilde{u}} = 3\}$ .

# A. Global harmonic coordinates

This section is dedicated to the proof of Theorem 4.3.

## A.1 Assumptions

Let  $\Sigma$  be a Riemannian manifold diffeomorphic to  $\mathbb{D}$  the unit coordinate disk of  $\mathbb{R}^3$ . Let  $C > 0$  be a (large) constant. We assume that the following Sobolev embeddings hold on  $\Sigma$

$$\begin{aligned} \|F\|_{L^6(\Sigma)} &\leq C \|\nabla^{\leq 1} F\|_{L^2(\Sigma)}, \\ \|F\|_{L^\infty(\Sigma)} &\leq C \|\nabla^{\leq 2} F\|_{L^2(\Sigma)}, \end{aligned} \tag{A.1}$$

for all  $\Sigma$ -tangent tensors  $F$ .

Assume that we have the following  $L^{3/2}(\Sigma) - L^1(\partial\Sigma)$ -Poincaré inequality

$$\|F\|_{L^2(\Sigma)} \leq C \left( \|\nabla F\|_{L^{3/2}(\Sigma)} + \|F\|_{L^1(\partial\Sigma)} \right). \tag{A.2}$$

for all tensors  $F$ .

Assume that we have the following trace estimate from  $\Sigma$  on  $\partial\Sigma$

$$\|F\|_{H^{1/2}(\partial\Sigma)} \leq C \|\nabla^{\leq 1} F\|_{L^2(\Sigma)}, \tag{A.3}$$

for all tensors  $F$ .

**Remark A.1.** *To obtain Poincaré estimate (A.2), we can rely on the following sequence of more standard Poincaré and Sobolev embeddings which hold true in the Euclidean disk case*

$$\begin{aligned} \|F\|_{L^1(\Sigma)} &\lesssim \|\nabla F\|_{L^1(\Sigma)} + \|F\|_{L^1(\partial\Sigma)}, \\ \|F\|_{L^{3/2}(\Sigma)} &\lesssim \|\nabla^{\leq 1} F\|_{L^1(\Sigma)}, \\ \|F\|_{L^2(\Sigma)} &\lesssim \|\nabla^{\leq 1} F\|_{L^{3/2}(\Sigma)}. \end{aligned}$$

## A.2 Definitions and preliminary results

In this section, we recall definitions from Section 2 and state preliminary (known) results.

### A.2.1 Definitions and identities on $\partial\Sigma$

**Definition A.2** (Conformal isomorphism and  $x^i$  on  $\partial\Sigma$ ). *A conformal isomorphism  $\Phi$  is a diffeomorphism from  $\partial\Sigma$  onto the Euclidean unit 2-sphere  $\mathbb{S}$ , such that there exists a conformal factor  $\phi > 0$  satisfying*

$$\Phi_{\#} g_{\partial\Sigma} = \phi^2 g_{\mathbb{S}},$$

where  $\Phi_{\#}$  denotes the push-forward by  $\Phi$ .

To a fixed conformal isomorphism, we associate a triplet of function  $(x^i)_{i=1,2,3}$  on  $\partial\Sigma$  to be the pull-back by  $\Phi$  of the standard Cartesian coordinates restricted on  $\mathbb{S}$ .

We moreover say that  $\Phi$  is centred if the following conditions hold for the functions  $x^i$

$$\int_{\partial\Sigma} x^i = 0, \quad i = 1, \dots, 3.$$

We have the following identities on  $\partial\Sigma$ .

**Lemma A.3** (Conformal identities on  $\partial\Sigma$ ). *On  $\partial\Sigma$ , we have the following identities*

$$\sum_{i=1}^3 (x^i)^2 = 1, \tag{A.4}$$

and

$$\Delta_{g_{\partial\Sigma}} x^i + 2x^i = 2x^i (1 - \phi^{-2}). \tag{A.5}$$

*Proof.* The first identity (A.4) is a direct consequence of Definition A.2. For the second identity (A.5), we have using the Euclidean Laplacian in spherical coordinates that

$$\Delta_{g_s} x^i + 2x^i = 0.$$

From the conformal invariance of the Laplacian, we have

$$\Delta_{g_s} x^i = \phi^2 \Delta_g x^i,$$

and the desired formula follows.  $\square$

### A.2.2 Uniformisation theorem on $\partial\Sigma$

**Lemma A.4** (Uniformisation Theorem on  $\partial\Sigma$ ). *Under the assumptions of Theorem 4.3, there exists a unique – up to isomorphisms of  $\mathbb{S}$  – centred conformal isomorphism of the boundary  $(\partial\Sigma, g)$  to the Euclidean sphere  $\mathbb{S}$  and we have the following quantitative bounds for the conformal factor  $\phi$*

$$\|\nabla^{\leq 1}(\phi - 1)\|_{H^{1/2}(\partial\Sigma)} \lesssim \varepsilon. \quad (\text{A.6})$$

Moreover, for all  $k \geq 0$ , we have the following higher regularity estimates

$$\|\nabla^{\leq k+1}(\phi - 1)\|_{H^{1/2}(\partial\Sigma)} \lesssim C_k \left( \|\nabla^{\leq k} \text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{\leq k}(\theta - g)\|_{H^{1/2}(\partial\Sigma)} + \varepsilon \right). \quad (\text{A.7})$$

*Proof.* Arguing as in [Czi19a], one can obtain from the  $L^2(\Sigma)$  and  $H^{1/2}(\partial\Sigma)$  bounds (4.5) that

$$\|K - 1\|_{H^{-1/2}(\partial\Sigma)} \lesssim \varepsilon,$$

where  $H^{-1/2}(\partial\Sigma)$  is an (appropriately defined) fractional Sobolev space on  $\partial\Sigma$  (see [Czi19a]). The lemma then follows from an adaptation of Theorem [KS19b, Theorem 3.1] (or from adaptations of the argument of [Sha14, Section 6]).  $\square$

### A.2.3 Definitions and identities on $\Sigma$

We now have the following definition for  $x^i$  on  $\Sigma$ .

**Definition A.5** ( $x^i$  on  $\Sigma$ ). *We define the  $x^i$  on  $\Sigma$  to be the solutions of the following Dirichlet problem*

$$\begin{aligned} \Delta_g x^i &= 0, \\ x^i|_{\partial\Sigma} &= x^i. \end{aligned} \quad (\text{A.8})$$

**Remark A.6.** *From the maximum principle, one has on  $\Sigma$*

$$|x^i| \leq 1. \quad (\text{A.9})$$

We have the following energy and Bochner identities in  $\Sigma$ .

**Lemma A.7** (Energy and Bochner identities in  $\Sigma$ ). *We have the following energy identity on  $\Sigma$*

$$\|\nabla x^i\|_{L^2(\Sigma)}^2 = \int_{\partial\Sigma} x^i N(x^i), \quad (\text{A.10})$$

and the following Bochner identity on  $\Sigma$

$$\|\nabla^2 x^i\|_{L^2(\Sigma)}^2 = - \int_{\Sigma} \text{Ric} \cdot \nabla x^i \cdot \nabla x^i + \int_{\partial\Sigma} \nabla x^i \cdot \nabla_N \nabla x^i, \quad (\text{A.11})$$

for all  $i = 1, 2, 3$  and where here and in the following,  $N$  denotes the outward-pointing unit normal to  $\partial\Sigma$ .

*Proof.* The energy identity (A.10) is obtained by multiplying Laplace equation (A.8) by  $x^i$  and integrating by part.

Bochner identity (A.11) is obtained by using the following commutation formula

$$\Delta \nabla x^i = \text{Ric} \cdot \nabla x^i,$$

contracted with  $\nabla x^i$  and integrating by part.  $\square$

### A.3 A refined Bochner identity

This section is dedicated to the proof of the following refined Bochner identity, which is at the centre of the proof of Theorem 4.3

$$\sum_{i=1}^3 \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 + 2 \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i)^2 = -3 \sum_{i=1}^3 \int_{\Sigma} \left( \text{Ric} - \frac{1}{3} Rg \right) \cdot \nabla x^i \cdot \nabla x^i + G, \quad (\text{A.12})$$

where

$$\begin{aligned} G := & \sum_{i=1}^3 \int_{\partial\Sigma} \left( (\theta_{ab} - \mathcal{G}_{ab}) x^i (\nabla_a \nabla_b x^i + 2x^i \mathcal{G}_{ab}) \right. \\ & \left. - (N(x^i) - x^i) (2(\mathcal{A} x^i + 2x^i) - (\text{tr}\theta - 2)(N(x^i) + x^i)) \right) \\ & + \frac{1}{2} \int_{\partial\Sigma} (\text{tr}\theta - 2)^2 - \int_{\partial\Sigma} |\hat{\theta}|^2 + 2 \sum_{i=1}^3 \int_{\partial\Sigma} \left( \text{Ric}_{NN} - \frac{1}{2} R \right) x^i (N(x^i) - x^i) \\ & - 2 \sum_{i=1}^3 \int_{\Sigma} x^i \left( \text{Ric} - \frac{1}{2} Rg \right) \cdot \nabla^2 x^i. \end{aligned}$$

Identity (A.12) is obtained by further computation starting from the original Bochner identity (A.11). Its proof is postponed to the end of this section and uses the following two lemmas.

**Lemma A.8.** *We have*

$$\sum_{i=1}^3 \int_{\partial\Sigma} \nabla x^i \cdot \nabla_N \nabla x^i = 2 \int_{\partial\Sigma} (\phi^{-2} - 1) - 2 \int_{\partial\Sigma} (\text{tr}\theta - 2) - 2 \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i)^2 + E, \quad (\text{A.13})$$

where

$$\begin{aligned} E := & \sum_{i=1}^3 \int_{\partial\Sigma} \left( (\theta_{ab} - \mathcal{G}_{ab}) x^i (\nabla_a \nabla_b x^i + 2x^i \mathcal{G}_{ab}) \right. \\ & \left. - (N(x^i) - x^i) (2(\mathcal{A} x^i + 2x^i) - (\text{tr}\theta - 2)(N(x^i) + x^i)) \right). \end{aligned}$$

*Proof.* We decompose the integrand in two parts as follows

$$\nabla x^i \cdot \nabla_N \nabla x^i = \nabla_a x^i \nabla_N \nabla_a x^i + \nabla_N x^i \nabla_N \nabla_N x^i. \quad (\text{A.14})$$

For the first term of (A.14), we have<sup>1</sup>

$$\begin{aligned} \nabla_a x^i \nabla_N \nabla_a x^i &= \nabla_a x^i \nabla_a \nabla_N x^i \\ &= \nabla_a x^i (\nabla_a N(x^i) - \theta_{ab} \nabla_b x^i) \\ &= \nabla_a x^i (\nabla_a (N(x^i) - x^i) - (\theta_{ab} - \mathcal{G}_{ab}) \nabla_b x^i). \end{aligned}$$

Integrating that relation on  $\partial\Sigma$  and integrating by part gives

$$\begin{aligned} & \int_{\partial\Sigma} \nabla_a x^i \nabla_N \nabla_a x^i \\ &= \int_{\partial\Sigma} \nabla_a x^i (\nabla_a (N(x^i) - x^i) - (\theta_{ab} - \mathcal{G}_{ab}) \nabla_b x^i) \\ &= \int_{\partial\Sigma} (-2x^i - \mathcal{A} x^i) (N(x^i) - x^i) - \int_{\partial\Sigma} (\theta_{ab} - \mathcal{G}_{ab}) \nabla_a x^i \nabla_b x^i \\ & \quad + \int_{\partial\Sigma} 2x^i (N(x^i) - x^i). \end{aligned}$$

<sup>1</sup>We recall that  $\theta_{ab} = \nabla_a N_b$  (see the definition in Theorem 4.3).



Integrating by part and using relation (A.4), we have

$$\begin{aligned}
 & \sum_{i=1}^3 - \int_{\partial\Sigma} (\theta_{ab} - \mathcal{J}_{ab}) \nabla_a x^i \nabla_b x^i \\
 &= \sum_{i=1}^3 \left( \int_{\partial\Sigma} \frac{1}{2} \nabla_a \theta_{ab} \nabla_b (x^i)^2 + \int_{\partial\Sigma} (\theta_{ab} - \mathcal{J}_{ab}) x^i \nabla_a \nabla_b x^i \right) \\
 &= - \int_{\partial\Sigma} (\text{tr}\theta - 2) + \sum_{i=1}^3 \int_{\partial\Sigma} (\theta_{ab} - \mathcal{J}_{ab}) x^i (\nabla_a \nabla_b x^i + x^i \mathcal{J}_{ab}).
 \end{aligned}$$

Summing for  $i$  from 1 to 3, we therefore infer

$$\begin{aligned}
 & \sum_{i=1}^3 \int_{\partial\Sigma} \nabla_a x^i \nabla_N \nabla_a x^i \\
 &= \sum_{i=1}^3 \int_{\partial\Sigma} (-2x^i - \mathcal{K} x^i) (N(x^i) - x^i) + \sum_{i=1}^3 \int_{\partial\Sigma} (\theta_{ab} - \mathcal{J}_{ab}) x^i (\nabla_a \nabla_b x^i + x^i \mathcal{J}_{ab}) \quad (\text{A.15}) \\
 &+ \sum_{i=1}^3 \int_{\partial\Sigma} 2x^i (N(x^i) - x^i) - \int_{\partial\Sigma} (\text{tr}\theta - 2).
 \end{aligned}$$

Let turn to the second term of (A.14). We first recall that Laplace equation (A.8) can be rewritten as

$$\nabla_N \nabla_N x^i + \mathcal{K} x^i + \text{tr}\theta N(x^i) = 0.$$

Using this, we compute

$$\begin{aligned}
 \nabla_N x^i \nabla_N \nabla_N x^i &= -N(x^i) (\mathcal{K} x^i + \text{tr}\theta N(x^i)) \\
 &= -N(x^i) ((\mathcal{K} x^i + 2x^i) + 2(N(x^i) - x^i) + (\text{tr}\theta - 2)N(x^i)) \\
 &= -x^i ((\mathcal{K} x^i + 2x^i) + 2(N(x^i) - x^i) + (\text{tr}\theta - 2)N(x^i)) \\
 &\quad - (N(x^i) - x^i) ((\mathcal{K} x^i + 2x^i) + 2(N(x^i) - x^i) + (\text{tr}\theta - 2)N(x^i)) \\
 &= -x^i (\mathcal{K} x^i + 2x^i) - 2x^i (N(x^i) - x^i) - (x^i)^2 (\text{tr}\theta - 2) \\
 &\quad - (N(x^i) - x^i) ((\mathcal{K} x^i + 2x^i) + 2(N(x^i) - x^i) + (\text{tr}\theta - 2)(N(x^i) + x^i))
 \end{aligned}$$

Integrating this relation on  $\partial\Sigma$ , gives

$$\begin{aligned}
 & \int_{\partial\Sigma} \nabla_N x^i \nabla_N \nabla_N x^i \\
 &= \int_{\partial\Sigma} -x^i (\mathcal{K} x^i + 2x^i) - \int_{\partial\Sigma} (x^i)^2 (\text{tr}\theta - 2) + \int_{\partial\Sigma} -2x^i (N(x^i) - x^i) \\
 &\quad - \int_{\partial\Sigma} (N(x^i) - x^i) ((\mathcal{K} x^i + 2x^i) + 2(N(x^i) - x^i) + (\text{tr}\theta - 2)(N(x^i) + x^i)).
 \end{aligned}$$

Using relations (A.5), (A.4), we have

$$\sum_{i=1}^3 - \int_{\partial\Sigma} x^i (\mathcal{K} x^i + 2x^i) = - \sum_{i=1}^3 2 \int_{\partial\Sigma} (x^i)^2 (1 - \phi^{-2}) = 2 \int_{\partial\Sigma} (\phi^{-2} - 1).$$

Summing over  $i$  and using this relation and (A.4), we infer

$$\begin{aligned}
 & \sum_{i=1}^3 \int_{\partial\Sigma} \nabla_N x^i \nabla_N \nabla_N x^i \\
 &= 2 \int_{\partial\Sigma} (\phi^{-2} - 1) - \int_{\partial\Sigma} (\text{tr}\theta - 2) - 2 \sum_{i=1}^3 \int_{\partial\Sigma} x^i (N(x^i) - x^i) \quad (\text{A.16}) \\
 &\quad - \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i) ((\mathcal{K} x^i + 2x^i) + 2(N(x^i) - x^i) + (\text{tr}\theta - 2)(N(x^i) + x^i)).
 \end{aligned}$$

Summing (A.15) and (A.16), we have

$$\begin{aligned} \sum_{i=1}^3 \int_{\partial\Sigma} \nabla x^i \cdot \nabla_N \nabla x^i &= 2 \int_{\partial\Sigma} (\phi^{-2} - 1) - 2 \int_{\partial\Sigma} (\text{tr}\theta - 2) - 2 \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i)^2 \\ &\quad + \sum_{i=1}^3 \int_{\partial\Sigma} (\theta_{AB} - \not\theta_{AB}) x^i (\nabla_A \nabla_B x^i + 2x^i \not\theta_{AB}) \\ &\quad - \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i) (2(\not\Delta x^i + 2x^i) + (\text{tr}\theta - 2)(N(x^i) + x^i)), \end{aligned}$$

which finishes the proof of the lemma.  $\square$

We have the following lemma, which treats the linear boundary term of Lemma A.8.

**Lemma A.9.** *We have*

$$2 \int_{\partial\Sigma} (\phi^{-2} - 1) - 2 \int_{\partial\Sigma} (\text{tr}\theta - 2) = -2 \sum_{i=1}^3 \int_{\Sigma} \left( \text{Ric} - \frac{1}{2} Rg \right) \cdot \nabla x^i \cdot \nabla x^i + F, \quad (\text{A.17})$$

where

$$\begin{aligned} F &:= \frac{1}{2} \int_{\partial\Sigma} (\text{tr}\theta - 2)^2 - \int_{\partial\Sigma} |\hat{\theta}|^2 + 2 \sum_{i=1}^3 \int_{\partial\Sigma} \left( \text{Ric}_{NN} - \frac{1}{2} R \right) x^i (N(x^i) - x^i) \\ &\quad - 2 \sum_{i=1}^3 \int_{\Sigma} x^i \left( \text{Ric} - \frac{1}{2} Rg \right) \cdot \nabla^2 x^i. \end{aligned}$$

*Proof.* We define the following Einstein tensor on  $\Sigma$

$$G := \text{Ric} - \frac{1}{2} Rg.$$

Using Gauss equation (see [CK93, equation 5.0.5d]), we obtain

$$G_{NN} = -K + \frac{1}{4} \text{tr}\theta^2 - \frac{1}{2} |\hat{\theta}|^2,$$

on  $\partial\Sigma$ . Integrating the above on  $\partial\Sigma$ , using Gauss-Bonnet formula, we infer

$$\begin{aligned} \int_{\partial\Sigma} G_{NN} &= -4\pi + \int_{\partial\Sigma} 1 + \int_{\partial\Sigma} (\text{tr}\theta - 2) + \frac{1}{4} \int_{\partial\Sigma} (\text{tr}\theta - 2)^2 - \frac{1}{2} \int_{\partial\Sigma} |\hat{\theta}|^2, \\ &= - \int_{\partial\Sigma} (\phi^{-2} - 1) + \int_{\partial\Sigma} (\text{tr}\theta - 2) + \frac{1}{4} \int_{\partial\Sigma} (\text{tr}\theta - 2)^2 - \frac{1}{2} \int_{\partial\Sigma} |\hat{\theta}|^2. \end{aligned}$$

We thus deduce that

$$2 \int_{\partial\Sigma} (\phi^{-2} - 1) - 2 \int_{\partial\Sigma} (\text{tr}\theta - 2) = -2 \int_{\partial\Sigma} G_{NN} + F^1, \quad (\text{A.18})$$

where

$$F^1 := \frac{1}{2} \int_{\partial\Sigma} (\text{tr}\theta - 2)^2 - \int_{\partial\Sigma} |\hat{\theta}|^2.$$

Using Bianchi identity on  $\Sigma$

$$\text{div}G = 0,$$

and applying Stokes formula, we have

$$\int_{\Sigma} G \cdot \nabla x^i \cdot \nabla x^i = \int_{\partial\Sigma} G_{lN} x^i \nabla_l x^i - \int_{\Sigma} x^i G \cdot \nabla^2 x^i. \quad (\text{A.19})$$

Summing over  $i$  and using (A.4), we have

$$\begin{aligned} \sum_{i=1}^3 \int_{\partial\Sigma} G_{Ni} x^i \nabla_i x^i &= \sum_{i=1}^3 \int_{\partial\Sigma} G_{NN} x^i N(x^i) + \frac{1}{2} \int_{\partial\Sigma} G_{Na} \nabla_a \left( \sum_{i=1}^3 (x^i)^2 \right) \\ &= \int_{\partial\Sigma} G_{NN} + \sum_{i=1}^3 \int_{\partial\Sigma} G_{NN} x^i (N(x^i) - x^i). \end{aligned} \quad (\text{A.20})$$

Combining (A.19) and (A.20), we infer

$$-2 \int_{\partial\Sigma} G_{NN} = -2 \sum_{i=1}^3 \int_{\Sigma} G \cdot \nabla x^i \cdot \nabla x^i + F^2, \quad (\text{A.21})$$

where

$$F^2 := 2 \sum_{i=1}^3 \int_{\partial\Sigma} G_{NN} x^i (N(x^i) - x^i) - 2 \sum_{i=1}^3 \int_{\Sigma} x^i G \cdot \nabla^2 x^i.$$

Combining (A.18) and (A.21) then gives the desired result.  $\square$

*Proof of (A.12).* Using the results of Lemmas A.8 and A.9, and Bochner identity (A.11), we have

$$\begin{aligned} \sum_{i=1}^3 \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 &= - \sum_{i=1}^3 \int_{\Sigma} \text{Ric} \cdot \nabla x^i \cdot \nabla x^i + \sum_{i=1}^3 \int_{\partial\Sigma} \nabla x^i \cdot \nabla_N \nabla x^i \\ &= - \sum_{i=1}^3 \int_{\Sigma} \text{Ric} \cdot \nabla x^i \cdot \nabla x^i - 2 \sum_{i=1}^3 \int_{\Sigma} \left( \text{Ric} - \frac{1}{2} Rg \right) \cdot \nabla x^i \cdot \nabla x^i \\ &\quad - 2 \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i)^2 + E + F \\ &= -3 \sum_{i=1}^3 \int_{\Sigma} \left( \text{Ric} - \frac{1}{3} Rg \right) \cdot \nabla x^i \cdot \nabla x^i - 2 \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i)^2 + E + F, \end{aligned}$$

and using the expressions of E, F this concludes the proof of (A.12).  $\square$

## A.4 Refined Bochner estimate

This section is dedicated to show that, using the refined Bochner identity (A.12) together with the assumptions (4.5) of Theorem 4.3 one has the following estimate

$$\sum_{i=1}^3 \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 + \sum_{i=1}^3 \int_{\partial\Sigma} (N(x^i) - x^i)^2 + \|B\|_{L^2(\Sigma)}^2 \lesssim \varepsilon^2, \quad (\text{A.22})$$

where  $B$  is the following 2-tensor on  $\Sigma$

$$B := \sum_{i=1}^3 \nabla x^i \otimes \nabla x^i - g,$$

where  $\otimes$  denotes the standard tensorial product on  $T\Sigma$ . As a consequence, we will also obtain the following estimates

$$\|\nabla x^i\|_{L^2(\Sigma)} + \|\nabla x^i\|_{L^6(\Sigma)} \lesssim 1. \quad (\text{A.23})$$

The proof of (A.22) and (A.23) is postponed to the end of this section. It relies on the following three lemmas.

**Lemma A.10.** *We have*

$$\begin{aligned} &\sum_{i=1}^3 \left( \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2 \right) \\ &\lesssim \varepsilon \left( \|\nabla^2 x^i\|_{L^2(\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right) \left( 1 + \|\nabla^{\leq 2} x^i\|_{L^2(\Sigma)} \right) + \varepsilon \|B\|_{L^2(\Sigma)} + \varepsilon^2. \end{aligned} \quad (\text{A.24})$$

*Proof.* Using the assumptions (4.5), we have

$$\begin{aligned} -3 \sum_{i=1}^3 \int_{\Sigma} \left( \text{Ric} - \frac{1}{3} Rg \right) \cdot \nabla x^i \cdot \nabla x^i &= -3 \int_{\Sigma} \left( \text{Ric} - \frac{1}{3} Rg \right) \cdot \mathbf{B} \\ &\lesssim \|\text{Ric}\|_{L^2(\Sigma)} \|\mathbf{B}\|_{L^2(\Sigma)} \\ &\lesssim \varepsilon \|\mathbf{B}\|_{L^2(\Sigma)}. \end{aligned} \quad (\text{A.25})$$

Using a trace estimate and Bianchi identity for Einstein tensor  $\text{div}G = 0$ , we claim that the following bilinear estimate holds for all scalar functions  $f$  on  $\partial\Sigma^2$

$$\left| \int_{\partial\Sigma} G_{NN} f \right| \lesssim \|G\|_{L^2(\Sigma)} \|f\|_{H^{1/2}(\partial\Sigma)}. \quad (\text{A.26})$$

Using (A.26) and the trace estimate of Section A.1, Laplace equation (A.5) for  $x^i$  on  $\partial\Sigma$ , and (A.9), we have the following estimates for the terms composing  $G$

$$\begin{aligned} \left| \int_{\partial\Sigma} \left( \text{Ric}_{NN} - \frac{1}{2} R \right) x^i (N(x^i) - x^i) \right| &\lesssim \|\text{Ric}\|_{L^2(\Sigma)} \|\nabla^2 x^i\|_{L^2(\Sigma)}, \\ \left| \int_{\Sigma} x^i \left( \text{Ric} - \frac{1}{2} Rg \right) \cdot \nabla^2 x^i \right| &\lesssim \|\text{Ric}\|_{L^2(\Sigma)} \|\nabla^2 x^i\|_{L^2(\Sigma)}, \\ \int_{\partial\Sigma} (\text{tr}\theta - 2)^2 + \int_{\partial\Sigma} |\hat{\theta}|^2 &\lesssim \|\theta - \mathcal{J}\|_{L^4(\partial\Sigma)}^2, \\ \left| \int_{\partial\Sigma} (\theta_{ab} - \mathcal{J}_{ab}) x^i (\nabla_a \nabla_b x^i + 2x^i \mathcal{J}_{ab}) \right| &\lesssim \|\theta - \mathcal{J}\|_{L^4(\partial\Sigma)} \|\nabla^{\leq 1}(\phi - 1)\|_{H^{1/2}(\mathbb{S})}, \\ \left| \int_{\partial\Sigma} (N(x^i) - x^i) (\mathcal{L} x^i + 2x^i) \right| &\lesssim \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \|\phi^{-2} - 1\|_{L^2(\partial\Sigma)}, \\ \left| \int_{\partial\Sigma} (N(x^i) - x^i) (\text{tr}\theta - 2) (N(x^i) + x^i) \right| &\lesssim \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \|\text{tr}\theta - 2\|_{L^4(\partial\Sigma)} \|\nabla^{\leq 1} x^i\|_{L^4(\partial\Sigma)} \\ &\lesssim \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \|\text{tr}\theta - 2\|_{L^4(\partial\Sigma)} \|\nabla^{\leq 2} x^i\|_{L^2(\Sigma)}. \end{aligned}$$

Thus, using estimates (4.5), (A.6) and (A.9), we obtain that

$$|G| \lesssim \varepsilon \left( \|\nabla^2 x^i\|_{L^2(\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right) \left( 1 + \|\nabla^{\leq 2} x^i\|_{L^2(\Sigma)} \right) + \varepsilon^2. \quad (\text{A.27})$$

Combining (A.25) and (A.27) and the refined Bochner identity (A.12), we have

$$\begin{aligned} &\sum_{i=1}^3 \left( \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2 \right) \\ &\lesssim \varepsilon \left( \|\nabla^2 x^i\|_{L^2(\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right) \left( 1 + \|\nabla^{\leq 2} x^i\|_{L^2(\Sigma)} \right) + \varepsilon \|\mathbf{B}\|_{L^2(\Sigma)} + \varepsilon^2, \end{aligned}$$

as desired □

**Lemma A.11.** *We have*

$$\|\nabla x^i\|_{L^2(\Sigma)} \lesssim \left( \left( \int_{\partial\Sigma} 1 \right) + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right)^{1/2},$$

and

$$\|\nabla x^i\|_{L^6(\Sigma)} \lesssim \left( \left( \int_{\partial\Sigma} 1 \right) + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right)^{1/2} + \|\nabla^2 x^i\|_{L^2(\Sigma)}.$$

*Proof.* The first estimate follows from the energy identity (A.10) and (A.9). The second follows from the first and the Sobolev embeddings of Section A.1. □

<sup>2</sup>This can be obtained by obtaining locally a geodesic foliation by 2-spheres extending the boundary  $\partial\Sigma$ , extending  $f$ , integrating along the normal direction using Bianchi identity and integration by part. See [Sze18] for similar ideas.

**Lemma A.12.** *We have*

$$\begin{aligned} \|\mathbf{B}\|_{L^2(\Sigma)} &\lesssim \left( \left( \int_{\partial\Sigma} 1 \right) + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right)^{1/2} + \|\nabla^2 x^i\|_{L^2(\Sigma)} \|\nabla^2 x^i\|_{L^2(\Sigma)} \\ &\quad + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2. \end{aligned} \quad (\text{A.28})$$

*Proof.* From Hölder estimate and the estimates of Lemma A.11, we first have

$$\begin{aligned} \|\nabla\mathbf{B}\|_{L^{3/2}(\Sigma)} &\lesssim \|\nabla x^i\|_{L^6(\Sigma)} \|\nabla^2 x^i\|_{L^2(\Sigma)} \\ &\lesssim \left( \left( \int_{\partial\Sigma} 1 \right) + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right)^{1/2} + \|\nabla^2 x^i\|_{L^2(\Sigma)} \|\nabla^2 x^i\|_{L^2(\Sigma)}. \end{aligned} \quad (\text{A.29})$$

Using (A.4), we have at the boundary  $\partial\Sigma$

$$\begin{aligned} \mathbf{B}_{NN} &= \sum_{i=1}^3 N(x^i)N(x^i) - 1 = 2 \sum_{i=1}^3 x^i(N(x^i) - x^i) + \sum_{i=1}^3 (N(x^i) - x^i)^2, \\ \mathbf{B}_{Na} &= \sum_{i=1}^3 N(x^i)\nabla_a(x^i) = \sum_{i=1}^3 (N(x^i) - x^i)\nabla_a(x^i), \\ \mathbf{B}_{ab} &= \sum_{i=1}^3 \nabla_a(x^i)\nabla_b(x^i) - g_{ab} = 0. \end{aligned}$$

where in the last line we used the conformal invariance and the fact that the corresponding identity holds true in the Euclidean case. Thus, using estimates (A.6), (A.9), we deduce

$$\|\mathbf{B}\|_{L^1(\partial\Sigma)} \lesssim \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2. \quad (\text{A.30})$$

Combining (A.29) and (A.30) and the Poincaré-type estimate (A.2), the desired result follows.  $\square$

*Proof of (A.22) and (A.23).* Combining the results of Lemmas A.10, A.11, A.12, we obtain the following estimate

$$\begin{aligned} &\sum_{i=1}^3 \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 + \sum_{i=1}^3 \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2 \\ &\lesssim \varepsilon \left( \|\nabla^2 x^i\|_{L^2(\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right) \left( 1 + \|\nabla^{\leq 2} x^i\|_{L^2(\Sigma)} \right) + \varepsilon \|\mathbf{B}\|_{L^2(\Sigma)} + \varepsilon^2 \\ &\lesssim \varepsilon \left( \|\nabla^2 x^i\|_{L^2(\Sigma)} + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right) \left( 1 + \left( \int_{\partial\Sigma} 1 \right) + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right)^{1/2} + \|\nabla^2 x^i\|_{L^2(\Sigma)} \\ &\quad + \varepsilon \left( \left( \int_{\partial\Sigma} 1 \right) + \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} \right)^{1/2} + \|\nabla^2 x^i\|_{L^2(\Sigma)} \\ &\quad + \varepsilon \|N(x^i) - x^i\|_{L^2(\partial\Sigma)} + \varepsilon \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2 + \varepsilon^2. \end{aligned}$$

Using Young's inequality and absorption or direct absorption for  $\varepsilon > 0$  sufficiently small, we obtain from the above estimate that

$$\sum_{i=1}^3 \|\nabla^2 x^i\|_{L^2(\Sigma)}^2 + \sum_{i=1}^3 \|N(x^i) - x^i\|_{L^2(\partial\Sigma)}^2 \lesssim \varepsilon^2.$$

Using this estimate and Lemmas A.11, A.12, we further obtain

$$\|\mathbf{B}\|_{L^2(\Sigma)} \lesssim \varepsilon, \quad \|\nabla x^i\|_{L^2(\Sigma)} + \|\nabla x^i\|_{L^6(\Sigma)} \lesssim 1.$$

This finishes the proof of (A.22) and (A.23).  $\square$

## A.5 Higher order estimates

In this section, we prove the following higher order estimate for  $\nabla^3 x^i$ . Estimates for higher derivatives will follow similarly (see Lemma A.13) and are left to the reader.

$$\|\nabla^3 x^i\|_{L^2(\Sigma)} \lesssim \|\text{Ric}\|_{L^2(\Sigma)} + \|\theta - \mathcal{G}\|_{H^{1/2}(\partial\Sigma)} + \|\nabla^{\leq 1}(\phi - 1)\|_{H^{1/2}(\partial\Sigma)} + \varepsilon. \quad (\text{A.31})$$

As a consequence, using the  $L^2$  assumptions (4.5), we will have

$$\|\nabla^3 x^i\|_{L^2(\Sigma)} \lesssim \varepsilon. \quad (\text{A.32})$$

We turn to the proof of (A.31). First, the commuted Laplace equation for  $\nabla^2 x^i$  takes the following schematic form

$$\Delta \nabla^2 x^i = \nabla \text{Ric} \cdot \nabla x^i + \text{Ric} \cdot \nabla^2 x^i. \quad (\text{A.33})$$

for all  $i = 1, 2, 3$ .

Since we have obtained  $L^2$  smallness estimates for  $\nabla^2 x^i$  in the previous section, estimates for derivatives of  $\nabla^2 x^i$  will follow from (A.33) and from the following standard higher order elliptic estimates.

**Lemma A.13.** *For any  $\Sigma$ -tangent tensor  $F$  satisfying*

$$\Delta F = \nabla G + H,$$

we have

$$\|\nabla F\|_{L^2(\Sigma)} \lesssim \|G\|_{L^2(\Sigma)} + \|H\|_{L^{4/3}(\Sigma)} + \|F\|_{L^2(\Sigma)} + \|\mathcal{H}\|_{H^{1/2}(\partial\Sigma)},$$

where  $\mathcal{H}$  denotes the projection of  $F$  as a  $\partial\Sigma$ -tangent tensor. Moreover, we have

$$\|\nabla^2 F\|_{L^2(\Sigma)} \lesssim \|\Delta F\|_{L^2(\Sigma)} + \|\nabla^{\leq 1} F\|_{L^2(\Sigma)} + \|\nabla^{\leq 1} \mathcal{H}\|_{H^{1/2}(\partial\Sigma)}.$$

**Remark A.14.** *The elliptic estimates from Lemma A.13 can either be taken as assumptions additional to the functional assumptions of Section A.1, or proved using the  $L^2$  bounds (4.5), Stokes formula, the trace estimates from Section A.1 and  $H^{-1/2} \times H^{1/2}$  estimates on the boundary  $\partial\Sigma$ .*

Using the results of Lemma A.13, Sobolev embeddings from Section A.1, we have

$$\begin{aligned} \|\nabla^3 x^i\|_{L^2(\Sigma)} &\lesssim \|\text{Ric} \cdot \nabla x^i\|_{L^2(\Sigma)} + \|\text{Ric} \cdot \nabla^2 x^i\|_{L^{4/3}} + \|\nabla_{a,b}^2 x^i\|_{H^{1/2}(\partial\Sigma)} \\ &\lesssim \|\text{Ric}\|_{L^2(\Sigma)} \left( \|\nabla^{\leq 3} x^i\|_{L^2(\Sigma)} \right) + \|\nabla_{a,b}^2 x^i\|_{H^{1/2}(\partial\Sigma)}. \end{aligned} \quad (\text{A.34})$$

We have

$$\nabla_{a,b}^2 x^i = \nabla_{a,b}^2 x^i + \theta_{ab} N(x^i) = (\nabla_{a,b}^2 x^i + x^i \mathcal{G}_{ab}) + x^i (\theta_{ab} - \mathcal{G}_{ab}) + (N(x^i) - x^i) \theta_{ab}.$$

Thus, we deduce using standard  $H^{1/2}$  product estimates (see [Sha14])

$$\begin{aligned} \|\nabla_{a,b}^2 x^i\|_{H^{1/2}(\partial\Sigma)} &\lesssim \|\nabla^{\leq 1}(\phi - 1)\|_{H^{1/2}(\partial\Sigma)} + \|\theta - \mathcal{G}\|_{H^{1/2}(\partial\Sigma)} \\ &\quad + \left( \|N(x^i) - x^i\|_{L^\infty(\partial\Sigma)} + \|\nabla(N(x^i) - x^i)\|_{L^2(\partial\Sigma)} \right) \|\theta\|_{H^{1/2}(\partial\Sigma)}. \end{aligned}$$

By Sobolev embeddings on  $\partial\Sigma$ , we have for the last terms

$$\begin{aligned} &\|N(x^i) - x^i\|_{L^\infty(\partial\Sigma)} + \|\nabla(N(x^i) - x^i)\|_{L^2(\partial\Sigma)} \\ &\lesssim \|\nabla(N(x^i) - x^i)\|_{H^{1/2}(\partial\Sigma)}^{1/2} \|N(x^i) - x^i\|_{H^{1/2}(\partial\Sigma)}^{1/2} + \|N(x^i) - x^i\|_{H^{1/2}(\partial\Sigma)}, \end{aligned}$$

thus from Young's inequality and the trace estimates of Section A.1, we finally obtain

$$\begin{aligned} &\|\nabla_{a,b}^2 x^i\|_{H^{1/2}(\partial\Sigma)} - c \|\nabla(N(x^i) - x^i)\|_{H^{1/2}(\partial\Sigma)} \\ &\lesssim \|\nabla^{\leq 1}(\phi - 1)\|_{H^{1/2}(\partial\Sigma)} + \|\theta - \mathcal{G}\|_{H^{1/2}(\partial\Sigma)} + \|N(x^i) - x^i\|_{H^{1/2}(\partial\Sigma)}. \end{aligned} \quad (\text{A.35})$$

Combining (A.34) and (A.35) with an absorption argument yields (A.31) as desired.

## A.6 The $x^i$ are local coordinates on $\Sigma$

In this section, we prove that the following estimate holds

$$\sum_{i,j=1}^3 |\mathbf{g}(\nabla x^i, \nabla x^j) - \delta_{ij}| \lesssim \varepsilon, \quad (\text{A.36})$$

uniformly on  $\Sigma$ . As a consequence, using the local inverse theorem we deduce that the  $x^i$  form a local coordinate system on  $\Sigma$ .

We turn to the proof of (A.36). Using the Sobolev embeddings from Section A.1, the estimates A.32, A.22 and A.23, we obtain for the tensor  $\mathbf{B} = \sum_{i=1}^3 \nabla x^i \otimes \nabla x^i - g$  defined in Section A.4

$$\|\mathbf{B}\|_{L^\infty(\Sigma)} \lesssim \varepsilon. \quad (\text{A.37})$$

We now have the following geometric lemma, which achieves the proof of (A.36).

**Lemma A.15.** *Assume that  $|\mathbf{B}| \lesssim \varepsilon$ . Then, for  $\varepsilon > 0$  sufficiently small, we have*

$$\sum_{i,j=1}^3 |\mathbf{g}(\nabla x^i, \nabla x^j) - \delta_{ij}| \lesssim \varepsilon.$$

*Proof.* Taking the trace in  $\mathbf{B}$ , we first have

$$\left| \sum_{i=1}^3 |\nabla x^i|^2 - 3 \right| \lesssim \varepsilon. \quad (\text{A.38})$$

Assume that  $\nabla x^1$  has the maximal norm among the  $\nabla x^i$ . Contracting  $\mathbf{B}$  with  $\nabla x^1$  gives

$$|\nabla x^1|^4 + |\nabla x^1 \cdot \nabla x^2|^2 + |\nabla x^1 \cdot \nabla x^3|^2 - |\nabla x^1|^2 \lesssim \varepsilon |\nabla x^1|^2.$$

Dividing by  $|\nabla x^1|^2 > 0$ , we deduce

$$|\nabla x^1|^2 + |\nabla x^1|^{-2} |\nabla x^1 \cdot \nabla x^2|^2 + |\nabla x^1|^{-2} |\nabla x^1 \cdot \nabla x^3|^2 - 1 \lesssim \varepsilon. \quad (\text{A.39})$$

We first infer from (A.39) that

$$|\nabla x^1|^2 - 1 \lesssim \varepsilon,$$

which using (A.38) further impose that

$$||\nabla x^i|^2 - 1| \lesssim \varepsilon, \quad \text{and} \quad ||\nabla x^i| - 1| \lesssim \varepsilon, \quad (\text{A.40})$$

for all  $i = 1, 2, 3$ . Injecting the above bounds in (A.39), we obtain

$$|\nabla x^1 \cdot \nabla x^2|^2 + |\nabla x^1 \cdot \nabla x^3|^2 \lesssim \varepsilon. \quad (\text{A.41})$$

Contracting  $\mathbf{B}$  with  $\nabla x^2 \otimes \nabla x^3$  gives

$$\left| \nabla x^2 \cdot \nabla x^3 - \sum_{i=1}^3 (\nabla x^2 \cdot \nabla x^i)(\nabla x^3 \cdot \nabla x^i) \right| \lesssim \varepsilon$$

which using (A.40) and (A.41) gives

$$|\nabla x^2 \cdot \nabla x^3 - 2(\nabla x^2 \cdot \nabla x^3)| \lesssim \varepsilon,$$

and the result of the lemma follows.  $\square$

## A.7 The $x^i$ are global coordinates from $\Sigma$ onto $\mathbb{D}$

We want to improve the result of Section A.6 and show that the map

$$\Phi : x \mapsto (x^i(x))$$

is a global diffeomorphism from  $\Sigma$  onto  $\mathbb{D}$ . This follows from the following two lemmas.

**Lemma A.16.** *We have that*

$$\Phi(\Sigma) = \mathbb{D}.$$

**Lemma A.17.** *We have that*

$$\forall p \in \Phi(\Sigma), \# \{ \Phi^{-1}(p) \} = 1.$$

*Proof of Lemma A.16.* Let first prove that  $\Phi(\Sigma) \subset \mathbb{D}$ . We argue by contradiction and suppose that there exists  $p \in \Sigma$  such that  $|\Phi(p)| > 1$ . Let define the function  $X$  on  $\Sigma$  by

$$X := \frac{1}{|\Phi(p)|} \sum_{i=1}^n x^i(p) x^i.$$

From the definitions of the  $x^i$ , the function  $X$  is harmonic,  $|X| \leq 1$  on  $\partial\Sigma$  and  $|X(p)| = |\Phi(p)|^2 > 1$ , which contradicts the maximum principle.

We then show that  $\Phi(\Sigma) = \mathbb{D}$ . Since  $\Phi(\Sigma)$  is closed in  $\mathbb{D}$  and  $\mathbb{D}$  is connected, the result will follow provided that we can prove that  $\Phi(\Sigma)$  is an open subset of  $\mathbb{D}$ . Applying the maximum principle, we can obtain that  $\Phi^{-1}(\partial\mathbb{D}) = \partial\Sigma$ . Applying the local inverse theorem then ensures that  $\Phi(\Sigma)$  is open in  $\mathbb{D}$ . This finishes the proof of Lemma A.16.  $\square$

*Proof of Lemma A.17.* By the local inverse theorem applied at all points  $p \in \Phi(\Sigma)$ , one has that

$$C(p) := \# \{ \Phi^{-1}(p) \}$$

is locally constant at each point  $p \in \Phi(\Sigma)$  (see [Mil97] and Remark A.19). Since  $\Phi(\Sigma)$  is connected, this implies that  $C$  is constant on  $\Phi(\Sigma)$ .

It therefore suffices to compute  $C$  at  $(1, 0, \dots, 0) \in \Phi(\Sigma)$ . By the maximum principle, one easily obtains that  $\Phi^{-1}(1, 0, \dots, 0) = \{(1, 0, \dots, 0)\}$ , which finishes the proof of Lemma A.17.  $\square$

**Remark A.18.** *If one is concerned with the applicability of the local inverse theorem at the boundary, one can just extend the smooth function by Borel's lemma and then apply the classical local inverse theorem.*

**Remark A.19.** *The proof in [Mil97] is for manifold without boundaries, but extends directly to the case with boundary by considering the map  $\Phi$  corestricted to its image (which is a manifold with boundary). Note that this is the reason why we apply this result only for points  $p \in \Phi(\Sigma)$ .*





## B. Axis limits

This section is dedicated to the proof of Theorem 4.4. In Section B.1 we define the Cartesian optical normal coordinates, and show that they form a local (smooth outside of the axis) coordinate system. We then perform a Cartesian to spherical coordinate change of coordinates and derive limits at the axis for the metric components and its derivatives. These limits are the key to obtain the limits for the metric, null connection, curvature tensors in Section B.3 using the expression of these tensors in spherical coordinates.

### B.1 Optical normal coordinates

Let  $(\mathcal{M}, \mathbf{g})$  be a smooth Lorentzian manifold. Let  $\phi$  be a timelike geodesic of  $\mathcal{M}$ . We assume that  $\phi$  is parametrised by its geodesic affine parameter, *i.e.*  $\mathbf{g}(\dot{\phi}, \dot{\phi}) = -1$  and that a parallel transported orthonormal frame  $(e_0 := \dot{\phi}, e_1, e_2, e_3)$  along  $\phi$  is given. We assume that the frame  $e_\mu$  is smoothly extended outside of  $\phi$ , and that its extension satisfies  $\mathbf{D}(e_\mu)|_\phi = 0$ .<sup>1</sup> We define the map  $\Psi : \mathcal{M} \times \mathbb{R}^4 \rightarrow \mathcal{M}$ , by

$$\Psi(p, \dot{p}^\nu) := \exp_p(\dot{p}^\nu e_\nu(p)).$$

Let  $O \in \phi$ . There exists a neighbourhood of  $O$  covered by smooth normal coordinates which we call  $(z^\mu)$ , such that  $O = \{z = 0\}$  and  $\partial_{z^\mu}|_O = e_\mu(O)$ . In this coordinate system the metric and its first derivatives are trivial at  $O$  (see [GHL90]), *i.e.*

$$\begin{aligned} \mathbf{g}_{\mu\nu}(O) &= \boldsymbol{\eta}_{\mu\nu}, \\ \partial \mathbf{g}_{\mu\nu}(O) &= 0, \end{aligned}$$

where  $\boldsymbol{\eta}_{\mu\nu}$  denotes the Cartesian coordinates components of the Minkowskian metric.

We define the map  $\underline{\Psi} := \underline{\Psi}_O : \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$  by identifying  $\mathcal{M}$  to (a neighbourhood of 0 in)  $\mathbb{R}^4$  using the normal coordinates  $(z^\mu)$  at  $O$ .

**Remark B.1.** *In the Minkowskian case  $\mathcal{M} = \mathbb{R}^4$ , we have  $\underline{\Psi}^{\text{Mink}}(z, \dot{z}) = z + \dot{z}$ .*

The map  $\underline{\Psi}$  satisfies the following properties.

**Lemma B.2.** *The map  $\underline{\Psi}$  is a smooth map from  $\mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}^4$ . Moreover, the following identities hold*

$$\begin{aligned} \underline{\Psi}|_{\mathbb{R}^4 \times \{0\}} &= \text{Id}_{\mathbb{R}^4}, \\ \underline{\Psi}|_{\{0\} \times \mathbb{R}^4} &= \text{Id}_{\mathbb{R}^4}, \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} d\underline{\Psi}(z, 0) \cdot (v, 0) &= v, \\ d\underline{\Psi}(0, \dot{z}) \cdot (0, \dot{v}) &= \dot{v}, \end{aligned} \tag{B.2}$$

and

$$d\underline{\Psi}(z, 0) \cdot (0, \dot{v}) = \dot{v}^\nu e_\nu(z). \tag{B.3}$$

and

$$d^2 \underline{\Psi}(0, 0) = 0, \tag{B.4}$$

as a bilinear map from  $(\mathbb{R}^4 \times \mathbb{R}^4)^2 \rightarrow \mathbb{R}^4$ .

*Proof.* The smoothness of  $\underline{\Psi}$  is a consequence of the smoothness of the exponential map (see [GHL90, p. 86]) and of the vectorfields  $e_\mu$ .

Identities (B.1) are straight-forward consequences of the definition of normal coordinates and the exponential map. Identities (B.2) then follow from identities (B.1).

<sup>1</sup>Using normal coordinates vectorfields, it is always possible to obtain such an extension.

Identity (B.3) is a consequence of the definition of the exponential map.

To derive the last identity (B.4) we first rewrite (B.3) as

$$\partial_{z^\nu} \underline{\Psi}^\lambda(z, 0) = (e_\nu(z))^\lambda,$$

which deriving along  $z^\mu$  and evaluating at 0 gives

$$\begin{aligned} \partial_{z^\mu} \partial_{z^\nu} \underline{\Psi}^\lambda(0, 0) &= (\partial_{z^\mu} (e_\nu(z))^\lambda) |_{z=0} \\ &= \mathbf{D}_{\partial_{z^\mu}} (e_\nu)^\lambda |_{z=0} + \Gamma_{\mu\delta}^\lambda (e_\nu(z))^\delta |_{z=0} \\ &= 0, \end{aligned}$$

by definition of the extension of  $e_\nu$  and since the Christoffel symbols  $\Gamma$  for the normal coordinates vanish at  $z = 0$ .

Deriving the identities of (B.2) also gives

$$\begin{aligned} \partial_{z^\mu} \partial_{z^\nu} \underline{\Psi}^\lambda(0, 0) &= 0, \\ \partial_{z^\mu} \partial_{z^\nu} \underline{\Psi}^\lambda(0, 0) &= 0. \end{aligned}$$

Using Schwarz theorem, since  $\underline{\Psi}$  is smooth, all second order partial derivatives vanish at  $(0, 0)$  and identity (B.4) follows. This finishes the proof of the lemma.  $\square$

We define the map  $\Phi : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \times \mathbb{R}^4$  by

$$\Phi(x^0, x^1, x^2, x^3) := \left[ \left( x^0 + \sqrt{\sum_{i=1}^3 (x^i)^2}, 0, 0, 0 \right), \left( -\sqrt{\sum_{i=1}^3 (x^i)^2}, x^1, x^2, x^3 \right) \right],$$

and we define the map  $\Theta : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  by

$$\Theta := \underline{\Psi} \circ \Phi.$$

**Remark B.3.** *In the Minkowskian case, we have*

$$\Theta^{\text{Mink}} = \text{Id}_{\mathbb{R}^4}.$$

*However, for a general Lorentzian manifold, the map  $\Theta$  is not smooth at the axis  $\{x^i = 0\}$ . See Lemma B.4.*

**Lemma B.4.** *The map  $\Theta$  is a  $\mathcal{C}^{1,1}$  diffeomorphism from a neighbourhood  $\mathcal{U}$  of  $0 \in \mathbb{R}^4$  onto a neighbourhood  $\mathcal{V}$  of  $0 \in \mathbb{R}^4$ . Moreover  $\Theta$  is smooth on  $\mathcal{U} \setminus \{x^i = 0\}$ . We call (Cartesian) optical normal coordinates the induced local coordinates  $(x^\mu)$  by  $\Theta$  in  $\mathcal{M}$ .*

*Proof.* From the explicit definition of  $\Phi$  and the smoothness of  $\underline{\Psi}$  obtained in Lemma B.2, the map  $\Theta$  is smooth on  $\mathbb{R}^4 \setminus \{x^i = 0\}$ .

To obtain that  $\Theta$  is a  $\mathcal{C}^{1,1}$  map at  $\{x^i = 0\}$ , and also that  $d\Theta((0, x^i)) = \text{Id}_{\mathbb{R}^4}$ , it is enough to show that

$$\|d\Theta(x) - \text{Id}_{\mathbb{R}^4}\| = O\left(\sqrt{\sum_{i=1}^3 (x^i)^2}\right),$$

when  $\sqrt{\sum_{i=1}^3 (x^i)^2} \rightarrow 0$ .

From the chain rule we have

$$\begin{aligned} d\Theta(x) - \text{Id}_{\mathbb{R}^4} &= d\underline{\Psi}(\Phi(x)) \cdot d\Phi(x) - \text{Id}_{\mathbb{R}^4} \\ &= (d\underline{\Psi}(\Phi(x)) - d\underline{\Psi}[(x^0, 0, 0, 0), 0]) \cdot d\Phi(x) \\ &\quad + d\underline{\Psi}[(x^0, 0, 0, 0), 0] \cdot d\Phi(x) - \text{Id}_{\mathbb{R}^4}. \end{aligned}$$

From properties (B.2) and (B.3) of Lemma B.2, and the explicit expression of  $\Phi$ , we have

$$\begin{aligned} & d\underline{\Psi} [(x^0, 0, 0, 0), 0] \cdot d\Phi(x) \cdot h \\ = & d\underline{\Psi} [(x^0, 0, 0, 0), 0] \cdot \left[ \left( h^0 + \left( \sum_{i=1}^3 (x^i)^2 \right)^{-1/2} \sum_{j=1}^3 x^j h^j, 0, 0, 0 \right), \left( - \left( \sum_{i=1}^3 (x^i)^2 \right)^{-1/2} \sum_{j=1}^3 x^j h^j, h^1, h^2, h^3 \right) \right] \\ = & \text{Id}_{\mathbb{R}^4} \cdot h. \end{aligned}$$

From the explicit expression of  $\Phi$ , we have

$$\begin{aligned} \Phi(x) - [(x^0, 0, 0, 0), 0] &= O \left( \sqrt{\sum_{i=1}^3 (x^i)^2} \right), \\ \|d\Phi(x)\| &= O(1), \end{aligned}$$

when  $\sqrt{\sum_{i=1}^3 (x^i)^2} \rightarrow 0$ . From the smoothness of  $\underline{\Psi}$ , we thus deduce

$$\| (d\underline{\Psi}(\Phi(x)) - d\underline{\Psi} [(x^0, 0, 0, 0), 0]) \cdot d\Phi(x) \| = O \left( \sqrt{\sum_{i=1}^3 (x^i)^2} \right),$$

as desired.

We thus have obtained that  $\Theta$  is a  $\mathcal{C}^{1,1}$  map, and that  $d\Theta(x)$  is invertible in a neighbourhood  $\mathcal{U}$  of  $0 \in \mathbb{R}^4$  (and more generally of  $\{x^i = 0\}$ ). The result of the lemma then follows from an application of the local inverse theorem.  $\square$

The following lemma is a direct consequence of the definition of  $\Theta$ .

**Lemma B.5.** *The level sets of the function*

$$x^0 + \sqrt{\sum_{i=1}^3 (x^i)^2}$$

are the ingoing null cones  $\underline{\mathcal{C}}$  emanating from  $\emptyset$ . The level sets of the function

$$x^0 - \sqrt{\sum_{i=1}^3 (x^i)^2}$$

intersect  $\underline{\mathcal{C}}$  at the 2-spheres of the geodesic foliation on  $\underline{\mathcal{C}}$ .

**Lemma B.6.** *The coordinate functions  $x^0 + t$  and  $x^i$  are independent of the choice of point  $O = \emptyset(t)$ . In the following, we redefine  $x^0 := x^0 + t$ .*

*Proof.* Let  $p = (x_1^0, x_1^i)_{O(t_1)} = (x_2^0, x_2^i)_{O(t_2)}$ . Since  $p$  belongs to the same cone emanating from  $\emptyset$ , we have

$$\emptyset \left( t_1 + x_1^0 - \sqrt{\sum_{j=1}^3 (x_1^j)^2} \right) = \emptyset \left( t_2 + x_2^0 - \sqrt{\sum_{j=1}^3 (x_2^j)^2} \right).$$

By injectivity of the exponential map and since the  $e_\mu$  are independent, we further have

$$\begin{aligned} -\sqrt{\sum_{j=1}^3 (x_1^j)^2} &= -\sqrt{\sum_{j=1}^3 (x_2^j)^2}, \\ x_1^i &= x_2^i, \end{aligned}$$

for all  $i = 1, 2, 3$ , from which we also infer  $t_1 + x_1^0 = t_2 + x_2^0$ .  $\square$

We define the *spherical* optical normal coordinates  $(\underline{u}, u, \vartheta, \varphi)$  on  $\mathcal{M}$ , to be

$$(\underline{u}, u, \vartheta, \varphi) := \Xi^{-1}(x^0, x^1, x^2, x^3),$$

where  $\Xi : \mathbb{R}_{\underline{u}} \times \mathbb{R}_u \times (0, \pi)_{\vartheta} \times (0, 2\pi)_{\varphi} \rightarrow \mathbb{R}^4$  is the following spherical-to-Cartesian coordinates map

$$\Xi(\underline{u}, u, \vartheta, \varphi) := \left( \frac{\underline{u} + u}{2}, \frac{\underline{u} - u}{2} \sin \vartheta \cos \varphi, \frac{\underline{u} - u}{2} \sin \vartheta \sin \varphi, \frac{\underline{u} - u}{2} \cos \vartheta \right).$$

## B.2 Axis limits for the metric $\mathbf{g}$ in optical normal coordinates

Let  $O \in \phi$ . By invariance in  $t$ , we shall assume that  $O = \phi(t = 0)$ . In this section, we use the classical normal coordinates  $z$  at  $O$  defined in Section B.1 and the change of coordinates  $\Theta$  from the classical  $z$  to the optical  $x$  normal coordinates to derive limits at  $O$  for (derivatives of) the optical normal coordinates components of the metric  $\mathbf{g}$  (see Lemma B.8). Since it will be shown that the metric and its first derivatives are trivial at  $O$  and therefore everywhere on the axis  $\phi$ , we will additionally infer strong limits for derivatives of the metric in the axis direction (see Lemma B.9).

We have the following limits when  $|x| \rightarrow 0$  for the  $(\underline{u}, u, \vartheta, \varphi)$ -derivatives of  $\Theta$ .

**Lemma B.7.** *We have*

$$\left( \partial_{\underline{u}}^k, \partial_u^l, \partial_a^m \right) \Theta \circ \Xi(\underline{u}, u, \vartheta, \varphi) = O(|x|^{1-k-l}), \quad (\text{B.5})$$

$$\left( \partial_{\underline{u}}^k, \partial_u^l, \partial_a^m \right) ((d\Theta) \circ \Xi(\underline{u}, u, \vartheta, \varphi) - \text{Id}_{\mathbb{R}^4}) = O(|x|^{2-k-l}) \quad (\text{B.6})$$

when  $|x| \rightarrow 0$ , for all  $k, l, m \geq 0$ , and where here and in the following

$$\left( \partial_{\underline{u}}^k, \partial_u^l, \partial_a^m \right)$$

denotes all combinations of partial derivatives containing respectively  $k, l, m$ -derivatives of  $\underline{u}, u$  and  $a = \vartheta, \varphi$ .

*Proof.* We have

$$\Theta(\Xi(\underline{u}, u, \vartheta, \varphi)) = \underline{\Psi} \left[ (\underline{u}, 0, 0, 0), \left( -\frac{\underline{u} - u}{2}, \frac{\underline{u} - u}{2} \sin \vartheta \cos \varphi, \frac{\underline{u} - u}{2} \sin \vartheta \sin \varphi, \frac{\underline{u} - u}{2} \cos \vartheta \right) \right],$$

and the limits (B.5) follow from  $\underline{\Psi}(z, \dot{z}) = O(z, \dot{z})$  and the smoothness of  $\underline{\Psi}$ .

We have

$$\begin{aligned} & (d\Theta(\Xi(\underline{u}, u, \vartheta, \varphi)) - \text{Id}_{\mathbb{R}^4}) \cdot (h^\mu) \\ &= (d\underline{\Psi}(\Theta(\Xi(\underline{u}, u, \vartheta, \varphi))) - d\underline{\Psi}(0, 0)) \cdot \left[ (h^0 + \sin \vartheta \cos \varphi h^1 + \sin \vartheta \sin \varphi h^2 + \cos \vartheta h^3, 0, 0, 0), \right. \\ & \quad \left. (-\sin \vartheta \cos \varphi h^1 - \sin \vartheta \sin \varphi h^2 - \cos \vartheta h^3, h^i) \right], \end{aligned}$$

and the second limits follow from  $\|d\underline{\Psi}(z, \dot{z}) - d\underline{\Psi}(0, 0)\| = O((z, \dot{z})^2)$ ,  $\|d^2\underline{\Psi}(z, \dot{z})\| = O((z, \dot{z}))$  which are consequences of Lemma B.2, the smoothness of  $\underline{\Psi}$ , and the previously obtained (B.5). This finishes the proof of the lemma.  $\square$

As a consequence of Lemma B.7, we have the following limits for the metric  $\mathbf{g}$  in the Cartesian optical normal coordinates.

**Lemma B.8.** *We have*

$$\left( \partial_{\underline{u}}^k, \partial_u^l, \partial_a^m \right) (\mathbf{g}_{\mu\nu} - \eta_{\mu\nu}) = O(|x|^{2-k-l}), \quad (\text{B.7})$$

when  $|x| \rightarrow 0$ , for all  $k, l, m \geq 0$ , and where  $\mathbf{g}_{\mu\nu}$  denotes the components of the metric  $\mathbf{g}$  in the coordinates  $x^\mu$ .

*Proof.* We have the formula

$$\mathbf{g}_{\mu\nu}^x(x) = \mathbf{g}_{\mu'\nu'}^z(\Theta(x)) \partial_{x^\mu} \Theta^{\mu'} \partial_{x^\nu} \Theta^{\nu'},$$

which we rewrite as

$$\begin{aligned} \mathbf{g}_{\mu\nu}^x(x) - \boldsymbol{\eta}_{\mu\nu} &= (\mathbf{g}_{\mu\nu}^z(\Theta(x)) - \boldsymbol{\eta}_{\mu\nu}) \\ &\quad + \mathbf{g}_{\mu'\nu'}^z(\Theta(x)) \left( \partial_{x^\mu} \Theta^{\mu'} - \delta_\mu^{\mu'} \right) \partial_{x^\nu} \Theta^{\nu'} \\ &\quad + \mathbf{g}_{\mu'\nu'}^z(\Theta(x)) \partial_{x^\mu} \Theta^{\mu'} \left( \partial_{x^\nu} \Theta^{\nu'} - \delta_\nu^{\nu'} \right). \end{aligned} \tag{B.8}$$

The metric components of the normal coordinates  $\mathbf{g}_{\mu\nu}^z(z)$  are smooth and we have

$$\begin{aligned} \mathbf{g}_{\mu\nu}^z(z) - \boldsymbol{\eta}_{\mu\nu} &= O(|z|^2), \\ \partial_z \mathbf{g}_{\mu\nu}^z(z) &= O(|z|), \\ \partial_z^{\geq 2} \mathbf{g}_{\mu\nu}^z(z) &= O(1), \end{aligned} \tag{B.9}$$

when  $z \rightarrow 0$ . The proof of (B.7) then follows from formula (B.8) and the limits given by (B.5), (B.6) and (B.9). This finishes the proof of the lemma.  $\square$

We have the following stronger limits derivatives along the axis.

**Lemma B.9.** *We have*

$$(\partial_u + \partial_{\underline{u}})^n \left( \partial_{\underline{u}}^k, \partial_u^l, \partial_a^m \right) (\mathbf{g}_{\mu\nu} - \boldsymbol{\eta}_{\mu\nu}) = O(|x|^{|2-k-l|}), \tag{B.10}$$

when  $|x| \rightarrow 0$  and for all  $n \geq 0$  and all  $k, l, m \geq 0$ .

*Proof.* The lemma follows from the smoothness in  $(\underline{u}, u, \vartheta, \varphi)$  and the fact that the limits (B.7) hold uniformly along the axis  $\{u = \underline{u}\}$ .  $\square$

In the following we combine (B.7) and (B.10) into the following equivalent limits

$$\left( \partial_u^k, (\partial_u + \partial_{\underline{u}})^l, \partial_a^m \right) (\mathbf{g}_{\mu\nu} - \boldsymbol{\eta}_{\mu\nu}) = O(|x|^{|2-k|}), \tag{B.11}$$

when  $|x| \rightarrow 0$  and for all  $k, l, m \geq 0$ .

Since our goal is to obtain vertex limits for the null connection coefficients and rotation vectorfields which are naturally expressed in terms of the metric components in spherical coordinates, we infer the following equivalent of (B.11) for the metric in spherical coordinates.

**Lemma B.10.** *We have*

$$\left( \partial_u^k, (\partial_u + \partial_{\underline{u}})^l, \partial_a^m \right) (\mathbf{g}_{\tilde{\mu}\tilde{\nu}} - \boldsymbol{\eta}_{\tilde{\mu}\tilde{\nu}}) = O(|x|^{|2-k|}), \tag{B.12}$$

when  $|x| \rightarrow 0$ , for all  $k, l, m \geq 0$ , and where from now on indices  $\mu, \nu$  (resp.  $a, b$ ) refer to evaluation of a tensor with respect to the coordinate vectorfields  $(\partial_{\underline{u}}, \partial_u, \partial_\vartheta, \partial_\varphi)$  (resp.  $(\partial_\vartheta, \partial_\varphi)$ ), whereas tilde indices  $\tilde{\mu}\tilde{\nu}$  (resp.  $\tilde{a}, \tilde{b}$ ) refer to evaluation with respect to the normalised coordinate vectorfields  $(\tilde{\partial}_{\underline{u}} := \partial_{\underline{u}}, \tilde{\partial}_u := \partial_u, \tilde{\partial}_\vartheta := \frac{2}{\underline{u}-u} \partial_\vartheta, \tilde{\partial}_\varphi := \frac{2}{\underline{u}-u} \partial_\varphi)$  (resp.  $(\tilde{\partial}_\vartheta, \tilde{\partial}_\varphi)$ ). The tensor  $\boldsymbol{\eta}$  denotes the Minkowski metric, which writes in  $(\underline{u}, u, \vartheta, \varphi)$  coordinates

$$\boldsymbol{\eta} = -d\underline{u}du + \left( \frac{\underline{u}-u}{2} \right)^2 d\vartheta^2 + \left( \frac{\underline{u}-u}{2} \right)^2 \sin^2 \vartheta d\varphi^2.$$

We also have

$$\left( \partial_u^k, (\partial_u + \partial_{\underline{u}})^l, \partial_a^m \right) (\mathbf{g}^{\tilde{\mu}\tilde{\nu}} - \boldsymbol{\eta}^{\tilde{\mu}\tilde{\nu}}) = O(|x|^{|2-k|}),$$

where  $\mathbf{g}^{\tilde{\mu}\tilde{\nu}} := (\mathbf{g}_{\tilde{\mu}\tilde{\nu}})^{-1}$  and also coincides with the renormalisation of  $(\mathbf{g}_{\mu\nu})^{-1}$ .

*Proof.* We have

$$\mathbf{g}_{\bar{\mu}\bar{\nu}} = (J_1)_{\bar{\mu}}^{\mu'} \mathbf{g}_{\mu'\nu'}^x (J_1)_{\bar{\nu}}^{\nu'},$$

where  $J_1$  is the normalised Jacobian matrix of the spherical to Cartesian coordinates change

$$J_1 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 \sin \vartheta \cos \varphi & -1/2 \sin \vartheta \cos \varphi & \cos \vartheta \cos \varphi & -\sin \vartheta \cos \varphi \\ 1/2 \sin \vartheta \sin \varphi & -1/2 \sin \vartheta \sin \varphi & \cos \vartheta \sin \varphi & \sin \vartheta \cos \varphi \\ 1/2 \cos \vartheta & -1/2 \cos \vartheta & -\sin \vartheta & 0 \end{pmatrix}$$

and (B.12) follows from (B.11). The limits for the inverse matrix follows by inverting the same formula.  $\square$

**Corollary B.11.** *We have*

$$(\partial_{\underline{u}}^k, (\partial_{\underline{u}} + \partial_{\underline{u}})^l, \partial_{\omega}^m) (\not{g}_{ab} - \not{h}_{ab}) = O(|x|^{4-k}),$$

when  $|x| \rightarrow 0$ , and for all  $k, l, m \geq 0$ . As a consequence, we have for the area radius  $r$  of  $S_{\underline{u}, \underline{u}}$

$$(\partial_{\underline{u}}^k, (\partial_{\underline{u}} + \partial_{\underline{u}})^l) \left( r(\underline{u}, \underline{u}) - \frac{\underline{u} - \underline{u}}{2} \right) = O(|x|^{3-k}). \quad (\text{B.13})$$

**Remark B.12.** *The limits for the null connection coefficients and the rotation vectorfields are performed in the coordinate chart  $(\underline{u}, u, \vartheta, \varphi)$ , in a domain where these chart remains regular, e.g.  $\pi/10 < \vartheta < 9\pi/10$ . By changing the axis of the spherical coordinates and performing the same computations, one obtains the same limits in the regions  $0 \leq \vartheta \leq \pi/10$ ,  $9\pi/10 \leq \vartheta \leq \pi$ .*

**Lemma B.13.** *Let  $F$  be a  $k$ -tensor. We say that  $F = \mathcal{O}(|x|^\gamma)$  if for all  $k, l, m \geq 0$ , we have*

$$\sum_{(\mu_1, \dots, \mu_k) \in \{\underline{u}, u, \vartheta, \varphi\}} \left| \left( ((\underline{u} - u) \partial_{\underline{u}})^k, (\partial_{\underline{u}} + \partial_{\underline{u}})^l, \partial_{\underline{u}}^m \right) (F_{\bar{\mu}_1 \dots \bar{\mu}_k}) \right| = O(|x|^\gamma),$$

when  $|x| \rightarrow 0$ . With this definition, we have

$$\mathbf{g} - \boldsymbol{\eta} = \mathcal{O}(|x|^2), \quad (\text{B.14a})$$

$$\Gamma(\mathbf{g}) - \Gamma(\boldsymbol{\eta}) = \mathcal{O}(|x|), \quad (\text{B.14b})$$

$$\mathbf{R} = \mathcal{O}(1), \quad (\text{B.14c})$$

when  $|x| \rightarrow 0$ , where we consider the Christoffel symbols  $\Gamma$  as 3-tensors.

*Proof.* The limits (B.14a) are direct consequences of (B.12). The limits (B.14b) follow from the definition of the Christoffel symbols and limits (B.14a). The limits (B.14c) follow from the expression of  $\mathbf{R}$  in terms of the (derivatives of) the Christoffel symbols and the limits (B.14b).  $\square$

**Remark B.14.** *The limits (B.12) are stronger than (B.14) for  $\mathbf{g}$ . But necessary for  $\Gamma(\mathbf{g})$  and  $\mathbf{R}$ .*

### B.3 Axis limits for the null connection coefficients

We recall that the null pair  $(e_3, e_4)$  for the optical normal coordinates is defined by  $e_3 := -\mathbf{D}\underline{u}$ . By construction of the optical normal spherical coordinates in Section B.1, the null pair  $(e_3, e_4)$  writes

$$e_3 = 2\partial_{\underline{u}}, \quad e_4 = -\mathbf{g}^{uv} \partial_{\nu} - \frac{1}{2} \mathbf{g}^{uu} \partial_u, \quad (\text{B.15})$$

which we also rewrite as

$$e_4 = 2\partial_{\underline{u}} - (\mathbf{g}^{uv} - \boldsymbol{\eta}^{uv}) \partial_{\nu} - \frac{1}{2} (\mathbf{g}^{uu} - \boldsymbol{\eta}^{uu}) \partial_u. \quad (\text{B.16})$$

From the above formulas, we deduce the following expressions of the null connection coefficients associated to the null pair  $(e_3, e_4)$  in terms of  $\mathbf{g}_{\bar{\mu}\bar{\nu}}$  and  $\partial \mathbf{g}_{\bar{\mu}\bar{\nu}}$ .

**Lemma B.15.** *We have*

$$\underline{\chi}_{ab} = -\frac{2}{\underline{u}-u} \mathbf{g}_{ab} + \frac{1}{4}(\underline{u}-u)^2 \partial_u (\mathbf{g}_{\bar{a}\bar{b}} - \boldsymbol{\eta}_{\bar{a}\bar{b}}),$$

$$\begin{aligned} \zeta_a &= \partial_a \mathbf{g}_{\underline{u}\bar{u}} + \frac{1}{2} \partial_u ((\underline{u}-u) \mathbf{g}_{\bar{a}\bar{b}}) - \frac{1}{2} (\mathbf{g}^{\bar{u}\bar{u}} - \boldsymbol{\eta}^{\bar{u}\bar{u}}) \left( \partial_a \mathbf{g}_{\underline{u}\bar{u}} + \frac{1}{2} \partial_u ((\underline{u}-u) \mathbf{g}_{\bar{a}\bar{u}}) \right) \\ &\quad - (\underline{u}-u)^{-1} (\mathbf{g}^{\bar{u}\bar{b}} - \boldsymbol{\eta}^{\bar{u}\bar{b}}) \left( \frac{1}{4} \partial_u ((\underline{u}-u)^2 \mathbf{g}_{\bar{a}\bar{b}}) \right), \end{aligned}$$

$$\boldsymbol{\eta} = -\frac{1}{2} (\mathbf{g}^{\bar{u}\bar{u}} - \boldsymbol{\eta}^{\bar{u}\bar{u}}),$$

$$\begin{aligned} \chi_{ab} &= \frac{2}{\underline{u}-u} \mathbf{g}_{ab} + \frac{1}{4}(\underline{u}-u)^2 \partial_{\underline{u}} (\mathbf{g}_{\bar{a}\bar{b}} - \boldsymbol{\eta}_{\bar{a}\bar{b}}) \\ &\quad - \frac{1}{4}(\underline{u}-u) \mathbf{g}_{\bar{v}\bar{b}} \partial_a (\mathbf{g}^{\bar{u}\bar{v}} - \boldsymbol{\eta}^{\bar{u}\bar{v}}) - \frac{1}{4}(\underline{u}-u) \mathbf{g}_{\bar{v}\bar{a}} \partial_b (\mathbf{g}^{\bar{u}\bar{v}} - \boldsymbol{\eta}^{\bar{u}\bar{v}}) \\ &\quad - \frac{1}{8} (\mathbf{g}^{\bar{u}\bar{u}} - \boldsymbol{\eta}^{\bar{u}\bar{u}}) \partial_u ((\underline{u}-u)^2 \mathbf{g}_{\bar{a}\bar{b}}) \\ &\quad - \frac{1}{2} (\mathbf{g}^{\bar{u}\bar{u}} - \boldsymbol{\eta}^{\bar{u}\bar{u}}) \left( \frac{1}{2} \partial_a ((\underline{u}-u) \mathbf{g}_{\bar{u}\bar{b}}) + \frac{1}{4} \partial_{\underline{u}} ((\underline{u}-u)^2 \mathbf{g}_{\bar{a}\bar{b}}) - \frac{1}{2} \partial_b ((\underline{u}-u) \mathbf{g}_{\bar{a}\bar{u}}) \right) \\ &\quad - \frac{1}{4} (\underline{u}-u) (\mathbf{g}^{\bar{u}\bar{c}} - \boldsymbol{\eta}^{\bar{u}\bar{c}}) (\partial_a \mathbf{g}_{\bar{c}\bar{b}} + \partial_c \mathbf{g}_{\bar{a}\bar{b}} - \partial_b \mathbf{g}_{\bar{a}\bar{c}}), \end{aligned}$$

together with  $\boldsymbol{\eta} = -\underline{\boldsymbol{\eta}} = \zeta$ ,  $\underline{\omega} = |\underline{\xi}| = 0$  and  $\omega = -\frac{1}{4}e_3(\boldsymbol{\eta})$ ,  $\xi = -\frac{1}{2}\nabla \boldsymbol{\eta}$ .

*Proof.* Using (B.15), we have

$$\underline{\chi}_{ab} = \underline{\chi}(\partial_a, \partial_b) = \mathbf{g}(\mathbf{D}_{\partial_a} 2\partial_u, \partial_b) = 2\mathbf{g}(\mathbf{D}_{\partial_u} \partial_a, \partial_b) = \mathbf{g}(\mathbf{D}_{\partial_u} \partial_a, \partial_b) + \mathbf{g}(\mathbf{D}_{\partial_u} \partial_b, \partial_a) = \partial_u(\mathbf{g}_{ab}),$$

where we used the symmetry of  $\underline{\chi}$ . We further have

$$\partial_u(\mathbf{g}_{ab}) = \partial_u \left( \left( \frac{\underline{u}-u}{2} \right)^2 \mathbf{g}_{\bar{a}\bar{b}} \right),$$

and the desired formula follows.

Using (B.15), we have

$$\begin{aligned} \zeta_a &= \mathbf{g} \left( \mathbf{D}_{\partial_a} \partial_u, -\mathbf{g}^{uv} \partial_\nu - \frac{1}{2} \mathbf{g}^{uu} \partial_u \right) \\ &= -\mathbf{g}^{uv} \mathbf{g}_{\mu\nu} \Gamma_{au}^\mu(\mathbf{g}) \\ &= -\Gamma_{au}^u(\mathbf{g}). \end{aligned}$$

Using that  $\mathbf{g}_{uu} = \mathbf{g}_{au} = 0$ , we further have

$$\begin{aligned} -\Gamma_{au}^u(\mathbf{g}) &= -\frac{1}{2} \mathbf{g}^{u\mu} (\partial_a \mathbf{g}_{\mu u} + \partial_u \mathbf{g}_{a\mu} - \partial_\mu \mathbf{g}_{au}) \\ &= -\frac{1}{2} \mathbf{g}^{\bar{u}\bar{u}} \left( \partial_a \mathbf{g}_{\underline{u}\bar{u}} + \frac{1}{2} \partial_u ((\underline{u}-u) \mathbf{g}_{\bar{a}\bar{u}}) \right) \\ &\quad - (\underline{u}-u)^{-1} \mathbf{g}^{\bar{u}\bar{b}} \left( \frac{1}{4} \partial_u ((\underline{u}-u)^2 \mathbf{g}_{\bar{a}\bar{b}}) \right) \\ &= \partial_a \mathbf{g}_{\underline{u}\bar{u}} + \frac{1}{2} \partial_u ((\underline{u}-u) \mathbf{g}_{\bar{a}\bar{b}}) - \frac{1}{2} (\mathbf{g}^{\bar{u}\bar{u}} - \boldsymbol{\eta}^{\bar{u}\bar{u}}) \left( \partial_a \mathbf{g}_{\underline{u}\bar{u}} + \frac{1}{2} \partial_u ((\underline{u}-u) \mathbf{g}_{\bar{a}\bar{u}}) \right) \\ &\quad - (\underline{u}-u)^{-1} (\mathbf{g}^{\bar{u}\bar{b}} - \boldsymbol{\eta}^{\bar{u}\bar{b}}) \left( \frac{1}{4} \partial_u ((\underline{u}-u)^2 \mathbf{g}_{\bar{a}\bar{b}}) \right), \end{aligned}$$



as desired.

By definition of  $\eta$ , we have

$$\eta = -\frac{1}{2}\mathbf{g}^{\mu\nu}\partial_\mu u\partial_\nu u = -\frac{1}{2}\mathbf{g}^{uu} = -\frac{1}{2}(\mathbf{g}^{\tilde{u}\tilde{u}} - \eta^{\tilde{u}\tilde{u}}),$$

as desired.

Using (B.15) and (B.16), we have

$$\begin{aligned}\chi_{ab} &= 2\mathbf{g}(\mathbf{D}_{\partial_a}\partial_{\underline{u}}, \partial_b) + \mathbf{g}\left(\mathbf{D}_{\partial_a}\left(-(\mathbf{g}^{uv} - \eta^{uv})\partial_\nu - \frac{1}{2}(\mathbf{g}^{uu} - \eta^{uu})\partial_u\right), \partial_b\right) \\ &= 2\mathbf{g}(\mathbf{D}_{\partial_a}\partial_{\underline{u}}, \partial_b) - \mathbf{g}_{\nu b}\partial_a((\mathbf{g} - \eta)^{uv}) - \mathbf{g}_{\mu b}(\mathbf{g}^{uv} - \eta^{uv})\Gamma_{a\nu}^\mu(\mathbf{g}) - \frac{1}{4}(\mathbf{g}^{uu} - \eta^{uu})\chi_{ab},\end{aligned}$$

which by symmetrising in  $a, b$  gives

$$\begin{aligned}\chi_{ab} &= \mathbf{g}(\mathbf{D}_{\partial_a}\partial_{\underline{u}}, \partial_b) + \mathbf{g}(\mathbf{D}_{\partial_b}\partial_{\underline{u}}, \partial_a) \\ &\quad - \frac{1}{2}\mathbf{g}_{\nu b}\partial_a((\mathbf{g} - \eta)^{uv}) - \frac{1}{2}\mathbf{g}_{\nu a}\partial_b((\mathbf{g} - \eta)^{uv}) \\ &\quad - \frac{1}{2}\mathbf{g}_{\mu b}(\mathbf{g}^{uv} - \eta^{uv})\Gamma_{a\nu}^\mu(\mathbf{g}) - \frac{1}{2}\mathbf{g}_{\mu a}(\mathbf{g}^{uv} - \eta^{uv})\Gamma_{b\nu}^\mu(\mathbf{g}) \\ &\quad - \frac{1}{4}(\mathbf{g}^{uu} - \eta^{uu})\chi_{ab}.\end{aligned}$$

We further have

$$\begin{aligned}\partial_{\underline{u}}(\mathbf{g}_{ab}) &= \frac{2}{\underline{u} - u}\mathbf{g}_{ab} + \frac{1}{4}(\underline{u} - u)^2\partial_{\underline{u}}(\mathbf{g}_{\tilde{a}\tilde{b}} - \eta_{\tilde{a}\tilde{b}}), \\ -\frac{1}{2}\mathbf{g}_{\nu b}\partial_a((\mathbf{g} - \eta)^{uv}) &= -\frac{1}{4}(\underline{u} - u)\mathbf{g}_{\tilde{\nu}\tilde{b}}\partial_a(\mathbf{g}^{\tilde{u}\tilde{\nu}} - \eta^{\tilde{u}\tilde{\nu}}), \\ -\frac{1}{2}\mathbf{g}_{\mu b}(\mathbf{g}^{uv} - \eta^{uv})\Gamma_{a\nu}^\mu(\mathbf{g}) &= -\frac{1}{4}(\mathbf{g}^{uv} - \eta^{uv})(\partial_a\mathbf{g}_{\nu b} + \partial_\nu\mathbf{g}_{ab} - \partial_b\mathbf{g}_{a\nu}) \\ &= -\frac{1}{4}(\mathbf{g}^{\tilde{u}\tilde{u}} - \eta^{\tilde{u}\tilde{u}})\partial_u(\mathbf{g}_{ab}) \\ &\quad - \frac{1}{4}(\mathbf{g}^{\tilde{u}\tilde{u}} - \eta^{\tilde{u}\tilde{u}})\left(\frac{1}{2}\partial_a((\underline{u} - u)\mathbf{g}_{\tilde{u}\tilde{b}}) + \frac{1}{4}\partial_{\underline{u}}((\underline{u} - u)^2\mathbf{g}_{\tilde{a}\tilde{b}}) - \frac{1}{2}\partial_b((\underline{u} - u)\mathbf{g}_{\tilde{a}\tilde{u}})\right) \\ &\quad - \frac{1}{8}(\underline{u} - u)(\mathbf{g}^{\tilde{u}\tilde{c}} - \eta^{\tilde{u}\tilde{c}})(\partial_a\mathbf{g}_{\tilde{c}\tilde{b}} + \partial_c\mathbf{g}_{\tilde{a}\tilde{b}} - \partial_b\mathbf{g}_{\tilde{a}\tilde{c}}),\end{aligned}$$

and by symmetrising in  $a, b$ , the desired formula follows. This finishes the proof of the lemma.  $\square$

The following lemma directly follows from the limits from Lemma B.10 and from the formulas from Lemma B.15.

**Lemma B.16.** *We have*

$$\begin{aligned}\underline{\chi} + \frac{2}{\underline{u} - u}\mathcal{J}, \zeta, \chi - \frac{2}{\underline{u} - u}\mathcal{J} &= \mathcal{O}(|x|), \\ \eta &= \mathcal{O}(|x|^2),\end{aligned}\tag{B.17}$$

when  $|x| \rightarrow 0$ .

**Lemma B.17.** *Let  $F$  be an  $S_{u, \underline{u}}$ -tangent tensor such that*

$$F = \mathcal{O}(|x|^\gamma),$$

when  $|x| \rightarrow 0$ . Then, the following limits hold

$$\left|((\underline{u} - u)\nabla_3)^k, (\nabla_3 + \nabla_4)^l, ((\underline{u} - u)\nabla)^m F\right| = \mathcal{O}(|x|^\gamma),$$

when  $|x| \rightarrow 0$  and for all  $k, l, m \geq 0$ .

*Proof.* We have the following formulas relating covariant to partial derivatives

$$\begin{aligned}\nabla_a F_b &= \partial_a (F_b) - \Gamma_{ab}^c(\underline{g}) F_c, \\ (\underline{u} - u) \nabla_3 F_a &= 2(\underline{u} - u) \partial_u (F_a) - (\underline{u} - u) \underline{\chi}_a^b F_b, \\ (\nabla_3 + \nabla_4) F_a &= 2(\partial_u + \partial_{\underline{u}})(F_a) - (\chi + \underline{\chi})_a^b F_b \\ &\quad - (\mathbf{g}^{uv} - \boldsymbol{\eta}^{uv}) \partial_\nu (F_a) - \frac{1}{2} (\mathbf{g}^{\tilde{u}\tilde{u}} - \boldsymbol{\eta}^{\tilde{u}\tilde{u}}) \partial_u (F_a).\end{aligned}$$

Using (B.14), (B.17), we obtain that

$$(\underline{u} - u) \nabla F, (\underline{u} - u) \nabla_3 F, (\nabla_3 + \nabla_4) F = \mathcal{O}(|x|^\gamma),$$

when  $|x| \rightarrow 0$ , and the result follows by iteration.  $\square$

From the limits (B.17) and Lemma B.17, we deduce the following lemma.

**Lemma B.18.** *We have*

$$\begin{aligned}\left| \left( (\underline{u} - u) \nabla_3 \right)^k, (\nabla_3 + \nabla_4)^l, \left( (\underline{u} - u) \nabla \right)^m \left( \underline{\chi} + \frac{2}{\underline{u} - u} \underline{g}, \zeta, \chi - \frac{2}{\underline{u} - u} g \right) \right| &= \mathcal{O}(|x|), \\ \left| \left( (\underline{u} - u) \nabla_3 \right)^k, (\nabla_3 + \nabla_4)^l, \left( (\underline{u} - u) \nabla \right)^m \boldsymbol{\eta} \right| &= \mathcal{O}(|x|^2),\end{aligned}\tag{B.18}$$

and

$$\left| \left( (\underline{u} - u) \nabla_3 \right)^k, (\nabla_3 + \nabla_4)^l, \left( (\underline{u} - u) \nabla \right)^m (\omega, \xi) \right| = \mathcal{O}(|x|),\tag{B.19}$$

when  $|x| \rightarrow 0$ , and for all  $k, l, m \geq 0$ .

*Proof.* The limits (B.18) follow directly from (B.17) and Lemma B.17. The limits (B.19) are obtained from the limits for  $\boldsymbol{\eta}$  and the relations of Lemma B.15.  $\square$

We also have the following limits for the null decomposition of the spacetime curvature tensor.

**Lemma B.19.** *We have*

$$\left| \left( (\underline{u} - u) \nabla_3 \right)^k, (\nabla_3 + \nabla_4)^l, \left( (\underline{u} - u) \nabla \right)^m (\alpha, \beta, \rho, \sigma, \underline{\beta}, \underline{\alpha}) \right| = \mathcal{O}(1),\tag{B.20}$$

when  $|x| \rightarrow 0$ , for all  $k, l, m \geq 0$ .

*Proof.* By definition of the null decomposition of  $\mathbf{R}$  and using the expressions (B.15) for  $e_4, e_3$  in terms of the coordinate vectorfields, we have

$$\begin{aligned}\alpha_{ab} &= \mathbf{R}(e_4, \partial_a, e_4, \partial_b) \\ &= \mathbf{g}^{uv} \mathbf{g}^{u'v'} R_{\nu a \nu' b} + \frac{1}{2} \mathbf{g}^{uv} R_{\nu a u b} + \frac{1}{2} \mathbf{g}^{u'v'} R_{u a \nu' b} + \frac{1}{4} \mathbf{g}^{uu} R_{u a u b}.\end{aligned}$$

We therefore deduce from the limits (B.14) for  $\mathbf{g}$  and  $\mathbf{R}$  that

$$\alpha = \mathcal{O}(1),$$

and using the result of Lemma B.17 the result follows. The other components of the null decomposition of  $\mathbf{R}$  are treated similarly.  $\square$

**Remark B.20.** *Using (B.13), one can replace  $(\underline{u} - u)/2$  by the area radius  $r$  of the 2-spheres  $S_{u, \underline{u}}$  in the limits (B.18), (B.19) and (B.20).*



## C. The canonical foliation

In this section we provide a proof Theorem 4.5, which goes along the same lines as the local existence result for the canonical foliation in [CG19a, Section 6]. We only resketch the argument, pointing out the slight modifications which are due to the difference of definition and to the vertex setting.

By translation invariance, we assume that  $\underline{u}^* = 0$  and we note  $z^\mu$  the normal coordinates at  $\varnothing(0)$  such that  $z^\mu|_{\varnothing(0)} = 0$  for all  $\mu = 0, 1, 2, 3$  and  $\partial_{z^0} = \dot{\varnothing}(0)$ . The normal coordinates are well defined in a coordinate ball of radius  $\varepsilon > 0$  and for all fixed integer  $k \geq 5$ , we have

$$|\partial^{\leq k+1}(\mathbf{g}_{\mu\nu} - \boldsymbol{\eta}_{\mu\nu})| \lesssim 1. \quad (\text{C.1})$$

We denote by  $s := z^0 - \sqrt{\sum_{i=1}^3 (z^i)^2}$  the geodesic affine parameter on  $\underline{\mathcal{C}}^* = \{z^0 + \sqrt{\sum_{i=1}^3 (z^i)^2} = 0\}$  and by  $(e_3', e_4')$  its associated null pair (such that  $e_3'(s) = 2$ ).

Our goal is to obtain a function  $(u, \omega) \in [0, \delta] \times \mathbb{S}^2 \mapsto (s(u, \omega), \omega) \in \underline{\mathcal{C}}^*$  such that  $\{s = s(u)\}$  defines the level sets of the canonical foliation of Theorem 4.5 on  $\underline{\mathcal{C}}^*$  (see the detailed geometric set up of [CG19a, Section 6]).

For  $u$  a function on  $\underline{\mathcal{C}}^*$ , we define the null lapse  $\Omega$  as

$$\Omega^{-1} := \partial_s u = \frac{1}{2} e_3'(u),$$

Assume that  $u$  defines a regular foliation on  $\underline{\mathcal{C}}^*$  and let  $(\lambda, \underline{f}, \underline{f})$  be the transition coefficients between the null frames  $(e_3, e_4)$  and  $(e_3', e_4')$  as defined in Section 2.15. From the relations of Lemma 2.31, we have

$$\lambda = \Omega, \quad \underline{f} = -\Omega^{-1} \nabla s, \quad \underline{f} = 0, \quad (\text{C.2})$$

where we recall that  $\lambda, \underline{f}, \underline{f}$  are  $S_u$ -tangent tensors. From a simple exact recomputation of the relations of Proposition 2.35, we have

$$\begin{aligned} \zeta_a &= \frac{1}{2} \mathbf{g}(\mathbf{D}_{e_a} e_4, e_3) \\ &= \frac{1}{2} \Omega \mathbf{g}(\mathbf{D}_{e_a} (\Omega^{-1} e_4' - \underline{f} - \frac{1}{4} |f|^2 e_3), e_3') \\ &= \zeta'_a - \Omega e_a(\Omega^{-1}) - \frac{1}{2} \Omega \mathbf{g}(\mathbf{D}_{e_a} \underline{f}, e_3') \\ &= (\zeta')_a^\dagger + \nabla_a(\log \Omega) + \frac{1}{2} \underline{f} \cdot \underline{\chi}, \end{aligned} \quad (\text{C.3})$$

and

$$\underline{\omega} = -2e_3(\log \Omega). \quad (\text{C.4})$$

Thus, we rewrite equations

$$d\sharp v \zeta + \rho = \bar{\rho}, \quad \bar{\omega} = 0,$$

defining the canonical foliation as

$$\begin{aligned} \Delta(\log \Omega) &= -d\sharp v (\zeta')^\dagger + (\bar{\rho} - \rho) - \frac{1}{2} \nabla \underline{f} \cdot \underline{\chi}, \\ \int_{S_u} \partial_u(\log \Omega) &= 0. \end{aligned} \quad (\text{C.5})$$

Using relations (C.2) and the relations of Proposition 2.34, we further rewrite the first equation of (C.5) as (see also [CG19a, Section 5])

$$\Delta_{g(s)}(\log \Omega) = F(s, \nabla s, \nabla^2 s),$$

with

$$F(s, \nabla s, \nabla^2 s) = F_1(s) + F_2(s) \cdot \nabla s + F_3(s) \cdot \nabla s \cdot \nabla s + F_4(s) \cdot \nabla^2 s,$$

for which using (C.1), we claim that

$$\begin{aligned} |(s\nabla)^{\leq k} F_1(s)| &\lesssim 1, & |(s\nabla)^{\leq k} F_2(s)| &\lesssim s^{-2}, \\ |(s\nabla)^{\leq k} F_3(s)| &\lesssim s^{-4}, & |(s\nabla)^{\leq k} F_4(s)| &\lesssim s^{-1}. \end{aligned} \quad (\text{C.6})$$

Using the initial conditions (4.10) for the canonical foliation, and the definition of the geodesic foliation  $s$ , we have the initial conditions at the vertex  $\phi(0)$

$$\Omega = 1, \quad s = u = 0.$$

Thus using additionally (2.39), equations (C.5) finally rewrite as the following system of quasilinear elliptic-transport equations

$$\begin{aligned} s &= \int_0^u \Omega \, du', \\ \Delta_{g(s)}(\log \Omega) &= F(s, \nabla s, \nabla^2 s), \\ \overline{\log \Omega}(u) &= \int_0^u \overline{(\text{tr} \underline{\chi} - \overline{\text{tr} \underline{\chi}})} \log \Omega(u'), \, du'. \end{aligned} \quad (\text{C.7})$$

To prove local existence for this system, we apply a Banach-Picard fixed point argument as in [CG19a] by defining similarly

$$s_0 = u, \quad \Omega_0 = 1,$$

and the iteration

$$\begin{aligned} s_{n+1} &:= \int_0^u \Omega_n \, du', \\ \Delta_{n+1}(\log \Omega_{n+1}) &:= F(s_{n+1}, \nabla s_{n+1}, \nabla^2 s_{n+1}), \\ \overline{\log \Omega_{n+1}} &:= \int_0^u \overline{(\text{tr} \underline{\chi}_n - \overline{\text{tr} \underline{\chi}_n})} \log \Omega_n(u'), \, du'. \end{aligned}$$

We leave to the reader to check that for  $\delta > 0$  sufficiently small, using the bounds (C.6), the following boundedness holds (see the argument for non-weighted norms in [CG19a, Section 6])

$$\forall u \in (0, \delta), \quad 0 < s_n < \varepsilon,$$

and

$$\|u^{-4}(u\nabla)^{\leq k}(s_n - u)\|_{L_u^\infty([0, \delta])L^2(S_u)} + \|u^{-3}(u\nabla)^{\leq k} \log \Omega_n\|_{L_u^\infty([0, \delta])L^2(S_u)} \lesssim 1,$$

and the following contraction holds

$$\begin{aligned} &\|u^{-4}(u\nabla)^{\leq k}(s_{n+1} - s_n)\|_{L_u^\infty([0, \delta])L^2(S_u)} + \|u^{-3}(u\nabla)^{\leq k}(\log \Omega_{n+1} - \log \Omega_n)\|_{L_u^\infty([0, \delta])L^2(S_u)} \\ &\leq \kappa \left( \|u^{-4}(u\nabla)^{\leq k}(s_n - s_{n-1})\|_{L_u^\infty([0, \delta])L^2(S_u)} + \|u^{-3}(u\nabla)^{\leq k}(\log \Omega_n - \log \Omega_{n-1})\|_{L_u^\infty([0, \delta])L^2(S_u)} \right), \end{aligned}$$

with  $\kappa < 1$ . The limit defines the desired canonical foliation, and moreover we have

$$\|u^{-4}(u\nabla)^{\leq k}(s - u)\|_{L_u^\infty([0, \delta])L^2(S_u)} + \|u^{-3}(u\nabla)^{\leq k} \log \Omega\|_{L_u^\infty([0, \delta])L^2(S_u)} < \infty. \quad (\text{C.8})$$

For all  $k \geq 5$ , estimates (C.8) holds true at the vertex, and thus from Sobolev embeddings we deduce that

$$|(u\nabla)^k s| = O(u^3), \quad |(u\nabla)^k \log \Omega| = O(u^2),$$

when  $u \rightarrow 0$  and for all  $k \geq 0$ . Commuting equation (C.7) by  $\nabla_3$ , one can further obtain

$$|(u\nabla)^{\leq k}(u\nabla_3)^l s| = O(u^3), \quad |(u\nabla)^{\leq k}(u\nabla_3)^l \log \Omega| = O(u^2), \quad (\text{C.9})$$

for all  $k, l \geq 0$ . Using the relations of Lemma 2.35 between null connection coefficients, relations (C.2) and the limits (C.9), we deduce that the limits of Theorem 4.4 also hold for the null connection coefficients of the canonical foliation. The spherical coordinates  $(s, \vartheta, \varphi)$  for the geodesic foliation from Theorem 4.4 also generate spherical coordinates  $(u, \vartheta, \varphi)$  for the canonical foliation. Moreover, since the coordinate vectorfields for  $\vartheta, \varphi$  only differ by null vectors, we have  $g'_{ab} = g_{ab}$ , where  $a, b \in \{\vartheta, \varphi\}$ . Thus the limits for the induced metric from Theorem 4.4 also holds for the canonical foliation. This finishes the proof of Theorem 4.5.

## D. Klainerman-Sobolev estimates

This section is dedicated to the proof of the Klainerman-Sobolev  $\tilde{H}^{1/2}$  estimates of Lemmas 6.6, 6.7, 8.4 and 9.6. In all cases we only treat the degenerate versions involving respectively  $u$  and  $q$  weights. The non-degenerate cases follow similarly, or by rescaling of standard  $H^1$  to  $H^{1/2}$  trace estimate.

We first prove the following Klainerman-Sobolev  $H^{1/2}$  estimates in the Euclidean case. The proofs of Lemmas 6.6, 6.7, 8.4 and 9.6 are deduced from Lemma D.1 at the end of this section.

**Lemma D.1** (Euclidean Klainerman-Sobolev  $H^{1/2}$  estimate). *Let  $S_r$  denote the Euclidean sphere of radius  $r$ , and let  $\tilde{H}^{1/2}(S_r)$  denote the scaling homogeneous fractional Sobolev space on  $S_r$ , Euclidean analogous to the definition of  $\tilde{H}^{1/2}$  from Definition 3.3. Let  $q(r)$  be a scalar function of  $r$  such that  $|q| \lesssim r$  and  $|\partial_r q| \lesssim 1$ . For all  $r, r' > 0$  and for all scalar function  $f$ , we have*

$$\left\| r_2^{1/2} q(r_2)^{1/2} f \right\|_{\tilde{H}^{1/2}(S_{r_2})} \lesssim \left\| (r_1)^{1/2} q(r_1)^{1/2} f \right\|_{\tilde{H}^{1/2}(S_{r_1})} + \|q(r)\partial_r f\|_{L^2(A_{r_1, r_2})} + \|(r\nabla)^{\leq 1} f\|_{L^2(A_{r_1, r_2})},$$

where  $A_{r_1, r_2}$  denotes the annulus region comprised between  $S_{r_1}$  and  $S_{r_2}$ .

*Proof.* For simplicity, we use the following characterisation of the fractional Sobolev space  $\tilde{H}^{1/2}(S_r)$

$$\left( r^{-1/2} \|f\|_{\tilde{H}^{1/2}(S_r)} \right)^2 \simeq \int \left| (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} f \right|^2 d\vartheta d\varphi. \quad (\text{D.1})$$

Deriving (D.1) in  $r$  and integrating from  $r_1$  to  $r_2$ , we infer

$$\begin{aligned} \left( r_2^{-1/2} \left\| r_2 q(r_2)^{1/2} f \right\|_{\tilde{H}^{1/2}(S_{r_2})} \right)^2 &\lesssim \left( (r_1)^{-1/2} \left\| r_1 q(r_1)^{1/2} f \right\|_{\tilde{H}^{1/2}(S_{r_1})} \right)^2 \\ &\quad + \left| \int_{r_1}^{r_2} \int (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} (r q^{1/2} f) (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} (\partial_r (r q^{1/2} f)) dr d\vartheta d\varphi \right|. \end{aligned}$$

Using the assumptions  $|q| \lesssim r$  and  $|\partial_r q| \lesssim 1$ , integrating by part and using Cauchy-Schwartz, we have

$$\begin{aligned} &\left| \int_{r_1}^{r_2} \int \left| (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} f \right|^2 (r q^{1/2} \partial_r (r q^{1/2})) dr d\vartheta d\varphi \right| \\ &\lesssim \left| \int_{r_1}^{r_2} \int \left| (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} f \right|^2 r^2 dr d\vartheta d\varphi \right| \\ &\lesssim \|(r\nabla)^{\leq 1} f\|_{L^2(A_{r_1, r_2})} \|f\|_{L^2(A_{r_1, r_2})}. \end{aligned}$$

Integrating by part and applying Cauchy-Schwartz, we also have

$$\begin{aligned} &\left| \int_{r_1}^{r_2} \int (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} (r q^{1/2} f) (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/4} (r q^{1/2} \partial_r f) dr d\vartheta d\varphi \right| \\ &\lesssim \left| \int_{r_1}^{r_2} \int \left( (\text{Id} - \mathbb{A}_{\vartheta, \varphi})^{1/2} f \right) (q \partial_r f) r^2 dr d\vartheta d\varphi \right| \\ &\lesssim \|(r\nabla)^{\leq 1} f\|_{L^2(A_{r_1, r_2})} \|q \partial_r f\|_{L^2(A_{r_1, r_2})}. \end{aligned}$$

Combining the above estimates finishes the proof of the lemma.  $\square$

*Proof of Lemma 6.7.* Let  $F$  be an  $S$ -tangent 1-tensor in  $\mathcal{M}^{\text{ext}}$ . Using the comparison between fractional Sobolev spaces of Lemma 3.35, we have for all 2-spheres  $S_{u, \underline{u}} \subset \Sigma_t^{\text{ext}}$

$$\left\| \underline{u}^{1/2} u^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} \simeq \sum_{(\vartheta, \varphi)} \sum_{a \in \{\vartheta, \varphi\}} \left\| r u^{1/2} (r^{-1} F)_a \right\|_{H_{\vartheta, \varphi}^{1/2}} \simeq \sum_{a \in \{\vartheta, \varphi\}} \left\| r^{1/2} q(r)^{1/2} (r^{-1} F)_a \right\|_{\tilde{H}^{1/2}(S_{r=(\underline{u}-u)/2})},$$

where here  $q(r) = u$ . Using the Klainerman-Sobolev Euclidean estimates of Lemma D.1 between the 2-spheres  $S_{u, \underline{u}}$  and  $S_{u+\underline{u}-\underline{u}^*, \underline{u}^*}$  in coordinates, we thus infer

$$\begin{aligned} \left\| \underline{u}^{1/2} u^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u, \underline{u}})} &\lesssim \left\| \underline{u}^{*1/2} u^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u+\underline{u}-\underline{u}^*, \underline{u}^*})} \\ &\quad + \sum_{a \in \{\vartheta, \varphi\}} \left( \|u(\partial_{\underline{u}} - \partial_u)(r^{-1} F)_a\|_{L^2(\Sigma_t^{\text{ext}})} + \|(r\nabla)^{\leq 1} (r^{-1} F)_a\|_{L^2(\Sigma_t^{\text{ext}})} \right). \end{aligned} \quad (\text{D.2})$$

We have  $\partial_{\underline{u}} - \partial_u = \frac{1}{2}(1 + \frac{1}{2}\eta)e_4 - \frac{1}{2}e_3$ . Therefore, using the bootstrap bound  $|\eta| \lesssim D\varepsilon$ , we infer

$$\begin{aligned} |(\partial_{\underline{u}} - \partial_u)(F_a)| &\lesssim |\nabla_4 F_a| + |\nabla_3 F_a| + |F \cdot \chi_a| + |F \cdot \underline{\chi}_a| \\ &\lesssim r (|\nabla_4 F| + |\nabla_3 F| + r^{-1}|F|). \end{aligned} \quad (\text{D.3})$$

Similarly, we have

$$|(r\nabla)(F_a)| \lesssim |(r\nabla)F_a| + |F \cdot \Gamma_a|,$$

where  $\Gamma$  are the Christoffel symbols for the metric  $g$  in coordinates  $\vartheta, \varphi$ . Using the Bootstrap Assumptions 3.26, we have  $|\Gamma| \lesssim 1$ , thus

$$|(r\nabla)(F_a)| \lesssim r|(r\nabla)^{\leq 1}F|. \quad (\text{D.4})$$

Combining (D.2), (D.3) and (D.4), and that  $r^{-1}u \lesssim 1$  in  $\mathcal{M}^{\text{ext}}$ , this concludes the proof of Lemma 6.7.  $\square$

*Proof of Lemmas 8.4 and 9.6.* Arguing similarly as in the proof of Lemma 6.7, we obtain for all 2-spheres  $S_{u_1, \underline{u}^*}$  and  $S_{u_2, \underline{u}^*}$  of  $\mathcal{C}^*$

$$\begin{aligned} \left\| r^{1/2} q^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u_2, \underline{u}^*})} &\lesssim \left\| r^{1/2} q^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u_1, \underline{u}^*})} + \|r^{-1}qF\|_{L^2(\mathcal{C}^*)} \\ &\quad + \|(r\nabla)F\|_{L^2(\mathcal{C}^*)} + \|q\nabla_3 F\|_{L^2(\mathcal{C}^*)}, \end{aligned} \quad (\text{D.5})$$

for  $q = \min(u, r)$ . Using the estimates of Lemma 3.36 we have

$$\begin{aligned} \left\| r^{1/2} q^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u_1, \underline{u}^*})} &\lesssim r^{-1/2} \left\| r^{1/2} q^{1/2} (r\nabla)^{\leq 1} F \right\|_{L^2(S_{u_1, \underline{u}^*})} \\ &\lesssim r q^{1/2} \left\| (r\nabla)^{\leq 1} F \right\|_{L^\infty(S_{u_1, \underline{u}^*})}, \end{aligned}$$

which by assumption tends to 0 when  $u_1 \rightarrow \underline{u}^*$ . Passing to the limit in (D.5), using additionally that  $r^{-1}q \lesssim 1$  on  $\mathcal{C}^*$ , this finishes the proof of Lemma 9.6. The proof of Lemma 8.4 is obtained along the same lines, restricting the argument to the top region  $\mathcal{C}^* \cap \mathcal{M}_{\text{bot}}^{\text{int}}$ .  $\square$

*Proof of Lemma 6.6.* Arguing as in the proof of Lemma 6.7, we obtain for all 2-spheres  $S_{u_1, \underline{u}^*}$  and  $S_{u_2, \underline{u}^*}$  of  $\mathcal{C}^* \cap \mathcal{M}^{\text{ext}}$

$$\begin{aligned} \left\| r^{1/2} u^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u_2, \underline{u}^*})} &\lesssim \left\| r^{1/2} u^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{u_1, \underline{u}^*})} + \|r^{-1}uF\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} \\ &\quad + \|(r\nabla)F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|u\nabla_3 F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})}, \end{aligned} \quad (\text{D.6})$$

for all  $S$ -tangent tensor  $F$ .

Let  $\phi : [0, 1] \rightarrow [0, 1]$  be a standard cut-off function such that  $\phi|_{[0, 1/4]} = 0$  and  $\phi|_{[3/4, 1]} = 1$  and let  $\tilde{\phi}(u) := \phi\left(\frac{u-1}{\tau\underline{u}^*-1}\right)$ . Applying (D.6) to  $\tilde{F}$ , where  $\tilde{F} := \tilde{\phi}F$ , with  $u_2 = \tau\underline{u}^*$  and  $u_1 = 1$ , we obtain

$$\left\| r^{1/2} u^{1/2} F \right\|_{\tilde{H}^{1/2}(S_{\tau\underline{u}^*, \underline{u}^*})} \lesssim \|r^{-1}uF\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|(r\nabla)F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} + \|u\nabla_3 F\|_{L^2(\mathcal{C}^* \cap \mathcal{M}^{\text{ext}})} \quad (\text{D.7})$$

where we used that  $|\tilde{\phi}| \lesssim 1$  and  $|u\nabla_3 \tilde{\phi}| \lesssim 1$ .<sup>1</sup> Using (D.6) with  $u_2 = u$  and  $u_1 = \tau\underline{u}^*$  and using (D.7), we obtain the desired result. This finishes the proof of Lemma 6.6.  $\square$

<sup>1</sup>The same argument holds for the non-degenerate estimate since we also have  $|r\nabla_3 \tilde{\phi}| \lesssim 1$ .

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## Le problème d'évolution en relativité générale

### Résumé

Cette thèse est consacrée à l'étude du problème de Cauchy pour les équations d'Einstein dans le vide de la relativité générale. On s'intéresse plus particulièrement à l'étude locale et globale en temps des solutions pour des données initiales prescrites sur une hypersurface de genre espace et une hypersurface de genre lumière/caractéristique. Nous obtenons pour ce problème de Cauchy spatial-caractéristique une généralisation du théorème de courbure  $L^2$  de Klainerman-Rodnianski-Szeftel. Nous obtenons également une généralisation du théorème de la stabilité asymptotique non-linéaire de l'espace-temps de Minkowski de Christodoulou-Klainerman. Le point commun entre ces deux généralisations est l'introduction de nouveaux choix de jauge, consistant en des feuilletages de l'espace-temps adaptés au problème de Cauchy spatial-caractéristique. Ceux-ci permettent de localiser et d'appliquer (en boîte noire ou à quelques modifications près) les théorèmes originaux correspondant à nos résultats. En particulier, nous introduisons ou généralisons l'étude de feuilletages par des cônes de lumière à sommets ou sections sphériques prescrits, par des hypersurfaces spatiales maximales à bord prescrits, ainsi que l'étude de coordonnées (canoniques, géodésiques ou harmoniques) sur ces hypersurfaces. Ces choix de jauge et l'analyse des équations d'Einstein sous ces conditions constituent le point central de cette thèse.

**Mots clés:** *Relativité générale, problème de Cauchy, espace de Minkowski, faible régularité, stabilité asymptotique, hypersurfaces maximales et caractéristiques.*

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## The evolution problem in general relativity

### Abstract

This thesis is devoted to study the Cauchy problem for Einstein vacuum equations of general relativity. More precisely, we investigate local and global existence of solutions when initial data are prescribed on a spacelike and on a characteristic/null hypersurface. For this spacelike-characteristic Cauchy problem, we obtain a generalisation of the bounded  $L^2$  curvature theorem of Klainerman-Rodnianski-Szeftel. We also obtain a generalisation of the global nonlinear stability of Minkowski space of Christodoulou-Klainerman. The common feature to these two generalisations is the introduction of new gauge choices which consist in spacetime foliations adapted to the spacelike-characteristic setting. This enables to apply (as a black box or up to some modification) the original theorems corresponding to our results. In particular, we introduce or generalise the study of foliations by null cones with prescribed vertices or prescribed spherical sections, maximal spacelike hypersurfaces with prescribed boundaries, as well as the study of coordinates (canonical, geodesic or harmonic) on these hypersurfaces. These gauge choices and the analysis of the Einstein equations under these conditions are the centre of this thesis.

**Keywords:** *General relativity, Cauchy problem, Minkowski space, low regularity, asymptotic stability, maximal and characteristic hypersurfaces.*